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Non-parametric model calibration in finance

Calibration non paramétrique de modèles en finance

Thèse présentée par

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Chapter 1

Introduction

La calibration des surfaces d'options vanilles est un problème primordial de la modélisation en finance. Dans le cas de trois différents modèles: volatilité locale et stochastique, corrélation locale et volatilité locale avec taux stochastiques, ce problème se résume à la résolution d'une équation intégral-différentielle partielle non linéaire. D'un point de vue mathématique, cette classe d'équations est particulièrement complexe, elle soulève de nombreuses questions, aussi bien d'un point de vue théorique que d'un point de vue pratique.

Considérations financières

Les mathématiques financières trouvent leur origine dans les travaux de Louis Bachelier au début du 20^{ème} siècle, avec sa thèse intitulée "Théorie de la Spéculation" [22]. Mais c'est seulement dans les années 70 qu'elles prirent réellement leur envol grâce aux célèbres articles de Black, Scholes et Merton [6] et [7]. Poursuivant l'idée de Bachelier selon laquelle un mouvement brownien est à même de capturer l'incertitude liée à l'évolution future du prix d'une action, ils choisirent de modéliser ce cours par l'équation différentielle stochastique suivante

$$\frac{dS_t}{S_t} = r(t)dt + \sigma dW_t \quad (1.1)$$

où S_t représente le prix de l'actif financier, $r(t)$ le taux d'intérêt, σ la volatilité et W_t un mouvement brownien standard. Cette équation et ses développements ont permis l'essor d'un immense marché de produits financiers dérivés d'une grande variété de valeurs économiques sous-jacentes. Le plus important de ces marchés est sans conteste celui des actions, sur lequel reposent également de multiples transactions d'options vanilles.

La complexité croissante des produits devant être évalués a progressivement rendu obsolète le modèle de Black, Scholes et Merton. Ce dernier, n'utilisant que le sous-jacent et la monnaie pour couvrir une vanille, est de fait incapable de rendre compte du "smile" de la volatilité implicite observé systématiquement sur les cours d'options vanilles [43]. Fondamentalement, cette limitation provient du caractère stochastique de la volatilité instantanée d'une action (Black et Scholes supposaient de leur côté cette quantité constante).

Le caractère stochastique de la variance instantanée du prix d'une action est particulièrement important lorsqu'il s'agit d'évaluer et de couvrir des options exotiques dépendant fortement de la volatilité: variance swaps, cliquets... Ces derniers ne peuvent être évalués correctement dans le cadre du modèle de Black et Scholes, leur valeur étant intimement liée aux mouvements de la volatilité.

Deux solutions bien connues existent pour traiter ce problème. La première, et plus simple, façon

de rendre la volatilité stochastique est de la faire dépendre de la valeur du sous-jacent. Dans ce contexte, σ devient une fonction du temps et du niveau de l'action $\sigma(t, S_t)$, la volatilité ainsi définie est appelée locale. La deuxième solution, les modèles à volatilité stochastique, bien plus ambitieuse d'un point de vue conceptuel consiste à ajouter à la volatilité un facteur stochastique qui lui serait propre. Ces deux approches possèdent leurs propres avantages. La volatilité implicite découlant des modèles à volatilité stochastique présente un intérêt particulier, de nombreuses références peuvent être trouvées à ce sujet [40] and [41]. Dans [20], les auteurs écrivent une équation aux dérivées partielles quasi-linéaire parabolique vérifiée par la volatilité implicite pour une classe de modèles assez générale, et introduisent la notion de volatilité effective qui nous sera utile plus tard. Dans [29], le comportement de cette quantité est étudié à proximité de la maturité.

L'un des problèmes les plus importants à résoudre lors de la confrontation de nos modèles avec la réalité des marchés est celui de la calibration. Comme dans de nombreuses situations en physique ou en ingénierie, lorsqu'un modèle a été suggéré, ses paramètres doivent être estimés grâce à des données externes. Dans le cas de la modélisation des produits dérivés en finance, ces données sont les valeurs des options les plus liquides, les vanilles. Il est bien connu depuis les travaux fondateurs de Litzenberger et Breeden [19] et leur célèbre extension par Bruno Dupire [13] que la connaissance des prix de vanilles pour tous les strikes et toutes les maturités (notons $C(T, K)$ une telle surface) est équivalente à la connaissance de la marginale, sous la probabilité risque-neutre¹, de l'actif sous-jacent, et plus précisément, qu'il existe une unique valeur pour la volatilité locale $\sigma(t, S_t)$ permettant de reproduire de telles marginales

$$\sigma_D^2(T, K) = 2 \frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{K^2 \frac{\partial^2 C}{\partial K^2}}$$

Toutefois, il est également bien connu que l'évolution en temps du modèle à volatilité locale n'est pas stable: le smile de la volatilité implicite générée par ce modèle s'aplatit au cours du temps ce qui contredit la persistance de ce phénomène observée sur les marchés. Ceci a conduit les chercheurs à créer une modélisation plus robuste, du type volatilité stochastique.

De nombreux modèles ont été écrits depuis plus de 20 ans. Parmi les plus populaires, on trouve ainsi Heston [17], le SABR [9], Fouque et al. [18] mais aussi [28]. Le modèle d'Heston, par exemple, est le suivant

$$\frac{dS_t}{S_t} = r(t)dt + y_t dW_t^1 \quad dy_t = \kappa (\alpha - y_t) dt + \gamma \sqrt{y_t} dW_t^2$$

Ces modèles offrent les dynamiques réalistes manquant à la volatilité locale. Néanmoins, le problème de leur calibration se pose. La solution la plus communément acceptée est l'écriture d'un modèle combinant les deux types de volatilité mentionnés précédemment: un modèle générique avec une volatilité ayant à la fois une composante locale et une composante stochastique. De tels modèles sont particulièrement utiles en pratique, ils offrent la souplesse et les dynamiques réalistes des modèles à volatilité stochastique ainsi que les propriétés de calibration des modèles à volatilité locale. Leurs équations de diffusion sont les suivantes

$$\frac{dS_t}{S_t} = r(t)dt + a(t, S_t)b(y_t)dW_t^1 \quad (1.2)$$

$$dy_t = \mu(t, y_t)dt + \xi(t, y_t)dW_t^2 \quad (1.3)$$

Le problème de la calibration de modèles à volatilité locale et stochastique a été résolu depuis quelques temps maintenant. Du point de vue des équations différentielles stochastiques, sa résolution se fonde sur les théorèmes de Gyongy [10]. L'équation vérifiée par la densité du couple

¹Dans [27], pour un marché sans arbitrages, les auteurs prouvent l'existence d'une mesure, appelée risque-neutre, sous laquelle les prix actualisés sont des martingales locales.

sous-jacent/volatilité, dans le cas $r = 0$, est l'équation intégral-différentielle non linéaire du second ordre

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{S^2}{2} \sigma_D^2 b^2 \frac{\int_{\mathbb{R}} p dy}{\int_{\mathbb{R}} b^2 p dy} p \right) - \frac{\partial^2}{\partial S \partial y} (\rho \sigma_D b \xi S \left(\frac{\int_{\mathbb{R}} p dy}{\int_{\mathbb{R}} b^2 p dy} \right)^{\frac{1}{2}} p) - \frac{\partial^2}{\partial y^2} \left(\frac{\xi^2}{2} p \right) + \frac{\partial}{\partial y} (\mu p) = 0 \quad (1.4)$$

$$p(0, S, y) = \delta(S = S_0, y = y_0) \quad (1.5)$$

La première question qui se pose est la suivante

Problème 1. *Peut-on prouver l'existence et l'unicité de solutions de l'équation de calibration (1.4-1.5) des modèles à volatilité locale et stochastique? Quels résultats obtient-on en pratique lors de sa résolution numérique?*

Un autre important domaine de recherche est celui de la corrélation entre produits financiers. Les mesures empiriques donnent des résultats satisfaisants pour des actifs donnés. Néanmoins, le cas d'un panier de sous-jacents pose problème. L'utilisation d'un modèle à volatilité locale pour la diffusion de chacun des sous-jacents du panier et de la corrélation empirique pour relier les browniens des différents actifs ne permet pas de reproduire les vanilles du panier cotées sur le marché. Ceci soulève des problèmes significatifs lors de la couverture de produits dépendants de plusieurs sous-jacents. L'une des approches pour résoudre ce problème est la "Corrélation Locale": la matrice de corrélation ρ des n sous-jacents est déformée en utilisant un paramètre λ , fonction du temps et du niveau du panier (d'où le nom de corrélation locale)

$$\tilde{\rho}_{ij} = \lambda + (1 - \lambda)\rho_{ij} = \rho_{ij} + \lambda(1 - \rho_{ij}) \quad (1.6)$$

Ceci nous mène au second problème

Problème 2. *Peut-on trouver la valeur de λ permettant de calibrer les vanilles du panier? De quel genre de résultats (existence, unicité...) dispose-t-on concernant l'équation découlant de cette calibration? Est-il possible de calibrer numériquement les vanilles d'un panier sur deux sous-jacents?*

Le dernier sujet sur lequel nous nous penchons est celui des taux d'intérêt. Le modèle le plus utilisé dans ce domaine est celui de Heath, Jarrow et Morton [26]. En ce qui nous concerne, nous étudions un modèle hybride à volatilité locale et taux stochastique

$$\frac{dS_t}{S_t} = r(t, y_t)dt + \sigma(t, S_t)dW_t^1 \quad dy_t = \mu(t, y_t)dt + \lambda(t, y_t)dW_t^2 \quad (1.7)$$

Grâce à une approche edp similaire à celle utilisée pour la volatilité locale et stochastique, il est possible d'écrire une équation de calibration pour les vanilles du modèle hybride. Le même genre de questions que précédemment apparaît

Problème 3. *Quelles sont les propriétés de l'équation de calibration? La résolution numérique de cette équation aux dérivées partielles permet-elle de reproduire efficacement les vanilles du marché?*

Approche théorique des équations de calibration

D'un point de vue mathématique, les problèmes précédents se réduisent à un type d'équations plus général

$$\frac{\partial p}{\partial t} - L_p p = 0 \quad \text{dans }]0, T[\times \Omega \quad (1.8)$$

$$p(0, x) = p_0(x) \quad \text{sur } \Omega \quad (1.9)$$

où l'opérateur L_q est égal à

$$L_q p = \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\ \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) p(t, x) \right) - \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^j p(t, x))$$

Les termes essentiels de cette équation sont les quotients

$$Q_{ij}(t, x) = \frac{\int f_{ij}(x^{\lambda_{ij}^c}) q_{ij}(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int q_{ij}(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}$$

Ils rendent l'équation à la fois non linéaire et intégral-différentielle. Dans [34], le terme non local se trouve à l'extérieur des dérivées, dans ce cas les auteurs parviennent à prouver l'existence et l'unicité de solutions. Néanmoins, dans notre cas, les Q_{ij} étant dérivés, nous sommes confrontés à un domaine particulièrement complexe des équations intégral-différentielles partielles. Le type d'équations le plus général étudié jusqu'à présent est à notre connaissance le suivant

$$\frac{\partial p}{\partial t} - F(t, x, p, Dp, D^2p, N(p)) = 0 \quad \text{in } Q^T =]0, T[\times \mathbb{R}^d \quad (1.10) \\ p(0, x) = p_0(x) \quad \text{on } \mathbb{R}^d$$

où F est une fonction de $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d) \times \mathbb{R}$ dans \mathbb{R} . Dp est le gradient de p et D^2p sa matrice hessienne. $\mathbb{S}(d)$ est l'espace des matrices $d \times d$ symétriques réelles. $N(p)$ est un terme non local. Dans [14], l'auteur prouve l'existence de solutions pour (1.10) sous certaines hypothèses. En particulier, $N(p)$ doit être défini pour toutes les fonctions p continues et bornées. De plus, l'opérateur N doit être Lipschitz pour une certaine norme. Ce genre de résultat peut être adapté au cas de l'équation adjointe de (1.8), les opérateurs Q_{ij} sont en effet bien définis et Lipschitz sur l'ensemble des fonctions minorées par une constante donnée strictement positive. Toutefois, l'équation (1.8) ne peut être traitée avec le même genre de méthodes, les dérivées du terme non local étant impliquées. De nouvelles questions voient ainsi le jour

Problème 4. *Quelle méthode peut-on utiliser pour prouver l'existence de solutions de (1.8)? Les Q_{ij} admettent-ils des propriétés exploitables à cet égard? Quel genre d'estimations a priori doit-on employer?*

Résolution numérique

La résolution numérique des équations de calibration est bien évidemment particulièrement intéressante d'un point de vue financier. Une méthode efficace pour résoudre des équations paraboliques du second ordre est le schéma Alternate Direction Implicit. Il est généralement appliqué à des équations linéaires classiques, et est décrit en profondeur dans [15]. De nombreuses questions voient le jour du fait du caractère non local du cadre de notre étude

Problème 5. *Est-il pertinent d'utiliser l'algorithme ADI pour résoudre nos équations de calibration? Comment peut-on traiter le terme non local? Le schéma aux différences finies choisi est-il consistant, et quel est l'ordre de l'erreur de troncature?*

La précision de la calibration obtenue en résolvant l'équation est satisfaisante pour les trois modèles financiers mentionnés précédemment. Néanmoins, le cas du modèle à volatilité locale et stochastique présente un intérêt certain, et soulève plusieurs questions. Certains praticiens s'accordent pour dire que la stabilité de la calibration est incertaine lorsque la volatilité de la volatilité (la fonction ξ dans le modèle (1.3)) est trop grande, ce qui nous amène au dernier problème étudié dans cette thèse

Problème 6. *Trouve-t-on en pratique une instabilité quand ξ est grand? Est-il possible de relier cette instabilité aux limitations rencontrées lors de l'étude du Problème 4? Sommes-nous capables de lui apporter une explication théorique?*

Plan et Résultats

Cette thèse comporte trois parties: la première traite les questions théoriques liées au Problème 4. Ces résultats sont ensuite appliqués aux questions financières ayant motivé notre travail. Pour conclure, nous nous intéressons aux aspects numériques découlant des parties précédentes.

Partie I: Etude théorique des équations de calibration

Dans le chapitre 3, nous commençons par introduire les espaces de base couramment utilisés dans le cadre des équations linéaires paraboliques: les espaces de Hölder. Nous donnons ensuite les résultats classiques concernant de telles équations: de quelles hypothèses a-t-on besoin, qu'appelle-t-on conditions de compatibilité... Pour finir, nous prouvons quelques estimations sur les solutions u de telles équations, par exemple

$$|u|_{H^{0,h,h/2}(D^t)} \leq t^{1-\frac{h}{2}} K_{H^1} |f|_{H^0(D^t)}$$

où f est le second membre de l'équation, et les conditions aux bords sont supposées nulles.

Le chapitre 4 est consacré à l'équivalence entre équations intégral-différentielles stochastiques et équations intégral-différentielles partielles. L'application du théorème de Gyongy au cas du modèle à volatilité locale et stochastique prouve que la diffusion calibrée de tels modèles suit une EIDS. L'équation de Kolmogorov progressive appliquée à la densité de solutions d'EDS permet de relier ces dernières au domaine des équations aux dérivées partielles. Toutefois, cette équivalence doit être étendue au cas des SIDE. Après un rapide tour d'horizon des équations de McKean-Vlasov, intimement liées à l'EIDS de calibration, nous prouvons que l'équivalence reste vraie dans notre cadre non linéaire: la densité calibrée du modèle à volatilité locale et stochastique suit l'équation intégral-différentielle partielle non linéaire (1.4).

Dans le chapitre 5, nous étudions l'équation généralisée (1.8). Comme souligné ci-dessus, cette équation est particulièrement complexe. Le fait que le terme non local soit dérivé rend inutiles les résultats déjà existants. La seule méthode venant à l'esprit pour surmonter cette difficulté est de construire un point fixe pour la solution de l'équation paramétrisée. Pour ce faire, il est nécessaire de disposer d'un certain contrôle sur les variations des quotients Q_{ij} .

Une propriété importante concernant ces quantités est leur trivialisations quand les fonctions f_{ij} sont constantes. Une telle observation permet de prouver un premier résultat: si les fonctions f_{ij} sont suffisamment proches de constantes, nous obtenons le contrôle requis sur les Q_{ij} , et prouvons l'existence de solutions pour (1.8) avec une condition aux bords (1.9) strictement positive. Ce résultat est vrai sur $]0, T]$ pour tout T , la distance f^* entre f_{ij} et ladite constante décroît toutefois

quand T grandit.

Dans la preuve du résultat précédent, f^* a deux raisons d'être. Premièrement, il nous assure du fait que la solution reste bornée inférieurement par une constante strictement positive fixée au préalable. Et deuxièmement, il garantit la stabilité de l'ensemble des solutions de l'équation paramétrisée. En modifiant la preuve d'existence, nous sommes capables de prouver un résultat auxiliaire, l'existence locale en temps: on trouve une autre constante f^{**} à priori plus grande que f^* , un temps T^* et une solution de l'équation sur $]0, T^*]$ pour des fonctions f_{ij} proches de constantes, la distance entre les deux étant plus petite que f^{**} .

Nous donnons également un résultat d'unicité faible, si deux fonctions sont solutions de (1.8) avec les mêmes conditions aux bords et produisent les mêmes quantités Q_{ij} , alors elles sont égales. Pour conclure, on s'intéresse à l'équation adjointe. Par une méthode de point fixe, on prouve l'existence de solutions en temps court pour des fonctions f_{ij} quelconques. Ce résultat, intéressant en lui-même, nous permet également de mieux comprendre les limitations qu'on rencontre pour l'équation initialement étudiée.

Partie II: Applications à la finance

Cette deuxième partie de ma thèse est consacrée aux trois modèles financiers introduits précédemment. Un premier chapitre traite du modèle à volatilité locale et stochastique. On commence par rappeler quelques propriétés classiques de ce modèle. On exhibe en particulier un graphique montrant les inconsistances de ses dynamiques. On utilise ensuite une approche équations aux dérivées partielles pour trouver la valeur de la partie locale de la volatilité permettant de calibrer les vanilles du modèle. Ce résultat n'est en lui-même pas original et peut être obtenu grâce au théorème de Gyongy comme mentionné précédemment. La méthode présente toutefois de l'intérêt et sera utilisée plus loin pour les deux autres modèles. On adapte ensuite les résultats théoriques de la Partie I pour prouver l'existence de solutions calibrées quand la fonction b gouvernant la partie stochastique de la volatilité est proche d'une constante. Pour conclure, on donne les résultats obtenus lors de la résolution numérique de l'équation de calibration pour deux exemples de volatilité stochastique: une volatilité lognormale et un CIR.

Le deuxième chapitre s'intéresse aux modèles à corrélation locale. En utilisant une méthode similaire au cas du modèle à volatilité locale et stochastique, nous parvenons à écrire la valeur du paramètre λ (dont la définition est donnée ci-dessus) permettant de calibrer les vanilles du panier. Ceci nous permet d'écrire une équation intégral-différentielle partielle vérifiée par la densité de notre modèle. Après avoir présenté un résultat d'existence théorique de solutions pour cette EIDP, nous la résolvons numériquement dans le cas d'un panier à deux sous-jacents pour tester l'efficacité de la calibration.

Le dernier chapitre est consacré à l'étude d'un modèle hybride à volatilité locale et taux stochastiques. Comme dans les exemples précédents, il est possible d'écrire une EIDP non linéaire sur la densité calibrée de la diffusion de notre modèle. Cette équation n'a pas exactement la même forme que l'équation généralisée (2.8). Une méthode similaire pour sa résolution peut toutefois être appliquée et permet d'obtenir un résultat d'existence. Pour ce qui est de l'aspect pratique de ce modèle, la calibration obtenue en résolvant numériquement l'équation permet d'obtenir des résultats plutôt satisfaisants.

Partie III: Etude numérique et instabilité de l'équation

La dernière partie de cette thèse est consacrée aux aspects pratiques de mon travail. Dans le chapitre 9, nous nous intéressons aux questions concernant le schéma ADI utilisé pour la résolution numérique de l'équation. On définit d'abord ses équations aux différences finies. Afin d'obtenir une meilleure approximation de la solution, un algorithme prédicteur-correcteur est implémenté. Nous prouvons la consistance de ces deux étapes et montrons que l'erreur de troncature est en $O(\Delta t^2)$, comme dans le cas linéaire. Pour conclure, on trace le graphe de l'erreur de convergence contre le nombre de pas de temps utilisé dans la résolution. Dans le schéma basique à une étape, l'erreur est en $O(\Delta t)$ comme attendu. Quand les étapes prédictrices et correctrices sont toutes deux utilisées, le graphe montre une erreur du second ordre, ce qui confirme nos résultats théoriques.

Pour conclure cette thèse, on étudie l'apparition d'instabilités dans la résolution numérique de la calibration du modèle à volatilité locale et stochastique. Comme mentionné précédemment, il a été observé que la calibration perd grandement en précision lorsque la volatilité de la volatilité devient trop grande. Ce phénomène est en fait plutôt en accord avec les limitations rencontrées lors de l'étude théorique de l'équation de calibration. En effet, un simple changement d'échelle montre l'équivalence entre un ξ grand et une fonction b variant beaucoup. Pour vérifier cette assertion, on trace le graphe de la densité du modèle pour une maturité donnée et ce pour différentes fonctions b . Une instabilité apparaît clairement quand b prend un large spectre de valeurs.

Pour expliquer ce phénomène, on se concentre sur l'instabilité dite de Hadamard. Dans un premier temps, on décrit cette instabilité d'un point de vue théorique et on donne quelques exemples connus d'opérateurs instables au sens de Hadamard. Ensuite, en remarquant que l'équation de calibration linéarisée peut s'écrire sous une forme plus générale, nous donnons un critère sur les coefficients de l'équation généralisée garantissant son instabilité. Finalement, on tente d'appliquer ce critère au cas de la volatilité locale et stochastique. Malheureusement, ledit critère n'étant pas vérifié, de nouvelles pistes doivent être étudiées.

On conclut dans le chapitre 11 avec un court résumé des résultats obtenus, ainsi que des questions toujours ouvertes.

Contents

1	Introduction	2
2	Introduction	11
2.1	Financial considerations	11
2.2	Theoretical approach of the calibration equations	13
2.3	Numerical Resolution	14
2.4	Outline and Results	15
2.5	Notations	17
I	Theoretical study of the calibration equations	18
3	Results concerning linear differential equations of parabolic type	19
3.1	Hölder spaces	20
3.2	Linear equations of parabolic type	20
4	Theoretical study of the SIDE and PIDE stemming from the calibration	25
4.1	McKean-Vlasov type equations	26
4.1.1	First example	26
4.1.2	Local dependence	26
4.1.3	General existence result	27
4.2	Stochastic Differential Equation with conditional expectations	27
4.2.1	The classic stochastic differential equation associated with (4.9)	28
4.2.2	Equivalence between existence results for (4.9) and existence results for (4.10)	29
5	Resolution of the Partial Integro-Differential Equation	32
5.1	Assumptions and study of the operator L	33
5.2	Implicit function theorem and compatibility conditions	36
5.3	Existence results for Equation (5.8)	37
5.3.1	Global existence in time	38
5.3.2	Local existence	41
5.4	The adjoint equation: a simpler case	43
II	Financial applications	47
6	Local and Stochastic Volatility models	48
6.1	Local volatility and Stochastic volatility models	48

6.1.1	Dupire's model	49
6.1.2	Mimicking theorem and Stochastic volatilities	51
6.2	Partial Differential Equation for the calibration of LSV models	52
6.2.1	Calibration equation	52
6.2.2	Application of the results from Part I	54
6.3	Numerical results	55
6.3.1	Lognormal volatility	56
6.3.2	CIR	57
7	Application to the "Local Correlation" model	58
7.1	Inconsistencies between stock and basket options	58
7.2	"Local Correlation" model	58
7.3	Resolution of the equation for the calibration of a basket	62
7.3.1	Theoretical study	62
7.3.2	Numerical Resolution	65
8	Application to stochastic interest rates	67
8.1	Hybrid Local Volatility models	67
8.1.1	Calibration equation	67
8.1.2	Theoretical study	69
8.2	Numerical calibration	70
III	Numerical studies and instabilities of the solutions	72
9	Algorithm for the resolution of the calibration equation	73
9.1	Alternating Direction Implicit scheme	73
9.2	Time Convergence Rate of the modified ADI algorithm	79
10	Instabilities of the solutions	81
10.1	Numerical explosion for "oscillating" volatilities	81
10.2	Hadamard instability and Linearized equation	84
11	Conclusions	91
A	Proof of Lemma 4	92
B	Estimates for Green's function	97
B.1	Space estimate B.1	97
B.2	Time estimate B.2	101
C	Data for the numerical simulations	102
C.1	LSV and Stochastic Interest Rates models	102
C.2	Local Correlation model	104
D	CIR	105

Chapter 2

Introduction

Consistently fitting vanilla option surfaces is an important issue when it comes to modeling in finance. In three different models: local and stochastic volatility, local correlation and hybrid local volatility with stochastic rates, this calibration boils down to the resolution of a nonlinear partial integro-differential equation. The mathematical resolution of that type of equation is particularly challenging and raises numerous questions both theoretical and numerical. In this thesis, we study those calibration equations.

2.1 Financial considerations

Financial mathematics find their origin in the work of Louis Bachelier at the beginning of the 20th century with his thesis "Théorie de la Spéculation" [22]. But it was only 70 years later that they gained a wider recognition, when Black, Scholes and Merton published their famous articles [6] and [7]. Pursuing the idea of Bachelier that a brownian motion can capture the uncertainty related to the future evolution of a stock price, they chose to model said price with the stochastic differential equation

$$\frac{dS_t}{S_t} = r(t)dt + \sigma dW_t \quad (2.1)$$

where S_t stands for the stock price, $r(t)$ is the interest rate, σ the volatility and W_t a standard brownian motion. This equation and its implications induced the development of huge financial derivative markets on a wide range of underlying economic quantities. One of the most visible market of underlyings is surely the equities. Index level and share price quotes are nowadays a common topic. Upon equities also rest deep exchange-based markets of vanilla derivatives on indices and single stocks.

The increasing complexity of the products that need to be evaluated and risk-managed made the model of Black, Scholes and Merton obsolete. While they only used the underlying stock price and the bond to hedge a derivative, this cannot be justified anymore: their model is not able to capture what is today known as the implied volatility "smile" of traded vanilla options [43]. The root of this discrepancy is that the volatility is not, as assumed in Black and Scholes' model, a constant but rather, by itself, stochastic. This stochastic nature of the instantaneous stock price variance is of particular relevance when it comes to pricing and hedging heavily volatility-dependent exotic options such as variance swaps or cliquet-type products. They cannot be priced properly in Black and Scholes' framework since their very risk lies in the movements of the volatility.

There are two well-known approaches to deal with this problem. The first and simplest way to render the volatility stochastic is making it depend on the stock price. In this context, σ becomes

a function of time and spot level $\sigma(t, S_t)$, giving rise to the so-called local volatility models. The second and more ambitious solution, the stochastic volatility models, consists in adding to the volatility a stochastic factor of it's own. Both approaches are of particular interest. The implied volatility stemming from stochastic volatilities is of particular interest, many references study the links between them like [40] and [41]. In [20], the authors derive a quasi-linear parabolic partial differential equation solved by the implied volatility in a general class of models and introduce the notion of effective volatility, that shall appear later in this work. In [29], the authors describe the behaviour of this quantity near expiry.

One of the most important challenge for real-life applications of a model to derivatives trading is the issue of calibration. Similar to common situations in many areas of physics and engineering, once a model has been suggested, its parameters have to be estimated using external data. In the case of derivative modeling, those data are the liquid (tradable) options, generally called vanilla products. It is well-known since the pioneering work of Litzenberger and Breeden [19] and its celebrated extension by Bruno Dupire [13] that the knowledge of market data such as the prices of vanilla options across all strikes and maturities (let $C(T, K)$ denote such surface) is equivalent to the knowledge of the risk-neutral marginals¹ of the underlying stock distribution, and moreover, that there is a unique value for the local volatility $\sigma(t, S_t)$ that recovers exactly such marginals

$$\sigma_D^2(T, K) = 2 \frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{K^2 \frac{\partial^2 C}{\partial K^2}}$$

However, it has also been well-known for almost as many years that the evolution in time of local volatility models is not stable: the implied volatility smile generated by the model flattens out over time which contradicts the persistent presence of this phenomenon in reality. Thereby leading researchers and financial engineers to look for a more robust, stochastic volatility type of modeling. Numerous models have been created for over 20 years. Among the most popular ones we find Heston [17], the SABR [9], Fouque et al. [18] or else [28]. For instance, Heston's model is the following

$$\frac{dS_t}{S_t} = r(t)dt + y_t dW_t^1 \qquad dy_t = \kappa(\alpha - y_t) dt + \gamma \sqrt{y_t} dW_t^2$$

These models offer the realistic dynamics that lack in local volatility ones. However, the problem of their calibration arises. To deal with it a common solution is considering a mix of the two kind of volatilities previously mentioned: a generic model with a local and a stochastic component for the volatility. Such models are very useful in practice, since they offer both the flexibility and realistic dynamics of stochastic volatility models, and the exact calibration properties of local volatility models. Their diffusion is the following

$$\frac{dS_t}{S_t} = r(t)dt + a(t, S_t)b(y_t)dW_t^1 \tag{2.2}$$

$$dy_t = \mu(t, y_t)dt + \xi(t, y_t)dW_t^2 \tag{2.3}$$

The problem of calibrating local and stochastic volatility models has been dealt with for quite some time now, its resolution from a stochastic differential equation point of view is based upon Gyongy's mimicking theorems [10]. The equation solved, in the case $r = 0$, by the density of the

¹In [27], and in the case of a arbitrage-free market, the authors prove the existence of a measure, called risk-neutral under which the discounted prices are local martingales

couple spot/volatility is the second order fully nonlinear partial integro-differential equation,

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{S^2}{2} \sigma_D^2 b^2 \frac{\int_{\mathbb{R}} p dy}{\int_{\mathbb{R}} b^2 p dy} \right) - \frac{\partial^2}{\partial S \partial y} \left(\rho \sigma_D b \xi S \left(\frac{\int_{\mathbb{R}} p dy}{\int_{\mathbb{R}} b^2 p dy} \right)^{\frac{1}{2}} p \right) - \frac{\partial^2}{\partial y^2} \left(\frac{\xi^2}{2} p \right) + \frac{\partial}{\partial y} (\mu p) = 0 \quad (2.4)$$

$$p(0, S, y) = \delta(S = S_0, y = y_0) \quad (2.5)$$

The first question that arises is

Problem 1. *Is it possible to prove existence and uniqueness of solutions for the calibration equation (2.4-2.5) of local and stochastic volatility models? What kind of numerical results do we get when solving this pde?*

Another important field of research in financial modeling is the correlation between assets. Empirical measures give satisfactory results. However, the case of a basket on multiple underlyings is problematic. If someone uses local volatility models for each underlying and correlates the brownian motions using the empirical correlation, the basket obtained will not reproduce the vanillas quoted on the market. This raises significant issues when hedging multiple underlying products. One of the solution for this problem is the known "local correlation" approach: the correlation matrix for the n underlyings is deformed using a parameter λ , function of the time and the basket level (hence the name local correlation)

$$\tilde{\rho}_{ij} = \lambda + (1 - \lambda)\rho_{ij} = \rho_{ij} + \lambda(1 - \rho_{ij}) \quad (2.6)$$

This leads to the second problem

Problem 2. *Can we find the value of λ that fits the vanillas of the basket? What kind of results (existence, uniqueness...) can we get on solutions of that equation? Is it possible to calibrate numerically the vanillas of a basket on two underlyings?*

The last financial topic we shall be interested in are interest rates. The most famous model in this area was written by Heath, Jarrow and Morton [26]. As far as we are concerned, we study a hybrid local volatility model with stochastic rates

$$\frac{dS_t}{S_t} = r(t, y_t)dt + \sigma(t, S_t)dW_t^1 \quad dy_t = \mu(t, y_t)dt + \lambda(t, y_t)dW_t^2 \quad (2.7)$$

Using a partial differential equation approach similar to the local correlation and the local and stochastic volatility, we write a calibration equation for the vanillas of this hybrid model. The same kind of question arises

Problem 3. *What are the properties of this calibration equation? Does the numerical resolution of the pde bring a satisfactory fit?*

2.2 Theoretical approach of the calibration equations

From a mathematical point of view, the previous problems boil down to the more general type of equations

$$\frac{\partial p}{\partial t} - L_p p = 0 \quad \text{in }]0, T[\times \Omega \quad (2.8)$$

$$p(0, x) = p_0(x) \quad \text{on } \Omega \quad (2.9)$$

where the operator L_q is equal to

$$L_q p = \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\ \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) p(t, x) \right) - \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^j p(t, x))$$

The essential terms of this equation are the quotients

$$Q_{ij}(t, x) = \frac{\int f_{ij}(x^{\lambda_{ij}^c}) q_{ij}(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int q_{ij}(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}$$

They make it both nonlinear and integro-differential. In [34], the nonlocal term is outside of the derivatives, the authors manage to prove existence and uniqueness of solutions. However, in our case, the Q_{ij} being differentiated, they lead us to a particularly intricate domain of partial integro-differential equations. The most general equation of the type studied until now was, to our knowledge, the following

$$\frac{\partial p}{\partial t} - F(t, x, p, Dp, D^2 p, N(p)) = 0 \quad \text{in } Q^T =]0, T[\times \mathbb{R}^d \quad (2.10) \\ p(0, x) = p_0(x) \quad \text{on } \mathbb{R}^d$$

where F is a function from $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d) \times \mathbb{R}$ into \mathbb{R} . Dp is the gradient of p and $D^2 p$ its hessian matrix. $\mathbb{S}(d)$ is the space of symmetric $d \times d$ real valued matrices. $N(p)$ is a nonlocal term. In [14], the author proves the existence of solutions for (2.10) under suitable assumptions. In particular, $N(p)$ needs to be defined for all continuous and bounded functions p . Moreover, the operator N must be Lipschitz for a certain norm. This kind of result can be adapted to the adjoint of equation (2.8), the operators Q_{ij} are indeed well-defined and Lipschitz on the sets of functions strictly positive, bounded away from 0. However, equation (2.8) can not be dealt with using those methods since the derivatives of the nonlocal term are involved. New questions have to be tackled with, we summarize them

Problem 4. *What method can we use to prove existence of solutions for (2.8)? Do the Q_{ij} have properties we could exploit? What kind of a priori estimates are required?*

2.3 Numerical Resolution

The numerical resolution of the calibration equations is obviously very interesting from a financial point of view. A very efficient method to solve second order parabolic equations is the Alternate Direction Implicit scheme. It is usually applied to classic linear equations and is thoroughly described in [15]. Being in a nonlinear non-local framework, many questions arises

Problem 5. *Is it relevant to use ADI algorithms to solve the equations stemming from our calibration problems? How should we deal with the nonlocal term? Is the finite difference scheme we chose consistent, and what is the order of the truncation error?*

The accuracy of the fit we obtain when solving the calibration equations is satisfactory in the three financial cases we mentioned. However, the case of the local and stochastic volatility model is of particular interest, and raises some questions. Practitioners seem to agree that the stability of the

calibration becomes uncertain when the volatility of the volatility (the function ξ in model (2.3)) is too large. This brings us to the last problem we shall consider in this thesis

Problem 6. *Do we get in practice an instability when ξ is large? Is it possible to link this instability to the limitations we faced when studying Problem 4? Can we find a theoretical explanation to this phenomenon?*

2.4 Outline and Results

This thesis is split into three consecutive parts: the first one deals with the theoretical questions raised in Problem 4. We then apply those results to the financial issues that motivated this work. Finally the third part discusses the numerical aspects we just expounded.

Part I: Theoretical study of the calibration equations

In Chapter 3, we start by introducing the basic spaces used in linear parabolic equations: Hölder spaces. We then give the classic results concerning such equations: what assumptions are required, what do we call compatibility conditions... At last, we shall prove some estimates on the solution u of said equations, for instance

$$|u|_{H^{0,h,h/2}(D^t)} \leq t^{1-\frac{h}{2}} K_{H^1} |f|_{H^0(D^t)}$$

where f is the second member of the equation, and the boundary conditions are assumed to be null.

Chapter 4 is dedicated to the equivalence between stochastic integro-differential equations and partial integro-differential equations. Applying Gyongy's theorem to the case of the local and stochastic volatility model shows that the calibrated diffusion of such models follows an SIDE. It is common knowledge that Kolmogorov forward equations can be written on the density of SDE solutions thus linking SDEs to PDEs. However, in the case of SIDE, this equivalence result needs to be extended. After a brief overview of McKean-Vlasov type equations, which are intimately related to the calibration SIDE, we prove that such an equivalence remains true in our framework. The density of the calibrated local and stochastic volatility model follows the nonlinear PIDE (2.4).

In Chapter 5, we study the general equation (2.8). As pointed out above, this equation is particularly intricate. The fact that the nonlocal term is differentiated renders useless the existing results. The only method that comes to mind to overcome those difficulties is to construct a fixed point for the solution of the parameterized equation. To do so, we need some control over the variations of the quotients Q_{ij} .

An essential feature of those quantities is that when the function f_{ij} is constant, the quotient is trivial (and equal to said constant). Such a fact enables us to prove a first result: if the functions f_{ij} are close enough to constants, we gain the control we needed on Q_{ij} , and show the existence of a solution for (2.8) with a strictly positive boundary condition (2.9). This result is true on $]0, T]$ for any T , the distance f^* between f_{ij} and the constant however decreases when T becomes larger.

In the proof of the previous result, f^* is used for two purposes. First, it guarantees that the solution remains bounded away from 0 by a constant previously determined. Second, it ensures us of the stability of the set of solutions for the parameterized equation. Modifying this proof, we are able to prove an auxiliary result, a local existence in time. We find a different constant f^{**} possibly larger than f^* , a time T^* and a solution for the equation on $]0, T^*]$ with functions f_{ij} close to a constant, the distance being smaller than f^{**} .

We also give a weak uniqueness result, if two functions are solutions of (2.8) with the same boundary conditions and produce the same quantities Q_{ij} , then they are equal. At last, we consider the adjoint equation. Using a fixed point method, we prove short-time existence of a solution for any functions f_{ij} . This result, by itself interesting, also gives us a better understanding of the limitations we face for the equation we were originally studying.

Part II: Financial applications

This part of the thesis is dedicated to the three financial topics we introduced earlier. A first chapter deals with local and stochastic volatility models. We start with a reminder of the properties of a local volatility model. In particular, we give a graph showing the inconsistencies of its dynamics. We then use a partial differential equation approach to find the value of the volatility's local part that fits the vanillas. In itself, the result is not original and can be obtained through Gyongy's theorem as mentioned before. The method however is of interest and will be used in the two other financial cases. We then adapt the theoretical results of Part I to prove existence of solutions when the function b that governs the stochastic part of the volatility is close to a constant. At last, we give the numerical results obtained when solving the calibration equation for two examples of stochastic volatility: a lognormal volatility and a CIR.

The second chapter deals with local correlation models. We start with a graph showing the discrepancy between the vanillas of the underlyings and of the basket. This justifies the use of a different correlation. We then write the value of the parameter λ (defined above) that fits the vanillas using the same kind of method than in the local and stochastic volatility model. This enables us to write a partial integro-differential equation solved by the density of our model. After expounding the existence result we can get concerning that PDE, we solve it numerically in the case of a basket on two underlyings and give the calibration results obtained in this way.

The last chapter is dedicated to the study of hybrid local volatility models with stochastic rates. Using the same approach than in the previous examples, we are able to write a nonlinear PIDE on the density of the diffusion. Due to some differences in the model, the equation does not have the same form as previously. A similar resolution method can however be applied and gives the same kind of existence results. As for the numerical computations they offer a very satisfactory fit of our model's vanillas.

Part III: Numerical studies and instabilities of the solutions

This last part is devoted to the numerical aspects of our work. In Chapter 9, we answer the questions concerning the ADI scheme. We define the finite difference equations used in the resolution. To get a better approximation of the solution, a predictor-corrector algorithm based upon ADI schemes was implemented. We prove the consistency of those two steps and get an error in $O(\Delta t^2)$, as in the classic linear case. To finish, we plot the convergence accuracy of the algorithm against the number of time steps. In the basic one step scheme, we get a $O(\Delta t)$ error as expected. When both the predictor and the corrector steps are used, the graph also shows the theoretical error, a second order convergence.

To conclude the thesis, we study the occurrence of instabilities in the numerical resolution of the LSV calibration problem. As mentioned earlier, it is commonly accepted that the fit loses its accuracy when the volatility of the volatility becomes large. This phenomenon is actually quite consistent with the limitations we faced in the theoretical parts of this work. Indeed, a simple

change of scale shows a direct link between a large ξ and a function b that varies a lot. To check this statement, we plot the density at a given maturity for different functions b . An instability clearly occurs when b takes a large range of values.

To explain this phenomenon, we focus on Hadamard instabilities. At first, we describe this instability and give a few examples of known Hadamard unstable operators. Then, we notice that the linearized calibration equation can be written in a more general form. We find a criterion on the coefficients of this new equation that guarantees its Hadamard instability. Unfortunately, said criterion is not verified by the local and stochastic volatility model.

We conclude in Chapter 11 with a short summary of our results, and of the open questions that still need to be dealt with.

2.5 Notations

Ω : bounded open subset of \mathbb{R}^d

$\partial\Omega$: boundary of Ω

$D :=]0, T[\times \Omega$, cylindrical parabolic domain of definition

$B := \{0\} \times \Omega$, base of the cylinder

$B^T := \{T\} \times \Omega$, top of the domain

$C :=]0, T[\times \partial\Omega$, face of the cylinder

$D^t :=]0, t[\times \Omega$, time-dependent cylinder

$B^t := \{t\} \times \Omega$, top of the time-dependent cylinder

$C^t :=]0, t[\times \partial\Omega$, face of the time-dependent cylinder

\bar{B} : closure of B

$d(P, Q) := [|x - x'|^2 + |t - t'|]^{\frac{1}{2}}$ where $P = (t, x), Q = (t', x') \in D$

$H^{2,1}(D)$: space of functions on D twice continuously differentiable in space and once in time

$H^{k,h,h/2}(D)$: space of Hölder functions on D with order k , exponent h in space and $h/2$ in time

$H^{k,h/2}(]0, T[)$:= space of time dependent Hölder functions in $]0, T[$ with order k and exponent $h/2$

$|u|_{H^{k,h,h/2}(D)}$: Hölder norm related to $H^{k,h,h/2}(D)$

$|u|_{H^0(D)} = \sup_D |u|$

$\delta_{,=x}$:= Dirac distribution in x

Part I

Theoretical study of the calibration equations

Chapter 3

Results concerning linear differential equations of parabolic type

The first part of my thesis is dedicated to the study of the stochastic differential equation

$$\begin{aligned} dX_t &= \left(V_{ij}(t, X_t, \frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}) \right)_{1 \leq i, j \leq d} dW_t + \mu_t dt \\ X_0 &= \phi \end{aligned} \quad (3.1)$$

and of the partial differential equation

$$\begin{aligned} \frac{\partial p}{\partial t} - L_q p &= 0 \\ p(0, x) &= p_0(x) \end{aligned} \quad (3.2)$$

where the operator L_q is equal to

$$\begin{aligned} L_q p &= \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^n V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ij}^c}) q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\ &\quad \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{ij}^c}) q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) p(t, x) \right) - \sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} (\mu_i^j p(t, x)) \end{aligned}$$

When considering the operator $L_q p$ with q frozen and taken as a parameter, the Equation (3.2) becomes a classic second order parabolic equation.

In this first chapter, we recall the existing theory concerning that kind of equation and prove two estimates that shall be needed. We have at our disposal many tools to study them. The usual spaces used for their resolution are Hölder spaces. In a brief first section, we define those spaces. The second section is dedicated to parabolic equations. We recall the known existence results concerning the linearized equation $\frac{\partial p}{\partial t} - L_q p = 0$ as well as Schauder's inequality. At last, we prove a few estimates on different Hölder norms of the solutions, estimates that contain a power of the time in the bound. They shall be used in the next chapters to prove short-time existence of solutions for (3.2).

3.1 Hölder spaces

Let D be a domain in \mathbb{R}^{d+1} , the first coordinate plays the part of the time, the d remaining variables are space variables, a point P in D can also be written as $(t, x) = (t, x_1, \dots, x_d) \in D$. We start as in [3] and [12] with the following notion of parabolic distance¹ $d(P, Q) = [|x - x'|^2 + |t - t'|]^{1/2}$ where $P = (t, x)$ and $Q = (t', x')$ belong to D and $|x|$ is the euclidian norm of the d -dimensional vector x . This formula defines a metric and can be used to define the concept of h -Hölder continuity (with h a number between 0 and 1 strictly). For a function u , we write

$$\begin{aligned} |u|_{H^0(D)} &= \sup_D |u| \\ H_h^D(u) &= \sup_{P, Q \in D} \frac{|u(P) - u(Q)|}{d(P, Q)^h} \\ |u|_{H^{0, h, h/2}(D)} &= |u|_{H^0(D)} + H_h^D(u) \end{aligned}$$

One says that u is uniformly Hölder with exponent h in D if and only if $|u|_{H^{0, h, h/2}(D)} < \infty$. We denote by $H^{0, h, h/2}(D)$ the set of all functions u for which $|u|_{H^{0, h, h/2}(D)} < \infty$. We now assume that u is differentiable with respect to t and twice differentiable with respect to the space variables, with all the derivatives continuous². The following norms can be defined

$$|u|_{H^{1, h, h/2}(D)} = |u|_{H^{0, h, h/2}(D)} + \sum_{1 \leq i \leq d} \left| \frac{\partial u}{\partial x_i} \right|_{H^{0, h, h/2}(D)} + |\partial_t u|_{H^{0, h, h/2}(D)} \quad (3.3)$$

$$|u|_{H^{2, 1}(D)} = |u|_{H^0(D)} + \sum_{1 \leq i \leq d} \left| \frac{\partial u}{\partial x_i} \right|_{H^0(D)} + \sum_{1 \leq i, j \leq d} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{H^0(D)} + |\partial_t u|_{H^0(D)} \quad (3.4)$$

$$|u|_{H^{2, h, h/2}(D)} = |u|_{H^{1, h, h/2}(D)} + \sum_{1 \leq i, j \leq d} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{H^{0, h, h/2}(D)} \quad (3.5)$$

We denote respectively by $H^{1, h, h/2}(D)$, $H^{2, 1}(D)$ and $H^{2, h, h/2}(D)$ the set of all functions u for which $|u|_{H^{1, h, h/2}(D)} < \infty$, respectively $|u|_{H^{2, 1}(D)} < \infty$ and $|u|_{H^{2, h, h/2}(D)} < \infty$. Those sets are Banach spaces and algebras for the norms given by definitions (3.3), (3.4) and (3.5). Indeed, for all u, v in $H^{0, h, h/2}(D)$, we see that

$$H_h^D(uv) \leq H_h^D(u)|v|_{H^0(D)} + H_h^D(v)|u|_{H^0(D)}$$

which clearly gives us

$$|uv|_{H^{k, h, h/2}(D)} \leq |u|_{H^{k, h, h/2}(D)}|v|_{H^{k, h, h/2}(D)} \quad (3.6)$$

for all u, v in $H^{k, h, h/2}(D)$ and $k = 0, 1, 2$. The spaces of higher order can be defined exactly in the same way. In the rest of this work, we shall need $H^{k, h, h/2}(D)$ for $k = 1, 2, 4$.

3.2 Linear equations of parabolic type

We are now interested in parabolic differential equations. Let us first define more precisely the domain of definition of the equations. D is in the following a bounded $(d + 1)$ dimensional domain,

¹The same notion obviously exists in an elliptic framework as defined in [25]

²We notice that this assumption is only needed for $H^{2, 1}(D)$ since h -Hölder continuity implies regular continuity.

lying between a domain B on $t=0$ and another domain B^T on $t = T$ (for a fixed T). The rest of the boundary of D , lying in the strip $0 < t \leq T$, is denoted by C . We also assume that D verifies the property

Property 1. *Smoothness of the boundary.* For every point Q of \bar{C} , there exists a $(d+1)$ -dimensional neighborhood V such that $V \cap \bar{C}$ can be represented, for some i ($1 \leq i \leq d$), in the form

$$x_i = r(t, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \quad (3.7)$$

with $r, \frac{\partial r}{\partial x}, \frac{\partial^2 r}{\partial x^2}, \frac{\partial r}{\partial t}$ Holder continuous (exponent h) and $\frac{\partial^2 r}{\partial x \partial t}, \frac{\partial^2 r}{\partial t^2}$ simply continuous.

Since we wish to impose boundary conditions, we also have to consider functions ψ defined on $\bar{B} \cup C$.

Definition. A function ψ is said to belong to $H^{k,h,h/2}(\partial D)$ if there exists a function Ψ in $H^{k,h,h/2}(D)$ and $\Psi = \psi$ on $\bar{B} \cup C$. We then define

$$|\psi|_{H^{k,h,h/2}(\partial D)} = \inf |\Psi|_{H^{k,h,h/2}(D)}$$

where the inf is taken on the set of functions Ψ in $H^{k,h,h/2}(D)$ that coincide with ψ on $\bar{B} \cup C$.

This process defines a norm on $H^{k,h,h/2}(\partial D)$. Let us now consider the operator

$$\mathcal{L}u := \sum_{i,j=1}^d a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u \quad (3.8)$$

The coefficients of \mathcal{L} are assumed to verify

- $\forall 1 \leq i, j \leq d, a_{ij}, b_i$ and c belong to $H^{0,h,h/2}(D)$ and

$$|a_{ij}|_{H^{0,h,h/2}(D)} \leq K_1 \quad |b_i|_{H^{0,h,h/2}(D)} \leq K_1 \quad |c|_{H^{0,h,h/2}(D)} \leq K_1 \quad (3.9)$$

- There exists $K_2 > 0$ such that $\forall (t,x) \in D$ and $\forall \xi \in \mathbb{R}^d$

$$\sum_{i,j=1}^d a_{ij}(x,t) \xi_i \xi_j \geq K_2 |\xi|^2 \quad (3.10)$$

An operator verifying (3.10) is said to be uniformly parabolic on D . Let us now consider the equation

$$\frac{\partial u}{\partial t} - \mathcal{L}u = f(x,t) \quad \text{in } D \cup B^T \quad (3.11)$$

$$u = \psi \quad \text{on } \bar{B} \cup C \quad (3.12)$$

The boundary condition and term source are assumed to verify

$$f \in H^{0,h,h/2}(D) \quad \psi \in H^{2,h,h/2}(\partial D) \quad (3.13)$$

In addition, imposing boundary values for the solution of the partial differential equation (3.11) implies some compatibility assumptions between \mathcal{L} and ψ . Indeed, let us consider a function Ψ extension of ψ in D . By continuity and given assumption (3.7) on B , the quantity $\frac{\partial \Psi}{\partial t}$ is properly

defined on C and does not depend there on the choice of Ψ . We are thus able to define on C the quantity $\frac{\partial \psi}{\partial t}$.

The same method allows us to define for all $1 \leq i, j \leq d$, $\frac{\partial \psi}{\partial x_i}$ and $\frac{\partial^2 \psi}{\partial x_i \partial x_j}$ on B , the base of our domain of definition. Hence, on ∂B , which is also a part of the boundary of C , the operator $\mathcal{L}\psi$ and the derivative $\frac{\partial \psi}{\partial t}$ are both defined. Since we wish a function u to solve $\frac{\partial u}{\partial t} - \mathcal{L}u = f(x, t)$ on D , with $u = \psi$ on $\overline{B} \cup C$, by continuity of all the quantities used here, we can write the following definition

Definition. *Compatibility condition.* We say that a boundary condition ψ defined on $\overline{B} \cup C$ verifies the compatibility condition with the source function f if

$$\frac{\partial \psi}{\partial t} - \mathcal{L}\psi = f(x, t) \quad \text{on } \partial B \quad (3.14)$$

We have given all the assumptions needed for the existence theorem, we can state it now

Theorem 1. *If ψ verifies the compatibility condition (3.14) and under the assumptions (3.9), (3.10) and (3.13), there exists a unique solution of the Equation (3.11), this solution belongs to $H^{2,h,h/2}(D)$ and we have the Schauder inequality (with K_{H^2} depending only on K_1 , on K_2 , on h and on D)*

$$|u|_{H^{2,h,h/2}(D)} \leq K_{H^2} (|\psi|_{H^{2,h,h/2}(\partial D)} + |f|_{H^{0,h,h/2}(D)}) \quad (3.15)$$

Proof. This result is classic, its proof can be found for instance in Chapter 3 of [12] by A. Friedman. \square

We now consider cylindrical domains and slightly different notations. Since we wish to prove time-dependent results, the height T of the domain of definition is now a variable. The base of the cylinder is still denoted by B , but we define the domain as $D^t =]0, t[\times \Omega$. We also write $C^t =]0, t[\times \partial B$ the side of the cylinder. We are interested in Equation (3.11) on D^t with the boundary condition $u = 0$ on $\overline{B} \cup C^t$. We prove the following original estimates that shall be useful to get short-time existence of solutions in the next chapters

Lemma 1. *When ψ is equal to 0 in the previous theorem, we can write bounds containing the time on norms of the solution. For all $0 < t \leq T$, the solution u of (3.11)-(3.12) on D^t verifies*

$$|u|_{H^0(D^t)} \leq t K_{H^0} |f|_{H^0(D^t)} \quad (3.16)$$

$$|u|_{H^{0,h,h/2}(D^t)} \leq t^{1-\frac{h}{2}} K_{H^1} |f|_{H^0(D^t)} \quad (3.17)$$

K_{H^0} and K_{H^1} only depends on K_1 , on K_2 , on h and on B .

Proof. To prove the first inequality, one needs a result from O. Ladyzhenskaya, V. Solonnikov and N. Ural'ceva [3] about volume potentials and representation of solutions of parabolic equations. It is the theorem (16.2) of section IV.16 we shall use. One reads there that the solution of Equation (3.11) with $\psi = 0$ can be written as

$$u(x, t) = \int_0^t \int_{\Omega} G(x, y, t, \tau) f(y, \tau) dy d\tau$$

where G is Green's function for the operator \mathcal{L} and verifies for all $x, y \in D$, $0 \leq \tau < t \leq T$ the inequality

$$|G(x, y, t, \tau)| \leq K(t - \tau)^{-\frac{d}{2}} \exp(-K' \frac{|x - y|^2}{t - \tau})$$

with K and K' two constants depending on the data of the problem but not on f , that might change from one line to another. Using both these results, we get, for all $t' \leq t$ and $x \in \Omega$

$$\begin{aligned} |u(x, t')| &\leq \int_0^{t'} \int_{\Omega} |G(x, y, t', \tau)| |f(y, \tau)| dy d\tau \\ &\leq |f|_{H^0(D')} \int_0^{t'} d\tau \int_{\Omega} K(t - \tau)^{-\frac{d}{2}} \exp(-K' \frac{|x-y|^2}{t-\tau}) dy \\ &\leq t |f|_{H^0(D')} K_{H^0} \end{aligned}$$

where K_{H^0} depends on K , on K' and on B . Eventually, we get the inequality

$$|u|_{H^0(D')} \leq t K_{H^0} |f|_{H^0(D')}$$

As for the second inequality, we need other estimates of Green's function: for all $x, x'', y \in D$, $0 \leq \tau < t \leq T$

$$|G(x, y, t, \tau) - G(x', y, t, \tau)| \leq K \frac{|x - x'|^h}{(t - \tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x'' - y|^2}{t - \tau})$$

with x'' the closest point to y between x and x' . And for all $x, y \in D$, $0 \leq \tau < t' < t \leq T$ the inequality

$$|G(x, y, t, \tau) - G(x, y, t', \tau)| \leq K \frac{|t - t'|^{\frac{h}{2}}}{(t' - \tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x - \xi|^2}{t - \tau})$$

The proof of those estimates can be found in appendix B. Now, for all $0 \leq t'' \leq t' \leq t$ and $x, x' \in \Omega$, we have

$$\begin{aligned} |u(x, t') - u(x', t')| &\leq \int_0^{t'} \int_{\Omega} |G(x, y, t', \tau) - G(x', y, t', \tau)| |f(y, \tau)| dy d\tau \\ &\leq K |f|_{H^0(D')} \int_0^{t'} \int_{\Omega} \frac{|x - x'|^h}{(t' - \tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x'' - y|^2}{t' - \tau}) dy d\tau \\ &\leq K |f|_{H^0(D')} \int_0^{t'} \frac{|x - x'|^h}{(t' - \tau)^{\frac{h}{2}}} d\tau \leq t'^{1-\frac{h}{2}} K |f|_{H^0(D')} |x - x'|^h \end{aligned}$$

We assume that $t' - t'' < t''/2$, this ensures us that $0 < 2t'' - t'$ and that $\forall 0 \leq \tau \leq 2t'' - t'$, we have $t' - \tau \leq 2(t'' - \tau)$. Let us compute

$$\begin{aligned} |u(x, t') - u(x, t'')| &\leq \int_0^{2t''-t'} \int_{\Omega} |G(x, y, t', \tau) - G(x, y, t'', \tau)| |f(y, \tau)| dy d\tau \\ &+ \int_{2t''-t'}^{t'} \int_{\Omega} |G(x, y, t', \tau)| |f(y, \tau)| dy d\tau + \int_{2t''-t'}^{t''} \int_{\Omega} |G(x, y, t'', \tau)| |f(y, \tau)| dy d\tau \\ &\leq K |f|_{H^0(D')} (\int_0^{2t''-t'} \int_{\Omega} \frac{|t' - t''|^{\frac{h}{2}}}{(t'' - \tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x-y|^2}{t' - \tau}) dy d\tau + |t' - t''|) \\ &\leq K |f|_{H^0(D')} |t' - t''|^{\frac{h}{2}} (\int_0^{2t''-t'} \int_{\Omega} \frac{2^{\frac{d+h}{2}}}{(t'' - \tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x-y|^2}{t' - \tau}) dy d\tau + |t' - t''|^{1-\frac{h}{2}}) \\ &\leq K |f|_{H^0(D')} |t' - t''|^{\frac{h}{2}} (t'^{1-\frac{h}{2}} + |t' - t''|^{1-\frac{h}{2}}) \end{aligned}$$

We now study the case $t' - t'' \geq t''/2$.

$$\begin{aligned}
|u(x, t') - u(x, t'')| &\leq \int_0^{t''} \int_{\Omega} |G(x, y, t', \tau) + G(x, y, t'', \tau)| |f(y, \tau)| dy d\tau \\
&+ \int_{t''}^{t'} \int_{\Omega} |G(x, y, t', \tau)| |f(y, \tau)| dy d\tau \\
&\leq K |f|_{H^0(D')} (t'' + |t' - t''|)
\end{aligned}$$

This concludes the proof. □

Remark 1. *The proof of inequality (3.15) in [3] shows that the constant K_{H^1} only depends on the coefficients of the operator \mathcal{L} through their norm, it is actually a nondecreasing function of those norms. Such a property is also true for the constant K_{H^2} .*

We shall also need a weak form of the maximum principle

Proposition 1. *Let us consider Equation (3.11) with the boundary condition (3.12). If $\psi \geq 0$ on $\overline{B} \cup C^T$ and $f \geq 0$ in $D \cup B^T$, then $u \geq 0$ in $D \cup B^T$.*

Proof. The proof of this property is classic, it can be found for instance on p.135 of Friedman [5] in the case where $c \geq 0$. The change of function $q = \exp(-c^*t)p$ with $c^* \geq c$ deals with the general case. □

Chapter 4

Theoretical study of the SIDE and PIDE stemming from the calibration

The calibration of diffusion models in finance (such as the Local and Stochastic Volatility model, or the Local Correlation model) raises very challenging theoretical problems. They can be studied from two - equivalent - points of view.

The first one is probabilistic, by the mean of Stochastic Differential Equations. Indeed, using Gyongy's theorem, one can prove (as shall be done in Chapter 6, Section 6.2) that for a given Local Volatility $\sigma_D(T, K)$, the diffusion model following the stochastic differential equation

$$\begin{aligned}\frac{dS_t}{S_t} &= \frac{\sigma_D(t, S_t)}{\sqrt{\mathbb{E}[b^2(y_t)|S_t]}} b(y_t) dW_t^1 \\ dy_t &= \xi(t, y_t) dW_t^2 + \mu(t, y_t) dt\end{aligned}\quad (4.1)$$

replicates the vanilla prices implicitly defined by $\sigma_D(T, K)$. W_t^1 and W_t^2 are two correlated brownian motions, we denote by ρ their correlation and assume it to be constant.

If we denote by p_t the time density of the couple (S_t, y_t) (assuming it exists), this last equation can also be seen formally as

$$\begin{aligned}\frac{dS_t}{S_t} &= \left(\frac{\int p_t(S_t, y) dy}{\int b^2(y) p_t(S_t, y) dy} \right)^{\frac{1}{2}} \sigma_D(t, S_t) b(y_t) dW_t^1 \\ dy_t &= \xi(t, y_t) dW_t^2 + \mu(t, y_t) dt\end{aligned}\quad (4.2)$$

Equivalently, such a diffusion can be studied from a Partial Differential Equation point of view by the mean of Kolmogorov forward equation, described for instance in [5] or [47]. The density p_t of the previous model satisfies the nonlinear integro-differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial S^2} (\sigma_D^2 b^2 \frac{\int p dy}{\int b^2 p dy} p) + \frac{\partial^2}{\partial S \partial y} (\rho \sigma_D b \xi (\frac{\int p dy}{\int b^2 p dy})^{\frac{1}{2}} p) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (\xi^2 p) - \frac{\partial}{\partial y} (\mu p) \quad (4.3)$$

The previous stochastic and partial differential equations can be formalized in a more general framework, where one can still prove that existence of solutions for one is equivalent to existence of solutions for the other. In this third chapter, we study theoretical aspects of those different types of equations. We start with a brief overview of McKean-Vlasov equations, which present many likenesses with our case. We then describe a generalized form of Equation (4.2) and expound the assumptions we shall need. At last, we prove the equivalence between this generalized equation

and a nonlinear partial integro-differential equation: existence of a strong solution for one of the two implies the same thing for the other.

4.1 McKean-Vlasov type equations

4.1.1 First example

We say an SDE is of McKean-Vlasov type when the coefficients of the diffusion depend on the distribution of the solution itself. It was suggested by Kac [31] and is a model for Vlasov kinetic equation of plasma [32] (hence its name). McKean first studied it in [44], two important references in the matter are [30] and [45]. A simple example we give here is

$$\begin{aligned} dX_t &= \left(\int b(X_t, y) \mu_t(dy) \right) dt + dW_t \\ X_0 &= x_0 \end{aligned} \quad (4.4)$$

where X_t is a \mathbb{R}^d process, W_t a \mathbb{R}^d brownian motion and $x_0 \in \mathbb{R}^d$. The function b is defined on $\mathbb{R}^d \times \mathbb{R}^d$. The important feature of this equation is the dependance on μ_t , the law of the random variable X_t . We call solution of the Equation (4.4) a couple (X_t, μ_t) verifying (4.4). In [1], the following theorem is proved

Theorem 2. *If b verifies the assumptions $|b(x, y)| \leq C(1 + |x|)$ and $\forall x, b(x, \cdot)$ is continuous, then there exists a strong solution to Equation (4.4).*

This previous equation is nonlocal: the dependence on the law of X_t has nonlocality in the sense that the drift term at a given point x depends on the entire distribution μ_t of the solution.

4.1.2 Local dependence

A different kind of McKean-Vlasov equations can be defined. We consider here a stochastic differential equation where both the drift and the volatility term depend on the density of the time marginal, the dependence in this case being local. It is written as follows

$$\begin{aligned} dX_t &= b(X_t, p_t(X_t)) dt + \sigma(X_t, p_t(X_t)) dW_t \\ X_0 &= x_0 \end{aligned} \quad (4.5)$$

where p_t is the density of X_t (with respect to Lebesgue measure). Here too, the processes considered belong to \mathbb{R}^d . The volatility matrix σ is a mapping from $\mathbb{R}^d \times \mathbb{R}$ into the $d \times d$ matrices which are symmetric and verify the condition

$$\forall x, p \in \mathbb{R}^d \times \mathbb{R}, \forall \xi \in \mathbb{R}^d, \xi^t \sigma(x, p) \xi \geq K_\sigma |\xi|^2$$

We also let $a = \sigma \sigma^*$ denote the square of matrix σ . The drift term is a function from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} . W_t is still a d -dimensional brownian motion¹. x_0 the initial condition is now a random variable on \mathbb{R}^d . In [2], B. Jourdain and S. Meleard prove, by placing this equation in the frame of Hölder spaces, the result

Theorem 3. *Under the stronger assumption*

$$\forall x, p \in \mathbb{R}^d \times \mathbb{R}, \forall \xi \in \mathbb{R}^d, \xi^t \left(a(x, p) + p \frac{\partial a}{\partial p}(x, p) \right) \xi \geq K_a |\xi|^2$$

¹In [33], the authors give some existence results in the case of SDEs driven by Levy processes.

the nonlinear stochastic differential equation (4.5) admits a unique strong solution (X_t, p_t) .

The method used to prove the theorem actually gives p as the solution in $H^{2,h,h/2}([0, T] \times \mathbb{R}^d)$ of the nonlinear partial differential equation

$$\begin{aligned} \frac{\partial p}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x, p(x))p(x)) + \sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} (b_i(x, p(x))p(x)) &= 0 \\ p_0(x) &= \xi(x) \end{aligned} \quad (4.6)$$

where ξ is the density of the random variable x_0 and is assumed to belong to $H^{2,h,h/2}$. The functions σ and b belong respectively to $H^{2,h,h/2}(\mathbb{R}^d \times \mathbb{R})$ and $H^{1,h,h/2}(\mathbb{R}^d \times \mathbb{R})$. The Hölder spaces $H^{k,h,h/2}$ are defined in section 3.1 of Chapter 3. The Equation (4.6) has been known to admit solutions since the work of O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'ceva [3]. We mention this previous example because the methods used in the article [2] are similar to ours.

4.1.3 General existence result

The previous result required assumptions about the derivatives of σ and b . Here, we state a theorem with weaker requirements. The usual assumption for classical stochastic differential equation is the Lipschitz growth of the coefficients. In the case of coefficients depending on the law of the process X_t , the same is true when the Lipschitz condition is defined with respect to Vasserstein metric on the space of probability laws on \mathbb{R}^d . The definition of this metric is the following

$$\rho(p, q) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| r(dx, dy) \right\}$$

where the inf is taken over all measures r with marginals p on x and q on y . In [4], K. Graham proves the

Theorem 4. *If for all $x, y \in \mathbb{R}^d$ and p, q probability laws on \mathbb{R}^d , we have $|\sigma(x, p) - \sigma(y, q)| + |b(x, p) - b(y, q)| \leq K(|x - y| + \rho(p, q))$, then there is strong existence and uniqueness for the stochastic differential equation (4.5).*

This result can not be applied to our equations: the quantity $\frac{\int p_t(S_t, y) dy}{\int b^2(y) p_t(S_t, y) dy}$ is not lipschitz with respect to the metric ρ .

4.2 Stochastic Differential Equation with conditional expectations

We are interested in the following d -dimensional stochastic differential equation

$$\begin{aligned} dX_t &= \sigma_t dW_t + \mu_t dt \\ X_0 &= \phi \end{aligned} \quad (4.7)$$

with W_t a Brownian motion in \mathbb{R}^d and μ_t a given drift. The initial condition ϕ is a given random variable. The volatility matrix σ_t is chosen as

$$\sigma_t^{ij} = V_{ij}(t, X_t, \mathbb{E}[f_{ij}(X_t^{\lambda_{ij}^c}) | X_t^{\lambda_{ij}}]) \quad (4.8)$$

where X_t^i is the i -th coordinate of X_t , λ_{ij} a subset of $\{1, \dots, d\}$ and $X_t^{\lambda_{ij}}$ the vector $(X_t^k)_{k \in \lambda_{ij}}$. In the case where X_t admits a density $p(t, \cdot)$, the stochastic differential equation becomes

$$\begin{aligned} dX_t &= \left(V_{ij}(t, X_t, \frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}) \right)_{1 \leq i, j \leq d} dW_t + \mu_t dt \\ X_0 &= \phi \end{aligned} \quad (4.9)$$

where $\int f(x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}$ is the integral against the variables x_{k_1}, \dots, x_{k_l} with (k^1, \dots, k^l) the complementary set of λ_{ij} in $\{1, \dots, d\}$, the domain of integration being \mathbb{R}^l .

We also write $p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) = p(t, x)$ with $x \in \mathbb{R}^d$ and $x_i = X_t^i$ if $i \in \lambda_{ij}$, $x_i = x_i^{\lambda_{ij}^c}$ otherwise. This last diffusion is a generalization of the equations that appear when calibrating the previously mentioned financial models. In part II of this thesis, one can find different examples of equations involving conditional expectations or quotients of the form

$$\frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}$$

In the current section, we prove a relationship between these last stochastic differential equations and the following nonlinear integro-differential equation

$$\begin{aligned} \frac{\partial p}{\partial t} - L_p p &= 0 && \text{in }]0, T[\times \mathbb{R}^d \\ p(0, x) &= p_0(x) && \text{on } \mathbb{R}^d \end{aligned} \quad (4.10)$$

where the operator L_q is defined as

$$\begin{aligned} L_q p &= \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) p_{ik}(t, x_t^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int p_{ik}(t, x_t^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\ &\quad \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) p_{jk}(t, x_t^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int p_{jk}(t, x_t^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) p(t, x) \right) - \sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} (\mu_t^i p(t, x)) \end{aligned}$$

and p_0 is the density of ϕ the initial condition for Equation (4.7).

Remark. • *An example of equation involving conditional expectations can be found in [52]. In that case however, the conditioning is taken with respect to another random variable, not the unknown process itself.*

- *For other articles concerning conditional expectations in the drift term, we refer the reader to [57] and its references.*

4.2.1 The classic stochastic differential equation associated with (4.9)

Let us first study the equation where p is considered as a parameter and not an unknown. As in [2], the functional spaces used are Hölder spaces, they are described thoroughly in Chapter 3. We make the following assumptions on the coefficients of the equation

- $\forall 1 \leq i \leq d, \mu_t^i \in H^{1, h, h/2}(D^T)$

- $\forall 1 \leq i, j \leq d, V_{ij}(t, x, e) \in C^3(\mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}_+^*)$ with all derivatives bounded on the domain of definition
- $\forall 1 \leq i, j \leq d, f_{ij} \in H^{2,h,h/2}(\mathbb{R}^d)$
- $\forall (t, x, e) \in \mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}_+^*, \forall (\xi_i)_{1 \leq i \leq d} \in \mathbb{R}^d,$

$$\sum_{1 \leq i, j \leq d} \xi_i \xi_j \sum_{k=1}^d V_{ik}(t, x, e) V_{jk}(t, x, e) \geq K |\xi|^2$$

We now let q denote a function belonging to $H^{2,h,h/2}([0, T[\times \mathbb{R}^d)$ such that

$$\forall 1 \leq i \leq d, \frac{\int f_{ij}(x^{\lambda_{ij}^c}) q_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int q_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}} \in H^{2,h,h/2}(D^T)$$

The classic stochastic differential equation we are interested in is

$$\begin{aligned} dX_t &= \left(V_{ij}(t, X_t, \frac{\int f_{ij}(x^{\lambda_{ij}^c}) q_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int q_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}) \right)_{1 \leq i, j \leq d} dW_t + \mu_t dt \\ X_0 &= \phi \end{aligned} \quad (4.11)$$

Using results from Friedman in Chapter VI of his book [5], and given the set of assumptions we just described, we have the

Proposition 2. *The Stochastic Differential Equation (4.11) admits a unique strong solution X_t . Moreover, there exists a function $p \in H^{2,h,h/2}([0, T[\times \mathbb{R}^d)$ such that $p(t, \cdot)$ is the density of X_t with respect to Lebesgue measure. This function p is the unique solution of the partial differential equation $\frac{\partial p}{\partial t} - L_q p = 0$ with the initial condition $p(0, \cdot) = p_0$.*

4.2.2 Equivalence between existence results for (4.9) and existence results for (4.10)

In this section, we are interested in the links between the Equations (4.9) and (4.10). We prove the following

Proposition 3. *If the Equation (4.9) admits a couple (X_t, p) as solution for $t \in [0, T]$, with $p(t, \cdot)$ the density of X_t and $p \in H^{2,h,h/2}([0, T[\times \mathbb{R}^d)$, then the function p is solution of the partial integro-differential equation (4.10).*

Reciprocally, if the Equation (4.10) has a solution $p \in H^{2,h,h/2}([0, T[\times \mathbb{R}^d)$, then there exists a process $(X_t)_{t \in [0, T]}$ such that (X_t, p) is a strong solution of Equation (4.9).

Proof. We start by assuming the existence of a couple (X_t, p) such that

$$\begin{aligned} dX_t &= \left(V_{ij}(t, X_t, \frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}) \right)_{1 \leq i, j \leq d} dW_t + \mu_t dt \\ X_0 &= \phi \end{aligned}$$

with $t \in [0, T]$, $p \in H^{2,h,h/2}([0, T[\times \mathbb{R}^d)$ and $p(t, \cdot)$ the density of the random variable X_t . Since $X_0 = \phi$, and the law of ϕ is assumed to have density $p_0(\cdot)$, we have $p(0, x) = p_0(x)$ on \mathbb{R}^d .

Now, let f be a function in $C_b^2(\mathbb{R}^d)$, Ito's lemma gives us

$$\begin{aligned}
df(X^t) &= \sum_{1 \leq i \leq d} \frac{\partial f}{\partial x_i} dX_t^i + \sum_{1 \leq i, j \leq d} \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} \langle dX_t^i, dX_t^j \rangle \\
&= \sum_{1 \leq i \leq d} \frac{\partial f}{\partial x_i} \mu_t^i dt + \sum_{1 \leq i, j \leq d} \frac{\partial f}{\partial x_i} V_{ij}(t, X_t, \frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}) dW_t^j \\
&\quad + \sum_{1 \leq i, j \leq d} \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} \left(\sum_{k=1}^n V_{ik}(t, X_t, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) p_{ik}(t, X_t^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int p_{ik}(t, X_t^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\
&\quad \left. V_{jk}(t, X_t, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) p_{jk}(t, X_t^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int p_{jk}(t, X_t^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) \right)
\end{aligned}$$

Integrating this equality and taking expectations, we have

$$\begin{aligned}
\mathbb{E}[f(X_t)] - \mathbb{E}[f(X_0)] &= \int_0^t \sum_{1 \leq i \leq d} \mathbb{E}[\frac{\partial f}{\partial x_i}(X_s) \mu_s^i] ds + \int_0^t \sum_{1 \leq i, j \leq d} \frac{1}{2} \mathbb{E}[\frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) \\
&\quad \left(\sum_{k=1}^n V_{ik}(s, X_s, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) p_{ik}(s, X_s^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int p_{ik}(s, X_s^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) V_{jk}(s, X_s, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) p_{jk}(s, X_s^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int p_{jk}(s, X_s^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) \right)] ds
\end{aligned}$$

Since $p(t, \cdot)$ is the density of X_t , the expectations can be written differently

$$\mathbb{E}[f(X_t)] - \mathbb{E}[f(X_0)] = \int_0^t \int_{\mathbb{R}^d} p \mathbb{L}_p f dx ds \tag{4.12}$$

where \mathbb{L}_p is defined as

$$\begin{aligned}
\mathbb{L}_p f(t, x) &= \sum_{1 \leq i \leq d} \frac{\partial f}{\partial x_i}(x) \mu_t^i + \sum_{1 \leq i, j \leq d} \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \left(\sum_{k=1}^n V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) p_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int p_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\
&\quad \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) p_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int p_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) \right)
\end{aligned}$$

In equality (4.12), we integrate by parts. The adjoint L_p of operator \mathbb{L}_p thus appears, with derivatives in the sense of distributions. Since p belongs to $H^{2,h,h/2}([0, T] \times \mathbb{R}^d)$, the derivatives are in fact classic and we get

$$\int_{\mathbb{R}^d} f(x) p(t, x) dx - \int_{\mathbb{R}^d} f(x) p(0, x) dx = \int_0^t \int_{\mathbb{R}^d} f(x) L_p p(s, x) dx ds$$

Differentiating this last line with respect to the time, it becomes

$$\int_{\mathbb{R}^d} f(x) \frac{\partial p}{\partial t}(t, x) dx = \int_{\mathbb{R}^d} f(x) L_p p(t, x) dx$$

The equality being true for all $f \in C_b^2(\mathbb{R}^d)$, we obtain p as solution of the equation $\frac{\partial p}{\partial t} - L_p p$ with initial condition $p(0, x) = p_0(x)$. This concludes the first part of the proof.

As for the reciprocal, let p belong to $H^{2,h,h/2}([0, T] \times \mathbb{R}^d)$ and solve $\frac{\partial p}{\partial t} - L_p p$ with $p(0, x) = p_0(x)$.

We use this function p as parameter in the stochastic differential equation (4.11). Proposition 2 gives us the strong existence of a process X_t solution of the stochastic differential equation

$$dX_t = \left(V_{ij}(t, X_t, \frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}(t, X_t^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}) \right)_{1 \leq i, j \leq d} dW_t + \mu_t dt \quad (4.13)$$

$$X_0 = \phi$$

It also gives us the density of this process X_t as the unique solution of equation $\frac{\partial \cdot}{\partial t} - L_p \cdot = 0$ with initial condition p_0 . Since p verifies this equation, the couple (X_t, p) is indeed solution of (4.9). This concludes the proof. \square

Chapter 5

Resolution of the Partial Integro-Differential Equation

We are now interested in the equations we presented in Chapter 4. The first one is the evolution equation associated to operator L_p

$$\frac{\partial p}{\partial t} - L_p p = 0 \quad \text{in }]0, T[\times \Omega \quad (5.1)$$

$$p(0, x) = p_0(x) \quad \text{on } \Omega \quad (5.2)$$

where the operator L_q is equal to

$$L_q p = \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) p(t, x) \right) - \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^t p(t, x))$$

This last Equation (5.1) is written in a form where the derivatives of the non local terms are involved. It is an extremely complicated endeavour to solve it, we are only able to prove two partial existence results. Both are based upon the following idea: if we consider constant functions f_{ij} , then the quotient of integrals simplifies itself and the equation becomes a classic linear parabolic equation. We thus consider a set of functions $(f_{ij})_{1 \leq i, j \leq d}$, a set of constants $(\bar{f}_{ij})_{1 \leq i, j \leq d}$ and assume some control over the gap between those two sets: for all $1 \leq i, j \leq d$

$$|f_{ij} - \bar{f}_{ij}| \leq f^* \quad (5.3)$$

Using fixed point methods in Hölder spaces and the previous statement, we prove two theorems:

- for any T , existence of a solution on $]0, T]$ for f^* small enough.
- existence of an $F^* > 0$ such that for all $f^* \leq F^*$, we have short time existence of a solution.

Another equation is of particular interest. The evolution equation for the adjoint \mathbb{L} of operator L is simpler, and quite similar to a few problems studied in the literature. Since the nonlocal terms are this time outside of the derivatives, we get more general results and a better understanding of the

original equation. We thus have another equation to deal with

$$\frac{\partial p}{\partial t} - \mathbb{L}_p p = 0 \quad \text{in }]0, T[\times \Omega \quad (5.4)$$

$$p(0, x) = p_0(x) \quad \text{on } \Omega \quad (5.5)$$

where the operator \mathbb{L}_q is equal to

$$\begin{aligned} \mathbb{L}_q p &= \frac{1}{2} \sum_{1 \leq i, j \leq d} \left(\sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\ &\quad \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) \right) \frac{\partial^2 p}{\partial x_i \partial x_j} + \sum_{1 \leq i \leq d} \mu_i^j \frac{\partial p}{\partial x_i} \end{aligned}$$

As far as Equation (5.4) is concerned, we do not need such assumptions over the set $(f_{ij})_{1 \leq i, j \leq d}$ to prove existence of a solution. We are actually able to prove short time existence for any $(f_{ij})_{1 \leq i, j \leq d}$.

Since the functions $(f_{ij})_{1 \leq i, j \leq d}$ play a particular part in this chapter, we use different notations for the operators. First, we let $\{f\}$ define the collection $(f_{ij})_{1 \leq i, j \leq d}$. Then, we write

$$\begin{aligned} L_{ij}(p, \{f\}) &:= \sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) p_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int p_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \\ &\quad V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) p_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int p_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) \end{aligned} \quad (5.6)$$

The operator L_p is now considered as a mapping taking a function p and a collection $\{f\}$ as arguments, it becomes

$$L(p, \{f\}) := \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (p L_{ij}(p, \{f\})) - \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^j p)$$

This chapter is organized as follows. First, we study the operator L , the assumptions required for it to be properly defined and its differentiability. We then prove the two theorems concerning Equation (5.1). And at last, we deal with the case of the adjoint equation (5.4).

5.1 Assumptions and study of the operator L

Keeping the notations from Chapter 3, we denote by $0 < t \leq T$ the time-variable and by $x = (x_1, x_2, \dots, x_d) \in \Omega \subset \mathbb{R}^d$ the d -dimensional space variable where Ω is an open bounded subset with a sufficiently smooth boundary, meaning that $D^T =]0, T[\times \Omega$ verifies Property 1. D^T is the domain of definition of the variable (t, x) . We also denote by $B = \{0\} \times \Omega$, $B^T = \{T\} \times \Omega$ and $C^T =]0, T[\times \partial\Omega$ the different parts of the boundary. Since we want to use the classic results from Chapter 3 concerning parabolic equations, we assume that all the coefficients of L belong to the appropriate Hölder spaces

- $\forall 1 \leq i \leq d, \mu_i^j \in H^{1, h, h/2}(D^T)$, $\forall 1 \leq i, j \leq d, f_{ij} \in H^{2, h, h/2}(\mathbb{R}^d)$ with K_0 a born on their norms
- $\forall 1 \leq i, j \leq d, V_{ij}(t, x, e) \in C^3(\mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}_+^*)$ and its three derivatives with respect to the last variable are bounded on $[\eta, +\infty[$ for all $\eta > 0$

Another necessary condition concerns the quotients

$$Q_{ij}(t, x) = \frac{\int_{\Omega_{ij}(x)} f_{ij}(x^{\lambda_{ij}^C}) p_{ij}(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^C}) dx^{\lambda_{ij}^C}}{\int_{\Omega_{ij}(x)} p_{ij}(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^C}) dx^{\lambda_{ij}^C}}$$

where

$$\Omega_{ij}(x) = \{y \in \mathbb{R}^{\text{Card}(\lambda_{ij}^C)} \text{ such as } (x^{\lambda_{ij}}, y) \in \Omega\}$$

Those quotients must belong to $H^{2,h,h/2}(D^T)$. Let us assume that p belongs to $H^{2,h,h/2}(D^T)$ and that p and f_{ij} are strictly positive on the closure of their domain of definition. This gives us two constants such that $0 < \delta \leq f_{ij} \leq \Delta$. Since p is nonnegative, we obtain that Q_{ij} is well-defined and

$$0 < \delta \leq Q_{ij} \leq \Delta$$

Let us now study the differentiability of Q_{ij} with the

Lemma 2. *The functions Q_{ij} belong to $H^{2,h,h/2}(D^T)$. Moreover, for any $V \in C^3(\mathbb{R}_+^*)$ with V and its three derivatives bounded on $[\eta, +\infty[$ for any η by a constant K_η , and any real number \bar{f} , we have*

$$|V(Q_{ij}) - V(\bar{f})|_{H^{2,h,h/2}(D^T)} \leq |f - \bar{f}|_{H^0(D^T)} P(|p|_{H^{2,h,h/2}(D^T)})$$

where P denotes a polynomial function, nondecreasing and strictly positive on \mathbb{R}^+ .

This lemma is essentially technical, its proof can be found in Appendix A. It requires a new assumption on Ω

Property 2. *Smoothness of the integration domains. There exists a constant $K > 0$, such that*

$$\forall 1 \leq i, j \leq d \text{ and } \forall x, x' \in \Omega, \left| \int_{\Omega_{ij}(x) \setminus \Omega_{ij}(x')} 1 dx^{\lambda_{ij}^C} \right| \leq K d(x, x')^h \quad (5.7)$$

Let us exhibit an example of domain that satisfies the previous condition. We choose $d = 2$ and define Ω as the set $(x, y) \in \mathbb{R}^2$ symmetric with respect to the y axis and that verifies

- $|y| \leq 2$ if x belongs to $[0, 1]$
- $|y| \leq 1 + \sqrt{1 - (x - 1)^2}$ if x belongs to $[1, 2]$

Ω is simply a square where the corners have been replaced by quadrants. We now choose to integrate against y , the condition (5.7) is trivial for $x \in [-1, 1]$. Let us choose $x < x' \in [1, 2]$ (the other cases are similar), we see that

$$\left| \int_{\Omega_{ij}(x) \setminus \Omega_{ij}(x')} 1 dx^{\lambda_{ij}^C} \right| = 2(\sqrt{1 - (x - 1)^2} - \sqrt{1 - (x' - 1)^2}) \leq K|x - x'|^{\frac{1}{2}}$$

For the property to be true, a condition appears on h , it must be smaller than $\frac{1}{2}$. We assume in the following that Ω and h are such that Property 2 is verified.

Lemma 2 enables us to choose the domain of definition for L as the following

$$\mathcal{D} = \{(p, \{f\}), p, f_{ij} \in H^{2,h,h/2}(D^T) \text{ and } \forall 1 \leq i, j \leq d, p, f_{ij} > 0 \text{ on } \overline{D^T}\}$$

We write \mathcal{D}_p and $\mathcal{D}_{\{f\}}$ the projections on p and $\{f\}$ of \mathcal{D} . We also notice that $L(p, \{f\})$ belongs to $H^{0,h,h/2}(D^T)$. Let us study the differentiability of L as an operator from \mathcal{D} into $H^{0,h,h/2}(D^T)$. For notational simplicity, in the rest of this chapter, we omit the arguments of functions when integrating against $dx^{\lambda_{ij}^c}$. We take $(p, \{f\}), (q, \{g\}) \in \mathcal{D}$ and compute

$$L(p, \{g\}) - L(q, \{g\}) = \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (pL_{ij}(p, \{g\}) - qL_{ij}(q, \{g\})) - \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^j h)$$

with $h = p - q$. We are reduced to the differentiability of L_{ij} as an operator from \mathcal{D} into $H^{2,h,h/2}(D^T)$.

$$L_{ij}(p, \{g\}) - L_{ij}(q, \{g\}) = \sum_{k=1}^d V_{ik}(t, x, \frac{\int g_{ik} p}{\int p}) V_{jk}(t, x, \frac{\int g_{jk} p}{\int p}) - V_{ik}(t, x, \frac{\int g_{ik} q}{\int q}) V_{jk}(t, x, \frac{\int g_{jk} q}{\int q})$$

which itself boils down to the linearization of $V_{ij}(t, x, \frac{\int g_{ij} p}{\int p})$. Since V_{ij} belongs to $C^3(\mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}_+^*)$ and

$$\frac{\int g_{ij}(q+h)}{\int q+h} - \frac{\int g_{ij} q}{\int q} = \frac{\int g_{ij} h}{\int q} - \frac{\int g_{ij} q \int h}{(\int q)^2} + o(|h|_{H^{2,h,h/2}(D^T)})$$

we eventually get

$$D_p L(q, \{g\}).h = \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (hL_{ij}(q, \{g\}) + qD_p L_{ij}(q, \{g\}).h) - \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^j h)$$

with

$$\begin{aligned} D_p L_{ij}(q, \{g\}).h &= \sum_{k=1}^d V_{ik}(t, x, \frac{\int g_{ik} q}{\int q}) \frac{\partial V_{jk}}{\partial e}(t, x, \frac{\int g_{jk} q}{\int q}) (\frac{\int g_{jk} h}{\int q} - \frac{\int g_{jk} q \int h}{(\int q)^2}) \\ &\quad + \frac{\partial V_{ik}}{\partial e}(t, x, \frac{\int g_{ik} q}{\int q}) (\frac{\int g_{ik} h}{\int q} - \frac{\int g_{ik} q \int h}{(\int q)^2}) V_{jk}(t, x, \frac{\int g_{jk} q}{\int q}) \end{aligned}$$

Similarly, writing $\{h\} = \{f\} - \{g\}$, we have

$$D_{\{f\}} L(q, \{g\}).\{h\} = -\frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (qD_{\{f\}} L_{ij}(q, \{g\}).\{h\})$$

with

$$\begin{aligned} D_{\{f\}} L_{ij}(q, \{g\}).\{h\} &= \sum_{k=1}^d V_{ik}(t, x, \frac{\int g_{ik} q}{\int q}) \frac{\partial V_{jk}}{\partial e}(t, x, \frac{\int g_{jk} q}{\int q}) \frac{\int h_{jk} q}{\int q} \\ &\quad + \frac{\partial V_{ik}}{\partial e}(t, x, \frac{\int g_{ik} q}{\int q}) \frac{\int h_{ik} q}{\int q} V_{jk}(t, x, \frac{\int g_{jk} q}{\int q}). \end{aligned}$$

$D_p L$ and $D_{\{f\}} L$ are clearly continuous. As a conclusion, L is differentiable.

5.2 Implicit function theorem and compatibility conditions

Let ψ be a strictly positive function defined on $\bar{B} \cup C^T$ and belonging to $H^{2,h,h/2}(\partial D^T)$. We are interested in the equation

$$\begin{aligned} \frac{\partial p}{\partial t} - L(p, \{f\}) &= 0 && \text{in } D^T \\ p &= \psi && \text{on } \bar{B} \cup C^T \end{aligned} \quad (5.8)$$

We consider the following operator

$$\begin{aligned} \mathcal{F} : (p, \{f\}) &\rightarrow \left(\frac{\partial p}{\partial t} - L(p, \{f\}), p|_{\bar{B} \cup C^T} - \psi \right) \\ \mathcal{S} &\rightarrow H^{0,h,h/2}(D^T) \times H^{2,h,h/2}(\partial D^T) \end{aligned}$$

Now, having noticed that the quotient $\frac{f_{ij} p}{f_{ij}}$ is trivial for a constant f_{ij} , let us take a collection $\{\bar{f}\}$ of strictly positive constants $\bar{f}_{ij} \in \mathbb{R}$ and a boundary condition ψ (verifying the previous assumptions) such that

$$\frac{\partial \psi}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (\psi \sum_{k=1}^d V_{ik}(t, x, \bar{f}_{ik}) V_{jk}(t, x, \bar{f}_{jk})) + \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^i \psi) = 0 \quad \text{on } \partial B \quad (5.9)$$

in the sense described in the preliminaries, Chapter 3, and such that $\forall (t, x) \in D^T$ and $\forall (\xi_i)_{1 \leq i \leq d} \in \mathbb{R}^d$,

$$\sum_{1 \leq i, j \leq d} \xi_i \xi_j \sum_{k=1}^d V_{ik}(t, x, \bar{f}_{ik}) V_{jk}(t, x, \bar{f}_{jk}) \geq K_\sigma |\xi|^2 \quad (5.10)$$

for a certain constant K_σ (this is a classic ellipticity assumption). One clearly has

$$\frac{\partial p}{\partial t} - L(p, \{\bar{f}\}) = \frac{\partial p}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (p \sum_{k=1}^d V_{ik}(t, x, \bar{f}_{ik}) V_{jk}(t, x, \bar{f}_{jk})) + \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_i^i p)$$

And the operator $\frac{\partial}{\partial t} - L(\cdot, \{\bar{f}\})$ acting on functions p is uniformly parabolic and linear. The classical results expounded in Chapter 3 give us the existence of a function $\bar{p} \in H^{2,h,h/2}(D^T)$ solution of $\frac{\partial p}{\partial t} - L(p, \{\bar{f}\}) = 0$ and $p = \psi$ on $\bar{B} \cup C^T$ and with $0 < \bar{p}_* \leq \bar{p}$ for a certain constant \bar{p}_* :

- the coefficients of the operator belong to the appropriate spaces.
- the compatibility condition is given by (5.9)
- the ellipticity is given by (5.10).
- the existence of \bar{p}^* is proved at the end of this section using Proposition 1 and the fact that $\psi > 0$ on $\bar{B} \cup C^T$.

This gives us $\mathcal{F}(\bar{p}, \{\bar{f}\}) = 0$. The first idea that comes to mind is an application of the implicit function theorem. Indeed, in our case, it would give us a neighborhood of $\{\bar{f}\}$ where a solution of $\frac{\partial}{\partial t} - L(\cdot, \{f\}) = 0$ exists. Since

$$D_p L_{ij}(\bar{p}, \{\bar{f}\}) = 0$$

we have

$$D_p L(\bar{p}, \{\bar{f}\}) \cdot h = \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2}{\partial x_i \partial x_j} (h L_{ij}(\bar{p}, \{\bar{f}\})) - \sum_{1 \leq i \leq n} \frac{\partial}{\partial x_i} (\mu_i^i h)$$

It is clear that \mathcal{F} is differentiable, with in particular $D_p \mathcal{F}(\bar{p}, \{\bar{f}\}) \cdot h = (\frac{\partial h}{\partial t} - D_p L(\bar{p}, \{\bar{f}\}) \cdot h, h|_{\bar{B} \cup C^T})$. To apply the implicit function theorem, we need to prove that $D_p \mathcal{F}(\bar{p}, \{\bar{f}\})$ is a bijection. This boils down to finding for all $q \in H^{0,h,h/2}(D^T)$ and $\chi \in H^{2,h,h/2}(\partial D^T)$ a function h in $H^{2,h,h/2}(D^T)$ such that

$$\begin{aligned} \frac{\partial h}{\partial t} - D_p L(\bar{p}, \{\bar{f}\}) \cdot h &= q && \text{in } D^T \\ h &= \chi && \text{on } \bar{B} \cup C^T \end{aligned}$$

A difficulty appears here, the compatibility condition (3.14) is required between q and χ for such a function h to exist, this condition is $\frac{\partial \chi}{\partial t} - D_p L(\bar{p}, \{\bar{f}\}) \cdot \chi = q$ on ∂B and is clearly not verified for any (q, χ) , the differential is therefore not a bijection.

This impossibility to apply the implicit function theorem enhances the necessity for compatibility conditions in our nonlinear case. For a given $\{f\}$, with boundary conditions ψ on $\bar{B} \cup C^T$, if p is solution of the problem $\frac{\partial p}{\partial t} - L(p, f) = 0$ and $p = \psi$ on $\bar{B} \cup C^T$, then necessarily

$$\frac{\partial \psi}{\partial t} - L(\psi, \{f\}) = 0 \quad \text{on } \partial B \quad (5.11)$$

With such an assumption, we shall prove in the next section two existence results. But first, let us prove the strict positivity of \bar{p}

Lemma 3. *The solution $\bar{p} \in H^{2,h,h/2}(D^T)$ of $\frac{\partial p}{\partial t} - L(p, \{\bar{f}\}) = 0$ and $p = \psi \geq \psi_* > 0$ on $\bar{B} \cup C^T$ verifies $0 < \bar{p}_* \leq \bar{p}$ for a certain constant \bar{p}_* .*

Proof. Developing the derivatives in $L(\cdot, \{\bar{f}\})$ enables us to write the equation as $\frac{\partial p}{\partial t} - Gp + cp = 0$ where G contains all the space derivatives and c is the term of order 0.

We start by noticing that if c is a positive constant, then the function $q = \exp(-ct)\psi_*$ is solution of $\frac{\partial q}{\partial t} - Gq + cq = 0$ with $\bar{p} - q = \psi - \exp(-ct)\psi_* \geq 0$ on $\bar{B} \cup C^T$. Applying Proposition 1 to $\bar{p} - q$ gives us $\bar{p} \geq q \geq \exp(-cT)\psi_* > 0$.

For any function c , let us now consider a constant $c^* \geq c$ and q verifying $\frac{\partial q}{\partial t} - Gq + c^*q = 0$ with $q = \psi$ on $\bar{B} \cup C^T$. We just proved that $q > \bar{p}_*$ for a given $\bar{p}_* > 0$. Now, we have

$$\frac{\partial \bar{p} - q}{\partial t} - G(\bar{p} - q) + c(\bar{p} - q) = (c^* - c)q$$

Proposition 1 guarantees that $\bar{p} \geq q > \bar{p}_*$ which concludes the proof. \square

5.3 Existence results for Equation (5.8)

The theorem we will prove in this section requires the assumptions made previously.

5.3.1 Global existence in time

Theorem 5. For every collection $\{\bar{f}\}$ in $\mathcal{D}_{\{f\}}$ with \bar{f}_{ij} a constant function verifying the compatibility assumptions

$$\frac{\partial \Psi}{\partial t} - L(\Psi, \{\bar{f}\}) = 0 \text{ on } \partial B \quad (5.12)$$

and uniform ellipticity ie $\exists K_\sigma > 0, \forall (\xi_i)_{1 \leq i \leq d} \in \mathbb{R}^d$,

$$\sum_{1 \leq i, j \leq n} \xi_i \xi_j \sum_{k=1}^n V_{ik}(t, x, \bar{f}_{ik}) V_{jk}(t, x, \bar{f}_{jk}) \geq K_\sigma |\xi|^2$$

there exists f^* , a strictly positive constant such that, for any collection $\{f\}$ in $\mathcal{D}_{\{f\}}$ with

$$|\{f\} - \{\bar{f}\}|_{(H^0(D^T))^{d^2}} \leq f^* \quad (5.13)$$

$$\frac{\partial \Psi}{\partial t} - L(\Psi, \{f\}) = 0 \quad \text{on } \partial B \quad (5.14)$$

there exists $p \in H^{2,h,h/2}(D^T)$ verifying $\frac{\partial p}{\partial t} - L(p, \{f\}) = 0$ on $D^T \cup B^T$ and $p = \Psi$ on $\bar{B} \cup C^T$

Proof. Let $\{f\}$ belong to $\mathcal{D}_{\{f\}}$ with the conditions (5.13) and (5.14) verified. We start with a technical lemma proved in Appendix A

Lemma 4. For a given $\varepsilon > 0$, there exists two constants $K_l, K'_l > 0$ depending on ε and on the data of the problem such as $\forall p, q \in H^{2,h,h/2}(D^T)$ with $p, q \geq \varepsilon$ on D^T and $\forall 1 \leq i, j \leq n$,

$$\begin{aligned} |L_{ij}(p, \{f\}) - L_{ij}(p, \{\bar{f}\})|_{H^{2,h,h/2}(D^T)} &\leq K_l (|\{f\}| + |\{\bar{f}\}|) |\{f\} - \{\bar{f}\}| P(|p|_{H^{2,h,h/2}(D^T)}) \\ |L_{ij}(p, \{f\}) - L_{ij}(p, \{\bar{f}\}) - (L_{ij}(q, \{f\}) - L_{ij}(q, \{\bar{f}\}))|_{H^{2,h,h/2}(D^T)} \\ &\leq K'_l |\{f\}| |\{f\} - \{\bar{f}\}| |p - q|_{H^{2,h,h/2}(D^T)} \\ &\quad P(|p|_{H^{2,h,h/2}(D^T)}) P(|q|_{H^{2,h,h/2}(D^T)}) \end{aligned}$$

with P a polynomial function, increasing and strictly positive on \mathbb{R}^+ , and all the norms involving $\{f\}$ or $\{\bar{f}\}$ are $(H^0(D^T))^{n^2}$.

This lemma gives us control over the variations of L_{ij} . When the functions f_{ij} are close to constants, the operator L is close to a classic linear one. We now remember from the previous section that there exists a function $\bar{p} \in H^{2,h,h/2}(D^T)$ solution of $\frac{\partial \bar{p}}{\partial t} - L(\bar{p}, \{\bar{f}\}) = 0$ and $\bar{p} = \Psi$ on $\bar{B} \cup C^T$ with $\bar{p}_* \leq \bar{p}$ for a certain strictly positive constant \bar{p}_* . The idea is to construct a sequence of functions $(p_n)_{n \in \mathbb{N}} \in \mathcal{D}_p$ that shall converge to a solution of Equation (5.8) We define $(p_n)_{n \in \mathbb{N}}$ as follows

- $p_0 = \bar{p}$
- p_{n+1} is solution of the equation $\frac{\partial p_{n+1}}{\partial t} - L(p_{n+1}, \{\bar{f}\}) = L(p_n, \{f\}) - L(p_n, \{\bar{f}\})$ with $p = \Psi$ on $\bar{B} \cup C^T$

The existence of p_{n+1} stems from the same points than the existence of \bar{p} . The only thing we have to do is verify the compatibility condition: from assumptions (5.12) and (5.14) the left-side term takes the values $\frac{\partial \Psi}{\partial t} - L(\Psi, \{\bar{f}\}) = 0$ on ∂B while the right-side term is equal to $L(\Psi, \{\bar{f}\}) - L(\Psi, \{f\}) = 0$ on ∂B .

Hence, we do have $\frac{\partial p_{n+1}}{\partial t} - L(p_{n+1}, \{\bar{f}\}) = L(p_n, \{f\}) - L(p_n, \{\bar{f}\})$. We notice that a function p

limit in an appropriate space of the sequence $(p_n)_{n \in \mathbb{N}}$ is solution of our problem (given the fact that L is continuous with respect to p in a certain topology). Let us prove the convergence of p_n by showing that it is a Cauchy sequence.

We consider the quantity $|p_{n+1} - p_n|_{H^{2,h,h/2}(D^T)}$. The idea is to show that this last norm can be controlled by a constant depending on f^* times the difference $|p_n - p_{n-1}|_{H^{2,h,h/2}(D^T)}$, choosing f^* small enough thus allows to prove two things: first that the p_n are strictly positive and second that p_n is a Cauchy sequence. Let us start with

$$\begin{aligned} \frac{\partial p_1}{\partial t} - L(p_1, \{\bar{f}\}) &= L(\bar{p}, \{f\}) - L(\bar{p}, \{\bar{f}\}) \\ \frac{\partial \bar{p}}{\partial t} - L(\bar{p}, \{f\}) &= 0 \end{aligned}$$

Recalling that $L(\cdot, \{\bar{f}\})$ is linear, we get

$$\frac{\partial p_1 - \bar{p}}{\partial t} - L(p_1 - \bar{p}, \{f\}) = L(\bar{p}, \{f\}) - L(\bar{p}, \{\bar{f}\})$$

with the clear boundary condition $p_1 - \bar{p} = 0$ on $\bar{B} \cup C^T$. We apply Schauder's inequality (3.15) which gives us, with K denoting a constant depending on the data of the problem (it might be different from one line to another)

$$\begin{aligned} |p_1 - \bar{p}|_{H^{2,h,h/2}(D^T)} &\leq K_{H^2} |L(\bar{p}, \{f\}) - L(\bar{p}, \{\bar{f}\})|_{H^{0,h,h/2}(D^T)} \\ &\leq K_{H^2} \left| \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2}{\partial x_i \partial x_j} (\bar{p}(L_{ij}(\bar{p}, \{f\}) - L_{ij}(\bar{p}, \{\bar{f}\}))) \right|_{H^{0,h,h/2}(D^T)} \\ &\leq K \sum_{1 \leq i, j \leq n} |\bar{p}(L_{ij}(\bar{p}, \{f\}) - L_{ij}(\bar{p}, \{\bar{f}\}))| \\ &\leq f^*(1 + f^*)KK_L |\bar{p}|_{H^{2,h,h/2}(D^T)} P(|\bar{p}|_{H^{2,h,h/2}(D^T)}) \end{aligned}$$

where the last line is an application of Lemma 4 with $\varepsilon = \bar{p}_*$. We are now able to choose f^* as the only strictly positive root of

$$f^*(1 + f^*) = \frac{\bar{p}_*}{4KK_L |\bar{p}|_{H^{2,h,h/2}(D^T)} P(|\bar{p}|_{H^{2,h,h/2}(D^T)})} \quad (5.15)$$

This gives us

$$|p_1 - \bar{p}|_{H^{2,h,h/2}(D^T)} \leq \frac{\bar{p}_*}{4}$$

Let us now prove by recursion that

$$|p_{n+1} - p_n|_{H^{2,h,h/2}(D^T)} \leq \frac{1}{2} |p_n - p_{n-1}|_{H^{2,h,h/2}(D^T)}$$

The recursion assumption and the inequality on $p_1 - \bar{p}$ give us (keeping in mind that $p_0 = \bar{p}$)

$$\forall k \leq n, |p_k - \bar{p}|_{H^{2,h,h/2}(D^T)} \leq \frac{\bar{p}_*}{2}$$

We compute

$$\begin{aligned}\frac{\partial p_{n+1}}{\partial t} - L(p_{n+1}, \{\bar{f}\}) &= L(p_n, \{f\}) - L(p_n, \{\bar{f}\}) \\ \frac{\partial p_n}{\partial t} - L(p_n, \{\bar{f}\}) &= L(p_{n-1}, \{f\}) - L(p_{n-1}, \{\bar{f}\})\end{aligned}$$

Since $L(\cdot, \{\bar{f}\})$ is linear, we get

$$\frac{\partial p_{n+1} - p_n}{\partial t} - L(p_{n+1} - p_n, \{\bar{f}\}) = L(p_n, \{f\}) - L(p_n, \{\bar{f}\}) - L(p_{n-1}, \{f\}) + L(p_{n-1}, \{\bar{f}\})$$

with $p_{n+1} - p_n = 0$ on $\bar{B} \cup C^T$. Once again, we apply Schauder's inequality (3.15)

$$\begin{aligned}|p_{n+1} - p_n|_{H^{2,h,h/2}(D^T)} &\leq K_{H^2} |L(p_n, \{f\}) - L(p_n, \{\bar{f}\}) - L(p_{n-1}, \{f\}) + L(p_{n-1}, \{\bar{f}\})|_{H^{0,h,h/2}(D^T)} \\ &\leq K_{H^2} \left| \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2}{\partial x_i \partial x_j} (p_n(L_{ij}(p_n, \{f\}) - L_{ij}(p_n, \{\bar{f}\})) \right. \\ &\quad \left. - p_{n-1}(L_{ij}(p_{n-1}, \{f\}) - L_{ij}(p_{n-1}, \{\bar{f}\}))) \right|_{H^{0,h,h/2}(D^T)} \\ &\leq K \sum_{1 \leq i, j \leq n} |p_n(L_{ij}(p_n, \{f\}) - L_{ij}(p_n, \{\bar{f}\})) \\ &\quad - p_{n-1}(L_{ij}(p_{n-1}, \{f\}) - L_{ij}(p_{n-1}, \{\bar{f}\}))|_{H^{2,h,h/2}(D^T)}\end{aligned}$$

Developing and applying Lemma 4 with $\varepsilon = \frac{\bar{p}_*}{2}$, we get

$$\begin{aligned}|p_{n+1} - p_n|_{H^{2,h,h/2}(D^T)} &\leq K \sum_{1 \leq i, j \leq n} |p_n - p_{n-1}|_{H^{2,h,h/2}(D^T)} |L_{ij}(p_n, \{f\}) - L_{ij}(p_n, \{\bar{f}\})|_{H^{2,h,h/2}(D^T)} \\ &\quad + |p_{n-1}|_{H^{2,h,h/2}(D^T)} |L_{ij}(p_{n-1}, \{f\}) - L_{ij}(p_{n-1}, \{\bar{f}\}) - L_{ij}(p_n, \{f\}) + L_{ij}(p_n, \{\bar{f}\})|_{H^{2,h,h/2}(D^T)} \\ &\leq f^*(1 + f^*) K K'_L |p_n - p_{n-1}|_{H^{2,h,h/2}(D^T)} P(|p_n|_{H^{2,h,h/2}(D^T)}) P(|p_{n-1}|_{H^{2,h,h/2}(D^T)})\end{aligned}$$

We now choose f^* as the minimum of its previous value and the only strictly positive root of

$$f^*(1 + f^*) = \frac{1}{2KK'_L P^2(\frac{\bar{p}_*}{2} + |\bar{p}|_{H^{2,h,h/2}(D^T)})} \quad (5.16)$$

(we recall that P is increasing, K'_L is a constant depending on the data of the problem, \bar{p}_* in particular, but not on n). This gives us the inequality we were looking for, and proves that p_n is a Cauchy sequence in $H^{2,h,h/2}(D^T)$. We let $p \in H^{2,h,h/2}(D^T)$ denote its limit. We also have

$$\forall n \in \mathbb{N}, |p_n - \bar{p}|_{H^{2,h,h/2}(D^T)} \leq \frac{\bar{p}_*}{2}$$

The last step of the proof is to make n tend to $+\infty$ in the equation

$$\frac{\partial p_{n+1}}{\partial t} - L(p_{n+1}, \{\bar{f}\}) = L(p_n, \{f\}) - L(p_n, \{\bar{f}\})$$

The only result needed is $V_{ij}(t, x, \frac{\int f_{ij} p_n}{\int p_n}) \rightarrow V_{ij}(t, x, \frac{\int f_{ij} p}{\int p})$. Since $|p_n|$ is smaller than $|\bar{p}|_{H^{2,h,h/2}(D^T)} + \frac{\bar{p}_*}{2}$, we can apply the dominated convergence theorem to $\int f_{ij} p_n$ and to $\int p_n$ and all the necessary

derivatives, this proves that

$$\int f_{ij} p_n \rightarrow \int f_{ij} p \quad \int p_n \rightarrow \int p$$

Moreover, because $p_n \geq \frac{\bar{p}^*}{2}$, the denominator $\int p_n$ is bounded away from 0. Using the continuity of V_{ij} , we obtain the convergence we were looking for. It is thus possible to take the limit in the previous equation

$$\frac{\partial p}{\partial t} - L(p, \{f\}) = L(p, \{f\}) - L(p, \{\bar{f}\})$$

ie $\frac{\partial p}{\partial t} - L(p, \{f\}) = 0$. Since $\forall n \in \mathbb{N}, p_n = \psi$ on $\bar{B} \cup C^T$, the boundary condition is also verified. p is the solution we were looking for. \square

5.3.2 Local existence

The theorem we just proved gives existence of a solution on the domain D^T for any T . Intuitively, the fact that T can be as large as we want, suggests a smaller f^* . It is possible to modify the proof of the previous theorem to give short time existence for possibly larger values of f^* .

Theorem 6. *For every collection $\{\bar{f}\}$ in $\mathcal{D}_{\{f\}}$ defined as in Theorem 5, there exists $T^* \leq T$ and f^* , two strictly positive constant such that, for all collection $\{f\}$ in $\mathcal{D}_{\{f\}}$ with*

$$\begin{aligned} |\{f\} - \{\bar{f}\}|_{(H^{2,h,h/2}(D^{T^*}))^{d^2}} &\leq f^* \\ \frac{\partial \psi}{\partial t} - L(\psi, \{f\}) &= 0 \quad \text{on } \partial B \end{aligned}$$

there exists $p \in H^{2,h,h/2}(D^{T^*})$ verifying $\frac{\partial p}{\partial t} - L(p, \{f\}) = 0$ on $D^{T^*} \cup B^{T^*}$ and $p = \psi$ on $\bar{B} \cup C^{T^*}$

Proof. We still use the function $\bar{p}_* \leq \bar{p} \leq \bar{p}^*$. But this time, the strict positivity of the p_n is proved using a short-time argument. Let $|\bar{p}|_{H^{2,h,h/2}(D^T)} \leq x$ and $t \leq T$ be two real numbers and let X_x^t denote the set

$$X_x^t = \left\{ p \in H^{2,h,h/2}(D^t), |p|_{H^{2,h,h/2}(D^t)} \leq x, \frac{\bar{p}_*}{2} \leq p \leq \bar{p}^* + \frac{\bar{p}_*}{2}, p = \psi \text{ on } \bar{B} \cup C^t \right\}$$

The set X_x^t clearly contains the function \bar{p} . We then consider the application M which takes a function $u \in X_x^t$ and sends it on $v \in H^{2,h,h/2}(D^t)$ solution of the equation

$$\begin{aligned} \frac{\partial v}{\partial t} - L(v, \{\bar{f}\}) &= L(u, \{f\}) - L(u, \{\bar{f}\}) && \text{in } D^t \\ v &= \psi && \text{on } \bar{B} \cup C^t \end{aligned}$$

The existence of v stems from the same arguments than in the previous theorem. We apply once again Schauder's inequality

$$\begin{aligned} |v|_{H^{2,h,h/2}(D^t)} &\leq K_{H^2} (|\psi|_{H^{2,h,h/2}(\partial D^t)} + |L(u, \{f\}) - L(u, \{\bar{f}\})|_{H^{0,h,h/2}(D^t)}) \\ &\leq K_{H^2} (|\psi|_{H^{2,h,h/2}(\partial D^t)} + K f^* (1 + f^*) Q(|u|_{H^{2,h,h/2}(D^t)})) \\ &\leq K_{H^2} (|\psi|_{H^{2,h,h/2}(\partial D^t)} + K f^* (1 + f^*) Q(x)) \end{aligned}$$

Choosing f^* such that

$$f^*(1 + f^*) = \max \frac{x - K_{H^2} |\Psi|_{H^{2,h,h/2}(\partial D^t)}}{K_{H^2} K Q(x)} \quad (5.17)$$

where the max is taken on the set of x larger than $|\bar{p}|_{H^{2,h,h/2}(D^t)}$ and x^* the value where the max is reached (the quotient goes to 0 when x tends to $+\infty$), we get that

$$|v|_{H^{2,h,h/2}(D^t)} \leq x^*$$

We now want to prove that M is a contraction, we see that for all $p, q \in X_x^t$

$$\frac{\partial M(p) - M(q)}{\partial t} - L(M(p) - M(q), \{\bar{f}\}) = L(p, \{f\}) - L(p, \{\bar{f}\}) - L(q, \{f\}) + L(q, \{\bar{f}\})$$

with $M(p) - M(q) = 0$ on $\bar{B} \cup C^T$. The exact same computations than earlier give the existence of a constant $F^* > 0$ such that $\forall f^* \leq F^*$ M is a contraction.

Let us now define the value of t . In the proof of the global existence, we used f^* to ascertain the strict positivity of the functions p_n . Here, we apply Theorem 1 to the function $\tilde{v} = v - \bar{p}$. v and \bar{p} are equal to ψ on $\bar{B} \cup C^t$, thus \tilde{v} is null on the boundary. Plus, we have

$$\frac{\partial \tilde{v}}{\partial t} - L(\tilde{v}, \{\bar{f}\}) = L(u, \{f\}) - L(u, \{\bar{f}\}) \quad \text{in } D^t$$

Hence

$$\begin{aligned} |\tilde{v}|_{H^0(D^t)} &\leq t K_{H^0} |L(u, \{f\}) - L(u, \{\bar{f}\})|_{H^0(D^t)} \\ &\leq t K_{H^0} K f^* (1 + f^*) Q(|u|_{H^{2,h,h/2}(D^t)}) \\ &\leq t K_{H^0} K f^* (1 + f^*) Q(x^*) \end{aligned}$$

If f^* is such that $T K_{H^0} K f^* (1 + f^*) Q(x^*) < \bar{p}_*$, then the result is exactly the same as in Theorem 5. On the contrary, if f^* is too large for this inequality to be verified, we control the sign of \tilde{v} using t . Taking

$$T^* = \frac{\bar{p}_*}{2 K_{H^0} K f^* (1 + f^*) Q(x^*)} \quad (5.18)$$

guarantees that $v \geq \frac{\bar{p}_*}{2}$. We have proved that $v = M(u)$ belongs to $X_{x^*}^{T^*}$ for the values just defined, and that M is a contraction on that set. This gives us the existence of a fixed point for M . We conclude as in Theorem 5. \square

Remark. *It is difficult to quantify precisely the gain in f^* obtained through this local existence theorem. In Part III however, we notice some numerical instabilities that seem to be linked to the theoretical limitations we just faced. In particular, the time at which those instabilities occur is inversely related to the magnitude of the variations of f .*

We now give a weak uniqueness result for our equation.

Proposition 4. *If two function p^1 and p^2 are solutions of*

$$\begin{aligned} \frac{\partial p}{\partial t} - L_p p &= 0 & \text{in }]0, T[\times \Omega \\ p &= \psi & \text{on } \bar{B} \cup C^T \end{aligned}$$

and furthermore, for all $1 \leq i, j \leq d$, we have

$$F_{ij}(t, x) := \frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}^1(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}^1(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}} = \frac{\int f_{ij}(x^{\lambda_{ij}^c}) p_{ij}^2(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}}{\int p_{ij}^2(t, x^{\lambda_{ij}}, x^{\lambda_{ij}^c}) dx^{\lambda_{ij}^c}} \quad (5.19)$$

Then $p^1 = p^2$.

Proof. From (5.19), we see that p_1 and p_2 are solutions of

$$\frac{\partial p}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^d V_{ik}(t, x, F_{ik}(t, x)) V_{jk}(t, x, F_{jk}(t, x)) p(t, x) \right) - \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_t^i p(t, x)) = 0$$

Both of these functions verify the same boundary condition $p = \psi$ on $\bar{B} \cup C^T$. The uniqueness result contained in Theorem 1 gives us the equality we were looking for. \square

5.4 The adjoint equation: a simpler case

We now study the adjoint equation for operator (5.8)

$$\begin{aligned} \frac{\partial p}{\partial t} - \mathbb{L}_p p &= 0 && \text{in }]0, T[\times \Omega \\ p(0, x) &= p_0(x) && \text{on } \Omega \end{aligned} \quad (5.20)$$

where we recall that operator \mathbb{L}_q is equal to

$$\begin{aligned} \mathbb{L}_q p &= \frac{1}{2} \sum_{1 \leq i, j \leq d} \left(\sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ik}^c}) q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}{\int q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^c}) dx^{\lambda_{ik}^c}}) \right. \\ &\quad \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{jk}^c}) q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}{\int q_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}^c}) dx^{\lambda_{jk}^c}}) \right) \frac{\partial^2 p}{\partial x_i \partial x_j} + \sum_{1 \leq i \leq d} \mu_t^i \frac{\partial p}{\partial x_i} \end{aligned}$$

We start by noticing that, since its coefficients are outside of the derivatives, this last equation has the same form than Equation (5.21) below

$$\begin{aligned} \frac{\partial u}{\partial t} - F(t, x, u, Du, D^2 u, N(u)) &= 0 \text{ in } Q^T =]0, T[\times \mathbb{R}^d \\ u(0, x) &= u_0(x) \text{ on } \mathbb{R}^d \end{aligned} \quad (5.21)$$

where F is a function from $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d) \times \mathbb{R}$ into \mathbb{R} . Du is the gradient of u and $D^2 u$ the hessian matrix of u . $\mathbb{S}(d)$ is the space of symmetric $d \times d$ real valued matrices. $N(u)$ is a nonlocal term. This last equation is thoroughly studied in [14]. The author makes a number of assumptions among which we find

1. For all $u \in C_b(\bar{Q}^T)$, the function $N(u)$ exists and belongs to $C_b(Q^T)$.
2. For all $u, v \in C_b(\bar{Q}^T)$ and $0 \leq t \leq T$,

$$\sup_{\tau \in [0, t]} |N(u)(\tau, \cdot) - N(v)(\tau, \cdot)|_\infty \leq K \sup_{\tau \in [0, t]} |u(\tau, \cdot) - v(\tau, \cdot)|_\infty$$

3. For all $u \in H_b^{0,h,h/2}(\overline{Q^T})$ and $0 \leq t \leq T$,

$$\sup_{\tau \in [0,t]} |N(u)(\tau, \cdot)|_{H^{0,h,h/2}(\mathbb{R}^d)} \leq K(1 + \sup_{\tau \in [0,t]} |u(\tau, \cdot)|_{H^{0,h,h/2}(\mathbb{R}^d)})$$

4. F is Lipschitz with respect to its last variable, independently of the others.

He then proves that there exists a unique $u \in H_b^{0,h,h/2}(\overline{Q^T})$ viscosity solution¹ of the problem (5.21). In our case, there are actually d^2 nonlocal terms, they are defined as

$$N_{ik}(p)(t, x) = \frac{\int f_{ik}(x^{\lambda_{ik}^C}) q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^C}) dx^{\lambda_{ik}^C}}{\int q_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}^C}) dx^{\lambda_{ik}^C}}$$

It is however impossible to apply (at least directly without modifications) the previous existence result. Handling the N_{ik} is intricate because of the denominator. For instance, Assumption 2 is not verified. We need to prove by ourselves that there are actual solutions for Equation (5.20).

We now relax the previous set of assumptions to

- $\forall 1 \leq i \leq d, \mu_i^i \in H^{0,h,h/2}(D^T), \forall 1 \leq i, j \leq d, f_{ij} \in H^{2,h,h/2}(\mathbb{R}^d)$ with K_0 a born on their norms
- $\forall 1 \leq i, j \leq d, V_{ij}(t, x, e) \in C^1(\mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}_+^*)$ with the derivative with respect to the last variable bounded on $[\eta, +\infty[$ for all $\eta > 0$.

and prove the

Theorem 7. *Given the assumptions above, for any collection $\{f\}$ in $\mathcal{D}_{\{f\}}$ with*

$$\frac{\partial \psi}{\partial t} - \mathbb{L}_\psi \psi = 0 \text{ on } \partial B$$

there exists $T^ \leq T$ a strictly positive constant and $p \in H^{2,h,h/2}(D^{T^*})$ such that $\frac{\partial p}{\partial t} - \mathbb{L}_p p = 0$ on $D^{T^*} \cup B^{T^*}$ and $p = \psi$ on $\overline{B} \cup C^{T^*}$.*

Remark. *This result does not need the assumption that $\{f\}$ is close to a constant, it is much more general. Its proof gives a better understanding of the theoretical limitations appearing in the first case we studied.*

Proof. Let us consider $\{f\} \in \mathcal{D}_{\{f\}}$ with $\frac{\partial \psi}{\partial t} - \mathbb{L}_\psi \psi = 0$ on ∂B .

This proof too is based upon a fixed point method. We let p_0 belong to $H^{2,h,h/2}(D^t)$ with $p_0 = \psi$ on $\overline{B} \cup C^{T^*}$. Such a p_0 exists, plus we can choose it so that $0 < \delta \leq p_0$ on D^T . Now, let p_n be defined by induction. We take p_{n+1} as the solution of the equation

$$\begin{aligned} \frac{\partial p_{n+1}}{\partial t} - \mathbb{L}_{p_n} p_{n+1} &= 0 \text{ on } D^t \cup B^t \\ p_{n+1} &= \psi \text{ on } \overline{B} \cup C^t \end{aligned} \quad (5.22)$$

It admits a solution from the same arguments than earlier, including the compatibility condition. Given the regularity assumptions on the coefficients, from Lemma 1 and inequality (3.17) in particular, we have for all $t \leq T$

$$|p_1 - p_0|_{H^{0,h,h/2}(D^t)} \leq t^{1-\frac{h}{2}} K_{H^1} \left| \frac{\partial p_0}{\partial t} - \mathbb{L}_{p_0} p_0 \right|_{H^0(D^t)}$$

¹We refer to [49] for a presentation of the theory of viscosity solutions.

Choosing

$$T^* = \left(\frac{\delta}{4K_{H^1} \left| \frac{\partial p_0}{\partial t} - \mathbb{L}_{p_0} p_0 \right|_{H^0(D^t)}} \right)^{\frac{1}{1-h/2}}$$

ensures us that

$$|p_1 - p_0|_{H^{0,h,h/2}(D^t)} \leq \frac{\delta}{4}$$

Let us now compute

$$\begin{aligned} \frac{\partial p_{n+1} - p_n}{\partial t} - \mathbb{L}_{p_n}(p_{n+1} - p_n) &= (\mathbb{L}_{p_n} - \mathbb{L}_{p_{n-1}})p_n \text{ on } D^t \cup B^t \\ p_{n+1} - p_n &= 0 \text{ on } \bar{B} \cup C^t \end{aligned}$$

The same inequality than above and Lemma 4 with $\varepsilon = \frac{\delta}{2}$ give us

$$\begin{aligned} |p_{n+1} - p_n|_{H^{0,h,h/2}(D^t)} &\leq t^{1-\frac{h}{2}} K_{H^1} |(\mathbb{L}_{p_n} - \mathbb{L}_{p_{n-1}})p_n|_{H^0(D^t)} \\ &\leq t^{1-\frac{h}{2}} K_{H^1} \left| \sum_{1 \leq i, j \leq d} (L_{ij}(p_n, \{f\}) - L_{ij}(p_{n-1}, \{f\})) \right|_{H^0(D^t)} |p_n|_{H^{2,1}(D^t)} \\ &\leq t^{1-\frac{h}{2}} K_{H^1} K_{H^2} K'_l K |p_n - p_{n-1}|_{H^{0,h,h/2}(D^t)} P(|p_n|_{H^{0,h,h/2}(D^t)}) P(|p_{n-1}|_{H^{0,h,h/2}(D^t)}) \end{aligned}$$

We recall from Remark 1 that K_{H^1} and K_{H^2} depend on the coefficients of \mathbb{L}_{p_n} only through their norms and in a nondecreasing way. From the regularity assumptions made on the V_{ij} and the f_{ij} and this statement, we deduce that K_{H^1} and K_{H^2} are functions of the norms of $\frac{\int f_{ik} p_n}{\int p_n}$. Using the study of those quotients's regularity made in Appendix A, we get that K_{H^1} and K_{H^2} are bounded from above by an increasing function of the quantity $|p_n|_{H^{0,h,h/2}(D^t)}$. The same statement applies to K'_l . Thus, if we assume by induction that $|p_n - p_0|_{H^{0,h,h/2}(D^t)} \leq \frac{\delta}{2}$ (it is in particular true for $n = 1$), then $K_{H^1} K_{H^2} K'_l K P(|p_n|_{H^{0,h,h/2}(D^t)}) P(|p_{n-1}|_{H^{0,h,h/2}(D^t)})$ is smaller than a constant depending on δ but not on n , let Δ denote that constant. All we have to do now is choose T^* as the minimum of its previous value and of $(\frac{1}{2\Delta})^{\frac{1}{1-h/2}}$ to get

$$|p_{n+1} - p_n|_{H^{0,h,h/2}(D^t)} \leq \frac{1}{2} |p_n - p_{n-1}|_{H^{0,h,h/2}(D^t)}$$

As in the proof of Theorem 5, this shows that p_n is a Cauchy sequence and that $|p_n - p_0|_{H^{0,h,h/2}(D^{T^*})} \leq \frac{\delta}{2}$ (we recall that this is a sufficient condition for the quotients of integrals to be well-defined and in $H^{2,h,h/2}(D^{T^*})$). We thus obtain a function p in $H^{0,h,h/2}(D^{T^*})$ limit of p_n in that space. Moreover, from Schauder's inequality (3.15) applied to (5.22), we see that

$$|p_{n+1}|_{H^{2,h,h/2}(D^{T^*})} \leq K_{H^2} |\Psi|_{H^{2,h,h/2}(\partial D^{T^*})}$$

where K_{H^2} depends only on $|p_n|_{H^{0,h,h/2}(D^{T^*})}$, which is smaller than a constant independent from n . p_n is thus a bounded sequence in $H^{2,h,h/2}(D^{T^*})$, let κ denote said bound. Repeated applications of Ascoli-Arzelà theorem give us a function $q \in H^{2,1}(D^{T^*})$ limit in $H^{2,1}(D^{T^*})$ of a subsequence of p_n .

Since

$$\sup_{n \in \mathbb{N}} \left\{ \frac{\left| \frac{\partial^2 p_n}{\partial x_i \partial x_j}(t, x) - \frac{\partial^2 p_n}{\partial x_i \partial x_j}(t', x') \right|}{(|x - x'|^2 + |t - t'|)^{h/2}}; (x, t), (x', t') \in D^{T^*} \right\} \leq \kappa$$

we have

$$\sup_{n \in \mathbb{N}} \left\{ \frac{\left| \frac{\partial^2 q}{\partial x_i \partial x_j}(t, x) - \frac{\partial^2 q}{\partial x_i \partial x_j}(t', x') \right|}{(|x - x'|^2 + |t - t'|)^{h/2}}; (x, t), (x', t') \in D^{T^*} \right\} \leq \kappa$$

And this computation being true for all the derivatives appearing in the norm $H^{2,h,h/2}$, we find that $q \in H^{2,h,h/2}(D^{T^*})$. By uniqueness of p_n 's limit in $H^{0,h,h/2}(D^{T^*})$, we clearly have $p = q$. Let $c(n)$ denote the extraction such that $p_{c(n)} \rightarrow p$ in $H^{2,h,h/2}(D^{T^*})$.

Since $p_{c(n)-1} \rightarrow p$ in $H^{0,h,h/2}(D^{T^*})$, we see that operator $\mathbb{L}_{p_{c(n)-1}}$ tends to \mathbb{L}_p (using the same considerations than in Theorem 5). Taking the limit in

$$\frac{\partial p_{c(n)}}{\partial t} - \mathbb{L}_{p_{c(n)-1}} p_{c(n)} = 0$$

shows that p is solution of Equation (5.20), which concludes the proof. \square

Remark. We notice that this method cannot be applied to Equation (5.8) because of the presence of the derivatives of p_n in the operator L_{p_n} . The norm $\left| \frac{\partial p_0}{\partial t} - L_{p_n} p_0 \right|_{H^0(D^t)}$ can only be controlled by $|p_n|_{H^{2,h,h/2}(D^t)}$, and equivalently $|p_{n+1}|_{H^{0,h,h/2}(D^t)}$ by $|p_n|_{H^{2,h,h/2}(D^t)}$.

Part II

Financial applications

Chapter 6

Local and Stochastic Volatility models

In the first part of this thesis, we have considered some theoretical Stochastic and Partial Differential equations. Those equations first appeared when studying the calibration of the vanillas in Local and Stochastic Volatility models. But more generally, the same kind of problems appears in different frameworks in Finance. This part of my work is dedicated to some of them.

At first, we go back to the original study of Local and Stochastic Volatility models. We expound the motivation for those models: combining the rich possibilities given by stochastic volatilities in term of dynamics with the main feature of local volatility, fitting the vanillas quoted on the market. We then write the calibration equations of those vanillas in a formal way and see in what measure results from Part I can be applied. At last, we give the results stemming from the numerical resolution of those equations when applied to different stochastic volatilities.

6.1 Local volatility and Stochastic volatility models

Financial mathematics find their origin in the work of Louis Bachelier at the beginning of the 20th century with his thesis "Théorie de la Spéculation". But it was Black, Scholes and Merton who gave them a new dimension in 1973 with their study of the evaluation and hedging of options [6], [7]. Their model gained a wide recognition among the practitioners, it is written as

$$\frac{dS_t}{S_t} = r(t)dt + \sigma dW_t$$

with $r(t)$ the deterministic interest rate and σ the volatility of the model, assumed constant in their framework. W_t is a standard brownian motion, the idea of using it as source of randomness actually came to Bachelier who noticed the similarities between its trajectories and the prices on a stock exchange.

The parameter σ is of special interest since it allows us to define the notion of implied volatility. Let us consider a given price for a certain financial product. Black-Scholes modelization generates a pricing formula related to this product¹. The implied volatility is the unique value of σ , the volatility parameter, which when used in the formula gives the initial price. The implied volatility is a dimensionless quantity, it is commonly used to quote prices and provides a way to compare products that have different characteristics.

In the introduction, we pointed out the major part played by vanilla options in the world of finance. Computing the implied volatility for vanillas with different strikes and maturities gives a surface of implied volatility. Had Black-Scholes model been perfect, this surface would have been constant

¹This formula is thoroughly described in the very useful book by Lamberton and Lapeyre [53].

at the height σ . Unfortunately it is not the case, in practice one gets a "smile", as can be seen for instance in [23].

This major drawback in Black-Scholes construction led to consider the volatility not as a constant but as a stochastic process. The first way to do so is to consider a volatility depending on the time and on the level of the underlying. This is what we call Local Volatility models, the concept was created by B. Dupire in 1993 [13], it is also attributed to Derman [8]. Dupire showed that such models can replicate exactly a given surface of vanillas. They are written as

$$\frac{dS_t}{S_t} = r(t)dt + \sigma_D(t, S_t)dW_t \quad (6.1)$$

The second way to create some randomness in the volatility is to add to the model another stochastic component, and then to consider the volatility as a function of this component. Those stochastic volatility models are more complex and are richer from a theoretical point of view. The Stochastic Differential Equation associated to them is the following

$$\begin{aligned} \frac{dS_t}{S_t} &= r(t)dt + b(y_t)dW_t^1 \\ dy_t &= \mu(t, y_t)dt + \xi(t, y_t)dW_t^2 \end{aligned}$$

6.1.1 Dupire's model

Dupire's formula

In this brief subsection, we recall the results from Dupire. Under the dynamics described in (6.1), if one considers an option with terminal payoff $h(S)$, then the price of this option $P(t, S)$ satisfies the following pde

$$\frac{\partial P}{\partial t} + \frac{\sigma^2(t, S)}{2} S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0 \quad (6.2)$$

with the terminal condition $P(T, S) = h(S)$. The case we are interested in is the one of vanilla options $h(S) = (S - K)^+$. Let us write P as a function of the four variables $C(t, S; T, K)$. Differentiating twice the pricing Equation (6.2) with respect to the strike gives a pde with terminal condition $\delta_{S=K}$ and unknown function $G = \frac{\partial^2 C}{\partial K^2}$. By uniqueness, G is thus the Green function associated to the linear equation (6.2). As a function of (T, K) it verifies the adjoint of equation (6.2), also known as Kolmogorov forward equation

$$\frac{\partial G}{\partial T} - \frac{\partial^2}{\partial K^2} \left(\frac{\sigma^2(T, K)}{2} K^2 G \right) + r \frac{\partial}{\partial K} (KG) + rG = 0$$

with the initial condition $G|_{T=t} = \delta_{S=K}$. Integrating twice with respect to K this last equation gives at last

$$\frac{\partial C}{\partial T} - \frac{\sigma^2(T, K)}{2} K^2 \frac{\partial^2 C}{\partial K^2} + rK \frac{\partial C}{\partial K} = 0 \quad (6.3)$$

with the initial condition $C(t, S; t, K) = (S - K)^+$.

Eventually, we take an initial state (t_0, S_0) and a sufficiently smooth surface of vanillas $C(T, K)$ for $T > t_0$ (the exact smoothness of the surface is not the question here, for details we refer the reader

to [13]). This allows us to take

$$\sigma^2(T, K) = 2 \frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{K^2 \frac{\partial^2 C}{\partial K^2}} \quad (6.4)$$

The uniqueness of the solution of (6.3) guarantees us that a model with the local volatility (6.4) gives the exact prices C . This was Dupire's original purpose. As pointed out in the Introduction, it is an essential feature: it feels comfortable hedging a product using vanillas that are properly fitted to the market.

Dynamics of Local Volatility models

Nevertheless, local volatility models have a major drawback, their dynamics with respect to spot shifts are inconsistent with the markets. Indeed, one can observe on market quotes that when the spot moves in a certain direction, the surface of vanillas moves in the same direction: if the spot increases, the smile at a given maturity shifts to the right. And inversely when the spot decreases. In the article [9], P. Hagan, D. Kumar, A. Lesniewski and D. Woodward use first-order approximations of the implied volatility in the case of a Local Volatility depending only on the spot to prove that the dynamics in such a model are wrong: the smile moves in the opposite direction as the price of the underlying asset.

To illustrate this fact, we conduct a numerical simulation. Let us take a surface of vanilla prices (one can find the exact surface in the Appendix C). Using a local volatility model and a Monte-Carlo algorithm, we compute the implied volatility for different strikes and maturities using an initial spot value of 1. We then make the same computations shifting the spot in both directions and plot the evolution of the smile after such changes.

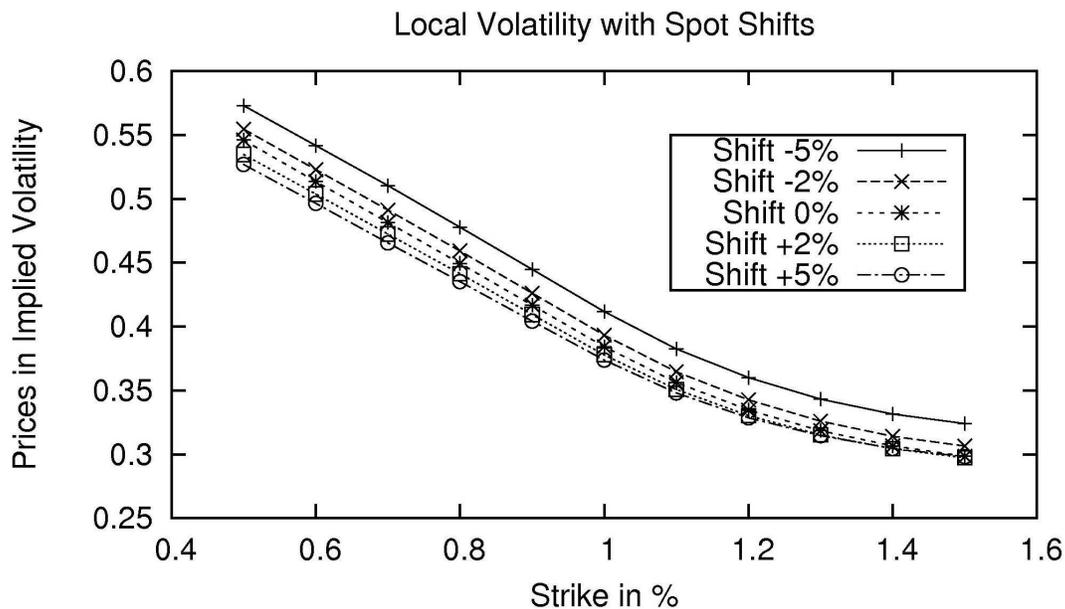


Figure 6.1: Implied Volatility for shifted initial spot values

We notice that the smile evolves accordingly to the predictions of Hagan, Kumar, Lesniewski and Woodward. In local volatility models, the smile exhibits the wrong sensitivity to spot shifts. This is potentially problematic with regard to hedges, they become unstable when the spot moves.

6.1.2 Mimicking theorem and Stochastic volatilities

This last statement led researchers to consider a more general class of models. The stochasticity of the volatility now stems from another stochastic component, different from the one of the spot. Those Stochastic Volatility models can be written as

$$\begin{aligned}\frac{dS_t}{S_t} &= r(t)dt + b(y_t)dW_t^1 \\ dy_t &= \mu(t, y_t)dt + \xi(t, y_t)dW_t^2\end{aligned}\tag{6.5}$$

where W_t^1 is a one-dimension standard brownian motion, y_t the stochastic factor of the volatility, W_t^2 a k -dimensional brownian motion, possibly correlated to W_t^1 and ξ the volatility matrix of the volatility factor. We let $X_t = (S_t, y_t)$ denote this $k + 1$ -dimensional process, we write its diffusion as

$$dX_t = \alpha_t dt + \beta_t dW_t\tag{6.6}$$

As was stressed in the Introduction, as far as vanillas are concerned, only the one-dimensional marginal of the first coordinate of X_t matters. To exploit this fact, we turn ourselves to a mimicking theorem from Gyongy [10]²

Theorem 8. *If X_t follows (6.6), with β_t verifying the uniform ellipticity condition $\beta_t^T \beta_t \geq KI_{k+1}$, then the functions*

$$\begin{aligned}\sigma(t, x) &= (\mathbb{E}[\beta_t^T \beta_t | X_t = x])^{\frac{1}{2}} \\ \gamma(t, x) &= \mathbb{E}[\alpha_t | X_t = x]\end{aligned}$$

are bounded, measurable, from $\mathbb{R}_+ \times \mathbb{R}^{k+1}$ into the space of $(k + 1) \times (k + 1)$ symmetric matrices and into \mathbb{R}^{k+1} respectively and there exists a weak solution to the sde

$$dx_t = \gamma(t, x_t)dt + \sigma(t, x_t)dW_t\tag{6.7}$$

That solution has the same one-dimensional marginals as X_t .

Remark. *One says that the stochastic differential equation (6.7) has a weak solution if there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an \mathcal{F}_t -Brownian motion \bar{W}_t and an \mathcal{F}_t -adapted stochastic process \bar{x}_t such that*

$$d\bar{x}_t = \gamma(t, \bar{x}_t)dt + \sigma(t, \bar{x}_t)d\bar{W}_t$$

The theorem we just cited allows to deal with the problem: if we consider the Stochastic Volatility model (6.5), then we are able to construct a Local Volatility model that has the exact same vanillas as the initial Stochastic Volatility model. Indeed, applying Gyongy to (6.6), tells us that the first coordinate of \bar{x}_t has the same law as S_t . And this first coordinate \bar{S}_t follows the stochastic differential equation

$$d\bar{S}_t = r(t)dt + \mathbb{E}[b^2(y_t) | S_t = s]_{s=\bar{S}_t}^{\frac{1}{2}} d\bar{W}_t^1$$

This "localization" of a stochastic volatility model is treated thoroughly in an article by M. Atlan, precisely entitled "Localizing volatilities" [11]. Let us now study the inverse problem: given a surface of vanillas, construct a stochastic volatility model that replicates them. In order to answer

²Krylov also studied in [46] the construction of stochastic differential equations whose solutions mimic certain features of the solutions of Ito processes.

this problem, we consider a mix of the two models previously described, a Local and Stochastic Volatility model.

6.2 Partial Differential Equation for the calibration of LSV models

The diffusion for our model is assumed to be the following

$$\frac{dS_t}{S_t} = r(t)dt + a(t, S_t)b(y_t)dW_t^1 \quad (6.8)$$

$$dy_t = \mu(t, y_t)dt + \xi(t, y_t)dW_t^2 \quad (6.9)$$

$(S_t, t \geq 0)$ is the stock price process and $(y_t, t \geq 0)$ the stochastic component of the volatility. The function b simply transforms that factor into a proper volatility. a is the local volatility part of the model, exactly as in Dupire's formula, we shall specify its value depending on the vanillas we want to calibrate. ξ is the volatility of the volatility factor (commonly called vovol) and μ is a drift term. W^1 and W^2 are one-dimensional standard brownian motions with correlation ρ .

6.2.1 Calibration equation

Let us now consider a surface of vanilla prices $C(T, K)$ and the corresponding Local Volatility σ_D . We make the following set of hypothesis

- (H1) for all $t \geq 0$, $S \in \mathbb{R}_+^*$ and $(y, \delta_1, \delta_2) \in \mathbb{R}^3$,

$$a^2(t, S)b^2(y)S^2\delta_1^2 + 2\rho a(t, S)b(y)S\xi(t, y)\delta_1\delta_2 + \xi^2(t, y)\delta_2^2 \geq K(\delta_1^2 + \delta_2^2)$$

for a certain constant $K > 0$ (uniform ellipticity)

- (H2) the functions a^2 , b^2 , $ab\xi$ and ξ^2 belong to $H^{2,h,h/2}(\mathbb{R}^2)$, the function b^2 belongs to $H^{2,h}(\mathbb{R})$
- (H3) $r(t)$ is in $H^{1,h/2}([0, T])$ and μ in $H^{1,h,h/2}(\mathbb{R})$

Proposition 5. *The diffusion model defined by (6.8-6.9) has a density with respect to Lebesgue measure, we let $p(t, S, y)$ denote said density. Moreover, if the model fits the surface of vanillas $C(T, K)$ then necessarily*

$$a^2(t, S) = \sigma_D^2(t, S) \frac{\int_{\mathbb{R}} p(t, S, y) dy}{\int_{\mathbb{R}} b^2(y) p(t, S, y) dy} \quad (6.10)$$

Proof. Given the regularity of the coefficients and the uniform ellipticity assumption (H1), we can apply results from the chapter VI of Friedman's book [5]. It ensures us that the transition probability of the couple (S_t, y_t) has a transition density with respect to Lebesgue measure.

Remark. *This existence does not need the coefficients to be differentiable with derivatives Holder. Those assumptions are made to give us the right to take the adjoint of Kolmogorov backward equation.*

We now assume that our model fits exactly the surface C. Letting (S_0, y_0) denote the initial state of

the system, the joint density $p(t, S, y)$ of the couple (S_t, y_t) verifies Kolmogorov forward equation

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} a^2 b^2 S^2 p \right) - \frac{\partial^2}{\partial S \partial y} (\rho a b \xi S p) - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \xi^2 p \right) + \frac{\partial}{\partial S} (r S p) + \frac{\partial}{\partial y} (\mu p) = 0$$

$$p(0, S, y) = \delta(S = S_0, y = y_0)$$

Applying Fubini, we can define $q(t, S) = \int_{\mathbb{R}} p(t, S, y) dy$ the first marginal density of our couple. We can then integrate the previous equation. Passing to the limit to nullify the integration constant, we obtain

$$\frac{\partial q}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} a^2 S^2 \left(\int_{\mathbb{R}} b^2(y) p(t, S, y) dy \right) \right) + \frac{\partial}{\partial S} (r S q) = 0$$

$$q(0, S) = \delta(S = S_0)$$

We now apply this equation to a local volatility model, equivalent to the case $b = 1$ and $a = \sigma_D$. q_D , the density of the spot in this model solves the equation

$$\frac{\partial q_D}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 S^2 q_D \right) + \frac{\partial}{\partial S} (r S q_D) = 0$$

$$q_D(0, S) = \delta(S = S_0)$$

Since we assumed that the vanillas of the LSV model are perfectly fitted, we have $q = q_D$. Identifying the terms in the two last formulas gives us the necessary value of a for the vanillas to be calibrated

$$a^2(t, S) = \sigma_D^2(t, S) \frac{q}{\int_{\mathbb{R}} b^2 p dy} = \sigma_D^2(t, S) \frac{\int_{\mathbb{R}} p(t, S, y) dy}{\int_{\mathbb{R}} b^2(y) p(t, S, y) dy}$$

□

Using this proposition, and reintroducing the value of a in Kolmogorov forward equation, we see that the joint density $p(t, S, y)$ is solution of the nonlinear partial integro-differential equation

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 b^2 S^2 \frac{\int_{\mathbb{R}} p dy}{\int_{\mathbb{R}} b^2 p dy} p \right) - \frac{\partial^2}{\partial S \partial y} (\rho \sigma_D b \xi S \left(\frac{\int_{\mathbb{R}} p dy}{\int_{\mathbb{R}} b^2 p dy} \right)^{\frac{1}{2}} p)$$

$$- \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \xi^2 p \right) + \frac{\partial}{\partial S} (r S p) + \frac{\partial}{\partial y} (\mu p) = 0 \quad (6.11)$$

$$p(0, S, y) = \delta_{S_0, y_0} \quad (6.12)$$

We thus have the equivalence between the existence of a model of the form (6.8-6.9) that calibrates perfectly the vanillas C and the existence of a solution p to the pde (6.11) such that the function a defined by (6.10) verifies the assumptions (H1) and (H2).

The reciprocal stems directly from using the solution p of the pde in the definition of a . The diffusion then becomes a classic linear stochastic differential equation.

Remark. This pde approach of the problem makes the quotient $\frac{\int_{\mathbb{R}} b^2 p dy}{\int_{\mathbb{R}} p dy}$ appear. Looking at it from a probabilistic point of view, we notice that it is nothing but the conditional expectation of the volatility squared, knowing the spot. Now, if we apply Gyongy theorem to the diffusion (6.8-6.9) where a is replaced by the value we just computed, we get that the Local Volatility that replicates

the vanillas of our LSV is equal to

$$\mathbb{E}[a^2(t, S_t) b^2(y_t) | S_t] = \mathbb{E}[\sigma_D^2(t, S_t) \frac{\int_{\mathbb{R}} p dy}{\int_{\mathbb{R}} b^2 p dy} b^2(y_t) | S_t] = \sigma_D^2(t, S_t)$$

which confirms that our vanillas are properly fitted. The result we just saw in itself is not original, the partial differential equation method however is unusual.

Furthermore, the quotient above is nothing else than the effective volatility mentioned in the introduction.

6.2.2 Application of the results from Part I

We are now interested in the resolution of Equation (6.11). Taking the notations from Chapter 5, we have, with $d = 2$ and $x = (S, y)$

$$\begin{aligned} V_{11}(t, S, y, I) &= \frac{S}{\sqrt{I}} \sigma_D(t, S) b(y) & V_{12} &= 0 \\ \lambda_{11} &= \{1\} & f_{11}(y) &= b^2(y) \\ V_{21}(t, S, y, I) &= \rho \xi(t, y) & V_{22}(t, S, y, I) &= \sqrt{1 - \rho^2} \xi(t, y) \\ \lambda_{12} &= \emptyset & \lambda_{21} &= \emptyset & \lambda_{22} &= \emptyset \\ \mu_1(t, S, y) &= r(t) S & \mu_2(t, S, y) &= \mu(t, y) \end{aligned}$$

We start by a theoretical study of the solvability of the equation. Because of the theoretical limitations faced in Part I, we are obliged to be in a bounded domain framework. We thus consider a rectangle $]S_m, S_M[\times]y_m, y_M[$ and let Ω denote this rectangle where the corners have been smoothed so that Ω verifies Property 1 and 2 (we replace the corners with crescents).

The domain of definition is $D^T =]0, T[\times \Omega$. We recall that $B = \{0\} \times \Omega$ and $C^T =]0, T[\times \partial\Omega$. We take an initial condition $p_0(S, y)$ and define a function Ψ on $\bar{B} \cup C^T$ by

$$\begin{aligned} \Psi &= p_0 & \text{on } \bar{B} \\ \Psi(t, S, y) &= p_0(S, y) & \text{for } (S, y) \in \partial\Omega \end{aligned}$$

As was done previously, we define two sets (the first one is trivial, the second one stems from the change we operated on the initial rectangle)

$$\begin{aligned} \Omega_S &= \{S \in \mathbb{R} \text{ such as } \exists y \in \mathbb{R} \text{ with } (S, y) \in \Omega\} =]S_m, S_M[\\ \Omega(S) &= \{y \in \mathbb{R} \text{ such as } (S, y) \in \Omega\} \end{aligned}$$

The integral over \mathbb{R} becomes, for $t > 0$ and $S \in \Omega_S$

$$\frac{\int_{\Omega(S)} p(t, S, y) dy}{\int_{\Omega(S)} b^2(y) p(t, S, y) dy}$$

We take any y_0 in $]y_m, y_M[$ and make the following assumptions

- (A1) Ψ belongs to $H^{2, h, h/2}(\partial D^T)$ and is strictly positive on $\bar{B} \cup C^T$
- (A2) σ_D belongs to $C^3(\mathbb{R}_+^* \times \mathbb{R}_+^*)$, b to $C^3(\mathbb{R})$ and ξ to $C^3(\mathbb{R}_+^* \times \mathbb{R})$

- (A3) for all $t \geq 0$, $(S, y) \in \Omega$ and $(\delta_1, \delta_2) \in \mathbb{R}^2$,

$$\frac{S^2}{b^2(y_0)} \sigma_D^2(t, S) b^2(y) \delta_1^2 + 2\rho \frac{S}{b(y_0)} \sigma_D(t, S) b(y) \xi(t, y) \delta_1 \delta_2 + \xi^2(t, y) \delta_2^2 \geq K(\delta_1^2 + \delta_2^2)$$

- (A4) Ψ verifies two different compatibility conditions, ie on ∂B we have

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 b^2 \frac{S^2}{b^2(y_0)} \Psi \right) + \frac{\partial^2}{\partial S \partial y} \left(\rho \sigma_D b \alpha \frac{S}{b(y_0)} \Psi \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \alpha^2 p \right) - \frac{\partial}{\partial S} (r S \Psi) - \frac{\partial}{\partial y} (\beta \Psi) \\ \frac{\partial \Psi}{\partial t} &= \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 b^2 S^2 \frac{\int \Psi dy}{\int b^2 \Psi dy} \Psi \right) + \frac{\partial^2}{\partial S \partial y} \left(\rho \sigma_D b \alpha S \left(\frac{\int \Psi dy}{\int b^2 \Psi dy} \right)^{\frac{1}{2}} \Psi \right) \\ &\quad + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \alpha^2 \Psi \right) - \frac{\partial}{\partial S} (r S \Psi) - \frac{\partial}{\partial y} (\beta \Psi) \end{aligned}$$

- (A5) $r(t)$ is in $H^{1, h/2}([0, T])$ and μ in $H^{1, h, h/2}(\mathbb{R})$

Proposition 6. For any $T > 0$, there exists a constant $b^* > 0$ such that, if $|b^2 - b^2(y_0)| \leq b^*$ on $]y_m, y_M[$, then the equation

$$\begin{aligned} \frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 b^2 S^2 \frac{\int_{\Omega(S)} p dy}{\int_{\Omega(S)} b^2 p dy} p \right) - \frac{\partial^2}{\partial S \partial y} \left(\rho \sigma_D b \alpha S \left(\frac{\int_{\Omega(S)} p dy}{\int_{\Omega(S)} b^2 p dy} \right)^{\frac{1}{2}} p \right) \\ - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \alpha^2 p \right) + \frac{\partial}{\partial S} (r S p) + \frac{\partial}{\partial y} (\beta p) = 0 \\ p = \Psi \text{ on } \bar{B} \cup C^T \end{aligned}$$

admits a solution on D^T , this solution belongs to $H^{2, h, h/2}(D^T)$.

Proof. This result is a direct application of Theorem 5. All the assumptions needed are verified, for instance, we do have $\frac{S}{\sqrt{I}} \sigma_D(t, S) b(y) \in C^3(\Omega \times \mathbb{R}_+^*)$ with the three derivatives with respect to the last variable bounded on $[\eta, +\infty[$ for all $\eta > 0$. \square

From a theoretical point of view, this result is not completely satisfying, for two reasons. The first one is the necessity of considering bounded domains. And the second is the initial condition that needs to be strictly positive and in $H^{2, h, h/2}$. We could indeed consider a sequence of such functions approximating the Dirac mass in (6.12). But the corresponding sequence of numbers b^* would have no reason to be bounded away from 0.

However, from a numerical point of view, we are obliged to consider bounded domains and an approximation of the Dirac as initial condition, so our result gives us at least the existence in such a case.

6.3 Numerical results

The theoretical limitations we ran into trying to prove the global existence of a solution and the partial result we were only able to obtain motivated the numerical resolution of that type of equation. Discussions with a few practitioners confirmed that instabilities occur in the calibration when the vol (in our notations the function ξ) is too large. This seems to confirm the idea that when the function b oscillates too much (a change of scale in the factor y_t clearly shows the equivalence between a b that moves a lot and a large ξ), the resolution of the equation is not guaranteed anymore.

In this section, we study the calibration that stems from solving the partial differential equation (6.11), for two stochastic volatilities: "lognormal" and "CIR". The details of the algorithm used for the resolution and a study of the instabilities can be found in Part III.

6.3.1 Lognormal volatility

We start with a simple mean reverting model for the volatility factor, the function b is chosen as an exponential

$$\frac{dS_t}{S_t} = r(t)dt + a(t, S_t)\exp(y_t)dW_t^1 \quad (6.13)$$

$$dy_t = \kappa(\delta - y_t)dt + \gamma dW_t^2 \quad (6.14)$$

with

$$a^2(t, S) = \sigma_D^2(t, S) \frac{\int_{\mathbb{R}} p(t, S, y) dy}{\int_{\mathbb{R}} \exp(2y) p(t, S, y) dy}$$

We solve the Equation (6.11) with the functions we just chose and the Local Volatility σ_D associated to the implied volatility surface described in Appendix C. Once we have found the function p , density of the couple (S_t, y_t) , we compute the vanilla prices for different strikes and maturities using this density. To have a point of comparison, we also calculate the same prices with the Local Volatility σ_D and compare all of them to the targeted prices (column TP).

Table 6.1: 6 months Vanillas

Strike	TP	LV	LSV
0.5	54.54	54.46	54.43
0.6	51.35	51.34	51.20
0.7	48.12	48.14	48.02
0.8	44.86	44.93	44.82
0.9	41.57	41.66	41.56
1.0	38.26	38.32	38.15
1.1	35.41	35.46	35.34
1.2	33.25	33.27	33.15
1.3	31.68	31.67	31.57
1.4	30.53	30.48	30.46
1.5	29.80	29.72	29.76

Table 6.2: 1 year Vanillas

Strike	TP	LV	LSV
0.5	49.82	49.79	49.71
0.6	47.23	47.22	47.15
0.7	44.61	44.60	44.56
0.8	41.97	41.98	41.92
0.9	39.30	39.32	39.29
1.0	36.63	36.61	36.59
1.1	34.31	34.30	34.29
1.2	32.35	32.38	32.36
1.3	30.75	30.77	30.75
1.4	29.43	29.44	29.44
1.5	28.41	28.42	28.40

We then plot the gaps between the original vanillas we wanted to calibrate and the ones obtained with the model.

We see that the calibration is quite efficient, the gaps are equivalent to the ones of the Local Volatility model. Reducing the grid, and using fewer time points gives a less accurate convergence, however, it makes the calculation quicker. A tradeoff has naturally to be found between the quality of calibration for long maturities, and calculation time. We now reproduce the same computations for another model of volatility.

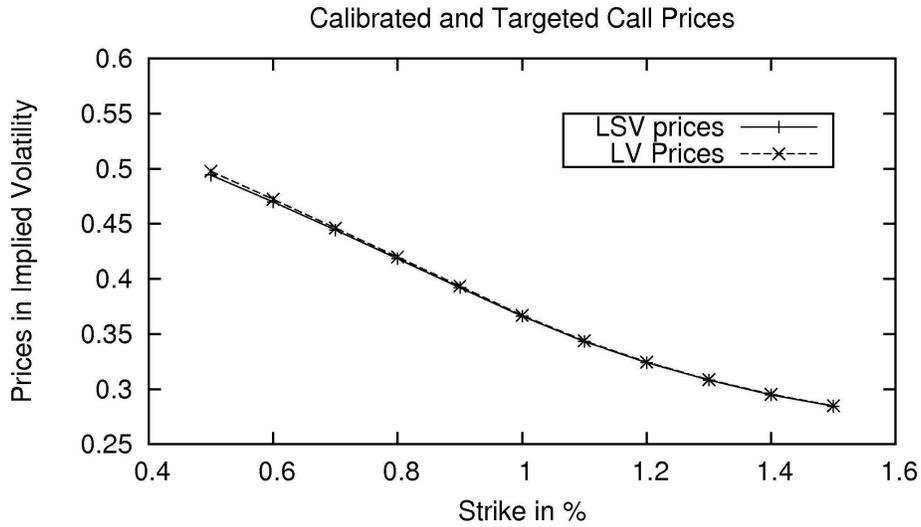


Figure 6.2: Smile for 1 year maturity calls, computed with LV and LSV

6.3.2 CIR

In this subsection, we are interested in the calibration of a CIR model, the diffusion is the following

$$\frac{dS_t}{S_t} = r(t)dt + a(t, S_t)y_t dW_t^1 \quad (6.15)$$

$$dy_t = \kappa(\alpha - y_t)dt + \gamma\sqrt{y_t}dW_t^2 \quad (6.16)$$

Detailed properties of this process are described in Appendix D. In particular, as long as $2\kappa\alpha > \gamma^2$, y_t is strictly positive a.e. Once again, we solve the Equation (6.11) with this stochastic volatility and plot the results of the calibration.

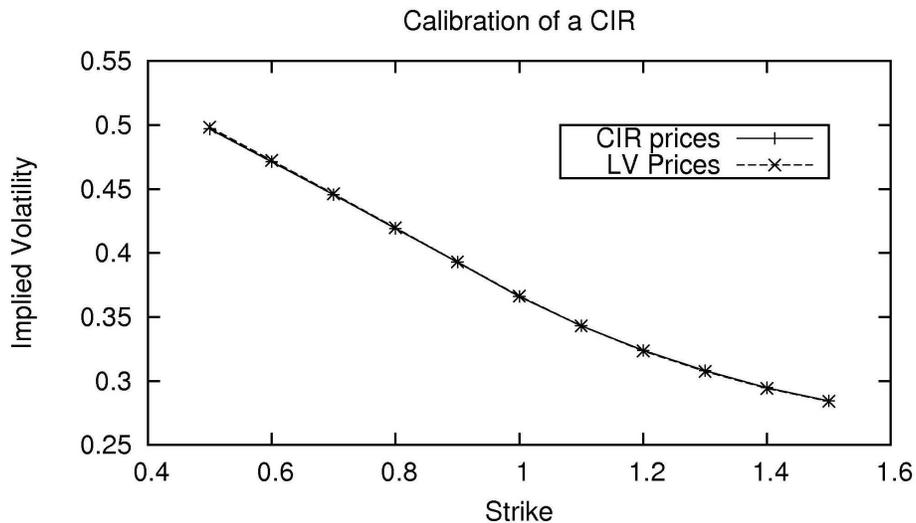


Figure 6.3: Smile for 1 year maturity vanillas computed with LV and calibrated CIR

The exact gap between the curves can be found in Appendix C. The accuracy of the fit is satisfactory.

Chapter 7

Application to the "Local Correlation" model

In this chapter, we are interested in the calibration of the vanillas of a basket on n underlyings. We consider a market with n stocks and a basket on those stocks. The purpose is to define a diffusion model for those underlyings that is able to reproduce their implied volatility surface as well as the one of the basket.

We start with notations, let $(S_t^i)_{1 \leq i \leq n}$ denote the n stocks involved in our problem. The basket's value is given by

$$B_t = \sum_{i=1}^n w_i S_t^i \quad (7.1)$$

where the set $(w_i)_{1 \leq i \leq n}$ stands for the weights of the different underlyings. They are assumed to be constant in the rest of our work. We also fix $n+1$ surfaces of vanillas $(C_i(T, K))_{1 \leq i \leq n}$ and $C_B(T, K)$.

7.1 Inconsistencies between stock and basket options

The naive approach to solve this problem is simply to consider n local volatility models

$$\frac{dS_t^i}{S_t^i} = r(t)dt + \sigma_i(t, S_t^i)dW_t^i \quad (7.2)$$

We know how to choose the functions σ_i to fit the surfaces $(C_i(T, K))_{1 \leq i \leq n}$ with this diffusion for the S^i . The correlation matrix $\rho = (\rho_{ij})_{1 \leq i, j \leq n}$ associated to the standard brownian motions W_t^i of each underlying can be estimated with historical data. The model is now entirely defined. By equation (7.1) of B_t , the vanilla prices for the basket are completely determined and are equal to $\mathbb{E}[(\sum w_i S_T^i - K)^+]$. However, there is no particular reason why the surface computed in this framework would be equal to $C_B(T, K)$. In fact, the skew of the basket is more pronounced on the market than in a model with constant correlations between the underlyings [50].

7.2 "Local Correlation" model

In the manner of B. Dupire who decided to let the volatility depend on the level of the spot, we add a degree of freedom to our model by distorting the matrix of correlation with a function of B_t the level of the basket. This method is involved in [51], where the spot level induces some feedback on

the values of the different underlyings. In our context, the new correlation matrix we take is written as a linear combination of ρ and of the constant matrix with only 1 for coefficients. This matrix $\tilde{\rho}$ is equal to

$$\tilde{\rho}_{ij} = \lambda + (1 - \lambda)\rho_{ij} = \rho_{ij} + \lambda(1 - \rho_{ij}) \quad (7.3)$$

We shall see while writing the calibration equation that λ has to be chosen as a function of the time and of B_t . The matrix $\tilde{\rho}_{ij}$ can be seen as an analogous of Dupire's Local Volatility, a "Local Correlation" so to speak.

Now that the model is properly defined, let us precise the assumptions needed to pursue the calibration

- (H1) the matrix ρ is a correlation matrix¹, ie is semidefinite, positive, with non-diagonal coefficients in $] - 1, 1[$ and diagonal coefficients equal to 1. In order to be in a well-posed pde framework, we have to strengthen this definition and assume that ρ is definite, we let K_ρ denote its smaller (necessarily strictly positive) eigenvalue.
- (H2) the functions σ_i belong to $H^{2,h,h/2}(\mathbb{R})$ and the $S^i\sigma_i(t, S^i)$ are bounded away from 0 on $\mathbb{R}_+ \times \mathbb{R}_+^*$.
- (H3) $r(t)$ is in $H^{1,h/2}([0, T])$

Given those assumptions, we have existence of a density in the more general case of a matrix ρ function of the couple (t, B) . We are also able to write a condition for the vanillas of the model to be fitted.

Proposition 7. *The diffusion model defined by (7.2), in the more general case of a correlation function of the couple (t, B) , has a density $p(t, S^1, \dots, S^n)$ with respect to Lebesgue measure. Furthermore, this model calibrates the surface $C_B(T, K)$ of B_t 's vanillas (represented by it's local volatility surface σ_B) if and only if*

$$\sigma_B^2(t, B)B^2 \int \tilde{p}(t, B, S^2, \dots, S^n) dS^2 \dots dS^n = \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij}(t, B) \int (\tilde{\beta}_i \tilde{\beta}_j \tilde{p})(t, B, S^2, \dots, S^n) dS^2 \dots dS^n \quad (7.4)$$

where

$$\tilde{p}(t, B, S^2, \dots, S^n) = p(t, S^1, S^2, \dots, S^n) \quad \tilde{\beta}_i(t, B, S^2, \dots, S^n) = S^i \sigma_i(t, S^i)$$

with

$$S^1 = \frac{1}{w_1} (B - \sum_{i=2}^n w_i S^i) \quad (7.5)$$

Proof. Using the same argument and the results from Friedman [5], we get the existence of the transition density $p(t, S^1, \dots, S^n)$. We can now write the calibration problem for the vanillas of the basket B_t . The density we just defined satisfies Kolmogorov forward equation

$$\frac{\partial p}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2}{\partial S^i \partial S^j} (\rho_{ij} S^i \sigma_i S^j \sigma_j p) + \sum_{1 \leq i \leq n} \frac{\partial}{\partial S^i} (r S^i p) + r p = 0$$

¹An interesting reference on the subject, with a different parameterization can be found in [42]

To ease the problem, it is useful to change the coordinates (S^1, \dots, S^n) into (B, S^2, \dots, S^n) with S^1 defined by (7.5). After computations, the equation becomes

$$\begin{aligned} \frac{\partial \tilde{p}}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq n} w_i w_j \frac{\partial^2}{\partial B^2} (\rho_{ij} \tilde{\beta}_i \tilde{\beta}_j \tilde{p}) - \frac{1}{2} \sum_{2 \leq i, j \leq n} \left(\frac{\partial^2}{\partial S^i \partial S^j} + w_i \frac{\partial^2}{\partial B \partial S^j} + w_j \frac{\partial^2}{\partial S^i \partial B} \right) (\rho_{ij} \tilde{\beta}_i \tilde{\beta}_j \tilde{p}) \\ - \sum_{2 \leq i \leq n} w_1 \frac{\partial^2}{\partial B \partial S^i} (\rho_{1i} \tilde{\beta}_1 \tilde{\beta}_i \tilde{p}) + \frac{\partial}{\partial B} (rB \tilde{p}) + \sum_{2 \leq i \leq n} \frac{\partial}{\partial S^i} (rS^i \tilde{p}) + r \tilde{p} = 0 \end{aligned}$$

where $\tilde{p}(B, S^2, \dots, S^n)$ and $\tilde{\beta}_i(B, S^2, \dots, S^n)$ are defined above. We integrate the equation against the variables (S^2, \dots, S^n) , and writing $q = \frac{1}{w_1} \int \tilde{p} dS^2 \dots dS^n$, we see that the density of the marginal law of B satisfies

$$\frac{\partial q}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq n} w_i w_j \frac{\partial^2}{\partial B^2} (\rho_{ij} \int \tilde{\beta}_i \tilde{\beta}_j \tilde{p} dS^2 \dots dS^n) + \frac{\partial}{\partial B} (rBq) + rq = 0$$

Comparing this equation to Dupire's equation for the local volatility σ_B

$$\frac{\partial q}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial B^2} (\sigma_D^2 B^2 q) + \frac{\partial}{\partial B} (rBq) + rq = 0$$

we see that if our model reproduces the vanillas σ_D then, we have the equality

$$\sigma_D^2 B^2 \int \tilde{p} dS^2 \dots dS^n = \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \int \tilde{\beta}_i \tilde{\beta}_j \tilde{p} dS^2 \dots dS^n$$

Reciprocally, the condition we just wrote is clearly sufficient for the options to be calibrated \square

Remark. Let us note that this condition is written as an equality between two functions of the time and of B. The other variables are no longer represented.

We now assume that the condition (7.4) is not verified, the model defined by (7.2) does not fit the vanillas of the basket B_t . We need to enrich it if we want to solve the calibration problem. Our choice is to distort the correlation matrix. We write the new matrix $\tilde{\rho}$ as described before (7.3). Hence, we denote by Θ the matrix $\Theta_{ij} = 1$ for all $1 \leq i, j \leq n$. We also notice that, the trace of ρ being equal to n, K_ρ is smaller than 1.

Lemma 5. The matrix $\tilde{\rho} = (1 - \lambda)\rho + \lambda\Theta$ is also a correlation matrix as long as λ is in $]r, 1[$ with

$$r = -\min\left(\max_{i \neq j} \frac{1 + \rho_{ij}}{1 - \rho_{ij}}, \frac{K_\rho}{n - K_\rho}\right) < 0 \quad (7.6)$$

Proof. Clearly, we have for all $(\xi_i, \xi_j) \in \mathbb{R}^2$

$$\sum_{1 \leq i, j \leq n} \xi_i ((1 - \lambda)\rho_{ij} + \lambda)\xi_j \geq K_\rho (1 - \lambda) |\xi|^2 + \lambda \left(\sum_{1 \leq i \leq n} \xi_i \right)^2$$

If λ is positive, since $K_\rho(1 - \lambda)$ is strictly positive, the matrix remains definite positive. Now, if $\lambda < 0$, we can apply Cauchy-Schwarz to get

$$\sum_{1 \leq i, j \leq n} \xi_i ((1 - \lambda)\rho_{ij} + \lambda)\xi_j \geq (K_\rho(1 - \lambda) + \lambda n) |\xi|^2$$

Since $K_\rho \leq 1$, we see that $\lambda > -\frac{K_\rho}{n - K_\rho}$ is enough for $K_\rho(1 - \lambda) + \lambda n$ to be strictly positive.

The diagonal coefficients of $\tilde{\rho}$ are still 1. As for the other terms, thanks to the first term in relation (7.6), they still belong to the interval $] - 1, 1[$. \square

We introduce the new correlation matrix in condition (7.4), this gives us the value of λ

$$\lambda(\tilde{\rho}) = \frac{\sigma_D^2 B^2 \int \tilde{\rho} dS^2 \dots dS^n - \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \int \tilde{\beta}_i \tilde{\beta}_j \tilde{\rho} dS^2 \dots dS^n}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int \tilde{\beta}_i \tilde{\beta}_j \tilde{\rho} dS^2 \dots dS^n} \quad (7.7)$$

$\lambda(\tilde{\rho})$ is a function of B and t . We can now use this value to write a pde on the density $\tilde{\rho}$. Any solution of this equation is a density that calibrates the vanillas of the basket. The equation is the following

$$\frac{\partial \tilde{\rho}}{\partial t} + L_1 \tilde{\rho} + L_2(\tilde{\rho}) = 0 \quad (7.8)$$

where L_1 is linear and verifies

$$\begin{aligned} L_1 \tilde{\rho} = & -\frac{1}{2} \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \frac{\partial^2}{\partial B^2} (\tilde{\beta}_i \tilde{\beta}_j \tilde{\rho}) - \frac{1}{2} \sum_{2 \leq i, j \leq n} \rho_{ij} \left(\frac{\partial^2}{\partial S^i \partial S^j} + w_i \frac{\partial^2}{\partial B \partial S^j} + w_j \frac{\partial^2}{\partial S^i \partial B} \right) (\tilde{\beta}_i \tilde{\beta}_j \tilde{\rho}) \\ & - \sum_{2 \leq i \leq n} w_1 \rho_{1i} \frac{\partial^2}{\partial B \partial S^i} (\tilde{\beta}_1 \tilde{\beta}_i \tilde{\rho}) + \frac{\partial}{\partial B} (rB \tilde{\rho}) + \sum_{2 \leq i \leq n} \frac{\partial}{\partial S^i} (rS^i \tilde{\rho}) + r \tilde{\rho} \end{aligned}$$

and L_2 is the nonlinear part of the equation

$$\begin{aligned} L_2^{\lambda(\tilde{\rho})}(\tilde{\rho}) = & -\frac{1}{2} \sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \frac{\partial^2}{\partial B^2} (\lambda(\tilde{\rho}) \tilde{\beta}_i \tilde{\beta}_j \tilde{\rho}) - \frac{1}{2} \sum_{2 \leq i, j \leq n} (1 - \rho_{ij}) \left(\frac{\partial^2}{\partial S^i \partial S^j} + w_i \frac{\partial^2}{\partial B \partial S^j} \right. \\ & \left. + w_j \frac{\partial^2}{\partial S^i \partial B} \right) (\lambda(\tilde{\rho}) \tilde{\beta}_i \tilde{\beta}_j \tilde{\rho}) - \sum_{2 \leq i \leq n} w_1 (1 - \rho_{1i}) \frac{\partial^2}{\partial B \partial S^i} (\lambda(\tilde{\rho}) \tilde{\beta}_1 \tilde{\beta}_i \tilde{\rho}) \end{aligned}$$

Remark. The operator $L_1 + L_2$ stems from a change of coordinates on a uniformly elliptic operator. It is also elliptic, uniformly on any domain where the $\tilde{\beta}_i$ are bounded away from 0 by a strictly positive constant.

Furthermore, the initial condition is

$$\tilde{\rho}(0, B, S^2, \dots, S^n) = \delta(\sum w_i S_0^i, S_0^2, \dots, S_0^n)$$

where S_0^i is the market value at instant 0 of the i -th stock. Applying this initial condition to (7.7), we see that the initial value of λ is

$$\lambda(\tilde{\rho})(0, B) = \frac{\sigma_D^2(0, B) B^2 - \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} S_0^i \sigma_i(0, S_0^i) S_0^j \sigma_j(0, S_0^j)}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) S_0^i \sigma_i(0, S_0^i) S_0^j \sigma_j(0, S_0^j)} \quad (7.9)$$

7.3 Resolution of the equation for the calibration of a basket

7.3.1 Theoretical study

We still have time $0 < t \leq T$ and space-variables $x = (B, S^2, \dots, S^n) \in \Omega \subset \mathbb{R}^n$. Ω remains an open subset with a sufficiently smooth boundary. Since the first space-variable plays a particular part, we write $U = (S^2, \dots, S^n)$. We also denote by $D^T =]0, T[\times \Omega$ the domain of definition and by $D_b = \{0\} \times \Omega$, $D_s^T = \{T\} \times \Omega$ and $C^T =]0, T[\times \partial\Omega$ the different parts of the boundary. We also specify $\Omega_B = \{B \in \mathbb{R} / \exists U \in \mathbb{R}^{n-1}, (B, U) \in \Omega\}$ and $\forall B \in \Omega_B, \Omega_U(B) = \{U \in \mathbb{R}^{n-1} / (B, U) \in \Omega\}$. We are interested in the following equation

$$\frac{\partial p}{\partial t} + L_1 p + L_2^{\lambda(p)}(p) = 0 \quad \text{on } D^T \cup D_s^T \quad (7.10)$$

We add the boundary condition $p = \Psi$ on $\overline{D_b} \cup C^T$ with Ψ constant on C^T . The complexity of the equation stems from the term λ

$$\lambda(p)(t, B) = \frac{\sigma_D^2 B^2 \int_{\Omega_U(B)} p dU - \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j p dU}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j p dU}$$

Remark. The equation (7.10) is very similar to the type of equations studied in Part I. The existence result is obtained through the same means, we only need control over λ to be sure that this quantity remains in the domain $[r, 1]$. Here too, we have to suppose that the β_i do not vary too much

The result we are interested in requires the following assumptions on the β_i and Ψ .

- (A1) $\forall 1 \leq i \leq n, \beta_i \in H^{2,h,h/2}(D^T)$ and there exists two constants $(\beta_{inf}, \beta_{sup}) \in \mathbb{R}^2$ such that

$$0 < \beta_{inf} \leq \beta_i \quad |\beta_i|_{H^{2,h,h/2}(D^T)} \leq \beta_{sup}$$

- (A2) the functions β_i are close to a constant ie $\forall 1 \leq i \leq n, \exists \beta_i^*, |\beta_i - \beta_i^*|_{H^0(D^T)} \leq b^*$ where b^* will be computed later
- (A3) Ψ is strictly positive and in $H^{2,h,h/2}(\partial D^T)$. Furthermore, the initial condition must verify the inequality of Lemma 5 ie for all $B \in \Omega_B$,

$$r < \lambda(\Psi) = \frac{\sigma_D^2 B^2 \int_{\Omega_U(B)} \Psi|_{D_b} dU - \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j \Psi|_{D_b} dU}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j \Psi|_{D_b} dU} < 1$$

- (A4) compatibility conditions are obviously still necessary

$$\begin{aligned} \frac{\partial \Psi}{\partial t} + L_1 \Psi + L_2^{\lambda(\Psi)}(\Psi) &= 0 && \text{on } \partial D_b \\ \frac{\partial \Psi}{\partial t} + L_1 \Psi + L_2^{\lambda^*}(\Psi) &= 0 && \text{on } \partial D_b \end{aligned}$$

where the quantity λ^* is defined by

$$\lambda^*(t, B) = \frac{\sigma_D^2 B^2 - \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \beta_i^* \beta_j^*}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \beta_i^* \beta_j^*} \quad (7.11)$$

We also define a function p_0 as $p_0(t, B, U) = \Psi(B, U)$ on Ω for any $t \geq 0$. By assumptions on Ψ , the function p_0 belongs to $H^{2,h,h/2}(D^T)$ and is strictly positive on $\overline{D^T}$. We write $\underline{p_0} = \inf p_0$ and $\overline{p_0} = \sup p_0$.

Theorem 9. *Under the previous assumptions, there exists $0 < b^*$, $0 < T^* \leq T$ and a solution $p \in H^{2,h,h/2}(D^{T^*})$ of the equation (7.10) on $D^{T^*} \cup B^{T^*}$ with boundary condition $p = \Psi$ on $\overline{D_b} \cup C^{T^*}$. Moreover, we have the inequality $r \leq \lambda(p) < 1$ on $D^{T^*} \cup B^{T^*}$ which guarantees us that the matrix $\tilde{\rho}$ of our model is indeed a correlation matrix.*

Proof. The proof of this result is almost identical to the one of Theorem 6. The assumption (A2) gives us control over the variations of λ . We have the

Lemma 6. *There exists two constants K_λ, K'_λ (depending only on $h, n, \beta_{inf}, \beta_{sup}, \underline{p_0}, \overline{p_0}$ and Ω) and a polynomial function P strictly positive and increasing on \mathbb{R}_+^* such that $\forall p, q \in H^{2,h,h/2}(D^T)$ verifying $\frac{p_0}{2} \leq p, q$, we have*

$$\begin{aligned} |\lambda(p) - \lambda^*|_{2+h}^{D^t} &\leq b^*(1+b^*)^3 K_\lambda P(|p|_{2+h}^{D^t}) \\ |\lambda(p) - \lambda(q)|_{2+h}^{D^t} &\leq b^*(1+b^*)^3 K'_\lambda |p - q|_{2+h}^{D^t} P(|p|_{2+h}^{D^t}) P(|q|_{2+h}^{D^t}) \end{aligned}$$

Remark. *As a consequence of this lemma, we see that $\forall p \in H^{2,h,h/2}(D^T)$ verifying $\frac{p_0}{2} \leq p$, $\lambda(p)$ belongs to $H^{2,h,h/2}(D^T)$.*

To solve Equation (7.10), we apply the same fixed point method than before and use Lemma 6 to get an upper bound on the second term. Let us consider the application M which takes a function $u \in X^t$ (where t will be defined in order for the distorted matrix $\tilde{\rho}$ to remain a correlation matrix) and sends it on $v \in H^{2,h,h/2}(D^t)$ solution of the equation

$$\frac{\partial v}{\partial t} + L_1 v + L_2^{\lambda^*}(v) = L_2^{\lambda^*}(u) - L_2^{\lambda(u)}(u) \quad \text{on } D^t \cup D_s^t \quad (7.12)$$

and $v = \Psi$ on $\overline{D_b} \cup C^t$. This application sends the set

$$X^t = \{p \in H^{2,h,h/2}(D^t), |p - p_0|_{2+h}^{D^t} \leq \frac{p_0}{2}, p = \Psi \text{ on } \overline{D_b} \cup C^t\}$$

into itself and is a contraction for a suitable b^* . We compute, with $v = M(u)$ and $q = M(p)$

$$\frac{\partial v - p_0}{\partial t} + L_1(v - p_0) + L_2^{\lambda^*}(v - p_0) = L_2^{\lambda^*}(u) - L_2^{\lambda(u)}(u) - \frac{\partial p_0}{\partial t} - L_1 p_0 - L_2^{\lambda^*}(p_0)$$

and

$$\frac{\partial v - q}{\partial t} + L_1(v - q) + L_2^{\lambda^*}(v - q) = L_2^{\lambda^*}(u - p) - (L_2^{\lambda(u)}(u) - L_2^{\lambda(p)}(p))$$

Exactly as in Chapter 5, we apply Schauder's inequality to get

$$\begin{aligned} |v - p_0|_{2+h}^{D'} &\leq KK_\lambda b^* (1 + b^*)^3 P(|u|_{2+h}^{D'}) \\ |v - q|_{2+h}^{D'} &\leq KK'_\lambda b^* (1 + b^*)^3 |u - p|_{2+h}^{D'} P(|u|_{2+h}^{D'}) P(|p|_{2+h}^{D'}) \end{aligned}$$

where P are polynomial functions, strictly positive and increasing on \mathbb{R}^+ . The first inequality allows to prove that v belongs to X^t for b^* small enough, the second that M is a contraction.

Let us now define the value of t . We apply Theorem 1 to the function $\tilde{v} = v - p_0$. v and p_0 are equal to ψ on $\overline{D_b} \cup C^t$, thus \tilde{v} is null on the boundary. Hence

$$|\tilde{v}|_{H^0(D^t)} \leq t K_{H^0} |L_2^{\lambda^*}(u) - L_2^\lambda(u) - \frac{\partial p_0}{\partial t} - L_1 p_0 - L_2^{\lambda^*}(p_0)|_{H^0(D^t)}$$

Since, $|\lambda(v) - \lambda(p_0)| \leq K|\tilde{v}|$ and $\lambda(p_0) = \lambda(\Psi)$, we get from assumption (A3) that

$$r \leq \lambda(v) < 1$$

for t small enough. There remains to prove Lemma 6. Let us compute

$$\begin{aligned} \lambda(p) - \lambda^* &= \frac{\sigma_D^2 B^2 \int_{\Omega_U(B)} p dU - \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j p dU}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j p dU} - \frac{\sigma_D^2 B^2 - \sum_{1 \leq i, j \leq n} w_i w_j \rho_{ij} \beta_i^* \beta_j^*}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \beta_i^* \beta_j^*} \\ &= (\sigma_D^2 B^2 T_1 - T_2) T_3 \end{aligned}$$

with

$$\begin{aligned} T_1 &= \sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} (\beta_i^* \beta_j^* - \tilde{\beta}_i \tilde{\beta}_j) p dU \\ T_2 &= \sum_{1 \leq i, j, k, l \leq n} w_i w_j w_k w_l \rho_{ij} (1 - \rho_{kl}) \int_{\Omega_U(B)} (\tilde{\beta}_i \tilde{\beta}_j \beta_k^* \beta_l^* - \beta_i^* \beta_j^* \tilde{\beta}_k \tilde{\beta}_l) p dU \end{aligned}$$

and

$$T_3 = \frac{1}{\sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j p dU \sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \beta_i^* \beta_j^*}$$

Those three terms can be computed exactly as in Appendix A. It gives us the result we wanted. We also have

$$\lambda(p) - \lambda(q) = (\sigma_D^2 B^2 T'_1 - T'_2) T'_3$$

with

$$\begin{aligned} T'_1 &= \sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j q dU \sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} (\beta_i^* \beta_j^* - \tilde{\beta}_i \tilde{\beta}_j) (p - q) dU \\ &\quad - \sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} (\beta_i^* \beta_j^* - \tilde{\beta}_i \tilde{\beta}_j) q dU \sum_{1 \leq i, j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j (p - q) dU \end{aligned}$$

$$\begin{aligned}
T'_2 = & \sum_{1 \leq i,j,k,l \leq n} w_i w_j w_k w_l \rho_{ij} (1 - \rho_{kl}) \int_{\Omega_U(B)} (\tilde{\beta}_i \tilde{\beta}_j \beta_k^* \beta_l^* - \beta_i^* \beta_j^* \tilde{\beta}_k \tilde{\beta}_l) (p - q) dU \\
& \sum_{1 \leq i,j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j q dU - \sum_{1 \leq i,j \leq n} w_i w_j (1 - \rho_{ij}) \int_{\Omega_U(B)} \tilde{\beta}_i \tilde{\beta}_j (p - q) dU \\
& \sum_{1 \leq i,j,k,l \leq n} w_i w_j w_k w_l \rho_{ij} (1 - \rho_{kl}) \int_{\Omega_U(B)} (\tilde{\beta}_i \tilde{\beta}_j \beta_k^* \beta_l^* - \beta_i^* \beta_j^* \tilde{\beta}_k \tilde{\beta}_l) q dU
\end{aligned}$$

and

$$T'_3 = \frac{1}{\sum_{1 \leq i,j \leq n} w_i w_j (1 - \rho_{ij}) \int \tilde{\beta}_i \tilde{\beta}_j p dU \sum_{1 \leq i,j \leq n} w_i w_j (1 - \rho_{ij}) \int \tilde{\beta}_i \tilde{\beta}_j q dU \sum_{1 \leq i,j \leq n} w_i w_j (1 - \rho_{ij}) \beta_i^* \beta_j^*}$$

which concludes the proof. \square

Remark. Once again, the existence result is partial. In the case of "Local Correlation", the proof requires for the β_i to be close to constants, the volatilities σ_i must behave like $\frac{1}{S_i}$.

7.3.2 Numerical Resolution

In this subsection, we present the results of the calibration for a basket on two underlyings. Let us consider two assets, we assume both of them generate the following implied volatility surface

Table 7.1: Implied Volatility Surface for the basket's underlyings

K \ T	0,088	0,167	0,25	0,50	0,75	1,0	1,1
80%	48,0%	47,9%	46,5%	45,0%	43,8%	43,0%	43,0%
90%	45,0%	44,9%	44,1%	43,0%	42,0%	41,4%	41,4%
100%	43,0%	42,9%	42,2%	41,5%	40,8%	40,4%	40,4%
110%	41,5%	41,4%	40,8%	40,3%	39,9%	39,6%	39,6%
120%	41,0%	40,9%	40,6%	40,2%	39,7%	39,4%	39,4%

Using a Monte-Carlo simulation and the Local Volatilities stemming from those surfaces, we compute the theoretical prices for the basket B_t , with weights $w_1 = w_2 = 0.5$ and a correlation $\rho_{12} = -0.5$

Table 7.2: Theoretical Implied Volatility Surface for the basket

K \ T	0,088	0,167	0,25	0,50	0,75	1,0	1,1
80%	22,90%	22,54%	21,80%	20,72%	20,00%	19,61%	19,54%
90%	21,52%	21,21%	20,73%	19,78%	19,34%	19,03%	19,04%
100%	20,01%	19,94%	19,83%	19,15%	18,91%	18,68%	18,71%
110%	18,73%	18,43%	18,77%	18,58%	18,47%	18,26%	18,31%
120%	17,91%	17,67%	18,04%	18,11%	18,15%	18,04%	18,13%

We distort this theoretical surface by a factor of 0.9 making the prices of the basket inconsistent with the prices of the underlyings and apply our calibration algorithm. Solving the partial integro-differential equation (7.8) gives the following vanillas (quoted in implied volatility)

K \ T	0,25		0,50		0,75		1,0	
	Theor.	LC	Theor.	LC	Theor.	LC	Theor.	LC
80%	19,62%	19,77%	18,65%	19,03%	18,00%	18,55%	17,65%	18,14%
90%	18,66%	18,83%	17,80%	18,21%	17,40%	17,91%	17,13%	17,70%
100%	17,85%	17,66%	17,24%	17,18%	17,02%	17,08%	16,81%	16,99%
110%	16,89%	16,66%	16,72%	16,54%	16,63%	16,34%	16,44%	16,35%
120%	16,23%	16,02%	16,29%	16,24%	16,34%	16,11%	16,24%	16,26%

Table 7.3: Calibration of the basket

The results are rather satisfactory, especially at the money. Due to lengthy computations, we chose to define a sparse initial surface, this is probably why the calibration is not better far from the money. However, the fitting method appears to be valid. In Appendix C, we give results for other correlations and weights. Now follows an outlook of the values taken by the new correlation $\tilde{\rho}$ at different maturities when the theoretical surface is distorted by factors 0.95 and 1.05. The parameters are: $\rho = 0.5$, $w_1 = 0.7$ and $w_2 = 0.3$.

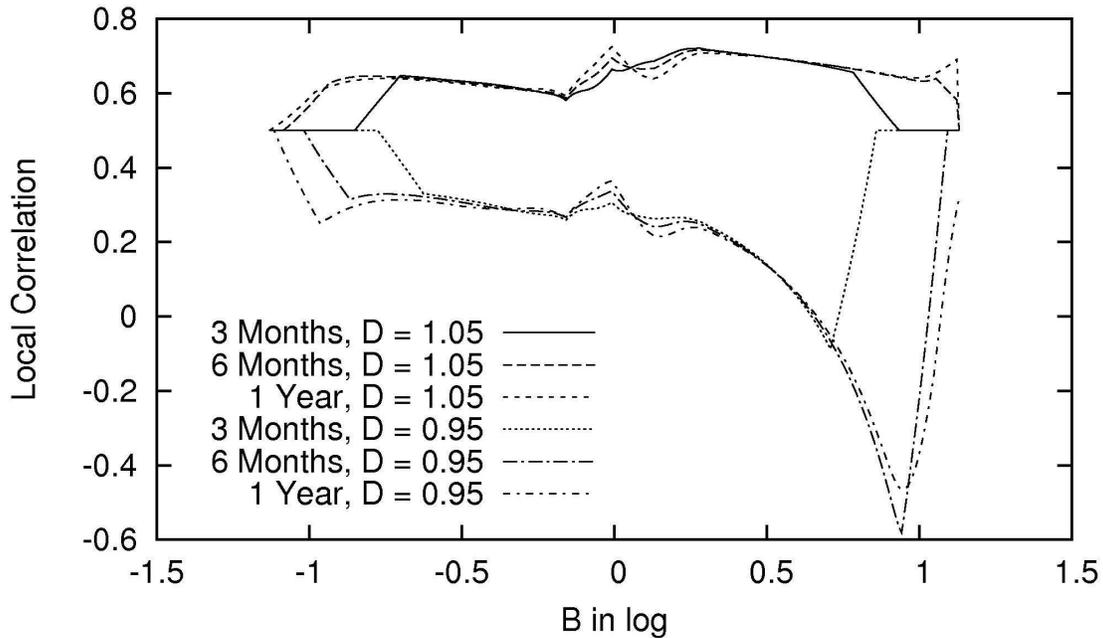


Figure 7.1: Local Correlation for different distortions and different maturities

As expected, we notice that the Local Correlation and the distortion factor evolve in the same direction. The underlyings must be more correlated when the implied volatility of the basket is higher, and reciprocally. Furthermore, it appears that in the case of the 0.95 distortion, the correlation has to violently decrease for high values of B : the two underlyings must be anti-correlated when they are both large.

As for the influence of the maturity, we must first explain that in the computation of λ , when the denominator is smaller than 10^{-6} , we chose not to change the correlation, to avoid numerical errors. It appears that, as long as B is in a zone where λ was actually computed, the framework we chose to test the calibration actually generates a Local Correlation constant in time.

Chapter 8

Application to stochastic interest rates

In this chapter, we are interested in hybrid local volatility models with stochastic rates. We assume that the interest rate is stochastic, it follows a diffusion equation. The volatility depends on the level of the spot exactly as in a local volatility model. We are looking for the exact value of the volatility that enables us to calibrate the vanillas in this model.

8.1 Hybrid Local Volatility models

8.1.1 Calibration equation

The risk-neutral diffusion of the model is written as

$$\begin{aligned}\frac{dS_t}{S_t} &= r(t, y_t)dt + \sigma(t, S_t)dW_t^1 \\ dy_t &= \mu(t, y_t)dt + \lambda(t, y_t)dW_t^2\end{aligned}\tag{8.1}$$

The two brownian motions are correlated, the value of the correlation is assumed to be constant, still equal to ρ . We make the classic assumptions on the coefficients of the diffusion

- (H1) for all $t \geq 0$, $S \in \mathbb{R}_+^*$ and $(y, \delta_1, \delta_2) \in \mathbb{R}^3$,

$$\sigma^2(t, S)\delta_1^2 + 2\rho\sigma(t, S)\lambda(t, y)\delta_1\delta_2 + \lambda^2(t, y)\delta_2^2 \geq K(\delta_1^2 + \delta_2^2)$$

- (H2) the functions r , σ and λ belong to $H^{2,h,h/2}(\]0, T[\times\mathbb{R})$
- (H3) μ is in $H^{1,h,h/2}(\]0, T[\times\mathbb{R})$

Proposition 8. *The diffusion model defined above has a transition density with respect to Lebesgue measure. The value of σ that fits the vanillas is given by*

$$\sigma^2(t, S) = \sigma_D^2(t, S) + 2 \frac{\bar{r}(t) \int_{\mathbb{R}} \int_S^{+\infty} p(t, s, y) ds dy - \int_{\mathbb{R}} \int_S^{+\infty} r(t, y) p(t, s, y) ds dy}{S \int_{\mathbb{R}} p(t, S, y) dy}\tag{8.2}$$

where $p(t, S, y)$ is the density of the couple (S_t, y_t) . And $\bar{r}(t)$ a deterministic curve of rates used in the computation of Dupire's local volatility σ_D .

Proof. The existence of the density $p(t, S, y)$ has already been proved. Let us prove Formula (8.2).

The function p solves the forward parabolic equation

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma^2 S^2 p \right) - \frac{\partial^2}{\partial S \partial y} (\rho \sigma \lambda S p) - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \lambda^2 p \right) + \frac{\partial}{\partial S} (r S p) + \frac{\partial}{\partial y} (\mu p) + r p = 0$$

with the initial condition $p(0, S, y) = \delta(S = S_0, y = y_0)$. As previously, we integrate the equation with respect to y , writing $q(t, S) = \int_{\mathbb{R}} p(t, S, y) dy$

$$\frac{\partial q}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma^2 S^2 q \right) + \frac{\partial}{\partial S} \left(S \int_{\mathbb{R}} r(t, y) p(t, S, y) dy \right) + \int_{\mathbb{R}} r(t, y) p(t, S, y) dy = 0$$

This equation needs to be matched with

$$\begin{aligned} \frac{\partial q_D}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 S^2 q_D \right) + \frac{\partial}{\partial S} (\bar{r} S q_D) + \bar{r} q_D &= 0 \\ q_D(0, S) &= \delta(S = S_0) \end{aligned}$$

Both of them can be written as

$$\begin{aligned} \frac{1}{2} \sigma^2 S^2 &= \frac{\int_0^{+\infty} (s - S)^+ \left(\frac{\partial q}{\partial t} + \frac{\partial}{\partial s} (s \int r p dy) + \int r p dy \right) ds}{q} \\ \frac{1}{2} \sigma_D^2 S^2 &= \frac{\int_0^{+\infty} (s - S)^+ \left(\frac{\partial q_D}{\partial t} + \frac{\partial}{\partial s} (s \bar{r} q_D) + \bar{r} q_D \right) ds}{q_D} \end{aligned}$$

We compute

$$\begin{aligned} \int_0^{+\infty} (s - S)^+ \frac{\partial}{\partial s} (s \int r p dy) ds &= \int_0^{+\infty} (s - S)^+ \left(s \frac{\partial}{\partial s} \left(\int r p dy \right) + \int r p dy \right) ds \\ &= - \int_0^{+\infty} \frac{\partial}{\partial s} ((s - S)^+) s \int r p dy ds \\ &= - \int_0^{+\infty} s 1_{s \geq S} \int r p dy ds \end{aligned}$$

where the second line stems from a simple integration by parts. Reintroducing this into the previous equations, we get

$$\begin{aligned} \frac{1}{2} \sigma^2 S^2 &= \frac{\int_0^{+\infty} (s - S)^+ \frac{\partial q}{\partial t} ds - S \int_S^{+\infty} \int r p dy ds}{q} \\ \frac{1}{2} \sigma_D^2 S^2 &= \frac{\int_0^{+\infty} (s - S)^+ \frac{\partial q_D}{\partial t} ds - \bar{r} S \int_S^{+\infty} q_D ds}{q_D} \end{aligned}$$

In order to calibrate the vanillas, all we need to do is match the marginal density q with q_D , this gives us the necessary condition

$$\frac{1}{2} \sigma^2 S^2 + \frac{S \int_S^{+\infty} \int r p dy ds}{q} = \frac{1}{2} \sigma_D^2 S^2 + \frac{\bar{r} S \int_S^{+\infty} q ds}{q}$$

The only thing that remains to be done now is replace q by $\int_{\mathbb{R}} p(t, S, y) dy$. This completes the proof. \square

8.1.2 Theoretical study

The calibration equation for the vanillas of our hybrid model is thus

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma^2 S^2 p \right) - \frac{\partial^2}{\partial S \partial y} (\rho \sigma \lambda S p) - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \lambda^2 p \right) + \frac{\partial}{\partial S} (r S p) + \frac{\partial}{\partial y} (\mu p) + r p = 0 \quad (8.3)$$

with σ given by formula (8.2). Using the same technics as in Part I, we are able to prove an existence result under a certain number of assumptions. The domain of definition $D^T =]0, T[\times \Omega$ for the equation still has to be bounded, we define

$$\begin{aligned} \Omega_S &= \{S \in \mathbb{R} \text{ such as } \exists y \in \mathbb{R} \text{ with } (S, y) \in \Omega\} \\ \Omega_y &= \{y \in \mathbb{R} \text{ such as } \exists S \in \mathbb{R} \text{ with } (S, y) \in \Omega\} \\ \Omega(S) &= \{y \in \mathbb{R} \text{ such as } (S, y) \in \Omega\} \\ \omega(S) &= \{(s, y) \in [S, +\infty[\times \mathbb{R} \text{ such as } (s, y) \in \Omega\} \end{aligned}$$

The definition (8.2) becomes, for $t > 0$ and $S \in \Omega_S$

$$\sigma^2(t, S) = \sigma_D^2(t, S) + 2 \frac{\int_{\omega(S)} (\bar{r}(t) - r(t, y)) p(t, s, y) ds dy}{S \int_{\Omega(S)} p(t, S, y) dy}$$

The boundary condition Ψ on $\bar{B} \cup C^T$ $p_0(S, y)$ is defined as earlier by

$$\Psi = p_0 \quad \text{on } \bar{B} \quad \Psi(t, S, y) = p_0(S, y) \quad \text{for } (S, y) \in \partial \Omega$$

with $p_0(S, y)$ the initial condition. The coefficients of equation (8.3) are assumed to verify the sufficient properties

- (A1) Ψ belongs to $H^{2,h,h/2}(\partial D^T)$ and is strictly positive on $\bar{B} \cup C^T$
- (A2) σ_D belongs to $C^3(\mathbb{R}_+^* \times \mathbb{R}_+^*)$ and λ to $C^3(\mathbb{R}_+^* \times \mathbb{R})$
- (A3) for all $t \geq 0$, $(S, y) \in \Omega$ and $(\delta_1, \delta_2) \in \mathbb{R}^2$,

$$\sigma_D^2(t, S) S^2 \lambda(t, y) \delta_1^2 + 2 \rho \sigma_D(t, S) S \lambda(t, y) \delta_1 \delta_2 + \lambda^2(t, y) \delta_2^2 \geq K(\delta_1^2 + \delta_2^2)$$

- (A4) Ψ verifies the two different compatibility conditions on ∂B

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 S^2 \Psi \right) + \frac{\partial^2}{\partial S \partial y} (\rho \sigma_D S \lambda \Psi) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \lambda^2 \Psi \right) - \frac{\partial}{\partial S} (r S \Psi) - \frac{\partial}{\partial y} (\mu \Psi) - r \Psi \\ \frac{\partial \Psi}{\partial t} &= \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \left(\sigma_D^2 + 2 \frac{\int_{\omega(S)} (\bar{r} - r) \Psi ds dy}{S \int_{\Omega(S)} \Psi dy} \right) S^2 \Psi \right) + \frac{\partial^2}{\partial S \partial y} \left(\rho \sqrt{\sigma_D^2 + 2 \frac{\int_{\omega(S)} (\bar{r} - r) \Psi ds dy}{S \int_{\Omega(S)} \Psi dy}} S \lambda \Psi \right) \\ &\quad + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \lambda^2 \Psi \right) - \frac{\partial}{\partial S} (r S \Psi) - \frac{\partial}{\partial y} (\mu \Psi) - r \Psi \end{aligned}$$

- (A5) \bar{r} is in $H^{1,h/2}(]0, T])$ and r in $H^{2,h,h/2}(]0, T[\times \mathbb{R})$

Proposition 9. *There exists a constant $r^* > 0$ such that, if*

$$\forall y \in \Omega_y, |r(\cdot, y) - \bar{r}(\cdot)|_{H^{1,h/2}(]0, T])} \leq r^* \quad (8.4)$$

then equation (8.3) has a solution on D^T , this solution belongs to $H^{2,h,h/2}(D^T)$.

Proof. The form of the equation is slightly different than previously. The same method works nevertheless. Here, all we need to prove is that

$$|\sigma^2 - \sigma_D^2|_{H^{2,h,h/2}(D^T)} = \left| 2 \frac{\int_{\omega(S)} (\bar{r} - r) p ds dy}{S \int_{\Omega(S)} p dy} \right|_{H^{2,h,h/2}(D^T)} \leq r^* P(|p|_{H^{2,h,h/2}(D^T)})$$

and that

$$\left| \frac{\int_{\omega(S)} (\bar{r} - r) p ds dy}{S \int_{\Omega(S)} p dy} - \frac{\int_{\omega(S)} (\bar{r} - r) q ds dy}{S \int_{\Omega(S)} q dy} \right|_{H^{2,h,h/2}(D^T)} \leq r^* |p - q|_{H^{2,h,h/2}(D^T)} P(|p|_{H^{2,h,h/2}(D^T)}) P(|q|_{H^{2,h,h/2}(D^T)})$$

It can be done exactly as in Appendix A. □

8.2 Numerical calibration

In this section, we apply the theoretical results above to calibrate a given diffusion model. We suppose that the instantaneous rate obeys a Vasicek model (or in other words is an Ornstein-Uhlenbeck process). This gives the diffusion equations

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t dt + \sigma(t, S_t) dW_t^1 \\ dr_t &= a(b - r_t) dt + \gamma dW_t^2 \end{aligned} \tag{8.5}$$

We apply the ADI algorithm described in Chapter 9 to Equation (8.3) with the coefficients associated to this diffusion. The initial condition is $p_0(S, r) = \delta(S = S_0, r = r_0)$.

As in the two previous chapters, this pde is solved with a variable change for the spot $x = \ln(S)$. The grid we choose is $[-10\sigma\sqrt{t}, 10\sigma\sqrt{t}] \times [-0.1, 0.2]$ with $\sigma = 0.2$ and $r_0 = 0.04$. We discretize it with 300 points in both the spot direction and the rate direction. The initial condition (Dirac mass at the point $(\ln(S_0), r_0)$) is approximated by a bivariate Gaussian centered at that point with a very small variance.

The following numerical values are taken for the diffusion

$$a = 0.5 \qquad b = 0.7 \qquad \gamma = 0.01 \qquad r_0 = 0.04$$

These values generate the interest rate

Table 8.1: Interest rates

Maturity	0.5	1.0	1.5	2.0	2.5	3.0	4.0	5.0
Rate	4.344%	4.634%	4.880%	5.090%	5.269%	5.423%	5.672%	5.861%

To assess the quality of the calibration, we calculate call and put options on the spot process by integration on the grid and compare them to the targeted prices (target columns). Convergence is quite satisfactory. For instance, for 6 months and 1 year maturity vanillas, we find

Table 8.2: 6 months options

Strike	Target	Call	Target	Put
0.5	49.30	49.31	0.39	0.40
0.6	40.14	40.15	1.01	1.02
0.7	31.49	31.51	2.15	2.16
0.8	23.55	23.57	3.99	4.00
0.9	16.52	16.55	6.74	6.76
1.0	10.62	10.65	10.62	10.64
1.1	6.16	6.19	15.95	15.96
1.2	3.22	3.24	22.79	22.79
1.3	1.53	1.54	30.87	30.87
1.4	0.67	0.68	39.79	39.79
1.5	0.28	0.28	49.18	49.17

Table 8.3: 1 year options

Target	Call	Target	Put
48.95	48.97	1.25	1.26
40.63	40.66	2.47	2.48
32.89	32.91	4.27	4.28
25.80	25.83	6.72	6.74
19.45	19.48	9.91	9.93
13.94	13.96	13.94	13.95
9.46	9.48	19.00	19.02
6.07	6.09	25.15	25.16
3.68	3.70	32.31	32.31
2.12	2.13	40.28	40.29
1.17	1.18	48.87	48.87

Part III

Numerical studies and instabilities of the solutions

Chapter 9

Algorithm for the resolution of the calibration equation

In the previous chapters, we treated the calibration equations from a theoretical point of view and gave graphs for their efficiency. In this part, we work from a numerical point of view: the empirical resolution of the nonlinear non-local partial differential equations mentioned in Part I and II. First, we describe the algorithm used to solve the equation: a classic Alternating-Direction Implicit scheme. The nonlinear term is handled using a forward induction.

The strong feature of an ADI scheme is its convergence rate in time and space: $O(\delta x^2) + O(\delta t^2)$. The nonlinearity of the equation challenges this assertion. In a second part, we study the convergence rate in time of the algorithm for the calibration.

9.1 Alternating Direction Implicit scheme

The calibration equation is a parabolic equation of the second-order. One of the most efficient method to solve such equations is a finite-difference approximation with alternating direction methods. For more informations on the subject, we refer the reader to the book from Richtmyer and Morton [15], numerous articles have also been published, in particular [37], [38] and [39]. We now consider the following equation

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial x^2} \left(\frac{1}{2} f^2 I^2(p) p \right) - \frac{\partial^2}{\partial x \partial y} (\rho f g I(p) p) - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} g^2 p \right) + \frac{\partial}{\partial x} (\alpha p) + \frac{\partial}{\partial y} (\beta p) + \gamma p = 0 \quad (9.1)$$

$$p(0, S, y) = \delta(S = S_0, y = y_0) \quad (9.2)$$

where the term $I(p)$ is the quotient of integrals

$$I^2(p)(t, x) = \frac{\int h^2(y) p(t, x, y) dy}{\int p(t, x, y) dy} \quad (9.3)$$

Remark. *We restricted ourselves to two-dimensional equations since they cover all the concrete examples studied previously. But the computations that follow are true in the general case. The cost in time however becomes an issue in higher dimensions.*

The domain for the numerical resolution is $]0, T[\times]x_*, x^*[\times]y_*, y^*[$. The first step is to take care of the initial condition. Instead of the Dirac, we choose for initial condition p_0 a gaussian distribution with very small variance . It obviously approximates our initial condition. It also verifies (on any bounded domain) the properties of regularity and strict positivity required in Part I to obtain existence (though in the present chapter, we are only interested in a numerical resolution). We define

the boundary condition by $p(t, x, y) = p_0(x, y)$ if $x = x_*, x^*$ or $y = y_*, y^*$.

The algorithm is based upon a predictor-corrector approach. Let Δx , Δy and Δt be increments of the variables x , y and t , where $\Delta x = \frac{x^* - x_*}{I}$, $\Delta y = \frac{y^* - y_*}{J}$ and $\Delta t = \frac{T}{N}$ with I , J and N integers. The sets of points in the x, y, t -plane is given by $x_n = x_* + i \frac{\Delta x}{x^* - x_*}$, $y_j = y_* + j \frac{\Delta y}{y^* - y_*}$ and $t_n = n \frac{\Delta t}{T}$, for $0 \leq i \leq I$, $0 \leq j \leq J$ and $0 \leq n \leq N$. We construct four sequences p_n , p_n^* , q_n and q_n^* of space-dependent functions with n between 0 and N .

A classic Alternating-Direction Implicit scheme functions as follows: we define the initial functions

$$p_0^*(i, j) = q_0(i, j) = q_0^*(i, j) = p_0(x_i, y_j)$$

and then by induction

$$\begin{aligned} \frac{q_{n+1}^* - p_n}{\Delta t} &= \delta_x^2 (I^2(p_n) \frac{f_{n+1}^2 q_{n+1}^* + f_n^2 p_n}{4}) + \delta_{xy}^2 (\rho f_n g_n I(p_n) p_n) + \delta_y^2 (\frac{1}{2} g_n^2 p_n) \\ &- \delta_x (\frac{\alpha_{n+1} q_{n+1}^* + \alpha_n p_n}{2}) - \delta_y (\beta_n p_n) - \frac{\gamma_{n+1} q_{n+1}^* + \gamma_n p_n}{2} \end{aligned} \quad (9.4)$$

$$\begin{aligned} \frac{q_{n+1} - p_n}{\Delta t} &= \delta_x^2 (I^2(p_n) \frac{f_{n+1}^2 q_{n+1}^* + f_n^2 p_n}{4}) + \delta_{xy}^2 (\rho f_n g_n I(p_n) p_n) + \delta_y^2 (\frac{g_{n+1}^2 q_{n+1} + g_n^2 p_n}{4}) \\ &- \delta_x (\frac{\alpha_{n+1} q_{n+1}^* + \alpha_n p_n}{2}) - \delta_y (\frac{\beta_{n+1} q_{n+1} + \beta_n p_n}{2}) - \frac{\gamma_{n+1} q_{n+1} + \gamma_n p_n}{2} \end{aligned} \quad (9.5)$$

where $f_n(i, j)$ designates $f(n\Delta t, x_i, y_j)$ (the same thing being true for the other coefficients of the equation). δ is a difference operator for the space derivatives. For instance, with $1 \leq i \leq I - 1$ and $1 \leq j \leq J - 1$

$$\begin{aligned} \delta_x^2 f_n &= \frac{f_n(i-1, j) - 2f_n(i, j) + f_n(i+1, j)}{\Delta x^2} \\ \delta_{xy}^2 f_n &= \frac{f_n(i-1, j-1) - f_n(i+1, j) - f_n(i, j+1) + f_n(i+1, j+1)}{4\Delta x \Delta y} \\ \delta_x f_n &= \frac{f_n(i+1, j) - f_n(i-1, j)}{2\Delta x} \\ &\vdots \end{aligned}$$

The equations (9.4) and (9.5) form two tridiagonal systems that can be solved very efficiently. A recursion formula can be computed on the functions q_n we just defined

$$\frac{q_{n+1} - q_{n+1}^*}{\Delta t} = \delta_y^2 (\frac{g_{n+1}^2 q_{n+1} - g_n^2 p_n}{4}) - \delta_y (\frac{\beta_{n+1} q_{n+1} - \beta_n p_n}{2}) - \gamma_{n+1} \frac{q_{n+1} - q_{n+1}^*}{2}$$

Thus

$$q_{n+1}^* = q_{n+1} - \frac{2\Delta t}{2 + \gamma_{n+1}\Delta t} (\delta_y^2 (\frac{g_{n+1}^2 q_{n+1} - g_n^2 p_n}{4}) - \delta_y (\frac{\beta_{n+1} q_{n+1} - \beta_n p_n}{2}))$$

And eventually

$$\begin{aligned}
\frac{q_{n+1} - p_n}{\Delta t} &= \delta_x^2(I^2(p_n) \frac{f_{n+1}^2 q_{n+1} + f_n^2 p_n}{4}) + \delta_{xy}^2(\rho f_n g_n I(p_n) p_n) + \delta_y^2(\frac{g_{n+1}^2 q_{n+1} + g_n^2 p_n}{4}) \\
&- \delta_x(\frac{\alpha_{n+1} q_{n+1} + \alpha_n p_n}{2}) - \delta_y(\frac{\beta_{n+1} q_{n+1} + \beta_n p_n}{2}) - \frac{\gamma_{n+1} q_{n+1} + \gamma_n p_n}{2} \\
&- \frac{\Delta t}{4 + 2\gamma_{n+1} \Delta t} \delta_x^2(I^2(p_n) f_{n+1}^2 (\delta_y^2(\frac{g_{n+1}^2 q_{n+1} - g_n^2 p_n}{4}) - \delta_y(\frac{\beta_{n+1} q_{n+1} - \beta_n p_n}{2}))) \\
&+ \frac{\Delta t}{2 + \gamma_{n+1} \Delta t} \delta_x(\alpha_{n+1} (\delta_y^2(\frac{g_{n+1}^2 q_{n+1} - g_n^2 p_n}{4}) - \delta_y(\frac{\beta_{n+1} q_{n+1} - \beta_n p_n}{2}))) \quad (9.6)
\end{aligned}$$

In the litterature concerning Alternating Direction Implicit schemes, the functions q_n are often called the predicted value of the solution p , a second "corrector" step usually follows. In our case, the equation being nonlinear, we start by chosing $p_{n+1} = q_{n+1}$ and study the finite-difference approximation that results.

Proposition 10. *This last finite-difference equation is consistent with the partial differential equation (9.1) on a bounded domain with smooth initial condition p_0 , the truncation error is $O(\Delta t) + O(\Delta x^2) + O(\Delta y^2) + O(\frac{\Delta x^3}{\Delta y})$.*

Proof. Let p be a classic solution of equation (9.1) on a bounded domain, with $p|_{t=0} = p_0$. We assume that p is strictly positive and sufficiently differentiable for all the quantities in the sequel to be properly defined (we know from the previous chapters that such a p exists when h is close enough to a constant and the coefficients of the equation are smooth enough). All the derivatives that we use are bounded as a consequence of the Hölderian nature of p . We now write $p_{i,j}^n = p^n(x_i, y_j) = p(t_n, x_i, y_j)$. A simple Taylor expansion with remainder gives

$$\frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} = \frac{\frac{\partial p}{\partial t}(t_n, x_i, y_j) + \frac{\partial p}{\partial t}(t_{n+1}, x_i, y_j)}{2} + \frac{\Delta t}{2} \left(\frac{\partial^2 p}{\partial t^2}(t_n + \theta, x_i, y_j) - \frac{\partial^2 p}{\partial t^2}(t_n + \theta^*, x_i, y_j) \right)$$

As for the space derivatives, we clearly have

$$\begin{aligned}
\delta_x^2 p^n &= \frac{\partial^2 p^n}{\partial x^2}(x_i, y_j) + \frac{\Delta x^2}{24} \left(\frac{\partial^4 p^n}{\partial x^4}(x_i + \theta_1, y_j) + \frac{\partial^4 p^n}{\partial x^4}(x_i - \theta_2, y_j) \right) \\
\delta_{xy}^2 p^n &= \frac{\partial^2 p^n}{\partial x \partial y}(x_i, y_j) + \frac{\Delta y^2}{12} \left(\frac{\partial^4 p^n}{\partial x \partial y^3}(x_i, y_j + \theta_1) + \frac{\partial^4 p^n}{\partial x \partial y^3}(x_i, y_j - \theta_2) \right) \\
&+ \frac{\Delta x^2}{24 \Delta y} \left[\frac{\partial^3 p^n}{\partial x^3}(x_i + \theta_3, y_{j-1}) - \frac{\partial^3 p^n}{\partial x^3}(x_i + \theta_4, y_{j-1}) \right. \\
&\left. + \frac{\partial^3 p^n}{\partial x^3}(x_i - \theta_5, y_{j+1}) - \frac{\partial^3 p^n}{\partial x^3}(x_i - \theta_6, y_{j+1}) \right] \\
\delta_x p^n &= \frac{\partial p^n}{\partial x}(x_i, y_j) + \frac{\Delta x^2}{12} \left(\frac{\partial^3 p^n}{\partial x^3}(x_i + \theta_1, y_j) + \frac{\partial^3 p^n}{\partial x^3}(x_i - \theta_2, y_j) \right)
\end{aligned}$$

where the different constants θ are between 0 and Δt , Δx or Δy depending on the context, they may

change from one formula to another. Let E denote the truncation error for scheme (9.6), we have

$$\begin{aligned}
E &= \frac{p^{n+1} - p^n}{\Delta t} - \delta_x^2(I^2(p^n) \frac{f_{n+1}^2 p^{n+1} + f_n^2 p^n}{4}) - \delta_{xy}^2(\rho f_n g_n I(p^n) p^n) - \delta_y^2(\frac{g_{n+1}^2 p^{n+1} + g_n^2 p^n}{4}) \\
&+ \delta_x(\frac{\alpha_{n+1} p^{n+1} + \alpha_n p^n}{2}) + \delta_y(\frac{\beta_{n+1} p^{n+1} + \beta_n p^n}{2}) + \frac{\gamma_{n+1} p^{n+1} + \gamma_n p^n}{2} \\
&+ \frac{\Delta t}{4 + 2\gamma_{n+1} \Delta t} \delta_x^2(I^2(p^n) f_{n+1}^2 (\delta_y^2(\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4}) - \delta_y(\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2}))) \\
&- \frac{\Delta t}{2 + \gamma_{n+1} \Delta t} \delta_x(\alpha_{n+1} (\delta_y^2(\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4}) - \delta_y(\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2})))
\end{aligned}$$

Applying the previous expansions and using the fact that p verifies equation (9.1) at times t_n and t_{n+1} gives us

$$E = e_1 - e_{21} - e_{22} - e_{23} - e_3 + e_{41} + e_{42} + e_5$$

with e_1 the error coming from the time-derivative

$$e_1 = \frac{\Delta t}{2} (\frac{\partial^2 p}{\partial t^2}(t_n + \theta, x_i, y_j) - \frac{\partial^2 p}{\partial t^2}(t_n + \theta^*, x_i, y_j))$$

e_{21} , e_{22} and e_{23} come from the second order space-derivatives

$$\begin{aligned}
e_{21} &= \frac{\Delta x^2}{96} [\frac{\partial^4}{\partial x^4}(I^2(p^n) f_n^2 p^n)(x_i + \theta_1, y_j) + \frac{\partial^4}{\partial x^4}(I^2(p^n) f_n^2 p^n)(x_i - \theta_2, y_j) \\
&+ \frac{\partial^4}{\partial x^4}(I^2(p^n) f_{n+1}^2 p^{n+1})(x_i + \theta_3, y_j) + \frac{\partial^4}{\partial x^4}(I^2(p^n) f_{n+1}^2 p^{n+1})(x_i - \theta_4, y_j)] \\
e_{22} &= \frac{\Delta y^2}{12} (\frac{\partial^4}{\partial x \partial y^3}(\rho f_n g_n I(p^n) p^n)(x_i, y_j + \theta_1) + \frac{\partial^4}{\partial x \partial y^3}(\rho f_n g_n I(p^n) p^n)(x_i, y_j - \theta_2)) \\
&+ \frac{\Delta x^2}{24 \Delta y} (\frac{\partial^3}{\partial x^3}(\rho f_n g_n I(p^n) p^n)(x_i + \theta_3, y_{j-1}) - \frac{\partial^3}{\partial x^3}(\rho f_n g_n I(p^n) p^n)(x_i + \theta_4, y_{j-1})) \\
&+ \frac{\partial^3}{\partial x^3}(\rho f_n g_n I(p^n) p^n)(x_i - \theta_5, y_{j+1}) - \frac{\partial^3}{\partial x^3}(\rho f_n g_n I(p^n) p^n)(x_i - \theta_6, y_{j+1})) \\
e_{23} &= \frac{\Delta y^2}{96} [\frac{\partial^4}{\partial y^4}(g_n^2 p^n)(x_i, y_j + \theta_1) + \frac{\partial^4}{\partial y^4}(g_n^2 p^n)(x_i, y_j - \theta_2) \\
&+ \frac{\partial^4}{\partial y^4}(g_{n+1}^2 p^{n+1})(x_i, y_j + \theta_3) + \frac{\partial^4}{\partial y^4}(g_{n+1}^2 p^{n+1})(x_i, y_j - \theta_4)]
\end{aligned}$$

e_3 allows to compensate for both the nondiagonal term and the nonlocal term $I(p^n)$ that cannot be computed implicitly

$$e_3 = \frac{\partial^2}{\partial x^2} [(I^2(p^n) - I^2(p^{n+1})) \frac{f_{n+1}^2 p^{n+1}}{4}] + \frac{\partial^2}{\partial x \partial y} [\frac{\rho}{2} (f_n g_n I(p^n) p^n - f_{n+1} g_{n+1} I(p^{n+1}) p^{n+1})]$$

e_{41} and e_{42} are the terms corresponding to the first order space-derivatives

$$\begin{aligned}
e_{41} &= \frac{\Delta x^2}{24} \left[\frac{\partial^3}{\partial x^3} (\alpha_n p^n)(x_i + \theta_1, y_j) + \frac{\partial^3}{\partial x^3} (\alpha_n p^n)(x_i - \theta_2, y_j) \right. \\
&\quad \left. + \frac{\partial^3}{\partial x^3} (\alpha_{n+1} p^{n+1})(x_i + \theta_3, y_j) + \frac{\partial^3}{\partial x^3} (\alpha_{n+1} p^{n+1})(x_i - \theta_4, y_j) \right] \\
e_{42} &= \frac{\Delta y^2}{24} \left[\frac{\partial^3}{\partial y^3} (\beta_n p^n)(x_i, y_j + \theta_1) + \frac{\partial^3}{\partial y^3} (\beta_n p^n)(x_i, y_j - \theta_2) \right. \\
&\quad \left. + \frac{\partial^3}{\partial y^3} (\beta_{n+1} p^{n+1})(x_i, y_j + \theta_3) + \frac{\partial^3}{\partial y^3} (\beta_{n+1} p^{n+1})(x_i, y_j - \theta_4) \right]
\end{aligned}$$

At last, e_5 is the correction term stemming from the Alternating Direction Implicit scheme

$$\begin{aligned}
e_5 &= \frac{\Delta t}{4 + 2\gamma_{n+1}\Delta t} \delta_x^2 (I^2(p^n) f_{n+1}^2 (\delta_y^2 (\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4}) - \delta_y (\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2}))) \\
&\quad - \frac{\Delta t}{2 + \gamma_{n+1}\Delta t} \delta_x (\alpha_{n+1} (\delta_y^2 (\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4}) - \delta_y (\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2})))
\end{aligned}$$

Thanks to the regularity of p and of the coefficients of (9.1), we have $e_1 \leq K\Delta t^2$, $e_{21} + e_{41} \leq K\Delta x^2$, $e_{23} + e_{42} \leq K\Delta y^2$ and $e_{22} \leq K(\Delta y^2 + \frac{\Delta x^3}{\Delta y})$. We can also easily prove that $e_5 \leq K\Delta t^2$. The term that prevents us from getting an error in $O(\Delta t^2)$ is e_3 . All we can get is $e_3 \leq K\Delta t$ (using Lemma 4 Part I, we know that $I(\cdot)$ is differentiable, with bounded derivatives). This concludes the proof. \square

Remark. In a case with no $I(p)$ term, the equation is a classic linear and parabolic one. In that case, when the off-diagonal¹ term is absent ($\rho = 0$ for instance), the previous scheme has an error in $O(\Delta t^2)$. To obtain such an error in the general case, a second "corrector" step is generally used: the predicted value q_{n+1} is introduced as an approximation of p_{n+1} in the cross-derivatives. Here, we try to use it in the nonlocal term too.

The correction step is the following

$$\begin{aligned}
\frac{p_{n+1}^* - p_n}{\Delta t} &= \delta_x^2 \left(\frac{I^2(q_{n+1}) f_{n+1}^2 p_{n+1}^* + I^2(p_n) f_n^2 p_n}{4} \right) + \delta_{xy}^2 \left(\rho \frac{(fg)_{n+1} I(q_{n+1}) q_{n+1} + (fg)_n I(p_n) p_n}{2} \right) \\
&\quad + \delta_y^2 \left(\frac{1}{2} g_n^2 p_n \right) - \delta_x \left(\frac{\alpha_{n+1} p_{n+1}^* + \alpha_n p_n}{2} \right) - \delta_y \left(\beta_n p_n \right) - \frac{\gamma_{n+1} p_{n+1}^* + \gamma_n p_n}{2} \\
\frac{p_{n+1} - p_n}{\Delta t} &= \delta_x^2 \left(\frac{I^2(q_{n+1}) f_{n+1}^2 p_{n+1}^* + I^2(p_n) f_n^2 p_n}{4} \right) + \delta_{xy}^2 \left(\rho \frac{(fg)_{n+1} I(q_{n+1}) q_{n+1} + (fg)_n I(p_n) p_n}{2} \right) \\
&\quad + \delta_y^2 \left(\frac{g_{n+1}^2 p_{n+1} + g_n^2 p_n}{4} \right) - \delta_x \left(\frac{\alpha_{n+1} p_{n+1}^* + \alpha_n p_n}{2} \right) - \delta_y \left(\frac{\beta_{n+1} p_{n+1} + \beta_n p_n}{2} \right) \\
&\quad - \frac{\gamma_{n+1} p_{n+1} + \gamma_n p_n}{2}
\end{aligned}$$

Here too, one can compute p_{n+1}^*

$$p_{n+1}^* = p_{n+1} - \frac{2\Delta t}{2 + \gamma_{n+1}\Delta t} \left(\delta_y^2 \left(\frac{g_{n+1}^2 p_{n+1} - g_n^2 p_n}{4} \right) - \delta_y \left(\frac{\beta_{n+1} p_{n+1} - \beta_n p_n}{2} \right) \right)$$

¹The case of mixed-derivatives in the equation is dealt with in [35]

Eventually, we have a Crank-Nicholson like formula

$$\begin{aligned}
\frac{p_{n+1} - p_n}{\Delta t} &= \delta_x^2 \left(\frac{I^2(q_{n+1})f_{n+1}^2 p_{n+1} + I^2(p_n)f_n^2 p_n}{4} \right) + \delta_{xy}^2 \left(\rho \frac{(fg)_{n+1}I(q_{n+1})q_{n+1} + (fg)_n I(p_n)p_n}{2} \right) \\
&+ \delta_y^2 \left(\frac{g_{n+1}^2 p_{n+1} + g_n^2 p_n}{4} \right) - \delta_x \left(\frac{\alpha_{n+1} p_{n+1} + \alpha_n p_n}{2} \right) - \delta_y \left(\frac{\beta_{n+1} p_{n+1} + \beta_n p_n}{2} \right) \\
&- \frac{\gamma_{n+1} p_{n+1} + \gamma_n p_n}{2} \\
&- \frac{\Delta t}{4 + 2\gamma_{n+1}\Delta t} \delta_x^2 \left(I^2(q_{n+1})f_{n+1}^2 \left(\delta_y^2 \left(\frac{g_{n+1}^2 p_{n+1} - g_n^2 p_n}{4} \right) - \delta_y \left(\frac{\beta_{n+1} p_{n+1} - \beta_n p_n}{2} \right) \right) \right) \\
&+ \frac{\Delta t}{2 + \gamma_{n+1}\Delta t} \delta_x \left(\alpha_{n+1} \left(\delta_y^2 \left(\frac{g_{n+1}^2 p_{n+1} - g_n^2 p_n}{4} \right) - \delta_y \left(\frac{\beta_{n+1} p_{n+1} - \beta_n p_n}{2} \right) \right) \right)
\end{aligned}$$

Let us study the consistency of this new scheme

Proposition 11. *The algorithm with a corrector step is also consistent. The truncation error is $O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2) + O(\frac{\Delta x^3}{\Delta y})$.*

Proof. To prove the consistency, we define q^{n+1} as

$$\begin{aligned}
\frac{q^{n+1} - p^n}{\Delta t} &= \delta_x^2 \left(I^2(p^n) \frac{f_{n+1}^2 q^{n+1} + f_n^2 p^n}{4} \right) + \delta_{xy}^2 \left(\rho f_n g_n I(p^n) p^n \right) + \delta_y^2 \left(\frac{g_{n+1}^2 q^{n+1} + g_n^2 p^n}{4} \right) \\
&- \delta_x \left(\frac{\alpha_{n+1} q^{n+1} + \alpha_n p^n}{2} \right) - \delta_y \left(\frac{\beta_{n+1} q^{n+1} + \beta_n p^n}{2} \right) - \frac{\gamma_{n+1} q^{n+1} + \gamma_n p^n}{2} \\
&- \frac{\Delta t}{4 + 2\gamma_{n+1}\Delta t} \delta_x^2 \left(I^2(p^n) f_{n+1}^2 \left(\delta_y^2 \left(\frac{g_{n+1}^2 q^{n+1} - g_n^2 p^n}{4} \right) - \delta_y \left(\frac{\beta_{n+1} q^{n+1} - \beta_n p^n}{2} \right) \right) \right) \\
&+ \frac{\Delta t}{2 + \gamma_{n+1}\Delta t} \delta_x \left(\alpha_{n+1} \left(\delta_y^2 \left(\frac{g_{n+1}^2 q^{n+1} - g_n^2 p^n}{4} \right) - \delta_y \left(\frac{\beta_{n+1} q^{n+1} - \beta_n p^n}{2} \right) \right) \right)
\end{aligned}$$

The computations are almost identical to the previous proposition. This time the error is equal to

$$\begin{aligned}
E^* &= \frac{p^{n+1} - p^n}{\Delta t} - \delta_x^2 \left(\frac{I^2(q^{n+1})f_{n+1}^2 p^{n+1} + I^2(p^n)f_n^2 p^n}{4} \right) \\
&- \delta_{xy}^2 \left(\rho \frac{(fg)_{n+1}I(q^{n+1})q^{n+1} + (fg)_n I(p^n)p^n}{2} \right) - \delta_y^2 \left(\frac{g_{n+1}^2 p^{n+1} + g_n^2 p^n}{4} \right) \\
&+ \delta_x \left(\frac{\alpha_{n+1} p^{n+1} + \alpha_n p^n}{2} \right) + \delta_y \left(\frac{\beta_{n+1} p^{n+1} + \beta_n p^n}{2} \right) + \frac{\gamma_{n+1} p^{n+1} + \gamma_n p^n}{2} \\
&+ \frac{\Delta t}{4 + 2\gamma_{n+1}\Delta t} \delta_x^2 \left(I^2(q^{n+1})f_{n+1}^2 \left(\delta_y^2 \left(\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4} \right) - \delta_y \left(\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2} \right) \right) \right) \\
&- \frac{\Delta t}{2 + \gamma_{n+1}\Delta t} \delta_x \left(\alpha_{n+1} \left(\delta_y^2 \left(\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4} \right) - \delta_y \left(\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2} \right) \right) \right)
\end{aligned}$$

We use the same decomposition. The errors e_1^* , e_{23}^* , e_{41}^* and e_{42}^* do not change. e_{21}^* is slightly

different but still in $O(\Delta x^2)$. As for e_{22}^* and e_5^* , we now have

$$\begin{aligned}
e_{22}^* &= \frac{e_{22}}{2} + \rho \frac{\Delta y^2}{24} \left(\frac{\partial^4}{\partial x \partial y^3} ((fg)_{n+1} I(q^{n+1}) q^{n+1})(x_i, y_j + \theta_1) + \frac{\partial^4}{\partial x \partial y^3} ((fg)_{n+1} I(q^{n+1}) q^{n+1})(x_i, y_j - \theta_2) \right) \\
&\quad + \frac{\Delta x^2}{24 \Delta y} \left(\frac{\partial^3}{\partial x^3} ((fg)_{n+1} I(q^{n+1}) q^{n+1})(x_i + \theta_3, y_{j-1}) - \frac{\partial^3}{\partial x^3} ((fg)_{n+1} I(q^{n+1}) q^{n+1})(x_i + \theta_4, y_{j-1}) \right) \\
&\quad + \frac{\partial^3}{\partial x^3} ((fg)_{n+1} I(q^{n+1}) q^{n+1})(x_i - \theta_5, y_{j+1}) - \frac{\partial^3}{\partial x^3} ((fg)_{n+1} I(q^{n+1}) q^{n+1})(x_i - \theta_6, y_{j+1}) \\
e_5^* &= \frac{\Delta t}{4 + 2\gamma_{n+1} \Delta t} \delta_x^2 (I^2(q^{n+1}) f_{n+1}^2 (\delta_y^2 (\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4}) - \delta_y (\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2}))) \\
&\quad - \frac{\Delta t}{2 + \gamma_{n+1} \Delta t} \delta_x (\alpha_{n+1} (\delta_y^2 (\frac{g_{n+1}^2 p^{n+1} - g_n^2 p^n}{4}) - \delta_y (\frac{\beta_{n+1} p^{n+1} - \beta_n p^n}{2})))
\end{aligned}$$

which also verify $e_{22}^* \leq K(\Delta y^2 + \frac{\Delta x^3}{\Delta y})$ and $e_5^* \leq K\Delta t^2$. The real difference can be seen in

$$e_3^* = \frac{\partial^2}{\partial x^2} [(I^2(q^{n+1}) - I^2(p^{n+1})) \frac{f_{n+1}^2 p^{n+1}}{4}] + \frac{\partial^2}{\partial x \partial y} [\frac{\rho}{2} ((fg)_{n+1} I(q^{n+1}) q^{n+1} - (fg)_{n+1} I(p^{n+1}) p^{n+1})]$$

The important feature of the predictor is that the difference $q^{n+1} - p^{n+1}$ is $O(\Delta t^2)$. This gives $e_3^* \leq K(\Delta t^2)$ and concludes the proof. \square

9.2 Time Convergence Rate of the modified ADI algorithm

In this brief section, we compare the convergence of the algorithm with the theoretical rates computed in the previous part. To do so, we compute the calibrated value of 1-year at-the-money vanillas for different number N of time steps. We then plot the error between this price and the targeted value against N. The next graph is obtained with the one-step predictor algorithm

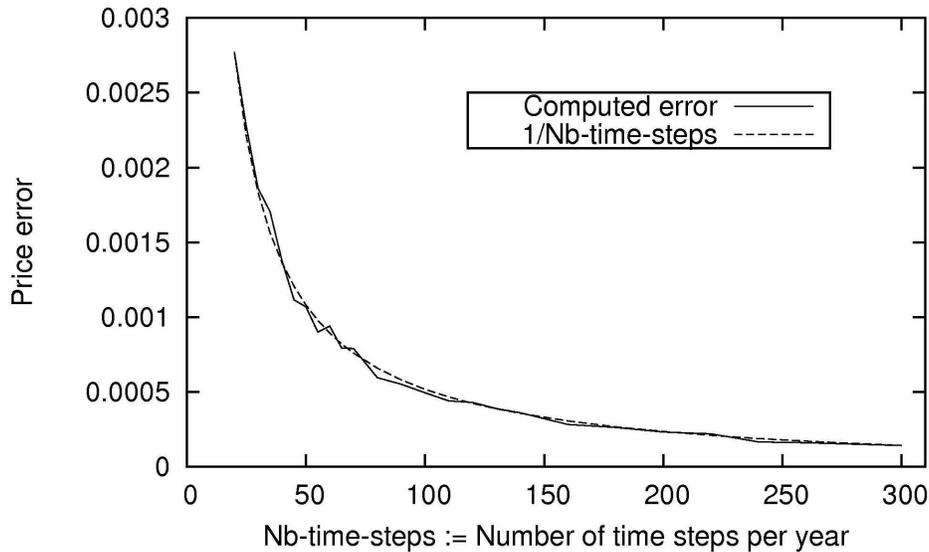


Figure 9.1: Vanilla price error for different number of time steps, predictor step only

The error is clearly in $O(\Delta t)$ as was proved in Proposition 10. We conduct the same experiment

with this time both the predictor and the corrector steps.

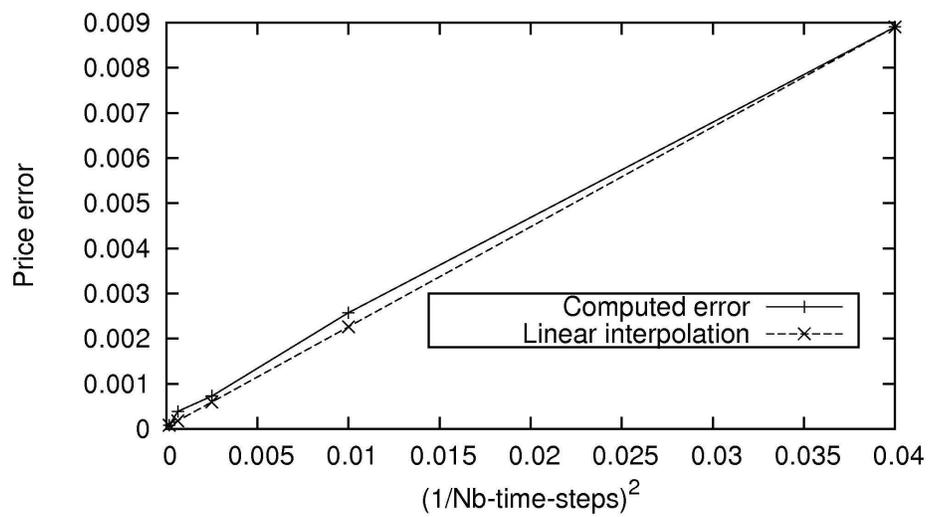


Figure 9.2: Vanilla price error for different number of time steps, predictor/corrector algorithm

This time too, numerical experiments seem to agree with theory. The error appears to be in $O(\Delta t^2)$. The predictor/corrector scheme serves its purpose.

Chapter 10

Instabilities of the solutions

In Chapter 9, we described the algorithm used to solve the different partial differential equations concerned by our work, and in Part I, we studied those equations from a theoretical point of view. Though a partial existence result for Equation (5.1) was found, we were still unable to prove it in the general case: strongly variable functions (f_{ij}) .

In this last chapter, we study the local and stochastic volatility model from that point of view. We start with the numerical resolution of the calibration for strongly variable functions b . Plotting the density, we see the appearance of what seems to be an instability of the solution¹. Such an instability does not occur in the case of the adjoint equation, which confirms us that the general equation 5.1 has some peculiar properties.

We then try to explain this instability from a theoretical point of view. Hadamard instability seems to be the logical explanation. Studying the linearized operator, we give a criterion for the instability of a certain type of equations. Unfortunately, that criterion is not fulfilled in the case of the local and stochastic volatility. New leads need to be found.

10.1 Numerical explosion for "oscillating" volatilities

At first, we go back to the equation for the calibration of a local and stochastic volatility model. For the sake of simplicity, we assume the interest rate to be zero. The equation is the following

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 b^2 S^2 \frac{\int p dy}{\int b^2 p dy} \right) - \frac{\partial^2}{\partial S \partial y} (\rho \sigma_D b \alpha S \left(\frac{\int p dy}{\int b^2 p dy} \right)^{\frac{1}{2}} p) - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \alpha^2 p \right) + \frac{\partial}{\partial y} (\beta p) = \alpha \quad (10.1)$$
$$p(0, S, y) = \delta(S = S_0, y = y_0)$$

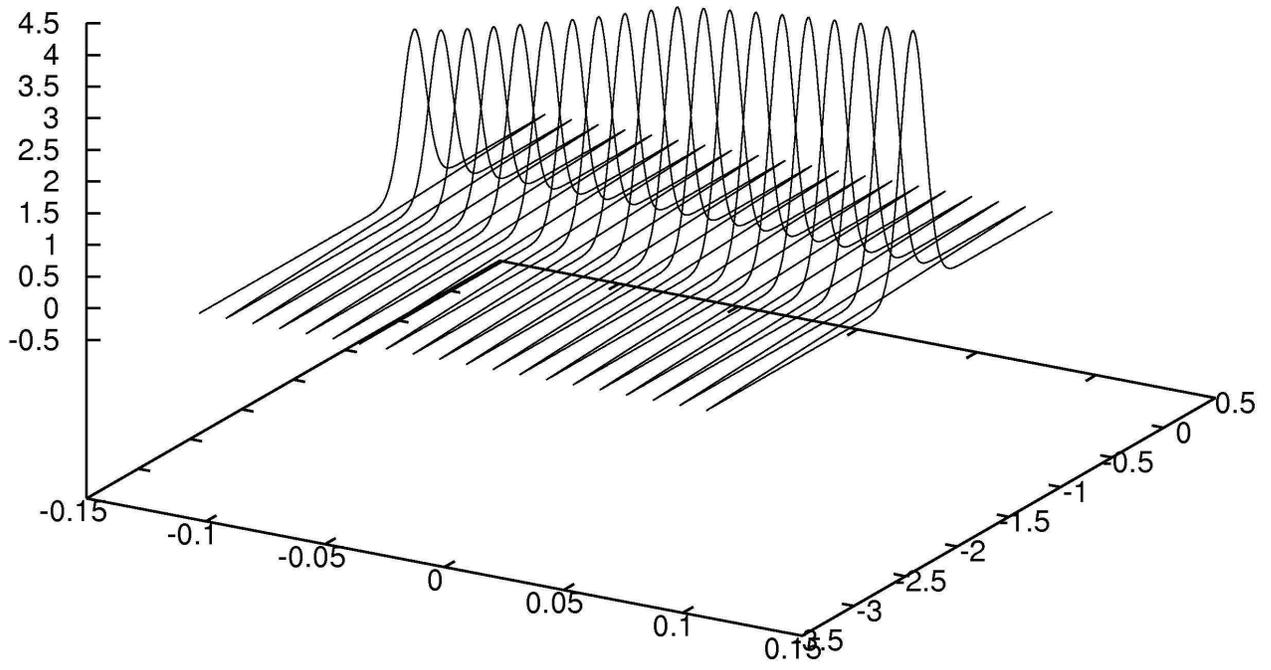
In Chapter 6, we saw the existence of a solution (on a bounded domain, with regularized boundary conditions) under certain assumptions on b , the essential one being: there exists a constant b^* such that $|b - b(y_0)| \leq b^*$ for a given y_0 . Using the algorithm described in the previous chapter, we are now interested in the behavior of the numerical solution of (10.1) when the function b violates the assumption.

The model chosen for this study is the mean-reverting volatility already expounded

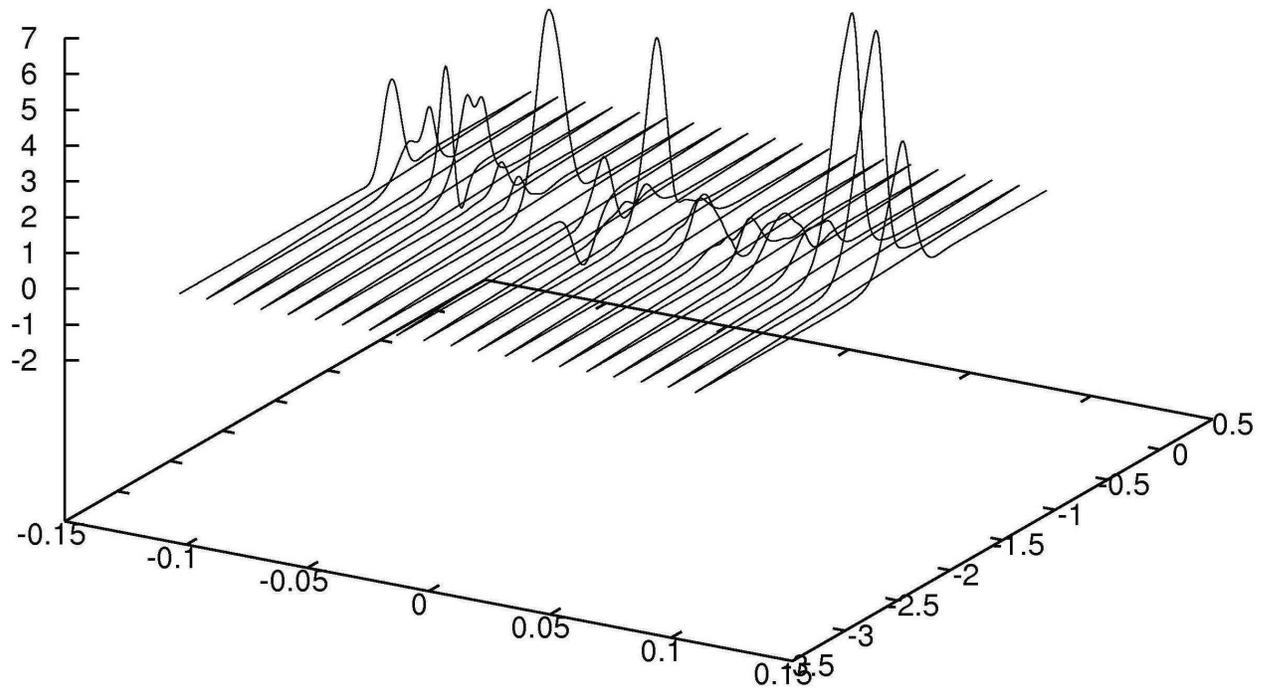
$$b(y) = \exp(y) \quad \alpha(t, y) = \gamma \quad \beta(t, y) = \kappa (\delta - y)$$

with γ , κ and δ three strictly positive constants. The value we chose in Part II for the different parameters of the model and of the algorithm led us to a satisfactory calibration. Here, we plot the density $p(T, x, y)$ for $T = 1$ year and $x = \ln(S/S_0)$ close to 0.

¹A study of the stability of linear finite difference schemes can be found in [36]



As expected, the solution p is perfectly smooth. Now, bouncing on the idea of strongly variable functions b , we plot the same thing with a function b equal to $b(y) = \exp(10y)$.



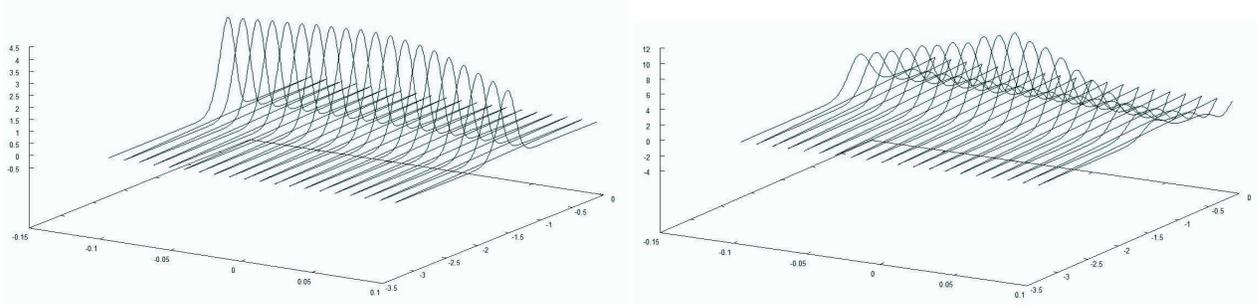
This time, the solution does not appear to be smooth anymore, on the contrary some kind of instability seem to occur. In the following, we focus our attention on a lead to explain this instability. In similar cases, an interesting approach is to consider the linearized operator, the bifurcation theory for instance, deals with some examples [24]. Here, we thus conduct a study of the linearization of equation (5.1). Computations are carried out on a simplified version of the linearized operator. We find a criterion under which this new operator is Hadamard unstable. Unfortunately, the LSV calibration equation does not verify this criterion.

Remark. We remember from Part I that we obtained local existence in the case of the adjoint equation, and this for any functions $(f_{ij})_{1 \leq i, j \leq d}$. Even if the adjoint has no meaning from a financial point of view, its numerical resolution is interesting to see the impact of an oscillating function b .

The adjoint equation for the local and stochastic volatility calibration is

$$\frac{\partial p}{\partial t} - \frac{1}{2} \sigma_D^2 b^2 S^2 \frac{\int p dy}{\int b^2 p dy} \frac{\partial^2 p}{\partial S^2} - \rho \sigma_D b \alpha S \left(\frac{\int p dy}{\int b^2 p dy} \right)^{\frac{1}{2}} \frac{\partial^2 p}{\partial S \partial y} - \frac{1}{2} \alpha^2 \frac{\partial^2 p}{\partial y^2} + \beta \frac{\partial p}{\partial y} = 0 \quad (10.2)$$

We make the same test and plot its numerical solution for $b(y) = \exp(Cy)$ with $C = 10$ and $C = 15$



On both these graphics, we cannot see any sign of instability. With higher values of C , for instance $C = 20$, the function p computed numerically takes meaningless values (10^{80}), but at no time does it start to oscillate. We also compute the solution p for a maturity of $T = 2$ years (until now, let us recall that $T = 1$ year) and $C = 15$, the graph shows values from -10^{18} to 10^{18} , we are probably outside of the local existence in time.

To sum up, all the results for the adjoint seem to be in agreement with the theoretical proofs of Part I.

As for the original equation, the numerical resolution points towards the same conclusion: an instability when b varies too much.

10.2 Hadamard instability and Linearized equation

In this section, we go back to the operator described in Part I and study its linearization

$$\begin{aligned} \frac{\partial p}{\partial t} - L(p, \{f\}) &:= \frac{\partial p}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{k=1}^d V_{ik}(t, x, \frac{\int f_{ik}(x^{\lambda_{ij}^C}) p_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}}) dx^{\lambda_{ik}^C}}{\int p_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}}) dx^{\lambda_{ik}^C}}) \right. \\ &\quad \left. V_{jk}(t, x, \frac{\int f_{jk}(x^{\lambda_{ij}^C}) p_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}}) dx^{\lambda_{jk}^C}}{\int p_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}}) dx^{\lambda_{jk}^C}}) p(t, x) \right) + \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_t^i p(t, x)) \end{aligned}$$

In Section 5.1 and 5.2, we considered a particular solution for the equation above $(\tilde{p}, \{\tilde{f}\})$ ie $\mathcal{F}(\tilde{p}, \{\tilde{f}\}) = 0$ (with \mathcal{F} defined in the proof of Theorem 5) and studied the differential of the operator L around this solution

$$D_p L(\tilde{p}, \{\tilde{f}\}) \cdot h = \frac{\partial h}{\partial t} - \frac{1}{2} \sum_{1 \leq i, j \leq d} \frac{\partial^2}{\partial x_i \partial x_j} (h L_{ij}(\tilde{p}, \{\tilde{f}\}) + \tilde{p} D_p L_{ij}(\tilde{p}, \{\tilde{f}\}) \cdot h) + \sum_{1 \leq i \leq d} \frac{\partial}{\partial x_i} (\mu_t^i h)$$

with

$$\begin{aligned} L_{ij}(\tilde{p}, \{\tilde{f}\}) &:= \sum_{k=1}^d V_{ik}(t, x, \frac{\int \tilde{f}_{ik}(x^{\lambda_{ij}^C}) \tilde{p}_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}}) dx^{\lambda_{ik}^C}}{\int \tilde{p}_{ik}(t, x^{\lambda_{ik}}, x^{\lambda_{ik}}) dx^{\lambda_{ik}^C}}) V_{jk}(t, x, \frac{\int \tilde{f}_{jk}(x^{\lambda_{ij}^C}) \tilde{p}_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}}) dx^{\lambda_{jk}^C}}{\int \tilde{p}_{jk}(t, x^{\lambda_{jk}}, x^{\lambda_{jk}}) dx^{\lambda_{jk}^C}}) \\ D_p L_{ij}(\tilde{p}, \{\tilde{f}\}) \cdot h &= \sum_{k=1}^d V_{ik}(t, x, \frac{\int \tilde{f}_{ik} \tilde{p}}{\int \tilde{p}}) \frac{\partial V_{jk}}{\partial e} (t, x, \frac{\int \tilde{f}_{jk} \tilde{p}}{\int \tilde{p}}) \left(\frac{\int \tilde{f}_{jk} h}{\int \tilde{p}} - \frac{\int \tilde{f}_{jk} \tilde{p} \int h}{(\int \tilde{p})^2} \right) \\ &\quad + \frac{\partial V_{ik}}{\partial e} (t, x, \frac{\int \tilde{f}_{ik} \tilde{p}}{\int \tilde{p}}) \left(\frac{\int \tilde{f}_{ik} h}{\int \tilde{p}} - \frac{\int \tilde{f}_{ik} \tilde{p} \int h}{(\int \tilde{p})^2} \right) V_{jk}(t, x, \frac{\int \tilde{f}_{jk} \tilde{p}}{\int \tilde{p}}) \\ &= \sum_{k=1}^d \left(\int \tilde{f}_{jk} h \int \tilde{p} - \int \tilde{f}_{jk} \tilde{p} \int h \right) \tilde{V}_{ijk} + \left(\int \tilde{f}_{ik} h \int \tilde{p} - \int \tilde{f}_{ik} \tilde{p} \int h \right) \tilde{V}_{jik} \end{aligned}$$

and

$$\tilde{V}_{ijk} := \frac{1}{(\int \tilde{p})^2} V_{ik}(t, x, \frac{\int \tilde{f}_{ik} \tilde{p}}{\int \tilde{p}}) \frac{\partial V_{jk}}{\partial e} (t, x, \frac{\int \tilde{f}_{jk} \tilde{p}}{\int \tilde{p}})$$

Developing the integral terms, we get

$$\int \tilde{f}_{jk} h \int \tilde{p} - \int \tilde{f}_{jk} \tilde{p} \int h = \int h(t, x^{\lambda_{jk}}, x^{\lambda_{jk}}) \left(\int \tilde{p}_{jk}(t, x^{\lambda_{jk}}, y^{\lambda_{jk}}) (\tilde{f}_{jk}(x^{\lambda_{jj}^C}) - \tilde{f}_{jk}(y^{\lambda_{jj}^C})) dy^{\lambda_{jk}^C} dx^{\lambda_{jk}^C} \right)$$

The coefficient of h in this integral is $C = \int \tilde{p}_{jk}(t, x^{\lambda_{jk}}, y^{\lambda_{jk}}) (\tilde{f}_{jk}(x^{\lambda_{jj}^C}) - \tilde{f}_{jk}(y^{\lambda_{jj}^C})) dy^{\lambda_{jk}^C}$. Two cases appear

- If \tilde{f}_{jk} is close to a constant then C is small. The operator $D_p L(\tilde{p}, \{\tilde{f}\}) \cdot h$ is a perturbation of an adjoint heat equation. This fact was used in Part I to prove existence of solutions.
- On the contrary if \tilde{f}_{jk} varies a lot, the coefficient C is negative for values of $x^{\lambda_{jj}^C}$ close to the point where the minimum of \tilde{f}_{jk} is attained.

In the second case, we do not obtain per say a backward heat equation. However, this property and the fact that the operator $D_p L(\tilde{p}, \{\tilde{f}\}) \cdot h$ resembles the heat operator led us to the study of a

simplified equation involving an integral term similar to the previous one

$$\frac{\partial h}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(t, y, z)(h(t, y, z) + v(t, y, z) \int \rho(t, y, z) h(t, y, z) dz)) - \frac{1}{2} \frac{\partial^2}{\partial z^2} (\xi(t, z) h) = 0 \quad (10.3)$$

Let us study the well-posedness of this last equation. The notion of ill and well-posed problems goes back to the beginning of the twentieth century with the work of J. Hadamard described for instance in [16]. A problem is said to be well-posed if it admits a unique solution and if this solution depends continuously on the data of the problem. On the contrary, Hadamard unstable operators are a class of ill-posed operators. This kind of instability is defined as

Definition. We say that a linear operator is Hadamard unstable with respect to a norm $\|\cdot\|$ when the parabolic equation associated to L lacks of continuous dependance on the initial condition. This boils down to the following property: for all $\varepsilon > 0$, we can find two functions p and ϕ verifying for a given t

$$\begin{aligned} \frac{\partial p}{\partial t} - Lp &= 0 & \|p(t, \cdot)\| &\geq 1 \\ p(0, x) &= \phi(x) & \|\phi\| &\leq \varepsilon \end{aligned}$$

The most basic example of Hadamard unstable operator is $L = -\Delta$, where Δ is the Laplacian². More generally, parabolic operators with "negative" diffusion matrix are ill-posed.

Remark. A few references study the so-called forward-backward heat equation and give existence results of solutions for $L = x\Delta$ with $x \in]-1, 1[$ and mixed initial-terminal conditions, for instance [55] or [56].

We show this property for the particular case $D^T =]0, T[\times]x, \bar{x}[$.

Lemma 7. The operator $L = a(t, x) \frac{\partial^2}{\partial x^2}$ with $a \in H^{2, h, h/2}(D^T)$ and $a(t_0, x_0) < 0$ for a given $(t_0, x_0) \in D^T$ is Hadamard unstable.

Proof. The proof is divided into three steps.

First Step. Let us first consider the case of a function $a(x) < 0$ for all $x \in]x, \bar{x}[$. Spectral theory gives us the existence of a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of positive eigenvalues of L diverging to $+\infty$. Let $(e_n)_{n \in \mathbb{N}}$ be corresponding eigenvectors verifying $|e_n|_\infty = 1$. For a given ε , we choose

$$p_n(t, x) = \varepsilon e_n(x) \exp(\lambda_n t)$$

Clearly, for all $n \in \mathbb{N}$, p_n is solution of $\frac{\partial p}{\partial t} - a(t, x) \frac{\partial^2 p}{\partial x^2} = 0$. Since $|e_n|_\infty = 1$, the initial condition verifies $|p_n(0, \cdot)|_\infty = \varepsilon$. And for any $t > 0$, $|p_n(t, \cdot)|_\infty = \varepsilon \exp(\lambda_n t)$. The λ_n being positive and non bounded, for n large enough, we have $|p_n(t, \cdot)|_\infty \geq 1$. In this case, the operator L is indeed ill-posed in the sense of Hadamard.

Second Step. We now relax the time hypothesis by assuming that $a(t, x) < 0$ for all $(t, x) \in D^T$. For any $\eta, \varepsilon > 0$, there exists $t_* > 0$ such that $|a(t, x) - a(0, x)| < \eta$ for all $(t, x) \in D^{t_*}$.

From the first step, we get two functions p and ϕ verifying

$$\begin{aligned} \frac{\partial p}{\partial t} - a(0, x) \frac{\partial^2 p}{\partial x^2} &= 0 & |p(t^*, \cdot)|_{H^2(]x, \bar{x}[)} &\geq 1 \\ p(0, x) &= \phi(x) & |\phi|_{H^2(]x, \bar{x}[)} &\leq \varepsilon \end{aligned}$$

²With a constraint on the terminal condition, this operator becomes well-posed as shown in [54]

By contradiction, if L was well-posed, and since

$$\frac{\partial p}{\partial t} - Lp = (a(0, x) - a(t, x)) \frac{\partial^2 p}{\partial x^2}$$

We would have, with K independent from t^* , thus from η ,

$$|p|_{H^2(D^{t^*})} \leq K(|(a - a(0, \cdot)) \frac{\partial^2 p}{\partial x^2}|_{H^0(D^{t^*})} + |\phi|_{H^2(]x, \bar{x}[)}) \leq K(\eta |p|_{H^2(D^{t^*})} + \varepsilon)$$

Hence, for η and ε small enough, $|p|_{H^2(D^{t^*})} \leq \frac{K\varepsilon}{1-K\eta} < 1$ which is impossible: the operator L is ill-posed.

Third Step. Let us now consider the case $a(0, x_0) < 0$ for a given x_0 . By continuity of a and the same kind of argument than in the previous step, we can assume that $a(t, x) < 0$ on $[0, T^*] \times]x_1, x_2[$ and $a(t, x_1) = a(t, x_2) = 0$ on $[0, T]$. From the previous step, we get for any $\varepsilon > 0$ two functions p on $[0, T] \times [x_1, x_2]$ and ϕ on $[x_1, x_2]$ verifying for a given $t^* \leq T^*$

$$\begin{aligned} \frac{\partial p}{\partial t} - a(t, x) \frac{\partial^2 p}{\partial x^2} &= 0 & |p(t^*, \cdot)|_{H^2(]x_1, x_2[)} &\geq 1 \\ p(0, x) &= \phi(x) & |\phi|_{H^2(]x_1, x_2[)} &\leq \varepsilon \end{aligned}$$

We extend ϕ by continuity to $]x, \bar{x}[$, let us call $\bar{\phi}$ this extension. We can clearly construct $\bar{\phi}$ such that $|\bar{\phi}|_{H^2(]x, \bar{x}[)} \leq 2\varepsilon$. By contradiction, if L was well-posed, we would have a function \bar{p} such that

$$\begin{aligned} \frac{\partial \bar{p}}{\partial t} - a(t, x) \frac{\partial^2 \bar{p}}{\partial x^2} &= 0 & \bar{p}(0, x) &= \bar{\phi}(x) \end{aligned}$$

and $|\bar{p}|_{H^2 D^T} \leq K\varepsilon$, with K independent from ε . Since $a(t, x_1) = a(t, x_2) = 0$, we have $\bar{p}(t, x_1) = \bar{\phi}(x_1)$ and $\bar{p}(t, x_2) = \bar{\phi}(x_2)$. By the maximum principle, we get $\bar{p} = p$ on $[0, T] \times]x_1, x_2[$, which is impossible. \square

We now consider the following partial differential equation system

$$\frac{\partial H}{\partial t} = M \frac{\partial^2 H}{\partial x^2} + N \frac{\partial H}{\partial x} + PH \quad (10.4)$$

where $H \in (H^{2,h,h/2}(D^T))^n$ is the unknown, M and N two coefficients matrices belonging to $(H^{0,h,h/2}(D^T))^{n^2}$ and P a vector in $(H^{0,h,h/2}(D^T))^n$.

Lemma 8. *The system 10.4 is well-posed if and only if the matrix M is definite positive.*

Proof. This result is well-known, it can be found for instance in [24]. \square

We are now interested in the well-posedness of equation (10.3) on the domain $\Omega =]0, T[\times B$ with $B =]y_{min}, y_{max}[\times]z_{min}, z_{max}[$.

We define an initial condition $h(0, y, z) = h_0(y, z) > 0$ on B . The boundary condition is completed with $h(t, y, z) = h_0(y, z)$ on ∂B . We write the compatibility assumption

$$\frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(t, y, z)(h_0(y, z) + \nu(t, y, z) \int \rho(t, y, z) h_0(y, z) dz)) + \frac{1}{2} \frac{\partial^2}{\partial z^2} (\xi(t, z) h_0(y, z)) = 0 \text{ on } \partial B$$

Futhermore, we assume that μ , ν , ρ and ξ belong to $H^{2,h,h/2}([0, T[\times B)$ and that μ , ν and ξ are strictly positive on $[0, T] \times \bar{B}$.

Let us study the operator associated to (10.3). We start with the case where μ , ν and $\frac{\partial^2}{\partial z^2}(\xi(t, z))$ are independent from z . We assume that $1 + \nu(t, y) \int \rho(t, y, z) dz < 0$ for some $(t, y) \in]0, T[\times]y_{min}, y_{max}[$. Our goal is to prove the theorem

Theorem 10. *For all $\varepsilon > 0$, there exists $h_0 \in H^{2,h,h/2}$ with $|h_0|_{H^{2,h,h/2}} \leq \varepsilon$ and $h \in H^{2,h,h/2}(\Omega)$ solution of (10.3) such that $h(T, y, z) \geq 1$ for some $(y, z) \in B$.*

Proof. We write the solution as $h(t, y, z) = h^y(t, y)h^z(t, z)$. The equation we obtain is

$$h^y \left[\frac{\partial h^z}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial z^2} (\xi(t, z) h^z(t, z)) \right] + h^z \left[\frac{\partial h^y}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(t, y) h^y(t, y)) \right] - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(t, y) \nu(t, y) h^y(t, y)) \int \rho(t, y, z) h^z(t, z) dz = 0$$

Given the assumption concerning ξ , the function $h^z(t) = \exp(\frac{1}{2} \int_0^t \frac{\partial^2}{\partial z^2} (\xi(s, z)) ds)$ is solution of $\frac{\partial h^z}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial z^2} (\xi(t, z) h^z(t, z)) = 0$ with $h^z(0) = 1$. It does not depend on z . Thus, we have to solve

$$\begin{aligned} h^z(t) \left[\frac{\partial h^y}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(t, y) h^y(t, y)) \right] - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(t, y) \nu(t, y) h^y(t, y)) \int \rho(t, y, z) h^z(t) dz \\ = h^z(t) \left[\frac{\partial h^y}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(t, y) h^y(t, y) (1 + \nu(t, y) \int \rho(t, y, z) dz)) \right] = 0 \end{aligned} \quad (10.5)$$

We see that if $1 + \nu(t, y) \int \rho(t, y, z) dz < 0$ for some $(t, y) \in]0, T[\times]y_{min}, y_{max}[$, the equation becomes locally a heat equation backward in time, which is known to be Hadamard unstable. We use the Lemma 7 to find a function $h_0^y(y)$ such that $|h_0^y|_{H^{2,h,h/2}} \leq \varepsilon$ and a solution $h^y(t, y)$ to the equation (10.5) with initial condition h_0^y and verifying $|h^y(t, \cdot)|_{H^{2,h,h/2}([y_{min}, y_{max}[)} \geq 1 / \inf(h^z(1, z))$. \square

We now consider the case where the functions μ , ν and ξ oscillate slowly with respect to z . This means that, for a given $z_0 \in]z_{min}, z_{max}[$ and some $Z > 0$ (Z will be computed later), we have

$$|\mu - \mu(\cdot, \cdot, z_0)|_{H^{2,h,h/2}(\Omega)} \leq Z \quad |\nu - \nu(\cdot, \cdot, z_0)|_{H^{2,h,h/2}(\Omega)} \leq Z \quad |\xi - \xi(\cdot, z_0)|_{H^{2,h,h/2}(\Omega)} \leq Z$$

We write

$$Lh(t, y, z) := \frac{\partial^2}{\partial y^2} (\mu(t, y, z) (h(t, y, z) + \nu(t, y, z) \int \rho(t, y, z) h(t, y, z) dz)) + \frac{\partial^2}{\partial z^2} (\xi(t, z) h)$$

We also assume that for some $(t, y) \in]0, T[\times]y_{min}, y_{max}[$, $1 + \nu(t, y, z_0) \int \rho(t, y, z) dz < 0$. We prove the

Theorem 11. *For Z small enough, the operator $\frac{\partial}{\partial t} - \frac{1}{2}L$ is ill-posed.*

Proof. The proof is based upon the fact that for Z small enough the operator L is close to the operator L_0 (defined two lines below) that is known to be ill-posed.

$$L_0 h(t, y, z) := \frac{\partial^2}{\partial y^2} (\mu(t, y, z_0) (h(t, y, z) + \nu(t, y, z_0) \int \rho(t, y, z) h(t, y, z) dz)) + \frac{\partial^2}{\partial z^2} (\xi(t, z_0) h)$$

We have

$$\begin{aligned}
|(L - L_0)h|_{H^{0,h,h/2}(\Omega)} &\leq |(\mu(t,y,z) - \mu(t,y,z_0))(h(t,y,z) + v(t,y,z) \int \rho(t,y,z)h(t,y,z)dz)|_{H^{2,h,h/2}(\Omega)} \\
&\quad + |\mu(t,y,z_0)(v(t,y,z) - v(t,y,z_0)) \int \rho(t,y,z)h(t,y,z)dz + (\xi(t,z) - \xi(t,z_0))h|_{H^{2,h,h/2}(\Omega)} \\
&\leq Z(2|h|_{H^{2,h,h/2}(\Omega)} + |(v(t,y,z) + \mu(t,y,z_0)) \int \rho(t,y,z)h(t,y,z)dz|_{H^{2,h,h/2}(\Omega)}) \\
&\leq ZK|h|_{H^{2,h,h/2}(\Omega)}
\end{aligned}$$

with K a constant depending on the norm of μ , v and ρ . The equation we are interested in is $\frac{\partial h}{\partial t} - \frac{1}{2}Lh = 0$. Let us assume by contradiction that this operator is well-posed. As a consequence, it depends continuously on the data of the problem. This means that for all $f \in H^{0,h,h/2}(\Omega)$ and $\phi \in H^{2,h,h/2}(B)$, there exists a unique $h \in H^{2,h,h/2}(\Omega)$ solution of $\frac{\partial h}{\partial t} - \frac{1}{2}Lh = f$ with $h|_{t=0} = \phi$. Plus, this function verifies

$$|h|_{H^{2,h,h/2}(\Omega)} \leq K_L(|f|_{H^{0,h,h/2}(\Omega)} + |\phi|_{H^{2,h,h/2}}), \quad (10.6)$$

where K_L is a constant depending on the norm of the coefficients of L .

Now, we notice that the operator $\frac{\partial}{\partial t} - \frac{1}{2}L_0$ falls under the assumptions of Theorem 10. Hence, there exists two functions $h_0^y(y)$ and $h^y(t,y)$ such that

$$\begin{aligned}
\bullet \quad \frac{\partial h^y}{\partial t} - \frac{1}{2}L_0 h^y &= 0 & h^y(0,y) &= h_0^y(y) \\
\bullet \quad |h_0^y|_{H^{2,h,h/2}} &\leq \varepsilon & |h^y|_{H^{2,h,h/2}(\Omega)} &\geq 1
\end{aligned}$$

Since $\frac{\partial h^y}{\partial t} - \frac{1}{2}L_0 h^y = 0$, we have $\frac{\partial h^y}{\partial t} - \frac{1}{2}Lh^y = \frac{1}{2}(L_0 - L)h^y$. We apply inequality (10.6) and obtain

$$|h^y|_{H^{2,h,h/2}(\Omega)} \leq K_L(|\frac{1}{2}(L_0 - L)h^y|_{H^{0,h,h/2}(\Omega)} + |h_0^y|_{H^{2,h,h/2}}) \leq K_L(\frac{1}{2}ZK|h^y|_{H^{0,h,h/2}(\Omega)} + \varepsilon)$$

If $Z < \frac{2}{KK_L}$, this last line gives us $|h^y|_{H^{2,h,h/2}(\Omega)} \leq \frac{2K_L\varepsilon}{2-ZKK_L}$. Choosing $\varepsilon < \frac{2-ZKK_L}{2K_L}$, we get $|h^y|_{H^{2,h,h/2}(\Omega)} < 1$ which is the contradiction we were looking for. This just proves that the operator is ill-posed. \square

We are now interested in a generalization of the method previously used. Let us consider functions μ and ξ independent from z and a function v polynomial in this variable. We look for a solution h under the same form

$$v(t,y,z) = \sum_{i=0}^l v_i(t,y)z^i \quad h(t,y,z) = \sum_{i=0}^l h_i(t,y)z^i$$

Writing equation (10.3) using this decomposition gives

$$\begin{aligned}
\sum_{i=0}^l \frac{\partial h_i}{\partial t} z^i - \frac{1}{2} \sum_{i=0}^l z^i \frac{\partial^2}{\partial y^2} (\mu(t,y)(h_i(t,y) + \sum_{k=0}^l v_i(t,y)h_k(t,y) \int \rho(t,y,z)z^k dz)) \\
- \frac{1}{2} \sum_{i=2}^l i(i-1)z^{i-2} \xi(t)h_i(t,y) = 0
\end{aligned}$$

Projecting this last computation on the powers of z , we obtain a system of parabolic equations

$$\begin{aligned} \frac{\partial h_i}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(h_i(1 + \int \rho z^i dz) + \sum_{k \neq i} v_i h_k \int \rho z^k dz)) - \frac{(i+1)(i+2)}{2} \xi h_{i+2} &= 0 \text{ for } i \leq l-2 \\ \frac{\partial h_i}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\mu(h_i(1 + \int \rho z^i dz) + \sum_{k \neq i} v_i h_k \int \rho z^k dz)) &= 0 \text{ otherwise} \end{aligned}$$

Remark. We notice that ξ could actually be a polynomial of degree up to 2, this slightly modifies the system, but only for the terms of order 0.

This system can be written in matrix form as

$$\left(\frac{\partial h_i}{\partial t} \right) = \frac{1}{2} M \left(\frac{\partial^2 \mu h_i}{\partial y^2} \right) + N \left(\frac{\partial h_i}{\partial y} \right) + P(h_i)$$

where, in particular,

$$M = \begin{pmatrix} 1 + v_0 \int \rho dz & v_0 \int z \rho dz & \dots & v_0 \int z^l \rho dz \\ \vdots & \vdots & \ddots & \vdots \\ v_l \int \rho dz & v_l \int z \rho dz & \dots & 1 + v_l \int z^l \rho dz \end{pmatrix}$$

The condition for such a system to be ill-posed is stated previously in Lemma 8: the matrix M must have at least one negative eigenvalue. It is possible to show that the eigenvalues of M are 1 with a multiplicity of $(l-1)$ and $1 + \sum_{i=0}^l v_i \int \rho z^i dz = 1 + \int \rho v dz$. Eventually, this gives us the following result that actually contains Theorems 10 and 11

Theorem 12. *There exists a Z such that, if $|\mu(\cdot, \cdot, \cdot) - \mu(\cdot, \cdot, z_0)|_{H^{2,h,h/2}(\Omega)} \leq Z$ and $|\xi - P|_{H^{2,h,h/2}(\Omega)} \leq Z$ where P is a polynomial in z of degree 2, and if*

$$\exists (t, y) \in]0, T[\times]y_{min}, y_{max}[, 1 + \int v(t, y, z) \rho(t, y, z) dz \leq 0 \quad (10.7)$$

Then, equation (10.3) is ill-posed in the sense of Hadamard.

Proof. For a given $Z > 0$, the density of polynomial functions in $H^{2,h,h/2}(\Omega)$ gives us l functions $(v_i)_{1 \leq i \leq l} \in (H^{2,h,h/2}(\Omega))^l$ such that

$$|v(t, y, z) - \sum_{i=0}^l v_i(t, y) z^i|_{H^{2,h,h/2}(\Omega)} \leq Z$$

The operator stemming from equation (10.3) can thus be approached by

$$L_0 h(t, y, z) := \frac{\partial^2}{\partial y^2} (\mu(t, y, z_0) (h(t, y, z) + \sum_{i=0}^l v_i(t, y) z^i \int \rho(t, y, z) h(t, y, z) dz)) + \frac{\partial^2}{\partial z^2} (\xi(t, z_0) h)$$

with an error bounded by Z . Condition (10.7) gives us the ill-posedness of this last operator. Using the proof of Theorem 11, we obtain the instability of equation (10.3) under these new assumptions. \square

In the case of the LSV calibration, we start with simplifying assumptions: we choose a correlation

equal to 0 and ignore the term of order 1. The linearized equation (taken in (p,b)) is

$$\frac{\partial h}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 b^2 S^2 \left(\frac{\int p dy}{\int b^2 p dy} h + \frac{\int h dy}{\int b^2 p dy} p - \frac{\int p dy \int b^2 h dy}{(\int b^2 p dy)^2} p \right) \right) - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \alpha^2 h \right) = 0 \quad (10.8)$$

Using the formalism previously described, we write

$$\frac{\partial h}{\partial t} - \frac{\partial^2}{\partial S^2} \left(\frac{1}{2} \sigma_D^2 b^2 S^2 \frac{\int p dy}{\int b^2 p dy} \left(h + p \left(\frac{\int h dy}{\int p dy} - \frac{\int b^2 h dy}{\int b^2 p dy} \right) \right) \right) - \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \alpha^2 h \right) = 0$$

This gives us

$$\mu = \sigma_D^2 b^2 S^2 \frac{\int p dy}{\int b^2 p dy} \quad \nu = p \quad \rho = \frac{1}{\int p dy} - \frac{b^2}{\int b^2 p dy}$$

Computing condition (10.7) in the case of the LSV brings

$$1 + \int \nu(t, S, y) \rho(t, S, y) dy = 1 + \int \frac{p}{\int p dy} - \frac{b^2 p}{\int b^2 p dy} = 1$$

Unfortunately, the calibration equation does not fall under the scope of Theorem 12. In this case, the functions ν and ρ are orthogonal.

Chapter 11

Conclusions

Calibrating the financial models we consider in this work brought us to study of a broad variety of problems. Let us summarize the results we attained

- The theoretical question of existence and uniqueness of solutions for the calibration PIDE has been partially solved, a strong assumption is required on certain coefficients of the equation.
- We also proved short-time existence of solutions for the adjoint equation of said PIDE. The assumption previously mentioned is not needed in that case.
- The generalization of the proof for the original PIDE is still an open problem.
- However, numerical experimentations pointed out the appearance of an instability phenomenon, which might explain our inability to extend the existence result.
- The study of the numerical problems we encountered led us to a linearized equation for which we gave an instability criterion.
- This criterion is not verified in the case we were originally interested in. Other leads need to be explored.
- From an algorithmic point of view, we managed to extend the ADI predictor-corrector algorithm to our nonlinear framework and gave consistency results for the scheme.
- We also applied our results to three financial models and obtained satisfactory calibrations.

Appendix A

Proof of Lemma 4

We recall that $D^T =]0, T[\times \Omega$ verifies Property (1). We need to integrate functions of (x, t) against a given number of variables $(x_i)_{i \in \lambda}$ where λ is a subset of $\{1, \dots, d\}$. To simplify the notations, we consider an integer $1 \leq k \leq d$ and the functions Q and R defined as

$$Q(t, z) = \frac{\int_{\Omega(z)} f(y) p(t, z, y) dy}{\int_{\Omega(z)} p(t, z, y) dy} \quad R(t, z) = \frac{\int_{\Omega(z)} f(y) q(t, z, y) dy}{\int_{\Omega(z)} q(t, z, y) dy}$$

where z belongs to Ω_z , y to \mathbb{R}^{d-k} and

$$\begin{aligned} \Omega_z &= \{z \in \mathbb{R}^k \text{ such as } \exists y \in \mathbb{R}^{d-k} \text{ with } (z, y) \in \Omega\} \\ \Omega(z) &= \{y \in \mathbb{R}^{d-k} \text{ such as } (z, y) \in \Omega\} \end{aligned}$$

Let \mathcal{F}_f be a bounded set containing $\overline{\Omega_z}$ for all $z \in \Omega_z$. f , p and q are assumed to belong respectively to $H^{2,h,h/2}(\mathcal{F}_f)$ and $H^{2,h,h/2}(D^T)$ (for p and q) and to be strictly positive on the closure of their domain of definition. This gives us two constants $0 < \eta < \xi$ such that $\eta \leq f \leq \xi$, hence Q and R are properly defined, with $\eta \leq Q, R \leq \xi$. Let us now study their differentiability

Lemma 9. *The functions Q and R belong to $H^{2,h,h/2}(D^T)$. Moreover, for any $V \in C^3(\mathbb{R}_+^*)$ with V and its three derivatives bounded on $[\eta, \xi]$, by a constant K_η , and any real number \bar{f} , we have*

$$\begin{aligned} |V(Q) - V(\bar{f})|_{H^{2,h,h/2}(D^T)} &\leq |f - \bar{f}|_{H^0(D^T)} P(|p|_{H^{2,h,h/2}(D^T)}) \\ |V(Q) - V(R)|_{H^{2,h,h/2}(D^T)} &\leq |f - \bar{f}|_{H^0(D^T)} |p - q|_{H^{2,h,h/2}(D^T)} P(|p|_{H^{2,h,h/2}(D^T)}) P(|q|_{H^{2,h,h/2}(D^T)}) \end{aligned}$$

where P denotes a polynomial function, nondecreasing and strictly positive on \mathbb{R}^+ .

Proof. Proving the inequality gives the whole result. We start with $V(X) = X$

$$\begin{aligned} |Q - \bar{f}|_{H^{2,h,h/2}(D^T)} &= \left| \frac{\int_{\Omega(z)} (f(y) - \bar{f}) p(t, z, y) dy}{\int_{\Omega(z)} p(t, z, y) dy} \right|_{H^{2,h,h/2}(D^T)} \\ &\leq \left| \int_{\Omega(z)} (f(y) - \bar{f}) p(t, z, y) dy \right|_{H^{2,h,h/2}(D^T)} \left| \frac{1}{\int_{\Omega(z)} p(t, z, y) dy} \right|_{H^{2,h,h/2}(D^T)} \end{aligned}$$

We study those two norms separately. Let $0 < t' < t \leq T$ and z, z' belong to Ω_z . In the following, K stands for a constant depending only on the domain Ω . We see that

$$\left| \int_{\Omega(z)} (f(y) - \bar{f}) p(t, z, y) dy \right| \leq K |f - \bar{f}|_{H^0(D^T)} |p|_{H^0(D^T)}$$

and

$$\begin{aligned}
& \left| \int_{\Omega(z)} (f(y) - \bar{f})p(t, z, y)dy - \int_{\Omega(z')} (f(y) - \bar{f})p(t', z', y)dy \right| \\
& \leq \left| \int_{\Omega(z) \cap \Omega(z')} (f(y) - \bar{f})(p(t, z, y) - p(t', z', y))dy \right| \\
& \quad + \left| \int_{\Omega(z) \setminus \Omega(z')} (f(y) - \bar{f})p(t, z, y)dy \right| + \left| \int_{\Omega(z') \setminus \Omega(z)} (f(y) - \bar{f})p(t', z', y)dy \right| \\
& \leq K|f - \bar{f}|_{H^0(D^T)}|p|_{H^{0,h,h/2}(D^T)}d((t, z), (t', z'))^h \\
& \quad + |f - \bar{f}|_{H^0(D^T)}|p|_{H^0(D^T)}\left(\int_{\Omega(z') \setminus \Omega(z)} 1dy + \int_{\Omega(z) \setminus \Omega(z')} 1dy\right)
\end{aligned}$$

By Property 2 of the boundary of our domain D^T , it is possible to find a constant K depending only on Ω such as $\forall z, z' \in \Omega_z, \int_{\Omega(z) \setminus \Omega(z')} 1dy \leq Kd(z, z')^h$. This gives us

$$\left| \int_{\Omega(z)} (f(y) - \bar{f})p(t, z, y)dy \right|_{H^{0,h,h/2}(D^T)} \leq K|f - \bar{f}|_{H^0(D^T)}|p|_{H^{0,h,h/2}(D^T)}$$

The exact same computations prove that

$$\begin{aligned}
\left| \frac{\partial}{\partial t} \left(\int_{\Omega(z)} (f(y) - \bar{f})p(t, z, y)dy \right) \right|_{H^{0,h,h/2}(D^T)} & \leq K|f - \bar{f}|_{H^0(D^T)} \left| \frac{\partial p}{\partial t} \right|_{H^{0,h,h/2}(D^T)} \\
& \leq K|f - \bar{f}|_{H^0(D^T)}|p|_{H^{1,h,h/2}(D^T)}
\end{aligned}$$

And so on for the other derivatives, eventually, we get

$$\left| \int_{\Omega(z)} (f(y) - \bar{f})p(t, z, y)dy \right|_{H^{2,h,h/2}(D^T)} \leq K|f - \bar{f}|_{H^0(D^T)}|p|_{H^{2,h,h/2}(D^T)}$$

There remains to study the norm $\left| \frac{1}{\int_{\Omega(z)} p(t, z, y)dy} \right|_{H^{2,h,h/2}(D^T)}$. We see that if g is a function in $H^{2,h,h/2}(D^T)$ with $g \geq \varepsilon$, then letting $0 < t' < t \leq T$ and x, x' belong to Ω , we have

$$\begin{aligned}
\left| \frac{1}{g}(t, x) \right| & \leq \frac{1}{\varepsilon} \\
\left| \frac{1}{g}(t, x) - \frac{1}{g}(t', x') \right| & \leq \frac{|g(t', x') - g(t, x)|}{\varepsilon^2} \leq \frac{|g|_{H^{0,h,h/2}(D^T)}}{\varepsilon^2} d((t, x), (t', x'))^h
\end{aligned}$$

The time derivative gives

$$\begin{aligned}
\left| -\frac{\partial g}{g^2} \right| & \leq \frac{|g|_{H^{2,1}(D^T)}}{\varepsilon^2} \\
\left| \frac{\partial g}{g^2}(t, x) - \frac{\partial g}{g^2}(t', x') \right| & \leq \frac{\left| \frac{\partial g}{\partial t}(t, x) - \frac{\partial g}{\partial t}(t', x') \right| g^2(t', x') + |g^2(t, x) - g^2(t', x')| \left| \frac{\partial g}{\partial t}(t', x') \right|}{\varepsilon^4} \\
& \leq \frac{|g|_{H^{1,h,h/2}(D^T)}^3}{\varepsilon^4} d((t, x), (t', x'))^h
\end{aligned}$$

As for the second space derivatives, we have

$$\begin{aligned} \left| -\frac{\partial^2 g}{\partial x_i \partial x_j} + 2\frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \right| &\leq \frac{|g|_{H^{2,1}(D^T)}}{\varepsilon^2} + \frac{|g|_{H^{2,1}(D^T)}^2}{\varepsilon^3} \\ \left| \frac{\partial^2 g}{\partial x_i \partial x_j}(t, x) - \frac{\partial^2 g}{\partial x_i \partial x_j}(t', x') \right| &\leq \frac{|g|_{H^{2,h,h/2}(D^T)}^3}{\varepsilon^4} d((t, x), (t', x'))^h \\ \left| \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}(t, x) - \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}(t', x') \right| &\leq \frac{|g|_{H^{2,h,h/2}(D^T)}^5}{\varepsilon^6} d((t, x), (t', x'))^h \end{aligned}$$

These inequalities boil down to

$$\left| \frac{1}{g} \right|_{H^{2,h,h/2}(D^T)} \leq K_\varepsilon P(|g|_{H^{2,h,h/2}(D^T)})$$

with $P(X) = 1 + X + X^2 + X^3 + X^5$ and K_ε a constant depending only on ε . Eventually, we have

$$\begin{aligned} \left| \frac{1}{\int_{\Omega(z)} p(t, z, y) dy} \right|_{H^{2,h,h/2}(D^T)} &\leq K_\varepsilon P\left(\left| \int_{\Omega(z)} p(t, z, y) dy \right|_{H^{2,h,h/2}(D^T)}\right) \\ &\leq K_\varepsilon P(K|p|_{H^{2,h,h/2}(D^T)}) \end{aligned}$$

To conclude, we see that

$$|Q - \bar{f}|_{H^{2,h,h/2}(D^T)} \leq K|f - \bar{f}|_{H^0(D^T)} |p|_{H^{2,h,h/2}(D^T)} P(K|p|_{H^{2,h,h/2}(D^T)})$$

Let us now prove

$$|Q - R|_{H^{2,h,h/2}(D^T)} \leq |f - \bar{f}|_{H^0(D^T)} |p - q|_{H^{2,h,h/2}(D^T)} P(|p|_{H^{2,h,h/2}(D^T)}) P(|q|_{H^{2,h,h/2}(D^T)})$$

We have

$$|Q - R|_{H^{2,h,h/2}(D^T)} = \left| \frac{\int_{\Omega(z)} f p dy \int_{\Omega(z)} q dy - \int_{\Omega(z)} f q dy \int_{\Omega(z)} p dy}{\int_{\Omega(z)} p dy \int_{\Omega(z)} q dy} \right|_{H^{2,h,h/2}(D^T)}$$

The denominator can be dealt with exactly as earlier, all we have to study is

$$\begin{aligned} \left| \int f p \int q - \int f q \int p \right|_{H^{2,h,h/2}(D^T)} &= \left| \int (f - \bar{f})(p - q) \int q - \int (f - \bar{f})q \int p - q \right|_{H^{2,h,h/2}(D^T)} \\ &\leq K|f - \bar{f}|_{H^0(D^T)} |p - q|_{H^{2,h,h/2}(D^T)} (|p|_{H^{2,h,h/2}(D^T)} + |q|_{H^{2,h,h/2}(D^T)}) \end{aligned}$$

which ends the case $V(X) = X$. We now consider any function V .

$$\begin{aligned} \left| V\left(\frac{\int f p}{\int p}\right)(t, x) - V(\bar{f}) \right| &\leq K_\eta \left| \frac{\int f p}{\int p}(t, x) - \bar{f} \right| \\ \left| V\left(\frac{\int f p}{\int p}\right)(t, x) - V\left(\frac{\int f p}{\int p}\right)(t', x') \right| &\leq K_\eta \left| \frac{\int (f - \bar{f})p}{\int p}(t, x) - \frac{\int (f - \bar{f})p}{\int p}(t', x') \right| \end{aligned}$$

Hence, we have

$$|V(Q) - V(\bar{f})|_{H^{0,h,h/2}(D^T)} \leq K_\eta |Q - \bar{f}|_{H^{0,h,h/2}(D^T)}$$

Thus

$$\begin{aligned}
\left| \frac{\partial}{\partial t} (V(Q) - V(\bar{f})) \right|_{H^{0,h,h/2}(D^T)} &\leq |V'(Q)|_{H^{0,h,h/2}(D^T)} \left| \frac{\partial Q}{\partial t} \right|_{H^{0,h,h/2}(D^T)} \\
&\leq K_\eta |Q|_{H^{0,h,h/2}(D^T)} \left| \frac{\partial}{\partial t} (Q - \bar{f}) \right|_{H^{0,h,h/2}(D^T)} \\
&\leq K_\eta |Q|_{H^{0,h,h/2}(D^T)} |Q - \bar{f}|_{H^{1,h,h/2}(D^T)}
\end{aligned}$$

As far as second derivatives are concerned, we see that

$$\begin{aligned}
\left| \frac{\partial^2}{\partial x_i \partial x_j} (V(Q) - V(\bar{f})) \right|_{H^{0,h,h/2}(D^T)} &\leq |V''(Q)|_{H^{0,h,h/2}(D^T)} \left| \frac{\partial Q}{\partial x_i} \right|_{H^{0,h,h/2}(D^T)} \left| \frac{\partial Q}{\partial x_j} \right|_{H^{0,h,h/2}(D^T)} \\
&\quad + |V'(Q)|_{H^{0,h,h/2}(D^T)} \left| \frac{\partial^2 Q}{\partial x_i \partial x_j} \right|_{H^{0,h,h/2}(D^T)} \\
&\leq K_\eta (|Q|_{H^{0,h,h/2}(D^T)} + |Q|_{H^{1,h,h/2}(D^T)}^2) |Q - \bar{f}|_{H^{2,h,h/2}(D^T)}
\end{aligned}$$

Eventually, we get

$$\begin{aligned}
|V(Q) - V(\bar{f})|_{H^{2,h,h/2}(D^T)} &\leq K |Q - \bar{f}|_{H^{2,h,h/2}(D^T)} P(|Q|_{H^{2,h,h/2}(D^T)}) \\
&\leq K |f - \bar{f}|_{H^0(D^T)} P(|p|_{H^{2,h,h/2}(D^T)})
\end{aligned}$$

The same computations obviously work for the second inequality of the lemma, which concludes the proof. \square

Remark. *The exact same computations in the case of a function $V(Q)$ alone clearly gives us the following inequality*

$$|V(Q)|_{H^{2,h,h/2}(D^T)} \leq |f|_{H^0(D^T)} P(|p|_{H^{2,h,h/2}(D^T)})$$

Using this last lemma, we can prove the auxiliary lemma used in the proofs of Part I.

Lemma. *For a given $\varepsilon > 0$, there exists a constant $K_l > 0$ depending on ε and on the data of the problem such that $\forall p, q \in H^{2,h,h/2}(D^T)$, $p, q \geq \varepsilon$ on D^T , $\forall 1 \leq i, j \leq n$,*

$$\begin{aligned}
|L_{ij}(p, \{f\}) - L_{ij}(p, \{\bar{f}\})|_{H^{2,h,h/2}(D^T)} &\leq K_l (|\{f\}| + |\{\bar{f}\}|) |\{f\} - \{\bar{f}\}| P(|p|_{H^{2,h,h/2}(D^T)}) \\
|L_{ij}(p, \{f\}) - L_{ij}(p, \{\bar{f}\}) - (L_{ij}(q, \{f\}) - L_{ij}(q, \{\bar{f}\}))|_{H^{2,h,h/2}(D^T)} \\
&\leq K'_l |\{f\}| |\{f\} - \{\bar{f}\}| |p - q|_{H^{2,h,h/2}(D^T)} \\
&\quad P(|p|_{H^{2,h,h/2}(D^T)}) P(|q|_{H^{2,h,h/2}(D^T)})
\end{aligned}$$

with P a polynomial function, increasing and strictly positive on \mathbb{R}^+ , and all the norms involving $\{f\}$ or $\{\bar{f}\}$ are $(H^0(D^T))^{n^2}$.

Proof. In this proof, we omit the variables t and x as arguments for V_{ij} since they clearly pose no problem. We compute

$$\begin{aligned}
|L_{ij}(p, \{f\}) - L_{ij}(p, \{\bar{f}\})|_{H^{2,h,h/2}(D^T)} &= \left| \sum_{k=1}^n \left(V_{ik} \left(\frac{\int f_{ik} p}{\int p} \right) V_{jk} \left(\frac{\int f_{jk} p}{\int p} \right) - V_{ik}(\bar{f}_{ik}) V_{jk}(\bar{f}_{jk}) \right) \right|_{H^{2,h,h/2}(D^T)} \\
&\leq \sum_{k=1}^n |V_{ik} \left(\frac{\int f_{ik} p}{\int p} \right) - V_{ik}(\bar{f}_{ik})|_{H^{2,h,h/2}(D^T)} |V_{jk} \left(\frac{\int f_{jk} p}{\int p} \right)|_{H^{2,h,h/2}(D^T)} \\
&\quad + |V_{ik}(\bar{f}_{ik})|_{H^{2,h,h/2}(D^T)} |V_{jk} \left(\frac{\int f_{jk} p}{\int p} \right) - V_{jk}(\bar{f}_{jk})|_{H^{2,h,h/2}(D^T)}
\end{aligned}$$

Given the assumptions on the functions f_{ij} and V_{ij} , we can apply Lemma 9 and the remark above to get

$$\begin{aligned}
|L_{ij}(p, \{f\}) - L_{ij}(p, \{\bar{f}\})|_{H^{2,h,h/2}(D^T)} &\leq K \sum_{k=1}^n |f_{jk}|_{H^0(D^T)} |f_{ik} - \bar{f}_{ik}|_{H^0(D^T)} P(|p|_{H^{2,h,h/2}(D^T)}) \\
&\quad + |\bar{f}_{ik}|_{H^0(D^T)} |f_{jk} - \bar{f}_{jk}|_{H^0(D^T)} P(|p|_{H^{2,h,h/2}(D^T)}) \\
&\leq K(|\{f\}|_{(H^0(D^T))^{n^2}} + |\{\bar{f}\}|_{(H^0(D^T))^{n^2}}) |\{f\} - \{\bar{f}\}|_{(H^0(D^T))^{n^2}} P(|p|_{H^{2,h,h/2}(D^T)})
\end{aligned}$$

Noticing that $L_{ij}(p, \{\bar{f}\}) = L_{ij}(q, \{\bar{f}\})$, let us now compute

$$\begin{aligned}
|L_{ij}(p, \{f\}) - L_{ij}(q, \{f\})|_{H^{2,h,h/2}(D^T)} &= \left| \sum_{k=1}^n \left(V_{ik} \left(\frac{\int f_{ik} p}{\int p} \right) V_{jk} \left(\frac{\int f_{jk} p}{\int p} \right) - V_{ik} \left(\frac{\int f_{ik} q}{\int q} \right) V_{jk} \left(\frac{\int f_{jk} q}{\int q} \right) \right) \right|_{H^{2,h,h/2}(D^T)} \\
&\leq \sum_{k=1}^n |V_{ik} \left(\frac{\int f_{ik} p}{\int p} \right) - V_{ik} \left(\frac{\int f_{ik} q}{\int q} \right)|_{H^{2,h,h/2}(D^T)} |V_{jk} \left(\frac{\int f_{jk} p}{\int p} \right)|_{H^{2,h,h/2}(D^T)} \\
&\quad + |V_{ik} \left(\frac{\int f_{ik} q}{\int q} \right)|_{H^{2,h,h/2}(D^T)} |V_{jk} \left(\frac{\int f_{jk} p}{\int p} \right) - V_{jk} \left(\frac{\int f_{jk} q}{\int q} \right)|_{H^{2,h,h/2}(D^T)}
\end{aligned}$$

Exactly as earlier, we apply Lemma 9 to conclude the proof. \square

Appendix B

Estimates for Green's function

In this technical appendix, we use the notations from [3]. The goal is to prove the following estimates on Green's function for operator (3.8): for all $x, x', y \in D$, $0 \leq \tau < t \leq T$

$$|G(x, y, t, \tau) - G(x', y, t, \tau)| \leq K \frac{|x - x'|^h}{(t - \tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x'' - y|^2}{t - \tau}) \quad (\text{B.1})$$

with x'' the closest point to y between x and x' . And for all $x, y \in D$, $0 \leq \tau < t' < t \leq T$ the inequality

$$|G(x, y, t, \tau) - G(x, y, t', \tau)| \leq K \frac{|t - t'|^{\frac{h}{2}}}{(t' - \tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x - y|^2}{t - \tau}) \quad (\text{B.2})$$

K and K' are two constants depending on the coefficients of operator (3.8), on h and on d . They might change in the rest of this appendix.

From (16.7) p.408, we write G as

$$G(x, y, t, \tau) = Z(x, y, t, \tau) - g(x, y, t, \tau)$$

with Z a fundamental solution for parabolic equation (3.11) and g defined in p.409. To prove (B.1) and (B.2), we show those exact same estimates on Z and g .

Lemma 10. For all $x, \xi \in \mathbb{R}^d$ and $0 \leq \tau < \lambda < t \leq T$

$$\int_{\mathbb{R}^d} \exp(-K' \frac{|x - y|^2}{t - \lambda}) \exp(-K' \frac{|y - \xi|^2}{\lambda - \tau}) dy \leq K \frac{(t - \lambda)^{\frac{d}{2}} (\lambda - \tau)^{\frac{d}{2}}}{(t - \tau)^{\frac{d}{2}}} \exp(-K' \frac{|x - \xi|^2}{t - \tau})$$

Lemma 11. For all $0 \leq \tau < t \leq T$ and $\alpha, \beta > 0$

$$\int_{\tau}^t (t - \lambda)^{\alpha-1} (\lambda - \tau)^{\beta-1} d\lambda \leq K (t - \tau)^{\alpha+\beta-1}$$

B.1 Space estimate B.1

We first prove estimate B.1 on the fundamental solution Z then on the single-layer potential g . The function Z is constructed using the parametrix method of E. Levy, it is written as

$$Z(x, \xi, t, \tau) = Z_0(x - \xi, \xi, t, \tau) + \int_{\tau}^t \int_{\mathbb{R}^d} Z_0(x - y, y, t, \lambda) Q(y, \xi, \lambda, \tau) dy d\lambda \quad (\text{B.3})$$

Z_0 stands for the heat kernel with frozen coefficients. One has the following lemma

Lemma 12. For all $x, x', y \in \mathbb{R}^d$ and $0 \leq \tau < t \leq T$

$$|Z_0(x-y, y, t, \tau) - Z_0(x'-y, y, t, \tau)| \leq K \frac{|x-x'|^h}{(t-\tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x''-y|^2}{t-\tau})$$

with x'' the closest point to y between x and x' .

Proof. From the definition (11.2) p.356, we have $\forall x, x', y \in \mathbb{R}^d, \forall 0 \leq \tau < t \leq T$

$$\text{Diff} = |Z_0(x-y, y, t, \tau) - Z_0(x'-y, y, t, \tau)| \leq K \frac{|x-x'|}{(t-\tau)^{\frac{d+1}{2}}} \exp(-K' \frac{|x''-y|^2}{t-\tau})$$

Indeed, from (11.3) p.357, with $r = 0$ and $s = 1$, we see that

$$|\frac{\partial Z_0}{\partial z}(z-y, y, t, \tau)| \leq K(t-\tau)^{-\frac{d+1}{2}} \exp(-K' \frac{|z-y|^2}{t-\tau})$$

Placing ourselves in the plane generated by the points x, x' and y , we let C denote the half-circle of diameter $[x, x']$, lying in the half-space created by (x, x') and that does not contain y ¹. The distance from y to C is clearly larger than $|x''-y|$. This allows us to write

$$\text{Diff} \leq \int_C |\frac{\partial Z_0}{\partial z}(z-y, y, t, \tau)| dz \leq K \frac{\pi|x-x'|}{(t-\tau)^{\frac{d+1}{2}}} \exp(-K' \frac{|x''-y|^2}{t-\tau})$$

As in p.361, we now use interpolation inequalities to get

$$\begin{aligned} \text{Diff} &\leq K \frac{|x-x'|^h}{(t-\tau)^{h\frac{d+1}{2}}} \exp(-K' \frac{|x''-y|^2}{t-\tau}) (|Z_0(x-y, y, t, \tau)| + |Z_0(x'-y, y, t, \tau)|)^{1-h} \\ &\leq K \frac{|x-x'|^h}{(t-\tau)^{h\frac{d+1}{2} + (1-h)\frac{d}{2}}} \exp(-K' \frac{|x''-y|^2}{t-\tau}) \\ &\leq K \frac{|x-x'|^h}{(t-\tau)^{\frac{d+h}{2}}} \exp(-K' \frac{|x''-y|^2}{t-\tau}) \end{aligned}$$

□

Given this lemma, we use (11.26) p.363

$$|Q(x, \xi, t, \tau)| \leq K \frac{1}{(t-\tau)^{\frac{d+2-h}{2}}} \exp(-K' \frac{|x-\xi|^2}{t-\tau})$$

to estimate

$$\begin{aligned} \text{Diff} &= \left| \int_{\tau}^t \int_{\mathbb{R}^d} (Z_0(x-y, y, t, \lambda) - Z_0(x'-y, y, t, \lambda)) Q(y, \xi, \lambda, \tau) dy d\lambda \right| \\ &\leq \int_{\tau}^t \int_{\mathbb{R}^d} K \frac{|x-x'|^h}{(t-\lambda)^{\frac{d+h}{2}}} \exp(-K' \frac{|x''-y|^2}{t-\lambda}) \frac{1}{(\lambda-\tau)^{\frac{d+2-h}{2}}} \exp(-K' \frac{|y-\xi|^2}{\lambda-\tau}) dy d\lambda \end{aligned}$$

¹If y belongs to (x, x') , both possibilities are fine.

Replacing $\exp(-K' \frac{|x''-y|^2}{t-\lambda})$ by $\exp(-K' \frac{|x-y|^2}{t-\lambda}) + \exp(-K' \frac{|x'-y|^2}{t-\lambda})$ and using the preliminary lemmas 10 and 11, we get

$$\begin{aligned} \text{Diff} &\leq K \exp(-K' \frac{|x''-\xi|^2}{t-\tau}) \int_{\tau}^t \frac{|x-x'|^h}{(t-\lambda)^{\frac{d+h}{2}} (\lambda-\tau)^{\frac{d+2-h}{2}}} \frac{(t-\lambda)^{\frac{d}{2}} (\lambda-\tau)^{\frac{d}{2}}}{(t-\tau)^{\frac{d}{2}}} d\lambda \\ &\leq K \frac{|x-x'|^h}{(t-\tau)^{\frac{d}{2}}} \exp(-K' \frac{|x''-\xi|^2}{t-\tau}) \int_{\tau}^t (t-\lambda)^{-\frac{h}{2}} (\lambda-\tau)^{\frac{h-2}{2}} d\lambda \\ &\leq K \frac{|x-x'|^h}{(t-\tau)^{\frac{d}{2}}} \exp(-K' \frac{|x''-\xi|^2}{t-\tau}) \end{aligned}$$

We now prove the inequality for function g . Using (16.10) p.409, we know that

$$g(x, y, t, \tau) = \int_{\tau}^t \int_{\partial\Omega} Z(x, \xi, t, \lambda) \omega(\xi, y, \lambda, \tau) dS_{\xi} d\lambda$$

with S_{ξ} the element of the surface $\partial\Omega$ taken in ξ and ω defined by (16.11) such that

$$\begin{aligned} |\omega(\xi, y, \lambda, \tau)| &\leq K (\lambda-\tau)^{-\frac{d+1}{2}} \exp(-K' \frac{|\xi-y|^2}{\lambda-\tau}) \\ \int_{\tau}^t \int_{\partial\Omega} |\omega(\xi, y, \lambda, \tau)| dS_{\xi} d\lambda &\leq K \end{aligned}$$

Let $x, x', y \in D$, $0 \leq \tau < t \leq T$. As in [3], we let σ denote the intersection of $\partial\Omega$ with $B(x, \frac{|x''-y|}{2}) \cup B(x', \frac{|x''-y|}{2})$. We compute

$$\begin{aligned} \text{Diff} &= |g(x, y, t, \tau) - g(x', y, t, \tau)| = \left| \int_{\frac{t+\tau}{2}}^t \int_{\partial\Omega} (Z(x, \xi, t, \lambda) - Z(x', \xi, t, \lambda)) \omega(\xi, y, \lambda, \tau) dS_{\xi} d\lambda \right| \\ &+ \left| \int_{\tau}^{\frac{t+\tau}{2}} \int_{\partial\Omega \setminus \sigma} (Z(x, \xi, t, \lambda) - Z(x', \xi, t, \lambda)) \omega(\xi, y, \lambda, \tau) dS_{\xi} d\lambda \right| \\ &+ \left| \int_{\tau}^{\frac{t+\tau}{2}} \int_{\sigma} (Z(x, \xi, t, \lambda) - Z(x', \xi, t, \lambda)) \omega(\xi, y, \lambda, \tau) dS_{\xi} d\lambda \right| \\ &\leq K (t-\tau)^{-\frac{d+1}{2}} \left| \int_{\frac{t+\tau}{2}}^t \int_{\partial\Omega} |Z(x, \xi, t, \lambda) - Z(x', \xi, t, \lambda)| \exp(-K' \frac{|\xi-y|^2}{\lambda-\tau}) dS_{\xi} d\lambda \right| \\ &+ K \sup_{\xi \in \partial\Omega \setminus \sigma, \tau \leq \lambda \leq \frac{t+\tau}{2}} |Z(x, \xi, t, \lambda) - Z(x', \xi, t, \lambda)| \int_{\tau}^{\frac{t+\tau}{2}} \int_{\partial\Omega \setminus \sigma} |\omega(\xi, y, \lambda, \tau)| dS_{\xi} d\lambda \\ &+ K \int_{\tau}^{\frac{t+\tau}{2}} \int_{\sigma} |Z(x, \xi, t, \lambda) - Z(x', \xi, t, \lambda)| (\lambda-\tau)^{-\frac{d+1}{2}} \exp(-K' \frac{|\xi-y|^2}{\lambda-\tau}) dS_{\xi} d\lambda \end{aligned}$$

Since

$$|Z(x, \xi, t, \lambda) - Z(x', \xi, t, \lambda)| \leq K \frac{|x-x'|^h}{(t-\lambda)^{\frac{d+h}{2}}} \exp(-K' \frac{|\xi'-\xi|^2}{t-\lambda})$$

with ξ' the closest point to ξ between x and x' . We also notice that $|\xi' - \xi| \geq \frac{1}{2}|x'' - y|$ on $\partial\Omega \setminus \sigma$

and $|y - \xi| \geq \frac{1}{2}|x'' - y|$ on σ . We get

$$\begin{aligned}
\text{Diff} &\leq K \frac{|x - x'|^h}{(t - \tau)^{\frac{d+1}{2}}} \int_{\frac{t+\tau}{2}}^t \int_{\partial\Omega} (t - \lambda)^{-\frac{d+h}{2}} \exp(-K' \frac{|\xi' - \xi|^2}{t - \lambda}) \exp(-K' \frac{|\xi - y|^2}{\lambda - \tau}) dS_\xi d\lambda \\
&+ K \frac{|x - x'|^h}{(t - \lambda)^{\frac{d+h}{2}}} \exp(-K' \frac{|x'' - y|^2}{t - \lambda}) \\
&+ K \frac{|x - x'|^h}{(t - \tau)^{\frac{d+1}{2}}} \int_{\tau}^{\frac{t+\tau}{2}} \int_{\sigma} (\lambda - \tau)^{-\frac{d+1}{2}} \exp(-K' \frac{|\xi' - \xi|^2}{t - \lambda}) \exp(-K' \frac{|\xi - y|^2}{\lambda - \tau}) dS_\xi d\lambda \\
&:= J_1 + J_2 + J_3
\end{aligned}$$

We split $\exp(-K' \frac{|\xi' - \xi|^2}{t - \lambda})$ into $\exp(-K' \frac{|x - \xi|^2}{t - \lambda}) + \exp(-K' \frac{|x' - \xi|^2}{t - \lambda})$ and compute for instance

$$\frac{|x - \xi|^2}{t - \lambda} + \frac{|\xi - y|^2}{\lambda - \tau} = \frac{t - \tau}{(t - \lambda)(\lambda - \tau)} \left| \xi - \frac{(\lambda - \tau)x + (t - \lambda)y}{t - \tau} \right|^2 + \frac{|x - y|^2}{t - \tau}$$

This allows us to write

$$\begin{aligned}
\int_{\partial\Omega} \exp(-K' \frac{|x - \xi|^2}{t - \lambda}) \exp(-K' \frac{|\xi - y|^2}{\lambda - \tau}) dS_\xi &= \int_{\partial\Omega} \theta^{\frac{d-1}{2}} \exp(-\theta |\xi - X|^2) dS_\xi \frac{\exp(-K' \frac{|x - y|^2}{t - \tau})}{\theta^{\frac{d-1}{2}}} \\
&\leq K \theta^{-\frac{d-1}{2}} \exp(-K' \frac{|x - y|^2}{t - \tau})
\end{aligned}$$

with $\theta = \frac{t - \tau}{(t - \lambda)(\lambda - \tau)}$ and $X = \frac{(\lambda - \tau)x + (t - \lambda)y}{t - \tau}$. Applying this last result and the preliminary lemma 11 gives

$$\begin{aligned}
J_1 &\leq K \frac{|x - x'|^h}{(t - \tau)^{\frac{d+1}{2}}} \int_{\frac{t+\tau}{2}}^t (t - \lambda)^{-\frac{d+h}{2}} \frac{(t - \lambda)^{\frac{d-1}{2}} (\lambda - \tau)^{\frac{d-1}{2}}}{(t - \tau)^{\frac{d-1}{2}}} \exp(-K' \frac{|x'' - y|^2}{t - \tau}) d\lambda \\
&\leq K \frac{|x - x'|^h}{(t - \tau)^{\frac{d+1}{2}}} \exp(-K' \frac{|x'' - y|^2}{t - \tau}) \int_{\frac{t+\tau}{2}}^t (t - \lambda)^{-\frac{1+h}{2}} d\lambda \\
&\leq K \frac{|x - x'|^h}{(t - \tau)^{\frac{d+1}{2}}} \exp(-K' \frac{|x'' - y|^2}{t - \tau}) (t - \tau)^{\frac{1-h}{2}}
\end{aligned}$$

The estimate we were looking for is proven for J_1 and J_2 . As far as J_3 is concerned, the exact same computations as in p.412 give us the result.

B.2 Time estimate B.2

We now want to prove the time inequality. Let $x, y \in D$ and $0 \leq \tau < t' < t \leq T$. First, we assume that $t - t' < \frac{t' - \tau}{4}$. Using Theorem 16.3 p.413, we compute

$$\begin{aligned}
 |G(x, y, t, \tau) - G(x, y, t', \tau)| &\leq \int_{t'}^t \left| \frac{\partial G}{\partial t}(x, y, \lambda, \tau) \right| \\
 &\leq K |t - t'| \sup_{t' \leq \lambda \leq t} |\lambda - \tau|^{-\frac{d+2}{2}} \exp\left(-K' \frac{|x-y|^2}{\lambda - \tau}\right) \\
 &\leq K \frac{|t - t'|}{|t' - \tau|^{\frac{d+2}{2}}} \exp\left(-K' \frac{|x-y|^2}{t - \tau}\right) \\
 &\leq K \frac{|t - t'|^{\frac{h}{2}}}{|t' - \tau|^{\frac{d+h}{2}}} \exp\left(-K' \frac{|x-y|^2}{t - \tau}\right)
 \end{aligned}$$

where we used in the last line $t - t' < \frac{t' - \tau}{4}$.

Let us now consider the case $t - t' \geq \frac{t' - \tau}{4}$. Inequality (16.16) p.412 gives us

$$\begin{aligned}
 |G(x, y, t, \tau) - G(x, y, t', \tau)| &\leq K(|t - \tau|^{-\frac{d}{2}} \exp(-K' \frac{|x-y|^2}{t - \tau}) + |t' - \tau|^{-\frac{d}{2}} \exp(-K' \frac{|x-y|^2}{t' - \tau})) \\
 &\leq K |t' - \tau|^{-\frac{d}{2}} \exp(-K' \frac{|x-y|^2}{t - \tau}) \\
 &\leq K \frac{|t - t'|^{\frac{h}{2}}}{|t' - \tau|^{\frac{d+h}{2}}} \exp(-K' \frac{|x-y|^2}{t - \tau})
 \end{aligned}$$

Appendix C

Data for the numerical simulations

In this appendix, we give more data concerning Part II and the numerical results obtained using the calibration method stemming from the partial differential equation (5.1).

C.1 LSV and Stochastic Interest Rates models

The surface of implied volatility used for the tests in Chapters 6 and 8 is the one of the EuroSTOXX 50 on the 2009/04/02. It was modified to have a spot of 100 and to nullify the impact of the forward, the interest rate is chosen constant equal to 3%. A functional is fitted to the existing quoted options, the rest of the surface is extrapolated using said functional. The surface thus obtained is the following

	0,003	0,088	0,167	0,250	0,500	0,750	1,000
5%	126,2%	79,21%	78,13%	72,65%	68,57%	63,00%	61,20%
10%	121,9%	77,19%	76,17%	70,94%	67,02%	61,68%	59,94%
15%	117,5%	75,16%	74,20%	69,23%	65,46%	60,36%	58,68%
20%	113,1%	73,13%	72,24%	67,53%	63,91%	59,04%	57,42%
30%	104,3%	69,08%	68,31%	64,11%	60,80%	56,41%	54,90%
40%	95,47%	65,01%	64,37%	60,68%	57,69%	53,76%	52,38%
50%	86,33%	60,87%	60,36%	57,21%	54,54%	51,09%	49,82%
60%	76,92%	56,66%	56,30%	53,71%	51,35%	48,39%	47,23%
70%	67,22%	52,39%	52,17%	50,15%	48,13%	45,66%	44,62%
80%	57,22%	48,04%	47,98%	46,55%	44,87%	42,91%	41,97%
90%	46,92%	43,64%	43,74%	42,90%	41,58%	40,14%	39,30%
100%	36,30%	39,18%	39,42%	39,25%	38,26%	37,37%	36,64%
110%	30,87%	36,70%	35,71%	36,39%	35,42%	34,97%	34,31%
120%	30,30%	35,76%	33,41%	34,22%	33,26%	32,93%	32,36%
130%	30,13%	35,38%	32,25%	32,64%	31,68%	31,26%	30,75%
140%	29,96%	35,03%	31,45%	31,81%	30,53%	29,92%	29,43%
150%	29,79%	34,65%	30,77%	31,36%	29,80%	28,90%	28,41%
160%	29,62%	34,27%	30,07%	30,91%	29,27%	28,14%	27,64%
170%	29,45%	33,89%	29,36%	30,44%	28,78%	27,65%	27,09%
180%	29,27%	33,48%	28,63%	29,97%	28,27%	27,17%	26,60%
190%	29,10%	33,08%	27,89%	29,49%	27,75%	26,70%	26,12%
200%	28,92%	32,66%	27,13%	28,99%	27,24%	26,22%	25,62%
210%	28,75%	32,23%	26,35%	28,50%	26,71%	25,73%	25,11%
220%	28,57%	31,79%	25,55%	27,99%	26,17%	25,23%	24,60%
230%	28,39%	31,35%	24,74%	27,48%	25,62%	24,74%	24,08%
240%	28,21%	30,89%	23,90%	26,96%	25,06%	24,23%	23,56%
250%	28,03%	30,42%	23,06%	26,42%	24,51%	23,71%	23,02%

	1,500	2,000	3,000	4,000	5,000	7,000	10,000
5%	54,25%	52,70%	47,80%	46,37%	44,22%	42,21%	40,36%
10%	53,31%	51,80%	47,14%	45,74%	43,69%	41,76%	39,99%
15%	52,36%	50,91%	46,47%	45,12%	43,15%	41,31%	39,62%
20%	51,42%	50,02%	45,80%	44,49%	42,61%	40,86%	39,25%
30%	49,53%	48,23%	44,46%	43,23%	41,54%	39,95%	38,50%
40%	47,63%	46,44%	43,12%	41,98%	40,46%	39,05%	37,75%
50%	45,71%	44,63%	41,75%	40,70%	39,37%	38,13%	37,00%
60%	43,77%	42,80%	40,36%	39,40%	38,28%	37,22%	36,24%
70%	41,81%	40,95%	38,95%	38,09%	37,16%	36,28%	35,48%
80%	39,83%	39,08%	37,50%	36,76%	36,04%	35,34%	34,71%
90%	37,82%	37,19%	36,04%	35,41%	34,89%	34,40%	33,94%
100%	35,80%	35,29%	34,55%	34,05%	33,74%	33,43%	33,16%
110%	33,92%	33,57%	33,16%	32,81%	32,65%	32,54%	32,44%
120%	32,30%	32,07%	31,98%	31,75%	31,72%	31,78%	31,81%
130%	30,93%	30,81%	30,98%	30,85%	30,94%	31,15%	31,28%
140%	29,79%	29,74%	30,17%	30,11%	30,29%	30,62%	30,84%
150%	28,86%	28,86%	29,51%	29,49%	29,76%	30,20%	30,47%
160%	28,13%	28,16%	28,96%	28,86%	29,23%	29,84%	30,10%
170%	27,60%	27,58%	28,40%	28,22%	28,70%	29,47%	29,75%
180%	27,20%	27,12%	27,79%	27,58%	28,15%	29,10%	29,37%
190%	26,80%	26,67%	27,19%	26,93%	27,59%	28,71%	29,00%
200%	26,41%	26,22%	26,56%	26,26%	27,02%	28,33%	28,62%
210%	26,01%	25,77%	25,93%	25,58%	26,45%	27,94%	28,25%
220%	25,60%	25,32%	25,28%	24,88%	25,85%	27,55%	27,85%
230%	25,19%	24,85%	24,62%	24,17%	25,26%	27,16%	27,46%
240%	24,78%	24,37%	23,95%	23,44%	24,64%	26,75%	27,05%
250%	24,36%	23,89%	23,26%	22,69%	24,02%	26,34%	26,65%

We now give the results of the calibration for other maturities. We compute the fitted prices for the two stochastic volatilities already studied in Chapter 6, the lognormal one "LNV" and the "CIR".

	50%	60%	70%	80%	90%	100%	110%	120%	130%	140%	150%
LNV	44,55%	42,75%	40,92%	39,08%	37,20%	35,19%	33,49%	32,00%	30,75%	29,71%	28,83%
BS	44,63%	42,80%	40,95%	39,08%	37,19%	35,29%	33,57%	32,07%	30,81%	29,74%	28,86%
CIR	44,53%	42,74%	40,91%	39,05%	37,16%	35,20%	33,50%	32,01%	30,76%	29,72%	28,85%

Table C.1: Calibrated prices for 2 year options

	50%	60%	70%	80%	90%	100%	110%	120%	130%	140%	150%
LNV	41,77%	40,37%	38,97%	37,57%	36,13%	34,58%	33,21%	32,04%	31,06%	30,27%	29,62%
BS	41,75%	40,36%	38,95%	37,50%	36,04%	34,55%	33,16%	31,98%	30,98%	30,17%	29,51%
CIR	41,73%	40,34%	38,94%	37,54%	36,11%	34,54%	33,17%	32,00%	31,03%	30,24%	29,60%

Table C.2: Calibrated prices for 3 year options

	50%	60%	70%	80%	90%	100%	110%	120%	130%	140%	150%
LNV	39,42%	38,27%	37,11%	35,99%	34,83%	33,76%	32,68%	31,76%	30,99%	30,35%	29,82%
BS	39,37%	38,28%	37,16%	36,04%	34,89%	33,74%	32,65%	31,72%	30,94%	30,29%	29,76%
CIR	39,29%	38,17%	37,04%	35,92%	34,78%	33,71%	32,65%	31,73%	30,96%	30,33%	29,81%

Table C.3: Calibrated prices for 5 year options

	50%	60%	70%	80%	90%	100%	110%	120%	130%	140%	150%
LNV	38,87%	37,84%	36,82%	35,82%	34,82%	33,43%	32,53%	31,76%	31,13%	30,61%	30,22%
BS	38,13%	37,22%	36,28%	35,34%	34,40%	33,43%	32,54%	31,78%	31,15%	30,62%	30,20%
CIR	38,03%	37,13%	36,21%	35,28%	34,36%	33,42%	32,17%	31,46%	30,87%	30,39%	30,02%

Table C.4: Calibrated prices for 7 year options

	50%	60%	70%	80%	90%	100%	110%	120%	130%	140%	150%
LNV	37,01%	36,24%	35,47%	34,72%	33,97%	33,22%	32,45%	31,87%	31,41%	31,04%	30,75%
BS	37,00%	36,24%	35,48%	34,71%	33,94%	33,16%	32,44%	31,81%	31,28%	30,84%	30,47%
CIR	36,97%	36,19%	35,43%	34,67%	33,91%	33,03%	32,36%	31,78%	31,31%	30,94%	30,64%

Table C.5: Calibrated prices for 10 year options

C.2 Local Correlation model

Here, we give some results for other tests. We calibrate a model with the following parameters $\rho = 0$, $w_1 = 0.3$ and $w_2 = 0.7$, the targeted surface is the theoretical one distorted with two factors: first 0.95 and second 1.05.

K \ T	0,25		0,50		0,75		1,0	
	Theor.	LC	Theor.	LC	Theor.	LC	Theor.	LC
80%	32.17%	32.10%	30.91%	30.88%	29.99%	29.96%	29.46%	29.43%
90%	31.49%	31.40%	30.43%	32.10%	29.60%	29.52%	29.22%	29.14%
100%	30.58%	30.42%	29.86%	29.62%	29.30%	29.02%	29.02%	28.76%
110%	30.01%	29.80%	29.68%	29.36%	29.04%	28.66%	28.93%	28.56%
120%	29.92%	29.74%	29.63%	29.30%	29.08%	28.68%	28.96%	28.52%

Table C.6: Calibration of the basket with $D = 0.95$, $\rho = 0$, $w_1 = 0.3$ and $w_2 = 0.7$

K \ T	0,25		0,50		0,75		1,0	
	Theor.	LC	Theor.	LC	Theor.	LC	Theor.	LC
80%	35.56%	35.36%	34.17%	34.05%	33.15%	33.01%	32.56%	32.42%
90%	34.81%	34.63%	33.64%	33.50%	32.73%	32.56%	32.30%	32.14%
100%	33.81%	33.60%	33.01%	32.75%	32.39%	32.07%	32.08%	31.75%
110%	33.17%	32.93%	32.81%	32.44%	32.10%	31.66%	31.98%	31.50%
120%	33.07%	32.81%	32.75%	32.35%	32.14%	31.65%	32.02%	31.47%

Table C.7: Calibration of the basket with $D = 1.05$, $\rho = 0$, $w_1 = 0.3$ and $w_2 = 0.7$

At last, we choose a different surface for the second underlying multiplying the first one (described in 7.1) by 0.9, the correlation is this time taken as 0.5.

K \ T	0,25		0,50		0,75		1,0	
	Theor.	LC	Theor.	LC	Theor.	LC	Theor.	LC
80%	40.78%	40.35%	39.16%	38.77%	37.87%	37.49%	37.16%	36.84%
90%	39.73%	39.51%	38.39%	38.10%	37.26%	36.97%	36.74%	36.46%
100%	38.44%	38.37%	37.52%	37.27%	36.71%	36.40%	36.32%	36.01%
110%	37.54%	37.40%	37.06%	36.78%	36.20%	35.85%	36.05%	35.71%
120%	37.31%	37.24%	36.89%	36.68%	36.10%	35.81%	35.98%	35.68%

Table C.8: Calibration of the basket with $D = 0.95$, $\rho = 0.5$, $w_1 = 0.3$ and $w_2 = 0.7$

Appendix D

CIR

Let us go back to the case of the CIR diffusion used in Chapter 6 as a volatility factor

$$dy_t = \kappa (\alpha - y_t) dt + \gamma \sqrt{y_t} dW_t$$

We let τ denote $\tau = \inf\{t \geq 0, y_t = 0\}$. In [48], one finds the

Proposition 12. *If α , κ and γ are positive, then there exists a unique positive solution y_t . Moreover, under the condition $2\kappa\alpha > \gamma^2$ then $P(\tau = \infty) = 1$.*

This last condition guarantees the strict positivity of the volatility process. Let us now simulate this process. The natural explicit euler scheme is

$$y_{t_{i+1}} = y_{t_i} + \kappa(\alpha - y_{t_i})(t_{i+1} - t_i) + \gamma\sqrt{y_{t_i}}(W_{t_{i+1}} - W_{t_i})$$

Unfortunately, it can lead to negative values since the gaussian increment is not negatively bounded. However, exactly as in [58], if we write

$$\begin{aligned} y_t &= y_0 + \lim_{n \rightarrow \infty} \left\{ \sum_{i, t_i < t} \kappa(\alpha - y_{t_{i+1}}) \frac{t}{n} + \gamma \sum_{i, t_i < t} \sqrt{y_{t_{i+1}}} (W_{t_{i+1}} - W_{t_i}) - \gamma \sum_{i, t_i < t} (\sqrt{y_{t_{i+1}}} - \sqrt{y_{t_i}}) (W_{t_{i+1}} - W_{t_i}) \right\} \\ &= y_0 + \lim_{n \rightarrow \infty} \left\{ \sum_{i, t_i < t} \left(\kappa(\alpha - y_{t_{i+1}}) - \frac{\gamma^2}{2} \right) \frac{t}{n} + \gamma \sum_{i, t_i < t} \sqrt{y_{t_{i+1}}} (W_{t_{i+1}} - W_{t_i}) \right\} \end{aligned}$$

The following implicit scheme appears quite naturally

$$y_{t_{i+1}} = y_{t_i} + \left(\kappa(\alpha - y_{t_{i+1}}) - \frac{\gamma^2}{2} \right) \frac{t}{n} + \gamma \sqrt{y_{t_{i+1}}} (W_{t_{i+1}} - W_{t_i})$$

More precisely, $\sqrt{V_{t_{i+1}}}$ can be chosen as the unique strictly positive root of the polynomial $P(x) = (1 + \kappa \frac{t}{n})x^2 - \gamma(W_{t_{i+1}} - W_{t_i})x - (V_{t_i} + (\kappa\alpha - \frac{\gamma^2}{2})\frac{t}{n})$. Indeed, the condition $2\kappa\alpha > \gamma^2$ gives $P(0) < 0$. Eventually, we obtain the formula used in the previous numerical computations

$$V_{t_{i+1}} = \left(\frac{\gamma(W_{t_{i+1}} - W_{t_i}) + \sqrt{\gamma^2(W_{t_{i+1}} - W_{t_i})^2 + 4(y_{t_i} + (\kappa\alpha - \frac{\gamma^2}{2})\frac{t}{n})(1 + \kappa\frac{t}{n})}}{2(1 + \kappa\frac{t}{n})} \right)^2$$

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