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Analyse d'Algorithmes Stochastiques Appliqués à la Finance

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À mes deux Bons Papas

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Résumé

Cette thèse porte sur l'analyse d'algorithmes stochastiques et leur application en Finance notamment et est composée de deux parties.

Dans la première partie, nous présentons un résultat de convergence pour des algorithmes stochastiques où les innovations vérifient une hypothèse de moyennisation avec une certaine vitesse. Nous l'appliquons ensuite à différents types d'innovations (suites i.i.d., suites à discrépance faible, chaînes de Markov homogènes, fonctionnelles de processus α-mélangeant) et nous l'illustrons à l'aide d'exemples motivés principalement par la Finance. Nous établissons ensuite un résultat de vitesse "universelle" de convergence dans le cadre d'innovations équiréparties dans $[0,1]^q$ et nous confrontons nos résultats à ceux obtenus dans le cadre i.i.d..

La seconde partie est consacrée aux applications. Nous présentons d'abord un problème d'allocation optimale appliqué au cas d'un nouveau type de place de trading : les *dark pools*. Ces places proposent un prix d'achat (ou de vente) certain, mais n'assurent pas le volume délivré. Le but est alors d'exécuter le maximum de la quantité souhaitée sur ces places. Ceci mène à la construction d'un algorithme stochastique sous contraintes à l'aide du Lagrangien que nous étudions dans les cadres d'innovations i.i.d. et moyennisantes. Le chapitre suivant présente un algorithme d'optimisation pour trouver la meilleure distance de placement d'ordres limites : il s'agit de minimiser le coût d'exécution d'une quantité donnée. Ceci mène à la construction d'un algorithme stochastique sous contraintes avec projection. Pour assurer l'existence et l'unicité de l'équilibre, des critères suffisants sur certains paramètres du modèle sont obtenus à l'aide d'un principe de monotonie opposée pour les diffusions unidimensionnelles.

Le chapitre suivant porte sur l'implicitation et la calibration de paramètres dans des modèles financiers. La première technique mène à un algorithme de recherche de zéro et la seconde à une méthode de gradient stochastique. Nous illustrons ces deux techniques par des exemples d'applications sur 3 modèles : le modèle de Black-Scholes, le modèle de Merton et le modèle pseudo-CEV. Enfin le dernier chapitre porte sur l'application des algorithmes stochastiques dans le cadre de modèles d'urnes aléatoires utilisés en essais cliniques. A l'aide des méthodes de l'EDO et de l'EDS, nous retrouvons les résultats de consistance (convergence $p.s.$) et de normalité asymptotique (TCL) de Bai et Hu mais sous des hypothèses plus faibles sur les matrices génératrices. Nous étudions aussi un modèle "multi-bras" pour lequel nous retrouvons le résultat de convergence $p.s.$ et nous montrons un nouveau résultat de normalité asymptotique par simple application du TCL pour les algorithmes stochastiques.

Mots-clés. Approximation stochastique, suites à discrépance faible, processus α-mélangeants, allocation d'actifs, ordres limites, carnet d'ordres, principe de co-monotonie, implicitation, calibration, urnes de Pòlya étendues, essais cliniques multibras.

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Introduction

Cette thèse porte sur l'étude d'algorithmes stochastiques et leur application en Finance notamment. Dans une première partie, nous donnons les aspects théoriques de ces procédures récursives stochastiques et dans la seconde nous appliquons cette théorie à divers problèmes rencontrés en Finance mais aussi à des modèles d'urnes aléatoires utilisés en essais cliniques.

0.1 Présentation

Dans la première partie, nous commençons par un chapitre introductif qui rappelle les principaux résultats "classiques" de la théorie de l'approximation stochastique en nous cantonnant cependant pour l'essentiel à un cadre markovien ou à innovations i.i.d.. Ces résultats nous serons utiles dans la seconde partie qui porte sur les applications, mais nous permettent aussi de faire un parallèle avec le cadre plus général présenté dans le second chapitre, où les innovations de l'algorithme mêlent des composantes aléatoires et déterministes.

Un algorithme stochastique est une procédure récursive de recherche de zéro d'une *fonction moyenne* admettant un représentation sous forme d'espérance. Son analogue déterministe est l'algorithme de Newton. Plus précisément le but d'un tel algorithme est de trouver

$$\theta^* \in \mathbb{R}^d \quad \text{tel que} \quad h(\theta^*) = 0 \quad \text{où} \quad h(\theta) = \int_{\mathbb{R}^q} H(\theta, y)\nu(dy),$$

où ν est une mesure de probabilité (simulable) et $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ est une fonction borélienne. Ceci nous mène donc à construire la procédure récursive suivante

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H\left(\theta_n, Y_{n+1}\right), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}^d, \tag{0.1}$$

où $(\gamma_n)_{n\geq 1}$ est une suite positive appelée *pas* de l'algorithme, $(Y_n)_{n\geq 1}$ est une suite de variables aléatoires i.i.d. de loi ν, indépendante de la variable initiale θ_0, appelée *innovations*. L'étude des algorithmes stochastiques a commencé dans les années 1950 avec les travaux de Robbins-Monro [103] et de Kiefer-Wolfowitz [72] (variante introduisant une méthode de différences finies à pas décroissant pour l'approximation d'une fonction moyenne de type gradient). Elle a fait l'objet de nombreux travaux liés au contrôle adaptatif (voir [74], [75]), aux estimations recursives (voir [77], [87]), etc. Des généralisations à des suites d'innovations non i.i.d. ont également été étudiées de façon approfondie : chaînes de Markov (voir [25]), processus mélangeants (voir [39]), suites à discrépances faibles (voir [81]), et de nombreuses applications dans des domaines variés ont également été considérées (physique, théorie des jeux, statistique par exemple).

Remarquons que la procédure récursive définie par (0.1) peut se réécrire sous la forme

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\left(h(\theta_n) + \Delta M_{n+1}\right), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}^d, \tag{0.2}$$

où $\Delta M_{n+1} = H\left(\theta_n, Y_{n+1}\right) - h(\theta_n)$ est un $\mathcal{F}_n$-accroissement de martingale locale (qui se révèle être une vraie martingale sous des hypothèses naturelles sur H) avec $\mathcal{F}_n := \sigma(\theta_0, Y_1, \ldots, Y_n)$ la filtration

associée à l'algorithme. Lorsque les innovations ne sont pas supposées i.i.d., la procédure récursive (0.2) peut se réécrire

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \left(h(\theta_n) + \Delta M_{n+1} + r_{n+1} \right), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}^d, \tag{0.3}$$

où $\Delta M_{n+1} = H\left(\theta_n, Y_{n+1}\right) - \mathbb{E}\left[H\left(\theta_n, Y_{n+1}\right) \mid \mathcal{F}_n\right]$ est un $\mathcal{F}_n$-accroissement de martingale locale et $r_{n+1} = \mathbb{E}\left[H\left(\theta_n, Y_{n+1}\right) \mid \mathcal{F}_n\right] - h(\theta_n)$ est un terme de reste $\mathcal{F}_{n+1}$-adapté. Dans le Chapitre 1, nous rappelons d'abord les résultats de convergence des procédures déterministes, puis ceux du cadre stochastique. Il existe deux méthodes pour montrer la convergence $p.s.$ de l'algorithme : l'une basée sur une approche martingale via le Lemme de Robbins-Siegmund (utile pour faire le parallèle avec le second chapitre) et l'autre fondée sur la méthode de l'EDO qui relie l'algorithme (0.2) au comportement asymptotique de l'équation différentielle $\dot{\theta} = -h(\theta)$. Un résultat de convergence pour les algorithmes contraints est aussi mentionné. Nous rappelons ensuite le résultat de vitesse de convergence, à savoir le TCL, obtenu par la méthode de l'EDS et clôturons ce chapitre par le principe de moyennisation de Ruppert et Polyak qui permet d'obtenir la variance asymptotique minimale.

Le reste de cette thèse est organisé comme suit : Chapitre 2, nous présentons un résultat de convergence pour des algorithmes stochastiques où les innovations vérifient une hypothèses de moyennisation avec une certaine vitesse. Nous l'appliquons ensuite à différents types d'innovations (suites i.i.d., suites à discrépance faible, chaînes de Markov homogènes, fonctionnelles de processus α-mélangeant) et nous l'illustrons à l'aide d'exemples motivés principalement par la Finance.

Chapitre 3 nous présentons un résultat de vitesse "universelle" de convergence dans le cadre d'innovations équiréparties dans $[0,1]^q$. Nous confrontons nos résultats à ceux obtenus dans le cadre i.i.d. et nous les illustrons à l'aide d'un exemple pour des suites i.i.d. et à discrépance faible.

Le Chapitre 4 présente un problème d'allocation optimale appliqué au cas d'un nouveau type de place de trading : les *dark pools*. Ces places proposent un prix d'achat (ou de vente) certain, mais n'assurent pas le volume délivré. Le but est alors d'exécuter le maximum de la quantité souhaitée sur ces places. Ceci mène à la construction d'un algorithme stochastique sous contraintes à l'aide du Lagrangien. Nous l'étudions ensuite dans deux cadres : celui d'innovations i.i.d. et celui où elles sont moyennisantes. Nous présentons ensuite une procédure alternative basée sur un principe de renforcement. Enfin les performances des deux algorithmes sont comparées dans différents cadres (i.i.d., α-mélangeant et données réelles) mais aussi à celles d'un agent initié.

Le Chapitre 5 présente un algorithme d'optimisation pour trouver la meilleure distance de placement dans un carnet d'ordre : il s'agit de minimiser le coût d'exécution d'une quantité donnée. Ceci mène à la construction d'un algorithme stochastique sous contraintes avec projection. Pour assurer l'existence et l'unicité de l'équilibre, nous avons besoin d'hypothèses sur la fonction de coût, ce qui aboutit à des critères suffisants sur certains paramètres du modèle obtenus en utilisant un principe de monotonie opposée pour les diffusions unidimensionnelles.

Le Chapitre 6 porte sur l'implicitation et la calibration de paramètres dans des modèles financiers. La première technique mène à un algorithme de recherche de zéro et la seconde à une méthode de gradient stochastique. Pour cette dernière, il existe plusieurs approches (à l'aide du processus tangent ou par différences finies) que nous présentons. Nous illustrons ces deux techniques par des exemples d'applications sur 3 modèles : le modèle de Black-Scholes, le modèle de Merton et le modèle pseudo-CEV.

Le Chapitre 7 porte sur l'application des algorithmes stochastiques dans le cadre de modèles d'urnes aléatoires utilisés en essais cliniques. A l'aide des méthodes de l'EDO et de l'EDS, nous retrouvons les résultats de consistance (convergence $p.s.$) et de normalité asymptotique (TCL) de [18] mais sous des hypothèses plus faibles sur les matrices génératrices. Nous étudions aussi les modèles "multi-bras", notamment le modèle introduit dans [19], pour lequel nous retrouvons le résultat de convergence $p.s.$ et nous montrons un nouveau résultat de normalité asymptotique par simple

application du TCL pour les algorithmes stochastiques.

Les principaux résultats de cette thèse sont les suivants

0.2 Algorithmes stochastiques avec innovations moyennisantes

Ce chapitre est basé sur un article accepté dans *Monte Carlo Methods and Applications*.

Le but de ce chapitre est d'établir un théorème de convergence pour des algorithmes stochastiques multidimensionnels quand les "innovations" vérifient des hypothèses de moyennisation "faibles" en présence d'une fonction de Lyapunov trajectorielle. Ces hypothèses de moyennisation nous permettent d'unifier plusieurs cadres : celui où les innovations sont simulées (possiblement déterministes comme pour les nombres quasi-aléatoires aussi connus sous le noms de suites à discrépance faible ou QMC) et celui où elles sont exogènes (comme des données de marché) avec des propriétés (de vitesse) d'ergodicité. Nous proposons plusieurs champs d'applications en termes d'innovations, à savoir des suites i.i.d., des nombres quasi-aléatoires, des fonctionnelles de processus α-mélangeant ou des chaînes de Markov homogènes.

0.2.1 Description du cadre

Dans ce chapitre, on considère un cadre général pour des algorithmes stochastiques de la forme

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\left(H(\theta_n, Y_n) + \Delta M_{n+1}\right), \quad n \geq 0, \tag{0.4}$$

où θ_0 est une variable aléatoire à valeurs dans $\mathbb{R}^d$, $(Y_n)_{n\geq 0}$ est une suite de variables aléatoires à valeurs dans $\mathbb{R}^q$ et ΔM_{n+1} est un accroissement de martingale, tous définis sur le même espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. De plus $\theta_0 \in L^1(\mathbb{P})$ et θ_0 est indépendant de $(Y_n)_{n\geq 0}$. La suite de pas $(\gamma_n)_{n\geq 1}$ est décroissante et H est une fonction borélienne de $\mathbb{R}^d \times \mathbb{R}^q$ dans $\mathbb{R}^d$. Nous noterons $\mathcal{F}_n = \sigma(\theta_0, Y_0, Y_1, \ldots, Y_n)$, $n \geq 0$, la filtration naturelle du processus des innovations $(Y_n)_{n\geq 0}$. Aucune hypothèse markovienne n'est faite *a priori* sur la suite $(Y_n)_{n\geq 0}$.

Nous dirons que la suite $(Y_n)_{n\geq 0}$ vérifie une hypothèse de ν-stabilité si

$$\mathbb{P}(d\omega)\text{-}p.s. \quad \frac{1}{n}\sum_{k=0}^{n-1} \delta_{Y_k(\omega)} \overset{(\mathbb{R}^q)}{\underset{n\to\infty}{\Longrightarrow}} \nu \tag{0.5}$$

où $\overset{(\mathbb{R}^q)}{\Longrightarrow}$ désigne la convergence étroite de mesures de probabilité définies sur $(\mathbb{R}^q, \mathcal{B}or(\mathbb{R}^q))$.

L'algorithme stochastique défini par (0.4) est alors une recherche récursive de zéro de la fonction moyenne (asymptotique)

$$h(\theta) := \int_{\mathbb{R}^q} H(\theta, y)\nu(dy). \tag{0.6}$$

Soient $p \in [1, \infty)$ et $(\varepsilon_n)_{n\geq 1}$ une suite positive telle que

$$\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{et} \quad \liminf_n n\varepsilon_n = 0. \tag{0.7}$$

Soit $\mathcal{V}_{\varepsilon_n,p}$ la classe de fonctions dont la vitesse de convergence, $p.s.$ et dans $L^p(\mathbb{P})$, dans le Théorème de Birkhoff est ε_n^{-1}, c'est-à-dire

$$\mathcal{V}_{\varepsilon_n,p} = \left\{ f \in L^p(\nu) \mid \frac{1}{n}\sum_{k=1}^{n} f(Y_k) - \int f d\nu \overset{\mathbb{P}\text{-}p.s. \ \&\ L^p(\mathbb{P})}{=} O(\varepsilon_n) \right\}.$$

Les principaux résultats obtenus sont les suivants.

0.2.2 Convergence *p.s.*

Le résultat principal de ce chapitre est le théorème suivant, analogue au Lemme de Robbins-Siegmund.

Théorème 0.1. (*a*) Bornitude *: Soient* $h : \mathbb{R}^d \to \mathbb{R}^d$ *vérifiant (0.6),* $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ *une fonction borélienne et* $(Y_n)_{n\geq 0}$ *une suite de variables aléatoires vérifiant (0.5). Supposons qu'il existe une fonction continûment différentiable* $L : \mathbb{R}^d \to \mathbb{R}^+$ *vérifiant*

$$\nabla L \text{ est lipschitzienne, continue et } |\nabla L|^2 \leq C\,(1+L)\,,$$

et que la pseudo-fonction moyenne H *vérifie l'hypothèse de* Lyapunov trajectorielle

$$\forall \theta \in \mathbb{R}^d \backslash \{\theta^*\},\ \forall y \in \mathbb{R}^q, \quad \langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq 0.$$

Soit $p \in [1, \infty)$ *et* $(\varepsilon_n)_{n\geq 1}$ *une suite positive vérifiant (0.7). Supposons que*

$$H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, p}.$$

De plus, supposons que H *vérifie l'hypothèse de croissance (quasi-)linéaire suivante*

$$\forall \theta \in \mathbb{R}^d, \forall y \in \mathbb{R}^q, \quad |H(\theta, y)| \leq C_H \phi(y)(1 + L(\theta))^{\frac{1}{2}}$$

et que la suite d'accroissements de martingale $(\Delta M_{n+1})_{n\geq 0}$ *vérifie pour tout* $n \geq 0$,

$$\mathbb{P}\text{-}p.s. \begin{cases} \mathbb{E}\left(|\Delta M_{n+1}|^{2\vee \frac{p}{p-1}} \mid \mathcal{F}_n\right) \leq C_M \phi(Y_n)^{2\vee \frac{p}{p-1}} (1 + L(\theta_n))^{1\vee \frac{p}{2(p-1)}} & si \quad p > 1, \\ \frac{|\Delta M_{n+1}|}{(1+L(\theta_n))^{\frac{1}{2}}} \leq C_M & si \quad p = 1 \end{cases}$$

où C_M *est une constante réelle positive et* $\sup_{n\geq 0} \|\phi(Y_n)\|_{2\vee \frac{p}{(p-1)}} < +\infty$.

Soit $\gamma = (\gamma_n)_{n\geq 1}$ *une suite positive décroissante de pas vérifiant*

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0, \quad et \quad \sum_{n\geq 1} n\varepsilon_n \max\left(\gamma_n^2, |\Delta\gamma_{n+1}|\right) < +\infty.$$

Alors, la procédure récursive définie par (0.4) vérifie $(L(\theta_n))_{n\geq 0}$ *est* L^1*-bornée,* $L(\theta_n) \underset{n\to\infty}{\longrightarrow} L_\infty < +\infty$ *p.s.,* $\theta_n - \theta_{n-1} \underset{n\to\infty}{\longrightarrow} 0$ *p.s. et*

$$\sum_{n\geq 1} \langle \nabla L(\theta_n) \,|\, H(\theta_n, Y_n) - H(\theta^*, Y_n)\rangle < +\infty.$$

(*b*) Convergence *p.s. : Si, de plus,* $\{\theta^*\}$ *est une composante connexe de* $\{L = L(\theta^*)\}$ *et que la pseudo-fonction moyenne* H *vérifie l'hypothèse de* Lyapunov trajectorielle stricte

$$\forall \delta > 0,\ \forall \theta \in \mathbb{R}^d \backslash \{\theta^*\},\ \forall y \in \mathbb{R}^q, \quad \langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq \chi_\delta(y)\Psi_\delta(\theta) \tag{0.8}$$

où $\nu(\chi_\delta) > 0$, Ψ_δ *est s.c.i. et positive sur* $\mathbb{R}^d \backslash \{\theta^*\}$ *et* $\bigcap_{\delta>0}\{\Psi_\delta = 0\} = \{\theta^*\}$, *alors*

$$\theta_n \underset{n\to\infty}{\overset{p.s.}{\longrightarrow}} \theta^*.$$

L'assertion (*a*) en établissant la bornitude de $(\theta_n)_{n\geq 0}$, par exemple via $L(\theta_n) \underset{n\to\infty}{\longrightarrow} L_\infty$ si $\lim_{|\theta|\to\infty} L(\theta) = +\infty$ et la convergence de $\sum_{n\geq 1} \gamma_n \Delta M_n$ (par exemple lorsque $p = 2$), est l'étape cruciale qui ouvre la possibilité d'appliquer la méthode de l'*EDO*.

0.2.3 Application du théorème de convergence à différents types de suites d'innovations

0.2.3.1 Suites à discrépance faible

Le cas des suites à discrépance faible, et plus généralement des suites "équiréparties", pour l'approximation stochastique a été introduit dans [81] dans le cas unidimensionnel avec H bornée. Nous généralisons ces résultats au cas multidimensionnel lorsque la pseudo-fonction moyenne H n'est pas bornée, en appliquant le Théorème 0.1 aux cas où H est à variation finie (dans le sens de Hardy et Krause, voir [95], ou au sens de la mesure) grâce à l'inégalité de Koksma-Hlawka et où H est lipschitzienne continue par le Théorème de Proinov (voir [102]). Rappelons tout d'abord la définition et une caractérisation de telles suites.

Définition 0.1. *Une suite $(\xi_n)_{n\geq 1}$ à valeurs dans $[0,1]^q$ est équirépartie sur $[0,1]^q$ si*

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\xi_k} \overset{(\mathbb{R}^q)}{\Longrightarrow} \mathcal{U}([0,1]^q) \quad \textit{quand } n\to\infty.$$

Proposition 0.1. *Soit $(\xi_n)_{n\geq 1}$ une suite à valeurs dans $[0,1]^q$. Alors $(\xi_n)_{n\geq 1}$ est équirépartie sur $[0,1]^q$ si et seulement si*

$$D_n^*(\xi) := \sup_{x\in[0,1]^q}\left|\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}_{[\![0,x]\!]}(\xi_k) - \prod_{i=1}^{q}x^i\right| \longrightarrow 0 \quad \textit{quand } n\to\infty,$$

où $D_n^*(\xi)$ est appelée discrépance à l'origine ou discrépance étoile.

Nous appliquons donc le Théorème 0.1 en posant $Y_n=\xi_{n+1}$, $\mathcal{F}_n=\{\emptyset,\Omega\}$, $n\geq 0$, $\Delta M_{n+1}\equiv 0$ avec l'hypothèse de Lyapunov trajectorielle stricte sur H (comme $p=1$, la fonction ϕ est inutile).

▷ **Cas à variation finie.** Supposons que $u\mapsto H(\theta^*,u)$ est à variation finie (au sens de la mesure ou de Hardy et Krause). L'inégalité de Koksma-Hlawka nous donne alors le résultat suivant : soit $\xi=(\xi_1,\dots,\xi_n)$ un n-uplet de vecteurs à valeurs dans $[0,1]^q$ et f une fonction à variation finie. Alors

$$\left|\frac{1}{n}\sum_{k=1}^{n}f(\xi_k) - \int_{[0,1]^q} f(u)\lambda_q(du)\right| \leq V(f)D_n^*(\xi).$$

Par conséquent, la classe de fonctions de l'hypothèse d'ergodicité s'écrit

$$\mathcal{V} = \{f:[0,1]^q\to\mathbb{R} \text{ t.q. } V(f)<+\infty\} \subset \mathcal{V}_{\varepsilon_n,1} \quad \text{avec} \quad \varepsilon_n = \frac{(\log n)^q}{n}$$

et l'on suppose que $\max_{1\leq k\leq n} k\left(D_k^*(\xi)\right) = O((\log n)^q)$. Les hypothèses sur la suite de pas sont alors

$$\sum_{n\geq 1}\gamma_n = +\infty,\ \gamma_n(\log n)^q\to 0 \quad \text{et} \quad \sum_{k\geq 1}\max(|\Delta\gamma_{n+1}|,\gamma_n^2)(\log n)^q < +\infty.$$

▷ **Cas Lipschitz continu.** Lorsque $q\geq 2$ augmente, il est difficile, voire illusoire, de vérifier qu'une fonction est dans $\mathcal{V}$. L'idée naturelle de régularité est de supposer que $u\mapsto H(\theta^*,u)$ est lipschitzienne continue. Par le Théorème de Proinov (voir [102]), on obtient que :

$$\left|\frac{1}{n}\sum_{k=1}^{n}H\left(\theta^*,\xi_k\right) - \int_{[0,1]^q}H(\theta^*,u)\lambda_q(du)\right| \leq C_q w_{H(\theta^*,\cdot)}\left(D_n^*\left(\xi_1,\dots,\xi_n\right)^{\frac{1}{q}}\right)$$

où $w_{H(\theta^*,\cdot)}$ est le module de continuité uniforme de $H(\theta^*,\cdot)$ (par rapport à la norme sup sur $\mathbb{R}^q$) et $C_q \in (0,\infty)$ est une constante universelle ne dépendant que de q. On a donc que $H(\theta^*,\cdot) \in \mathcal{V}_{\varepsilon_n,1}$ avec $\varepsilon_n = \frac{\log n}{n^{\frac{1}{q}}}$ et l'hypothèse sur la suite de pas devient alors

$$\sum_{n\geq 1}\gamma_n = +\infty, \quad \gamma_n(\log n)n^{1-\frac{1}{q}} \to 0 \quad \text{et} \quad \sum_{k\geq 1}\max(|\Delta\gamma_{n+1}|\,,\gamma_n^2)(\log n)n^{1-\frac{1}{q}} < +\infty.$$

Notons au passage que le Théorème de Proinov montre que les suites à discrépance faible n'échappent pas à la malédiction de la dimension dès lors qu'on les utilise avec des fonctions ayant une régularité de type Lipchitz.

0.2.3.2 Fonctionnelle d'un processus α-mélangeant

Le but ici est de relaxer autant que possible les hypothèses sur $(Y_n)_{n\geq 0}$ pour pouvoir appliquer l'approximation stochastique à des données exogènes (par exemple des données de marché).

Soit $X = (X_k)_{k\in\mathbb{Z}}$ un processus stationnaire à valeurs dans $\mathbb{R}^q$ avec les filtrations associées $\mathcal{F}_n = \mathcal{F}_n^X := \sigma(X_k; k \leq n)$ et $\mathcal{G}_n = \mathcal{G}_n^X := \sigma(X_k; k \geq n)$. Les coefficients d'$\alpha$-mélange sont définis comme suit

$$\alpha_n = \sup\left\{|\mathbb{P}(U\cap V) - \mathbb{P}(U)\mathbb{P}(V)| \,|\, U \in \mathcal{F}_k, V \in \mathcal{G}_{k+n}, k \geq 0\right\}.$$

Soient f une fonctionnelle définie sur $(\mathbb{R}^q)^{\mathbb{Z}}$ à valeurs dans $\mathbb{R}^q$ et posons

$$\forall n \in \mathbb{Z}, \quad Y_n := f(\cdots, X_{n-1}, X_n).$$

Alors $(Y_n)_{n\geq 0}$ est un processus stationnaire de distribution marginale $\nu = \mathcal{L}(Y_0)$. La proposition suivante nous permet de donner un critère d'appartenance aux ensembles $\mathcal{V}_{n^{-\beta},2}$, $\beta \in [0, 1/2[$.

Proposition 0.2. *Supposons que $g \in L^{2+\delta}(\nu)$, $\delta > 0$ et que l'une de ces deux hypothèses soit verifiée*

1. *Pour tout $n \in \mathbb{Z}$, $Y_n := f(\cdots, X_{n-1}, X_n)$ et X est un processus stationnaire α-mélangeant vérifiant la condition suivante*
$$\sum_{k=1}^{\infty}\sqrt{\frac{\alpha_k^{\frac{\delta}{2+\delta}}}{k}} < +\infty.$$
2. *$Y_n = X_n$ pour tout $n \geq 0$ et X est un processus stationnaire α-mélangeant vérifiant la condition suivante*
$$\sum_{k\geq 0}\alpha_k^{\frac{\delta}{2+\delta}} < +\infty.$$

Alors

$$g \in \mathcal{V}_{\varepsilon_n^{(\eta)},2}, \quad \text{avec } \varepsilon_n^{(\eta)} = (\log n)^{\frac{3}{2}+\eta}\, n^{-\frac{1}{2}}, \text{ pour tout } \eta > 0.$$

En particulier g appartient à $\mathcal{V}_{n^{-\beta},2}$ pour tout $\beta \in (0, \frac{1}{2})$.

La preuve de la Proposition 0.2 repose sur le Théorème de Gàl-Koksma (voir [54]) établi et prouvé dans un cadre probabiliste dans [3]. Les conditions sur X et $(H(\theta^*, Y_n))_{n\geq 0}$ proviennent d'un résultat établi dans [40].

0.2.3.3 Innovations à dynamique markovienne homogène

Supposons que le processus des innovations $(Y_n)_{n\geq 0}$ soit une chaîne de Markov homogène à valeurs dans $\mathbb{R}^q$ dont la transition est $(P(y,dz))_{y\in\mathbb{R}^q}$ et de loi initiale $\mu = \mathcal{L}(Y_0)$. Par commodité, nous supposerons que la chaîne vit dans son espace canonique $((\mathbb{R}^q)^{\mathbb{N}}, \mathcal{B}or(\mathbb{R}^q)^{\otimes\mathbb{N}})$. Par contraste avec l'approche très fouillée développée dans [25], nous n'avons pas besoin ici de résoudre l'équation de Poisson reliée à la pseudo-transition.

On applique alors le Théorème 0.1 en réécrivant l'algorithme stochastique sous la forme

$$\theta_{n+1} = \theta_n - \gamma_{n+1} K(\theta_n, Y_{n+1}), \quad n \geq 0,$$

où $K : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ est une fonction borélienne vérifiant (0.9) ci-dessous et $\theta_0 : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ est indépendant de $(Y_n)_{n\geq 0}$. Notons que $(Y_n)_{n\geq 0}$ est toujours une chaîne de Markov par rapport à la filtration $\mathcal{F}_n = \sigma(\theta_0, Y_0, \ldots, Y_n)$, $n \geq 0$.

Posons $H(\theta, y) := P\left(K(\theta, .)\right)(y)$ et $\Delta M_{n+1} := K(\theta_n, Y_{n+1}) - \mathbb{E}\left[K(\theta_n, Y_{n+1}) \,|\, \mathcal{F}_n\right]$. Alors la procédure a la forme canonique (0.4) par rapport à la filtration $(\mathcal{F}_n)_{n\geq 0}$.

Soit $p \in [1, \infty)$ et posons $r = 2 \vee \frac{p}{p-1} \in [2, +\infty]$. Supposons que K vérifie

$$\forall \theta \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^q, \quad |K(\theta, y)| \leq C_K \widetilde{\phi}(y) \sqrt{1 + L(\theta)} \tag{0.9}$$

où $\sup_{n\geq 0} \left\|\widetilde{\phi}(Y_n)\right\|_r < +\infty$. En revanche, l'hypothèse de Lyapunov (0.8) reste sur H.

Alors H vérifie l'hypothèse de croissance sous-linéaire avec

$$\phi(y) = P\widetilde{\phi}(y) = \mathbb{E}_y \widetilde{\phi}(Y_1) \leq \left\|\widetilde{\phi}\right\|_{L^r(P(y,dz))} < +\infty$$

et l'hypothèse sur ΔM_{n+1} est vérifiée avec $\phi(y) = \left\|\widetilde{\phi}\right\|_{L^r(P(y,dz))}$. Finalement on peut choisir $\phi(y) = \left\|\widetilde{\phi}\right\|_{L^r(P(y,dz))}$, ayant en tête que $\|\phi(Y_n)\|_r = \left\|\widetilde{\phi}(Y_{n+1})\right\|_r$.

Si $(Y_n)_{n\geq 0}$ est ν-ergodique, *i.e.* si pour toute fonction borélienne bornée $f : \mathbb{R}^q \to \mathbb{R}$,

$$\mathbb{P}_\mu\text{-}p.s. \quad \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^q} f d\nu,$$

alors ν est nécessairement invariante pour P, et donc $(Y_n)_{n\geq 0}$ est stationnaire sous $\mathbb{P}_\nu$. De plus, la chaîne est ergodique sous $\mathbb{P}_\nu$ pour l'opérateur de décalage Θ (sur l'espace canonique), *i.e.*, pour toute fonctionnelle $F : (\mathbb{R}^{\mathbb{N}}, \mathcal{B}or(\mathbb{R}^{\mathbb{N}})) \to \mathbb{R}$, $F \in L^r(\nu)$,

$$\frac{1}{n}\sum_{k=1}^{n} F \circ \Theta^k \underset{n\to\infty}{\longrightarrow} \mathbb{E}_\nu(F) \quad \mathbb{P}_\nu\text{-}p.s. \text{ et dans } L^r(\nu).$$

Typiquement, si ν est une distribution invariante extrêmale pour P, alors la chaîne est ergodique sous $\mathbb{P}_\nu$ (c'est évidemment le cas si ν est unique). Le Théorème de Birkhoff nous donne alors que

$$\mathcal{V}_{0^+,p}(\mathbb{P}_\nu) = L^p(\nu).$$

On établit enfin dans ce cadre la proposition suivante qui permet de s'affranchir du cadre stationnaire pour les innovations.

Proposition 0.1. *Soient $p \in [1, +\infty)$ et $p' \in (0, p]$. Si $(Y_n)_{n\geq 0}$ est $\mathbb{P}_\nu$-ergodique et $P(y, dz) = g(y, z)\nu(dz)$, $y \in \mathbb{R}^q$ où $g : (\mathbb{R}^q)^2 \to \mathbb{R}_+$ vérifie $\nu(dz)$-p.p., $g(\cdot, z) > 0$. Alors ν est l'unique mesure invariante de P et pour toute suite $(\varepsilon_n)_{n\geq 1}$ vérifiant $\varepsilon_n \to 0$ et $\lim_n \inf n\varepsilon_n = 0$,*

$$\forall y \in \mathbb{R}^q, \quad g(y, \cdot) \in L^{\frac{p}{p-p'}}(\nu) \Longrightarrow \mathcal{V}_{\varepsilon_n,p'}(\mathbb{P}_y) \supset \mathcal{V}_{\varepsilon_n,p}(\mathbb{P}_\nu).$$

0.3 Vitesse de convergence dans le cas des suites à discrépance faible

Nous reprenons ici le cadre de la section précédente dans le cas d'innovations équiréparties. Considérons une procédure récursive quasi-stochastique de la forme suivante

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, \xi_{n+1}), \quad \theta_0 \in \mathbb{R}, \tag{0.10}$$

où $H : \mathbb{R} \times [0,1]^q \to \mathbb{R}$, $(\xi_n)_{n\geq 1}$ est une suite de nombres quasi-aléatoires à valeurs dans $[0,1]^q$, $(\gamma_n)_{n\geq 1}$ est une suite de pas décroissants vérifiant $\gamma_n \underset{n\to+\infty}{\longrightarrow} 0$ et $\sum_{n\geq 1} \gamma_n = +\infty$.

Supposons que $\theta_n \to \theta^*$ où

$$\theta^* = \{h = 0\} \quad \text{avec} \quad h(\theta) = \int_{[0,1]^q} H(\theta, \xi) \lambda_q(d\xi),$$

où λ_q est la mesure de Lebesgue sur $\mathbb{R}^q$. La convergence de la suite $(\theta_n)_{n\geq 0}$ est obtenue en appliquant le Théorème 0.1.

Pour établir la vitesse de convergence de l'algorithme (0.10), nous avons besoin d'hypothèses supplémentaires. Tout d'abord la pseudo-fonction moyenne H doit admettre une sorte de développement de Taylor : nous supposons que $H(\cdot, \xi)$ est différentiable en θ^* uniformément en ξ, à savoir

$$H(\theta, \xi) = H(\theta^*, \xi) + \partial_\theta H(\theta^*, \xi)(\theta - \theta^*) + \psi(\theta, \xi) \left|\theta - \theta^*\right|^{1+\eta}, \quad \eta > 0, \tag{0.11}$$

où

$$\sup_{\xi \in [0,1]^q} |\partial_\theta H(\theta^*, \xi)| < +\infty \quad \text{et} \quad \sup_{\xi \in [0,1]^q, u \in \mathbb{R}} |\psi(u, \xi)| < +\infty.$$

Nous avons également besoin d'hypothèses de moyennisation : nous supposons que la pseudo-fonction moyenne H et sa dérivée par rapport à sa première variable prises en la cible θ^* appartiennent à la classe de fonctions de $L^1(\lambda_q)$ dont la vitesse de moyennisation est ε_n, *i.e.*

$$H(\theta^*, \cdot) \quad \text{et} \quad \partial_\theta H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, 1} = \left\{ f \in L^1(\lambda_q) \mid \frac{1}{n} \sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^q} f \lambda_q = O(\varepsilon_n) \right\}, \tag{0.12}$$

où $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$ et $\liminf_n n\varepsilon_n = 0$..

Nous avons aussi besoin d'une condition d'"attractivité" sur $h'(\theta^*)$, typiquement $h'(\theta^*) > 0$ (avec nos conventions).

Théorème 0.2. *Soient $H : \mathbb{R} \times [0,1]^q \to \mathbb{R}$ et $h : \mathbb{R} \to \mathbb{R}$ deux fonction boréliennes définies comme ci-dessus telles que (0.11) et(0.12) soient vérifiées. Supposons $\theta_n \to \theta^*$ et que $h'(\theta^*) > 0$. Supposons de plus que la suite de pas $(\gamma_n)_{n\geq 1}$ vérifie*

$$n\varepsilon_n\gamma_n \underset{n\to+\infty}{\longrightarrow} 0 \quad \textit{et} \quad \sum_{n\geq 1} n\varepsilon_n |\Delta\gamma_{n+1}| < +\infty. \tag{0.13}$$

1. *Supposons que la suite de pas admet la représentation suivante*

$$\gamma_n = \frac{c}{n}, \quad c > 0, \quad n \geq 1.$$

(a) *S'il existe $\alpha > 0$ tel que $\sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)} < +\infty$, alors, pour tout $\epsilon > 0$,*

$$\theta_n - \theta^* = o\left(\left(\varepsilon_n \vee n^{-ch'(\theta^*)}\right) n^{\epsilon}\right).$$

(a') *S'il existe* $\alpha > 0$ *tel que*

$$\sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)} < +\infty \quad \text{et} \quad \varepsilon_n n^{c(1+\alpha)h'(\theta^*)} \underset{n\to+\infty}{\longrightarrow} 0,$$

alors la suite $\left(n^{ch'(\theta^*)}(\theta_n - \theta^*)\right)_{n\geq 0}$ *est bornée.*

(b) *Si* $\sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)} = +\infty$ *pour tout* $\alpha > 0$, *alors, pour tout* $\epsilon > 0$,

$$\theta_n - \theta^* = O\left(\varepsilon_n n^{\epsilon}\right).$$

2. *Supposons que la suite de pas admet la représentation suivante*

$$\gamma_n = \frac{c}{n^\rho}, \quad c > 0, \quad 1/2 < \rho < 1, \quad n \geq 1.$$

Alors

$$\forall \epsilon > 0, \quad \theta_n - \theta^* = o\left(\varepsilon_n n^{1-\rho+\epsilon}\right).$$

Remarque. Le résultat 1.(a') correspond dans le cas aléatoire au résultat de vitesse non-TCL (c.f. [44]).

0.4 Allocation optimale parmi différents pools de liquidité

Ce chapitre fait l'objet d'un papier accepté dans *SIAM Journal on Financial Mathematics*.

Les évolutions du paysage de trading ont conduit depuis peu à ouvrir la capacité d'échanger le même instrument financier sur différentes places. A cause des problèmes de liquidité, les entreprises de trading découpent les ordres de grande taille et les envoient sur plusieurs destinations de trading pour optimiser leur exécution. Pour résoudre ce problème, nous construisons deux procédures récursives stochastiques d'apprentissage qui ajustent les proportions de l'ordre à envoyer aux différentes places, l'une basée sur un principe d'optimisation, l'autre sur un principe de renforcement. Les deux procédures sont étudiées d'un point de vue théorique : nous montrons la convergence *p.s.* de l'algorithme d'optimisation sous des hypothèses de faible ergodicité (ou de "moyennisation") sur le processus des données en entrée. Aucune hypothèse markovienne n'est requise. Quand les données sont i.i.d., nous montrons que la vitesse de convergence est régie par un théorème central limite.

Ce chapitre se concentre sur le problème de découpage d'ordre dans le cas de places particulières : les *dark pools*. Sur ces places, le prix d'achat (plus avantageux que sur le marché primaire) est garanti mais pas la quantité. Supposons qu'il y ait N dark pools ($N \geq 2$). Soit $V > 0$ le volume aléatoire à exécuter et $\theta_i \in (0,1)$ le *rabais* proposé par le dark pool $i \in \mathcal{I}_N := \{1,\dots,N\}$. Soit r_i le pourcentage de V envoyé au dark pool i et $D_i \geq 0$ la quantité d'actifs délivrée par le dark pool i au prix $\theta_i S$ où S est le prix d'achat du marché primaire. Le reste de l'ordre sera exécuté sur le marché primaire au prix S. Ainsi le coût d'exécution C est donné par

$$C = S\sum_{i=1}^{N} \theta_i \min(r_i V, D_i) + S\left(V - \sum_{i=1}^{N} \min(r_i V, D_i)\right) = S\left(V - \sum_{i=1}^{N} \rho_i \min(r_i V, D_i)\right)$$

où $\rho_i = 1 - \theta_i > 0$, $i = 1,\dots,N$. Il est commode d'inclure le prix S dans les volumes aléatoires en posant $\widetilde{V} := V\,S$ et $\widetilde{D}_i := D_i S$ à la place de V et D_i ce qui revient au problème d'optimisation du coût moyen C suivant

$$\max\left\{\sum_{i=1}^{N} \rho_i \mathbb{E}\left(\min(r_i V, D_i)\right),\ r \in \mathcal{P}_N\right\}.$$

0.4.1 Allocation optimale : un algorithme stochastique Lagrangien

Décrivons tout d'abord le comportement précis de la fonction moyenne d'exécution $\varphi : [0,1] \to \mathbb{R}_+$ d'un seul dark pool. Soit (V, D) un vecteur aléatoire à valeurs dans $\mathbb{R}_+^2$ défini sur un espace de probabilités $(\Omega, \mathcal{A}, \mathbb{P})$. Tout au long du paragraphe nous ferons l'hypothèse de consistance suivante

$$V > 0 \quad \mathbb{P}\text{-}p.s. \quad \text{et} \quad \mathbb{P}(D > 0) > 0. \tag{0.14}$$

Supposons que $V \in L^1(\mathbb{P})$. Alors la *fonction moyenne d'exécution* $\varphi : [0,1] \to \mathbb{R}_+$ du dark pool est définie par

$$\forall\, r \in [0,1], \qquad \varphi(r) = \rho\, \mathbb{E}(\min(rV, D)) \tag{0.15}$$

où $\rho > 0$. Si

$$\text{la fonction de distribution (continue à droite) de } \tfrac{D}{V} \text{ est continue sur } \mathbb{R}_+ \tag{0.16}$$

(*i.e.* la distribution de $\frac{D}{V}$ n'a pas d'atome sauf éventuellement en 0), alors φ est différentiable partout sur $[0,1]$ avec

$$\varphi'(r) = \rho\, \mathbb{E}\left(\mathbf{1}_{\{rV \le D\}} V\right), \text{ si } r \in (0,1], \quad \text{et} \quad \varphi'(0) = \rho\, \mathbb{E}(V \mathbf{1}_{\{D>0\}}) > 0. \tag{0.17}$$

0.4.2 Construction de l'algorithme

Supposons que pour tout $i \in \mathcal{I}_N$, (V, D_i) vérifie l'hypothèse de consistance (0.14) et φ_i désigne la fonction moyenne d'exécution du dark pool i de type (0.15). Posons pour tout $r = (r_1, \dots, r_N) \in \mathcal{P}_N$, $\Phi(r_1, \dots, r_N) := \sum_{i=1}^N \varphi_i(r_i)$. On étend ensuite toutes les fonctions moyennes d'exécution φ_i sur toute la droite réelle, ce qui nous permet d'étendre Φ sur tout l'hyperplan affine engendré par $\mathcal{P}_N$ à savoir

$$\mathcal{H}_N := \{r \in \mathbb{R}^N \mid \sum_{i=1}^N r_i = 1\}.$$

Regardons maintenant l'approche du Lagrangien qui prend seulement en compte les contraintes affines qui sont ici $\max_r \Phi(r) - \lambda \sum_{i \in \mathcal{I}_N} r_i$. Des calculs formels directs suggèrent que

$$r^* \in \operatorname{argmax}_{\mathcal{P}_N} \Phi \quad \Longleftrightarrow \quad \forall\, i \in \mathcal{I}_N, \quad \varphi_i'(r_i^*) = \frac{1}{N} \sum_{j=1}^N \varphi_j'(r_j^*).$$

Comme la méthode du Lagrangien ne donne pas de condition nécessaire et suffisante pour qu'un point soit le maximum d'une fonction (concave), nous avons besoin d'une hypothèse supplémentaire.

Proposition 0.3. *Supposons que (V, D_i) vérifie les hypothèses de consistance (0.14) et (0.16) pour tout $i \in \mathcal{I}_N$. Supposons que les fonctions φ_i, $i \in \mathcal{I}_N$, définies par (0.15) vérifient l'hypothèse suivante*

$$(\mathcal{C}) \quad \equiv \quad \min_{i \in \mathcal{I}_N} \varphi_i'(0) \ge \max_{i \in \mathcal{I}_N} \varphi_i' \left(\frac{1}{N-1} \right).$$

Alors $\operatorname{argmax}_{\mathcal{P}_N} \Phi$ *est compact convexe et*

$$\operatorname{argmax}_{\mathcal{H}_N} \Phi = \operatorname{argmax}_{\mathcal{P}_N} \Phi = \{r \in \mathcal{P}_N, \mid \varphi_i'(r_i) = \varphi_1'(r_1),\ i = 1, \dots, N\}.$$

Nous allons maintenant construire l'algorithme stochastique d'allocation optimale. Par la représentation (0.17) pour toutes les dérivées φ_i', $i \in \mathcal{I}_N$, si $(\mathcal{C})$ est vérifiée, alors $\operatorname{argmax}_{\mathcal{H}_N}\Phi = \operatorname{argmax}_{\mathcal{P}_N}\Phi$ et

$$r^* \in \operatorname{argmax}_{\mathcal{H}_N}\Phi \Longleftrightarrow \forall\, i \in \{1,\dots,N\},\ \mathbb{E}\left(V\left(\rho_i \mathbf{1}_{\{r_i^* V \le D_i\}} - \frac{1}{N}\sum_{j=1}^{N} \rho_j \mathbf{1}_{\{r_j^* V \le D_j\}}\right)\right) = 0.$$

Par conséquent, ceci nous donne la procédure suivante de recherche de zéro

$$r^n = r^{n-1} + \gamma_n H(r^{n-1}, V^n, D_1^n, \dots, D_N^n),\ n \ge 1, \quad r^0 \in \mathcal{P}_N, \tag{0.18}$$

où, pour tout $i \in \mathcal{I}_N$, tout $r \in \mathcal{H}_N$, tout $V > 0$ et tout $D_1, \dots, D_N \ge 0$,

$$H_i(r, V, (D_i)_{i\in\mathcal{I}_N}) = V\Big(\rho_i \mathbf{1}_{\{r_i V \le D_i\}\cap\{r_i\in[0,1]\}} - \frac{1}{N}\sum_{j=1}^{N} \rho_j \mathbf{1}_{\{r_j V \le D_j\}\cap\{r_j\in[0,1]\}} + R_i(r, V, (D_i)_{i\in\mathcal{I}_N})\Big)$$

et le processus des "innovations" $(V^n, D_1^n, \dots, D_N^n)_{n\ge1}$ est une suite de vecteurs aléatoires à composantes positives telle que, pour tout $n \ge 1$, $(V^n, D_i^n, i = 1,\dots,N) \overset{(d)}{=} (V, D_i, i = 1,\dots,N)$. Le terme de reste R_i a un effet de retour à la moyenne (voir [74, 75]) pour pousser l'algorithme dans $\mathcal{P}_N$. On aurait pu tout aussi bien le remplacer par une projection.

0.4.3 Cadre (IID) : convergence *p.s.* et TCL

Théorème 0.3. *Supposons que (V, D) vérifie (0.14), que $V \in L^2(\mathbb{P})$ et que l'hypothèse $(\mathcal{C})$ soit vérifiée. Supposons de plus que la distribution de $\frac{D}{V}$ vérifie l'hypothèse de continuité (0.16). Soit $\gamma := (\gamma_n)_{n\ge1}$ une suite de pas vérifiant l'hypothèse usuelle de pas décroissant*

$$\sum_{n\ge1}\gamma_n = +\infty \quad \text{et} \quad \sum_{n\ge1}\gamma_n^2 < +\infty.$$

Soit $(V^n, D_1^n, \dots, D_N^n)_{n\ge1}$ une suite i.i.d. définie sur un espace de probabilités $(\Omega, \mathcal{A}, \mathbb{P})$. Alors, il existe une variable aléatoire r^* à valeurs dans $\operatorname{argmax}_{\mathcal{P}_N}\Phi$ telle que

$$r^n \longrightarrow r^* \quad p.s.$$

Supposons que la fonction moyenne h admette un unique zéro r^* (ce qui est le cas si toutes les fonctions φ_i, $i \in \mathcal{I}_N$, sont (strictement) décroissantes). Pour assurer l'existence et le calcul de la différentielle de h, nous avons besoin de supposer que la fonction de distribution de (V, D) *sachant* $\{D > 0\}$ est absolument continue de densité f définie sur $(0, +\infty)^2$ et vérifiant

$$\begin{cases} (i) & \text{pour tout } v > 0,\ u \mapsto f(v,u) \text{ est continue et positive sur } (0,\infty), \\ (ii) & \forall\, \varepsilon \in (0,1), \quad \sup\limits_{\varepsilon V \le u \le V/\varepsilon} f(V,u)V^2 \in L^1(\mathbb{P}). \end{cases} \tag{0.19}$$

Théorème 0.4. *Supposons que (V, D_i) vérifie (0.14) et (0.16) pour tout $i \in \mathcal{I}_N$ et que $\operatorname{argmax}\Phi$ est réduit à un seul point $r^* \in \mathcal{P}_N$ donc $r^n \to r^*$ $\mathbb{P}$-p.s. quand $n \to \infty$. Supposons de plus que (0.19) soit vérifiée pour tout (V, D_i), $i \in \mathcal{I}_N$, et que $V \in L^{2+\delta}(\mathbb{P})$, $\delta > 0$. Posons*

$$\gamma_n = \frac{c}{n},\ n \ge 1 \quad \textit{avec} \quad c > \frac{1}{2\Re e(\lambda_{\min})}$$

où $\lambda_{\min}$ *est la valeur propre de* $A^\infty := -Dh(r^*)_{|\mathbf{1}^\perp}$ *de plus petite partie réelle. Alors*

$$\sqrt{n}(r^n - r^*) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0; \sqrt{c}\,\Sigma^\infty)$$

où la matrice de covariance asymptotique Σ^∞ *est donnée par*

$$\Sigma^\infty = \int_0^\infty e^{u(A^\infty - \frac{Id}{2c})} C^\infty e^{u(A^\infty - \frac{Id}{2c})^t} du$$

où

$$C^\infty = \mathbb{E}\left(H(r^*, V, D_1, \dots, D_N) H(r^*, V, D_1, \dots, D_N)^t\right)_{|\mathbf{1}^\perp}$$

et $(A^\infty - \frac{Id}{2c})^t$ *désigne l'*opérateur transposé *de* $\left(A^\infty - \frac{Id}{2c}\right)_{\mathbf{1}^\perp} \in \mathcal{L}(\mathbf{1}^\perp)$.

0.4.4 Cadre (ERG) : convergence

Supposons que $\operatorname{argmax}_{\mathcal{P}_N} \Phi = \{r^*\} \subset int(\mathcal{P}_N)$ et que la suite $(V^n, D_i^n, i = 1, \dots, N)_{n\geq 1}$ vérifie

$$(ERG)_\nu \equiv \begin{cases} (i) & \mathbb{P}\text{-}p.s. \quad \dfrac{1}{n}\displaystyle\sum_{k=1}^n \delta_{(V^k, D_1^k, \dots, D_N^k)} \overset{(\mathbb{R}_+^{N+1})}{\Longrightarrow} \nu, \\ (ii) & \sup_n \mathbb{E}(V^n)^2 < +\infty \end{cases}$$

avec une distribution limite ν. Nous avons besoin d'une hypothèse supplémentaire : il existe $\varepsilon_0 > 0$ tel que

$$\begin{cases} (i) & \mathbb{P}(V \geq \varepsilon_0) > 0 \\ (ii) & \operatorname{supp}\Big(\mathcal{L}\Big(\frac{D_i}{V}, i = 1, \dots, N \,|\, \{V \geq \varepsilon_0\}\Big)\Big) \text{ est un voisinage de } \mathcal{P}_N \text{ dans } \mathbb{R}_+^N. \end{cases} \tag{0.20}$$

Supposons qu'il existe un exposant $\alpha_i \in (0, 1]$ tel que

$$\forall\, u \in \mathbb{R}_+, \quad \frac{1}{n}\sum_{k=1}^n V^k \mathbf{1}_{\{uV^k < D_i^k\}} - \mathbb{E}(V \mathbf{1}_{\{uV < D_i\}}) = O(n^{-\alpha_i}) \quad p.s. \text{ et dans } L^2(\mathbb{P}). \tag{0.21}$$

Théorème 0.5. *Soit* $(V^n, D_1^n, \dots, D_N^n)_{n\geq 0}$ *une suite vérifiant* $(ERG)_\nu$ *et telle que, pour tout* $i \in \mathcal{I}_N$, *la distribution marginale* $\nu_i = \mathcal{L}(V, D_i)$ *vérifie les hypothèses (0.14) et (0.16). Supposons de plus que la suite* $(V^n, D_i^n)_{n\geq 1}$ *vérifie (0.21). Si la suite de pas* $(\gamma_n)_{n\geq 1}$ *satisfait*

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad \gamma_n = o(n^{\underline{\alpha}-1}) \quad \text{et} \quad \sum_{n\geq 1} n^{1-\underline{\alpha}} \max(\gamma_n^2, |\gamma_n - \gamma_{n+1}|) < +\infty$$

où $\underline{\alpha} := \min_{i\in\mathcal{I}_N} \alpha_i \in (0, 1]$, *alors l'algorithme défini par (0.18) converge p.s. vers* $r^* = \operatorname{argmax}_{\mathcal{P}_N} \Phi$.

0.4.5 Une procédure alternative basée sur un principe de renforcement.

Cette procédure construite par Berenstein et Lehalle est fondée sur un mécanisme intuitif de renforcement. Soit I_i^n le profit induit par l'exécution de l'ordre envoyé au dark pool i au temps n. La proportion r_i^n de l'ordre V^{n+1} à envoyer au *dark pool* i pour exécution au temps $n+1$ est définie comme proportionnelle à son profit *i.e.* par

$$\forall\, i \in \mathcal{I}_N, \qquad r_i^n := \frac{I_i^n}{\sum_{j\in\mathcal{I}_N} I_j^n}.$$

La mise à jour du vecteur aléatoire I^n se fait comme suit

$$\forall\, n \geq 0,\ \forall\, i \in \mathcal{I}_N, \quad I_i^{n+1} = I_i^n + \rho_i \min\left(r_i^n V^{n+1}, D_i^{n+1}\right), \quad I_i^0 = 0,$$

que l'on peut réécrire de manière récursive en posant $X^n = \frac{I^n}{n}$, $n \geq 1$, de la manière suivante

$$X_i^{n+1} = X_i^n - \frac{1}{n+1}\left(X_i^n - \rho_i \min\left(r_i^n V^{n+1}, D_i^{n+1}\right)\right), \quad i \in \mathcal{I}_N.$$

Il s'agit d'un algorithme stochastique standard (de pas $\gamma_n = \frac{1}{n}$). Si la suite $(V^n, D_1^n, \ldots, D_N^n)_{n\geq 1}$ est i.i.d., alors l'algorithme est un $(\mathcal{F}_n)$-processus de Markov à temps discret (non homogène), donc il admet la représentation canonique suivante

$$X_i^{n+1} = X_i^n - \gamma_{n+1}\left(X_i^n - \varphi_i(r_i^n)\right) + \gamma_{n+1}\Delta M_i^{n+1} \quad i \in \mathcal{I}_N, \quad n \geq 0,$$

où $\gamma_n = \frac{1}{n}$ et $\Delta M_i^n = \rho_i \min\left(r_i^{n-1} V^n, D_i^n\right) - \varphi_i(r_i^{n-1})$, $i \in \mathcal{I}_N$, $n \geq 1$, est un $\mathcal{F}_n$-accroissement de martingale. Dans ce cadre, la fonction moyenne de l'algorithme est

$$h : x \longmapsto \left(x_i - \varphi_i\left(\frac{x_i}{\sum_j x_j}\right)\right)_{1\leq i\leq N}.$$

Nous allons maintenant donner des conditions sur h qui assurent l'existence d'un équilibre x^*.

Proposition 0.4. *Soit $N \geq 1$. Supposons que (0.14) soit vérifiée pour tout (V, D_i), $i \in \mathcal{I}_N$.*
(a) Il existe un $x^* \in \mathbb{R}_+^N$ tel que

$$\sum_{i\in\mathcal{I}_N} x_i^* > 0 \quad \text{et} \quad \varphi_i\left(\frac{x_i^*}{\sum_{j\in\mathcal{I}_N} x_j^*}\right) = x_i^*, \quad i \in \mathcal{I}_N. \tag{0.22}$$

(b) Soit $\psi_i(u) := \frac{\varphi_i(u)}{u}$, $u > 0$, $\psi_i(0) = \varphi_i'(0)$, $i \in \mathcal{I}_N$. Supposons que pour tout $i \in \mathcal{I}_N$, ψ_i est (continue et) décroissante sur $[0, \infty)$ et que

$$\sum_{i\in\mathcal{I}_N} \psi_i^{-1}(\min_{i\in\mathcal{I}_N} \varphi_i'(0)) < 1.$$

Alors il existe $x^* \in int(\mathcal{P}_N)$ vérifiant (0.22).

Corollaire 0.1. *Supposons que toutes les fonctions ψ_i sont continues et décroissantes. Si, de plus, les coefficients de rabais ρ_i sont égaux (à 1) et si $\mathbb{P}(D_i = 0) = 0$ pour tout $i \in \mathcal{I}_N$, alors il existe un équilibre dans $int(\mathcal{P}_N)$.*

La question naturelle suivante est de savoir quand un équilibre x^* de l'algorithme est (au moins) une cible pour ce dernier, *i.e.* est attractif pour l'EDO associée $\dot{x} = -h(x)$.

Proposition 0.5. *Un équilibre x^* vérifiant (0.22) est localement uniformément attractif dès que*

$$\sum_{j\in\mathcal{I}_N} \frac{x_j^*}{(\bar{x}^*)^2}\varphi_j'\left(\frac{x_j^*}{\bar{x}^*}\right) < 1 - \frac{1}{\bar{x}^*}\max_{i\in\mathcal{I}_N}\varphi'\left(\frac{x_i^*}{\bar{x}^*}\right), \quad \bar{x}^* = \sum_{i\in\mathcal{I}_N} x_i^*.$$

En ce qui concerne la convergence de l'algorithme, on ne dispose pas de résultat plus précis que la convergence conditionnelle "à la Kushner-Clark" ou les résultats génériques "à la Benaïm" issus de la méthode de l'EDO.

0.5 Distance optimale de placement d'ordres limites

Ce chapitre présente une procédure récursive stochastique sous contraintes pour trouver la distance optimale de placement d'ordre d'achat dans un carnet d'ordre afin de minimiser le coût d'exécution. Nous montrons la convergence *p.s.* de l'algorithme sous des hypothèses sur la fonction de coût et donnons des critères pour les praticiens sur les paramètres du modèle qui assurent que les conditions pour appliquer l'algorithme sont remplies (utilisant notamment le principe de co-monotonie).

On considère sur une courte période $[0,T]$, quelques secondes par exemple, un processus de Poisson d'"exécution" d'ordres d'achat

$$\left(N_t^{(\delta)}\right)_{0\leq t\leq T} \quad \text{d'intensité} \quad \Lambda_T(\delta,S) := \int_0^T \lambda(S_t-(S_0-\delta))dt$$

où $0 \leq \delta \leq \delta_{\max}$ avec $\delta_{\max} \in (0,S_0)$ représentant la profondeur du carnet d'ordre et $(S_t)_{t\geq 0}$ le processus stochastique modélisant la dynamique du prix d'un actif. On suppose que la fonction λ est définie sur toute la droite réelle comme fonction finie décroissante convexe.

Sur la période $[0,T]$, on veut exécuter un portefeuille de taille $Q_T \in \mathbb{N}$ investi dans l'actif S. Le coût d'exécution pour une distance δ est alors $\mathbb{E}\left[(S_0-\delta)\left(Q_T \wedge N_T^{(\delta)}\right)\right]$. On ajoute à ce coût d'exécution une pénalisation dépendant de la quantité restant à exécuter à la fin de la période : comme l'on veut avoir en portefeuille Q_T actifs à la fin de la période $[0,T]$, on achète la quantité restante $\left(Q_T - N_T^{(\delta)}\right)_+$ au prix S_T. On introduit alors une fonction de pénalisation $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$ croissante convexe avec $\Phi(0)=0$ pour modéliser le coût supplémentaire de l'exécution de la quantité restante (incluant l'impact de marché). Ainsi le coût d'exécution global sur la période $[0,T]$ s'écrit

$$C(\delta) := \mathbb{E}\left[(S_0-\delta)\left(Q_T \wedge N_T^{(\delta)}\right) + \kappa S_T \Phi\left(\left(Q_T - N_T^{(\delta)}\right)_+\right)\right]$$

où $\kappa > 0$. Quand $\Phi = \mathrm{id}$, on considère simplement que l'on achète la quantité non exécutée au prix final S_T, mais on peut aussi considérer une fonction de pénalisation $\Phi(x) = (1+\eta(x))x$, où η modélise l'impact de marché induit par l'exécution de $\left(Q_T - N_T^{(\delta)}\right)_+$ à la fin de la période. Notre but est alors de minimiser ce coût, à savoir résoudre le problème d'optimisation suivant

$$\min_{0\leq\delta\leq\delta_{\max}} C(\delta).$$

Notre stratégie pour résoudre numériquement ce problème d'optimisation en utilisant une base de données assez grande est de concevoir un algorithme stochastique. Pour être plus précis, on montre que sous des hypothèses naturelles sur la quantité à exécuter et sur le paramètre κ, la fonction C est deux fois différentiable, strictement convexe sur $[0,\delta_{\max}]$ avec $C'(0) < 0$. Par conséquent $\mathrm{argmin}_{\delta\in[0,\delta_{\max}]} C(\delta) = \{\delta^*\}$ d'une part et $\delta^* = \delta_{\max}$ si et seulement si C est décroissante sur $[0,\delta_{\max}]$. Des critères assurant ces propriétés sont donnés sous forme de majoration du paramètre d'ajustement κ. Avant eux, on établit que les trois fonctions C, C' et C'' admettent des représentations comme espérances. En particulier on montre qu'il existe une fonctionnelle borélienne

$$H : [0,\delta_{\max}] \times \mathbb{D}([0,T],\mathbb{R}) \longrightarrow \mathbb{R}$$

telle que

$$\forall \delta \in [0,\delta_{\max}], \quad C'(\delta) = \mathbb{E}\left[H\left(\delta,(S_t)_{t\in[0,T]}\right)\right].$$

La fonctionnelle H s'avère avoir une forme explicite mettant en jeu des intégrales sur $[0,T]$ de l'intensité $\lambda(S_t - S_0 + \delta)$ du processus de Poisson $(N_t)_{t\in[0,T]}$. Ainsi toute quantité $H\left(\delta,(S_t)_{t\in[0,T]}\right)$

peut être simulée par une approximation naturelle discrète, soit avec une base de données réelles (d'ordres exécutés passés), soit par le schéma d'Euler d'une diffusion modélisant le prix $(S_t)_{t\in[0,T]}$.

On peut alors construire un algorithme stochastique contraint à rester dans $[0,\delta_{\max}]$ qui converge *p.s.* vers δ^*. En procédant selon les méthodes de projection d'algorithmes contraints (voir [74] et [75]), on pose

$$\delta_{n+1} = \mathrm{Proj}_{[0,\delta_{\max}]}\left(\delta_n - \gamma_{n+1} H\left(\delta_n, \left(\bar{S}_t\right)_{0\le t\le T}\right)\right), \quad \delta_0 \in [0,\delta_{\max}], \tag{0.23}$$

où $\mathrm{Proj}_{[0,\delta_{\max}]}$ désigne la projection sur $[0,\delta_{\max}]$, $\left(\bar{S}_t\right)_{0\le t\le T}$ est le schéma d'Euler de $(S_t)_{0\le t\le T}$ et la suite positive décroissante de pas $(\gamma_n)_{n\ge1}$ vérifie l'hypothèse standard

$$\sum_{n\ge1}\gamma_n = +\infty \quad \text{et} \quad \gamma_n \to 0. \tag{0.24}$$

Les théorèmes suivants donnent la convergence *p.s.* de la procédure récursive (0.23) : le premier pour des suites i.i.d. et le second pour des innovations "moyennisantes".

Théorème 0.6. *Supposons que C soit strictement convexe avec $C'(0)<0$, $\left((\bar{S}_{t_i}^{(n)})_{0\le i\le m}\right)_{n\ge1}$ est une suite i.i.d. à valeurs dans $\mathbb{R}^{m+1}$ de copies d'un schéma d'Euler à pas constant de $(S_t)_{0\le t\le T}$ de pas $\frac{T}{m}$. De plus supposons que la suite de pas de l'algorithme vérifie l'hypothèse standard*

$$\sum_{n\ge1}\gamma_n = +\infty \quad et \quad \sum_{n\ge1}\gamma_n^2 < +\infty.$$

Alors la procédure récursive définie par (0.23) converges p.s. vers sa cible $\delta^* = \underset{\delta\in[0,\delta_{\max}]}{\mathrm{argmin}}\, C(\delta)$:

$$\delta_n \xrightarrow{p.s.} \delta^*.$$

Définition 0.2. *Soit $q\in\mathbb{N}$. Une suite $(\xi_n)_{n\ge1}$ à valeurs dans $[0,L]^q$ est ν-moyennisante si*

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\xi_k} \overset{(\mathbb{R}^q)}{\Longrightarrow} \nu \quad quand\ n\to\infty.$$

Alors $(\xi_n)_{n\ge1}$ vérifie

$$D_n^*(\xi) := \sup_{x\in[0,L]^q}\left|\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}_{[\![0,x]\!]}(\xi_k) - \nu([\![0,x]\!])\right| \longrightarrow 0 \quad quand\ n\to\infty.$$

Dans ce cadre, on a besoin de l'existence d'une fonction de Lyapunov *trajectorielle*, ce qui signifie dans ce cadre unidimensionnel que $H(\cdot,(y_{t_i})_{0\le i\le m})$ est monotone, de monotonie indépendante de $(y_{t_i})_{0\le i\le m}\in\mathbb{R}_+^{m+1}$.

Théorème 0.7. *Soit $\lambda(x) = Ae^{-kx}$, $A>0$, $k>0$. Supposons que C est strictement convexe avec $C'(0)<0$ et $C'(\delta_{\max})>0$, $\left(\bar{S}^{(n)}\right)_{n\ge1}$ est une suite à valeurs dans $[0,L]^q$ ν-moyennisante et $(\gamma_n)_{n\ge1}$ est une suite positive décroissante vérifiant*

$$\sum_{n\ge1}\gamma_n = +\infty, \quad nD_n^*(\xi)\gamma_n \underset{n\to\infty}{\longrightarrow} 0, \quad et \quad \sum_{n\ge1} nD_n^*(\xi)\max\left(\gamma_n^2, |\Delta\gamma_{n+1}|\right) < +\infty.$$

De plus, supposons que

$$Q_T \geq 2T\lambda(-S_0) \quad \text{et} \quad \begin{cases} \kappa \leq \dfrac{1+k(S_0-\delta_{\max})}{k\,\|S\|_\infty} & \text{si} \quad \Phi \neq \mathrm{id} \\ \kappa \leq \dfrac{1+k(S_0-\delta_{\max})}{k\,\|S\|_\infty\,(\Phi(Q_T)-\Phi(Q_T-1))} & \text{si} \quad \Phi = \mathrm{id} \end{cases}$$

Alors la procédure récursive définie par (0.23) converges p.s. vers sa cible $\delta^* = \underset{\delta\in[0,\delta_{\max}]}{\operatorname{argmin}}\, C(\delta)$ *:*

$$\delta_n \xrightarrow{p.s.} \delta^*.$$

Pour appliquer ce théorème à notre problème d'optimisation, on a besoin de vérifier que la fonction de coût est convexe avec $C'(0) < 0$. Ceci aboutit à des bornes pour le paramètre κ, mais ces bornes ne sont pas faciles à calculer en pratique puisqu'il s'agit de rapport d'espérance. On propose donc d'approcher ces bornes et on donne des critères suffisants qui assurent que C vérifient les propriétés requises. Ces critères impliquent des conditions sur κ, Q_T et Φ. Nous obtenons ces résultats en appliquant un principe de monotonie opposée pour les diffusions unidimensionnelles à l'aide de la transformation de Lamperti.

Soit I un intervalle ouvert non vide de $\mathbb{R}$. Considérons le processus de diffusion réel

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t, \quad X_0 = x_0 \in I, \quad t\in[0,T]. \tag{0.25}$$

Définition 0.3. *La* transformation de Lamperti *du processus de diffusion (0.25) est la fonction* $L : [0,T]\times I \to \mathbb{R}$ *définie pour tout* $(t,x)\in[0,T]\times I$ *par*

$$L(t,x) := \int_{x_1}^{x} \frac{d\xi}{\sigma(t,\xi)} \tag{0.26}$$

où x_1 *est une valeur arbitraire fixée de* I*. Soit* $t\in[0,T]$*. L'inverse de* $L(t,\cdot)$ *sera notéc* $L^{-1}(t,\cdot)$*.*

Définition 0.4. *Le processus de diffusion (0.25) est* admissible *si*

(i) $\sigma \in \mathcal{C}^1([0,T]\times I, I)$,

(ii) $\forall (t,x)\in[0,T]\times I$, $|b(t,x)| \leq C(1+|x|)$ *et* $0 < \sigma(t,x) \leq C(1+|x|)$,

(iii) $\forall\, x\in I$, $\int_{(-\infty,x]\cap I} \frac{d\xi}{\sigma(t,\xi)} = \int_{[x,+\infty,)\cap I} \frac{d\xi}{\sigma(t,\xi)} = +\infty$,

(iv) pour toute valeur initiale $x_0 \in I$*, (0.25) a une unique solution faible qui vit dans* I *jusqu'à* $t = +\infty$,

(v) la fonction β *définie par*

$$\beta(t,y) := \left(\frac{b}{\sigma} - \int_{x_1}^{\cdot} \frac{1}{\sigma^2(t,\xi)}\frac{\partial\sigma}{\partial t}(t,\xi)d\xi - \frac{1}{2}\frac{\partial\sigma}{\partial x}\right)(t, L^{-1}(t,y)),$$

est continue sur $[0,T]\times\mathbb{R}$*, croissante en* y *pour tout* $t\in[0,T]$ *et vérifie*

$$\exists\, K > 0 \;\; \text{tel que} \;\; |\beta(t,y)| \leq K(1+|y|),\; t\in[0,T],\; y\in\mathbb{R}.$$

Supposons que Λ qui dépend du paramètre δ soit *essentiellement exponentielle* au sens suivant

$$0 < \underline{k}_1 := \operatorname*{infess}_{\delta\in[0,\delta_{\max}]} \left(-\frac{\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)}{\Lambda_T(\delta,S)}\right) \leq \overline{k}_1 := \operatorname*{supess}_{\delta\in[0,\delta_{\max}]} \left(-\frac{\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)}{\Lambda_T(\delta,S)}\right) < +\infty,$$

$$0 < \underline{k}_2 := \operatorname*{infess}_{\delta\in[0,\delta_{\max}]} \left(-\frac{\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)}{\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)}\right) \leq \overline{k}_2 := \operatorname*{supess}_{\delta\in[0,\delta_{\max}]} \left(-\frac{\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)}{\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)}\right) < +\infty.$$

Notons que l'hypothèse ci-dessus implique

$$k_0 := \operatorname{infess}\left(-\frac{\frac{\partial}{\partial\delta}\Lambda_T(0,S)}{\Lambda_T(0,S)}\right) \geq \underline{k}_1 > 0.$$

Théorème 0.8. *Supposons que la dynamique $(S_t)_{t\geq 0}$ du prix S est une diffusion brownienne à valeurs dans $(0,\infty)$*

$$dS_t = b(t,S_t)dt + \sigma(t,S_t)dW_t, \quad S_0 > 0,$$

admissible pour $I=(0,\infty)$ au sens de la Définition 0.4. Supposons que la fonction λ soit essentiellement exponentielle. Alors on a les critères suivants.

(a) Monotonie à l'origine : *$C'(0) < 0$ dès que*

$$Q_T \geq 2T\lambda(-S_0) \quad \text{et} \quad \kappa \leq \frac{S_0}{\mathbb{E}\left[S_T\right](\Phi(Q_T)-\Phi(Q_T-1))} + \frac{1}{k_0\mathbb{E}\left[S_T\right](\Phi(Q_T)-\Phi(Q_T-1))}.$$

(b) Convexité. *Soit $\rho_Q \in \left(0, 1-\frac{\mathbb{P}(N^\mu=Q_T-1)}{\mathbb{P}(N^\mu\leq Q_T-1)}_{|\mu=T\lambda(-S_0)}\right)$. Si $\Phi \neq \mathrm{id}$, supposons que Φ vérifie*

$$\forall x \in [1, Q_T-1], \quad \Phi(x)-\Phi(x-1) \leq \rho_Q(\Phi(x+1)-\Phi(x)).$$

Si

$$Q_T \geq \left(2 \vee \left(1+\frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\right)\right)T\lambda(-S_0) \quad \text{et} \quad \kappa \leq \frac{2\underline{k}_1}{\overline{k}_1\overline{k}_2\mathbb{E}\left[S_T\right]\Phi'(Q_T)},$$

alors $C''(\delta) \geq 0$, $\delta \in [0,\delta_{\max}]$, donc C est convexe sur $[0,\delta_{\max}]$.

0.6 Implicitation et calibration de paramètres dans des modèles financiers

Ce chapitre illustre comment utiliser les algorithmes stochastiques pour impliciter ou calibrer des paramètres dans des modèles financiers. Le problème de l'implicitation aboutit à une procédure récursive de recherche de zéro et la calibration à un problème d'optimisation résolu par une méthode de gradient stochastique. Nous illustrons nos résultats sur trois modèles différents, à savoir le modèle standard de Black-Scholes, le modèle pseudo-CEV et le modèle de Merton.

Soit $\Theta \subset \mathbb{R}^d$ un ouvert convexe de $\mathbb{R}^d$. Considérons q actifs risqués dont la dynamique est régie par un processus de diffusion dont les coefficients dépendent d'un paramètre $\theta \in \Theta$, *i.e.*

$$dX_t(\theta) = b(\theta,t,X_t(\theta))dt + \sigma(\theta,t,X_t(\theta)).dW_t, \quad X_0(\theta) = x_0(\theta) \in \mathbb{R}^q, \tag{0.27}$$

où $b : \Theta \times [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$, $\sigma : \Theta \times [0,T] \times \mathbb{R}^q \to \mathcal{M}_{q,\ell}(\mathbb{R})$ et W est un mouvement brownien ℓ-dimensionnel défini sur un espace de probabilité $(\Omega,\mathcal{A},\mathbb{P})$.

0.6.1 Implicitation

Supposons que la dynamique de q actifs risqués $(X_t(\theta))_{t\geq 0}$ est régie par (0.27) où les coefficients b et σ sont des fonctions continues et lipschitziennes en x uniformément par rapport à $(\theta,t) \in \Theta \times [0,T]$. Ces hypothèses assurent l'existence d'une unique solution forte à l'EDS ci-dessus. Soit $X_t^0 = e^{rt}$ l'actif sans risque.

Le cadre considéré est le suivant : Soit $\varphi : \mathbb{R}^q \to \mathbb{R}^d$ une fonction de payoff et $P_0 \in \mathbb{R}^d$ un vecteur de prix de marché. Le but est de résoudre à l'aide d'un algorithme stochastique le problème inverse en $\theta \in \Theta$

$$e^{-rT}\mathbb{E}\left[\varphi(X_T(\theta))\right] = P_0.$$

Supposons que l'équation ci-dessus en θ a au moins une solution notée θ^* et que P_0 est consistant, mais n'est pas une valeur extremale de $\theta \mapsto e^{-rT}\mathbb{E}\left[\varphi(X_T(\theta))\right]$. Ainsi on cherche un zéro de la fonction moyenne $h : \mathbb{R}^d \to \mathbb{R}^d$ définie par

$$h(\theta) = e^{-rT}\mathbb{E}\left[\varphi(X_T(\theta))\right] - P_0 = \mathbb{E}\left[H(\theta, Z)\right],$$

où

$$H(\theta, z) = e^{-rT}\varphi(X_T(\theta)) - P_0$$

et Z est un vecteur de variables aléatoires i.i.d. utilisées pour calculer $X_T(\theta)$, soit exactement, soit par un schéma d'Euler.

0.6.2 Calibration

Soit $\Theta \subset \mathbb{R}^d$ un ouvert convexe de $\mathbb{R}^d$ et

$$\begin{array}{rcl} Y : (\Theta \times \Omega, \mathcal{B}or(\Theta) \otimes \mathcal{A}) & \longrightarrow & (\mathbb{R}^p, \mathcal{B}or(\mathbb{R}^p)) \\ (\theta, \omega) & \longmapsto & Y_\theta(\omega) \end{array}$$

un processus bi-dimensionnel indexé par $\theta \in \Theta$. Supposons de plus que

$$\forall \theta \in \Theta, \quad Y_\theta \in L^1_{\mathbb{R}^p}(\Omega, \mathcal{A}, \mathbb{P}).$$

Typiquement Y_θ est un vecteur de p payoffs (actualisés) recentrés par leur prix de marché portant sur q actifs risqués dont la dynamique est régie par (0.27)

$$Y_\theta = (F_i(\theta, X_{T_i}(\theta)))_{i=1,\dots,p}.$$

Soit $|\cdot|_S$ la norme euclidienne sur $\mathbb{R}^p$ dérivant d'une matrice définie positive S. Le produit scalaire résultant est défini par

$$\forall u, v \in \mathbb{R}^p, \quad \langle u|v\rangle_S = u^t S v,$$

où u^t est la transposée du vecteur colonne $u \in \mathbb{R}^p$.

Le *paradigme de la calibration* est alors le problème de minimisation suivant

$$(C) \equiv \mathrm{argmin}_{\theta\in\Theta}|\mathbb{E}Y_\theta|_S = \mathrm{argmin}_{\theta\in\Theta}\left[L(\theta) := \frac{1}{2}|\mathbb{E}Y_\theta|_S^2\right]$$

qui consiste à trouver le paramètre $\theta^* \in \Theta$ qui donne les primes du portefeuille de p produits dérivés les plus proches de leurs prix de marché au sens des moindres carrés pour la norme $|\cdot|_S$.

Remarque. On peut ajouter une fonction de pénalisation à la fonction L, typiquement $\frac{1}{2}|\theta - \theta_{-1}|^2$, où θ_{-1} est une valeur de référence comme celle, obtenue par calibration, de la veille. Ceci aboutit au problème de minimisation suivant

$$(C') \equiv \mathrm{argmin}_{\theta\in\Theta} L(\theta) + \frac{\kappa}{2}|\theta - \theta_{-1}|^2,$$

où $\kappa > 0$ est un coefficient de pénalisation.

Pour résoudre ce problème par simulation, deux approches peuvent être implémentées :

- celle de Robbins-Siegmund qui nécessite de connaître la représentation du gradient de L (supposé exister).
- celle de Kiefer-Wolfowitz qui est la contrepartie de celle de Robbins-Siegmund basée sur une méthode de différences finies (à pas décroissant).

▷ **L'approche de Robbins-Siegmund.** Supposons que pour tout $\theta_0 \in \Theta$,

(i) $\mathbb{P}(d\omega)$-$p.s.$, $\theta \mapsto Y_\theta(\omega)$ est différentiable en θ_0 de Jacobiennne $J_{Y(\omega)}(\theta_0) := \left[\frac{\partial Y_\theta^i(\omega)}{\partial \theta^j}\right]_{1\le i\le p, 1\le j\le d}$,

(ii) $\exists \mathcal{U}_{\theta_0}$, voisinage de θ_0 dans Θ, tel que $\left(\frac{|Y_\theta - Y_{\theta_0}|}{|\theta-\theta_0|}\right)_{\theta\in\mathcal{U}_{\theta_0}\setminus\{\theta_0\}}$ est uniformément intégrable.

On vérifie que $\theta \mapsto \mathbb{E}Y_\theta$ est différentiable sur Θ et que sa Jacobienne est donnée par

$$J_{\mathbb{E}Y_\cdot}(\theta) = \mathbb{E}\left[J_{Y(\omega)}(\theta)\right].$$

Alors, la fonction L est différentiable partout sur Θ et son gradient (par rapport à la norme euclidienne canonique) est donné par

$$\forall \theta \in \Theta, \quad \nabla L(\theta) = \mathbb{E}\left[J_Y(\theta)\right]^t S\, \mathbb{E}Y_\theta = \mathbb{E}\left[(J_Y(\theta))^t\right] S\, \mathbb{E}Y_\theta.$$

Maintenant nous avons besoin d'une représentation de $\nabla L(\theta)$ sous forme d'espérance. A cette fin, on construit pour tout $\theta \in \Theta$, une copie indépendante $\widetilde{Y}_\theta$ de Y_θ définie comme suit : on considère l'espace de probabilités produit $(\Omega^2, \mathcal{A}^{\otimes 2}, \mathbb{P}^{\otimes 2})$ et on pose, pour tous $(\omega, \widetilde{\omega}) \in \Omega^2$, $Y_\theta(\omega, \widetilde{\omega}) = Y_\theta(\omega)$ (extension de Y_θ sur Ω^2 encore notée Y_θ) et $\widetilde{Y}_\theta(\omega, \widetilde{\omega}) = Y_\theta(\widetilde{\omega})$. On a directement par le théorème des mesures produits que les deux familles $(Y_\theta)_{\theta\in\Theta}$ et $\left(\widetilde{Y}_\theta\right)_{\theta\in\Theta}$ sont indépendantes et de même loi.

On peut donc écrire

$$\begin{aligned} \forall \theta \in \Theta, \quad \nabla L(\theta) &= \mathbb{E}\left[(J_Y(\theta))^t\right] S\, \mathbb{E}\widetilde{Y}_\theta \\ &= \mathbb{E}\left[(J_Y(\theta))^t\, S\, \widetilde{Y}_\theta\right]. \end{aligned}$$

Par conséquent, la différentiabilité trajectorielle de Y_θ requiert celle de $X_t(\theta)$ par rapport à θ. Cette question est liée au processus θ-tangent $\left(\frac{\partial X_t(\theta)}{\partial \theta}\right)_{t\ge 0}$ de $X(\theta)$. Des propriétés de différentiabilité sont aussi nécessaires sur les fonctions F_i pour remplir l'hypothèse (i). Finalement, on peut écrire formellement

$$\left(J_{Y(\omega)}(\theta)\right)^t S\, \widetilde{Y}_\theta(\omega) = H(\theta, W(\omega))$$

où W désigne les innovations de l'algorithme (soit $W = (W_{T_1}, \ldots, W_{T_p})$ si $X_{T_i}(\theta)$ est simulable exactement, soit $W = \left(\left(\Delta W_{\frac{kT_1}{N}}\right)_{1\le k\le N}, \ldots, \left(W_{\frac{kT_p}{N}}\right)_{1\le k\le N}\right)$ suite d'accroissements de mouvement brownien de pas T_i/N, $i = 1, \ldots, p$, si $X_{T_i}(\theta)$ est approché par son schéma d'Euler à pas constant). On peut alors définir récursivement l'algorithme de recherche de zéro pour ∇L en posant

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, W_{n+1}), \quad \text{avec} \quad H(\theta, W) = (J_Y(\theta))^t\, S\, \widetilde{Y}_\theta,$$

où $(W_n)_{n\ge 1}$ est une suite i.i.d. de copies de W et $(\gamma_n)_{n\ge 1}$ est la suite positive décroissante de pas vérifiant

$$\sum_{n\ge 1} \gamma_n = +\infty \quad \text{et} \quad \sum_{n\ge 1} \gamma_n^2 < +\infty.$$

Reste alors à vérifier que les hypothèses du Lemme de Robbins-Siegmund sont vérifiées pour en déduire la convergence $p.s.$ de l'algorithme de gradient stochastique vers un minimum local de L (ces questions techniques sont détaillées dans le chapitre et quelques astuces données pour les résoudre).

▷ **Autre méthode : dérivation au sens "faible".** Si le processus $(X_t(\theta))_{t\geq 0}$ est à trajectoires càdlàg, alors le processus tangent trajectoriel n'existe pas. Dans certains cas, on peut dériver directement la fonction moyenne par rapport à θ et avoir la représentation sous forme d'espérance de sa dérivée. Nous illustrons cette méthode quand θ est l'intensité d'un processus de Poisson en utilisant une formule sur les variables aléatoires de Poisson qui peut être vue comme un exemple élémentaire de calcul de Malliavin pour les processus de Poisson.

Pour être précis, si $\varphi(\lambda) = \mathbb{E}\left[\Phi(\lambda, N^\lambda)\right]$, où $(N^\lambda)_{\lambda\geq 0}$ est une famille de variables aléatoires de Poisson de paramètre λ. Alors, sous des conditions naturelles sur Φ et en utilisant une propriété des variables de Poisson, on a

$$\varphi'(\lambda) = \mathbb{E}\left[\frac{\partial}{\partial\lambda}\Phi(\lambda, N^\lambda)\right] + \mathbb{E}\left[\Phi(\lambda, N^\lambda + 1) - \Phi(\lambda, N^\lambda)\right].$$

▷ **L'approche de Kieffer-Wolfowitz.** Des implémentations pratiques de l'approche de Robbins-Siegmund indiquent une seconde difficulté technique : la fonction aléatoire $\theta \mapsto Y_\theta$ n'est pas toujours différentiable trajectoriellement (ni au sens $L^r(\mathbb{P})$ qui pourrait être suffisant). Plus important d'une certaine façon, même si Y_θ est différentiable, l'expression résultant pour $\nabla_\theta Y_\theta$ peut être assez compliquée, requérant plus de soins à programmer. Une alternative est donnée par l'algorithme de Kiefer-Wolfowitz (K-W) qui combine le principe d'approximation stochastique récursive avec une approche par différences finies pour la différentiation. L'idée est simplement d'approcher le gradient ∇L par

$$\frac{\partial L}{\partial\theta_i}(\theta) \approx \frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta^i}, \quad 1 \leq i \leq p,$$

où $(e_i)_{1\leq i\leq p}$ désigne la base canonique de $\mathbb{R}^p$ et $\eta = (\eta^i)_{1\leq i\leq p}$. Ce terme de différence finie a une représentation intégrale donnée par

$$\frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta^i} = \mathbb{E}\frac{\Lambda(\theta + \eta^i e_i, W) - \Lambda(\theta - \eta^i e_i, W)}{2\eta^i},$$

où

$$\Lambda(\theta, W) := Y(\theta, W)^t S Y(\theta, W),$$

($Y(\theta, W)$ est reliée à l'innovation W). Partant de cette représentation, on peut déduire une formule de mise à jour récursive pour θ_n comme suit

$$\theta^i_{n+1} = \theta^i_n - \gamma_{n+1}\frac{\Lambda(\theta_n + \eta^i_{n+1}e_i, W^{n+1}) - \Lambda(\theta_n - \eta^i_{n+1}e_i, W^{n+1})}{2\eta^i_{n+1}}, \quad 1 \leq i \leq p.$$

On reproduit ci-dessous un résultat classique de convergence pour les procédures K-W (qui est la contrepartie naturelle du Lemme de Robbins-Siegmund dans le cadre de gradient stochastique).

Théorème 0.9. *Supposons que la fonction $\theta \mapsto L(\theta)$ soit deux fois différentiable de Hessienne lipschitzienne. Supposons que*

$$\theta \mapsto Lambda(\theta, W) \text{ est Lipschitz dans } L^2$$

et que les suites de pas vérifient

$$\sum_{n\geq 1}\gamma_n = \sum_{n\geq 1}\eta^i_n = +\infty, \quad \sum_{n\geq 1}\gamma_n^2 < +\infty, \quad \eta_n \to 0 \quad \text{et} \quad \sum_{n\geq 1}\left(\frac{\gamma_n}{\eta^i_n}\right)^2 < +\infty.$$

Alors, θ_n converge p.s. vers une composante connexe de $\{L = \ell\} \cap \{\nabla L = 0\}$ pour un certain niveau $\ell \geq 0$.

0.7 Modèles d'urne aléatoire par algorithmes stochastiques (article soumis)

Ce chapitre présente le lien entre algorithmes stochastiques et essais cliniques basés sur des modèles d'urnes aléatoires étudiés dans [17, 18, 19]. Nous reformulons les dynamiques de la composition de l'urne et du nombre de fois que chaque traitement est administré comme des algorithmes stochastiques standards avec reste. Nous en déduisons alors la convergence *p.s.* (consistance forte) et la normalité asymptotique (Théorème Central Limit TCL) des procédures normalisées sous des hypothèses plus faibles en utilisant les méthodes de l'EDO et de l'EDS. Dans un second temps, nous étudions une famille de modèle très répandue, connue sous le nom d'essais cliniques multi-bras, où la mise à jour de la composition de l'urne dépend des performances passées des traitements. En augmentant la dimension du vecteur d'état, notre approche par algorithme stochastique fournit cette fois un nouveau résultat de normalité asymptotique.

La modélisation repose sur les urnes de Pólya généralisée (GPU). L'idée est que l'urne contient des boules de d types différents représentant les traitements. Toutes les variables aléatoires impliquées dans le modèle sont supposées être définies sur un même espace de probabilités $(\Omega, \mathcal{A}, \mathbb{P})$. Soit $Y_0 = (Y_0^i)_{i=1,\dots,d} \in \mathbb{R}_+^d \setminus \{0\}$ la composition initiale de l'urne, où Y_0^i désigne le nombre de boules de type i, $i = 1, \dots, d$. L'allocation des traitements est séquentielle et la composition de l'urne au tirage n est notée $Y_n = (Y_n^i)_{i=1,\dots,d}$. Quand le patient n se présente, on tire aléatoirement (*i.e.* uniformément) une boule de l'urne avec remise. Si la boule est de type j, alors le traitement j est administré au patient n, $i = 1, \dots, d$, $n \geq 1$. La composition de l'urne est mise à jour en prenant en compte la réaction du patient n au traitement j, ou les réactions de tous les patients jusqu'au $n^{\text{ème}}$ (*i.e.* l'efficacité du traitement administré), à savoir en ajoutant D_n^{ij} boules de type i, $i = 1, \dots, d$. Par conséquent, plus il y a de boules d'un type donné dans l'urne, plus le traitement est efficace. La procédure est itérée tant que des patients se présentent. La composition de l'urne à l'étape n est modélisée par un vecteur Y_n à valeurs dans $\mathbb{R}^d$ dont la dynamique est la suivante

$$Y_n = Y_{n-1} + D_n X_n, \quad Y_0 \in \mathbb{R}_+^d \setminus \{0\}, \tag{0.28}$$

avec $D_n = (D_n^{ij})_{1 \leq i,j \leq d}$ la matrice des règles d'ajout et $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to \{e^1, \cdots, e^d\}$ le résultat du $n^{\text{ème}}$ tirage ($\{e^1, \cdots, e^d\}$ désigne la base canonique de $\mathbb{R}^d$ et e^i symbolise le traitement i). On modélise le tirage dans l'urne en posant

$$X_n = \sum_{i=1}^{d} \mathbb{1}_{\left\{ \frac{\sum_{\ell=1}^{i-1} Y_{n-1}^{\ell}}{\sum_{\ell=1}^{d} Y_{n-1}^{\ell}} < U_n \leq \frac{\sum_{\ell=1}^{i} Y_{n-1}^{\ell}}{\sum_{\ell=1}^{d} Y_{n-1}^{\ell}} \right\}} e^i, \quad n \geq 1, \tag{0.29}$$

où $(U_n)_{n \geq 1}$ est i.i.d. de loi $U_1 \overset{\mathcal{L}}{\sim} \mathcal{U}_{[0,1]}$.

Soit $\mathcal{F}_n = \sigma(Y_0, U_k, D_k, 1 \leq k \leq n)$ la filtration de la procédure. Les *matrices génératrices* sont définies comme le $\mathcal{F}_n$-compensateur de la suite des règles d'ajout *i.e.*

$$H_n = \left(\mathbb{E}\left[D_n^{ij} \,|\, \mathcal{F}_{n-1} \right] \right)_{1 \leq i,j \leq d}, \; n \geq 1.$$

0.7.1 Convergence et premier résultat de vitesse

Pour établir la convergence *p.s.* de la composition de l'urne renormalisée, nous avons besoin des hypothèses suivantes

(H1) Les matrices génératrices $H_n = (H_n^{ij})_{1 \leq i,j \leq d}$, $n \geq 1$, vérifient *p.s.*

$$\forall\, i, j \in \{1, \dots, d\}, \quad H_n^{ij} \geq 0 \quad \text{et} \quad \forall\, j \in \{1, \dots, d\}, \quad \sum_{i=1}^{d} H_n^{ij} = 1 > 0.$$

(H2) La règle d'ajout D_n est indépendante de la procédure de tirage X_n conditionnellement à $\mathcal{F}_{n-1}$ et vérifie

$$\forall 1 \leq j \leq d, \quad \sup_{n\geq 1} \mathbb{E}\left[\left\|D_n^{\cdot j}\right\|^2 \mid \mathcal{F}_{n-1}\right] < +\infty.$$

(H3) Supposons qu'il existe une matrice $d \times d$ irréductible H (à coefficients positifs) telle que

$$H_n \xrightarrow[n\to\infty]{p.s.} H.$$

Cette hypothèse garantit par le Théorème de Perron-Frobenius (voir [30]) que 1 est la valeur propre maximale de H et que les composantes de son vecteur propre (à droite) v^* peuvent être choisies toutes positives. Ainsi, nous normalisons ce vecteur v^* tel que $\mathrm{Tr}(v^*) = 1$. De plus **(H3)** est plus faible que l'hypothèse faite dans [18], à savoir

$$\sum_{n\geq 1} \frac{\|H_n - H\|_\infty}{n} < +\infty \quad p.s.$$

Nous allons reformuler les dynamiques de (0.28)-(0.29) sous forme d'algorithmes stochastiques pour pouvoir appliquer la méthode de l'EDO et en déduire la convergence $p.s.$. Partant de (0.28) avec $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$ et utilisant les propriétés des matrices génératrices H_n, on obtient la procédure récursive suivante pour $\widetilde{Y}_n = \dfrac{Y_n}{n}$, $n \geq 1$,

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\left(I_d - H\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + r_{n+1}\right)$$

de pas $\gamma_n = \frac{1}{n}$ avec un $\mathcal{F}_n$ accroissement de martingale

$$\Delta M_{n+1} := D_{n+1}X_{n+1} - \mathbb{E}\left[D_{n+1}X_{n+1} \mid \mathcal{F}_n\right]$$

et un terme de reste donné par

$$r_{n+1} := \left(\frac{n}{\mathrm{Tr}(Y_n)} - 1\right) H_{n+1}\widetilde{Y}_n + (H_{n+1} - H)\widetilde{Y}_n.$$

De plus, pour établir la bornitude $p.s.$ de $(\widetilde{Y}_n)_{n\geq 0}$, on utilise l'équation récursive suivante vérifiée par $\mathrm{Tr}(Y_n)$:

$$\mathrm{Tr}(Y_{n+1}) = \mathrm{Tr}(Y_n) + 1 + \mathrm{Tr}(\Delta M_{n+1}).$$

L'*EDO* associée à la procédure récursive est

$$ODE_{I_d - H} \equiv \dot{y} = -(I_d - H)y.$$

Ainsi, une fois la décomposition appropriée établie pour $\widetilde{Y}_n$ (avec le reste), la méthode de l'*EDO* pour les algorithmes stochastiques s'applique directement et on obtient alors le Théorème suivant :

Théorème 0.10. *Sous les hypothèses* **(H1)**, **(H2)** *et* **(H3)**,

(a) $\dfrac{\mathrm{Tr}(Y_n)}{n} \xrightarrow[n\to\infty]{p.s.} 1$ *et* $\dfrac{Y_n}{\mathrm{Tr}(Y_n)} \xrightarrow[n\to\infty]{p.s.} v^*$.

(b) $\widetilde{N}_n := \dfrac{N_n}{n} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{p.s.} v^*$.

0.7.1.1 Vitesse de convergence

Nous allons établir un "TCL joint" pour le couple $\theta_n := (\widetilde{Y}_n, \widetilde{N}_n)^t$ avec une distribution gaussienne jointe asymptotique explicite (incluant les covariances). En réécrivant les dynamiques des deux composantes pour que le reste vérifie les hypothèses adéquates, on obtient

$$\theta_{n+1} = \theta_n - \frac{1}{n+1} h(\theta_n) + \frac{1}{n+1} \left(\Delta \mathbf{M}_{n+1} + R_{n+1} \right),$$

avec $\Delta \mathbf{M}_{n+1} := \begin{pmatrix} \Delta M_{n+1} \\ \Delta \widetilde{M}_{n+1} \end{pmatrix}$, où $\Delta \widetilde{M}_{n+1} := X_{n+1} - \mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n \right]$, $R_{n+1} := \begin{pmatrix} \bar{r}_{n+1} \\ \widetilde{r}_{n+1} \end{pmatrix}$, où

$$\bar{r}_{n+1} := \left(\frac{H_{n+1} - H}{\mathrm{Tr}(\widetilde{Y}_n)} + \frac{(\mathrm{Tr}(\widetilde{Y}_n) - 1)^2}{\mathrm{Tr}(\widetilde{Y}_n)} H \right) \widetilde{Y}_n \quad \text{et} \quad \widetilde{r}_{n+1} := \frac{(\mathrm{Tr}(\widetilde{Y}_n) - 1)^2}{\mathrm{Tr}(\widetilde{Y}_n)} \widetilde{Y}_n,$$

et

$$\forall\, \theta = \begin{pmatrix} y \\ \nu \end{pmatrix},\ y \in \mathbb{R}^d,\ \nu \in \mathbb{R}^d, \quad h(\theta) := \begin{pmatrix} (I_d - (2 - \mathrm{Tr}(y))H)y \\ \nu - (2 - \mathrm{Tr}(y))y \end{pmatrix} \quad \text{avec} \quad h(\theta^*) = 0.$$

La fonction h est différentiable sur $\mathbb{R}^{2d}$ et sa différentielle au point θ^* est donnée par

$$Dh(\theta^*) = \begin{pmatrix} I_d - H + v^* \mathbf{1}^t & 0_{\mathcal{M}_d(\mathbb{R})} \\ v^* \mathbf{1}^t - I_d & I_d \end{pmatrix}.$$

Pour établir un TCL pour la suite $(\theta_n)_{n \geq 1}$, on doit faire les hypothèses supplémentaires suivantes :

(H4) Les règles d'ajout D_n vérifient $p.s.$

$$\forall 1 \leq j \leq d, \quad \begin{cases} \sup_{n \geq 1} \mathbb{E}\left[\| D_n^{\cdot j} \|^{2+\delta} \,|\, \mathcal{F}_{n-1} \right] \leq C < +\infty \quad \text{pour un } \delta > 0, \\ \mathrm{Cov}\left[D_n^{\cdot j} (D_n^{\cdot j})^t \,|\, \mathcal{F}_{n-1} \right] \longrightarrow C^j, \end{cases}$$

où $C^j = (C^j_{il})_{1 \leq i,l \leq d}$, $j = 1, \ldots, d$, sont des matrices $d \times d$ définies positives.

(H5) La matrice H vérifie

$$n\, \mathbb{E}\left[|||H_n - H|||^2 \right] \underset{n \to \infty}{\longrightarrow} 0.$$

Le Théorème suivant est établi par simple application du TCL pour les algorithmes stochastiques. On retrouve ainsi "gratuitement" un résultat établi dans [18] par une méthode spécifique.

Théorème 0.11. *Supposons que* **(H1)**, **(H3)**, **(H4)** *et* **(H5)** *soient vérifiées et que*

$$\mathcal{R}e\left(\mathrm{Sp(H)} \setminus \{1\} \right) < 1/2. \tag{0.30}$$

Alors, $\theta_n \to \theta^*$ *p.s. quand* $n \to \infty$ *et*

$$\sqrt{n}\left(\theta_n - \theta^* \right) \underset{n \to \infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0, \Sigma) \qquad \textit{avec} \qquad \Sigma = \int_0^{+\infty} e^{u(Dh(\theta^*) - \frac{I}{2})} \Gamma e^{u(Dh(\theta^*) - \frac{I}{2})^t} du$$

$$\textit{et} \quad \Gamma = \begin{pmatrix} \displaystyle\sum_{k=1}^{d} v^{*k} C^k & H\left(\mathrm{diag}(v^*) - v^*(v^*)^t \right) \\ \left(\mathrm{diag}(v^*) - v^*(v^*)^t \right)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t \end{pmatrix} = p.s.\text{-} \lim_{n \to \infty} \mathbb{E}\left[\Delta \mathbf{M}_n \Delta \mathbf{M}_n^t \,|\, \mathcal{F}_{n-1} \right].$$

0.7.2 Application au modèle d'urne multi-bras de Bai-Hu-Shen

Considérons le modèle introduit dans [19] (et considéré encore dans [18]) et définissons un *indicateur de succès* comme suit : soit $(T_n^i)_{n\geq 1}$, $1 \leq i \leq d$, d suites indépendantes de variables aléatoires i.i.d. de Bernoulli à valeurs dans $\{0,1\}$, indépendantes de la suite i.i.d. $(U_n)_{n\geq 1}$, telles que

$$\mathbb{E}\left[T_n^i\right] = p^i, \quad 0 < p^i < 1, \quad 1 \leq i \leq d.$$

La filtration de la procédure est $\mathcal{F}_n = \sigma\left(Y_0, U_k, T_k, 1 \leq k \leq n\right)$, $n \geq 0$. Soit $N_n = (N_n^1, \ldots, N_n^d)^t$ et $S_n = (S_n^1, \ldots, S_n^d)^t$, où $N_n^i = N_{n-1}^i + X_n^i$, $n \geq 1$, désigne toujours le nombre de fois que le traitement i est choisi parmi les n premiers tirages et

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \quad n \geq 1,$$

désigne le *nombre de succès* du traitement i parmi les N_n^i essais, $i = 1, \ldots, d$. Cependant, pour éviter la dégénérescence de la procédure, nous faisons l'hypothèse d'initialisation suivante

$$N_0^i = 1, \; S_0^i = 1, \quad i = 1, \ldots, d.$$

Définissons $Q_n = (Q_n^1, \ldots, Q_n^d)^t$, où $Q_n^i = \frac{S_n^i}{N_n^i}$, $i = 1, \ldots, d$ et $E_n = \sum_{i=1}^d Q_n^i$. Dans [19] les auteurs considèrent la règle d'ajout suivante,

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{Q_n^1(1-T_{n+1}^2)}{E_n - Q_n^2} & \cdots & \frac{Q_n^1(1-T_{n+1}^d)}{E_n - Q_n^d} \\ \frac{Q_n^2(1-T_{n+1}^1)}{E_n - Q_n^1} & T_{n+1}^2 & \cdots & \frac{Q_n^2(1-T_{n+1}^d)}{E_n - Q_n^d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{Q_n^d(1-T_{n+1}^1)}{E_n - Q_n^1} & \frac{Q_n^d(1-T_{n+1}^2)}{E_n - Q_n^2} & \cdots & T_{n+1}^d \end{pmatrix},$$

i.e. au tirage $n+1$, si la réaction au traitement i est un succès, alors une boule de type i est ajoutée dans l'urne. Sinon, $\frac{Q_n^j}{E_n - Q_n^i}$ boules (virtuelles) de type j, $j \neq i$, sont ajoutées. Alors, on peut facilement vérifier que les matrices génératrices sont données par

$$H_{n+1} = \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right] = \begin{pmatrix} p^1 & \frac{Q_n^1}{E_n - Q_n^2} q^2 & \cdots & \frac{Q_n^1}{E_n - Q_n^d} q^d \\ \frac{Q_n^2}{E_n - Q_n^1} q^1 & p^2 & \cdots & \frac{Q_n^2}{E_n - Q_n^d} q^d \\ \vdots & \vdots & \ddots & \vdots \\ \frac{Q_n^d}{E_n - Q_n^1} q^1 & \frac{Q_n^d}{E_n - Q_n^2} q^2 & \cdots & p^d \end{pmatrix}.$$

Dès que $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$, $H_n \xrightarrow{p.s.} H$ qui s'écrit

$$H = \begin{pmatrix} p^1 & \frac{p^1}{E - p^2} q^2 & \cdots & \frac{p^1}{E - p^d} q^d \\ \frac{p^2}{E - p^1} q^1 & p^2 & \cdots & \frac{p^2}{E - p^d} q^d \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^d}{E - p^1} q^1 & \frac{p^d}{E - p^2} q^2 & \cdots & p^d \end{pmatrix} \quad \text{où } E = p^1 + \cdots + p^d.$$

La matrice H est clairement irréductible puisque $0 < p^i < 1$, $1 \leq i \leq d$. Alors, par le Théorème 0.10 on obtient

$$\widetilde{Y}_n = \frac{Y_n}{n} \xrightarrow[n\to\infty]{p.s.} v^* \quad \text{et} \quad \widetilde{N}_n = \frac{N_n}{n} \xrightarrow[n\to\infty]{p.s.} v^*.$$

Pour déduire un TCL, ce qui est nouveau et laissé en conjecture dans [18], on doit augmenter la dimension du vecteur d'état de la procédure en considérant la suite aléatoire $3d$-dimensionnelle

$$\widetilde{\theta}_n = \begin{pmatrix} \widetilde{Y}_n \\ \widetilde{N}_n \\ \widetilde{S}_n \end{pmatrix} \quad \text{où} \quad \widetilde{S}_n = \frac{S_n}{n}, \quad n \geq 1.$$

La première étape est de remarquer que la matrice génératrice H_{n+1} peut s'écrire comme fonction de $\widetilde{S}_n$ et $\widetilde{N}_n$, *i.e.* $H_{n+1} = \Phi(\widetilde{S}_n, \widetilde{N}_n)$, où $\Phi : \mathbb{R}_+^d \times (0,\infty)^d \to \mathcal{M}_d(\mathbb{R})$ est une fonction différentiable définie par

$$\Phi(s,\nu) = \left(\Phi^{ij}(s,\nu)\right)_{1\leq i,j\leq d} \quad \text{où} \quad \begin{cases} \Phi^{ii}(s,\nu) = p^i & 1 \leq i \leq d \\ \Phi^{ij}(s,\nu) = \frac{s^i/\nu^i}{\sum_{k\neq j} s^k/\nu^k}\, q^j & 1 \leq i,j \leq d,\ i \neq j. \end{cases}$$

On montre ensuite que $\widetilde{S}_n$ satisfait la procédure récursive suivante

$$\widetilde{S}_{n+1} = \widetilde{S}_n - \frac{1}{n+1}\left(\widetilde{S}_n - \mathrm{diag}(p)(2 - \mathrm{Tr}(\widetilde{Y}_n))\widetilde{Y}_n\right) + \frac{1}{n+1}\left(\Delta\widehat{M}_{n+1} + \widehat{r}_{n+1}\right)$$

$$\text{où} \quad \Delta\widehat{M}_{n+1} := \mathrm{diag}(T_{n+1})X_{n+1} - \mathbb{E}\left[\mathrm{diag}(T_{n+1})X_{n+1} \,|\, \mathcal{F}_n\right] = \mathrm{diag}(T_{n+1})X_{n+1} - \mathrm{diag}(p)\frac{\widetilde{Y}_n}{\mathrm{Tr}(Y_n)}$$

est un $\mathcal{F}_n$-accroissement de martingale et $\widehat{r}_{n+1} = \mathrm{diag}(p)\frac{\left(\mathrm{Tr}(\widetilde{Y}_n)-1\right)^2}{\mathrm{Tr}(\widetilde{Y}_n)}\widetilde{Y}_n$. En réécrivant la dynamique de $\widetilde{Y}_n$ sous la forme

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\left(I_d - (2 - \mathrm{Tr}(\widetilde{Y}_n))H_{n+1}\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + \check{r}_{n+1}\right),$$

où $\check{r}_{n+1} := \dfrac{\left(\mathrm{Tr}(\widetilde{Y}_n) - 1\right)^2}{\mathrm{Tr}(\widetilde{Y}_n)} H_{n+1}\widetilde{Y}_n$, on obtient la procédure récursive suivante pour $\widetilde{\theta}_n$

$$\widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \frac{1}{n+1}\widetilde{h}(\widetilde{\theta}_n) + \frac{1}{n+1}\left(\Delta\widetilde{\mathbf{M}}_{n+1} + \widetilde{R}_{n+1}\right), \quad n \geq 1,$$

où, pour tout $\widetilde{\theta} = (y,\nu,s)^t \in \mathbb{R}_+^{3d}$,

$$\widetilde{h}(\widetilde{\theta}) := \begin{pmatrix} (I_d - (2 - \mathrm{Tr}(y))\Phi(s,\nu))y \\ \nu - (2 - \mathrm{Tr}(y))y \\ s - (2 - \mathrm{Tr}(y))\mathrm{diag}(p)y \end{pmatrix}, \ \Delta\widetilde{\mathbf{M}}_{n+1} := \begin{pmatrix} \Delta M_{n+1} \\ \Delta\widetilde{M}_{n+1} \\ \Delta\widehat{M}_{n+1} \end{pmatrix} \ \text{ et } \ \widetilde{R}_{n+1} := \begin{pmatrix} \check{r}_{n+1} \\ \widetilde{r}_{n+1} \\ \widehat{r}_{n+1} \end{pmatrix}.$$

La fonction Φ étant différentiable à l'équilibre $\widetilde{\theta}^*$, on a

$$D\widetilde{h}(\widetilde{\theta}^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & -\frac{\partial}{\partial\nu}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} & -\frac{\partial}{\partial s}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} \\ v^*\mathbf{1}^t - I_d & I_d & 0_{\mathcal{M}_d(\mathbb{R})} \\ \mathrm{diag}(p)\left(v^*\mathbf{1}^t - I_d\right) & 0_{\mathcal{M}_d(\mathbb{R})} & I_d \end{pmatrix}$$

qui est inversible. On obtient alors, par application directe du TCL pour les algorithmes stochastiques, le Théorème suivant qui établit la normalité asymptotique de la procédure.

Théorème 0.12. *Supposons que* **(H1)**, **(H4)**, **(H5)** *et (0.30) soient vérifiées. Alors, dès que* $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$,

$$\widetilde{\theta}_n \xrightarrow[n\to\infty]{p.s.} \widetilde{\theta}^* \quad \text{et} \quad \sqrt{n}\left(\widetilde{\theta}_n - \widetilde{\theta}^*\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \widetilde{\Sigma}\right),$$

où

$$\widetilde{\theta}^* := (v^*, v^*, \mathrm{diag}(p)v^*)^t, \quad \widetilde{\Sigma} = \int_0^{+\infty} e^{u(D\widetilde{h}(\widetilde{\theta}^*)-\frac{I}{2})}\widetilde{\Gamma}e^{u(D\widetilde{h}(\widetilde{\theta}^*)-\frac{I}{2})^t}du$$

avec $\widetilde{\Gamma} = p.s.\ \lim_{n\to+\infty} \mathbb{E}\left[\Delta\widetilde{\mathbf{M}}_{n+1}\Delta\widetilde{\mathbf{M}}_{n+1}^t \,|\, \mathcal{F}_n\right]$ *qui s'écrit*

$$\widetilde{\Gamma} = \begin{pmatrix} \displaystyle\sum_{k=1}^{d} v^{*k}C^k & H\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right) & \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)\mathrm{diag}(p) \\ \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t & \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)\mathrm{diag}(p) \\ \mathrm{diag}(p)\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t & \mathrm{diag}(p)\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t & \mathrm{diag}(p)\left(v^* - v^*v^{*t}\mathrm{diag}(p)\right) \end{pmatrix}.$$

Première partie

Quelques outils pour l'approximation stochastique, classiques et moins classiques

Chapitre 1

Quelques rappels sur les algorithmes stochastiques

1.1 Introduction

Deux problèmes courants rencontrés en analyse numérique sont la recherche de l'ensemble des points où une fonction traverse un niveau donné et la recherche des points où une fonction atteint son minimum. Ces deux problèmes sont liés puisqu'on peut les ramener à des recherche de zéros d'une fonction. Dans le cadre déterministe, la procédure récursive de zéro s'écrit

$$\forall n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) \quad (0 < \gamma_n \leq \gamma_0) \tag{1.1}$$

où $h : \mathbb{R}^d \to \mathbb{R}^d$ est un champ de vecteur continu satisfaisant une hypothèse de croissance sous-linéaire à l'infini. Sous certaines hypothèses appropriées de "retour à la moyenne", on montre qu'une telle procédure est bornée et éventuellement converge vers un zéro θ^* de h. Un exemple est l'algorithme de Newton-Raphson ($h \leftarrow Dh(x)^{-1}h(x)$ et $\gamma_n = 1$).

L'approximation stochastique est une extension de cette méthode dans le cas où la fonction dont on cherche le zéro admet une représentation sous forme d'espérance, *i.e.* $h(\theta) = \mathbb{E}\left[H(\theta, Y)\right]$, où Y est un vecteur aléatoire (simulable). On construit alors une procédure récursive stochastique de la forme

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H\left(\theta_n, Y_{n+1}\right), \quad Y_{n+1} \text{ i.i.d. de loi } \mu, \quad n \geq 0, \tag{1.2}$$

qui converge aussi vers un zéro θ^* de h au moins sous des hypothèses appropriées, spécifiées par la suite, sur H et la suite de gain $\gamma = (\gamma_n)_{n\geq 1}$.

Dans ce chapitre, nous commençons par rappeler les résultats de convergence pour l'algorithme déterministe, via une approche directe et par la méthode de l'EDO. Puis nous donnons deux résultats de convergence *p.s.* pour la version stochastique : le premier utilisant une approche martingale via le Lemme de Robbins-Siegmund, le second considérant l'algorithme comme une approximation bruitée d'une EDO.

Nous rappelons aussi une alternative à la méthode de gradient, via l'utilisation de différences finies à pas décroissant (méthode de Kiefer-Wolfowitz).

Puis nous finissons par rappeler les résultats de vitesse de convergence : tout d'abord le théorème central limite (TCL) pour les algorithmes stochastiques, puis le principe de moyennisation de Ruppert et Polyak qui permet d'obtenir la variance asymptotique minimale dans le TCL.

1.2 Méthodes récursives déterministes

1.2.1 Recherche de zéro d'une fonction

Soit $h : \mathbb{R}^d \to \mathbb{R}^d$ un champ de vecteur. On considère la procédure récursive de recherche de zéro suivante

$$\forall n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n), \quad \theta_0 \in \mathbb{R}^d, \tag{1.3}$$

où $(\gamma_n)_{n\geq 1}$ est une suite de réels strictement positifs.

Théorème 1.1. *(a) Si $\theta \mapsto \theta - \gamma h(\theta)$ est (localement) contractante et $\gamma_n = \gamma > 0$ alors*

$$\theta_n \underset{n\to\infty}{\longrightarrow} \theta^* \quad \textit{unique zéro de } h.$$

(b) Si h est continue à croissance linéaire, $\theta \mapsto h(\theta)$ est séparante en θ^* i.e.

$$\forall \theta \neq \theta^*, \qquad \langle h(\theta) | \theta - \theta^* \rangle > 0,$$

et

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{et} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty, \tag{1.4}$$

alors

$$\theta_n \underset{n\to\infty}{\longrightarrow} \theta^* \quad \textit{unique zéro de } h.$$

Remarque. Un exemple du cas (a) est la *méthode de Newton* : on remplace alors h par $\widetilde{h} = (Dh)^{-1}h$ si $Dh(\theta)$ est inversible au voisinage de θ^*.

Démonstration. (b) En utilisant la procédure récursive (1.3), on a

$$\begin{aligned} |\theta_{n+1} - \theta^*|^2 &= |\theta_n - \theta^*|^2 - 2\gamma_{n+1}\langle h(\theta_n)|\theta_n - \theta^*\rangle + \gamma_{n+1}^2 |h(\theta_n)|^2 \\ &\leq |\theta_n - \theta^*|^2 - 2\gamma_{n+1}\langle h(\theta_n)|\theta_n - \theta^*\rangle + C\gamma_{n+1}^2(1 + |\theta_n - \theta^*|^2) \end{aligned}$$

ce qui implique que la suite $(s_n)_{n\geq 0}$ définie par

$$s_n = \frac{|\theta_n - \theta^*|^2 + \sum_{k=1}^n \gamma_k \langle h(\theta_{k-1})|\theta_{k-1} - \theta^*\rangle + C\sum_{k\geq n+1}\gamma_k^2}{\prod_{k=1}^n (1 + C\gamma_k^2)}$$

est une suite décroissante et positive donc convergente vers une limite finie s_∞. On en déduit alors que

$$|\theta_n - \theta^*|^2 \underset{n\to\infty}{\longrightarrow} \ell_\infty \quad \text{et} \quad \sum_{k=1}^n \gamma_k \langle h(\theta_{k-1})|\theta_{k-1} - \theta^*\rangle < +\infty.$$

Comme $\sum_{n\geq 1}\gamma_n = +\infty$, la seconde assertion implique alors que

$$\liminf_n \langle h(\theta_n)|\theta_n - \theta^*\rangle = 0.$$

On déduit de la première assertion que la suite $(\theta_n)_{n\geq 0}$ est bornée. Donc, quitte à extraire deux fois, il existe $(\theta_{n'})_{n'\geq 0}$ telle que $\theta_{n'} \underset{n'\to\infty}{\longrightarrow} \theta_\infty$ et $\langle h(\theta_{n'})|\theta_{n'} - \theta^*\rangle \to 0$. Par continuité de h, on en déduit que $\langle h(\theta_\infty)|\theta_\infty - \theta^*\rangle = 0$. D'où $\theta_\infty = \theta^*$ car h est séparante en θ^*. D'où $\ell_\infty = 0$ ce qui entraîne

$$\theta_n \underset{n\to\infty}{\longrightarrow} \theta^*. \qquad \square$$

Remarque. La fonction $L : \theta \mapsto |\theta - \theta^*|^2$ n'intervient pas dans le calcul des θ_n : c'est un simple outil de démonstration.

1.2.2 Le point de vue de l'optimisation

On peut interpréter ce qui précède comme la recherche du minimum de la fonction (convexe) $L : \theta \mapsto |\theta - \theta^*|^2$. Le même type de démonstration conduit de façon plus générale au résultat ci-dessous relatif à une fonction potentielle L.

Théorème 1.2. *Soit $L : \mathbb{R}^d \to \mathbb{R}_+$* essentiellement quadratique, *i.e. L est $\mathcal{C}^1$, ∇L est Lipschitz,*

$$|\nabla L|^2 \leq C(1 + L) \quad \textit{et} \quad \lim_{|\theta| \to +\infty} L(\theta) = +\infty.$$

Si h est continue, à croissance $\sqrt{L}$-linéaire, i.e.

$$|h| \leq C(1 + L)^{\frac{1}{2}}$$

et est à force de rappel "séparante" en θ^*, i.e.

$$\forall\, \theta \neq \theta^*, \qquad \langle \nabla L | h \rangle(\theta) > 0,$$

et si la suite de pas $(\gamma_n)_{n \geq 1}$ vérifie (1.4), alors

(i) $\{h = 0\} = \{\nabla L = 0\} = \operatorname{argmin}_\theta L(\theta) = \{\theta^*\}$,

(ii) la procédure (1.3) vérifie

$$\theta_n \underset{n \to \infty}{\longrightarrow} \theta^*.$$

Exemples. **1.** Descente de gradient : $h(x) = \nabla L(x)$ ou $h(x) = \rho(x)\nabla L(x)$, avec $\rho > 0$ bornée.

2. Cadre convexe : Si $V : \mathbb{R}^d \to \mathbb{R}_+$ est une fonction convexe différentiable ayant un unique minimum en θ^*, alors on peut substituer à V

$$L(\theta) = |\theta - \theta^*|^2$$

car

$$\forall\, \theta \neq \theta^*, \qquad \langle \nabla V(\theta) | \theta - \theta^* \rangle > 0.$$

L'avantage est alors que L est trivialement essentiellement quadratique.

1.2.3 Méthode de l'*EDO*

L'idée de cette approche est de faire le lien entre la procédure récursive (1.3) et l'Equation Différentielle Ordinaire (*EDO*) suivante

$$EDO_h \;\equiv\; \dot{\theta} = -h(\theta).$$

En effet, si l'on pose $\theta_{\Gamma_n} = \theta_n$ où $\Gamma_n = \gamma_1 + \cdots + \gamma_n$, $n \geq 1$, alors l'équation

$$\frac{\theta_{n+1} - \theta_n}{\gamma_{n+1}} = -h(\theta_n)$$

peut être lue, lorsque $n \to \infty$, comme un *schéma d'Euler* (à pas décroissant) de EDO_h puisque

$$\dot{\theta}(\Gamma_n) \approx \frac{\theta(\Gamma_{n+1}) - \theta(\Gamma_n)}{\Gamma_{n+1} - \Gamma_n} = -h(\theta(\Gamma_n)).$$

L'avantage de cette méthode est aussi qu'elle complète l'approche directe : en effet, si $\langle \nabla L | h \rangle$ n'est pas séparante ou si $\{\langle \nabla L | h \rangle = 0\}$ est un ensemble gros et compliqué, l'approche précédente ne s'applique pas. L'idée ici est de remplacer l'hypothèse de séparation par une de semi-continuité inférieure.

Théorème 1.3. *Soit $L : \mathbb{R}^d \to \mathbb{R}_+$* essentiellement quadratique, *i.e. L est $\mathcal{C}^1$, ∇L est Lipschitz,*

$$|\nabla L|^2 \leq C(1+L) \quad \textit{et} \quad \lim_{|\theta| \to +\infty} L(\theta) = +\infty.$$

Si h est continue telle que

$$|h| \leq C(1+L)^{\frac{1}{2}}$$

et

$$\theta \mapsto \langle \nabla L | h \rangle \textit{ positive et semi-continue inférieurement (s.c.i.)},$$

alors l'ensemble des valeurs d'adhérence de la suite $(\theta_n)_{n \geq 0}$ noté Θ^∞ est une composante connexe de $\{L = \ell_\infty\} \cap \{\langle \nabla L | h \rangle = 0\}$.

Si, de plus, h est continue, alors Θ^∞ est un sous-ensemble compact connexe stable par EDO_h et EDO_h^*, où

$$EDO_h^* \ \equiv \ \dot{\theta} = h(\theta).$$

Remarque. Si l'on suppose *a priori* que la suite $(\theta_n)_{n \geq 0}$ est bornée, on peut remplacer la condition de pas (1.4) par celle moins contraignante

$$\gamma_n \underset{n \to \infty}{\longrightarrow} 0 \quad \text{et} \quad \sum_{n \geq 1} \gamma_n = +\infty.$$

Démonstration. En utilisant la démonstration de l'approche directe, on a que

(*i*) $L(\theta_n) \underset{n \to \infty}{\longrightarrow} \ell_\infty \in [0, \infty[$ donc $(\theta_n)_{n \geq 0}$ bornée et $\theta_{n+1} - \theta_n \underset{n \to \infty}{\longrightarrow} 0$.

(*ii*) $\displaystyle\sum_{n \geq 1} \gamma_{n-1} \langle \nabla L | h \rangle (\theta_n) < +\infty.$

Ainsi l'ensemble des valeurs d'adhérence Θ^∞ de la suite $(\theta_n)_{n \geq 0}$ est un sous-ensemble compact connexe de $\{L = \ell_\infty\} \cap \{\langle \nabla L | h \rangle = 0\}$ (car "bien enchaîné").

Posons

$$\theta^{(0)}_{\Gamma_n} := \theta_n, \quad \theta^{(0)}_t := \theta_0 - \int_0^t h(\theta^{(0)}_{\underline{s}}) ds$$

où, par interpolation linéaire, $\underline{s} := \Gamma_n$ sur $[\Gamma_n, \Gamma_{n+1}[$. On va maintenant étudier l'asymptotique de tout l'avenir à partir de Γ_n lorque $n \to \infty$ *via* le temps continu. Au lieu d'étudier $(\theta_n)_{n \geq 0}$ on s'intéresse donc à la suite de fonctions

$$\theta^{(n)}_t := \theta^{(0)}_{\Gamma_n + t}, \quad t \geq 0.$$

Comme pour tous $s, t \in [0, T]$,

$$|\theta^{(n)}_t - \theta^{(n)}_s| \leq \sup_k |h(\theta_k)| |t - s|,$$

on a que la suite $(\theta^{(n)})_{n \geq 0}$ est uniformément relativement compacte pour la topologie de la convergence uniforme sur les compacts de $\mathbb{R}_+$ (notée U_K). Soit $\theta_\infty \in \Theta^\infty$. On peut supposer que

$$\theta_{\varphi(n)} \underset{n \to \infty}{\longrightarrow} \theta_\infty \quad \text{et} \quad \theta^{(\varphi(n))} \underset{n \to \infty}{\overset{U_K}{\longrightarrow}} \theta^{(\infty)}, \quad \theta^{(\infty)}_0 = \theta_\infty.$$

Or,

$$\sum_{n\geq 1}\gamma_n\langle\nabla L|h\rangle(\theta_n)=\int_0^\infty\langle\nabla L|h\rangle(\theta_t^{(0)})dt<+\infty$$

et

$$\int_0^\infty\langle\nabla L|h\rangle(\theta_t^{(n)})dt=\int_{\Gamma_n}^\infty\langle\nabla L|h\rangle(\theta_t^{(0)})dt=\sum_{k\geq n}\gamma_k(\nabla L|h)(\theta_k)\underset{n\to\infty}{\longrightarrow}0$$

Donc, la semi-continuité inférieure de $\langle\nabla L|h\rangle$ et le lemme de Fatou entraînent successivement

$$\begin{aligned}0\leq\int_0^\infty\langle\nabla L|h\rangle(\theta_t^{(\infty)})dt &\leq \int_0^\infty\liminf_n\langle\nabla L|h\rangle(\theta_t^{(\varphi(n))})dt\\ &\leq \liminf_n\int_0^\infty\langle\nabla L|h\rangle(\theta_t^{(\varphi(n))})dt\\ &= 0.\end{aligned}$$

Par suite,

$$\langle\nabla L|h\rangle(\theta_t^{(\infty)})=0\quad dt\text{-}p.p.$$

donc comme $\langle\nabla L|h\rangle$ est s.c.i.

$$\langle\nabla L|h\rangle(\theta_\infty)=0.$$

On en conclut donc que Θ^∞ est une composante connexe de $\{L=\ell_\infty\}\cap\{(\nabla L|h)=0\}$.

Si h est continue, on déduit de

$$\theta_t^{(n)}:=\theta_n-\int_0^t h(\theta_{\tilde{s}}^{(n)})ds,\quad \tilde{s}:=\underline{s+\Gamma_n}-\Gamma_n\approx s$$

que tout valeur d'adhérence de $(\theta_n)_{n\geq 0}$ satisfait l'équation différentielle ordinaire EDO_h et prend ses valeurs dans l'ensemble Θ^∞. Par conséquent Θ^∞ est un sous ensemble compact connexe invariant par EDO_h et EDO_h^*. □

1.3 Approximation stochastique

Maintenant supposons qu'aucun accès direct aux valeurs numériques de $h(\theta)$ n'est possible mais que h a une représentation intégrale par rapport à un vecteur aléatoire Y à valeurs dans $\mathbb{R}^d$, soit

$$h(\theta)=\mathbb{E}\left[H(\theta,Y)\right],\quad H:\mathbb{R}^d\times\mathbb{R}^q\overset{Borel}{\longrightarrow}\mathbb{R}^d,\quad Y\overset{(d)}{=}\mu,\tag{1.5}$$

(satisfaisant $\mathbb{E}\left|H(\theta,Y)\right|<+\infty$ pour tout $\theta\in\mathbb{R}^d$). Si $H(\theta,y)$ est facile à calculer pour n'importe quel couple (θ,y) et que la distribution μ de Y est facile à simuler, une idée peut être de rendre aléatoire la procédure déterministe de recherche de zéro (1.3) en utilisant à chaque étape une simulation Monte-Carlo pour approcher $h(\theta_n)$.

Une idée plus sophistiquée est d'essayer de faire les deux simultanément en utilisant d'une part $H\left(\theta_n,Y_{n+1}\right)$ au lieu de $h(\theta_n)$ où $(Y_n)_{n\geq 1}$ est une suite de variables aléatoires i.i.d. de même loi que Y.

Basés sur cette analyse heuristique, nous pouvons raisonnablement espérer que la procédure récursive

$$\theta_{n+1}=\theta_n-\gamma_{n+1}H\left(\theta_n,Y_{n+1}\right),\quad Y_{n+1}\text{ i.i.d. de loi }\mu,\quad n\geq 0,\tag{1.6}$$

converge aussi vers un zéro θ^* de h au moins sous des hypothèses appropriées, spécifiées par la suite, sur H et la suite de gain $\gamma=(\gamma_n)_{n\geq 1}$. Dans ce cadre, les fonctions de Lyapunov sont sollicitées pour assurer la stabilité de la procédure.

1.3.1 Convergence *p.s.* : l'approche martingale

L'approximation stochastique fournit des théorèmes variés qui garantissent la convergence *p.s.* et/ou dans L^p des procédures d'approximation stochastique. Nous donnons ci-dessous un résultat préliminaire général (multidimensionnel) connu comme le Lemme de Robbins-Siegmund à partir duquel les principaux résultats de convergence seront facilement déduits (bien qu'il n'en contienne aucun stricto sensu).

Dans ce qui suit, la fonction H et la suite $(Y_n)_{n\geq 1}$ sont définies par (1.5) et h est le champ de vecteur de $\mathbb{R}^d$ dans $\mathbb{R}^d$ défini par $h(\theta) = \mathbb{E}\left[H(\theta, Y_1)\right]$.

Théorème 1.4. LEMME DE ROBBINS-SIEGMUND. *Soit $h : \mathbb{R}^d \to \mathbb{R}^d$ et $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ satisfaisant (1.5). Supposons qu'il existe une fonction $L : \mathbb{R}^d \to \mathbb{R}^+$ continûment différentiable satisfaisant*

$$\nabla L \quad \textit{est Lipschitz continue et} \quad |\nabla L|^2 \leq C(1+L) \tag{1.7}$$

*telle que h satisfasse l'*hypothèse de retour à la moyenne

$$\langle \nabla L | h \rangle \geq 0. \tag{1.8}$$

De plus, supposons que H satisfasse l'hypothèse suivante de croissance (pseudo-)linéaire

$$\forall \theta \in \mathbb{R}^d, \quad \|H(\theta, Y)\|_2 \leq C\sqrt{1 + L(\theta)} \tag{1.9}$$

(ce qui implique que $|h| \leq C\sqrt{1+L}$).

Soit $\gamma = (\gamma_n)_{n\geq 1}$ une suite de pas satisfaisant

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{et} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty \tag{1.10}$$

Finalement, supposons θ_0 indépendant de $(Y_n)_{n\geq 1}$ et $\mathbb{E}\left[L(\theta_0)\right] < +\infty$.

Alors, la procédure récursive définie par (1.6) satisfait : $\theta_n - \theta_{n-1} \overset{\mathbb{P}\text{-}p.s.\&L^2(\mathbb{P})}{\longrightarrow} 0$, $(L(\theta_n))_{n\geq 0}$ est $L^1(\mathbb{P})$-bornée,

$$L(\theta_n) \underset{n\to\infty}{\overset{p.s.}{\longrightarrow}} L_\infty \in L^1(\mathbb{P}) \quad \textit{et} \quad \sum_{n\geq 1} \gamma_n \langle \nabla L | h \rangle(\theta_{n-1}) < +\infty \quad p.s.$$

Remarques.

- Si la fonction L satisfait aussi $\lim_{|\theta|\to\infty} L(\theta) = +\infty$, alors L est souvent appelée fonction de Lyapunov du système comme dans la Théorie des Equations Différentielles Ordinaires.

- Notons que l'hypothèse (1.7) sur L implique que $\nabla\sqrt{1+L}$ est bornée donc que $\sqrt{L}$ a au plus une croissance linéaire, *i.e.* L a au plus une croissance quadratique.

- Une lecture attentive de la preuve ci-dessous montre que l'hypothèse $\sum_{n\geq 1}\gamma_n = +\infty$ n'est pas nécessaire. Cependant, nous la laissons en l'état parce qu'elle est indispensable pour n'importe quelle application de ce Lemme. Ces hypothèses sont connues comme "hypothèses de Robbins-Siegmund".

- Si l'on suppose que les innovations ne sont pas i.i.d. mais seulement $\mathcal{F}_n$-adaptées, on obtient alors un algorithme stochastique de la forme suivante

$$\forall\, n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1}\left(\Delta M_{n+1} + r_{n+1}\right),$$

où $\Delta M_{n+1} = H(\theta_n, Y_{n+1}) - \mathbb{E}\left[H(\theta_n, Y_{n+1})|\mathcal{F}_n\right]$ est un $\mathcal{F}_n$-accroissement de martingale et $r_{n+1} = \mathbb{E}\left[H(\theta_n, Y_{n+1})|\mathcal{F}_n\right] - h(\theta_n)$ est un terme de reste $\mathcal{F}_{n+1}$-adapté. Dans ce cas, si l'on suppose que

$$\sum_{n\geq 1} \gamma_n \left|r_n\right|^2 < +\infty \quad p.s.,$$

alors les conclusions du Lemme de Robbins-Siegmund restent vraies.

La clé de la preuve est le théorème de convergence pour les sur-martingales positives : si $(S_n)_{n\geq 0}$ est une sur-martingale positive ($S_n \in L^1(\mathbb{P})$ et $\mathbb{E}\left[S_{n+1}|\mathcal{F}_n\right] \leq S_n$ $p.s.$) alors, S_n converge $\mathbb{P}$-$p.s.$ vers une v.a. intégrable (positive) S_∞.

Démonstration. Posons $\mathcal{F}_n := \sigma(\theta_0, Y_1, \ldots, Y_n), n \geq 1$, et pour simplifier les notations posons $\Delta\theta_n = \theta_n - \theta_{n-1}$, $n \geq 1$. Il vient de la formule fondamentale du calcul qu'il existe $\xi_{n+1} \in (\theta_n, \theta_{n+1})$ (intervalle géométrique) tel que

$$\begin{aligned}
L(\theta_{n+1}) &= L(\theta_n) + \langle \nabla L(\xi_{n+1})|\Delta\theta_{n+1}\rangle \\
&\leq L(\theta_n) + \langle \nabla L(\theta_n)|\Delta\theta_{n+1}\rangle + [\nabla L]_{Lip}|\Delta\theta_{n+1}|^2 \\
&= L(\theta_n) - \gamma_{n+1}\langle \nabla L(\theta_n)|H(\theta_n, Y_{n+1})\rangle + [\nabla L]_{Lip}\gamma_{n+1}^2|H(\theta_n, Y_{n+1})|^2 \qquad (1.11)\\
&\leq L(\theta_n) - \gamma_{n+1}\langle \nabla L(\theta_n)|h(\theta_n)\rangle - \gamma_{n+1}\langle \nabla L(\theta_n)|\Delta M_{n+1}\rangle \qquad (1.12)\\
&\quad + [\nabla L]_{Lip}\gamma_{n+1}^2|H(\theta_n, Y_{n+1})|^2
\end{aligned}$$

où

$$\Delta M_{n+1} = H(\theta_n, Y_{n+1}) - \mathbb{E}\left[H(\theta_n, Y_{n+1})|\mathcal{F}_n\right] = H(\theta_n, Y_{n+1}) - h(\theta_n).$$

Notre but est maintenant de montrer que ΔM_{n+1} est un $\mathcal{F}_n$-accroissement d'une martingale (de carré intégrable) satisfaisant $\mathbb{E}|\Delta M_{n+1}|^2 \leq C(1 + L(\theta_n))$ pour une constante appropriée $C > 0$.

Notons tout d'abord que pour tout $n \geq 0$, $L(\theta_n) \in L^1(\mathbb{P})$ et $H(\theta_n, Y_{n+1}) \in L^2(\mathbb{P})$: ceci vient de (1.12) et d'une récurrence puisque

$$\mathbb{E}|\langle \nabla L(\theta_n)|H(\theta_n, Y_{n+1})\rangle| \leq \frac{1}{2}\left(\mathbb{E}|\nabla L(\theta_n)|^2 + \mathbb{E}|H(\theta_n, Y_{n+1})|^2\right) \leq C(1 + \mathbb{E}L(\theta_n))$$

(où on utilise que $|\langle a|b\rangle| \leq \frac{1}{2}(|a|^2 + |b|^2)$, $a, b \in \mathbb{R}$). Comme θ_n est $\mathcal{F}_n$-mesurable et que Y_{n+1} est indépendant de $\mathcal{F}_n$, on a

$$\mathbb{E}\left[H(\theta_n, Y_{n+1}) \,|\, \mathcal{F}_n\right] = \mathbb{E}\left[H(t, Y_1)\right]_{|t=\theta_n} = h(\theta_n).$$

Par conséquent, $\mathbb{E}\left[\Delta M_{n+1} \,|\, \mathcal{F}_n\right] = 0$. L'inégalité pour $\mathbb{E}\left[|\Delta M_{n+1}|^2 \,|\, \mathcal{F}_n\right]$ vient du fait que $|\Delta M_{n+1}|^2 \leq 2\left(|H(\theta_n, Y_{n+1})|^2 + |h\theta_n)|^2\right)$ et de l'hypothèse (1.9). Maintenant, on déduit des hypothèses (1.10) et (1.13) qu'il existe deux constantes réelles positives $C_L = C[\nabla L]_{Lip} > 0$ telles que

$$S_n = \frac{L(\theta_n) + \sum_{k=0}^{n-1}\gamma_{k+1}\langle \nabla L(\theta_k)|h(\theta_k)\rangle + C_L\sum_{k\geq n+1}\gamma_k^2}{\prod_{k=1}^n(1 + C_L\gamma_k^2)}$$

est une sur-martingale (positive) avec $S_0 = L(\theta_0) \in L^1(\mathbb{P})$. Ceci utilise que $\langle \nabla L|h\rangle \geq 0$. Donc, $\mathbb{P}$-$p.s.$ converge vers une v.a. intégrable S_∞. Par conséquent, en utilisant que $\sum_{k\geq n+1}\gamma_k^2 \to 0$, on obtient

$$L(\theta_n) + \sum_{k=0}^{n-1}\gamma k + 1\langle \nabla L|h\rangle(\theta_k) \xrightarrow{p.s.} \tilde{S}_\infty = S_\infty\prod_{n\geq 1}(1 + C_L\gamma_n^2) \in L^1(\mathbb{P}). \qquad (1.13)$$

La sur-martingale $(S_n)_{n\geq 0}$ étant $L^1(\mathbb{P})$-bornée, on déduit de la même façon que $(L(\theta_n))_{n\geq 0}$ est L^1-bornée puisque

$$L(\theta_n) \leq \prod_{k=1}^{n}(1 + C_L\gamma_k^2)S_n, \quad n \geq 0.$$

Maintenant, une série à termes positifs qui est bornée supérieurement par une suite convergeant $p.s.$ converge $p.s.$ dans $\mathbb{R}^+$, donc

$$\sum_{n\geq 0}\gamma_{n+1}\langle \nabla L|h\rangle(\theta_n) < +\infty \quad \mathbb{P}\text{-}p.s.$$

Il s'ensuit de (1.13) que, $\mathbb{P}$-$p.s.$, $L(\theta_n) \underset{n\to\infty}{\longrightarrow} L_\infty$ qui est intégrable puisque $(L(\theta_n))_{n\geq 0}$ est L^1-bornée. Finalement,

$$\sum_{n\geq 1}\mathbb{E}(|\Delta\theta_n|^2) \leq \sum_{n\geq 1}\gamma_n^2\mathbb{E}\left(|H(\theta_{n-1}, Y_n)|^2\right) \leq C\sum_{n\geq 1}\gamma_n^2(1 + \mathbb{E}L(\theta_{n-1})) < +\infty$$

donc $\sum_{n\geq 1}\mathbb{E}(|\Delta\theta_n|^2) < +\infty$ $p.s.$ ce qui implique $\theta_n - \theta_{n-1} \to 0$ $p.s.$. □

Remarque. Le même argument qui montre que $(L(\theta_n))_{n\geq 1}$ est $L^1(\mathbb{P})$-bornée montre que

$$\mathbb{E}\left(\sum_{n\geq 1}\gamma_n\langle \nabla L|h\rangle(\theta_{n-1})\right) < +\infty \ p.s.$$

Corollaire 1.1. (a) Algorithme de Robbins-Monro. *Supposons que la fonction moyenne h soit continue et vérifie*

$$\forall\theta \in \mathbb{R}^d,\ \theta \neq \theta^*,\ \langle\theta - \theta^*|h(\theta)\rangle > 0 \tag{1.14}$$

(ce qui implique que $\{h = 0\} = \{\theta^*\}$). Supposons de plus que $\theta_0 \in L^2(\mathbb{P})$ et que H vérifie

$$\forall\theta \in \mathbb{R}^d, \quad \mathbb{E}\left[|H(\theta, Y)|^2\right] \leq C(1 + |\theta|^2).$$

Supposons que la suite de pas $(\gamma_n)_{n\geq 1}$ vérifie (1.10). Alors

$$\theta_n \underset{n\to\infty}{\overset{p.s.}{\longrightarrow}} \theta^*$$

et dans tout L^p, $p \in (0, 2)$.

(b) Gradient Stochastique. *Soit $L : \mathbb{R}^d \to \mathbb{R}_+$ une fonction différentiable vérifiant (1.7), $\lim_{|\theta|\leftarrow\infty} L(\theta) = +\infty$ et $\{\nabla L = 0\} = \{\theta^*\}$. Supposons que la fonction moyenne h soit donnée par $h = \nabla L$ et que H vérifie*

$$\forall\theta \in \mathbb{R}^d, \quad \mathbb{E}\left[|H(\theta, Y)|^2\right] \leq C(1 + |\theta|^2),$$

et que $\theta_0 \in L^1(\mathbb{P})$. Supposons que la suite de pas $(\gamma_n)_{n\geq 1}$ vérifie (1.10). Alors $L(\theta^*) = \min_{\mathbb{R}^d} L$,

$$\theta_n \underset{n\to\infty}{\overset{p.s.}{\longrightarrow}} \theta^*$$

et $\nabla L(\theta_n)$ converge vers 0 dans tout L^p, $p \in (0, 2)$.

Démonstration. (a) L'hypothèse (1.14) est alors l'hypothèse de retour à la moyenne associée à la fonction de Lyapunov $L(\theta) = \frac{1}{2}|\theta - \theta^*|^2$. L'hypothèse sur H est clairement l'hypothèse de croissance linéaire (1.9) pour cette fonction L. Par conséquent, il vient du Lemme de Robbins-Siegmund que

$$|\theta_n - \theta^*|^2 \xrightarrow[n\to\infty]{p.s.} L_\infty \in L^1(\mathbb{P}) \quad \text{et} \quad \sum_{n\geq 1} \gamma_n \langle |\theta_n - \theta^*| | h(\theta_{n-1}) \rangle < +\infty \quad p.s.$$

De plus, $(|\theta - \theta^*|^2)_{n\geq 0}$ est $L^1(\mathbb{P})$-bornée.

Soit $\omega \in \Omega$ tel que $|\theta_n(\omega) - \theta^*|^2$ converge dans $\mathbb{R}^+$ et $\sum_{n\geq 1} \gamma_n \langle \theta_{n-1}(\omega) - \theta^* | h(\theta_{n-1}) \rangle < +\infty$. Puisque $\sum_{n\geq 1} \gamma_n \langle \theta_{n-1}(\omega) - y^* | h(\theta_{n-1}(\omega)) \rangle < +\infty$, il s'ensuit que

$$\liminf_n \langle \theta_{n-1}(\omega) - y^* | h(\theta_{n-1}(\omega)) \rangle = 0.$$

On peut supposer l'existence d'une sous-suite $(\phi(n,\omega))_n$ telle que

$$\langle \theta_{\phi(n,\omega)}(\omega) - th^* | h(\theta_{\phi(n,\omega)}(\omega)) \rangle \underset{n\to\infty}{\longrightarrow} 0 \quad \text{et} \quad \theta_{\phi(n,\omega)}(\omega) \underset{n\to\infty}{\longrightarrow} \theta_\infty.$$

Si $\liminf_n \langle \theta_{n-1}(\omega) - y^* | h(\theta_{n-1}(\omega)) \rangle > 0$, la convergence de la série induit une contradiction avec $\sum_{n\geq 1} \gamma_n = +\infty$. Maintenant, jusqu'à une extraction supplémentaire, on peut supposer puisque $(\theta_n(\omega))_{n\geq 0}$ est bornée, que $\theta_{\phi(n,\omega)}(\omega) \underset{n\to\infty}{\longrightarrow} \theta_\infty$. Il s'ensuit que $\langle \theta_\infty - \theta^* | h(\theta_\infty) \rangle = 0$ qui implique que $\theta_\infty = \theta^*$. Maintenant

$$\lim_n |\theta_n(\omega) - \theta^*|^2 = \lim_n |\theta_{\phi(n,\omega)}(\omega) - \theta^*|^2 = 0.$$

Finalement, pour tout $p \in (0,2)$, $(|\theta_n(\omega) - \theta^*|^p)_{n\geq 0}$ est $L^{\frac{2}{p}}(\mathbb{P})$-bornée, donc uniformément intégrable. Par conséquent, la convergence a lieu dans $L^1(\mathbb{P})$, *i.e.* $\theta_n \underset{n\to\infty}{\longrightarrow} \theta^*$ dans $L^p(\mathbb{P})$.

(b) On peut appliquer le lemme de Robbins-Siegmund avec L comme fonction de Lyapunov puisque $\langle \nabla L | h \rangle(\theta) = |\nabla L(\theta)|^2 \geq 0$. L'hypothèse sur H est alors celle de croissance quadratique. Par conséquent,

$$L(\theta_n) \xrightarrow[n\to\infty]{p.s.} L_\infty \in L^1(\mathbb{P}) \quad \text{et} \quad \sum_{n\geq 1} \gamma_n |\nabla L(\theta_{n-1})|^2 < +\infty \quad p.s.$$

Soit $\omega \in \Omega$ tel que $L(\theta_n(\omega))$ converge dans $\mathbb{R}_+$ et $\sum_{n\geq 1} \gamma_n |\nabla L(\theta_{n-1}(\omega))|^2 < +\infty$ (et $\theta_n(\omega) - \theta_{n-1}(\omega) \to 0$). Le même argument que ci-dessus montre que

$$\liminf_n |\nabla L(\theta_n(\omega))|^2 = 0.$$

A partir de la convergence de $L(\theta_n(\omega))$ vers $L_\infty(\omega)$ et de $\lim_{|\theta|\to\infty} L(\theta) = +\infty$, on déduit la bornitude de $(\theta_n(\omega))_{n\geq 0}$. Alors il existe une sous-suite $(\phi(n,\omega))_{n\geq 1}$ telle que $\theta_{\phi(n,\omega)} \to \tilde{\theta}$ et

$$\lim_n \nabla L(\theta_{\phi(n,\omega)}(\omega)) = 0 \text{ et } L(\theta_{\phi(n,\omega)}(\omega)) \to L_\infty(\omega).$$

Alors $\nabla L(\tilde{\theta}) = 0$ ce qui implique que $\tilde{\theta} = \theta^*$. Alors $L_\infty(\omega) = L(\theta^*)$. La fonction L étant positive, différentiable et tendant vers l'infini en l'infini, cela implique que L atteint son unique minimum global en θ^*. En particulier $\{L = L(\theta^*)\} = \{\nabla L = 0\} = \{\theta^*\}$. Par conséquent, la seule valeur limite possible pour la suite bornée $(\theta_n(\omega))_{n\geq 1}$ est θ^* *i.e.* $\theta_n(\omega)$ converge vers θ^*.

La convergence $L^p(\mathbb{P})$ vers 0 de $|\nabla L(\theta_n)|$, $p \in (0,2)$, vient du même argument d'uniforme intégrabilité que dans (a). □

1.3.2 Point de vue de l'optimisation : l'approche de Kiefer-Wolfowitz

On revient au problème d'optimisation, à savoir $\min_{\mathbb{R}^d} L$, où $L(\theta) = \mathbb{E}[\Lambda(\theta, Y)]$. S'il n'y a pas de gradient local $\frac{\partial \Lambda}{\partial \theta}(\theta, y)$ ou si le calcul de $\frac{\partial \Lambda}{\partial \theta}(\theta, y)$ n'est pas compétitif vis à vis de $\Lambda(\theta, x)$ par exemple, il existe une alternative aux méthodes de gradient, les approches par différences finies.

L'idée est simplement d'approcher le gradient ∇L par

$$\frac{\partial L}{\partial \theta_i}(\theta) \approx \frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta^i}, \quad 1 \leq i \leq d,$$

où $(e_i)_{1 \leq i \leq d}$ désigne la base canonique de $\mathbb{R}^d$ et $\eta = (\eta^i)_{1 \leq i \leq d}$. Ce terme de différence finie admet une représentation intégrale donnée par

$$\frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta^i} = \mathbb{E}\frac{\Lambda(\theta + \eta^i e_i, Y) - \Lambda(\theta - \eta^i e_i, Y)}{2\eta^i}.$$

Partant de cette représentation, on peut déduire une procédure récursive stochastique pour θ_n comme suit

$$\theta^i_{n+1} = \theta^i_n - \gamma_{n+1} \frac{\Lambda(\theta_n + \eta^i_{n+1} e_i, Y_{n+1}) - \Lambda(\theta_n - \eta^i_{n+1} e_i, Y_{n+1})}{2\eta^i_{n+1}}, \quad 1 \leq i \leq d.$$

Nous donnons ci-dessous le résultat de convergence pour les procédures de Kiefer-Wolfowitz (qui est la contrepartie naturelle du Lemme de Robbins-Siegmund dans le cadre du gradient stochastique).

Théorème 1.5. *Supposons que la fonction $\theta \mapsto L(\theta)$ soit deux fois différentiable de Hessienne Lipschitz. Supposons que*

$$\theta \mapsto \Lambda(\theta, Y) \text{ est Lipschitz dans } L^2$$

et que les suites de pas vérifient

$$\sum_{n \geq 1} \gamma_n = \sum_{n \geq 1} \eta^i_n = +\infty, \quad \sum_{n \geq 1} \gamma_n^2 < +\infty, \quad \eta_n \to 0 \quad \text{et} \quad \sum_{n \geq 1} \left(\frac{\gamma_n}{\eta^i_n}\right)^2 < +\infty, \quad 1 \leq i \leq d.$$

Alors, θ_n converge p.s. vers une composante connexe de $\{L = \ell\} \cap \{\nabla L = 0\}$ pour un certain niveau $\ell \geq 0$.

1.3.3 Résultat de convergence pour les algorithmes contraints

Le but est de déterminer $\{\theta \in \Theta : h(\theta) = \mathbb{E}[H(\theta, Y)] = 0\}$ où $\Theta \subset \mathbb{R}^d$ est un ensemble convexe fermé, $h : \mathbb{R}^d \to \mathbb{R}^d$ et $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$. Pour $\theta_0 \in \Theta$, on considère la suite $(\theta_n)_{n \geq 0}$ à valeurs dans $\mathbb{R}^d$ définie par

$$\theta_{n+1} = \Pi_\Theta\left(\theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1})\right), \quad n \geq 0, \tag{1.15}$$

où $(Y_n)_{n \geq 1}$ est une suite i.i.d. de même loi que Y et Π_Θ désigne la projection euclidienne sur Θ. La procédure récursive (1.15) peut se réécrire comme suit

$$\theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) - \gamma_{n+1} \Delta M_{n+1} + \gamma_{n+1} p_{n+1}, \quad n \geq 0, \tag{1.16}$$

où $\Delta M_{n+1} = H(\theta_n, Y_{n+1}) - h(\theta_n)$ est un accroissement de martingale et

$$p_{n+1} = \frac{1}{\gamma_{n+1}} \Pi_\Theta\left(\theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1})\right) - \frac{1}{\gamma_{n+1}} \theta_n + H(\theta_n, Y_{n+1}).$$

Théorème 1.6. *(voir [74] et [75]) Soit $(\theta_n)_{n\geq 0}$ une suite définie par (1.16). Supposons qu'il existe un unique $\theta^* \in \Theta$ tel que $h(\theta^*) = 0$ et que la fonction moyenne vérifie l'hypothèse de retour à la moyenne sur Θ, à savoir*

$$\forall \theta \neq \theta^* \in \Theta, \quad \langle h(\theta) \,|\, \theta - \theta^* \rangle \,. \tag{1.17}$$

Supposons que la suite de pas $(\gamma_n)_{n\geq 1}$ vérifie

$$\sum_{n\geq 1} \gamma_n = +\infty \quad et \quad \sum_{n\geq 1} \gamma_n^2 < +\infty. \tag{1.18}$$

De plus, si la fonction H vérifie

$$\forall \theta \in \Theta, \quad \mathbb{E}\left[|H(\theta, Y)|^2\right] \leq K(1 + |\theta|^2), \quad K > 0, \tag{1.19}$$

alors

$$\theta_n \xrightarrow[n\to+\infty]{p.s.} \theta^*.$$

1.3.4 Convergence *p.s.* : la méthode de l'*EDO*

Considérons la procédure récursive suivante définie sur un espace de probabilités filtré $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$

$$\forall\, n \geq n_0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1} \left(\Delta M_{n+1} + r_{n+1}\right), \tag{1.20}$$

où $h : \mathbb{R}^d \to \mathbb{R}^d$ est une fonction localement Lipschitz et continue, θ_{n_0} est un vecteur aléatoire $\mathcal{F}_{n_0}$-mesurable fini et, pour tout $n \geq n_0$, ΔM_{n+1} est un $\mathcal{F}_n$-accroissement de martingale et r_n est un terme de reste $\mathcal{F}_n$-adapté.

Théorème 1.7. CONVERGENCE *p.s.* AVEC LA MÉTHODE DE L'*EDO* *(c.f. e.g. [25, 27, 26, 44, 75, 48, 23]). Supposons que h soit localement Lipschitz, que*

$$r_n \xrightarrow[n\to\infty]{p.s.} 0 \quad et \quad \sup_{n\geq n_0} \mathbb{E}\left[|\Delta M_{n+1}|^2 \,|\, \mathcal{F}_n\right] < +\infty \quad p.s., \tag{1.21}$$

et que $(\gamma_n)_{n\geq 1}$ soit une suite positive vérifiant

$$\sum_{n\geq 1} \gamma_n = +\infty \quad et \quad \sum_{n\geq 1} \gamma_n^2 < +\infty. \tag{1.22}$$

Alors l'ensemble Θ^∞ des valeurs d'adhérence de $(\theta_n)_{n\geq n_0}$ est p.s. un ensemble compact connexe, stable par le flot de

$$EDO_h \equiv \dot{\theta} = -h(\theta),$$

et par

$$EDO_h^* \equiv \dot{\theta} = h(\theta).$$

De plus, si $\theta^* \in \Theta^\infty$ est l'unique équilibre uniformément stable sur Θ^∞ de EDO_h, alors

$$\theta_n \xrightarrow[n\to\infty]{p.s.} \theta^*.$$

COMMENTAIRE. Par uniformément stable, on entend

$$\sup_{\theta\in\Theta^\infty} |\theta(\theta_0, t) - \theta^*| \longrightarrow 0 \quad \text{quand} \quad t \to +\infty,$$

où $\theta(\theta_0, t)_{\theta_0\in\Theta^\infty, t\in\mathbb{R}_+}$ est le flot de EDO_h sur Θ^∞.

Démonstration. Cela débouche sur des liens plus fins entre le comportement asymptotique de l'algorithme et celui d'EDO_h puisque (c.f. [23]) Θ_∞ s'avère être un ensemble "intérieurement chaîne récurrent" pour (le flot d') EDO_h.

On met en œuvre la *méthode de l'EDO*. Cette fois encore, on part de

$$\theta_n = \theta_0 - \sum_{k=1}^{n} \gamma_k \left(h(\theta_{k-1}) - \Delta M_k - r_k \right)$$

que l'on réécrit en temps continu en posant $\theta^{(0)}_{\Gamma_n} = \theta_n$, pour tout $n \geq 0$ puis, pour tout $t \geq 0$,

$$\theta^{(0)}_t = \theta_0 - \int_0^t h(\theta^{(0)}_{\underline{s}})ds + \sum_{k=1}^{N(t)} \gamma_k \Delta M_k + \sum_{k=1}^{N(t)} \gamma_k r_k$$

où

$$N(t) = \min\{k \in \mathbb{N} \,|\, \Gamma_{k+1} > t\}.$$

On définit à nouveau pour tout $n \geq 0$, les *fonctions décalées*

$$\begin{aligned} \theta^{(n)}_t &= \theta^{(0)}_{\Gamma_n + t} \\ &= \theta_n - \int_{\Gamma_n}^{\Gamma_n + t} h(\theta^{(0)}_{\underline{s}})ds + \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \Delta M_k + \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k r_k. \end{aligned}$$

Comme $L(\theta_n) \longrightarrow L_\infty \in L^1(\mathbb{P})$ implique que $K_\infty(h) = \sup_n |h(\theta_n)| < +\infty \quad p.s.$, on a toujours

$$\left| \int_{\Gamma_n}^{\Gamma_n + t} h(\theta^{(0)}_{\underline{u}})du - \int_{\Gamma_n}^{\Gamma_n + s} h(\theta^{(0)}_{\underline{u}})du \right| \leq K_\infty(h)|t - s|.$$

D'une part, par (1.21) et (1.22), on déduit que la série $\sum_{n\geq 1} \gamma_n \Delta M_n$ converge $p.s.$ et, via la propriété de Cauchy,

$$\sup_{m \geq n} \left| \sum_{k=n+1}^{m} \gamma_k \Delta M_k \right| \xrightarrow{p.s.} 0 \quad \text{quand } n \to \infty.$$

D'autre part

$$\sup_{t\in[0,T]} \left| \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k r_k \right| \leq \max_{k \geq n} |r_k| \sum_{k=n+1}^{N(\Gamma_n+T)} \gamma_k = \max_{k \geq n} |r_k| \, T \xrightarrow[n\to\infty]{p.s.} 0.$$

Ainsi la suite de fonctions $(\theta^{(n)}_t)_{t\geq 0}$ est U_K-relativement compacte et Θ^∞ est un compact connexe. De plus, comme h est localement Lipschitz, Θ^∞ est flot invariant.

Si $\theta^* \in \Theta^\infty$ est l'unique équilibre uniformément stable, alors $\Theta^\infty = \{\theta^*\}$. D'où le résultat. □

Remarque. La convergence de la martingale est inutile, il suffit que $\sum_n \gamma_n = +\infty$, $\gamma_n \to 0$ et

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \Delta M_k \right| = 0 \quad p.s.$$

Cette condition découle des critères suivants

- $\sup_{n\geq 0} \mathbb{E}\left(|\Delta M_{n+1}|^q\right) < +\infty$ et $\sum_{n\geq 1} \gamma_n^{1+\frac{q}{2}} < +\infty$ pour $q \geq 2$ (c.f. [93]).

- Il existe $\lambda > 0$ tel que, pour tout $x \in \mathbb{R}^d$,

$$\mathbb{E}\left[\exp\left(\langle x|\Delta M_{n+1}\rangle\right) | \mathcal{F}_n\right] \leq e^{\frac{\lambda}{2}|x|^2}$$

et $\sum_{n\geq 1} e^{-c/\gamma_n} < +\infty$ pour tout $c > 0$ (c.f. [24] ou [75]). Ce cadre contient les variables aléatoires bornées via l'inégalité de Hoeffding.

1.3.5 Vitesse de convergence : le TCL

Dans des cadres standards, un algorithme stochastique converge vers sa cible à une vitesse $\sqrt{\gamma_n}$ (ce qui suggère d'utiliser des pas $\gamma_n = \frac{c}{n}$, $c > 0$). Pour être précis, $\frac{\theta_n - \theta^*}{\sqrt{\gamma_n}}$ converge en loi vers une certaine loi gaussienne de matrice de covariance dépendant de $Dh(\theta^*)$. Nous nous plaçons ici dans le cadre d'algorithme stochastique avec terme de reste, *i.e.* du type (1.20).

Théorème 1.8. *TCL (c.f. e.g. [25, 44, 75]). Soit θ^* un équilibre de $\{h = 0\}$. Supposons que la fonction h soit différentiable au point θ^* et que toutes les valeurs propres de $Dh(\theta^*)$ aient une partie réelles positives. Supposons que, pour un certain $\delta > 0$,*

$$\sup_{n\geq n_0} \mathbb{E}\left[|\Delta M_{n+1}|^{2+\delta} | \mathcal{F}_n\right] < +\infty \, p.s., \quad \mathbb{E}\left[\Delta M_{n+1}\Delta M_{n+1}^t | \mathcal{F}_n\right] \xrightarrow[n\to\infty]{p.s.} \Gamma, \tag{1.23}$$

où Γ est une matrice déterministe symétrique définie positive et pour un $\epsilon > 0$,

$$\mathbb{E}\left[(n+1)\,|r_{n+1}|^2 \mathbb{1}_{\{|\theta_n - \theta^*| \leq \epsilon\}}\right] \xrightarrow[n\to\infty]{} 0. \tag{1.24}$$

Spécifions la suite de pas comme suit

$$\forall n \geq 1, \quad \gamma_n = \frac{\alpha}{n}, \quad \alpha > \frac{1}{2\mathcal{R}e(\lambda_{min})} \tag{1.25}$$

où λ_{min} désigne la valeur propre de $Dh(\theta^*)$ de plus petite partie réelle. Alors, la convergence p.s. est régie sur l'ensemble de convergence $\{\theta_n \xrightarrow{p.s.} \theta^*\}$ par le Théorème Central Limit suivant

$$\sqrt{n}\,(\theta_n - \theta^*) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \alpha\Sigma) \quad \textit{avec} \quad \Sigma := \int_0^{+\infty} \left(e^{-\left(Dh(\theta^*) - \frac{I_d}{2\alpha}\right)u}\right)^t \Gamma e^{-\left(Dh(\theta^*) - \frac{I_d}{2\alpha}\right)u} du.$$

1.3.6 Principe de moyennisation (Ruppert-Polyak)

L'idée originelle était de "lisser" le comportement d'un algorithme stochastique convergeant en considérant la moyenne arithmétique des valeurs passées plutôt que la valeur calculée à la $n^{ème}$ itération. De façon surprenante, si cette procédure de moyennisation est combinée à un changement d'échelle de la suite de pas γ_n, on atteint gratuitement la meilleure vitesse de convergence possible.

Pour être précis : soit $(\gamma_n)_{n\geq 1}$ une suite de pas vérifiant

$$\gamma_n \sim \left(\frac{c}{b+n}\right)^{\vartheta}, \qquad \vartheta \in (1/2, 1).$$

Alors, on implémente la procédure standard

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1})$$

et on pose

$$\bar{\theta}_n := \frac{\theta_0 + \cdots + \theta_{n-1}}{n}.$$

Sous des hypothèses naturelles (c.f. [98]), on montre dans différents cas étudiés précédemment (gradient stochastique, Robbins-Monro) que

$$\bar{\theta}_n \xrightarrow{p.s.} \theta^*$$

où θ^* est la cible de l'algorithme et de plus

$$\sqrt{n}(\bar{\theta}_n - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0; \Sigma^*_{\min})$$

où $\Sigma^*_{\min}$ est la matrice de variance-covariance la plus petite : ainsi si $d = 1$,

$$\Sigma^*_{\min} = \frac{\mathrm{Var}(H(\theta^*, Z))}{h'(\theta^*)^2}.$$

Théorème 1.9. RUPPERT & POLYAK *(c.f. e.g. [43]) Supposons que la procédure définie récursivement*

$$\theta_{n+1} = \theta_n - \gamma_{n+1}(h(\theta_n) + \Delta M_{n+1})$$

où h est une fonction borélienne, continue en son unique zéro θ^*, vérifiant

$$\forall\, \theta \in \mathbb{R}^d, \quad h(\theta) = Dh(\theta^*)(\theta - \theta^*) + O(|\theta - \theta^*|^2)$$

où toutes la valeurs propres de $Dh(\theta^*)$ ont une partie réelle positive. Supposons de plus que, pour une certaine constante $C > 0$,

$$\mathbb{E}\left[\Delta M_{n+1} \,|\, \mathcal{F}_n\right] \mathbf{1}_{\{|\theta_n - \theta^*| \le C\}} = 0 \quad p.s.$$

et qu'il existe un exposant $\delta > 0$ tel que

$$\mathbb{E}\left[\Delta M_{n+1} (\Delta M_{n+1})^t \,|\, \mathcal{F}_n\right] \xrightarrow{p.s.} \mathbf{1}_{\{|\theta_n - \theta^*| \le C\}} \Gamma > 0 \ \in \ \mathcal{S}(d, \mathbb{R})$$

$$\sup_n \mathbb{E}\left[|\Delta M_{n+1}|^{2+\delta} \,|\, \mathcal{F}_n\right] \mathbf{1}_{\{|\theta_n - \theta^*| \le C\}} < +\infty.$$

Alors, si $\gamma_n = \frac{c}{n^\alpha}$, $n \ge 1$, $1/2 < \alpha < 1$, la suite des moyennes empiriques

$$\bar{\theta}_n = \frac{\theta_0 + \cdots + \theta_{n-1}}{n}$$

vérifie sur $\{\theta_n \xrightarrow{p.s.} \theta^*\}$, le TCL avec la variance optimale

$$\sqrt{n}(\bar{\theta}_n - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0; Dh(\theta^*)^{-1} \Gamma Dh(\theta^*)) \qquad \text{sur } \{\theta_n \xrightarrow{p.s.} \theta^*\}.$$

Remarque. La moyenne empirique vérifie la procédure récursive suivante

$$\forall\, n \ge 0, \quad \bar{\theta}_{n+1} = \bar{\theta}_n - \frac{1}{n+1}(\bar{\theta}_n - \theta_n), \qquad \bar{\theta}_0 = 0.$$

Chapter 2

Stochastic Approximation with Averaging Innovation Applied to Finance

This chapter is based on a paper accepted for publication in *Monte Carlo Methods and Applications*.

The aim of the chapter is to establish a convergence theorem for multi-dimensional stochastic approximation when the "innovations" satisfy some "light" averaging properties in the presence of a pathwise Lyapunov function. These averaging assumptions allow us to unify apparently remote frameworks where the innovations are simulated (possibly deterministic like in Quasi-Monte Carlo simulation) or exogenous (like market data) with ergodic properties. We propose several fields of applications and illustrate our results on five examples mainly motivated by Finance.

2.1 Introduction

The aim of this chapter is to establish a convergence theorem for multi-dimensional recursive stochastic approximation in a non-standard framework (compared to the huge literature on this field, see [25], [43], [75], [12], etc): we will significantly relax our assumption on the innovation process by only asking for some natural "light" ergodic or simply averaging assumptions, compensated by a reinforcement of the mean reversion assumption since we will require the existence of a *pathwise* Lyapunov function. We will show that this approach unifies seemingly remote settings: those where the innovations are simulated or even deterministic (quasi-Monte Carlo simulation) and those where the innovations are exogenous data (like market data). Especially in the latest case it may be not realistic to make a priori too stringent assumptions on the dynamics of such data process, like mixing or Markov. On the other hand, the pathwise Lyapunov assumption is definitely an intrinsic limitation to the kind of problem we can deal with, compared to the procedures extensively investigated in [25] or more recently in [41] where innovations are Markovian and share mixing properties.

However, we provide several examples, mainly inspired by Finance, to illustrate the fact that the field of application of our framework is rather wide and can solve efficiently various kinds of problems, some of them having already been considered in the literature.

Let us be more specific: this chapter presents convergence results for $\mathbb{R}^d$-valued stochastic approximation procedures of Robbins-Monro type (see [103] for the original paper), namely

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_n), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}^d, \tag{2.1}$$

where $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ is a Borel function, $(\gamma_n)_{n\geq 1}$ a sequence of positive steps and the "innovation" sequence $(Y_n)_{n\geq 0}$ satisfies some "elementary" averaging assumptions (θ_0 is assumed to be

deterministic in this introduction for convenience simplicity). In fact, we will consider a slightly more general setting which includes an extra noisy term

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\left(H(\theta_n, Y_n) + \Delta M_{n+1}\right), \quad n \geq 0, \tag{2.2}$$

where $(\Delta M_n)_{n\geq 1}$ is a sequence of $\mathbb{R}^d$-valued $\mathcal{F}_n$-adapted martingale increments for a filtration $\mathcal{F}_n$.

To establish the *a.s.* convergence of the sequence $(\theta_n)_{n\geq 0}$ toward its "target" θ^* (to be specified later on), the idea is to make the assumption that the innovation sequence $(Y_n)_{n\geq 0}$ satisfies an *averaging* property in a "linear" setting: typically that, for a wide enough class $\mathcal{V}$ of integrable functions (with respect to a probability measure ν),

$$\forall f \in \mathcal{V}, \quad \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^q} f d\nu \tag{2.3}$$

at a common rate of convergence to be specified further on. If $\mathcal{V} \supset \mathcal{C}_b(\mathbb{R}^q, \mathbb{R})$, this implies

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{Y_k} \underset{n\to\infty}{\overset{(\mathbb{R}^q)}{\Longrightarrow}} \nu \quad a.s.$$

by a separability argument ($\overset{(\mathbb{R}^q)}{\Longrightarrow}$ denotes the weak convergence of probability measures). Such a sequence $(Y_n)_{n\geq 0}$ is often called "stable" in the literature, at least when it is a Markov chain. If $\mathcal{V} = L^1(\nu)$, the sequence $(Y_n)_{n\geq 0}$ may be called in short "ergodic" although no true ergodic framework comes in the game at this stage. The target of our recursive procedure (2.2) is then, as expected, a zero, if any, of the (asymptotic) *mean function* of the algorithm defined as

$$h(\theta) := \int_{\mathbb{R}^q} H(\theta, y)\nu(dy).$$

The key assumption is the existence of *pathwise Lyapunov function* with respect to the innovation *i.e.* a function L satisfying

$$\langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq 0$$

for every θ and y. This assumption may look very stringent but in fact, it embodies standard framework of Stochastic Approximation with Markov representation of the form (2.1) when the $(Y_n)_{n\geq 0}$ is i.i.d. since, under appropriate integrability assumptions, it can be rewritten as follows in canonical form

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\left(\widetilde{H}(\theta_n, Y_n) + \Delta\widetilde{M}_{n+1}\right), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}^d,$$

where $\widetilde{H}(\theta, \cdot) = h(\theta)$ and $\Delta\widetilde{M}_{n+1} = H(\theta_n, Y_n) - h(\theta_n)$, $n \geq 0$. Then $(\Delta\widetilde{M}_n)_{n\geq 1}$ is a sequence of $\sigma(Y_0, \ldots, Y_{n-1})$-martingale increments (under appropriate integrability assumptions). Finally $\widetilde{H}(\theta, \cdot) = h(\theta)$ does not depend on y so that the above notion of pathwise Lyapunov function reduces to the standard one. The above canonical form has been extensively investigated (and extended) in many textbooks on Stochastic Approximation (see [25], [43], [74], [75]).

Our main theorem (Theorem 2.1) let us retrieve almost entirely the classical results about L^p-boundedness and *a.s.* convergence of this procedure under standard Lyapunov assumption. Many extensions have been developed when $(Y_n)_{n\geq 0}$ or even $(\theta_n, Y_n)_{n\geq 0}$ have a Markovian dynamics (see the seminal textbook [25] and more recent contributions like [41] and several references therein). The main constraint induced by such an approach is that the existence as well as assumptions on the solution of the Poisson equation related to this chain are needed.

As a first field of applications, we are interested in quasi-random numbers. The original idea of replacing by uniformly distributed sequences (with low discrepancy) i.i.d. innovations in recursive stochastic approximation procedures goes back to the early 1990's in [81], leading to "Quasi-Stochastic Approximation" (QSA, referring to QMC for Quasi-Monte Carlo). The framework in [81] was purely one-dimensional whereas many numerical tests h proved the efficiency of QSA in a multi-dimensional setting. The aim is to establish a convergence theorem in this higher dimensional setting under natural regularity assumptions (*i.e.* based on Lipschitz regularity rather than finite variation in the Hardy & Krause or in the measure sense, often encountered in the QMC world). As concerns the low discrepancy sequences, our framework is probably close to the most general one to get pointwise *a.s.* convergence of stochastic approximation.

As a second setting, we consider the case when $(Y_n)_{n\geq 0}$ is a functional of α-mixing process satisfying a priori no Markov assumption. These processes are stationary and dependent, so more realistic to model inputs made of real data. To describe the class of functions $\mathcal{V}$ we need to prove the convergence of the series of covariance coefficients of the innovations. To this end we use some results in [97] and the covariance inequality for α-mixing process (see [42]). Next with the probabilistic version of the Gàl-Koksma theorem (see [54] and [23, 4, 5]) we prove that this class is large enough ($L^{2+\delta}(\nu) \subset \mathcal{V}$, $\delta > 0$). Finally we examine the case of homogeneous Markov chain with (unique) invariant distribution ν. Several convergence results of stochastic approximation have been proved in this setting in [25], but they all rely on the existence (and some regularity properties) of a solution to Poisson equation. To describe $\mathcal{V}$ we add an assumption on the transition of the chain which allows us to prove that this class does not depend on the initial value of the chain.

Finally we propose several examples of applications illustrated with numerical experiment. They can be parted in two classes: the first one devoted to simulated innovations (*i.e.* Numerical Probability methods) and the second one deals with the applications involving real data. Primarily we present a simple case of calibration: the search, for a derivative product in a financial model, of an implicit model parameter fitting with its market value. We implement the algorithm with both an i.i.d. sequence and a quasi-Monte Carlo sequence to compare their respective rates of convergence. The second example is devoted to the recursive computation of risk measures commonly considered in energy portfolio management: the Value-at-Risk and the Conditional-Value-at-Risk. We design a stochastic gradient and a companion procedure to compute risk measures (like in [20, 53]) and we show that they can be successfully implemented in a QSA framework. In the third example, we solve numerically a "toy" long term investment problem leading to a static potential minimization derived from an ergodic control problem (see [90]). The potential is related to the invariant measure of a diffusion so that the innovation relies on (inhomogeneous Markov) Euler schemes with decreasing step introduced in [77] (see also [87]). These three stochastic approximation procedures rely on *simulated* data. The fourth example is the so-called two-armed bandit introduced in learning automata and mathematical psychology in the 1950's (see [94]). Its *a.s.* behaviour in the i.i.d. setting has been extensively investigated in [80] and [78] and then partially extended in [110] to a more general ergodic framework. We show how the starting point of this extension appears as a consequence our multiplicative case (Theorem 2.2). The last example describes a model of asset allocation across liquidity pools fully developed in [82] involving exogenous real market data, *a priori* sharing no Markov property but on which an averaging assumption seems natural (at least within a medium laps of time).

The chapter is organized as follows: in Section 2.2 are stated and proved the two main results: Theorem 2.1 and its counterpart Theorem 2.2, for multiplicative noise. Section 2.3 is devoted to quasi-Stochastic Approximation, *i.e.* the case where the innovation process is an uniformly distributed deterministic sequence over $[0,1]^q$. Section 2.4 is devoted to applications to random innovations, namely additive noise, mixing process (functionals of α-mixing process), ergodic homogeneous Markov chain. Section 2.5 presents five examples of applications including numerical

illustrations, mostly in connection with Finance: implicit correlation search, recursive computation of VaR and CVaR, long term investment evaluation, two-armed bandit algorithm and optimal allocation problem (more developed in [82]).

Notations $\langle \cdot \,|\, \cdot \rangle$ denotes the canonical Euclidean inner product and $|\cdot|$ its related norm on $\mathbb{R}^d$. The almost sure convergence will be denoted by $\xrightarrow{a.s.}$ and $\xRightarrow{(\mathbb{R}^q)}$ will denote the weak convergence of probability measures on $(\mathbb{R}^q, \mathcal{B}or(\mathbb{R}^q))$. $\Delta a_n = a_n - a_{n-1}$ for every sequence $(a_n)_n$.

2.2 Algorithm design and main theoretical result

In this chapter, we consider the following general framework for recursive stochastic algorithms of the following form

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\left(H(\theta_n, Y_n) + \Delta M_{n+1}\right), \quad n \geq 0, \tag{2.4}$$

where $(Y_n)_{n\geq 0}$ is an $\mathbb{R}^q$-valued sequence of $\mathcal{F}_n$-adapted random variables and $(\Delta M_n)_{n\geq 1}$ is a sequence of $\mathcal{F}_n$-adapted martingale increment, all defined on a same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$. Moreover $\theta_0 \in L^1_{\mathbb{R}^d}(\Omega, \mathcal{F}_0, \mathbb{P})$ and θ_0 is independent of $(Y_n, \Delta M_{n+1})_{n\geq 0}$. The positive step sequence $(\gamma_n)_{n\geq 1}$ is non-increasing and H is a Borel function from $\mathbb{R}^d \times \mathbb{R}^q$ to $\mathbb{R}^d$.

In the following, we adopt a kind of compromise by assuming that $(Y_n)_{n\geq 0}$ is a process satisfying some averaging properties and that the function $H(\theta^*, \cdot)$ belongs to a class of functions (to be specified further on) for which a rate of convergence ($a.s.$ and in L^p) holds in (2.3). Moreover we need to reinforce the Lyapunov condition on the pseudo-mean function H which limits, at least theoretically, the range of application of the method.

2.2.1 Framework and assumptions

Let $(Y_n)_{n\geq 0}$ be an $\mathbb{R}^q$-valued random variables sequence. We will say that the sequence $(Y_n)_{n\geq 0}$ satisfies a *ν-stability assumption* or equivalently is *ν-averaging* if

$$\mathbb{P}(d\omega)\text{-}a.s. \quad \frac{1}{n}\sum_{k=0}^{n-1} \delta_{Y_k(\omega)} \xRightarrow[n\to\infty]{(\mathbb{R}^q)} \nu. \tag{2.5}$$

We will see that the stochastic approximation procedure defined by (2.4) is a recursive zero search of the (asymptotic) mean function

$$h(\theta) := \int_{\mathbb{R}^q} H(\theta, y)\nu(dy). \tag{2.6}$$

Let $p \in [1, \infty)$ and let $(\varepsilon_n)_{n\geq 1}$ be a sequence of nonnegative numbers such that

$$\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \liminf_n n\varepsilon_n = 0. \tag{2.7}$$

We denote by $\mathcal{V}_{\varepsilon_n, p}$ the class of functions which convergence rate in (2.3) in both $a.s.$ and in $L^p(\mathbb{P})$ sense is ε_n^{-1}, namely

$$\mathcal{V}_{\varepsilon_n,p} = \left\{ f \in L^p(\nu) \,|\, \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) - \int f d\nu \overset{\mathbb{P}\text{-}a.s. \ \&\ L^p(\mathbb{P})}{=} O(\varepsilon_n) \right\}. \tag{2.8}$$

2.2.2 Main result

Now we are in a position to state an *a.s.*-convergence theorem "à la" Robbins-Siegmund.

Theorem 2.1. (*a*) Boundedness. *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ satisfying (2.6), $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ a Borel function and let $(Y_n)_{n\geq 0}$ be a ν-stable sequence (i.e. satisfying (2.5)). Assume there exists a continuously differentiable function $L : \mathbb{R}^d \to \mathbb{R}_+$ satisfying*

$$\nabla L \text{ is Lipschitz continuous and } |\nabla L|^2 \leq C\,(1+L) \tag{2.9}$$

and that the pseudo-mean function H satisfies the pathwise Lyapunov *assumption*

$$\forall \theta \in \mathbb{R}^d \backslash \{\theta^*\},\ \forall y \in \mathbb{R}^q, \quad \langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq 0. \tag{2.10}$$

Let $p \in [1, \infty)$ and let $(\varepsilon_n)_{n\geq 1}$ be a sequence satisfying (2.7). Assume that

$$H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, p}. \tag{2.11}$$

Moreover, assume that H satisfies the following (quasi-)linear growth assumption

$$\forall \theta \in \mathbb{R}^d,\ \forall y \in \mathbb{R}^q, \quad |H(\theta, y)| \leq C_H \phi(y)(1 + L(\theta))^{\frac{1}{2}} \tag{2.12}$$

and that the martingale increments sequence $(\Delta M_n)_{n\geq 1}$ satisfies for every $n \geq 0$,

$$\mathbb{P}\text{-}a.s. \begin{cases} \mathbb{E}\left(|\Delta M_{n+1}|^{2\vee \frac{p}{p-1}} \,|\, \mathcal{F}_n\right) \leq C_M \phi(Y_n)^{2\vee \frac{p}{p-1}} (1 + L(\theta_n))^{1\vee \frac{p}{2(p-1)}} & \text{if} \quad p > 1, \\ \frac{|\Delta M_{n+1}|}{(1+L(\theta_n))^{\frac{1}{2}}} \leq C_M & \text{if} \quad p = 1 \end{cases} \tag{2.13}$$

where C_M is a positive real constant and $\sup_{n\geq 0} \|\phi(Y_n)\|_{2\vee \frac{p}{p-1}} < +\infty$.
Let $\gamma = (\gamma_n)_{n\geq 1}$ be a nonnegative non-increasing sequence of "admissible" gain parameters satisfying

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0, \quad \text{and} \quad \sum_{n\geq 1} n\varepsilon_n \max\left(\gamma_n^2, |\Delta\gamma_{n+1}|\right) < +\infty. \tag{2.14}$$

Then, the recursive procedure defined by (2.4) satisfies $(L(\theta_n))_{n\geq 0}$ is L^1-bounded, $L(\theta_n) \underset{n\to\infty}{\longrightarrow} L_\infty < +\infty$ a.s., $\Delta\theta_n \underset{n\to\infty}{\longrightarrow} 0$ a.s. and

$$\sum_{n\geq 1} \langle \nabla L(\theta_n) \,|\, H(\theta_n, Y_n) - H(\theta^*, Y_n)\rangle < +\infty.$$

(*b*) *A.s.* convergence toward θ^*. *Furthermore, if $\{\theta^*\}$ is a connected component of $\{L = L(\theta^*)\}$ and the pseudo-mean function H satisfies the* strict pathwise Lyapunov *assumption*

$$\forall \delta > 0,\ \forall \theta \in \mathbb{R}^d \backslash \{\theta^*\},\ \forall y \in \mathbb{R}^q, \quad \langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq \chi_\delta(y)\Psi_\delta(\theta) \tag{2.15}$$

where $\nu(\chi_\delta) > 0$, Ψ_δ is l.s.c. and positive on $\mathbb{R}^d \backslash \{\theta^*\}$ and $\bigcap_{\delta>0}\{\Psi_\delta = 0\} = \{\theta^*\}$, then

$$\theta_n \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \theta^*.$$

Remark. The conditions on the step sequence $\gamma = (\gamma_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$ are satisfied for example by

$$\varepsilon_n = n^{-\beta}, \quad \beta \in (0, 1], \quad \text{and} \quad \gamma_n = \frac{c}{n^a}, \quad 1 - \beta < a \leq 1, \quad c > 0. \tag{2.16}$$

Proof. ***First step***: We introduce the function

$$\Lambda(\theta) := \sqrt{1 + L(\theta)}$$

as a Lyapunov function instead of $L(\theta)$ like in the classical case. It follows from the fundamental formula of calculus that there exists $\xi_{n+1} \in (\theta_n, \theta_{n+1})$ such that

$$\begin{aligned}\Lambda(\theta_{n+1}) &= \Lambda(\theta_n) + \langle \nabla\Lambda(\theta_n) \,|\, \Delta\theta_{n+1}\rangle + \langle \nabla\Lambda(\xi_{n+1}) - \nabla\Lambda(\theta_n) \,|\, \Delta\theta_{n+1}\rangle \\ &\leq \Lambda(\theta_n) + \langle \nabla\Lambda(\theta_n) \,|\, \Delta\theta_{n+1}\rangle + |\nabla\Lambda(\xi_{n+1}) - \nabla\Lambda(\theta_n)||\Delta\theta_{n+1}|.\end{aligned}$$

Lemma 2.1. *The new Lyapunov function Λ satisfies the two following properties*

(i) $\nabla\Lambda$ *is bounded (so that* Λ *is Lipschitz).*

(ii) $\forall \theta, \theta' \in \mathbb{R}^d, \quad |\nabla\Lambda(\theta') - \nabla\Lambda(\theta)| \leq C_L \frac{|\theta' - \theta|}{\Lambda(\theta)}$.

Proof of Lemma 2.1. *(i)* $\nabla\Lambda = \frac{\nabla L}{2\sqrt{1+L}}$ is bounded by (2.9), consequently Λ is Lipschitz.

(ii) Let $\theta, \theta' \in \mathbb{R}^d$,

$$\begin{aligned}\left|\nabla\Lambda(\theta) - \nabla\Lambda(\theta')\right| &\leq \frac{|\nabla L(\theta) - \nabla L(\theta')|}{2\sqrt{1 + L(\theta)}} + \frac{|\nabla L(\theta')|}{2}\left|\frac{\sqrt{1 + L(\theta')} - \sqrt{1 + L(\theta)}}{\sqrt{1 + L(\theta)}\sqrt{1 + L(\theta')}}\right| \\ &\leq \frac{[\nabla L]_{\mathrm{Lip}}}{2\sqrt{1 + L(\theta)}}\left|\theta - \theta'\right| + \frac{C}{2\sqrt{1 + L(\theta)}}[\Lambda]_{\mathrm{Lip}}\left|\theta - \theta'\right| \\ &\leq \frac{1}{2}\left([\nabla L]_{\mathrm{Lip}} + C[\Lambda]_{\mathrm{Lip}}\right)\frac{|\theta - \theta'|}{\Lambda(\theta)} \\ &= C_L\frac{|\theta - \theta'|}{\Lambda(\theta)}.\end{aligned}$$

□

Thus, applying the above lemma to $\theta = \theta_n$ and $\theta' = \xi_{n+1}$, and noting that $|\xi_{n+1} - \theta_n| \leq |\Delta\theta_{n+1}|$ yields

$$\begin{aligned}\Lambda(\theta_{n+1}) &\leq \Lambda(\theta_n) - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta_n, Y_n)\rangle - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, \Delta M_{n+1}\rangle + C_L\frac{|\Delta\theta_{n+1}|^2}{\sqrt{1 + L(\theta_n)}} \\ &= \Lambda(\theta_n) - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta_n, Y_n) - H(\theta^*, Y_n)\rangle - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta^*, Y_n)\rangle \\ &\quad -\gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, \Delta M_{n+1}\rangle + C_L\gamma_{n+1}^2\frac{|H(\theta_n, Y_n) + \Delta M_{n+1}|^2}{\sqrt{1 + L(\theta_n)}}.\end{aligned}$$

We have for every $n \geq 0$,

$$|\gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta^*, Y_n)\rangle| \leq C_\Lambda\gamma_{n+1}\phi(Y_n) \in L^1(\mathbb{P})$$

since $\nabla\Lambda$ is bounded. Besides $\mathbb{E}\left[\langle \nabla\Lambda(\theta_n) \,|\, \Delta M_{n+1}\rangle \,|\, \mathcal{F}_n\right] = 0$, $n \geq 0$, since ΔM_n is a true martingale increment and $\nabla\Lambda$ is bounded. Furthermore, owing to (2.12) and (2.13)

$$\mathbb{E}\left[\frac{|H(\theta_n, Y_n) + \Delta M_{n+1}|^2}{\sqrt{1 + L(\theta_n)}} \,|\, \mathcal{F}_n\right] \leq C\phi^2(Y_n)\Lambda(\theta_n)$$

(where conditional expectation is extended to positive random variables). Consequently,

$$\begin{aligned}\mathbb{E}\left[\Lambda(\theta_{n+1}) \,|\, \mathcal{F}_n\right] &\leq \Lambda(\theta_n)\left(1 + C_L'\gamma_{n+1}^2\phi(Y_n)^2\right) - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta_n, Y_n) - H(\theta^*, Y_n)\rangle \\ &\quad -\gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta^*, Y_n)\rangle.\end{aligned} \tag{2.17}$$

We set $V_n := \dfrac{A_n}{B_n}$, $n \geq 1$ where

$$A_n := \Lambda(\theta_n) + \sum_{k=0}^{n-1} \gamma_{k+1} \langle \nabla\Lambda(\theta_k) \mid H(\theta_k, Y_k) - H(\theta^*, Y_k) \rangle, \quad B_n := \prod_{k=1}^{n} \left(1 + C_L' \gamma_k^2 \phi(Y_{k-1})^2\right).$$

Using the mean-reverting assumption (2.15) implies that $(A_n)_{n\geq 0}$ is a nonnegative process and B_n is $\mathcal{F}_{n-1}$-adapted, $n \geq 1$. Elementary computations first show that

$$\mathbb{E}\left[A_{n+1} \mid \mathcal{F}_n\right] \leq A_n \frac{B_{n+1}}{B_n} - \gamma_{n+1} \langle \nabla\Lambda(\theta_n) \mid H(\theta^*, Y_n) \rangle$$

which finally yields

$$\forall n \geq 0, \quad \mathbb{E}\left[V_{n+1} \mid \mathcal{F}_n\right] \leq V_n - \Delta W_{n+1}, \tag{2.18}$$

where $W_n := \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \langle \nabla\Lambda(\theta_k) \mid H(\theta^*, Y_k) \rangle$ with $\widetilde{\gamma}_n := \frac{\gamma_n}{B_n}$, $n \geq 0$.

Second step: Now our aim is to prove that the sequence $(W_n)_{n\geq 0}$ is L^1-bounded and $a.s.$ converges. To this end we set $S_n^* := \sum_{k=0}^{n-1} H(\theta^*, Y_k)$, then it follows

$$W_n = \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \left\langle \nabla\Lambda(\theta_k) \mid \Delta S_{k+1}^* \right\rangle = \widetilde{\gamma}_n \langle \nabla\Lambda(\theta_{n-1}) \mid S_n^* \rangle - \sum_{k=1}^{n-1} \langle S_k^* \mid \widetilde{\gamma}_{k+1} \nabla\Lambda(\theta_k) - \widetilde{\gamma}_k \nabla\Lambda(\theta_{k-1}) \rangle.$$

First, since $\nabla\Lambda$ is bounded, note that

$$\widetilde{\gamma}_n \left|\nabla\Lambda(\theta_{n-1})\right| |S_n^*| \leq \|\nabla\Lambda\|_\infty n\varepsilon_n \widetilde{\gamma}_n \frac{|S_n^*|}{n\varepsilon_n} \leq \|\nabla\Lambda\|_\infty n\varepsilon_n \gamma_n \frac{|S_n^*|}{n\varepsilon_n}$$

which $a.s.$ goes to 0 as n goes to infinity since $n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0$ by (2.14) and $\left(\dfrac{S_n^*}{n\varepsilon_n}\right)_{n\geq 1}$ remains $a.s.$ bounded. Moreover

$$\mathbb{E}\left[\widetilde{\gamma}_n \left|\nabla\Lambda(\theta_{n-1})\right| |S_n^*|\right] \leq n\varepsilon_n\gamma_n \|\nabla\Lambda\|_\infty \left\| \frac{S_n^*}{n\varepsilon_n} \right\|_1$$

which converges to 0 in L^1 because $n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0$ and $H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, p}$. On the other hand,

$$\sum_{k=1}^{n-1} \langle S_k^* \mid \widetilde{\gamma}_{k+1} \nabla\Lambda(\theta_k) - \widetilde{\gamma}_k \nabla\Lambda(\theta_{k-1}) \rangle = \sum_{k=1}^{n-1} \langle S_k^* \mid \nabla\Lambda(\theta_k) \rangle \, \Delta\widetilde{\gamma}_{k+1} + \sum_{k=1}^{n-1} \widetilde{\gamma}_k \langle S_k^* \mid \nabla\Lambda(\theta_k) - \nabla\Lambda(\theta_{k-1}) \rangle.$$

As $\nabla\Lambda = \dfrac{\nabla L}{2\sqrt{1+L}}$ is bounded by construction, we have

$$\sum_{k=1}^{n} |\Delta\widetilde{\gamma}_{k+1} \langle S_k^* \mid \nabla\Lambda(\theta_k) \rangle| \leq \sum_{k=1}^{n} |\Delta\widetilde{\gamma}_{k+1}| \, |S_k^*| \, \|\nabla\Lambda\|_\infty \leq \|\nabla\Lambda\|_\infty \sum_{k=1}^{n} k\varepsilon_k \, |\Delta\widetilde{\gamma}_{k+1}| \left| \frac{S_k^*}{k\varepsilon_k} \right|.$$

Now, using that $\frac{a}{1+a} \leq \sqrt{a}$, $a > 0$,

$$|\Delta\widetilde{\gamma}_{k+1}| \leq |\Delta\gamma_{k+1}| + \gamma_k \frac{C_L' \gamma_{k+1}^2 \phi(Y_k)^2}{B_{k+1}} \leq |\Delta\gamma_{k+1}| + \gamma_k \frac{C_L' \gamma_{k+1}^2 \phi(Y_k)^2}{1 + C_L' \gamma_{k+1}^2 \phi(Y_k)^2} \leq |\Delta\gamma_{k+1}| + C_L' \gamma_k \gamma_{k+1} \phi(Y_k).$$

Hence

$$\sum_{k=1}^{n} |\Delta\widetilde{\gamma}_{k+1} \langle S_k^* \mid \nabla\Lambda(\theta_k) \rangle| \leq \|\nabla\Lambda\|_\infty \left(\sum_{k=1}^{n} k\varepsilon_k \, |\Delta\gamma_{k+1}| \left| \frac{S_k^*}{k\varepsilon_k} \right| + C_L' \sum_{k=1}^{n} k\varepsilon_k \gamma_k \gamma_{k+1} \phi(Y_k) \left| \frac{S_k^*}{k\varepsilon_k} \right| \right).$$

By Hölder's Inequality

$$\mathbb{E}\left(\phi(Y_k)\left|\frac{S_k^*}{k\varepsilon_k}\right|\right) \leq \|\phi(Y_k)\|_{\frac{p}{p-1}} \left\|\frac{S_k^*}{k\varepsilon_k}\right\|_p.$$

As $\left(\frac{S_k^*}{k\varepsilon_k}\right)_{n\geq 0}$ is bounded, γ is admissible and $\sup_{k\geq 0}\|\phi(Y_k)\|_{\frac{p}{p-1}} < +\infty$, then

$$\text{the series } \sum_{k=1}^n \Delta\widetilde{\gamma}_k \langle S_k^* \mid \nabla\Lambda(\theta_k)\rangle \text{ is absolutely converging in } L^1(\mathbb{P}).$$

We study now the series $\sum_{k=1}^n \widetilde{\gamma}_k \langle S_k^* \mid \nabla\Lambda(\theta_k) - \nabla\Lambda(\theta_{k-1})\rangle$. We have

$$|\nabla\Lambda(\theta_k) - \nabla\Lambda(\theta_{k-1})| \leq C_L' \frac{|\Delta\theta_k|}{\sqrt{1+L(\theta_{k-1})}} \leq C_L'\gamma_k \frac{|H(\theta_{k-1},Y_{k-1})| + |\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}.$$

We are interested in the L^1-convergence of the series

$$\sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|H(\theta_{k-1},Y_{k-1})|}{\sqrt{1+L(\theta_{k-1})}} \quad \text{and} \quad \sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}.$$

For the first sum, as $\frac{|H(\theta_{k-1},Y_{k-1})|}{\sqrt{1+L(\theta_{k-1})}} \leq C_H\phi(Y_{k-1})$, we then come to $\sum_{k=1}^n C_H\gamma_k^2\mathbb{E}\left[|S_k^*|\,|\phi(Y_{k-1})|\right]$ and by Hölder's inequality we obtain

$$\mathbb{E}\left[|S_k^*|\,|\phi(Y_{k-1})|\right] \leq \|S_k^*\|_p \|\phi(Y_{k-1})\|_{\frac{p}{p-1}} < +\infty$$

because $\|S_k^*\|_p = O(k\varepsilon_k)$ by (2.8) and $\sup_{n\geq 0}\|\phi(Y_n)\|_{\frac{p}{p-1}} < +\infty$. Furthermore, as $\sum_{k\geq 1} k\varepsilon_k\gamma_k^2 < +\infty$ by (2.14), then the series $\sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|H(\theta_{k-1},Y_{k-1})|}{\sqrt{1+L(\theta_{k-1})}}$ converges in L^1.
For the second sum, we derive from Hölder's inequality (with p and $\frac{p}{p-1}$) and (2.13) that

$$\mathbb{E}\left[|S_k^*|\frac{|\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}\right] \leq \|S_k^*\|_p \left\|\frac{|\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}\right\|_{\frac{p}{p-1}} \leq C_M \|S_k^*\|_p \|\phi(Y_{k-1})\|_{\frac{p}{p-1}\vee 2} < +\infty.$$

This yields that $\sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}$ converges in L^1 too. Finally we then obtain that $W_n \xrightarrow[n\to\infty]{a.s.} W_\infty$ and $\sup_{n\geq 0}\|W_n\|_1 < +\infty$. Thus we have that

$$(V_n + W_n)^- \leq W_n^- \leq |W_n| \in L^1(\mathbb{P}) \quad \text{since} \quad \sup_{n\geq 0}\|W_n\|_1 < +\infty.$$

As $V_0 = \Lambda(\theta_0) \leq C(1+|\theta_0|) \in L^1$, it follows by induction from (2.18) that, for every $n \geq 0$, $\mathbb{E}\,V_n < +\infty$. Hence $S_n := V_n + W_n$, $n \geq 0$, is a true supermartingale with an L^1-bounded negative part. We then deduce that

$$S_n \xrightarrow[n\to\infty]{a.s.} S_\infty \in L^1.$$

Now $W_n \xrightarrow[n\to\infty]{a.s.} W_\infty$ implies $V_n \xrightarrow[n\to\infty]{a.s.} V_\infty < +\infty$ $a.s.$

Third step: Now we show that the product B_n converges $a.s.$ to derive that A_n converges $a.s.$. In fact

$$\sum_{n\geq 1}\gamma_n^2\phi^2(Y_{n-1}) < +\infty \ a.s.,$$

since $\sup_{n\geq 0}\mathbb{E}\left[\phi^2(Y_n)\right] < +\infty$ and $\sum_{n\geq 1}\gamma_n^2 < +\infty$ by combining (2.7) and (2.14), which in turn implies that $B_n \xrightarrow[n\to\infty]{a.s.} B_\infty < +\infty$. As a consequence $A_n \xrightarrow[n\to\infty]{a.s.} A_\infty < +\infty$. Therefore using the mean reverting property (2.15) of H with respect to $\nabla\Lambda$, we classically derive that

$$\sum_{n\geq 1}\gamma_n \left\langle \nabla\Lambda(\theta_{n-1}) \,|\, H(\theta_{n-1},Y_{n-1}) - H(\theta^*,Y_{n-1})\right\rangle < +\infty \quad a.s. \tag{2.19}$$

Consequently

$$\Lambda(\theta_n) \xrightarrow[n\to\infty]{a.s.} \Lambda_\infty < +\infty \quad a.s.$$

As $\lim_{|\theta|\to+\infty} L(\theta) = +\infty$, $\lim_{|\theta|\to+\infty}\Lambda(\theta) = +\infty$, then the sequence $(\theta_n)_{n\geq 0}$ is $a.s.$-bounded and

$$L(\theta_n) \xrightarrow[n\to\infty]{a.s.} L_\infty < +\infty \quad a.s.$$

Now let us show that $\Delta\theta_n \underset{n\to+\infty}{\longrightarrow} 0$. In fact, $|\Delta\theta_{n+1}|^2 \leq C\gamma_{n+1}^2\left(|H(\theta_n,Y_n)|^2 + |\Delta M_{n+1}|^2\right)$, so that

$$\mathbb{E}\left[|\Delta\theta_{n+1}|^2 \,|\, \mathcal{F}_n\right] \leq C\gamma_{n+1}^2\phi(Y_n)^2(1+L(\theta_n))$$

and $(L(\theta_n))_{n\geq 0}$ being $a.s.$ bounded,

$$\sum_{n\geq 0}\mathbb{E}\left[|\Delta\theta_{n+1}|^2 \,|\, \mathcal{F}_n\right] < +\infty. \ a.s.$$

which classically implies that $\sum_{n\geq 0}|\Delta\theta_{n+1}|^2 < +\infty \ a.s.$

Fourth step: To prove the convergence of θ_n toward θ^*, we use Assumptions (2.15) and (2.19) to deduce that

$$\sum_{n\geq 1}\gamma_n\chi_\delta(Y_{n-1})\Psi_\delta(\theta_{n-1}) < +\infty \quad a.s. \tag{2.20}$$

Now

$$\sum_{k=0}^{n}\gamma_{k+1}\chi_\delta(Y_k) = \sum_{k=0}^{n}\gamma_{k+1}\Delta S_k^\chi = \gamma_{n+1}S_n^\chi - \sum_{k=1}^{n-1}\Delta\gamma_{k+2}S_k^\chi$$

where $S_n^\chi = \sum_{k=0}^n \chi_\delta(Y_k)$ and we set $S_0^\chi = 0$ and $\Delta S_0^\chi = 0$.
By Assumption (2.5), $\dfrac{S_n^\chi}{n} \to \nu(\chi_\delta) > 0$ as $n\to\infty$. Let n_0 be the smallest integer such that

$$\forall n \geq n_0, \quad \frac{S_n^\chi}{n} \geq \epsilon_0 = \frac{\nu(\chi_\delta)}{2} > 0.$$

Then, a standard discrete integration by part yields

$$\forall n \geq n_0, \quad \sum_{k=n_0}^{n}\gamma_{k+1}\chi_\delta(Y_k) = n\gamma_{n+1}\frac{S_n^\chi}{n} - C_{n_0} + \sum_{k=n_0}^{n-1}k(-\Delta\gamma_{k+2})\frac{S_k^\chi}{k} \quad a.s.,$$

where $C_{n_0} = \gamma_{n_0+1} S^{\chi}_{n_0-1}$. Therefore, using that the sequence $(-\Delta\gamma_n)_{n\geq 1}$ is nonnegative,

$$\sum_{k=n_0}^{n} \gamma_{k+1}\chi_\delta(Y_k) \geq n\gamma_{n+1}\epsilon_0 - C_{n_0} + \sum_{k=n_0}^{n-1} k(-\Delta\gamma_{k+2})\epsilon_0 = \epsilon_0 \left(n\gamma_{n+1} + \sum_{k=n_0}^{n-1} k(-\Delta\gamma_{k+2}) \right) - C_{n_0}$$
$$= \epsilon_0 \left(\gamma_{n+1} + n_0\gamma_{n_0+1} + \sum_{k=n_0+1}^{n-1} \gamma_{k+1} \right) - C_{n_0}$$

by a reverse discrete integration by parts. Finally

$$\sum_{k=n_0}^{n} \gamma_{k+1}\chi_\delta(Y_k) \geq \epsilon_0 \left(\gamma_{n+1} + \sum_{k=n_0+1}^{n-1} \gamma_{k+1} \right) - C_{n_0} \to \infty \quad \text{as} \quad n \to \infty$$

since $\sum_{n\geq 1} \gamma_n = +\infty$. We have then shown that

$$\sum_{k\geq 0} \gamma_{k+1}\chi_\delta(Y_k) = +\infty \quad a.s.$$

Combining this fact with (2.20) classically implies that

$$\liminf_n \widetilde{\Psi}_\delta(\theta_n) = 0.$$

Let Θ_∞ be the set of limiting points of the sequence $(\theta_n)_{n\geq 0}$. Θ_∞ is a compact connected set since $(\theta_n)_{n\geq 0}$ is bounded and $\Delta\theta_n \underset{n\to\infty}{\longrightarrow} 0$. Moreover $\{\Psi_\delta = 0\}$ is closed because $\Psi_\delta \leq 0$ and l.s.c. and Θ_∞ is closed too. So $\Theta_\infty \cap \{\Psi_\delta = 0\}$ is a family of nonempty compact sets which decreases as $\delta \searrow 0$ since it is bounded. As a consequence,

$$\bigcap_{\delta>0} (\Theta_\infty \cap \{\Psi_\delta = 0\}) \neq \emptyset.$$

The other assumption on Ψ_δ implies

$$\bigcap_{\delta>0} (\Theta_\infty \cap \{\Psi_\delta = 0\}) \subset \bigcap_{\delta>0} \{\Psi_\delta = 0\} = \{\theta^*\},$$

so that in fact it is reduced to θ^*. Hence θ^* is a limiting point of $(\theta_n)_{n\geq 0}$ which implies that $L(\theta_n)$ converges towards $L(\theta^*)$. By the assumption on the Lyapunov function L, $\{\theta^*\}$ is a connected component of $\{L = L(\theta^*)\}$ and as Θ_∞ is connected, $\Theta_\infty = \{\theta^*\}$. Therefore

$$\theta_n \xrightarrow{a.s.} \theta^* \quad \text{as} \quad n \to \infty. \qquad \square$$

BACK TO THE I.I.D. INNOVATION SETTING. ▷ Theorem 2.1 contains the *standard martingale approach* "à la" Robbins-Siegmund in the i.i.d. setting. Indeed, we consider the recursive procedure

$$\theta_{n+1} = \theta_n - \gamma_{n+1} K(\theta_n, Y_{n+1}), \quad n \geq 0,$$

where $(Y_n)_{n\geq 1}$ is i.i.d. with distribution ν and θ_0 is independent of $(Y_n)_{n\geq 1}$ (all defined on $(\Omega, \mathcal{A}, \mathbb{P})$). We set $\mathcal{F}_n = \sigma(\theta_0, Y_1, \ldots, Y_n)$, $n \geq 0$, $p = 2$,

$$H(\theta, y) = h(\theta) \quad \text{with} \quad h(\theta) = \int_{\mathbb{R}^q} K(\theta, y)\nu(dy) \quad \text{and} \quad \Delta M_{n+1} = K(\theta_n, Y_{n+1}) - \mathbb{E}\left[K(\theta_n, Y_{n+1}) \mid \mathcal{F}_n\right].$$

Assume that

$$\forall \theta \in \mathbb{R}^d, \quad \|K(\theta, Y_1)\|_2 \leq C_K(1 + L(\theta))^{\frac{1}{2}}.$$

Then Assumption (2.12) is satisfied by h and (2.13) holds (with $\phi \equiv 1$). Furthermore, by combining (2.7) and (2.14), we retrieve the step assumption in the standard Robbins-Monro Theorem, namely

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

▷ Another (naive) way to apply Theorem 2.1 in this i.i.d. setting is to focus, under the above assumption on the averaging property so that: then $H = K$ and $\Delta M_n \equiv 0$. We still consider the above procedure but we assume furthermore the existence of a pathwise Lyapunov function. By noticing that $(K(\theta^*, Y_n))_{n\geq 1}$ is i.i.d. and in L^2, it follows from the quadratic law of large numbers (at rate $n^{-\frac{1}{2}}$) and the Law of Iterated Logarithm at rate $O(\varepsilon_n)$ with $\varepsilon_n = \sqrt{\frac{\log\log n}{n}}$ that $K(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n,2}$. As a consequence the condition (2.14) is clearly more restrictive than the above regular one, however any step of the form $\gamma_n = \frac{c}{n^\alpha}$, $c > 0$, $\frac{3}{4} < \alpha \leq 1$ satisfies (2.14).

2.2.3 The case of multiplicative noise

If we assume that the function H in (2.4) is of the following form

$$\forall \theta \in \mathbb{R}^d, \forall y \in \mathbb{R}^q, \quad H(\theta, y) = \chi(y)h(\theta) + H(\theta^*, y), \tag{2.21}$$

where χ is a Borel function such that $\nu(\chi) = 1$, $\chi \in \mathcal{V}_{\varepsilon_n,p}$ and $\sup_{n\geq 0} \|\chi(Y_n)\|_{2\vee\frac{p}{p-1}} < +\infty$, $H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n,p}$ and $\sup_{n\geq 0} \|H(\theta^*, Y_n)\|_{2\vee\frac{p}{p-1}} < +\infty$, h is Lipschitz bounded with $h(\theta^*) = 0$, then we replace the growth assumption (2.12) on H by one on the mean function h, *i.e.*

$$\forall \theta \in \mathbb{R}^d, \forall y \in \mathbb{R}^q, \quad |h(\theta)| \leq C_h\sqrt{1 + L(\theta)} \tag{2.22}$$

and the pathwise mean-reverting assumption (2.15) is the classical

$$\forall \theta \in \mathbb{R}^d \setminus \{\theta^*\}, \quad \langle \nabla L \mid h \rangle (\theta) > 0. \tag{2.23}$$

Theorem 2.2. *The recursive procedure (2.4) with the function H defined by (2.21) and the previous assumptions on χ and (2.22)-(2.23) on h satisfies*

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^*.$$

Proof. ***First step***: This setting cannot be reduced to the general setting. We use the same notations as in the proof of Theorem 2.1. With the new form of the function H, we obtain

$$\begin{aligned}\Lambda(\theta_{n+1}) \quad \leq \quad & \Lambda(\theta_n) - \gamma_{n+1} \langle \nabla\Lambda(\theta_n) \mid \chi(Y_n)h(\theta_n)\rangle - \gamma_{n+1} \langle \nabla\Lambda(\theta_n) \mid H(\theta^*, Y_n)\rangle \\ & -\gamma_{n+1} \langle \nabla\Lambda(\theta_n) \mid \Delta M_{n+1}\rangle + C_L\gamma_{n+1}^2 \frac{|H(\theta_n, Y_n) + \Delta M_{n+1}|^2}{\sqrt{1 + L(\theta_n)}}.\end{aligned}$$

By the same arguments as before we get

$$\mathbb{E}\left[\Lambda(\theta_{n+1}) \mid \mathcal{F}_n\right] \leq \Lambda(\theta_n)\left(1 + C'_L\gamma_{n+1}^2\phi(Y_n)^2\right) - \gamma_{n+1} \langle \nabla\Lambda(\theta_n) \mid \chi(Y_n)h(\theta_n)\rangle - \gamma_{n+1} \langle \nabla\Lambda(\theta_n) \mid H(\theta^*, Y_n)\rangle.$$

We set $V_n := \dfrac{A_n}{B_n}$, where $A_n := \Lambda(\theta_n) + \sum_{k=0}^{n-1} \gamma_{k+1} \langle \nabla\Lambda \mid h\rangle (\theta_k)$ and $B_n := \prod_{k=1}^n \left(1 + C'_L\gamma_k^2\phi(Y_{k-1})^2\right)$. Using the mean-reverting assumption (2.23) implies that $(A_n)_{n\geq 0}$ is a nonnegative process whereas $(B_n)_{n\geq 0}$ is still $\mathcal{F}_{n-1}$-adapted. Elementary computations show that

$$\mathbb{E}\left[A_{n+1} \mid \mathcal{F}_n\right] \leq A_n\frac{B_{n+1}}{B_n} - \gamma_{n+1} \langle \nabla\Lambda(\theta_n) \mid H(\theta^*, Y_n)\rangle - \gamma_{n+1}\widetilde{\chi}(Y_n) \langle \nabla\Lambda \mid h\rangle (\theta_n)$$

where $\widetilde{\chi}(Y_n) := \chi(Y_n) - \nu(\chi)$, $n \geq 0$. Finally we have

$$\forall n \geq 0, \quad \mathbb{E}\left[V_{n+1} \,|\, \mathcal{F}_n\right] \leq V_n - \Delta W_{n+1} - \Delta Z_{n+1}, \tag{2.24}$$

where $W_n := \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \left\langle \nabla\Lambda(\theta_k) \,|\, H(\theta^*, Y_k) \right\rangle$ and $Z_n := \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \widetilde{\chi}(Y_k) \left\langle \nabla\Lambda \,|\, h \right\rangle (\theta_k)$ with $\widetilde{\gamma}_n := \frac{\gamma_n}{B_n}$, $n \geq 0$.

Second step: Following the lines of the proof of Theorem 2.1 we show that the sequence $(W_n)_{n\geq 0}$ is L^1-bounded and $a.s.$ converges. Now our aim is to prove the same results for the sequence $(Z_n)_{n\geq 0}$. To this end we set $S_n^{\widetilde{\chi}} := \sum_{k=0}^{n-1} \widetilde{\chi}(Y_k)$, then it follows

$$\begin{aligned} Z_n &= \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \Delta S_{k+1}^{\widetilde{\chi}} \left\langle \nabla\Lambda \,|\, h \right\rangle (\theta_k) \\ &= \widetilde{\gamma}_n S_n^{\widetilde{\chi}} \left\langle \nabla\Lambda \,|\, h \right\rangle (\theta_{n-1}) - \sum_{k=1}^{n-1} S_k^{\widetilde{\chi}} \left(\widetilde{\gamma}_{k+1} \left\langle \nabla\Lambda \,|\, h \right\rangle (\theta_k) - \widetilde{\gamma}_k \left\langle \nabla\Lambda \,|\, h \right\rangle (\theta_{k-1}) \right). \end{aligned}$$

By the same methods as for the sequence $(W_n)_{n\geq 0}$ (*i.e.* using assumptions on H, Λ and $(\gamma_n)_{n\geq 1}$), we obtain that

$$Z_n \xrightarrow[n\to\infty]{a.s.} Z_\infty \quad \text{and} \quad \sup_{n\geq 0} \|Z_n\|_1 < +\infty.$$

Thus we have that

$$(V_n + W_n + Z_n)^- \leq (W_n + Z_n)^- \leq |W_n + Z_n| \in L^1(\mathbb{P}) \quad \text{since} \quad \sup_{n\geq 0} \|W_n + Z_n\|_1 < +\infty.$$

As $V_0 = \Lambda(\theta_0) \leq C(1 + |\theta_0|) \in L^1$, it follows by induction from (2.18) that, for every $n \geq 0$, $\mathbb{E}\, V_n < +\infty$. Hence $S_n := V_n + W_n + Z_n$, $n \geq 0$, is a true supermartingale with a L^1-bounded negative part. We then deduce that

$$S_n \xrightarrow[n\to\infty]{a.s.} S_\infty \in L^1.$$

Now $W_n \xrightarrow[n\to\infty]{a.s.} W_\infty$ and $Z_n \xrightarrow[n\to\infty]{a.s.} Z_\infty$ imply that $V_n \xrightarrow[n\to\infty]{a.s.} V_\infty < +\infty$ $a.s.$

Third step: Like in the proof of Theorem 2.1, we have that $B_n \xrightarrow[n\to\infty]{a.s.} B_\infty < +\infty$ which implies that $A_n \xrightarrow[n\to\infty]{a.s.} A_\infty < +\infty$. Therefore using the mean-reverting property (2.23) of h with respect to $\nabla\Lambda$, we classically derive that

$$\sum_{n\geq 0} \gamma_{n+1} \nu(\chi) \left\langle \nabla\Lambda \,|\, h \right\rangle (\theta_n) < +\infty \quad a.s. \tag{2.25}$$

The end of the proof follows the lines of the one of Theorem 2.1. □

2.3 Application to quasi-stochastic approximation

This section is devoted to quasi-random innovations: the innovation sequence $(Y_n)_{n\geq 0}$ becomes a deterministic uniformly distributed (u.d.) sequence $(\xi_{n+1})_{n\geq 0}$ over a unit hypercube $[0,1]^q$, *i.e.*

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, \xi_{n+1}), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}^d.$$

We extend the one-dimensional result first introduced in [81] to a general multi-dimensional setting with unbounded function H. We first recall few definitions and properties of u.d. sequences (see [95] and the reference therein). We emphasize how to apply Theorem 2.1 when H has "bounded variation" on $[0,1]^q$ and when H is Lipschitz continuous.

2.3.1 Definition and characterisation

Definition 2.1. *A $[0,1]^q$-valued sequence $(\xi_n)_{n\geq 1}$ is uniformly distributed (u.d.) on $[0,1]^q$ if*

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\xi_k} \overset{(\mathbb{R}^q)}{\Longrightarrow} \mathcal{U}([0,1]^q) \quad \text{as } n\to\infty.$$

The proposition below provides a characterisation of uniform distribution for a sequence $(\xi_n)_{n\geq 1}$.

Proposition 2.1. *(a) Let $(\xi_n)_{n\geq 1}$ be a $[0,1]^q$-valued sequence. Then $(\xi_n)_{n\geq 1}$ is uniformly distributed on $[0,1]^q$ if and only if*

$$D_n^*(\xi) := \sup_{x\in[0,1]^q}\left|\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}_{[\![0,x]\!]}(\xi_k) - \prod_{i=1}^{q}x^i\right| \longrightarrow 0 \quad \text{as } n\to\infty,$$

where $D_n^*(\xi)$ is called the discrepancy at the origin or star discrepancy.
(b) There exists sequences, called sequences with low discrepancy *such that $D_n^*(\xi) = \left(\frac{\log^d n}{n}\right)$. We refer to [95, 32] for examples of such sequences (like Halton, Kakutani, Sobol' sequences, etc).*

2.3.2 Standard classes $\mathcal{V}_{\varepsilon_n,1}$ for quasi-stochastic approximation

We set here $Y_n = \xi_{n+1}$, $\mathcal{F}_n = \{\emptyset,\Omega\}$ and $\Delta M_{n+1}\equiv 0$, $n\geq 0$. The strong Lyapunov condition on H is crucial here. Note that the function ϕ becomes useless since we always consider the case $p=1$. To apply Theorem 2.1, we mainly need to specify the accessible classes $\mathcal{V}_{\varepsilon_n,1}$ in such a framework.

▷ **Function with finite variation.** A function $f:[0,1]^q\to\mathbb{R}$ has finite variation in the measure sense if there exists a signed measure ν on $([0,1]^d,\mathcal{B}or([0,1]^q)$ such that $\nu(\{0\})=0$ and

$$\forall\, x\in[0,1]^q,\quad f(x) = f(\mathbf{1}) + \nu([\![0,\mathbf{1}-x]\!])$$

where $[\![x,y]\!] = \prod_{i=1}^{q}[x^i,y^i]$ if $x\leq y$ (componentwise) and is empty otherwise and $\mathbf{1}=(1,\ldots,1)$. The variation $V(f)$ of $f$is then defined as $|\nu|([0,1]^q)$ where ν denotes the total variation measure attached to ν. For further details on this notion of variation, see [32]. When $q=1$ this notion coincide with left continuous functions with finite variations. As concerns the slightly more general notion of finite variation in the the Hardy and Krause sense, see [95] and the references therein). The role of finite variation is emphasized by the following error bound.

Proposition 2.2 (Koksma-Hlawka Inequality)**.** *Let $\xi=(\xi_1,\ldots,\xi_n)\in([0,1]^q)^n$ and let f be a function with finite variation $V(f)$, either in the Hardy & Krause or in the measure sense. Then*

$$\left|\frac{1}{n}\sum_{k=1}^{n}f(\xi_k) - \int_{[0,1]^q}f(u)\lambda_q(du)\right| \leq V(f)D_n^*(\xi).$$

Hence, if $(\xi_n)_{n\geq 1}$ has a low discrepancy, the class $\mathcal{V}=\{f:[0,1]^q\to\mathbb{R}\text{ s.t. }V(f)<+\infty\}$ of functions with finite variations satisfies $\mathcal{V}\subset\mathcal{V}_{\varepsilon_n,1}$ with $\varepsilon_n=\frac{(\log n)^q}{n}$. Consequently, if $H(\theta,\cdot)\in\mathcal{V}$, the assumptions on admissible (non-increasing) step sequences $(\gamma_n)_{n\geq 1}$ in Theorem 2.1 reads

$$\sum_{n\geq 1}\gamma_n=+\infty,\quad \gamma_n(\log n)^q\to 0 \quad\text{and}\quad \sum_{k\geq 1}\max(|\Delta\gamma_{n+1}|,\gamma_n^2)(\log n)^q<+\infty.$$

so that the choice of $\gamma_n := \frac{c}{n^\rho}$, $\frac{1}{2}<\rho\leq 1$, is admissible (like in the i.i.d. setting).

▷ **Lipschitz continuous functions.** If $q\geq 2$ it is difficult to check whether f has finite variations in any sense: in fact these functions become "rare" as q increases. If we look for more natural regularity assumption to be satisfied by $H(\theta^*,\cdot)$ like Lipschitz continuity, the following theorem due to Proinov (see [102]) provides an alternative (but less "attractive") error bound.

Theorem 2.3. *(Proinov) Assume $\mathbb{R}^q$ is equipped with the ℓ^∞-norm $\left|(x^1,\ldots,x^q)\right|_\infty := \max_{1\le i\le q}\left|x^i\right|$. Let $(\xi_1,\ldots,\xi_n)\in([0,1]^q)^{\otimes n}$. For every continuous function $f:[0,1]^q\to\mathbb{R}$,*

$$\left|\frac{1}{n}\sum_{k=1}^{n} f\left(\xi_k\right)-\int_{[0,1]^q} f(u)\lambda_q(du)\right|\le C_q w_f\left(D_n^*\left(\xi_1,\ldots,\xi_n\right)^{\frac{1}{q}}\right)$$

where $w_f(\delta):=\sup_{x,y\in[0,1]^q,|x-y|_\infty\le\delta}|f(x)-f(y)|$, $\delta\in(0,1)$, is the ℓ^∞-uniform continuity modulus of f and $C_q\in(0,\infty)$ is a universal constant only depending on q. If $q=1$, $C_q=1$ and if $q\ge 2$, $C_q\in[1,4]$.

As a consequence $\mathrm{Lip}([0,1]^q,\mathbb{R})\subset\mathcal{V}_{\varepsilon_n,1}$ with $\varepsilon_n=\frac{\log n}{n^{\frac{1}{q}}}$ (with obvious extensions to Hölder functions). Consequently, if $H(\theta^*,\cdot)\in\mathcal{V}$, the assumptions on admissible (non-increasing) step sequences $(\gamma_n)_{n\ge1}$ in Theorem 2.1 reads

$$\sum_{n\ge1}\gamma_n=+\infty,\quad \gamma_n(\log n)n^{1-\frac{1}{q}}\to 0\quad\text{and}\quad\sum_{k\ge1}\max(\left|\Delta\gamma_{n+1}\right|,\gamma_n^2)(\log n)n^{1-\frac{1}{q}}<+\infty.$$

so that the choice of $\gamma_n:=\frac{c}{n}$ is always admissible (more generally $\gamma_n=cn^{-\rho}$, $1-\frac{1}{q}<\rho\le 1$). An application of "quasi-Stochastic Approximation" is proposed in Section 2.5.1 (see also [53]).

2.4 Applications to different types of random innovations

This section is devoted to some first applications of the above theorem. By applications, we mean here printing out some classes of random innovation processes $(Y_n)_{n\ge0}$ for which the averaging rate assumption (2.8) is naturally satisfied by "large" class $\mathcal{V}_{\varepsilon_n,p}$.

First we present a simple framework of stochastic approximation where the noise is additive which is studied in [39] with some mixing properties, but here we only need (2.5). We showed in [82] how easily our result applies to real life stochastic optimization problem (as far as convergence is concerned).

Afterwards we focus on mixing innovations: we consider that the sequence $(Y_n)_{n\ge0}$ is a functional of a stationary α-mixing process (satisfying condition on the summability of the mixing coefficients).

The last application is the case of an homogeneous Markov chain which can be seen as a possible more elementary counterpart of some (convergence) result obtained *e.g.* in [25]. Some (quasi-optimal) *a.s.* rate of convergence can be obtained if H is smooth enough in θ (see Chapter 3), but to establish a regular Central Limit Theorem it is most likely that we cannot avoid to deal with the Poisson equation.

2.4.1 Recursive procedure with additive noise

We consider here the case where the function H is the sum of the mean function h and a noise, namely

$$\forall\theta\in\mathbb{R}^d,\quad\forall y\in\mathbb{R}^q,\quad H(\theta,y)=h(\theta)+y,\quad\text{and}\quad\Delta M_{n+1}\equiv 0.$$

In this framework, the Lyapunov assumption (2.15) becomes classical involving only the mean function h, namely

$$\forall\theta\in\mathbb{R}^d\setminus\{\theta^*\}\qquad\langle\nabla L\,|\,h\rangle(\theta)>0.$$

Likewise, the growth control assumption (2.12) amounts to

$$\forall\theta\in\mathbb{R}^d,\quad|h(\theta)|\le C_h\sqrt{1+L(\theta)},$$

provided the moment assumption $\sup_n \|Y_n\|_{\frac{p}{p-1}} < +\infty$, for some $p \in (1, \infty]$, is satisfied (take $\phi(y) := |y| \vee 1$). The martingale is vanishing in this example. Finally the step assumption (2.14) is ruled by the averaging rate of the sequence $(Y_n)_{n\geq 0}$.

2.4.2 Functional of a stationary α-mixing process

Here we provide a short background on α-mixing processes and their functionals. Our motivation here is to relax as much as possible our assumption on $(Y_n)_{n\geq 0}$ in order to apply stochastic approximation methods to exogenous possibly non Markovian stationary data.

We aim now at applying our convergence theorem to input sequences $(Y_n)_{n\geq 0}$ which are (causal) functionals of an α-mixing process. Consider a stationary $\mathbb{R}^q$-valued process $X = (X_k)_{k\in\mathbb{Z}}$, its natural filtration $\mathcal{F}_n = \mathcal{F}_n^X := \sigma(X_k,\, k \leq n)$ and $\mathcal{G}_n = \mathcal{G}_n^X := \sigma(X_k,\, k \geq n)$. The α-mixing coefficients are defined as follows

$$\alpha_n = \sup\left\{|\mathbb{P}(U \cap V) - \mathbb{P}(U)\mathbb{P}(V)|\,,\; U \in \mathcal{F}_k, V \in \mathcal{G}_{k+n}, k \geq 0\right\}. \tag{2.26}$$

Let f be a measurable mapping from $(\mathbb{R}^q)^{\mathbb{Z}}$ to $\mathbb{R}$. Let $(Y_k)_{k\in\mathbb{Z}}$ be a causal functional of X, *i.e.*

$$\forall n \in \mathbb{Z}, \quad Y_n := f(\ldots, X_{n-1}, X_n).$$

Then $(Y_n)_{n\geq 0}$ is a stationary process with marginal distribution $\nu = \mathcal{L}(Y_0)$.

The proposition below show that if $(X_n)_{n\in\mathbb{Z}}$ is α-mixing "fast enough" then $H(\theta^*, \cdot)$ "almost" lies in $\mathcal{V}_{n^{-\frac{1}{2}},2}$ (up to logarithmic factor) as soon as $\mathbb{E}|H(\theta^*, Y_0)|^{2+\delta} < +\infty$ for a $\delta > 0$ (so is true for $H(\theta, \cdot)$ since we do not know θ^* *a priori*).

Proposition 2.3. *Assume $g \in L^{2+\delta}(\nu)$, $\delta > 0$, and that one of the following assumptions holds*

(a) For all $n \in \mathbb{Z}$, $Y_n := f(\ldots, X_{n-1}, X_n)$ where X is a stationary α-mixing process satisfying

$$\sum_{k\geq 1} \frac{\alpha_k^{\frac{\delta}{2(2+\delta)}}}{\sqrt{k}} < +\infty. \tag{2.27}$$

(b) $Y_n = X_n$, $n \geq 0$, and X is a stationary α-mixing process satisfying the condition

$$\sum_{k\geq 1} \alpha_k^{\frac{\delta}{2+\delta}} < +\infty. \tag{2.28}$$

Then

$$g \in \mathcal{V}_{\varepsilon_n^{(\eta)},2}, \quad \text{with } \varepsilon_n^{(\eta)} = (\log n)^{\frac{3}{2}+\eta}\, n^{-\frac{1}{2}}, \text{ for every } \eta > 0. \tag{2.29}$$

In particular g lies in $\mathcal{V}_{n^{-\beta},2}$ for every $\beta \in (0, \frac{1}{2})$.

Remark. Condition (2.28) is satisfied when the underlying process X is geometrically α-mixing. Slightly refined results could be obtained by calling upon Philip and Stout's Law of Iterated Logarithm but the resulting claims would be significantly more technical to state for little practical benefit.

An applicationof this result (based on real data) is briefly developed in Section 2.5.5. The proof of Proposition2.3 relies on the Gàl-Koksma Theorem (see [54] and [23] for a probabilistic version). We state it here in a stationary framework.

Theorem 2.4. *(Gàl-Koksma's Theorem) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(Z_n)_{n\geq 1}$ be a sequence of random variables belonging to L^p, $p \geq 1$, satisfying*

$$\mathbb{E}\,|Z_1 + Z_2 + \cdots + Z_N|^p = O(\Psi(N))$$

where $\frac{\Psi(N)}{N}$, $N \geq 1$, is a nondecreasing sequence. Then for every $\eta > 0$,

$$Z_1(\omega) + Z_2(\omega) + \cdots + Z_N(\omega) = o\left((\Psi(N)(\log(N))^{p+1+\eta})^{\frac{1}{p}}\right) \quad \mathbb{P}(d\omega)\text{-}a.s.$$

Remark. The conditions on X and Z come from a result established in [40]: by setting $P_0(Z_k) := \mathbb{E}\,[Z_k \mid \mathcal{F}_0] - \mathbb{E}\,[Z_k \mid \mathcal{F}_{-1}]$, if

$$\sum_{k\in\mathbb{Z}} \|P_0(Z_k)\|_2 < +\infty \quad \text{then} \quad \sum_{k\in\mathbb{Z}} |\mathrm{Cov}(Z_0, Z_k)| < +\infty. \tag{2.30}$$

Moreover using [97], condition (2.30) is satisfied as soon as

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|\mathbb{E}\,[Z_0 \mid \mathcal{G}_k]\|_2 < +\infty. \tag{2.31}$$

Proof of Proposition 2.3. Let $Z_n = g(Y_n) - \int_{\mathbb{R}^q} g(y)\nu(dy)$, $n \in \mathbb{Z}$. Without loss of generality we may assume $\int_{\mathbb{R}^q} g(y)\nu(dy) = 0$.

(a) We will rely on the above Gal-Koksma Theorem (Theorem 2.4).

First, we evaluate $\mathbb{E}\,|Z_0 + \cdots + Z_{n-1}|^2$. Setting $S_k^Z = \sum_{j=1}^k \mathbb{E}\,[Z_j Z_0]$, $k \in \mathbb{N}$, elementary computations lead to

$$\mathbb{E}\,|Z_0 + \cdots + Z_{n-1}|^2 = n\mathbb{E}Z_0^2 + 2\sum_{k=1}^{n-1}\sum_{j=1}^{k} \mathbb{E}\,[Z_j Z_0] = n\mathbb{E}Z_0^2 + 2\sum_{k=1}^{n-1} S_k^Z = n\left(\mathbb{E}Z_0^2 + \frac{2}{n}\sum_{k=1}^{n-1} S_k^Z\right).$$

To establish that S_n^Z converges, we will establish (2.31). Set $B_2(\mathcal{G}_k) := \{W \in \mathcal{G}_k : \|W\|_2 \leq 1\}$. Then

$$\|\mathbb{E}\,(Z_0 \mid \mathcal{G}_k)\|_2 = \sup_{W\in B_2(\mathcal{G}_k)} \mathbb{E}\,(WZ_0) \leq 8\alpha_k^{\frac{1}{r}} \|g(Y_0)\|_p$$

owing to the classical covariance inequality for α-mixing process (see [42], Theorem 3(1), p.9) with $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, $r, p > 2$. As $g \in L^{2+\delta}(\nu)$, $\delta > 0$, we may set $p = 2+\delta$, and $r = \frac{2(2+\delta)}{\delta}$. As a consequence

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|\mathbb{E}\,(Z_0 \mid \mathcal{G}_k)\|_2 < +\infty,$$

which implies (owing to (2.30)) that S_k^Z converges. Now, by Cesaro's Lemma we have

$$\mathbb{E}\,|Z_0 + \cdots + Z_{n-1}|^2 = O(n) \quad \text{or equivalently} \quad \left\|\frac{1}{n}(Z_0 + \cdots + Z_{n-1})\right\|_2 = O\left(n^{-\frac{1}{2}}\right).$$

Thus, one concludes by by Gal-Koksma's Theorem since, for every $\eta > 0$,

$$\frac{Z_0 + \cdots + Z_{n-1}}{n} = o\left(\frac{(\log n)^{\frac{3}{2}+\eta}}{\sqrt{n}}\right) \quad \mathbb{P}\text{-}a.s.$$

(b) If we assume that $Y_n = X_n$, $n \geq 0$, we directly use the covariance inequality for α-mixing process

$$|\mathrm{Cov}\,(Z_j, Z_0)| \ \leq 8\alpha_j^{\frac{1}{r}} \|Z_0\|_p \|Z_0\|_q ,$$

where $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1$. By symmetry, we take $p = q > 2$ and we get

$$|\mathbb{E}\,(Z_j Z_0)| \leq 8\alpha_j^{1-\frac{2}{p}} \|Z_0\|_p^2 .$$

As $g \in L^{2+\delta}$, $\delta > 0$, we may set $p = 2 + \delta$ and we obtain $\alpha_j^{1-\frac{2}{2+\delta}} = \alpha_j^{\frac{\delta}{2+\delta}}$. The condition (2.27) can be replaced by the less stringent Ibragimov's condition (2.28) to complete the proof. □

2.4.3 Homogeneous Markov chain

Assume that the innovation process $(Y_n)_{n\geq 0}$ is an $\mathbb{R}^q$-valued homogeneous Markov chain which transition is $(P(y, dz))_{y\in\mathbb{R}^q}$ and starting distribution $\mu = \mathcal{L}(Y_0)$. For convenience we will assume that the chain lives on its canonical space $((\mathbb{R}^q)^{\mathbb{N}}, \mathcal{B}or(\mathbb{R}^q)^{\otimes\mathbb{N}})$.

2.4.3.1 Application of the convergence theorem

We consider the classical Markov stochastic approximation procedure procedure

$$\theta_{n+1} = \theta_n - \gamma_{n+1} K(\theta_n, Y_{n+1}), \quad n \geq 0, \tag{2.32}$$

where $K : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ is a Borel function satisfying (2.33) below and $\theta_0 : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ is independent of $(Y_n)_{n\geq 0}$. Note that $(Y_n)_{n\geq 0}$ is still a Markov chain with respect to $\mathcal{F}_n = \sigma(\theta_0, Y_0, \ldots, Y_n)$, $n \geq 0$.

Set $H(\theta, y) := P\,(K(\theta, .))\,(y)$ and $\Delta M_{n+1} := K(\theta_n, Y_{n+1}) - \mathbb{E}\,[K(\theta_n, Y_{n+1}) \,|\, \mathcal{F}_n]$. Then the procedure has the canonical form (2.2) with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$.

Remark. If we consider that the Markov chain starts from Y_1, then $\mathcal{F}_n = \sigma(\theta_0, Y_1, \ldots, Y_n)$ and $\mathbb{E}\,[K(\theta_0, Y_1)|\mathcal{F}_0] = \mathbb{E}\,[K(\theta, Y_1)]_{|\theta=\theta_0} = \mu P(K(\theta, .))_{|\theta=\theta_0}$ since θ_0 and Y_1 are independent.

Let $p \in [1, \infty)$ and set $r = 2 \vee \frac{p}{p-1} \in [2, +\infty]$. We make the following growth assumption on the function K

$$\forall \theta \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^q, \quad |K(\theta, y)| \leq C_K \widetilde{\phi}(y) \sqrt{1 + L(\theta)} \tag{2.33}$$

where $L : \mathbb{R}^d \to \mathbb{R}_+$ satisfies (2.9) and $\sup_{n\geq 0} \left\|\widetilde{\phi}(Y_n)\right\|_r < +\infty$.

Then H satisfies (2.12) with $\phi(y) = P\widetilde{\phi}(y) = \mathbb{E}_y \widetilde{\phi}(Y_1) \leq \left\|\widetilde{\phi}\right\|_{L^r(P(y,dz))} < +\infty$ and ΔM_{n+1} satisfies (2.13) with $\phi(y) = \left\|\widetilde{\phi}\right\|_{L^r(P(y,dz))}$ so that, finally, we may choose $\phi(y) = \left\|\widetilde{\phi}\right\|_{L^r(P(y,dz))}$, having in mind that $\|\phi(Y_n)\|_r = \left\|\widetilde{\phi}(Y_{n+1})\right\|_r$. Now, the proposition below straightforwardly follows from Theorem 2.1.

Proposition 2.4. *Let $p \in [1, \infty)$ and $\theta^* \in \mathbb{R}^d$. If K satisfies (2.33) and H satisfies the strict pathwise Lyapunov assumption (2.15), if $(\gamma_n)_{n\geq 1}$ satisfies (2.14) for a sequence $(\varepsilon_n)_{n\geq 1}$ satisfying (2.7) and $H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, p}$, then the recursive procedure with Markov innovations defined by (2.32) converges, i.e.*

$$\theta_n \xrightarrow[n\to+\infty]{a.s.} \theta^*.$$

2.4.3.2 Ergodic framework description

We will say that the Markov chain $(Y_n)_{n\geq 0}$ (starting from $Y_0 \overset{\mathcal{L}}{\sim} \mu$) is ν-ergodic (resp. ν-stable) under $\mathbb{P}_\mu$ if for every bounded Borel (resp. continuous) function $f : \mathbb{R}^q \to \mathbb{R}$,

$$\mathbb{P}_\mu\text{-}a.s. \quad \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^q} f d\nu. \tag{2.34}$$

As soon as the transition $(P(y,dz))_{y\in\mathbb{R}^q}$ of $(Y_n)_{n\geq 0}$ is Feller, the above ν-stability property implies that ν is an invariant distribution of the chain, *i.e.* $\nu P = \nu$. In case of ν-ergodicity the same conclusion holds unconditionally. As a consequence the whole sequence $(Y_n)_{n\geq 0}$ is stationary under $\mathbb{P}_\nu$.

Let us focus on the case $\mu = \nu$. If (2.34) holds (with $\mu = \nu$), it is classical background that the whole chain is ergodic under $\mathbb{P}_\nu$ (on the canonical space) for the shift operator Θ, *i.e.* by Birkhoff's theorem, for every functional $F : ((\mathbb{R}^q)^{\mathbb{N}}, \mathcal{B}or(\mathbb{R}^q)^{\otimes\mathbb{N}}) \to \mathbb{R}$, $F \in L^p(\mathbb{P}_\nu)$,

$$\frac{1}{n}\sum_{k=0}^{n-1} F\circ\Theta^k \underset{n\to\infty}{\longrightarrow} \mathbb{E}_\nu(F) \quad \mathbb{P}_\nu\text{-}a.s. \text{ and in } L^p(\mathbb{P}_\nu),$$

so that by considering $F((y_n)_{n\geq 0}) = f(y_0)$, $f \in L^p(\nu)$, we finally get that

$$\mathcal{V}_{0^+,p}(\mathbb{P}_\nu) = L^p(\nu).$$

Note that if the set of invariant distributions for P (convex and) weakly compact and if ν is extremal in it (so will be *e.g.* the case if ν is unique!) then the chain is ergodic under $\mathbb{P}_\nu$ so that the above equality still holds. Furthermore, we know by a straightforward application of Gàl-Koksma Theorem that for any $g \in L^2(\nu)$ for which the related Poisson Equation $g - \nu(g) = \varphi_g - P\varphi_g$ has a solution $\varphi_g \in L^2(\nu)$, then

$$\begin{aligned}
\mathbb{E}_\nu|g(Y_0)+\cdots+g(Y_{n-1})-n\nu(g)|^2 &= \mathbb{E}_\nu|\varphi_g(Y_0)-P\varphi_g(Y_{n-1})+\sum_{1\leq k\leq n-2}\varphi_g(Y_k)-P\varphi_g(Y_{k-1})|^2 \\
&\leq 6\nu(\varphi_g^2)+3(n-2)\nu\left((\varphi_g-P\varphi_g)^2\right)=O(n)
\end{aligned}$$

so that $\displaystyle\bigcap_{\beta\in(0,\frac{1}{2})} \mathcal{V}_{n^{-\beta},2}(\mathbb{P}_\nu) \supset L^2(\nu).$

Now we will make a connection between the classes $\mathcal{V}_{\varepsilon_n,p}(\mathbb{P}_\nu)$ and $\mathcal{V}_{\varepsilon_n,p'}(\mathbb{P}_y)$ which will provide examples of non-stationary (Markovian) innovations that can be "plugged" in stochastic Approximation procedures in the spirit of Theorem 2.1.

Proposition 2.5. *Let $p \in [1,+\infty)$ and let $p' \in (0,p]$. If $(Y_n)_{n\geq 0}$ is $\mathbb{P}_\nu$-ergodic and $P(y,dz) = g(y,z)\nu(dz)$, $y \in \mathbb{R}^q$ where $g : (\mathbb{R}^q)^2 \to \mathbb{R}_+$ satisfies $\nu(dz)$-a.e., $g(\cdot,z) > 0$. Then ν is the unique invariant distribution of P and for every sequence $(\varepsilon_n)_{n\geq 0}$ satisfying (2.7),*

$$\forall y \in \mathbb{R}^q, \quad g(y,\cdot) \in L^{\frac{p}{p-p'}}(\nu) \Longrightarrow \mathcal{V}_{\varepsilon_n,p'}(\mathbb{P}_y) \supset \mathcal{V}_{\varepsilon_n,p}(\mathbb{P}_\nu).$$

Proof. It follows form the assumption that any invariant distribution ν' is equivalent to ν which implies classically uniqueness.

▷ *The a.s. rate.* Let $f \in L^p(\nu)$, $y \in \mathbb{R}^q$ and $A_f := \left\{ \omega : \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k(\omega)) - \int_{\mathbb{R}^q} f d\nu = O(\varepsilon_n) \right\}$. Since $\liminf_n n\varepsilon_n > 0$, if Θ denotes the shift operator on the canonical space of the chain $(Y_n)_{n\geq 0}$, A_f clearly satisfies $A_f = \Theta^{-1}(A_f)$ *i.e.* $\mathbb{1}_{A_f} = \mathbb{1}_{A_f} \circ \Theta$. Therefore

$$\mathbb{P}_y(A_f) = \mathbb{E}_y(\mathbb{1}_{A_f}) = \mathbb{E}_y(\mathbb{1}_{A_f} \circ \Theta) = \mathbb{E}_y(\mathbb{P}_{Y_1}(A_f)).$$

Assume now $f \in \mathcal{V}_{\varepsilon_n,p}(\mathbb{P}_\nu)$. By assumption $\mathbb{P}_\nu(A_f) = 1$. Let $y \in \mathbb{R}^q$. Then

$$\mathbb{P}_\nu(A_f) = \int_{\mathbb{R}^q} \nu(dz)\mathbb{P}_z(A_f) = 1 \quad \text{so that} \quad \nu(dz)\text{-a.s.} \quad \mathbb{P}_z(A_f) = 1.$$

Now $P(y,dz) \ll \nu(dz)$ implies $\int_{\mathbb{R}^q} P(y,dz)\mathbb{P}_z(A_f) = 1$ *i.e.* $\mathbb{E}_y\left[\mathbb{P}_{Y_1}(A_f)\right] = 1$ or equivalently $\mathbb{P}_y(A_f) = 1$.

▷ *The $L^{p'}$-rate.* Let $p' \in (0,p]$, let $f \in \mathcal{V}_{\varepsilon_n,p}(\mathbb{P}_\nu) \subset L^p(\nu)$ and

$$\varphi_{n,p'}(y) := \left\| \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k(\omega)) - \int_{\mathbb{R}^q} f d\nu \right\|_{L^{p'}(\mathbb{P}_y)}.$$

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k(\omega)) - \int_{\mathbb{R}^q} f d\nu \right\|_{L^p(\nu)} = O(\varepsilon_n) \quad \text{so that} \quad \int_{\mathbb{R}^q} \phi_n^p(y)\nu(dy) = O(\varepsilon_n^p). \tag{2.35}$$

Assume temporarily that $p' \geq 1$. Consequently, Minkowski inequality implies

$$\begin{aligned}
\varphi_{n,p'}(y) &\leq \frac{\left|f(y) - \int_{\mathbb{R}^q} f d\nu\right|}{n} + \left(1 - \frac{1}{n}\right) \left\| \frac{1}{n-1}\sum_{k=1}^{n-1} f(Y_k) - \int_{\mathbb{R}^q} f d\nu \right\|_{L^{p'}(\mathbb{P}_y)} \\
&= \frac{\left|f(y) - \int_{\mathbb{R}^q} f d\nu\right|}{n} + \left(1 - \frac{1}{n}\right) \left(\mathbb{E}_y \left[\mathbb{E}\left(\left| \frac{1}{n-1}\sum_{k=1}^{n-1} f(Y_k) - \int_{\mathbb{R}^q} f d\nu \right|^{p'} \mid \mathcal{F}_1 \right) \right] \right)^{\frac{1}{p'}} \\
&= \frac{\left|f(y) - \int_{\mathbb{R}^q} f d\nu\right|}{n} + \left(1 - \frac{1}{n}\right) \left\| \varphi_{n-1,p'}(Y_1) \right\|_{L^{p'}(\mathbb{P}_y)}.
\end{aligned}$$

where we used the Markov property in the last equality. Since $P(y,dz) = g(y,z)\nu(dz)$, we derive from Hölder's Inequality (applied to $r = \frac{p}{p'}$ and $s = \frac{p}{p-p'}$)

$$\begin{aligned}
\mathbb{E}_y \varphi_{n-1}(Y_1)^{p'} &= \int_{\mathbb{R}^q} \varphi_{n-1}(z)^{p'} P(y,dz) = \int_{\mathbb{R}^q} \varphi_{n-1}(z)^{p'} g(y,z)\nu(dz) \\
&\leq \|g(y,\cdot)\|_{L^{\frac{p}{p-p'}}(\nu)} \left(\int_{\mathbb{R}^q} \varphi_{n-1}(y)^p \nu(dy) \right)^{\frac{p'}{p}} \\
&\leq \|g(y,\cdot)\|_{L^{\frac{p}{p-p'}}(\nu)} O(\varepsilon_n^{p'}) \quad \text{owing to (2.35)}.
\end{aligned}$$

Finally

$$\varphi_n(y) \leq \frac{c}{n} + \left(1 - \frac{1}{n}\right) \|g(y,\cdot)\|_{\frac{p}{p-p'}} O(\varepsilon_n) = O(\varepsilon_n) \quad i.e. \quad f \in \mathcal{V}_{\varepsilon_n,p'}(\mathbb{P}_y).$$

The case $p' \in (0,1)$ follows by the usual adjustments (pseudo-Minkowski inequality, etc). □

COMMENTS. By contrast with the approach of [25], it is not mandatory to solve the Poisson equation related to the pseudo-transition

$$\Pi_{\theta_n}(Y_n, dz) = \mathbb{P}\left(Y_{n+1} \in dz \,|\, \mathcal{F}_n\right)$$

of the algorithm. Indeed, they assume there exists a function $v_\theta := v(\theta, \cdot)$ solution to

$$Id - \Pi_\theta v_\theta = H(\theta, \cdot) - h(\theta) \tag{2.36}$$

(Assumption (H_4) in [25] p. 220). The target θ^* is then a zero of the mean function h (not canonically defined at this stage in [25]). In our setting, $\Pi_\theta(y, dz) = P(y, dz)$ since the dynamics of $(Y_n) - n \geq 0$ does not depend upon θ, so that Condition (2.36) reads

$$v(\theta, y) - \int_{\mathbb{R}^q} v(\theta, z) P(y, dz) = H(\theta, y) - h(\theta)$$

where the mean function is naturally defined by

$$h(\theta) = \int_{\mathbb{R}^q} H(\theta, y)\nu(dy)$$

(ν is the unique invariant probability measure for P). Then the family of Poisson equations (indexed by the parameter θ) reads

$$v(\theta, y) - Pv(\theta, y) = H(\theta, y) - h(\theta).$$

A *formal* solution is given by $v(\theta, y) = \sum_{k \geq 0} P^k \left(H(\theta, \cdot) - h(\theta)\right)(y)$, but the point is precisely to establish its existence and its properties by using the mixing properties of the semi-group P (see [25]).

2.5 Applications and numerical examples

This section is devoted to several examples (mainly inspired by in Finance) of application of convergence theorems in the different frameworks developed in Section 2.3 and 2.4.

2.5.1 Application to implicit correlation search by quasi-stochastic approximation

Consider a 2-dimensional Black-Scholes model *i.e.* $X_0^t = e^{rt}$ (riskless asset) and

$$\forall t \geq 0, \quad X_t^i = x_0^i e^{(r - \frac{\sigma_i^2}{2})t + \sigma_i W_t^i}, \quad x_0^i > 0, \quad i = 1, 2,$$

for the two risky assets where $\left\langle W^1, W^2 \right\rangle_t = \rho t$, $\rho \in [-1, 1]$. Consider a best-of call option characterized by its payoff

$$\left(\max\left(X_T^1, X_T^2\right) - K\right)_+.$$

We will use a stochastic recursive procedure to solve the inverse problem in ρ

$$P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) = P_0^{market}$$

where P_0^{market} is the quoted premium of the option (mark-to-market) with

$$\begin{aligned} P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) &:= e^{-rT}\mathbb{E}\left[\left(\max\left(X_T^1, X_T^2\right) - K\right)_+\right] \\ &= e^{-rT}\mathbb{E}\left[\left(\max\left(x_0^1 e^{\mu_1 T + \sigma_1\sqrt{T}Z^1}, x_0^2 e^{\mu_2 T + \sigma_2\sqrt{T}\left(\rho Z^1 + \sqrt{1-\rho^2}Z^2\right)}\right) - K\right)_+\right] \end{aligned}$$

where $\mu_i = r - \frac{\sigma_i^2}{2}$, $i = 1, 2$, $Z = (Z^1, Z^2) \stackrel{d}{=} \mathcal{N}(0, I_2)$. We assume from now on that this equation (in ρ) has at least one solution, say ρ^*. The most convenient way to prevent edge effects due to the fact that $\rho \in [-1, 1]$ is to use a trigonometric parametrization of the correlation by setting $\rho = \cos\theta, \theta \in \mathbb{R}$. This introduces an over-parametrization since θ and $2\pi - \theta$ yield the same solution inside $[0, 2\pi]$, but this is not at all a significant problem for practical implementation (a careful examination shows that in fact one equilibrium is repulsive and one is attractive). From now on, for convenience, we will just mention the dependence of the premium function in the variable θ, namely

$$\theta \longmapsto P(\theta) := P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \cos(\theta), T).$$

The function P is a 2π-periodic continuous function. Extracting the implicit correlation from the market amounts to solving

$$P(\theta) = P_0^{market} \quad (\text{with } \rho = \cos\theta).$$

We need the following additional assumption

$$P_0^{market} \in (\min_\theta P, \max_\theta P)$$

i.e. that P_0^{market} is not an extremal value of P. It is natural to set for every $\theta \in \mathbb{R}$ and every $z = (z^1, z^2) \in \mathbb{R}^2$

$$H(\theta, z) = e^{-rT}\left(\max\left(x_0^1 e^{\mu_1 T + \sigma_1\sqrt{T}z^1}, x_0^2 e^{\mu_2 T + \sigma_2\sqrt{T}(z^1\cos\theta + z^2\sin\theta)}\right) - K\right)_+ - P_0^{market}$$

and to define the recursive procedure

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Z_{n+1}), \quad n \geq 0, \quad \text{where} \quad Z_{n+1} \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0, I_2),$$

and the gain parameter sequence satisfies (2.14). For every $z \in \mathbb{R}^2$, $\theta \longmapsto H(\theta, z)$ is continuous and 2π-periodic which implies that the mean function $h(\theta) := \mathbb{E}H(\theta, Z_1) = P(\theta) - P_0^{market}$ and $\theta \longmapsto \mathbb{E}\left[H^2(\theta, Z_1)\right]$ are both continuous and 2π-periodic as well (hence bounded).

The main difficulty to apply Theorem 2.1 is to find out the appropriate Lyapunov function. The quoted value P_0^{market} is not an extremum of the function P, hence $\int_0^{2\pi} h^\pm(\theta)d\theta > 0$ where $h^\pm := \max(\pm h, 0)$. We consider θ_0^* any (fixed) solution to the equation $h(\theta) = 0$ and two real numbers $\beta_\pm$ such that

$$0 < \beta_+ < \frac{\int_0^{2\pi} h_+(\theta)d\theta}{\int_0^{2\pi} h_-(\theta)d\theta} < \beta_-$$

and we set

$$g(\theta) := \begin{cases} \mathbb{1}_{\{h>0\}}(\theta) + \beta_+ \mathbb{1}_{\{h<0\}}(\theta) & \text{if} \quad \theta \geq \theta_0^* \\ \mathbb{1}_{\{h>0\}}(\theta) + \beta_- \mathbb{1}_{\{h<0\}}(\theta) & \text{if} \quad \theta < \theta_0^*. \end{cases}$$

The function

$$\theta \longmapsto g(\theta)h(\theta) = h_+ - \beta_\pm h_-$$

is continuous and "positively" 2π-periodic on $[\theta_0^*, \infty)$ and "negatively" 2π-periodic on $(-\infty, \theta_0^*]$. Moreover, $gh(\theta) = 0$ iff $h(\theta) = 0$ so that $gh(\theta_0^*) = gh(\theta_0^*-) = 0$ which ensures on the way the continuity of gh on $\mathbb{R}$. Furthermore $\int_0^{2\pi} gh(\theta)d\theta > 0$ and $\int_{-2\pi}^0 gh(\theta)d\theta < 0$ so that, on the one hand,

$$\lim_{\theta\to\pm\infty} \int_0^\theta gh(u)du = +\infty$$

and, on the other hand, there exists a real constant $C > 0$ such that the function

$$L(\theta) = \int_0^\theta gh(u)du + C$$

is nonnegative. Its derivative is given by $L' = gh$ so that $L'h = gh^2 \geq 0$ and $\{L'h = 0\} = \{h = 0\}$. It remains to prove that L' is Lipschitz continuous. One checks by applying the usual differentiation theorem for functions defined by an integral that, if $\sigma_1 \neq \sigma_2$ or $x_1 \neq x_2$, then P is differentiable on the whole real line, otherwise it is differentiable only on $\mathbb{R} \setminus 2\pi\mathbb{Z}$, and in both cases

$$P'(\theta) = \sigma_2\sqrt{T}\,\mathbb{E}\left(\mathbb{1}_{\{X_T^2 > \max(X_T^1, K)\}} X_T^2(\cos(\theta)Z^2 - \sin(\theta)Z^1)\right).$$

Furthermore, with obvious notations, as soon as $P'(\theta)$ exists,

$$\left|P'(\theta)\right| \leq \mathbb{E}\left|X_T^2(\cos(\theta)Z^2 - \sin(\theta)Z^1)\right|.$$

The right handside of the inequality defined a 2π-periodic continuous function, hence bounded on the real line. Consequently $|P'(\theta)|$ is bounded. It follows that the 2π-periodic functions h and $h_\pm$ are Lipschitz continuous which implies in turn that $L' = gh$ is Lipschitz as well.

Moreover, one can show that the equation $P(\theta) = P_0^{market}$ market has finitely many solutions on every interval of length 2π. One may apply Theorem 2.1 to derive that θ_n will converge toward a solution θ^* of the equation $P(\theta) = P_0^{market}$.

Numerical illustration. We set the model parameters to the following values

$$x_0^1 = x_0^2 = 100,\ r = 0.10,\ \sigma_1 = \sigma_2 = 0.30,\ \rho = -0.50$$

and the payoff parameters

$$T = 1,\ K = 100.$$

The implicit correlation search recursive procedure is implemented with a sequence of some quasi-random normal numbers, namely

$$(\zeta_n^1, \zeta_n^2) = \left(\sqrt{-2\log(\xi_n^1)}\sin\left(2\pi\xi_n^2\right), \sqrt{-2\log(\xi_n^1)}\cos\left(2\pi\xi_n^2\right)\right),$$

where $\xi_n = (\xi_n^1, \xi_n^2)$, $n \geq 1$, is simply a regular 2-dimensional Halton sequence (see [95] for a definition).

The Black-Scholes reference price 30.75 is used as a market price so that the target of the stochastic algorithm is $\theta^* \in \arccos(-0.5)$. The parameters of the stochastic approximation procedure are

$$\theta_0 = 0,\ n = 10^5,\ \gamma_n = \frac{8}{n},\ n \geq 1.$$

The choice of θ_0 is "blind" on purpose (see Figure 2.1).

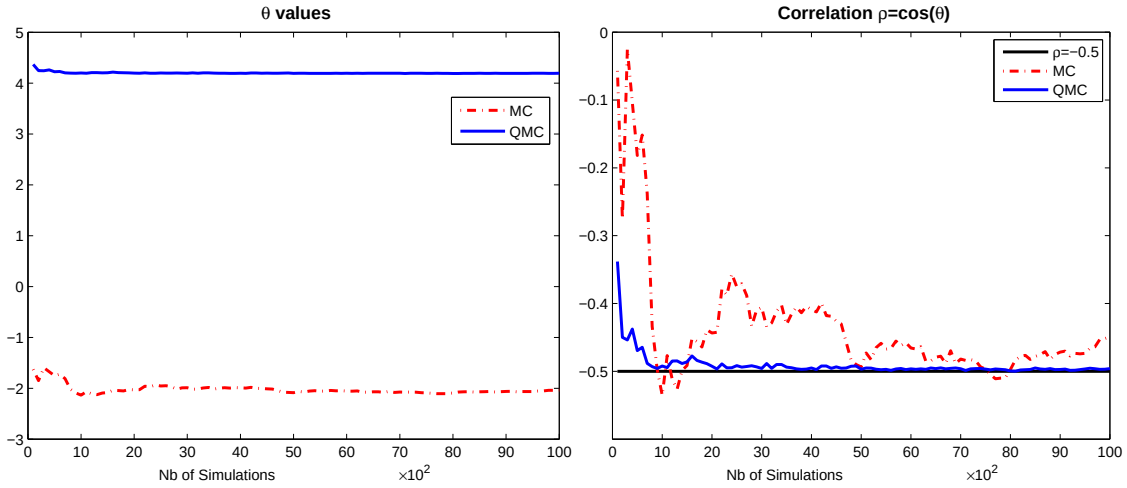

Figure 2.1: B-S Best-of-Call option. $T = 1$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.30$, $x_0^1 = x_0^2 = 100$, $K = 100$. *Left*: convergence of θ_n toward a θ^* (up to $n = 10000$). *Right*: convergence of $\rho_n := cos(\theta_n)$ toward -0.5.

2.5.2 Recursive computation of the VaR and the CVaR

Another example of application is the recursive computation of financial risk measure which are the best known and the most common: the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR). This risk measures evaluate the extreme losses of a portfolio potentially faced by traders. The recursive computation of the VaR and the CVaR was introduced in [20], based on the formulation as an optimization problem (see [104]) and on an unconstrained importance sampling procedure developed in [89]. These variance reduction aspects are not investigated here.

2.5.2.1 Definitions and formulation

Let $Y : (\Omega, A, \mathbb{P}) \to \mathbb{R}$ be a random variable representative of a loss ($Y \geq 0$ is a loss equal to Y).

Definition 2.2. *The Value at Risk (at confidence level $\alpha \in (0,1)$, $\alpha \approx 1$) of a given portfolio is the (lowest) α-quantile of the distribution Y i.e.*

$$VaR_\alpha(Y) := \inf \{\theta \,|\, \mathbb{P}(Y \leq \theta) \geq \alpha\}.$$

As soon as the distribution function of Y has no atom, the value at risk satisfies $P(Y \leq VaR_\alpha(Y)) = \alpha$ and if the distribution function F_Y of Y is also increasing (strictly) then, it is the unique solution. As this risk measure is not consistent (see [47]), another consistent risk measure is provided by the Conditional value at Risk when $Y \in L^1(\mathbb{P})$ with a continuous distribution (no atom).

Definition 2.3. *Let $Y \in L^1(\mathbb{P})$ with an atomless distribution. The Conditional value at Risk (at level α) is the conditional expectation of the portfolio loss Y beyond $VaR_\alpha(Y)$, i.e.*

$$CVaR_\alpha(Y) := \mathbb{E}\left[Y | Y \geq VaR_\alpha(Y)\right].$$

The following formulation of the $VaR_\alpha(Y)$ and $CVaR_\alpha(Y)$ as solutions to an optimization problem is due to Rockafellar and Uryasev in [104].

Proposition 2.6. *(Rockafellar and Uryasev) Let $Y \in L^1(\mathbb{P})$ with an atomless distribution. The function $V : \theta \mapsto \theta + \frac{1}{1-\alpha}\mathbb{E}\,(Y-\theta)_+$ is convex, and*

$$CVaR_\alpha(Y) = \min_\theta \left(\theta + \frac{1}{1-\alpha}\mathbb{E}\,(Y-\theta)_+\right) \text{ with } VaR_\alpha(Y) = \inf \arg\min_\theta \left(\theta + \frac{1}{1-\alpha}\mathbb{E}(Y-\theta)_+\right).$$

2.5.2.2 Stochastic gradient for the computation of both $VaR_\alpha(Y)$ and $CVaR_\alpha(Y)$

▷ *Computation of the $VaR_\alpha(Y)$.* What precedes suggests to implement a stochastic gradient descent derived from the above *convex* objective function $V(\theta) = \theta + \frac{1}{1-\alpha}\mathbb{E}(Y-\theta)_+$. Assume that $Y \in L^1(\mathbb{P})$ with a continuous increasing distribution function F_Y (for the sake of simplicity, see [20] for a slightly more general framework). Let $\nu = \mathcal{L}(Y)$. We check that

$$\lim_{\theta\to+\infty} \frac{V(\theta)}{\theta} = 1 \quad \text{and} \quad \lim_{\theta\to+\infty} \frac{V(-\theta)}{\theta} = \frac{\alpha}{1-\alpha} \quad \text{hence} \quad \lim_{\theta\to\pm\infty} V(\theta) = +\infty.$$

so that $\{VaR_\alpha(Y)\} = \operatorname{argmin}_{\mathbb{R}} V$. We check that $V'(\theta) = \mathbb{E}[H(\theta, Y)]$ where

$$H(\theta, y) := 1 - \frac{1}{1-\alpha}\mathbb{1}_{\{y\geq\theta\}}.$$

Note that H is uniformly bounded by $1 \vee \frac{\alpha}{1-\alpha}$. This leads to devise the stochastic gradient descent

$$\theta_{n+1} = \theta_n - \gamma_{n+1}H(\theta_n, Y_n), \quad n \geq 0, \quad \theta_0 \in L^1(\mathbb{P}).$$

whose unique target is $\theta^* = VaR_\alpha(Y)$. It is clear that, for every $y \in \mathbb{R}$, $\theta \mapsto H(\theta, y)$ is nondecreasing so that $L(\theta) = \frac{1}{2}(\theta-\theta^*)^2$ is a good candidate as a Lyapunov function. In fact it is even a strict pathwise Lyapunov function in the sense of (2.15) by setting for every $\delta > 0$, $\Psi_\delta(\theta) = \delta\mathbb{1}_{|\theta-\theta^*|>\delta}$ and $\chi_\delta(y) = \mathbb{1}_{|y-\theta^*|\leq\delta}$.

As soon as $(Y_n)_{\geq 0}$ is ν-averaging, there exists a sequence $(\varepsilon_n)_{n\geq 1}$ such that, for every $\theta \in \mathbb{R}$, $\mathbb{1}_{\{y\geq\theta\}} \in \mathcal{V}_{\varepsilon_n,2}$ since the empirical distribution measure *a.s.* (and subsequently in L^2) converges uniformly toward F_Y. Finally, as soon as the step sequence $(\gamma_n)_{n\geq 1}$ is admissible for $(\varepsilon_n)_{n\geq 1}$, Theorem 2.1 implies that

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^* = VaR_\alpha(Y).$$

In practice $\gamma_n = c/n$, $c > 0$, is always admissible given the rate of convergence of the empirical measure in usual applications. Of course, when the Y_n are i.i.d., standard martingale arguments "à la Robbins-Monro" make things straightforward under less stringent assumptions on the step sequence.

▷ *Computation of the $CVaR_\alpha(Y)$.* The idea to compute the $CVaR_\alpha(Y)$ is to devise a companion procedure of the above stochastic gradient by setting, $\zeta_0 = 0$ and for every $n \geq 0$,

$$\zeta_{n+1} = \zeta_n - \frac{1}{n+1}\left(\zeta_n - v(\theta_n, Y_n)\right) \quad \text{with} \quad v(\theta, y) := \theta + \frac{(y-\theta)_+}{1-\alpha}.$$

One checks that for every $n \geq 0$,

$$\zeta_n = \frac{1}{n}\sum_{k=0}^{n-1} v(\theta_k, Y_k) = \frac{1}{n}\sum_{k=0}^{n-1} v(\theta^*, Y_k) + \frac{1}{n}\sum_{k=0}^{n-1} v(\theta_k, Y_k) - v(\theta^*, Y_k)$$

Using that v is Lipschitz continuous in θ uniformly in y, we derive that the second term in the right hand side of the above equality goes to 0 *a.s.* as $\theta_n \to 0$ *a.s.*

As concerns the first term, still in right hand side, first note that $v(\theta^*, y)$ has a linear growth in y so it will *a.s.* go to $\mathbb{E}\, v(\theta^*, Y) = V(\theta^*) = CVaR_\alpha(Y)$ as soon as, *e.g.*, $\sup_{n\geq 1} \frac{1}{n}\sum_{k=0}^{n-1} |Y_k|^{1+\eta} < +\infty$ *a.s.* for an $\eta > 0$ by combining standard uniform integrability arguments (with respect to the empirical measure) and the ν-stability of $(Y_n)_{n\geq 0}$. In practice one must keep in mind that an adaptive importance sampling procedure like that detailed in [20] should be added. For a *QMC* implementation of the procedure, see [53].

2.5.3 Long term investment evaluation (and inhomogeneous Markov innovations)

In this example we deal with averaging *inhomogeneous* Markov innovations, namely the Euler scheme with decreasing step of a Brownian diffusion. To describe the functional class $\mathcal{V}_{\varepsilon_n,p}$, we rely on an approach developed in [77] and [87] to compute the invariant measure of a diffusion.

2.5.3.1 Computation of the invariant distribution of a diffusion

We consider a stochastic recursive algorithm for the computation of the invariant distribution ν introduced in [77] of a Brownian diffusion process

$$dY_t = b(Y_t)dt + \sigma(Y_t)dW_t \tag{2.37}$$

where $b : \mathbb{R}^q \to \mathbb{R}^q$ and $\sigma : \mathbb{R}^q \to \mathcal{M}_{q,\ell}(\mathbb{R})$ (matrices with q rows and ℓ columns) are Lipschitz continuous, and W is a ℓ-dimensional Brownian motion. We denote by $\mathcal{A}$ its infinitesimal generator and by $(P_t)_{t\geq 0}$ its transition semi-group.

First, we introduce the Euler discretization of (2.37) with a step γ_n vanishing to 0, *i.e.*

$$\forall n \in \mathbb{N}, \quad \bar{Y}_{n+1} = \bar{Y}_n + \bar{\gamma}_{n+1} b(\bar{Y}_n) + \sqrt{\bar{\gamma}_{n+1}}\sigma(\bar{Y}_n)U_{n+1}, \tag{2.38}$$

where $\bar{Y}_0 \in L^0_{\mathbb{R}^q}(\Omega, \mathcal{F}, \mathbb{P})$ and $(U_n)_{n\geq 1}$ is $\mathbb{R}^\ell$-valued normalized white noise defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent of $\bar{Y}_0$. The step sequence $\bar{\gamma} := (\bar{\gamma}_n)_{n\geq 1}$ satisfies the conditions

$$\forall n \geq 1, \quad \bar{\gamma}_n \geq 0, \quad \lim_{n\to\infty} \bar{\gamma}_n = 0 \quad \text{and} \quad \bar{\Gamma}_n := \sum_{k=1}^{n} \bar{\gamma}_k \underset{n\to\infty}{\longrightarrow} +\infty. \tag{2.39}$$

For every $n \geq 1$ and every $\omega \in \Omega$, set

$$\nu_n(\omega, dy) := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\bar{Y}_k(\omega)}. \tag{2.40}$$

We will use $\nu_n(\omega, f)$ which can be compute recursively to approximate $\nu(f)$.

Definition 2.4. *(Strong condition of stability) A diffusion with generator $\mathcal{A}$ satisfies a strong stability condition of type (V, α) if there exists a (so-called Lyapunov) function $V \in \mathcal{C}^2(\mathbb{R}^q, [1, +\infty[)$ such that $\lim_{|y|\to+\infty} V(y) = +\infty$ and $\exists\, \alpha > 0$, $\exists\, \beta \geq 0$ such that $\mathcal{A}V \leq -\alpha V + \beta$.*

Remark. If the (V, α)-strong stability condition holds then (2.37) admits a strong solution starting from any $y \in \mathbb{R}^d$ and admits at least one invariant distribution ν (*i.e.* $\nu P_t = \nu$, $t \geq 0$).

Definition 2.5. (*a*) *A couple $(\bar{\gamma}, \eta)$ is an* averaging step-weight system *if the sequences $(\bar{\gamma}_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$ are nonnegative, general terms of a non-converging series and such that*

$$\lim_n \bar{\gamma}_n = 0, \quad \sum_{n\geq 1} \frac{1}{H_n}\left(\Delta\frac{\eta_n}{\bar{\gamma}_n}\right)_+ < +\infty \quad \text{and} \quad \sum_{n\geq 1}\left(\frac{\eta_n}{H_n\sqrt{\bar{\gamma}_n}}\right)^2 < +\infty,$$

where $H_n = \sum_{k=1}^n \eta_k$.
(*b*) *In particular, if* $\eta_n \equiv 1$, *then* $(\bar\gamma, 1)$ *is an* averaging step-weight system *if*

$$\lim_n \bar\gamma_n = 0, \quad \sum_{n\geq 1} \frac{1}{n}\left(\frac{1}{\bar\gamma_n} - \frac{1}{\bar\gamma_{n+1}}\right) < +\infty \quad \textit{and} \quad \sum_{n\geq 1} \frac{1}{n^2\bar\gamma_n} < +\infty.$$

The terminology "averaging" refers here to the fact that if $\mathcal{A}$ is (V,α)-stable (and the invariant distribution ν is unique for the sake of simplicity) then, as soon as $(\bar\gamma, \eta)$ is averaging (see *e.g.* [77], [87] or [88]), then

$$\sup_{n\geq 0} \mathbb{E}V(\bar Y_n) < +\infty \quad \text{and} \quad \mathbb{P}(d\omega)\text{-}a.s. \quad \nu_n^\eta(\omega, dy) := \frac{1}{H_n}\sum_{k=0}^{n-1} \eta_k \delta_{\bar Y_k(\omega)} \overset{(\mathbb{R}^d)}{\Longrightarrow} \nu.$$

Example. If $\bar\gamma_n = \frac{\bar\gamma_0}{n^r}$, $0 < r < 1$, and $\eta_n \equiv 1$, then $(\bar\gamma, 1)$ is averaging.

We assume that the diffusion $(Y_t)_{t\geq 0}$ satisfies a strong condition of stability of type (V,α) with V sub-quadratic and that the invariant measure ν is unique. Besides the coefficients b and σ satisfy $|b|^2 + \mathrm{Tr}(\sigma\sigma^t) = O(V)$. Then the Euler scheme with decreasing step $(\bar Y_n)_{n\geq 0}$ defined by (2.38) satisfies a strong condition of stability of type (W, n_0) where W is a function depending upon V and the moments of U_1, namely

$$\forall n \geq n_0, \quad \mathbb{E}\left[W(\bar Y_{n+1}) \,|\, \sigma\left(\bar Y_k, 0 \leq k \leq n\right)\right] \leq (1 - \alpha\,\bar\gamma_{n+1})\, W(\bar Y_n) + \beta.$$

$\triangleright$ If U_1 is sub-normal (typically if U_1 *is* normal), and $\mathrm{Tr}(\sigma\sigma^*) \leq C_\sigma V^{1-\zeta}$, we may choose $W = \exp(\lambda V^\zeta)$ with λ small enough (see [87] Proposition III.2 p. 36).

$\triangleright$ If U_1 has a moment of order $2(p+1)$, $p \geq 2$, then $W = V^{p+1}$ (*idem*).

Assume that the function $f : \mathbb{R}^q \to \mathbb{R}$ admits a regular enough solution ϕ to the Poisson equation

$$\mathcal{A}\phi = -(f - \nu(f)), \tag{2.41}$$

i.e. belonging to the set

$$\mathcal{E}_{p,W} := \left\{\phi \in \mathcal{C}^p(\mathbb{R}^q, \mathbb{R}),\, \forall j \in \{0,\ldots,p\},\, \forall y \in \mathbb{R}^q,\, \left|D^j\phi(y)\right|^2 = o\left(\frac{W(y)}{V^j(y)}\right)\right\}$$

and satisfying $D^p\phi$ Lipschitz. For such functions ϕ, let us to define the functions D_q, $3 \leq q \leq p$, by

$$\forall y \in \mathbb{R}^q, \quad D_q(y) = \sum_{j\geq q/2}^{q} \frac{C_j^{q-j}}{j!} D^j\phi(y) \cdot \left((b(y))^{\otimes(q-j)}, \mathbb{E}\left[(\sigma(y)U_1)^{\otimes(2j-q)}\right]\right).$$

They will appear in the development of the error of order p, $p \geq 3$.

Theorem 2.5. *Let* $p \geq 2$ *such that* $U_1 \in L^{2(p+1)}$ *and* $\phi \in \mathcal{E}_{p,W}$ *solution to Poisson equation (2.41) such that* $D^p\phi$ *is Lipschitz. Define* q^* *by*

$$q^* = \min_{q\in\{3,\ldots,p\}} \{D_q \neq 0\} \wedge (p+1).$$

(Note that if $U_1 \overset{\mathcal{L}}{\sim} \mathcal{N}(0, I_q)$ *then* $q^* = 4$*). Let* $\bar\Gamma_n^{(\beta)} = \sum_{k=1}^n \bar\gamma_k^\beta$, $\beta \in \mathbb{R}$. *Assume that the couples* $(\bar\gamma, 1)$ *and* $(\bar\gamma, \frac{1}{\bar\gamma})$ *are averaging and that* $(\bar\gamma_n)_{n\geq 1}$ *is non-increasing. If* $q^* \leq p$ *and*

$$\frac{\bar\Gamma_n^{(q^*/2-1)}}{\sqrt{\bar\Gamma_n^{(-1)}}} \underset{n\to+\infty}{\longrightarrow} \xi \in]0, +\infty], \qquad \left(\left(\bar\Gamma_n^{(q^*/2-1)}\bar\gamma_n\right)^{-1}\right)_{n\geq 1} \textit{ is non-increasing,}$$

$$\sum_{n\geq 1}\frac{1}{\bar\Gamma_n^{(q^*/2-1)}}\left|\Delta\frac{1}{\bar\gamma_n}\right|<+\infty\quad\text{and}\quad\sum_{n\geq 1}\frac{1}{\bar\gamma_n\left(\bar\Gamma_n^{(q^*/2-1)}\right)^2}<+\infty,$$

then

$$f\in\mathcal{V}_{\varepsilon_n,2}\quad\text{with}\quad\varepsilon_n=\frac{\bar\Gamma_n^{(q^*/2-1)}}{n}\underset{n\to+\infty}{\longrightarrow}0.$$

Corollary 2.1. *If $\bar\gamma_n=\frac{\bar\gamma_0}{n^r}$, $\bar\gamma_0>0$, the above theorem holds true when $0<r\leq\frac{1}{q^*-1}$ and $\varepsilon_n=n^{-r(q^*/2-1)}$. In particular, for a Gaussian Euler scheme, $\varepsilon_n=n^{-r}$.*

Sketch of proof of Theorem 2.5. Proposition V.4 in [87] gives

$$\left\|\frac{n}{\bar\Gamma_n^{(q^*/2-1)}}\left(\frac{1}{n}\sum_{k=0}^{n-1}f(\bar Y_k)-\nu(f)\right)\right\|_2=\|M_n+S_n\|_2+o(1)$$

where $M_n=\dfrac{1}{\bar\Gamma_n^{(q^*/2-1)}}\displaystyle\sum_{k=1}^{n}\sqrt{\frac{1}{\bar\gamma_k}}\left\langle\nabla\phi(\bar Y_{k-1})\,|\,\sigma(\bar Y_{k-1})U_k\right\rangle$ and $S_n=\displaystyle\sum_{q=q^*}^{p}\frac{1}{\bar\Gamma_n^{(q^*/2-1)}}\sum_{k=1}^{n}\bar\gamma_k^{\frac{q}{2}-1}D_q(\bar Y_{k-1})$.

Using that $\phi\in\mathcal{E}_{p,W}$, *i.e.* that for every q, $|D_q|^2=o(W)$, and that $\sup_n\mathbb{E}\,W(\bar Y_n)<+\infty$ (according to the stability condition of the Euler scheme), we get (see the remark after Proposition V.1 p.62 in [87]),

$$\|M_n\|_2\leq\frac{\sqrt{\bar\Gamma_n^{(-1)}}}{\bar\Gamma_n^{(q^*/2-1)}}\sup_{0\leq k\leq n-1}\left\|\sigma^*\nabla\phi(\bar Y_k)\right\|_2<+\infty$$

since $\sqrt{\bar\Gamma_n^{(-1)}}/\bar\Gamma_n^{(q^*/2-1)}\underset{n\to+\infty}{\longrightarrow}\xi^{-1}\in]0,+\infty[$ and $\|\sup_n S_n\|_2<+\infty$. □

2.5.3.2 Application to the minimization of a potential

The aim is to minimize a convex potential $V:\mathbb{R}^q\to\mathbb{R}$ having a minimum (*e.g.* because $\lim_{|\theta|\to+\infty}V(\theta)=+\infty$) assumed to be unique. We also assume that V has a representation as the expectation with respect to the invariant distribution ν of an ergodic diffusion, say Y defined above. Typically V appears as the long run limit (under appropriate assumptions) of a functional through Birkhoff's Theorem:

$$V(\theta)=\lim_{t\to+\infty}\frac{1}{T}\int_0^T v(\theta,Y_{t+s})\,ds=\mathbb{E}_\nu\left(\frac{1}{T}\int_0^T v(\theta,Y_s\,ds\right)=\int_{\mathbb{R}^q}v(\theta,y)\nu(dy).$$

We make the following assumptions

- (i) Integrability: $\forall y\in\mathbb{R}^q,\ \theta\mapsto v(\theta,y)$ is convex.
- (ii) Pathwise convexity: $\forall\theta\in\mathbb{R}^d,\ v(\theta,\cdot)\in L^1(\nu)$.
- (iii) Differentiability: $\forall\theta\in\mathbb{R}^d,\ \nabla_\theta v(\theta,y)$ exists.
- (iv) Uniform integrability: $\forall\theta\in\mathbb{R}^d,\ \left(\frac{|v(\theta,y)-v(\theta',y)|}{|\theta-\theta'|}\right)_{\theta'\in[\theta-\eta_\theta,\theta+\eta_\theta]\setminus\{\theta\}}$, $\eta_\theta>0$, is uniformly integrable.

Then (using uniqueness of θ^*),

$$\theta^*=\mathrm{argmin}_{\theta\in\mathbb{R}^d}\int_{\mathbb{R}^q}v(\theta,y)\nu(dy)\ \text{ iff }\ \int_{\mathbb{R}^q}\nabla_\theta v(\theta^*,y)\nu(dy)=0,$$

At this stage the idea is to devise a stochastic gradient (gradient based recursive zero search) using the Gaussian Euler scheme $(\bar{Y}_n)_{n\geq 0}$ with decreasing step $\bar{\gamma}_n = \bar{\gamma}_0 n^{-\frac{1}{3}}$, $\bar{\gamma}_0 > 0$, of Y as an ν-averaging innovation process with rate $\varepsilon_n = \bar{\Gamma}_n/n \to 0$:

$$\forall n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1}\nabla_\theta v(\theta_n, \bar{Y}_n),\ \theta_0 \in \mathbb{R}^q.$$

Let $p \in [1,\infty)$ such that $\nabla_\theta v$ satisfies the growth assumption (2.12) with $L(\theta) = |\theta - \theta^*|^2$, $\nabla_\theta v(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n,p}$ and $(\gamma_n)_{n\geq 1}$ is admissible for ε_n given by Corollary 2.1, then Theorem 2.1 implies that $\theta_n \to \theta^*$ *a.s.*

Toy numerical example. We consider a long-term investment project (see the example in [90]) which yields payoff at a rate that depends on the installed capacity level and on the value of an underlying state process modeled with an ergodic diffusion. The process Y represents an economic indicator such as the asset demand or its discounted price. Our aim is to determine the capacity expansion strategy that *maximizes* the long-term average payoff resulting from the project operation. So it is an ergodic control problem in a microeconomic framework. In [90] is shown that this dynamical optimization problem is equivalent (see above) to a static optimization problem involving the stationary distribution ν of Y and the (concave) running payoff function C, namely, still following [90],

$$\forall\, \theta \in \mathbb{R}_+,\ \forall\, y \in \mathbb{R}_+, \quad C(\theta, y) = y^\alpha \theta^\beta - c\,\theta \quad \text{where} \ \ \alpha, \beta \in (0,1) \ \ \text{and} \ \ c \in (0,\infty).$$

The term $y^\alpha\theta^\beta$ can be identified to the so-called *Cobb-Douglas production function*, while the term $c\,\theta$ measures the cost of capital use. Our task is to *minimize* $\displaystyle\int_{\mathbb{R}^q} (-1)\left(y^\alpha\theta^\beta - c\,\theta\right)\nu(dy)$ (so that of course $\theta^* = \left(\frac{\beta \mathbb{E} Y_1^\alpha}{c}\right)^{\frac{1}{1-\beta}}$!). Since $\nabla_\theta\, C(\theta, y)$ is singular at $\theta = 0$, we will introduce the increasing convex with linear growth change of variable $\theta = \left(\tilde{\theta} + (\tilde{\theta}^2+1)^{1/2}\right)^{\rho(\tilde{\theta})}$, $\rho(\tilde{\theta}) = \frac{1}{1-\beta}\mathbb{1}_{\tilde{\theta}<0} + \mathbb{1}_{\tilde{\theta}\geq 0}$, from $\mathbb{R}$ onto $(0,\infty)$ and we consider

$$\nabla_\theta v(\widetilde{\theta}, y) = -\nabla_\theta C\left(\left(\tilde{\theta} + (\tilde{\theta}^2+1)^{1/2}\right)^{\rho(\tilde{\theta})}, y\right),\ \tilde{\theta} \in \mathbb{R},\ y \in \mathbb{R}_+.$$

Still following [90], the dynamics of the underlying state process Y is modeled by the one-dimensional CIR diffusion (whose diffusion coefficient is unfortunately not Lipschitz), namely

$$dY_t = \kappa\,(\vartheta - Y_t)\,dt + \sigma\sqrt{|Y_t|}dW_t, \quad Y_0 > 0, \tag{2.42}$$

where $\kappa, \vartheta, \sigma > 0$ are constants satisfying $2\kappa\vartheta > \sigma^2$ so that $(Y_t)_{t\geq 0}$ is $(0,\infty)$-valued.

The resulting stochastic gradient procedure with step $(\widetilde{\gamma}_n)_{n\geq 1}$ reads

$$\forall n \geq 0, \quad \widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \gamma_{n+1}\nabla_\theta v(\widetilde{\theta}_n, \bar{Y}_n), \quad \widetilde{\theta}_0 \in \mathbb{R},$$

where $(\gamma_n)_{n\geq 1}$ is admissible with respect to $\varepsilon_n = \bar{\Gamma}_n/n$ and $(\bar{Y}_n)_{n\geq 0}$ the Euler scheme with step $\bar{\gamma}_n = \bar{\gamma}_0 n^{-1/3}$ ($L(\widetilde{\theta}) = |\widetilde{\theta} - \widetilde{\theta}^*|^2$ is still a pathwise Lyapunov function). One checks that $\nabla_\theta v$ satisfies (2.12) with $\phi(y) \equiv c_{\alpha,\beta} y^\alpha$, $c_{\alpha,\beta} > 0$ and $p = 2$ since $\sup_n \mathbb{E}\,\bar{Y}_n^2 < +\infty$ and $\alpha \in (0,1)$.

The invariant distribution of Y is a Gamma law which density is given by

$$\nu(dy) = \frac{1}{\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}\right)} y^{\frac{2\kappa\vartheta}{\sigma^2}-1} \exp\left(\frac{2\kappa}{\sigma^2}\left[\vartheta \log\left(\frac{2\kappa}{\sigma^2} - y\right)\right]\right)\mathbb{1}_{\{y>0\}},$$

where Γ is the gamma function. Thus we can compute the previous integral, namely

$$\int_{\mathbb{R}_+} y^\alpha \nu(dy) = \frac{\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}+\alpha\right)}{\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}\right)}\left(\frac{\sigma^2}{2\kappa}\right)^\alpha < +\infty,$$

so we have in fact a closed form for θ^* given by $\theta^* = \left(\frac{\beta\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}+\alpha\right)}{c\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}\right)}\left(\frac{\sigma^2}{2\kappa}\right)^\alpha\right)^{\frac{1}{1-\beta}}$. Figure 2.2 illustrates the convergence of the algorithm (the parameters are specified in the caption).

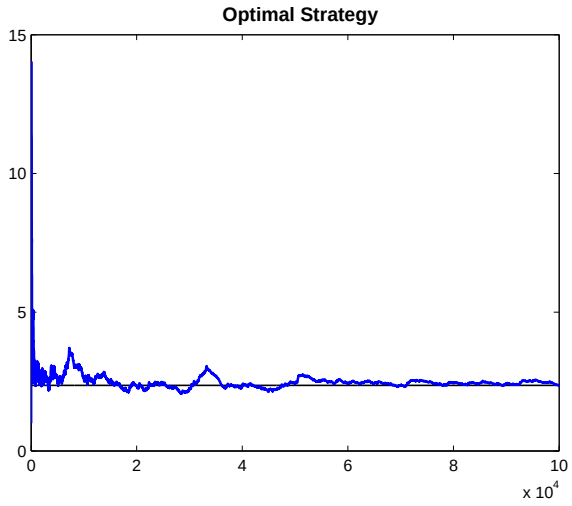

Figure 2.2: Convergence towards the optimal capacity level of the investment project: $\kappa = 1$, $\vartheta = 1$, $\sigma = 1.5$, $\alpha = 0.8$, $\beta = 0.7$, $c = 0.5$, $n = 10^5$, $\widetilde{\gamma}_n = \frac{5}{n}$.

If one considers a basket of assets modeled by a Wishart process (see [35] and [59]), a similar long-term ergodic control process can be devised. Closed forms are no longer available for the static optimization problem. However, our numerical approach can be extended straightforwardly (provided one has at hand an efficient method of simulation for Wishart process, like that proposed in [59]).

2.5.4 The ergodic two-armed bandit

An application of the multiplicative setting is the so-called two-armed bandit algorithm introduced in mathematical psychology, learning automata (see [96, 94]) and more recently asset allocation ([80]). Criteria on *a.s.* convergence under pure i.i.d. assumptions were obtained in [80, 78] and under ergodic assumptions in [110]. A penalized version of this algorithm is also studied in [79].

This algorithm is defined as follows: at each step $n \geq 0$, one plays arm A (resp. arm B) at random with probability θ_n (resp. $1-\theta_n$), where $\theta_0 = \theta \in (0,1)$ and θ_n is updated according the following "*rewarding*" rule: for every $n \geq 0$,

$$\theta_{n+1} = \theta_n + \gamma_{n+1}\left((1-\theta_n)\mathbb{1}_{\{U_{n+1}\leq\theta_n\}\cap A_{n+1}} - \theta_n\mathbb{1}_{\{U_{n+1}>\theta_n\}\cap B_{n+1}}\right) \tag{2.43}$$

where $(U_n)_{n\geq 1}$ is an i.i.d. sequence of uniform random variables, independent of $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ which are two sequences of (possibly dependent) events evaluating the performances of the arms A and B respectively (A_n is the event "A's performance is satisfactory at time n idem for B_n and B).

This stochastic procedure can be rewritten in a canonical form as follows

$$\theta_{n+1} = \theta_n + \gamma_{n+1}\left(\mathbb{1}_{A_{n+1}} - \mathbb{1}_{B_{n+1}}\right) h(\theta_n) + \gamma_{n+1}\Delta M_{n+1}, \quad \theta_0 = \theta \in (0,1) \tag{2.44}$$

where $h(\theta) = \theta(1-\theta)$, $M_n := \sum_{k=1}^n m_k$, $M_0 := 0$, with

$$m_k := \mathbb{1}_{A_k}(1-\theta_{k-1})\left(\mathbb{1}_{\{U_{n+1}\le\theta_n\}} - \theta_{k-1}\right) + \mathbb{1}_{B_k}\theta_{k-1}\left((1-\theta_{k-1}) - \mathbb{1}_{\{U_{n+1}>\theta_n\}}\right).$$

We make the assumption that A outperforms B in average *i.e.* that $\nu(A) > \nu(B)$ where

$$\frac{1}{n}\sum_{k=1}^n \mathbf{1}_{A_n} \underset{n\to\infty}{\longrightarrow} \nu(A) \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^n \mathbf{1}_{B_n} \underset{n\to\infty}{\longrightarrow} \nu(B)$$

and that these convergences hold at rate ε_n satisfying (2.7). Then applying Theorem 2.2 with $Y_k := \mathbb{1}_{A_{k+1}} - \mathbb{1}_{B_{k+1}}$, $k \ge 0$, and $\chi(y) = \frac{y}{\nu(A)-\nu(B)}$, we get a first convergence result: as soon as $(\gamma_n)_{n\ge1}$ is admissible in the sense of (2.14) for the sequence $(\varepsilon_n)_{n\ge1}$,

$$\theta_n \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \theta^* \in \{0,1\}.$$

where 1 is the target and 0 is a trap. Further investigations on θ^* are carried in [110] in this ergodic framework to analyze the fallibility of the algorithm which extend former results established in [80, 78] in the purely i.i.d. setting.

2.5.5 Optimal split of orders across liquidity pools

This is an example of application in Finance to be implemented exclusively on real data. It is an optimal allocation problem which solved by a stochastic Lagrangian approach originally developed in [82]. Here, only numerical results with real market data are presented.

2.5.5.1 Model description

The principle of a *Dark pool* is to propose a price with no guarantee of executed quantity at the occasion of an OTC transaction. Usually this price is lower than the one offered on the regular market. So one can model the impact of the existence of N dark pools ($N \ge 2$) on a given transaction as follows: let $V > 0$ be the random volume to be executed, let $\theta_i \in (0,1)$ be the *discount factor* proposed by the dark pool i. Let r_i denote the percentage of V sent to the dark pool i for execution. Let $D_i \ge 0$ be the quantity of securities that can be delivered (or made available) by the dark pool i at price $\theta_i S$.

The remainder of the order is to be executed on the regular market, at price S. Then the cost C of the whole executed order is given by

$$C = S\sum_{i=1}^N \theta_i \min(r_i V, D_i) + S\left(V - \sum_{i=1}^N \min(r_i V, D_i)\right) = S\left(V - \sum_{i=1}^N \rho_i \min(r_i V, D_i)\right)$$

where $\rho_i = 1-\theta_i \in (0,1), i = 1,\dots,N$. Minimizing the mean execution cost, *given the price* S, amounts to solving the following maximization problem

$$\max\left\{\sum_{i=1}^N \rho_i \mathbb{E}\left(S\min(r_i V, D_i)\right), r \in \mathcal{P}_N\right\} \tag{2.45}$$

where $\mathcal{P}_N := \left\{r = (r_i)_{1\le i\le N} \in \mathbb{R}_+^N \mid \sum_{i=1}^N r_i = 1\right\}$. It is then convenient to *include the price* S *into both random variables* V *and* D_i by considering $\widetilde{V} := V\,S$ and $\widetilde{D}_i := D_i S$ instead of V and D_i.

Let $\mathcal{I}_N = \{1, \ldots, N\}$. We set for all $r = (r_1, \ldots, r_N) \in \mathcal{P}_N$, $\Phi(r_1, \ldots, r_N) := \sum_{i=1}^N \varphi_i(r_i)$, where

$$\forall i \in \mathcal{I}_N, \quad \varphi_i(u) := \rho_i \mathbb{E}\left(\min\left(uV, D_i\right)\right), \quad u \in [0,1].$$

We assume that for all $i \in \mathcal{I}_N$,

$$V > 0 \ \mathbb{P}\text{-}a.s., \quad \mathbb{P}(D_i > 0) > 0 \text{ and the distribution function of } \tfrac{D_i}{V} \text{ is continuous on } \mathbb{R}_+, \tag{2.46}$$

then φ_i, $i \in \mathcal{I}_N$, are everywhere differentiable on the unit interval $[0,1]$ with

$$\varphi_i'(u) = \rho_i \, \mathbb{E}\left(\mathbb{1}_{\{uV \le D_i\}} V\right), \quad u \in (0,1], \tag{2.47}$$

and one extends φ_i, $i \in \mathcal{I}_N$, on the whole real line into a concave nondecreasing function with $\lim_{\pm\infty} \varphi_i = \pm\infty$. So we can formally extend Φ on the whole affine hyperplane spanned by $\mathcal{P}_N$ *i.e.* $\mathcal{H}_N := \left\{ r = (r_1, \ldots, r_N) \in \mathbb{R}^N \mid \sum_{i=1}^N r_i = 1 \right\}$.

2.5.5.2 Design of the recursive procedure

We aim at solving the following maximization problem $\max_{r \in \mathcal{P}_N} \Phi(r)$. The Lagrangian associated to the sole affine constraint suggests that any $r^* \in \arg\max_{\mathcal{P}_N} \Phi$ iff $\varphi_i'(r_i^*)$ is constant when i runs over $\mathcal{I}_N$ or equivalently if $\varphi_i'(r_i^*) = \frac{1}{N} \sum_{j=1}^N \varphi_j'(r_j^*)$, $i \in \mathcal{I}_N$.

We set $Y^n := (V^n, D_1^n, \ldots, D_N^n)_{n \ge 1}$. Then using the representation of the derivatives φ_i' yields

$$r^* \in \arg\max_{\mathcal{P}_N} \Phi \Longleftrightarrow \forall i \in \{1, \ldots, N\}, \ \mathbb{E}\left(V \left(\rho_i \mathbb{1}_{\{r_i^* V < D_i\}} - \frac{1}{N} \sum_{j=1}^N \rho_j \mathbb{1}_{\{r_j^* V < D_j\}} \right) \right) = 0.$$

Consequently, this leads to the following recursive zero search procedure

$$r_i^{n+1} = r_i^n + \gamma_{n+1} H_i(r^n, Y^{n+1}), \quad r^0 \in \mathcal{P}_N, \quad n \ge 0, \quad i \in \mathcal{I}_N, \tag{2.48}$$

where for every $i \in \mathcal{I}_N$, $r \in \mathcal{P}_N$, every $V > 0$ and every $D_1, \ldots, D_N \ge 0$,

$$H_i(r, Y) = V \left(\rho_i \mathbb{1}_{\{r_i V < D_i\}} - \frac{1}{N} \sum_{j=1}^N \rho_j \mathbb{1}_{\{r_j V < D_j\}} \right)$$

where $(Y^n)_{n \ge 1}$ is a sequence of random vectors with nonnegative components such that, for every $n \ge 1$, $(V^n, D_i^n, i = 1, \ldots, N) \overset{d}{=} (V, D_i, i = 1, \ldots, N)$.

The underlying idea of the algorithm is to reward the dark pools which outperform the mean of the N dark pools by increasing the allocated volume sent at the next step (and conversely). For sake of simplicity that $\operatorname{argmax}_{\mathcal{P}_N} \Phi = \{r^*\} \subset \operatorname{int}(\mathcal{P}_N)$. Our "light" ν-averaging assumption is to assume that there exists an exponent $\eta \in (0,1]$ such that for every $u \in \mathbb{R}_+$ and every $i \in \mathcal{I}_N$

$$\frac{1}{n} \sum_{k=1}^n V^k \mathbf{1}_{\{u < \frac{D_i^k}{V^k}\}} - \mathbb{E}(V \mathbf{1}_{\{u < \frac{D_i}{V}\}}) = O(n^{-\eta}) \quad a.s. \text{ and in } L^2(\mathbb{P}) \tag{2.49}$$

(which hold under geometric α-mixing assumptions on $(D^n, V^n)_{n \ge 1}$). Under additional technical assumptions on the support of $\mathcal{L}(Y^n)$ (see [82]), we can apply Theorem 2.1: if the sequence $(\gamma_n)_{n \ge 1}$ satisfies (2.14), we get that the algorithm defined by (2.48) *a.s.* converges towards $r^* = \operatorname{argmax}_{\mathcal{P}_N} \Phi$.

2.5.5.3 Numerical Tests

We consider the shortage setting, *i.e.* $\mathbb{E}V > \sum_{i=1}^{N} \mathbb{E}D_i$ because it is the most interesting case and the most common in the market. Now, we introduce an index to measure the performances of our recursive allocation procedure.

▷ **Relative cost reduction (w.r.t. the regular market):** it is defined as the ratios between the cost reduction of the execution using dark pools and the cost resulting from an execution on the regular market, *i.e.*, for every $n \geq 1$,

$$\frac{CR^{algo}}{V^n} = \frac{\sum_{i=1}^{N} \rho_i \min\left(r_i^n V^n, D_i^n\right)}{V^n}.$$

We have considered for V the traded volumes of a very liquid security – namely the asset BNP – during an 11 day period. Then we selected the N most correlated assets (in terms of traded volumes) with the original asset. These assets are denoted S_i, $i = 1, \ldots, N$ and we considered their traded volumes during the same 11 day period. Finally, the available volumes of each dark pool i have been modeled as follows using the mixing function

$$\forall\, 1 \leq i \leq N, \quad D_i := \beta_i \left((1 - \alpha_i)V + \alpha_i S_i \frac{\mathbb{E}V}{\mathbb{E}S_i} \right)$$

where $\alpha_i,\ i = 1, \ldots, N$ are the recombining coefficients, $\beta_i,\ i = 1, \ldots, N$ some scaling factors and $\mathbb{E}\,V$ and $\mathbb{E}\,S_i$ stand for the empirical mean of the data sets of V and S_i. The simulations presented here have been made with four dark pools ($N = 4$). Since the data used here cover 11 days, it is clear that, unlike the simulated data, these pseudo-real data are not stationary: in particular they are subject to daily changes of trend and volatility (at least). To highlight the resulting changes in the response of the algorithms, we have specified the days by drawing vertical doted lines. The dark pool pseudo-data parameters are set to $\beta = (0.1, 0.2, 0.3, 0.2)^t$, $\alpha = (0.4, 0.6, 0.8, 0.2)^t$ and the dark pool trading (rebate) parameters are set to $\rho = (0.0, 0.02, 0.04, 0.06)^t$.

We benchmarked – see Figure 2.3 – the algorithm on the whole data set (11 days) as though it were stationary. In particular, the running means of the performances are computed from the very beginning for the first 1500 data, and then by a moving average computed on a window of 1500 data.

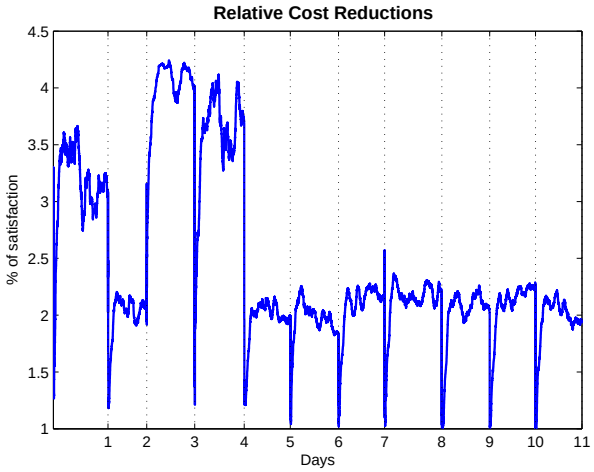

Figure 2.3: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0.2 < \alpha_i \leq 0.8$ and $r_i^0 = 1/N$, $1 \leq i \leq N$.

Chapter 3

Rate of Convergence for Quasi-Stochastic Approximation

This chapter provides "universal" rates of convergence for Quasi-Stochastic Approximation (QSA) in the framework of averaging innovations. Under additional assumptions on the pseudo-mean function (derivability, Taylor approximation, averaging), we give rates of convergence of the stochastic algorithm towards its target for two different types of gain parameter. According to the form of the step, these rates depend on the derivative of the mean function at the equilibrium point, the averaging rate or the step sequence.

3.1 Introduction

Plugging quasi-random numbers into a recursive stochastic approximation procedure instead of pseudo-random numbers is a rather natural idea given the performances of QMC methods for numerical integration. It goes back, to our knowledge, to the early 1990's. As expected, several numerical tests showed that it may significantly accelerate the convergence of the procedure like it does in Monte Carlo simulations.

The convergence of Quasi-Stochastic Approximation (QSA) is investigated from a theoretical viewpoint in several papers. In [81], based on an extension of uniformly distributed sequences on unit hypercubes called averaging systems, the authors proved the convergence under a contraction assumption on hand hand and under a monotone assumption which requires stringent conditions on the pseudo-mean function on the other hand. These two results are one-dimensional.

In Chapter 2, we generalize this result to a multi-dimensional framework with no boundedness of the pseudo-mean function using averaging assumption and pathwise Lyapunov property. We also give the averaging rate when the pseudo-mean function is assumed either to have finite variation (using Koksma-Hlawka inequality) or to be Lipschitz continuous (using Proinov Theorem).

In [106] is proved convergence and established rate of convergence for QSA algorithm. Their goal is to reduce the "curse of variance" observed in these algorithms, and to reduce computational cost by avoiding the generation of random numbers. Motivation for their work came in part from the Q-learning algorithm for approximate optimal control introduced in [91]. Their analysis takes place in continuous time, so they introduce a differential equation with an "average" vector field. Stability and convergence of the procedure are proved. They establish the rate of convergence under linear assumption on the pseudo-mean function and under the fact that the real parts of the eigenvalues of the matrix appeared in the linear expression are all smaller than -1 (which is more stringent than the classical framework which requires -1/2 instead of -1). Thus, in this deterministic setting, they find that the convergence is $O(t^{-1})$.

In this chapter, we consider the framework of averaging innovation presented in Chapter 2 for low discrepancy sequences. The convergence of the recursive procedure is provided by the previous chapter. Our goal is to establish "universal" rates of convergence for QSA. To this end, we need some additional assumption on the pseudo-mean function: we assume that it is is differentiable at the target uniformly in the second variable, *i.e.* it admits a Taylor approximation, and furthermore that its partial derivative with respect to the first argument, taken at the equilibrium point, belongs to the same class of function as the pseudo-mean function. Then we consider two different types of gain parameter $\gamma = (\gamma_n)_{n\geq 1}$: if $\gamma_n = \frac{c}{n}$, $c > 0$, we find that the rate of convergence depend on the derivative of the mean function at the target and the rate of averaging, if $\gamma_n = \frac{c}{n^\rho}$, $c > 0$, $0 < \rho < 1$, this rate depend on the power ρ of the step sequence and the averaging rate.

This chapter is organized as follows: Section 3.2 contains the results on the rate of convergence of QSA and their proofs and Section 3.4 gives an application to Finance which illustrates by numerical experiment the improvement of rate of convergence by using quasi-random numbers instead pseudo-random numbers.

3.2 Main result

In this chapter, we consider a quasi-stochastic approximation procedure whose form is as follows

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, \xi_{n+1}), \quad \theta_0 \in \mathbb{R}, \tag{3.1}$$

where $H : \mathbb{R} \times [0,1]^q \to \mathbb{R}$, $(\xi_n)_{n\geq 1}$ is $[0,1]^q$-valued sequence of quasi-random numbers, $(\gamma_n)_{n\geq 1}$ is a sequence of gain parameter satisfying $\gamma_n \underset{n\to+\infty}{\longrightarrow} 0$ and $\sum_{n\geq 1} \gamma_n = +\infty$.

We assume that $\theta_n \to \theta^*$ where

$$\theta^* = \{h = 0\} \quad \text{with} \quad h(\theta) = \int_{[0,1]^q} H(\theta, \xi)\lambda_q(d\xi),$$

where λ_q is the Lebesgue measure on $\mathbb{R}^q$. The convergence of the sequence $(\theta_n)_{n\geq 0}$ can be obtained by two methods: either directly by applying the convergence theorem for stochastic approximation with averaging innovation of the previous chapter, or by a two step strategy: first simply establish that $(\theta_n)_{n\geq 0}$ is bounded by the same approach and then applying the celebrated ODE method.

To establish the rate of convergence of the procedure (3.1), we need some additional assumptions. First we require a Taylor like development of the pseudo-mean function H: we assume that $H(\cdot, \xi)$ is differentiable at θ^* *uniformly* in ξ, namely

$$H(\theta, \xi) = H(\theta^*, \xi) + \partial_\theta H(\theta^*, \xi)(\theta - \theta^*) + \psi(\theta, \xi)\,|\theta - \theta^*|^{1+\eta}, \quad \eta > 0, \tag{3.2}$$

where

$$\sup_{\xi\in[0,1]^q} |\partial_\theta H(\theta^*, \xi)| < +\infty \quad \text{and} \quad \sup_{\xi\in[0,1]^q, u\in\mathbb{R}} |\psi(u, \xi)| < +\infty. \tag{3.3}$$

We also need some averaging assumptions, *i.e.* that the pseudo-mean function H and its partial derivative (with respect to the first component) at the target θ^* belong to the class of functions presented in the previous chapter. To be precise, we assume that

$$H(\theta^*, \cdot) \quad \text{and} \quad \partial_\theta H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n,1} = \left\{ f \in L^1(\lambda_q) \mid \frac{1}{n}\sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^q} f\lambda_q = O(\varepsilon_n) \right\}, \tag{3.4}$$

where $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$ and $\lim_n \inf n\varepsilon_n = 0$.

We need an "attractivity" condition on $h'(\theta^*)$, typically $h'(\theta^*) > 0$ (with our conventions).

Theorem 3.1. *Let $H : \mathbb{R} \times [0,1]^q \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be Borel functions defined as above such that (3.2) and (3.4) hold. Assume that $\theta_n \to \theta^*$ and that $h'(\theta^*) > 0$. Furthermore assume that the sequence $(\gamma_n)_{n\geq 1}$ satisfies*

$$n\varepsilon_n\gamma_n \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \sum_{n\geq 1} n\varepsilon_n|\Delta\gamma_{n+1}| < +\infty. \tag{3.5}$$

1. *Assume that the sequence of gain parameter has the following representation*

$$\gamma_n = \frac{c}{n}, \quad c > 0, \quad n \geq 1.$$

(a) *If there exists $\alpha > 0$ such that $\sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)} < +\infty$, then for every $\epsilon > 0$,*

$$\theta_n - \theta^* = o\left(\left(\varepsilon_n \vee n^{-ch'(\theta^*)}\right) n^{\epsilon}\right).$$

(a′) *If there exists $\alpha > 0$ such that*

$$\sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)} < +\infty \quad \text{and} \quad \varepsilon_n n^{c(1+\alpha)h'(\theta^*)} \underset{n\to+\infty}{\longrightarrow} 0,$$

then the sequence $\left(n^{ch'(\theta^*)}(\theta_n - \theta^*)\right)_{n\geq 0}$ is bounded.

(b) *If $\sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)} = +\infty$ for any $\alpha > 0$, then for every $\epsilon > 0$,*

$$\theta_n - \theta^* = O\left(\varepsilon_n n^{\epsilon}\right).$$

2. *Assume that the sequence of gain parameter has the following representation*

$$\gamma_n = \frac{c}{n^\rho}, \quad c > 0, \quad 1/2 < \rho < 1, \quad n \geq 1.$$

Then

$$\forall \epsilon > 0, \quad \theta_n - \theta^* = o\left(\varepsilon_n n^{1-\rho+\epsilon}\right).$$

Remark. The result 1.(a′) corresponds in the random case to the non-CLT rate result (see [44]).

Proof. Using the representation of the recursive procedure (3.1), we have

$$\theta_{n+1} - \theta^* = \theta_n - \theta^* - \gamma_{n+1}H(\theta^*, \xi_{n+1}) - \gamma_{n+1}\partial_\theta H(\theta^*, \xi_{n+1})(\theta_n - \theta^*) - \gamma_{n+1}\psi(\theta_n, \xi_{n+1})\,|\theta_n - \theta^*|^{1+\eta}.$$

By setting $z_n = \theta_n - \theta^*$, $n \geq 1$, and $s_n = \operatorname{sign}(\theta_n - \theta^*)$, $n \geq 1$, we obtain

$$z_{n+1} = z_n\left(1 - \gamma_{n+1}\left[\partial_\theta H(\theta^*, \xi_{n+1}) + \psi(\theta_n, \xi_{n+1})\,|z_n|^\eta\, s_n\right]\right) - \gamma_{n+1}H(\theta^*, \xi_{n+1}).$$

Furthermore, let set

$$\Pi_n := \prod_{k=1}^{n}\left(1 - \gamma_k\left[\partial_\theta H(\theta^*, \xi_k) + \psi(\theta_{k-1}, \xi_k)\,|z_{k-1}|^\eta\, s_{k-1}\right]\right).$$

As a consequence,

$$\frac{z_{n+1}}{\Pi_{n+1}} = \frac{z_n}{\Pi_n} - \frac{\gamma_{n+1}}{\Pi_{n+1}}H(\theta^*, \xi_{n+1}).$$

With the obvious notations $\tilde{z}_n = \dfrac{z_n}{\Pi_n}$, $\tilde{\gamma}_n = \dfrac{\gamma_n}{\Pi_n}$, we have

$$\tilde{z}_{n+1} = \tilde{z}_n - \tilde{\gamma}_{n+1} H(\theta^*, \xi_{n+1}) = \tilde{z}_n - \tilde{\gamma}_{n+1} \Delta S^H_{n+1}$$

where $S^H_n := \displaystyle\sum_{k=1}^{n} H(\theta^*, \xi_k)$, *i.e.*

$$\begin{aligned} z_n &= \Pi_n \left(\tilde{z}_0 - \sum_{k=1}^{n} \tilde{\gamma}_k \Delta S^H_k \right) = \Pi_n \left(\tilde{z}_0 - \left[\tilde{\gamma}_n S^H_n - \sum_{k=1}^{n-1} \Delta \tilde{\gamma}_{k+1} S^H_k \right] \right) \\ &= \Pi_n z_0 - \gamma_n S^H_n + \Pi_n \sum_{k=1}^{n-1} \Delta \tilde{\gamma}_{k+1} S^H_k \end{aligned}$$

with $\tilde{z}_0 = z_0$ and where $\Delta\tilde{\gamma}_n = \Delta\gamma_n \Pi_n^{-1} + \gamma_{n-1} \Delta \Pi_n^{-1}$ and

$$\begin{aligned} \Delta \Pi_n^{-1} &= \Pi_{n-1}^{-1} \left(\frac{1}{1 - \gamma_n \left[\partial_\theta H(\theta^*, \xi_n) + \psi(\theta_{n-1}, \xi_n) \left| z_{n-1} \right|^\eta s_{n-1} \right]} - 1 \right) \\ &= \Pi_{n-1}^{-1} \frac{\gamma_n \left[\partial_\theta H(\theta^*, \xi_n) + \psi(\theta_{n-1}, \xi_n) \left| z_{n-1} \right|^\eta s_{n-1} \right]}{1 - \gamma_n \left[\partial_\theta H(\theta^*, \xi_n) + \psi(\theta_{n-1}, \xi_n) \left| z_{n-1} \right|^\eta s_{n-1} \right]} \\ &= \Pi_n^{-1} \gamma_n \left[\partial_\theta H(\theta^*, \xi_n) + \psi(\theta_{n-1}, \xi_n) \left| z_{n-1} \right|^\eta s_{n-1} \right]. \end{aligned}$$

Consequently

$$\Delta\tilde{\gamma}_n = \Pi_n^{-1} \left(\Delta\gamma_n + \gamma_n \gamma_{n-1} \left[\partial_\theta H(\theta^*, \xi_n) + \psi(\theta_{n-1}, \xi_n) \left| z_{n-1} \right|^\eta s_{n-1} \right] \right).$$

As $\sup_{\xi \in [0,1]^q} |\partial_\theta H(\theta^*, \xi)| < +\infty$, then

$$\sup_{n \geq 1} \left| \partial_\theta H(\theta^*, \xi_n) + \psi(\theta_{n-1}, \xi_n) \left| z_{n-1} \right|^\eta s_{n-1} \right| < +\infty.$$

Hence

$$\begin{aligned} z_n &= \Pi_n z_0 - \gamma_n S^H_n + \Pi_n \sum_{k=1}^{n-1} S^H_k \Pi_{k+1}^{-1} \left(\Delta\gamma_{k+1} + \gamma_k \gamma_{k+1} \left[\partial_\theta H(\theta^*, \xi_{k+1}) + \psi(\theta_k, \xi_{k+1}) \left| z_k \right|^\eta s_k \right] \right) \\ &= \Pi_n z_0 - n\varepsilon_n \gamma_n \frac{S^H_n}{n\varepsilon_n} \\ &\quad + \Pi_n \sum_{k=1}^{n-1} \frac{S^H_k}{k\varepsilon_k} k\varepsilon_k \Pi_{k+1}^{-1} \left(\Delta\gamma_{k+1} + \gamma_k \gamma_{k+1} \left[\partial_\theta H(\theta^*, \xi_{k+1}) + \psi(\theta_k, \xi_{k+1}) \left| z_k \right|^\eta s_k \right] \right). \end{aligned} \tag{3.6}$$

Now we must inspect the behavior of Π_n itself, namely

$$\Pi_n = \prod_{k=1}^{n} (1 - \gamma_k \left[\partial_\theta H(\theta^*, \xi_k) + \psi(\theta_{k-1}, \xi_k) \left| z_{k-1} \right|^\eta s_{k-1} \right]).$$

Let set

$$\omega_k = \psi(\theta_{k-1}, \xi_k) \left| z_{k-1} \right|^\eta s_{k-1}, \quad k \geq 1.$$

Let $n_0 \geq 1$. Therefore, we have

$$\begin{aligned} -\log\left(\frac{\Pi_n}{\Pi_{n_0}}\right) &= \sum_{k=n_0+1}^{n} \gamma_k \left(\partial_\theta H(\theta^*, \xi_k) + \omega_k\right) + O\left(\left(\gamma_k \left(\partial_\theta H(\theta^*, \xi_k) + \omega_k\right)\right)^2\right) \\ &= \sum_{k=n_0+1}^{n} \gamma_k \left(\partial_\theta H(\theta^*, \xi_k) + \widetilde{\omega}_k\right) \end{aligned}$$

where $\widetilde{\omega}_k = \omega_k + \gamma_k \left(\partial_\theta H(\theta^*, \xi_k) + \omega_k\right)^2 \rho_k$ with $(\rho_k)_{k\geq 1}$ bounded. We have that $\widetilde{\omega}_k \underset{k\to+\infty}{\longrightarrow} 0$ since we assume that $z_n \to 0$ and $\psi(\cdot,\cdot)$ is bounded.

Let $\alpha \in]0,1[$ and $n_0 = n_0(\alpha) \geq 1$ such that

$$\forall k \geq n_0, \quad |\widetilde{\omega}_k| \leq \alpha h'(\theta^*).$$

Consequently

$$-\log\left(\frac{\Pi_n}{\Pi_{n_0}}\right) = \left(\Gamma_n - \Gamma_{n_0}\right) h'(\theta^*) + \sum_{k=n_0+1}^{n} \gamma_k \left(\partial_\theta H(\theta^*, \xi_k) - h'(\theta^*) + \widetilde{\omega}_k\right). \tag{3.7}$$

where $\Gamma_n := \sum_{k=1}^{n} \gamma_k$.

Now let us deal with the series (which is in fact a rather "usual" term in our framework). Set $f(\xi) = \partial_\theta H(\theta^*, \xi) - h'(\theta^*)$. We have $\int_{[0,1]^q} f d\lambda_q = 0$, so that

$$\begin{aligned} \sum_{k=1}^{n} \gamma_k f(\xi_k) &= \sum_{k=1}^{n} \gamma_k \Delta S_k^f = \gamma_n S_n^f - \sum_{k=1}^{n-1} S_k^f \Delta\gamma_{k+1} \\ &= n\varepsilon_n \gamma_n \frac{S_n^f}{n\varepsilon_n} - \sum_{k=1}^{n-1} \frac{S_k^f}{k\varepsilon_k} k\varepsilon_k \Delta\gamma_{k+1} \end{aligned}$$

where $\dfrac{S_k^f}{k\varepsilon_k}$ is bounded. Therefore

$$\left|\sum_{k=1}^{n-1} \frac{S_k^f}{k\varepsilon_k} k\varepsilon_k \Delta\gamma_{k+1}\right| \leq \sup_{n\geq 1} \frac{\left|S_n^f\right|}{n\varepsilon_n} \sum_{k=1}^{n-1} k\varepsilon_k \left(-\Delta\gamma_{k+1}\right).$$

Consequently, using (3.5), we have that the series $\displaystyle\sum_{k=1}^{n} \gamma_k f(\xi_k)$ converges towards L_∞. Then, for every $n \geq n_0$,

$$\left(\Gamma_n - \Gamma_{n_0}\right)(1-\alpha)h'(\theta^*) + L_\infty + \eta_n \leq -\log\left(\frac{\Pi_n}{\Pi_{n_0}}\right) \leq \left(\Gamma_n - \Gamma_{n_0}\right)(1+\alpha)h'(\theta^*) + L_\infty + \eta_n,$$

where $\eta_n \underset{n\to+\infty}{\longrightarrow} 0$.

1. First Case : $\gamma_n = \dfrac{c}{n}$, $c > 0$. Then $\Gamma_n = c\log n + c_1' + \delta_n$, $c_1' > 0$, $\delta_n \underset{n\to+\infty}{\longrightarrow} 0$, so that

$$c(1-\alpha)h'(\theta^*)\log n - c_2' \leq -\log\left(\frac{\Pi_n}{\Pi_{n_0}}\right) - L_\infty - \eta_n \leq c(1+\alpha)h'(\theta^*)\log n + c_2'$$

with $c_2' > 0$, *i.e.*

$$K_1' n^{c(1-\alpha)h'(\theta^*)} \leq \frac{\Pi_{n_0}}{\Pi_n} \leq K_1 n^{c(1+\alpha)h'(\theta^*)}.$$

(a) ***First Sub-case***: Our aim is to show that

$$\sum_{n\geq 1} \frac{S_n^H}{n\varepsilon_n} n\varepsilon_n \left(\Delta\gamma_{n+1} + \gamma_n\gamma_{n+1}\left[\partial_\theta H(\theta^*, \xi_{n+1}) + \psi(\theta_n, \xi_{n+1})\,|z_n|^\eta\, s_n\right]\right)\Pi_{n+1}^{-1} < +\infty,$$

which means that the rate of convergence of the algorithm (3.1) is $\Pi_n \vee \varepsilon_n$. Since

$$\frac{S_n^H}{n\varepsilon_n} \quad \text{and} \quad \partial_\theta H(\theta^*, \xi_{n+1}) + \psi(\theta_n, \xi_{n+1})\,|z_n|^\eta\, s_n \quad \text{are bounded,}$$

the convergence of the above series follows from

$$\sum_{n\geq 1} n\varepsilon_n \left(\Delta\gamma_{n+1} + \gamma_n\gamma_{n+1}\right)\Pi_{n+1}^{-1} < +\infty,$$

i.e. we need that, there exists $\alpha > 0$ such that

$$\sum_{n\geq 1} n\varepsilon_n \frac{1}{n^2} n^{c(1+\alpha)h'(\theta^*)} = \sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)} < +\infty, \tag{3.8}$$

which gives us a condition on the step parameter c. Then (keeping in mind that α may be chosen arbitrary small), as announced, we obtain

$$\forall \epsilon > 0, \quad \theta_n - \theta^* = o\left(\left(n^{-ch'(\theta^*)} \vee \varepsilon_n\right) n^\epsilon\right).$$

(b) ***Second Sub-case***: Assume that the series $\sum_{n\geq 1} \varepsilon_n n^{-1+c(1+\alpha)h'(\theta^*)}$ does not converge. Then

$$\begin{aligned}
&\sum_{k=1}^{n-1} \left| \frac{S_k^H}{k\varepsilon_k} k\varepsilon_k \left(\Delta\gamma_{k+1} + \gamma_k\gamma_{k+1}\left[\partial_\theta H(\theta^*, \xi_{k+1}) + \psi(\theta_k, \xi_{k+1})\,|z_k|^\eta\, s_k\right]\right)\Pi_{k+1}^{-1} \right| \\
\leq\ & C \sup_{m\geq 1} \left| \frac{S_m^H}{m\varepsilon_m} \right| \sum_{k=1}^{n-1} \varepsilon_k k^{-1+c(1+\alpha)h'(\theta^*)} \\
\lesssim\ & C \sup_{m\geq 1} \left| \frac{S_m^H}{m\varepsilon_m} \right| \varepsilon_n n^{c(1+\alpha)h'(\theta^*)}
\end{aligned}$$

where C is a positive constant, so that for any $\epsilon > 0$,

$$\begin{aligned}
&\Pi_n \left| \sum_{k=1}^{n-1} \frac{S_k^H}{k\varepsilon_k} k e_k \left(\Delta\gamma_{k+1} + \gamma_k\gamma_{k+1}\left[\partial_\theta H(\theta^*, \xi_{k+1}) + \psi(\theta_k, \xi_{k+1})\,|z_k|^\eta\, s_k\right]\right)\Pi_{k+1}^{-1} \right| \\
\lesssim\ & C_{H,\varepsilon,\alpha,c} \varepsilon_n n^{c(1+\alpha)h'(\theta^*) - c(1-\alpha)h'(\theta^*)} \\
=\ & C_{H,\varepsilon,\alpha,c} \varepsilon_n n^{2c\alpha h'(\theta^*)} = O(\varepsilon_n n^\epsilon),
\end{aligned}$$

which finally yields, by choosing α small enough, a rate

$$\forall \epsilon > 0, \quad \theta_n - \theta^* = O(\varepsilon_n n^\epsilon).$$

(a') In the sub-case of (a), the assumption $\varepsilon_n n^{c(1+\alpha)h'(\theta^*)} \underset{n\to+\infty}{\longrightarrow} 0$ implies that $\frac{\varepsilon_n}{\Pi_n} \underset{n\to+\infty}{\longrightarrow} 0$. Plugging this in (3.6) yields the following more precise result

$$\frac{z_n}{\Pi_n} \underset{n\to+\infty}{\longrightarrow} \text{finite limit}$$

But if one plugs the rate $o\left(n^{-ch'(\theta^*)+\epsilon}\right)$ into the expression of Π_n, one can control the behaviour of the sequence $(\widetilde{\omega}_n)_{n\geq 1}$. In fact

$$\forall \epsilon > 0, \quad \widetilde{\omega}_n = O\left(n^{-\delta-ch'(\theta^*)+\epsilon}\right), \quad \delta > 0,$$

so that $\sum_{n\geq 1} \gamma_n \widetilde{\omega}_n$ converges, which in turn should imply by tracing the above proof that

$$K_1' n^{ch'(\theta^*)} \leq \frac{\Pi_{n_0}}{\Pi_n} \leq K_1 n^{ch'(\theta^*)},$$

which finally yields

$$\left(n^{ch'(\theta^*)}(\theta_n - \theta^*)\right)_{n\geq 0} \quad \text{is a bounded sequence.}$$

2. Second Case: $\gamma_n = \dfrac{c}{n^\rho}$, $c > 0$, $1/2 < \rho < 1$. We have to evaluate the convergence rate of

$$\sum_{k=1}^{n-1} k\varepsilon_k \left(|\Delta\gamma_{k+1}| + \gamma_k\gamma_{k+1}\left[\partial_\theta H(\theta^*, \xi_{k+1}) + \psi(\theta_k, \xi_{k+1})\,|z_k|^\eta\, s_k\right]\right)\Pi_{k+1}^{-1}.$$

Because of the representation of the gain parameter, we have $\Delta\gamma_k \asymp -\rho k^{-\rho-1}$, $\gamma_k\gamma_{k-1} \asymp k^{-2\rho}$, so that

$$|\Delta\gamma_k| + \gamma_k\gamma_{k-1} \asymp k^{-2\rho}.$$

Now, going back to (3.7) one derives that

$$\log\left(\frac{\Pi_0}{\Pi_n}\right) = (\Gamma_n + o(\Gamma_n))\, h'(\theta^*) + L_\infty + \eta_n,$$

where $\eta_n \underset{n\to+\infty}{\longrightarrow} 0$, since the gain parameter satisfies (3.5). Now

$$\Gamma_n = c\sum_{k=1}^{n} k^{-\rho} \asymp \frac{c}{1-\rho} n^{1-\rho}.$$

We are concerned with the rate of convergence of

$$\Pi_n \sum_{k=1}^{n} \varepsilon_k k^{1-2\rho} \Pi_{k+1}^{-1}.$$

Elementary computations show (using (3.7)) that

$$\frac{\Pi_n}{\Pi_{k+1}} \leq \exp\left(-C_{\rho,c} \sum_{\ell=k+2}^{n} \ell^{-\rho}\right) =: \frac{A_{k+1}}{A_n}$$

so that

$$\Pi_n \sum_{k=1}^{n-1} \varepsilon_k k^{1-2\rho} \Pi_{k+1}^{-1} \leq \frac{1}{A_n} \sum_{k=1}^{n-1} \varepsilon_k k^{1-2\rho} A_{k+1}.$$

Now,

$$A_{k+1} - A_k = A_k \left(e^{K(k+1)^{-\rho}} - 1 \right) \geq K A_k (k+1)^{-\rho} \geq K^{'} A_k k^{-\rho}.$$

Hence, for any $\epsilon > 0$, we have

$$\begin{aligned}
\Pi_n \sum_{k=1}^{n-1} \varepsilon_k k^{1-2\rho} \Pi_{k+1}^{-1} &\leq \frac{1}{K' A_n} \sum_{k=1}^{n-1} \varepsilon_k k^{1-\rho} \left(A_{k+2} - A_{k+1} \right) \\
&\approx \frac{1}{K' A_n} \left[A_{n+1} \varepsilon_n n^{1-\rho} + K_2 \sum_{k=1}^{n} \varepsilon_k k^{-\rho} A_{k+1} \right] \\
&\approx \frac{1}{K'} \left[\varepsilon_n n^{1-\rho} + \frac{K_2 \varepsilon_n n^{1-\rho+\epsilon}}{A_n \varepsilon_n n^{1-\rho+\epsilon}} \sum_{k=1}^{n} \varepsilon_k k^{1-\rho+\epsilon} A_{k+1} \frac{1}{k^{1+\epsilon}} \right].
\end{aligned}$$

$\sum_{k \geq 1} \frac{1}{k^{1+\epsilon}} < +\infty$ and Kronecker Lemma implies that

$$\Pi_n \sum_{k=1}^{n} \varepsilon_k k^{1-2\rho} \Pi_{k+1}^{-1} = o\left(\varepsilon_n n^{1-\rho+\epsilon} \right).$$

Consequently, the conclusion to the second case is

$$\forall \epsilon > 0, \quad z_n = o(\varepsilon_n n^{1-\rho+\epsilon}). \qquad \square$$

Remark. In a "probabilistic" framework, *i.e.* where $(\xi_n)_{n \geq 1}$ is an i.i.d. sequence, ε_n is typically equal to $\sqrt{\frac{\log \log n}{n}}$ (from the LIL). Then (3.8) reads

$$\exists\, \alpha > 0 \text{ such that } \sum_{n \geq 1} \sqrt{\log \log n} n^{-3/2 + c(1+\alpha) h'(\theta^*)} < +\infty,$$

which holds true as soon as $-3/2 + c(1+\alpha) h'(\theta^*) < -1$, *i.e.*, since α is arbitrary, $c h'(\theta^*) < 1/2$ which is the usual critical level for c.

3.3 Application to parametric averaging rate

3.3.1 I.i.d. framework

Assume that the sequence $(\xi_n)_{n \geq 1}$ is i.i.d. to compare our results to the i.i.d. setting for stochastic approximation. Then, by the Law of Iterated Logarithm (LIL), we have that the averaging rate is $\varepsilon_n = \sqrt{\frac{\log \log n}{n}}$. Consequently we obtain the following results for convergence rate:

- Assume that $\gamma_n = \frac{c}{n}$, $c > 0$.
  - If $c < \frac{1}{2h'(\theta^*)}$, then $\theta_n - \theta^* = o\left(n^{-c h'(\theta^*) + \epsilon} \right)$, $\epsilon > 0$.
  - If $c \geq \frac{1}{2h'(\theta^*)}$, then $\theta_n - \theta^* = o\left(\sqrt{\frac{\log \log n}{n}} n^{\epsilon} \right) = o\left(n^{-\frac{1}{2} + \epsilon} \right)$, $\epsilon > 0$.

  Notice that we find the same critical level for the constant c: indeed, if $c > \frac{1}{2h'(\theta^*)}$, a CLT holds for the algorithm (see [44]).
- Assume that $\gamma_n = \frac{c}{n^{\rho}}$, $c > 0$, $\frac{1}{2} < \rho < 1$. Then $\theta_n - \theta^* = o\left(\sqrt{\log \log n} n^{1/2 - \rho + \epsilon} \right)$, $\epsilon > 0$.

3.3.2 Low discrepancy sequences framework

Assume that the sequence $(\xi_n)_{n\geq 1}$ is uniformly distributed with low discrepancy. Then, by Koksma-Hlawka inequality (if $H(\theta^*,\cdot)$ or $\partial_\theta H(\theta^*,\cdot)$ has finite variation) or by Proinov Theorem (if $H(\theta^*,\cdot)$ or $\partial_\theta H(\theta^*,\cdot)$ are Lipschitz continuous), we have that the averaging rate is $\varepsilon_n = \frac{\log n}{n}$. Consequently we obtain the following results for convergence rate:

- Assume that $\gamma_n = \frac{c}{n}$, $c > 0$.
  - If $c < \frac{1}{h'(\theta^*)}$, then $\theta_n - \theta^* = o\left(n^{-ch'(\theta^*)+\epsilon}\right)$, $\epsilon > 0$.
  - If $c \geq \frac{1}{h'(\theta^*)}$, then $\theta_n - \theta^* = o\left(\frac{\log n}{n} n^{\epsilon}\right) = o\left(n^{-1+\epsilon}\right)$, $\epsilon > 0$.
- Assume that $\gamma_n = \frac{c}{n^\rho}$, $c > 0$, $\frac{1}{2} < \rho < 1$. Then $\theta_n - \theta^* = o\left(\log n n^{-\rho+\epsilon}\right)$, $\epsilon > 0$.

3.4 Numerical Example

In this section we give an example of application in Finance to show the improvement of rate of convergence by using quasi-random numbers instead of pseudo-random numbers. The details of the implicitation procedure are given in Chapter 6, so we just pose the problem without technical points.

3.4.1 Application to implicit correlation search

Consider a 2-dimensional Black-Scholes model *i.e.* $X_0^t = e^{rt}$ (riskless asset) and

$$X_t^i = x_0^i e^{(r-\frac{\sigma_i^2}{2})t+\sigma_i W_t^i}, \quad x_0^i > 0, \quad i = 1,2,$$

for the two risky assets where $\left\langle W^1, W^2 \right\rangle_t = \rho t$, $\rho \in [-1,1]$. Consider a best-of call option characterized by its payoff

$$\left(\max\left(X_T^1, X_T^2\right) - K\right)_+.$$

We will use a stochastic recursive procedure to solve the inverse problem in ρ

$$P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) = P_0^{market}$$

where P_0^{market} is the quoted premium of the option (mark-to-market) with

$$\begin{aligned} P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) &:= e^{-rT}\mathbb{E}\left[\left(\max\left(X_T^1, X_T^2\right) - K\right)_+\right] \\ &= e^{-rT}\mathbb{E}\left[\left(\max\left(x_0^1 e^{\mu_1 T+\sigma_1\sqrt{T}Z^1}, x_0^2 e^{\mu_2 T+\sigma_2\sqrt{T}(\rho Z^1+\sqrt{1-\rho^2}Z^2)}\right) - K\right)_+\right] \end{aligned}$$

where $\mu_i = r - \frac{\sigma_i^2}{2}$, $i = 1,2$, $Z = (Z^1, Z^2) \overset{(d)}{=} \mathcal{N}(0, I_2)$. We use a trigonometric parametrization by setting $\rho = \cos\theta$, $\theta \in \mathbb{R}$. Then we devise the following stochastic recursive procedure

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Z_{n+1}), \quad \theta_0 \in \mathbb{R}, \quad Z_{n+1} \overset{d}{=} \mathcal{N}(0, I_2),$$

where

$$H(\theta, Z) = e^{-rT}\mathbb{E}\left[\left(\max\left(x_0^1 e^{\mu_1 T+\sigma_1\sqrt{T}Z^1}, x_0^2 e^{\mu_2 T+\sigma_2\sqrt{T}(\cos\theta Z^1+\sin\theta Z^2)}\right) - K\right)_+\right] - P_0^{market}.$$

Numerical experiment. We set the model parameters to the following values

$$x_0^1 = x_0^2 = 100,\ r = 0.10,\ \sigma_1 = \sigma_2 = 0.30,\ \rho = -0.50$$

and the payoff parameters

$$T = 1,\ K = 100.$$

The implicit correlation search recursive procedure is implemented with a sequence of some quasi-random normal numbers, namely

$$(\zeta_n^1, \zeta_n^2) = \left(\sqrt{-2\log(\xi_n^1)}\sin\left(2\pi\xi_n^2\right), \sqrt{-2\log(\xi_n^1)}\cos\left(2\pi\xi_n^2\right)\right),$$

where $\xi_n = (\xi_n^1, \xi_n^2),\ n \geq 1$, is simply a regular 2-dimensional Halton sequence.

The reference Black-Scholes price 30.75 is used as a market price so that the target of the stochastic algorithm is $\theta^* \in \arccos(-0.5)$. The stochastic approximation procedure parameters are

$$\theta_0 = 0,\ n = 10^6.$$

The choice of θ_0 is blind on purpose. Finally we set $\gamma_n = \frac{8}{n}$.

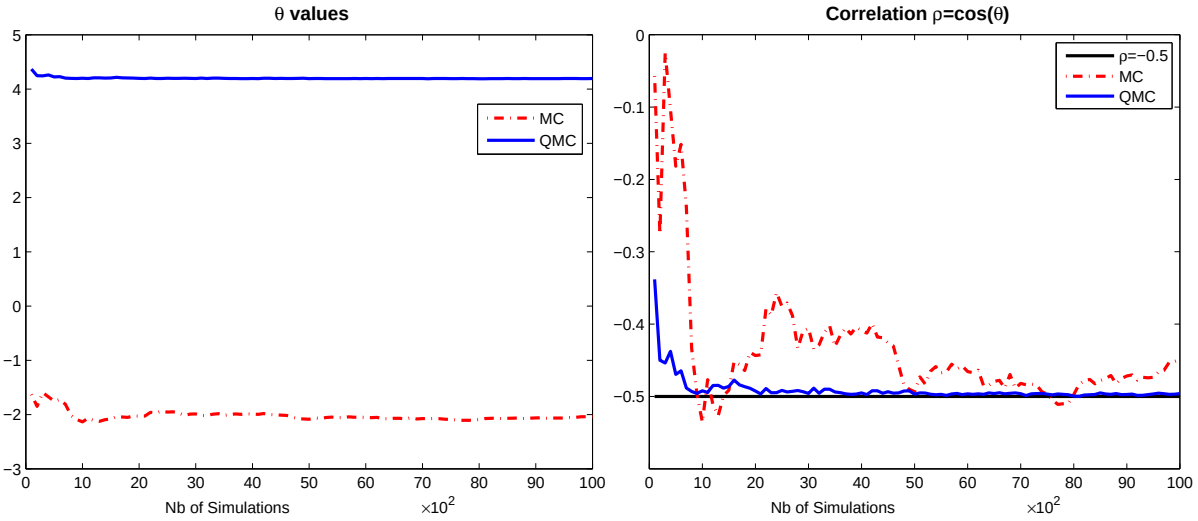

Figure 3.1: B-S Best-of-Call option. $T = 1$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.30$, $x_0^1 = x_0^2 = 100$, $K = 100$. Left: convergence of θ_n toward a θ^*. (up to $n = 10000$). Right: convergence of $\rho_n := cos(\theta_n)$ toward -0.5

By simulation we approximate the value of the derivative at the equilibrium point and we obtain that $h'(\theta^*) \approx 4.04$.

▷ If we set $\gamma_n = \frac{c}{n}$, $c < \frac{1}{2h'(\theta^*)}$, then the rate of convergence is $n^{-ch'(\theta^*)+\epsilon}$, $\epsilon > 0$. We compare the rate of convergence of MC and QMC simulations.

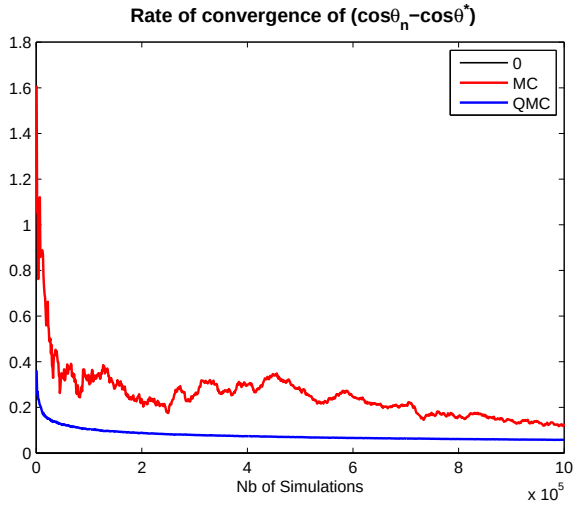

Figure 3.2: Convergence of $n^{-ch'(\theta^*)+\epsilon}(\cos(\theta_n) + 0.5)$, $c = 0.12$, $\epsilon = 0.2$.

In this case, the rate obtained for QMC is better because we need that $c < \frac{1}{h'(\theta^*)}$. We obtain the following result for QMC in this case.

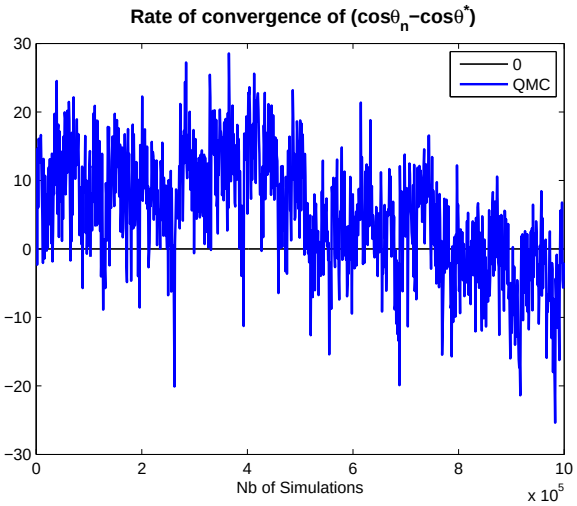

Figure 3.3: Convergence of $n^{-ch'(\theta^*)+\epsilon}(\cos(\theta_n) + 0.5)$, $c = 0.24$, $\epsilon = 0.01$.

$\triangleright$ If we set $\gamma_n = \frac{1}{n}$, then the rate of convergence is $\varepsilon_n n^{\epsilon}$, $\epsilon > 0$. We compare the rate of convergence of MC and QMC simulations for $\epsilon_n = \sqrt{\frac{\log\log n}{n}}$.

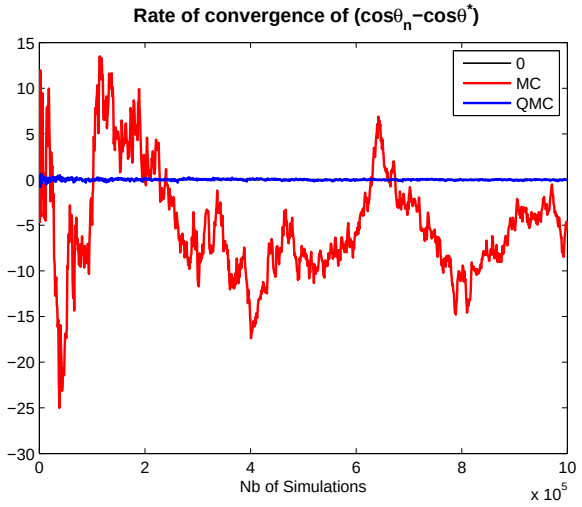

Figure 3.4: Convergence of $n^{\frac{1}{2}-\epsilon}(\cos(\theta_n) + 0.5)$, $\epsilon = 0.01$.

In this case, the rate obtained for QMC depends on $\epsilon_n = \frac{\log n}{n}$. We obtain the following result for QMC in this case.

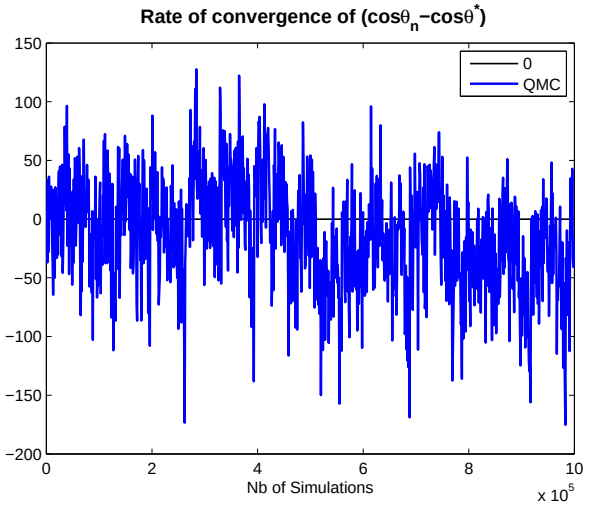

Figure 3.5: Convergence of $n^{1-\epsilon}(\cos(\theta_n) + 0.5)$, $\epsilon = 0.01$.

$\triangleright$ If we set $\gamma_n = \frac{1}{n^\rho}$, $1/2\rho < 1$, then the rate of convergence is $\varepsilon_n n^{1-\rho+\epsilon}$, $\epsilon > 0$. We compare the rate of convergence of MC and QMC simulations for $\epsilon_n = \sqrt{\frac{\log\log n}{n}}$.

Case $\rho = 0.6$

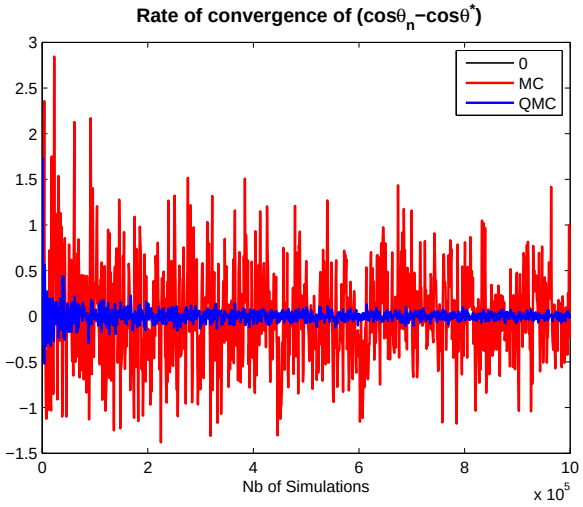

Figure 3.6: Convergence of $n^{\rho-\frac{1}{2}-\epsilon}(\cos(\theta_n) + 0.5)$, $\epsilon = 0.01$.

In this case, the rate obtained for QMC depends on $\epsilon_n = \frac{\log n}{n}$. We obtain the following result for QMC in this case.

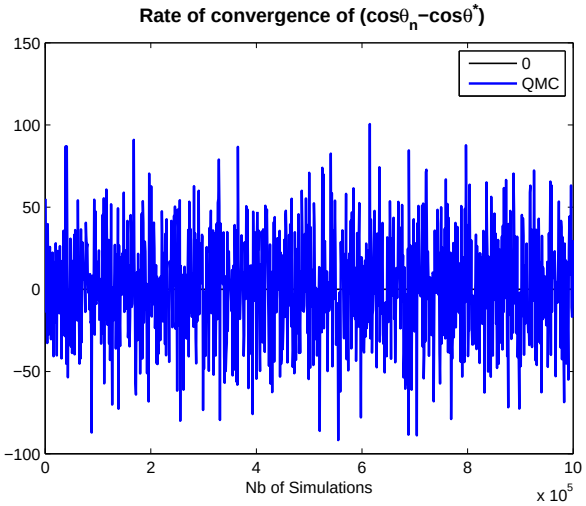

Figure 3.7: Convergence of $n^{\rho-\epsilon}(\cos(\theta_n) + 0.5)$, $\epsilon = 0.01$.

Case $\rho = 0.9$

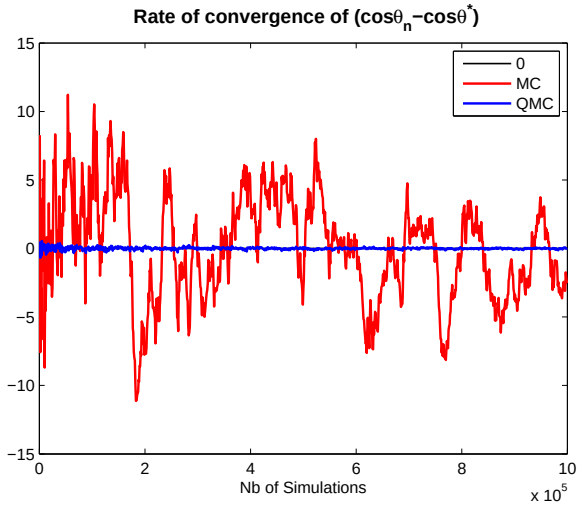

Figure 3.8: Convergence of $n^{\rho-\frac{1}{2}-\epsilon}(\cos(\theta_n)+0.5)$, $\epsilon = 0.01$.

In this case, the rate obtained for QMC depends on $\epsilon_n = \frac{\log n}{n} n$. We obtain the following result for QMC in this case.

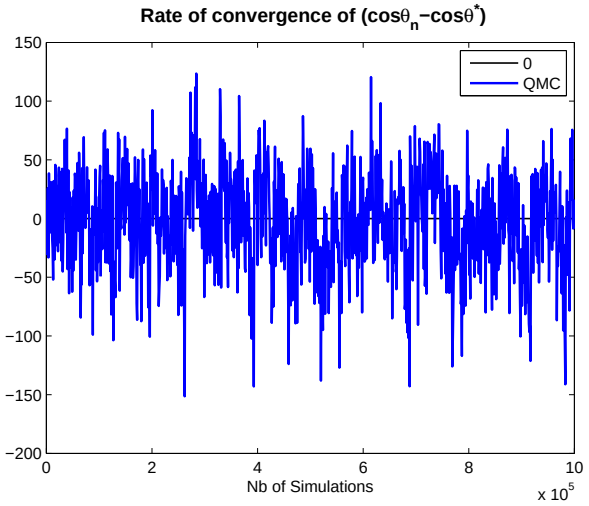

Figure 3.9: Convergence of $n^{\rho-\epsilon}(\cos(\theta_n)+0.5)$, $\epsilon = 0.01$.

Deuxième partie

Applications à la Finance et aux essais cliniques

Chapter 4

Optimal split of orders across liquidity pools: a stochastic algorithm approach

This chapter is based on a paper accepted for publication in *SIAM Journal on Financial Mathematics*.

Evolutions of the trading landscape lead to the capability to exchange the same financial instrument on different venues. Because of liquidity issues, the trading firms split large orders across several trading destinations to optimize their execution. To solve this problem we devised two stochastic recursive learning procedures which adjust the proportions of the order to be sent to the different venues, one based on an optimization principle, the other on some reinforcement ideas. Both procedures are investigated from a theoretical point of view: we prove *a.s.* convergence of the optimization algorithm under some light ergodic (or "averaging") assumption on the input data process. No Markov property is needed. When the inputs are i.i.d. we show that the convergence rate is ruled by a Central Limit Theorem. A variant including some market impact effect is also proposed. Finally, the mutual performances of both algorithms are compared on simulated and real data with respect to an "oracle" strategy devised by an "insider" who *a priori* knows the executed quantities by every venues.

4.1 Introduction

The trading landscape have seen a large number of evolutions following two regulations: Reg NMS in the US and MiFID in Europe. One of their consequences is the capability to exchange the same financial instrument on different trading venues. New trading destinations appeared to complement the trading capability of primary markets like the NASDAQ and the NYSE in the US, or EURONEXT, the London Stock Exchange and Xetra in Europe. Such alternative venues are called "Electronic Communication Network" (ECN) in the US and Multilateral Trading Facilities (MTF) in Europe. Each trading venue differentiates from the others at any time because of the fees or rebate it demands to trade and the behaviour of the liquidity it offers.

From a regulatory viewpoint, these changes have been driven by a run for quality for the price formation process. Economists have studied optimal liquidity supply and demand policies for long (see for instance [37]) and their findings took part in the recent regulatory evolutions. This chapter will not address the impact of the optimal behaviour of each agent on the efficiency of the market micro-structure (like in [86]). We will focus here on the optimality of the process of liquidity supply and demand in Dark Pools of one isolated agent with respect to its own criteria.

Since the concerns about liquidity consumption increased with the financial crisis, trading firms use Smart Order Routers (SOR) like a key element in the process of optimizing their execution of

large orders. Such devices are dedicated to split orders across trading destinations as a complement to the temporal slicing coming from the well-known balance between trading rapidly (to minimize market risk) and trading slowly (to avoid market impact).

If the temporal slicing has been studied since the end of the nineties [9] with recent advances to adapt it to sophisticated investment strategies [85], this kind of spatial slicing (across trading destinations) has been mainly studied by economists from the point of view of its global market efficiency [51] rather than from one investor's point of view.

The complexity of spreading an order between N trading destinations comes from the fact that you never know the quantity D_i available on the i^{th} trading venue to execute your order of size V during a time interval Δt at your given price. If the fraction $r_i V$ of the order that you sent to the i^{th} liquidity pool is higher than D_i: you will loose time and the opportunity to execute $r_i V - D_i$ in an another pool; on the other hand, if $r_i V$ is lower than D_i: you will loose money if this pool is too expensive (for a buy order), and an opportunity to execute more quantity there (if cheap). The only way to optimize such a split in real time is to adjust on the fly the proportions $(r_i)_{1 \leq i \leq N}$ according to the result of your previous executions.

This chapter offers a partial answer to the optimal split of an order across several trading venues, and especially "dark pools". Some existing frameworks for optimal trading are built on the top of explicit "*market impact*" models linking the volume of a liquidity consuming order to a move of the market price. The Almgren-Chriss [9] framework is typical: two market impact models are proposed, one for the "*permanent market impact*", which is usually considered to be linear and capture the long term price moves of the traded instrument following the controlled trade, and another for the "*immediate impact*" which is a statistical estimate of the usual "*slippage*" of an order of this volume with respect to the liquidity of the traded instrument and its volatility. The rule of thumb used by traders for decades is to consider that the immediate impact is in square root of the volume (see [111]). Besides, a microscopic model of the immediate market impact is proposed in [6] and [100]; it relies on the shape of the order book and the nature of its resilience rather than on an expected slippage in a statistical sense. The immediate market impact is deduced from the proposed order book dynamics that way: if the volume at the ask price S is $\eta(S)$ and the best ask price is S_a, then a buy order "at any price" of V shares will move the price by δS such that $V = \int_{S_a}^{S_a+\delta S} \eta(S)\, dS$. This gives an implicit formula for the immediate market impact $\delta_S(V)$ of an order of size V. This quantitative literature on market impact models relies on studies conducted for long by economists, from Kyle [76] who proposed in the 1980's a linear model, to Engle, Ferstenberg and Russell [45] who conducted an in-depth econometric study to link the market impact with market context variables.

Practically, a trading algorithm is sending to the market liquidity consuming and liquidity providing orders. On the one hand the formers obviously move the price as far as their size is greater than the first limit of the market according to a mechanism close to the one just described. On the other hand the effect of liquidity providing (*i.e.* passive) orders is not that clear. One could argue that providing too much liquidity could create a "*resistance*" to the price moves that is close to what should be called "*implicit market impact*", others could answer that as far as the orders are "small enough", they are not constraining the price formation process at all. The combined effect of liquidity consuming and liquidity providing orders is better understood and modeled today than the effect of interacting with Dark Pool order-books. For instance, in Gatheral's no-dynamic arbitrage framework [57] the liquidity consuming orders can be seen as the first ones moving immediately the price whereas the liquidity providing ones are inducing decay to the order-books dynamics. Impact of orders traded on multiple pools has been also studied by economists, using huge data bases like in [92], where Menkveld captures statistical properties of the impact of such trading, or more qualitatively using stylized models to obtain conclusions on the efficiency of fragmented trading (see for instance Amihud and Mendelson [10]). At this stage no model is available at the very short term needed by our framework, we will consequently assume that "our orders are small enough" not to

disturb too much the role of price formation process in the considered cost function. The optimal trading framework proposed for market makers by Avellaneda and Stoikov [16] can be taken as an illustration of a model for passive orders. Since market makers are mainly contributing to the price formation process via passive (*i.e.* liquidity providing) orders, no market impact is modeled. The moves of the price, which follows a Brownian motion, are not constrained at all by the fact that the market maker provides from time to time "too much" liquidity to the market. The obtained results for market making are nevertheless quite interesting and realistic, by contrast to the results obtained via an explicit microscopic market impact model based on order book which associated optimal trading curve can oscillate in an unrealistic manner as exposed in [7].

As concerns the modeling of the market behaviour part, the starting point of this chapter is close to Avellaneda and Stoikov's approach, but at a smaller time scale because we will consider only one "*child order*" coming for instance from their optimal market making strategy. Instead of a flow of liquidity consuming orders model using a point process which intensity is a power law of the distance to the mid point, we will consider a general case of a random variable of "*accessible liquidity*" D_i^t on the trading pool number i at time t. An order of size V^t posted on this pool will consequently obtain a trade of size $\min(D_i^t, V^t)$. In their framework D_i^t would be written $N_i^{t+\Delta t} - N_i^t$ where N_i is the point process modeling the trading flow "hitting" passive order of size V^t exposed to the trading pool during Δt. Typically we have in mind very short resting time Δt or even $\Delta t = 0$ if practitioners only consider "*Immediate Or Cancel*" (IOC) orders. IOCs are sent to a trading pool to instantaneously capture all the liquidity available within a price range, and the remaining quantity is immediately canceled by the destination market.

Because we are focusing on the optimal splitting tactic and not on the strategy generating the orders to split, we will model the price change between the time the order is sent to the trading pools i and the time it comes back *via* a multiplier of the price: the shares are bought at price S_t^i on the trading pool i such that $\mathbb{E}(S_t^i \,|\, \mathcal{F}_t) = \theta_i S_t$ where S_t is the price on the "last resort" reference (or primary) pool at time t (and $\mathcal{F}_t$ is the available information on this market, see the case of dark pools below). This parameter θ_i can be seen as the impact of the elapsed time before execution but also as a kind of "rebate" (like on dark pools) for IOC orders.

In such a context, this chapter is an in depth analysis of the optimal immediate split of small orders and not an optimal scheduling of a large order. The laws followed by the price and volumes are kept as generic as possible. Note that, at the end of the chapter, we propose a slight variant of our approach that takes into account a possible market impact resulting from the execution on the reference pool.

The illustrations and most of the vocabulary come from the "*Dark pool*" case, where the price S is not chosen by the trader (it is the "mid price" of an external reference market, like in most European Dark Pools, but can more generally be seen as any price inside the bid-ask spread, different for each Dark Pool with respect to its liquidity offer). IOC orders for which the answer of the pool is immediate correspond to a resting time $\Delta t = 0$. More generally, these results apply to the case of Dark pools that do not publish pre-trade informations (*i.e.* the accessible liquidity D_i^t is never observed), so an efficient use of the results of the previous executions (namely the realizations of the $\min(D_i^t, r_i^t V^t)$ for any i and all t in the past) is crucial. *The results exposed here solve the problem of simultaneously splitting orders and using the information coming back from the pools to adjust the proportions to send for the next order*, according to a criterion linked to the overall executed quantity (*i.e.* a linear combination of the $\min(D_i, r_i V)$).

For Dark Pools (when no public information is available before sending an order to the trading pool) or when the information provided by a snapshot is too blurred by high frequency updates to be taken as reliable, it is mandatory to estimate available quantities and to optimize the trading and splitting tactics accordingly. In a white paper Almgren (see [8]) proposed a way to estimate the hidden quantities in an order-book using a reinforcement scheme. In a technical report Ganchev et

al. (see [55]) use a two steps approach: first online estimate the available quantities in each Dark Pool adapting well-known censored statistics results, then optimize the split of an independent incoming order across the pools using the statistical properties of the obtained estimates. Here we propose an integrated approach: as usual for stochastic algorithms, the estimation and optimization steps are intricate and simultaneous.

The resulting trading tactic (which optimality is proven here) can be considered as the best choice to apply to child orders that are generated using an optimal scheduling method like in [8] and [31]. It may also be related to the class of multi-armed bandit recursive learning procedures, recently brought back to light in several papers (see [80, 109], [78, 79]; which in turn belongs to the wide family of "recursive stochastic algorithms" also known as "stochastic approximation" and extensively investigated in the applied probability literature (see [75], [25], [43], etc)).

In fact, we introduce two learning algorithms one based on an optimization under constraints principle and a second algorithm based on a reinforcement principle for which we establish the existence of an equilibrium. We extensively investigate the first one, considering successively the classical – although unrealistic – case where the inputs (requests, answers) are i.i.d. and a setting in which the input only share some averaging properties. In the i.i.d. setting we establish $a.s.$ convergence of the procedure and a Central Limit Theorem relying on classical results from Stochastic Approximation Theory. By *averaging setting* (also referred as *ergodic setting*), we mean that the inputs of the procedure has $a.s.$ an averaging property with respect to a distribution ν at a given rate, say $n^{-\beta}$, $0 < \beta \leq 1$, for a wide enough class of Borel functions $\mathcal{V}_{\beta,p}$. Typically, in our problem, these inputs are the successive $N+1$-tuples $(V^n, D_i^n, i = 1, \ldots, N)$, $n \geq 1$. Typically, if we denote this input sequence inputs by $(Y_n)_{n\geq 1}$, we will assume that, for every $f \in \mathcal{V}_{\beta,p}$,

$$\frac{1}{n}\sum_{k=1}^{n} f(Y_k) - \int_{\mathbb{R}_+^{N+1}} f d\nu = O(n^{-\beta}) \quad \mathbb{P}\text{-}a.s \text{ and in } L^p(\mathbb{P}).$$

Usually, $\mathcal{V}_{\beta,p}$ is supposed, at least for $\beta \in (0, 1/2)$, to contain at least bounded continuous function $g : \mathbb{R}_+^{N+1} \to \mathbb{R}$ and subsequently all bounded ν-$a.s.$ continuous functions. This will be enough for our purpose in this chapter. But the key point to be noted here is that *no Markov assumption is needed on this input sequence* $(Y_n)_{n\geq 1}$. These assumptions are hopefully light enough to be satisfied by real data since it can be seen as a kind of "light" ergodicity at a given rate. In a Markovian framework it could be related to the notion of "stability" in the literature, see [43].

Thus, this setting includes stationary α-mixing processes (satisfying an Ibragimov condition) like those investigated in [42] (in [39] weaker dependence assumptions are made in the chapter devoted to stochastic approximation but the perturbation is supposed to be additive and non causal which is not at all the case in our problem). As concerns the second procedure for which no Lyapunov function seems to be (easily) made available, we establish the existence of an equilibrium and show the ODE related to the algorithm is a competitive system in the terminology of monotonous differential systems extensively studied by Hirsch et al. (see *e.g.* [65]). The behaviour of such competitive systems is known to be the most challenging, even when the equilibrium point is unique (which is not the case here).

Both procedures are compared in the final section, using simulated and real data. Further numerical tests and applications are ongoing works in CA Cheuvreux.

The chapter is organized as follows: in Section 4.2, we make precise the modeling of splitting orders among several venues in the framework of *Dark pools*, first in static then in a dynamic way. This leads to an optimization problem under constraints. In Section 4.3, we study the execution function of one dark pool and introduce the recursive stochastic algorithm resulting from the optimization problem. In Section 4.4 we analyze in depth this algorithm ($a.s.$ convergence and weak rate) when the "innovations" (data related to the orders, the executed quantities and the market

price) are assumed i.i.d. In Section 4.5 we extend the *a.s.* convergence result to a more realistic framework where these innovations are supposed to share some appropriate averaging properties (*e.g.* satisfied by α-mixing processes satisfying Ibragimov's condition). Section 4.6 is devoted to the second learning procedure, based this time on reinforcement principle, introduced in [28]. We make a connection with the theory of (competitive) monotonous dynamical systems. Finally, in Section 4.7, we present several simulations results on simulated and real data to evaluate the performances of both procedures with respect to an "oracle" strategy of an "insider" who could know *a priori* the executed quantities by every dark pool. In Section 4.8, we first propose a variant of the optimal splitting procedure that take into a account the market impact on the reference primary pool when completing the execution of the stock of shares.We conclude by showing how to take advantage of two procedures with close performances to make a dynamic calibration.

NOTATIONS:

- For every $N \geq 1$, set $\mathcal{I}_N := \{1, 2, \ldots, N\}$, $\mathcal{P}_N := \{r = (r_i)_{1\leq i\leq n} \in \mathbb{R}_+^N \mid \sum_{i=1}^N r_i = 1\}$. Let $\mathbf{1}^\perp := \{u \in \mathbb{R}^N \mid \sum_{i\in\mathcal{I}_N} u_i = 0\}$.
- δ_{ij} denotes the Kronecker symbol.
- $\langle . | . \rangle$ denotes the canonical inner product on $\mathbb{R}^d$ and $|\,.\,|$ the derived Euclidean norm.
- $int(A)$ denotes the interior of a subset A of $\mathbb{R}^d$.
- δ_a denotes the Dirac mass at $a \in \mathbb{R}^d$.

4.2 A simple model for the execution of orders by dark pools

4.2.1 The execution model

As mentioned in the introduction, we will focus in this chapter on the splitting order problem in the case of (competing) *dark pools*. The execution policy of a dark pool differs from a primary market: thus a dark pool proposes bid/ask prices with no guarantee of executed quantity at the occasion of an over the counter transaction. Its bid and ask prices are slightly different from those offered on the primary market (practically, it is often a small modification of the mid-price). Let us temporarily focus on a buy order to be split across several dark pools. We do not aim at specifying a given dynamics for the bid and ask prices, but just to make some reasonable generic assumptions. We mean assumptions not too far from reality which makes possible to connect the problem with a stochastic optimization that can be solved *on line* (hence in a forward way).

One can model the impact of the existence of N dark pools ($N \geq 2$) on a given transaction started at time t as follows: let $V(t) > 0$ be the random volume to be executed between time t and $t + \Delta t$. If S_t denotes the (bid) price of the asset at time t on the primary market and S_t^i the one of dark pool i, we assume that

$$(H_1) \equiv \forall\, t \geq 0, \quad \mathbb{E}(S_t^i \,|\, S_t) = \theta_i S_t,\ i = 1, \ldots, N.$$

where $\theta_i \in (0, 1)$ can be seen as the mean *discount factor* proposed by the dark pool $i \in \{1, \ldots, N\}$ with respect to the best opposite. Let $r_i(t)$ denote the percentage of $V(t)$ sent to the dark pool i for execution and let $D_i(t) \geq 0$ be the quantity of securities that can be delivered (or made available) by the dark pool i at price S_t^i. In case of of a remaining quantity on the order, what is left is sent at time $t + \Delta t$ for aggressive execution on the primary market, at price $S_{t+\Delta t}$. Then the total cost $C(t)$ of the executed order is given by

$$C(t) \;=\; \sum_{i=1}^N S_t^i \min(r_i(t)V, D_i(t)) + S_{t+\Delta t}\left(V(t) - \sum_{i=1}^N \min(r_i(t)V(t), D_i(t))\right).$$

The second assumption is that, at the scale we consider (say around 10 minutes with an execution every few seconds):

$(H_2) \equiv$ The price process $(S_t)_t$ and $(V(t), D_1(t), \ldots, D_{_N}(t))_t$ are independent,

$(H_3) \equiv$ The process $(V(t), D_1(t), \ldots, D_{_N}(t))_t$ is stationary with marginal distribution ν.

(In practice it is enough for this process to be mean-reverting and to converge toward a steady regime). We will also need in the next section that it shares some ν-ergodic/averaging property under this steady regime.

Finally we assume that $(S_t)_t$ is either a martingale or more generally

$(H_4) \equiv \forall\, t \geq 0, \quad \mathbb{E}\big(S_t \,|\, \mathcal{F}_s^S\big) = a_{t-s} S_s$, $a_u \in (0, \infty)$. where a_u satisfies $\lim_{u \to 0} a_u = 1$. (Typically, one may think at $a_u =$ as a discount factor like $e^{-\mu u}$).

Then elementary computations show that

$$\mathbb{E}(C_t) = \mathbb{E}(S_{t+\Delta t})\mathbb{E}(V(t)) - a_{\Delta t}\mathbb{E}(S_t) \sum_{i=1}^{N} \Big(1 - \frac{\theta_i}{a_{\Delta t}}\Big)\mathbb{E}\Big(\min(r_i(t)V(t), D_i(t))\Big).$$

At this stage, one may wish to minimize the mean execution cost $\mathbb{E}(C_t)$. Taking into account the stationarity assumption (H_3), this amounts to solving the following (conditional) maximization problem

$$\max \left\{ \sum_{i=1}^{N} \rho_i \mathbb{E}\left[\min(r_i V, D_i)\right],\ r \in \mathcal{P}_{_N} \right\}. \tag{4.1}$$

where the expectation is taken with respect to the marginal distribution ν and $\rho_i = 1 - \frac{\theta_i}{a_{\Delta t}}$, $i = 1, \ldots, N$. If one of the ρ_i is negative, dark pool i is out of the game and the problem is ill-posed. So we make the assumption that Δt is chosen small enough so that

$$\max_{1 \leq i \leq N} \theta_i < a_{\Delta t}$$

which implies in turn that $\rho_i > 0$, $i = 1, \ldots, N$.

If one considers symmetrically a sell order to be executed, the dark pool is supposed to propose a higher price $\theta_i S$, $\theta_i > 1$, than the best opposite of the visible order book. The seller aims at maximizing the execution global (mean) price of the transaction. This leads to the same formal optimization problem, this time with $\rho_i = \theta_i - a_{\Delta t}$, $i = 1, \ldots, N$, assumed to be all positive which requires the assumption $\min_i \theta_i > a_{\Delta_t}$.

In both cases, the restriction on Δt means that, if $a_u \leq 1$ where $u \to 0$, one must complete the execution of the order fast enough to limit the exposure to the market risk in order to take advantage of dark pool executions.

All these considerations lead us to focus on the abstract optimal allocation problem (4.1) in which the price variable S no longer appears in what follows. In fact this is a *sine qua non* condition to define an *on line* approach to the optimal splitting based on a learning procedure.

4.2.2 The dynamical aspect

In practice, there is no *a priori* assumption – or information available – on the joint distribution of $(V, D_1, \ldots, D_N)$ under $\mathbb{P}$. So the only reasonable way to provide a procedure to solve this allocation problem is to devise an *on-line learning algorithm based on historical data*, namely the results of former transactions with the dark pools on this security executed in the past. This underlines that our agent dealing with the dark pools is a financial institution like an investment bank, a broker or possibly a large investor which often – that means at least daily – faces some large scale execution problems on the same securities.

This means that we will have to make some assumptions on the dynamics of these transactions *i.e.* on the data input sequence $(V^n, D_1^n, \ldots, D_N^n)_{n\geq 1}$ supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (where superscript n stands for an execution launched at time t_n (to be consistent we assume that the execution are made successively that is $t_n > t_{n-1} + \Delta t$).

Our basic assumption on the sequence $(D_i^n, V^n, i = 1, \leq, N)_{n\geq 1}$ is of statistical – or ergodic – nature: we ask this sequence to be ν-averaging ($a.s.$ and in $L^p(\mathbb{P})$), at least on bounded continuous functions, where ν is a distribution on $(\mathbb{R}_+^{N+1}, \mathcal{B}or(\mathbb{R}_+^{N+1}))$. This leads to the following formal assumption:

$$(ERG)_\nu \equiv \begin{cases} (i) & \text{the sequence } (V^n, D_i^n, i = 1, \ldots, N)_{n\geq 1} \text{ is averaging } i.e. \\ & \mathbb{P}\text{-}a.s. \quad \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \delta_{(V^k, D_1^k, \ldots, D_N^k)} \overset{(\mathbb{R}_+^{N+1})}{\Longrightarrow} \nu, \\ (ii) & \sup_n \mathbb{E}(V^n)^2 < +\infty. \end{cases}$$

where $\overset{(\mathbb{R}_+^{N+1})}{\Longrightarrow}$ denotes the weak convergence of probability measures on $\mathbb{R}_+^{N+1}$. For convenience, we will denote $(V, D_1, \ldots, D_N)$ the canonical random vector on $\mathbb{R}_+^{N+1}$ so that we can write $\nu = \mathcal{L}(V, D_1, \ldots, D_N)$.

Assumption (ii) on the marginal distribution of the sequence $(V^n)_{n\geq 1}$ is mainly technical. In fact standard arguments from weak convergence theory show that combining (i) and (ii) implies

$$\frac{1}{n}\sum_{k=1}^{n} V^k \longrightarrow \mathbb{E}\, V \quad \text{as} \quad n \to \infty$$

($\sup_n \mathbb{E}(V^n)^{1+\varepsilon} < +\infty$ would be enough). An important sub case is the the (IID) setting

$$(IID) \equiv \begin{cases} (i) & \text{the sequence } (V^n, D_1^n, \ldots, D_N^n)_{n\geq 1} \text{ is i.i.d.} \\ & \text{with distribution } \nu = \mathcal{L}(V, D_1, \ldots, D_N), \\ (ii) & V \in L^2(\mathbb{P}). \end{cases}$$

This more restrictive assumption is undoubtedly less realistic from a modeling point of view but it remains acceptable as a first approximation. It is the most common framework to apply the standard Stochastic Approximation machinery ($a.s.$ convergence, asymptotically normal fluctuations, etc). So, its interest may be considered at least as pedagogical. The (ERG) setting is slightly more demanding in terms of assumptions and needs more specific methods of proof. It will be investigated as a second step, using some recent results established in [83] which are well suited to the specificities of our problem (in particular we will not need to assume the existence of a solution to the Poisson equation related to the procedure like in the reference book [25]).

4.3 Optimal allocation: a stochastic Lagrangian algorithm

4.3.1 The mean execution function of a dark pool

In view of the modeling section, we need to briefly describe the precise behaviour of the mean execution function $\varphi : [0,1] \to \mathbb{R}_+$ of a single dark pool.

Let (V, D) be an $\mathbb{R}_+^2$-valued random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ representing the *global volume* to be executed and the *deliverable quantity* (by the dark pool) respectively. Throughout this chapter we will assume the following consistency assumption

$$V > 0 \quad \mathbb{P}\text{-}a.s. \quad \text{and} \quad \mathbb{P}(D > 0) > 0. \tag{4.2}$$

The $a.s.$ positivity of V means that we only consider true orders. The fact that D is not identically 0 means that the dark pool does exist in practice. The "rebate" coefficient ρ is specific to the dark pool.

To define in a consistent way the mean execution function of a dark pool we only need to assume that $V \in L^1(\mathbb{P})$ (although more stringent integrability assumptions are made throughout the chapter).

Here the *mean execution function* $\varphi : [0,1] \to \mathbb{R}_+$ of the dark pool is defined by

$$\forall\, r \in [0,1], \qquad \varphi(r) = \rho\, \mathbb{E}(\min(rV, D)) \tag{4.3}$$

where $\rho > 0$. The function φ is finite, non-identically 0. It is clearly a concave nondecreasing bounded function. Furthermore, one easily checks that its right and left derivatives are given at every $r \in [0,1]$ by

$$\varphi_l'(r) = \rho\, \mathbb{E}\left(\mathbf{1}_{\{rV \le D\}} V\right) \qquad \text{and} \qquad \varphi_r'(r) = \rho\, \mathbb{E}\left(\mathbf{1}_{\{rV < D\}} V\right). \tag{4.4}$$

In particular,

$$\varphi'(0) = \rho\, \mathbb{E}(V \mathbf{1}_{\{D>0\}}) > 0$$

and if

$$\textit{the (right continuous) distribution function of } \tfrac{D}{V} \textit{ is continuous on } \mathbb{R}_+, \tag{4.5}$$

then

$$\varphi \textit{ is everywhere differentiable on the unit interval } [0,1] \textit{ with } \varphi' = \varphi_l' \textit{ on } (0,1].$$

Assumption (4.5) means that the distribution of $\frac{D}{V}$ has no atom except possibly at 0. It can be interpreted as the fact that a dark pool has no "quantized" answer to an order.

More general models of execution functions in which the rebate ρ and the deliverable quantity D may depend upon the quantity to be executed rV are briefly discussed further on.

4.3.2 Design of the stochastic Lagrangian algorithm

Let V be the quantity to be executed by N dark pools. For every dark pool $i \in \mathcal{I}_N$ the available quantity D_i is defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ as V. We assume that all couples (V, D_i) satisfy the consistency assumption (4.2).

To each dark pool $i \in \mathcal{I}_N$ is attached a (bounded concave) mean execution function φ_i of type (4.3), introduced in Section 4.3.1. or the execution function introduced and studied in Section 4.8.

Then for every $r = (r_1, \dots, r_N) \in \mathcal{P}_N$,

$$\Phi(r_1, \dots, r_N) := \sum_{i=1}^{N} \varphi_i(r_i). \tag{4.6}$$

In order to design the algorithm we will need to extend the mean execution function φ (whatever its form is) as a concave function on the whole real line by setting

$$\varphi(r) = \left(r - \frac{r^2}{2}\right)\varphi'(0) \quad \text{if } r < 0 \quad \text{and} \quad \varphi(r) = \varphi(1) + \varphi'(1)\log r \quad \text{if } r > 1. \tag{4.7}$$

Based on the extension of the functions φ_i defined by (4.7), we can formally extend Φ on the whole affine hyperplane spanned by $\mathcal{P}_N$ *i.e.*

$$\mathcal{H}_N := \{r \in \mathbb{R}^N \mid \sum_i r_i = 1\}.$$

As announced, we aim at solving the following maximization problem

$$\max_{r \in \mathcal{P}_N} \Phi(r)$$

but we will also have to deal for algorithmic purpose with the same maximization problem when r runs over $\mathcal{H}_N$.

Before stating a rigorous result, let us a have a look at a Lagrangian approach that only takes into account the affine constraint that is $\max_r \Phi(r) - \lambda \sum_i r_i$. Straightforward formal computations suggest that

$$r^* \in \operatorname{argmax}_{\mathcal{P}_N} \Phi \text{ iff } \varphi_i'(r_i^*) \text{ is constant when } i \text{ runs over } \mathcal{I}_N$$

or equivalently if

$$\forall\, i \in \mathcal{I}_N, \qquad \varphi_i'(r_i^*) = \frac{1}{N}\sum_{j=1}^{N} \varphi_j'(r_j^*). \tag{4.8}$$

In fact this statement is not correct in full generality because the Lagrangian method does not provide a necessary and sufficient condition for a point to be a maximum of a (concave) function; thus, it does not take into account the case where Φ reaches its maximum on the boundary $\partial\mathcal{P}_N$ where the above condition on the derivatives may fail. So, an additional assumption is necessary to make it true as established in the proposition below.

Proposition 4.1. *Assume that (V, D_i) satisfies the consistency assumptions (4.2) and (4.5) for every $i \in \mathcal{I}_N$.*

(a) Assume that the functions φ_i defined by (4.3) satisfy the following assumption

$$(\mathcal{C}) \quad \equiv \quad \min_{i \in \mathcal{I}_N} \varphi_i'(0) \geq \max_{i \in \mathcal{I}_N} \varphi_i'\left(\frac{1}{N-1}\right).$$

Then $\operatorname{argmax}_{\mathcal{P}_N} \Phi$ is a compact convex set and

$$\operatorname{argmax}_{\mathcal{P}_N} \Phi = \{r \in \mathcal{P}_N, \mid \varphi_i'(r_i) = \varphi_1'(r_1),\ i = 1, \dots, N\}.$$

Furthermore $\operatorname{argmax}_{\mathcal{H}_N}\Phi = \operatorname{argmax}_{\mathcal{P}_N}\Phi$.

(*b*) *If the functions* φ_i *satisfy the slightly more stringent assumption,*

$$(\mathcal{C}_<) \quad \equiv \quad \min_{i\in\mathcal{I}_N}\varphi_i'(0) > \max_{i\in\mathcal{I}_N}\varphi_i'\left(\frac{1}{N-1}\right).$$

then

$$\operatorname{argmax}_{\mathcal{H}_N}\Phi = \operatorname{argmax}_{\mathcal{P}_N}\Phi \subset int(\mathcal{P}_N).$$

Remarks. • If $N = 2$, one checks that Assumption $(\mathcal{C})$ is also necessary to derive the conclusion of item (a).

• As a by-product of the proof below we have the following more precise result on the optimal allocation r^*: if $r^* \in \operatorname{argmax}_{\mathcal{P}_N}$ and $\mathcal{I}_0(r^*) := \{i \in \mathcal{I}_N \,|\, r_i^* = 0\}$, then

$$\max_{i\in\mathcal{I}_0(r^*)}\varphi_i'(0) \le \min_{i\in\mathcal{I}_0(r^*)^c}\varphi_i'(0).$$

Interpretation and comments: • In the case of a "regular" mean execution function, Assumption $(\mathcal{C})$ is a kind of *homogeneity assumption on the rebates* made by the involved dark pools. If we assume that $\mathbb{P}(D_i = 0) = 0$ for every $i \in \mathcal{I}_N$ (all dark pools buy or sell at least one security with the announced rebate), then $(\mathcal{C})$ reads

$$\min_{i\in\mathcal{I}_N}\rho_i \ge \max_{i\in\mathcal{I}_N}\left(\rho_i\frac{\mathbb{E}\,V\mathbf{1}_{\{\frac{V}{N-1}\le D_i\}}}{\mathbb{E}\,V}\right)$$

since $\varphi_i'(0) = \rho_i\,\mathbb{E}\,V$. In particular,

Assumption $(\mathcal{C})$ *is always satisfied when all the* ρ_i*'s are equal*

i.e. all dark pools propose the same rebates which usually is usually the mid-price (defined as the middle between bid and ask prices).

• Assumption $(\mathcal{C})$ is in fact our main assumption in terms of modeling. It may look somewhat difficult to satisfy when the rebates are not equal. But the crucial fact in order to preserve the generality of what follows is that it contains *no assumption about the dependence between the volume* V *and the "answers"* D_i *from the dark pools.*

Proof. (a) The function Φ is continuous on a compact set hence $\operatorname{argmax}_{\mathcal{P}_N}\Phi$ is not empty. Let $r \in \operatorname{argmax}_{\mathcal{P}_N}\Phi$ and $\mathcal{I}_0(r) := \{i \in \mathcal{I}_N \,|\, r_i = 0\}$. Clearly $\mathcal{I}_0(r) \neq \mathcal{I}_N$ so that $\operatorname{card}\mathcal{I}_0(r) \le N-1$. Let $u \in \mathbf{1}^\perp$ such that $u_i > 0$, $i \in \mathcal{I}_0(r)$. Then $t \mapsto \Phi(r+tu)$ defined on the right neighbourhood of 0 reaches its maximum at 0 so that its derivative at 0 is non-positive. Specifying the vector u yields

$$\forall\, i \in \mathcal{I}_0(r),\ \forall\, j \in \mathcal{I}_0(r)^c, \quad \varphi_i'(0) \le \varphi_j'(r_j).$$

Now if $u \in \mathbf{1}^\perp$ with $u_i = 0$, $i \in \mathcal{I}_0(r)$, then the $t \mapsto \Phi(r+tu)$ is defined on a neighbourhood of 0 and reaches its maximum at $t = 0$ so that its derivative is 0 at 0; specifying the vector u yields

$$\forall\, i,\ j \in \mathcal{I}_0(r)^c, \quad \varphi_i'(r_i) = \varphi_j'(r_j).$$

Now, there exists at least one index $i_1 \in \mathcal{I}_0(r)^c$ such that $r_{i_1} \ge \frac{1}{|I_0(r)^c|} \ge \frac{1}{N-1}$. Hence $\varphi_{i_1}'(r_{i_1}) \le \varphi_{i_1}'(\frac{1}{N-1})$ which implies in turn that for every $i_0 \in \mathcal{I}_0(r)$, $\varphi_{i_0}'(0) \le \varphi_{i_1}'(r_{i_1}) \le \varphi_{i_1}'(\frac{1}{N-1})$. Finally Assumption $(\mathcal{C})$ implies that these inequalities hold as equalities so that

$$\forall\, i \in \mathcal{I}_N, \quad \varphi_i'(r_i) = \varphi_1'(r_1).$$

Conversely, let $r \in \mathcal{P}_N$ satisfying the above equalities. Then, for every $r' \in \mathcal{P}_N$, the function $t \mapsto \Phi(tr' + (1-t)r)$ is concave on $[0,1]$ with a right derivative equal to 0 at $t = 0$. So it is maximum at $t = 0$ *i.e.* $\Phi(r) \geq \Phi(r')$.

Now we pass to the the maximization over $\mathcal{H}_N$. Since it is an affine space and Φ is concave, it is clear, *e.g.* by considering Φ as a function of $(r_1, \ldots, r_{N-1})$, that

$$\operatorname{argmax}_{\mathcal{H}_N} \Phi = \{r \in \mathcal{H}_N, \mid \varphi_i'(r_i) = \varphi_1'(r_1),\ i = 1, \ldots, N\}$$

(which is non-empty since it contains at least $\operatorname{argmax}_{\mathcal{P}_N}$). Now let $r \in \mathcal{H}_N \setminus \mathcal{P}_N$. Assume there exists $i_0 \in \mathcal{I}_N$ such that $r_{i_0} < 0$. Then there always exists an index $i_1 \in \mathcal{I}_N$ such that $r_{i_1} \geq \frac{1-r_{i_0}}{N-1} > \frac{1}{N-1}$. Consequently

$$\varphi_{i_0}'(r_{i_0}) = (1 - r_{i_0})\varphi_{i_0}'(0) > \varphi_{i_0}'(0) \geq \min_i \varphi_i'(0) \geq \max_i \varphi_i' \left(\frac{1}{N-1}\right) \geq \varphi_{i_1}' \left(\frac{1}{N-1}\right) \geq \varphi_{i_1}'(r_{i_1})$$

which contradicts the equality of these two derivatives. Consequently all r_i's are non-negative so that $r \in \mathcal{P}_N$.

(*b*) If $\mathcal{C}_<$ holds, the above proof shows that $\mathcal{I}_0(r) = \emptyset$ so that $\operatorname{argmax}_{\mathcal{P}_N} \Phi_N \subset int(\mathcal{P}_N)$. □

4.3.3 Design of the stochastic algorithm

Now we are in position to devise the stochastic algorithm for the optimal allocation among the dark pools, taking advantage of the characterization of $\operatorname{argmax}_{\mathcal{P}_N} \Phi$. In fact we will simply use the obvious remark that N numbers $a_1, \ldots, a_N$ are equal if and only if they are all equal to their arithmetic mean $\frac{a_1 + \cdots + a_N}{N}$.

We consider the mean execution function as defined by (4.3). We assume from now on that the continuity assumption (4.5) holds so that the representation (4.4) of its derivative can be taken as its right or its left derivative on $(0, 1]$ (and its right derivative only at 0).

Using this representation (4.4) for all the derivatives φ_i' yields that, if Assumption $(\mathcal{C})$ is satisfied, then $\operatorname{argmax}_{\mathcal{H}_N} \Phi = \operatorname{argmax}_{\mathcal{P}_N} \Phi$ and

$$r^* \in \operatorname{argmax}_{\mathcal{P}_N} \Phi \Longleftrightarrow \forall\, i \in \{1, \ldots, N\},\ \mathbb{E}\left(V\left(\rho_i \mathbf{1}_{\{r_i^* V \leq D_i\}} - \frac{1}{N}\sum_{j=1}^N \rho_j \mathbf{1}_{\{r_j^* V \leq D_j\}}\right)\right) = 0.$$

However, the set $\mathcal{P}_N$ is not stable for the "naive" zero search algorithm naturally derived from the above characterization, we are led to devise the procedure on the hyperplane $\mathcal{H}_N$.

Consequently, this leads to devise the following zero search procedure

$$r^n = r^{n-1} + \gamma_n H(r^{n-1}, V^n, D_1^n, \ldots, D_N^n),\ n \geq 1, \quad r^0 \in \mathcal{P}_N, \tag{4.9}$$

where, for every $i \in \mathcal{I}_N$, every $r \in \mathcal{H}_N$, every $V > 0$ and every $D_1, \ldots, D_N \geq 0$,

$$\begin{aligned} H_i(r, V, D_1, \ldots, D_N) \quad = \quad & V\Big(\rho_i \mathbf{1}_{\{r_i V \leq D_i\} \cap \{r_i \in [0,1]\}} - \frac{1}{N}\sum_{j=1}^N \rho_j \mathbf{1}_{\{r_j V \leq D_j\} \cap \{r_j \in [0,1]\}} \\ & + R_i(r, V, D_1, \ldots, D_N)\Big) \end{aligned} \tag{4.10}$$

and the "innovation" $(V^n, D_1^n, \ldots, D_N^n)_{n \geq 1}$ is a sequence of random vectors with non negative components such that, for every $n \geq 1$, $(V^n, D_i^n, i = 1, \leq, N) \stackrel{d}{=} (V, D_i, i = 1, \leq, N)$ and the remainder terms R_i have a mean-reverting effect to pull back the algorithm into $\mathcal{P}_N$. They are designed from

the extension (4.7) of the derivative functions φ_i' outside the unit interval $[0,1]$; to be precise, for every $i \in \mathcal{I}_N$,

$$\begin{aligned} R_i(r, V, D_1, \ldots, D_N) &= \rho_i \left((1-r_i)\mathbf{1}_{\{D_i>0\}\cap\{r_i<0\}} + \frac{1}{r_i}\mathbf{1}_{\{V\le D_i\}\cap\{r_i>1\}} \right) \\ &\quad - \frac{1}{N}\sum_{j=1}^{N} \rho_j \left((1-r_j)\mathbf{1}_{\{D_j>0\}\cap\{r_j<0\}} + \frac{1}{r_j}\mathbf{1}_{\{V\le D_j\}\cap\{r_j>1\}} \right). \end{aligned}$$

4.3.4 Interpretation and implementability of the procedure

▷ Implementability. The vector $(r_i^n)_{1\le i\le N}$ in (4.9) represents the dispatching of the orders among the N dark pools to be sent at time $n+1$ by the investor. It is computed at time n. On the other hand V^n represents the volume to be executed (or its monetary value if one keeps in mind that we "plugged" the price into the volume) and the D_i^n the "answer" of dark pool i, still at time n.

The point is that the investor does have no access to the quantities D_i^n. However, he/she knows what he/she receives from dark pool i, *i.e.* $\min(D_i^n, r_i^{n-1}V^n)$. As a consequence, the investor has access to the event

$$\{\min(D_i^n, r_i^{n-1}V^n) = r_i^{n-1}V^n\} = \{r_i^{n-1}V^n \le D_i^n\}$$

which in turn makes possible the updating of the procedure although he/she has no access to the true value of D_i^n.

So, except for edge effects outside the simplex $\mathcal{P}_N$, the procedure as set can be implemented on real data.

▷ Interpretation. As long as r is a true allocation vector, *i.e.* lies in the simplex $\mathcal{P}_N$, the interpretation of the procedure is the following: assume first that all the factors ρ_i are equal (to 1). Then the dark pools which fully executed the sent orders ($r_iV \le D_i$) are rewarded proportionally to the numbers of dark pools which *did not fully executed* the request they received. Symmetrically, the dark pools which could not execute the whole request are penalized proportionally to the number of dark pools which satisfied the request.

Thus, if only one dark pool, say dark pool 1, fully executes the request at time n, its percentage will be increased for the request at time $n+1$ by $\gamma_n(1-\frac{1}{N})V^n$ *i.e.* it will asked to execute $r_1^n = r_1^{n-1} + \gamma_n(1-\frac{1}{N})V^n$ % of the total order V^{n+1}. The other $N-1$ dark pools will be penalized symmetrically: the pourcentage r_i^n of the total request V^{n+1} each of them will receive at time $n+1$ will be reduced by $\gamma_n\frac{1}{N}V^n$.

If k dark pools totally execute their request at time n and the $N-k$ other fail, the percentages of V^{n+1} that the "successful" dark pools will receive for execution at time $n+1$ will be increased by $\gamma_n(1-\frac{k}{N})V^n$, each of the $N-k$ "failing dark pools" being reduced by $\gamma_n\frac{k}{N}V^n$.

If no dark pool was able to satisfy their received request at time n, none will be penalized and if all dark pools fully execute the received orders, none will be rewarded.

In short, the dark pools are rewarded or penalized by comparing their mutual performances. When the "attractivity" coefficients ρ_i are not equal, the reasoning is the same but weighted by these attractivities.

▷ Practical implementation. One may force the above procedure to stay in the simplex $\mathcal{P}_N$ by projecting, once updated, the procedure on $\mathcal{P}_N$ each time it exits the simplex. This amounts to replace the possibly negative r_i by 0, the $r_i > 1$ by 1 and to renormalize the vector r by dividing it by the sum of its terms.

Furthermore, to avoid that the algorithm leaves too often the simplex, one may simply normalize the step γ_n by considering the predictable step

$$\tilde{\gamma}_n = \gamma_n \times \frac{n-1}{V^1 + \cdots + V^{n-1}} \approx \frac{\gamma_n}{\mathbb{E}\, V}.$$

4.4 The (IID) setting: $a.s.$ convergence and CLT

In this section, we make the assumption that the sequence $(V^n, D^n)_{n\geq 1}$ of successive requests and dark pool answers are independent with a common distribution. Such a restriction may be discussed. However, under some circumstances, it is not so unrealistic if one thinks to an agent who receives "exogenous" orders to be executed at every time period. In this setting, we will be able to prove a Central Limit Theorem which rules the weak convergence rate of the algorithm and provides Gaussian confidence intervals. A point to be noted is that no significant restriction is made on the correlations or dependence between the sent volumes and the dark pool answers.

Theorem 4.1. *Assume that (V, D) satisfy (4.2), that $V \in L^2(\mathbb{P})$ and that Assumption $(\mathcal{C})$ holds. Assume furthermore that the distribution of $\frac{D}{V}$ satisfies the continuity Assumption (4.5). Let $\gamma := (\gamma_n)_{n\geq 1}$ be a step sequence satisfying the usual decreasing step assumption*

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

Let $(V^n, D_1^n, \ldots, D_N^n)_{n\geq 1}$ be an i.d.d. sequence defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, there exists an $\mathrm{argmax}_{\mathcal{P}_N}\Phi$-valued random variable r^* such that

$$r^n \longrightarrow r^* \quad a.s.$$

If the functions φ_i satisfy $(\mathcal{C}_<)$ then $\mathrm{argmax}_{\mathcal{P}_N}\Phi \subset int(\mathcal{P}_N)$.

Proof of the theorem. In this setting, the algorithm is (non homogeneous) Markov discrete time process with respect to the natural filtration $\mathcal{F}_n := \sigma(r^0, (V^k, D_1^k, \ldots, D_N^k),\, 1 \leq k \leq n)$ with the following canonical representation

$$\begin{aligned} r^{n+1} &= r^n + \gamma_{n+1} H(r^n, V^{n+1}, D_1^{n+1}, \ldots, D_N^{n+1}),\ r^0 \in \mathcal{P}_N \\ &= r^n + \gamma_{n+1} h(r^n) + \gamma_{n+1} \Delta M_{n+1} \end{aligned}$$

where, for every $r \in \mathcal{H}_N$,

$$h(r) := \mathbb{E}\, H(r, V, D_1, \ldots, D_N) = \left(\varphi_i'(r_i) - \frac{1}{N} \sum_{j=1}^{N} \varphi_j'(r_j) \right)_{1\leq i\leq N}$$

is the so-called *mean* function of the algorithm, and

$$\begin{aligned} \Delta M_{n+1} &= H(r^n, V^{n+1}, D_1^{n+1}, \ldots, D_N^{n+1}) - \mathbb{E}(H(r^n, V^{n+1}, D_1^{n+1}, \ldots, D_N^{n+1}) \,|\, \mathcal{F}_n) \\ &= H(r^n, V^{n+1}, D_1^{n+1}, \ldots, D_N^{n+1}) - h(r^n) \end{aligned}$$

since $(V^{n+1}, D_1^{n+1}, \ldots, D_N^{n+1})$ is independent of $\mathcal{F}_n$.

One derives from Proposition 4.1(*a*) that the mean function h of the algorithm satisfies $\{h = 0\} = \mathrm{argmax}_{\mathcal{P}_N}$ and that, for every $r \in \mathcal{H}_N \setminus \{h = 0\}$ and every $r^* \in \{h = 0\}$,

$$\langle h(r) \,|\, r - r^* \rangle = \langle h(r) - h(r^*) \,|\, r - r^* \rangle = \sum_{i=1}^{N} \underbrace{(\varphi_i'(r_i) - \varphi_i'(r_i^*))(r_i - r_i^*)}_{\leq 0} < 0 \tag{4.11}$$

simply because each function φ_i' is non-increasing which implies that each term of the sum is non-positive. The sum is not zero otherwise $\varphi'(r_i) = \varphi'(r_i^*)$ as soon as $r_i \neq r_i^*$ which would imply $h(r) = 0$.

The random vector V being square integrable, it is clear that $H_i(r, V, D_1, \ldots, D_N)$ satisfies the linear growth assumption

$$\forall\, i \in \mathcal{I}_N,\ \forall\, r \in \mathcal{H}_N, \quad \|H_i(r, V, D_1, \ldots, D_N)\|_2 \leq 2\,(\max_j \rho_j)\|V\|_2(N + |r|)$$

At this stage one may conclude using a simple variant of the standard Robbins-Monro Theorem (like that established in [89]): there exists a random variable r^* taking values in $\{h = 0\}$ such that $r^n \to r^*$. □

4.4.1 Weak rate of convergence

In this section, for the sake of simplicity, we will assume that the mean function h has a single zero denoted r^*. The aim of this section is to establish that the algorithm satisfies, under slightly more stringent assumptions, a Central Limit Theorem *i.e.*

$$\mathbb{P}\Big(|r^n - r^*| \leq \frac{a}{\sqrt{n}}\Big) \to \mathbb{P}(|\Sigma Z| \leq a) \quad \text{as} \quad n \to \infty \ \text{ where } \ Z \overset{d}{=} \mathcal{N}(0; I_N)$$

and Σ is an explicit covariance matrix. This result is established in Theorem 4.2 below by fulfilling the assumptions of the regular Central Limit Theorem (*CLT*) for stochastic approximation. For a precise statement, we refer (among others) to [25] (Theorem 13 p.332). For the sake of simplicity, we will assume that the mean function h has a single zero denoted r^*. The following short proposition provides a simple criterion to ensure this uniqueness.

Proposition 4.2. *Assume* $(\mathcal{C}_<)$ *and that all the functions* φ_i', $i \in \mathcal{I}_N$, *are decreasing (strictly). Then*

$$\{h = 0\} = \mathrm{argmax}_{\mathcal{P}_N} \Phi = r^* \in \mathit{int}(\mathcal{P}_N).$$

Proof. As $(\mathcal{C}_<)$ is satisfied so that $\mathrm{argmax}_{\mathcal{P}_N} \Phi \subset \mathit{int}(\mathcal{P}_N)$. If r, $r' \in \{h = 0\}$, $r \neq r'$, it follows from (4.11) that $\varphi_i'(r_i) = \varphi_i'(r_i')$ for some index i such that $r_i \neq r_i'$. □

A first step toward this *CLT* is to show the existence and to compute the differential of h (Hessian of Φ) at r^*. To this end, we will make one further assumption on distribution of (V, D), keeping in mind that $\mathbb{P}(D > 0) > 0$, but that $\mathbb{P}(D = 0)$ may possibly be positive too. Namely, assume that the distribution function of (V, D) *given* $\{D > 0\}$ is absolutely continuous with a probability density f defined on $(0, +\infty)^2$. Furthermore we make the following assumptions on f:

$$\begin{cases} (i) & \text{for every } v > 0,\ u \mapsto f(v, u) \text{ is continuous and positive on } (0, \infty), \\ (ii) & \forall\, \varepsilon \in (0, 1), \quad \sup_{\varepsilon V \leq u \leq V/\varepsilon} f(V, u)V^2 \in L^1(\mathbb{P}). \end{cases} \tag{4.12}$$

Note that (ii) is clearly always satisfied when $V \in L^2(\mathbb{P})$ and f is bounded. The conditional distribution function of D given $\{D>0\}$ and V is given by

$$F_{D}(u \,|\, V=v, D>0) := \mathbb{P}(D \le u \,|\, V=v, D>0) = \int_0^u f(v,u')du', \ u>0, \ v>0,$$

Lemma 4.1. (a) *Assume (V,D) satisfies the above assumption (4.12). Then the mean execution function $\varphi(u) := \rho\,\mathbb{E}(\min(uV, D))$ is concave, twice differentiable on $\mathbb{R}_+$ and for every $u>0$,*

$$\varphi''(u) = -\rho\,\mathbb{E}\left(V^2 \mathbf{1}_{\{D>0\}} f(V, uV)\right) < 0.$$

(b) *If (V, D_i) satisfies the above assumption (4.12) for every $i \in \mathcal{I}_{N}$, then the function $\tilde{h}$ defined on $\mathbb{R}_+^N$ by $\tilde{h}(u_1,\dots,u_{N}) = \left(\varphi_i'(u_i) - \frac{1}{N}\sum_{1\le j\le N}\varphi_j'(u_j)\right)_{1\le i\le N}$ is differentiable on $(0,\infty)^N$ and admits a continuous extension on $\mathbb{R}_+^N$ given by*

$$D\tilde{h}(u) = -\frac{1}{N}\Big[-a_j(u_j) + N a_i(u_i)\,\delta_{ij}\Big]_{1\le i,j\le N} \qquad \text{with} \qquad a_i(u) = -\varphi_i''(u) > 0.$$

(c) *Let $A := [-a_j + N a_i \delta_{ij}]_{1\le i,j\le N}$, $a_1,\dots,a_{N} > 0$ and let $\underline{a} = \min_i a_i$. Its kernel $\mathrm{Ker}(A)$ is one dimensional, $A(\mathbb{R}^N) = \mathbf{1}^\perp$ and $A_{|\mathbf{1}^\perp}$ is bijective. Every non-zero eigenvalue λ (with eigenspace E_λ) satisfies*

$$\Re(\lambda) \ge N \times \underline{a} \qquad \text{and} \qquad E_\lambda \subset \mathbf{1}^\perp.$$

Proof. (a) is a straightforward application of the Lebesgue differentiation Theorem for expectation.
(b) is a consequence of (a).
(c) The transpose A^t of A has a strict dominating diagonal structure *i.e.* $A^t_{ii} > 0$, $A^t_{ij} < 0$, $i \neq j$ and $\sum_j A^t_{ij} = 0$ for every i. Consequently, it follows from Gershgorin's Lemma (see [56]) that 0 is an eigenvalue of order 1 of A^t (with $\mathbf{1}$ as an eigenvector and that all other eigenvalues have (strictly) positive real parts). Consequently $\mathrm{Ker}(A)$ is one dimensional. The fact that $A(\mathbb{R}^N) \subset \mathbf{1}^\perp$ is obvious so that this inclusion holds as an equality by the dimension formula. Hence all the eigenvectors not in $\mathrm{Ker}(A)$ are in $\mathbf{1}^\perp$. Set $\tilde{a}_i - a_i - \underline{a} \ge 0$, $i = 1,\dots,N$. Then $\tilde{A}^t$ has a dominating diagonal structure so that all its eigenvalues have non-negative real parts. Now if λ is an eigenvalue of A, it is obvious that $\lambda - N\underline{a}$ is an eigenvalue of $\tilde{A}$. Consequently $\Re(\lambda) \ge N\underline{a}$. □

Now we re in position to prove the *CLT* for our algorithm.

Theorem 4.2. *Assume that the assumptions of Theorem 4.1 holds and that* $\operatorname{argmax}\Phi$ *is reduced to a single point $r^* \in \mathcal{P}_{N}$ so that $r^n \to r^*$ $\mathbb{P}$-a.s. as $n \to \infty$. Furthermore, suppose that Assumption (4.12) holds for every (V, D_i), $i \in \mathcal{I}_{N}$ and that $V \in L^{2+\delta}(\mathbb{P})$, $\delta > 0$. Set*

$$\gamma_n = \frac{c}{n}, \ n \ge 1 \ \text{ with } \ c > \frac{1}{2\Re e(\lambda_{\min})}$$

where $\lambda_{\min}$ denotes the eigenvalue of $A^\infty := -Dh(r^*)_{|\mathbf{1}^\perp}$ with the lowest real part. Then

$$\sqrt{n}(r^n - r^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0; \sqrt{c}\,\Sigma^\infty)$$

where the asymptotic covariance matrix Σ^∞ is given by

$$\Sigma^\infty = \int_0^\infty e^{u(A^\infty - \frac{Id}{2c})} C^\infty e^{u(A^\infty - \frac{Id}{2c})^t} du$$

where

$$C^\infty = \mathbb{E}\left(H(r^*, V, D_1, \dots, D_{N}) H(r^*, V, D_1, \dots, D_{N})^t\right)_{|\mathbf{1}^\perp}$$

and $(A^\infty - \frac{Id}{2c})^t$ stands for the transpose operator *of $A^\infty - \frac{Id}{2c} \in \mathcal{L}(\mathbf{1}^\perp)$.*

Remark. The above claim is consistent since $u \mapsto H(r, v, \delta_1, \ldots, \delta_{\mathcal{N}})^t\, u$ preserves $\mathbf{1}^\perp$.

Proof. First note that, since $r^* \in int(\mathcal{P}_{\mathcal{N}})$, the above Lemma 4.1$(b)$ shows that (still making the confusion between the linear operator $Dh(r^*)$ and its matrix representation in the canonical basis)

$$Dh(r^*) = -\frac{1}{N}\left[-a_j(r_j^*) + N a_i(r_i^*)\,\delta_{ij}\right]_{1\le i,j\le N} \qquad \text{with } a_i(r) = \rho_i \mathbb{E}(V^2 \mathbf{1}_{\{D_i>0\}} f(V, rV)) > 0$$

Then, Lemma 4.1 (c) implies that $-Dh(r^*)_{|\mathbf{1}^\perp}$ has eigenvalues with positive real parts, all lower bounded by $\min_i a_i(r_j^*) > 0$.

At this stage, one can apply the CLT for stochastic algorithms defined on $\mathbf{1}^\perp$ (see *e.g.* [25], p.341). □

4.5 The (ERG) setting: convergence

For the sake of simplicity, although it is not really necessary, we will assume throughout this section that all the execution functions φ_i are decreasing so that by Proposition 4.2

$$\text{argmax}_{\mathcal{P}_{\mathcal{N}}}\, \Phi = \{r^*\} \subset int(\mathcal{P}_{\mathcal{N}}).$$

The aim of this section is to relax the stringent i.i.d. assumption made on the sequence $(V^n, D^n)_{n\ge1}$ in Section 4.4. We will establish the convergence of the algorithm in Theorem 4.3 under a dramatically lighter ν-averaging property on this sequence which is close to plain ergodicity, taking full advantage of the fact that this algorithm satisfies a *pathwise Lyapunov* assumption to apply a non-classic *a.s.* convergence theorem established in [83].

So we assume that the sequence $(V^n, D_i^n, i = 1, \ldots, N)_{n\ge1}$ satisfies $(ERG)_\nu$ with a limiting distribution ν such that, for every $i \in \mathcal{I}_{\mathcal{N}}$, its marginal $\nu_i = \mathcal{L}(V, D_i)$ satisfies the consistency assumption (4.2) and the continuity assumption (4.5). We will also need to make a specific assumption: there exists $\varepsilon_0 > 0$ such that

$$\begin{cases} (i) & \mathbb{P}(V \ge \varepsilon_0) > 0 \\ (ii) & \text{supp}\Big(\mathcal{L}\Big(\frac{D_i}{V}, i = 1, \ldots, N \,|\, \{V \ge \varepsilon_0\}\Big)\Big) \text{ is a neighbourhood of } \mathcal{P}_{\mathcal{N}} \text{ in } \mathbb{R}_+^N. \end{cases} \tag{4.13}$$

This assumption means that *all allocations across the pools lying in the neighbourhood of* $\mathcal{P}_{\mathcal{N}}$ can be executed.

On the other hand, it follows from $(ERG)_\nu$ and some standard weak convergence arguments that

$$\forall\, i \in \mathcal{I}_{\mathcal{N}},\ \forall\, u \in \mathbb{R}_+, \quad \frac{1}{n}\sum_{k=1}^{n} V^k \mathbf{1}_{\{uV^k \le D_i^k\}} - \mathbb{E}(V \mathbf{1}_{\{uV \le D_i\}}) \overset{a.s.\&L^2}{\longrightarrow} 0 \quad \text{as } n \to \infty,$$

since the (non-negative) functions $f_u(v, \delta) := v \mathbf{1}_{\{uv \le \delta\}}$, $u > 0$, are $\mathbb{P}_{(V,D_i)}$-*a.s.* continuous and $O(v)$ as $v \to +\infty$ by (4.5). Moreover this *a.s.* convergence holds uniformly on compact sets with respect to u since $u \mapsto \mathbb{E}\, V \mathbf{1}_{\{uV \le D_i\}}$ is continuous, still owing to (4.5). Our specific assumption is to require a rate in the above *a.s.* and $L^2(\mathbb{P})$-convergence. Namely, we assume that there exists an exponent $\alpha_i \in (0, 1]$ such that

$$\forall\, u \in \mathbb{R}_+, \quad \frac{1}{n}\sum_{k=1}^{n} V^k \mathbf{1}_{\{uV^k < D_i^k\}} - \mathbb{E}(V \mathbf{1}_{\{uV < D_i\}}) = O(n^{-\alpha_i}) \quad a.s. \text{ and in } L^2(\mathbb{P}). \tag{4.14}$$

This assumption *e.g.* from the more general assumption that, for every $i \in \mathcal{I}_N$, the marginal $\nu_i = \mathcal{L}(V, D_i)$ satisfies (4.5) and

$$(V^n, D_i^n) \text{ is } \nu_i\text{-averaging at rate } \alpha_i$$

on a subspace $\mathcal{V}_{\alpha_i,2}$ containing all the functions f_u.

Note that, when the sequence $(V^n, D_i^n, i = 1, \dots, N)_{n\geq 1}$ is i.i.d. with distribution ν then elementary martingale arguments show that the whole sequence is ν-averaging at rate $\frac{1}{2} - \eta$ for every $\eta \in (0, 1/2)$ on $\mathcal{V}_{\frac{1}{2}-\eta,2} = L^2(\nu)$ (and all $f_u \in L^2(\nu)$, $u > 0$, since $V \in L^2(\mathbb{P})$). So, the theorem below almost embodies the *a.s.* convergence theorem established in the (IID) setting (except for the integrability assumption on V).

Now we are in position to state the main convergence result of this section. We rely on the extension of Robbins-Siegmund Lemma proposed in [83] (see Chapter 2).

Theorem 4.3. *Let $(V^n, D_1^n, \dots, D_N^n)_{n\geq 0}$ be a sequence of input satisfying $(ERG)_\nu$ and such that, for every $i \in \mathcal{I}_N$, the marginal distribution $\nu_i = \mathcal{L}(V, D_i)$ satisfies the consistency assumptions (4.2) and (4.5). Suppose furthermore that, the sequence $(V^n, D_i^n)_{n\geq 1}$ satisfies the rate assumption (4.14). If the step sequence $(\gamma_n)_{n\geq 1}$ satisfies*

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad \gamma_n = o(n^{\underline{\alpha}-1}) \quad \text{and} \quad \sum_{n\geq 1} n^{1-\underline{\alpha}} \max(\gamma_n^2, |\gamma_n - \gamma_{n+1}|) < \infty$$

where $\underline{\alpha} := \min_{i\in\mathcal{I}_N} \alpha_i \in (0, 1]$, then the algorithm defined by (4.10) a.s. converges towards $r^* = \operatorname{argmax}_{\mathcal{P}_N} \Phi$.

TECHNICAL COMMENT. The above condition on the step sequence $(\gamma_n)_{n\geq 1}$ is satisfied as soon as $\gamma_n = \frac{c}{n^\beta}$ with $\beta \in (1 - \underline{\alpha}, 1]$.

Proof. STEP 1. First, we aim at applying the extended Robbins-Siegmund Lemma established in [83] (see Chapter 2, Theorem 2.1 for its statement) for stochastic algorithms with ν-averaging inputs dynamics in presence of a Lyapunov function. We will consider the case $p = 2$ and $\beta \in (0, \underline{\alpha}]$. We set $G = -H$ and $\Delta M^n \equiv 0$ and we consider the input $Y^n = (V^{n+1}, D_1^{n+1}, \dots, D_N^{n+1})$, $n \geq 0$. Let $L(r) = \frac{1}{2}|r - r^*|^2$ be our candidate as a Lyapunov function.

First note that it follows from (4.10) that the function H satisfies the growth assumption (2.12) since

$$\forall r \in \mathcal{H}_n, \ \forall y \in \mathbb{R}^{N+1}, \quad |H(r, y)| \leq C_H g(y)(1 + |r|)$$

where $C_H > 0$ and $g(v, \delta_1, \dots, \delta_N) = v$.

In view of the ergodic assumption (4.14) and the fact that r^* lies in $\mathcal{P}_N$, it is clear from its definition that $H(r^*, .) \in \mathcal{V}_{\beta,2}$ for every $\beta \in (0, \underline{\alpha}]$.

At this stage it remains to check the "weak local Lyapunov" assumption (2.15) for $G = -H$. This fact is obvious since, for every $r \in \mathcal{H}_N$ and every input $y = (v, \delta_1, \dots, \delta_N) \in (0, +\infty) \times \mathbb{R}^N$,

$$\langle H(r, y) - H(r^*, y) | r - r^* \rangle = \sum_{i=1}^N (\widetilde{H}_i(r_i, v, \delta_i) - \widetilde{H}_i(r_i^*, v, \delta_i))(r_i - r_i^*) \leq 0 \tag{4.15}$$

where

$$\widetilde{H}_i(u, v, \delta_i) = \rho_i v \left(\mathbf{1}_{\{uv\leq\delta_i\}} \mathbf{1}_{[0,1]}(u) + (1 - u)\mathbf{1}_{\{\delta_i>0, u<0\}} + \frac{1}{u}\mathbf{1}_{\{v\leq\delta_i, u>1\}} \right), \ i \in \mathcal{I}_N \tag{4.16}$$

is clearly non-increasing with respect to u.

At this stage, using that $\sup_{n\geq 1} \mathbb{E}(V^n)^2 < +\infty$, we can apply our extended Robbins-Siegmund lemma also that

$$|r^n - r^*| \xrightarrow{a.s.} L_\infty < +\infty \ a.s. \ \text{ and } \quad \sum_{n\geq 1} \gamma_n \langle r^n - r^* \,|\, G(r^n, Y^n) - G(r^*, Y^n) \rangle < +\infty \ a.s. \tag{4.17}$$

STEP 2. At this stage it suffices to show that r^* is *a.s.* a limiting point of $(r^n)_{n\geq 0}$ since $|r^n - r^*|$ converges to $L_\infty < +\infty$ *a.s.*

Let η denote a positive real number such that, for every $i \in \mathcal{I}_N$, $[r_i^* - \eta, r_i^* + \eta] \subset (0,1)$. One derives from (4.16) and the monotonicity of $\widetilde{H}_i(u,v,\delta)$ in $u \in \mathbb{R}$ that for every $i \in \mathcal{I}_N$ and every $r \in \mathcal{H}_N$,

$$(\widetilde{H}_i(r_i, v, \delta_i) - \widetilde{H}_i(r_i^*, v, \delta_i))(r_i^* - r_i) \geq \rho_i v \eta \mathbf{1}_{\{r_i > r_i^* + \eta\}} \mathbf{1}_{\{\delta/v \in J_\eta\}}$$

where $J_\eta = (r_i^*, r_i^* + \eta)$. As a consequence

$$\langle G(r,y) - G(r^*,y) \,|\, r - r^* \rangle \geq \varepsilon_0 \underline{\rho}\, \eta \mathbf{1}_{\{v \geq \varepsilon_0\}} \mathbf{1}_{\{y \in O_\eta\}} \sum_{i \in \mathcal{I}_N} \mathbf{1}_{r_i > r_i^* + \eta}.$$

where $\underline{\rho} = \min_i \rho_i$ and the open set O_η is defined by

$$O_\eta = \left\{ y = (v, \delta_1, \ldots, \delta_N) \in (\varepsilon_0, +\infty) \times \mathbb{R}_+^N \text{ s.t. } \frac{\delta_i}{v} \in J_\eta,\, i \in \mathcal{I}_N \right\}.$$

Now, one derives from (4.17) that

$$\sum_n \gamma_n \mathbf{1}_{O_\eta}(Y^n) \sum_{i \in \mathcal{I}_N} \mathbf{1}_{\{r_i^n > r_i^* + \eta\}} < +\infty \quad a.s.$$

Now Assumption (4.13) implies that $\nu(O_\eta) > 0$. Furthermore $\nu(\partial O_\eta) = 0$ owing to the continuity assumption so that $(ERG)_\nu$ implies

$$\frac{1}{n} \sum_{k=1}^n \mathbf{1}_{O_\eta}(Y^k) \longrightarrow \nu(O_\eta) > 0 \quad a.s.$$

An Abel transform and the facts that the sequence γ_n is non-increasing and $\sum_{n\geq 1} \gamma_n = +\infty$ classically implies that

$$\sum_{n \geq 1} \gamma_n (\mathbf{1}_{O_\eta}(Y^k) - \nu(O_\eta)) \quad a.s. \text{ converge}$$

so that

$$\sum_n \gamma_n \mathbf{1}_{O_\eta}(Y^n) = +\infty \quad a.s.$$

In turn, this implies that

$$\liminf_n \sum_{i \in \mathcal{I}_N} \mathbf{1}_{\{r_i^n > r_i^* + \eta\}} = 0 \quad a.s.$$

This holds of course for a sequence of real numbers η^ℓ decreasing to 0.

Let $\mathcal{R}_\infty$ be the set of limiting values of the sequence $(r^n)_{n\geq 0}$. It is is *a.s.* non-empty since the sequence $(r^n)_{n\geq 0}$ is bounded. Then $\mathcal{R}_\infty$ is *a.s.* compact and it follows from what precedes that $\mathcal{R}_\infty \cap \prod_{1\leq i\leq N}(-\infty, r_i^* + \eta^\ell] \neq \emptyset$ (and is compact). Hence a decreasing intersection of non-empty compact sets being a (non-empty) compact set $\mathcal{R}_\infty \cap \prod_{1\leq i\leq N}(-\infty, r_i^*] \neq \emptyset$. On the other hand $\mathcal{R}_\infty \subset \mathcal{H}_N$ since the algorithm is $\mathcal{H}_N$-valued. But $\prod_{1\leq i\leq N}(-\infty, r_i^*] \cap \mathcal{H}_N = \{r^*\}$. Consequently r^* is a limiting point of the algorithm which implies that it is its true *a.s.* limit. □

APPLICATION TO α-MIXING STATIONARY DATA. If $(V^n, D_i^n, i = 1, \ldots, N)_{n\geq 1}$ is a stationary α-mixing sequence which mixing coefficients $(\alpha_n)_{n\geq 1}$ satisfy Ibragimov's condition for some $\delta > 0$:

$$\sum_{n \geq 1} \alpha_n^{\frac{2}{2+\delta}} < +\infty$$

(which is satisfied in case of geometric α-mixing) then the sequence $(V^n, D_i^n, i = 1, \ldots, N)_{n\geq 1}$ is ν-averaging where ν is the stationary marginal distribution of the sequence (supposed to satisfy (4.2) and (4.5)) at rate β for every $\beta \in (0, 1/2)$. To be precise, $L^2(\nu) \subset \mathcal{V}_{0^+,2}$ and

$$L^{2+\delta}(\nu) \subset \bigcap_{0<\beta<\frac{1}{2}} \mathcal{V}_{\beta,2}.$$

In particular, all the functions $f_u(v, \delta) := v\mathbf{1}_{\{uv\leq\delta\}}$, $u \geq 0$, lie in every $\mathcal{V}_{\beta,2}$, $0 < \beta < \frac{1}{2}$, so that the rate condition (4.14) is satisfied.

As concerns the stationary assumption on the input data sequence, it can be considered as realistic if one think of execution objectives given on a daily basis.

Example: An exponential discrete time Ornstein-Ulhenbeck model for $(V^n, D_1^n, \ldots, D_N^n)$.

$$V^n = v^0 e^{X_0^n}, \qquad D_i^n = d_i^0 e^{X_i^n},\ i = 1, \ldots, N,\ n \geq 1,$$

where v^0, d_1^0, $\ldots$, d_N^0 are positive real numbers and the sequence $(X^n)_{n\geq 1}$ satisfies the linear auto-regressive dynamics

$$X^{n+1} = m + AX^n + B\,\Xi^{n+1}, n \geq 1,$$

with $m \in \mathbb{R}^{N+1}$, $A \in \mathcal{M}(N+1, N+1, \mathbb{R})$, $|||A||| < 1$, $B \in \mathcal{M}(N+1, M)$ with $\mathrm{rank}(B) = N+1$ $(\leq M)$ and $(\Xi^n)_{n\geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0; Id_M)$-distributed random variables. We assume that the sequence is stationary *i.e.* that the distribution of X^1 is the (Gaussian) invariant distribution (with covariance matrix C solution to the Lyapunov equation $C - ACA^t = BB^t$ where ^{t} stands for transpose). Then (see [42], p.99), the sequence $(X^n)_{n\geq 0}$ is geometrically α-mixing and subsequently so is $(V^n, D_1^n, \ldots, D_N^n)_{n\geq 1}$ (with respect to its natural filtration). Furthermore, it is clear that its distribution ν satisfies the dispersion assumption (4.13) the process $(X_0^1, X_i^1 - X_0^1, i = 1, \ldots, N)$ is a non-degenerate Gaussian distribution over $\mathbb{R}^N$ since B has full rank $N+1$.

4.6 An alternative procedure based on a reinforcement principle

Recently, inspired by the discussion developed by Almgren and Harts in [8] about liquidity estimation, Berenstein and Lehalle devised a "smart routing" recursive procedure of requests to be executed by a pool of N dark pools (see [28]). This procedure is not based on the optimization of a potential function but on a intuitive reinforcement mechanism. Let I_i^n be the profit induced by the execution of the order sent to dark pool i at time n. The proportion r_i^n of the global order V^{n+1} to be sent to dark pool i for execution at time $n+1$ is defined as proportional to this profit *i.e.* by

$$\forall\, i \in \mathcal{I}_N, \qquad r_i^n := \frac{I_i^n}{\sum_j I_j^n}.$$

The updating of the random vector I^n is as follows

$$\forall\, n \geq 0,\ \forall\, i \in \mathcal{I}_N, \quad I_i^{n+1} = I_i^n + \rho_i \min\left(r_i^n V^{n+1}, D_i^{n+1}\right), \quad I_i^0 = 0.$$

The first equation models the idea of "reinforcement" since the proportion of orders sent for execution to dark pool i is proportional to the historical performances of this dark pool since the beginning of the procedure.

The second equation describes in a standard way – like in the optimization algorithm – the way dark pools execute orders.

Elementary computations show that the algorithm can be written directly in a recursive way in terms of a new vector valued variable

$$X^n = \frac{I^n}{n}, \qquad n \geq 1,$$

since

$$X_i^{n+1} = X_i^n - \frac{1}{n+1}\left(X_i^n - \rho_i \min\left(r_i^n V^{n+1}, D_i^{n+1}\right)\right), \quad i \in \mathcal{I}_N.$$

This is a standard of form a stochastic algorithm (with step $\gamma_n = \frac{1}{n}$).

Furthermore, note that setting $\underline{\rho} := \min_i \rho_i$,

$$\sum_{i\in\mathcal{I}_N} I_i^n \geq \underline{\rho} \min\left(\frac{1}{N}V^n, \min_{i\in\mathcal{I}_N} D_i^n\right)$$

since $r_i^n \geq \frac{1}{N}$ for at least one dark pool $i \in \mathcal{I}_N$. Consequently, as soon as the sequence $(V^n, D_1^n, \ldots, D_N^n)$ is stationary and ergodic

$$\liminf_n \sum_{i\in\mathcal{I}_N} X_i^n \geq \underline{\rho} \lim_n \frac{1}{n}\sum_{k=1}^n \min\left(\frac{1}{N}V^k, \min_{i\in\mathcal{I}_N} D_i^k\right) = \underline{\rho}\,\mathbb{E}\min\left(\frac{1}{N}V, \min_{i\in\mathcal{I}_N} D_i\right) \ a.s.$$

So if we make the natural assumption that

$$\mathbb{E}\min\left(\frac{1}{N}V, \min_{i\in\mathcal{I}_N} D_i\right) > 0$$

then, *a.s.*, the algorithm X^n cannot converge to 0.

If we make the additional assumption that the sequence $(V^n, D_1^n, \ldots, D_N^n)$ is *i.i.d.* then the algorithm is a discrete time (non homogeneous) $(\mathcal{F}_n)$-Markov process with respect to the filtration $\mathcal{F}_n = \sigma(V^k, D_1^k, \ldots, D_N^k, k = 1, \ldots, n)$, $n \geq 0$, so that it admits the canonical representation

$$X_i^{n+1} = X_i^n - \gamma_{n+1}\left(X_i^n - \varphi_i(r_i^n)\right) + \gamma_{n+1}\Delta M_i^{n+1} \quad i \in \mathcal{I}_N, \quad n \geq 0,$$

where $\gamma_n = \frac{1}{n}$ and

$$\Delta M_i^n = \rho_i \min\left(r_i^{n-1}V^n, D_i^n\right) - \varphi_i(r_i^{n-1}), \quad i \in \mathcal{I}_N, \quad n \geq 1,$$

is an $\mathcal{F}_n$-martingale increment. Furthermore it is L^2-bounded as soon as $V \in L^2$.

In fact the specific difficulties induced by this algorithm are more in relation with its mean function

$$h : x \longmapsto \left(x_i - \varphi_i\left(\frac{x_i}{\sum_j x_j}\right)\right)_{1\leq i\leq N} \tag{4.18}$$

than with the martingale "disturbance term" $\gamma_{n+1}\Delta M^{n+1}$. Our first task will be to prove under natural assumptions the existence of a non degenerate equilibrium point that is zero of the function h. Then we will show why this induces the existence of many parasitic equilibrium points.

4.6.1 Existence of an equilibrium

In this section, we will need to introduce a new function associated to a generic order V and a generic dark pool with characteristics (ρ, D).

$$\psi(u) := \frac{\varphi(u)}{u}, \quad u > 0, \qquad \psi(0) = \varphi'(0) = \rho \mathbb{E}\, V \mathbf{1}_{\{D>0\}}. \tag{4.19}$$

If Assumption (4.2) holds then $\psi(0) < +\infty$ and ψ is continuous at 0. It follows from the concavity of φ and $\varphi(0) = 0$ that ψ is non-increasing. It is continuous as soon as φ is *e.g.* if Assumption (4.5) holds true.

Proposition 4.3. *Let $N \geq 1$. Assume that Assumption (4.2) holds for every couple (V, D_i), $i \in \mathcal{I}_N$.*

(a) There exists a $x^* \in \mathbb{R}_+^N$ such that

$$\sum_{i\in\mathcal{I}_N} x_i^* > 0 \quad \text{and} \quad \varphi_i\left(\frac{x_i^*}{\sum_{j\in\mathcal{I}_N} x_j^*}\right) = x_i^*, \quad i \in \mathcal{I}_N. \tag{4.20}$$

(b) Let ψ_i be the functions associated to dark pool $i \in \mathcal{I}_N$ by (4.19). Assume that for every $i \in \mathcal{I}_N$, ψ_i is (continuous and) decreasing on $[0, \infty)$ and that

$$\sum_{i\in\mathcal{I}_N} \psi_i^{-1}(\min_{i\in\mathcal{I}_N} \varphi_i'(0)) < 1. \tag{4.21}$$

Then there exists $x^* \in int(\mathcal{P}_N)$ satisfying (4.20).

Proof. *(a)* We define for every $r = (r_1, \ldots, r_N) \in \mathcal{P}_N$

$$\Psi(r) := \left(\frac{\varphi_i(r_i)}{\sum_{j\in\mathcal{I}_N} \varphi_j(r_j)}\right)_{i\in\mathcal{I}_N}.$$

This function maps the compact convex set $\mathcal{P}_N$ into itself. Furthermore it is continuous since, on the one hand, for every $i \in \mathcal{I}_N$, φ_i is continuous owing to the fact that (V, D_i) satisfies (4.2) and, on the other hand,

$$\sum_{j\in\mathcal{I}_N} \varphi_j(r_j) \geq \min_{j\in\mathcal{I}_N} \varphi_j\left(\frac{1}{N}\right) > 0.$$

Indeed, for every $i \in \mathcal{I}_N$,

$$\varphi_j\left(\frac{1}{N}\right) \geq \frac{1}{N}\mathbb{E}\,\min(V, D_i) > 0$$

since $V > 0$ $\mathbb{P}$-*a.s.* and $\mathbb{P}(D_j = 0) < 1$. Then it follows from the Brouwer Theorem that the function Ψ has a fixed point r^*. Set for every $i \in \mathcal{I}_N$,

$$x_i^* = r_i^* \sum_{j\in\mathcal{I}_N} \varphi_j(r_j).$$

It follows immediately from this definition that

$$\forall\, i \in \mathcal{I}_N, \quad x_i^* = \varphi_i(r_i^*)$$

which in turn implies that $\sum_{j\in\mathcal{I}_N} \varphi_j(r_j^*) = \sum_{j\in\mathcal{I}_N} x_j^*$ so that $r_i^* = \frac{x_i^*}{\sum_{j\in\mathcal{I}_N} x_j^*}$, $i \in \mathcal{I}_N$.

(b) For every $i \in \mathcal{I}_N$ we consider the inverse of ψ_i defined on the interval $(0, \varphi_i'(0)]$. This function is decreasing continuous and $\lim_{v\to 0} \psi_i^{-1}(v) = +\infty$. Then, let Θ be the continuous function defined by

$$\forall\, \theta \in (0, \min_{i\in\mathcal{I}_N} \varphi_i'(0)], \quad \Theta(\theta) = \sum_{i\in\mathcal{I}_N} \psi_i^{-1}(\theta).$$

We know that $\lim_{\theta\to 0} \Theta(\theta) = +\infty$ and we derive from Assumption (4.21) that $\Theta(\min_{i\in\mathcal{I}_N} \varphi_i'(0)) \le 1$. So, owing to the (strict) monotonicity of θ, there exists $\theta^* \in (0, \min_{\in\mathcal{I}_N} \varphi_i'(0))$ such that $\Theta(\theta^*) = 1$. Set

$$r_i^* = \psi_i^{-1}(\theta^*), \quad i \in \mathcal{I}_N.$$

Then $r^* := (r_1^*, \ldots, r_N^*) \in int(\mathcal{P}_N)$ since $\sum_i r_i^* = 1$ by definition of θ^*. If $r_{i_0}^* = 0$, then $\theta^* = \psi_{i_0}(0) = \min_{i\in\mathcal{I}_N} \varphi_i'(0)$ which is impossible. □

Corollary 4.1. *Assume that all the functions ψ_i are continuous and decreasing. If furthermore, the rebate coefficients ρ_i are equal (to 1) and if $\mathbb{P}(D_i = 0) = 0$ for every $i \in \mathcal{I}_N$ then there exists an equilibrium point lying in $int(\mathcal{P}_N)$.*

Proof. Under the above assumptions $\varphi_i'(0) = \mathbb{E}\, V > 0$. Consequently

$$\psi_i^{-1}(\min_{i\in\mathcal{I}_N} \varphi_i'(0)) = \psi_i^{-1}(\psi_i(0)) = 0 < 1.$$ □

COMMENTS. Unfortunately there is no hope to prove that all the equilibrium points lie in the interior of $\mathcal{P}_N$ since one may always adopt an execution strategy which boycotts a given dark pool or, more generally, N_0 dark pools. So it seems hopeless to get uniqueness of the equilibrium point. To be more precise, under the assumptions of claim (b) of the above Proposition 4.3, there exists at least one strategy involving a subset of $N - N_0$ dark pools $N_0 = 0, \ldots, N-1$ (one dark pool is needed at least). Elementary combinatorial arguments show that there are *at least* $2^N - 1$ equilibrium points.

So, from a theoretical point of view, we are facing a situation where there may be many parasitic equilibrium points, some of them being clearly parasitic. However it is quite difficult to decide *a priori*, even if we make the unrealistic assumption that we know all the involved distributions, which equilibrium points are parasitic.

This is a typical situation encountered when dealing with procedures devised from a reinforcement principle.

However, one may reasonably hope that some of them are so-called "traps", that means equilibrium points which are repulsive at least in one noisy direction so that the algorithm escapes from it. Another feature described below suggests that a theoretical study of the convergence behaviour of this procedure would need a specific extra work.

The next natural question is to wonder whether an equilibrium x^* of the algorithm – namely a zero of h – is (at least) a target for the algorithm *i.e.* is attractive for the companion *ODE*, $\dot{x} = -h(x)$.

Proposition 4.4. *An equilibrium x^* satisfying (4.20) is locally uniformly attractive as soon as*

$$\sum_{j\in\mathcal{I}_N} \frac{x_j^*}{(\bar{x}^*)^2} \varphi_j'\left(\frac{x_j^*}{\bar{x}^*}\right) < 1 - \frac{1}{\bar{x}^*} \max_{i\in\mathcal{I}_N} \varphi'\left(\frac{x_i^*}{\bar{x}^*}\right)$$

where $\bar{x}^* = \sum_{i\in\mathcal{I}_N} x_i^*$.

Remark. In fact the following inequalities are satisfied by any equilibrium x^*:

$$1 - \frac{1}{\bar{x}^*}\varphi_i'\left(\frac{x_i^*}{\bar{x}^*}\right) > 0, \quad i \in \mathcal{I}_N.$$

This follows from the convexity of the function $\xi \mapsto \xi - \varphi_i\left(\frac{\xi}{\bar{x}^*}\right)$ which is zero at x_i^* with positive derivative. As a consequence, the right hand side in the above sufficient condition is always positive which makes this criterion more realistic.

Proof. Elementary computations show that the differential $Dh(x)$ of h at $x \in \mathbb{R}_+^N$ is given by

$$\forall\, i,\, j \in \mathcal{I}_N, \qquad \frac{\partial h_i}{\partial x_j}(x) = \delta_{ij}\left(1 - \frac{1}{\bar{x}}\varphi_i'\left(\frac{x_i}{\bar{x}}\right)\right) + \frac{x_i}{\bar{x}^2}\varphi_i'\left(\frac{x_i}{\bar{x}}\right).$$

As a consequence all the diagonal terms of $Dh(x^*)$ are positive. The above condition for all the eigenvalues of $Dh(x)$ to have positive real parts follows from a standard application of Gershgorin's Lemma to the transpose of $Dh(x)$. □

4.6.2 A competitive system

But once again, even if we could show that all equilibrium points are noisy traps, the convergence would not follow for free since this algorithm is associated to a so-called *competitive system*. A competitive differential system $\dot{x} = h(x)$ is a system in which the field $h : \mathbb{R}^N \to \mathbb{R}^N$ is differentiable and satisfies

$$\forall x \in \mathbb{R}^N,\ \forall\, i,\, j \in \mathcal{I}_N,\ i \neq j, \qquad \frac{\partial h_i}{\partial x_j}(x) > 0.$$

As concerns Almgren and Harts's algorithm, the mean function h is given by (4.18), and under the standard differentiability assumption on the functions φ_i's,

$$\forall x \in \mathbb{R}^N, \quad \frac{\partial h_i}{\partial x_j}(x) = \varphi_i'\left(\frac{x_i}{x_1 + \cdots + x_N}\right)\frac{x_i}{(x_1 + \cdots + x_N)^2} > 0.$$

These systems are known to have possibly a non converging behaviour even in presence of a single (attracting) equilibrium. This is to be compared to their *cooperative* counterparts (with negative non-diagonal partial derivatives) whose flow converge uniformly on compact sets toward the single equilibrium in that case. This property can be transferred to the stochastic procedure by the mean if the so-called ODE method which shows that the algorithm almost behaves like some trajectories of the Ordinary differential Equation associated to its mean field h (see *e.g.* [75, 25, 43] for an introduction). Cooperativeness and competitiveness are in fact some criteria which ensure some generalized monotonicity properties on the flow of the ODE viewed as a function of its space variable. For some background on cooperative and competitive systems we refer to [64, 65] and the references therein.

4.7 Numerical tests

The aim of this section is to compare the behaviour of both algorithms on different data sets : simulated i.i.d. data, simulated α-mixing data and (pseudo-)real data.

Two natural situations of interest can be considered *a priori*: *abundance* and *shortage*. By "abundance" we mean $\mathbb{E}\,V \leq \sum_{i=1}^N \mathbb{E}D_i$ (in average, the requested volume is lower than the available one). The "shortage" setting is the reverse situation where $\mathbb{E}\,V > \sum_{i=1}^N \mathbb{E}D_i$.

In fact, in the "abundance" setting, both our procedures (optimization and reinforcement) tend to remain "frozen" at their starting allocation value (usually uniform allocation) and they do not provide a significant improvement with respect to more naive approaches. By contrast the shortage setting is by far more commonly encountered on true markets and turns out to be much more challenging for our allocation procedures, so from now on we will focus on this situation.

Our first task is to define a reference strategy. To this end, we introduce an "oracle strategy" devised by an insider who knows all the values V^n and D_i^n before making his/her optimal execution requests to the dark pools. It can be described as follows: assume for simplicity that the rebates are ordered *i.e.* $\rho_1 > \rho_2 > \cdots > \rho_N$. Then, it is clear that the "oracle" strategy yields the following cost reduction (CR) of the execution at time $n \geq 1$,

$$\mathrm{CR}^{oracle} := \begin{cases} \displaystyle\sum_{i=1}^{i_0-1} \rho_i D_i^n + \rho_{i_0} \left(V^n - \sum_{i=1}^{i_0-1} D_i^n \right), & \text{if } \displaystyle\sum_{i=1}^{i_0-1} D_i^n \leq V^n < \sum_{i=1}^{i_0} D_i^n \\ \displaystyle\sum_{i=1}^{N} \rho_i D_i^n, & \text{if } \displaystyle\sum_{i=1}^{N} D_i^n < V^n. \end{cases}$$

Now, we introduce indexes to measure the performances of our recursive allocation procedures.

- **Relative cost reduction (w.r.t. the primary market):** they are defined as the ratios between the cost reduction of the execution using dark pools and the cost resulting from an execution on the primary market for the three algorithms, *i.e.*, for every $n \geq 1$,
  - *Oracle:* $\dfrac{CR^{oracle}}{V^n}$
  - *Recursive "on-line" algorithms:* $\dfrac{CR^{algo}}{V^n} = \dfrac{\sum_{i=1}^N \rho_i \min\left(r_i^n V^n, D_i^n\right)}{V^n}$ (with $algo = opti, reinf$).
- **Performances (w.r.t. the oracle):** the ratios between the relative cost reductions of our allocation algorithms and that of the oracle, *i.e.* for every $n \geq 1$

$$\frac{CR^{opti}}{CR^{oracle}} \qquad \text{and} \qquad \frac{CR^{reinf}}{CR^{oracle}}$$

which seems a more realistic measure of the performance of our allocation procedures since the oracle strategy cannot be beaten.

Since these relative cost reductions are strongly fluctuating (with variables V^n and D_i^n in fact), we will plot *the moving average* of these ratios (on the running period of interest) and express them in percentage.

Moreover, when we simulate the data, we have chosen 10^4 simulations because it corresponds approximatively to the number of pseudo-real data observed within a day.

The choice of the gain parameter is the following (in the different settings considered below)

$$\gamma_n = \frac{c}{n}, \quad n \geq 1$$

where c equals to few units.

4.7.1 The (IID) setting

We consider here simulated data in the i.i.d. setting, where the quantity V and D_i, $i \in \mathcal{I}_N$, are log-normal variables and $N = 3$. The variables V and D_i, $i \in \mathcal{I}_N$, satisfy the assumptions of the CLT and we have the rate of convergence at least of the optimization algorithm.

The shortage setting is specified as follows:

$$\mathbb{E}\,V = \frac{3}{2}\sum_{i=1}^{N}\mathbb{E}D_i$$

with

$$\mathbb{E}D_i = i,\ 1 \le i \le N, \quad \mathrm{Var}(V) = 1, \mathrm{Var}(D_i) = 1,\ 1 \le i \le N \quad \text{and } \rho = \begin{pmatrix} 0.01 \\ 0.03 \\ 0.05 \end{pmatrix}.$$

The running means of the performances are computed from the very beginning for the first 100 data, and by a moving average on a window of 100 data.

The initial value for both algorithms is set at $r_i^0 = \frac{1}{N}$, $1 \le i \le N$.

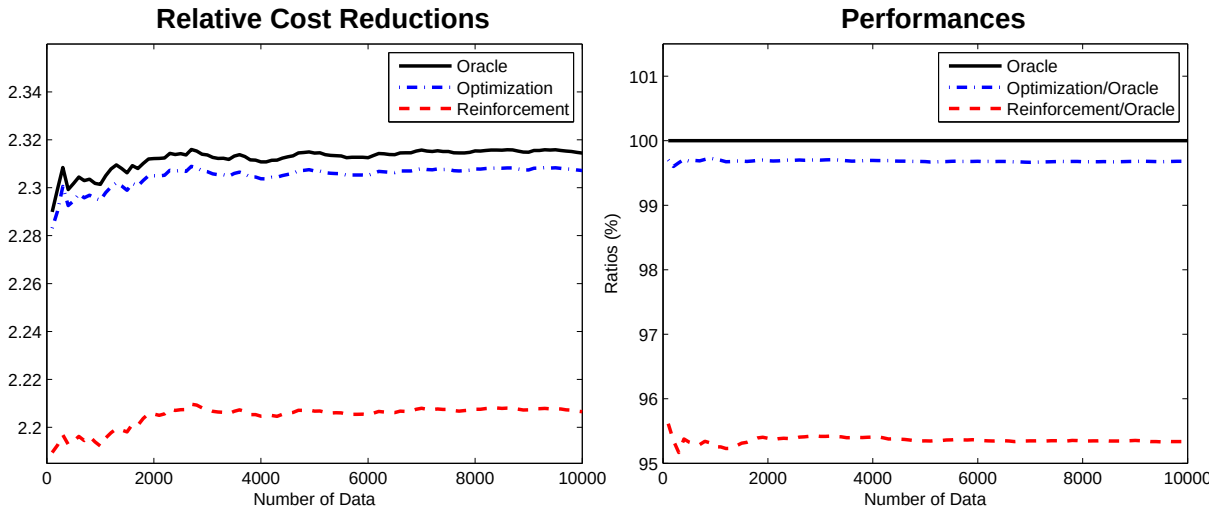

Figure 4.1: *Shortage setting*: Case $N = 3$, $m_V = \frac{3}{2}\sum_{i=1}^{N} m_{D_i}$, $m_{D_i} = i$, $\sigma_V = 1$, $\sigma_{D_i} = 1$, $1 \le i \le N$.

As expected, the optimization procedure outperforms the reinforcement one and both procedures quickly converge (see Figure 4.1) with respect to the data set size. Note that the allocation coefficients (not reproduced here) generated by the two algorithms are significantly different. A more interesting feature is that the performances of the optimization procedure almost replicate those of the "oracle". Further simulations suggest that the optimization algorithm also seems more robust when the variances of the random variables fluctuate.

4.7.2 The (ERG) setting

We consider here simulated data in the ergodic setting, where the quantity V and D_i, $i \in \mathcal{I}_N$, are exponentials of an Ornstein-Uhlenbeck process, *i.e.*

$$X^{n+1} = m + AX^n + B\Xi^{n+1},$$

where $\|A\| < 1$, $BB^* \in GL(d, \mathbb{R})$ and

$$m = \begin{pmatrix} m_1 \\ \vdots \\ m_{N+1} \end{pmatrix} \in \mathbb{R}^{N+1}, \quad \Xi^{n+1} = \begin{pmatrix} \Xi_1^{n+1} \\ \vdots \\ \Xi_{N+1}^{n+1} \end{pmatrix} \sim \mathcal{N}(0, I_{N+1}) \text{ i.i.d.}, \quad e^{X^n} = \begin{pmatrix} V^n \\ D_1^n \\ \vdots \\ D_N^n \end{pmatrix}.$$

We are still interested in the shortage situation. The initial value of the algorithms is $r_i^0 = \frac{1}{N}$, $1 \leq i \leq N$ and we set

$$\rho = \begin{pmatrix} 0.01 \\ 0.03 \\ 0.05 \end{pmatrix}, \quad m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.7 & 0.01 & 0.01 & 0.01 \\ 0.01 & 0.3 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.2 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.02 & 0 & 0 & 0 \\ 0.01 & 0.9 & 0 & 0 \\ 0.01 & 0.01 & 0.6 & 0 \\ 0.01 & 0.01 & 0.01 & 0.3 \end{pmatrix}.$$

The running means of the performances are computed from the very beginning for the first 100 data, and by a moving average on a window of 100 data.

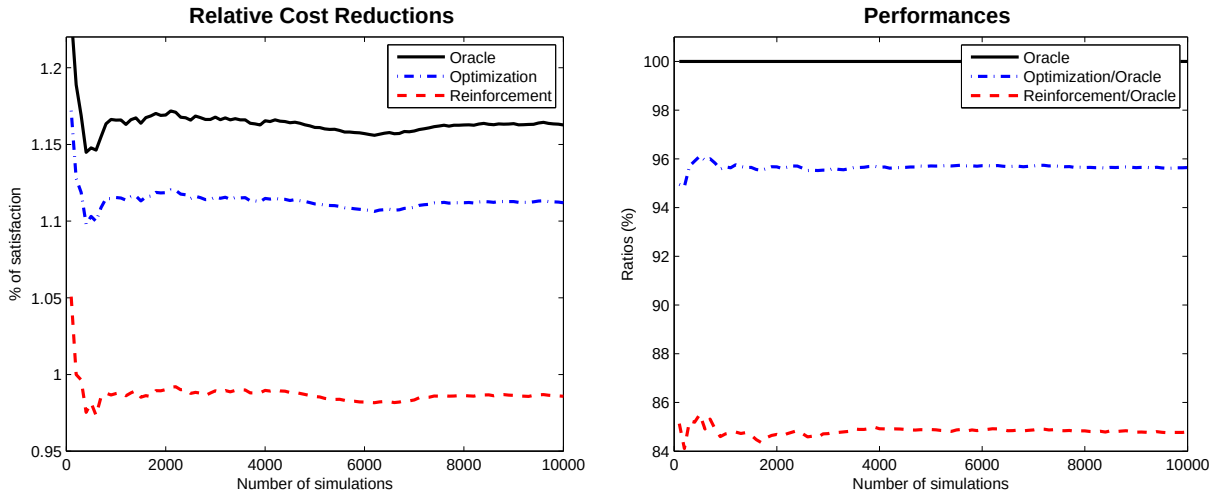

Figure 4.2: *Shortage setting*: Case $N = 3$, $m_V \geq \sum_{i=1}^N m_{D_i}$, $\sigma_V = 1.21$, $\sigma_D = (8.21, 3.05, 1.07)^t$.

We observe in this ergodic setting a very similar behaviour to the i.i.d. one, with maybe a more significant advantage for the optimization approach (see Figure 4.2 right): the difference between the performances of both algorithms reaches 11% in favour of the optimization algorithm.

4.7.3 The pseudo-real data setting

Firstly we explain how the data have been created. We have considered for V the traded volumes of a very liquid security – namely the asset BNP – during an 11 day period. Then we selected the N most correlated assets (in terms of traded volumes) with the original asset. These assets are denoted S_i, $i = 1, \ldots, N$ and we considered their traded volumes during the same 11 day period. Finally, the available volumes of each dark pool i have been modeled as follows using the mixing function

$$\forall 1 \leq i \leq N, \quad D_i := \beta_i \left((1 - \alpha_i)V + \alpha_i S_i \frac{\mathbb{E}\, V}{\mathbb{E} S_i} \right)$$

where $\alpha_i \in (0, 1)$, $i = 1, \ldots, N$ are the recombining coefficients, β_i, $i = 1, \ldots, N$ are some scaling factors and $\mathbb{E}\, V$ and $\mathbb{E} S_i$ stand for the empirical mean of the data sets of V and S_i.

The shortage situation corresponds to $\sum_{i=1}^N \beta_i < 1$ since it implies $\mathbb{E}\left[\sum_{i=1}^N D_i\right] < \mathbb{E}\, V$.

The simulations presented here have been made with four dark pools ($N = 4$). Since the data used here covers 11 days and it is clear that unlike the simulated data, these pseudo-real data are not stationary: in particular they are subject to daily changes of trend and volatility (at least). To highlight these resulting changes in the response of the algorithms, we have specified the days by

drawing vertical doted lines. The dark pool pseudo-data parameters are set to

$$\beta = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.2 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} 0.4 \\ 0.6 \\ 0.8 \\ 0.2 \end{pmatrix}$$

and the dark pool trading (rebate) parameters are set to $\rho = \begin{pmatrix} 0.01 \\ 0.02 \\ 0.04 \\ 0.06 \end{pmatrix}$.

The mean and variance characteristics of the data set of $(V^n)_{n\geq 1}$ and $(D_i^n)_{n\geq 1}$, $i = 1, \ldots, 4$ are the following:

	V	D_1	D_2	D_3	D_4
Mean	955.42	95.54	191.08	286.63	191.08
Variance	2.01×10^6	9.05×10^3	4.29×10^4	4.73×10^5	5.95×10^4

Table 4.1: The mean and variance characteristics of the data set.

Firstly, we benchmarked both algorithms on the whole data set (11 days) as though it were stationary without any resetting (step, starting allocation, etc.). In particular, the running means of the satisfactions ratios are computed from the very beginning for the first 1500 data, and by a moving average on a window of 1500 data. As a second step, we proceed on a daily basis by resetting the parameters of both algorithms (initial allocation for both and the step parameter γ_n of the optimization procedure) at the beginning of every day.

▷ **Long-term optimization** We observe that, except for the first and the fourth days where they behave similarly, the optimization algorithm is more performing than the reinforcement one. Its performance is approximately 30 % higher on average (see Figure 4.3).

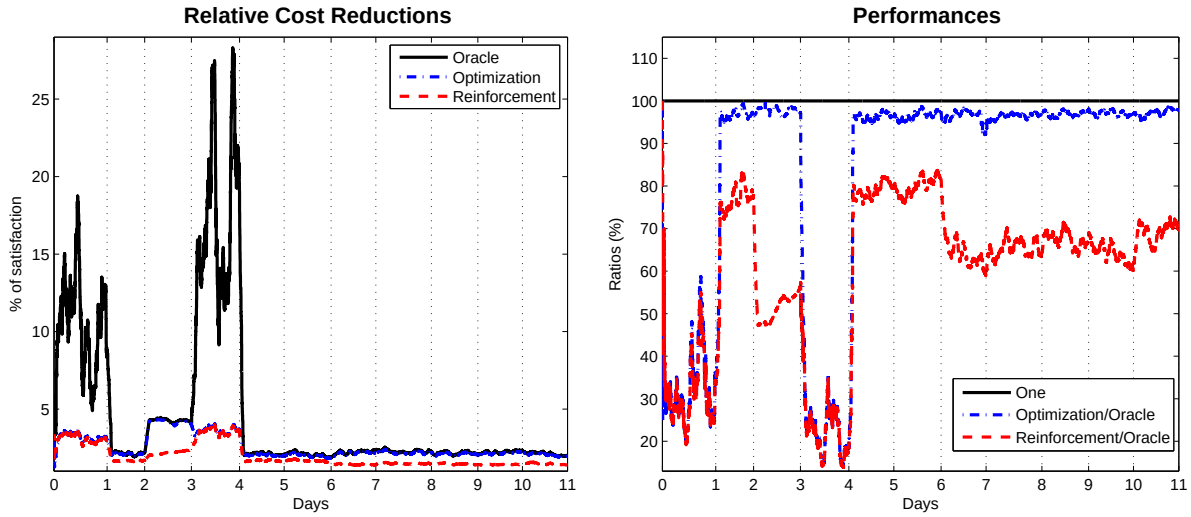

Figure 4.3: *Long term optimization*: Case $N = 4$, $\sum_{i=1}^N \beta_i < 1$, $0.2 \leq \alpha_i \leq 0.8$ and $r_i^0 = 1/N$, $1 \leq i \leq N$.

This test confirms that the statistical features of the data are strongly varying from one day to another (see Figure 4.3), so there is no hope that our procedures converge in standard sense on a long term period. Consequently, it is necessary to switch to a short term monitoring by resetting the parameters of the algorithms on a daily basis as detailed below.

▷ **Daily resetting of the procedure** We consider now that we reset on a daily basis all the parameters of the algorithm, namely we reset the step γ_n at the beginning of each day and the satisfaction parameters and we keep the allocation coefficients of the preceding day. We obtain the following results

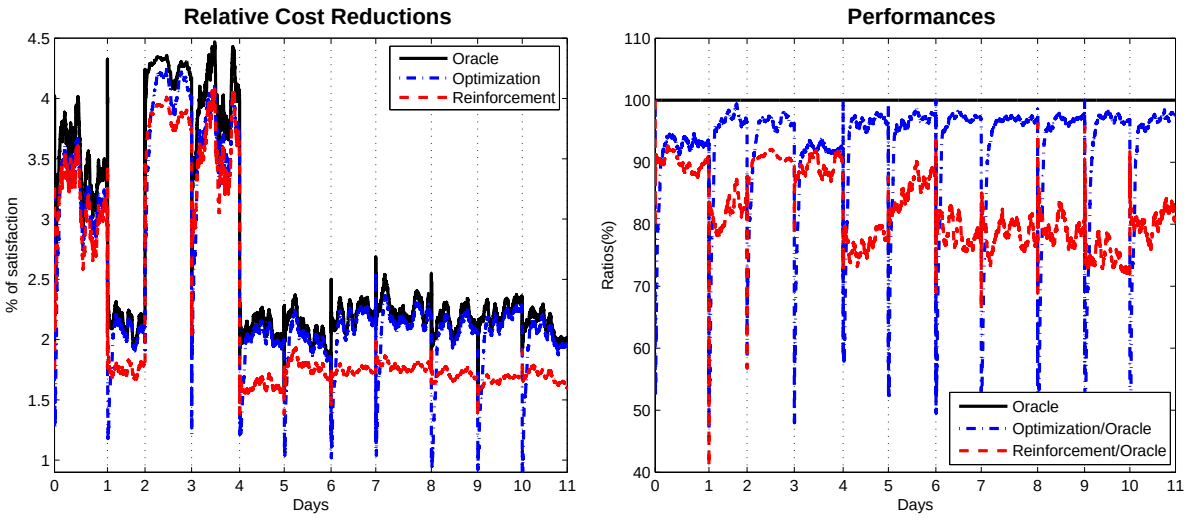

Figure 4.4: *Daily resetting of the algorithms parameters*: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0.2 \leq \alpha_i \leq 0.8$ and $r_i^0 = 1/N$ $1 \leq i \leq N$.

We observe (see Figure 4.4) that the optimization algorithm still significantly outperforms the reinforcement one, reaching more 95 % of the performance of the oracle. Furthermore, although not represented here, the allocation coefficients look more stable.

4.8 Provisional remarks

4.8.1 A variant including some market impact effect

In order to include some market impact effect we introduce a differentiable nondecreasing function $\eta : \mathbb{R}_+ \to [1, +\infty)$ such that

$$\eta(0) = 1.$$

We make the assumption that buying a volume, say W on the primary market will make the agent go deeper in the order book so that he/she will pay the security at price $S\,\eta(W)$ instead of the quoted price S. Typically, one may set $\eta(u) = e^{\kappa u}$, $u \geq 0$, $\kappa > 0$.

To keep our problem meaningful, we make the additional integrability assumption

$$\mathbb{E}\Big(V\eta(V) + V^2 \sup_{0\leq v\leq V} |\eta'(v)|\Big) < +\infty.$$

Then, going back to our original dynamical modeling (Section 2.2) one easily checks that under the assumptions (H_1)-(H_4) the optimization problem amounts to **minimizing** the function Ψ_η

defined by

$$\Psi_\eta : r \longmapsto \mathbb{E}\left[\sum_{i=1}^N \theta_i \min\left(r_i V, D_i\right) + a_{\Delta t}\eta\Big(V - \sum_{i=1}^N \min(r_i V, D_i)\Big)\Big(V - \sum_{i=1}^N \min(r_i V, D_i)\Big)\right].$$

Differentiating this function yields

$$\frac{\partial \Psi_\eta}{\partial r_i} = a_{\Delta t}\mathbb{E}\Big(\widetilde{H}_{\eta,i}(r,V,D)\Big)$$

where $$\widetilde{H}_{\eta,i}(r,V,D) = V\mathbf{1}_{\{r_i V \le D_i\}}\left(\frac{\theta_i}{a_{\Delta t}} - \mathbf{n}'\Big(V - \sum_{i=1}^N \min(r_i V, D_i)\Big)\right)$$

with $\mathbf{n}(u) := u\eta(u)$ so that $\mathbf{n}'(u) = \eta(u) + u\eta'(u)$. Then relying again on the first order Lagrangian heuristic to say that all partial derivatives need to be equal at the minimum, this leads to the following stochastic algorithm

$$r_i^{n+1} = r_i^n - \gamma_{n+1}\Big(\widetilde{H}_{\eta,i}(r^n, V^n, D^n) - \frac{1}{N}\sum_{j=1}^N \widetilde{H}_{\eta,j}(r^n, V^n, D^n)\Big), \quad n \ge 0,$$

and $r^0 \in \mathcal{P}_N$ (if $\eta \equiv 1$ we fall upon our former algorithm).

Numerical Example: We consider the case of pseudo-real data with $a_{\Delta t} \equiv 1$ and $\eta(u) = e^{\kappa u}$, $u \ge 0$, $\kappa > 0$ with a resetting on a daily basis.

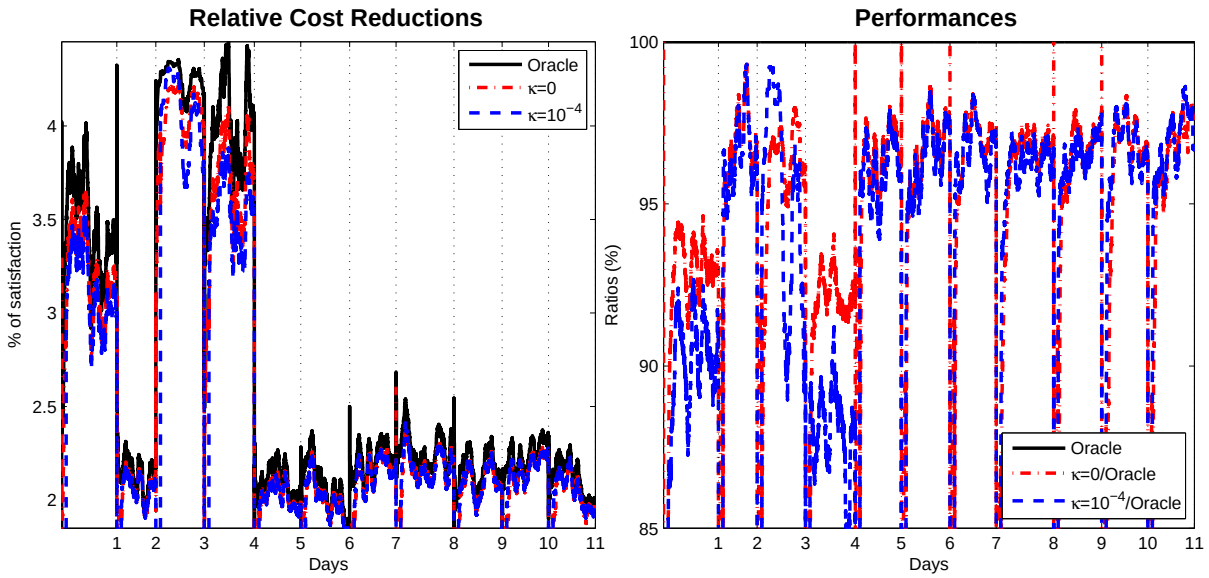

Figure 4.5: *Daily resetting of the algorithms parameters*: Case $N = 4$, $\sum_{i=1}^N \beta_i < 1$, $0.2 \le \alpha_i \le 0.8$ and $r_i^0 = 1/N$ $1 \le i \le N$.

4.8.2 Taking advantage of the duality "Optimization *vs* reinforcement"?

For practical implementation what conclusions can be drawn from our investigations on both procedures. Both reach quickly a stabilization/convergence phase close to optimality. The reinforcement algorithm leaves the simplex structurally stable which means the proposed dispatching

at each time step is realistic whereas the stochastic Lagrangian algorithm in its present form sometimes needs to be corrected from time to time. This can be corrected by adding a projection on the simplex at each step.We did not consider this variant from a theoretical point of view to keep our convergence proofs more elementary.

In a high volatility context, the stochastic Lagrangian algorithm clearly prevails with performances that turn out to be significantly better. This optimization procedure also relies on established convergence results in a rather general framework. However, given the computational cost of each procedure which is close to zero, a possible strategy is to implement both in parallel to get a synergistic effect *i.e*; a *dynamical calibration*. In particular, one may use the reinforcement algorithm – which step parameter is structurally fixed equal to $\frac{1}{n}$ – can be used to help tuning the constant c of the gain parameter $\gamma_n = \frac{c}{n}$ of the stochastic Lagrangian. Doing so one may start with a small constant c that could be slowly increased, preventing an explosion of the variance of the procedure. Then based one may increase slowly this constant until the Lagrangian outperforms the reinforcement procedure.

Chapter 5

Optimal posting distance of limit orders: a stochastic algorithm approach

This chapter is based on a preprint.

This chapter presents a stochastic recursive procedure under constraints to find the optimal distance at which an agent must post his order to minimize his execution cost. We prove the *a.s.* convergence of the algorithm under assumptions on the cost function and give some practical criteria on model parameters to ensure that the conditions to use the algorithm are fulfilled (using notably principle of opposite monotony). We illustrate our results with numerical experiments on simulated data but also by using a financial market dataset.

5.1 Introduction

In the recent years, with the growth of electronic trading, most of the transactions in the markets occur in *Limit Order Books*. During the matching of electronic orders, traders send orders of two kinds to the market: passive (*i.e.* limit or patient orders) which will not give birth to a trade but will stay in the order book (sell orders at a higher price than the *higher bid price* or buy orders at a higher price than the *lower ask price* are passive orders) and aggressive orders (*i.e.* market or impatient orders) which will generate a trade (sell orders at a lower price than the higher remaining buy price or buy orders at a higher price than the lowest remaining price). When a trader has to buy or sell a large number of shares, he cannot just send this large order at once (the price moves during the execution procedure); he has first to schedule his trading rate to make balance between the market risk and the market impact cost of being too aggressive (too many orders exhaust the order book and makes the price move). Several theoretical frameworks have been proposed for optimal scheduling of large orders (see [9], [31], [100], [6]). Once this optimal trading rate is known, the trader has to send smaller orders in the electronic book by alternating limit (*i.e.* patient) orders and market (*i.e.* urgent) orders. The optimal mix of limit and market orders for a trader has not been studied in the quantitative literature even if it has been studied from a global economic efficiency viewpoint (see for instance [50] and not from the viewpoint of one trader trying to optimize its own interactions with other market participants). One of the difficulties from the trader prospective is that transactions obtained by inserting a limit order in an electronic book is a functional of its distance to the mid-price, giving birth to a large number of possible tactics in terms of reassessment of the price of such orders.

In this chapter, we study the optimal distance to submit limit orders in an electronic order book, without needing a model of the limit order book dynamics (see [1] or [61] for such models of limit order books).

Optimal submission strategies have been studied in the microstructure literature using utility framework and optimal control (see [16], [60], [21], [62] and [63]). The authors consider an agent who plays the role of a dealer, *i.e.* he provides liquidity on the exchange by quoting bid and ask prices at which he is willing to buy and sell a specific quantity of assets. Strategies for bid and ask orders are derived by maximizing his utility function.

Our approach is different: we consider an agent who wants to buy (or sell) on a short period $[0, T]$ a quantity Q_T of traded assets and we look for the optimal distance where he must post his order to minimize the execution cost.

We are typically at a smaller time scale than in optimal liquidation frameworks (or any market making framework coming from a backward stochastic control process) and order posting strategies derived from the viewpoint presented here can be "plugged" into any larger scale strategy.

If a stochastic algorithm approach has been already proposed by the authors for optimal spacial split of an order across different Dark Pools, here the purpose is not to control fractions of the size of an order, but to adjust successive posting prices to converge to an optimal price. Qualitatively, this framework can be used as soon as a trader wants to trade a given quantity Q_T over a given time interval $[0, T]$. The trader can post his order very close to the "*fair price*"(that can be seen as the fundamental price, the mid price of the available trading venues or any other reference price). In this case he will be exposed to the risk to trade too fast at a "bad price", or he can post it far away from the fair price, and in this case he will be exposed to never obtain a transaction for the quantity Q_T, but only a part of it (say the positive part of $Q_T - N_T$, where N_T is the quantity that the trading flow allowed to trade). He will then have to consume aggressively liquidity with the remaining quantity, disturbing the market and paying not only the current market price S_T, but also market impact (i.e. $S_T\,\Phi(Q_T - N_T)$).

The approach presented here follows the mechanism of a "learning trader". He will try to guess the optimal posting distance to the fair price achieving the balance between being too demanding in price and too impatient, by trials and errors. The optimal recursive procedure derived from our framework gives the best price adjustment to apply to an order given the observed past on the market. We provide proofs of the convergence of the procedure and of its optimality.

To this end we model the execution process of orders by a Poisson process $(N_t^{(\delta)})_{0\leq t\leq T}$ whose intensity $\Lambda_T(\delta, S)$ depends on the fair price $(S_t)_{t\geq 0}$ and the distance of order submission δ. The execution cost results from the sum of the price of the executed quantity and a penalization function depending on the remaining quantity to be executed at the end of the period $[0, T]$. This penalty models the over-cost induced by crossing the spread and the market impact of this execution. The aim is to find the optimal distance $\delta^* \in [0, \delta_{\max}]$, where $\delta_{\max}$ is the depth of the limit order book, which minimize the execution cost. This leads to an optimization problem under constraints which we solve by using a recursive stochastic procedure with projection (This particular class of algorithm is studied in [74] and [75]). We prove the *a.s.* convergence of the constrained algorithm under additional assumptions on the execution cost function. From a practical point of view, it is not easy to check the conditions on the cost function. So we give sufficient criteria on model parameters to ensure the viability of the algorithm which relies on a principle of opposite and co-monotony for diffusion process. Our approach consists to start from the co-monotony principle for n-tuples of independent variables established for functions where marginals are co-monotonic component by component. We first apply this result to the Euler scheme of the diffusion, under appropriate conditions on the drift and the diffusion coefficient. As a second step, we use a weak functional convergence theorem to transfer the principle to co(or anti-)-monotonic functionals of the diffusion (for a given functional F on $\mathcal{C}([0, T], \mathbb{R})$, monotonic should be understood hence as

$$(\forall t \in [0, T], \quad \alpha(t) \leq \beta(t)) \Longrightarrow F(\alpha) \leq F(\beta).$$

We conclude this chapter by some numerical experiments with simulated and real data. We consider

the Poisson intensity presented in [16] and use a Brownian motion to model the fair price dynamics. We plot the cost function and its derivative and show the convergence of the algorithm to its target δ^*.

The chapter is organized as follows: In Section 5.2, we first propose a model for the execution process of posted orders, then we define a penalized cost function (including the market impact at the terminal execution date). Then we devise a constrained stochastic recursive procedure to solve the resulting optimization problem in terms of optimal posting distance on the limit order book. We state the main convergence result and provide operating criteria that ensure this convergence, based on a monotony principle for one dimensional diffusions. Section 5.3 establishes the representations as expectations of the cost function and its derivatives which allow to define the mean function of the algorithm. Section 5.4 presents the principle of opposite monotony which is used in Section 5.5 to derive the convergence criteria (which ensure that the optimization is well-posed). Finally Section 5.6 illustrates with numerical experiments the convergence result of the recursive procedure towards its target.

Notations. • $(x)_+ = \max\{x, 0\}$ denotes the positive part of x, $\lfloor x \rfloor = \max\{k \in \mathbb{N} : k \leq x\}$.
• $\mathcal{C}([0,T], A) := \{f : [0,T] \to A \text{ continuous}\}$ and $\mathbb{D}([0,T], A) := \{f : [0,T] \to A \text{ càdlàg}\}$ where $A = \mathbb{R}^q$, $\mathbb{R}^q_+$, etc.

5.2 Design of the execution procedure and main results

5.2.1 Modeling and design of the algorithm

We consider on a short period T, say a dozen of seconds, a Poisson process modeling the execution of posted passive buy orders on the market

$$\left(N_t^{(\delta)}\right)_{0\leq t\leq T} \quad \text{with intensity} \quad \Lambda_T(\delta, S) := \int_0^T \lambda(S_t - (S_0 - \delta))dt \tag{5.1}$$

where $0 \leq \delta \leq \delta_{\max}$ with $\delta_{\max} \in (0, S_0)$ denotes the depth of the order book and $(S_t)_{t\geq 0}$ is a stochastic process modeling the dynamics of the "fair price" of a security stock (from an economic point of view). In practice one may consider that S_t represents the best opposite price. We assume that the function λ is defined on the whole real line as a finite non-increasing convex function. Its specification will rely on parametric or non parametric statistical estimation based on former transactions (see Figure 5.1 below and Section 5.6). At time $t = 0$, buy orders are posted in the limit order book at price $S_0 - \delta$. Between t and $t + \Delta t$, the probability for such an order to be executed is $\lambda(S_t - (S_0 - \delta))\Delta t$ where $S_t - (S_0 - \delta)$ is the distance to the current fair price of our posted order at time t. The further the order is at time t, the lower is the probability for this order to be executed since λ is decreasing on $[-S_0, +\infty)$. Empirical tests strongly confirm this kind of relationship with a convex function λ (even close to an exponential shape, see Figure 5.1). Over the period $[0, T]$, we aim at executing a portfolio of size $Q_T \in \mathbb{N}$ invested in the asset S. The execution cost for a distance δ is $\mathbb{E}\left[(S_0 - \delta)\left(Q_T \wedge N_T^{(\delta)}\right)\right]$. We add to this execution cost a penalization depending on the remaining quantity to execute, namely at the end of the period T, we want to have Q_T assets in the portfolio, so we buy the remaining quantity $\left(Q_T - N_T^{(\delta)}\right)_+$ at price S_T. Then we introduce a *market impact penalization function* $\Phi : \mathbb{R} \mapsto \mathbb{R}_+$, nondecreasing and convex, with $\Phi(0) = 0$ to model the additional cost of the execution of the remaining quantity (including the market impact). Then the resulting cost of execution on a period $[0, T]$ reads

$$C(\delta) := \mathbb{E}\left[(S_0 - \delta)\left(Q_T \wedge N_T^{(\delta)}\right) + \kappa\, S_T\, \Phi\left(\left(Q_T - N_T^{(\delta)}\right)_+\right)\right] \tag{5.2}$$

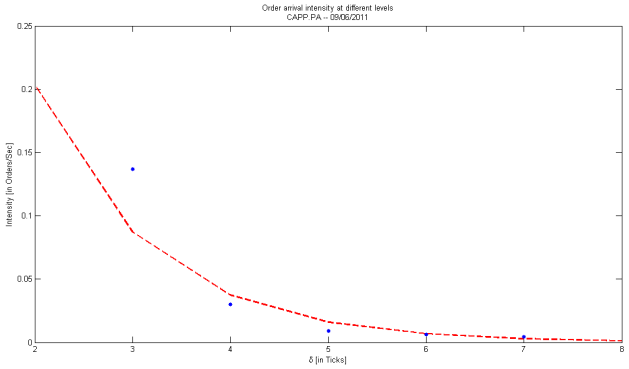

Figure 5.1: Empirical probabilities of execution (blue stars) and its fit with an exponential law (red dotted line) with respect to the distance to the "fair price".

where $\kappa > 0$ is free tuning parameter. When $\Phi = \mathrm{id}$, we just consider that we buy the remaining quantity at the end price S_T, but introducing a market impact penalization function $\Phi(x) = (1 + \eta(x))x$, where $\eta \geq 0$, $\eta \not\equiv 0$, models the market impact induced by the execution of $\left(Q_T - N_T^{(\delta)}\right)_+$ at the end of the period whereas we neglect the market impact of the execution process over $[0, T]$. Our aim is then to minimize this cost, namely to solve the following optimization problem

$$\min_{0 \leq \delta \leq \delta_{\max}} C(\delta). \tag{5.3}$$

Our strategy to solve numerically (5.3) using a large enough dataset is to take advantage of the representation of C and its first two derivatives as expectations to devise a recursive stochastic algorithm, namely a stochastic gradient procedure, to find the minimum of the (penalized) cost function. To be more precise we will show that under natural assumptions on the quantity Q_T to be executed and on the parameter κ, the function C is twice differentiable, *strictly convex on* $[0, \delta_{\max}]$ *with* $C'(0) < 0$. Consequently,

$$\mathrm{argmin}_{\delta \in [0, \delta_{\max}]} C(\delta) = \{\delta^*\}, \quad \delta^* \in (0, \delta_{\max}]$$

and

$$\delta^* = \delta_{\max} \quad \text{iff} \quad C \text{ is non-increasing on } [0, \delta_{\max}].$$

Criteria involving κ and based on both the risky asset S and the trading process especially the execution intensity λ, are established further on in Proposition 5.7 and Proposition 5.8. We specify representations as expectations of the function C and its derivatives C' and C''. In particular we will show that there exists a Borel functional

$$H : [0, \delta_{\max}] \times \mathbb{D}([0, T], \mathbb{R}) \longrightarrow \mathbb{R}$$

such that

$$\forall \delta \in [0, \delta_{\max}], \quad C'(\delta) = \mathbb{E}\left[H\left(\delta, (S_t)_{t \in [0,T]}\right)\right].$$

The functional H has an explicit form given in Proposition 5.2, Equations (5.15) or (5.17)), involving integrals over $[0, T]$ of the intensity $\lambda(S_t - S_0 + \delta)$ of the Poisson process $(N_t^{(\delta)})_{t \in [0,T]}$. In particular, any quantity $H\left(\delta, (S_t)_{t \in [0,T]}\right)$ can be simulated, up to a natural time discretization, either from a true dataset (of past executed orders) or from the stepwise constant Euler scheme (with step $\frac{T}{m}$) of a formerly calibrated diffusion process modeling $(S_t)_{t \in [0,T]}$. This will lead us to replace in our

implementations the continuous time process $(S_t)_{t\in[0,T]}$ over $[0,T]$, by a finite dimensional $\mathbb{R}^{m+1}$-valued random vector $(S_{t_i})_{0\le i\le m}$ (where $t_0 = 0$ and $t_m = T$) with the implicit assumption that $S_t = S_{t_i}$ on $[t_i, t_{i+1})$, $i = 0,\dots,m$.

Based on this representation (5.15) of C', we can formally devise a recursive stochastic gradient descent *a.s.* converging toward δ^*. However to make it consistent, we need to introduce constrain so that it lives in $[0,\delta_{\max}]$. In the classical literature on Stochastic Approximation Theory (see [74] and [75]) this amounts to consider a variant *with projection on the "order book depth interval"* $[0,\delta_{\max}]$, namely

$$\delta_{n+1} = \mathrm{Proj}_{[0,\delta_{\max}]}\left(\delta_n - \gamma_{n+1}H\left(\delta_n, \left(\bar S_{t_i}^{(n+1)}\right)_{0\le i\le m}\right)\right),\ n\ge 0,\ \delta_0\in[0,\delta_{\max}], \tag{5.4}$$

where $\mathrm{Proj}_{[0,\delta_{\max}]}$ denotes the projection on the (nonempty closed convex) $[0,\delta_{\max}]$, $(\gamma_n)_{n\ge1}$ is a positive step sequence satisfying at least the minimal *decreasing step assumption*

$$\sum_{n\ge1}\gamma_n = +\infty \quad\text{and}\quad \gamma_n\to 0. \tag{5.5}$$

The input sequence $\left\{\left(\bar S_{t_i}^{(n)}\right)_{0\le i\le m}, n\ge 0\right\}$ showing up in the above procedure is the innovation process and needs to be specified. In a first (naive) approach one may imagine that it is a sequence of i.i.d. copies $\left(\bar S_{t_i}^{(n)}\right)_{0\le i\le m}$ of the true underlying dynamics of $(S_{t_i})_{0\le i\le m}$ or at least of its Euler scheme. Of course this not at all realistic in the perspective of performing this procedure on real data. An alternative is to assume that, at least within a laps of a few minutes, the dynamics of the asset S is *stationary* and, say, *α-mixing*. In such a framework, on can consider backward shifted samples of $(S_t)_{0\le t\le T}$: if $\Delta t > 0$ denotes a fixed time shift parameter such that $t_i - t_{i-1} = \Delta t$, we set

$$\forall\, t\in[0,T],\quad \bar S_{t_i}^{(n)} = \bar S_{t_i - n\Delta t} = \bar S_{t_{i-n}}$$

so that the sequence $\bar S^{(n)} = (\bar S_{t_i}^{(n)})_{0\le i\le m}$ shares some averaging properties in the sense of Chapter 2. Here $\bar S$ may represent either a stepwise constant Euler scheme when thinking of simulated data or a historical high frequency data base of best opposite prices of the asset S.

5.2.2 Main convergence results

The following theorems give *a.s.* convergence result for the stochastic procedure (5.4): the first one for i.i.d. sequences and the second one for "averaging" sequences (see Chapter 2).

5.2.2.1 I.i.d. simulated data from a formerly calibrated model

Theorem 5.1. *Assume that C is strictly convex with $C'(0) < 0$, $\left((\bar S_{t_i}^{(n)})_{0\le i\le m}\right)_{n\ge1}$ is an $\mathbb{R}^{m+1}$-valued sequence of i.i.d. copies of the stepwise constant Euler scheme of $(S_t)_{0\le t\le T}$ with step $\frac{T}{m}$. Furthermore, assume that the step sequence satisfies the standard assumption*

$$\sum_{n\ge1}\gamma_n = +\infty \quad \textit{and} \quad \sum_{n\ge1}\gamma_n^2 < +\infty. \tag{5.6}$$

Then the recursive procedure defined by (5.4) converges a.s. towards its target $\delta^* = \operatorname*{argmin}_{\delta\in[0,\delta_{\max}]} C(\delta)$:

$$\delta_n \xrightarrow{a.s.} \delta^*.$$

This theorem is a straightforward application of the classical *a.s.* convergence for constrained stochastic algorithms (see Chapter 1 Theorem 1.6).

5.2.2.2 Direct implementation on a dataset sharing averaging properties

In this case, we assume that the sequence $(\bar{S}_{t_i}^{(n)})_{0\le i\le m}$ shares an averaging properties with respect to a distribution ν on $(\mathbb{R}^{m+1}, \mathcal{B}or(\mathbb{R}^{m+1}))$ as developed in Chapter 2.

Definition 5.1. *Let $q\in\mathbb{N}$. A $[0,L]^q$-valued sequence $(\xi_n)_{n\ge1}$ is ν-averaging if*

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\xi_k}\overset{(\mathbb{R}^q)}{\Longrightarrow}\nu \quad \text{as } n\to\infty.$$

Then $(\xi_n)_{n\ge1}$ satisfies

$$D_n^*(\xi):=\sup_{x\in[0,L]^q}\left|\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}_{[\![0,x]\!]}(\xi_k)-\nu([\![0,x]\!])\right|\longrightarrow 0 \quad \text{as } n\to\infty,$$

where $D_n^*(\xi)$ is called the discrepancy at the origin or star discrepancy.

In this setting, we will require the existence of a *pathwise* Lyapunov function, which means in this one dimensional setting that $H(\cdot,(y_{t_i})_{0\le i\le m})$ is monotonic with a monotony independent of $(y_{t_i})_{0\le i\le m}\in\mathbb{R}_+^{m+1}$.

Theorem 5.2. *Let $\lambda(x)=Ae^{-kx}$, $A>0$, $k>0$. Assume that C is strictly convex with $C'(0)<0$ and $C'(\delta_{\max})>0$, $\left(\bar{S}^{(n)}\right)_{n\ge1}$ is a $[0,L]^q$-valued ν-averaging sequence and $(\gamma_n)_{n\ge1}$ is a positive non-increasing sequence satisfying*

$$\sum_{n\ge1}\gamma_n=+\infty,\quad nD_n^*(\xi)\gamma_n\underset{n\to\infty}{\longrightarrow}0,\quad \text{and}\quad \sum_{n\ge1}nD_n^*(\xi)\max\left(\gamma_n^2,|\Delta\gamma_{n+1}|\right)<+\infty. \tag{5.7}$$

Furthermore, assume that

$$Q_T\ge 2T\lambda(-S_0)\quad \text{and}\quad \begin{cases}\kappa\le\dfrac{1+k(S_0-\delta_{\max})}{k\left\|S\right\|_\infty} & \text{if}\quad \Phi\neq\mathrm{id}\\[2ex] \kappa\le\dfrac{1+k(S_0-\delta_{\max})}{k\left\|S\right\|_\infty\left(\Phi(Q_T)-\Phi(Q_T-1)\right)} & \text{if}\quad \Phi=\mathrm{id}\end{cases} \tag{5.8}$$

Then the recursive procedure defined by (5.4) converges a.s. towards its target $\delta^*=\underset{\delta\in[0,\delta_{\max}]}{\operatorname{argmin}}\,C(\delta)$:

$$\delta_n\overset{a.s.}{\longrightarrow}\delta^*.$$

Proof. We will apply Theorem 2.1 in a QMC framework similar to Section 3 *c.f.* Chapter 2. First we set the integrability parameter p to $p=1$. Note that $\delta^*\in(0,\delta_{\max})$ since $C'(0)<0$ and $C'(\delta_{\max})>0$ so we can extend C' as a convex function on the whole real line. Moreover, by using the proof of Proposition 5.7, we prove that if $Q_T\ge 2T\lambda(-S_0)$ and (5.8) is satisfied, then H is nondecreasing in δ so H satisfies the strict pathwise Lyapunov assumption with $L(\delta)=\frac{1}{2}\left|\delta-\delta^*\right|^2$. It remains to check the averaging rate assumption for $H(\delta^*,\cdot)$. As it is a nondecreasing function on $[0,L]^q$, then it has finite variation and by using the Koksma-Hlawka Inequality, we get

$$\left|\frac{1}{n}\sum_{k=1}^{n}H(\delta^*,\xi_k)-\int_{[0,L]^q}H(\delta^*,u)\nu(du)\right|\le\left(H(\delta^*,L)-H(\delta^*,0)\right)D_n^*(\xi),$$

so that $H(\delta^*,\cdot)$ is ν-averaging at rate $\varepsilon_n=D_n^*(\xi)$. Finally, Theorem 2.1 from Chapter 2 yields

$$\delta_n\overset{a.s.}{\longrightarrow}\delta^*.$$

□

5.2.3 Criteria for the convexity and monotony at the origin

Checking that the assumptions on the function C (*i.e.* C convex with $C'(0) < 0$) in Theorem 5.1 are satisfied on $[0, \delta_{\max}]$ is a nontrivial task: in fact, as emphasized further on in Figures 5.2 and 5.7 in Section 5.6, the function C in (5.2) is never convex on the whole nonnegative real line, so we need reasonably simple criteria involving the market impact function Φ, Q_T and the parameter κ and others quantities related to the asset dynamics which ensure that the required conditions are fulfilled by the function C. These criteria should take the form of upper bounds on the free parameter κ.

Their original form, typically those derived by simply writing $C'(0) < 0$ and $C''(0) \geq 0$, are not really operating since they involve ratios of expectations of functionals combining both the dynamics of the asset S and the execution parameters in a highly nonlinear way. A large part of this chapter is be devoted to establish simpler criteria (although slightly more conservative) based on a monotony principle for one-dimensional diffusions introduced in further details in Section 5.4. Since the statement of this criteria do not require any knowledge on this monotony principle, we present them in the theorem 5.3 below.

We still need an additional assumption, this time on the function λ. Roughly speaking we need that the functional Λ depends on the distance parameter δ *essentially exponentially* in the following sense:

$$0 < \underline{k}_1 := \operatorname*{infess}_{\delta \in [0,\delta_{\max}]} \left(-\frac{\frac{\partial}{\partial \delta}\Lambda_T(\delta, S)}{\Lambda_T(\delta, S)} \right) \leq \overline{k}_1 := \operatorname*{supess}_{\delta \in [0,\delta_{\max}]} \left(-\frac{\frac{\partial}{\partial \delta}\Lambda_T(\delta, S)}{\Lambda_T(\delta, S)} \right) < +\infty, \tag{5.9}$$

$$0 < \underline{k}_2 := \operatorname*{infess}_{\delta \in [0,\delta_{\max}]} \left(-\frac{\frac{\partial^2}{\partial \delta^2}\Lambda_T(\delta, S)}{\frac{\partial}{\partial \delta}\Lambda_T(\delta, S)} \right) \leq \overline{k}_2 := \operatorname*{supess}_{\delta \in [0,\delta_{\max}]} \left(-\frac{\frac{\partial^2}{\partial \delta^2}\Lambda_T(\delta, S)}{\frac{\partial}{\partial \delta}\Lambda_T(\delta, S)} \right) < +\infty. \tag{5.10}$$

Note that the above assumption implies

$$k_0 := \operatorname{infess} \left(-\frac{\frac{\partial}{\partial \delta}\Lambda_T(0, S)}{\Lambda_T(0, S)} \right) \geq \underline{k}_1 > 0$$

Although this assumption is stated on the functional Λ (and subsequently depends on S), this is mainly an assumption on the intensity function λ. In particular, both above assumptions are satisfied by intensity functions of the form

$$\lambda_k(x) = e^{-kx}, \quad x \in \mathbb{R}, \quad k \in (0, +\infty).$$

For λ_k, one checks that $\underline{k}_1 = \overline{k}_1 = \underline{k}_2 = \overline{k}_2 = k$.

Theorem 5.3. *Assume that the asset dynamics $(S_t)_{t \geq 0}$ of the asset S is a $(0, \infty)$-Brownian diffusion*

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_0 > 0,$$

admissible for this open interval in the sense of Definition 5.3 below (the geometric Brownian motion is admissible, see Section 5.4). Assume that the function λ is essentially exponential in the sense of (5.9) and (5.10) above. Then the following criteria hold true.

(a) Monotony at the origin: *The derivative $C'(0) < 0$ as soon as*

$$Q_T \geq 2T\lambda(-S_0) \quad \text{and} \quad \kappa \leq \frac{S_0}{\mathbb{E}\left[S_T\right](\Phi(Q_T) - \Phi(Q_T - 1))} + \frac{1}{k_0 \mathbb{E}\left[S_T\right](\Phi(Q_T) - \Phi(Q_T - 1))}.$$

(b) Convexity. *Let $\rho_Q \in \left(0, 1 - \frac{\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)}_{|\mu = T\lambda(-S_0)}\right)$. If $\Phi \neq \mathrm{id}$, assume that Φ satisfies*

$$\forall x \in [1, Q_T - 1], \quad \Phi(x) - \Phi(x - 1) \leq \rho_Q(\Phi(x + 1) - \Phi(x)).$$

If

$$Q_T \geq \left(2 \vee \left(1 + \frac{\overline{k}_1^2}{\underline{k}_1 \underline{k}_2}\right)\right) T\lambda(-S_0) \quad \textit{and} \quad \kappa \leq \frac{2\underline{k}_1}{\overline{k}_1 \overline{k}_2 \mathbb{E}\left[S_T\right] \Phi'(Q_T)},$$

then $C''(\delta) \geq 0$, $\delta \in [0, \delta_{\max}]$, *so that* C *is convex on* $[0, \delta_{\max}]$.

Remark. These conditions on the model parameters are conservative. Indeed, "sharper" criteria can be given whose bounds involve ratios of expectation which can be evaluated only by Monte Carlo simulations:

$$C'(0) < 0 \Longleftrightarrow \kappa < b_2,$$

where

$$b_2 = \frac{\mathbb{E}\left[-Q_T \mathbb{P}^{(0)}\left(N^\mu > Q_T\right) + \left(S_0 \frac{\partial}{\partial \delta} \Lambda_T(0,S) - \Lambda_T(0,S)\right) \mathbb{P}^{(0)}\left(N^\mu \leq Q_T - 1\right)\right]}{\mathbb{E}\left[S_T \frac{\partial}{\partial \delta} \Lambda_T(0,S) \varphi^{(0)}(\mu)\right]}$$

and

$$C \text{ is convex on } [0, \delta_{\max}] \Longleftrightarrow \max_{\delta \in \mathcal{D}_-} \frac{A(\delta)}{B(\delta)} < \kappa < \min_{\delta \in \mathcal{D}_+} \frac{A(\delta)}{B(\delta)}$$

where

$$\begin{aligned} A(\delta) \quad = \quad & \mathbb{E}\left[\left((S_0 - \delta) \frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) - 2 \frac{\partial}{\partial \delta} \Lambda_T(\delta, S)\right) \mathbb{P}^{(\delta)}\left(N^\mu \leq Q_T - 1\right)\right. \\ & \left. - (S_0 - \delta) \left(\frac{\partial}{\partial \delta} \Lambda_T(\delta, S)\right)^2 \mathbb{P}^{(\delta)}\left(N^\mu = Q_T - 1\right)\right], \end{aligned}$$

$$B(\delta) = \mathbb{E}\left[S_T \left(\frac{\partial^2}{\partial \delta^2} \Lambda_T(\delta, S) \varphi^{(\delta)}(\mu) - \left(\frac{\partial}{\partial \delta} \Lambda_T(\delta, S)\right)^2 \psi^{(\delta)}(\mu)\right)\right],$$

$$\mathcal{D}_- = \{\delta \in [0, \delta_{\max}] \mid B(\delta) < 0\} \quad \text{and} \quad \mathcal{D}_+ = \{\delta \in [0, \delta_{\max}] \mid B(\delta) > 0\}.$$

5.2.4 Introduction to monotony principle for diffusions

Let I be a non-empty open interval of $\mathbb{R}$. Consider the real-valued diffusion process

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x_0 \in I, \quad t \in [0, T]. \tag{5.11}$$

Definition 5.2. *The* Lamperti transform *of the diffusion process (5.11) is the function* $L : [0,T] \times I \to \mathbb{R}$ *defined for every* $(t,x) \in [0,T] \times I$ *by*

$$L(t,x) := \int_{x_1}^{x} \frac{d\xi}{\sigma(t,\xi)} \tag{5.12}$$

where x_1 *is an arbitrary fixed value lying in* I. *Let* $t \in [0,T]$. *The inverse of* $L(t,\cdot)$ *will be denoted* $L^{-1}(t,\cdot)$.

Definition 5.3. *The diffusion process (5.11) is* admissible *if*

(*i*) $\sigma \in \mathcal{C}^1([0,T] \times I, I)$,

(*ii*) $\forall (t,x) \in [0,T] \times I$, $|b(t,x)| \leq C(1 + |x|)$ *and* $0 < \sigma(t,x) \leq C(1 + |x|)$,

(*iii*) $\forall\, x \in I$, $\int_{(-\infty,x] \cap I} \frac{d\xi}{\sigma(t,\xi)} = \int_{[x,+\infty) \cap I} \frac{d\xi}{\sigma(t,\xi)} = +\infty$,

(*iv*) *for every starting value* $x_0 \in I$, *(5.11) has a unique weak solution which lives in* I *up to* $t = +\infty$ *(see Proposition 5.5 for criteria),*

(*v*) *the function* β *defined by*

$$\beta(t,y) := \left(\frac{b}{\sigma} - \int_{x_1}^{\cdot} \frac{1}{\sigma^2(t,\xi)} \frac{\partial \sigma}{\partial t}(t,\xi) d\xi - \frac{1}{2}\frac{\partial \sigma}{\partial x}\right)(t, L^{-1}(t,y)),$$

is continuous on $[0,T]\times\mathbb{R}$*, nondecreasing in* y *for every* $t\in[0,T]$ *and satisfies*

$$\exists\, K>0 \;\text{ such that }\; |\beta(t,y)| \le K(1+|y|),\; t\in[0,T],\; y\in\mathbb{R}.$$

Definition 5.4. *Let* $F : \mathbb{D}([0,T],\mathbb{R}) \to \mathbb{R}$ *be a functional.*

(*i*) *The functional* F *is nondecreasing (resp. non-increasing) on* $\mathbb{D}([0,T],\mathbb{R})$ *if*

$$\forall \alpha_1,\alpha_2\in\mathbb{D}([0,T],\mathbb{R}),\quad (\forall t\in[0,T],\; \alpha_1(t)\le\alpha_2(t)) \Rightarrow F(\alpha_1)\le F(\alpha_2) \text{ (resp. } F(\alpha_1)\ge F(\alpha_2)).$$

(*ii*) *The functional* F *is continuous at* $\alpha\in\mathcal{C}([0,T],\mathbb{R})$ *if*

$$\forall \alpha_n\in\mathbb{D}([0,T],\mathbb{R}),\quad \alpha_n \xrightarrow{U} \alpha\in\mathcal{C}([0,T],\mathbb{R}),\quad F(\alpha_n)\to F(\alpha).$$

where U *denotes the uniform convergence of functions on* $[0,T]$*. The functional* F *is* C*-continuous if it is continuous at every* $\alpha\in\mathcal{C}([0,T],\mathbb{R})$*.*

(*iii*) *The functional* F *has polynomial growth if there exists a positive real number* $r>0$ *such that*

$$\forall \alpha\in\mathbb{D}([0,T],\mathbb{R}),\quad |F(\alpha)| \le K\left(1+\|\alpha\|_{\sup}^r\right). \tag{5.13}$$

Remark. Any C-continuous functional in the above sense is in particular $\mathbb{P}_Z$-*a.s.* continuous for every process Z with continuous paths.

Theorem 5.4. *Assume that the diffusion process (5.11) is admissible. Let* $F,G : \mathbb{D}([0,T,\mathbb{R}) \to \mathbb{R}$ *be two functionals* C*-continuous with polynomial growth and opposite monotony. Then*

$$\mathrm{Cov}\left(F\left((X_t)_{t\in[0,T]}\right), G\left((X_t)_{t\in[0,T]}\right)\right) \le 0.$$

5.3 Representations as expectations of C and its derivatives

First we briefly recall for convenience few basic facts on Poisson distributed variables that will be needed to compute the cost function C and its derivatives C' and C'' (proofs are left to the reader).

Proposition 5.1. (Classical formulas). *Let* $(N^\mu)_{\mu>0}$ *be a family of Poisson distributed random variables with parameter* $\mu>0$*.*

(*i*) *For every function* $f:\mathbb{N}\to\mathbb{R}_+$ *such that* $\log f(n) = O(n)$*,*

$$\frac{d}{d\mu}\mathbb{E}\left[f(N^\mu)\right] = \mathbb{E}\left[f(N^\mu+1)-f(N^\mu)\right] = \mathbb{E}\left[f(N^\mu)\left(\frac{N^\mu}{\mu}-1\right)\right].$$

In particular, for any $k\in\mathbb{N}$, $\frac{d}{d\mu}\mathbb{P}\left(N^\mu\le k\right) = -\mathbb{P}\left(N^\mu = k\right)$*.*

For any $k\in\mathbb{N}^*$*,*

(*ii*) $\mathbb{E}\left[k\wedge N^\mu\right] = k\mathbb{P}\left(N^\mu>k\right) + \mu\mathbb{P}\left(N^\mu\le k-1\right)$ *and* $\frac{d}{d\mu}\mathbb{E}\left[k\wedge N^\mu\right] = \mathbb{P}\left(N^\mu\le k-1\right)$*,*

(*iii*) $\mathbb{E}\left[(k-N^\mu)_+\right] = k\mathbb{P}\left(N^\mu\le k\right) - \mu\mathbb{P}\left(N^\mu\le k-1\right)$*,*

(*iv*) $k\mathbb{P}\left(N^\mu=k\right) = \mu\mathbb{P}\left(N^\mu=k-1\right)$*.*

To compute the cost function (or its gradient), it is convenient to proceed a pre-conditioning with respect to $\mathcal{F}_T^S := \sigma(S_t, 0 \le t \le T)$. We come down to compute the above quantity when $N^{(\delta)}$ is replaced by N^μ, a standard Poisson random variable with parameter μ. Therefore we have

$$\begin{aligned} C(\delta) &= \mathbb{E}\left[(S_0-\delta)\left(Q_T \wedge N_T^{(\delta)}\right) + \kappa S_T \Phi\left(\left(Q_T - N_T^{(\delta)}\right)_+\right)\right] \\ &= \mathbb{E}\left[(S_0-\delta)\mathbb{E}\left[\left(Q_T \wedge N_T^{(\delta)}\right) \mid \mathcal{F}_T^S\right] + \kappa S_T \mathbb{E}\left[\Phi\left(\left(Q_T - N_T^{(\delta)}\right)_+\right) \mid \mathcal{F}_T^S\right]\right] \\ &= \mathbb{E}\left[(S_0-\delta)\mathbb{E}\left[Q_T \wedge N^\mu\right]_{|\mu=\Lambda_T(\delta,S)} + \kappa S_T \mathbb{E}\left[\Phi\left((Q_T - N^\mu)_+\right)\right]_{|\mu=\Lambda_T(\delta,S)}\right] \\ &= \mathbb{E}\left[\widetilde{C}\left(\delta, \Lambda_T(\delta,S), (S_t)_{0\le t\le T}\right)\right], \end{aligned} \tag{5.14}$$

where for every $x \in \mathcal{C}([0,T],\mathbb{R}_+)$ and every $\mu \in \mathbb{R}_+$,

$$\widetilde{C}(\delta,\mu,x) = (x_0-\delta)\left(Q_T\mathbb{P}\left(N^\mu > Q_T\right) + \mu\mathbb{P}\left(N^\mu \le Q_T - 1\right)\right) + \kappa x_T \mathbb{E}\left[\Phi\left((Q_T - N^\mu)_+\right)\right].$$

We introduce some notations for reading convenience: we set

$$\mathbb{P}^{(\delta)}\left(N^\mu > Q_T\right) = \mathbb{P}\left(N^\mu > Q_T\right)_{|\mu=\Lambda_T(\delta,S)} \quad \text{and} \quad \mathbb{E}^{(\delta)}\left[f(\mu)\right] = \mathbb{E}\left[f(\mu)\right]_{|\mu=\Lambda_T(\delta,S)}.$$

Now we are in position to compute the first and second derivatives of the cost function C.

Proposition 5.2. *(a) If $\Phi \neq \mathrm{id}$, then $C'(\delta) = \mathbb{E}\left[H(\delta,S)\right]$ with*

$$\begin{aligned} H(\delta,S) &= -Q_T\mathbb{P}^{(\delta)}\left(N^\mu > Q_T\right) + \left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)(S_0-\delta) - \Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu \le Q_T - 1\right) \\ &\quad -\kappa S_T \frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\varphi^{(\delta)}(\mu) \end{aligned} \tag{5.15}$$

where $\varphi^{(\delta)}(\mu) = \mathbb{E}^{(\delta)}\left[\left(\Phi\left(Q_T - N^\mu\right) - \Phi\left(Q_T - N^\mu - 1\right)\right)\mathbb{1}_{\{N^\mu \le Q_T - 1\}}\right]$ and

$$\begin{aligned} C''(\delta) &= \mathbb{E}\left[\left((S_0-\delta)\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S) - 2\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu \le Q_T - 1\right) - \kappa S_T \frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)\varphi^{(\delta)}(\mu)\right. \\ &\quad \left. -(S_0-\delta)\left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)^2\mathbb{P}^{(\delta)}\left(N^\mu = Q_T - 1\right) + \kappa S_T\left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)^2\psi^{(\delta)}(\mu)\right] \end{aligned} \tag{5.16}$$

where $\psi^{(\delta)}(\mu) = \mathbb{E}^{(\delta)}\left[\Phi\left((Q_T - N^\mu - 2)_+\right) - 2\Phi\left((Q_T - N^\mu - 1)_+\right) + \Phi\left((Q_T - N^\mu)_+\right)\right]$.

(b) If $\Phi = \mathrm{id}$, then $C'(\delta) = \mathbb{E}\left[H(\delta,S)\right]$ with

$$H(\delta,S) = -Q_T\mathbb{P}^{(\delta)}\left(N^\mu > Q_T\right) + \left((S_0-\delta-\kappa S_T)\frac{\partial}{\partial\delta}\Lambda_T(\delta,S) - \Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu \le Q_T - 1\right) \tag{5.17}$$

$$\begin{aligned} \text{and} \quad C''(\delta) &= \mathbb{E}\left[\left((S_0-\delta-\kappa S_T)\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S) - 2\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu \le Q_T - 1\right)\right. \\ &\quad \left. -(S_0-\delta-\kappa S_T)\left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)^2\mathbb{P}^{(\delta)}\left(N^\mu = Q_T - 1\right)\right] \end{aligned} \tag{5.18}$$

Proof. Interchanging derivation and expectation in the representation (5.14) implies

$$C'(\delta) = \mathbb{E}\left[\frac{\partial}{\partial\delta}\widetilde{C}\left(\delta, \Lambda_T(\delta,S), (S_t)_{0\le t\le T}\right) + \frac{\partial}{\partial\mu}\widetilde{C}\left(\delta, \Lambda_T(\delta,S), (S_t)_{0\le t\le T}\right)\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right].$$

(a) We come down to compute the partial derivatives of $\widetilde{C}(\delta,\mu,x)$.

$$\frac{\partial\widetilde{C}}{\partial\delta}(\delta,\mu,x) = -\mathbb{E}\left[(Q_T \wedge N^\mu)\right] = -Q_T\mathbb{P}\left(N^\mu > Q_T\right) - \mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right),$$

$$\frac{\partial\widetilde{C}}{\partial\mu}(\delta,\mu,x) = -(x_0-\delta)\frac{\partial}{\partial\mu}\mathbb{E}\left[(Q_T - N^\mu)_+\right] + \kappa x_T\frac{\partial}{\partial\mu}\mathbb{E}\left[\Phi\left((Q_T - N^\mu)_+\right)\right].$$

We have

$$\begin{aligned}\frac{\partial}{\partial\mu}\mathbb{E}\left[(Q_T - N^\mu)_+\right] &= -Q_T\mathbb{P}\left(N^\mu = Q_T\right) - \mathbb{P}\left(N^\mu \leq Q_T - 1\right) + \mu\mathbb{P}\left(N^\mu = Q_T - 1\right)\\ &= -\mathbb{P}\left(N^\mu \leq Q_T - 1\right) \quad \text{thanks to } (iv) \text{ in Proposition 5.1}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\mu}\mathbb{E}\left[\Phi\left((Q_T - N^\mu)_+\right)\right] &= \mathbb{E}\left[\Phi\left((Q_T - N^\mu - 1)_+\right) - \Phi\left((Q_T - N^\mu)_+\right)\right]\\ &= \mathbb{E}\left[\left(\Phi\left(Q_T - N^\mu - 1\right) - \Phi\left(Q_T - N^\mu\right)\right)\mathbb{1}_{\{N^\mu \leq Q_T - 1\}}\right] := -\varphi(\mu)\end{aligned}$$

owing to (v) in Proposition 5.1. Therefore

$$\frac{\partial\widetilde{C}}{\partial\mu}(\delta,\mu,x) = (x_0-\delta)\,\mathbb{P}\left(N^\mu \leq Q_T - 1\right) - \kappa x_T\varphi(\mu).$$

Consequently

$$\begin{aligned}C'(\delta) &= \mathbb{E}\left[-Q_T\mathbb{P}^{(\delta)}\left(N^\mu > Q_T\right) + \left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)(S_0-\delta) - \Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu \leq Q_T - 1\right)\right.\\ &\quad \left. - \kappa S_T\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\varphi^{(\delta)}(\mu)\right]\\ &= \mathbb{E}\left[\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\right], \qquad (5.19)\end{aligned}$$

where $\varphi^{(\delta)}(\mu) := \varphi(\mu)_{\,|\mu=\Lambda_T(\delta,S)}$ and for every $x\in\mathcal{C}([0,T],\mathbb{R}_+)$ and every $\mu,\nu\in\mathbb{R}_+$,

$$\widehat{C}(\delta,\mu,\nu,x) = -Q_T\mathbb{P}\left(N^\mu > Q_T\right) + \left(\nu(x_0-\delta)-\mu\right)\mathbb{P}\left(N^\mu \leq Q_T - 1\right) - \kappa x_T\nu\varphi(\mu).$$

Interchanging derivation and expectation in the representation (5.19) implies

$$\begin{aligned}C''(\delta) &= \mathbb{E}\left[\frac{\partial}{\partial\delta}\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\right.\\ &\quad + \frac{\partial}{\partial\mu}\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\\ &\quad \left. + \frac{\partial}{\partial\nu}\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)\right].\end{aligned}$$

We deal now with the partial derivatives of $\widehat{C}(\delta,\mu,\nu,x)$.

$$\begin{aligned}\frac{\partial\widehat{C}}{\partial\delta}(\delta,\mu,\nu,x) &= -\nu\mathbb{P}\left(N^\mu \leq Q_T - 1\right),\\ \frac{\partial\widehat{C}}{\partial\mu}(\delta,\mu,\nu,x) &= -\mathbb{P}\left(N^\mu \leq Q_T - 1\right) - (x_0-\delta)\nu\mathbb{P}\left(N^\mu = Q_T - 1\right) + \kappa x_T\nu\psi(\mu),\\ \frac{\partial\widehat{C}}{\partial\nu}(\delta,\mu,\nu,x) &= (x_0-\delta)\mathbb{P}\left(N^\mu \leq Q_T - 1\right) - \kappa x_T\varphi(\mu).\end{aligned}$$

Consequently

$$\begin{aligned} C''(\delta) &= \mathbb{E}\left[\left((S_0-\delta)\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)-2\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu\leq Q_T-1\right)-\kappa S_T\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)\varphi^{(\delta)}(\mu)\right. \\ &\quad \left.-(S_0-\delta)\left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)^2\mathbb{P}^{(\delta)}\left(N^\mu=Q_T-1\right)+\kappa S_T\left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)^2\psi^{(\delta)}(\mu)\right]. \end{aligned}$$

(b) If $\Phi=\mathrm{id}$ so that $\frac{\partial}{\partial\mu}\mathbb{E}\left[\Phi\left((Q_T-N^\mu)_+\right)\right]=-\mathbb{P}\left(N^\mu\leq Q_T-1\right)$. Therefore

$$\frac{\partial\widetilde{C}}{\partial\delta}(\delta,\mu,x)=-Q_T\mathbb{P}\left(N^\mu>Q_T\right)-\mu\mathbb{P}\left(N^\mu\leq Q_T-1\right)$$

and

$$\frac{\partial\widetilde{C}}{\partial\mu}(\delta,\mu,x)=(x_0-\delta-\kappa x_T)\mathbb{P}\left(N^\mu\leq Q_T-1\right).$$

Consequently

$$\begin{aligned} C'(\delta) &= \mathbb{E}\left[-Q_T\mathbb{P}^{(\delta)}\left(N^\mu>Q_T\right)+\left((S_0-\delta-\kappa S_T)\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)-\Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu\leq Q_T-1\right)\right] \\ &= \mathbb{E}\left[\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\right], \end{aligned} \tag{5.20}$$

where for every $x\in\mathcal{C}([0,T],\mathbb{R}_+)$ and every $\mu,\nu\in\mathbb{R}_+$,

$$\widehat{C}(\delta,\mu,\nu,x)=-Q_T\mathbb{P}\left(N^\mu>Q_T\right)+\left((x_0-\delta-\kappa x_T)\nu-\mu\right)\mathbb{P}\left(N^\mu\leq Q_T-1\right).$$

Interchanging derivation and expectation in the the representation (5.20) implies

$$\begin{aligned} C''(\delta) &= \mathbb{E}\left[\frac{\partial}{\partial\delta}\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\right. \\ &\quad +\frac{\partial}{\partial\mu}\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\frac{\partial}{\partial\delta}\Lambda_T(\delta,S) \\ &\quad \left.+\frac{\partial}{\partial\nu}\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)\right]. \end{aligned}$$

We come down to compute the partial derivatives of $\widehat{C}\left(\delta,\Lambda_T(\delta,S),\frac{\partial}{\partial\delta}\Lambda_T(\delta,S),(S_t)_{0\leq t\leq T}\right)$.

$$\frac{\partial\widehat{C}}{\partial\delta}(\delta,\mu,\nu,x)=-\nu\mathbb{P}\left(N^\mu\leq Q_T-1\right),\quad \frac{\partial\widehat{C}}{\partial\nu}(\delta,\mu,\nu,x)=(x_0-\delta-\kappa x_T)\mathbb{P}\left(N^\mu\leq Q_T-1\right),$$

$$\frac{\partial\widehat{C}}{\partial\mu}(\delta,\mu,\nu,x)=-\mathbb{P}\left(N^\mu\leq Q_T-1\right)-(x_0-\delta-\kappa x_T)\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\mathbb{P}\left(N^\mu=Q_T-1\right).$$

Consequently

$$\begin{aligned} C''(\delta) &= \mathbb{E}\left[\left((S_0-\delta-\kappa S_T)\frac{\partial^2}{\partial\delta^2}\Lambda_T(\delta,S)-2\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)\mathbb{P}^{(\delta)}\left(N^\mu\leq Q_T-1\right)\right. \\ &\quad \left.-(S_0-\delta-\kappa S_T)\left(\frac{\partial}{\partial\delta}\Lambda_T(\delta,S)\right)^2\mathbb{P}^{(\delta)}\left(N^\mu=Q_T-1\right)\right]. \end{aligned}$$

□

5.4 Monotony principle for a class of one-dimensional diffusions

In this section, we present the principle of co- and opposite monotony, first for random vectors taking values in a nonempty interval I, then for one-dimensional diffusions lying in I.

5.4.1 Case of random variables and random vectors

First we recall a classical result for random variables.

Proposition 5.3. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be two monotonic functions with opposite monotony. Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to I$ be a real valued random variable such that $f(X), g(X) \in L^2(\mathbb{P})$. Then*

$$\mathrm{Cov}(f(X), g(X)) \leq 0.$$

Proof. Let X, Y be two independent random variables defined on the same probability space with the same distribution $\mathbb{P}_X$. Then

$$(f(X) - f(Y))(g(X) - g(Y)) \leq 0$$

hence its expectation. Consequently

$$\mathbb{E}\left[f(X)g(X)\right] - \mathbb{E}\left[f(X)g(Y)\right] - \mathbb{E}\left[f(Y)g(X)\right] + \mathbb{E}\left[f(Y)g(Y)\right] \leq 0$$

so using that $Y \overset{(d)}{=} X$ and Y, X are independent

$$2\mathbb{E}\left[f(X)g(X)\right] \leq \mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(Y)\right] + \mathbb{E}\left[f(Y)\right]\mathbb{E}\left[g(X)\right] = 2\mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(X)\right]$$

that is $\mathrm{Cov}(f(X), g(X)) \leq 0$. □

Proposition 5.4. *Let $F, G : \mathbb{R}^d \to \mathbb{R}$ be two monotonic functions with opposite monotony in each of their variables, i.e. for every $i \in \{1, \ldots, d\}$, $x_i \longmapsto F(x_1, \ldots, x_i, \ldots, x_n)$ and $x_i \longmapsto G(x_1, \ldots, x_i, \ldots, x_n)$ are monotonic with an opposite monotony which may depend on i (but does not depend on $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{d-1}$). Let $X_1, \ldots, X_d$ be independent real valued random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $F(X_1, \ldots, X_d), G(X_1, \ldots, X_d) \in L^2(\mathbb{P})$. Then*

$$\mathrm{Cov}\left(F(X_1, \ldots, X_d), G(X_1, \ldots, X_d)\right) \leq 0.$$

Proof. The proof of the above proposition is made by induction on d. The case $d = 1$ is given by Proposition 5.3. We give here the proof for $d = 2$ for notational convenience, but the general case of dimension d follows straightforwardly. By opposite monotonic assumption on F and G, we have for every $x_2 \in \mathbb{R}$, $X_1' \overset{d}{=} X_1$ with X_1', X_1 independent, that

$$\left(F(X_1, x_2) - F(X_1', x_2)\right)\left(G(X_1, x_2) - G(X_1', x_2)\right) \leq 0,$$

which implies that (see Proposition 5.3)

$$\mathrm{Cov}\left(F(X_1, x_2)G(X_1, x_2)\right) \leq 0.$$

If X_1 and X_2 are independent, using Fubini Theorem and what precedes, we have

$$\begin{aligned}\mathbb{E}\left[F(X_1, X_2)G(X_1, X_2)\right] &= \int_{\mathbb{R}} \mathbb{P}_{X_2}(dx_2)\mathbb{E}\left[F(X_1, x_2)G(X_1, x_2)\right] \\ &\leq \int_{\mathbb{R}} \mathbb{P}_{X_2}(dx_2)\mathbb{E}\left[F(X_1, x_2)\right]\mathbb{E}\left[G(X_1, x_2)\right].\end{aligned}$$

By setting $\varphi(x_2) = \mathbb{E}\left[F(X_1, x_2)\right]$ and $\psi(x_2) = \mathbb{E}\left[G(X_1, x_2)\right]$ and using the monotonic assumptions on F and G, we have

$$\int_{\mathbb{R}} \mathbb{P}_{X_2}(dx_2)\mathbb{E}\left[F(X_1, x_2)\right]\mathbb{E}\left[G(X_1, x_2)\right] = \mathbb{E}\left[\varphi(X_2)\psi(X_2)\right] \leq \mathbb{E}\left[\varphi(X_2)\right]\mathbb{E}\left[\psi(X_2)\right],$$

i.e. $\mathrm{Cov}\left(F(X_1, X_2)G(X_1, X_2)\right) \leq 0$. □

5.4.2 Case of (one-dimensional) diffusions

This framework corresponds to the infinite dimensional case and we can not apply straightforwardly the result of Proposition 5.3: indeed, if we define the following order relation on the process of $\mathbb{D}([0,T],\mathbb{R})$, namely

$$\forall \alpha_1,\alpha_2 \in \mathbb{D}([0,T],\mathbb{R}), \quad \alpha_1 \leq \alpha_2 \Longleftrightarrow (\forall t \in [0,T],\ \alpha_1(t) \leq \alpha_2(t)),$$

this order relation is partial and not total which makes the formal proof of Proposition 5.3 collapse. To establish a principle of opposite monotony on diffusions, we proceed in two steps: first, we use the Lamperti transform to "force" the diffusion coefficient to be equal to 1 and we establish the opposite monotony principle for this kind of diffusions. Then, by the inverse Lamperti transform, we go back to the original process.

In this section, we first present the framework in which we place. Then we recall some weak convergence results for diffusion with diffusion coefficient equal to 1. Afterwards we present the Lamperti transform and we conclude by the general result on opposite monotony principle.

Let I be a nonempty open interval of $\mathbb{R}$. One considers a real-valued Brownian diffusion process

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t, \quad X_0 = x_0 \in I, \quad t \in [0,T], \tag{5.21}$$

where b, $\sigma : [0,T] \times I \to \mathbb{R}$ are Borel functions with at most linear growth such that the above Equation (5.21) admits at least one (weak) solution over $[0,T]$ and W is a Brownian motion defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$. We assume that the diffusion X *a.s.* does not explode and lives in the interval I. This implies assumptions on the function b and σ especially in the neighbourhood (in I) of the endpoints of I that we will not detail here. At a finite endpoint of I, these assumptions are strongly connected with the Feller classification for which we refer to [70] with $\sigma(t,\cdot) > 0$ for every $t \in [0,T]$. We will simply make some classical linear growth assumption on b and σ (which prevent explosion at a finite time) that will be used for different purpose in what follows.

To "remove" the diffusion coefficient of the diffusion X, we will introduce the so-called *Lamperti transform* which requires additional assumptions on the drift b and the diffusion coefficient σ, namely assumptions (i)-(iii) in Definition 5.3.

Remark. Condition (iii) clearly does not depend on $x \in I$. Furthermore, if $I = \mathbb{R}$, (iii) follows from (ii) since $\frac{1}{\sigma(t,\xi)} \geq \frac{1}{C}\frac{1}{1+|\xi|}$.

Before passing to a short background on the Lamperti transform which will lead to study a new diffusion deduced from (5.21) whose diffusion coefficient is equal to 1, we need to recall (and adapt) some background on solution and discretization of SDE.

5.4.2.1 Background on diffusions with $\sigma \equiv 1$ (weak solution, discretization).

The following proposition gives condition on the drift for the existence and the uniqueness of a weak solution of a SDE with $\sigma \equiv 1$ (see [69] Proposition 3.6, Chap. 5, p. 303 and Corollary 3.11, Chap. 5, p. 305).

Proposition 5.5. *Consider the stochastic differential equation*

$$dY_t = \beta(t,Y_t)dt + dW_t, \quad t \in [0,T], \tag{5.22}$$

where T is a fixed positive number, W is a one-dimensional Brownian motion and $\beta : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a Borel-measurable function satisfying

$$|\beta(t,y)| \leq K(1+|y|), \quad t \in [0,T], \quad y \in \mathbb{R}, \quad K > 0.$$

For any probability measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, equation (5.22) has a weak solution with initial distribution ν.

If, furthermore, the drift term b satisfies one of the following conditions

(i) β is bounded on $[0,T]\times\mathbb{R}$,

(ii) β is continuous, locally Lipschitz in $y\in\mathbb{R}$ uniformly in $t\in[0,T]$,

then this weak solution is unique (in fact (ii) is a uniqueness assumption for the existing strong solution).

Now we introduce the stepwise constant (Brownian) Euler scheme $\bar{Y}^n = \left(\bar{Y}_{\frac{kT}{n}}\right)_{0\le k\le n}$ with step $\frac{T}{n}$ of the process $Y=(Y_t)_{t\ge 0}$ defined by (5.22). It is defined by

$$\bar{Y}_{t_{k+1}^n} = \bar{Y}_{t_k^n} + \beta(t_k^n, \bar{Y}_{t_k^n})\frac{T}{n} + \sqrt{\frac{T}{n}}U_{k+1}, \quad \bar{Y}_0 = Y_0 = y_0, \quad k=0,\dots,n-1, \tag{5.23}$$

where $t_k^n = \frac{kT}{n}$, $k=0,\dots,n$, and $(U_k)_{0\le k\le n}$ denotes a sequence of i.i.d. $\mathcal{N}(0,1)$-distributed random variables given by

$$U_k = \sqrt{\frac{n}{T}}\left(W_{t_k^n} - W_{t_{k-1}^n}\right), \qquad k=1,\dots,n.$$

The following theorem gives a weak convergence result for the stepwise constant Euler scheme (5.23). Its proof is a straightforward consequence of the functional limit theorems for semi-martingales (see Theorem 3.39, Chap. IX, p. 551 in [66]).

Theorem 5.5. *Let $\beta:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying*

$$\exists K>0, \quad |\beta(t,y)|\le K(1+|y|), \quad t\in[0,T], \quad y\in\mathbb{R}.$$

Assume that the weak solution of equation (5.22) is unique. Then, the stepwise constant Euler scheme of (5.22) with step $\frac{T}{n}$ satisfies

$$\bar{Y}^n \xrightarrow{\mathcal{L}} Y \quad \text{for the Skorokhod topology as } n\to\infty.$$

In particular, for every $\mathbb{P}_Y$-a.s. continuous functional $F:\mathbb{D}([0,T],\mathbb{R})\to\mathbb{R}$ with polynomial growth, we have

$$\mathbb{E}F(\bar{Y}^n) \underset{n\to\infty}{\longrightarrow} \mathbb{E}F(Y).$$

5.4.2.2 Background on the Lamperti transform

We will introduce a new diffusion $Y_t := L(t,X_t)$ which will satisfy a new SDE whose diffusion coefficient will be constant equal to 1. This function L defined on $[0,T]\times I$ is known in the literature as the Lamperti transform. It is defined for every $(t,x)\in[0,T]\times I$ by

$$L(t,x) := \int_{x_1}^x \frac{d\xi}{\sigma(t,\xi)} \tag{5.24}$$

where x_1 is an arbitrary fixed value lying in I. The Lamperti transform clearly depends on the choice of x_1 in I but not its properties of interest. First, under Definition 5.3 (i)-(ii), $L\in\mathcal{C}^{1,2}([0,T]\times I)$ with

$$\frac{\partial L}{\partial t}(t,x) = -\int_{x_1}^x \frac{1}{\sigma^2(t,\xi)}\frac{\partial\sigma}{\partial t}(t,\xi)d\xi, \quad \frac{\partial L}{\partial x}(t,x) = \frac{1}{\sigma(t,x)} > 0 \quad \text{and} \quad \frac{\partial^2 L}{\partial x^2}(t,x) = -\frac{1}{\sigma^2(t,x)}\frac{\partial\sigma}{\partial x}(t,x).$$

Let $t\in[0,T]$, $L(t,\cdot)$ is an increasing $\mathcal{C}^2$-diffeomorphism from I onto $\mathbb{R}=L(t,I)$ (the last claim follows from Definition 5.3 (iii)). Its inverse will be denoted $L^{-1}(t,\cdot)$.

Notice that, $(t,y)\mapsto L^{-1}(t,y)$ is continuous on $[0,T]\times I$ since both sets

$$\big\{(t,y)\in[0,T]\times I\,:\,L^{-1}(t,y)\le c\big\}=\{(t,y)\in[0,T]\times\mathbb{R}\,:\,L(t,c)\ge y\}$$

and

$$\big\{(t,y)\in[0,T]\times I\,:\,L^{-1}(t,y)\ge c\big\}=\{(t,y)\in[0,T]\times\mathbb{R}\,:\,L(t,c)\le y\}$$

are both closed for every $c\in\mathbb{R}$. Therefore, if Definition 5.3 (i)-(iii) holds, the function $\beta:[0,T]\times I\mapsto\mathbb{R}$ defined by

$$\beta(t,y):=\left(\frac{b}{\sigma}-\int_{x_1}^{\cdot}\frac{1}{\sigma^2(t,\xi)}\frac{\partial\sigma}{\partial t}(t,\xi)d\xi-\frac{1}{2}\frac{\partial\sigma}{\partial x}\right)(t,L^{-1}(t,y)). \tag{5.25}$$

is a Borel function, continuous as soon as b is.

Now, we set

$$\forall\, t\in[0,T],\quad Y_t:=L(t,X_t).$$

Itô formula straightforwardly yields

$$dY_t=\beta(t,Y_t)dt+dW_t,\quad Y_0=L(0,x_0)=:y_0\in\mathbb{R}. \tag{5.26}$$

Remarks. • In the homogeneous case, which is the most important case for our applications,

$$dX_t=b(X_t)dt+\sigma(X_t)dW_t,\quad X_0=x_0\in\mathbb{R},\quad t\in[0,T], \tag{5.27}$$

we have

$$L(x):=\int_{x_1}^{x}\frac{d\xi}{\sigma(\xi)},$$

then by setting $Y_t:=L(X_t)$, we obtain

$$dY_t=\beta(Y_t)dt+dWt,\quad Y_0=L(x_0)=:y_0\quad\text{with}\quad\beta:=\left(\frac{b}{\sigma}-\frac{\sigma'}{2}\right)\circ L^{-1}.$$

Note that β is bounded as soon as $\frac{b}{\sigma}-\frac{\sigma'}{2}$ is.

• If the partial derivative b'_x exists on $[0,T]\times I$, one easily checks that, using $(L^{-1})'_y(t,y)=\sigma(t,L^{-1}(t,y))$, for every $(t,y)\in[0,T]\times I$,

$$\beta'_y(t,y)=\left(b'_x-\frac{b\sigma'_x+\sigma'_t}{\sigma}-\frac{\sigma\sigma''_{x^2}}{2}\right)(t,L^{-1}(t,y)). \tag{5.28}$$

As a consequence, one derives that β satisfies the linear growth assumption as soon as the function

$$b'_x-\frac{b\sigma'_x+\sigma'_t}{\sigma}-\frac{\sigma\sigma''_{x^2}}{2}\ \text{ is bounded on }\ [0,T]\times I \tag{5.29}$$

Definition 5.5. *The* functional Lamperti transform*, denoted Λ, is a functional from $\mathcal{C}([0,T],I)$ to $\mathcal{C}([0,T],\mathbb{R})$ defined by*

$$\forall\,\alpha\in\mathcal{C}([0,T],I),\quad\Lambda(\alpha)=L(\cdot,\alpha(\cdot)).$$

Proposition 5.6. *If the diffusion coefficient σ satisfies Definition 5.3 (i)-(iii), the functional Lamperti transform is an homeomorphism from $\mathcal{C}([0,T],I)$ onto $\mathcal{C}([0,T],\mathbb{R})$.*

Proof. Let $\alpha \in \mathcal{C}([0,T],I)$. Since σ is bounded away from 0 on the compact set $[0,T]\times\alpha([0,T])$, standard arguments based on Lebesgue domination theorem, imply that $\Lambda(\alpha)\in\mathcal{C}([0,T],\mathbb{R})$. Conversely, as $L(t,\cdot): I\to\mathbb{R}$ is an homeomorphism for every $t\in[0,T]$, Λ admits an inverse defined by

$$\forall\,\xi\in\mathcal{C}([0,T],\mathbb{R}),\quad \Lambda^{-1}(\xi) := \left(t\mapsto L^{-1}(t,\xi(t))\right)\in\mathcal{C}([0,T],I).$$

Let U_K denote the topology of the convergence on compact sets of I on $\mathcal{C}([0,T],I)$.

$\triangleright$ *U_K-Continuity of Λ:* If $\alpha_n \xrightarrow{U_K} \alpha_\infty$, the set $K=[0,T]\times\bigcup_{n\in\overline{\mathbb{N}}}\alpha_n([0,T])$ is a compact set included in I. Hence σ is bounded away from 0 on K so that

$$\forall\, t\in[0,T],\quad |L(t,\alpha_n(t))-L(t,\alpha_\infty(t))|\le\frac{1}{\inf_K\sigma}|\alpha_n(t)-\alpha_\infty(t)|$$

i.e.
$$\|\Lambda(\alpha_n)-\Lambda(\alpha_\infty)\|_{\sup}\le\frac{1}{\inf_K\sigma}\|\alpha_n-\alpha_\infty\|_{\sup}.$$

$\triangleright$ *U_K-Continuity of Λ^{-1}:* by using Definition 5.3 (ii), we have for a fixed $t\in[0,T]$,

$$\forall x,x'\in I,\quad \left|L(t,x)-L(t,x')\right|\ge\frac{1}{C}\int_{x\wedge x'}^{x\vee x'}\frac{d\xi}{1+|\xi|}=\frac{1}{C}\left|\Phi(x)-\Phi(x')\right|,$$

where $\Phi(z)=\operatorname{sign}(z)\log(1+|z|)$. Thus,

$$\forall y,y'\in\mathbb{R},\quad \left|\Phi(L^{-1}(t,y))-\Phi(L^{-1}(t,y'))\right|\le C\left|y-y'\right|.$$

Let $(\xi_n)_{n\ge1}$ be a sequence of functions of $\mathbb{D}([0,T],\mathbb{R})$ such that $\xi_n \xrightarrow[n\to+\infty]{U}\xi\in\mathcal{C}([0,T],\mathbb{R})$. Then, for every $t\in[0,T]$ and $n\ge1$,

$$\left|\Phi(L^{-1}(t,\xi_n(t)))-\Phi(L^{-1}(t,0))\right|\le C\left|\xi_n(t)\right|\le C\left(\|\xi_n(t)-\xi\|+\|\xi\|\right)+|\Phi(x_0)|\le C',$$

since $L^{-1}(t,0)=x_0$. Consequently, for every $t\in[0,T]$ and every $n\ge1$, $L^{-1}(t,\xi_n(t))\in K':=\Phi^{-1}([-C',C'])$. The set K' is compact (because the function Φ is continuous and proper $(\lim_{|z|\to\infty}|\Phi(z)|=+\infty)$). As $\inf_{K'}\Phi'>0$, we deduce that there exists $\eta_0>0$ such that

$$\forall x,y\in I,\quad |\Phi(x)-\Phi(y)|>\eta_0|x-y|,$$

i.e.
$$\forall t\in[0,T],\ \forall u,v\in L(t,I),\quad \left|L^{-1}(t,u)-L^{-1}(t,v)\right|\le C''\left|u-v\right|,\quad C''>0.$$

Hence, one concludes that

$$\|\Lambda^{-1}(\xi_n)-\Lambda^{-1}(\xi_\infty)\|_{\sup}\le C''\|\xi_n-\xi_\infty)\|_{\sup}.$$ □

5.4.2.3 Opposite monotony principle for diffusion

Theorem 5.6. *Assume that the real-valued diffusion process (5.21) is admissible (see Defintion 5.3). Let $F,G:\mathbb{D}([0,T],\mathbb{R})\to\mathbb{R}$ be two C-continuous functionals, satisfying (5.13), with opposite monotony. Then*

$$\operatorname{Cov}\left(F\left((X_t)_{t\in[0,T]}\right),G\left((X_t)_{t\in[0,T]}\right)\right)\le 0.$$

Remark. In the homogeneous case (see (5.27)), as L is increasing,

$$\beta \text{ is nondecreasing iff } \frac{b}{\sigma} - \frac{\sigma'}{2} \text{ is nondecreasing.}$$

If $b, \sigma' \in D(I)$, this condition is equivalent to

$$\left(\frac{b}{\sigma}\right)' \geq \frac{\sigma''}{2}.$$

Before passing to the proof, we state few lemmas: one is a key step to transfer opposite monotony from the Euler scheme to the diffusion process, the other aims at transferring uniqueness property for weak solutions.

Lemma 5.1. *For every $\alpha \in \mathbb{D}([0,T],\mathbb{R})$, set*

$$\alpha^{(n)} = \sum_{k=0}^{n-1} \alpha(t_k^n) \mathbb{1}_{[t_k^n, t_{k+1}^n)} + \alpha(T)\mathbb{1}_{\{T\}}, \quad n \geq 1, \tag{5.30}$$

with $t_k^n := \frac{kT}{n}$, $k = 0, \ldots, n$. Then $\alpha^{(n)} \xrightarrow{U} \alpha$ as $n \to \infty$.

If $F : \mathbb{D}([0,T],\mathbb{R}) \to \mathbb{R}$ is C-continuous and nondecreasing (resp. non-increasing), then the unique function $F_n : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying $F(\alpha^{(n)}) = F_n(\alpha(t_k^n), k = 0, \ldots, n)$ is continuous and nondecreasing (resp. non-increasing) in each of its variables. Furthermore, if F satisfies a polynomial growth assumption of the form

$$\forall\, \alpha \in \mathbb{D}([0,T],\mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\sup}^r)$$

then, for every $n \geq 1$,

$$|F_n(x_0, \ldots, x_n)| \leq C(1 + \max_{0 \leq k \leq n} |x_k|^r)$$

with the same real constant $C > 0$.

Lemma 5.2. *Let (S,d), (T,δ) be two Polish spaces and let $\Phi : S \mapsto T$ be a continuous injective function. Let μ and μ' be two probability measures on $(S, \mathcal{B}or(S))$. If $\mu \circ \Phi^{-1} = \mu' \circ \Phi^{-1}$, then $\mu = \mu'$.*

Proof of Lemma 5.2. For every Borel set A of S, $\mu(A) = \sup\{\mu(K),\ K \subset A,\ K \text{ compact}\}$. Let $A \in \mathcal{B}or(S)$ such that $\mu(A) \neq \mu'(A)$. Then there exists a compact set K of A such that $\mu(K) \neq \mu'(K)$. But $\Phi(K)$ is a compact set of S because Φ is continuous, so $\Phi^{-1}(\Phi(K))$ is a Borel set of S which contains K. As Φ is injective, $\Phi^{-1}(\Phi(K)) = K$. Therefore $\mu\left(\Phi^{-1}(\Phi(K))\right) \neq \mu'\left(\Phi^{-1}(\Phi(K))\right)$. We deduce that $\mu \circ \Phi^{-1} \neq \mu' \circ \Phi^{-1}$. □

Proof of Theorem 5.6. First we consider the Lamperti transform $(Y_t)_{t \geq 0}$ (see (5.24)) of the diffusion X solution to (5.22) with $X_0 = x_0 \in I$. Using the homeomorphism property of Λ and calling upon the above Lemma 5.2 with Λ^{-1} and Λ, we see that existence and uniqueness assumptions on Equation (5.22) can be transferred to (5.26) since Λ is a one-to-one mapping between the solutions of these two SDE's.

Then we introduce the stepwise constant (Brownian) Euler scheme $\bar{Y}^n = \left(\bar{Y}_{\frac{kT}{n}}\right)_{0 \leq k \leq n}$ with step $\frac{T}{n}$ (defined by (5.23)) of $Y = (Y_t)_{t \geq 0}$. It is clear by induction on k that there exists for every $k \in \{1, \ldots, n\}$ a function $\Theta_k : \mathbb{R}^{k+1} \to \mathbb{R}$ such that

$$\bar{Y}_{t_k^n} = \Theta_k(y_0, \Delta W_{t_1^n}, \ldots, \Delta W_{t_k^n})$$

where for $(y_0, z_1, \ldots, z_k) \in \mathbb{R}^{k+1}$,

$$\Theta_k(y_0, z_1, \ldots, z_k) = \Theta_{k-1}(y_0, z_1, \ldots, z_{k-1}) + \beta(t^n_{k-1}, \Theta_{k-1}(y_0, z_1, \ldots, z_{k-1}))\frac{T}{n} + z_k.$$

Thus for every $i \in \{1, \ldots, k\}$, $z_i \mapsto \Theta_k(y_0, z_1, \ldots, z_i, \ldots, z_k)$ is nondecreasing because β is nondecreasing. We deduce that if $F_n : \mathbb{R}^{n+1} \to \mathbb{R}$ is nondecreasing in each variables, then, for every $i \in \{1, \ldots, k\}$,

$$z_i \mapsto F_n\left(y_0, \Theta_1(y_0, z_1), \ldots, \Theta_n(y_0, z_1, \ldots, z_n)\right) \text{ is nondecreasing.}$$

By the same reasoning, we deduce that for $G_n : \mathbb{R}^{n+1} \to \mathbb{R}$, non-increasing in each variables, we have for every $i \in \{1, \ldots, k\}$,

$$z_i \mapsto G_n\left(y_0, \Theta_1(y_0, z_1), \ldots, \Theta_n(y_0, z_1, \ldots, z_n)\right) \text{ is non-increasing.}$$

Let F_n and G_n be the functions defined on $\mathbb{R}^{n+1}$ associated to F and G respectively by Lemma 5.1. As β has linear growth, Y and its Euler scheme have polynomial moments at any order $p > 0$. Then we can apply Proposition 5.4 to deduce that

$$\begin{aligned}\mathbb{E}\left[FG\left(\bar{Y}^n\right)\right] &= \mathbb{E}\left[F_n\left(\left(\bar{Y}_{\frac{kT}{n}}\right)_{0\le k\le n}\right) G_n\left(\left(\bar{Y}_{\frac{kT}{n}}\right)_{0\le k\le n}\right)\right] \\ &\le \mathbb{E}\left[F_n\left(\left(\bar{Y}_{\frac{kT}{n}}\right)_{0\le k\le n}\right)\right]\mathbb{E}\left[G_n\left(\left(\bar{Y}_{\frac{kT}{n}}\right)_{0\le k\le n}\right)\right] = \mathbb{E}\left[F\left(\bar{Y}^n\right)\right]\mathbb{E}\left[G\left(\bar{Y}^n\right)\right].\end{aligned}$$

Note that if F and G are C-continuous with polynomial growth, then FG too. We derive from Theorem 5.5 that

$$\mathbb{E}\left[FG\left(\bar{Y}^n\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}FG(Y), \quad \mathbb{E}\left[F\left(\bar{Y}^n\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}F(Y), \quad \mathbb{E}\left[G\left(\bar{Y}^n\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}G(Y),$$

therefore

$$\mathrm{Cov}\left(F(Y), G(Y)\right) \le 0.$$

To conclude the proof, we need to go back to the process X by using the inverse Lamperti transform. Indeed, for every $t \in [0, T]$, $X_t = L^{-1}(t, Y_t)$, where Y satisfies (5.26). Let $F : \mathbb{D}([0, T], \mathbb{R}) \to \mathbb{R}$. Set

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad \widetilde{F}(\alpha) := F\left(\left(L^{-1}(t, \alpha_t)\right)_{t\in[0,T]}\right).$$

Assume first that F and G are bounded. The functional $\widetilde{F}$ is C-continuous owing to Proposition 5.6, nondecreasing (resp. non-increasing) since $L^{-1}(t, .)$ is for every $t \in [0, T]$ and is bounded. Consequently,

$$\mathrm{Cov}\left(F(X), G(X)\right) = \mathrm{Cov}\left(\widetilde{F}(Y), \widetilde{G}(Y)\right) \le 0.$$

To conclude one may approximate in a robust way with respect to the constraints on the functionals, F and G by a canonical truncation procedure, say

$$F_M := \max\left((-M), \min\left(F, M\right)\right), \quad M \in \mathbb{N}.$$

If F and G have polynomial growth, it is clear that $\mathrm{Cov}\left(F_M(X), G_M(X)\right) \to \mathrm{Cov}\left(F(X), G(X)\right)$ as $M \to \infty$. □

Examples of admissible diffusions. • *The Bachelier model*: This simply means that $X_t = \mu t + \sigma W_t$, $\sigma > 0$, clearly fulfills the assumptions of Theorem 5.6.

• *The Black-Scholes model*: The diffusion process X is a geometric Brownian motion, solution to the SDE

$$dX_t = rX_t dt + \vartheta X_t dW_t, \quad X_0 = x_0 > 0,$$

where $r \in \mathbb{R}$ and $\vartheta > 0$ are real numbers. The geometric Brownian motion lives in the open interval $I = (0, +\infty)$ and $\beta(y) = \frac{r}{\vartheta} - \frac{\vartheta}{2}$ is constant. One checks that $L(x) = \frac{1}{\sigma} \log\left(\frac{x}{x_1}\right)$ where $x_1 \in (0, +\infty)$ is fixed.

• *The Hull-White model*: It is an elementary improvement of the Black-Scholes model where $\vartheta : [0, T] \to (0, +\infty)$ is a deterministic positive function *i.e.* the diffusion process X is a geometric Brownian motion solution the SDE

$$dX_t = rX_t dt + \vartheta(t) X_t dW_t, \quad X_0 = x_0 > 0.$$

Then, elementary stochastic calculus shows that

$$X_t = x_0 e^{rt - \frac{1}{2}\int_0^t \vartheta^2(s)ds + \int_0^t \vartheta(s) dW_s} = x_0 e^{rt - \frac{1}{2}\int_0^t \vartheta^2(s)ds + B_{\int_0^t \vartheta^2(s)ds}}$$

where $(B_u)_{u \geq 0}$ is a standard Brownian motion (the second equality follows form the Dambins-Dubins-Schwarz theorem).

Consequently $X_t = \varphi\left(t, B_{\int_0^t \vartheta^2(s)ds}\right)$ where the functional $\xi \mapsto \left(t \mapsto \varphi\left(t, \xi\left(\int_0^{\cdot} \vartheta^2(s)ds\right)\right)\right)$ defined on $\mathbb{D}([0, T_\vartheta], \mathbb{R})$, $T_\vartheta = \int_0^T \vartheta^2(t)dt$, is C-continuous on $\mathcal{C}([0, T_\vartheta], \mathbb{R})$. Hence for any C-continuous $\mathbb{R}$-functional on $\mathbb{D}([0, T], \mathbb{R})$, the $\mathbb{R}$-valued functional $\widetilde{F}$ defined by

$$\widetilde{F}(\xi) = F\left(\varphi\left(t, \xi\left(\int_0^{\cdot} \vartheta^2(s)ds\right)\right)\right)$$

is C-continuous on $\mathbb{D}([0, T_\vartheta], \mathbb{R})$. Then, on can transfer the opposite monotony property form B to X.

• *Local volatility model (elliptic case)*: More generally, it applies still with $I = (0, +\infty)$ to some usual extensions like the models with local volatility

$$dX_t = rX_t dt + \vartheta(X_t) X_t dW_t, \quad X_0 = x_0 > 0,$$

where $\vartheta : \mathbb{R} \to (\vartheta_0, +\infty)$, $\vartheta_0 > 0$, is a bounded, non-increasing, twice differentiable function satisfying $x \mapsto x\vartheta(x)$ is concave, $|\vartheta'(x)| \leq \frac{C}{1+x}$ and $|\vartheta''(x)| \leq \frac{C}{1+x^2}$, $x \in (0, +\infty)$.

Note that the family of functions defined for every $x \in (0, +\infty)$ by

$$\vartheta_{\vartheta_0, \eta_0, a, \alpha}(x) = \vartheta_0 + \frac{a}{(x + \eta_0)^\alpha},\ \eta_0,\ a > 0,\ \alpha \in (0, 1)$$

satisfy the above assumptions.

In this case $I = (0, +\infty)$ and, $x_1 \in I$ being fixed, one has for every $x \in I$,

$$L(x) = \int_{x_1}^x \frac{d\xi}{\xi\vartheta(\xi)}$$

which clearly defines an increasing homeomorphism from I onto $\mathbb{R}$ since $\vartheta$is bounded. Then $\beta = \widetilde{\beta} \circ L^{-1}$ with

$$\widetilde{\beta}(x) = \frac{r}{\vartheta(x)} - \frac{1}{2}(I_{(0,+\infty)} \times \vartheta)'(x)$$

where $I_{(0,+\infty)}$ denotes the identity function in $(0,+\infty)$.As a consequence

$$\beta \text{ is nondecreasing on } \mathbb{R} \text{ iff } \frac{r}{\vartheta} - \frac{1}{2}(I_{(0,+\infty)} \times \vartheta)' \text{ is nondecreasing on } (0,+\infty).$$

Now ϑ non-increasing implies that $\frac{r}{\vartheta}$ is nondecreasing and $I_{(0,\infty)} \times \vartheta$ being concave, $(I_{(0,\infty)} \times \vartheta)' \leq 0$. Finally , under the above assumptions, β is nondecreasing.

Furthermore, one easily derives from the explicit form (5.28) and the condition (5.29) that β has linear growth as soon as the function

$$x \mapsto rx\frac{\vartheta'}{\vartheta}(x) + \frac{x^2\vartheta\vartheta''(x)}{2} + x\vartheta\vartheta'(x) \text{ is bounded on } (0,\infty)$$

which easily follows from the assumptions made on ϑ.

Extension to other classes of diffusions and models. This general approach does not embody all situations: thus the true CEV model does not fulfill the above assumptions. The *CEV* model is a diffusion process X following the SDE

$$dX_t = rX_t dt + \vartheta X_t^\alpha dW_t, \quad X_0 = x_0,$$

where $\vartheta > 0$ and $0 < \alpha < 1$ are real numbers.

So this *CEV* model, for which $I = (0,+\infty)$, does not fulfill Definition 5.3 (iii). As a consequence $L(t,I) \neq \mathbb{R}$ is an open interval (depending on the choice of x_1. To be precise, if $x_1 \in (0,+\infty)$ is fixed,

$$L(x) = \frac{1}{\vartheta(1-\alpha)}\left(x^{1-\alpha} - x_1^{1-\alpha}\right), \quad x \in (0,+\infty)$$

so that, if we set

$$J_{x_1} := L(I) = \left(-\frac{x_1^{1-\alpha}}{\vartheta(1-\alpha)}, +\infty\right),$$

L defines an homeomorphism from $I = (0,+\infty)$ onto J_{x_1}. Finally the function β defined by

$$\beta(y) = \frac{r}{\vartheta}\left(\vartheta(1-\alpha)y + x_1^{1-\alpha}\right) - \frac{\alpha\vartheta}{2}\frac{1}{(\vartheta(1-\alpha)y + x_1^{1-\alpha})}, \quad y \in J_{x_1}$$

is nondecreasing with linear growth at $+\infty$. Now, tracing the lines of the above proof, in particular establishing weak existence and uniqueness of the solution of the EDS (5.22) in that setting, leads to the same positive conclusion concerning the covariance inequalities for co-monotonic or anti-monotonic functionals.

5.5 Convexity and monotony criteria for the cost function C

To ensure that the optimization problem is well-posed, namely that the cost function C has a minimum on $[0, \delta_{\max}]$, we need some additional assumptions: the cost function C must be convex with $C'(0) < 0$. This leads to define bounds for the parameter κ and this section is devoted to give sufficient condition on κ to ensure that this two properties are satisfied. The computations of the bounds given below rely on the co- (and opposite) monotony principle introduced in the previous Section 5.2.4 and Section 5.4.

5.5.1 Criteria for local and global monotony

The above proposition gives bounds for the parameter κ which ensure that the cost function has a minimum. The aim of this subsection is to obtain sufficient bounds, easy to compute, namely depending only of the model parameters.

Proposition 5.7. *(a)* Monotony at the origin. *$C'(0) < 0$ as soon as $Q_T \geq 2T\lambda(-S_0)$, $k_0 = \operatorname{infess}\left(-\frac{\frac{\partial}{\partial\delta}\Lambda_T(0,S)}{\Lambda_T(0,S)}\right) > 0$ and*

$$\kappa \leq \frac{S_0}{\mathbb{E}\left[S_T\right]\left(\Phi(Q_T) - \Phi(Q_T - 1)\right)} + \frac{1}{k_0\mathbb{E}\left[S_T\right]\left(\Phi(Q_T) - \Phi(Q_T - 1)\right)}.$$

In particular, when $\Phi \equiv \mathrm{id}$, the condition reduces to

$$\kappa \leq \frac{S_0}{\mathbb{E}\left[S_T\right]} + \frac{1}{k_0\mathbb{E}\left[S_T\right]}.$$

(b) Global monotony (exponential intensity). *Assume that $s^* := \| \sup_{t\in[0,T]} S_t\|_{L^\infty} < +\infty$. If $\lambda(x) = Ae^{-kx}$, $A > 0$, $k > 0$, $Q_T \geq 2T\lambda(-S_0)$ and*

$$\kappa \leq \frac{1 + k(S_0 - \delta_{\max})}{k\, s^*} \quad \text{if } \Phi \neq \mathrm{id}, \quad \kappa \leq \frac{1 + k(S_0 - \delta_{\max})}{k\, s^*(\Phi(Q_T) - \Phi(Q_T - 1))} \quad \text{if } \Phi = \mathrm{id}, \tag{5.31}$$

then $H(\cdot, (y_i)_{0\leq i\leq m})$ is nondecreasing on $[0, \delta_{\max}]$ for every $(y_i)_{0\leq i\leq m} \in [0, s^*]^{m+1}$.

To prove this result, we need to establish the monotony of several functions of μ which appear in the expression of C'.

Lemma 5.3. *(i) The function $\mu \longmapsto \mu\mathbb{P}\left(N^\mu \leq Q\right)$ is nondecreasing on $\left[0, \left\lfloor \frac{Q+1}{2} \right\rfloor\right]$.*

(ii) The function $\mu \longmapsto \Theta(Q, \mu) := \mathbb{E}\left[\Phi(Q - N^\mu) - \Phi(Q - N^\mu - 1) \mid N^\mu \leq Q - 1\right]$ is non-increasing.

Proof of Lemma 5.3. *(i)* We have $\frac{d}{d\mu}\left(\mu\mathbb{P}\left(N^\mu \leq Q\right)\right) = \mathbb{P}\left(N^\mu \leq Q\right) - \mu\mathbb{P}\left(N^\mu = Q\right)$. Consequently

$$\frac{d}{d\mu}\left(\mu\mathbb{P}\left(N^\mu \leq Q\right)\right) \geq 0 \quad \text{iff} \quad \sum_{k=0}^{Q} \frac{\mu^k}{k!} \geq \mu\frac{\mu^Q}{Q!}.$$

But $k \longmapsto \frac{\mu^k}{k!}$ is nondecreasing on $\{0, 1, \ldots, \lfloor\mu\rfloor\}$ and non-increasing on $\{\lfloor\mu\rfloor, \ldots\}$.

Hence $\sum_{k=0}^{Q} \frac{\mu^k}{k!} \geq \sum_{k=\lfloor\mu\rfloor}^{Q} \frac{\mu^Q}{Q!} = (Q - \lfloor\mu\rfloor)\frac{\mu^Q}{Q!}$, so that $\sum_{k=0}^{Q} \frac{\mu^k}{k!} \geq \mu\frac{\mu^Q}{Q!}$ as soon as $Q \geq 2\lfloor\mu\rfloor + 1$.

(ii) The function Φ is nondecreasing, non-negative and convex with $\Phi(0) = 0$. If we look at the representation of $\mu \mapsto N^\mu$ by

$$N^\mu(\omega) = \max\left\{n \in \mathbb{N} \mid \prod_{i=1}^{n} U_i(\omega) > e^{-\mu}\right\},$$

where U_i are i.i.d. uniformly distributed random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then $\mu \mapsto N^\mu$ is clearly nondecreasing, so $\mu \mapsto Q - N^\mu$ is non-increasing and $\mu \mapsto \varphi(\mu) = \Phi(Q - N^\mu) - \Phi(Q - N^\mu - 1)$ too (because of the convexity of Φ). □

Remark. If $\mu \in (0,1)$, then $\mu \longmapsto \mu\,\mathbb{P}\left(N^\mu \leq Q\right)$ is always nondecreasing. If $\mu \in [1,2)$, then the function $\mu \longmapsto \mu\mathbb{P}\left(N^\mu = 0\right) = \mu e^{-\mu}$ is not always nondecreasing clearly, but only on $[0,1]$.

Proof of Proposition 5.7. (a) In our problem the intensity parameter $\mu = \displaystyle\int_0^T \lambda(S_t - S_0 + \delta)dt$ is continuous, non-increasing to zero when δ tends to $+\infty$ and bounded by assumption $(\lambda(-S_0) < +\infty)$. Hence $\mu \in [0, \lambda(-S_0)T]$.
(i) From (5.19), we have for $\delta = 0$,

$$\begin{aligned}
&\widehat{C}\left(0, \Lambda_T(0,S), \frac{\partial}{\partial\delta}\Lambda_T(0,S), (S_t)_{0\leq t\leq T}\right)\\
\leq\ &\left(S_0\frac{\partial}{\partial\delta}\Lambda_T(0,S) - \Lambda_T(0,S)\right)\mathbb{P}^{(0)}\left(N^\mu \leq Q_T - 1\right) - \kappa S_T\frac{\partial}{\partial\delta}\Lambda_T(0,S)\varphi^{(0)}(\mu)\\
=\ &\left(\frac{\partial}{\partial\delta}\Lambda_T(0,S)\left(S_0 - \kappa S_T\Theta(Q_T, \Lambda_T(0,S))\right) - \Lambda_T(0,S)\right)\mathbb{P}^{(0)}\left(N^\mu \leq Q_T - 1\right),
\end{aligned}$$

because $-Q_T\mathbb{P}\left(N^\mu > Q_T\right)_{|\mu=\Lambda_T(0,S)} < 0$ and small if Q_T is large. Set $k_0 = \text{infess}\left(-\frac{\frac{\partial}{\partial\delta}\Lambda_T(0,S)}{\Lambda_T(0,S)}\right) > 0$ $a.s.$ by assumption, *i.e.* $\frac{\partial}{\partial\delta}\Lambda_T(0,S) \leq -k_0\Lambda_T(0,S)$ $a.s.$. Then

$$\begin{aligned}
&\widehat{C}\left(0, \Lambda_T(0,S), \frac{\partial}{\partial\delta}\Lambda_T(0,S), (S_t)_{0\leq t\leq T}\right)\\
\leq\ &-\left(1 + k_0(S_0 - \kappa S_T\Theta(Q_T, \Lambda_T(0,S)))\right)\left(\mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\right)_{|\mu=\Lambda_T(0,S)}.
\end{aligned}$$

Now, by Lemma 5.3,

$$\mu \longmapsto \Theta(Q_T, \mu) \quad \text{is non-increasing.}$$

Consequently $$\Theta(Q_T, \mu) \leq \Theta(Q_T, 0) = \Phi(Q) - \Phi(Q-1) = \varphi(0).$$

Therefore

$$\begin{aligned}
&\widehat{C}\left(0, \Lambda_T(0,S), \frac{\partial}{\partial\delta}\Lambda_T(0,S), (S_t)_{0\leq t\leq T}\right)\\
\leq\ &-\left(1 + k_0\left(S_0 - \kappa S_T\left(\Phi(Q_T) - \Phi(Q_T - 1)\right)\right)\right)\left(\mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\right)_{|\mu=\Lambda_T(0,S)}.
\end{aligned}$$

By Lemma 5.3, if $Q_T \geq 2T\lambda(-S_0)$, then $\mu \longmapsto \mu\mathbb{P}\left(N^\mu \leq Q - 1\right)$ is nondecreasing. Moreover, the functional

$$\begin{aligned}
F : \mathbb{D}([0,T,\mathbb{R}) &\to \mathbb{R}\\
\alpha &\mapsto \Lambda_T(0,\alpha) = \int_0^T \lambda(\alpha(t) - S_0 + \delta)dt
\end{aligned}$$

is non-increasing and

$$\alpha(T) \longmapsto \left(-1 - k_0(S_0 - \kappa\alpha(T)\left(\Phi(Q_T) - \Phi(Q_T - 1)\right)\right) \quad \text{is nondecreasing.}$$

Therefore, by opposite monotony principle for diffusion (see Theorem 5.6)

$$\begin{aligned}
&\mathbb{E}\left[\left(-1 - k_0(S_0 - \kappa S_T\left(\Phi(Q_T) - \Phi(Q_T - 1)\right)\right)\left(\mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\right)_{|\mu=\Lambda_T(0,S)}\right]\\
&\leq \mathbb{E}\left[-1 - k_0(S_0 - \kappa S_T\left(\Phi(Q_T) - \Phi(Q_T - 1)\right)\right]\mathbb{E}\left[\left(\mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\right)_{|\mu=\Lambda_T(0,S)}\right],
\end{aligned}$$

and we obtain

$$C'(0) \leq \mathbb{E}\left[-1 - k_0(S_0 - \kappa S_T\left(\Phi(Q_T) - \Phi(Q_T - 1)\right)\right]\mathbb{E}\left[\left(\mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\right)_{|\mu=\Lambda_T(0,S)}\right].$$

As $\mathbb{E}\left[\left(\mu\mathbb{P}\left(N^{\mu}\leq Q_T-1\right)\right)_{|\mu=\Lambda_T(0,S)}\right]\geq 0$, then $C'(0)\leq 0$ as soon as

$$\mathbb{E}\left[-1-k_0(S_0-\kappa S_T\left(\Phi(Q_T)-\Phi(Q_T-1)\right))\right]\leq 0,$$

i.e.
$$\kappa<\frac{S_0}{\mathbb{E}\left[S_T\right]\left(\Phi(Q_T)-\Phi(Q_T-1)\right)}+\frac{1}{k_0\mathbb{E}\left[S_T\right]\left(\Phi(Q_T)-\Phi(Q_T-1)\right)}.$$

(b) From (5.19), the form of λ and (a), we get that for every $\delta\in[0,\delta_{\max}]$, $S=(S_i)_{1\leq i\leq m}\in\mathbb{R}^{m+1}$,

$$H(\delta,S)=-Q_T\mathbb{P}^{(\delta)}\left(N^{\mu}>Q_T\right)+f(\delta,S)\left(\mu\mathbb{P}\left(N^{\mu}\leq Q_T-1\right)\right)_{|\mu=\Lambda_T(\delta,S)},$$

where $f(\delta,S)=-1-k(S_0-\delta-\kappa S_T\Theta(Q_T-1,\Lambda_T(\delta,S)))$ if $\Phi\neq\mathrm{id}$ and $f(\delta,S)=-1-k(S_0-\delta-\kappa S_T)$ if $\Phi=\mathrm{id}$. Since $\delta\mapsto -Q_T\mathbb{P}^{(\delta)}\left(N^{\mu}>Q_T\right)$ is nondecreasing, $\delta\mapsto\left(\mu\mathbb{P}\left(N^{\mu}\leq Q_T-1\right)\right)_{|\mu=\Lambda_T(\delta,S)}$ is non-increasing and non-positive owing to Lemma 5.3 (i) and $\delta\mapsto f(\delta,S)$ is nondecreasing owing to Lemma 5.3 (ii), $\delta\mapsto H(\delta,S)$ is nondecreasing if $f(\delta,S)>0$, $\delta\in[0,\delta_{\max}]$, $S\in\mathbb{R}^{m+1}$, which leads to (5.31). □

5.5.2 Sufficient condition for the convexity condition

Proposition 5.8. (i) *If* $\Phi\neq\mathrm{id}$, *assume that there exists* $\rho_Q\in\left(0,1-\frac{\mathbb{P}(N^{\mu}=Q_T-1)}{\mathbb{P}(N^{\mu}\leq Q_T-1)}_{|\mu=T\lambda(-S_0)}\right)$ *such that*

$$\forall x\in[1,Q_T-1],\quad \Phi(x)-\Phi(x-1)\leq\rho_Q(\Phi(x+1)-\Phi(x)).$$

If
$$Q_T\geq\left(2\vee\left(1+\frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\right)\right)T\lambda(-S_0)\qquad\text{and}\qquad\kappa\leq\frac{2\underline{k}_1}{\overline{k}_1\overline{k}_2\mathbb{E}\left[S_T\right]\Phi'(Q_T)},$$

then $C''(\delta)\geq 0$, $\delta\in[0,\delta_{\max}]$, *so that* C *is convex on* $[0,\delta_{\max}]$.

(ii) *When* $\Phi=\mathrm{id}$, *the condition reads*

$$\kappa\leq\frac{2\underline{k}_1}{\overline{k}_1\overline{k}_2\mathbb{E}\left[S_T\right]}.$$

To prove the above Proposition, we need the following results

Lemma 5.4. *If* $\mu\leq Q-1$, *then* $\mu\mapsto\frac{\mathbb{P}(N^{\mu}=Q-1)}{\mathbb{P}(N^{\mu}\leq Q-1)}$ *is nondecreasing.*

Proof of Lemma 5.4.

$$\begin{aligned}\frac{d}{d\mu}\frac{\mathbb{P}\left(N^{\mu}=Q-1\right)}{\mathbb{P}\left(N^{\mu}\leq Q-1\right)} &= \frac{\mathbb{P}\left(N^{\mu}=Q-2\right)-\mathbb{P}\left(N^{\mu}=Q-1\right)}{\mathbb{P}\left(N^{\mu}\leq Q-1\right)}+\frac{\mathbb{P}\left(N^{\mu}=Q-1\right)^2}{\mathbb{P}\left(N^{\mu}\leq Q-1\right)^2}\\ &= \frac{\mathbb{P}\left(N^{\mu}=Q-1\right)\left(\mathbb{P}\left(N^{\mu}\leq Q-1\right)\left(\frac{Q-1}{\mu}-1\right)+\mathbb{P}\left(N^{\mu}=Q-1\right)\right)}{\mathbb{P}\left(N^{\mu}\leq Q-1\right)^2}\\ &\geq 0\quad\text{if}\quad\mu\leq Q-1.\end{aligned}$$

□

Lemma 5.5. *Assume that* $\Phi\neq\mathrm{id}$. *If there exists* $\rho_Q\in\left(0,1-\frac{\mathbb{P}(N^{\mu}=Q_T-1)}{\mathbb{P}(N^{\mu}\leq Q_T-1)}_{|\mu=T\lambda(-S_0)}\right)$ *such that*

$$\forall x\in[1,Q_T-1],\quad \Phi(x)-\Phi(x-1)\leq\rho_Q(\Phi(x+1)-\Phi(x)),$$

then $\mu\mapsto\frac{\varphi(\mu)}{\mathbb{P}(N^{\mu}\leq Q_T-1)}$ *is non-increasing.*

Remark. If $\Phi = \mathrm{id}$, then $\mu \mapsto \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)} \equiv 1$, therefore we do not need the previous lemmas.

Proof of Lemma 5.5. We have

$$\frac{d}{d\mu}\frac{\varphi(\mu)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)} = \frac{\mathbb{P}\left(N^\mu = Q_T - 1\right)\varphi(\mu)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)^2} - \frac{\psi(\mu)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)} \leq 0$$

iff $$-\psi(\mu)\mathbb{P}\left(N^\mu \leq Q_T - 1\right) + \mathbb{P}\left(N^\mu = Q_T - 1\right)\varphi(\mu) \leq 0$$

iff $$\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\mathbb{E}\left[\Phi\left((Q_T - N^\mu - 1)_+\right) - \Phi\left((Q_T - N^\mu - 2)_+\right)\right] \leq \mathbb{P}\left(N^\mu \leq Q_T - 2\right)\varphi(\mu).$$

But

$$\begin{aligned}
\Phi\left((Q_T - N^\mu - 1)_+\right) &- \Phi\left((Q_T - N^\mu - 2)_+\right) \\
&\leq \rho_Q\left(\Phi\left((Q_T - N^\mu)_+\right) - \Phi\left((Q_T - N^\mu - 1)_+\right)\right)\mathbb{1}_{\{N^\mu \leq Q_T - 2\}} \\
&= \rho_Q\left(\Phi\left((Q_T - N^\mu)_+\right) - \Phi\left((Q_T - N^\mu - 1)_+\right) - (\Phi(1) - \Phi(0))\,\mathbb{1}_{\{N^\mu = Q_T - 1\}}\right) \\
&\leq \rho_Q\left(\Phi\left((Q_T - N^\mu)_+\right) - \Phi\left((Q_T - N^\mu - 1)_+\right)\right)
\end{aligned}$$

since $$(\Phi(1) - \Phi(0))\,\mathbb{1}_{\{N^\mu = Q_T - 1\}} \geq 0 \quad a.s.$$

Consequently

$$\begin{aligned}
&\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\mathbb{E}\left[\Phi\left((Q_T - N^\mu - 1)_+\right) - \Phi\left((Q_T - N^\mu - 2)_+\right)\right] - \mathbb{P}\left(N^\mu \leq Q_T - 2\right)\varphi(\mu) \\
\leq\ & \left(\rho_Q\mathbb{P}\left(N^\mu \leq Q_T - 1\right) - \mathbb{P}\left(N^\mu \leq Q_T - 2\right)\right)\varphi(\mu) \\
=\ & \left(\rho_Q - \left(1 - \frac{\mathbb{P}\left(N^\mu = Q_T - 1\right)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)}\right)\right)\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\varphi(\mu) \leq 0 \quad \text{if} \quad \rho_Q \leq 1 - \frac{\mathbb{P}\left(N^\mu = Q_T - 1\right)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)}.
\end{aligned}$$ □

Proof of Proposition 5.8. By using the notation (5.9)-(5.10), we obtain the following minoration for the second derivative of the cost function

$$\begin{aligned}
C''(\delta) \geq\ & \mathbb{E}\Big[2\underline{k}_1\Lambda_T(\delta, S)\mathbb{P}^{(\delta)}\left(N^\mu \leq Q_T - 1\right) \\
&+(S_0 - \delta)\underline{k}_1\underline{k}_2\Lambda_T(\delta, S)\left(\mathbb{P}^{(\delta)}\left(N^\mu \leq Q_T - 1\right) - \frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\Lambda_T(\delta, S)\mathbb{P}^{(\delta)}\left(N^\mu = Q_T - 1\right)\right) \\
&-\kappa S_T\Lambda_T(\delta, S)\left(\overline{k}_1\overline{k}_2\phi^{(\delta)}(\mu) - \underline{k}_1^2\Lambda_T(\delta, S)\psi^{(\delta)}(\mu)\right)\Big].
\end{aligned}$$

By adapting the result of Lemma 5.3, we obtain that, if $Q_T \geq \left(1 + \frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\right)T\lambda(-S_0)$, then

$$\mathbb{E}\left[\left(\mathbb{P}\left(N^\mu \leq Q_T - 1\right) - \frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\mu\mathbb{P}\left(N^\mu = Q_T - 1\right)\right)_{|\mu = \Lambda_T(\delta, S)}\right] \geq 0$$

and by convexity of the penalty function Φ, we have $\psi^{(\delta)}(\mu) \geq 0$ $a.s.$. Then we obtain the following upper bound for κ,

$$\kappa \leq \frac{2\underline{k}_1\mathbb{E}\left[\left(\mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\right)_{|\mu = \Lambda_T(\delta, S)}\right]}{\overline{k}_1\overline{k}_2\mathbb{E}\left[S_T\Lambda_T(\delta, S)\varphi^{(\delta)}(\mu)\right]}.$$

By Lemma 5.5, $\mu \mapsto \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)}$ is non-increasing and by Lemma 5.3 $\mu \mapsto \mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)$ is nondecreasing for $Q_T \geq 2\lfloor\mu\rfloor - 1$. Furthermore $\alpha \mapsto \Lambda_T(\delta, \alpha)$ is non-increasing. By applying the principle of opposite monotony (see Theorem 5.6), we then have, for $Q_T \geq 2T\lambda(-S_0)$, that

$$\mathbb{E}\left[S_T\Lambda_T(\delta, S)\varphi^{(\delta)}(\mu)\right] \leq \mathbb{E}\left[\left(\mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)\right)_{|\mu = \Lambda_T(\delta, S)}\right]\mathbb{E}\left[\left(\frac{\varphi(\mu)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)}\right)_{|\mu = \Lambda_T(\delta, S)}\right].$$

Therefore

$$\kappa \leq \frac{2\underline{k}_1}{\overline{k}_1\overline{k}_2\mathbb{E}\left[S_T\left(\frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)}\right)_{|\mu=\Lambda_T(\delta,S)}\right]}.$$

As, by Lemma 5.5, $\mu \mapsto \frac{\varphi(\mu)}{\mathbb{P}(N^\mu \leq Q_T - 1)}$ is non-increasing, then

$$\left(\frac{\varphi(\mu)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)}\right)_{|\mu=\Lambda_T(\delta,S)} \leq \left(\frac{\varphi(\mu)}{\mathbb{P}\left(N^\mu \leq Q_T - 1\right)}\right)_{|\mu=0} = \Phi((Q)_+) - \Phi((Q-1)_+) \leq \Phi'(Q),$$

i.e.

$$\kappa \leq \frac{2\underline{k}_1}{\overline{k}_1\overline{k}_2\mathbb{E}\left[S_T\right]\Phi'(Q)}.$$

□

Remark. As $\delta \in [0, \delta_{\max}]$, then $(S_0 - \delta) \in [S_0 - \delta_{\max}, S_0]$ and

$$\begin{aligned}
& (S_0 - \delta)\underline{k}_1\underline{k}_2\Lambda_T(\delta,S)\left(\mathbb{P}^{(\delta)}\left(N^\mu \leq Q_T - 1\right) - \frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\Lambda_T(\delta,S)\mathbb{P}^{(\delta)}\left(N^\mu = Q_T - 1\right)\right) \\
\geq\ & (S_0 - \delta_{\max})\underline{k}_1\underline{k}_2\Lambda_T(\delta,S)\left(\mathbb{P}^{(\delta)}\left(N^\mu \leq Q_T - 1\right) - \frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\Lambda_T(\delta,S)\mathbb{P}^{(\delta)}\left(N^\mu = Q_T - 1\right)\right) \\
=\ & (S_0 - \delta_{\max})\underline{k}_1\underline{k}_2\left[\mu\left(\mathbb{P}\left(N^\mu \leq Q_T - 1\right) - \frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\mu\mathbb{P}\left(N^\mu = Q_T - 1\right)\right)\right]_{|\,\mu=\Lambda_T(\delta,S)}.
\end{aligned}$$

Unfortunately we cannot use the opposite monotony principle for diffusion to improve the bound because, for $Q \geq \left(2 \vee \left(1 + \frac{\overline{k}_1^2}{\underline{k}_1\underline{k}_2}\right)\right)T\lambda(-S_0)$, the function $\mu \mapsto \mu\mathbb{P}\left(N^\mu \leq Q_T - 1\right)$ is nondecreasing and $\mu \mapsto 1 - \frac{\mu\mathbb{P}(N^\mu = Q_T - 1)}{\mathbb{P}(N^\mu \leq Q_T - 1)}$ is non-increasing, and we need to obtain a lower bound for this expression but we get an upper bound.

5.6 Numerical experiments

In this section, we present numerical results with simulated data. We first present the chosen model for the price dynamic and the penalization function. Within the numerical examples, we are modeling the optimal behaviour of a "learning trader" reassessing the price of his passive order every 5 units of time (can be seconds or minutes) in the order books to adapt to the characteristics of the market (fair price moves S_t and order flow dynamics N_t). During each n^{th} slice of 5 seconds, he posts his order of size Q_5 in the book at a distance δ of the best opposite price (δ lower than the best ask for a buy order), and waits 5 seconds. If the order is not completely filled after these 5 seconds (say at time T), the trader cancel the remaining quantity $(Q_5 - N_5)_+$ and buys it using a market order at price S_T plus a market impact; he will buy at $\kappa S_T\left(1 + \eta((Q_5 - N_5)_+)\right)$. Then he can reproduce the experiment choosing another value for the distance to the best opposite δ.

The reassessment procedure used here is the one of formula (5.4) using the expectation representation of C' given by property 5.2 as the proper form for H.

Then we plot the cost function and its derivative for the chosen penalization function and for the identity. We conclude by the results of the recursive procedure for each case of Φ on one hand simulated data and on the other hand real data obtained by replaying the market.

5.6.1 Simulated data

We assume that

$$dS_t = \sigma dW_t, \quad S_0 = s_0 \quad \text{and} \quad \Lambda_T(\delta, S) = A\int_0^T e^{-k(S_t - S_0 + \delta)}dt$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion and $\sigma, A, k > 0$ (this means that $\lambda(x) = Ae^{-kx}$). We denote by $(\bar{S}_t)_{t\geq 0}$ the Euler scheme with step $\frac{T}{m}$ of $(S_t)_{t\geq 0}$ defined by

$$\bar{S}_{k+1} := \bar{S}_k + \sigma\sqrt{\frac{T}{m}}Z_{k+1}, \quad \bar{S}_0 = s_0, \quad Z_{k+1} \sim \mathcal{N}(0,1), \quad k \geq 0,$$

and we approximate $\Lambda_T(\delta, S)$ by $\bar{\Lambda}_T(\delta, S) = A\frac{T}{m}\sum_{k=0}^{m} e^{-k(\bar{S}_k - S_0 + \delta)}$. The market impact penalization function is $\Phi(x) = (1 + \eta(x))x$ with $\eta(x) = A'e^{k'x}$. Now we present the cost function and its derivative for the following parameters:

- parameters of the asset dynamics: $s_0 = 100$ and $\sigma = 0.01$,
- parameters of the intensity of the execution process: $A = 5$ and $k = 1$,
- parameters of the execution: $T = 5$ and $Q = 10$,
- parameters of the penalization function: $\kappa = 1$, $A' = 0.1$ and $k' = 0.05$.

We use $N = 20$ for the Euler scheme and $M = 10000$ simulations Monte Carlo.

Setting 1 ($\eta \not\equiv 0$)

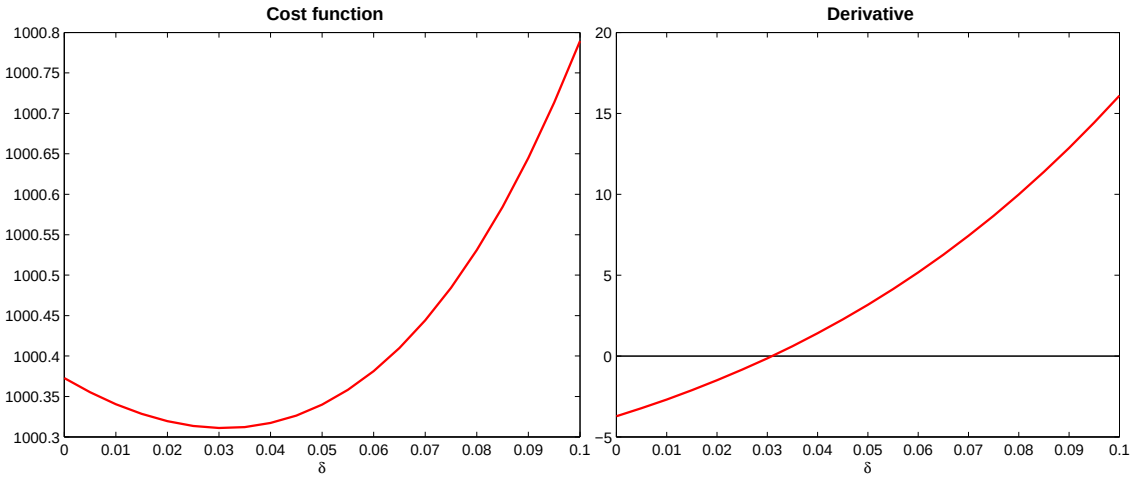

Figure 5.2: $\eta \not\equiv 0$: $T = 5$, $A = 5$, $k = 1$, $s_0 = 100$, $\sigma = 0.01$, $Q = 10$, $\kappa = 6$, $A' = 1$, $k' = 0.01$, $N = 20$ and $M = 10000$.

Setting 2 ($\eta \equiv 0$)

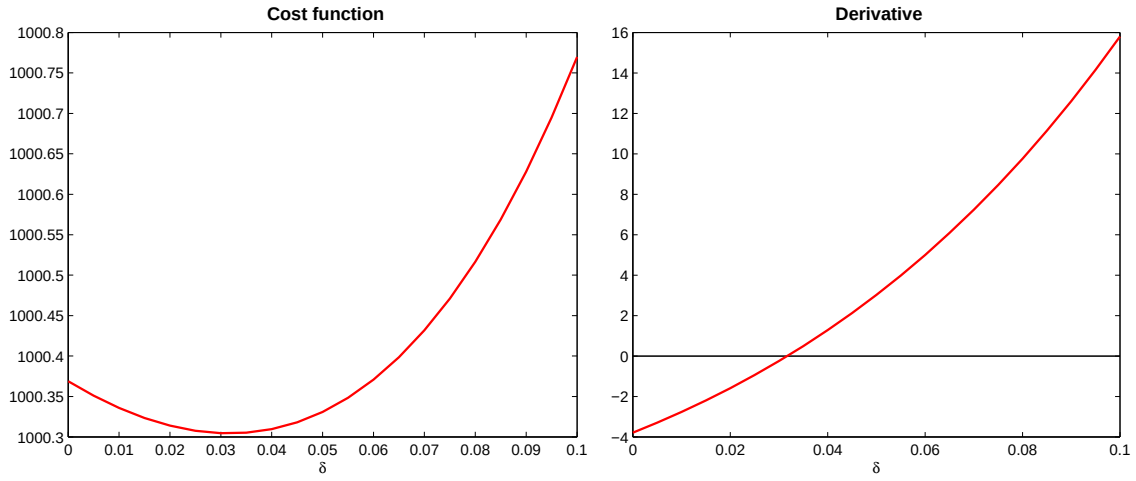

Figure 5.3: $\eta \equiv 0$: $T = 5$, $A = 5$, $k = 1$, $s_0 = 100$, $\sigma = 0.01$, $Q = 10$, $\kappa = 12$, $N = 20$ and $M = 10000$.

Now we present the results of our stochastic recursive procedure for the two cases with

$$n = 100 \quad \text{and} \quad \gamma_n = \frac{1}{100n}.$$

Setting 1 ($\eta \not\equiv 0$)

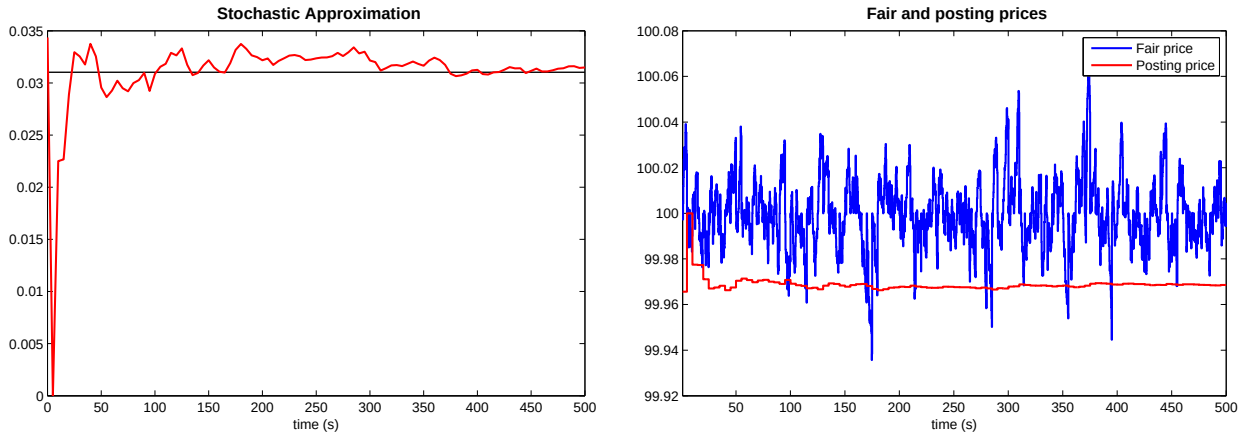

Figure 5.4: $\eta \not\equiv 0$: $T = 5$, $A = 5$, $k = 1$, $s_0 = 100$, $\sigma = 0.01$, $Q = 10$, $\kappa = 6$, $A' = 1$, $k' = 0.01$, $N = 20$ and $n = 100$

Setting 2 ($\eta \equiv 0$)

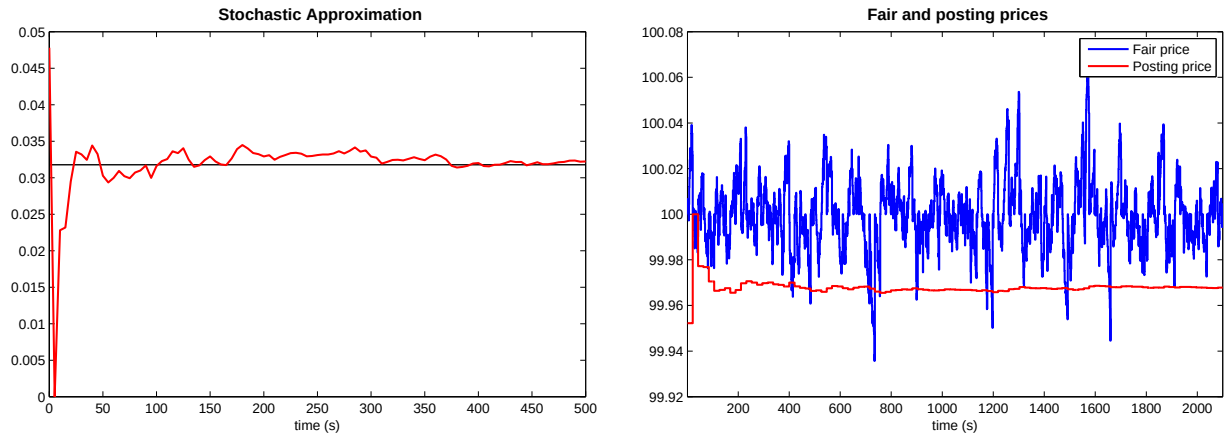

Figure 5.5: $\eta \equiv 0$: $T = 5$, $A = 5$, $k = 1$, $s_0 = 100$, $\sigma = 0.01$, $Q = 10$, $\kappa = 12$, $N = 20$ and $n = 100$.

5.6.2 Market data

The auto adaptiveness nature of this recurrence procedure allows to use it on real data, even if they are not exactly following the models.

In the numerical example of this section, the trader reassess his order using the previously exposed recurrence procedure not on simulated data following exactly the models, but on real data on which the parameters of the models have been fit.

As market data, we use the bid prices of Accor SA (ACCP.PA) of 11/11/2010 for the fair price process $(S_t)_{t\in[0,T]}$. We divide the day into periods of 15 trades which will denote steps of the stochastic procedure. Let M_{cycles} be the number of these periods. For every $m \in M_{\text{cycles}}$, we have a sequence of bid prices $(S^m_{t_i})_{1\leq i\leq 15}$ and we approximate the jump intensity of the Poisson process $\Lambda_{T^m}(\delta, S)$, where $T^m = \sum_{i=1}^{15} t_i$, by

$$\forall m \in M_{\text{cycles}}, \quad \Lambda_{T^m}(\delta, S) = A \sum_{i=2}^{15} e^{-k(S^m_{t_i} - S_{t_1} + \delta)}(t_i - t_{i-1}).$$

The empirical mean of the intensity function

$$\bar{\Lambda}(\delta, S) = \frac{1}{M_{\text{cycles}}} \sum_{n=1}^{M_{\text{cycles}}} \Lambda_{T^m}(\delta, S)$$

is plotted on Figure 5.6.

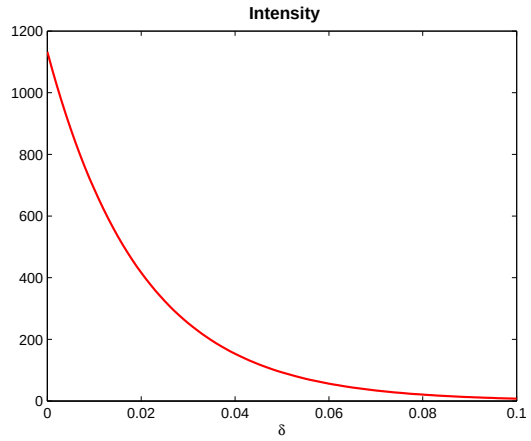

Figure 5.6: Fit of the exponential model on real data (Accor SA (ACCP.PA) 11/11/2010): $A = 1/50$, $k = 50$ and $M_{\text{cycles}} = 220$.

The penalization function is of the following form

$$\Phi(x) = (1 + \eta(x))x \quad \text{with} \quad \eta(x) = A' e^{k'x}.$$

Now we present the cost function and its derivative for the following parameters: $A = 1/50$, $k = 50$, $Q = 100$, $A' = 0.001$ and $k' = 0.0005$.

Setting1 ($\eta \not\equiv 0$)

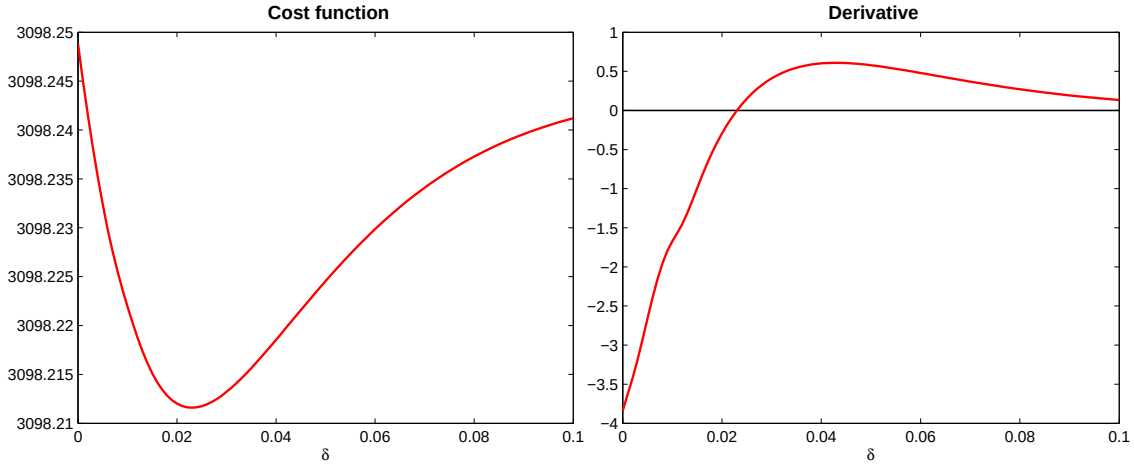

Figure 5.7: $\eta \not\equiv 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1$, $A' = 0.001$, $k' = 0.0005$ and $M_{\text{cycles}} = 220$.

Setting2 ($\eta \equiv 0$)

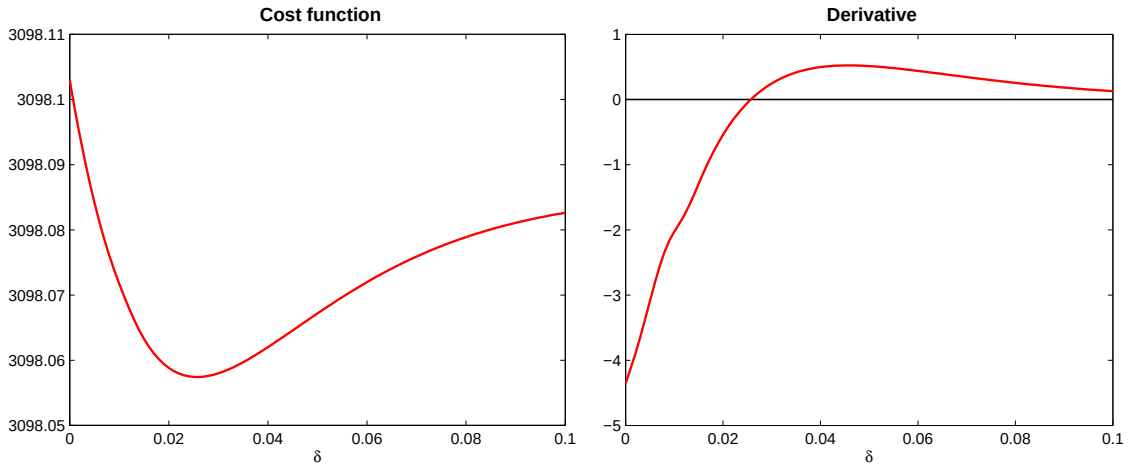

Figure 5.8: $\eta \equiv 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1.001$ and $M_{\text{cycles}} = 220$.

Now we present the results of the stochastic recursive procedure for two cases. To smoothen the behaviour of the stochastic algorithm, we use the averaging principle of Ruppert and Polyak (see Theorem 1.9 in Chapter 1). In short, this principle is two-folded:

– *Phase 1*: Implement the original zero search procedure with $\gamma_n = \frac{\gamma_1}{n^\rho}$, $\frac{1}{2} < \rho < 1$, $\gamma_1 > 0$,

– *Phase 2*: Compute the arithmetic mean at each step n of all the past values of the procedure, namely

$$\bar{\delta}_n = \frac{1}{n+1} \sum_{k=0}^{n} \delta_k, \ n \geq 1.$$

It has been shown by several authors that this procedure under appropriate assumptions is ruled by a CLT having a minimal asymptotic variance (among recursive procedures).

Setting 1 ($\eta \not\equiv 0$)

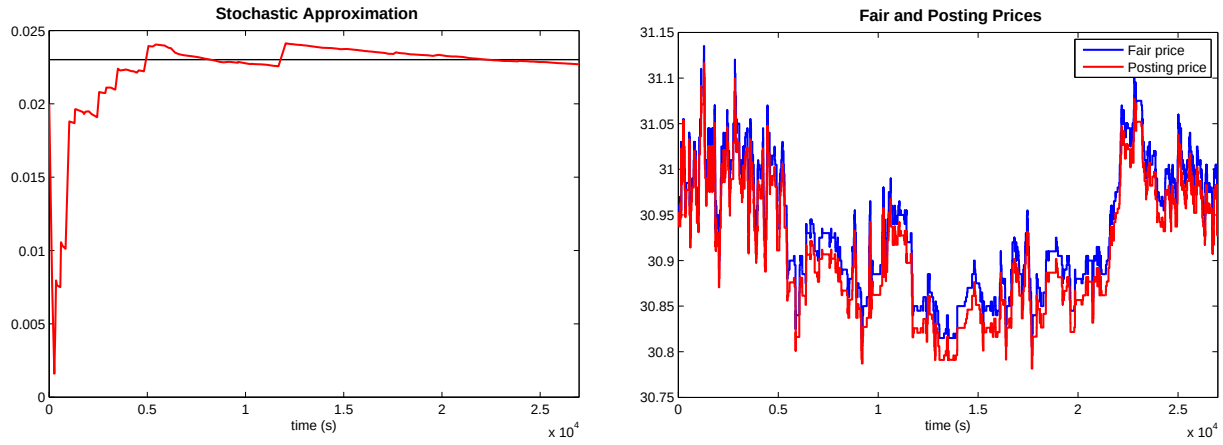

Figure 5.9: $\eta \not\equiv 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1$, $A' = 0.001$, $k' = 0.0005$ and $M_{\text{cycles}} = 220$. Crude algorithm with $\gamma_n = \frac{1}{550n}$.

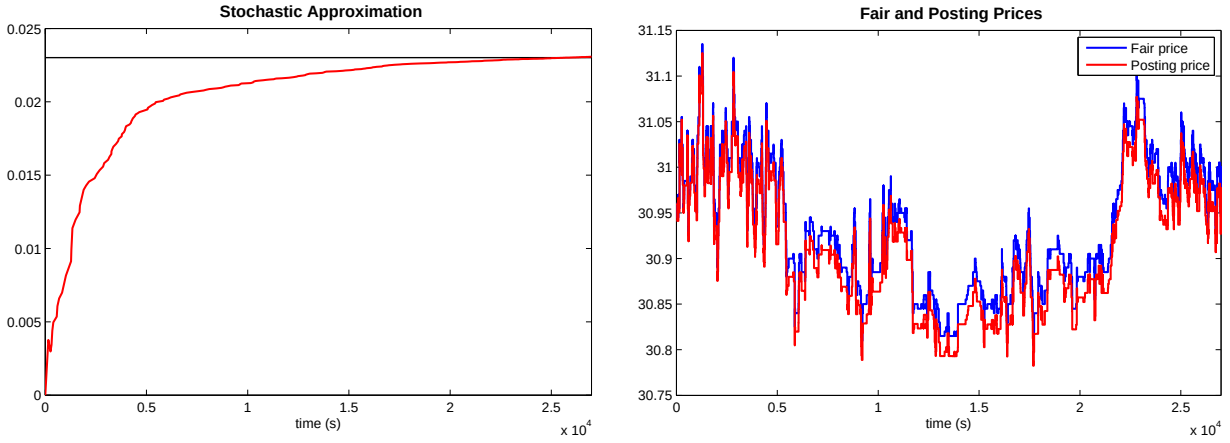

Figure 5.10: $\eta \not\equiv 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1$, $A' = 0.001$, $k' = 0.0005$ and $M_{\text{cycles}} = 220$. Averaging algorithm (Ruppert and Polyak) with $\gamma_n = \frac{1}{550 n^{0.95}}$.

Setting 2 ($\eta \equiv 0$)

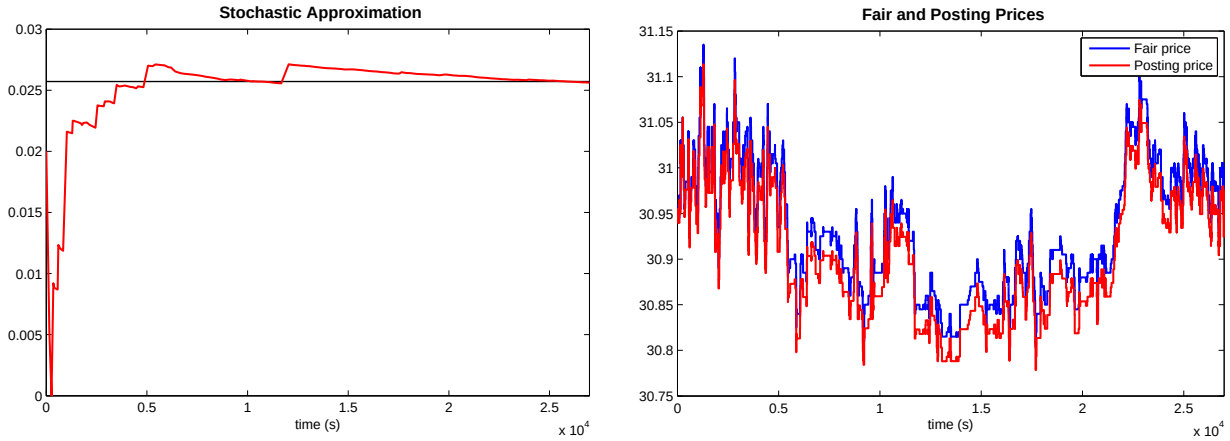

Figure 5.11: $\eta \equiv 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1.001$, $\gamma_n = \frac{1}{450 n}$ and $M_{\text{cycles}} = 220$.

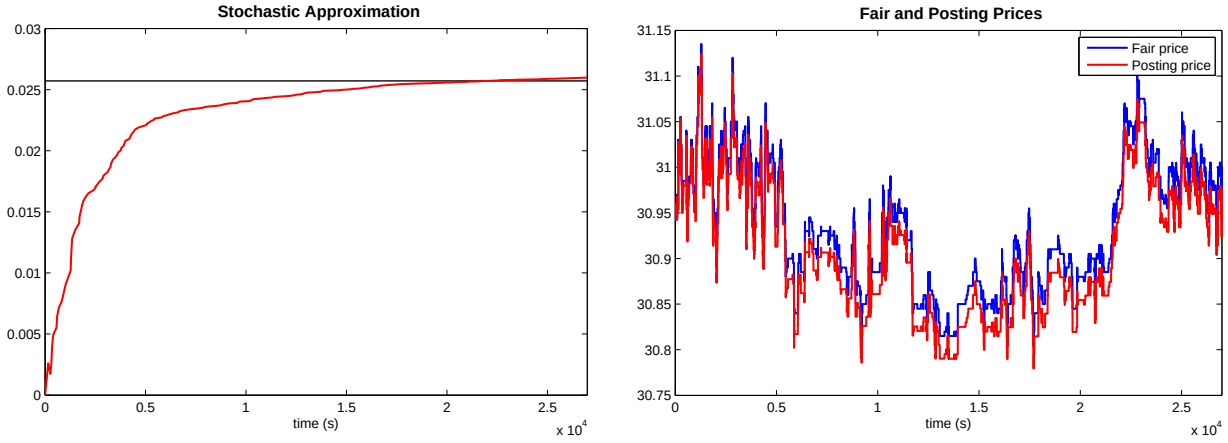

Figure 5.12: $\eta \not\equiv 0$: $A = 1/50$, $k = 50$, $Q = 100$, $\kappa = 1.001$ and $M_{\text{cycles}} = 220$. Averaging algorithm (Ruppert and Polyak) with $\gamma_n = \frac{1}{450n^{0.95}}$.

We see on Figures 5.10 (for $\eta \not\equiv 0$) and 5.12 (for $\eta \equiv 0$) that the recursive procedures converge toward their respective targets, namely the minimum of the execution cost functions presented in Figures 5.7 (for $\eta \not\equiv 0$) and 5.8 (for $\eta \equiv 0$).

Chapter 6

Parameter Implicitation and Calibration for Financial Models

This chapter illustrates how using stochastic recursive procedure to perform implicitation or calibration of (multi-)parameter model. The first one leads to a zero search procedure and the second one to a stochastic gradient algorithm. We illustrate our results on two different models, namely the standard Black-Scholes model where we look for implied volatility and implicit correlation and the pseudo-CEV model where we search the power.

6.1 Introduction

The aim of this chapter is to provide some first examples of model calibration in Finance using stochastic optimization methods, typically Stochastic Approximation (SA). The aim of Stochastic Approximation is to devise and analyze the convergence of recursive stochastic algorithms in order to solve non linear inverse or optimization problems. It goes back to Robbins and Monro in the 1950's (see [103]) who designed a first stochastic "dosage" procedure whose aim was to solve the equation $h(\theta) = c$ where h was an increasing real-valued function defined on the real line having a representation as an expectation *i.e.*

$$h(\theta) = \mathbb{E}\, H(\theta, X), \quad \theta \in \mathbb{R},\ X \text{ random variable with distribution } \mathbb{P}_X.$$

They proposed a step-by step randomized procedure of the form

$$\theta_{n+1} = \theta_n - \gamma_{n+1}(H(\theta_n, X_{n+1}) - c), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}$$

where $(X_n)_{n\geq 1}$ is an i.i.d. sequence of random variables with distribution $\mathbb{P}_X$ and $(\gamma_n)_{n\geq 1}$ a sequence of *step parameters* satisfying $\sum_n \gamma_n = +\infty$ and $\sum_n \gamma_n^2 < +\infty$. Such an approach can be seen as the extension of the Monte Carlo method to the resolution of nonlinear problems. Since this seminal paper, Stochastic Approximation has known a major development in the second part of the XXth century: extension to higher dimensional setting (h vector field on $\mathbb{R}^d$, $h = \nabla V$ where V is a real valued function on $\mathbb{R}^d$, switching of i.i.d. innovations to innovations having a Markovian dynamics, etc). For deeper insight on Stochastic Approximation, we refer to the various textbooks on this fields (see [75],[43] and [25]).

As concerns our precise needs in this chapter, we refer to Chapter 1 where the main result are briefly presented and commented. In particular this kind of procedure $a.s.$ converges toward their target under assumptions which are not significantly more stringent than those required on the mean function h in their deterministic counterpart. All this must be understood under the

meta-assumption that the distribution $\mathbb{P}_X$ – whatever it is supported by $\mathbb{R}^d$ or a path space – is simulable on a computer at a reasonably low cost.

In Finance, practitioners are facing basically two types of problems when they want to infer the values of the parameters of their model dynamics on "simple" derivative products (called *vanilla products*) for which liquidity is sufficient to guarantee the consistency of the quoted price. Typically for a stock, vanilla products are the *Call* and *Put* options.

The first problem is *implicitation*, which consists in solving an inverse problem: one has to extract the value of d parameters from exactly d quoted prices of derivatives with different features (strike prices, maturities, etc). Typically the extraction of the (Black-Scholes) implicit volatility from an option quoted price (or mark-to-market price) is an implicitation problem. The extraction of the *implicit correlation parameter* between the yield of two assets from the quoted price of an option involving the two assets (exchange option, Best-of Call option, etc) is a second simple example of implicitation problem, provided all the other parameters of both dynamics have been estimated before on their own vanilla derivative market. From general non-arbitrage theory , vanilla options (and not only vanilla...) have a representation as an expectation of their payoff under the risk-neutral probability (or more generally a martingale measure). In the world of Stochastic Approximation, the resulting procedures can be seen as zero search procedures and requires some generalized monotony properties on the so-called *mean function* h.

The second class of problems is know as *calibration*. It is a somewhat different problem since in that case one has - quantitatively speaking - more information or data from the market than the dimension of the model parameter vector. In practice this means that one has to fit an $\mathbb{R}^d$-parameter from q quoted prices of vanilla derivatives, $q > d$. In that case one performs a regression of the mark-to-market prices on the model prices in order to minimize a global error criterion, usually a weighted (possibly penalized) mean square criterion.

The next step is then to price more exotic products using the calibrated model but this is not our objective here. Implicitation and Calibration are among the major tasks to be performed in Finance and they have given rise to a huge literature, often technical but also academic (see [2, 15, 22, 29, 38, 108] among many others). Our aim is to illustrate how to use Stochastic Approximation to solve this kind of problem. Our stochastic optimization approach is clearly universal in that it can be applied, up to easy-to-implement adaptations, to any model whose dynamics can be simulated at least through its Euler scheme. Its rate of convergence is that of standard SA *i.e.* $n^{-\frac{1}{2}}$ if n is the number of iterations of the algorithms.

However, as everybody who knows the connections between SA and the asymptotic behaviour of differential dynamical systems, it may suffers from its non linearity, especially in its "exploration" phase. Furthermore, like their deterministic counterparts, it also suffers from instability: small changes in the price levels may induce major move of the "optimal calibrator". But this is far from being specific to SA, since it is in fact the major drawback of the way model are (daily) calibrated in Finance, motivating the introduction of an artificial "close-to-yesterday" penalization.

Overall, the rather poor rate of convergence of the approach – which is in fact just comparable to that of the plain Monte Carlo method and can be speeded up similarly by using QMC methods (see Chapter 3) – is balanced by this remarkable universality feature since it can be implemented as soon as the underlying innovation process can be simulated. More often than it should be, models are chosen in view of the efficiency of their static calibration process rather than on any other feature. Systematic implementation of SA implicitation and calibration procedure could help changing this custom.

This chapter is organized as follows: in Section 6.2, we present the problem of parameter implicitation and show how to solve it using a stochastic recursive procedure. Then we give several examples with numerical experiments for three financial models: the Black-Scholes model, the Mer-

ton model and the pseudo-CEV model. The parameter of interest are the implied volatility, the correlation between the yields of two risky assets and the other parameters of these dynamics. In Section 6.3 we introduce the paradigm of calibration and give three methods using SA to solve it: one based on Robbins-Siegmund Lemma (see Theorem 1.4) using tangent process, one based on derivation on the state space and the last one based on Kiefer-Wolfowitz algorithm. Then we illustrate these three approaches on the three models mentioned above for all dynamic parameters.

6.2 Parameter implicitation

The aim of this section is to present the implicitation of diffusion parameters using stochastic approximation. Let $\Theta \subset \mathbb{R}^d$ be an open convex set of $\mathbb{R}^d$. Consider the price dynamics $(X_t(\theta))_{t\geq 0}$ depending on a parameter $\theta \in \Theta$ of q traded risky assets

$$dX_t(\theta) = b(\theta, t, X_t(\theta))dt + \sigma(\theta, t, X_t(\theta)).dW_t, \quad X_0(\theta) = x_0(\theta) \in \mathbb{R}^q,$$

where $b : \Theta \times [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ and $\sigma : \Theta \times [0,T] \times \mathbb{R}^q \to \mathcal{M}_{q,\ell}(\mathbb{R})$ are continuous functions and W is a ℓ-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that b and σ are Lipschitz continuous in x uniformly with respect to $(\theta, t) \in \Theta \times [0,T]$. These assumptions ensure that there exists a unique solution for the above SDE. Let $X_t^0 = e^{rt}$ denote the riskless asset.

Consider the payoff function $\varphi : \mathbb{R}^q \to \mathbb{R}^d$ and a vector of market prices $P_0^{\text{market}} \in \mathbb{R}^d$. We will use a stochastic procedure to solve the inverse problem in $\theta \in \Theta$

$$e^{-rT}\mathbb{E}\left[\varphi(X_T(\theta))\right] = P_0^{\text{market}}.$$

Assume that this equation in θ has at least one solution, denote by θ^*. Furthermore, assume that P_0^{market} is consistent but is not an extremal value of $\theta \mapsto e^{-rT}\mathbb{E}\left[\varphi(X_T(\theta))\right]$. Thus we are looking for a zero of the mean function $h : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$h(\theta) = e^{-rT}\mathbb{E}\left[\varphi(X_T(\theta))\right] - P_0^{\text{market}} = \mathbb{E}\left[H(\theta, Z)\right],$$

where $H(\theta, z) = e^{-rT}\varphi(X_T(\theta)) - P_0^{\text{market}}$ and Z is a vector of i.i.d. random variables used to compute $X_T(\theta)$, either exactly, or by an Euler scheme.

To apply Robbins-Siegmund Lemma (see Theorem 1.4 in Chapter 1) we need to find a Lyapunov function for the algorithm and to check that the functions h and H satisfy the required assumptions (mean reversion, growth control, etc.).

The following subsections gives examples of parameter implicitation in Black-Scholes model, pseudo-CEV model or Merton model. First, we implicitate the volatility, then the correlation between two assets (only for the two first models) and finally the other paremeters for the two last models.

6.2.1 Application to implied volatility extraction

6.2.1.1 The Black-Scholes model

We devise a recursive procedure to compute the implied volatility in a standard Black-Scholes model. The dynamic of the risky asset is then

$$X_t = xe^{(r-\frac{\sigma^2}{2})t+\sigma W_t}, \quad x > 0,$$

and the riskless asset is $X_0^t = e^{rt}$.

Set

$$C_{BS}(\sigma) = \mathbb{E}\left[\left(xe^{-\frac{\sigma^2}{2}T+\sigma W_T} - e^{-rT}K\right)_+\right].$$

Then C_{BS} is non-decreasing on $[0, +\infty)$ and continuous as a function of the volatility. Furthermore

$$\lim_{\sigma\to 0} C_{BS}(\sigma) = \left(x - e^{-rT}K\right)_+ \quad \text{and} \quad \lim_{\sigma\to\infty} C_{BS}(\sigma) = x.$$

We deduce that for any market price $P_0^{\text{market}} \in \left[\left(x - e^{-rT}K\right)_+, x\right]$, there is a unique Black-Scholes implicit volatility for this price. We then construct the following recursive procedure

$$\sigma_{n+1} = \sigma_n - \gamma_{n+1}H(\sigma_n, Z_{n+1}), \quad n \geq 0, \quad \sigma_0 \geq 0,$$

where $H(\sigma, Z) = \left(xe^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}Z} - e^{-rT}K\right)_+ - P_0^{\text{market}}$.

NUMERICAL EXPERIMENT. We set the model parameters to the following values $x_0 = 100$, $r = 0.10$, $\sigma = 0.30$ and the payoff parameters $T = 1$, $K = 100$. The reference Black-Scholes price 16.78 is used as a market price. The mean function of this stochastic procedure is given by Figure 6.1.

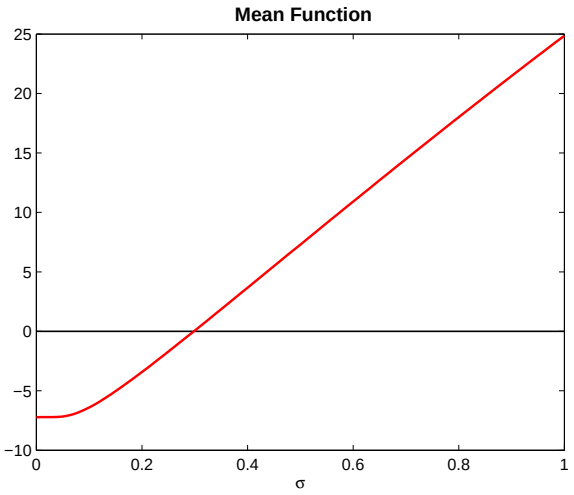

Figure 6.1: Mean function depending on the volatility.

The stochastic approximation procedure parameters are

$$\sigma_0 = 0, \ n = 10^5.$$

The choice of σ_0 is blind on purpose. Finally we set $\gamma_n = \frac{1}{10n}$.

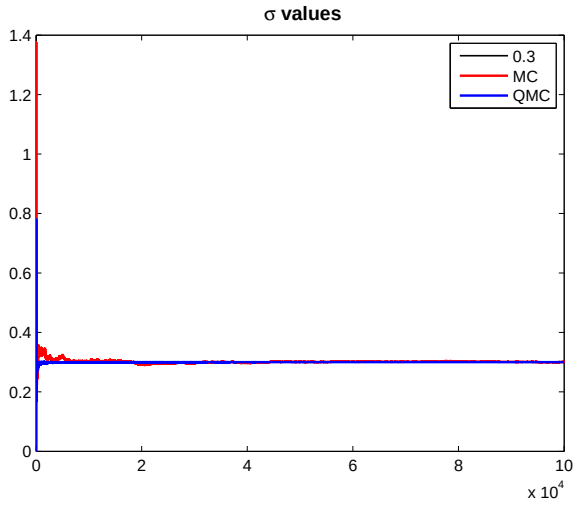

Figure 6.2: Implicitation of Black-Scholes volatility by Call option. $T = 1$, $K = 100$, $r = 0.10$, $x = 100$.

n	σ_n^{MC}	σ_n^{QMC}
100	0.2956	0.2499
1000	0.3452	0.2955
10 000	0.3013	0.2993
100 000	0.3006	0.2999

Table 6.1: Convergence of σ_n^{MC} and σ_n^{QMC} toward $\sigma^* = 0.3$.

6.2.1.2 The Pseudo-CEV model

Such a recursive procedure can be applied to other asset dynamic. Consider the pseudo-CEV model, namely

$$dX_t = rX_t dt + \vartheta \frac{X_t^{1+\beta}}{\sqrt{X_t^2 + \epsilon^2}} dW_t, \quad X_0 = x_0 > 0 \tag{6.1}$$

where $r > 0$, $\vartheta > 0$, $0 < \beta < 1$ and $0 < \epsilon < 1$.

The recursive procedure reads in this case

$$\vartheta_{n+1} = \vartheta_n - \gamma_{n+1} H(\vartheta_n, Z_{n+1}), \quad n \geq 0, \quad \vartheta_0 > 0,$$

where $H : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ is defined by $H(\vartheta, Z) = \left(\bar{X}_T - e^{-rT}K\right)_+ - P_0^{\text{market}}$ with $\bar{X}_T$ the end value of the Euler scheme $(\bar{X}_{\frac{kT}{N}})_{0 \leq k \leq N}$ of the diffusion (6.1).

Numerical experiment. We set the model parameters to the following values

$$x_0 = 100, \ r = 0.10, \ \vartheta = 0.30, \ \beta = 0.80$$

and the payoff parameters

$$T = 1, \ K = 100.$$

The reference price 10.86 is used as a market price. The mean function of this stochastic procedure is given by Figure 6.3.

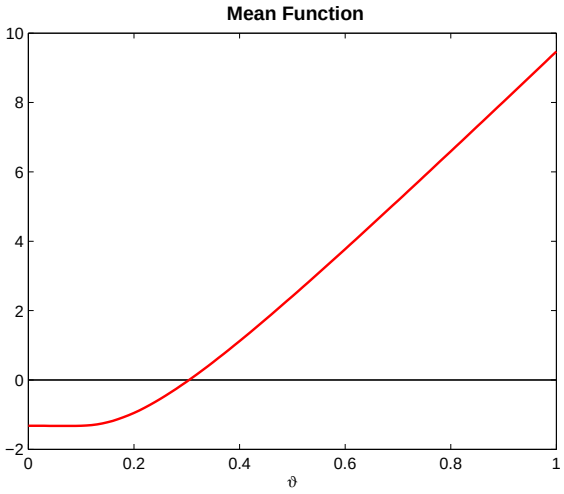

Figure 6.3: Mean function depending on the volatility.

The stochastic approximation procedure parameters are

$$\vartheta_0 = 0,\ n = 10^5.$$

The choice of ϑ_0 is blind on purpose. Finally we set $\gamma_n = \frac{1}{10n}$. We compare the rate of convergence of the recursive procedure using pseudo-random numbers and quasi-random numbers (Halton sequence).

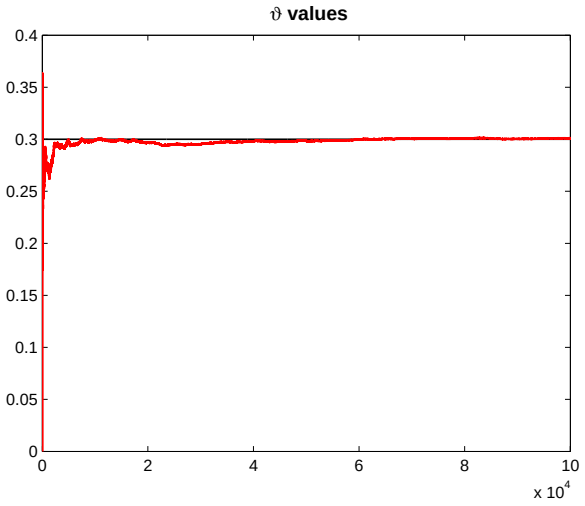

Figure 6.4: Implicitation of Pseudo-CEV volatility by Call option. $T = 1$, $K = 100$, $r = 0.10$, $x_0 = 100$, $\beta = 0.80$.

n	σ_n
100	0.2390
1000	0.2735
10 000	0.2996
100 000	0.3011

Table 6.2: Convergence of σ_n toward $\sigma^* = 0.3$.

6.2.1.3 The Merton model

Set, for every $x, \sigma, \lambda > 0$, $r \in \mathbb{R}$, and $a \in (0,1)$

$$X_t = xe^{(r-\frac{\sigma^2}{2}+\lambda a)t+\sigma W_t}(1-a)^{N_t}, \quad t \geq 0, \tag{6.2}$$

where W is a standard Brownian motion and $N = (N_t)_{t\geq 0}$ is a standard Poisson process with jump intensity λ.

Note, by using that W and N are independent, that

$$\begin{aligned} \|X_T\|_2^2 &= \mathbb{E}\left[x^2 e^{(2r-\sigma^2+2\lambda a)T+2\sigma W_T}(1-a)^{2N_T}\right] \\ &= x^2 e^{2rT}\mathbb{E}\left[e^{-\sigma^2 T+2\sigma W_T}\right]\mathbb{E}\left[e^{2\lambda a T}(1-a)^{2N_T}\right] \\ &= x^2 e^{(2r+\sigma^2+\lambda a^2)T}. \end{aligned} \tag{6.3}$$

Consequently the function H does not satisfy the required sub-linear growth assumption, we should multiply the pseudo-mean function by an appropriate function depending on the parameter to calibrate. For example, we should normalize H by replacing it by

$$\widetilde{H}(\sigma_n, Z) = e^{-\frac{\sigma^2}{2}T}H(\sigma_n, Z).$$

This is usually not necessary in practice. So is the case the simulations below because of the values of the parameters to find.

We devise the following recursive procedure

$$\sigma_{n+1} = \sigma_n - \gamma_{n+1}H(\sigma_n, Z_{n+1}), \quad n \geq 0, \quad \sigma_0 > 0,$$

where

$$H(\sigma, Z) = \left(xe^{(-\frac{\sigma^2}{2}+\lambda a)T+\sigma\sqrt{T}Z^1}(1-a)^{Z^2} - Ke^{-rT}\right)_+ - P_0^{\text{market}}$$

with $Z^1 \overset{\mathcal{L}}{\sim} \mathcal{N}(0,1)$ and $Z^2 \overset{\mathcal{L}}{\sim} \mathcal{P}(\lambda T)$.

Numerical experiment. We set the model parameters to the following values

$$x_0 = 100,\ r = 0.10,\ \sigma = 0.30,\ a = 0.05,\ \lambda = 3$$

and the payoff parameters

$$T = 1,\ K = 100.$$

The reference price 17.13 is used as a market price. The mean function of the algorithm is given by Figure 6.5.

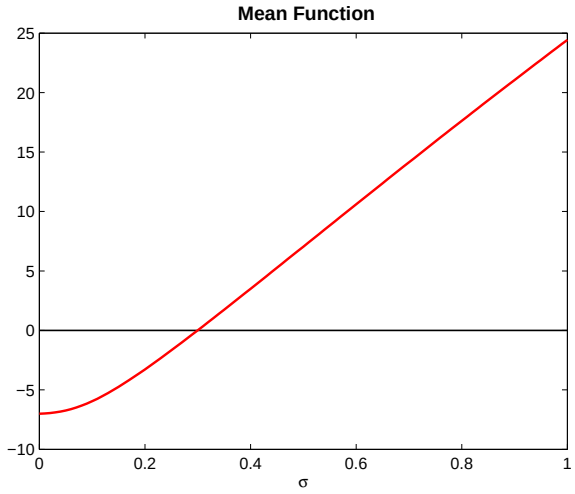

Figure 6.5: Mean function depending on the volatility.

The stochastic approximation procedure parameters are

$$\sigma_0 = 0,\ n = 10^5.$$

The choice of σ_0 is blind on purpose. Finally we set $\gamma_n = \frac{1}{50n}$.

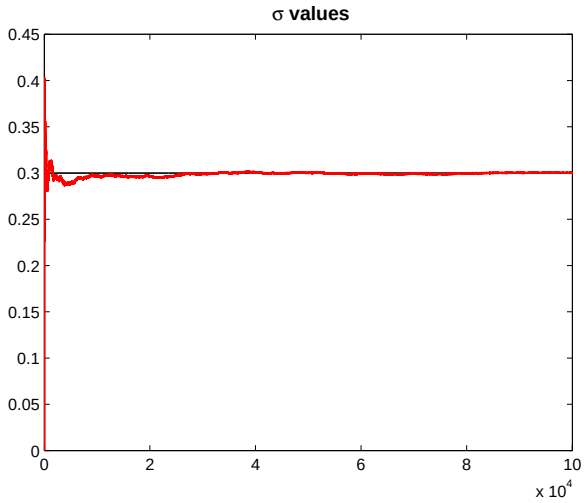

Figure 6.6: Implicitation of Merton volatility by Call option. $T = 1$, $K = 100$, $r = 0.10$, $x_0 = 100$, $a = 0.05$ et $\lambda = 3$.

n	σ_n
100	0.2848
1000	0.3082
10 000	0.2970
100 000	0.3004

Table 6.3: Convergence of σ_n toward $\sigma^* = 0.3$.

6.2.2 Application to implicit correlation search

Consider a 2-dimensional local volatility model *i.e.* $X_0^t = e^{rt}$ (riskless asset) and

$$dX_t^i = X_t^i \left(rdt + \sigma_i(X_t^i) dW_t^i \right), \quad X_0^i = x_0^i > 0, \quad i = 1, 2,$$

for the two risky assets where he functions $x \mapsto x\sigma_i(x)$ are Lipschitz continuous and $\left\langle W^1, W^2 \right\rangle_t = \rho t$, $\rho \in [-1, 1]$, denotes the correlation between W^1 and W^2 (that is the correlation between the yields of the risky assets X^1 and X^2). This ensures the existence and uniqueness of strong solutions for this SDE.

In this market, we consider a European best-of call option with maturity $T > 0$ characterized by its payoff

$$\left(\max \left(X_T^1, X_T^2 \right) - K \right)_+ .$$

A market of such best-of calls is a market of the correlation ρ. The volatility functions σ_i are supposed to have been calibrated on the market of tradable vanilla options on each risky assets X^i, $i = 1, 2$. Assume that we know how to simulate (X_T^1, X_T^2), either exactly (at a reasonable cost, as usual...), or at least approximatively by an Euler scheme $(\bar{X}^1_{\frac{kT}{N}}, \bar{X}^2_{\frac{kT}{N}})_{0 \le k \le N}$ from a $2N$-dimensional vector $Z = (Z^1, \ldots, Z^{2N}) \overset{(d)}{=} \mathcal{N}(0, I_{2N})$, namely for $0 \le k \le N-1$,

$$\begin{array}{rcl} \bar{X}^1_{\frac{(k+1)T}{N}} & = & \bar{X}^1_{\frac{kT}{N}} + r\bar{X}^1_{\frac{kT}{N}} \frac{T}{N} + \sigma_1 \left(\bar{X}^1_{\frac{kT}{N}} \right) \bar{X}^1_{\frac{kT}{N}} \sqrt{\frac{T}{N}} Z^{2k+1} \\ \bar{X}^2_{\frac{(k+1)T}{N}} & = & \bar{X}^2_{\frac{kT}{N}} + r\bar{X}^2_{\frac{kT}{N}} \frac{T}{N} + \sigma_2 \left(\bar{X}^2_{\frac{kT}{N}} \right) \bar{X}^2_{\frac{kT}{N}} \sqrt{\frac{T}{N}} \left(\rho Z^{2k+1} + \sqrt{1-\rho^2} Z^{2k+2} \right). \end{array} \tag{6.4}$$

We will use a stochastic recursive procedure to solve the inverse problem in ρ

$$\begin{array}{ll} P_{BoC}(x_0^1, x_0^2, K, r, \rho, T) = P_0^{market} & \text{if } (X_T^1, X_T^2) \text{ can be exactly simulated,} \\ \bar{P}_{BoC}(x_0^1, x_0^2, K, r, \rho, T) = P_0^{market} & \text{if we use an Euler scheme,} \end{array} \tag{6.5}$$

where P_0^{market} is the quoted premium of the option (mark-to-market) and

$$P_{BoC}(x_0^1, x_0^2, K, r, \rho, T) := e^{-rT} \mathbb{E} \left[\left(\max \left(X_T^1, X_T^2 \right) - K \right)_+ \right].$$

and

$$\bar{P}_{BoC}(x_0^1, x_0^2, K, r, \rho, T) := e^{-rT} \mathbb{E} \left[\left(\max \left(\bar{X}_T^1, \bar{X}_T^2 \right) - K \right)_+ \right].$$

In the following, we will favour the approach with the Euler scheme. We assume from now on that the mark-to-market price is consistent *i.e.* Equation (6.5) in ρ has at least one solution, say ρ^*.

The most convenient way to prevent edge effects due to the fact that $\rho \in [-1, 1]$ is to use a trigonometric parametrization of the correlation by setting

$$\rho = \cos\theta, \quad \theta \in \mathbb{R}.$$

At this stage one notes that $\sqrt{1 - \cos^2\theta} Z^{2k+2} = |\sin\theta| Z^{2k+2} \overset{(d)}{=} \sin\theta Z^{2k+2}$ since $Z^{2k+2} \overset{(d)}{=} -Z^{2k+2}$ and the $(Z^l)_{1 \le l \le 2N}$ are i.i.d., the Euler scheme in $(\bar{X}^1_{\frac{kT}{N}})_{0 \le k \le N}$ is unchanged and the second component $(\bar{X}^2_{\frac{kT}{N}})_{0 \le k \le N} = (\bar{X}^2_{\frac{kT}{N}}(\theta))_{0 \le k \le N}$ is now updated as follows

$$\bar{X}^2_{\frac{(k+1)T}{N}} = \bar{X}^2_{\frac{kT}{N}} + r\bar{X}^2_{\frac{kT}{N}} \frac{T}{N} + \sigma_2 \left(\bar{X}^2_{\frac{kT}{N}} \right) \bar{X}^2_{\frac{kT}{N}} \sqrt{\frac{T}{N}} \left(\cos\theta Z^{2k+1} + \sin\theta Z^{2k+2} \right) \tag{6.6}$$

and has the same distribution as that defined by (6.4) when $\cos\theta = \rho$.

Remark. If (X_T^1, X_T^2) can be exactly simulated, we do the same reasoning directly on the Brownian motion because $\left(W_t^1, \rho W_t^1 + \sqrt{1-\rho^2} W_t^2\right) \overset{(d)}{=} \left(W_t^1, \cos\theta W_t^1 + \sin\theta W_t^2\right)$ if $\cos\theta = \rho$.

From now on we will always consider the Euler scheme. Now, it follows from a straightforward induction that, for every $k \in \{1, \ldots, N\}$, there exists a functional

$$\begin{array}{rcl} \Xi = (\Xi_1, \Xi_2) : \mathbb{R} \times \mathbb{R}^{2k} & \longrightarrow & \mathbb{R}^2 \\ (\theta, z_1, \ldots, z_{2k}) & \longmapsto & \Xi(\theta, z_1, \ldots, z_{2k}) \end{array}$$

such that

$$X^i_{\frac{kT}{N}} = \Xi_i\left(\theta, Z^1, \ldots, Z^{2k}\right), \quad i = 1, 2$$

(still with $\rho = \cos\theta$). This introduces in general an over-parametrization (inside $[0, 2\pi]$) since θ and $2\pi - \theta$ yield the same solution owing to the fact that $\left(Z^{2k+1}, Z^{2k+2}\right) \overset{(d)}{=} \left(Z^{2k+1}, -Z^{2k+2}\right)$, but this is not at all a significant problem for practical implementation (a more careful examination would show that in fact one equilibrium is repulsive for our procedure and one is attractive). From now on, for convenience, we will just mention the dependence of the premium function in the variable θ, namely

$$\theta \longmapsto \bar{P}(\theta) := e^{-rT}\mathbb{E}\left[\left(\max\left(\bar{X}_T^1, \bar{X}_T^2\right) - K\right)_+\right]$$

when $\bar{X}_T^2$ is given by (6.6). The function P is a 2π-periodic continuous function. Extracting the implicit correlation from the market amounts to solving

$$\bar{P}(\theta) = P_0^{market}.$$

We need the following additional assumption

$$P_0^{market} \in (\min_\theta \bar{P}, \max_\theta \bar{P})$$

i.e. that P_0^{market} is not only consistent but is also not an extremal value of $\bar{P}$. So we are looking for a zero of the function h defined on $\mathbb{R}$ by

$$h(\theta) = \bar{P}(\theta) - P_0^{market}.$$

This function admits a representation as an expectation given by

$$h(\theta) = \mathbb{E}H(\theta, Z)$$

where $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is defined for every $\theta \in \mathbb{R}$ and every $z = (z^1, \ldots, z^{2N}) \in \mathbb{R}^{2N}$

$$H(\theta, z) = e^{-rT}\left(\max\left(\Xi_1(\theta, z), \Xi_2(\theta, z)\right) - K\right)_+ - P_0^{market}.$$

As a consequence, one derives the stochastic zero search recursive procedure by

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Z_{n+1}), \quad n \geq 0, \quad \text{with} \quad (Z_n)_{n\geq 1} \text{ i.i.d., } Z^1 \overset{(d)}{=} \mathcal{N}(0, I_{2N}), \quad \theta_0 \in \mathbb{R} \tag{6.7}$$

Proposition 6.1. *Assume that the gain parameter sequence $(\gamma_n)_{n\geq 1}$ satisfies*

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

As the quoted value P_0^{market} is not an extremum of the function $\bar{P}$, then the recursive procedure defined by (6.7) a.s. converges to θ^* solution of the impliciation problem $\bar{P}(\theta) = P_0^{market}$

Proof. For every $z \in \mathbb{R}^{2N}$, $\theta \longmapsto H(\theta, z)$ is continuous, 2π-periodic and dominated by an integrable random variable (obtained by replacing $Z^{2k+1}\cos\theta + Z^{2k+2}\sin\theta$ by $|Z^{2k+1}| + |Z^{2k+2}|$ in the above formula for H). One derives that the mean function h and $\theta \longmapsto \mathbb{E}\left[H^2(\theta, Z)\right]$ are both continuous and 2π-periodic as well (hence bounded).

The main difficulty to apply the Robbins-Siegmund Lemma (see Theorem 1.4in Chapter 1) is to find out an appropriate Lyapunov function.

The quoted value P_0^{market} is not an extremum of the function $\bar{P}$, hence $\int_0^{2\pi} h^{\pm}(\theta)d\theta > 0$ where $h^{\pm} := \max(\pm h, 0)$. Let θ_0^* be any (fixed) solution to the equation $h(\theta) = 0$ and two real numbers $\beta_{\pm}$ such that

$$0 < \beta_+ < \frac{\int_0^{2\pi} h_+(\theta)d\theta}{\int_0^{2\pi} h_-(\theta)d\theta} < \beta_-$$

and we set

$$g(\theta) := \left\{ \begin{array}{lll} \mathbb{1}_{\{h>0\}}(\theta) + \beta_+ \mathbb{1}_{\{h<0\}}(\theta) & if & \theta \geq \theta_0^* \\ \mathbb{1}_{\{h>0\}}(\theta) + \beta_- \mathbb{1}_{\{h<0\}}(\theta) & if & \theta < \theta_0^*. \end{array} \right.$$

The function

$$\theta \longmapsto g(\theta)h(\theta) = h_+ - \beta_{\pm} h_-$$

is continuous and 2π-periodic on (θ_0^*, ∞) and 2π-periodic (*sic*) on $(-\infty, \theta_0^*)$. Furthermore $gh(\theta) = 0$ iff $h(\theta) = 0$ so that $gh(\theta_0^*) = gh(\theta_0^*-) = 0$ which ensures on the way the continuity of gh on the whole real line. Furthermore

$$\int_0^{2\pi} gh(\theta)d\theta > 0 \quad \text{and} \quad \int_{-2\pi}^{0} gh(\theta)d\theta < 0$$

so that, on the one hand,

$$\lim_{\theta \to \pm\infty} \int_0^{\theta} gh(u)du = +\infty$$

and, on the other hand, there exists a real constant $C > 0$ such that the function

$$L(\theta) = \int_0^{\theta} gh(u)du + C$$

is nonnegative. Its derivative is given by $L' = gh$ so that $L'h = gh^2 \geq 0$ and $\{L'h = 0\} = \{h = 0\}$. It remains to prove that L' is Lipschitz continuous. One checks by applying the usual differentiation theorem for functions defined by an integral that,

- if $\sigma_1 \neq \sigma_2$ or $x_1 \neq x_2$, then $\bar{P}$ is differentiable on the whole real line,
- otherwise it is differentiable only on $\mathbb{R} \setminus 2\pi\mathbb{Z}$,

and in both cases

$$\bar{P}'(\theta) = \mathbb{E}\left[\bar{Y}_T \mathbb{1}_{\{\bar{X}_T^2 > \max(\bar{X}_T^1, K)\}}\right]$$

where $\bar{Y}_t := \frac{\partial \bar{X}_t^2}{\partial \theta}$, $t \in [0, T]$, is the tangent process satisfying

$$\begin{aligned} \bar{Y}_{\frac{(k+1)T}{N}} &= \bar{Y}_{\frac{kT}{N}} + r\bar{Y}_{\frac{kT}{N}}\frac{T}{N} + \sigma_2\left(\bar{X}_{\frac{kT}{N}}^2\right)\bar{X}_{\frac{kT}{N}}^2\sqrt{\frac{T}{N}}\left(\cos\theta Z^{2k+2} - \sin\theta Z^{2k+1}\right) \\ &\quad + \left(\sigma_2'(\bar{X}_{\frac{kT}{N}}^2)\bar{X}_{\frac{kT}{N}}^2 + \sigma_2\left(\bar{X}_{\frac{kT}{N}}^2\right)\right)\bar{Y}_{\frac{kT}{N}}\sqrt{\frac{T}{N}}\left(\cos\theta Z^{2k+1} + \sin\theta Z^{2k+2}\right). \end{aligned}$$

The differentiability of $\bar{P}$ is illustrated in Figures 6.7 and 6.8. Furthermore, with obvious notations, as soon as $\bar{P}'(\theta)$ exists,

$$\left|\bar{P}'(\theta)\right| \leq \mathbb{E}\left|\bar{Y}_T(\theta)\right|.$$

The right handside of the inequality defined a 2π-periodic continuous function, hence bounded on the real line. Consequently $\left|\bar{P}'(\theta)\right|$ is bounded either on $\mathbb{R}\setminus 2\pi\mathbb{Z}$ or on the whole real line. It follows that the 2π-periodic functions h and $h_\pm$ are Lipschitz continuous which implies in turn that $L' = gh$ is Lipschitz as well.

Moreover, one can show that the equation $\bar{P}(\theta) = P_0^{market}$ market has finitely many solutions on every interval of length 2π. One may apply Corollary 1.1(b) to derive that θ_n will converge toward a solution θ^* of the equation $\bar{P}(\theta) = P_0^{market}$. □

6.2.2.1 Application to a best-of call in a Black-Scholes model

Consider a 2-dimensional Black-Scholes model *i.e.* $X_0^t = e^{rt}$ (riskless asset) and

$$X_t^i = x_0^i e^{(r-\frac{\sigma_i^2}{2})t+\sigma_i W_t^i}, \quad x_0^i > 0, \quad i = 1, 2,$$

for the two risky assets where $\left\langle W^1, W^2\right\rangle_t = \rho t$, $\rho \in [-1, 1]$. In this 2-dimensional B-S setting there is a closed formula for the premium involving the bi-variate standard normal distribution (see [67]) and the true process can be simulated at every time $t \in \mathbb{R}_+$.

This is a toy example since in this very setting, some more efficient (and deterministic) procedures could be called upon, based on the closed form for the option.

So we are looking for a zero of the function h defined on $\mathbb{R}$ by

$$h(\theta) = P(\theta) - P_0^{market}.$$

This function admits a representation as an expectation given by

$$h(\theta) = \mathbb{E}H(\theta, Z)$$

where $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is defined for every $\theta \in \mathbb{R}$ and every $z = (z^1, z^2) \in \mathbb{R}^2$

$$H(\theta, z) = e^{-rT}\left(\max\left(x_0^1 e^{\mu_1 T+\sigma_1\sqrt{T}z^1}, x_0^2 e^{\mu_2 T+\sigma_2\sqrt{T}(z^1\cos\theta+z^2\sin\theta)}\right) - K\right)_+ - P_0^{market}$$

and $Z = (Z^1, Z^2) \overset{(d)}{=} \mathcal{N}(0, I_2)$. As a consequence, one derives the stochastic zero search recursive procedure by

$$\theta_{n+1} = \theta_n - \gamma_{n+1}H(\theta_n, Z_{n+1}), \quad n \geq 0, \quad \text{with} \quad (Z_n)_{n\geq 1} \text{ i.i.d. } \mathcal{N}(0, I_2).$$

Numerical experiment. We set the model parameters to the following values

$$x_0^1 = x_0^2 = 100,\ r = 0.10,\ \sigma_1 = \sigma_2 = 0.30,\ \rho = -0.50$$

and the payoff parameters

$$T = 1,\ K = 100.$$

The reference Black-Scholes price 30.74 is used as a market price so that the target of the stochastic algorithm is $\theta^* \in \arccos(-0.5)$.

Figure 6.7 shows the effect of the trigonometric parametrization on the mean function of the stochastic procedure.

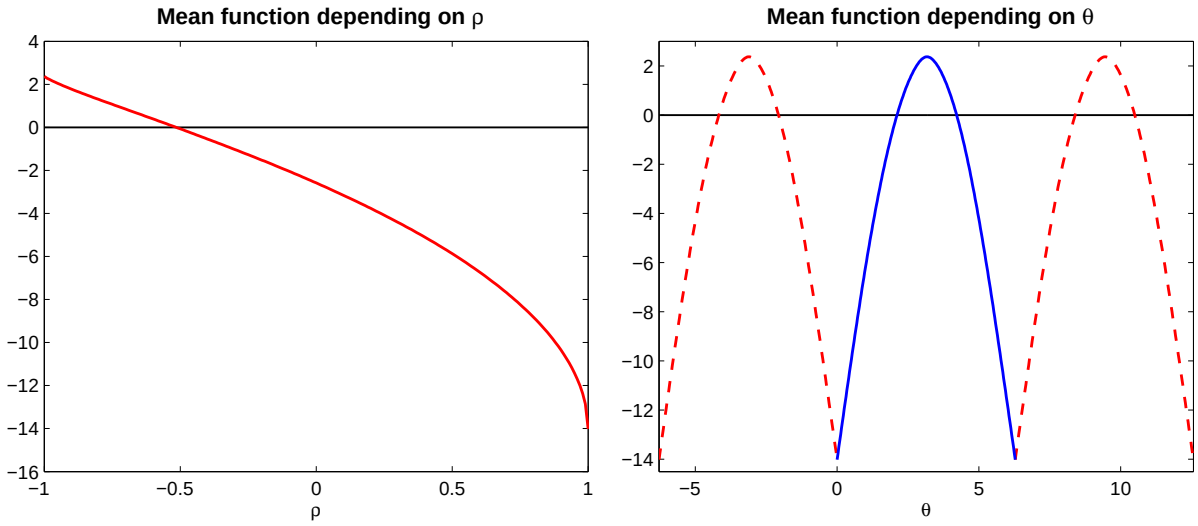

Figure 6.7: Mean function. *Left*: mean function depending on ρ. *Right*: mean function depending on θ.

If $\sigma_1 \neq \sigma 2$, then the mean function (with trigonometric parametrization) is continuous.

$$x_0^1 = 100,\ x_0^2 = 90,\ r = 0.10,\ \sigma_1 = 0.30,\ \sigma_2 = 0.25,\ \rho = -0.50$$

and the payoff parameters

$$T = 1,\ K = 100.$$

The reference Black-Scholes price is 30.77. Figure 6.8 shows that the mean function h is differentiable when $\sigma_1 \neq \sigma_2$.

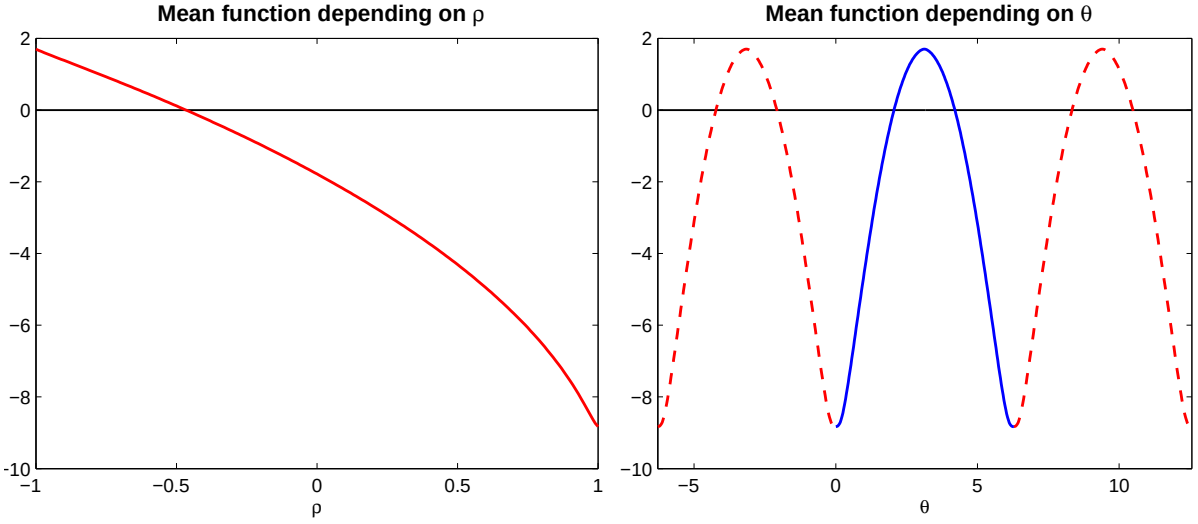

Figure 6.8: Mean function. *Left*: mean function depending on ρ. *Right*: mean function depending on θ.

The stochastic approximation procedure parameters are

$$\theta_0 \overset{(d)}{=} \mathcal{U}_{[0,2\pi]} (\text{which means that the initial value is random}),\ n = 10^5.$$

The choice of θ_0 is blind on purpose. Finally we set $\gamma_n = \frac{0.5}{n}$. The implicit correlation search recursive procedure is implemented with two sequences: one of pseudo-random normal numbers and one of some quasi-random normal numbers, namely

$$(\zeta_n^1, \zeta_n^2) = \left(\sqrt{-2\log(\xi_n^1)}\sin\left(2\pi\xi_n^2\right), \sqrt{-2\log(\xi_n^1)}\cos\left(2\pi\xi_n^2\right)\right),$$

where $\xi_n = (\xi_n^1, \xi_n^2),\ n \geq 1$, is simply a regular 2-dimensional Halton sequence.

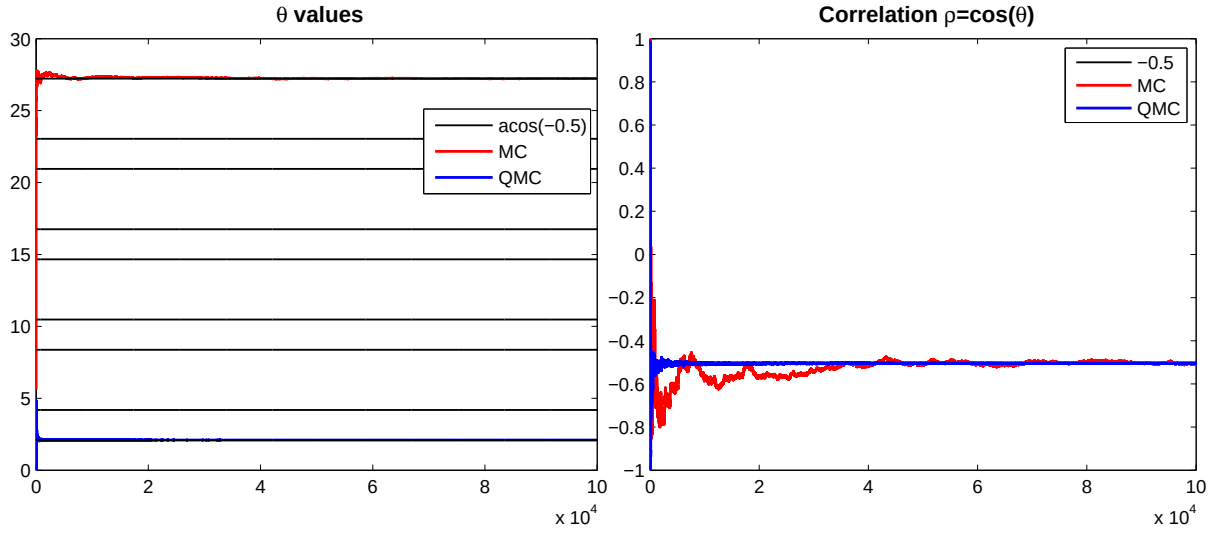

Figure 6.9: B-S Best-of-Call option. $T = 1$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.30$, $x_0^1 = x_0^2 = 100$, $K = 100$. *Left*: convergence of θ_n toward a θ^*. (up to $n = 10000$). *Right*: convergence of $\rho_n := \cos(\theta_n)$ toward -0.5.

n	θ_n^{MC}	$\rho_n^{MC} = \cos\theta_n^{MC}$	θ_n^{QMC}	$\rho_n^{QMC} = \cos\theta_n^{QMC}$
100	27.2301	-0.5025	2.7353	-0.9186
1000	27.4174	-0.6548	2.0971	-0.5024
10 000	27.3168	-0.5756	2.0998	-0.5047
100 000	27.2382	-0.5095	2.1001	-0.5050

Table 6.4: Convergence of ρ_n^{MC} and ρ_n^{QMC} toward $\rho^* = -0.5$.

6.2.2.2 Application to a best-of call in a pseudo-CEV model

We consider that the risky asset dynamics are governed by the following SDEs

$$dX_t^i = rX_t^i dt + \vartheta_i \frac{(X_t^i)^{1+\beta_i}}{\sqrt{(X_t^i)^2 + \epsilon^2}} dW_t^i, \quad x_0^i > 0, \quad i = 1, 2, \tag{6.8}$$

where $r > 0$, $\vartheta_i > 0$, $0 < \beta_i < 1$, $i = 1, 2$, $0 < \epsilon < 1$ and $\langle W^1, W^2 \rangle_t = \rho t$, $\rho \in [-1, 1]$. This model is also called "pseudo-CEV" which takes the advantage that the discounted asset is a local martingale (to the contrary of the real CEV model).

We consider the Euler scheme of the above diffusion $(\bar{X}^1_{\frac{kT}{N}}, \bar{X}^2_{\frac{kT}{N}})_{0 \leq k \leq N}$. One derives the stochastic zero search recursive procedure by

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Z_{n+1}), \quad n \geq 0, \quad \text{with} \quad (Z_n)_{n \geq 1} \ \text{i.i.d.} \ \mathcal{N}(0, I_{2N}),$$

where

$$H(\theta, z) = e^{-rT}\left(\max\left(\bar{X}_T^1, \bar{X}_T^2\right) - K\right)_+ - P_0^{market}.$$

NUMERICAL EXPERIMENT. We set the model parameters to the following values

$$x_0^1 = x_0^2 = 100,\ r = 0.10,\ \vartheta_1 = \vartheta_2 = 0.30,\ \beta_1 = \beta_2 = 0.80,\ \rho = -0.50$$

and the payoff parameters

$$T = 1,\ K = 100.$$

The reference price 17.78 is used as a market price so that the target of the stochastic algorithm is $\theta^* \in \arccos(-0.5)$. The stochastic approximation procedure parameters are

$$\theta_0 \overset{(d)}{=} \mathcal{U}_{[0,2\pi]},\ n = 10^5.$$

Figure 6.10 shows the effect of the trigonometric parametrization on the mean function of the stochastic procedure.

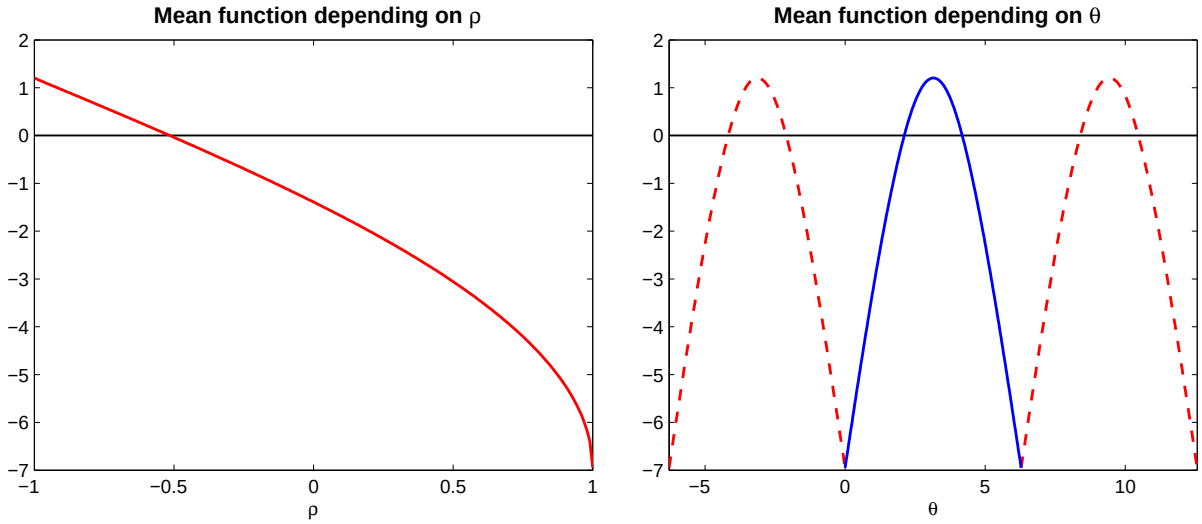

Figure 6.10: Mean function. *Left*: mean function depending on ρ. *Right*: mean function depending on θ.

The choice of θ_0 is blind on purpose. Finally we set $\gamma_n = \frac{0.3}{n}$.

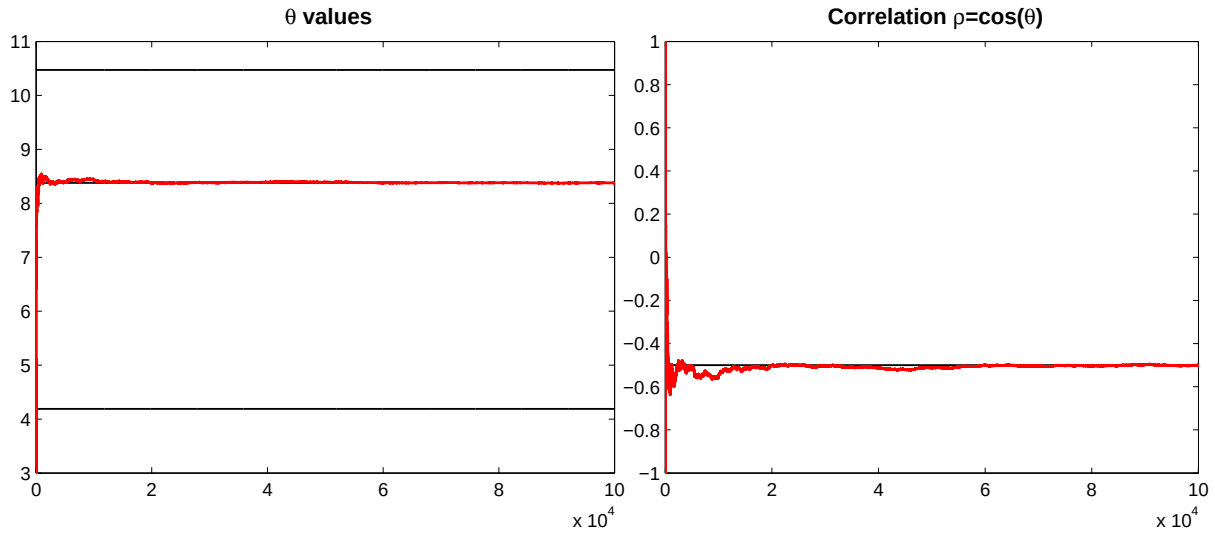

Figure 6.11: B-S Best-of-Call option. $T = 1$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.30$, $x_0^1 = x_0^2 = 100$, $K = 100$. *Left*: convergence of θ_n toward a θ^*. (up to $n = 10000$). *Right*: convergence of $\rho_n := \cos(\theta_n)$ toward -0.5.

n	θ_n	$\rho_n = \cos\theta_n$
100	8.0240	-0.1692
1000	8.4990	-0.6012
10 000	8.4396	-0.5527
100 000	8.3765	-0.4991

Table 6.5: Convergence of ρ_n toward $\rho^* = -0.5$.

6.2.3 Application to power extraction in the pseudo-CEV model

We can also use such a recursive procedure to compute the value of the power β in the pseudo-CEV model. As $0 < \beta < 1$, we use again (as in the case of correlation) a trigonometric parametrization of the power to prevent the edge effects by setting $\beta = \frac{1+\cos\theta}{2}$, $\theta \in \mathbb{R}$.

We obtain the following recursive procedure

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Z_{n+1}), \quad n \geq 0, \quad \theta_0 \in \mathbb{R},$$

where $H : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ defined by $H(\theta, Z) = \left(\bar{X}_T(\theta) - e^{-rT}K\right)_+ - P_0^{\text{market}}$ with $\bar{X}_T(\theta)$ the end value of the Euler scheme $(\bar{X}_{\frac{kT}{N}}(\theta))_{0\leq k\leq N}$ of the diffusion (6.1) with $\beta = \frac{1+\cos\theta}{2}$.

Numerical experiment. We set the model parameters to the following values

$$x_0 = 100, \ r = 0.10, \ \vartheta = 0.30, \ \beta = 0.80$$

and the payoff parameters

$$T = 1, \ K = 100.$$

The reference price 10.86 is used as a market price so that the target of the stochastic algorithm is $\theta^* \in \arccos(2\beta - 1) = \arccos(0.6)$.. The stochastic approximation procedure parameters are

$$\theta_0 = \frac{\pi}{2}, \ n = 10^5.$$

Figure 6.12 shows the effect of the trigonometric parametrization on the mean function of the stochastic procedure.

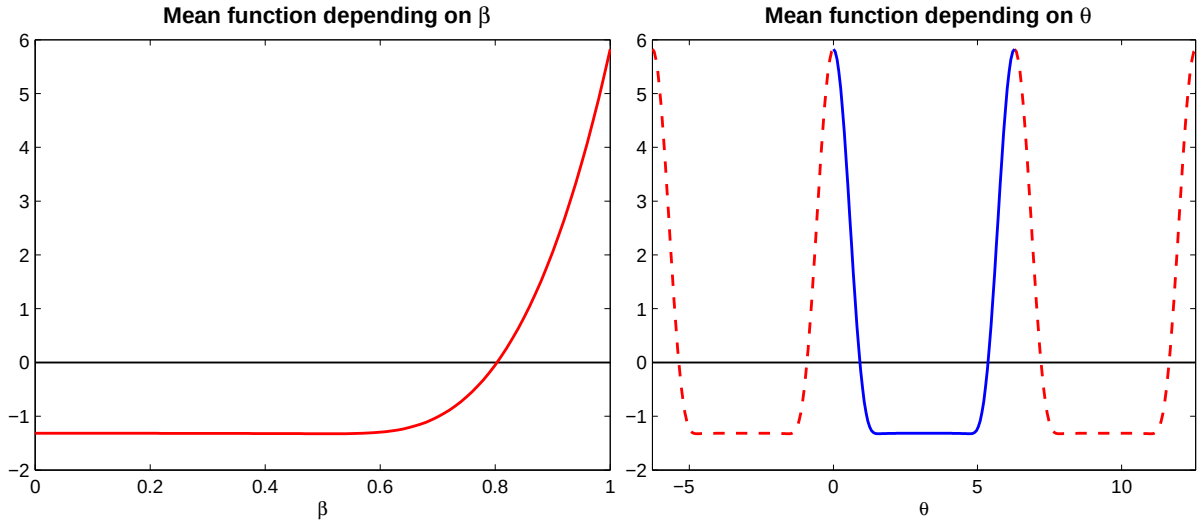

Figure 6.12: Mean function. *Left*: mean function depending on β. *Right*: mean function depending on θ.

The choice of θ_0 is blind on purpose. Finally we set $\gamma_n = \frac{1}{2n}$.

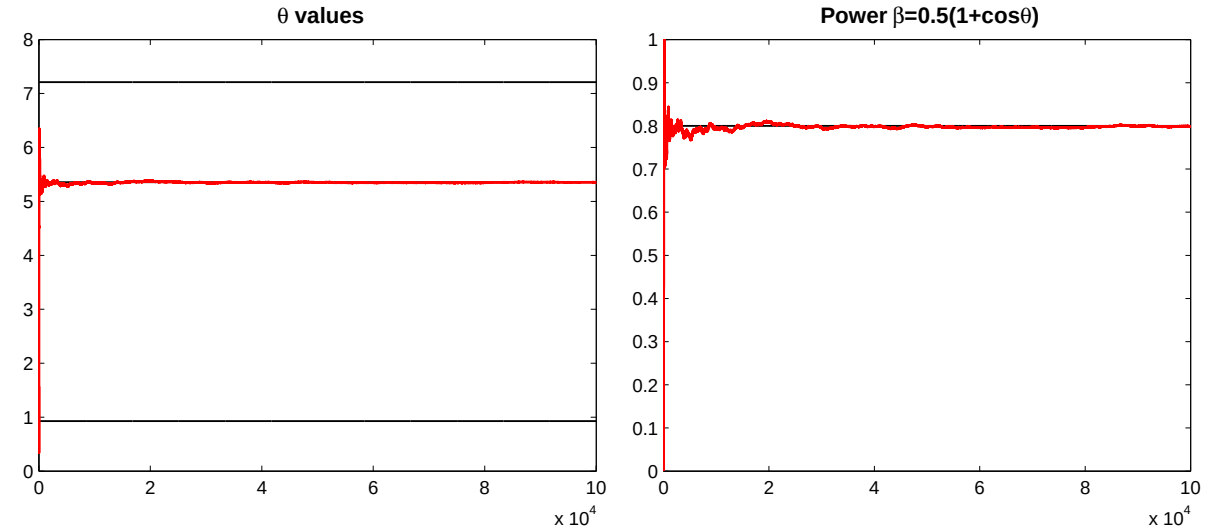

Figure 6.13: Pseudo-CEV Call option. $T = 1$, $K = 100$, $r = 0.10$, $x_0 = 100$, $\vartheta = 0.30$.Left: convergence of θ_n toward a θ^*. Right: convergence of $\beta_n := \frac{1+\cos(\theta_n)}{2}$ toward 0.80.

n	θ_n	$\beta_n = 0.5(1 + \cos\theta_n)$
100	5.8318	0.9499
1000	5.3301	0.7896
10 000	5.3445	0.7954
100 000	5.3524	0.7986

Table 6.6: Convergence of β_n toward $\beta^* = 0.8$.

6.2.4 Application to simultaneous implicitation of both power and volatility in a pseudo-CEV model

To compute the values of the implicit volatility ϑ and the power β in the pseudo-CEV model, we can also construct a bi-dimensional recursive procedure as follows

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Z_{n+1}), \quad n \geq 0, \quad \theta_0 \in \mathbb{R}_+ \times [0,1],$$

where the first component θ^1 is for the implicit volatility and the second one θ^2 denotes the trigonometric parametrization for the power β, *i.e.* $\beta = \frac{1+\cos\theta^2}{2}$, $\theta^2 \in \mathbb{R}$, and $H : \mathbb{R}_+^2 \times \mathbb{R}^N \to \mathbb{R}^2$ defined by $H(\theta, Z) = \left(\left(\bar{X}_{T_i}(\theta) - e^{-rT_i} K_i \right)_+ - P_{0,i}^{\text{market}} \right)_{i=1,2}$ with $\bar{X}_{T_i}(\theta)$ the end value of the Euler scheme $(\bar{X}_{\frac{kT_i}{N}}(\theta))_{0 \leq k \leq N}$ of the diffusion (6.1) for $i = 1, 2$.

NUMERICAL EXPERIMENT. The difficulty of this procedure is to find appropriate vanilla option to implicit the parameters. Indeed we need to make sure that there is a unique zero θ^* for the above problem. This leads to consider call options with very different maturities (typically a month and a year or a few years).

Figure 6.14 illustrates the behaviour of the call price depending on ϑ and β.

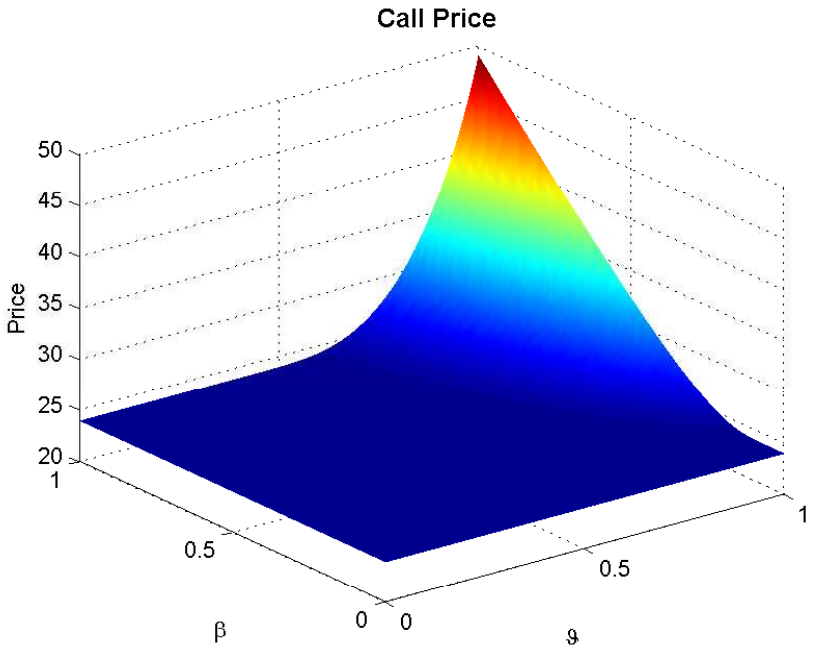

Figure 6.14: Call Price for $T = 1$ and $K = 80$ depending on ϑ and β.

Figure 6.15 illustrates this problem: for each couple of maturity and strike, we have computed the call option prices for $(\vartheta, \beta) \in [0,1]^2$ and we plot the solution couples of the inverse problem in (ϑ, β) for a given price (chosen for $\vartheta = 0.8$ and $\beta = 0.3$).

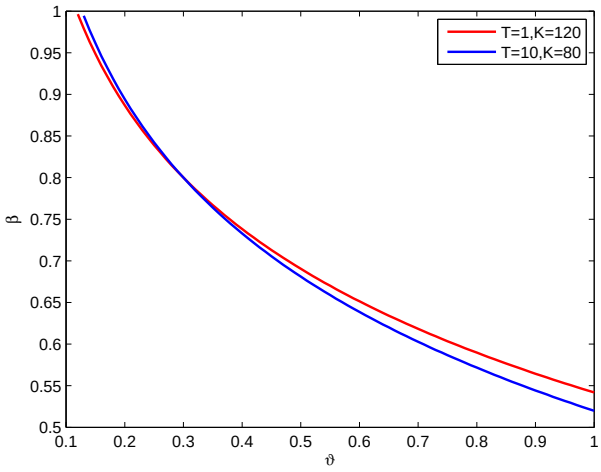

Figure 6.15: Solution couples for the problem $\bar{P}(\vartheta, \beta) = \bar{P}(0.8, 0.3)$ for different maturities and strikes.

This figure shows why the above recursive procedure does not work properly, because the curves are too close and the algorithm does not succeed to converge to the right solution.

6.2.5 Implicitation of the two other parameters of a Merton model

We have already devise a stochastic algorithm to find the implied volatility in a Merton model. Here we present two stochastic procedures: the first one for the loss parameter a and the second one for the intensity parameter λ.

6.2.5.1 Loss parameter implicitation

We devise the following recursive procedure

$$a_{n+1} = a_n - \gamma_{n+1} H(a_n, Z_{n+1}), \quad n \geq 0, \quad a_0 \in (0,1),$$

where $H(a, Z) = \left(x e^{(-\frac{\sigma^2}{2} + \lambda a)T + \sigma\sqrt{T} Z^1} (1-a)^{Z^2} - K e^{-rT} \right)_+ - P_0^{\text{market}}$ with $Z^1 \overset{\mathcal{L}}{\sim} \mathcal{N}(0,1)$ and $Z^2 \overset{\mathcal{L}}{\sim} \mathcal{P}(\lambda T)$.

Remark. By using (6.3), we should normalize H to satisfy the sub-linear quadratic growth assumption by replacing it by

$$\widetilde{H}(\sigma_n, Z) = e^{-\frac{\lambda a^2}{2} T} H(\sigma_n, Z). \tag{6.9}$$

This is not necessary in the below simulations because of the values of the parameters to find.

Numerical experiment. We set the model parameters to the following values

$$x_0 = 100, \ r = 0.10, \ \sigma = 0.30, \ a = 0.05, \ \lambda = 3$$

and the payoff parameters

$$T = 1, \ K = 100.$$

The reference price 17.13 is used as a market price. The mean function of the algorithm is given by Figure 6.16.

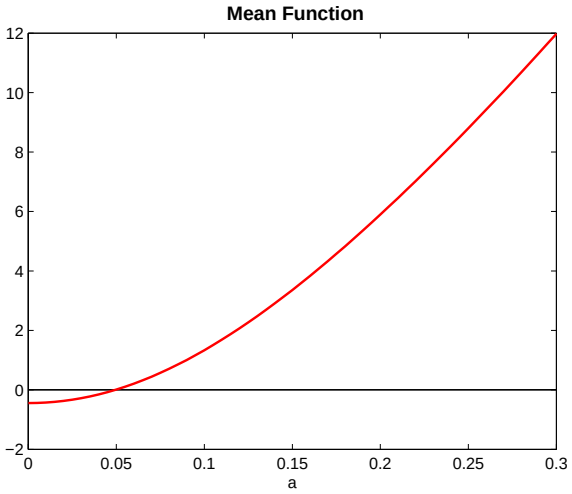

Figure 6.16: Mean function depending on a.

The stochastic approximation procedure parameters are

$$a_0 = 0,\ n = 10^5.$$

The choice of a_0 is blind on purpose. Finally we set $\gamma_n = \frac{1}{10n}$.

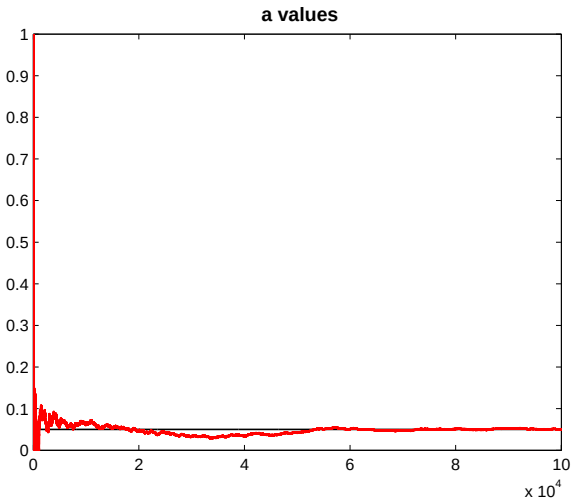

Figure 6.17: Implicitation of a by Call option.

n	a_n
100	0.1032
1000	0.0162
10 000	0.0641
100 000	0.0492

Table 6.7: Convergence of a_n toward $a^* = 0.05$.

6.2.5.2 Jump intensity implicitation

We devise the following recursive procedure

$$\lambda_{n+1} = \lambda_n - \gamma_{n+1} H(\lambda_n, Z_{n+1}), \quad n \geq 0, \quad \lambda_0 > 0,$$

where $H(\lambda, Z) = \left(x e^{(-\frac{\sigma^2}{2} + \lambda a)T + \sigma\sqrt{T}Z^1} (1-a)^{Z^2} - K e^{-rT} \right)_+ - P_0^{\text{market}}$ with $Z^1 \overset{\mathcal{L}}{\sim} \mathcal{N}(0,1)$ and $Z^2 \overset{\mathcal{L}}{\sim} \mathcal{P}(\lambda T)$ (the same remark for the normalization of H holds (see (6.9))).

Numerical experiment. We set the model parameters to the following values

$$x_0 = 100, \ r = 0.10, \ \sigma = 0.30, \ a = 0.05, \ \lambda = 3$$

and the payoff parameters

$$T = 1, \ K = 100.$$

The reference price 17.13 is used as a market price.

The mean function of the algorithm is given by Figure 6.18.

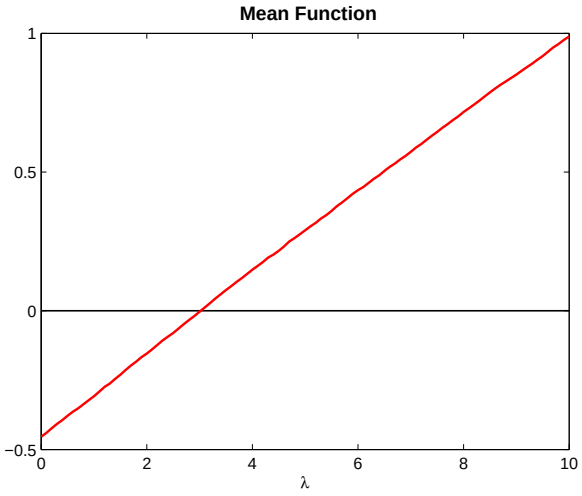

Figure 6.18: Mean function depending on λ.

The stochastic approximation procedure parameters are

$$\lambda_0 = 0, \ n = 10^5.$$

The choice of λ_0 is blind on purpose. Finally we set $\gamma_n = \frac{1}{0.8n}$.

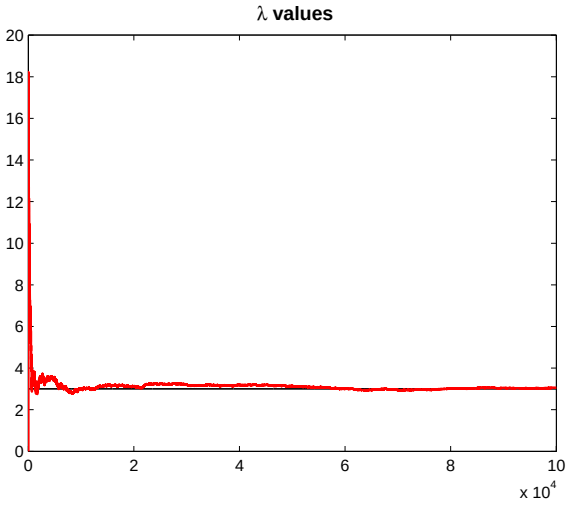

Figure 6.19: Implicitation of λ by Call option.

n	λ_n
100	12.0353
1000	3.4586
10 000	2.9798
100 000	3.0520

Table 6.8: Convergence of λ_n toward $\lambda^* = 3$.

6.3 The paradigm of calibration by simulation

Let $\Theta \subset \mathbb{R}^d$ be a convex open set of $\mathbb{R}^d$. Let

$$\begin{array}{rcl} Y : (\Theta \times \Omega, \mathcal{B}or(\Theta) \otimes \mathcal{A}) & \longrightarrow & (\mathbb{R}^p, \mathcal{B}or(\mathbb{R}^p)) \\ (\theta, \omega) & \longmapsto & Y_\theta(\omega) = \left(Y_\theta^1(\omega), \ldots, Y_\theta^p(\omega)\right) \end{array}$$

be a random vector representative of p payoffs, "re-centered" by their mark-to-market price (see examples below). In particular, for every $i \in \{1, \ldots, p\}$, $\mathbb{E}Y_\theta^i$ is representative of the error between the "theoretical" price obtained with parameter θ and the quoted price. To make the problem consistent we assume throughout this section that

$$\forall \theta \in \Theta, \quad Y_\theta \in L^1_{\mathbb{R}^p}(\Omega, \mathcal{A}, \mathbb{P}).$$

Let $S \in \mathcal{S}(p, \mathbb{R}) \cap GL(p, \mathbb{R})$ be a (positive definite) matrix. The resulting inner product is defined by

$$\forall u, v \in \mathbb{R}^p, \quad \langle u|v\rangle_S := u^t S v,$$

and the associated Euclidean norm $|\cdot|_S$ by $|u|_S := \sqrt{\langle u|u\rangle_S}$, where u^t is the transpose of the column vector $u \in \mathbb{R}^p$.

A natural choice of the matrix S can be a simple diagonal matrix $S = \mathrm{diag}(w_1, \ldots, w_p)$ with "weights" $w_i > 0$, $i = 1, \ldots, p$.

The *paradigm of model calibration* is to find the parameter θ^* that minimizes the "aggregated error" with respect to the $|\cdot|_S$-norm. This leads to the following minimization problem

$$(C) \equiv \text{argmin}_{\theta\in\Theta}|\mathbb{E}Y_\theta|_S = \text{argmin}_{\theta\in\Theta}\left[L(\theta) := \frac{1}{2}|\mathbb{E}Y_\theta|_S^2\right].$$

We will also have to make simulability assumptions on Y_θ and if necessary its derivatives with respect to θ (see below) otherwise our simulation based approach would be meaningless.

Remark. • Many extensions can be devised, in particular concerning the matrix S which can be assumed to depend on θ as well.

• We may add a penalization function to the function L, typically $\frac{1}{2}|\theta-\theta_{-1}|^2$, where θ_{-1} is a reference value like the optimal "calibration" of the previous day, leading to

$$(C') \equiv \text{argmin}_{\theta\in\Theta}L(\theta) + \frac{\kappa}{2}|\theta-\theta_{-1}|^2,$$

where $\kappa > 0$ is a penalization coefficient.

At this stage, basically two approaches can be implemented to solve this problem by simulation:
- A Robbins-Siegmund approach which needs to have access to a *representation of the gradient* - supposed to exist - of the function L.
- A more direct treatment based on the so-called *Kiefer-Wolfowitz* procedure which is a counterpart of the Robbins-Siegmund approach based on a finite difference method (with decreasing step) which does not require the existence of a representation of ∇L as an expectation.

6.3.1 The Robbins-Siegmund approach

We make the following assumption: for every $\theta_0 \in \Theta$,
(i) $\mathbb{P}(d\omega)$-$a.s.$, $\theta \mapsto Y_\theta(\omega)$ is differentiable at θ_0 with Jacobian $J_{Y(\omega)}(\theta_0) := \left[\frac{\partial Y_\theta^i(\omega)}{\partial\theta^j}\right]_{1\le i\le p, 1\le j\le d}$,
(ii) $\exists\mathcal{U}_{\theta_0}$, neighbourhood of θ_0 in Θ, such that $\left(\frac{|Y_\theta - Y_{\theta_0}|}{|\theta-\theta_0|}\right)_{\theta\in\mathcal{U}_{\theta_0}\setminus\{\theta_0\}}$ is uniformly integrable.

One checks that $\theta \mapsto \mathbb{E}Y_\theta$ is differentiable on Θ and that its Jacobian is given by

$$J_{\mathbb{E}Y.}(\theta) = \mathbb{E}\left[J_{Y(\omega)}(\theta)\right].$$

Then, the function L is differentiable everywhere on Θ and its gradient (with respect to the canonical Euclidean norm) is given by

$$\forall\theta\in\Theta, \quad \nabla L(\theta) = \mathbb{E}\left[J_Y(\theta)\right]^t S\,\mathbb{E}Y_\theta = \mathbb{E}\left[(J_Y(\theta))^t\right] S\,\mathbb{E}Y_\theta.$$

At this stage we need a representation of $\nabla L(\theta)$ as an expectation. To this end, we construct for every $\theta\in\Theta$, an independent copy $\widetilde{Y}_\theta$ of Y_θ defined as follows: we consider the product probability space $(\Omega^2, \mathcal{A}^{\otimes 2}, \mathbb{P}^{\otimes 2})$ and we set, for every $(\omega,\widetilde{\omega})\in\Omega^2$, $Y_\theta(\omega,\widetilde{\omega}) = Y_\theta(\omega)$ (extension of Y_θ on Ω^2 still denoted Y_θ) and $\widetilde{Y}_\theta(\omega,\widetilde{\omega}) = Y_\theta(\widetilde{\omega})$. It is straightforward by the product measure Theorem that the two families $(Y_\theta)_{\theta\in\Theta}$ and $\left(\widetilde{Y}_\theta\right)_{\theta\in\Theta}$ are independent with the same distribution. From now on we will make the usual abuse of notation consisting in assuming that these two independent copies live on the probability space $(\Omega,\mathcal{A},\mathbb{P})$.

Now, one can write

$$\begin{aligned}\forall\theta\in\Theta, \quad \nabla L(\theta) &= \mathbb{E}\left[(J_Y(\theta))^t\right] S\,\mathbb{E}\widetilde{Y}_\theta \\ &= \mathbb{E}\left[(J_Y(\theta))^t\, S\,\widetilde{Y}_\theta\right].\end{aligned}$$

The standard situation, as announced above, is that Y_θ is a vector of payoffs written on q traded risky assets, recentered by their respective quoted prices. The model dynamics of the q risky assets depends on the parameter $\theta \in \Theta$, say

$$Y_\theta = (F_i(\theta, X_{T_i}(\theta)))_{i=1,\dots,p} \tag{6.10}$$

where the price dynamics $(X_t(\theta))_{t\geq 0}$ of the q traded assets reads

$$dX_t(\theta) = b(\theta, t, X_t(\theta))dt + \sigma(\theta, t, X_t(\theta)).dW_t, \quad X_0(\theta) = x_0(\theta) \in \mathbb{R}^q, \tag{6.11}$$

with W an $\mathbb{R}^\ell$-valued standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, b an $\mathbb{R}^q$-valued vector field defined on $\Theta \times [0,T] \times \mathbb{R}^q$ and σ an $\mathcal{M}_{q,\ell}(\mathbb{R})$-valued field defined on the same product space, both satisfying appropriate regularity assumptions.

The *pathwise* differentiability of Y_θ in θ needs that of $X_t(\theta)$ with respect to θ. This question is closely related to the θ-tangent process - to be precise the θ-Jacobian process - $\left(\frac{\partial X_t^i(\theta)}{\partial \theta^j}\right)_{t\geq 0}$, $1 \leq i \leq q$, $1 \leq j \leq d$, of $X(\theta)$. A precise statement is provided in the following subsection which ensures that if b and σ are smooth enough with respect to the variable θ, then such a θ-tangent process does exist and is solution to a linear SDE (involving $X(\theta)$ in its coefficients). Differentiability assumptions are also needed on the functions F_i. To be precise we need that, for every $\theta \in \Theta$, the set $ND_\theta^i = \{x \text{ s.t. } F_i \text{ is not differentiable at } (\theta, x)\}$ is $\mathbb{P}_{X_{T_i}(\theta)}$-negligible. So is the case if $X_{T_i}(\theta)$ has an absolutely continuous distribution and all the sets ND_θ^i are Lebesgue negligible. Note that of course, the function L can be differentiable, in fact $\theta \mapsto \mathbb{E}Y_\theta$, even if the path of the process $(X_t(\theta))_{t\in[0,T]}$ are not. So is the case when $(X_t(\theta))_{t\in[0,T]}$ is a jump process. This case will be investigated on an example further on (see Section 3.4).

6.3.1.1 The tangent process of a diffusion

In a general setting, an alternative to finite difference methods is to rely on the tangent processes. The main result on which we rely is due to Kunita (see [73], Theorem 3.1).

Theorem 6.1. *Let $b : \mathbb{R}^q \to \mathbb{R}^q$, $\sigma : \mathbb{R}^q \to \mathcal{M}_{q\times\ell}(\mathbb{R})$, with regularity $\mathcal{C}_b^1$ (continuously differentiable with bounded first partial derivatives) ([1]). Let $X^x = (X_t^x)_{t\geq 0}$ denotes the unique strong solution of the SDE*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^q,$$

where $W = (W^1, \dots, W^\ell)$ is a ℓ-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then at every $t \in \mathbb{R}_+$, the mapping $x \mapsto X_t^x$ is a.s. continuously differentiable and its Jacobian $J_{X_t}(x) := \left[\frac{\partial (X_t^x)^i}{\partial x^j}\right]_{1\leq i,j\leq q}$ satisfies the linear stochastic differential system

$$\frac{\partial (X_t^x)^i}{\partial x^j} = \delta_{ij} + \sum_{\ell=1}^d \int_0^t \frac{\partial b^i}{\partial y^\ell}(X_s^x)\frac{\partial (X_s^x)^\ell}{\partial x^j}ds + \sum_{\ell=1}^d\sum_{k=1}^q \int_0^t \frac{\partial \sigma_{ik}}{\partial y^\ell}(X_s^x)\frac{\partial (X_s^x)^\ell}{\partial x^j}dW_s^k, \quad 1 \leq i,j \leq q,$$

where δ_{ij} denote the Kronecker symbol.

Remarks. • This result includes the non-homogeneous case by considering $X_t' = (t, X_t)$ instead of X_t at least as long as one wishes to differentiate with respect to the space variable x.
• Higher order differentiability properties hold true if b and σ are smoother.

1. In fact, the exact assumption from [73] is that b and σ are $\mathcal{C}^1$ with bounded α-Hölder derivatives for an $\alpha > 0$.

▷ **Computation by simulation.** One uses these formulate to compute some sensibility by Monte Carlo simulations since sensitivity parameters are functions of the couple made by the diffusion process X^x solution to the SDE starting at x and its tangent process $\nabla_x X^x$ at x: it suffices to consider the Euler scheme of this couple $(X_t^x, \nabla_x X_t^x)$ over $[0,T]$ with step $\frac{T}{N}$.

Assume $q = 1$ for notational convenience:

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0^x = x \in \mathbb{R},$$

$$dY_t = Y_t(b'(X_t^x)dt + \sigma'(X_t^x)dW_t), \quad Y_0 = 1.$$

In fact, in 1-dimension, one can take advantage of the semi-closed formula

$$\frac{dX_t^x}{dx} = \exp\left(\int_0^t \left(b'(X_s^x) - \frac{\sigma'(X_s^x)^2}{2}\right)ds + \int_0^t \sigma'(X_s^x)dWs\right).$$

▷ **Extension to a parameter θ.** All the theoretical results obtained for the δ, *i.e.* for the differentiation of the flow of an SDE with respect to its initial value can be extended to any parameter provided no ellipticity is required. This follows from the remark that if the coefficient(s) b and/or σ of a diffusion depend(s) on a parameter θ then the couple (X_t, θ) is still diffusion process, namely

$$dX_t = b(\theta, X_t)dt + \sigma(\theta, X_t)dW_t, \quad X_0 = x,$$

$$d\theta_t = 0, \quad \theta_0 = \theta.$$

Set $x' := (x, \theta)$ and $X_t'^{x'} := (X_t^x, \theta_t)$. Thus, following Theorems 3.1 and 3.3 of Section 3 in [73], if $(\theta, x) \mapsto b(\theta, x)$ and $(\theta, x) \mapsto \sigma(\theta, x)$ are $\mathcal{C}_b^{k+\alpha}$ $(0 < \alpha < 1)$ with respect to x and θ ([2]) then the solution of the SDE at a given time t will be $\mathcal{C}^{k+\beta}$ $(0 < \beta < \alpha)$ as a function of (x and θ). A more specific approach would show that some regularity in the sole variable θ would be enough but then this result does not follow for free from the general theorem of the differentiability of the flows.

Assume $b = b(\theta, \cdot)$ and $\sigma = \sigma(\theta, \cdot)$ and the initial value $x = x(\theta)$, $\theta \in \Theta$, $\Theta \subset \mathbb{R}^d$ (open set). One can also differentiate a SDE with respect to this parameter θ. We can assume that $d = 1$ (by considering a partial derivative if necessary). Then namely

$$\begin{aligned}\frac{\partial X_t(\theta)}{\partial \theta} &= \frac{\partial x(\theta)}{\partial \theta} + \int_0^t \left(\frac{\partial b}{\partial \theta}(\theta, X_s(\theta)) + \frac{\partial b}{\partial x}(\theta, X_s(\theta))\frac{\partial X_s(\theta)}{\partial \theta}\right)ds \\ &\quad + \int_0^t \left(\frac{\partial \sigma}{\partial \theta}(\theta, X_s(\theta)) + \frac{\partial \sigma}{\partial x}(\theta, X_s(\theta))\frac{\partial X_s(\theta)}{\partial \theta}\right)dW_s.\end{aligned}$$

Once coupled with the original diffusion process X^x, this yields some expressions for the the sensitivity with respect to the parameter θ, possibly closed, but usually computable by a Monte Carlo simulation of the Euler scheme of the couple $\left(X_t(\theta), \frac{\partial X_t(\theta)}{\partial \theta}\right)$.

As a conclusion let us mention that this tangent process approach is close to the finite difference method applied to $F(x, \omega) = h(X_T^x(\omega))$: it appears as a limit case of the finite difference method.

2. A function, g has a $\mathcal{C}_b^{k+\alpha}$ regularity if g is $\mathcal{C}^k$ with k-th order partial derivatives globally α-Hölder and all partial derivatives up to k-th order bounded.

6.3.1.2 Back to Robbins-Siegmund approach for parameter calibration

Some differentiability properties are also required on the functions F_i in order to fulfill the above Assumption (i). As model calibrations on vanilla derivative products performed in Finance, F_i is never everywhere differentiable - typically $F_i(y) := e^{-rT_i}(y - K_i)_+ - P^{\text{market}}(T_i, K_i)$ - but, if $X_t(\theta)$ has an absolutely continuous distribution (*i.e.* a probability density) for every time $t > 0$ and every $\theta \in \Theta$, then F_i only needs to be differentiable outside a Lebesgue negligible subset of $\mathbb{R}_+$. Finally we can write formally

$$\left(J_{Y(\omega)}(\theta)\right)^t S\,\widetilde{Y}_\theta(\omega) = H(\theta, W(\omega))$$

where W stands for an abstract random innovation taking values in an appropriate space. We denote by the capital letter W the innovation because when the underlying dynamics is a Brownian diffusion or its Euler-Maruyama scheme, it refers to a finite-dimensional functional of the Brownian motion on an interval $[0, T]$: either (two independent copies of) $(W_{T_1}, \ldots, W_{T_p})$ or (two independent copies of) the sequence $(\Delta W_{\frac{kT}{N}})_{1\le k\le N}$ of Brownian increments with step T/N over the interval $[0, T]$. Thus, these increments naturally appear in the simulation of the Euler scheme $\left(\Delta \bar{X}^n_{\frac{kT}{N}}(\theta)\right)_{1\le k\le N}$ of the process $(X_t(\theta))_{t\in[0,T]}$ when the latter cannot be simulated directly. Of course other situations may occur especially when dealing with with jump diffusions where W usually becomes the increment process of the driving Lévy process.

Whatsoever, we make the following reasonable meta-assumption that

- W is simulable,
- the functional $H(\theta, w)$ can be easily computed for any input (θ, w).

Then, one may define recursively the following zero search algorithm for ∇L, by setting

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, W_{n+1}), \quad n \ge 0, \quad \text{with} \quad H(\theta, W) = (J_Y(\theta))^t\, S\,\widetilde{Y}_\theta, \tag{6.12}$$

with Y_θ defined by (6.10) and (6.11) and where $(W_n)_{n\ge1}$ is an i.i.d. sequence of copies of W and $(\gamma_n)_{n\ge1}$ is a sequence of steps satisfying the usual decreasing step assumptions

$$\sum_{n\ge1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\ge1} \gamma_n^2 < +\infty.$$

In such a general framework, of course, one cannot ensure that the function H will satisfy the basic assumptions needed to make stochastic gradient algorithms converge, typically

$$\nabla L \text{ is Lipschitz continuous and } \quad \forall \theta \in \Theta, \quad \|H(\theta, \cdot)\|_2 \le C(1 + \sqrt{L(\theta)}) \tag{6.13}$$

or one of their numerous variants (see *e.g.* [25] for a large overview of possible assumptions). However, in many situations, one can make the problem fit into a converging setting either by an appropriate change of variable on θ or by modifying the function L and introducing an appropriate explicit (strictly) positive "weight function" $\chi(\theta)$ that makes the product $\chi(\theta)H(\theta, W(\omega))$ fit with these requirements.

Even though, the topological structure of $\{\nabla L = 0\}$ can be nontrivial, in particular disconnected. Nonetheless, one can show (using the ODE method), under natural assumptions, that

$$\theta_n \text{ converges to a connected component of } \{\chi\,|\nabla L|^2 = 0\} = \{\nabla L = 0\}.$$

The next step is that, if ∇L has several zeros, they cannot all be local minima of L especially when there more than two of them (this is a consequence of the well-known Mountain-Pass Lemma, see [71]). Some are local maxima or saddle points of various kinds. These equilibrium points which are not local minima are called *traps*. An important fact is that, under some non-degeneracy

assumptions on H of the noise at such a parasitic equilibrium point θ_∞ (say $\mathbb{E}^t H(\theta_\infty, W)H(\theta_\infty, W)$ is positive definite at least in the direction of an instable manifold of h at θ_∞), the algorithm will *a.s.* never converge toward such trap. This question has been extensively investigated in the literature in various settings for many years (see [84, 34, 99, 23, 49]).

▷ **Constrained version.** A final problem may arise due to the incompatibility between the geometry of the parameter set Θ and the above recursive algorithm: to be really defined by the above recursion, we need Θ to be left stable by (almost) all the mappings $\theta \mapsto \theta - \gamma H(\theta, w)$ at least for γ small enough. If not the case, we need to introduce some constraints on the algorithm by projecting it on Θ whenever θ_n skips outside Θ. This question has been originally investigated in [36] when Θ is a convex set. This leads to the following algorithm

$$\theta_{n+1} = \Pi_\Theta\left(\theta_n - \gamma_{n+1} H(\theta_n, W_{n+1})\right), \quad n \geq 0, \quad \theta_0 \in \Theta, \tag{6.14}$$

which *a.s.* converges to the target θ^* under natural assumption (see Theorem 1.6 in Chapter 1).

Once all these technical questions have been circumvented, we may state the following metatheorem which says that θ_n *a.s.* converges toward a local minimum of L.

At this stage it is clear that calibration looks like quite a generic problem for stochastic approximation and that almost all difficulties arising in the field of Stochastic Approximation can be encountered when implementing such a (pseudo-)stochastic gradient to solve it.

6.3.1.3 A toy example: calibration of the implied volatility in a Black-Scholes model

Set, for any $x, \sigma > 0$, $r \in \mathbb{R}$,

$$X_t(\sigma) = x e^{(r - \frac{\sigma^2}{2})t + \sigma W t}, \quad t \geq 0,$$

where W is a standard Brownian motion. Then let $(K_i, T_i)_{i=1,\dots,p}$ be p couples "maturity-strike price". Set

$$\theta := \sigma, \quad \Theta := (0, +\infty)$$

and

$$Y_\sigma := \left(e^{-rT_i}\left(X_{T_i}(\sigma) - K_i\right)_+ - P_{\text{market}}(T_i, K_i)\right)_{i=1,\dots,p},$$

where $P_{\text{market}}(T_i, K_i)$ is the mark-to-market price of the option with maturity T_i and strike price K_i.

To devise the recursive procedure (6.12), we need the pathwise σ-tangent process which reads

$$\frac{\partial X_t(\sigma)}{\partial \sigma} = (W_t - \sigma t)\, X_t(\sigma).$$

Numerical experiment. We set the model parameters to the following values

$$x = 100, \ r = 0.10, \ \sigma = 0.30$$

and we consider 3 calls with strikes $K_1 = 80$, $K_2 = 80$, $K_3 = 80$ and with the same maturity $T = 1$. The Black-Scholes reference prices $P_{\text{market}}(T, K_1) = 29.43$, $P_{\text{market}}(T, K_2) = 16.74$ and $P_{\text{market}}(T, K_3) = 8.61$ are used as market prices.

Figure 6.20 shows the calibration function to be minimized and its derivative depending on the implied volatility σ.

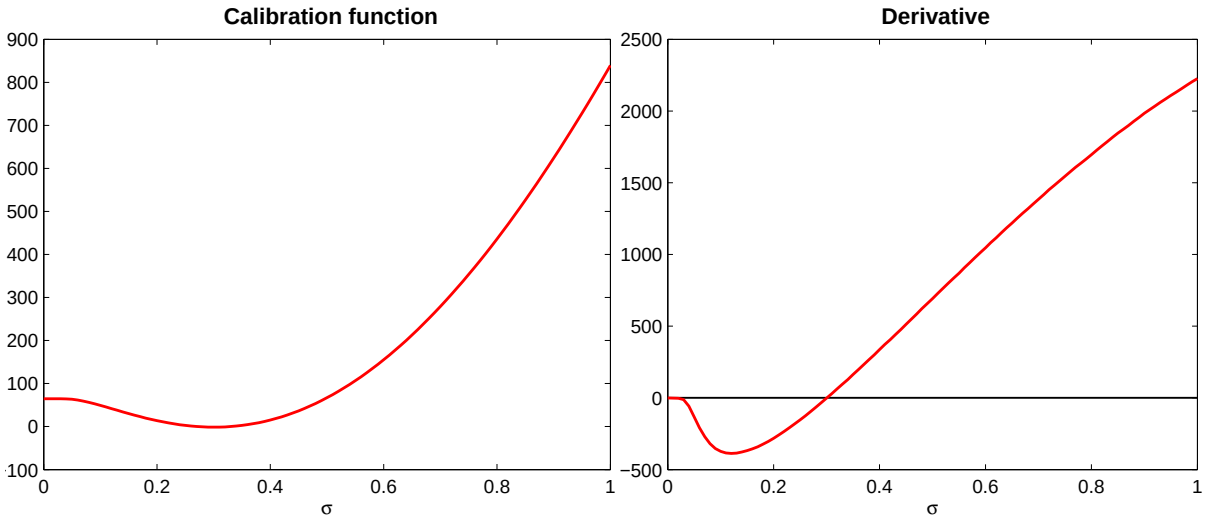

Figure 6.20: Calibration function and its derivative for the volatility.

The stochastic approximation procedure parameters are

$$\sigma_0 = 0,\ n = 10^5.$$

Finally we set $\gamma_n = \frac{1}{5000n}$. The recursive procedure for implied volatility calibration is implemented with two sequences: one of some pseudo-random normal numbers and one of some quasi-random normal numbers (created with an Halton sequence and the Box-Muller transform).

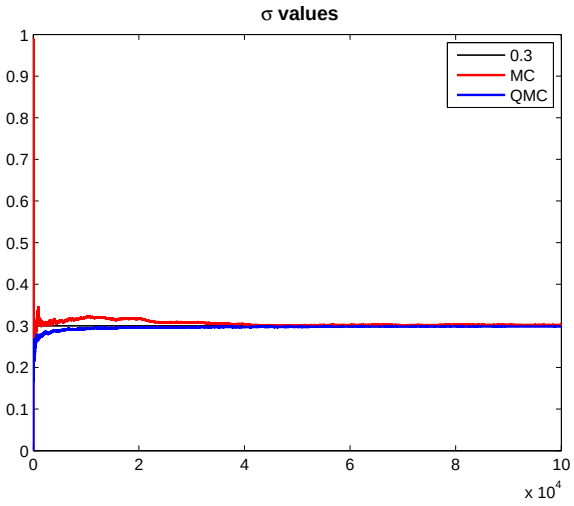

Figure 6.21: Calibration of Black-Scholes volatility by Call options. $T = 1$, $K = (80, 100, 120)^t$, $r = 0.10$, $x = 100$.

n	σ_n^{MC}	σ_n^{QMC}
100	0.2581	0.2183
1000	0.3194	0.2762
10 000	0.3201	0.2927
100 000	0.3025	0.2984

Table 6.9: Convergence of σ_n^{MC} and σ_n^{QMC} toward $\sigma^* = 0.3$.

6.3.1.4 Calibration of a pseudo-CEV model

Consider the pseudo-CEV model defined by (6.1).

▷ *Implied volatility calibration.* Set $\theta := \sigma$, $\Theta := (0, +\infty)$ and

$$Y_\sigma := \left(e^{-rT_i}\left(X_{T_i}(\sigma) - K_i\right)_+ - P_{\text{market}}(T_i, K_i)\right)_{i=1,\dots,p}.$$

To devise the recursive procedure (6.12), we need the σ-tangent process $S_t := \frac{\partial X_t(\sigma)}{\partial \sigma}$ which satisfies the following SDE

$$dS_t = rS_t dt + \frac{X_t^\beta(\sigma)}{\sqrt{\epsilon^2 + X_t^2(\sigma)}}\left(X_t(\sigma) + \vartheta\left(1 + \beta - \frac{X_t^2(\sigma)}{\epsilon^2 + X_t^2(\sigma)}\right)S_t\right)dW_t, \quad S_0 = 0.$$

We devise two stepwise constant Euler scheme to approximate the terminal values X_T and S_T with step $\frac{T}{N}$.

Numerical experiment. We set the model parameters to the following values

$$x = 100,\ r = 0.10,\ \sigma = 0.30,\ \beta = 0.80$$

and we consider 3 calls with the same maturity $T = 1$ and with strikes $K_1 = 80$, $K_2 = 100$ and $K_3 = 120$. The market prices are $P_{\text{market}}(T, K_1) = 27.55$, $P_{\text{market}}(T, K_2) = 10.82$ and $P_{\text{market}}(T, K_3) = 1.77$.

The function to be minimized and its derivative (the mean function of the algorithm) are given by Figure 6.22. Notice that when $0 \leq \sigma < 0.1$ the calibration function is blind to the variation of

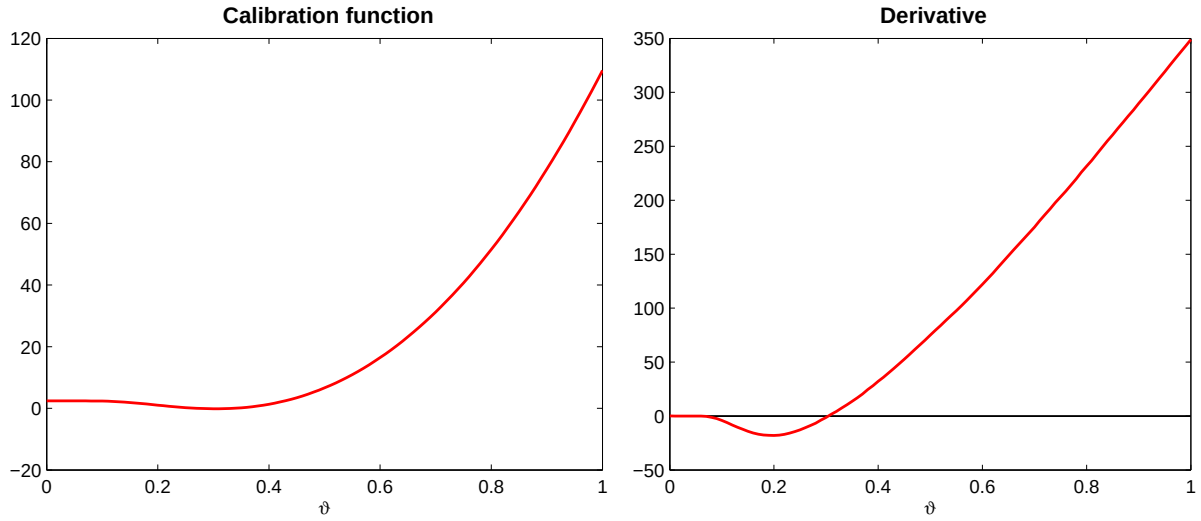

Figure 6.22: Calibration function and its derivative for the volatility.

the parameter. We then modify the parameter set $\Theta := (0.1, +\infty)$.

The stochastic approximation procedure parameters are

$$\sigma_0 \overset{(d)}{=} \mathcal{U}_{[0.1,1]}, \ n = 10^5.$$

Finally we set $\gamma_n = \frac{1}{300n}$.

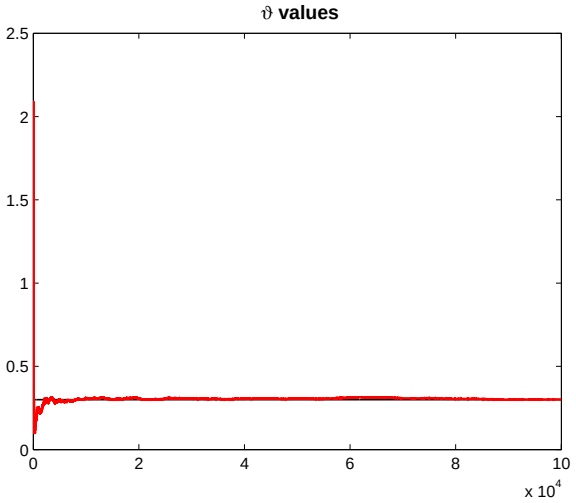

Figure 6.23: Calibration of implied volatility by Call options. $T = 1$, $K = (80, 100, 120)^t$, $r = 0.10$, $\beta = 0.80$, $x = 100$.

n	ϑ_n
100	0.2468
1000	0.2360
10 000	0.3077
100 000	0.3011

Table 6.10: Convergence of ϑ_n toward $\vartheta^* = 0.3$.

▷ *Calibration of the power* β. Set $\theta := \beta$, $\Theta := [0, 1]$. The model parameter are the same as above.

The associated tangent process $B_t := \frac{\partial X_t(\beta)}{\partial \beta}$ satisfies the following SDE

$$dB_t = rB_t dt + \vartheta \frac{X_t^\beta(\beta)}{\sqrt{\epsilon^2 + X_t^2(\beta)}} \left(X_t(\beta) \log X_t(\beta) + \left(1 + \beta - \frac{X_t^2(\beta)}{\epsilon^2 + X_t^2(\beta)} \right) B_t \right) dW_t, \quad B_0 = 0.$$

NUMERICAL EXPERIMENT. We devise two stepwise constant Euler scheme to approximate the terminal values X_T and B_T with step $\frac{T}{N}$, $N = 10$.

The function to be minimized and its derivative (the mean function of the algorithm) are given by Figure 6.24. Notice that when $0 \leq \beta < 0.5$ the calibration function is blind to the variation of

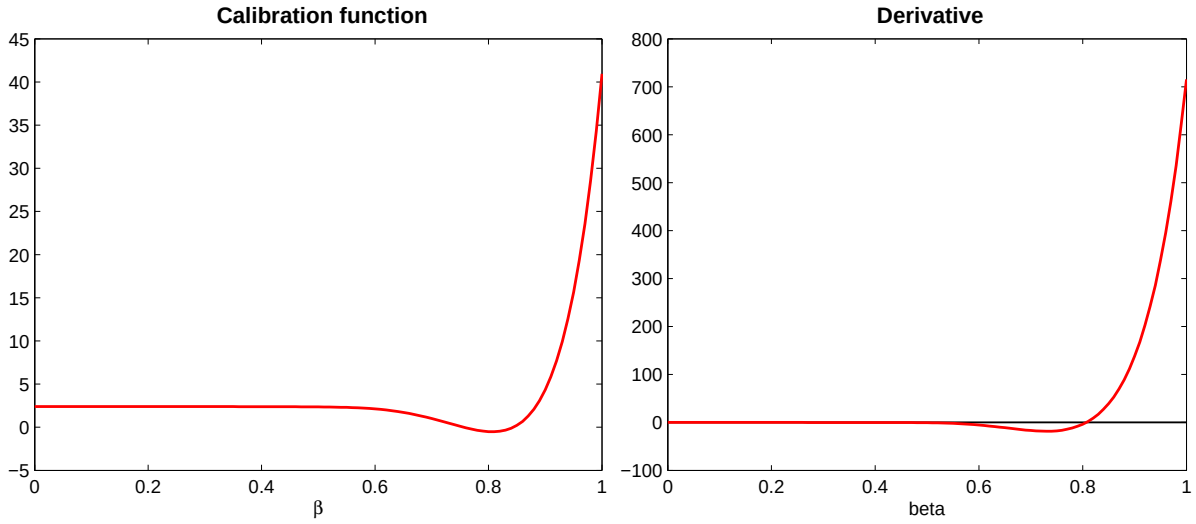

Figure 6.24: Calibration function and its derivative for the power β.

the parameter. We then modify the parameter set $\Theta := [0.5, 1]$.

The stochastic approximation procedure parameters are

$$\beta_0 \overset{(d)}{=} \mathcal{U}_{[0.5,1]}, \ n = 10^5.$$

Finally we set $\gamma_n = \frac{1}{100n}$.

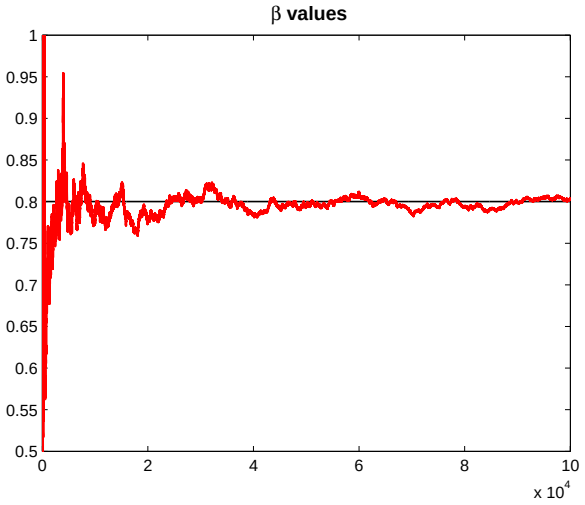

Figure 6.25: Calibration of β by Call options. $T = 1$, $K = (80, 100, 120)^t$, $r = 0.10$, $\sigma = 0.3$, $x = 100$.

n	β_n
100	0.5461
1000	0.6893
10 000	0.7771
100 000	0.8019

Table 6.11: Convergence of β_n toward $\beta^* = 0.8$.

$\triangleright$ *Simultaneous calibration of the volatility ϑ and the power β.* Set $\theta := (\vartheta, \beta)$, $\Theta := [0.1, +\infty) \times [0.5, 1]$. The model parameter are the same as above and we use 7 call prices to calibrate with $T = 1$ and $K = (70, 80, 90, 100, 110, 120, 130)$.

NUMERICAL EXPERIMENT. We set the model parameters to the following values

$$x = 100,\ r = 0.10.$$

We devise two stepwise constant Euler scheme to approximate the terminal values X_T with step $\frac{T}{N}$, $N = 10$.

The stochastic approximation procedure parameters are

$$\theta_0 \overset{(d)}{=} \mathcal{U}_{[0,1]\times[0.5,1]},\ n = 10^5.$$

Finally we set $\gamma_n^1 = \frac{1}{650n}$.

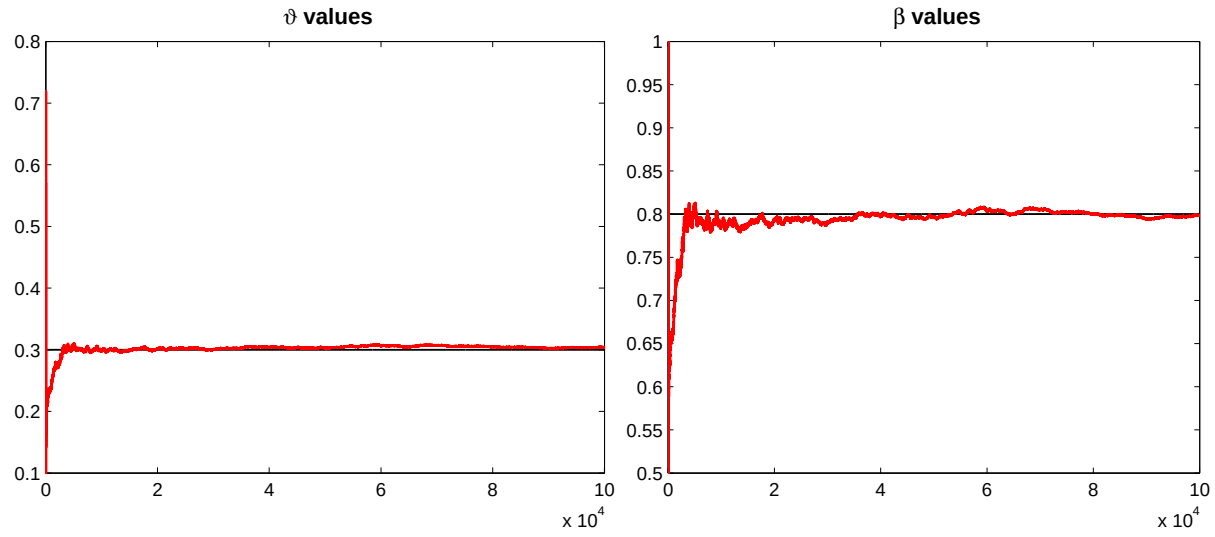

Figure 6.26: Calibration of ϑ and β by Call options. $T = 1$, $K = (80, 100, 120)^t$, $r = 0.10$, $x = 100$.

n	ϑ_n	β_n
100	0.1965	0.5998
1000	0.24	0.6692
10 000	0.2986	0.7870
100 000	0.3039	0.7981

Table 6.12: Convergence of (ϑ_n, β_n) toward $(\vartheta^*, \beta^*) = (0.3, 0.8)$.

6.3.1.5 Calibration of a Merton model

This is a toy model to illustrate the calibration using stochastic approximation since there exists closed formula for Call and Put prices (see (6.15)).

Assume that the asset price process $(X_t)_{t\geq 0}$ follows the Merton model defined by (6.2). Then let $(K_i, T_i)_{i=1,\dots,p}$ be p couples "maturity-strike price" and let $P_{\text{market}}(T_i, K_i)$ be the mark-to-market price of the option with maturity T_i and strike price K_i, $i = 1, \dots, p$.

Notice, by using (6.3), that we should normalize the pseudo-mean function H into

$$\widetilde{H}(\theta, W) = e^{-\left(\sigma^2+\lambda a^2\right)\frac{T}{2}} H(\theta, W),$$

where $\theta = \sigma$, $\theta = a$ or $\theta = \lambda$ according to the successive calibration procedure. This is not necessary in the below simulations because of the values of the parameters to find.

▷ *Implicit volatility calibration.* Set $\theta := \sigma$, $\Theta := (0, +\infty)$ and

$$Y_\sigma := \left(e^{-rT_i}\left(X_{T_i}(\sigma) - K_i\right)_+ - P_{\text{market}}(T_i, K_i)\right)_{i=1,\dots,p}.$$

To devise the recursive procedure (6.12), we need the σ-tangent process which reads

$$\frac{\partial X_t(\sigma)}{\partial \sigma} = (W_t - \sigma t)\, X_t(\sigma).$$

Numerical experiment. We set the model parameters to the following values

$$x = 100,\ r = 0.10,\ \sigma = 0.30,\ \lambda = 3,\ a = 0.05.$$

We consider 3 calls with strikes $K_1 = 80$, $K_2 = 100$, $K_3 = 120$ and with the same maturity $T = 1$. The market prices are $P_{\text{market}}(T, K_1) = 29.56$, $P_{\text{market}}(T, K_2) = 17.11$ and $P_{\text{market}}(T, K_3) = 9.11$ respectively.

The function to be minimized and its derivative (the mean function of the algorithm) are given by Figure 6.27.

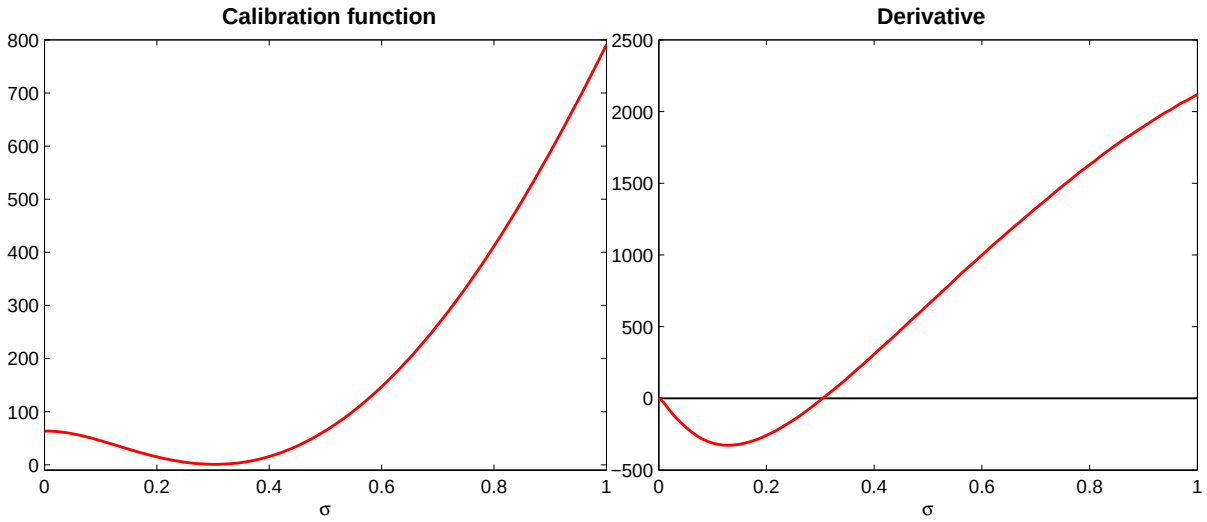

Figure 6.27: Calibration function and its derivative for the volatility.

The stochastic approximation procedure parameters are

$$\sigma_0 \overset{(d)}{=} \mathcal{U}_{[0,1]}, \ n = 10^5.$$

Finally we set $\gamma_n = \frac{1}{4000n}$.

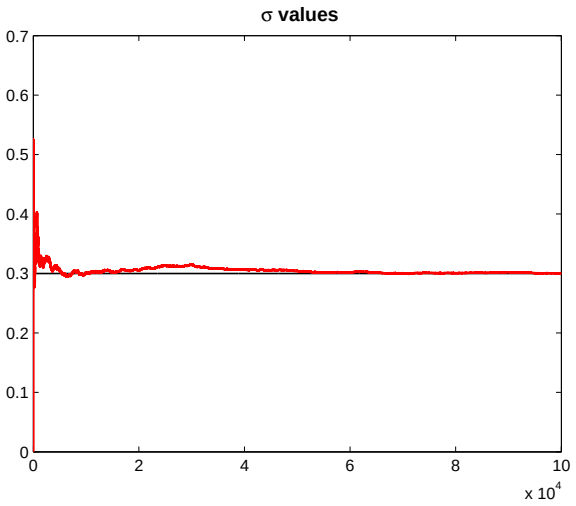

Figure 6.28: Calibration of implicit volatility on Call options. $T = 1$, $K = (80, 100, 120)^t$, $r = 0.10$, $a = 0.05$, $\lambda = 3$, $x = 100$.

n	σ_n
100	0.3091
1000	0.3377
10 000	0.3020
100 000	0.2999

Table 6.13: Convergence of σ_n toward $\sigma^* = 0.3$.

▷ *Calibration of the loss parameter a.* Set $\theta := a$, $\Theta := (0, 1)$.
The associated tangent process reads

$$\frac{\partial X_t(a)}{\partial a} = \left(\lambda t - \frac{N_t}{1-a}\right) X_t(a).$$

NUMERICAL EXPERIMENT. The model parameter are the same as above. The function to be minimized and its derivative (the mean function of the algorithm) are given by Figure 6.29.

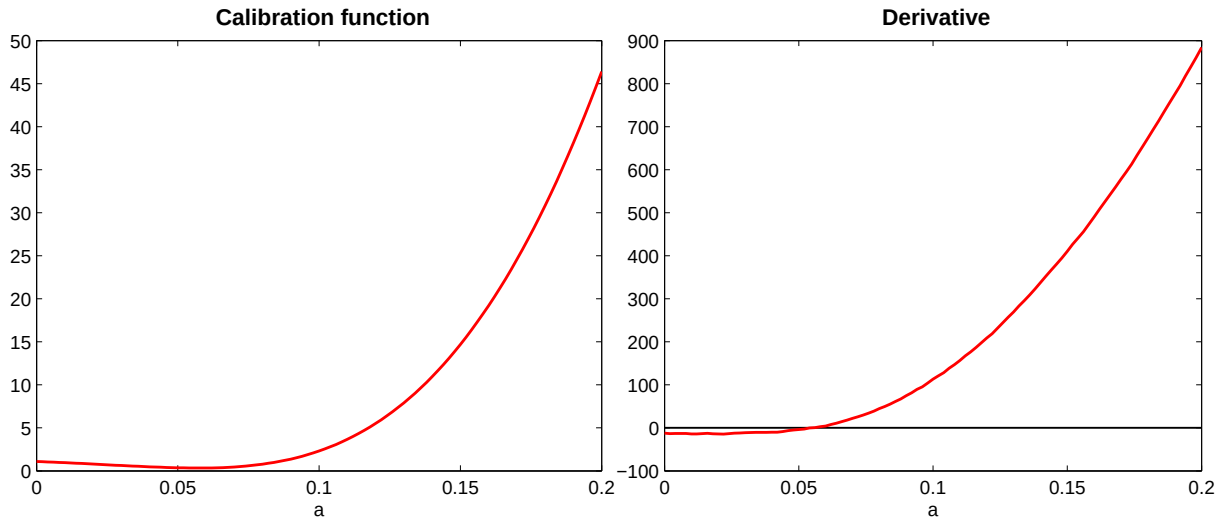

Figure 6.29: Calibration function and its derivative for the loss parameter a.

The stochastic approximation procedure parameters are

$$a_0 = 0, \ n = 10^5.$$

Finally we set $\gamma_n = \frac{1}{6000n}$.

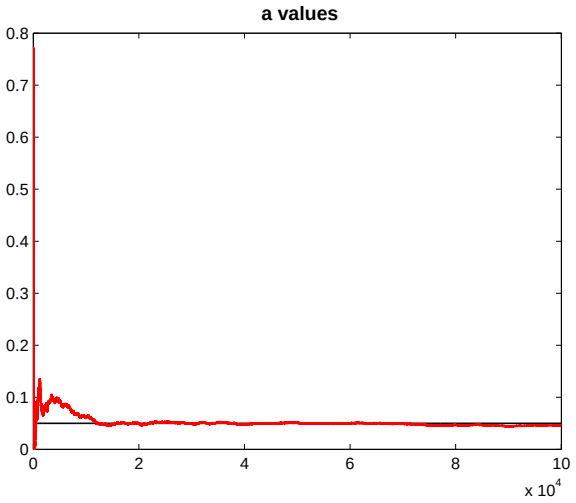

Figure 6.30: Calibration of a on Call options. $T = 1$, $K = (80, 100, 120)^t$, $r = 0.10$, $\sigma = 0.3$, $\lambda = 3$, $x = 100$.

n	a_n
100	0.0698
1000	0.1081
10 000	0.0605
100 000	0.0463

Table 6.14: Convergence of a_n toward $a^* = 0.05$.

▷ *Joint calibration of several parameters.* To check the price sensitivity to the different parameters and the existence of a unique equilibrium point, we study the price dependence to the intensity parameter λ when a and the total (squared) volatility $\sigma^2 + \lambda a^2$ are constant.

To this end we use the development of the price in series with Black-Scholes coefficients. Indeed by using that W and N are independent, we have for the Call price

$$\begin{aligned} e^{-rT}\mathbb{E}\left[(X_T - K)_+\right] &= \mathbb{E}\left[\left(xe^{\lambda aT}e^{-\frac{\sigma^2}{2}T+\sigma W_T}(1-a)^{N_T} - Ke^{-rT}\right)_+\right] \\ &= \sum_{j=0}^{+\infty} \mathbb{E}\left[\left(xe^{\lambda aT}(1-a)^j e^{-\frac{\sigma^2}{2}T+\sigma W_T} - Ke^{-rT}\right)_+\right] \mathbb{P}\left(N_T = j\right) \\ &= \sum_{j=0}^{+\infty} C_{BS}(xe^{\lambda aT}(1-a)^j, \sigma, r, K, T)\frac{(\lambda T)^j}{j!}e^{-\lambda T}, \end{aligned} \tag{6.15}$$

where $C_{BS}(x_0, \sigma, r, K, T)$ is the Call price in a Black-Scholes model with starting value x_0, volatility σ, interest rate r, strike K and maturity T.

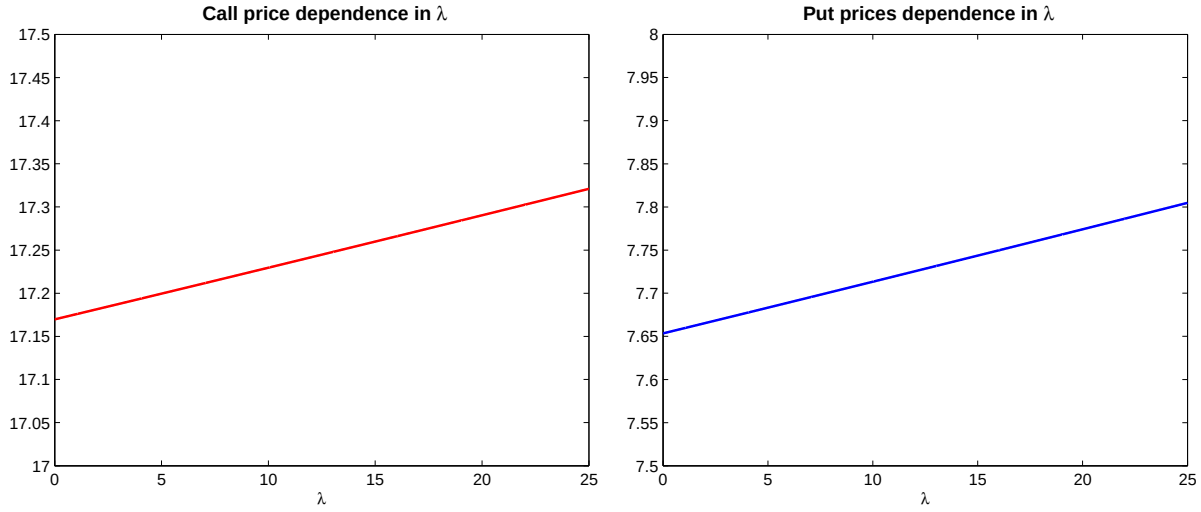

Figure 6.31: Call and put prices depending on λ with $K = 100$, $T = 1$, $a = 0.05$ and $\sigma^2 + \lambda a^2 = 0.0975$.

Note that while λ fluctuates between 0 and 25, the prices fluctuate between 17.17 and 17.32 for the call and between 7.65 and 7.8 for the put. Similar behaviours are observed for other couples of maturity-strike when the loss parameter a is small.

If we increase the value of the loss parameter a, we obtain the following result

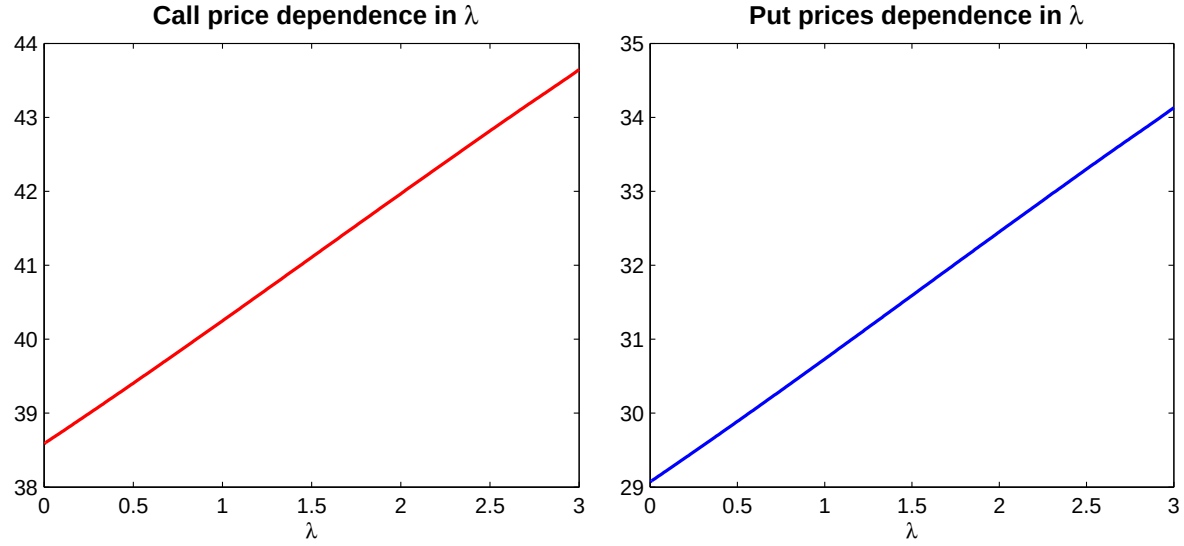

Figure 6.32: Call and put prices depending on λ with $K = 100$, $T = 1$, $a = 0.5$ and $\sigma^2 + \lambda a^2 = 0.84$.

Note that while λ fluctuates between 0 and 3, the prices fluctuate between 38.58 and 44.80 for the call and between 29.07 and 35.28 for the put.

We deduce from these numerical experiments that when the size of the jumps (modeled by a) is small, the vanilla option prices are not very sensitive to variations of the jump intensity λ, and the small jumps can be assimilated to a Brownian motion. When a is high, the prices variations are more significant, but the parameter value is unrealistic (even in financial crisis, 50% of loss on an asset is utopian). Consequently, this kind of models including small jumps are very difficult to calibrate, because they are too close to pure Brownian models with a slightly higher volatility.

6.3.2 Other method: weak approach to differentiation

If the process $(X_t)_{t\geq 0}$ has càdlàg paths, then the naive pathwise tangent process does not exist. In some cases, we can straightforwardly obtain the derivative of the mean function with respect to θ as the expectation. We illustrate this method when $\theta = \lambda$ is the intensity of a Poisson process by using formula on Poisson distributed random variables which can be seen as very elementary example of Malliavin calculus for Poisson process (see [101]).

6.3.2.1 Calibration of jump intensity λ in a Merton model

Assume that $(X_t)_{t\geq 0}$ satisfies (6.2). Set $\theta := \lambda$, $\Theta := (0, +\infty)$. The model parameter are the same as above. In this case, we cannot use the (pathwise) tangent process, because it does not exist. We will use some results on Poisson distributed random variables introduced in the Chapter on optimal distance in a limit order book because $N_T \overset{\mathcal{L}}{\sim} \mathcal{P}(\lambda T)$. To be precise, if $\varphi(\lambda) = \mathbb{E}\left[\Phi(\lambda, N^\lambda)\right]$, where $(N^\lambda)_{\lambda\geq 0}$ is a family of Poisson distributed random variables with parameter λ. Then, under natural condition on Φ (to derive under the sign $\mathbb{E}$) and by using property of Poisson distributed random variable, we have

$$\varphi'(\lambda) = \mathbb{E}\left[\frac{\partial}{\partial\lambda}\Phi(\lambda, N^\lambda)\right] + \mathbb{E}\left[\Phi(\lambda, N^\lambda + 1) - \Phi(\lambda, N^\lambda)\right].$$

Consequently, if

$$\Phi(\lambda, N^{\lambda T}) = \left(xe^{(-\frac{\sigma^2}{2}+\lambda a)T+\sigma W_T}\left(1-a\right)^{N^{\lambda T}} - Ke^{-rT}\right)_+ - P_0^{\text{market}},$$

then

$$\varphi'(\lambda) = \mathbb{E}\left[aTX_T\mathbb{1}_{\{X_T\geq Ke^{-rT}\}}\right] + T\mathbb{E}\left[\Phi(\lambda, N^\lambda + 1) - \Phi(\lambda, N^\lambda)\right].$$

Numerical experiment. The model parameter are the same as above. The function to be minimized and its derivative (the mean function of the algorithm) are given by Figure 6.33.

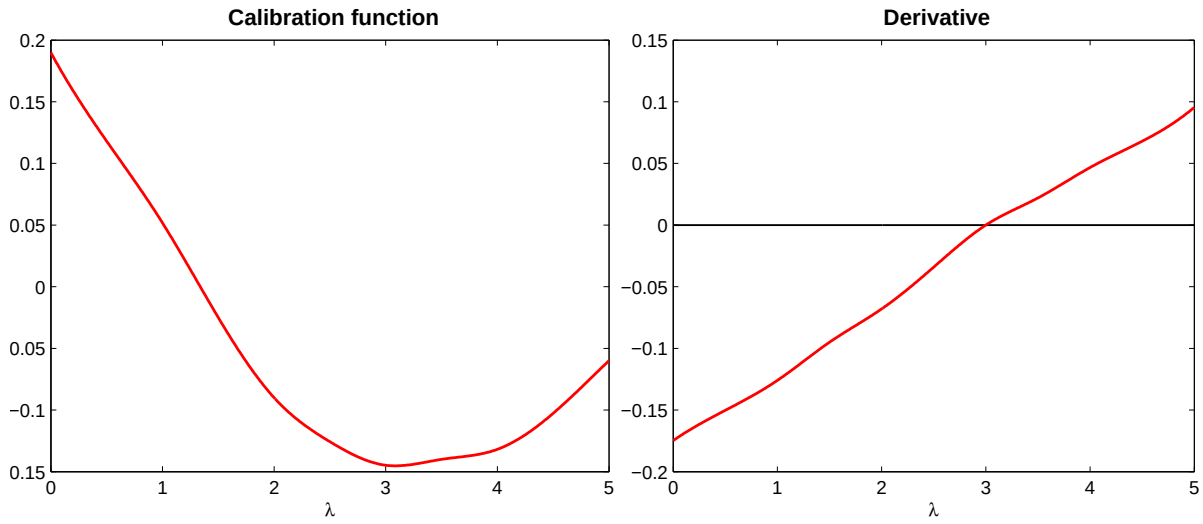

Figure 6.33: Calibration function and its derivative for the intensity.

The stochastic approximation procedure parameters are $\lambda_0 = 1$, $n = 10^5$. Finally we set $\gamma_n = \frac{1}{n}$.

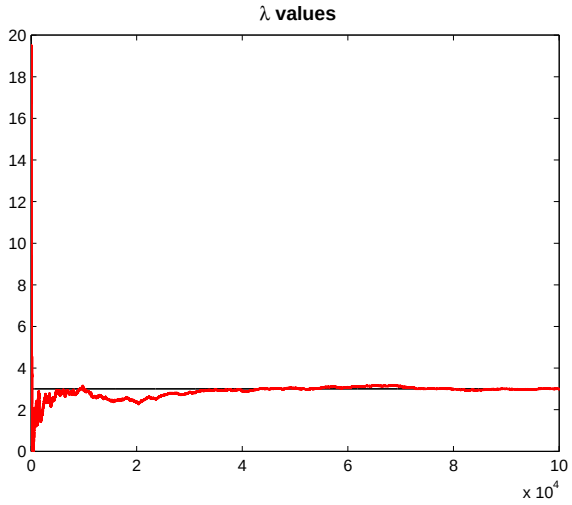

Figure 6.34: Calibration of Merton intensity on Call options. $T = 1$, $K = (80, 100, 120)^t$, $\sigma = 0.3$, $a = 0.05$, $r = 0.10$, $x = 100$.

n	λ_n
100	4.4235
1000	2.1533
10 000	2.9125
100 000	2.9894

Table 6.15: Convergence of λ_n toward $\lambda^* = 3$.

6.3.3 The Kieffer-Wolfowitz approach

Practical implementations of the Robbins-Siegmund approach point out a second technical difficulty: the random function $\theta \mapsto Y_\theta(\omega)$ is not always pathwise differentiable (nor in the $L^r(\mathbb{P})$-sense which could be enough). More important in some way, even if one shows that $\theta \mapsto \mathbb{E}Y_\theta$ is differentiable, possibly by calling upon other techniques (log-likelihood method, Malliavin weights, etc), the resulting expression for $J_{Y(\omega)}(\theta)$ may turn out to be difficult to simulate, requiring ing much programming care, whereas the random vectors Y_θ can be simulated in a standard way. In such a setting, an alternative is provided by the Kiefer-Wolfowitz algorithm (K-W) which combines the recursive stochastic approximation principle with a finite difference approach to differentiation. The idea is simply to approximate the gradient ∇L by

$$\frac{\partial L}{\partial \theta_i}(\theta) \approx \frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta^i}, \quad 1 \le i \le p,$$

where $(e_i)_{1 \le i \le p}$ denotes the canonical basis of $\mathbb{R}^p$ and $\eta = (\eta^i)_{1 \le i \le p}$. This finite difference term has an integral representation given by

$$\frac{L(\theta + \eta^i e_i) - L(\theta - \eta^i e_i)}{2\eta^i} = \mathbb{E}\frac{\Lambda(\theta + \eta^i e_i, W) - \Lambda(\theta - \eta^i e_i, W)}{2\eta^i},$$

where, with obvious temporary notations,

$$\Lambda(\theta, W) := \langle Y(\theta, W) \,|\, Y(\theta, W) \rangle_S = Y(\theta, W)^t S Y(\theta, W),$$

($Y(\theta, W)$ is related to the innovation W). Starting form this representation, we may derive a recursive updating formula for θ_n as follows

$$\theta_{n+1}^i = \theta_n^i - \gamma_{n+1} \frac{\Lambda(\theta_n + \eta_{n+1}^i e_i, W^{n+1}) - \Lambda(\theta_n - \eta_{n+1}^i e_i, W^{n+1})}{2\eta_{n+1}^i}, \quad n \geq 0, \quad 1 \leq i \leq p.$$

We reproduce below a typical convergence result for K-W procedures (which is the natural counterpart for the Robbins-Siegmund Lemma in a stochastic gradient framework).

Theorem 6.2. *Assume that the function $\theta \mapsto L(\theta)$ is twice differentiable with a Lipschitz Hessian. We assume that*

$$\theta \mapsto \Lambda(\theta, W) \text{ is Lipschitz in } L^2$$

and that the step sequences satisfy

$$\sum_{n\geq 1} \gamma_n = \sum_{n\geq 1} \eta_n^i = +\infty, \quad \sum_{n\geq 1} \gamma_n^2 < +\infty, \quad \eta_n \to 0 \quad \text{and} \quad \sum_{n\geq 1} \left(\frac{\gamma_n}{\eta_n^i}\right)^2 < +\infty.$$

Then, θ_n a.s. converges to a connected component of $\{L = \ell\} \cap \{\nabla L = 0\}$ for some level $\ell \geq 0$.

A special case of this procedure in a linear framework is the decreasing step finite difference method for greeks computation. The traps problem for the K-W algorithm (convergence toward a local minimum of L) has been more specifically investigated in [84].

Users must keep in mind that this procedure needs some care in the tuning of the step parameters γ_n and η_n. This may need some preliminary numerical experiments. Of course, all the recommendations made for the R-S procedures remain valid. For more details on the K-W procedure we refer to [25] and the references therein.

6.3.3.1 Application to the calibration of the implied volatility in a Black-Scholes model

NUMERICAL EXPERIMENT. We set the model parameters to the following values

$$x = 100, \ r = 0.10, \ \sigma = 0.30$$

and we consider 3 calls with strikes $K_1 = 80$, $K_2 = 80$, $K_3 = 80$ and with the same maturity $T = 1$. The Black-Scholes reference prices $P_{\text{market}}(T, K_1) = 20.42$, $P_{\text{market}}(T, K_2) = 16.71$ and $P_{\text{market}}(T, K_3) = 8.60$ are used as market prices.

The stochastic approximation procedure parameters are

$$\sigma_0 = 0, \ n = 10^5.$$

Finally we set $\gamma_n = \frac{3}{1000n}$ and $\eta_n = \frac{1}{n^{0.45}}$.

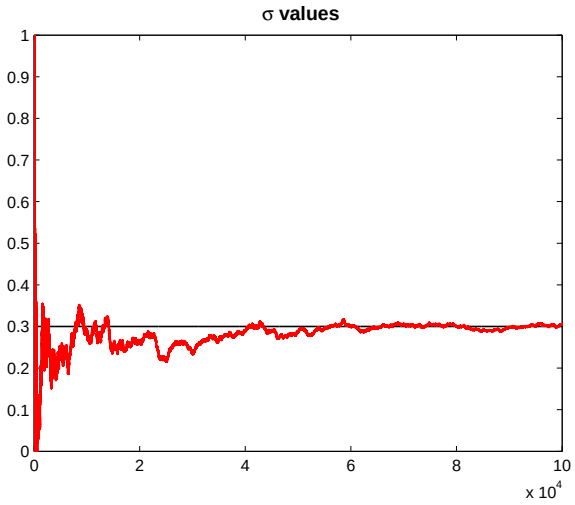

Figure 6.35: Calibration of Black-Scholes volatility on Call options. $T = 1$, $K = (80, 100, 120)^t$, $r = 0.10$, $x = 100$.

n	σ_n
100	0.8193
1000	0.0711
10 000	0.2862
100 000	0.3051

Table 6.16: Convergence of σ_n toward $\sigma^* = 0.3$.

6.3.4 Conclusion

Stochastic approximation is a useful tool to calibrate financial models on market data. When there exist closed formula for the interest model, the rate of convergence is slower than the one of deterministic methods because it is ruled by the Monte Carlo approach. Nevertheless when deterministic methods does not exist for the considered model, this technique is very practical and quite easy to implement.

Considering the difficulties encountered to calibrate several parameters in the models introduced above, the study done in this section leads us to wonder about the robustness of such models.

Chapter 7

Randomized Urn Models revisited using Stochastic Approximation

This chapter is based on a submitted paper.

This chapter presents the link between stochastic approximation and clinical trials based on randomized urn models investigated in [17, 18, 19]. We reformulate the dynamics of both the urn composition and the assigned treatments as standard stochastic approximation (SA) algorithms with remainder. Then, we derive the $a.s.$ convergence and the asymptotic normality (Central Limit Theorem CLT) of the normalized procedure under less stringent assumptions by calling upon the ODE and SDE methods. As a second step, we investigate a more involved family of models, known as multi-arm clinical trials, where the urn updating depends on the past performances of the treatments. By increasing the dimension of the state vector, our SA approach provides this time a new asymptotic normality result.

7.1 Introduction

The aim of this chapter is to illustrate the efficiency of Stochastic Approximation (SA) Theory by revisiting several recent results on randomized urn models applied to clinical trials (especially [17, 18, 19]). We will first retrieve the $a.s.$ convergence (strong consistency) and asymptotic normality results obtained in these papers under less stringent assumptions. Then we will take advantage of this more synthetic approach to establish a new Central Limit Theorem (CLT) in the more sophisticate randomized urn model known as "multi-arm clinical test". In this model, the urn updating which produces the adaptive design is based on statistical estimators of the past efficiency of the assigned treatments.

In these adaptive models, the starting point is the equation which governs the urn composition updated after each new treated patient. Basically, we will show that a normalized version of this urn composition can be formulated as a classical recursive stochastic algorithm with step $\gamma_n = \frac{1}{n}$ which classical Stochastic Approximation Theory deals with. Doing so we will be in position to establish the $a.s.$ convergence of the procedure by calling upon the so-called Ordinary Differential Equation Method (ODE method) and to derive the asymptotic normality - a CLT, to be precise - from the standard CLT for stochastic algorithms (sometimes called the Stochastic Differential Equation Method (SDE method), see $e.g.$ [44, 25]). These two main theoretical results are recalled in a self-contained form in Chapter 1. They can be found in all classical textbooks on SA ([25], [43], [44], [75]) and go back to [74] and [33].

Clinical trials essentially deal with the asymptotic behaviour of the patient allocation to several treatments during the procedure. Adaptive designs in clinical trials aim at detecting "on line" which

treatment should be assigned to more patients, while keeping randomness enough to preserve the basis of treatments. This adaptive approach relies on the cumulative information provided by the responses to treatments of previous patients in order to adjust treatment allocation to the new patients. To this end, many urn models have been suggested in the literature (see [68], [113], [112], [46] and [105]). The most widespread random adaptive model is the Generalized Friedman Urn (GFU) (see [13]), also called Generalized Pólya Urn (GPU). The idea of this modeling is that the urn contains balls of d different types representative of the treatments. All random variables involved in the model are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote $Y_0 = (Y_0^i)_{i=1,\dots,d} \in \mathbb{R}_+^d \setminus \{0\}$ the initial composition of the urn, where Y_0^i denotes the number of balls of type i, $i = 1, \dots, d$. The allocation of the treatments is sequential and the urn composition at draw n is denoted by $Y_n = (Y_n^i)_{i=1,\dots,d}$. When the n^{th} patient presents, one draws randomly (*i.e.* uniformly) a ball from the urn with instant replacement. If the ball is of type i, then the treatment i is assigned to the n^{th} patient, $i = 1, \dots, d$, $n \geq 1$. The urn composition is updated by taking into account the response of the n^{th} patient to the treatment i, or the responses of all patients up to the n^{th} one (*i.e.* the efficiency of the assigned treatment), namely by adding D_n^{ij} balls of type j, $j = 1, \dots, d$. The procedure is iterated as long as patients present. Consequently the larger the number of balls of a given type is, the more efficient the treatment is. The urn composition at n, modeled by an $\mathbb{R}^d$-valued vector Y_n satisfies the following recursive procedure:

$$Y_n = Y_{n-1} + D_n X_n, \quad n \geq 1, \quad Y_0 \in \mathbb{R}_+^d \setminus \{0\}, \tag{7.1}$$

with $D_n = (D_n^{ij})_{1 \leq i,j \leq d}$ is the addition rule matrix and X_n is the result of the n^{th} draw and $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to \{e^1, \cdots, e^d\}$ models the selected treatment ($\{e^1, \cdots, e^d\}$ denotes the canonical basis of $\mathbb{R}^d$ and e^i stands for treatment i). We model the drawing in the urn by setting

$$X_n = \sum_{i=1}^{d} \mathbb{1}_{\left\{ \frac{\sum_{\ell=1}^{i-1} Y_{n-1}^\ell}{\sum_{\ell=1}^{d} Y_{n-1}^\ell} < U_n \leq \frac{\sum_{\ell=1}^{i} Y_{n-1}^\ell}{\sum_{\ell=1}^{d} Y_{n-1}^\ell} \right\}} e^i, \quad n \geq 1, \tag{7.2}$$

where $(U_n)_{n \geq 1}$ is i.i.d. with distribution $U_1 \overset{\mathcal{L}}{\sim} \mathcal{U}_{[0,1]}$.

Let $\mathcal{F}_n = \sigma(Y_0, U_k, D_k, 1 \leq k \leq n)$ be the filtration of the procedure. The *generating matrices* are defined as the $\mathcal{F}_n$-compensator of the additions rule sequence $i; e.$

$$H_n = \left(\mathbb{E}\left[D_n^{ij} \,|\, \mathcal{F}_{n-1} \right] \right)_{1 \leq i,j \leq d}, \; n \geq 1.$$

The first designs under consideration were the homogeneous GFU models where the addition rules D_n are i.i.d. and the so-called generating matrices $H_n = H = \mathbb{E}D_n$ are identical, non-random, nonnegative and irreducible. Hence by the Perron-Frobenius theorem H has a unique and positive maximal eigenvalue and an eigenvector with positive components (see [13, 14, 52, 58]). But the homogeneity of the generating matrix is often not satisfied in practice and inhomogeneous GFU models have been introduced (see [17]) in which H_n are not random but converge to a deterministic limit H, under the assumption that the total number of balls added at each stage is constant. As a third step, the homogeneous Extended Pòlya Urn (EPU) models have been introduced in [107] in which only the mean total number of balls added at each stage is constant.

Finally, in [18] the authors proposed a nonhomogeneous EPU model because in applications, the addition rule D_n depends on the past history of previous trials (see [11]), so that the general generating matrix H_n is usually random. Thus the entries of H may not be all nonnegative (*e.g.*, when there is no replacement after the draw), and they assume that the matrix H has a unique maximal eigenvalue λ with associated (right) eigenvector $v^* = (v^{*,i})_{i=1,\dots,d}$ with $\sum_{i=1}^{d} v^{*,i} = 1$. Furthermore the conditional expectation of the total number of balls added at each stage were constant.

The first theoretical investigations on these models focused on the asymptotic properties of the urn composition (consistency and asymptotic normality). However, for practical matter, it is clear that the asymptotic behaviour of the vector $N_n := \sum_{k=1}^{n} X_k$ which stores the treatment allocation among the first n patients is of high interest, especially its variance structure in order to compare several adaptive designs. Thus, in [18] is proved the strong consistency of both (normalized) quantities Y_n/n and N_n/n (under a summability assumption on the generating matrices).

By considering an appropriate recursive procedure for the normalized urn composition derived from (7.1) we prove by the ODE method its $a.s.$ convergence toward v^* under a significantly less stringent assumption, namely the minimal requirement that $H_n \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} H$. The $a.s.$ convergence of the treatment allocation frequency $\frac{N_n}{n}$ toward the same v^*.

As concerns asymptotic normality, separate results on these two quantities are obtained in [18] under an additional assumption on the rate of convergence of the generating matrices H_n toward H. On our side we propose to consider a stochastic approximation procedure with remainder satisfied by the higher dimensional vector $(\frac{Y_n}{n}, \frac{N_n}{n})$. Then, the standard CLT for SA procedures with remainder directly provides the expected asymptotic normality result for the whole vector under an assumption on the L^2-rate of convergence of the generating matrices towards their limit (namely *i.e.* $|||H_n - H||| = o\left(n^{-\frac{1}{2}}\right)$) which is again slightly less stringent than the original one. As a result, we obtain the asymptotic joint distribution with an explicit global covariance structure matrix.

In the end of [18], an application to multi-arm clinical trials randomized urn models is proposed. This adaptive design has already been introduced in [19] with first consistency results. This kind of models is clearly the most interesting for practitioners since it takes into account the past results of the assigned treatments in the addition rule matrices, denoted S_n at time n (S_n^i denotes the number of cured patients by treatment i among the N_n^i treated ones). The above strong consistency results apply but none of the asymptotic normality works as stated since the generating matrices H_n do not – in fact *cannot* as we will emphasize – converge at the requested rate since they themselves satisfy a CLT. However by increasing again the structural dimension of the problem by considering the triplet $(\frac{Y_n}{n}, \frac{N_n}{n}, \frac{S_n}{n})$ which can be shown again to satisfy a SA algorithm with remainder for which $a.s.$ convergence and the CLT hold (provided the limiting generating matrix is still irreducible, etc). Thus we illustrate on this example that SA Theory is a powerful tool to investigate this kind of adaptive design problem. The main difficulty is to exhibit the appropriate form for the recursion by making *a priori* the balance between significant asymptotic terms and remainder terms.

The chapter is organized as follows. We rewrite the dynamics (7.1) of the urn composition as a stochastic approximation procedure with state variable for $\widetilde{Y}_n := \frac{Y_n}{n}$ in Section 7.2.1. In Section 7.2.2 the $a.s.$ convergence of $\frac{1}{n}\sum_{i=1}^{d} Y_n^i$ is established which implies that of $\widetilde{Y}_n$ and $\widetilde{N}_n := \frac{N_n}{n}$ by using the ODE method of SA under slightly lighten assumption than in [18]. The rate of convergence is investigated in Section 7.2.3: we obtain a CLT, once again under slightly less stringent assumptions on the limit generating matrix H than in [18]. Section 7.3 is devoted to multi-arm clinical tests. In Section 7.3.1 we briefly recall the Wei GFU model introduced [112, 19] where the generating matrices H_n are not random. In this case, the strong consistency and the asymptotic normality follow from the results of Section 7.2 (like in [18]). In Section 7.3.2 we study the adaptive design proposed in [19] where the addition rule matrices depend on the responses of all the past patients. We use result in Section 7.2.2 to prove the strong consistency. We prove in Section 7.3.3 a new CLT for this model, when the generating matrix H_n satisfies itself a CLT, which relies again on Stochastic Approximation techniques.

NOTATIONS $\forall u = (u^i)_{i=1,\dots,d} \in \mathbb{R}^d$, $\|u\|$ denotes the canonical Euclidean norm of the column vector u on $\mathbb{R}^d$, $\mathrm{Tr}(u) = \sum_{k=1}^{d} u^d$ denotes its "trace", u^t denotes its transpose; $|||A|||$ denotes the operator norm of the matrix $A \in \mathcal{M}_{d,q}(\mathbb{R})$ with d rows and q columns with respect to canonical Euclidean

norms. When $d=q$, $\mathrm{Sp}(A)$ denotes the set of eigenvalues of A. $\mathbf{1}=(1\cdots 1)^t$ denotes the unit column vector in $\mathbb{R}^d$ and I_d denotes the $d\times d$ identity matrix.

7.2 Convergence and first rate result

With the notations and definitions described in the introduction, we then formulate the main assumptions to establish the *a.s.* convergence of the urn composition.

(A1) The generating matrices $H_n = (H_n^{ij})_{1\leq i,j\leq d}$, $n\geq 1$, satisfies *a.s.*

$$\forall\, i,j\in\{1,\ldots,d\},\quad H_n^{ij}\geq 0\quad\text{and}\quad \forall\, j\in\{1,\ldots,d\},\quad \sum_{i=1}^{d} H_n^{ij} = c > 0. \tag{7.3}$$

We may assume up to a renormalization of Y_n without loss of generality that $c=1$.

(A2) The addition rule D_n is conditionally independent of the drawing procedure X_n given $\mathcal{F}_{n-1}$ and satisfies

$$\forall 1\leq j\leq d,\quad \sup_{n\geq 1}\mathbb{E}\left[\left\|D_n^{\cdot j}\right\|^2 \,|\, \mathcal{F}_{n-1}\right] < +\infty. \tag{7.4}$$

(A3) Assume that there exists an irreducible $d\times d$ matrix H (with non-negative entries) such that

$$H_n \xrightarrow[n\to\infty]{a.s.} H. \tag{7.5}$$

H is called the limit generating matrix.

This assumption guarantees by the Perron-Frobenius Theorem (see [30]) that 1 is the maximal eigenvalue of H and that the components of its right eigenvector v can be chosen all positive. Therefore, we may normalize this vector v^* such that $\mathrm{Tr}(v^*)=1$.

7.2.1 The dynamics as a stochastic approximation procedure

Our aim in this section is to reformulate the dynamics (7.1)-(7.2) into a recursive stochastic algorithm. Then we aim at applying the most powerful tools of SA, namely the "ODE" and the "SDE" methods to elucidate the asymptotic properties (*a.s.* convergence and weak rate) of both the urn composition and the treatment allocation. We start from (7.1) with $Y_0\in\mathbb{R}_+^d\setminus\{0\}$.

$$Y_{n+1} = Y_n + D_{n+1}X_{n+1} = Y_n + \mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right] + \Delta M_{n+1}, \tag{7.6}$$

where

$$\Delta M_{n+1} := D_{n+1}X_{n+1} - \mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]$$

is an $\mathcal{F}_n$-martingale increment. By the definition of the generating matrix H_n, we have

$$\begin{aligned}\mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right] &= \sum_{i=1}^{d}\mathbb{E}\left[D_{n+1}\mathbb{1}_{\{X_{n+1}=e^i\}}e^i\,|\,\mathcal{F}_n\right] = \sum_{i=1}^{d}\mathbb{E}\left[D_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{P}\left(X_{n+1}=e^i\,|\,\mathcal{F}_n\right)e^i\\ &= H_{n+1}\sum_{i=1}^{d}\frac{Y_n^i}{\mathrm{Tr}(Y_n)}e^i = H_{n+1}\frac{Y_n}{\mathrm{Tr}(Y_n)}\end{aligned}$$

so that $$Y_{n+1} = Y_n + H_{n+1}\frac{Y_n}{\mathrm{Tr}(Y_n)} + \Delta M_{n+1}.$$

Now we can derive a stochastic approximation for the normalized urn composition Y_n. First we have for every $n \geq 1$,

$$\frac{Y_{n+1}}{n+1} = \frac{Y_n}{n} + \frac{1}{n+1}\left(H_{n+1}\frac{Y_n}{\mathrm{Tr}(Y_n)} - \frac{Y_n}{n}\right) + \frac{\Delta M_{n+1}}{n+1}.$$

Consequently, $\widetilde{Y}_n = \dfrac{Y_n}{n}$, $n \geq 1$, satisfies a canonical recursive stochastic approximation procedure

$$\begin{aligned}\widetilde{Y}_{n+1} &= \widetilde{Y}_n + \frac{1}{n+1}\left(H_{n+1} - I_d\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + \left(\frac{n}{\mathrm{Tr}(Y_n)} - 1\right) H_{n+1}\widetilde{Y}_n\right) \\ &= \widetilde{Y}_n - \frac{1}{n+1}\left(I_d - H\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + r_{n+1}\right) \qquad (7.7)\end{aligned}$$

with step $\gamma_n = \frac{1}{n}$ and a remainder term given by

$$r_{n+1} := \left(\frac{n}{\mathrm{Tr}(Y_n)} - 1\right) H_{n+1}\widetilde{Y}_n + (H_{n+1} - H)\widetilde{Y}_n. \qquad (7.8)$$

Furthermore, in order to establish the $a.s.$ boundedness of $(\widetilde{Y}_n)_{n\geq 1}$ we will rely on the following recursive equation satisfied by $\mathrm{Tr}(Y_n)$:

$$\mathrm{Tr}(Y_{n+1}) = \mathrm{Tr}(Y_n) + \frac{\mathrm{Tr}(H_{n+1}Y_n)}{\mathrm{Tr}(Y_n)} + \mathrm{Tr}(\Delta M_{n+1}).$$

By the properties of the generating matrix H_{n+1}, we obtain

$$\mathrm{Tr}(H_{n+1}Y_n) = \sum_{i=1}^{d}(H_{n+1}Y_n)_i = \sum_{i=1}^{d}\sum_{j=1}^{d} H_{n+1}^{ij}Y_n^j = \sum_{j=1}^{d}\left(\sum_{i=1}^{d} H_{n+1}^{ij}\right)Y_n^j = \mathrm{Tr}(Y_n).$$

Consequently

$$\mathrm{Tr}(Y_{n+1}) = \mathrm{Tr}(Y_n) + 1 + \mathrm{Tr}(\Delta M_{n+1}). \qquad (7.9)$$

7.2.2 Convergence results

Theorem 7.1. *Under the assumptions* **(A1)**, **(A2)** *and* **(A3)**,

(a) $\frac{\mathrm{Tr}(Y_n)}{n} \xrightarrow[n\to\infty]{a.s.} 1$ *and* $\dfrac{Y_n}{\mathrm{Tr}(Y_n)} \xrightarrow[n\to\infty]{a.s.} v^*$.

(b) $\widetilde{N}_n := \dfrac{N_n}{n} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{a.s.} v^*$.

Remarks. • We simply need that $H_n \xrightarrow[n\to\infty]{a.s.} H$ whereas the assumption in [18], namely

$$\sum_{n\geq 1}\frac{\|H_n - H\|_\infty}{n} < +\infty \quad a.s.$$

• Assumption **(A3)** is not necessary to prove that $\frac{\mathrm{Tr}(Y_n)}{n} \xrightarrow[n\to\infty]{a.s.} 1$.

Proof. We will first prove that $(a) \Rightarrow (b)$, then we will prove (a).

$(a) \Rightarrow (b)$. We have

$$\mathbb{E}\left[X_n \,|\, \mathcal{F}_{n-1}\right] = \sum_{i=1}^{d} \frac{Y_{n-1}^i}{\mathrm{Tr}(Y_{n-1})} e^i = \frac{Y_{n-1}}{\mathrm{Tr}(Y_{n-1})}$$

and, by construction $\|X_n\|^2 = 1$ so that $\mathbb{E}\left[\|X_n\|^2 \,|\, \mathcal{F}_{n-1}\right] = 1$. Hence the martingale

$$\widetilde{M}_n = \sum_{k=1}^{n} \frac{X_k - \mathbb{E}\left[X_k \,|\, \mathcal{F}_{k-1}\right]}{k} \overset{a.s.\&\, L^2}{\underset{n\to\infty}{\longrightarrow}} \widetilde{M}_\infty \in L^2,$$

and by the Kronecker Lemma we obtain

$$\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} \frac{Y_{k-1}}{\mathrm{Tr}(Y_{k-1})} \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} 0.$$

This yields the announced implication owing to the Cesaro Lemma.

(a) First Step: We have

$$D_{n+1}X_{n+1} = \sum_{j=1}^{d} D_{n+1}^{\cdot j} \mathbb{1}_{\{X_{n+1}=e^j\}}$$

where $D_{n+1}^{\cdot j} = (D_{n+1}^{ij})_{i=1,\ldots,d}$. Therefore

$$\begin{aligned}
\|D_{n+1}X_{n+1}\|^2 &= \sum_{j=1}^{d} \left\|D_{n+1}^{\cdot j}\right\|^2 \mathbb{1}_{\{X_{n+1}=e^j\}}, \\
\text{so that} \qquad \mathbb{E}\left[\|D_{n+1}X_{n+1}\|^2 \,|\, \mathcal{F}_n\right] &= \sum_{j=1}^{d} \mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \,|\, \mathcal{F}_n\right] \mathbb{P}\left(X_{n+1} = e^j \,|\, \mathcal{F}_n\right) \\
&\le d \sup_{n\ge 0} \sup_{1\le j\le d} \mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \,|\, \mathcal{F}_n\right] < +\infty \quad a.s.
\end{aligned}$$

Consequently $\sup_{n\ge1} \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \,|\, \mathcal{F}_n\right] < +\infty$. Therefore thanks to the strong law of large numbers for conditionally L^2-bounded martingale increments, we have $\frac{M_n}{n} \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} 0$. Consequently it follows from (7.9) that

$$\frac{\mathrm{Tr}(Y_n)}{n} = 1 + \frac{\mathrm{Tr}(Y_0) - 1}{n} + \frac{\mathrm{Tr}(M_n)}{n} \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} 1. \tag{7.10}$$

Second Step: Since the components of $\widetilde{Y}_n$ are non-negative and $\mathrm{Tr}(\widetilde{Y}_n) = \frac{\mathrm{Tr}(Y_n)}{n} \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} 1$, it is clear that $(\widetilde{Y}_n)_{n\ge1}$ is $a.s.$ bounded and that $a.s.$ the set $\mathcal{Y}_\infty$ of all its limiting value is contained in

$$\mathcal{V} = \mathrm{Tr}^{-1}\{1\} = \left\{u \in \mathbb{R}_+^d \,|\, \mathrm{Tr}(u) = 1\right\}.$$

So we may try applying the ODE method (see Chapter 1 Theorem 1.7). Since $\widetilde{Y}_n$ and $H_{n+1}\widetilde{Y}_n$ are $a.s.$ bounded, (7.10) and **(A3)** imply that $r_n \overset{a.s.}{\underset{n\to\infty}{\longrightarrow}} 0$.

The ODE associated to the recursive procedure reads

$$ODE_{I_d-H} \equiv \dot{y} = -(I_d - H)y.$$

Owing to Assumption **(A3)**, $I_d - H$ admits v^* as unique zero in $\mathcal{V}$. The restriction of ODE_{I_d-H} to the affine hyperplane $\mathcal{V}$ is the linear system $\dot{z} = -(I_d - H)z$, where $z = y - v^*$ takes values in $\mathcal{V}_0 =$

$\left\{u \in \mathbb{R}^d \,|\, \mathrm{Tr}(u) = 0\right\}$. Since $\mathrm{Sp}\left((I_d - H)_{|\mathcal{V}_0}\right) \subset \{\lambda \in \mathbb{C},\, \mathcal{R}e(\lambda) > 0\}$, owing to Assumption **(A3)**. As a consequence v^* is an uniformly stable equilibrium for the restriction of ODE_{I_d-H} to $\mathcal{V}$, the whole hyperplane, as an attracting area. The fundamental result derived from the ODE method (see Theorem 1.7 in Chapter 1) yields the expected result

$$\widetilde{Y}_n \xrightarrow[n\to\infty]{a.s.} v^*. \qquad \square$$

Remark: If we assume that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy **(A1)**, then we can directly write a stochastic approximation for $\frac{Y_n}{\mathrm{Tr}(Y_n)}$ in which the remainder simply reads $(H_{n+1} - H)\frac{Y_n}{\mathrm{Tr}(Y_n)}$ and prove the $a.s.$ convergence under the same assumptions.

COMMENTS. We could apply directly the ODE method because we first proved that $(\widetilde{Y}_n)_{n\geq 1}$ is $a.s.$ bounded without using the standard Lyapunov machinery developed in SA Theory. That is why the assumption on the remainder sequence $(r_n)_{n\geq 1}$ simply reads

$$r_n \xrightarrow[n\to\infty]{a.s.} 0.$$

Another approach is the martingale one. It relies on the existence of a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}_+$ associated to the algorithm satisfying

$$\exists\, a > 0, \quad \forall y \in \mathbb{R}^d, , \quad y \neq v^*, \quad \langle \nabla V \,|\, I_d - H\rangle\,(y) > 0 \quad \text{and} \quad \langle \nabla V \,|\, I_d - H\rangle > a\,|\nabla V|^2\,. \quad (7.11)$$

In this framework the existence of a Lyapunov function can be established. Hence, the natural condition on the remainder sequence $(r_n)_{n\geq 1}$ reads (see [43])

$$\sum_{n\geq 1} \frac{\|r_n\|^2}{n} < +\infty \; a.s.$$

In that perspective, the assumption on the generating matrices would read $\displaystyle\sum_{n\geq 1} \frac{|||H_n - H|||^2}{n} < +\infty$ $a.s.$ which is still slightly less stringent than assumption on the generating matrices made in [18].

7.2.3 Rate of convergence

In the previous section we proved the $a.s.$ convergence of both quantities of interest, namely $\widetilde{Y}_n$ and $\widetilde{N}_n$, toward v^*. In this section we establish a "joint CLT" for the couple $\theta_n := (\widetilde{Y}_n, \widetilde{N}_n)^t$ with an explicit asymptotic joint normal distribution (including covariances). To this end we will show that θ_n satisfies a SA recursive procedure which ($a.s.$ converges toward $\theta^* = (v^*, v^*)^t$ and) fulfills the assumptions of the CLT Theorem 1.8 for SA algorithms (see Chapter 1), with a special attention paid to Condition (1.24) about the remainder term. As concerns $\widetilde{Y}_n$, we derive from (7.7) that

$$\forall\, n \geq 1, \qquad \widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\left(I_d - (2 - \mathrm{Tr}(\widetilde{Y}_n))H\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + \bar{r}_{n+1}\right),$$

where $$\bar{r}_{n+1} := \left(\frac{H_{n+1} - H}{\mathrm{Tr}(\widetilde{Y}_n)} + \frac{(\mathrm{Tr}(\widetilde{Y}_n) - 1)^2}{\mathrm{Tr}(\widetilde{Y}_n)}H\right)\widetilde{Y}_n.$$

For $\widetilde{N}_n$ we have, still for every $n \geq 1$,

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\left(\widetilde{N}_n - (2 - \mathrm{Tr}(\widetilde{Y}_n))\widetilde{Y}_n\right) + \frac{1}{n+1}\left(\Delta\widetilde{M}_{n+1} + \widetilde{r}_{n+1}\right)$$

with $\Delta\widetilde{M}_{n+1} := X_{n+1} - \mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right] = X_{n+1} - \dfrac{Y_n}{\mathrm{Tr}(Y_n)}$ and $\widetilde{r}_{n+1} := \dfrac{(\mathrm{Tr}(\widetilde{Y}_n)-1)^2}{\mathrm{Tr}(\widetilde{Y}_n)}\widetilde{Y}_n$.
Thus, we obtain a new recursive SA procedure, still with step $\gamma_n = \frac{1}{n}$, namely

$$\theta_{n+1} = \theta_n - \frac{1}{n+1}h(\theta_n) + \frac{1}{n+1}\left(\Delta\mathbf{M}_{n+1} + R_{n+1}\right), \quad n \geq 1,$$

with $\Delta\mathbf{M}_{n+1} := \begin{pmatrix} \Delta M_{n+1} \\ \Delta\widetilde{M}_{n+1} \end{pmatrix}$, $R_{n+1} := \begin{pmatrix} \bar{r}_{n+1} \\ \widetilde{r}_{n+1} \end{pmatrix}$ and

$$\forall\, \theta = \begin{pmatrix} y \\ \nu \end{pmatrix},\ y \in \mathbb{R}^d,\ \nu \in \mathbb{R}^d, \quad h(\theta) := \begin{pmatrix} (I_d - (2 - \mathrm{Tr}(y))H)y \\ \nu - (2 - \mathrm{Tr}(y))y \end{pmatrix} \quad \text{with} \quad h(\theta^*) = 0.$$

The function h is differentiable on $\mathbb{R}^{2d}$ and its differential at point θ^* is given by

$$Dh(\theta^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & 0_{\mathcal{M}_d(\mathbb{R})} \\ v^*\mathbf{1}^t - I_d & I_d \end{pmatrix}.$$

To establish a CLT for the sequence $(\theta_n)_{n\geq 1}$ we need to make the following additional assumptions:

(A4) The addition rules D_n *a.s.* satisfy

$$\forall 1 \leq j \leq d, \quad \begin{cases} \sup_{n\geq 1} \mathbb{E}\left[\|D_n^{\cdot j}\|^{2+\delta} \,|\, \mathcal{F}_{n-1}\right] \leq C < \infty \quad \text{for a } \delta > 0, \\ \mathrm{Cov}\left[D_n^{\cdot j}(D_n^{\cdot j})^t \,|\, \mathcal{F}_{n-1}\right] \underset{n\to\infty}{\longrightarrow} C^j, \end{cases}$$

where $C^j = (C^j_{il})_{1\leq i,l\leq d}$, $j = 1, \ldots, d$, are $d \times d$ positive definite matrices.

(A5) The matrix H satisfies

$$n\,\mathbb{E}\left[|||H_n - H|||^2\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{7.12}$$

Theorem 7.2. *Assume* **(A1)**, **(A3)**, **(A4)** *and* **(A5)** *and*

$$\mathcal{R}e\left(\mathrm{Sp(H)} \setminus \{1\}\right) < 1/2 \tag{7.13}$$

Then, $\theta_n \to \theta^*$ *a.s. as* $n \to \infty$ *and*

$$\sqrt{n}\left(\theta_n - \theta^*\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}\left(0, \Sigma\right) \qquad \textit{with} \qquad \Sigma = \int_0^{+\infty} e^{u(Dh(\theta^*)-\frac{I}{2})}\Gamma e^{u(Dh(\theta^*)-\frac{I}{2})^t}du$$

$$\textit{and} \quad \Gamma = \begin{pmatrix} \displaystyle\sum_{k=1}^{d} v^{*k}C^k & H\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right) \\ \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t \end{pmatrix} = a.s.\text{-}\lim_{n\to\infty} \mathbb{E}\left[\Delta\mathbf{M}_n\Delta\mathbf{M}_n^t \,|\, \mathcal{F}_{n-1}\right]. \tag{7.14}$$

Proof. We will check the three assumptions of the CLT for SA algorithms recalled in Chapter 1 (Theorem 1.8). Firstly, the condition (1.25) on the spectrum of $Dh(\theta^*)$ requested for algorithms with step $\frac{1}{n}$ in Theorem 1.8 reads $\mathcal{R}e\left(\mathrm{Sp}(Dh(\theta^*))\right) > \frac{1}{2}$. This follows from our Assumption (7.13) since by decomposing $\mathbb{R}^d = \mathbb{R}v^* \oplus \mathrm{Ker(Tr)}$, one checks that

$$\mathrm{Sp}(Dh(\theta^*)) = \{1\} \cup \{1 - \lambda,\ \lambda \in \mathrm{Sp}(H) \setminus \{1\}\}.$$

Secondly Assumption **(A4)** ensures that Condition (1.23) is satisfied since

$$\sup_{n\geq 1}\mathbb{E}\left[\|\Delta\mathbf{M}_n\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right]<+\infty\quad a.s.\quad\text{and}\quad\mathbb{E}\left[\Delta\mathbf{M}_n\Delta\mathbf{M}_n^t\,|\,\mathcal{F}_{n-1}\right]\xrightarrow[n\to\infty]{a.s.}\Gamma\quad\text{as}\quad n\to\infty,$$

where Γ is the symmetric nonnegative matrix given by (7.14) as established below. To this end we have to determine three blocks since Γ reads

$$\Gamma=\begin{pmatrix}\Gamma_1 & \Gamma_{12}\\ \Gamma_{12}^t & \Gamma_2\end{pmatrix}\quad\text{where}\quad\Gamma_1,\Gamma_2,\Gamma_{12}\in\mathcal{M}_d(\mathbb{R}).$$

Computation of Γ_1.

$$\begin{aligned}\mathbb{E}\left[\Delta M_{n+1}\Delta M_{n+1}^t\,|\,\mathcal{F}_n\right]&=\sum_{q=1}^{d}\mathbb{P}(X_{n+1}=e^q\,|\,\mathcal{F}_n)\big(\mathbb{E}\left[D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t\,|\,\mathcal{F}_n\right]\\&\quad-\mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]^t\big)\\&=\sum_{q=1}^{d}\frac{Y_n^q}{\mathrm{Tr}(Y_n)}\mathrm{Cov}\left(D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t\,|\,\mathcal{F}_n\right)\xrightarrow[n\to\infty]{a.s.}\Gamma_1=\sum_{q=1}^{d}v^{*q}C^q.\end{aligned}$$

Computation of Γ_2.

$$\begin{aligned}\mathbb{E}\left[\Delta\widetilde{M}_{n+1}\Delta\widetilde{M}_{n+1}^t\,|\,\mathcal{F}_n\right]&=\mathbb{E}\left[X_{n+1}X_{n+1}^t\,|\,\mathcal{F}_n\right]-\frac{Y_n}{\mathrm{Tr}(Y_n)}\left(\frac{Y_n^q}{\mathrm{Tr}(Y_n)}\right)^t\\&=\mathrm{diag}\left(\frac{Y_n}{\mathrm{Tr}(Y_n)}\right)-\frac{Y_n}{\mathrm{Tr}(Y_n)}\left(\frac{Y_n^q}{\mathrm{Tr}(Y_n)}\right)^t\xrightarrow[n\to\infty]{a.s.}\Gamma_2=\mathrm{diag}(v^*)-v^*(v^*)^t.\end{aligned}$$

Computation of Γ_{12}.

$$\begin{aligned}\mathbb{E}\left[\Delta M_{n+1}\Delta\widetilde{M}_{n+1}^t\,|\,\mathcal{F}_n\right]&=\mathbb{E}\left[D_{n+1}X_{n+1}X_{n+1}^t\,|\,\mathcal{F}_n\right]-\mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{E}\left[X_{n+1}\,|\,\mathcal{F}_n\right]^t\\&=\mathbb{E}\left[D_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{E}\left[X_{n+1}X_{n+1}^t\,|\,\mathcal{F}_n\right]\\&\quad-\mathbb{E}\left[D_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{E}\left[X_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{E}\left[X_{n+1}\,|\,\mathcal{F}_n\right]^t\\&=H_{n+1}\mathrm{diag}\left(\frac{Y_n}{\mathrm{Tr}(Y_n)}\right)-H_{n+1}\frac{Y_n}{\mathrm{Tr}(Y_n)}\left(\frac{Y_n}{\mathrm{Tr}(Y_n)}\right)^t\\&\xrightarrow[n\to\infty]{a.s.}\Gamma_{12}=H\left(\mathrm{diag}(v^*)-v^*(v^*)^t\right).\end{aligned}$$

Finally, it remains to check that the remainder sequence $(R_n)_{n\geq 1}$ satisfies (1.24) for an $\epsilon>0$:

$$\mathbb{E}\left[(n+1)\,\|R_{n+1}\|^2\,\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\epsilon\}}\right]\xrightarrow[n\to\infty]{}0.\tag{7.15}$$

We note that $\|R_{n+1}\|^2=\|\bar{r}_{n+1}\|^2+\|\widetilde{r}_{n+1}\|^2$. It follows from the definition of $\bar{r}_{n+1}$ and the elementary facts $\|\widetilde{Y}_n-v^*\|\leq\|\theta_n-\theta^*\|$ and $\mathrm{Tr}(\widetilde{Y}_n)\geq\|\widetilde{Y}_n\|$ that

$$\begin{aligned}\|\bar{r}_{n+1}\|^2\,\mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}&\leq 2\left(\frac{(\mathrm{Tr}(\widetilde{Y}_n)-1)^4}{\frac{\|v^*\|}{2}}+\frac{|||H_{n+1}-H|||^2}{\frac{\|v^*\|}{2}}\right)\frac{3}{2}\|v^*\|\mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}\\&\leq 6\left((\mathrm{Tr}(\widetilde{Y}_n)-1)^4+|||H_{n+1}-H|||^2\right)\mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}.\end{aligned}$$

But $\mathrm{Tr}(\widetilde{Y}_n)-1=\frac{\mathrm{Tr}(\Delta M_n)}{n}$ where $\sup_{n\geq 0}\mathbb{E}\left[|\mathrm{Tr}(\Delta M_{n+1})|^{2+\delta}\,|\,\mathcal{F}_n\right]\leq C'$, $\delta>0$, owing to (A4). Now using that $|\mathrm{Tr}(y)|\leq C_d\|y\|$,

$$\mathbb{E}\left[n\left|\mathrm{Tr}(\widetilde{Y}_n)-1\right|^4\mathbb{1}_{\left\{\|\theta_n-\theta^*\|\leq\frac{\|v^*\|}{2}\right\}}\right]\leq C_\delta^* n\mathbb{E}\left[\left|\mathrm{Tr}(\widetilde{Y}_n)-1\right|^{2+\delta}\right]=\frac{C_d}{n^{1+\delta}}\mathbb{E}\left[|\mathrm{Tr}(\Delta M_n)|^{2+\delta}\right]\leq\frac{C_d'}{n^{1+\delta}},$$

where $C_\delta^* > 0$ is a real constant. Consequently

$$\mathbb{E}\left[\left|\mathrm{Tr}(\widetilde{Y}_n) - 1\right|^4 \mathbb{1}_{\left\{\|\theta_n - \theta^*\| \leq \frac{\|v^*\|}{2}\right\}}\right] = o\left(\frac{1}{n}\right).$$

Thus, by (A5) we obtain

$$\mathbb{E}\left[\|\bar{r}_{n+1}\|^2 \mathbb{1}_{\left\{\|\theta_n - \theta^*\| \leq \frac{\|v^*\|}{2}\right\}}\right] = o\left(\frac{1}{n}\right).$$

The same argument yields $\mathbb{E}\left[\|\widetilde{r}_{n+1}\|^2 \mathbb{1}_{\left\{\|\theta_n - \theta^*\| \leq \frac{\|v^*\|}{2}\right\}}\right] = o\left(\frac{1}{n}\right)$, therefore (7.15) is satisfied. □

7.3 Application to urn models for multi-arm clinical trials

7.3.1 The Wei *GFU* Model

We consider here the model presented in [112] and in [19], where balls are added depending on the success probabilities of each treatment. Define an *efficiency indicator* as follows: let $(T_n^i)_{n\geq 1}$, $1 \leq i \leq d$, be d independent sequences of $[0,1]$-valued i.i.d. random variables, independent of the i.i.d.*sampling* sequence $(U_n)_{n\geq 1}$ so that

$$\mathbb{E}\left[T_n^i\right] = p^i, \quad 0 < p^i < 1, \quad 1 \leq i \leq d. \tag{7.16}$$

Remark. If $(T_n^i)_{n\geq 1}$, $1 \leq i \leq d$, is simply a *success indicator*, namely d independent sequences of i.i.d. $\{0,1\}$-valued Bernoulli trials with respective parameter p^i, then the convention is to set $T_n^i = 1$ to indicate that the response of the i^{th} treatment in the n^{th} trial is a success and $T_n^i = 0$ otherwise.

In this framework one considers the filtration $\mathcal{F}_n = \sigma\left(Y_0, U_k, T_k, 1 \leq k \leq n\right)$, $n \geq 0$. Consider the following addition rules: a success on the treatment i adds a ball of type i to the urn and a failure on the treatment i adds $\frac{1}{d-1}$ balls for each of the other $d-1$ types. Thus the addition rule proposed in [112] is as follows

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{1-T_{n+1}^2}{d-1} & \cdots & \frac{1-T_{n+1}^d}{d-1} \\ \frac{1-T_{n+1}^1}{d-1} & T_{n+1}^2 & \cdots & \frac{1-T_{n+1}^d}{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1-T_{n+1}^1}{d-1} & \frac{1-T_{n+1}^2}{d-1} & \cdots & T_{n+1}^d \end{pmatrix}$$

so that

$$H_{n+1} = \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right] = \mathbb{E}\, D_{n+1} = H = \begin{pmatrix} p^1 & \frac{q^2}{d-1} & \cdots & \frac{q^d}{d-1} \\ \frac{q^1}{d-1} & p^2 & \cdots & \frac{q^d}{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{q^1}{d-1} & \frac{q^2}{d-1} & \cdots & p^d \end{pmatrix},$$

where $q^i = 1 - p^i$, $1 \leq i \leq d$. The strong consistency has been first established in [14], then redone in [18] and in this chapter with Theorem 7.1, and the asymptotic normality

$$\frac{Y_n - nv^*}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

results from Theorem 3.2 in [18] and from Theorem 7.2 of this chapter where we obtain a joint CLT for $(\widetilde{Y}_n, \widetilde{N}_n)$. Furthermore we know that

$$v^{*i} = \frac{1/q^i}{\sum_{j=1}^d 1/q^j}, \quad 1 \leq i \leq d.$$

Note that if $p^i > p^j$, then $v^{*i} > v^{*j}$. Hence the components v^{*i} are ordered according to the increasing efficiency p^i of the treatments. Furthermore, it is clear that, if $p^i \uparrow 1$ and all other probabilities p^j stand still, then

$$\lim_{p^i \to 1} v^{*j} = \delta_{ij}$$

where δ_{ij} denotes the Kronecker symbol. Consequently, since v^{*i} is the asymptotic probability of assigning treatment i to a patient, the procedure asymptotically allocates more patients to the most efficient treatment(s). Following the practitioners, the fact that a marginal allocation of less efficient treatments is preserved is justified by some comparison matter.

However this model only takes into account in the addition rule matrix D_n the response of the n^{th} patient without considering the ones of past patients. This led the author of [19] to introduce a new model based on statistical observations of the efficiency of the assigned treatments to past patients.

7.3.2 The Bai-Hu-Shen *GFU* Model

We consider now the model introduced in [19] (and considered again in [18]) where $(T_n^i)_{n\geq 1}$,$1 \leq i \leq d$, are d independent sequences of i.i.d. $\{0,1\}$-valued Bernoulli trials satisfying (7.16) and the filtration $(\mathcal{F}_n)_{n\geq 0}$ is defined as in the previous section. Let $N_n = (N_n^1, \ldots, N_n^d)^t$ and $S_n = (S_n^1, \ldots, S_n^d)^t$, where $N_n^i = N_{n-1}^i + X_n^i$, $n \geq 1$, still denotes the number of times the i^{th} treatment is selected among the first n stages and

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \quad n \geq 1,$$

denotes the *number of successes* of the i^{th} treatment among these N_n^i trials, $i = 1, \ldots, d$. However, to avoid degeneracy of the procedure, we will make the following initialization assumption

$$N_0^i = 1, \ S_0^i = 1, \quad i = 1, \ldots, d$$

(which makes the above interpretation of these quantities correct "up to one unit").

Remark. Like with the Wei model, we can simply assume that T_n^i is a $[0,1]$-valued efficiency indicator.

Define $Q_n = (Q_n^1, \ldots, Q_n^d)^t$, where $Q_n^i = \frac{S_n^i}{N_n^i}$, $i = 1, \ldots, d$ and $E_n = \sum_{i=1}^d Q_n^i$. In [19] the authors consider the following addition rule matrices,

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{Q_n^1(1-T_{n+1}^2)}{E_n - Q_n^2} & \cdots & \frac{Q_n^1(1-T_{n+1}^d)}{E_n - Q_n^d} \\ \frac{Q_n^2(1-T_{n+1}^1)}{E_n - Q_n^1} & T_{n+1}^2 & \cdots & \frac{Q_n^2(1-T_{n+1}^d)}{E_n - Q_n^d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{Q_n^d(1-T_{n+1}^1)}{E_n - Q_n^1} & \frac{Q_n^d(1-T_{n+1}^2)}{E_n - Q_n^2} & \cdots & T_{n+1}^d \end{pmatrix},$$

i.e. at stage $n+1$, if the response of the i^{th} treatment is a success, then one ball of type i is added in the urn. Otherwise, $\frac{Q_n^j}{E_n - Q_n^i}$ (virtual) balls of type j, $j \neq i$, are added. Then, one easily checks that the generating matrices are given by

$$H_{n+1} = \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right] = \begin{pmatrix} p^1 & \frac{Q_n^1}{E_n - Q_n^2} q^2 & \cdots & \frac{Q_n^1}{E_n - Q_n^d} q^d \\ \frac{Q_n^2}{E_n - Q_n^1} q^1 & p^2 & \cdots & \frac{Q_n^2}{E_n - Q_n^d} q^d \\ \vdots & \vdots & \ddots & \vdots \\ \frac{Q_n^d}{E_n - Q_n^1} q^1 & \frac{Q_n^d}{E_n - Q_n^2} q^2 & \cdots & p^d \end{pmatrix}.$$

As soon as $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$, $H_n \xrightarrow{a.s.} H$ (see Lemma 7.1 below or [19] when $Y_0 \in (0,\infty)^d$) where

$$H = \begin{pmatrix} p^1 & \frac{p^1}{E - p^2} q^2 & \cdots & \frac{p^1}{E - p^d} q^d \\ \frac{p^2}{E - p^1} q^1 & p^2 & \cdots & \frac{p^2}{E - p^d} q^d \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^d}{E - p^1} q^1 & \frac{p^d}{E - p^2} q^2 & \cdots & p^d \end{pmatrix} \quad \text{where } E = p^1 + \cdots + p^d.$$

The matrix H is clearly irreducible since $0 < p^i < 1$, $1 \leq i \leq d$. Then Theorem 7.1 (or following the direct proof from [19]) we obtain

$$\widetilde{Y}_n = \frac{Y_n}{n} \xrightarrow[n\to\infty]{a.s.} v^* \quad \text{and} \quad \widetilde{N}_n = \frac{N_n}{n} \xrightarrow[n\to\infty]{a.s.} v^*. \tag{7.17}$$

Note that the normalizes maximal eigenvector v^* (associated to the eigenvalue 1) is given by

$$v^{*i} = \frac{p^i \, \frac{E - p^i}{1 - p^i}}{\sum_{1 \leq j \leq d} p^j \, \frac{E - p^j}{1 - p^j}}, \quad i = 1, \ldots, d.$$

Note that if $p^i > p^j$, $\frac{p^i}{p^j} \frac{E - p^i}{E - p^j} > 1$ and $\frac{1 - p^j}{1 - p^i} > 1$ so that $v^{*i} > v^{*j}$. Hence the entries v^{*i} are ordered according to the increasing efficiency p^i of the treatments. This model can be considered as more ethical than the Wei model since a better treatment will be administrated to more patients. Indeed, when $d > 2$, for any $i \neq j$, $1 \leq i, j \leq d$, if $p^i > p^j$,

$$\frac{v_{BHS}^{*i}}{v_{BHS}^{*j}} > \frac{v_W^{*i}}{v_W^{*j}} > 1$$

(when $d = 2$ both matrices H coincide).

Remark. Note that in that model the "balls" in the urn become virtual since there exists no $N \in \mathbb{N}$ such that, for every $n \geq 1$, $ND_n \in \mathcal{M}_d(\mathbb{N})$.

7.3.3 Asymptotic normality for multi-arm clinical trials for the BHS *GFU* model

In order to derive a *CLT*, not with the bias $\mathbb{E}Y_n$ but with nv^*, from their own general asymptotic normality result (which statement is similar to Theorem 7.2) they need to fulfill the following

convergence rate assumption for H_n

$$\sum_{n\geq 1} \frac{|||H_n - H|||_\infty}{\sqrt{n}} < \infty \quad a.s. \tag{7.18}$$

In [19], the $a.s.$ rate of decay $|||H_n - H|||_\infty = o(n^{-\frac{1}{4}})$ is shown which is clearly not fast enough to fulfill (7.18).

However, by enlarging the dimension of the structure process of the procedure by considering the $3d$-dimensional random sequence

$$\widetilde{\theta}_n = \begin{pmatrix} \widetilde{Y}_n \\ \widetilde{N}_n \\ \widetilde{S}_n \end{pmatrix} \quad \text{where} \quad \widetilde{S}_n = \frac{S_n}{n}, \quad n \geq 1,$$

we will establish that a CLT does hold for the BHS GFU model.

The first step is to notice that the generating matrix H_{n+1} can may be written as a function depending on $\widetilde{S}_n$ and $\widetilde{N}_n$, *i.e.* $H_{n+1} = \Phi(\widetilde{S}_n, \widetilde{N}_n)$, where $\Phi : \mathbb{R}_+^d \times (0,\infty)^d \to \mathcal{M}_d(\mathbb{R})$ is a differentiable function defined by

$$\Phi(s,\nu) = \left(\Phi^{ij}(s,\nu)\right)_{1\leq i,j\leq d} \quad \text{where} \quad \left\{ \begin{array}{ll} \Phi^{ii}(s,\nu) = p^i & 1 \leq i \leq d \\ \Phi^{ij}(s,\nu) = \frac{s^i/\nu^i}{\sum_{k\neq j} s^k/\nu^k}\, q^j & 1 \leq i,j \leq d,\ i \neq j. \end{array} \right.$$

Then the following strong consistency and CLT hold for $(\widetilde{\theta}_n)_{n\geq 1}$.

Theorem 7.3. *Assume that* **(A1)**, **(A4)**, **(A5)** *and (7.13) hold. Then, as soon as* $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$,

$$\widetilde{\theta}_n \xrightarrow[n\to\infty]{a.s.} \widetilde{\theta}^* \quad \text{and} \quad \sqrt{n}\left(\widetilde{\theta}_n - \widetilde{\theta}^*\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \widetilde{\Sigma}\right),$$

where

$$\widetilde{\theta}^* := (v^*, v^*, \mathrm{diag}(p)v^*)^t, \quad \widetilde{\Sigma} = \int_0^{+\infty} e^{u(D\widetilde{h}(\widetilde{\theta}^*)-\frac{I}{2})}\widetilde{\Gamma}e^{u(D\widetilde{h}(\widetilde{\theta}^*)-\frac{I}{2})^t}du$$

with $\widetilde{\Gamma} = a.s.\ \lim_{n\to+\infty} \mathbb{E}\left[\Delta\widetilde{\mathbf{M}}_{n+1}\Delta\widetilde{\mathbf{M}}_{n+1}^t \,|\, \mathcal{F}_n\right]$ *(with* $(\Delta\widetilde{\mathbf{M}}_n)_{n\geq 1}$ *a sequence of martingale increments defined in the proof below) reads*

$$\widetilde{\Gamma} = \begin{pmatrix} \displaystyle\sum_{k=1}^d v^{*k}C^k & H\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right) & \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)\mathrm{diag}(p) \\ \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t & \left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)\mathrm{diag}(p) \\ \mathrm{diag}(p)\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t & \mathrm{diag}(p)\left(\mathrm{diag}(v^*) - v^*(v^*)^t\right)^t & \mathrm{diag}(p)\left(v^* - v^*v^{*t}\mathrm{diag}(p)\right) \end{pmatrix}$$

and

$$D\widetilde{h}(\widetilde{\theta}^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & -\frac{\partial}{\partial\nu}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} & -\frac{\partial}{\partial s}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} \\ v^*\mathbf{1}^t - I_d & I_d & 0_{\mathcal{M}_d(\mathbb{R})} \\ diag(p)\left(v^*\mathbf{1}^t - I_d\right) & 0_{\mathcal{M}_d(\mathbb{R})} & I_d \end{pmatrix}$$

which is invertible.

Proof. Step 1(*Strong consistency*). We will show with Lemma 7.1 that $\widetilde{S}_n \xrightarrow[n\to\infty]{a.s.} \mathrm{diag}(p)v^*$ and that $H_n \xrightarrow[n\to\infty]{a.s.} H$ so that, using what precedes $\widetilde{\theta}_n \xrightarrow[n\to\infty]{a.s.} \widetilde{\theta}^*$.

Lemma 7.1. *If the assumptions (7.1), (7.3), (7.4) hold and $Y_0 \in \mathbb{R}_+^d$ such that $\mathrm{Tr}(Y_0) > 0$, then,*

$$Q_n \xrightarrow{a.s.} p = (p^1, \ldots, p^d) \quad \text{as } n \to \infty$$

so that Assumption (7.5) holds i.e. $H_n \xrightarrow[n\to\infty]{a.s.} H$.

Remark. If we assume that $Y_0^i > 0$, $1 \le i \le d$, then we can prove that $\lim_n N_n^i = +\infty$ *a.s.*, $1 \le i \le d$, faster than below by using that $Y_n^i \ge Y_0^i$, $1 \le i \le d$, $n \ge 1$. The following proof consider the more general case where $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$.

Proof of the lemma. *Step 1.* It follows from the dynamics (7.1) and the definition of D_{n+1} and H_{n+1} that, for every $n \ge 0$, $\mathrm{Tr}(Y_n) = \mathrm{Tr}(Y_0) + n$ and that, for every $i \in \{1, \ldots, d\}$,

$$Y_{n+1}^i = Y_n^i + \sum_{j=1}^d H_{n+1}^{ij} \frac{Y_n^i}{\mathrm{Tr} Y_n} + \Delta M_{n+1}^i$$

where $(\Delta M_n^i)_{n\ge1}$ is a sequence of martingale increments satisfying $\sup_n \mathbb{E}\left[|\Delta M_n^i|^2 \,|\, \mathcal{F}_{n-1}\right] < +\infty$ owing to (7.4). Now using that $S_0^i = N_0^i = 1$ by convention, one derives that

$$\forall\, i \ne j,\ H_{n+1}^{ij} \ge \frac{\kappa_0}{n}, \quad \text{with} \quad \kappa_0 = \frac{1}{2d} \min_{1\le i\le d} \left(p^i, q^i\right) > 0$$

so that, using that $H_{n+1}^{ii} = p^i$, there exists a deterministic integer n_0 such that for every $n \ge n_0$,

$$\begin{aligned} Y_{n+1}^i &\ge \Big(1 + \frac{p_i}{n} - \frac{\kappa_0}{n\mathrm{Tr} Y_n}\Big) Y_n^i + \frac{\kappa_0}{n} + \Delta M_{n+1}^i \\ &\ge \Big(1 + \frac{p_i}{2\mathrm{Tr}(Y_n)}\Big) Y_n^i + \frac{\kappa_0}{n} + \Delta M_{n+1}^i. \end{aligned}$$

Standard computations show that, setting $a_n^i = \prod_{k=n_0}^{n-1} (1 + \frac{p_i}{2\mathrm{Tr}(Y_n)}\Big)$, $i = 1, \ldots, d$,

$$\forall\, n \ge n_0, \quad \frac{Y_n^i}{a_n^i} \ge \frac{Y_{n_0}^i}{a_{n_0}^i} + \sum_{k=n_0+1}^n \frac{\kappa_0}{a_k^i} + \sum_{k=n_0+1}^n \frac{\Delta M_k^i}{a_k^i}$$

Since there exists κ_1, $\kappa_2 > 0$ such that $\kappa_1 n^{\frac{p^i}{2}} \le a_n^i \le \kappa_2 n^{\frac{p^i}{2}}$, one has

$$\forall\, \eta > 0, \quad \sum_{k=n_0+1}^n \frac{\Delta M_k^i}{a_k^i} = o\big(n^{\frac{1-p^i+\eta}{2}}\big).$$

Finally, there exists a positive real constant c' such that, for every $i = 1, \ldots, d$,

$$Y_n^i \ge c' n^{\frac{p^i}{2}} \sum_{k=n_0+1}^n k^{-\frac{p^i}{2}} + o\big(n^{\frac{1+\eta}{2}}\big)$$

so that

$$\forall\, i \in \{1, \ldots, d\}, \quad \liminf_n \widetilde{Y}_n^i \ge c' \int_0^1 u^{-\frac{p^i}{2}}\, du > 0$$

and, as a consequence, $\sum_{n\geq 1} \widetilde{Y}_n^i = +\infty$-$a.s.$ Now using that for every $i = 1, \ldots, d$,

$$N_n^i = \sum_{k=1}^{n} \mathbf{1}_{\{X_k = e^i\}} \quad \text{and} \quad \mathbb{P}(X_n = e^i \,|\, \mathcal{F}_{n-1}) = \widetilde{Y}_{n-1}^i \left(1 - \frac{\mathrm{Tr}(Y_0)}{\mathrm{Tr}(Y_{n-1})}\right), \quad n \geq 1,$$

we get by the conditional Borel-Cantelli Lemma that $N_\infty^i = \lim_n N_n^i = +\infty$ $a.s.$

Step 2. First we note that

$$Q_n^i = \frac{\sum_{k=1}^n T_k^i \Delta N_k^i}{N_n^i}$$

and we introduce the sequence $(\widetilde{Q}_n)_{n\geq 1}$ defined by

$$\widetilde{Q}_n^i = \sum_{k=1}^{n} (T_k^i - p^i) \frac{\Delta N_k^i}{N_{k-1}^i + 1}, \quad n \geq 1.$$

It is an $\mathcal{F}_n$-martingale since, T_k^i being independent of $\mathcal{F}_{k-1}$ and X_k,

$$\mathbb{E}\Big((T_k^i - p^i)\Delta N_k^i \,|\, \mathcal{F}_{k-1}\Big) = \mathbb{E}(T_k^i - p^i)\mathbb{P}(X_k = e^i \,|\, \mathcal{F}_{k-1}) = 0.$$

It has bounded increments since $|T_k^i - p^i| \leq 1$ and

$$\langle \widetilde{Q}^i \rangle_n \leq \sum_{k=1}^{n} \frac{\mathbb{E}((\Delta N_k^i)^2 \,|\, \mathcal{F}_{k-1})}{(N_{k-1}^i + 1)^2}.$$

It follows, using $(\Delta N_k^i)^2 = \Delta N_k^i$, that, for every $n \geq 1$,

$$\mathbb{E}\langle \widetilde{Q}^i \rangle_n \;\leq\; \mathbb{E}\Big(\sum_{k=1}^{n} \frac{\Delta N_k^i}{(N_{k-1}^i + 1)^2}\Big) \leq \mathbb{E}\Big(\sum_{k=1}^{n} \frac{\Delta N_k^i}{N_{k-1}^i N_k^i}\Big) \leq \frac{1}{N_0^i} = 1.$$

Consequently $\widetilde{Q}_n^i \to \widetilde{Q}_\infty^i \in L^1(\mathbb{P})$ $a.s.$ as $n \to \infty$. This in turn implies by Kronecker's Lemma that

$$Q_n^i \xrightarrow{a.s.} p^i \quad \text{as} \quad n \to \infty$$

since $N_n^i \to \infty$ by the first step. □

It follows from the lemma and Theorem 7.1 that $(\widetilde{Y}_n, \widetilde{N}_n) \to (v^*, v^*)$. Furthermore $\mathrm{diag}(\widetilde{S}_n) = \mathrm{diag}(Q_n)\widetilde{N}_n \to \mathrm{diag}(p)v^* = u^*$ so that $\widetilde{\theta}_n \to \widetilde{\theta}^*$ as $n \to \infty$.

Step 2 (*Asymptotic normality*). We will show now that $(\widetilde{\theta}_n)_{n\geq 1}$ satisfies an appropriate recursion to apply Theorem 1.8 (*CLT*). First, we write a recursive procedure for $\widetilde{S}_n$. Having in mind that $S_n = 1 + \sum_{1\leq k\leq n} \mathrm{diag}(T_k)X_k$, we get

$$\begin{aligned}
\widetilde{S}_{n+1} &= \widetilde{S}_n - \frac{1}{n+1}\Big(\widetilde{S}_n - \mathrm{diag}(T_{n+1})X_{n+1}\Big) \\
&= \widetilde{S}_n - \frac{1}{n+1}\left(\widetilde{S}_n - \mathrm{diag}(p)\frac{\widetilde{Y}_n}{\mathrm{Tr}(\widetilde{Y}_n)}\right) + \frac{1}{n+1}\Delta\widehat{M}_{n+1} \\
&= \widetilde{S}_n - \frac{1}{n+1}\Big(\widetilde{S}_n - \mathrm{diag}(p)(2 - \mathrm{Tr}(\widetilde{Y}_n))\widetilde{Y}_n\Big) + \frac{1}{n+1}\Big(\Delta\widehat{M}_{n+1} + \widehat{r}_{n+1}\Big) \qquad (7.19)
\end{aligned}$$

where $\Delta\widehat{M}_{n+1} := \mathrm{diag}(T_{n+1})X_{n+1} - \mathbb{E}\left[\mathrm{diag}(T_{n+1})X_{n+1} \,|\, \mathcal{F}_n\right] = \mathrm{diag}(T_{n+1})X_{n+1} - \mathrm{diag}(p)\dfrac{\widetilde{Y}_n}{\mathrm{Tr}(Y_n)}$

is an $\mathcal{F}_n$-martingale increment and $\widehat{r}_{n+1} = \mathrm{diag}(p)\frac{\left(\mathrm{Tr}(\widetilde{Y}_n)-1\right)^2}{\mathrm{Tr}(\widetilde{Y}_n)}\widetilde{Y}_n$. Then we rewrite the dynamics satisfied by $\widetilde{Y}_n$ as follows

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\left(I_d - (2-\mathrm{Tr}(\widetilde{Y}_n))H_{n+1}\right)\widetilde{Y}_n + \frac{1}{n+1}\left(\Delta M_{n+1} + \check{r}_{n+1}\right), \tag{7.20}$$

where $\check{r}_{n+1} := \dfrac{\left(\mathrm{Tr}(\widetilde{Y}_n)-1\right)^2}{\mathrm{Tr}(\widetilde{Y}_n)}H_{n+1}\widetilde{Y}_n$. Finally, we get the following recursive procedure for $\widetilde{\theta}_n$

$$\widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \frac{1}{n+1}\widetilde{h}(\widetilde{\theta}_n) + \frac{1}{n+1}\left(\Delta\widetilde{\mathbf{M}}_{n+1} + \widetilde{R}_{n+1}\right), \quad n \geq 1,$$

where, for every $\widetilde{\theta} = (y,\nu,s)^t \in \mathbb{R}_+^{3d}$,

$$\widetilde{h}(\widetilde{\theta}) := \begin{pmatrix} (I_d - (2-\mathrm{Tr}(y))\Phi(s,\nu))y \\ \nu - (2-\mathrm{Tr}(y))y \\ s - (2-\mathrm{Tr}(y))\mathrm{diag}(p)y \end{pmatrix}, \ \Delta\widetilde{\mathbf{M}}_{n+1} := \begin{pmatrix} \Delta M_{n+1} \\ \Delta\widetilde{M}_{n+1} \\ \Delta\widehat{M}_{n+1} \end{pmatrix} \text{ and } \widetilde{R}_{n+1} := \begin{pmatrix} \check{r}_{n+1} \\ \widetilde{r}_{n+1} \\ \widehat{r}_{n+1} \end{pmatrix}.$$

The function Φ being differentiable at the equilibrium point $\widetilde{\theta}^*$, we have

$$D\widetilde{h}(\widetilde{\theta}^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & -\frac{\partial}{\partial\nu}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} & -\frac{\partial}{\partial s}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} \\ v^*\mathbf{1}^t - I_d & I_d & 0_{\mathcal{M}_d(\mathbb{R})} \\ \mathrm{diag}(p)\left(v^*\mathbf{1}^t - I_d\right) & 0_{\mathcal{M}_d(\mathbb{R})} & I_d \end{pmatrix}$$

which is invertible since by Schur complement we have $\det(D\widetilde{h}(\widetilde{\theta}^*)) = \det(I_d - H + v^*\mathbf{1}^t)$ thanks to $\frac{\partial}{\partial\nu}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*} = -\mathrm{diag}(p)\frac{\partial}{\partial s}\left(\Phi(s,\nu)y\right)_{|\widetilde{\theta}=\widetilde{\theta}^*}$.

At this stage, the proof follows the lines of that of Theorem 7.2: the computation of the covariance matrix $\widetilde{\Gamma}$ and the treatment of the remainder term uses the same tools as before. The details are left to the reader. □

Remarks.

- The asymptotic variances of $\widetilde{Y}_n$ and $\widetilde{N}_n$ in Theorem 7.3 are different from those in Theorem 7.2 because the differential matrices $Dh(\theta^*)$ and $D\widetilde{h}(\widetilde{\theta}^*)$ are not the same.
- In the SA Theory, there also exists some results on the rate of convergence without CLT. For the sake of simplicity, in this chapter we only treat the case where $L > \frac{1}{2}$, with $L = \min\{\mathcal{R}e(\lambda);\ \lambda \in \mathrm{Sp}\left(Dh(\theta^*)\right)\}$. When $L = \frac{1}{2}$, $\sqrt{n}$ can be replaced by $\sqrt{\frac{n}{(\ln n)^\alpha}}$, $\alpha > 0$, and when $L < \frac{1}{2}$, we have an $a.s.$ convergence toward a finite random variable at a rate n^{-L} (see [44]).

Corollary 7.1. *Under the assumptions of Theorem 7.3,*

$$\sqrt{n}\big(H_n - H\big) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0;\Gamma_H)$$

where Γ_H is a $d^2 \times d^2$ matrix given by $\Gamma_H = D\Phi(u^*,v^*)[\widetilde{\Sigma}_{i+d,j+d}]_{1\leq i,j\leq 2d}D\Phi(u^*,v^*)^t$.

Proof. This is an easy consequence of the so-called Δ-method since

$$H_n = \Phi(\widetilde{S}_n, \widetilde{N}_n) = \Phi(u^*, v^*) + D\Phi(u^*, v^*).(\widetilde{S}_n - u^*, \widetilde{N}_n - v^*) + \|(\widetilde{S}_n - u^*, \widetilde{N}_n - v^*)\|\varepsilon(\widetilde{S}_n, \widetilde{N}_n)$$

with $\lim_{y\to(u^*,v^*)} \varepsilon(y) = 0$. Consequently

$$\sqrt{n}\big(H_n - H\big) = D\Phi(u^*, v^*).(\sqrt{n}(\widetilde{S}_n - u^*), \sqrt{n}(\widetilde{N}_n - v^*)) + \varepsilon_{\mathbb{P}}(n)$$

where $\varepsilon_{\mathbb{P}}(n)$ goes to 0 in probability (as the product of a tight sequence and an $a.s.$ convergent sequence). This concludes the proof. □

Remark. This corollary shows *a posteriori* that it was hopeless to try applying Theorem 7.2 in its standard form to establish asymptotic normality for multi-arm clinical trials since the assumption $\mathbb{E}|||H_n - H|||^2 = o(n^{-1})$ cannot be satisfied. Our global SA approach breaks the vicious circle.

Numerical Example: BHS model. We consider the case $d = 2$, so v^* as the same form as in the example in Subsection 7.2.3.

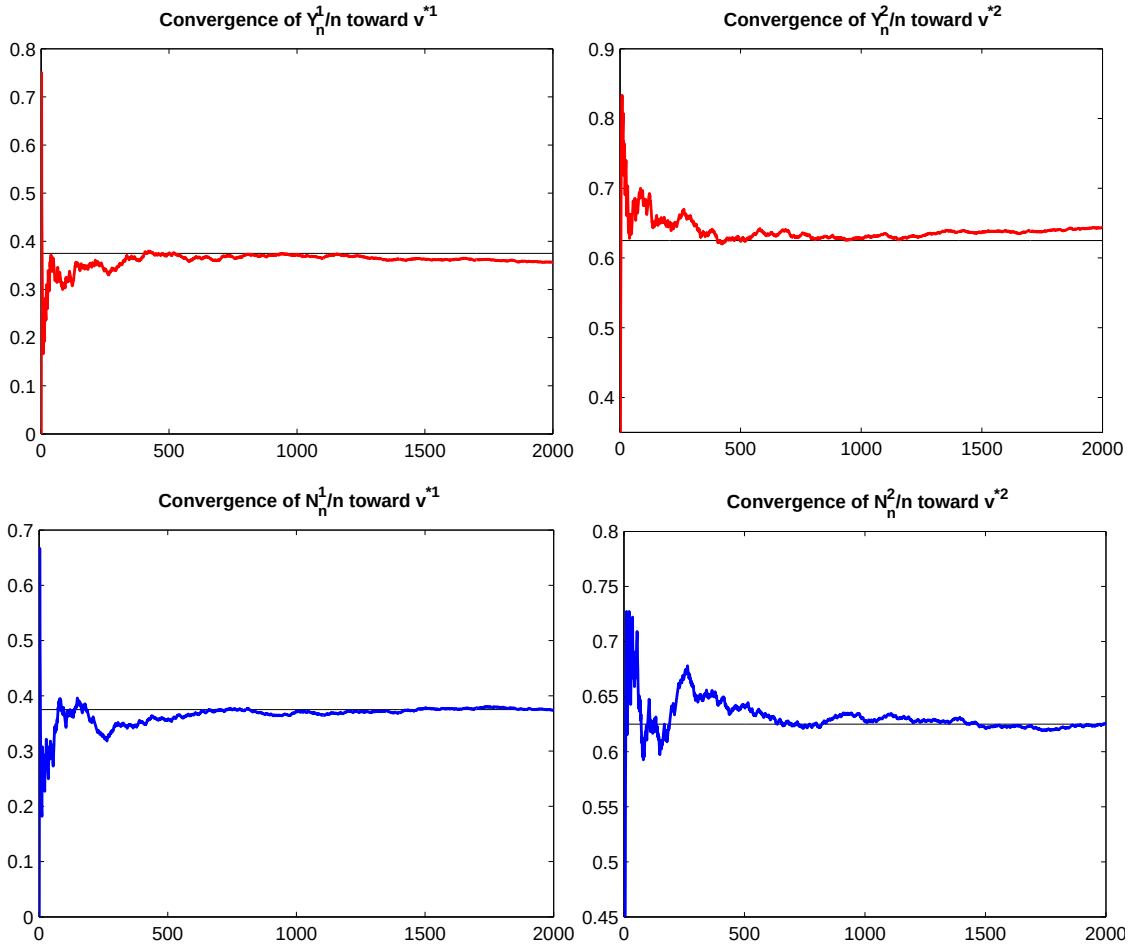

Figure 7.1: Convergence of $\frac{Y_n}{n}$ toward v^* (up-windows) and of $\frac{N_n}{n}$ toward v^* (down-windows): $d = 2$, $n = 2.10^3$, $p^1 = 0.5$, $p^2 = 0.7$, $Y_0 = (0.5, 0.5)^t$ and $N_0 = (1, 1)^t$.

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