Stochastic Variational Inequalities and Applications to Random Vibrations and Mechanical Structures

PhD defense

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- Remerciements aux examinateurs : Pierre Bernard (Université Blaise Pascal, Clermont Ferrand) et George Papanicolaou (Stanford University)

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- Random vibrations of mechanical structures
  $\longrightarrow$ **stochastic system with memory**

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- **Mathematical Interest:** Theoretical and numerical aspects of Stochastic Variational Inequalities (SVI)

Outline of the presentation

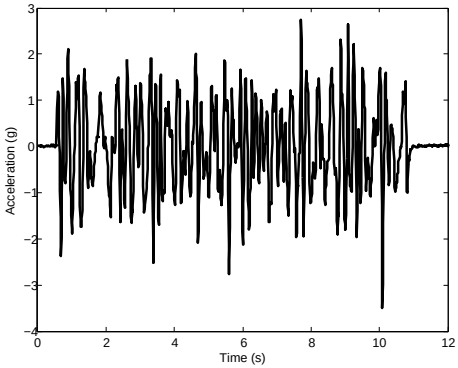
Motivations: Elasto-plastic problems in probabilistic engineering mechanics

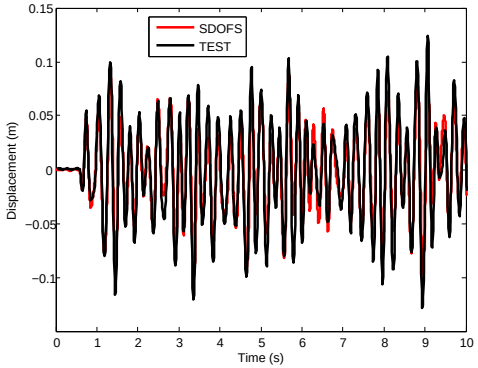
Illustrative example: piping system

For a class of structures: A one dimensional (1d) model

- global behavior of the structure

Seismic excitation (left) / Mass displacement (right): Test *vs* 1d model results





- nonlinear oscillator with memory

Elastic behavior: start with a linear oscillator ...

$c_0 > 0$: damping coefficient

$k > 0$: stiffness

$\text{“}\frac{dw(t)}{dt}\text{”}$: external force white noise

$x(t)$: response of the oscillator

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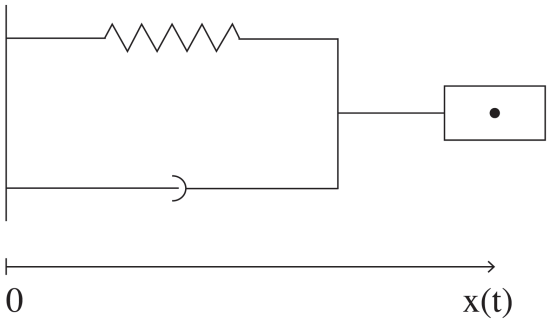
$x(t)$: response of the oscillator

linear case: $x(t)$ solves

$$\ddot{x}(t) + c_0\dot{x}(t) + F(t) = \text{“}\frac{dw(t)}{dt}\text{”}, \quad F(t) = kx(t)$$

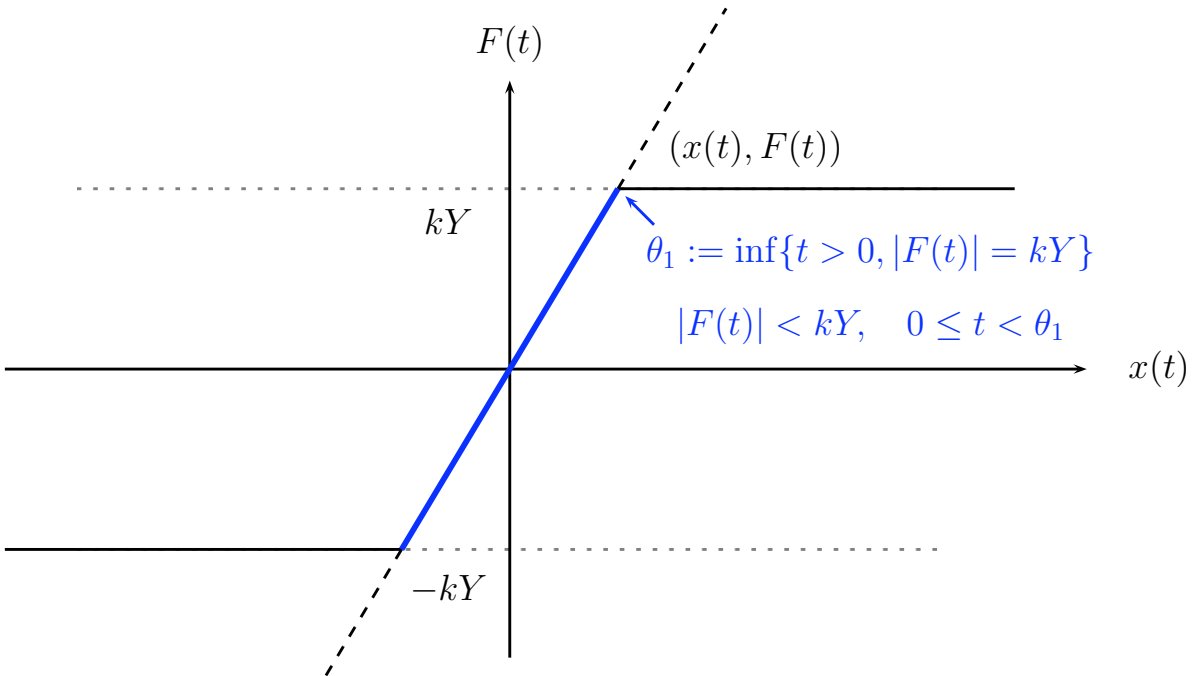


Elastic-perfectly-plastic behavior

elasto-perfectly-plastic case: $x(t)$ solves

$$\ddot{x}(t) + c_0\dot{x}(t) + F(t) = \text{“}\frac{dw(t)}{dt}\text{”}$$

- $|F(t)| \leq kY$, Y : elasto-plastic bound
  θ_1 : first time going into plastic phase



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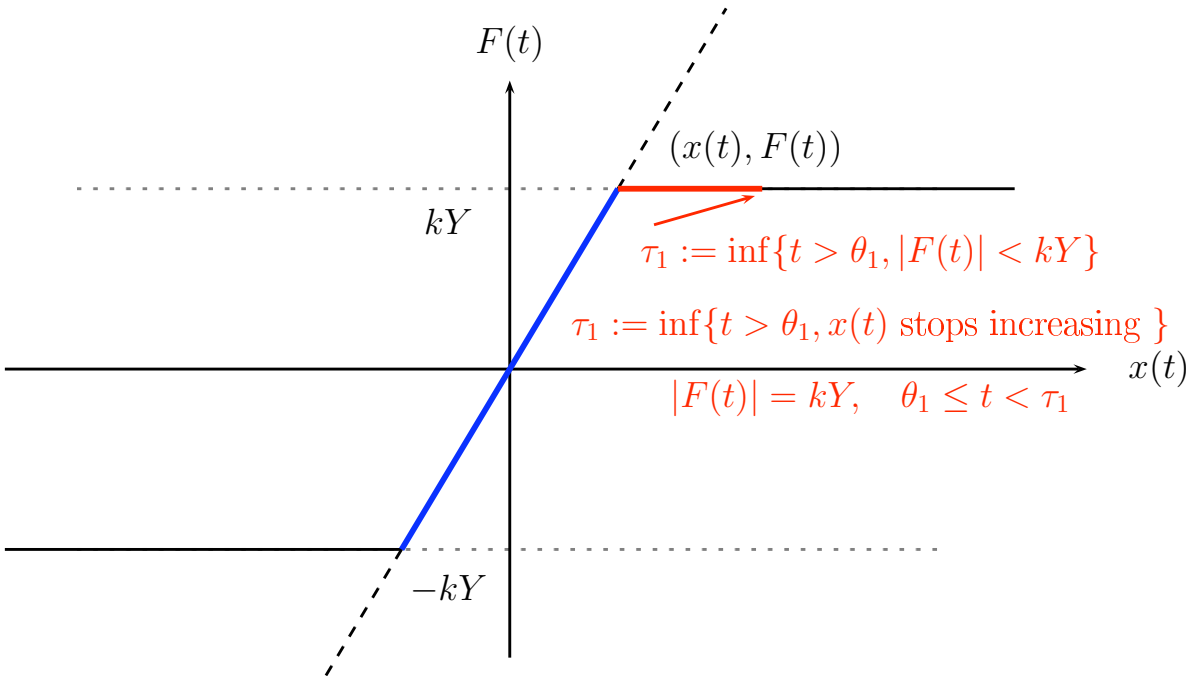
$$\ddot{x}(t) + c_0\dot{x}(t) + F(t) = \text{“}\frac{dw(t)}{dt}\text{”}, \qquad F(t) = k(x(t) - \Delta(t))$$

- $\longrightarrow$ a plastic deformation $\Delta(t)$ occurs in $x(t)$ when $|F(t)| = kY$.
  τ_1: first time going out of plastic phase



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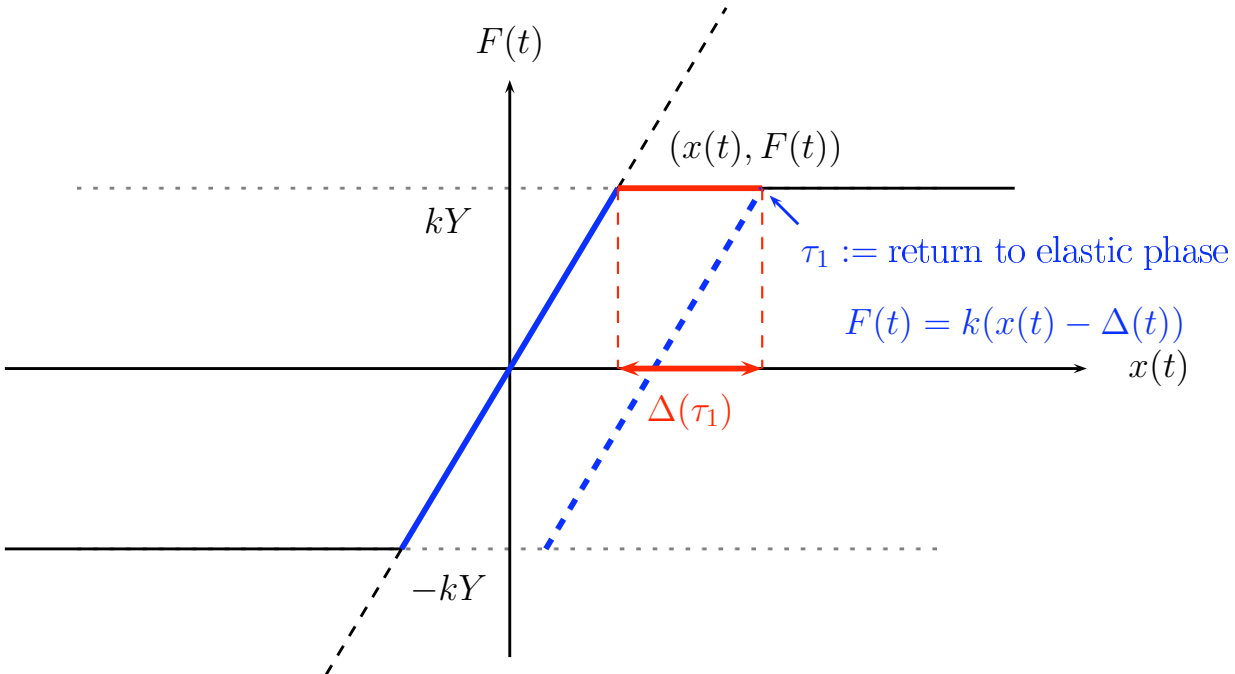
$\Delta(t)$ stops increasing
when $x(t)$ stops increasing.



Balance between elastic and plastic

Denote

$$y(t) = \dot{x}(t) \quad , \quad z(t) = x(t) - \Delta(t)$$

then

$$\ddot{x}(t) + c_0\dot{x}(t) + kz(t) = \text{“}\frac{dw(t)}{dt}\text{”}$$

becomes

- **elastic** $|z(t)| < Y$**:**

$$\begin{cases} \dot{y}(t) &= -(c_0 y(t) + kz(t)) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ \dot{z}(t) &= y(t) \end{cases}$$

- **plastic** $z(t) = Y, y(t) > 0$ **or** $z(t) = -Y, y(t) < 0$**:**

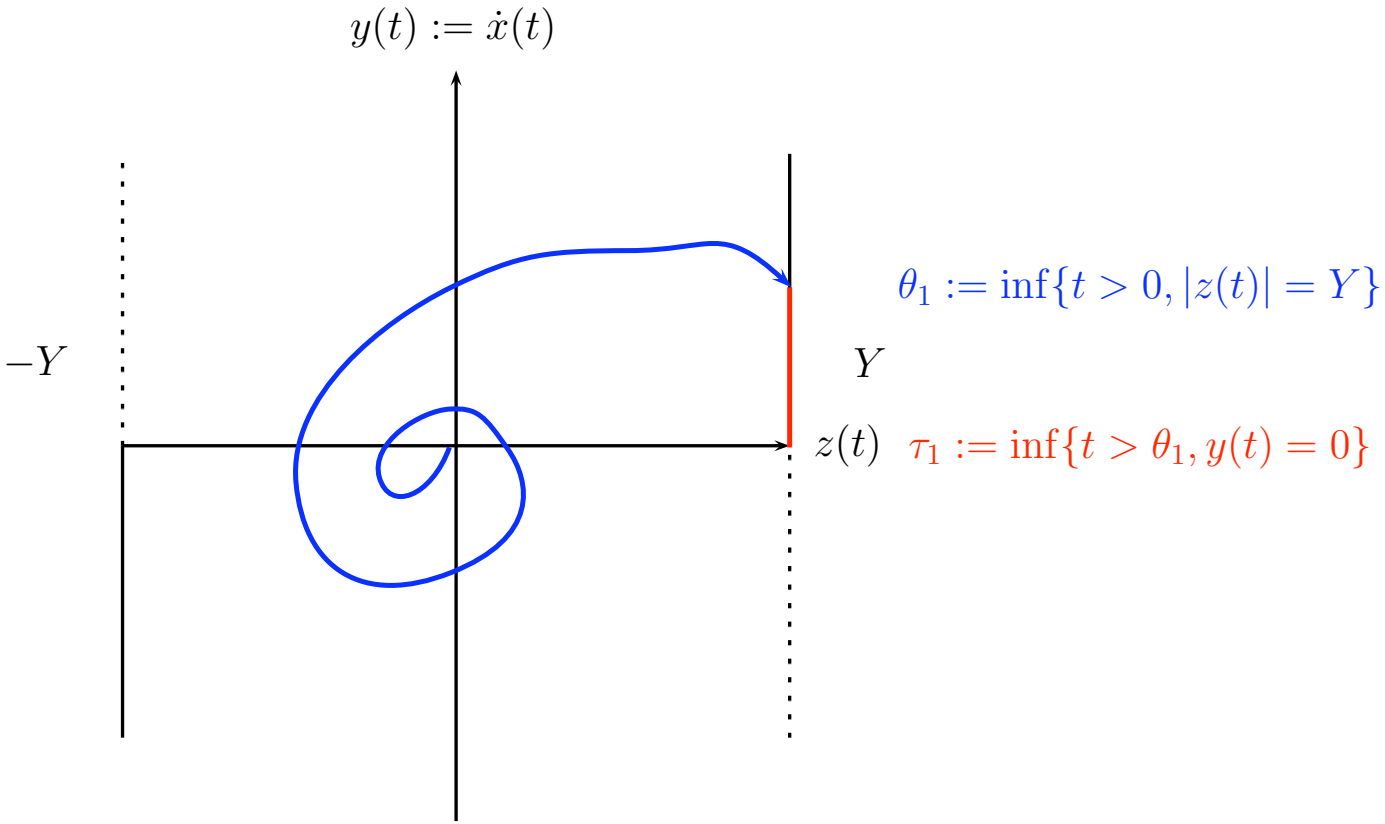
$$\begin{cases} \dot{y}(t) &= -(c_0 y(t) \pm kY) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ \dot{z}(t) &= 0 \end{cases}$$

Key idea: Switching between elastic and plastic phases

An example of phases transition

$$\begin{cases} 1^{st} \text{ elastic phase: } [0, \theta^1) \\ \dot{y}(t) = -(c_0 y(t) + k z(t)) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ \dot{z}(t) = y(t) \end{cases}$$

$$\begin{cases} 1^{st} \text{ plastic phase: } [\theta^1, \tau^1) \\ \dot{y}(t) = -(c_0 y(t) + k Y) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ z(t) = Y \end{cases}$$



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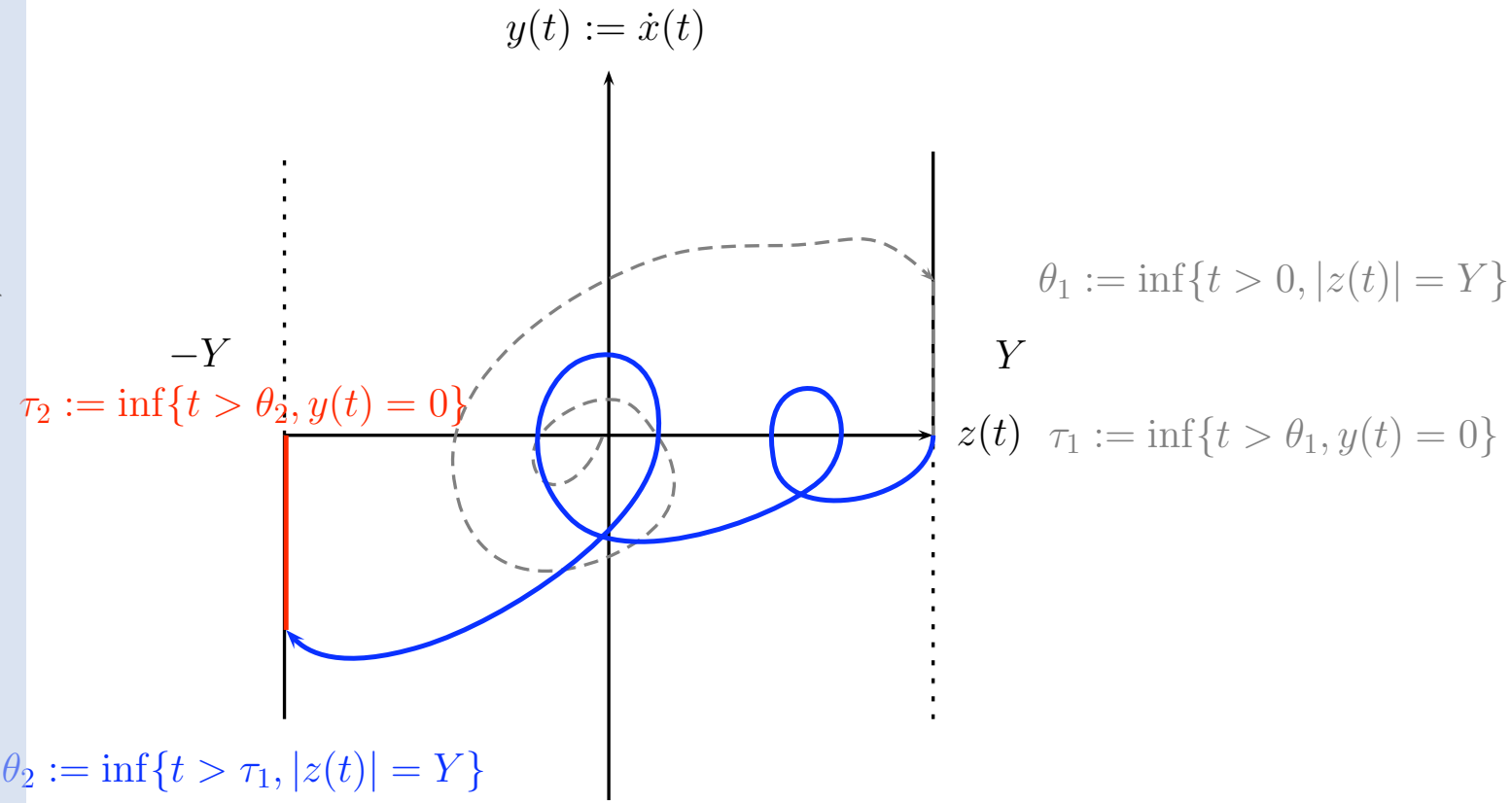
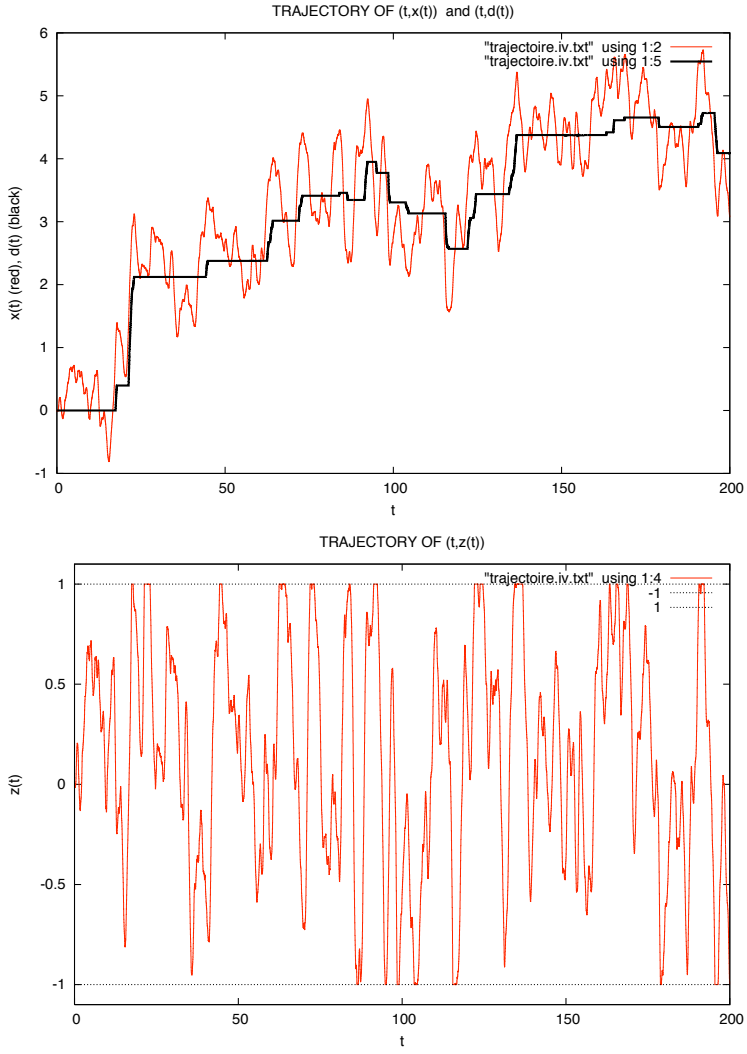
$$\begin{cases} 2^{st} \text{ elastic phase: } [\tau_1, \theta^2) \\ \dot{y}(t) = -(c_0 y(t) + k z(t)) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ \dot{z}(t) = y(t) \end{cases} \qquad \begin{cases} 2^{st} \text{ plastic phase: } [\theta^2, \tau^2) \\ \dot{y}(t) = -(c_0 y(t) - kY) + \text{“}\frac{dw(t)}{dt}\text{”}, \\ z(t) = -Y \end{cases}$$



An example of the plastic drift of the oscillator

Figure: on the top $t, x(t)$ (red) $t, \Delta(t)$ (black : plastic deformation) and at the bottom $t, z(t)$ for $c_0 = 1, k = 1, Y = 1$



Part 2: Stochastic variational inequality for the elasto-plastic problem

Mathematical tool to describe the right dynamic : the stochastic variational inequality

Reformulation of the problem without the plastic deformation $\Delta(t)$ and without the instants of phase transition.

Theorem (Bensoussan-Turi 2007)

The process $(y(t), z(t))$ is the unique solution of the stochastic variational inequality (SVI) defined by the following conditions

$$dy(t) = -(c_0 y(t) + kz(t))dt + dw(t),$$
$$(dz(t) - y(t)dt)(\phi - z(t)) \geq 0, \quad \forall |\phi| \leq Y, \quad |z(t)| \leq Y$$

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- $(y(t), z(t))$ reflected diffusion, $\Delta(t)$: reflection process
  References:
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- References:
  Deterministic VI for elasto-plasticity: [Duvaut-Lions1977].
  $\rightarrow$ Finally, it is not surprising to see arising stochastic variational inequalities in the framework of random vibrations.

Characterization of the stationary state (balance between elastic and plastic state)

Theorem (Bensoussan-Turi 2007)

$(y(t), z(t))$ *ergodic Markov process*

- *unique invariant probability distribution* ν *for* $(y(t), z(t))$ *and* $(y(t), z(t)) \xrightarrow[t\to\infty]{\mathcal{L}} \nu$ *(independently of the initial condition).*

- elastic domain: $D := (-\infty, +\infty) \times (-Y, Y)$

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- plastic domains: $D^+ := (-\infty, 0) \times \{-Y\}$ and $D^- := (0, +\infty) \times \{Y\}$
- ν has a density denoted by m is characterized by: $\forall \varphi$ smooth,

$$\int_D m(y,z)\{y\varphi_z - (c_0 y + kz)\varphi_y + \frac{1}{2}\varphi_{yy}\}dydz$$
$$+ \int_{D^+} m(y,Y)\{-(c_0 y + kY)\varphi_y(y,Y) + \frac{1}{2}\varphi_{yy}(y,Y)\}dy$$
$$+ \int_{D^-} m(y,-Y)\{-(c_0 y - kY)\varphi_y(y,-Y) + \frac{1}{2}\varphi_{yy}(y,-Y)\}dy = 0$$

Alternative method to the Monte-Carlo simulation (1): Start of my PhD research

From ergodic theory, we know the limiting behavior of $(y(t), z(t))$.

- For all bounded function f and $\forall (y_0, z_0) \in \bar{D}$,

$$\lim_{t\to\infty} \mathbb{E} f(y^{y_0}(t), z^{z_0}(t)) = \int_D f(y,z)m(y,z)dydz + \int_{D^+} f(Y,y)m(Y,y)dy + \int_{D^-} f(-Y,y)m(-Y,y)dy.$$

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- But, it is also well known that

$$\lim_{t\to\infty} \mathbb{E} f(y^{y_0}(t), z^{z_0}(t)) = \lim_{\lambda\to 0} \lambda \int_0^\infty e^{-\lambda t} \mathbb{E} f(y^{y_0}(t), z^{z_0}(t))dt.$$

Alternative method to the Monte-Carlo simulation (2)

- Denote $u_\lambda(y_0, z_0; f) = \mathbb{E}\left[\int_0^\infty \exp(-\lambda t) f(y^{y_0}(t), z^{z_0}(t)) dt\right]$.

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- Equivalent characterization of the asymptotic limit:

$$\begin{aligned} \lambda u_\lambda + A u_\lambda &= f(y,z) \quad \text{in } D \\ \lambda u_\lambda + B_+ u_\lambda &= f(y,Y) \quad \text{in } D^+ \\ \lambda u_\lambda + B_- u_\lambda &= f(y,-Y) \quad \text{in } D^- \end{aligned}$$

Nonlocal problem : $y \to u_\lambda(y, \pm Y; f)$ are continuous.

$\forall (y_0, z_0) \in \bar{D}$

$$\lim_{\lambda \to 0} \lambda u_\lambda(y_0, z_0; f) = \int_D f(y,z) m(y,z) dy dz + \int_{D^+} f(y,Y) m(y,Y) dy + \int_{D^-} f(y,-Y) m(y,-Y) dy$$

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- This result is fundamental for the numerical resolution of m : alternative method to Monte-Carlo, that requires simulations for long durations.
  Publication: [Bensoussan, Mertz, Pironneau,Turi 2009], SIAM Journal on Numerical Analysis , Volume 47 Issue 5

Alternative method to the Monte-Carlo simulation (3)

Superposition of three local problems:

- $$\begin{cases} \lambda v_\lambda + A v_\lambda = f & \text{in } D, \\ \lambda v_\lambda + B_+ v_\lambda = f_+ & \text{in } D^+, \\ \lambda v_\lambda + B_- v_\lambda = f_- & \text{in } D^-, \end{cases}$$
  with $v_\lambda(0^+, Y) = 0, v_\lambda(0^-, -Y) = 0,$

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  $$\text{with } \pi^+(0^+, Y) = 0, \pi^-(0^-, -Y) = 1.$$

Alternative method to the Monte-Carlo simulation (4)

- We look for p_+ and p_-:

$$v_\lambda + p_+ \pi_\lambda^+ + p_- \pi_\lambda^- \quad \text{continuous in } (0, \pm Y)$$

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- We look for p_+ and p_-:

$$v_\lambda + p_+\pi_\lambda^+ + p_-\pi_\lambda^- \quad \text{continuous in } (0, \pm Y)$$

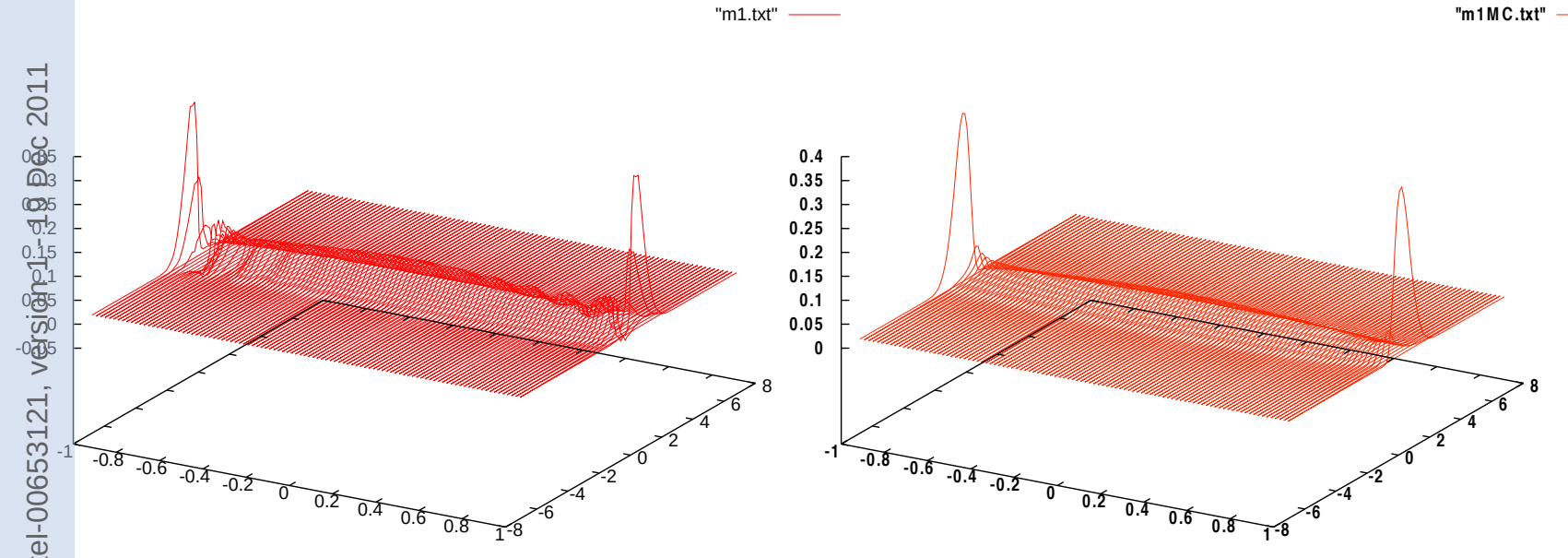
- Finally, we solve the following linear system:
with

$$\Pi := \begin{pmatrix} \pi^+(0^+, Y) - \pi^+(0^-, Y) & \pi^-(0^+, Y) - \pi^-(0^-, Y) \\ \pi^+(-0^+, -Y) - \pi^+(0^-, -Y) & \pi^-(0^+, -Y) - \pi^-(0^-, -Y) \end{pmatrix}$$

then

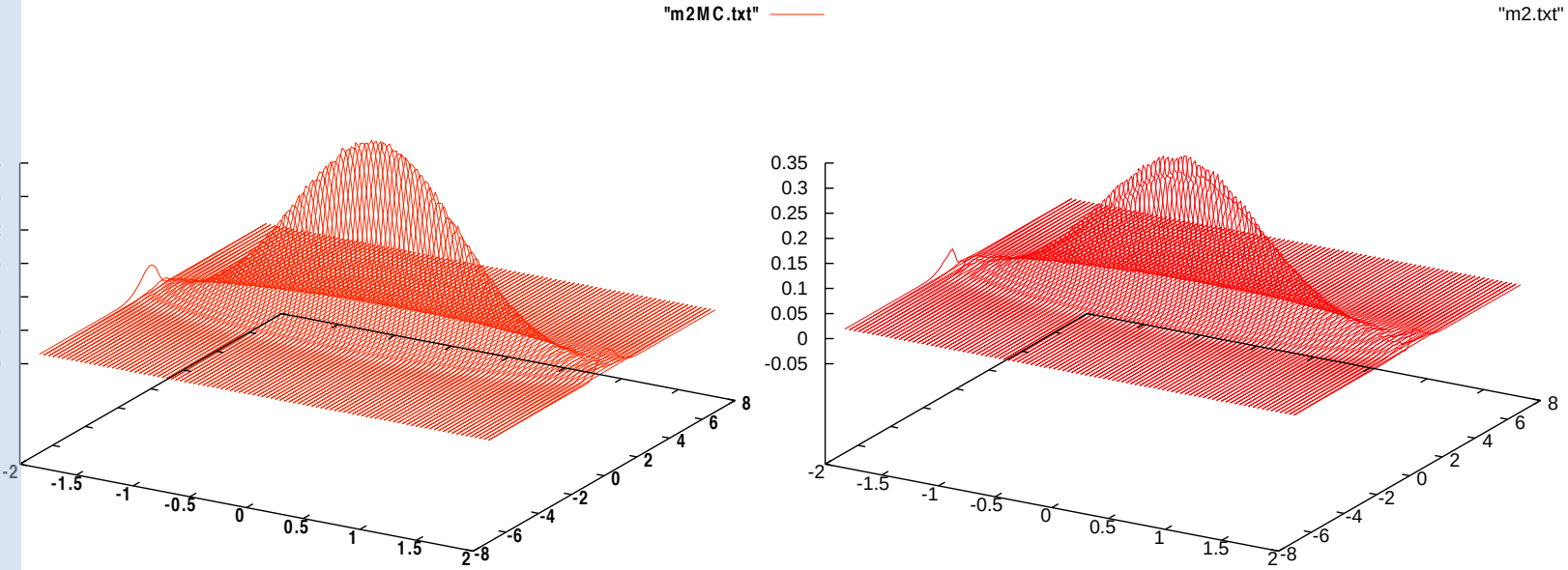
$$\Pi \begin{pmatrix} p_+ \\ p_- \end{pmatrix} = \begin{pmatrix} v_\lambda(0^-, Y) - v_\lambda(0^+, Y) \\ v_\lambda(0^-, -Y) - v_\lambda(0^+, -Y) \end{pmatrix}$$

Numerical result Vs Monte Carlo method : $Y = 1$



- **left:** plot of m with the deterministic method,
- **right:** plot of m with the Monte Carlo method, $T = 10, MC = 10^7$(number of trajectories), $-1 \leq z \leq 1, -7 \leq y \leq 7$.

Numerical result Vs Monte Carlo method : $Y = 2$



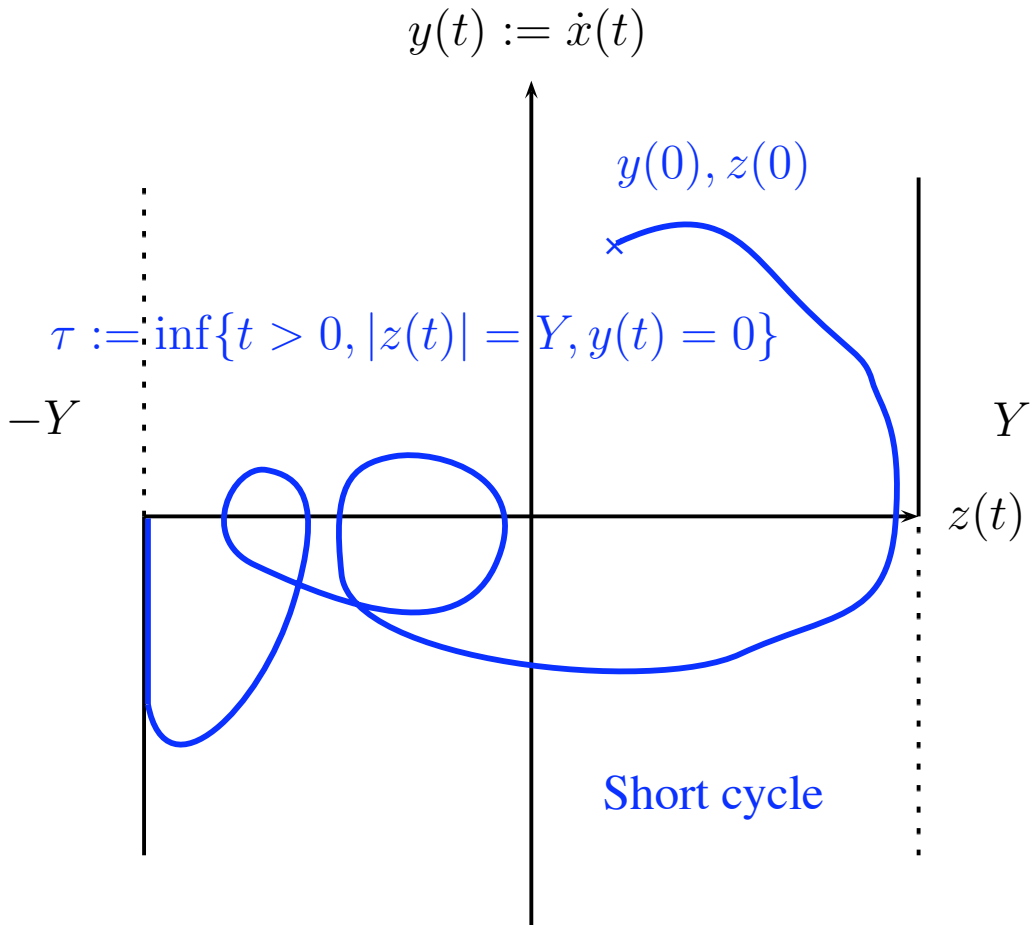
- **left:** plot of m with the deterministic method,
- **right:** plot of m with the Monte Carlo method, $T = 10, MC = 10^7$(number of trajectories), $-2 \leq z \leq 2, -7 \leq y \leq 7$.

Part 3: Short cycles related to the stochastic variational inequality

Short cycle behavior

Short cycle: path, solution of the SVI, starting from a point $(y, z) \in D$ and which contains

- only one phase evolving in D (elastic domain)
- and only one phase evolving in D^+ or D^- (plastic domains).



Short cycle behavior

- Short cycles: a new analytic characterization to the invariant measure of $(y(t), z(t))$.

- Key finding: connection between local problems and nonlocal problems.
  $\longrightarrow$ interpretation of local problems in terms of trajectory of $(y(t), z(t))$.

Definition and analysis of short cycles

- Let f be a bounded function on D, define $v(y, z; f)$ the solution of

$$Av = f \text{ in } D, \quad B_+ v = f \text{ in } D^+, \quad B_- v = f \text{ in } D^- \qquad (P_v)$$

with the local boundary conditions

$$v(0^+, Y) = 0 \text{ and } v(0^-, -Y) = 0$$

We call $v(y, z; f)$ a short cycle.

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We call $v(y, z; f)$ a short cycle.

Theorem (Analysis of short cycles, A. Bensoussan, L.M.)

There exists a unique solution to (P_v) of the form

$$v(y, z; f) = \varphi^+(y; f)\mathbf{1}_{\{y>0\}} + \varphi^-(y; f)\mathbf{1}_{\{y<0\}} + w(y, z; f)$$

where

- *w is a bounded function*
- $B_+\varphi^+ = f(y, Y), \quad y > 0, \quad \varphi^+(0^+; f) = 0$
- $B_-\varphi^- = f(y, -Y), \quad y < 0, \quad \varphi^-(0^-; f) = 0.$

New ergodic theorem : Statement of the result

Theorem (New ergodic theorem,A. Bensoussan, L. M.)

- *A new characterization of the invariant distribution:*

$$\nu(f) = \frac{v(0^-, Y; f) + v(0^+, -Y; f)}{2v(0^-, Y; 1)}$$

submitted: [Bensoussan, Mertz 2011] An analytic approach to the ergodic theory of stochastic variational inequalities

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- *Expansion of* u_λ

$$u_\lambda(y, z; f) = u(y, z; f) + \frac{\nu(f)}{\lambda} + o(\frac{1}{\lambda})$$

where

$Au = f - \nu(f)$ *in* D	$B_+u = f - \nu(f)$ *in* D^+	$B_-u = f - \nu(f)$ *in* D^-

with the non local boundary condition given by the fact that $y \to u(y, \pm Y; f)$ *are continuous.*

submitted: [Bensoussan, Mertz 2011] An analytic approach to the ergodic theory of stochastic variational inequalities

Probabilistic interpretation of the new characterization

Figure: $Av = f, B_+ v = f, B_- v = f$ and $v(0^+, -Y) = 0, v(0^-, Y) = 0$

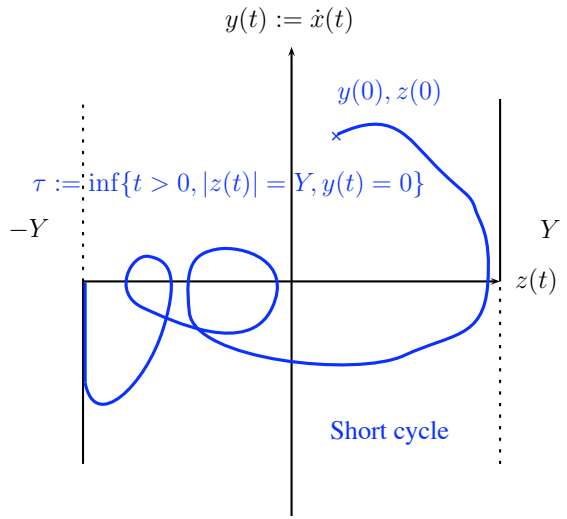

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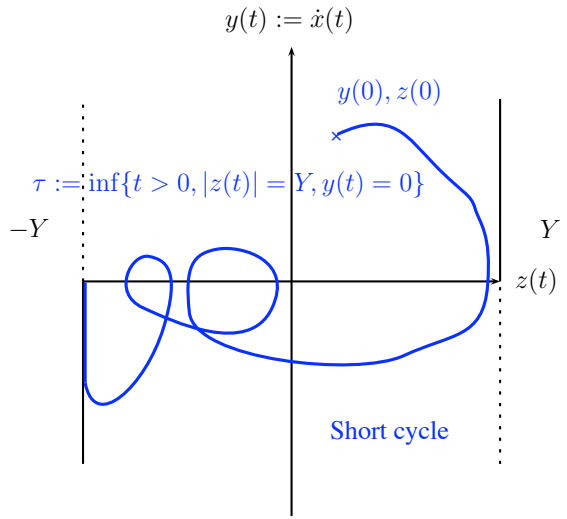
Figure: $Av = f, B_+v = f, B_-v = f$ and $v(0^+, -Y) = 0, v(0^-, Y) = 0$



- A new characterization of the invariant distribution:

$$\nu(f) = \frac{v(0^-, Y; f) + v(0^+, -Y; f)}{2v(0^-, Y; 1)}$$

- means “formally"

$$\nu(f) = \text{“}\frac{\frac{1}{2}\mathbb{E}_{(0^-,Y)}\left(\int_0^\tau f(y(s), z(s))ds\right) + \frac{1}{2}\mathbb{E}_{(0^+,-Y)}\left(\int_0^\tau f(y(s), z(s))ds\right)}{\frac{1}{2}\mathbb{E}_{(0^-,Y)}(\tau) + \frac{1}{2}\mathbb{E}_{(0^+,-Y)}(\tau)}\text{”}$$

Part 4 : Long cycles related to the stochastic variational inequality

Long cycle behavior

- Independent sequences in the trajectory.

Long cycle behavior

- Independent sequences in the trajectory.
- Long cycle: path, solution of the SVI, starting and ending in one of the two points of $\{(0, Y), (0, -Y)\}$, knowing that the trajectory had a stop by the other point.

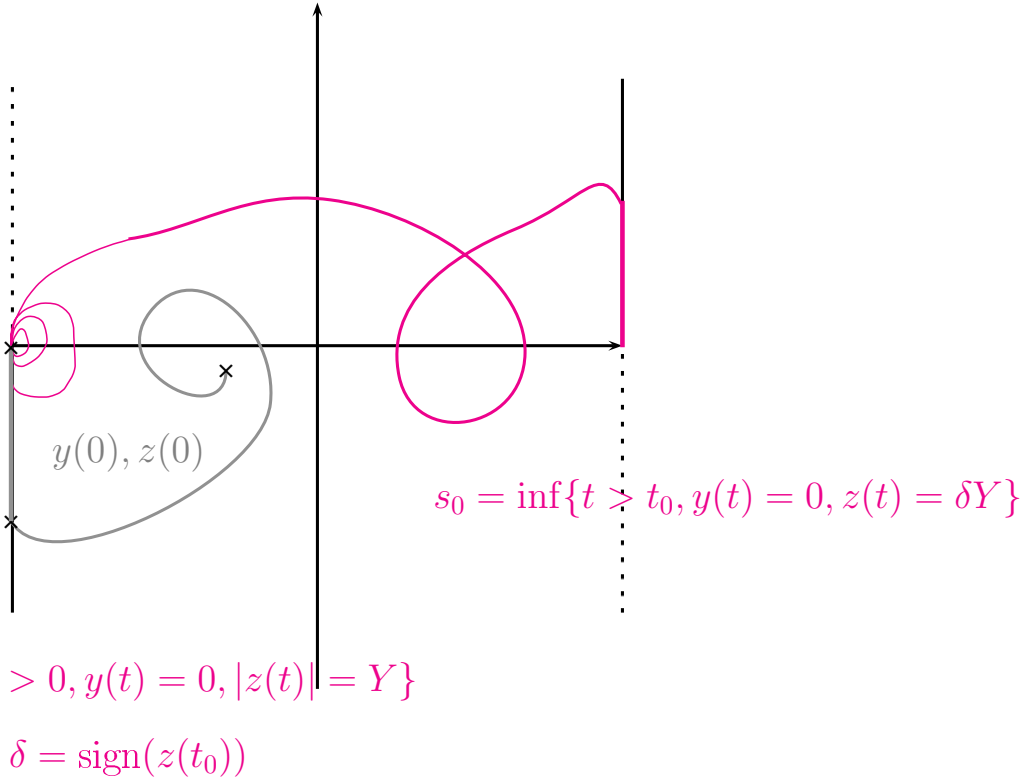
Long cycle behavior

- Independent sequences in the trajectory.
- Long cycle: path, solution of the SVI, starting and ending in one of the two points of $\{(0, Y), (0, -Y)\}$, knowing that the trajectory had a stop by the other point.
- Long cycles help to characterize the plastic behavior.

Definition and analysis of long cycles

- Define

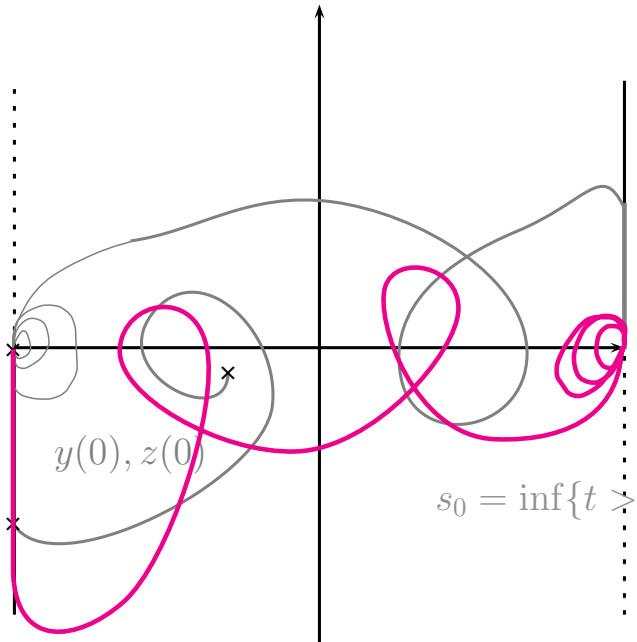
$$\begin{cases} t_0 = \inf\{t > 0, \quad y(t) = 0, |z(t)| = Y\}, \\ \delta = \text{sign}(z(t_0)), \\ s_0 = \inf\{t > t_0, \quad y(t) = 0, z(t) = -\delta Y\}. \end{cases}$$



Definition and analysis of long cycles

- Define

$$t_1 = \inf\{t > s_0, \quad y(t) = 0, z(t) = \delta Y\},$$



$t_0 = \inf\{t > 0, y(t) = 0, |z(t)| = Y\}$

$\delta = \text{sign}(z(t_0))$

$t_1 = \inf\{t > 0, y(t) = 0, z(t) = \delta Y\}$

Long cycle

Definition and analysis of long cycles

- Then in a recurrent manner, knowing t_n, we can define for $n \geq 0$

$$\begin{cases} t_{n+1} = \inf\{t > s_n, \quad y(t) = 0, z(t) = \delta Y\}, \\ s_{n+1} = \inf\{t > t_{n+1}, \quad y(t) = 0, z(t) = -\delta Y\}. \end{cases}$$

- Accordingly to these settings, we can define the n-th long cycle as the piece of trajectory enclosed by $[t_n, t_{n+1})$.

Definition and analysis of long cycles

Theorem (Long cycle behavior, A. Bensoussan, L.M.)

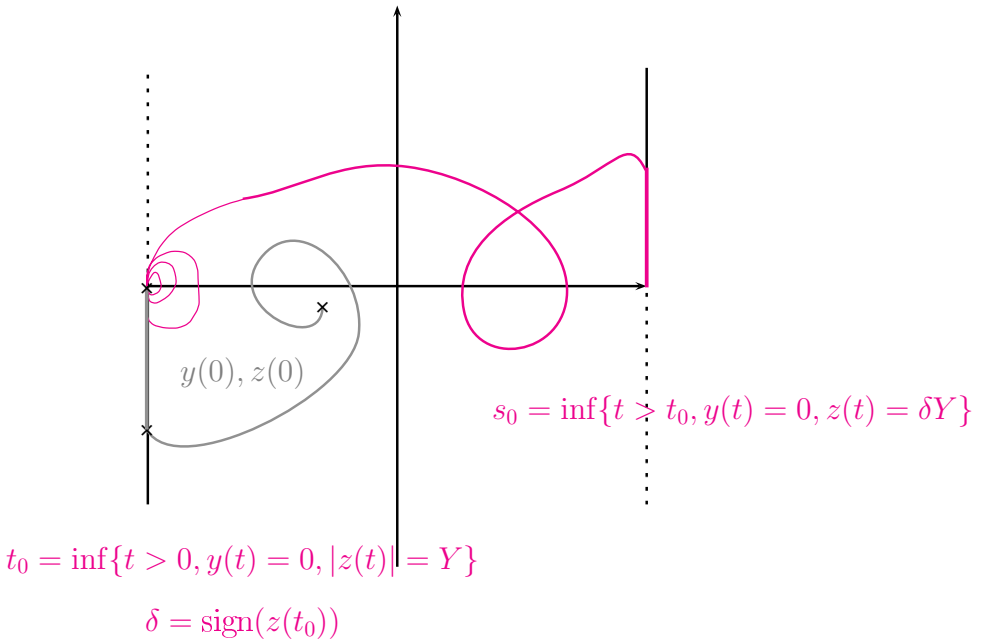
In this context, we have

$$\lim_{t \to +\infty} \frac{\sigma^2(x(t))}{t} = \frac{\mathbb{E}(\int_{t_0}^{t_1} y(s)ds)^2}{\mathbb{E}(t_1 - t_0)}$$

Key idea: PDEs related to long cycles and connection with short cycles.

submitted: [Bensoussan, Mertz 2011] Behavior of the plastic deformation of an elasto-perfectly-plastic oscillator with noise

PDEs related to Long cycles (type one way)

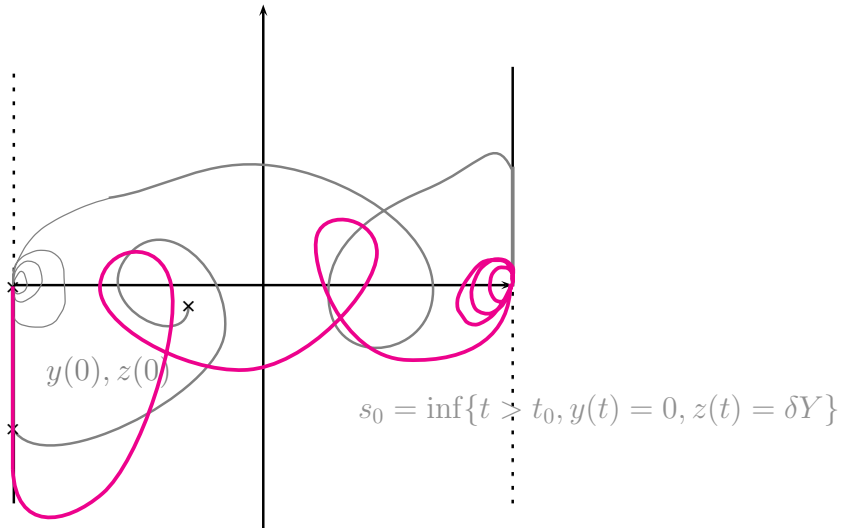


$$A\bar{v} = f, \quad B_+\bar{v} = f, \quad B_-\bar{v} = f \text{ and } \bar{v}(0^-, Y) = 0$$

where nonlocal problem : $y \longrightarrow \bar{v}(y, -Y)$ is continuous

$$\bar{v}(0, -Y) = \text{“}\mathbb{E}_{(0,-Y)}\left(\int_{t_0}^{s_0} f(y(s), z(s))ds\right)\text{”}$$

PDEs related to Long cycles (type return)



$$A\underline{v} = f, \quad B_+\underline{v} = f, \quad B_-\underline{v} = f \text{ and } \underline{v}(0^+, -Y) = 0$$

where nonlocal problem : $y \longrightarrow \underline{v}(y, Y)$ is continuous

$$\underline{v}(0, Y) = \text{“}\mathbb{E}_{(0,Y)}\left(\int_{s_0}^{t_1} f(y(s), z(s))ds\right)\text{”}$$

Numerical results in support of our result

In this section, we provide computational results which confirm our theoretical results.

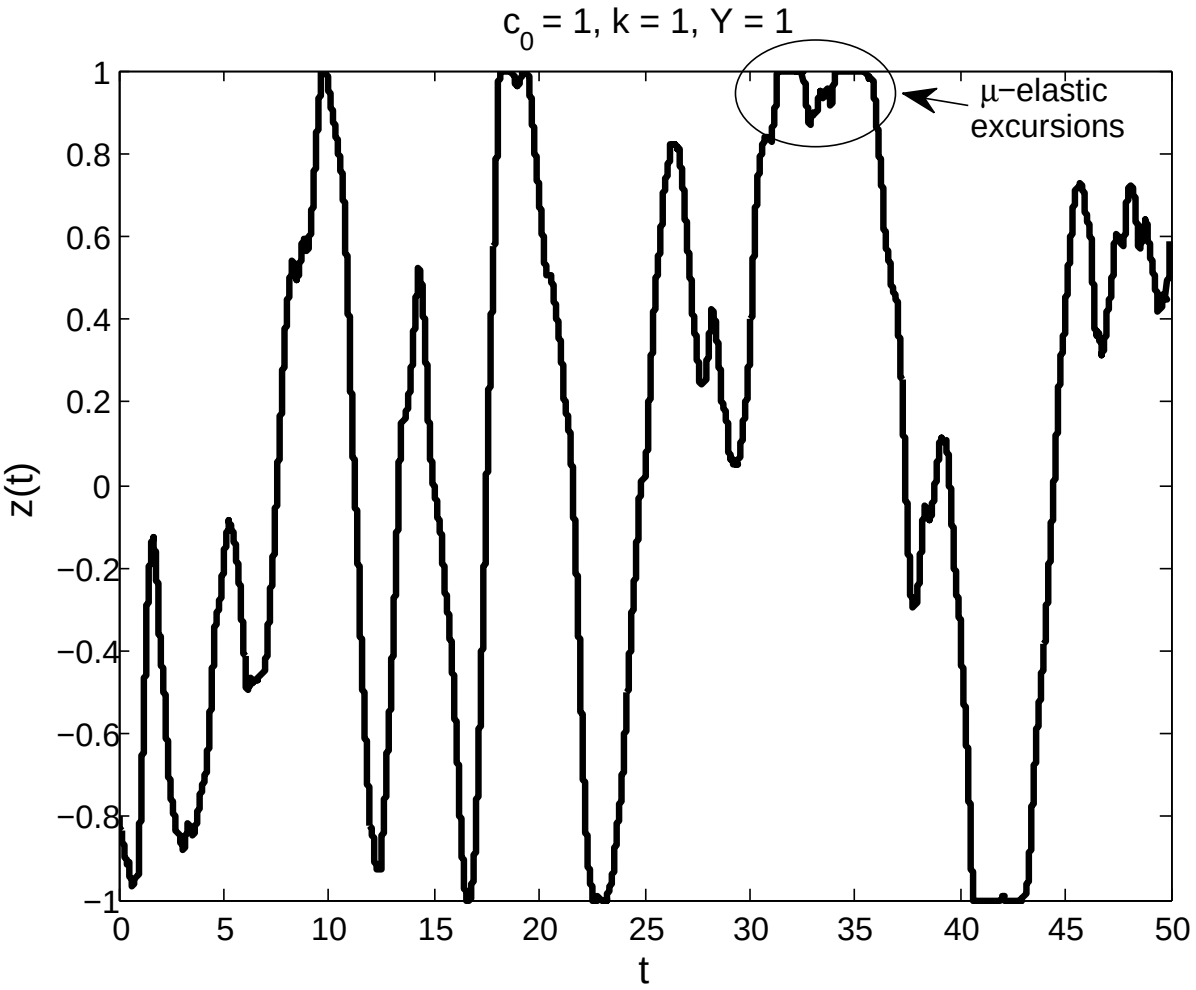
$c_0 = 1, k = 1$				
Y	$\frac{\sigma^2(x(t))}{t}, t = 500$	$\frac{\mathbb{E}(\int_{t_0}^{t_1} y(s)ds)^2}{\mathbb{E}(t_1 - t_0)}$	$\mathbb{E}(t_1 - t_0)$	Relative error %
0.1	$0.807^{\pm 0.031}$	$0.834^{\pm 0.069}$	$6.61^{\pm 0.11}$	3.2
0.2	$0.649^{\pm 0.026}$	$0.624^{\pm 0.047}$	$8.74^{\pm 0.13}$	3.8
0.3	$0.493^{\pm 0.020}$	$0.464^{\pm 0.034}$	$10.45^{\pm 0.16}$	5.8
0.4	$0.361^{\pm 0.014}$	$0.355^{\pm 0.026}$	$12.12^{\pm 0.18}$	1.7
0.5	$0.266^{\pm 0.011}$	$0.257^{\pm 0.019}$	$13.80^{\pm 0.21}$	3.3
0.6	$0.195^{\pm 0.008}$	$0.198^{\pm 0.014}$	$16.15^{\pm 0.26}$	1.5
0.7	$0.137^{\pm 0.005}$	$0.149^{\pm 0.011}$	$18.84^{\pm 0.31}$	8
0.8	$0.103^{\pm 0.004}$	$0.112^{\pm 0.008}$	$22.80^{\pm 0.39}$	8
0.9	$0.071^{\pm 0.003}$	$0.086^{\pm 0.006}$	$26.79^{\pm 0.47}$	15

Table: Monte-Carlo simulations $t = 500$, $\delta t = 10^{-4}$ and $MC = 5000$.

Part 5 : Stochastic variational inequality with jumps for artificial elastic-plastic phasing

Why introducing jumps? 1/2

Micro-elastic phasing $\longrightarrow$ difficulties to compute plastic deformations. (Frequency of deformations tends to be overestimated because of the noise)



Why introducing jumps? 2/2

- Micro-elastic phases $\rightarrow$ negligible in deformations but they are very small as well as numerous.

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- **a SVI with artificial jumps $\rightarrow$ to help phase transition.**

Why introducing jumps? 2/2

- Micro-elastic phases $\rightarrow$ negligible in deformations but they are very small as well as numerous.
- **a SVI with artificial jumps $\rightarrow$ to help phase transition.**
- $\rightarrow$ an empirical criterion which could be useful to calibrate the size of the jump. submitted: [Feau, Mertz 2011] An empirical study on plastic deformations of an elasto-plastic problem with noise

Model definition of a model with jumps

- For some $\epsilon > 0$, we define the process $(y^\epsilon(t), z^\epsilon(t))$ on each interval of time $[\tau_n^\epsilon, \tau_{n+1}^\epsilon)$ where

$$\tau_n^\epsilon := \inf\{t > \tau_{n-1}^\epsilon \mid y^\epsilon(t) = 0 \text{ and } |z^\epsilon(t)| = Y\}$$

- as follows:

$$\begin{cases} \dot{y}^\epsilon(t) = -(c_0 y^\epsilon(t) + k z^\epsilon(t)) + \dot{w}(t), \\ (\dot{z}^\epsilon(t) - y^\epsilon(t))(\phi - z^\epsilon(t)) \geq 0, \quad \forall\, |\phi| \leq Y \quad ; \quad |z^\epsilon(t)| \leq Y \end{cases}$$

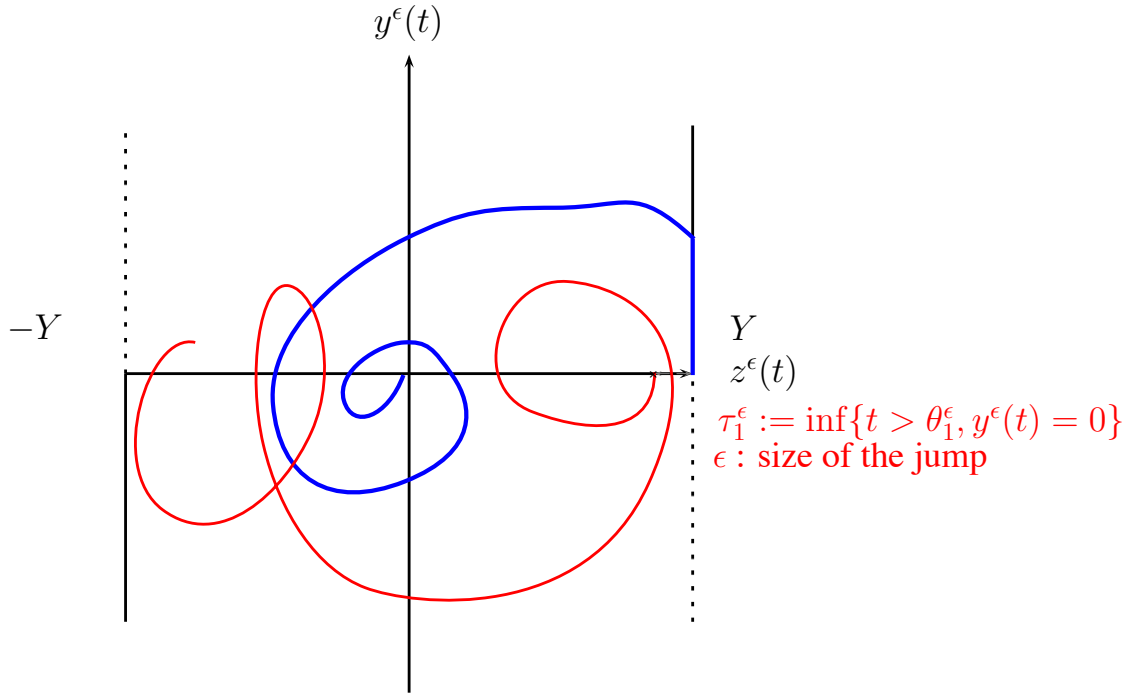
- with the following jump-conditions
  $y^\epsilon(\tau_n^\epsilon-) = 0, \qquad z^\epsilon(\tau_n^\epsilon-) = \pm Y$, and
  $y^\epsilon(\tau_n^\epsilon) = 0, \qquad z^\epsilon(\tau_n^\epsilon) = \text{sign}(z^\epsilon(\tau_n^\epsilon-))(Y - \epsilon)$.

Model definition : example of a jump

jump-conditions:

BEFORE THE JUMP : $y^\epsilon(\tau_1^\epsilon-) = 0, \qquad z^\epsilon(\tau_1^\epsilon-) = \pm Y$

AFTER THE JUMP : $y^\epsilon(\tau_1^\epsilon) = 0, \qquad z^\epsilon(\tau_1^\epsilon) = \text{sign}(z^\epsilon(\tau_1^\epsilon-))(Y-\epsilon).$



before the jump: $z^\epsilon(\tau_1^\epsilon-) = Y$

after the jump: $z^\epsilon(\tau_1^\epsilon) = Y - \epsilon$

Convergence to the continuous case as $\epsilon \to 0$: Statement of the result

Theorem (Convergence to the continuous case as $\epsilon \to 0$ A. Bensoussan, H. Jasso-Fuentes, L.M., S. Menozzi)

Assume $k > \frac{1}{2}\left(-\frac{c_0}{3} + c_0\sqrt{\frac{1}{9} + \frac{2c_0}{3}}\right)$. *Fix* $T > 0$ *and consider the processes* $(y(t), z(t))$ *and* $(y^\epsilon(t), z^\epsilon(t))$. *Then,*

$$\frac{1}{\epsilon}\mathbb{E}\left[\sup_{0\le t\le T}\left\{|y(t) - y^\epsilon(t)|^2 + k\,|z(t) - z^\epsilon(t)|^2\right\}\right] \to 0 \quad \text{as} \quad \epsilon \downarrow 0.$$

submitted: [Bensoussan, Jasso-Fuentes, Menozzi, Mertz. 2011] Asymptotic analysis of stochastic variational inequalities modeling an elasto-plastic problem with vanishing jumps

Convergence to the continuous case as $\epsilon \to 0$: Proof 1/3

- First, we define

$$\theta^\epsilon = \inf\{t > 0, \quad |z^\epsilon(t)| = Y\}$$

where

$$y^\epsilon(0) = 0 \text{ and } z^\epsilon(0) = Y - \epsilon$$

or

$$y^\epsilon(0) = 0 \text{ and } z^\epsilon(0) = -Y + \epsilon$$

with probability $1/2$.

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with probability $1/2$.

- We state that $\boxed{\text{if } \frac{\epsilon}{\mathbb{P}(\theta^\epsilon > T)} \to 0 \text{ we get the result.}}$

Convergence to the continuous case as $\epsilon \to 0$: Proof 2/3

- Chapman-Kolmogorov equation : on p^ϵ and p^0 i.e

$$p_t - \frac{1}{2}p_{yy} - ((c_0 y + kz)p)_y + yp_z = 0,$$

where $p^\epsilon(y, z, 0) = \delta_{0,Y-\epsilon}(y, z)$ and $p^0(y, z, 0) = \delta_{0,Y}(y, z)$.

Convergence to the continuous case as $\epsilon \to 0$: Proof 2/3

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- Hitting boundary problem $u(t, y, z)$ related to $\mathbb{P}_{(y,z)}(\theta^\epsilon > T - t)$.

Convergence to the continuous case as $\epsilon \to 0$: Proof 2/3

- Chapman-Kolmogorov equation : on p^ϵ and p^0 i.e

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- Hitting boundary problem $u(t, y, z)$ related to $\mathbb{P}_{(y,z)}(\theta^\epsilon > T - t)$.
- By testing p^ϵ and p^0 with the hitting boundary problem, we can justify the following identity:

$$\begin{aligned}\mathbb{P}(\theta^\epsilon > T) = &\int_D \left[p^\epsilon(y, z, T) - p^0(y, z, T)\right] \mathrm{d}y\mathrm{d}z \\ &+ \int_0^T \int_{-\infty}^0 yu(y, Y, t) \left[p^\epsilon(y, Y, t) - p^0(y, Y, t)\right] \mathrm{d}t\mathrm{d}y \\ &- \int_0^T \int_0^\infty yu(y, -Y, t) \left[p^\epsilon(y, -Y, t) - p^0(y, -Y, t)\right] \mathrm{d}t\mathrm{d}y.\end{aligned}$$

Convergence to the continuous case as $\epsilon \to 0$: Proof 3/3

Lemma

- $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_D \left[p^\epsilon(y,z,T) - p^0(y,z,T) \right] dydz$ *is finite,*
- $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_0^\infty yu(y,-Y,t) \left[p^\epsilon(y,-Y,t) - p^0(y,-Y,t) \right] dtdy$ *is finite,*
- *If we assume that* $k > \frac{1}{2}\left(-\frac{c_0}{3} + c_0 \sqrt{\frac{1}{9} + \frac{2c_0}{3}} \right)$ *then*

  $\liminf_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{-\infty}^0 yu(y,Y,t) \left[p^\epsilon(y,Y,t) - p^0(y,Y,t) \right] dtdy = +\infty$

To conclude, we have

$$\lim_{\epsilon \to 0} \frac{\epsilon}{\mathbb{P}(\theta^\epsilon > T)} = 0.$$

that proves the theorem.

Part 6: Long time behavior of an elastic-plastic oscillator excited by a filtered noise

Why considering a filtered noise?

- Engineering point of view: a filtered noise $\longrightarrow$ more realistic.
- Mathematical point of view: a generalization of the method proposed by Bensoussan-Turi.

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- Mathematical point of view: a generalization of the method proposed by Bensoussan-Turi.

Consider $x(t)$ an Ornstein-Ulhenbeck process reflected by ξ_t:

$$dx(t) = \underbrace{-\alpha x(t)dt + dw(t)}_{\text{Ornstein-Ulhenbeck}} + \underbrace{\mathbf{1}_{\{|x(t)|=L\}} d\xi_t}_{\text{reflection}}$$

We have $-L \leq x(t) \leq L$, $L > 0$ and $\alpha > 0$, $w(t)$ is a Wiener process.

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- Mathematical point of view: a generalization of the method proposed by Bensoussan-Turi.

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We have $-L \leq x(t) \leq L$, $L > 0$ and $\alpha > 0$, $w(t)$ is a Wiener process.

We consider the external force given by

$$-\alpha x(t) + \underbrace{d\tilde{w}(t)}_{\substack{\text{uncorrelated} \\ \text{noise of } w(t)}}$$

Model definition

- The stochastic variational inequality is given by

$$\begin{cases} dx(t) = -\alpha x(t)dt + dw(t) + \mathbf{1}_{\{x(t)=-L\}} d\xi_t^1 - \mathbf{1}_{\{x(t)=-L\}} d\xi_t^2 \\ dy(t) = -(\boxed{\alpha x(t)} + c_0 y(t) + kz(t))dt + d\tilde{w}(t), \\ (dz(t) - y(t)dt)(\phi - z(t)) \geq 0, \quad \forall |\phi| \leq Y, \quad |z(t)| \leq Y \end{cases} \tag{$\star$}$$

3D case

Model definition

- The stochastic variational inequality is given by

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3D case

- Note that If $\alpha = 0$, $x(t)$ is not involved in the dynamic of $y(t)$ and then $(y(t), z(t))$ satisfy the elasto-plastic problem studied by Bensoussan-Turi.

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3D case

- Note that If $\alpha = 0$, $x(t)$ is not involved in the dynamic of $y(t)$ and then $(y(t), z(t))$ satisfy the elasto-plastic problem studied by Bensoussan-Turi.
- Denote

$$\mathbf{R}u = \frac{1}{2}u_{xx} - \alpha x u_x - \alpha x u_y$$

  In this case, the infinitesimal generator is given by :
  - $\mathbf{A}u := Au + \mathbf{R}u$, on $(-L, L) \times \mathbb{R} \times (-Y, Y)$
  - $\mathbf{B}_+u := B_+u + \mathbf{R}u$, on $(-L, L) \times (0, \infty) \times \{Y\}$
  - $\mathbf{B}_-u := B_-u + \mathbf{R}u$, on $(-L, L) \times (-\infty, 0) \times \{-Y\}$

Ergodic Theorem: Statement of the result

Denote

- $\mathbf{D} := (-L, L) \times \mathbb{R} \times (-Y, Y)$
- $\mathbf{D}^+ := (-L, L) \times (0, +\infty) \times \{Y\}$
- $\mathbf{D}^- := (-L, L) \times (-\infty, 0) \times \{-Y\}$.

As the main result of this work we prove the following:

Theorem (Ergodic theorem, A. Bensoussan, L.M.)

There exists one and only one probability measure ν on $\mathbf{D} \cup \mathbf{D}^- \cup \mathbf{D}^+$ satisfying

$$\int_{\mathbf{D}} \mathbf{A}\phi d\nu + \int_{\mathbf{D}^-} \mathbf{B}_-\phi d\nu + \int_{\mathbf{D}^+} \mathbf{B}_+\phi d\nu = 0, \quad \forall \phi \quad \text{smooth.}$$

Moreover, ν has a probability density function m such that

$$\int_{\mathbf{D}} m(x, y, z)dxdydz + \int_{\mathbf{D}^+} m(x, y, Y)dxdy + \int_{\mathbf{D}^-} m(x, y, -Y)dxdy = 1.$$

submitted: [Bensoussan, Mertz 2011] Degenerate Dirichlet problems related to the ergodic theory for an elasto-plastic oscillator excited by a filtered white noise

Part 6 : Conclusion & Perspectives

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- 5: Extension of the Bensoussan-Turi method for proving ergodicity of an elasto-plastic problem with a filtered noise.

Now, we wish to extend our work to the following (coming soon):

In preparation (with A. Bensoussan, H. Jasso-Fuentes, S. Menozzi): invariant distribution for the case with jumps (ν^ϵ) and convergence $\nu^\epsilon \longrightarrow \nu$,

$\longrightarrow$ short cycle applied to the problem with jump.

- A characterization of the invariant distribution for the problem with jump:

$$\nu^\epsilon(f) = \frac{v(0, Y-\epsilon; f) + v(0, -Y+\epsilon; f)}{2v(0, Y-\epsilon; 1)}, \quad \nu^\epsilon(f) \to \nu(f)$$

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- means “formally"

$$\nu(f) = \text{“}\frac{\frac{1}{2}\mathbb{E}_{(0,Y-\epsilon)}\left(\int_0^\tau f(y(s), z(s))ds\right) + \frac{1}{2}\mathbb{E}_{(0,-Y+\epsilon)}\left(\int_0^\tau f(y(s), z(s))ds\right)}{\frac{1}{2}\mathbb{E}_{(0,Y-\epsilon)}(\tau) + \frac{1}{2}\mathbb{E}_{(0,-Y+\epsilon)}(\tau)}\text{”}$$

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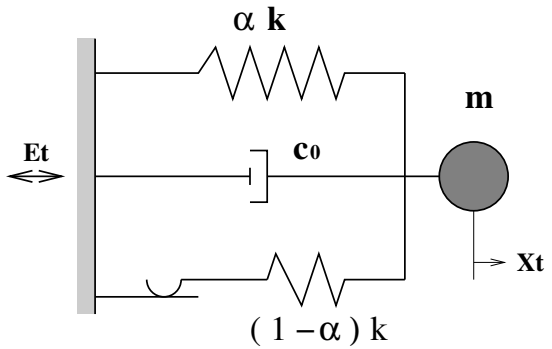
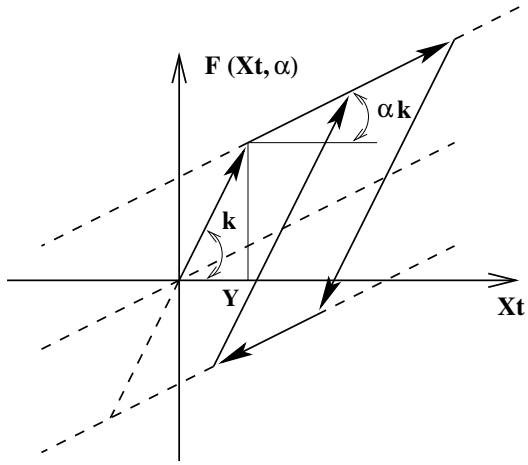
- Short and long cycles in the case of the elastic-plastic problem excited with a filtered noise?

Now, we wish to extend our work to the following :

- Ergodicity for an elasto-plastic oscillator with bilinear force?
- Morally, seems to be more ergodic, but much more challenging mathematically.





Fin de la thèse.

Merci de votre attention.