Stochastic flows on graphs

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Flots stochastiques sur les graphes

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Résumé

Dans cette thèse nous étudions des équations différentielles stochastiques sur quelques graphes simples dont les solutions sont des flots de noyaux au sens de Le Jan et Raimond.

Dans une première partie, nous définissons une extension de l'équation de Tanaka sur un nombre fini de demi-droites orientées et issues de l'origine. Utilisant certaines propriétés de régularité du flot associé au mouvement brownien biaisé, nous donnons une description complète de toutes les solutions.

S'appuyant sur une transformation discrète introduite par Csaki et Vincze, nous donnons dans un cas d'orientation particulière (qui couvre déjà l'équation de Tanaka usuelle) une approche discrète à quelques solutions.

La dernière partie de ce travail est effectuée avec O.Raimond. Par une méthode de couplage des flots, nous classifions les solutions de l'équation de Tanaka sur le cercle. Nous établissons aussi que ces flots sont coalescents.

Mots-clefs : Mouvement brownien de Walsh, mouvement brownien sur le cercle, flots stochastiques, transformation de Csaki et Vincze, équation de Tanaka.

Abstract

In this thesis we study stochastic differential equations on some simple graphs whose solutions are stochastic flows of kernels in the sense of Le Jan and Raimond.

In the first part, we define an extension of Tanaka's equation on a finite number of oriented half-lines issuing from the origin. Using some regularity properties of the skew Brownian motion flow, we give a complete description of all the solutions.

Based on a discrete transformation introduced by Csaki and Vincze, we give for a particular orientation (which already covers the usual Tanaka's equation) a discrete approach to some solutions.

The last part of this work is carried out with O. Raimond. By a method of coupling flows, we classify the solutions of Tanaka's equation on the circle. We also establish that all these flows are coalescing.

keywords Walsh Brownian motion, Brownian motion on the circle, stochastic flows, Csaki-Vincze transformation, Tanaka's equation.

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Chapter 1

Introduction

Dans cette thèse nous nous intéressons à des équations différentielles stochastiques dont les solutions sont des flots de noyaux introduits par Le Jan et Raimond [35]. Le point de départ consiste à écrire l'équation de Tanaka usuelle sur deux demi-droites ayant la même origine et deux orientations opposées. Dans une première partie, nous considérons une équation plus générale définie sur un nombre fini de demi-droites orientées arbitrairement. Le mouvement à 1 point associé aux diverses solutions est le mouvement brownien de Walsh [50]. En utilisant une version du mouvement brownien biaisé (Skew Brownian motion) étudiée par Burdzy-Kaspi [3], nous donnons une classification complète des solutions à travers les mesures déterministes sur les simplexes. Dans une deuxième partie, nous étudions dans un cas particulier qui couvre déjà l'équation de Tanaka classique deux solutions particulières de notre équation: le flot de Wiener, qui est une fonction simple du mouvement brownien initial et le flot d'applications ayant une expression plus compliquée. Ce dernier, dont l'existence et l'unicité sont dûes à Watanabe [51], a été construit par Le Jan et Raimond dans le cas réel en attachant de l'aléa supplémentaire aux minimas locaux du mouvement brownien [36]. Partant d'un flot discret simple, nous donnons une autre construction de ce flot d'applications. Notre outil principal est une transformation discrète introduite par Csaki et Vincze [44] que nous étudions en détail et relevons ses liens forts avec l'equation de Tanaka. La dernière partie de ce travail

est en collaboration avec Olivier Raimond. Nous définissons une équation de Tanaka sur le cercle et donnons une classification complète de ses solutions. Notre approche consiste à projeter les flots de Le Jan et Raimond [36] sur le cercle et s'inspirer en même temps du modèle discret pour achever la construction. En outre, nous montrons que tous les flots associés sont coalescents.
La structure de la thèse est la suivante:

(i) L'introduction présente la problématique et énonce les résultats importants de la thèse.

(ii) Dans le chapitre 2, nous introduisons le mouvement brownien biaisé, ainsi que celui de Walsh que nous utiliserons fréquemment au cours des chapitres 4 et 5.

(iii) Le Chapitre 3 reprend les deux articles de Le Jan et Raimond [35] et [36] et rappelle certains objets importants pour cette thèse.

(iv) Le chapitre 4 est une version détaillée d'un article intitulé "Stochastic flows related to Walsh Brownian motion "paru dans Electronic journal of probability. Nous étendons le travail de Le Jan et Raimond [36] et relions les flots stochastiques au mouvement brownien de Walsh.

(v) Le chapitre 5 est une étude approfondie du flot de Wiener et du flot d'applications de l'équation de Tanaka associée au mouvement brownien de Walsh. Il s'agit aussi d'un article intitulé "Discrete approximations to solution flows of Tanaka's equation related to Walsh Brownian motion ", accepté pour publication dans le Séminaire de probabilité.

(vi) Le chapitre 6 est un travail en collaboration avec Olivier Raimond. Nous appliquons les résultats de [36] pour l'étude de l'équation de Tanaka sur le cercle.

Nous donnons maintenant un descriptif plus détaillé du contenu de ce manuscrit en respectant l'ordre des chapitres.

1.1 Mouvements browniens: biaisé et de Walsh

Considérons un mouvement brownien réfléchi R_t et un paramètre $\alpha \in [0, 1]$. Itô et McKean [25] montrent comment construire ce qu'ils appellent le mouvement brownien biaisé à partir de R_t. Considérons les intervalles d'excursion de R_t hors 0. Puis, changeons le signe de chaque excursion indépendamment de sorte qu'une excursion donnée est positive avec la probabilité α et négative avec la probabilité $1 - \alpha$. Le processus resultant est appelé mouvement brownien biaisé de paramètre α. Lorsque $\alpha = \frac{1}{2}$, il s'agit du brownien ordinaire et le cas $\alpha \in \{0, 1\}$ correspond au brownien réfléchi. Si $\alpha \neq \frac{1}{2}$, ce processus est une diffusion qui se comporte comme un brownien ordinaire loin de l'origine, et qui traverse l'origine plus aisément dans une direction qu'une autre. Dans [50], Walsh calcule le semigroupe associé à cette diffusion et montre qu'il s'agit d'une semimartingale continue ayant un temps local discontinu en espace lorsque $\alpha \neq \frac{1}{2}$. Ceci a généré de nouvelles recherches menant à des articles fondamentaux sur le mouvement brownien biaisé (voir [3], [10], [11], [22]). L'article [38] résume différentes façons de construire ce processus et montre les liens entre eux. Il ajoute aussi des applications récentes en matière de modélisation et des simulations numériques.

Il est connu que le mouvement brownien possède la propriété de représentation prévisible pour sa filtration naturelle. Dans [52], Yor posait la question réciproque: les filtrations ayant la propriété de représentation prévisible sont-elles les filtrations naturelles d'un autre mouvement brownien?

Vers la fin de son article [50], Walsh propose d'étudier une généralisation du mouvement brownien biaisé qui répondra ultérieurement par la négative à la question de Yor (réponse fournie par Tsirelson [49]). Walsh introduit son mouvement comme suit:

"The idea is to take each excursion of R_t and, instead of giving it a random sign, to assign it a random variable θ with a given distribution in $[0, 2\pi[$, and to do this independently for each excursion. That is, if the excursion occurs during the interval

(u, v), we replace R_t by the pair (R_t, θ) for $u \leq t < v$, θ being a random variable with values in $[0, 2\pi[$. This provides a process $\{(R_t, \theta_t), t \geq 0\}$, where θ_t is constant during each excursion from 0, has the same distribution as θ, and is independent for different excursions. We then consider $Z_t = (R_t, \theta_t)$ as a process in the plane, expressed in polar coordinates. It is a diffusion which, when away from the origin, is a Brownian motion along a ray, but which has what might be called a roundhouse singularity at the origin: when the process enters it, it, like Stephen Leacock's hero, immediately rides off in all directions at once."

Walsh ne prouve pas que son processus est une diffusion sur $\mathbb{R}^2$. Cependant, depuis l'introduction de ce processus, plusieurs autres constructions ont été proposées: en utilisant les résolvantes [46], à partir du générateur infinitésimal [7] ou encore via la théorie des excursions [48]. En 1989 Barlow, Pitman, et Yor [4] produisent une autre construction en utilisant le semigroupe. Leur approche consiste à utiliser un peu d'intuition pour écrire un semigroupe possible puis vérifier que l'éventuel semigroupe réunit les conditions nécessaires. La théorie générale permet de produire un processus canonique associé au semigroupe donné. Les auteurs vérifient ensuite que ce processus possède les caractéristiques désirées du processus de Walsh. Dans le premier chapitre, nous adoptons la définition de Barlow, Pitman et Yor du mouvement brownien de Walsh $W(\alpha_1, \cdots, \alpha_N)$ sur le graphe G suivant (Figure 1.1) où $N \in \mathbb{N}^*$, $\alpha_1, \cdots, \alpha_N > 0$ avec $\sum_{i=1}^{N} \alpha_i = 1$. Nous commençons par fournir une preuve constructive de l'existence du mouvement brownien de Walsh. Nous allons le construire directement à partir d'un mouvement brownien réfléchi comme cela a été décrit par Walsh. Plus précisément, on a la proposition suivante:

Proposition 1.1. *Soient* $\mathbb{D}_n = \{\frac{k}{2^n}, n \geq 0,\ k \in \mathbb{N}\}$ *et* $\mathbb{D} = \cup_{n\in\mathbb{N}}\mathbb{D}_n$*. Pour* $0 \leq u < v$*, définissons*

$$n(u, v) = \inf\{n \in \mathbb{N} : \mathbb{D}_n \cap]u, v[\neq \emptyset\}, \quad f(u, v) = \inf \mathbb{D}_{n(u,v)} \cap]u, v[.$$

Soient B *un mouvement brownien standard sur un espace de probabilité* $(\Omega, \mathcal{A}, \mathbb{P})$ *et*

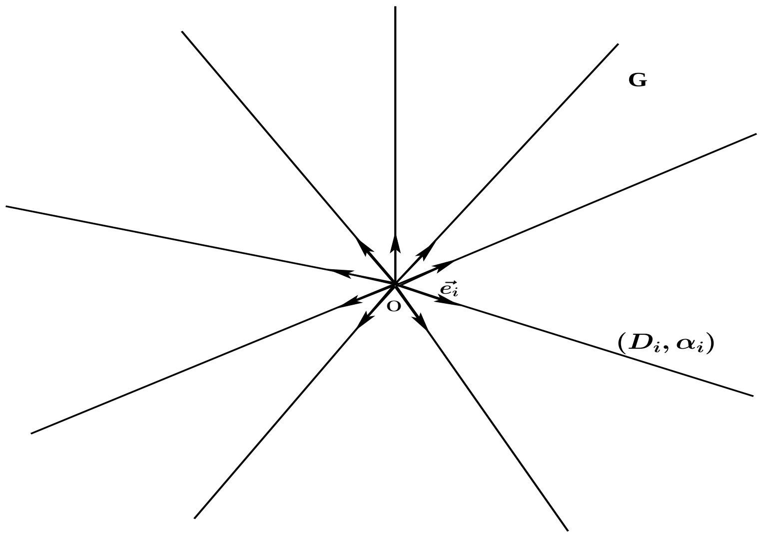

Figure 1.1: Graphe G

$(\vec{\gamma}_r, r \in \mathbb{D})$ *une suite i.i.d. de v.a. de loi* $\sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$, *qui est indépendante aussi de* B. *Définissons*

$$B_t^+ = B_t - \min_{u \in [0,t]} B_u,\ \ g_t = \sup\{r \le t : B_r^+ = 0\},\ \ d_t = \inf\{r \ge t : B_r^+ = 0\},$$

Enfin, posons

$$Z_t = \vec{\gamma}_r B_t^+, r = f(g_t, d_t)\ \ si\ B_t^+ > 0, \quad Z_t = 0 \quad si \quad B_t^+ = 0.$$

Alors

$(Z_t, t \ge 0)$ *est un* $W(\alpha_1, \cdots, \alpha_N)$ *processus sur* G *issu de* 0.

Ceci nous amènera à prouver le théorème de Donsker suivant, généralisant celui de [15] traitant le cas $\alpha_1 = \cdots = \alpha_N = \frac{1}{N}$, ainsi que le théorème de Donsker pour le brownien biaisé ([12],[29],[22], [38]).

Proposition 1.2. *Soit* $M = (M_n)_{n \ge 0}$ *une chaine de Markov sur* G *issue de* 0 *et dont les lois de transitions sont décrites par la matrice* Q *suivante:*

$$Q(0, \vec{e}_i) = \alpha_i, \quad Q(n\vec{e}_i, (n \pm 1)\vec{e}_i) = \frac{1}{2} \quad \forall i \in [1, N], \quad n \in \mathbb{N}^*. \tag{1.1}$$

Soient $t \longmapsto M(t)$ *l'interpolation linéaire de* $(M_n)_{n\geq 0}$ *et* $M_t^n = \frac{1}{\sqrt{n}}M(nt), n \geq 1$. *Alors*

$$(M_t^n)_{t\geq 0} \xrightarrow[n\to+\infty]{loi} (Z_t)_{t\geq 0}$$

dans $C([0,+\infty[,G)$ *où* Z *est un* $W(\alpha_1,\cdots,\alpha_N)$ *processus issu de* 0.

Pour prouver ce résultat nous plongeons une chaine de Markov ayant la même loi que M dans la trajectoire d'un processus de Walsh Z. Nous associons à Z N mouvements browniens biaisés $(Z^i)_{1\leq i\leq N}$ qui engendrent la filtration de Z et qui sont définis par

$$Z_t^i = |Z_t|1_{\{Z_t\in D_i\}} - |Z_t|1_{\{Z_t\notin D_i\}},\ i \in [1,N].$$

L'inégalité simple suivante

$$d(Z_t,Z_s) \leq \sum_{i=1}^N |Z_t^i - Z_s^i|,$$

où d est la distance du plus court chemin sur G permet toujours de définir une modification continue du processus Z. Ceci donne un sens à la convergence énoncée dans la Proposition 1.2 et permet d'en déduire le cas général à partir du théorème de Donsker connu pour le brownien biaisé.

Soit Z un $W(\alpha_1,\cdots,\alpha_N)$ processus sur G et associons à Z le mouvement brownien

$$B_t = |Z_t| - \tilde{L}_t(|Z|) - |z| \text{ où } \tilde{L}_t(|Z|) = \lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_0^t 1_{\{|Z_u|\leq\varepsilon\}}du.$$

Barlow, Pitman et Yor montrent que toute $(\mathcal{F}_t^Z)$ martingale locale est de la forme

$$M_t = \int_0^t H_s dB_s$$

pour un certain processus H, $(\mathcal{F}_t^Z)$ prévisible. Le mouvement brownien B sera d'un intérêt particulier dans ce manuscrit grâce à la formule d'Itô suivante dûe à Freidlin-Sheu [17]. Avant d'énoncer cette formule, nous introduisons les notations suivantes: soit $G^* = G \setminus \{0\}$ et désignons par $C_b^2(G^*)$ l'espace des applications $f : G \longrightarrow \mathbb{R}$ telles que

(i) f est continue sur G.

(ii) f est deux fois dérivable sur G^* et admet deux dérivées, f' et f'' bornées sur G^* (ici $f'(z)$ est la dérivée de f en z suivant la direction $\vec{e_i}$ pour tout $z \in D_i \setminus \{0\}$).

(iii) $\lim_{z\to 0, z\in D_i, z\neq 0} f'(z)$ et $\lim_{z\to 0, z\in D_i, z\neq 0} f''(z)$ existent pour tout $i \in [1, N]$.

Théorème 1.1. *[17] Soit $(Z_t)_{t\geq 0}$ un $W(\alpha_1, \cdots, \alpha_N)$ processus sur G issu de z. Alors:*

(i) $(|Z|_t)_{t\geq 0}$ est un mouvement brownien réfléchi issu de $|z|$.

(ii) $B_t = |Z_t| - \tilde{L}_t(|Z|) - |z|$ est un mouvement brownien où

$$\tilde{L}_t(|Z|) = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_0^t 1_{\{|Z_u|\leq \varepsilon\}} du.$$

(iii) $\forall f \in C_b^2(G^*)$:

$$f(Z_t) = f(z) + \int_0^t f'(Z_s)dB_s + \frac{1}{2}\int_0^t f''(Z_s)ds + (\sum_{i=1}^N \alpha_i \lim_{z\to 0, z\in D_i, z\neq 0} f'(z))\tilde{L}_t(X).$$

Nous donnons une preuve simple de cette formule en utilisant de nouveau les processus $(Z^i)_{1\leq i\leq N}$ et l'approximation du temps local par le nombre de montées. Noter que la partie martingale locale de $f(Z_t)$ est une intégrale de B ce qui correspond à la propriété de représentation prévisible établie dans [4]. Lorsque $z = 0$, une application du lemme de Skorokhod ([45] page 239) montre que

$$|Z_t| = B_t^+ \ (:= B_t - \inf_{u\in[0,t]} B_u).$$

Partant de B, on peut ainsi d'après la Proposition 1.1 construire un mouvement brownien de Walsh Z issu de 0 tel que

$$df(Z_t) = f'(Z_t)dB_t + \frac{1}{2}f''(Z_t)dt, f \in D(\alpha_1, \cdots, \alpha_N)$$

où

$$D(\alpha_1, \cdots, \alpha_N) = \{f \in C_b^2(G^*) : \sum_{i=1}^N \alpha_i \lim_{z\to 0, z\in D_i, z\neq 0} f'(z) = 0\}.$$

Lorsque $N = 2, \alpha_1 = \alpha_2 = \frac{1}{2}$, la projection de Z sur $\mathbb{R}$:

$$X_t = |Z_t|1_{\{Z_t\in D_1\}} - |Z_t|1_{\{Z_t\in D_2\}}$$

satisfait l'équation de Tanaka

$$dX_t = \mathrm{sgn}(X_t)dB_t.$$

La proposition suivante montre que le domaine $D(\alpha_1, \cdots, \alpha_N)$ est suffisamment riche pour caractériser le semigroupe du mouvement de Walsh.

Proposition 1.3. *Soit $Q = (Q_t)_{t\geq 0}$ un semigroupe de Feller tel que:*

$$Q_t f(x) = f(x) + \frac{1}{2}\int_0^t Q_u f''(x)du \quad \forall f \in D(\alpha_1, \cdots, \alpha_N).$$

Alors, Q est le semigroupe de $W(\alpha_1, \cdots, \alpha_N)$ processus.

Ce travail est loin de la théorie des filtrations, mais signalons finalement que pour $N \geq 3$, il n'existe aucun mouvement brownien W tel que $\mathcal{F}_t^Z = \mathcal{F}_t^W$ [49].

1.2 Flots stochastiques et équation de Tanaka

1.2.1 Flots stochastiques

Considérons une équation différentielle ordinaire sur $\mathbb{R}^d$

$$\frac{dx}{dt} = f(x,t). \tag{1.2}$$

où $f(x,t)$ est continue en (x,t) et est lipschitzienne en x. Notons $\varphi_{s,t}(x)$ la solution de (1.2) vérifiant la condition initiale $x(s) = x$. Il est bien connu que $\varphi_{s,t}(x)$ satisfait les propriétés suivantes:

(a) $\varphi_{s,t}(x)$ est continue en (s,t,x).

(b) $\varphi_{s,u}(x) = \varphi_{t,u} \circ \varphi_{s,t}(x)$ pour tout $s \leq t \leq u$, $x \in \mathbb{R}^d$.

(c) $\varphi_{s,s} = \mathrm{Id}$ pour tout s.

(d) $\varphi_{s,t}$ est un homéomorphisme de $\mathbb{R}^d$ dans $\mathbb{R}^d$ pour tout $s \leq t$.

$\varphi_{s,t}$ est appelé flot déterministe d'homéomorphismes sur $\mathbb{R}^d$. À l'origine, un flot stochastique d'homéomorphismes sur un espace de probabilité $(\Omega, \mathcal{A}, \mathbb{P})$ est la donnée d'une famille $(\varphi_{s,t}(x, \omega))_{\omega \in \Omega}$ d'homéomorphismes déterministes à acroissements indépendants: pour tout $t_1 < t_2 < \cdots < t_n$, la famille $\{\varphi_{t_i, t_{i+1}}, 1 \leq i \leq n-1\}$ est indépendante. Une classe importante des flots stochastiques d'homéomorphismes est construite en résolvant l'ÉDS

$$dx(t) = \sum_{k=1}^{r} F_k(x,t) dB_t^k + F_0(x,t)dt \tag{1.3}$$

où $F_0(x,t), F_1(x,t), \cdots, F_r(x,t)$ sont continues en (x,t) et Lipschitziennes en x et $(B^1, \cdots, B^r)$ est un mouvement brownien en dimension r (voir [14], [39], [9], [24], [28]). La loi d'un flot stochastique solution de (1.3) est décrite par la loi des mouvements à k points

$$t \mapsto (\varphi_{0,t}(x_1), \cdots, \varphi_{0,t}(x_k)) \in \mathbb{R}^{dk},$$

pour tout $k \geq 1$, $x_1, \cdots, x_k \in \mathbb{R}^d$ qui dépendent seulement du générateur à 1 point L et d'une matrice de covariance C, données par

$$Lf(x) = \lim_{t \to 0+} \frac{E[f(\varphi_{0,t}(x_k)) - f(x)]}{t}, \; x \in \mathbb{R}^d,$$

$$C^{p,q}(x,y) = \lim_{t \to 0+} \frac{E[(\varphi_{0,t}^p(x) - x^p)(\varphi_{0,t}^q(y) - y^q)]}{t}, \; x, y \in \mathbb{R}^d.$$

Ces paramètres sont déterminés par les coefficiens de (1.3) et s'appellent les caractéristiques locales du flot [37].

Un autre problème naturel consiste à construire des flots stochastiques dont les caractéristiques sont moins régulières. Il faut ainsi abandonner le cadre des flots d'homéomorphismes et des ÉDS qui n'admettent pas toujours des solutions fortes. Un premier exemple important est donné sur $\mathbb{R}$, par le flot d'Arratia [1]. Ici, deux points suivent des trajectoires browniennes indépendantes jusqu'au moment où elles se rencontrent. Alors elles se transforment en un seul mouvement brownien: on parle de flot coalescent.

Récemment, Le Jan et Raimond ont étendu ce procédé en considérant des flots stochastiques de noyaux $(K_{s,t}(x,dy))$. Un noyau aléatoire K agit sur les fonctions test f par la relation $Kf(x) = \int f(y)K(x,dy)$. Étant donné un flot de noyaux K,

$$P_t^{\otimes n} f(x) = \int f(y_1, \cdots, y_n) E[K_{0,t}(x_1, dy_1) \cdots K_{0,t}(x_n, dy_n)], n \geq 1$$

définit une famille compatible de semigroupes felleriens, c'est à dire telle que la distribution marginale de tout k composantes du mouvement à n points est celle du mouvement à k points. Réciproquement, par une extension du théorème de De Finetti, Le Jan et Raimond associent à tout système compatible de semigroupes felleriens un flot stochastique de noyaux. Lorsque

$$P_t^2 f^{\otimes 2}(x,x) = P_t f^2(x)$$

pour tout f, x, t, le flot de noyaux est un flot d'applications $K_{s,t}(x,dy) = \delta_{\varphi_{s,t}(x)}(dy)$. Dans cette thèse, nous allons considérer des ÉDS sans solutions fortes au sens usuel mais admettant des solutions fortes sous forme des mesures aléatoires. Nous commençons par donner un apperçu des résultats de Le Jan et Raimond sur l'équation de Tanaka.

1.2.2 Équation de Tanaka

Soit $(W_t)_{t\in\mathbb{R}}$ un mouvement brownien sur la droite réelle défini sur un espace de probabilité $(\Omega, \mathcal{A}, \mathbb{P})$, c'est à dire $(W_t)_{t\geq 0}$ et $(W_{-t})_{t\geq 0}$ sont deux mouvements browniens réels indépendants. Tout au long de cette thèse, nous utilisons la notation suivante

$$W_{s,t} = W_t - W_s, \quad s \leq t.$$

Considérons l'équation de Tanaka

$$\varphi_{s,t}(x) = x + \int_s^t \operatorname{sgn}(\varphi_{s,u}(x))dW_u, \quad s \leq t, x \in \mathbb{R}, \tag{1.4}$$

Ceci est l'exemple classique des ÉDS n'admettant pas de solution forte mais ayant une solution faible. Dans [51], Watanabe établit l'existence et l'unicité d'un flot d'applications solution de (1.4). Ce flot a la forme suivante

$$\varphi_{s,t}(x) = (x + \operatorname{sgn}(x)W_{s,t})1_{\{t\leq\tau_s(x)\}} + \varepsilon_{s,t}W_{s,t}^+ 1_{\{t>\tau_s(x)\}},$$

où $\tau_s(x) = \inf\{r \geq s : \ W_{s,r} = -|x|\}$ et $W^+_{s,t} = W_{s,t} - \min_{s\leq r\leq t} W_{s,r}$. De plus, il doit satisfaire: $\varepsilon_{s,t}$ est indépendante de W pour (s,t) fixés. Il est clair que la propriété de cocycle de φ, faisant intervenir s, t et u va induire certaines relations entre les processus $(\varepsilon_{s,t})$ et W. Le Jan et Raimond [36] montrent après qu'il suffit d'attacher les variables $(\varepsilon_{s,t})$ aux minimas locaux de W pour construire φ.
Si K est un flot de noyaux, on dit que K est solution de l'équation de Tanaka si: pour tout $s \leq t, x \in \mathbb{R}$, $f \in C^2_b(\mathbb{R})$ (f est C^2 et f', f'' sont bornées)

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(f'\text{sgn})(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du \quad p.s. \tag{1.5}$$

Lorsque $K = \delta_\varphi$, est un flot d'applications, (1.5) est équivalente à (1.4). Le Jan et Raimond montrent qu'il existe une seule solution forte de (1.5) obtenue en divisant la masse en deux à l'origine

$$K^W_{s,t}(x) = (x + \text{sgn}(x)W_{s,t})1_{\{t\leq\tau_s(x)\}} + \frac{1}{2}(\delta_{W^+_{s,t}} + \delta_{-W^+_{s,t}})1_{\{t>\tau_s(x)\}}.$$

L'équation (1.5) admet encore des solutions faibles K (telles que $\mathcal{F}^W_{0,t} \neq \mathcal{F}^K_{0,t}$). Ces noyaux ont la forme suivante

$$K_{s,t}(x) = \delta_{x+\text{sgn}(x)W_{s,t}}1_{\{t\leq\tau_s(x)\}} + (U_{s,t}\delta_{W^+_{s,t}} + (1-U_{s,t})\delta_{-W^+_{s,t}})1_{\{t>\tau_s(x)\}}.$$

où $U_{s,t}$ est indépendante de W et de loi m ne dépendant pas de $t-s$. La classification donnée par Le Jan et Raimond est la suivante:

Théorème 1.2. *[36] (a) Toute mesure de probabilité m sur $[0,1]$ de moyenne $\frac{1}{2}$ définit un flot stochastique de noyaux K^m solution de (1.5).*

- *À $\delta_{\frac{1}{2}}$ est associée une solution Wiener K^W.*
- *À $\frac{1}{2}(\delta_0 + \delta_1)$ est associé un flot d'applications coalescent φ.*

(b) Pour tout flot de noyaux K solution de (1.5) il existe une unique mesure m de moyenne $\frac{1}{2}$ telle que $K \overset{loi}{=} K^m$.

1.3 Flots stochastiques reliés au mouvement brownien de Walsh

1.3.1 Résultats

Fixons $N \geq 1$, $\alpha_1, \cdots, \alpha_N > 0$ tels que $\displaystyle\sum_{i=1}^{N} \alpha_i = 1$. Considérons le graphe G de la figure 1.1 et associons à chaque demi-droite D_i un signe $\varepsilon_i \in \{-1, 1\}$. Puis, définissons

$$\varepsilon(x) = \varepsilon_i \text{ si } \ x \in D_i, x \neq 0 \ (= \varepsilon_N \text{ si } x = 0).$$

Cette fonction joue le rôle de la fonction sgn sur le graphe G. Pour simplifier les notations, nous supposons que $\varepsilon_1 = \cdots = \varepsilon_p = 1, \ \ \varepsilon_{p+1} = \cdots = \varepsilon_N = -1$ pour un certain $p \leq N$. Posons

$$G^+ = \bigcup_{1 \leq i \leq p} D_i, \quad G^- = \bigcup_{p+1 \leq i \leq N} D_i. \text{ Alors } \ G = G^+ \bigcup G^- \text{ (Figure 1.2).}$$

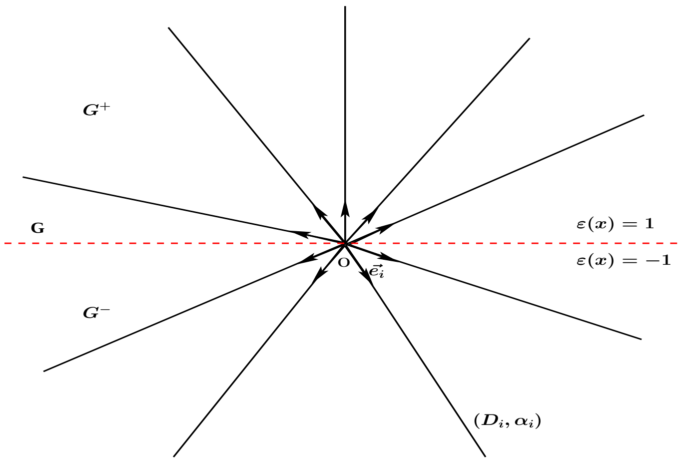

Figure 1.2: Graphe G.

Définissons aussi $\vec{e}(z) = \vec{e}_i$ si $z \in D_i, z \neq 0$ (convention $\vec{e}(0) = \vec{e}_N$) et $\alpha^+ = \sum_{i=1}^{p} \alpha_i$. Dans le chapitre 4, nous étudions l'équation suivante

Définition 1.1. *(Équation (E)).*
Sur un espace de probabilité $(\Omega, \mathcal{A}, \mathbb{P})$, soient W un mouvement brownien sur la droite réelle et K un flot stochastique de noyaux sur G. On dit que (K, W) est une solution de (E) si pour tout $s \leq t, f \in D(\alpha_1, \cdots, \alpha_N), x \in G$,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(\varepsilon f')(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du \quad p.s.$$

Le cas $N = 2, p = 2, \varepsilon_1 = \varepsilon_2 = 1, \alpha_1 = \alpha_2 = \frac{1}{2}$ (Figure 1.3) correspond à l'équation de Tanaka (1.5) [36].

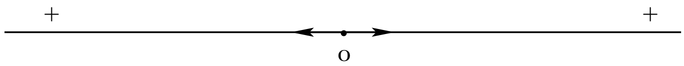

Figure 1.3: Équation de Tanaka.

Le cas $N = 2, p = 1, \varepsilon_1 = 1, \varepsilon_2 = -1$ (Figure 1.4) correspond à l'équation du mouvement brownien biaisé comme solution forte de W,

$$X_t^{s,x} = x + W_{s,t} + (2\alpha - 1)\tilde{L}_{s,t}^x, \quad t \geq s, x \in \mathbb{R}, \tag{1.6}$$

où

$$\tilde{L}_{s,t}^x = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1_{|X_u^{s,x}| \leq \varepsilon} du \quad \text{(Temps local symétrique).}$$

Dans [20], nous démontrons essentiellement les deux résultats suivants:

Théorème 1.3. *Soient W un mouvement brownien sur la droite réelle et $X_t^{s,x}$ le flot stochastique associé à (1.6) avec $\alpha = \alpha^+$. Définissons $Z_{s,t}(x) = X_t^{s,\varepsilon(x)|x|}, s \leq t, x \in G$ et*

$$\begin{aligned} K_{s,t}^W(x) &= \delta_{x+\vec{e}(x)\varepsilon(x)W_{s,t}} 1_{\{t \leq \tau_{s,x}\}} \\ &+ \Big(\sum_{i=1}^{p} \frac{\alpha_i}{\alpha^+} \delta_{\vec{e}_i|Z_{s,t}(x)|} 1_{\{Z_{s,t}(x)>0\}} + \sum_{i=p+1}^{N} \frac{\alpha_i}{\alpha^-} \delta_{\vec{e}_i|Z_{s,t}(x)|} 1_{\{Z_{s,t}(x)\leq 0\}}\Big) 1_{\{t>\tau_{s,x}\}}, \end{aligned}$$

où $\tau_{s,x} = \inf\{r \geq s : x + \vec{e}(x)\varepsilon(x)W_{s,r} = 0\}$. Alors, K^W est l'unique solution Wiener de (E). C'est à dire, K^W résout (E) et si K est une autre solution Wiener de (E), alors pour tout $s \leq t, x \in G$, $K_{s,t}^W(x) = K_{s,t}(x)$ p.s.

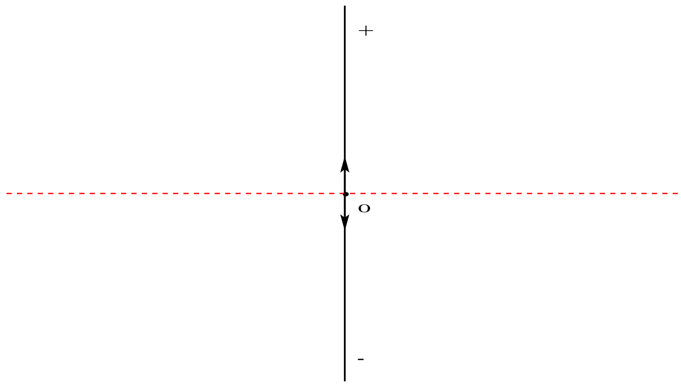

Figure 1.4: Équation du mouvement brownien biaisé.

Théorème 1.4. *(1) Soit*

$$\Delta_k = \big\{u = (u_1, \cdots, u_k) \in [0,1]^k : \sum_{i=1}^{k} u_i = 1\big\}, \quad k \geq 1.$$

Supposons que $\alpha^+ \neq \frac{1}{2}$.

(a) Soient m^+ et m^- deux mesures de probabilités respectivement sur Δ_p et Δ_{N-p} telles que:

(+) $\displaystyle\int_{\Delta_p} u_i m^+(du) = \frac{\alpha_i}{\alpha^+}, \forall 1 \leq i \leq p,$

(-) $\displaystyle\int_{\Delta_{N-p}} u_j m^-(du) = \frac{\alpha_{j+p}}{\alpha^-}, \ \forall 1 \leq j \leq N-p.$

Alors, (m^+, m^-) définit un flot de noyaux K^{m^+,m^-} solution de (E).

- *À $(\delta_{(\frac{\alpha_1}{\alpha^+}, \cdots, \frac{\alpha_p}{\alpha^+})}, \delta_{(\frac{\alpha_{p+1}}{\alpha^-}, \cdots, \frac{\alpha_N}{\alpha^-})})$ est associée une solution Wiener K^W.*
- *À $\left(\displaystyle\sum_{i=1}^{p} \frac{\alpha_i}{\alpha^+} \delta_{0,..,0,1,0,..,0}, \sum_{i=p+1}^{N} \frac{\alpha_i}{\alpha^-} \delta_{0,..,0,1,0,..,0}\right)$ est associé un flot d'applications coalescent φ.*

(b) Pour tout flot de noyaux K solution de (E), il existe un unique couple de mesures (m^+, m^-) satisfaisant les conditions $(+)$ et $(-)$ tel que $K \overset{loi}{=} K^{m^+,m^-}$.

(2) Si $\alpha^+ = \frac{1}{2}, N > 2$, alors (E) admet une seule solution (Wiener).

Donnons les grandes lignes de la preuve de ces résultats.

1.3.2 Construction

Supposons que φ est un flot d'applications solution de (E), alors $\varphi_{0,t} := \varphi_{0,\cdot}(0)$ est un mouvement brownien de Walsh et par suite: $\forall f \in C_b^2(G^*)$

$$f(\varphi_{0,t}) = f(0)+\int_0^t f'(\varphi_{0,s})dB_s+\frac{1}{2}\int_0^t f''(\varphi_{0,s})ds+\left(\sum_{i=1}^N \alpha_i \lim_{z\to 0, z\in D_i, z\neq 0} f'(z)\right)\tilde{L}_t(|\varphi_{0,\cdot}|).$$

où $B_t = |\varphi_{0,t}| - \tilde{L}_t(|\varphi_{0,\cdot}|)$. Si $f(x) = \varepsilon(x)|x|$, alors $Y_t = f(\varphi_{0,t})$ est un mouvement brownien biaisé de paramètre α^+ et par application de f dans la dernière formule, on trouve

$$Y_t = \int_0^t \varepsilon(\varphi_{0,s})dB_s + (2\alpha^+ - 1)\tilde{L}_t(Y).$$

En comparant la formule de Freidlin-Sheu avec l'équation (E), on peut espérer avoir $B_t = \int_0^t \varepsilon(\varphi_{0,s})dW_{0,s}$. En conclusion, la norme algébrique Y de $\varphi_{0,\cdot}$ vérifie l'ÉDS

$$Y_t = W_{0,t} + (2\alpha^+ - 1)\tilde{L}_t(Y). \tag{1.7}$$

Le processus Y détermine si $\varphi_{0,\cdot}$ appartient à G^+ où G^- sans indiquer la branche exacte sur laquelle se trouve $\varphi_{0,\cdot}$. Vu que $\varphi_{0,\cdot}$ est un $W(\alpha_1, \cdots, \alpha_N)$ processus, la seule construction possible de $\varphi_{0,\cdot}$ à partir de Y consiste à associer indépendamment à chaque excursion positive (resp. négative) de Y une v.a. de loi

$$\sum_{i=1}^p \frac{\alpha_i}{\alpha^+}\delta_{\vec{e}_i} \text{ (resp. } \sum_{j=p+1}^N \frac{\alpha_j}{\alpha^-}\delta_{\vec{e}_j}).$$

Pour construire tout un flot d'applications $(\varphi_{s,t}(x))$ solution de (E), nous partons alors d'un flot associé à l'ÉDS (1.6) avec $\alpha = \alpha^+$. Une version récemment étudiée par Burdzy-Kaspi [3] permet d'achever la construction. À partir de φ, on peut définir une solution Wiener de (E) en posant $K_{s,t}^W(x) = E[\delta_{\varphi_{s,t}(x)}|\sigma(W)]$. La formule explicite de $K_{s,t}^W(x)$ est donnée par le Théorème 3. En suivant [32], nous montrons alors qu'une solution Wiener de (E) est unique dans sa décomposition en chaos ce qui permet de conclure le Théorème 3. La construction de la solution K^{m^+,m^-} est similaire à celle de φ: on attache aux excursions positives (resp. négatives) du mouvement brownien biaisé des v.a. indépendantes de loi m^+ (resp. m^-). Nous construisons φ et K^{m^+,m^-} sur le même espace de probabilité de sorte que

$$K_{s,t}^{m^+,m^-}(x) = E[\delta_{\varphi_{s,t}(x)}|\sigma(K^{m^+,m^-})]$$

et $\hat{K}_{s,t}(x,y) = K_{s,t}^{m^+,m^-}(x) \otimes \delta_{\varphi_{s,t}}(y)$ est un flot de noyaux sur G^2. Selon la terminologie de Le Jan et Raimond, nous disons que φ domine faiblement K^{m^+,m^-}.

1.3.3 Unicité

Soit (K, W) une solution quelconque de (E). On cherche ainsi deux mesures m^+ et m^- telles que $K \overset{loi}{=} K^{m^+,m^-}$. Définissons

$$V_t^{+,i} = K_{0,t}(0)(D_i),\ V_t^{-,j} = K_{0,t}(0)(D_j) \text{ for all} 1 \leq i \leq p,\ p+1 \leq j \leq N,$$

$$V_t^+ = (V_t^{+,i})_{1\leq i\leq p}, V_t^- = (V_t^{-,i})_{p+1\leq i\leq N}.$$

Soit Y l'unique solution forte de (1.7). En suivant Le Jan et Raimond, nous montrons que V_t^+ est indépendant de W conditionnellement à $(Y_t > 0)$ et est de loi m^+ ne dépendant pas de $t > 0$. Nous définissons d'une façon similaire la mesure m^- et on a un résultat analogue: V_t^- est indépendant de W conditionnellement à $(Y_t < 0)$ et est de loi m^- ne dépendant pas de $t > 0$. Ceci entraîne que $(K_{0,t}(0), W) \overset{loi}{=} (K_{0,t}^{m^+,m^-}(0), W')$ pour tout $t > 0$ où (K^{m^+,m^-}, W') est la solution associée à (m^+, m^-) construite avant. En partant de x quelconque, on a aussi $(K_{0,t}(x), W) \overset{loi}{=} (K_{0,t}^{m^+,m^-}(x), W')$. Pour conclure que

$$(K_{0,t}(x_1), \cdots, K_{0,t}(x_n)) \overset{loi}{=} (K_{0,t}^{m^+,m^-}(x_1), \cdots, K_{0,t}^{m^+,m^-}(x_n))$$

pour tout $(x_1, \cdots, x_n) \in G^n$, nous prouvons la proposition suivante:

Proposition 1.4. *Soit $\mathbb{P}_{t,x_1,\cdots,x_n}$ la loi de $(K_{0,t}(x_1), \cdots, K_{0,t}(x_n), W)$ où $t \geq 0$ et $x_1, \cdots, x_n \in G$. Alors, $\mathbb{P}_{t,x_1,\cdots,x_n}$ est uniquement determinée par $\{\mathbb{P}_{u,x}, u \geq 0, x \in G\}$.*

Le cas $\alpha^+ = \frac{1}{2}$. Dans ce cas on a $Y_t = W_{0,t}$. Nous montrons que toutes les solutions de (E) ont le même mouvement à n points pour tout $n \geq 1$. Il s'en suit alors que (E) admet une unique solution, qui est le flot de Wiener.

1.4 Approche discrète à quelques flots de l'équation de Tanaka-Walsh

1.4.1 Résultats

Dans le chapitre 5, nous considérons l'équation (E) dans le cas $\varepsilon = 1$. Notre nouvelle équation est alors la suivante:

Définition 1.2. *(Équation (T)).*
On considère le graphe G de la figure 1.1. Soient W un mouvement brownien sur la droite réelle et K un flot de noyaux définis sur un espace de probabilité $(\Omega, \mathcal{A}, \mathbb{P})$. On dit que (K, W) est une solution de (T) si pour tout $s \leq t, f \in D(\alpha_1, \cdots, \alpha_N), x \in G$,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}f'(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du \quad p.s.$$

D'après le Théorème 3, l'unique solution Wiener de (T) est simplement

$$K_{s,t}^W(x) = \delta_{x+\vec{e}(x)W_{s,t}} 1_{\{t \leq \tau_{s,x}\}} + \sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i W_{s,t}^+} 1_{\{t > \tau_{s,x}\}}.$$

où

$$\tau_{s,x} = \inf\{r \geq s : x + \vec{e}(x)W_{s,r} = 0\}.$$

Cependant, l'expression de l'unique flot d'applications solution de (T) est plus compliquée. Nous nous proposons ici de construire ce flot en partant d'un jeu de pile ou face. Soient $G_{\mathbb{N}} = \{x \in G; |x| \in \mathbb{N}\}$ et $\mathcal{P}(G)$ (resp. $\mathcal{P}(G_{\mathbb{N}})$) l'espace des probabilités sur G (resp. $G_{\mathbb{N}}$). D'abord, nous introduisons les flots discrets comme suit

Définition 1.3. *(Flots discrets) On dit qu'un processus $\psi_{p,q}(x)$ (resp. $N_{p,q}(x)$) indexé par $\{p \leq q \in \mathbb{Z}, x \in G_{\mathbb{N}}\}$ et à valeurs dans $G_{\mathbb{N}}$ (resp. $\mathcal{P}(G_{\mathbb{N}})$) est un flot discret* *d'applications (resp. de noyaux) sur $G_{\mathbb{N}}$ si:*

(i) La famille $\{\psi_{i,i+1}; i \in \mathbb{Z}\}$ (resp. $\{N_{i,i+1}; i \in \mathbb{Z}\}$) est indépendante.

(ii)$\forall p \in \mathbb{Z}, x \in G_{\mathbb{N}}, \psi_{p,p+2}(x) = \psi_{p+1,p+2}(\psi_{p,p+1}(x))$ (resp. $N_{p,p+2}(x) = N_{p,p+1}N_{p+1,p+2}(x)$) p.s. où

$$N_{p,p+1}N_{p+1,p+2}(x, A) := \sum_{y \in G_{\mathbb{N}}} N_{p+1,p+2}(y, A) N_{p,p+1}(x, \{y\}) \textit{ pour tout } A \subset G_{\mathbb{N}}.$$

On dit que $S = (S_n)_{n \in \mathbb{Z}}$ est une marche aléatoire simple sur $\mathbb{Z}$ lorsque $(S_n)_{n \in \mathbb{N}}$ et $(S_{-n})_{n \in \mathbb{N}}$ sont deux marches aléatoires indépendantes sur $\mathbb{Z}$. Soient S une marche aléatoire simple sur $\mathbb{Z}$ et $(\vec{\eta}_i)_{i \in \mathbb{Z}}$ une suite i.i.d. de v.a. de loi $\sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$ qui est indépendante aussi de S. Soient

$$S_{p,n} = S_n - S_p, \; S^+_{p,n} = S_n - \min_{h \in [p,n]} S_h.$$

Pour $p \in \mathbb{Z}, x \in G_{\mathbb{N}}$, définissons

$$\Psi_{p,p+1}(x) = x + \vec{e}(x) S_{p,p+1} \text{ si } \; x \neq 0, \Psi_{p,p+1}(0) = \vec{\eta}_p S^+_{p,p+1}.$$

$$K_{p,p+1}(x) = \delta_{x + \vec{e}(x) S_{p,p+1}} \text{ si } \; x \neq 0, K_{p,p+1}(0) = \sum_{i=1}^{N} \alpha_i \delta_{S^+_{p,p+1} \vec{e}_i}.$$

On étend cette définition pour tout $p \le n \in \mathbb{Z}$ en posant

$$\Psi_{p,n}(x) = x 1_{\{p=n\}} + \Psi_{n-1,n} \circ \Psi_{n-2,n-1} \circ \cdots \circ \Psi_{p,p+1}(x) 1_{\{p>n\}},$$

$$K_{p,n}(x) = \delta_x 1_{\{p=n\}} + K_{p,p+1} \cdots K_{n-2,n-1} K_{n-1,n}(x) 1_{\{p>n\}}.$$

On munit $\mathcal{P}(G)$ de la topologie de la convergence faible suivante:

$$\beta(P, Q) = \sup \left\{ |\int g dP - \int g dQ|, \|g\|_\infty + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|} \le 1, g(0) = 0 \right\}.$$

Nous construisons (φ, K^W), partant de (Ψ, K) et montrons en particulier le résultat suivant:

Théorème 1.5. *(1) Ψ (resp. K) est un flot discret d'applications (resp. noyaux) sur $G_{\mathbb{N}}$.*

(2) Il existe une réalisation (ψ, N, φ, K^W) *sur un espace de probabilité commun* $(\Omega, \mathcal{A}, \mathbb{P})$ *telle que*

(i) $(\psi, N) \overset{loi}{=} (\Psi, K)$.

(ii) (φ, W) *(resp.* (K^W, W)*) est l'unique flot d'applications (resp. Wiener) solution de (T).*

(iii) Pour tout $s \in \mathbb{R}, T > 0, x \in G, x_n \in \frac{1}{\sqrt{n}}G_{\mathbb{N}}$ *telle que* $\lim_{n\to\infty} x_n = x$*, on a*

$$\lim_{n\to\infty} \sup_{s\le t\le s+T} |\frac{1}{\sqrt{n}}\psi_{\lfloor ns\rfloor,\lfloor nt\rfloor}(\sqrt{n}x_n) - \varphi_{s,t}(x)| = 0 \quad p.s. \tag{1.8}$$

et

$$\lim_{n\to\infty} \sup_{s\le t\le s+T} \beta(K_{\lfloor ns\rfloor,\lfloor nt\rfloor}(\sqrt{n}x_n)(\sqrt{n}.), K^W_{s,t}(x)) = 0 \quad p.s.$$

Ce théorème entraine le corollaire suivant:

Corollaire 1.1. *Pour tout* $s \in \mathbb{R}, x \in G_{\mathbb{N}}$*, soient* $t \longmapsto \Psi(t)$ *l'interpolation linéaire de* $\big(\Psi_{\lfloor ns\rfloor,k}(x), k \ge \lfloor ns\rfloor\big)$ *et* $\Psi^n_{s,t}(x) := \frac{1}{\sqrt{n}}\Psi(nt)$, $K^n_{s,t}(x) = E[\delta_{\Psi^n_{s,t}(x)}|\sigma(S)], t \ge s, n \ge 1$*. Pour tout* $1 \le p \le q$, $(x_i)_{1\le i\le q} \subset G$*, soit* $x^n_i \in \frac{1}{\sqrt{n}}G_{\mathbb{N}}$ *tel que* $\lim_{n\to\infty} x^n_i = x_i$*. Définissons*

$$Y^n = \Big(\Psi^n_{s_1,\cdot}(\sqrt{n}x^n_1), \cdots, \Psi^n_{s_p,\cdot}(\sqrt{n}x^n_p), K^n_{s_{p+1},\cdot}(\sqrt{n}x^n_{p+1}), \cdots, K^n_{s_q,\cdot}(\sqrt{n}x^n_q)\Big).$$

Alors

$$Y^n \xrightarrow[n\to+\infty]{law} Y \quad dans \quad \prod_{i=1}^{p} C([s_i, +\infty[, G) \times \prod_{j=p+1}^{q} C([s_j, +\infty[, \mathcal{P}(G))$$

où

$$Y = \Big(\varphi_{s_1,\cdot}(x_1), \cdots, \varphi_{s_p,\cdot}(x_p), K^W_{s_{p+1},\cdot}(x_{p+1}), \cdots, K^W_{s_q,\cdot}(x_q)\Big).$$

1.4.2 Étapes de la preuve

Notre preuve du Théorème 1.5 est basée sur le résultat suivant de Csaki et Vincze

Théorème 1.6. *([44] page 109) Soit* $S = (S_n)_{n\ge 0}$ *une marche aléatoire simple sur* $\mathbb{Z}$*. Alors, il existe une marche aléatoire simple sur* $\mathbb{Z}$, $\overline{S} = (\overline{S}_n)_{n\ge0}$ *telle que:*

$$\overline{Y}_n := \max_{k\le n}\overline{S}_k - \overline{S}_n \Rightarrow |\overline{Y}_n - |S_n|| \le 2 \quad \forall n \in \mathbb{N}.$$

Ce théorème décrit l'équation de Tanaka en temps discret. Nous montrons la conséquence suivante: soient S une marche aléatoire simple sur $\mathbb{Z}$ et ε une v.a. de Bernoulli indépendante de S (juste une !). Alors, il existe une marche aléatoire simple sur $\mathbb{Z}$, notée M, telle que

$$\sigma(M) = \sigma(\varepsilon, S)$$

et

$$(\frac{1}{\sqrt{n}}S(nt), \frac{1}{\sqrt{n}}M(nt))_{t\geq 0} \xrightarrow[n\to+\infty]{\text{loi}} (B_t, W_t)_{t\geq 0} \ \text{ dans } C([0,\infty[, \mathbb{R}^2).$$

où $t \longmapsto S(t)$ (resp. $M(t)$) est l'interpolation linéaire de S (resp. M) et B, W sont deux mouvements browniens satisfaisants

$$dW_t = \text{sgn}(W_t)dB_t.$$

Soient $S = (S_n)_{n\geq 0}$ une marche aléatoire simple sur $\mathbb{Z}$ et $Y_n := \max_{k\leq n} S_k - S_n$. Pour $0 \leq p < q$, on dit que $E = [p, q]$ est un intervalle d'excursion pour Y si les propriétés suivantes sont satisfaites (avec la convention $Y_{-1} = 0$):

- $Y_p = Y_{p-1} = Y_q = Y_{q+1} = 0$.
- $\forall\ p \leq j < q, Y_j = 0 \Rightarrow Y_{j+1} = 1$.

Si $E = [p, q]$ est un intervalle d'excursion pour Y, on définit $e(E) := p, \quad f(E) := q$. Soit $(E_i)_{i\geq 1}$ l'ensemble aléatoire de tous les intervalles d'excursion de Y ordonnés tels que: $e(E_i) < e(E_j)\ \forall i < j$. Nous apellons E_i la i-ième excursion de Y.

Proposition 1.5. *Sur un espace de probabilité $(\Omega, \mathcal{A}, P)$, considérons les processus indépendants suivants:*

- $\vec{\eta} = (\vec{\eta}_i)_{i\geq 1}$*, une suite i.i.d. de v.a. de loi* $\sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$.
- $(\overline{S}_n)_{n\in\mathbb{N}}$ *une marche aléatoire simple sur* $\mathbb{Z}$.

Alors, sur une extension de $(\Omega, \mathcal{A}, P)$, il existe une chaine de Markov $(M_n)_{n\in\mathbb{N}}$ issue de 0 de matrice stochastique donnée par (1.1) telle que:

$$\overline{Y}_n := \max_{k\leq n} \overline{S}_k - \overline{S}_n \Rightarrow |M_n - \vec{\eta}_i \overline{Y}_n| \leq 2$$

sur la i-ième excursion de $\overline{Y}$.

Avec les notations de la dernière proposition, soit $(\vec{\eta}.\overline{Y})$ la chaine de Markov définie par $(\vec{\eta}.\overline{Y})_n = \vec{\eta}_i \overline{Y}_n$ sur la i-ième excursion de $\overline{Y}$ et $(\vec{\eta}.\overline{Y})_n = 0$ si $\overline{Y}_n = 0$. Nous montrons après que pour tout $p \in \mathbb{Z}$, $\Psi_{p,p+\cdot}(0) \overset{loi}{=} (\vec{\eta}.\overline{Y})$. En utilisant la dernière proposition et le théorème de Donsker pour le mouvement brownien de Walsh, on parvient à prouver que $(\Psi^{(n)}_{s,\cdot})_{s\in\mathbb{D}}$ converge en loi vers un processus $(\psi_{s,\cdot})_{s\in\mathbb{D}}$ le long d'une sous suite (voir le Corollaire 1.1 pour la définition de $\Psi^{(n)}_{s,\cdot}$). Le processus limite est indépendant de la sous suite choisie et se prolonge naturellement en un flot ψ solution de (T). Quitte à changer l'espace de probabilité, on peut supposer que $(\Psi^{(n)}_{s,\cdot})_{n\in\mathbb{N},s\in\mathbb{R}}$ et ψ sont définis sur le même espace initial et que la convergence précédente est presque sûre. Alors, dans ce cas (1.8) est vérifiée. Les convergences des flots de noyaux sont plus simples et se déduisent immédiatement.

1.5 Équation de Tanaka sur le cercle

Par analogie avec l'équation de Tanaka associée au mouvement brownien de Walsh, nous définissons dans ce chapitre une équation de Tanaka sur un autre graphe simple qui est le cercle.

Considérons le cercle unité $\mathscr{C}$ de la figure 1.5. On dit qu'une fonction f définie sur $\mathscr{C}$ est dérivable en $z_0 \in \mathscr{C}$ si

$$f'(z_0) := \lim_{h\to 0} \frac{f(z_0 e^{ih}) - f(z_0)}{h}$$

existe. Soit $C^2(\mathscr{C})$ l'espace des fonctions f définies sur $\mathscr{C}$ telles que f' et f'' existent et sont continues sur $\mathscr{C}$.

Définition 1.4. *Fixons $l \in]0,\pi]$ et définissons*

$$\epsilon(z) := 1_{\{arg(z)\in[0,l]\}} - 1_{\{arg(z)\in]l,2\pi[\}}, z \in \mathscr{C}.$$

Soient W un mouvement brownien sur la droite réelle et K un flot stochastique de noyaux sur $\mathscr{C}$ définis sur $(\Omega, \mathcal{A}, \mathbb{P})$. On dit que (K, W) est une solution de l'équation

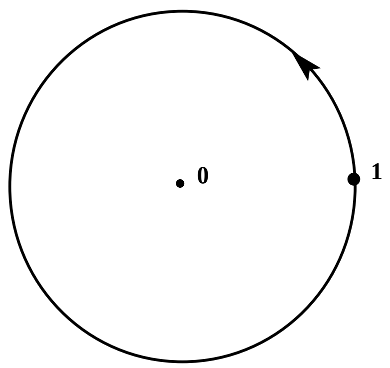

Figure 1.5: Le cerle $\mathscr{C}$.

de Tanaka sur $\mathscr{C}$*, notée* $(T_{\mathscr{C}})$ *si pour tout* $s \leq t, f \in C^2(\mathscr{C}), x \in \mathscr{C}$,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(\epsilon f')(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du \quad p.s.$$

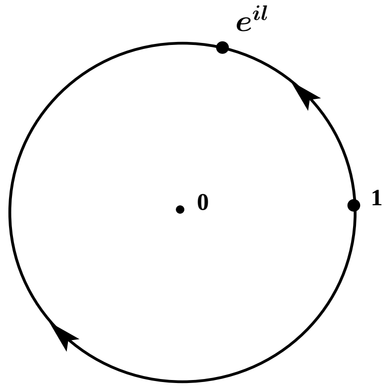

Figure 1.6: Équation de Tanaka sur $\mathscr{C}$.

1.5.1 Classification

Notre premier résultat concernant $(T_{\mathscr{C}})$ donne une classification complète de toutes les solutions:

Théorème 1.7. *Soient m^+ et m^- deux mesures de probabilités sur $[0,1]$ de moyennes $\frac{1}{2}$. Alors*

(1) (m^+, m^-) définit un flot stochastique de noyaux K^{m^+,m^-} solution de $(T_{\mathscr{C}})$.

- *À $m^+ = m^- = \delta_{\frac{1}{2}}$ est associée une solution Wiener K^W.*
- *À $m^+ = m^- = \frac{1}{2}(\delta_0 + \delta_1)$ est associé un flot d'applications φ.*

(2) Pour tout flot de noyaux K solution de $(T_{\mathscr{C}})$, il existe un unique couple de mesures (m^+, m^-) de moyennes $\frac{1}{2}$ tel que $K \overset{loi}{=} K^{m^+,m^-}$.

Pour prouver ce théorème, on va s'inspirer du modèle discret associé: l'expression d'une solution quelconque de $(T_{\mathscr{C}})$ est facile à déterminer sur des petits intervalles de temps connaissant les flots de Tanaka de Le Jan et Raimond. Soit K une solution de l'équation de Tanaka (Figure 1.3) et désignons par $\tilde{K}_{s,t}(1)$ la mesure image de $K_{s,t}(0)$ par l'application $x \mapsto e^{ix}$. Alors $\tilde{K}_{s,\cdot}(1)$ vérifie $(T_{\mathscr{C}})$ sur un petit intervalle de temps. Si K est une solution de (1.5) en remplaçant W par $-W$ et $\tilde{K}_{s,t}(e^{il})$ est la mesure image de $K_{s,t}(0)$ par l'application $x \mapsto e^{ix}$, alors $\tilde{K}_{s,\cdot}(e^{il})$ vérifie $(T_{\mathscr{C}})$ sur un petit intervalle de temps aussi. Ajoutons que toute solution K de $(T_{\mathscr{C}})$ vérifie

$$K_{s,t}(x) = \delta_{xe^{i\epsilon(x)W_{s,t}}}, \text{ si } \ s \le t \le \tau_s(x)$$

où $\tau_s(x) = \inf\{r \ge s, \ xe^{i\epsilon(x)W_{s,r}} = 1 \text{ ou } e^{il}\}$. Il s'agit maintenant de composer soigneusement ces noyaux pour construire une solution. Pour prouver la partie 2 du Théorème 1.7, nous montrons l'unicité des solutions sur des petits intervalles de temps ce qui est suffisant d'après la propriété du flot.

1.5.2 Coalescence

Si K est une solution de $(T_{\mathscr{C}})$, alors d'après le Théorème 1.7, $K \overset{loi}{=} K^{m^+,m^-}$ pour un certain couple de mesures (m^+, m^-). Nous allons construire (φ, K^{m^+,m^-}) sur le même espace de probabilité tel que la proposition suivante est vérifiée:

Proposition 1.6. *(1) Il existe une suite croissante* $(T_k)_{k\geq 1}$ *de* $(\mathcal{F}^W_{0,t})_{t\geq 0}$*-temps d'arrêts avec* $\lim_{k\to\infty} T_k = +\infty$ *p.s. et telle que p.s.* $\varphi_{0,T_k}(\mathscr{C}) = e^{il}, K^{m^+,m^-}_{0,T_k}(\mathscr{C}) = \delta_{e^{il}}$ *pour tout* $k \geq 1$.
(2) Il existe une suite croissante $(S_k)_{k\geq 1}$ *de* $(\mathcal{F}^W_{0,t})_{t\geq 0}$*-temps d'arrêts avec* $\lim_{k\to\infty} S_k = +\infty$ *p.s. et telle que p.s.* $\varphi_{0,S_k}(\mathscr{C}) = 1, K^{m^+,m^-}_{0,S_k}(\mathscr{C}) = \delta_1$ *pour tout* $k \geq 1$.

Cette proposition signifie en particulier que les flots sont coalescents.

Chapter 2

Skew and Walsh Brownian motions.

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2.1 Introduction

In this chapter we first summarize various equivalent ways to construct SBM. In Section 2.3 we introduce Walsh Brownian motion. We begin by providing a constructive proof of the existence of Walsh Brownian motion. We will construct it directly from the sample paths of a reflecting Brownian motion and a sequence of i.i.d. vectors with common distribution. Then after verifying that the constructed process has continuous paths, satisfies the simple Markov property, and has the semigroup $(P_t)_{t\geq 0}$ as proposed in [4], we use their result to conclude that the process is a Feller diffusion satisfying the definition of Walsh Brownian motion. Section 2.3 contains also a proof of Donsker theorem for Walsh Brownian motion as limit of a particular Markov chain. Our result extends that of [15] who treated the case $\alpha_1 = \cdots = \alpha_N = \frac{1}{N}$ and of course the Donsker theorem for the SBM which may be found for example in [12], [22], [38]. The rest of Section 2.3 will be devoted to Itô's formula proved by Freidlin and Sheu in [17] for a general class of diffusion processes defined by means of their generators. Here we first establish this formula using simple arguments and then deduce the characterization of Walsh Brownian motion by means of its generator (Proposition 2.5).

2.2 Skew Brownian motion

2.2.1 Local time

Before introducing SBM, we recall the definition of the local time of a continuous semimartingale from [45]. For the most part of this manuscript we will use the symmetric local time which is defined as follows

Definition 2.1. *Let*

$$\widetilde{sgn}(x) = 1_{\{x>0\}} - 1_{\{x<0\}}.$$

If X is a continuous semimartingale, then for any real number a, there exists an increasing continuous process $\tilde{L}^a(X)$ called the symmetric local time of X in a such that,

$$|X_t - a| = |X_0 - a| + \int_0^t \widetilde{sgn}(X_s - a)dX_s + \tilde{L}_t^a(X).$$

This is Tanaka's formula. Furthermore,

$$\tilde{L}_t^a(X) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t 1_{]-\varepsilon,+\varepsilon[}(X_s - a)d\langle X\rangle_s.$$

Some basic properties:

(i) $\tilde{L}_t^0(X) = \tilde{L}_t^0(|X|)$.

(ii) The measure $d\tilde{L}_t^a(X)$ is carried by the set $\{t : X_t = a\}$.

(iii) The "non symmetric "local time $L_t^a(X)$ is defined by considering $\mathrm{sgn}(x) = 1_{\{x>0\}} - 1_{\{x\leq 0\}}$ and satisfies also (ii). We also have

$$\tilde{L}^a(X) = \frac{1}{2}(L^a(X) + L^{a-}(X)).$$

The following lemma due to Skorokhod gives in some contexts a more explicit expression for the local time

Lemma 2.1. *Let y be a real-valued continuous function on $[0,\infty[$ such that $y(0) \geq 0$. There exists a unique pair (z,a) of functions on $[0,\infty[$ such that*

(i) $z = y + a$,

(ii) z is positive,

(iii) a is increasing, continuous, vanishing at zero and the corresponding measure $da(s)$ is carried by $\{s : z(s) = 0\}$.

The function a is moreover given by $a(t) = \sup_{s\leq t}(-y(s) \vee 0)$.

Let us give an application of this result. Consider a standard Brownian motion B and set $\beta_t = \int_0^t \mathrm{sgn}(B_s)dB_s = \int_0^t \widetilde{\mathrm{sgn}}(B_s)dB_s$. Then by P.Lévy's characterization, β is also a standard Brownian motion. Tanaka's formula together with the previous lemma imply that

$$|B_t| = \beta_t + \tilde{L}_t^0(B)$$

where $\tilde{L}_t^0(B) = -\inf_{s\leq t}\beta_s$. It is known that $\beta_t^+ = \beta_t - \inf_{s\leq t}\beta_s$ and β define the same canonical filtration i.e $\mathcal{F}_t^\beta = \mathcal{F}_t^{\beta+}$ (see also Theorem 2.1 below). Hence $\mathcal{F}_t^\beta = \mathcal{F}_t^{\beta+} = \mathcal{F}_t^{|B|}$ which is strictly smaller than $\mathcal{F}_t^B$. Therefore if B and β are two Brownian motions satisfying $B_t = \int_0^t \mathrm{sgn}(B_s)d\beta_s$, then necessarily $\mathcal{F}_t^\beta$ is strictly contained in $\mathcal{F}_t^B$.

Since $\tilde{L}_t^0(\beta^+) = \tilde{L}_t^0(|B|) = \tilde{L}_t^0(B) = -\inf_{s\leq t}\beta_s$, β^+ satisfies

$$\beta_t^+ = \beta_t + \tilde{L}_t^0(\beta^+). \tag{2.1}$$

2.2.2 The associated semigroup

Let $\alpha \in [0,1]$ and p_t be the semigroup of Brownian motion. Define

$$p_t^\alpha(0,y) = 2\alpha p_t(0,y)1_{\{y>0\}} + 2(1-\alpha)p_t(0,y)1_{\{y<0\}},$$

$$\begin{aligned} p_t^\alpha(x,y) &= (p_t(x,y) + (2\alpha-1)p_t(x,-y))1_{\{x>0,y>0\}} + 2(1-\alpha)p_t(x,y)1_{\{x>0,y<0\}} \\ &+ (p_t(x,y) + (1-2\alpha)p_t(x,-y))1_{\{x<0,y<0\}} + 2\alpha p_t(x,y)1_{\{x<0,y>0\}}. \end{aligned}$$

In the special case $\alpha = \frac{1}{2}$, we recover the semigroup of Brownian motion. The case $\alpha = 1$ (respectively $\alpha = 0$) corresponds to the semigroup of the reflecting Brownian motion above 0 (respectively below 0) , $|B|$ (respectively $-|B|$) where B is a Brownian motion. More generally, we have the following

Definition 2.2. *[50] p^α is a Feller semigroup on $C_0(\mathbb{R})$. A strong Markov process X with state space $\mathbb{R}$ and semigroup p^α and such that X is càdlàg is by definition the skew Brownian motion of parameter α ($SBM(\alpha)$).*

2.2.3 The associated SDE

In [22], Harrison and Shepp connected SBM with a particular stochastic equation as follows

Theorem 2.1. *[22] Let B be a Brownian motion. The SDE*

$$X_t = x + B_t + (2\alpha - 1)\tilde{L}_t^0(X),$$

has a pathwise unique solution if and only if $\alpha \in [0,1]$. In this case the unique solution is distributed as $SBM(\alpha)$.

By (2.1), $X_t = B_t^+$ is the unique solution if $x = 0, \alpha = 1$. If $x = 0, \alpha = 0$,

$$-X_t = -B_t + \tilde{L}_t^0(X)$$

and therefore $X_t = B_t - \sup_{r \leq t} B_r$. Since $d\tilde{L}_t^0(X)$ is carried by the set $\{t : X_t = 0\}$, X_t is simply given by $x + B_t$ before it hits the origin (say at time τ_x). Let $X_t^{(x)} = X_{t+\tau_x}$ and $B_t^{(x)} = B_{t+\tau_x} - B_{\tau_x}, t \geq 0$. Then

$$X_t^{(x)} = B_t^{(x)} + (2\alpha - 1)\tilde{L}_t^0(X^{(x)}).$$

This allows to deduce the solutions in the cases $\alpha = 0, 1$.

Remark 2.1. *If X is a $SBM(\alpha)$, then $L_t^0(X) = 2\alpha\tilde{L}_t^0(X) = 2\alpha\tilde{L}_t^0(|X|)$ by identifying Tanaka's formulas for symmetric and non symmetric local time for X.*

2.2.4 Approximation by random walks

As one can expect, $SBM(\alpha)$ may be approximated by the following random walk: Let $(S_k)_{k\geq 0}$ be a random walk starting from 0 with transition probabilities

$$\mathbb{P}(S_{k+1} = i \pm 1 | S_k = i) = \frac{1}{2} \quad \text{if} \quad i \neq 0,$$

$$\mathbb{P}(S_{k+1} = 1 | S_k = 0) = 1 - \mathbb{P}(S_{k+1} = -1 | S_k = 0) = \alpha.$$

We set for all $t \geq 0$ and any integer n,

$$X_t^n = \frac{1}{n} S_{\lfloor n^2 t \rfloor} + \frac{n^2 t - \lfloor n^2 t \rfloor}{n} (S_{1+\lfloor n^2 t \rfloor} - S_{\lfloor n^2 t \rfloor}).$$

The following theorem may be found in [12], [22], [38] or in a more general context in [29].

Theorem 2.2. *The sequence $(X^n)_{n \in \mathbb{N}}$ converges in distribution in the space of continuous functions to the $SBM(\alpha)$.*

2.3 Walsh Brownian motion

2.3.1 The associated semigroup

We review the existence of Walsh Brownian motion (WBM) established in [4]. Fix $N \in \mathbb{N}^*$ and $\alpha_1, \cdots, \alpha_N > 0$ such that $\sum_{i=1}^{N} \alpha_i = 1$. In the sequel G will denote the graph below consisting of N half lines $(D_i)_{1 \leq i \leq N}$ emanating from 0 (see Figure 2.1).

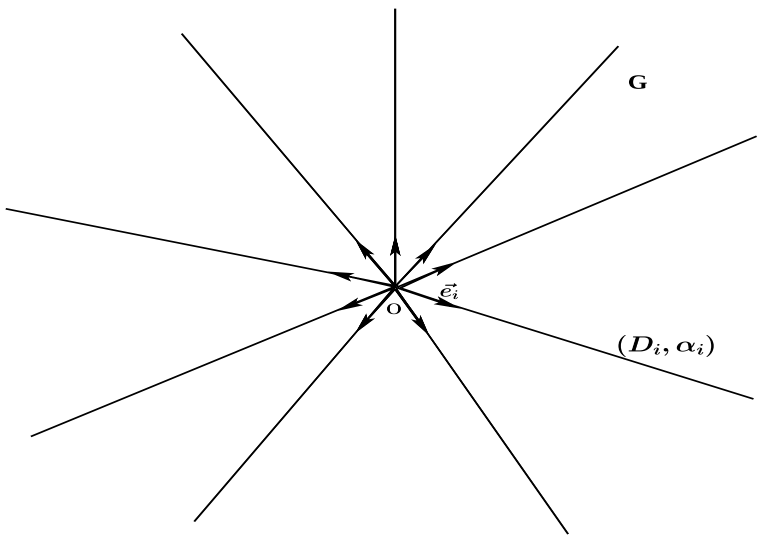

Figure 2.1: Graph G

Let $\vec{e}_i$ be a vector of modulus 1 such that $D_i = \{h\vec{e}_i, h \geqslant 0\}$ and define for all function $f : G \longrightarrow \mathbb{R}$ and $i \in [1, N]$, the mappings:

$$\begin{array}{cccc} f_i & : & \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ & & h & \longmapsto & f(h\vec{e}_i) \end{array}$$

From now on, we extend this definition on $\mathbb{R}$ by setting $f_i = 0$ on $]-\infty, 0[$. Define the following distance on G:

$$d(h\vec{e}_i, h'\vec{e}_j) = \begin{cases} h + h' & \text{if } i \neq j, (h, h') \in \mathbb{R}_+^2, \\ |h - h'| & \text{if } i = j, (h, h') \in \mathbb{R}_+^2. \end{cases}$$

For $x \in G$, we will use the simplified notation $|x| := d(x, 0)$.

We equip G with its Borel σ-field $\mathcal{B}(G)$. On $C_0(G)$, consider:

$$P_t f(h\vec{e}_j) = 2\sum_{i=1}^{N} \alpha_i p_t f_i(-h) + p_t f_j(h) - p_t f_j(-h), h > 0, \quad P_t f(0) = 2\sum_{i=1}^{N} \alpha_i p_t f_i(0).$$

where $(p_t)_{t>0}$ is the heat kernel of standard Brownian motion.

Definition 2.3. *[4] $(P_t)_{t\geq 0}$ is a Feller semigroup on $C_0(G)$. A strong Markov process Z with state space G and semigroup P_t, and such that Z is càdlàg is by definition $W(\alpha_1, \cdots, \alpha_N)$, the WBM on G.*

2.3.2 Construction by flipping Brownian excursions

For all $n \geq 0$, let $\mathbb{D}_n = \{\frac{k}{2^n},\ k \in \mathbb{N}\}$ and $\mathbb{D} = \cup_{n\in\mathbb{N}} \mathbb{D}_n$. For $0 \leq u < v$, define

$$n(u, v) = \inf\{n \in \mathbb{N} : \mathbb{D}_n \cap]u, v[\neq \emptyset\}, \quad f(u, v) = \inf \mathbb{D}_{n(u,v)} \cap]u, v[.$$

Let B be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\vec{\gamma}_r, r \in \mathbb{D})$ be a sequence of independent random variables with the same law $\sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$ which is also independent of B. We define

$$B_t^+ = B_t - \min_{u\in[0,t]} B_u,\ g_t = \sup\{r \leq t : B_r^+ = 0\},\ d_t = \inf\{r \geq t : B_r^+ = 0\},$$

and finally

$$Z_t = \vec{\gamma}_r B_t^+, r = f(g_t, d_t) \text{ if } B_t^+ > 0, \quad Z_t = 0 \text{ if } B_t^+ = 0.$$

Then we have the following

Proposition 2.1. *$(Z_t, t \geq 0)$ is a $W(\alpha_1, \cdots, \alpha_N)$ process on G started at 0.*

Proof. We use the notations

$$min_{s,t} = \min_{u \in [s,t]} B_u, \ \vec{e}_{0,t} = \vec{e}(Z_t), \mathcal{F}_s = \sigma(\vec{e}_{0,u}, B_u; 0 \leq u \leq s).$$

We now fix $0 \leq s < t$ and denote by $E_{s,t} = \{min_{0,s} = min_{0,t}\}(= \{g_t \leq s\} \ a.s)$. Let $h : G \longrightarrow \mathbb{R}$ be a bounded measurable function. Then

$$E[h(Z_t)|\mathcal{F}_s] = E[h(Z_t)1_{E_{s,t}}|\mathcal{F}_s] + E[h(Z_t)1_{E_{s,t}^c}|\mathcal{F}_s].$$

We first show that

$$E[h(Z_t)1_{E_{s,t}^c}|\mathcal{F}_s] = \sum_{i=1}^{N} E[h_i(B_t^+)1_{\{g_t > s, \vec{e}_{0,t} = \vec{e}_i\}}|\mathcal{F}_s] = \sum_{i=1}^{N} \alpha_i E[h_i(B_t^+)1_{\{g_t > s\}}|\mathcal{F}_s].$$

Note that, for all $r_1 \leq \cdots \leq r_p < r_{p+1} \in \mathbb{D}$, $\sigma(B, \vec{\gamma}_{r_1}, \cdots, \vec{\gamma}_{r_p})$ is independent of $\vec{\gamma}_{r_{p+1}}$. Fix $s_1 < \cdots < s_p \leq s$ and define $R_i = f(g_{s_i}, d_{s_i})$ $(i \in [1, p])$, $R_{p+1} = f(g_t, d_t)$. Then on $(g_t > s)$, we have

$$(\vec{e}_{0,s_1}, \cdots, \vec{e}_{0,s_p}, \vec{e}_{0,t}) = (\vec{\gamma}_{R_1}, \cdots, \vec{\gamma}_{R_p}, \vec{\gamma}_{R_{p+1}}) \text{ and } R_1 \leq \cdots \leq R_p < R_{p+1} \ a.s.$$

By summing over all possible cases, this shows that $\sigma(B, \vec{\gamma}_{R_1}, \cdots, \vec{\gamma}_{R_p})$ is independent of $\vec{\gamma}_{R_{p+1}}$ conditionally to $(g_t > s)$ which clearly proves our claim.
If $B_{s,r} = B_r - B_s$, then the density of $(\min_{r \in [s,t]} B_{s,r}, B_{s,t})$ with respect to the Lebesgue measure is given by:

$$g(x, y) = \frac{2}{\sqrt{2\pi(t-s)^3}}(-2x + y) \exp(\frac{-(-2x + y)^2}{2(t-s)})1_{\{y > x, x < 0\}} \text{ (see [31] page 28).}$$

Now, we show that $(B_{s,r}, r \geq s)$ is independent of $\mathcal{F}_s$. To this end, it will be convenient to set

$$\vec{\gamma}_u = \partial \notin \{\vec{e}_i, i \in [1, N]\} \text{ if } u \in \mathbb{R}_+ \setminus \mathbb{D}.$$

Define $\mathcal{F} = \sigma(\vec{\gamma}_{u\wedge g_s}, u \in \mathbb{D})$. Then for all $i_1, \cdots, i_p \in [1, N], u_1 \leq \cdots \leq u_p$, we have

$$(\vec{\gamma}_{u_1\wedge g_s} = \vec{e}_{i_1}, \cdots, \vec{\gamma}_{u_p\wedge g_s} = \vec{e}_{i_p}) = (\vec{\gamma}_{u_1} = \vec{e}_{i_1}, \cdots, \vec{\gamma}_{u_p} = \vec{e}_{i_p}) \cap (u_p < g_s) \ a.s.$$

By remarking that $(u_p < g_s) \in \sigma(B_r, r \leq s)$, it easily comes that $B_{s,\cdot}$ is independent of $\sigma(B_r, r \leq s) \vee \mathcal{F} \vee \sigma(\vec{\gamma}_{f(g_s,d_s)})$ which contains $\mathcal{F}_s$. This proves our claim and yields

$$\begin{aligned}
E[h_i(B_t^+)1_{\{g_t>s\}}|\mathcal{F}_s] &= E[h_i(B_{s,t} - \min_{r\in[s,t]} B_{s,r})1_{\{\min_{r\in[0,s]} B_{s,r} > \min_{r\in[s,t]} B_{s,r}\}}|\mathcal{F}_s] \\
&= \int_{\mathbb{R}} 1_{\{-B_s^+>x\}}(\int_{\mathbb{R}} h_i(y-x)g(x,y)dy)dx \\
&= 2\int_{\mathbb{R}_+} h_i(u)p_{t-s}(B_s^+, -u)du \quad (u = y - x)
\end{aligned}$$

and so

$$E[h(Z_t)1_{E_{s,t}^c}|\mathcal{F}_s] = 2\sum_{i=1}^{N}\alpha_i p_{t-s}h_i(-B_s^+).$$

On the other hand

$$E[h(Z_t)1_{E_{s,t}}|\mathcal{F}_s] = E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1_{E_{s,t}\cap(B_t>min_{0,s})}|\mathcal{F}_s]$$

$$= E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1_{\{B_t>min_{0,s}\}}|\mathcal{F}_s] - E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1_{E_{s,t}^c\cap(B_t>min_{0,s})}|\mathcal{F}_s].$$

Clearly on $\{\vec{e}_{0,s} = \vec{e}_k\}$,

$$\begin{aligned}
E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1_{\{B_t>min_{0,s}\}}|\mathcal{F}_s] &= E[h_k(B_{s,t} + B_s^+)1_{\{B_{s,t}+B_s^+>0\}}|\mathcal{F}_s] \\
&= p_{t-s}h_k(B_s^+),
\end{aligned}$$

and

$$E[h(\vec{e}_{0,s}(B_t - min_{0,s}))1_{\{E_{s,t}^c\cap(B_t>min_{0,s})\}}|\mathcal{F}_s] = E[h_k(B_{s,t}+B_s^+)1_{\{-B_s^+>\min_{r\in[s,t]} B_{s,r}, B_{s,t}+B_s^+>0\}}|\mathcal{F}_s].$$

$$= \int_{\mathbb{R}} h_k(y + B_s^+)1_{\{y+B_s^+>0\}}\left(\int_{\mathbb{R}} 1_{\{-B_s^+>x\}}g(x,y)dx\right)dy = p_{t-s}h_k(-B_s^+).$$

As a result

$$E[h(Z_t)|\mathcal{F}_s] = P_{t-s}h(Z_s).$$

where (P_t) is the semigroup of $W(\alpha_1, \cdots, \alpha_N)$. $\square$

Proposition 2.2. *Let $M = (M_n)_{n\geq 0}$ be a Markov chain on G started at 0 with stochastic matrix Q given by:*

$$Q(0, \vec{e}_i) = \alpha_i, \;\; Q(n\vec{e}_i, (n+1)\vec{e}_i) = Q(n\vec{e}_i, (n-1)\vec{e}_i) = \frac{1}{2} \;\; \forall i \in [1, N], \;\; n \in \mathbb{N}^*. \quad (2.2)$$

Then, for all $0 \leq t_1 < \cdots < t_p$, we have

$$(\frac{1}{2^n} M_{\lfloor 2^{2n} t_1 \rfloor}, \cdots, \frac{1}{2^n} M_{\lfloor 2^{2n} t_p \rfloor}) \xrightarrow[n \to +\infty]{law} (Z_{t_1}, \cdots, Z_{t_p}),$$

where Z is a $W(\alpha_1, \cdots, \alpha_N)$ process started at 0.

Proof. Let B be a standard Brownian motion and define for all $n \geq 1$: $T_0^n(B) = T_0^n(|B|) = 0$ and for $k \geq 0$

$$\begin{aligned} T_{k+1}^n(B) &= \inf\{r \geq T_k^n(B), |B_r - B_{T_k^n}| = \frac{1}{2^n}\}, \\ T_{k+1}^n(|B|) &= \inf\{r \geq T_k^n(|B|), ||B_r| - |B_{T_k^n}|| = \frac{1}{2^n}\}. \end{aligned}$$

Then, clearly $T_k^n(B) = T_k^n(|B|)$ and so $(T_k^n(|B|))_{k\geq 0} \overset{law}{=} (T_k^n(B))_{k\geq 0}$. It is known (see e.g.[30] page 31) that $\lim_{n\to+\infty} T_{\lfloor 2^{2n} t \rfloor}^n(B) = t$ a.s. uniformly on compact sets. Then, the result holds also for the reflected Brownian motion $|B|$. Now, let Z be the $W(\alpha_1, \cdots, \alpha_N)$ process started at 0 constructed from the reflected Brownian motion B^+ as before. Let $T_k^n = T_k^n(B^+)$ (defined analogously to $T_k^n(|B|)$) and $Z_k^n = 2^n Z_{T_k^n}$. Then clearly $(Z_k^n, k \geq 0)$ has the same law as M for all $n \geq 0$ (this will be rigorously justified in a more general setting in the proof of Proposition 4.4). Since a.s. $t \longmapsto Z_t$ is continuous, it comes that a.s. $\forall t \geq 0, \lim_{n\to+\infty} \frac{1}{2^n} Z_{\lfloor 2^{2n} t \rfloor}^n = Z_t$. $\square$

2.3.3 Donsker theorem for WBM

Proposition 2.3. *Under the conditions of Proposition 2.2, let $t \longmapsto M(t)$ be the linear interpolation of $(M_n)_{n\geq 0}$ and define $M_t^n = \frac{1}{\sqrt{n}} M(nt), n \geq 1$. Then*

$$(M_t^n)_{t\geq 0} \xrightarrow[n \to +\infty]{law} (Z_t)_{t\geq 0}$$

in $C([0, +\infty[, G)$ where Z is a $W(\alpha_1, \cdots, \alpha_N)$ process started at 0.

Proof. Let $(Z_t)_{t\geq 0}$ be a $W(\alpha_1, \cdots, \alpha_N)$ process on G started at 0 and

$$Z_t^i = |Z_t|1_{\{Z_t \in D_i\}} - |Z_t|1_{\{Z_t \notin D_i\}},\ i \in [1, N].$$

Then $Z_t^i = \Phi^i(Z_t)$ where $\Phi^i(x) = |x|1_{\{x\in D_i\}} - |x|1_{\{x\notin D_i\}}$. Let p^{α_i} be the semigroup of $SBM(\alpha_i)$ (see Section 2.2.2). Then the following relation is easy to check: $P_t(f\circ\Phi^i) = p_t^{\alpha_i} f \circ \Phi^i$ for all bounded measurable function f defined on $\mathbb{R}$ which shows that Z^i is a $SBM(\alpha_i)$ started at 0. For $n \geq 1, i \in [1, N], k \geq 0$, define

$$T_0^n = 0,\ \ T_{k+1}^n = \inf\{r \geq 0 : |Z_r - Z_{T_k^n}| = \frac{1}{\sqrt{n}}\}.$$

$$T_0^{n,i} = 0,\ \ T_{k+1}^{n,i} = \inf\{r \geq 0 : |Z_r^i - Z^i_{T_k^{n,i}}| = \frac{1}{\sqrt{n}}\}.$$

Remark that

$$T_{k+1}^n = T_{k+1}^{n,i} = \inf\{r \geq 0 : ||Z_r| - |Z_{T_k^n}|| = \frac{1}{\sqrt{n}}\}.$$

Furthermore if $Z_t \in D_i$, then obviously $d(Z_t, Z_s) = |Z_t^i - Z_s^i|$ for all $s \geq 0$ and consequently

$$d(Z_t, Z_s) \leq \sum_{i=1}^{N} |Z_t^i - Z_s^i|. \tag{2.3}$$

Now define $Z_k^n = \sqrt{n} Z_{T_k^n}, Z_k^{n,i} = \sqrt{n} Z^i_{T_k^{n,i}}$. Then by the proof of Proposition 2.2, $(Z_k^n, k \geq 0) \overset{law}{=} M$. Furthermore for all $T > 0$,

$$\sup_{t\in[0,T]} d(Z_t, \frac{1}{\sqrt{n}} Z^n_{\lfloor nt\rfloor}) \leq \sum_{i=1}^{N} \sup_{t\in[0,T]} |Z_t^i - \frac{1}{\sqrt{n}} Z^{n,i}_{\lfloor nt\rfloor}| \xrightarrow[n\to+\infty]{} 0 \text{ in probability}$$

by Lemma 4.4 [12] which proves our result. □

Proposition 2.4. *The $W(\alpha_1, \cdots, \alpha_N)$ process admits a modification having continuous paths.*

Proof. This is a consequence of (2.3) as well as the continuity of SBM. □

2.3.4 Freidlin-Sheu formula

Set $G^* = G \setminus \{0\}$ and define:

$C_b^2(G^*) = \big\{ f \in C(G) : \forall i \in [1, N], f_i$ is twice derivable on $\mathbb{R}_+^*$, $f_i', f_i'' \in C_b(\mathbb{R}_+^*)$ and both have finite limits at $0+ \big\}$.

For $f \in C_b^2(G^*)$, we use the conventions $f'(0) = f_N'(0+), f''(0) = f_N''(0+)$.

Theorem 2.3. *[17] Let $(Z_t)_{t\geq 0}$ be a $W(\alpha_1, \cdots, \alpha_N)$ process on G started at z and let $X_t = |Z_t|$. Then*

(i) $(X_t)_{t\geq 0}$ is a reflecting Brownian motion started at $|z|$.

(ii) $B_t = X_t - \tilde{L}_t(X) - |z|$ is a standard Brownian motion where

$$\tilde{L}_t(X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t 1_{\{|X_u| \leq \varepsilon\}} du.$$

(iii) $\forall f \in C_b^2(G^*)$,

$$f(Z_t) = f(z) + \int_0^t f'(Z_s) dB_s + \frac{1}{2} \int_0^t f''(Z_s) ds + (\sum_{i=1}^N \alpha_i f_i'(0+)) \tilde{L}_t(X). \qquad (2.4)$$

Remark 2.2. *For $N \geq 3$, the filtration $(\mathcal{F}_t^Z)$ has the martingale representation property with respect to B [4], but there is no Brownian motion W such that $\mathcal{F}_t^Z = \mathcal{F}_t^W$ [49].*

Proof. Let p^1 be the semigroup of the reflecting Brownian motion on $\mathbb{R}$ and define $\Phi(x) = |x|$. Then $X_t = \Phi(Z_t)$ and it can be easily checked that $P_t(f \circ \Phi) = p_t^1 f \circ \Phi$ for all bounded measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which proves (i). (ii) is an easy consequence of Tanaka's formula for local time.

(iii) Set $\tau_z = \inf\{r \geq 0, Z_r = 0\}$. For $t \leq \tau_z$, (2.4) follows from Itô's formula applied to the semimartingale X. By discussing the cases $t \leq \tau_z$ and $t > \tau_z$, one can assume that $z = 0$ and so in the sequel we take $z = 0$.

For all $i \in [1, N]$, $Z_t^i = |Z_t| 1_{\{Z_t \in D_i\}} - |Z_t| 1_{\{Z_t \notin D_i\}}$ is a $SBM(\alpha_i)$ started at 0 by the proof of Proposition 2.3. We use the notation $(\mathbb{P})$ to denote the convergence in probability. For $\delta > 0$, define $\tau_0^\delta = \theta_0^\delta = 0$ and for $n \geqslant 1$

$$\theta_n^\delta = \inf\{r \geq \tau_{n-1}^\delta, |Z_r| = \delta\}, \quad \tau_n^\delta = \inf\{r \geq \theta_n^\delta, Z_r = 0\}.$$

Let $f \in C_b^2(G^*)$ and $t > 0$. Then

$$f(Z_t) - f(0) = \sum_{n=0}^{\infty} f(Z_{\theta_{n+1}^\delta \wedge t}) - f(Z_{\theta_n^\delta \wedge t}) = Q_1^\delta + Q_2^\delta + Q_3^\delta$$

where

$$Q_1^\delta = \sum_{n=0}^{\infty}(f(Z_{\theta_{n+1}^\delta \wedge t}) - f(Z_{\tau_n^\delta \wedge t})) - \sum_{i=1}^{N}\sum_{n=0}^{\infty} \delta f_i'(0+) 1_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}},$$

$$Q_2^\delta = \sum_{i=1}^{N}\sum_{n=0}^{\infty} \delta f_i'(0+) 1_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}}, \quad Q_3^\delta = \sum_{n=0}^{\infty} f(Z_{\tau_n^\delta \wedge t}) - f(Z_{\theta_n^\delta \wedge t}).$$

We first show that $Q_1^\delta \xrightarrow[\delta \to 0]{} 0$ $(\mathbb{P})$ and for this write $Q_1^\delta = Q_{(1,1)}^\delta + Q_{(1,2)}^\delta$ with

$$Q_{(1,1)}^\delta = \sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(Z_{\theta_{n+1}^\delta}) - f(Z_{\tau_n^\delta}) - \delta f_i'(0+)) 1_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}},$$

$$Q_{(1,2)}^\delta = \sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(Z_t) - f(Z_{\tau_n^\delta \wedge t})) 1_{\{\theta_{n+1}^\delta > t, Z_{\theta_{n+1}^\delta} \in D_i\}}.$$

Since $f \in C_b^2(G^*)$, we have

(i)$\forall i \in [1, N]$; $\quad \displaystyle\int_0^\delta (f_i'(u) - f_i'(0+))du = f_i(\delta) - f_i(0) - \delta f_i'(0+)$.

(ii) There exists $M > 0$ such that $\forall i \in [1, N],\ u \geq 0 : |f_i'(u) - f_i'(0+)| \leq Mu$.

Consequently

$$\begin{aligned} |Q_{(1,1)}^\delta| &= |\sum_{n=0}^{\infty}\sum_{i=1}^{N}(f_i(\delta) - f_i(0) - \delta f_i'(0+)) 1_{\{\theta_{n+1}^\delta \leq t, Z_{\theta_{n+1}^\delta} \in D_i\}}| \\ &\leq \frac{NM\delta^2}{2}\sum_{n=0}^{\infty} 1_{\{\theta_{n+1}^\delta \leq t\}}. \end{aligned}$$

It is known that $\delta \displaystyle\sum_{n=0}^{\infty} 1_{\{\theta_{n+1}^\delta \leq t\}} \xrightarrow[\delta \to 0]{} \frac{1}{2}L_t(X)$ $(\mathbb{P})$ [45] where L_t is the nonsymmetric local time in zero and therefore $Q_{(1,1)}^\delta \xrightarrow[\delta \to 0]{} 0$ $(\mathbb{P})$.

Let $C > 0$ such that $\forall i \in [1, N],\ u \geq 0 : |f_i(u) - f_i(0)| \leq Cu$. Then

$$\begin{aligned} |Q_{(1,2)}^\delta| &= |\sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(Z_t) - f(Z_{\tau_n^\delta \wedge t})) 1_{\{\theta_{n+1}^\delta > t, Z_{\theta_{n+1}^\delta} \in D_i\}}| \\ &\leqslant \sum_{n=0}^{\infty}\sum_{i=1}^{N} |f_i(X_t) - f_i(0)| 1_{\{\tau_n^\delta < t < \theta_{n+1}^\delta, Z_{\theta_{n+1}^\delta} \in D_i\}} \\ &\leqslant CX_t \sum_{n=0}^{\infty} 1_{\{\tau_n^\delta < t < \theta_{n+1}^\delta\}} \leq C\delta \end{aligned}$$

which shows that $Q^\delta_{(1,2)} \xrightarrow[\delta\to 0]{} 0$ a.s. and so $Q^\delta_1 \xrightarrow[\delta\to 0]{} 0 \ \ (\mathbb{P})$.

Now define $Q^\delta_{(2,i)} = \delta\sum_{n=0}^{\infty} 1_{\{\theta^\delta_{n+1}\le t, Z_{\theta^\delta_{n+1}}\in D_i\}}$. Since $\sum_{n=0}^{\infty} 1_{\{\theta^\delta_{n+1}\le t, Z_{\theta^\delta_{n+1}}\in D_i\}}$ is the number of upcrossings of Z^i from 0 to δ before time t, we have $Q^\delta_{(2,i)} \xrightarrow[\delta\to 0]{} \frac{1}{2}L_t(Z^i)$ $(\mathbb{P})$.

Using Remark 2.1, we see that $Q^\delta_2 \xrightarrow[\delta\to 0]{} (\sum_{i=1}^{N}\alpha_i f_i'(0+))\tilde{L}_t(X)$ $(\mathbb{P})$.

We now establish that $Q^\delta_3 \xrightarrow[\delta\to 0]{} \int_0^t f'(Z_s)dB_s + \dfrac{1}{2}\int_0^t f''(Z_s)ds \ \ (\mathbb{P})$. For this write $Q^\delta_3 = Q^\delta_{(3,1)} + Q^\delta_{(3,2)}$ with

$$Q^\delta_{(3,1)} = \sum_{n=0}^{\infty}(f(Z_{\tau^\delta_n}) - f(Z_{\theta^\delta_n}))1_{\{\tau^\delta_n\le t\}} = \sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(0)-f_i(\delta))1_{\{\tau^\delta_n\le t, Z_{\theta^\delta_n}\in D_i\}},$$

$$Q^\delta_{(3,2)} = \sum_{i=1}^{N}(f(Z_t)-f_i(\delta))\sharp\{n\in\mathbb{N} : \theta^\delta_n < t < \tau^\delta_n, Z_{\theta^\delta_n}\in D_i\}.$$

It is clear that $\sharp\{n\in\mathbb{N} : \theta^\delta_n < t < \tau^\delta_n, Z_{\theta^\delta_n}\in D_i\} \xrightarrow[\delta\to 0]{} 1_{\{Z_t\in D_i\setminus\{0\}\}}$ a.s. and so $Q^\delta_{(3,2)}$ converges to $f(Z_t) - f(0)$ as $\delta\to 0$ a.s. Define $\tau^{\delta,i}_0 = \theta^{\delta,i}_0 = 0$ and

$$\theta^{\delta,i}_n = \inf\{r\ge \tau^{\delta,i}_{n-1}, Z_r = \delta\vec{e}_i\};\quad \tau^{\delta,i}_n = \inf\{r\ge\theta^{\delta,i}_n, Z_r = 0\}, n\geqslant 1.$$

Using $\sum_{n=0}^{\infty} 1_{\{\tau^\delta_n\le t, Z_{\theta^\delta_n}\in D_i\}} = \sum_{n=0}^{\infty} 1_{\{\tau^{\delta,i}_n\le t\}}$, it follows that

$$Q^\delta_{(3,1)} = \sum_{n=0}^{\infty}\sum_{i=1}^{N}(f(0)-f_i(\delta))1_{\{\tau^{\delta,i}_n\le t\}}.$$

On the other hand

$$f_i(X_{\tau^{\delta,i}_n\wedge t}) - f_i(X_{\theta^{\delta,i}_n\wedge t}) = (f_i(X_{\tau^{\delta,i}_n}) - f_i(X_{\theta^{\delta,i}_n}))1_{\{\tau^{\delta,i}_n\le t\}} + (f_i(X_t)-f_i(0))1_{\{\theta^{\delta,i}_n<t<\tau^{\delta,i}_n\}}$$

and therefore

$$Q^\delta_{(3,1)} = \sum_{i=1}^{N}\sum_{n=0}^{\infty}(f_i(X_{\tau^{\delta,i}_n\wedge t}) - f_i(X_{\theta^{\delta,i}_n\wedge t})) - \sum_{i=1}^{N}(f_i(X_t)-f_i(0))\times\sharp\{n\in\mathbb{N}, \theta^{\delta,i}_n < t < \tau^{\delta,i}_n\}.$$

Since $\sharp\{n\in\mathbb{N}, \theta^{\delta,i}_n < t < \tau^{\delta,i}_n\} \xrightarrow[\delta\to 0]{} 1_{\{Z_t\in D_i\setminus\{0\}\}}$ a.s., we deduce that

$$Q^\delta_3 \overset{\delta\to 0}{=} \sum_{i=1}^{N}\sum_{n=0}^{\infty}(f_i(X_{\tau^{\delta,i}_n\wedge t}) - f_i(X_{\theta^{\delta,i}_n\wedge t})) + o(1) \ \ a.s.$$

For all $i \in [1,N]$, let $\tilde{f}_i$ be C^2 on $\mathbb{R}$ such that $\tilde{f}_i = f_i$ on $\mathbb{R}_+$, $\tilde{f}'_i = f'_i$, $\tilde{f}''_i = f''_i$ on $\mathbb{R}^*_+$. Now a.s.

$$\forall s \in [0,t], i \in [1,N], \sum_{n=0}^{\infty} 1_{[\theta_n^{\delta,i}\wedge t, \tau_n^{\delta,i}\wedge t[}(s) \xrightarrow[\delta \to 0]{} 1_{\{Z_s \in D_i \setminus \{0\}\}}.$$

By dominated convergence for stochastic integrals (see [45] page 142),

$$\sum_{n=0}^{\infty} \tilde{f}_i(X_{\tau_n^{\delta,i}\wedge t}) - \tilde{f}_i(X_{\theta_n^{\delta,i}\wedge t}) = \int_0^t \sum_{n=0}^{\infty} 1_{[\theta_n^{\delta,i}\wedge t, \tau_n^{\delta,i}\wedge t[}(s) d\tilde{f}_i(X_s) \xrightarrow[\delta \to 0]{} \int_0^t 1_{\{Z_s \in D_i \setminus \{0\}\}} d\tilde{f}_i(X_s) \ (\mathbb{P}).$$

Finally

$$\begin{aligned}
\int_0^t f'(Z_s)dB_s + \frac{1}{2}\int_0^t f''(Z_s)ds &= \sum_{i=1}^{N} \int_0^t 1_{\{Z_s \in D_i \setminus \{0\}\}} (f'_i(X_s)dB_s + \frac{1}{2} f''_i(X_s)ds) \\
&= \sum_{i=1}^{N} \int_0^t 1_{\{Z_s \in D_i \setminus \{0\}\}} d\tilde{f}_i(X_s).
\end{aligned}$$

by Itô's formula and using the fact that $d\tilde{L}_s(X)$ is carried by $\{s : Z_s = 0\}$. Now the proof of Theorem 2.3 is complete. $\square$

The foregoing theorem entails the following characterization of the $W(\alpha_1, \cdots, \alpha_N)$ process by means of the generator of its semigroup.

Proposition 2.5. *Let*

- $D(\alpha_1, \cdots, \alpha_N) = \{f \in C_b^2(G^*) : \sum_{i=1}^{N} \alpha_i f'_i(0+) = 0\}$.
- $Q = (Q_t)_{t \geq 0}$ *be a Feller semigroup satisfying:*

$$Q_t f(x) = f(x) + \frac{1}{2}\int_0^t Q_u f''(x) du \quad \textit{for all } f \in D(\alpha_1, \cdots, \alpha_N).$$

Then, Q is the semigroup of the $W(\alpha_1, \cdots, \alpha_N)$ process.

Proof. Denote by P be the semigroup of the $W(\alpha_1, \cdots, \alpha_N)$ process, A' and $D(A')$ being respectively its generator and its domain on $C_0(G)$. Let

$$D'(\alpha_1, \cdots, \alpha_N) = \{f \in C_0(G) \bigcap D(\alpha_1, \cdots, \alpha_N), f'' \in C_0(G)\}. \tag{2.5}$$

Then it is enough to prove the statements:

(i) $\forall t > 0, \quad P_t(C_0(G)) \subset D'(\alpha_1, \cdots, \alpha_N)$.

(ii) $D'(\alpha_1, \cdots, \alpha_N) \subset D(A')$ and $A'f(x) = \frac{1}{2}f''(x)$ on $D'(\alpha_1, \cdots, \alpha_N)$.

(iii) $D'(\alpha_1, \cdots, \alpha_N)$ is dense in $C_0(G)$ for $||.||_\infty$.

(iv) If R and R' are respectively the resolvents of Q and P, then

$$R_\lambda = R'_\lambda \text{ for all } \lambda > 0 \text{ on } D'(\alpha_1, \cdots, \alpha_N).$$

(i) Pick $t > 0$ and $f \in C_0(G)$. Since P is Feller, we have $P_t f \in C_0(G)$. Set

$$H_i(y,h) = 2\sum_{j=1}^{N} \alpha_j f_j(y) p_t(h,-y) + f_i(y)(p_t(h,y) - p_t(h,-y)), \quad h > 0, y \geqslant 0.$$

For $h \geq 0, x = h\vec{\varepsilon}_i$,

$$P_t f(x) = \int_{\mathbb{R}_+} H_i(y,h)dy \text{ if } h > 0; \; P_t f(0) = 2\sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}_+} f_i(y) p_t(h,-y)dy \text{ if } h = 0.$$

It is clear that $h \longmapsto H^i(y,h)$ is C^∞ on $\mathbb{R}_+^*$ and furthermore $\forall y \geq 0, h > 0$,

$$\frac{\partial}{\partial h} H_i(y,h) = 2\sum_{j=1}^{N} \alpha_j f_j(y) \frac{\partial}{\partial h} p_t(h,-y) + f_i(y)(\frac{\partial}{\partial h} p_t(h,y) - \frac{\partial}{\partial h} p_t(h,-y)),$$

$$\frac{\partial}{\partial h^2} H^i(y,h) = 2\sum_{j=1}^{N} \alpha_j f_j(y) \frac{\partial}{\partial h^2} p_t(h,-y) + f_i(y)(\frac{\partial}{\partial h^2} p_t(h,y) - \frac{\partial}{\partial h^2} p_t(h,-y).$$

For some constant $M \in \mathbb{R}$, we have

$$\begin{aligned} |\frac{\partial}{\partial h} H_i(y,h)| &\leq M(|\frac{\partial}{\partial h} p_t(h,-y)| + |\frac{\partial}{\partial h} p_t(h,y)|) \\ &\leq \frac{M}{t}\{(y+h)p_t(h,-y) + |y-h| p_t(h,y)\}. \end{aligned}$$

Clearly $(P_t f)'_i$ exists on $\mathbb{R}_+^*$ and

$$\forall h > 0, \quad (P_t f)'_i(h) = \int_{\mathbb{R}_+} \frac{\partial}{\partial h} H_i(y,h)dy.$$

The last equality shows that $h \longmapsto (P_t f)'_i(h)$ is bounded on $\mathbb{R}_+^*$ since

$$\sup_{h>0} \int_0^{+\infty} (y+h)p_t(h,-y)dy < +\infty; \; \sup_{h>0} \int_0^{+\infty} |y-h| p_t(h,y)dy < +\infty. \quad (2.6)$$

An easy application of dominated convergence yields

$$(P_tf)_i'(h) \xrightarrow[h \to 0+]{} \frac{-2}{t}\sum_{j=1}^{N}\alpha_j \int_{\mathbb{R}_+} f_j(y)yp_t(0,y)dy + \frac{2}{t}\int_{\mathbb{R}_+} f_i(y)yp_t(0,y)dy$$

and in particular $\sum_{i=1}^{N}\alpha_i(P_tf)_i'(0+) = 0$. For the second derivative, we have for some constant $M \in \mathbb{R}$,

$$|\frac{\partial}{\partial h^2}H^i(y,h)| \leq M(Q_1(h,y)p_t(h,-y) + Q_2(h,y)p_t(h,y))$$

where Q_1, Q_2 are two polynomials in two variables (h,y) and both having positive coefficients. Using

$$\frac{\partial}{\partial h^2}p_t(h,u) = \frac{-1}{t}p_t(h,u) + \frac{(u-h)^2}{t^2}p_t(h,u) \xrightarrow[h\to 0+]{} p_t(0,u)(\frac{u^2}{t^2} - \frac{1}{t})$$

and dominated convergence, we deduce that

$$(P_tf)_i''(h) \xrightarrow[h \to 0+]{} 2\sum_{j=1}^{N}\alpha_j \int_{\mathbb{R}_+} f_j(y)(\frac{y^2-t}{t^2})p_t(0,y)dy \text{ (no longer depends } i \in [1,N]).$$

To conclude that $P_tf \in D'(\alpha_1,\cdots,\alpha_N)$, it remains to show that $\lim_{h\to+\infty}(P_tf)_i''(h) = 0$. For this, write $(P_tf)_i'' = I_1 + I_2 - I_3$ where

$$I_1(h) = 2\sum_{j=1}^{N}\alpha_j \int_{\mathbb{R}_+} f_j(y)\frac{\partial}{\partial h^2}p_t(h,-y)dy,$$

$$I_2(h) = \int_{\mathbb{R}_+} f_i(y)\frac{\partial}{\partial h^2}p_t(h,y)dy, \quad I_3(h) = \int_{\mathbb{R}_+} f_i(y)\frac{\partial}{\partial h^2}p_t(h,-y)dy.$$

Clearly, there exists a polonomial D with positive coefficients satisfying

$$|I_1(h)| \leq \int_{\mathbb{R}_+} D(h+y)e^{-\frac{(h+y)^2}{2t}}dy \xrightarrow[h \to +\infty]{} 0.$$

Likewise, $I_3(h) \xrightarrow[h \to +\infty]{} 0$. Finally

$$I_2(h) = \frac{-1}{t}\int_{\mathbb{R}_+} f_i(y)p_t(h,y)dy + \frac{1}{t^2}\int_{\mathbb{R}_+} f_i(y)(y-h)^2p_t(h,y)dy$$

$$= C_1\int_{\mathbb{R}} f_i(y+h)1_{\{y>-h\}}e^{-\frac{y^2}{2t}}dy + C_2\int_{\mathbb{R}} f_i(y+h)1_{\{y>-h\}}y^2e^{-\frac{y^2}{2t}}dy.$$

Since $f \in C_0(G)$, $I_2(h) \xrightarrow[h \to +\infty]{} 0$ by dominated convergence and so $\lim_{h \to +\infty} (P_t f)''_i(h) = 0$.

(ii) easily comes from 2.4. (iii) Let $f \in C_0(G)$, then $P_{\frac{1}{n}} f \in D'(\alpha_1, \cdots, \alpha_N)$ for all $n \geq 1$ by (i) and since P is Feller, we get $||P_{\frac{1}{n}} f - f||_\infty \xrightarrow[n \to +\infty]{} 0$.

(iv) Let A be the generator of P. By assumption, $D'(\alpha_1, \cdots, \alpha_N) \subset D(A)$ and $Af = \frac{1}{2} f''$ on $D'(\alpha_1, \cdots, \alpha_N)$. For $f \in D'(\alpha_1, \cdots, \alpha_N)$, $R_\lambda f$ is the unique element of $D(A)$ such that

$$(\lambda I - A)(R_\lambda f) = f.$$

In order to complete the proof, we will show that

$$(a)\ R'_\lambda f \in D(A), \quad (b) \lambda R'_\lambda f - A(R'_\lambda f) = f.$$

(a) Since $R'_\lambda f(x) = \displaystyle\int_0^{+\infty} e^{-\lambda t} P_t f(x) dt$ and $P_t f \in D'(\alpha_1, \cdots, \alpha_N)$, dominated convergence to gether with (ii) show that $R'_\lambda f \in D'(\alpha_1, \cdots, \alpha_N) \subset D(A)$. As $R'_\lambda f \in D'(\alpha_1, \cdots, \alpha_N)$, it comes that

$$A'(R_\lambda f) = A(R_\lambda f) = \frac{1}{2}(R_\lambda f)''.$$

Now $(\lambda I - A')(R'_\lambda f) = f$ yields $(\lambda I - A)(R'_\lambda f) = f$ which proves (b) and (iv). By (iii), we see that $R_\lambda = R'_\lambda$, in other words $P = Q$. $\square$

Exercice. Consider the full WBM in the plane $\mathcal{P} = \mathbb{R}^2$ as defined by its semigroup in [4]. Let $\mathcal{P}^* = \mathcal{P} \setminus \{0\}$ where $0 = 0_{\mathbb{R}^2}$. We will use polar co-ordinates (r, θ) to denote points in $\mathcal{P}$. For f defined on $\mathcal{P}$ and $\theta \in [0, 2\pi[$, let $f_\theta(h) = f(h, \theta), h \geq 0$. We say that f is continuous on $\mathcal{P}$ if f_θ is continuous on $\mathbb{R}_+$ for all $\theta \in [0, 2\pi[$. Define analogously

$C_b^2(\mathcal{P}^*) = \big\{ f \in C(\mathcal{P}) : \forall \theta \in [0, 2\pi[, f_\theta$ is twice derivable on $\mathbb{R}_+^*, f'_\theta, f''_\theta \in C_b(\mathbb{R}_+^*)$ and both have finite limits at $0+ \big\}$.

For $f \in C_b^2(\mathcal{P}^*)$, $z = (r, \theta) \in \mathcal{P}$ with $z \neq 0$, set $f'(z) = f'_\theta(r), f''(z) = f''_\theta(r)$. We use the conventions $f'(0) = f'_0(0+), f''(0) = f''_0(0+)$. Let Z be a full WBM in the plane started at z and $X_t = |Z_t|$. Using a similar decomposition of $f(Z_t) - f(0)$ as in the proof of Theorem 2.3 and the approximation of local time by the number of upcrossings in L^p spaces, show that:

(i) $(X_t)_{t\geq 0}$ is a reflecting Brownian motion started at $|z|$.

(ii) $B_t = X_t - \tilde{L}_t(X) - |z|$ is a standard Brownian motion where

$$\tilde{L}_t(X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t 1_{\{|X_u| \leq \varepsilon\}} du.$$

(iii) For all suitable functions $f \in C_b^2(\mathcal{P}^*)$:

$$f(Z_t) = f(z) + \int_0^t f'(Z_s)dB_s + \frac{1}{2}\int_0^t f''(Z_s)ds + \frac{1}{2\pi}(\int_{[0,2\pi[} f'_\theta(0+)d\theta)\tilde{L}_t(X).$$

Then deduce the analogous of Proposition 2.5. A large part of this thesis can be generalized to the full WBM in the plane.

Chapter 3

Stochastic flows and Tanaka's SDE

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3.1 Introduction

In this chapter we introduce basic objects for this thesis. We begin with stochastic flows: flows of mappings and more generally flows of kernels according to Le Jan and Raimond. A stochastic flow of kernels K can be viewed as the transition probabilities of a Markov process in a random environment. To K is associated a compatible family of Feller semigroups $(P^n)_{n\geq 1}$: namely, the marginal distribution of any k components of an n motion is necessarily a k point motion. Conversely Le Jan and Raimond have proved that any compatible family of Feller semigroups gives a unique stochastic flow of kernels. We briefly review these notions in Section 3.2. In Section 3.3, we define the noise filtration associated with any flow of kernels and recall the notion of "filtering with respect to a subnoise filtration ". Then we study coalescing flows which can be obtained from flows whose two point motion hits the diagnoal. Then the original flow can be recovered by filtering the coalescing flow with respect to a subnoise filtration. In Section 3.4, we apply this theory and extend Tanaka's equation to kernels. We recall the construction of flows associated to the extended Tanaka's equation from [36]. At the end of Section 3.4, a new flow of kernels will appear giving rise to a more general Tanaka's equation as well as to new difficulties. This has been the starting point of this thesis.

3.2 Stochastic flows

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and M be any locally compact separable metric space. Let us recall some fundamental definitions and results from [35].

3.2.1 Stochastic flows of mappings

Let $(F, \mathcal{F})$ be the space of all measurable mappings from M into M endowed with the σ-field $\mathcal{F}$ generated by $\varphi \longmapsto \varphi(x), x \in M$ and $C_0(M)$ be the space of all continuous functions on M which vanish at infinity.

Definition 3.1. *A family $(\varphi_{s,t})_{s\leq t}$ of $(F, \mathcal{F})$-valued random variables is called a stochastic flow of mappings if $\forall s \leqslant t$, the mapping*

$$\begin{array}{rcrcl} \varphi_{s,t} & : & (M \times \Omega, \mathcal{B}(M) \otimes \mathcal{A}) & \longrightarrow & (M, \mathcal{B}(M)) \\ & & (x, \omega) & \longmapsto & \varphi_{s,t}(x, \omega) \end{array}$$

is measurable and if it satisfies the following properties:

(1) $\forall s < t < u, x \in M, \mathbb{P}$-a.s., $\varphi_{s,u}(x) = \varphi_{t,u}(\varphi_{s,t}(x))$ (cocycle or flow property).

(2) $\forall s \leqslant t$, the law of $\varphi_{s,t}$ only depends on $t - s$ (stationarity).

(3) For all $t_1 < t_2 < \cdots < t_n$, the family $\{\varphi_{t_i,t_{i+1}}, 1 \leq i \leq n-1\}$ is independent.

(4) $\forall t \geq 0, x \in M, f \in C_0(M)$, $\lim_{y\to x} E[(f(\varphi_{0,t}(x)) - f(\varphi_{0,t}(y)))^2] = 0$.

(5) $\forall t \geq 0, f \in C_0(M)$, $\lim_{x\to+\infty} E[f^2(\varphi_{0,t}(x))] = 0$.

(6) $\forall x \in M, f \in C_0(M)$, $\lim_{t\to 0+} E[(f(\varphi_{0,t}(x)) - f(x))^2] = 0$.

A family of Feller semigroups $(P^n)_{n\geq 1}$ defined on M^n and acting on $C_0(M^n)$ is said to be compatible as soon as, for all $k \leq n$,

$$P_t^k f(x_1, \cdots, x_k) = P_t^n g(y_1, \cdots, y_n),$$

where f and g are in $C_0(M^n)$ such that

$$g(y_1, \cdots, y_n) = f(y_{i_1}, \cdots, y_{i_k})$$

with $\{i_1, \cdots, i_k\} \subset \{1, \cdots, n\}$ and $(x_1, \cdots, x_k) = (y_{i_1}, \cdots, y_{i_k})$. The following proposition associates to a stochastic flow of mappings a compatible system of Feller semigroups.

Proposition 3.1. *Consider a stochastic flow of mappings* $(\varphi_{s,t})_{s\le t}$*. For* $f \in C_0(M^n)$*,* $x = (x_1, \cdots, x_n)$*, set*

$$P_t^n f(x) = E[f(\varphi_{0,t}(x_1), \cdots, \varphi_{0,t}(x_n))].$$

Then $(P^n)_{n\ge 1}$ *is a compatible family of Feller semigroups acting respectively on* $C_0(M^n)$ *satisfying*

$$P_t^2 f^{\otimes 2}(x,x) = P_t f^2(x) \text{ for all } f \in C_0(M), x \in M, t \ge 0. \tag{3.1}$$

Moreover, the n *point motion (or Markov process) started at* $(x_1, \cdots, x_n) \in M^n$ *associated to* P^n *is given by* $(\varphi_{0,\cdot}(x_1), \cdots, \varphi_{0,\cdot}(x_n))$*.*

Proof. The second assertion is clear. See Proposition 3.3 below for the first claim. □

Stochastic differential equations have been for a long time a powerful tool to construct stochastic flows [28]. This is for example the case of SBM as we will see in the next chapter. However the approach of Le Jan and Raimond is different and is of type Kolmogorov extension theorem. Before reminding their first result, we need to introduce Feller convolution semigroups on $(F, \mathcal{F})$ and begin by

Definition 3.2. *A probability measure* Q *on* $(F, \mathcal{F})$ *is called regular if there exists a measurable mapping* $\mathcal{J} : (F, \mathcal{F}) \longrightarrow (F, \mathcal{F})$ *such that*

$$\begin{array}{rcl} (M \times F, \mathcal{B}(M) \otimes \mathcal{F}) & \longrightarrow & (M, \mathcal{B}(M)) \\ (x, \varphi) & \longmapsto & \mathcal{J}(\varphi)(x), \end{array}$$

is measurable and, for every $x \in M$*,*

$$Q(d\varphi) - a.s., \quad \mathcal{J}(\varphi)(x) = \varphi(x),$$

that is, $\mathcal{J}$ *is a measurable modification of the identity mapping on* $(F, \mathcal{F}, Q)$*. We call it a measurable representation of* Q*.*

Proposition 3.2. *Let Q_1 and Q_2 be two probability measures on $(F, \mathcal{F})$. Assume Q_1 is regular. Let $\mathcal{J}$ be a measurable presentation of Q_1. Then the mapping*

$$\begin{array}{rcl} (F^2, \mathcal{F}^{\otimes 2}) & \longrightarrow & (F, \mathcal{F}) \\ (\varphi_1, \varphi_2) & \longmapsto & \mathcal{J}(\varphi_1) \circ \varphi_2, \end{array}$$

is measurable. Moreover, if $\mathcal{J}'$ is another measurable presentation of Q_1, then for every $x \in M$,

$$Q_1(d\varphi_1) \otimes Q_2(d\varphi_2) - a.s., \quad \mathcal{J}(\varphi_1) \circ \varphi_2(x) = \mathcal{J}'(\varphi_1) \circ \varphi_2(x).$$

We denote $Q_1 \star Q_2$, and we call the convolution product of Q_1 and Q_2, the law of the random variable $(\varphi_1, \varphi_2) \longmapsto \mathcal{J}(\varphi_1) \circ \varphi_2$ defined on the probability space $(F^2, \mathcal{F}^{\otimes 2}, Q_1 \otimes Q_2)$.

It is now possible to give the following

Definition 3.3. *A convolution semigroup on $(F, \mathcal{F})$ is a family $(Q_t)_{t \geq 0}$ of regular probability measures on $(F, \mathcal{F})$ such that, for all nonnegative s and t, $Q_{s+t} = Q_s \star Q_t$. We say that $(Q_t)_{t \geq 0}$ is Feller as soon as*

(i) $\forall t \geq 0, x \in M, f \in C_0(M)$,

$$\lim_{y \to x} \int (f \circ \varphi(x) - f \circ \varphi(y))^2 Q_t(d\varphi) = 0.$$

(ii) $\forall t \geq 0, f \in C_0(M)$,

$$\lim_{x \to +\infty} \int f^2(\varphi(x)) Q_t(d\varphi) = 0.$$

(iii) $\forall x \in M, f \in C_0(M)$,

$$\lim_{t \to 0+} \int (f \circ \varphi(x) - f(x))^2 Q_t(d\varphi) = 0.$$

We now turn to the first fundamental result of Le Jan and Raimond

Theorem 3.1. *(i) Let $(P_t^n, n \geq 1)$ be a compatible family of Feller semigroups on M satisfying (3.1). Then there exists a unique Feller convolution semigroup $(Q_t)_{t\geq 0}$ on $(F, \mathcal{F})$ such that, for all $n \geq 1, t \geq 0, f \in C_0(M^n)$ and $x \in M^n$,*

$$P_t^n f(x) = \int f \circ \varphi^{\otimes n}(x) Q_t(d\varphi).$$

(ii) For every Feller convolution semigroup $(Q_t)_{t\geq 0}$ on $(F, \mathcal{F})$, there exists a stochastic flow of mappings $(\varphi_{s,t})_{s\leq t}$ such that for all $s \leq t$, the law of $\varphi_{s,t}$ is Q_{t-s}.

The following lemma gives a sufficient condition for a compatible family of Markovian kernels semigroups to be constituted of Feller semigroups.

Lemma 3.1. *(i) A compatible family $(P_t^n, n \geq 1)$ of semigroups of Markovian kernels is constituted of Feller semigroups when the following condition is satisfied:*
(C) For all $f \in C_0(M)$ and $x \in M$, $\lim_{t\to 0} P_t^1 f(x) = f(x)$ and for all $x \in M$, $\epsilon > 0$ and $t > 0$, $\lim_{y\to x} P_t^2 d_\epsilon(x, y) = 0$, where $d_\epsilon(x, y) = 1_{\{d(x,y)>\epsilon\}}$.
(ii) When (C) is satisfied, (3.1) holds and so a stochastic flow of mappings is associated with this family of semigroups.

Proof. (i) This is Lemma 1.11 in [35]. (ii) Let $f_\epsilon(x, y) = f(x)f(y)d_\epsilon(x, y)$. We have

$$P_t^2 f^{\otimes 2}(x, x) = P_t^2 f_\epsilon(x, x) + P_t^2(f^{\otimes 2} - f_\epsilon)(x, x).$$

By hypothesis $P_t^2 d_\epsilon(x, x) = 0$ and so $P_t^2 f_\epsilon(x, x) = 0$. On the other hand

$$(f^{\otimes 2} - f_\epsilon)(z_1, z_2) = f(z_1)f(z_2)1_{\{d(z_1,z_2)\leq\epsilon\}}$$

converges pointwise to $f^2(z_1)1_{\{z_1=z_2\}}$ as $\epsilon \to 0$. By dominated convergence

$$\begin{aligned}
\lim_{\epsilon\to 0} P_t^2(f^{\otimes 2} - f_\epsilon)(x, x) &= \int f^2(z_1)1_{\{z_1=z_2\}} P_t^2((x, x), dz_1dz_2) \\
&= P_t f^2(x) - \int f^2(z_1)1_{\{z_1\neq z_2\}} P_t^2((x, x), dz_1dz_2).
\end{aligned}$$

Again from $P_t^2 d_\epsilon(x, x) = 0$,

$$\int f^2(z_1)1_{\{z_1\neq z_2\}} P_t^2((x, x), dz_1dz_2) = 0,$$

which finishes the proof. □

3.2.2 Coalescing flows

Let φ be a stochastic flow of mappings. We say that φ is coalescing as soon as, for all $(x, y) \in M^2$, with probability 1, $T_{x,y} = \inf\{t \geq 0, \varphi_{0,t}(x) = \varphi_{0,t}(y)\} < \infty$ and $\varphi_{0,t}(x) = \varphi_{0,t}(y)$ for all $t \geq T_{x,y}$. Suppose that for each $x \in M$, $\varphi_{0,\cdot}(x)$ has a continuous version which will be usually the case in this thesis. Then, by Proposition 3.1, the following lemma holds.

Lemma 3.2. *(The strong Markov property.) For all $(x_1, \cdots, x_n) \in M^n$, denote by $\mathbb{P}_{x_1,\cdots,x_n}$ the law of $(\varphi_{0,\cdot}(x_1), \cdots, \varphi_{0,\cdot}(x_n))$ in $C(\mathbb{R}_+, M^n)$. Let T be a finite $(\mathcal{F}_t)$-stopping time where $\mathcal{F}_t = \sigma(\varphi_{0,u}, u \leq t), t \geq 0$. Then the law of $(\varphi_{0,T+\cdot}(x_1), \cdots, \varphi_{0,T+\cdot}(x_n))$ knowing $\mathcal{F}_T$ is given by $\mathbb{P}_{\varphi_{0,T}(x_1),\cdots,\varphi_{0,T}(x_n)}$.*

Applying the previous lemma at $T = T_{x,y}$, we see that φ is a coalescing flow if and only if, for all $(x, y) \in M^2$, with probability 1, $T_{x,y} < \infty$.

3.2.3 Stochastic flows of kernels

Let $\mathcal{P}(M)$ be the space of all probability measures on M and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions dense in $\{f \in C_0(M), ||f||_\infty \leq 1\}$. We equip $\mathcal{P}(M)$ with the distance $d(\mu, \nu) = (\sum_n 2^{-n}(\int f_n d\mu - \int f_n d\nu)^2)^{\frac{1}{2}}$ for all μ and ν in $\mathcal{P}(M)$. Thus, $\mathcal{P}(M)$ is a locally compact separable metric space. A kernel K on M is a measurable mapping from M into $\mathcal{P}(M)$. We denote by E the space of all kernels on M and we equip E with the σ-field $\mathcal{E}$ generated by the mappings $K \longmapsto \mu K, \mu \in \mathcal{P}(M)$, with μK the probability measure defined as $\mu K(A) = \int_M \mu(dx)K(x, A)$.

Let Γ denote the space of measurable mappings on $\mathcal{P}(M)$. We equip Γ with the σ-field generated by the mappings $\Phi \mapsto \Phi(\mu)$ for all $\mu \in \mathcal{P}(M)$. Note that $(\Gamma, \mathcal{G}) = (F, \mathcal{F})$ once we have replaced M by $\mathcal{P}(M)$.

Definition 3.4. *A family of $(E, \mathcal{E})$-valued random variables $(K_{s,t})_{s\leq t}$ is called a* *stochastic flow of kernels if,* $\forall s \leqslant t$ *the mapping*

$$\begin{array}{rcrcc} K_{s,t} & : & (M \times \Omega, \mathcal{B}(M) \otimes \mathcal{A}) & \longrightarrow & (\mathcal{P}(M), \mathcal{B}(\mathcal{P}(M))) \\ & & (x, \omega) & \longmapsto & K_{s,t}(x, \omega) \end{array}$$

is measurable and if it satisfies the following properties:

(1) $\forall s < t < u, x \in M$ *a.s.,*

$$\forall f \in C_0(M), K_{s,u}f(x) = K_{s,t}(K_{t,u}f)(x) \text{ (cocycle or flow property).}$$

(2) $\forall s \leqslant t$, *the law of* $K_{s,t}$ *only depends on* $t - s$ *(stationarity).*

(3) For all $t_1 < t_2 < \cdots < t_n$, *the family* $\{K_{t_i,t_{i+1}}, 1 \leq i \leq n-1\}$ *is independent.*

(4) $\forall t \geq 0, x \in M, f \in C_0(M)$, $\lim_{y \to x} E[(K_{0,t}f(x) - K_{0,t}f(y))^2] = 0.$

(5) $\forall t \geq 0, f \in C_0(M)$, $\lim_{x \to +\infty} E[(K_{0,t}f(x))^2] = 0.$

(6) $\forall x \in M, f \in C_0(M)$, $\lim_{t \to 0+} E[(K_{0,t}f(x) - f(x))^2] = 0.$

Note that δ_φ is a stochastic flow of kernels as soon as φ is a stochastic flow of mappings. In Section 3.4, we shall give other examples of stochastic flows of kernels in connection wih Tanaka's equation.

Proposition 3.3. *Let* K *be a stochastic flow of kernels on* M. *For* $f \in C_0(M^n), x = (x_1, \cdots, x_n) \in M^n$, *set*

$$K_{0,t}^{\otimes n} f(x) = \int_M f(y_1, \cdots, y_n) K_{0,t}(x_1, dy_1) \cdots K_{0,t}(x_n, dy_n)$$

and $P_t^n f(x) = E[K_{0,t}^{\otimes n} f(x)]$. *Then,* $(P^n)_{n \geq 1}$ *is a compatible family of Feller semigroups acting respectively on* $C_0(M^n)$.

Proof. We will show that:

(i) $P_{t+s}^n = P_t^n(P_s^n)$, (ii) $P_t^n(C_0(M^n)) \subset C_0(M^n)$,

(iii) $\forall x \in G^n, f \in C_0(M^n)$, $\lim_{t \to 0+} P_t^n f(x) = f(x)$.

(i) is an easy consequence of the cocyle property satisfied by K.

(ii) Let $f \in C_0(M^n)$ such that $f = f_1 \otimes \cdots \otimes f_n$ with $f_i \in C_0(M)$. We have

$$|\prod_{i=1}^{n} K_{0,t}f_i(x_i) - \prod_{i=1}^{n} K_{0,t}f_i(y_i)| \leq Q_1 + Q_2,$$

where

$$Q_1 = |\prod_{i=1}^{n} K_{0,t}f_i(x_i) - K_{0,t}f_1(y_1)\prod_{i=2}^{n} K_{0,t}f_i(x_i)|$$

and

$$Q_2 = |K_{0,t}f_1(y_1)\prod_{i=2}^{n} K_{0,t}f_i(x_i) - \prod_{i=1}^{n} K_{0,t}f_i(y_i)|.$$

There exist $C_1, C_2 \in \mathbb{R}$ such that

$$Q_1 \leq C_1|K_{0,t}f_1(x_1) - K_{0,t}f_1(y_1)|, \quad Q_2 \leq C_2|\prod_{i=2}^{n} K_{0,t}f_i(x_i)] - \prod_{i=2}^{n} K_{0,t}f_i(y_i)|.$$

Consequently, for some constant C_3, we have

$$|P_t^n f(x) - P_t^n f(y)| \leq C_3 \sum_{i=1}^{n} E[|K_{0,t}f_i(x_i) - K_{0,t}f_i(y_i)|].$$

By property (4) in the definition of flows, we get:

$$\lim_{y \longrightarrow x} P_t^n f(y) = P_t^n f(x) \text{ for all } x \in M^n. \tag{3.2}$$

This remains true for a finite linear combination of functions of the previous form. Now, if $f \in C_0(M^n)$, there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset C_0(M^n)$, such that f_k satisfy (3.2) for all k and $||f_k - f||_\infty \underset{k\to+\infty}{\longrightarrow} 0$.

It is clear that $|P_t^n f(x) - P_t^n f(y)| \leqslant 2||f_k - f||_\infty + |P_t^n f_k(x) - P_t^n f_k(y)|$. By letting $y \longrightarrow x$ and then $k \longrightarrow \infty$, we conclude that

$$\limsup_{y \longrightarrow x} |P_t^n f(x) - P_t^n f(y)| = 0.$$

On the other hand, if $f = f_1 \otimes \cdots \otimes f_n$, with $f_i \in C_0(M)$, then for all $i \in [1, n]$, there exists $C_i' \geq 0$ such that

$$|P_t^n f(x)| \leq C_i' E[|K_{0,t}f_i(x_i)|].$$

By property (5) in the definition of K, we get $\lim_{x\to+\infty} P_t^n f(x) = 0$ and the result extends for all $f \in C_0(M^n)$ as preceded.

(iii) If $f = f_1 \otimes \cdots \otimes f_n$, with $f_i \in C_0(M)$ for all $i \in [1, n]$, then similarly to (ii), there exists a constant C_4 with

$$|P_t^n f(x) - f(x)| \leq C_4 \sum_{i=1}^{n} E[|K_{0,t} f_i(x_i) - f_i(x_i)|].$$

Consequently, $\lim_{t\to 0+} P_t^n f(x) = f(x)$, by property (6) in the definition of K. Now, the result easily extends for all $f \in C_0(M^n)$ by the previous argument of density. $\square$

Let $\mathcal{I}$ denote the measurable mapping from $(E, \mathcal{E})$ on $\Gamma, \mathcal{G})$ defined by $\mathcal{I}(K)(\mu) = \mu K$.

Definition 3.5. *(1) A probability measure ν on $(E, \mathcal{E})$ is called regular if $\mathcal{I}^*(\nu)$ is a regular probability measure on $(\Gamma, \mathcal{G})$.*

(2) A convolution semigroup on $(E, \mathcal{E})$ is a family $(\nu_t)_{t\geq 0}$ of regular probability measures on $(E, \mathcal{E})$ such that $(\mathcal{I}^*(\nu_t))_{t\geq 0}$ is a convolution semigroup on $(\Gamma, \mathcal{G})$.

(3) A convolution semigroup $(\nu_t)_{t\geq 0}$ on $(E, \mathcal{E})$ is called Feller if

(i) $\forall t \geq 0, x \in M, f \in C_0(M)$,

$$\lim_{y\to x} \int (Kf(x) - Kf(y))^2 \nu_t(dK) = 0.$$

(ii) $\forall t \geq 0, f \in C_0(M)$,

$$\lim_{x\to+\infty} \int (Kf(x))^2 \nu_t(dK) = 0.$$

(iii) $\forall x \in M, f \in C_0(M)$,

$$\lim_{t\to 0+} \int (Kf(x) - f(x))^2 \nu_t(dK) = 0.$$

The following result extends Theorem 3.1.

Theorem 3.2. *(i) Let* $(P_t^n, n \geq 1)$ *be a compatible family of Feller semigroups on* M*. Then there exists a unique Feller convolution semigroup* $(\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$ *such that, for all* $n \geq 1, t \geq 0, f \in C_0(M^n)$ *and* $x \in M^n$*,*

$$P_t^n f(x) = \int K^{\otimes n} f(x) \nu_t(dK).$$

(ii) For every Feller convolution semigroup $(\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$*, there exists a stochastic flow of kernels* $(K_{s,t})_{s\leq t}$ *such that for all* $s \leq t$*, the law of* $K_{s,t}$ *is* ν_{t-s}*.*

Remarks 3.1. *(i) When (3.1) is satisfied, the stochastic flow of kernels* K *is induced by a stochastic flow of mappings.*
(ii) Let $\Omega^0 = \prod_{s\leqslant t} E, \quad \mathcal{A}^0 = \otimes_{s\leq t}\mathcal{E}$ *and*

$$\begin{array}{rcccc} K & : & (\Omega, \mathcal{A}, P) & \longrightarrow & (\Omega^0, \mathcal{A}^0) \\ & & \omega & \longmapsto & (K_{s,t}(\omega))_{s\leq t} \end{array}$$

be a stochastic flow of kernels. Then the law of K *is a probability measure on* $(\Omega^0, \mathcal{A}^0)$ *which is uniquely determined by the family* $(P_t^n, n \geq 1)$*. Denote by* ν_t *the law of* $K_{0,t}$*. Then the law of* K *is equivalently uniquely determined by* $(\nu_t)_{t\geq 0}$*.*

3.3 Noise filtration and coalescence

The content of this paragraph is borrowed from [35].

3.3.1 Noise filtration

Definition 3.6. *Let* $(\Omega, \mathcal{A}, \mathbb{P})$ *be a probability space. A noise filtration on* $(\Omega, \mathcal{A}, \mathbb{P})$ *is a family* $(\mathcal{F}_{s,t})_{-\infty\leq s\leq t\leq\infty}$ *of* σ*-fields of* $\mathcal{A}$ *such that*

(i) $\mathcal{F}_{s,t}$ *and* $\mathcal{F}_{t,u}$ *are independent for all* $s \leq t \leq u$*.*

(ii) $\mathcal{F}_{s,t} \vee \mathcal{F}_{t,u} = \mathcal{F}_{s,u}$ *for all* $s \leq t \leq u$*.*

(iii) For all $s \leq t$*,* $\mathcal{F}_{s,t}$ *contains all* $\mathbb{P}$*-negligeable sets of* $\mathcal{F}_{-\infty,\infty}$*.*

Let us give the following examples of noises

(1) **White noise filtration.** Consider a Brownian motion on the real line $(W_t)_{t\in\mathbb{R}}$ on $(\Omega,\mathcal{A},\mathbb{P})$, that is $(W_t)_{t\geq 0}$ and $(W_{-t})_{t\geq 0}$ are two independent standard Brownian motions. Throughout this thesis, we will use the notation

$$W_{s,t}=W_t-W_s,\ \ s\leq t.$$

Let

$$\mathcal{F}^W_{s,t}=\sigma(W_{u,v},s\leq u\leq v\leq t)\text{ for all }-\infty\leq s\leq t\leq\infty.$$

We complete $\mathcal{F}^W_{s,t}$ by the set of $\mathbb{P}$-negligeable sets of $\mathcal{F}^W_{-\infty,\infty}$ for all $-\infty\leq s\leq t\leq\infty$. Then $(\mathcal{F}^W_{s,t})$ is by definition the noise filtration associated to W.

(2) **Noise filtration associated to a stochastic flow.** Let K be a stochastic flow of kernels on $(\Omega,\mathcal{A},\mathbb{P})$ and define

$$\mathcal{F}^K_{s,t}=\sigma(K_{u,v},s\leq u\leq v\leq t)\text{ for all }-\infty\leq s\leq t\leq\infty.$$

We complete $\mathcal{F}^K_{s,t}$ by the set of $\mathbb{P}$-negligeable sets of $\mathcal{F}^K_{-\infty,\infty}$ for all $-\infty\leq s\leq t\leq\infty$. Then $(\mathcal{F}^K_{s,t})$ is by definition the noise filtration associated to K.

Given two noise filtrations $\mathcal{F}^1=(\mathcal{F}^1_{s,t})$ and $\mathcal{F}^2=(\mathcal{F}^2_{s,t})$, we say that $\mathcal{F}^1$ is a subnoise filtration of $\mathcal{F}^2$ provided $\mathcal{F}^1_{s,t}\subset\mathcal{F}^2_{s,t}$ for all $-\infty\leq s\leq t\leq\infty$. We now introduce the notion of filtering with respect to a subnoise filtration.

Proposition 3.4. *Let K be a stochastic flow of kernels and denote by $(\mathcal{F}^K_{s,t})$ the noise filtration associated to K. Let $\overline{\mathcal{F}}$ be a subnoise filtration of $\mathcal{F}^K$. Then there exits a stochastic flow of kernels $\overline{K}$ such that for all $s\leq t$ and $x\in M$,*

$$\overline{K}_{s,t}(x)=E[K_{s,t}(x)|\overline{\mathcal{F}}_{s,t}]=E[K_{s,t}(x)|\overline{\mathcal{F}}_{-\infty,\infty}]\ a.s.$$

We say that $\overline{K}$ is obtained by filtering K with respect to $\overline{\mathcal{F}}$.

3.3.2 Construction of a family of coalescent semigroups

Let $(P^n,n\geq 1)$ be a compatible family of Feller semigroups on M. Another important result of Le Jan and Raimond is the following

Theorem 3.3. *Let*

$$\Delta_n = \{x \in M^n, \exists i \neq j, x_i = x_j\} \text{ and } T_{\Delta_n} = \inf\{t \geq 0, X_t^n \in \Delta_n\}$$

where X^n denotes the n-point motion associated to P^n. There exists a unique compatible family $(P_t^{n,c}, n \geq 1)$ of Markovian semigroups on M such that if $X^{n,c}$ is the associated n-point motion and $T_{\Delta_n}^c = \inf\{t \geq 0, X_t^{n,c} \in \Delta_n\}$, then

(i) $(X_t^{n,c}, t \leq T_{\Delta_n}^c)$ is equal in law to $(X_t^n, t \leq T_{\Delta_n})$,

(ii) for $t \geq T_{\Delta_n}^c, X_t^{n,c} \in \Delta_n$.

Moreover, when the following condition (F) is satisfied, this family is constituted of Feller semigroups.
(F) For all $t > 0, \epsilon > 0$ and $x \in M$,

$$\lim_{y \to x} P_{(x,y)}^2[\{T_{\Delta_2} > t\} \cap \{d(X_t, Y_t) > \epsilon\}] = 0,$$

where $(X_t, Y_t) = X_t^2$. In this case, $(P_t^{n,c}, n \geq 1)$ satisfies (3.1) and is associated with a stochastic flow of mappings.
If $P_{(x,y)}^2[T_{\Delta_2} < \infty] = 1$ for all x and y in M, then the associated flow is coalescing.

Proof. Fix $(x_1, \cdots, x_n) \in M^n$ and let Y^n be the n point motion started at $(x_1, \cdots, x_n)$ associated to P^n. We denote the ith coordinate of Y_t^n by $Y_t^n(i)$. Let

$$T_1 = \inf\{u \geq 0, \exists i < j,\ Y_u^n(i) = Y_u^n(j)\}\ (\in [0, +\infty]), \quad Y_t^{n,c} := Y_t^n, t \in [0, T_1].$$

Suppose that $Y_{T_1}^n(i) = Y_{T_1}^n(j)$ with $i < j$. Then define the process

$$Y_t^{n,1}(h) = Y_t^n(h) \text{ for } h \neq j, Y_t^{n,1}(j) = Y_t^{n,1}(i), t \geq T_1.$$

Now set

$$T_2 = \inf\{u \geq T_1, \exists h < k, h \neq j, k \neq j,\ Y_u^{n,1}(h) = Y_u^{n,1}(k)\}.$$

For $t \in [T_1, T_2]$, we define $Y_t^{n,c} = Y_t^{n,1}$ and so on (see Figure 3.1).

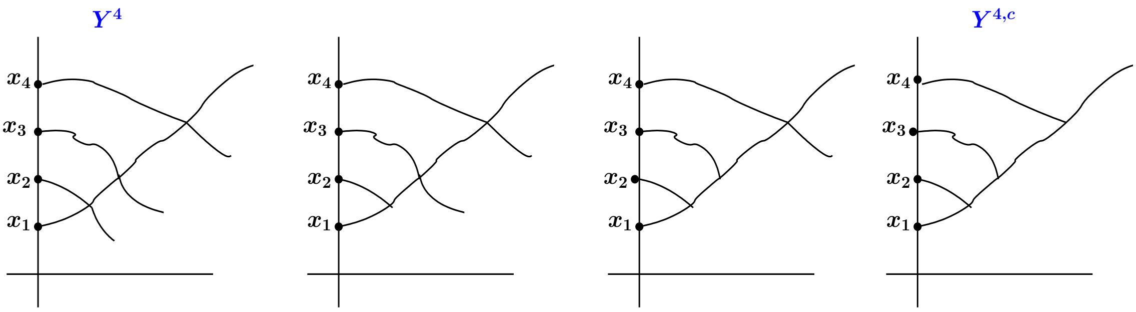

Figure 3.1: The coalescent semigroup.

In this way, we construct a Markov process $Y^{n,c}$ such that for all $i, j \in [1, n]$, $Y^{n,c}(i)$ and $Y^{n,c}(j)$ collide whenever they meet. Define $P_t^{n,c}(x_1, \cdots, x_n, dy)$ as the law of $Y_t^{n,c}$. Then $(P_t^{n,c}, n \geq 1)$ is a compatible family of Markovian semigroups on M. Let $X^{n,c}$ be the associated n-point motion. Then (i) and (ii) are clear. Note that when $P^2_{(x,y)}[T_{\Delta_2} < \infty] = 0$ for all $x \neq y$, then $P^{n,c} = P^n$ obviously. For every positive ϵ, we have

$$P^{2,c}_{(x,y)}[d(X_t, Y_t) > \epsilon] \leq P^2_{(x,y)}[\{T_{\Delta_2} > t\} \cap \{d(X_t, Y_t) > \epsilon\}]$$

which converges toward 0 as $y \to x$ when (F) is satisfied. Now the result holds from Lemma 3.1 (ii). The last claim is obvious. $\square$

Let $\nu = (\nu_t)_{t \geq 0}$ be the associated Feller convolution semigroup to $(P^n, n \geq 1)$. Suppose that $(P^{n,c}, n \geq 1)$ is constituted of Feller semigroups (which is true when (F) holds). We denote by ν^c the associated Feller convolution semigroup to $(P^{n,c}, n \geq 1)$. We close this section by the following

Theorem 3.4. *There exists a joint realization (K^c, K) where K^c and K are two stochastic flows of kernels associated respectively to ν^c and ν such that:*

(i) $\hat{K}_{s,t}(x, y) = K^c_{s,t}(x) \otimes K_{s,t}(y)$ *is a stochastic flow of kernels on* $M \times M$,

(ii) *For all* $s \leq t, x \in M$, $K_{s,t}(x) = E[K^c_{s,t}(x)|K]$ *a.s.*

Sketch of the proof. Fix $(x_1, y_1), \cdots, (x_n, y_n)$ in $M \times M$ and denote by X^{2n} the $2n$ point motion associated to P^{2n} started at $(x_1, y_1, \cdots, x_n, y_n)$. Let $X^{n,c}$ be the process obtained from $(X^{2n}(1), X^{2n}(3), \cdots, X^{2n}(2n-1))$ as in the proof of Theorem 3.3. Now set

$$\hat{X}^n = (X^{n,c}(1), X^{2n}(2), \cdots, X^{n,c}(n), X^{2n}(2n))$$

and define $\hat{P}^n_t((x_1, y_1, \cdots, x_n, y_n), dz)$ as the law of $\hat{X}^n_t$. Then $(\hat{P}^n, n \geq 1)$ is a compatible family of Feller semigroups on $(M \times M)^n$ to which is associated a stochastic flow of kernels $\hat{K}$ by Theorem 3.2. Then $\hat{K}$ is a tensor product

$$\hat{K}_{s,t}(x, y) = K^c_{s,t}(x) \otimes K_{s,t}(y)$$

and it is clear that the Feller convolution semigroup associated to K^c (respectively K) is ν^c (respectively K).

3.4 Tanaka's SDE and stochastic flows.

3.4.1 Tanaka's SDE

For a given Brownian motion W, Tanaka's equation driven by W is the one-dimensional stochastic differential equation

$$dX_t = \text{sgn}(X_t) dW_t, \quad X_0 = 0 \tag{3.3}$$

where $\text{sgn}(x) = 1_{\{x>0\}} - 1_{\{x\leq 0\}}$. The signum function does not satisfy the Lipschitz continuity condition required for the usual theorems guaranteeing existence and uniqueness of strong solutions. In fact, the Tanaka's equation admits a weak solution but has no strong solution, i.e. one for which X is adapted to the filtration generated by W (see the application of Lemma 2.1). In what follows X is a fixed weak solution of (3.3) and $(\mathcal{F}^X_t)$ is its natural filtration. Obviously $-X$ solves also (3.3). A natural question is the following

Can we describe all the solutions which are $(\mathcal{F}^X_t)$ adapted ?

The Balayage formula (see e.g. [45] page 260) supplies an answer to this question. Denote by g_t and d_t respectively the last zero of X before t and the first zero of X after t, namely:

$$g_t = \sup\{u < t : X_u = 0\},$$

$$d_t = \inf\{u > t : X_u = 0\}.$$

Let $(\varepsilon_u, u \geq 0)$ be a predictable process with respect to $(\mathcal{F}_t^X)$, and takes only values $+1$ and -1. Then by the Balayage formula $Y_t = \varepsilon_{g_t} X_t$ is a Brownian motion and more precisely

$$Y_t = \int_0^t \varepsilon_{g_s} dX_s.$$

Hence

$$\int_0^t \mathrm{sgn}(Y_s) dY_s = \int_0^t \mathrm{sgn}(\varepsilon_{g_s} X_s) \varepsilon_{g_s} dX_s = \int_0^t \mathrm{sgn}(X_s) dX_s = W_t.$$

In other words, Y solves (3.3) too. Conversely, Azéma and Yor (see [2] page 268) showed that any $(\mathcal{F}_t^X)$ continuous martingale Z such that $|Z| = |X|$ has the form $\eta_{g_t} X_t$ where η is an $(\mathcal{F}_t^X)$ predictable process. Since

$$|X_t| = |Y_t| = W_t - \inf_{0 \leq u \leq t} W_u,$$

we have therefore an answer to the above question.

3.4.2 Flows associated to Tanaka's SDE

In this paragraph, we consider a more general Tanaka's equation with variable initial conditions

$$\varphi_{s,t}(x) = x + \int_s^t \mathrm{sgn}(\varphi_{s,u}(x)) dW_u, \quad s \leq t, x \in \mathbb{R}, \tag{3.4}$$

where $(W_t)_{t \in \mathbb{R}}$ is a Brownian motion on the real line defined on a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

If K is a stochastic flow of kernels, then by definition, (K, W) is a solution of Tanaka's SDE if for all $s \leq t, x \in \mathbb{R}$, $f \in C_b^2(\mathbb{R})$ (f is C^2 on $\mathbb{R}$ and f', f'' are bounded)

$$K_{s,t} f(x) = f(x) + \int_s^t K_{s,u}(f' \mathrm{sgn})(x) dW_u + \frac{1}{2} \int_s^t K_{s,u} f''(x) du \quad a.s. \tag{3.5}$$

When $K = \delta_\varphi$, is a flow of mappings, (3.5) is then equivalent to (3.4). In [36] Le Jan and Raimond have classified all solution flows of (3.5) by probability measures on $[0,1]$ and have showed the following

Theorem 3.5. *(a) Let m be a probability measure on $[0,1]$ with mean $\frac{1}{2}$. Then, to m is associated a stochastic flow of kernels K^m solution of (3.5).*

- *To $\delta_{\frac{1}{2}}$ is associated a Wiener solution K^W.*
- *To $\frac{1}{2}(\delta_0 + \delta_1)$ is associated a coalescing stochastic flow of mappings φ.*

(b) For all stochastic flow of kernels K solution of (3.5) there exists a unique measure m with mean $\frac{1}{2}$ such that $K \stackrel{law}{=} K^m$.

We will review the construction of all solutions of (3.5) according to [36] in the next paragraph.

3.4.3 The construction

There exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which one can construct a process $(\varepsilon_{s,t}, U_{s,t}, W_{s,t})_{s\le t}$ indexed by $\{(s,t) \in \mathbb{R}^2, s \le t\}$ taking values in $\{-1,1\} \times [0,1] \times \mathbb{R}$ such that:

(i) $W_{s,t} := W_t - W_s, s \le t$ where W is a Brownian motion on the real line.

(ii) For a fixed $s < t$, $(\varepsilon_{s,t}, U_{s,t})$ is independent of W and

$$(\varepsilon_{s,t}, U_{s,t}) \stackrel{law}{=} m(du)(u\delta_1 + (1-u)\delta_{-1}).$$

In particular

$$\mathbb{P}(\varepsilon_{s,t} = 1 | U_{s,t}) = U_{s,t} \text{ and the law of } U_{s,t} \text{ is } m.$$

Set for all $s < t, \min_{s,t} = \inf\{W_u : u \in [s,t]\}$. Then

(iii) For all $s < t$ and $\{(s_i, t_i); 1 \le i \le n\}$ with $s_i < t_i$, the law of $(\varepsilon_{s,t}, U_{s,t})$ knowing $(\varepsilon_{s_i,t_i}, U_{s_i,t_i})_{1\le i\le n}$ and W is given by

$$m(du)(u\delta_1 + (1-u)\delta_{-1})$$

when $\min_{s,t} \notin \{\min_{s_i,t_i}; 1 \le i \le n\}$ and is given by

$$\sum_{i=1}^{n} \delta_{\varepsilon_{s_i,t_i}, U_{s_i,t_i}} \times \frac{1_{\{\min_{s,t}=\min_{s_i,t_i}\}}}{\mathrm{Card}\{i; \min_{s_i,t_i} = \min_{s,t}\}}$$

otherwise.

Note that (i)-(iii) uniquely define the law of $(\varepsilon_{s_1,t_1}, U_{s_1,t_1}, \cdots, \varepsilon_{s_n,t_n}, U_{s_n,t_n}, W)$ for all $s_i < t_i, 1 \le i \le n$. By construction, for all $s < t, u < v$, if $\mathbb{P}(\min_{s,t} = \min_{u,v}) > 0$, then

$$\mathbb{P}(\varepsilon_{s,t} = \varepsilon_{u,v}, U_{s,t} = U_{u,v} | \min_{s,t} = \min_{u,v}) = 1.$$

For $s, x \in \mathbb{R}$, define

$$\tau_s(x) = \inf\{r \ge s : \ W_{s,r} = -|x|\}$$

and for $x \in \mathbb{R}, s \le t$, let $W^+_{s,t} = W_t - \min_{s,t}$,

$$\varphi_{s,t}(x) = (x + \mathrm{sgn}(x)W_{s,t})1_{\{t\le\tau_s(x)\}} + \varepsilon_{s,t}W^+_{s,t}1_{\{t>\tau_s(x)\}},$$

$$K^m_{s,t}(x) = \delta_{x+\mathrm{sgn}(x)W_{s,t}}1_{\{t\le\tau_s(x)\}} + (U_{s,t}\delta_{W^+_{s,t}} + (1-U_{s,t})\delta_{-W^+_{s,t}})1_{\{t>\tau_s(x)\}}.$$

We will denote $W^+_{0,t}$ simply by W^+_t. Le Jan and Raimond showed that φ is a stochastic flow of mappings (see Lemma 4.3 [36]). For all $0 < s < t$, we have $\varepsilon_{0,t} = \varepsilon_{0,s}$ on $\{\min_{0,t} = \min_{0,s}\}$ and the law of $\varepsilon_{0,t}$ knowing $\sigma(\varepsilon_{0,u}, 0 \le u \le s) \vee \sigma(W)$ is $\frac{1}{2}(\delta_{-1} + \delta_1)$ on $\{\min_{0,t} < \min_{0,s}\}$. By Proposition 2.1, $\varphi_{0,\cdot}(0)$ is a Brownian motion. Apply Tanaka's formula so that

$$|\varphi_{0,t}(0)| = \int_0^t \mathrm{sgn}(\varphi_{0,u}(0))d\varphi_{0,u}(0) + L_t$$

where L_t is the nonsymmetric local time at 0 of $\varphi_{0,\cdot}(0)$. But $|\varphi_{0,t}(0)| = W^+_t$ and by identification, we get

$$\varphi_{0,t}(0) = \int_0^t \mathrm{sgn}(\varphi_{0,u}(0))dW_u.$$

Notice that φ and K^m are linked by the relation $K^m_{s,t}(x) = E[\delta_{\varphi_{s,t}(x)}|\sigma(U,W)]$ for all $x \in \mathbb{R}$ and $s < t$ which entails that K^m solves (3.5). To $m = \delta_{\frac{1}{2}}$, is associated the unique $\mathcal{F}^W$ adapted solution (Wiener flow) of (3.5):

$$K^W_{s,t}(x) = \delta_{x+\mathrm{sgn}(x)W_{s,t}} 1_{t\le\tau_s(x)} + \frac{1}{2}(\delta_{W^+_{s,t}} + \delta_{-W^+_{s,t}})1_{t>\tau_s(x)}.$$

Note that the one point motion of a solution of (3.5) is the Brownian motion and if $\int xm(x) \neq \frac{1}{2}$, this cannot be the case.

Remark 3.1. *Let ψ be a stochastic flow of mappings solving (3.4). Then*

(i) For all $x \in \mathbb{R}$,

$$\psi_{0,t}(x) = x + sgn(x)W_t = x + sgn(x)\int_0^t sgn(\psi_{0,u}(0))d\psi_{0,u}(0)\ \forall\ t \le \tau_0(x),$$

(ii) $|\psi_{0,t}(0)| = W^+_t$, for all $t \ge 0$ (see the application after Lemma 2.1).

In particular $\psi_{0,t}(0) = \psi_{0,t}(x) = 0$ at $t = \tau_0(x)$. Using (i) and (ii), one easily shows that in fact we have

$$\tau_0(x) = \inf\{r \ge 0, \psi_{0,r}(x) = \psi_{0,r}(0)\}.$$

Therefore ψ is a coalescing flow and $\psi_{0,r}(x) = \psi_{0,r}(0)$ for all $r \ge \tau_0(x)$. We have shown that for all $x \in \mathbb{R}$, $\psi_{0,\cdot}(x)$ is a measurable function of $\psi_{0,\cdot}(0)$. Since $\psi_{0,\cdot}(0)$ is a Brownian motion, the law of $(\psi_{0,\cdot}(x_1), \cdots, \psi_{0,\cdot}(x_n))$ is unique for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$. Assuming that φ constructed above is a flow associated to (3.4), then this is the unique flow of mappings solving Tanaka's equation.

We will review the content of this paragraph in a more general context as well as part (b) of Theorem 3.5 in the coming chapters. Let m be a probability measure on $[0,1]$ with mean $\frac{1}{2}$. We denote K^m simply by K and define $P^n_t = E[K^{\otimes n}_{0,t}]$. For all $1 \le i \le n$ and $(f_i)_{1\le i\le n}$ a family of measurbale bounded functions on $\mathbb{R}$, we have

$$P^n_t(f_1 \otimes \cdots \otimes f_n)(x_1, \cdots, x_n) = \int_0^1 P^{n,\alpha}_t(f_1 \otimes \cdots \otimes f_n)(x_1, \cdots, x_n)m(d\alpha),$$

where

$$P_t^{n,\alpha}(f_1 \otimes \cdots \otimes f_n)(x_1, \cdots, x_n) = E\big[\prod_{i=1}^{n} K_{0,t}^{\alpha} f_i(x_i)\big],$$

and

$$K_{s,t}^{\alpha}(x) := \delta_{x+\mathrm{sgn}(x)W_{s,t}} 1_{\{t\le\tau_s(x)\}} + (\alpha\delta_{W_{s,t}^{+}} + (1-\alpha)\delta_{-W_{s,t}^{+}})1_{\{t>\tau_s(x)\}}, s \le t, x \in \mathbb{R}.$$

Then one can easily check that K^{α} is a stochastic flow of kernels (this will be also justified in the next chapter). Remark also that $K^{\frac{1}{2}} = K^{W}$. It is now natural to ask what is the SDE satisfied by K^{α} for $\alpha \in [0, 1]$?. This will be of course a more general Tanaka's equation depending on α. We postpone the answer to the next chapter.

Chapter 4

Stochastic flows related to Walsh Brownian motion

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4.1 Introduction and main results

In [32], [35] Le Jan and Raimond have extended the classical theory of stochastic flows to include flows of probability kernels. Using the Wiener chaos decomposition, it was shown that non Lipschitzian stochastic differential equations have a unique Wiener measurable solution given by random kernels. Later, the theory was applied in [36] to the study of Tanaka's equation (3.4). The extension to kernels of (3.4) is defined by (3.5). Each solution flow of (3.5) can be characterized by a probability measure on $[0,1]$ which entirely determines its law. Among solutions of (3.5), there is only one flow of mappings which has been already studied in [51].

We now fix $\alpha \in [0,1]$ and consider the following SDE driven by a Brownian motion on the real line W:

$$X_t^{s,x} = x + W_{s,t} + (2\alpha - 1)\tilde{L}_{s,t}^x, \quad t \geq s, x \in \mathbb{R}, \tag{4.1}$$

where $W_{s,t} = W_t - W_s$, $s \leq t$ and

$$\tilde{L}_{s,t}^x = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1_{|X_u^{s,x}| \leq \varepsilon} du \quad \text{(The symmetric local time)}.$$

Equation (4.1) was introduced in [22]. For a fixed initial condition, it has a pathwise unique solution which is distributed as the $SBM(\alpha)$ (see Section 2.2). It was shown in [3] that when $\alpha \neq \frac{1}{2}$, flows associated to (4.1) are coalescing and a deeper study of (4.1) was provided later in [10] and [11]. Now, consider the following generalization of (3.4):

$$X_{s,t}(x) = x + \int_s^t \operatorname{sgn}(X_{s,u}(x)) dW_u + (2\alpha - 1)\tilde{L}_{s,t}^x(X), \quad s \leq t, x \in \mathbb{R}, \tag{4.2}$$

where

$$\tilde{L}_{s,t}^x(X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1_{|X_{s,u}(x)| \leq \varepsilon} du.$$

Each solution of (4.2) is distributed as the $SBM(\alpha)$. By Tanaka's formula for symmetric local time

$$|X_{s,t}(x)| = |x| + \int_s^t \widetilde{\mathrm{sgn}}(X_{s,u}(x))dX_{s,u}(x) + \tilde{L}^x_{s,t}(X).$$

By combining the last identity with (4.2), we have

$$|X_{s,t}(x)| = |x| + W_{s,t} + \tilde{L}^x_{s,t}(X). \tag{4.3}$$

The uniqueness of solutions of the Skorokhod equation (Lemma 2.1), entails

$$|X_{s,t}(x)| = |x| + W_{s,t} - \min_{s\le u\le t}[(|x| + W_{s,u}) \wedge 0]. \tag{4.4}$$

Clearly (4.3) and (4.4) imply that $\sigma(|X_{s,u}(x)|; s \le u \le t) = \sigma(W_{s,u}; s \le u \le t)$ which is strictly smaller than $\sigma(X_{s,u}(x); s \le u \le t)$ and so $X_{s,\cdot}(x)$ cannot be a strong solution of (4.2). For these reasons, we call (4.2) Tanaka's SDE related to $SBM(\alpha)$. Now recall the definitions

- $C^2_b(\mathbb{R}^*) = \{f \in C(\mathbb{R}) : f$ is twice derivable on $\mathbb{R}^*$, $f', f'' \in C_b(\mathbb{R}^*)$, $f'_{|]0,+\infty]}, f''_{|]0,+\infty]}$ (resp. $f'_{|]-\infty,0[}, f''_{|]-\infty,0[}$) have right (resp. left) limit in $0\}$.
- $D_\alpha = \{f \in C^2_b(\mathbb{R}^*) : \alpha f'(0+) = (1-\alpha)f'(0-)\}$.

For $f \in D_\alpha$, we set by convention $f'(0) = f'(0-), f''(0) = f''(0-)$. By Itô-Tanaka formula (see [38] page 432) or Freidlin-Sheu formula (Theorem 2.3) and Proposition 2.5, both extensions to kernels of (4.1) and (4.2) may be defined by

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(\varepsilon f')(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du, \quad f \in D_\alpha, \tag{4.5}$$

where $\varepsilon(x) = 1$ (respectively $\varepsilon(x) = \mathrm{sgn}(x)$) in the first (respectively second) case, but due to the pathwise uniqueness of (4.1), the unique solution of (4.5) when $\varepsilon(x) = 1$, is $K_{s,t}(x) = \delta_{X^{s,x}_t}$ (this can be justified by (6.15)). Our first aim is to define an extention of (4.5) related to WBM in general. We begin by defining our graph.

Definition 4.1. *(Graph G)*

Fix $N \geq 1$ and $\alpha_1, \cdots, \alpha_N > 0$ such that $\sum_{i=1}^{N} \alpha_i = 1$.

We consider the graph G defined in Section 2.3.1. Recall the definitions of

- $C_b^2(G^*) = \{f \in C(G) : \forall i \in [1, N], f_i$ *is twice derivable on* $\mathbb{R}_+^*, f_i', f_i'' \in C_b(\mathbb{R}_+^*)$ *and both have finite limits at* $0+\}$.
- $D(\alpha_1, \cdots, \alpha_N) = \{f \in C_b^2(G^*) : \sum_{i=1}^{N} \alpha_i f_i'(0+) = 0\}$.

For all $x \in G$, we define $\vec{e}(x) = \vec{e}_i$ if $x \in D_i, x \neq 0$ (convention $\vec{e}(0) = \vec{e}_N$). For $f \in C_b^2(G^*)$, $x \neq 0$, let $f'(x)$ be the derivative of f at x relatively to $\vec{e}(x)$ ($= f_i'(|x|)$ if $x \in D_i$) and $f''(x) = (f')'(x)$ ($= f_i''(|x|)$ if $x \in D_i$). We use the conventions $f'(0) = f_N'(0+), f''(0) = f_N''(0+)$. Now, associate to each ray D_i a sign $\varepsilon_i \in \{-1, 1\}$ and then define

$$\varepsilon(x) = \begin{cases} \varepsilon_i & \text{if } x \in D_i, x \neq 0 \\ \varepsilon_N & \text{if } x = 0 \end{cases}$$

To simplify, we suppose that $\varepsilon_1 = \cdots = \varepsilon_p = 1, \quad \varepsilon_{p+1} = \cdots = \varepsilon_N = -1$ for some $p \leq N$. Set

$$G^+ = \bigcup_{1 \leq i \leq p} D_i, \quad G^- = \bigcup_{p+1 \leq i \leq N} D_i. \text{ Then } G = G^+ \bigcup G^- \text{ (Figure 4.1).}$$

We also put $\alpha^+ = 1 - \alpha^- := \sum_{i=1}^{p} \alpha_i$.

Remark 4.1. *Our graph can be simply defined as N pieces of $\mathbb{R}_+$ in which the N origins are identified. The values of the $\vec{e}_i$ will not have any effect in the sequel.*

Definition 4.2. *(Equation (E)).*

On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let W be a Brownian motion on the real line and K be a stochastic flow of kernels on G. We say that (K, W) solves (E) if for all $s \leq t, f \in D(\alpha_1, \cdots, \alpha_N), x \in G$,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(\varepsilon f')(x) dW_u + \frac{1}{2} \int_s^t K_{s,u} f''(x) du \quad a.s.$$

If $K = \delta_\varphi$ is a solution of (E), we simply say that (φ, W) solves (E).

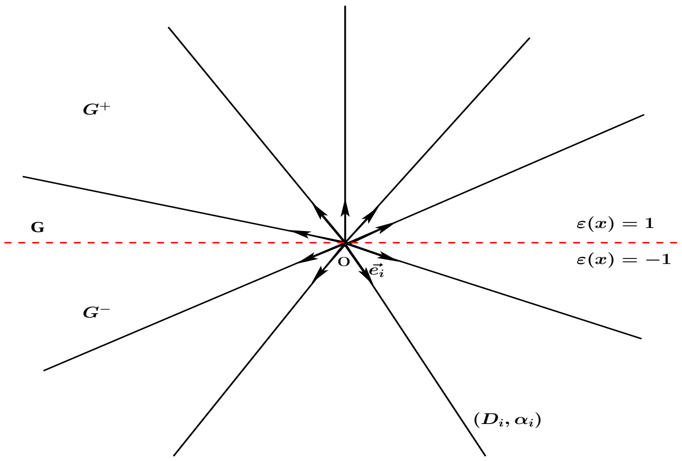

Figure 4.1: Graph G.

Remarks 4.1. *(1) Our graph can be simply defined as N pieces of $\mathbb{R}_+$ in which the N origins are identified. The values of the $\vec{e}_i$ will not have any effect in the sequel.*
(2) If (K, W) solves (E), then $\sigma(W) \subset \sigma(K)$ (see Corollary 4.2) below. So, one can simply say that K solves (E).
(3) The case $N = 2, p = 2, \varepsilon_1 = \varepsilon_2 = 1$ (Figure 4.2) corresponds to Tanaka's SDE related to SBM and includes in particular the usual Tanaka's SDE [36]. In fact, let $(K^{\mathbb{R}}, W)$ be a solution of (4.5) with $\alpha = \alpha_1, \varepsilon(y) = sgn(y)$ and define $\psi(y) = |y|(\vec{e}_1 1_{y \geq 0} + \vec{e}_2 1_{y<0}), y \in \mathbb{R}$. For all $x \in G$, define $K^G_{s,t}(x) = \psi(K^{\mathbb{R}}_{s,t}(y))$ with $y = \psi^{-1}(x)$. Let $f \in D(\alpha_1, \alpha_2), x \in G$ and g be defined on $\mathbb{R}$ by $g(z) = f(\psi(z))$ $(g \in D_{\alpha_1})$. Since $K^{\mathbb{R}}$ satisfies (4.5) in $(g, \psi^{-1}(x))$ (g is the test function and $\psi^{-1}(x)$ is the starting point), it easily comes that K^G satisfies (E) in (f, x). Similarly, if K^G solves (E), then $K^{\mathbb{R}}$ solves (4.5).

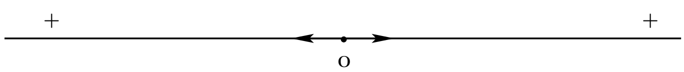

Figure 4.2: Tanaka's SDE.

(4) As in (2), the case $N = 2, p = 1, \varepsilon_1 = 1, \varepsilon_2 = -1$ (Figure 4.3) corresponds to (4.1).

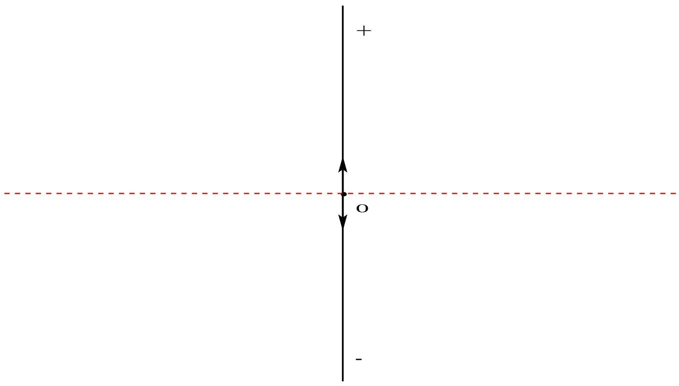

Figure 4.3: SBM equation.

(5) Equation (E) *can be defined differently. We call* f' *the derivative of* f *in the sense* $0 \rightarrow$ *(resp.* $0 \leftarrow$*) on* G^+ *(resp. on* G^-*) (Figure 4.4). Set* $\varepsilon_i = 1_{i\in[1,p]} - 1_{i\in[p+1,N]}$. *Then* (E) *is equivalent to*

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}f'(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du,$$

where $f \in C_b^2(G^*), \sum_{i=1}^{N} \alpha_i \varepsilon_i \lim_{z\to 0, z\in D_i, z\neq 0} f'(z) = 0.$

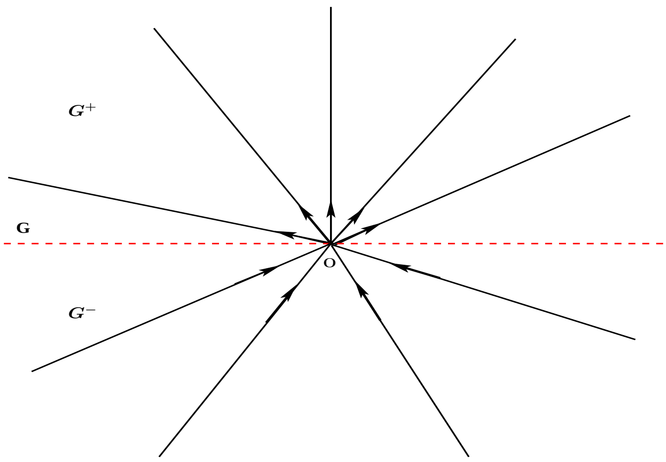

Figure 4.4: Graph G.

In this chapter, we classify all solutions of (E) by means of probability measures. We now state the first

Theorem 4.1. *Let W be a Brownian motion on the real line and $X_t^{s,x}$ be the flow associated to (4.1) with $\alpha = \alpha^+$. Define $Z_{s,t}(x) = X_t^{s,\varepsilon(x)|x|}, s \le t, x \in G$ and*

$$\begin{aligned} K_{s,t}^W(x) &= \delta_{x+\vec{e}(x)\varepsilon(x)W_{s,t}} 1_{\{t \le \tau_{s,x}\}} \\ &+ \Big(\sum_{i=1}^{p} \frac{\alpha_i}{\alpha^+} \delta_{\vec{e}_i |Z_{s,t}(x)|} 1_{\{Z_{s,t}(x)>0\}} + \sum_{i=p+1}^{N} \frac{\alpha_i}{\alpha^-} \delta_{\vec{e}_i |Z_{s,t}(x)|} 1_{\{Z_{s,t}(x)\le 0\}}\Big) 1_{\{t > \tau_{s,x}\}}, \end{aligned}$$

where $\tau_{s,x} = \inf\{r \ge s : x + \vec{e}(x)\varepsilon(x)W_{s,r} = 0\}$. Then, K^W is the unique Wiener solution of (E). This means that K^W solves (E) and if K is another Wiener solution of (E), then for all $s \le t, x \in G$, $\quad K_{s,t}^W(x) = K_{s,t}(x)$ a.s.

The proof of this theorem follows [32] (see also [41] for more details) with some modifications adapted to our case. We will use Freidlin-Sheu formula for WBM to check that K^W solves (E). Unicity will be justified by means of the Wiener chaos decomposition (Proposition 4.5). Besides the Wiener flow, there are also other weak solutions associated to (E) which are fully described by the following

Theorem 4.2. *(1) Define*

$$\Delta_k = \Big\{u = (u_1, \cdots, u_k) \in [0,1]^k : \sum_{i=1}^{k} u_i = 1\Big\}, \quad k \ge 1.$$

Suppose $\alpha^+ \ne \frac{1}{2}$.
(a) Let m^+ and m^- be two probability measures respectively on Δ_p and Δ_{N-p} satisfying:

$$(+) \int_{\Delta_p} u_i m^+(du) = \frac{\alpha_i}{\alpha^+}, \forall 1 \le i \le p,$$

$$(-) \int_{\Delta_{N-p}} u_j m^-(du) = \frac{\alpha_{j+p}}{\alpha^-}, \ \forall 1 \le j \le N-p.$$

Then, to (m^+, m^-) is associated a stochastic flow of kernels K^{m^+,m^-} solution of (E).

- *To $\big(\delta_{(\frac{\alpha_1}{\alpha^+}, \cdots, \frac{\alpha_p}{\alpha^+})}, \delta_{(\frac{\alpha_{p+1}}{\alpha^-}, \cdots, \frac{\alpha_N}{\alpha^-})}\big)$ is associated a Wiener solution K^W.*

- *To* $(\sum_{i=1}^{p}\frac{\alpha_i}{\alpha^+}\delta_{0,..,0,1,0,..,0}, \sum_{i=p+1}^{N}\frac{\alpha_i}{\alpha^-}\delta_{0,..,0,1,0,..,0})$ *is associated a coalescing stochastic flow of mappings* φ.

(b) For all stochastic flow of kernels K solution of (E) there exists a unique pair of measures (m^+, m^-) satisfying conditions $(+)$ and $(-)$ such that $K \stackrel{law}{=} K^{m^+,m^-}$.
(2) If $\alpha^+ = \frac{1}{2}, N > 2$, then there is just one solution of (E) which is a Wiener solution.

Remarks 4.2. *(1)If $\alpha^+ = 1$, solutions of (E) are characterized by a unique measure m^+ satisfying condition $(+)$ instead of a pair (m^+, m^-) and a similar remark applies if $\alpha^- = 1$.*
(2) The case $\alpha^+ = \frac{1}{2}, N = 2$ does not appear in the last theorem since it corresponds to $dX_t = W(dt)$.

This chapter follows ideas of [36] in a more general context and is organized as follows. In Section 4.2, we use a "specific" $SBM(\alpha^+)$ flow introduced by Burdzy-Kaspi and excursion theory to construct all solutions of (E). Unicity of solutions is proved in Section 4.3.

4.2 Construction of flows associated to (E)

In this section, we prove (a) of Theorem 4.2 and we show that K^W given in Theorem 4.1 solves (E).

4.2.1 Flow of Burdzy-Kaspi associated to SBM

Definition

We are looking for flows associated to the SDE (4.1). The flow associated to $SBM(1)$ which solves (4.1) is the reflected Brownian motion above 0 given by

$$Y_{s,t}(x) = (x + W_{s,t})1_{\{t\le\tau_{s,x}\}} + (W_{s,t} - \inf_{u\in[\tau_{s,x},t]} W_{s,u})1_{\{t>\tau_{s,x}\}},$$

where

$$\tau_{s,x} = \inf\{r \geq s : x + W_{s,r} = 0\}. \tag{4.6}$$

and a similar expression holds for the $SBM(0)$ which is the reflected Brownian motion below 0. These flows satisfy all properties of the $SBM(\alpha), \alpha \in]0,1[$ we will mention below such that the "strong"flow property (Proposition 4.1) and the strong comparison principle (4.7). When $\alpha \in]0,1[$, we follow Burdzy-Kaspi [11]. In the sequel, we will be interested in $SBM(\alpha^+)$ and so we suppose in this paragraph that $\alpha^+ \notin \{0,1\}$.

With probability 1, for all rationals s and x simultaneously, equation (4.1) has a unique strong solution with $\alpha = \alpha^+$. Define

$$Y_{s,t}(x) = \inf_{\substack{u,y \in \mathbb{Q} \\ u<s, x<X_s^{u,y}}} X_t^{u,y}, \quad L_{s,t}(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_s^t 1_{\{|Y_{s,u}(x)| \leq \varepsilon\}} du.$$

Then, $(s,t,x,\omega) \longmapsto Y_{s,t}(x,\omega)$ is measurable from $\{(s,t,x,\omega), s \leq t, x \in \mathbb{R}, \omega \in \Omega\}$ into $\mathbb{R}$. It is easy to see that a.s.

$$Y_{s,t}(x) \leq Y_{s,t}(y) \quad \forall s \leq t, x \leq y. \tag{4.7}$$

This implies that $x \longmapsto Y_{s,t}(x)$ is increasing and càdlàg for all $s \leq t$ a.s.

According to [11] (Proposition 1.1), $t \longmapsto Y_{s,t}(x)$ is Hölder continuous for all s, x a.s. and with probability equal to 1: $\forall s, x \in \mathbb{R}, \quad Y_{s,\cdot}(x)$ satisfies (4.1). We first check that Y is a flow of mappings and start by the following flow property:

Proposition 4.1. *$\forall\ t \geq s$ a.s.*

$$Y_{s,u}(x) = Y_{t,u}(Y_{s,t}(x)) \quad \forall u \geq t, x \in \mathbb{R}\ .$$

Proof. It is known, since pathwise uniqueness holds for the SDE (4.1), that for a fixed $s \leq t \leq u, x \in \mathbb{R}$, we have $Y_{s,u}(x) = Y_{t,u}(Y_{s,t}(x)) \ \ a.s.$ ([28] page 161). Now, using the regularity of the flow, the result extends clearly as desired. ☐

To conclude that Y is a stochastic flow of mappings, it remains to show the following

Lemma 4.1. $\forall t \geq s,\ x \in \mathbb{R},\ f \in C_0(\mathbb{R})$

$$\lim_{y\to x} E[(f(Y_{s,t}(x)) - f(Y_{s,t}(y)))^2] = 0.$$

Proof. We take $s = 0$. For $g \in C_0(\mathbb{R}^2)$, set

$$P_t^{(2)} g(x) = E[g(Y_{0,t}(x_1), Y_{0,t}(x_2))], \quad x = (x_1, x_2).$$

If $\varepsilon > 0, \quad f_\varepsilon(x,y) = 1_{\{|x-y|\geq\varepsilon\}}$, then by Theorem 10 in [38], $P_t^{(2)} f_\varepsilon(x,y) \xrightarrow[y \to x]{} 0$. For all $f \in C_0(\mathbb{R})$, we have

$$E[(f(Y_{0,t}(x)) - f(Y_{0,t}(y)))^2] = P_t^{(2)} f^{\otimes^2}(x,x) + P_t^{(2)} f^{\otimes^2}(y,y) - 2P_t^{(2)} f^{\otimes^2}(x,y).$$

To conclude the lemma, we need only to check that

$$\lim_{y\to x} P_t^{(2)} f(y) = P_t^{(2)} f(x),\ \forall x \in \mathbb{R}^2, \quad f \in C_0(\mathbb{R}^2).$$

Let $f = f_1 \otimes f_2$ with $f_i \in C_0(\mathbb{R})$, $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then

$$|P_t^{(2)} f(y) - P_t^{(2)} f(x)| \leq M \sum_{k=1}^{2} P_t^{(2)} (|1 \otimes f_k - f_k \otimes 1|)(y_k, x_k),$$

where $M > 0$ is a constant. For all $\alpha > 0$, $\exists \varepsilon > 0$, $|u - v| < \varepsilon \Rightarrow \forall 1 \leq k \leq 2 :$ $|f_k(u) - f_k(v)| < \alpha$. As a result

$$|P_t^{(2)} f(y) - P_t^{(2)} f(x)| \leq 2M\alpha + 2M \sum_{k=1}^{2} ||f_k||_\infty P_t^{(2)} f_\varepsilon(x_k, y_k),$$

and we arrive at $\limsup_{y\to x} |P_t^{(2)} f(y) - P_t^{(2)} f(x)| \leq 2M\alpha$ for all $\alpha > 0$ which means that $\lim_{y\to x} P_t^{(2)} f(y) = P_t^{(2)} f(x)$. Now this easily extends by a density argument for all $f \in C_0(\mathbb{R}^2)$. ∎

In the coming section, we present some properties related to the coalescence of Y we will require in Section 4.2.2 to construct solutions of (E).

Coalescence of the Burdzy-Kaspi flow

In this section, we suppose $\frac{1}{2} < \alpha^+ < 1$. The analysis of the case $0 < \alpha^+ < \frac{1}{2}$ requires an application of symmetry. Define

$$T_{x,y} = \inf\{r \geq 0,\ Y_{0,r}(x) = Y_{0,r}(y)\}, \quad x, y \in \mathbb{R}.$$

By the fundamental result of [3], $T_{x,y} < \infty$ *a.s.* for all $x, y \in \mathbb{R}$. Due to the local time, coalescence always occurs in 0; $Y_{0,r}(x) = Y_{0,r}(y) = 0$ if $r = T_{x,y}$. Recall the definition of $\tau_{s,x}$ from (4.6). Then $T_{x,y} > \sup(\tau_{0,x}, \tau_{0,y})$ a.s. ([3] page 203). Set

$$L_t^x = x+(2\alpha^+-1)L_{0,t}(x),\ \ U(x,y) = \inf\{z \geq y : L_t^x = L_t^y = z \text{ for some } t \geq 0\}, y \geq x.$$

According to [10] (Theorem 1.1), there exists $\lambda > 0$ such that

$$\forall u \geq y > 0, \quad \mathbb{P}(U(0,y) \leq u) = (1 - \frac{y}{u})^\lambda.$$

Thus for a fixed $0 < \gamma < 1$, we get $\lim_{y\to 0+} \mathbb{P}(U(0,y) \leq y^\gamma) = \lim_{y\to 0+}(1 - y^{1-\gamma})^\lambda = 1$.

From Theorem 1.1 [10], we have $U(x,y) - x \overset{law}{=} U(0, y-x)$ for all $0 < x < y$ and so

$$\lim_{y\to x+} \mathbb{P}(U(x,y) - x \leq (y-x)^\gamma) = 1,\ \forall x \geq 0. \tag{4.8}$$

Lemma 4.2. *For all $x \in \mathbb{R}$, we have $\lim_{y\to x} T_{x,y} = \tau_{0,x}$ in probability.*

Proof. For simplicity, we will write only Y_t instead of $Y_{0,t}(0)$. We first establish the result for $x = 0$. For all $t > 0$, we have

$$\mathbb{P}(t \leq T_{0,y}) \leq \mathbb{P}(L_{0,t}(0) \leq L_{0,T_{0,y}}(0)) = \mathbb{P}(L_t^0 \leq U(0,y))$$

since $(2\alpha^+ - 1)L_{0,T_{0,y}}(0) = U(0,y)$. The right-hand side converges to 0 as $y \to 0+$ by (4.8). On the other hand, by the strong Markov property at time $\tau_{0,y}$ for $y < 0$,

$$G_t(y) := \mathbb{P}(t \leq T_{0,y}) = \mathbb{P}(t \leq \tau_{0,y}) + E[1_{\{t>\tau_{0,y}\}} G_{t-\tau_{0,y}}(Y_{\tau_y})].$$

For all $\epsilon > 0$,

$$E[1_{\{t>\tau_{0,y}\}}G_{t-\tau_{0,y}}(Y_{\tau_{0,y}})] = E[1_{\{t-\tau_{0,y}>\epsilon\}}G_{t-\tau_{0,y}}(Y_{\tau_{0,y}})] + E[1_{\{0<t-\tau_{0,y}\leq\epsilon\}}G_{t-\tau_{0,y}}(Y_{\tau_{0,y}})]$$

$$\leq E[G_\epsilon(Y_{\tau_{0,y}})] + \mathbb{P}(0 < t - \tau_{0,y} \leq \epsilon).$$

From previous observations, we have $Y_{\tau_{0,y}} > 0$ $a.s.$ for all $y < 0$ and consequently $Y_{\tau_{0,y}} \longrightarrow 0+$ as $y \to 0-$. Since $\lim_{z\to 0+} G_\epsilon(z) = 0$, by letting $y \to 0-$ and using dominated convergence, then $\epsilon \to 0$, we get $\limsup_{y\to 0-} G_t(y) = 0$ as desired for $x = 0$. Now, the lemma easily holds after remarking that

$$T_{x,y} - \tau_{0,x} \overset{law}{=} T_{0,y-x} \text{ if } 0 \leq x < y \text{ and } T_{x,y} - \tau_{0,x} \overset{law}{=} T_{0,x-y} \text{ if } x < y \leq 0.$$

□

For $s \leqslant t, x \in \mathbb{R}$, define

$$g_{s,t}(x) = \sup\{u \in [s,t] : Y_{s,u}(x) = 0\} \quad (\sup(\emptyset) = -\infty). \tag{4.9}$$

Recall that $t \longmapsto Y_{s,t}(x)$ is Hölder continuous for all s, x a.s. Then for all $s \in \mathbb{R}$, $g_{s,t}(x,\omega)$ is measurable with respect to (t, x, ω). In fact, $1_{\{t\geq\tau_{s,x}\}}g_{s,t}(x,\omega) = 1_{\{t\geq\tau_{s,x}\}}(\tau_{s,x} + f \circ h(t,x,\omega))$ where $h(t,x,\omega) = (Y_{s,\tau_{s,x}+\cdot}(x), (t-\tau_{s,x}) \vee 0) \in H \times \mathbb{R}_+$, $H = \{f \in C([0,+\infty[,\mathbb{R}) : f(0) = 0\}$ and $h(f,t) = \sup\{0 \leq u \leq t : f(u) = 0\}$ for all $(f,t) \in H \times \mathbb{R}_+$. It is clear that f and h are measurable (for a fixed f, $h(f,\cdot)$ is right-continuous), which proves our claim.

We use Lemma 4.2 to prove

Lemma 4.3. *Fix $s \leq t, x \in \mathbb{R}$. Then, there exists an $\mathcal{F}^W_{s,t}$-measurable random variable $(v,y) \in \mathbb{Q}^2$, which depends on (s,t,x), such that on $\{t > \tau_{s,x}\}$, we have*

$$s \leq v < g_{s,t}(x) \text{ and } Y_{s,t}(x) = Y_{v,t}(y).$$

Proof. For $(s,x) \in \mathbb{Q}^2$, this is evident. For all $n \geq 0$, let $\mathbb{D}_n = \{\frac{k}{2^n},\ k \in \mathbb{Z}\}$ and $\mathbb{D}$ be the set of all dyadic numbers: $\mathbb{D} = \cup_{n\in\mathbb{N}}\mathbb{D}_n$. For $u < v$, define $n(u,v) = \inf\{n \in$

$\mathbb{N} : \mathbb{D}_n \cap]u, v[\neq \emptyset\}$ and $f(u, v) = \inf \mathbb{D}_{n(u,v)} \cap]u, v[$.

In the sequel, we assume that $(s, x) \notin \mathbb{Q}^2$. First take $x = 0$ and denote by $T^s_{a,b}$ the coalescing time of $Y_{s,\cdot}(a)$ and $Y_{s,\cdot}(b)$ for all $a, b \in \mathbb{R}$. Then for all $\epsilon > 0$,

$$\mathbb{P}(\exists\ \eta > 0 : Y_{s,t}(\eta) = Y_{s,t}(-\eta)) \geq \mathbb{P}(T^s_{-\epsilon,\epsilon} \leq t).$$

From $\mathbb{P}(t < T^s_{-\epsilon,\epsilon}) \leq \mathbb{P}(t < T^s_{0,\epsilon}) + \mathbb{P}(t < T^s_{0,-\epsilon})$ and the previous lemma, we have $\lim_{\epsilon\to 0} \mathbb{P}(t < T^s_{-\epsilon,\epsilon}) = 0$ and therefore $\mathbb{P}(\exists\ \eta > 0 : Y_{s,t}(\eta) = Y_{s,t}(-\eta)) = 1$. We will define (v, y) on $\tilde{\Omega} = \{\exists p \geq 1 : Y_{s,t}(\frac{1}{p}) = Y_{s,t}(-\frac{1}{p})\}$ and give an arbitrary value to (v, y) on $\tilde{\Omega}^c$. Let p be the smallest integer such that $Y_{s,t}(\frac{1}{p}) = Y_{s,t}(-\frac{1}{p})$ and $v = f(s, T^s_{-\frac{1}{p},\frac{1}{p}})$. Then $Y_{s,v}(\frac{1}{p}) > Y_{s,v}(-\frac{1}{p})$. Let $y = f(Y_{s,v}(-\frac{1}{p}), Y_{s,v}(\frac{1}{p}))$. By (4.7), for all $u \geq v$,

$$Y_{v,u}(Y_{s,v}(-\frac{1}{p})) \leq Y_{v,u}(y) \leq Y_{v,u}(Y_{s,v}(\frac{1}{p})).$$

The flow property (Proposition 4.1) yields $Y_{s,u}(-\frac{1}{p}) \leq Y_{v,u}(y) \leq Y_{s,u}(\frac{1}{p})$ for all $u \geq v$. So necessarily $Y_{s,t}(0) = Y_{s,t}(\frac{1}{p}) = Y_{s,t}(-\frac{1}{p}) = Y_{v,t}(y)$. For $x > 0$ and ϵ sufficiently small, we have

$$\mathbb{P}(Y_{s,t}(x + \epsilon) > Y_{s,t}(x), t > \tau_{s,x}) \leq \mathbb{P}(\tau_{s,x} < t < T^s_{x,x+\epsilon}).$$

This shows that $\lim_{\epsilon\to 0} \mathbb{P}(Y_{s,t}(x+\epsilon) > Y_{0,t}(x)|t > \tau_{s,x}) = 0$ by Lemma 4.2. Similarly, for ϵ small

$$\mathbb{P}(Y_{s,t}(x - \epsilon) < Y_{s,t}(x), t > \tau_{s,x}) \leq \mathbb{P}(\tau_{s,x} < t < T^s_{x-\epsilon,x}).$$

Lemma 4.2 states that the right-hand side converges to 0 as $\epsilon \longrightarrow 0$ and so $\lim_{\epsilon\to 0} \mathbb{P}(Y_{s,t}(x) > Y_{s,t}(x - \epsilon)|t > \tau_{s,x}) = 0$. Since

$$\{Y_{s,t}(x + \epsilon) > Y_{s,t}(x - \epsilon)\} \subset \{Y_{s,t}(x + \epsilon) > Y_{s,t}(x)\} \cup \{Y_{s,t}(x) > Y_{s,t}(x - \epsilon)\},$$

we get $\mathbb{P}(\exists\epsilon > 0 : Y_{s,t}(x - \epsilon) = Y_{s,t}(x + \epsilon)|t > \tau_{s,x}) = 1$. Following the same steps as the case $x = 0$, we define (v, y) on $\{t > \tau_{s,x}\}$ and give an arbitrary value to (v, y) on $\{t \leq \tau_{s,x}\}$. $\square$

Remark 4.2. *The preceding lemma implies in particular that for a fixed* (s, x)*, with probability* 1*, for all* $t > \tau_{s,x}$*, there exists* $(v, y) \in \mathbb{Q}^2$ *such that* $s \le v < g_{s,t}(x)$ *and* $Y_{s,t}(x) = Y_{v,t}(y)$*. This is clear by taking a rational* $t' \in]\tau_{s,x}, t[$*.*

We close this section by the

Lemma 4.4. *With probability 1, for all* $(s_1, x_1) \neq (s_2, x_2) \in \mathbb{Q}^2$ *simultaneously*
(i) $T^{x_1,x_2}_{s_1,s_2} := \inf\{r \ge sup(s_1, s_2) : Y_{s_1,r}(x_1) = Y_{s_2,r}(x_2)\} < \infty$,
(ii) $T^{x_1,x_2}_{s_1,s_2} > \sup(\tau_{s_1,x_1}, \tau_{s_2,x_2})$,
(iii) $Y_{s_1,T^{x_1,x_2}_{s_1,s_2}}(x_1) = Y_{s_2,T^{x_1,x_2}_{s_1,s_2}}(x_2) = 0$,
(iv) $Y_{s_1,r}(x_1) = Y_{s_2,r}(x_2)\ \forall r \ge T^{x_1,x_2}_{s_1,s_2}$.

Proof. (i) is a consequence of Proposition 4.1, the independence of increments and the coalescence of Y.(ii) Fix $(s_1, x_1) \neq (s_2, x_2) \in \mathbb{Q}^2$ with $s_1 \le s_2$. By the comparison principle (4.7) and Proposition 4.1, $Y_{s_1,t}(x_1) \ge Y_{s_2,t}(x_2)$ for all $t \ge s_2$ or $Y_{s_1,t}(x_1) \le Y_{s_2,t}(x_2)$ for all $t \ge s_2$. Suppose for example that $0 < z := Y_{s_1,s_2}(x_1) < x_2$ and take a rational $r \in]z, x_2[$. Then $T^{x_1,x_2}_{s_1,s_2} > \tau_{s_2,z} \ge \tau_{s_1,x_1}$ and $T^{x_1,x_2}_{s_1,s_2} \ge T^{r,x_2}_{s_2,s_2} > \tau_{s_2,x_2}$. (iii) is clear since coalescence occurs in 0. (iv) is an immediate consequence of the pathwise uniqueness of (4.1). $\square$

4.2.2 Construction of solutions associated to (E)

We now extend the notations given in Section 2.3.2. For all $n \ge 0$, let $\mathbb{D}_n = \{\frac{k}{2^n},\ k \in \mathbb{Z}\}$ and $\mathbb{D}$ be the set of all dyadic numbers: $\mathbb{D} = \cup_{n\in\mathbb{N}}\mathbb{D}_n$. For $u < v$, define $n(u, v) = \inf\{n \in \mathbb{N} : \mathbb{D}_n \cap]u, v[\neq \emptyset\}$ and $f(u, v) = \inf \mathbb{D}_{n(u,v)} \cap]u, v[$. Denote by $G_{\mathbb{Q}} = \{x \in G : |x| \in \mathbb{Q}_+\}$. We also fix a bijection $\psi : \mathbb{N} \longrightarrow \mathbb{Q} \times G_{\mathbb{Q}}$ and set $(s_i, x_i) = \psi(i)$ for all $i \ge 0$.

Construction of a stochastic flow of mappings φ solution of (E)

Let W be a Brownian motion on the real line and Y be the flow of the $SBM(\alpha^+)$ constructed from W in the previous section. We first construct $\varphi_{s,\cdot}(x)$ for all $(s, x) \in$

$\mathbb{Q}\times G_{\mathbb{Q}}$ and then extend this definition for all $(s,x)\in\mathbb{R}\times G$. We begin by $\varphi_{s_0,\cdot}(x_0)$, then $\varphi_{s_1,\cdot}(x_1)$ and so on. To define $\varphi_{s_0,\cdot}(x_0)$, we flip excursions of $Y_{s_0,\cdot}(\varepsilon(x_0)|x_0|)$ suitably. Then let $\varphi_{s_1,t}(x_1)$ be equal to $\varphi_{s_0,t}(x_0)$ if $Y_{s_0,t}(\varepsilon(x_0)|x_0|)=Y_{s_1,t}(\varepsilon(x_1)|x_1|)$. Before coalescence of $Y_{s_0,\cdot}(\varepsilon(x_0)|x_0|)$ and $Y_{s_1,\cdot}(\varepsilon(x_1)|x_1|)$, we define $\varphi_{s_1,\cdot}(x_1)$ by flipping excursions of $Y_{s_1,\cdot}(\varepsilon(x_1)|x_1|)$ independently of what happens to $\varphi_{s_0,\cdot}(x_0)$ and so on. In what follows, we translate this idea rigorously. Let $\vec{\gamma}^+,\vec{\gamma}^-$ be two independent random variables on any probability space such that

$$\vec{\gamma}^+\overset{law}{=}\sum_{i=1}^{p}\frac{\alpha_i}{\alpha^+}\delta_{\vec{e}_i},\quad \vec{\gamma}^-\overset{law}{=}\sum_{j=p+1}^{N}\frac{\alpha_j}{\alpha^-}\delta_{\vec{e}_j}. \tag{4.10}$$

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space rich enough and W be a Brownian motion on the real line defined on it. For all $s\le t, x\in G$, let $Z_{s,t}(x):=Y_{s,t}(\varepsilon(x)|x|)$ where Y is the flow of Burdzy-Kaspi constructed from W as in Section 4.2.1 if $\alpha^+\notin\{0,1\}$ (= the reflecting Brownian motion associated to (4.1) if $\alpha^+\in\{0,1\}$).

We retain the notations $\tau_{s,x}, g_{s,t}(x)$ of the previous section (see (4.6) and (4.9)). For $s\in\mathbb{R}, x\in G$ define, by abuse of notations

$$\tau_{s,x}=\tau_{s,\varepsilon(x)|x|},\ \ g_{s,\cdot}(x)=g_{s,\cdot}(\varepsilon(x)|x|)\ \text{and}\ \ d_{s,t}(x)=\inf\{r\ge t: Z_{s,r}(x)=0\}.$$

It will be convenient to set $Z_{s,r}(x)=\infty$ if $r<s$. For all $q\ge 1, u_0,\cdots,u_q\in\mathbb{R}, y_0,\cdots,y_q\in G$ define

$$T^{y_0,\cdots,y_q}_{u_0,\cdots,u_q}=\inf\{r\ge\tau_{u_q,y_q}: Z_{u_q,r}(y_q)\in\{Z_{u_i,r}(y_i), i\in[1,q-1]\}\}.$$

Let $\{(\vec{\gamma}^+_{s_0,x_0}(r),\vec{\gamma}^-_{s_0,x_0}(r)), r\in\mathbb{D}\cap[s_0,+\infty[\}$ be a family of independent copies of $(\vec{\gamma}^+,\vec{\gamma}^-)$ which is independent of W. We define $\varphi_{s_0,\cdot}(x_0)$ by

$$\varphi_{s_0,t}(x_0)=\begin{cases} x_0+\vec{e}(x_0)\varepsilon(x_0)W_{s_0,t} & \text{if } s_0\le t\le\tau_{s_0,x_0}\\ 0 & \text{if } t>\tau_{s_0,x_0}, Z_{s_0,t}(x_0)=0\\ \vec{\gamma}^+_{s_0,x_0}(f_0)|Z_{s_0,t}(x_0)|,\quad f_0=f(g_{s_0,t}(x_0),d_{s_0,t}(x_0)) & \text{if } t>\tau_{s_0,x_0}, Z_{s_0,t}(x_0)>0\\ \vec{\gamma}^-_{s_0,x_0}(f_0)|Z_{s_0,t}(x_0)|,\quad f_0=f(g_{s_0,t}(x_0),d_{s_0,t}(x_0)) & \text{if } t>\tau_{s_0,x_0}, Z_{s_0,t}(x_0)<0\end{cases}$$

Now, suppose that $\varphi_{s_0,\cdot}(x_0), \cdots, \varphi_{s_{q-1},\cdot}(x_{q-1})$ are defined and let $\{(\vec{\gamma}^+_{s_q,x_q}(r), \vec{\gamma}^-_{s_q,x_q}(r)), r \in \mathbb{D} \cap [s_q, +\infty[\}$ be a family of independent copies of $(\vec{\gamma}^+, \vec{\gamma}^-)$ which is also independent of $\sigma\left(\vec{\gamma}^+_{s_i,x_i}(r), \vec{\gamma}^-_{s_i,x_i}(r), r \in \mathbb{D} \cap [s_i, +\infty[, 1 \le i \le q-1, W\right)$.
Since $T^{x_0,\cdots,x_q}_{s_0,\cdots,s_q} < \infty$, let $i \in [1, q-1]$ and (s_i, x_i) such that $Z_{s_q,t_0}(x_q) = Z_{s_i,t_0}(x_i)$ with $t_0 = T^{x_0,\cdots,x_q}_{s_0,\cdots,s_q}$. We define $\varphi_{s_q,\cdot}(x_q)$ by

$$\varphi_{s_q,t}(x_q) = \begin{cases} x_q + \vec{e}(x_q)\varepsilon(x_q)W_{s_q,t} & \text{if } s_q \le t \le \tau_{s_q,x_q} \\ 0 & \text{if } t > \tau_{s_q,x_q}, Z_{s_q,t}(x_q) = 0 \\ \vec{\gamma}^+_{s_q,x_q}(f_q)|Z_{s_q,t}(x_q)|, \quad f_q = f(g_{s_q,t}(x_q), d_{s_q,t}(x_q)) & \text{if } t \in [\tau_{s_q,x_q}, t_0], Z_{s_q,t}(x_q) > 0 \\ \vec{\gamma}^-_{s_q,x_q}(f_q)|Z_{s_q,t}(x_q)|, \quad f_q = f(g_{s_q,t}(x_q), d_{s_q,t}(x_q)) & \text{if } t \in [\tau_{s_q,x_q}, t_0], Z_{s_q,t}(x_q) < 0 \\ \varphi_{s_i,t}(x_i) & \text{if } t \ge t_0 \end{cases}$$

In this way, we construct $(\varphi_{s,\cdot}(x), s \in \mathbb{Q}, x \in G_{\mathbb{Q}})$.
Now fix $s \in \mathbb{R}$. For all $x \in G, t \ge s$, set $\varphi_{s,t}(x) = x + \vec{e}(x)\varepsilon(x)W_{s,t}$ if $s \le t \le \tau_{s,x}$. If $t > \tau_{s,x}$ and there exist $s \le v < g_{s,t}(x), v \in \mathbb{Q}, y \in G_{\mathbb{Q}}$ such that $Z_{s,t}(x) = Z_{v,t}(y)$, then define $\varphi_{s,t}(x) = \varphi_{v,t}(y)$. We set $\varphi_{s,t}(x) = 0$ in the other case. In particular, $\varphi_{s,t}(x,\omega)$ is measurable with respect to (t, x, ω) (recall that $g_{s,t}(x,\omega)$ is measurable) and has independent increments by Lemma 4.3. Later, we will show that φ is a coalescing solution of (E).

Construction of a stochastic flow of kernels K^{m^+,m^-} solution of (E)

Let m^+ and m^- be two probability measures respectively on Δ_p and Δ_{N-p}. Let $\mathcal{U}^+$,$\mathcal{U}^-$ be two independent random variables on any probability space such that

$$\mathcal{U}^+ \overset{law}{=} m^+, \; \mathcal{U}^- \overset{law}{=} m^-. \tag{4.11}$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space rich enough and W be a Brownian motion on the real line defined on it. We retain the notations introduced in the previous paragraph for all functions of W. We consider a family $\{(\mathcal{U}^+_{s_0,x_0}(r), \mathcal{U}^-_{s_0,x_0}(r)), r \in \mathbb{D} \cap [s_0, +\infty[\}$ of independent copies of $(\mathcal{U}^+,\mathcal{U}^-)$ which is independent of W.

If $t > \tau_{s_0,x_0}$ and $Z_{s_0,t}(x_0) > 0$ (resp. $Z_{s_0,t}(x_0) < 0$), let

$$U^+_{s_0,t}(x_0) = \mathcal{U}^+_{s_0,x_0}(f_0) \quad (\text{resp. } U^-_{s_0,t}(x_0) = \mathcal{U}^-_{s_0,x_0}(f_0)),\ f_0 = f(g_{s_0,t}(x_0), d_{s_0,t}(x_0)).$$

Write $U^+_{s_0,t}(x_0) = (U^{+,i}_{s_0,t}(x_0))_{1\le i\le p}$ (resp. $U^-_{s_0,t}(x_0) = (U^{-,i}_{s_0,t}(x_0))_{p+1\le i\le N}$) if $Z_{s_0,t}(x_0) > 0, t > \tau_{s_0,x_0}$ (resp. $Z_{s_0,t}(x_0) < 0, t > \tau_{s_0,x_0}$) and now define

$$K^{m^+,m^-}_{s_0,t}(x_0) = \begin{cases} \delta_{x_0+\vec{e}(x_0)\varepsilon(x_0)W_{s_0,t}} & \text{if } s_0 \le t \le \tau_{s_0,x_0} \\ \sum_{i=1}^p U^{+,i}_{s_0,t}(x_0)\delta_{\vec{e}_i|Z_{s_0,t}(x_0)|} & \text{if } t > \tau_{s_0,x_0}, Z_{s_0,t}(x_0) > 0 \\ \sum_{i=p+1}^N U^{-,i}_{s_0,t}(x_0)\delta_{\vec{e}_i|Z_{s_0,t}(x_0)|} & \text{if } t > \tau_{s_0,x_0}, Z_{s_0,t}(x_0) < 0 \\ \delta_0 & \text{if } t > \tau_{s_0,x_0}, Z_{s_0,t}(x_0) = 0 \end{cases}$$

Suppose that $K^{m^+,m^-}_{s_0,\cdot}(x_0), \cdots, K^{m^+,m^-}_{s_{q-1},\cdot}(x_{q-1})$ are defined and let $\{(\mathcal{U}^+_{s_q,x_q}(r), \mathcal{U}^-_{s_q,x_q}(r)), r \in \mathbb{D} \cap [s_q, +\infty[\}$ be a family of independent copies of $(\mathcal{U}^+, \mathcal{U}^-)$ which is also independent of $\sigma\big(\mathcal{U}^+_{s_i,x_i}(r), \mathcal{U}^-_{s_i,x_i}(r), r \in \mathbb{D} \cap [s_i, +\infty[, 1 \le i \le q-1, W\big)$.
If $t > \tau_{s_q,x_q}$ and $Z_{s_q,t}(x_q) > 0$ (resp. $Z_{s_q,t}(x_q) < 0$), we define $U^+_{s_q,t}(x_q) = (U^{+,i}_{s_q,t}(x_q))_{1\le i\le p}$ (resp. $U^-_{s_q,t}(x_q) = (U^{-,i}_{s_q,t}(x_q))_{p+1\le i\le N}$) by analogy to $q = 0$. Let $i \in [1, q-1]$ and (s_i, x_i) such that $Z_{s_q,t_0}(x_q) = Z_{s_i,t_0}(x_i)$ with $t_0 = T^{x_0,\cdots,x_q}_{s_0,\cdots,s_q}$. Then, define

$$K^{m^+,m^-}_{s_q,t}(x_q) = \begin{cases} \delta_{x_q+\vec{e}(x_q)\varepsilon(x_q)W_{s_q,t}} & \text{if } s_q \le t \le \tau_{s_q,x_q} \\ \sum_{i=1}^p U^{+,i}_{s_q,t}(x_q)\delta_{\vec{e}_i|Z_{s_q,t}(x_q)|} & \text{if } t_0 > t > \tau_{s_q,x_q}, Z_{s_q,t}(x_q) > 0 \\ \sum_{i=p+1}^N U^{-,i}_{s_q,t}(x_q)\delta_{\vec{e}_i|Z_{s_q,t}(x_q)|} & \text{if } t_0 > t > \tau_{s_q,x_q}, Z_{s_q,t}(x_q) < 0 \\ \delta_0 & \text{if } t_0 \ge t > \tau_{s_q,x_q}, Z_{s_q,t}(x_q) = 0 \\ K^{m^+,m^-}_{s_i,t}(x_i) & \text{if } t > t_0 \end{cases}$$

In this way, we construct $(K^{m^+,m^-}_{s,}(x), s \in \mathbb{Q}, x \in G_{\mathbb{Q}})$.
Now fix $s \in \mathbb{R}$. For $x \in G, t \ge s$, set $K^{m^+,m^-}_{s,t}(x) = \delta_{x+\vec{e}(x)\varepsilon(x)W_{s,t}}$ if $s \le t \le \tau_{s,x}$. If $t > \tau_{s,x}$ and there exist $s \le v < g_{s,t}(x), v \in \mathbb{Q}, y \in G_{\mathbb{Q}}$ such that $Z_{s,t}(x) = Z_{v,t}(y)$, then define $K^{m^+,m^-}_{s,t}(x) = K^{m^+,m^-}_{v,t}(y)$. We set $K^{m^+,m^-}_{s,t}(x) = \delta_0$ in the other case. In particular, $K^{m^+,m^-}_{s,t}(x,\omega)$ is measurable with respect to (t,x,ω) and has independent increments by Lemma 4.3. In the next section we will show that K^{m^+,m^-} is a stochastic flow of kernels which solves (E).

Construction of (K^{m^+,m^-},φ) by filtering

Let m^+ and m^- be two probability measures as in Theorem 4.2 and $(\vec{\gamma}^+,\mathcal{U}^+), (\vec{\gamma}^-,\mathcal{U}^-)$ be two independent random variables satisfying

$$\mathcal{U}^+ = (\mathcal{U}^{+,i})_{1\le i\le p} \overset{law}{=} m^+, \quad \mathcal{U}^- = (\mathcal{U}^{-,j})_{p+1\le j\le N} \overset{law}{=} m^-,$$

$$\mathbb{P}(\vec{\gamma}^+ = \vec{e}_i|\mathcal{U}^+) = \mathcal{U}^{+,i}, \ \forall i \in [1,p], \tag{4.12}$$

and

$$\mathbb{P}(\vec{\gamma}^- = \vec{e}_j|\mathcal{U}^-) = \mathcal{U}^{-,j}, \ \forall j \in [p+1,N]. \tag{4.13}$$

Then, in particular $(\vec{\gamma}^+,\vec{\gamma}^-)$ and $(\mathcal{U}^+,\mathcal{U}^-)$ satisfy respectively (4.10) and (4.11). On a probability space $(\Omega,\mathcal{A},\mathbb{P})$ consider the following independent processes

- W is a Brownian motion on the real line.
- $\{(\vec{\gamma}^+_{s,x}(r),\mathcal{U}^+_{s,x}(r)), r \in \mathbb{D}\cap[s,+\infty[, (s,x) \in \mathbb{Q}\times G_{\mathbb{Q}}\}$ a family of independent copies of $(\vec{\gamma}^+,\mathcal{U}^+)$.
- $\{(\vec{\gamma}^-_{s,x}(r),\mathcal{U}^-_{s,x}(r)), r \in \mathbb{D}\cap[s,+\infty[, (s,x) \in \mathbb{Q}\times G_{\mathbb{Q}}\}$ a family of independent copies of $(\vec{\gamma}^-,\mathcal{U}^-)$.

Now, let φ and K^{m^+,m^-} be the processes constructed in Section 4.2.2 respectively from $(\vec{\gamma}^+,\vec{\gamma}^-,W)$ and $(\mathcal{U}^+,\mathcal{U}^-,W)$. Let $\sigma(\mathcal{U}^+,\mathcal{U}^-,W)$ be the σ-field generated by $\{\mathcal{U}^+_{s,x}(r),\mathcal{U}^-_{s,x}(r), r \in \mathbb{D}\cap[s,+\infty[, (s,x)\in\mathbb{Q}\times G_{\mathbb{Q}}\}$ and W. We then have the

Proposition 4.2. *(i) For all measurable bounded function f on G, $s \le t \in \mathbb{R}, x \in G$, with probability 1,*

$$K^{m^+,m^-}_{s,t} f(x) = E[f(\varphi_{s,t}(x))|\sigma(\mathcal{U}^+,\mathcal{U}^-,W)].$$

(ii) For all $s \in \mathbb{R}, x \in G$, with probability 1, $\forall t \ge s$,

$$|\varphi_{s,t}(x)| = |Z_{s,t}(x)|, \quad \varphi_{s,t}(x) \in G^+ \Leftrightarrow Z_{s,t}(x) \ge 0 \ \textit{and} \ \ \varphi_{s,t}(x) \in G^- \Leftrightarrow Z_{s,t}(x) \le 0.$$

(iii) For all $s, x \neq y$, with probability 1

$$t_0 := \inf\{r \geq s : \varphi_{s,r}(x) = \varphi_{s,r}(y)\} = \inf\{r \geq s : Z_{s,r}(x) = Z_{s,r}(y) = 0\}$$

and $\varphi_{s,r}(x) = \varphi_{s,r}(y), \quad \forall r \geq t_0$.

Proof. (i) comes from (4.12), (4.13) for $(s, x) \in \mathbb{Q} \times G_{\mathbb{Q}}$ and from Lemma 4.3 for all (s, x), (ii) is clear by construction for s rational, and then for all s using Remark 4.2. (iii) is an immediate consequence of (ii). □

Next we will prove that φ is a stochastic flow of mappings. It remains to prove that properties (1) and (4) in the definition are satisfied. As in Lemma 4.1, property (4) can be derived from the following

Lemma 4.5. *$\forall t \geq s, \epsilon > 0, x \in G$, we have*

$$\lim_{y \to x} \mathbb{P}(d(\varphi_{s,t}(x), \varphi_{s,t}(y)) \geq \epsilon) = 0.$$

Proof. We take $s = 0$. Notice that for all $z \in \mathbb{R}$, we have

$$Y_{0,t}(z) = z + W_t \text{ if } \; 0 \leq t \leq \tau_{0,z}.$$

Fix $\epsilon > 0, x \in \; G^+ \setminus \{0\}$ and y in the same ray as x with $|y| > |x|, d(y, x) \leq \frac{\epsilon}{2}$. Then $d(\varphi_{0,t}(x), \varphi_{0,t}(y)) = d(x, y) \leq \frac{\epsilon}{2}$ for $0 \leq t \leq \tau_{0,|x|} \wedge \tau_{0,|y|}$ ($= \tau_{0,|x|}$ in our case). Proposition 4.2 (iii) states that $\varphi_{0,t}(x) = \varphi_{0,t}(y)$ if $t \geq T_{|x|,|y|}$. This shows that

$$\{d(\varphi_{0,t}(x), \varphi_{0,t}(y)) \geq \epsilon\} \subset \{\tau_{0,|x|} < t < T_{|x|,|y|}\} \; a.s.$$

By Lemma 4.2,

$$\mathbb{P}(d(\varphi_{0,t}(x), \varphi_{0,t}(y)) \geq \epsilon) \leq \mathbb{P}(\tau_{0,|x|} < t < T_{|x|,|y|}) \to 0 \text{ as } y \to x, |y| > |x|.$$

By the same way,

$$\mathbb{P}(d(\varphi_{0,t}(x), \varphi_{0,t}(y)) \geq \epsilon) \leq \mathbb{P}(\tau_{0,|y|} < t < T_{|x|,|y|}) \to 0 \text{ as } y \to x, |y| < |x|.$$

The case $x \in G^-$ holds by the same reasoning. □

Proposition 4.3. *$\forall s < t < u, x \in G$, a.s.*

$$\varphi_{s,u}(x) = \varphi_{t,u}(\varphi_{s,t}(x)).$$

Proof. We begin with the following remark: for all $s < u, t < u, (x, y) \in G^2$, a.s.

$$\{Z_{t,u}(y) = Z_{s,u}(x)\} \cap \{t < g_{s,u}(x)\} \subset \{Z_{t,r}(y) = Z_{s,r}(x) \text{ for all } r \in [g_{s,u}(x), u]\}.$$

This is clear since coalescence of $Z_{t,\cdot}(y)$ and $Z_{s,\cdot}(x)$ must occur in zero.
Now fix $s < t < u, x \in G$ and set $y = \varphi_{s,t}(x)$. By Proposition 4.1, with probability 1, $\forall r \geq t, \ \ Y_{s,r}(\varepsilon(x)|x|) = Y_{t,r}(Y_{s,t}(\varepsilon(x)|x|))$. Applying Proposition 4.2 (ii), we get, a.s. $\forall r \geq t$,

$$Z_{t,r}(y) = Y_{t,r}(\varepsilon(y)|y|) = Y_{t,r}(Y_{s,t}(\varepsilon(x)|x|)) = Y_{s,r}(\varepsilon(x)|x|) = Z_{s,r}(x).$$

The event $\{\exists r \in [t, g_{t,u}(y)[\cap\mathbb{Q}, z \in G_{\mathbb{Q}} : Z_{t,u}(y) = Z_{r,u}(z)\}$ is a measurable function of $(Z_{r,h}, t \leq r \leq h \leq u)$ and y. By the independence of increments of φ and Lemma 4.3, a.s.

$$\{u > \tau_{t,y}\} \subset \{\exists r \in [t, g_{t,u}(y)[\cap\mathbb{Q}, z \in G_{\mathbb{Q}} : Z_{t,u}(y) = Z_{r,u}(z)\}. \tag{4.14}$$

All the equalities below hold a.s.

• On the event $\{u \leq \tau_{s,x}\}$, we have $\tau_{t,y} = \inf\{r \geq t, \ Z_{t,r}(y) = 0\} = \inf\{r \geq t, \ Z_{s,r}(x) = 0\} = \tau_{s,x}$. Consequently $u \leq \tau_{t,y}$ and it is easy to check that $\varphi_{s,u}(x) = \varphi_{t,u}(\varphi_{s,t}(x))$.

• On the event $\{t \leq \tau_{s,x} < u\}$, we still have $\tau_{t,y} = \tau_{s,x}$ and so $g_{t,u}(y) = g_{s,u}(x)$. Choose $r \in [t, g_{t,u}(y)[\cap\mathbb{Q}$ and $z \in G_{\mathbb{Q}}$ such that $Z_{t,u}(y) = Z_{r,u}(z)$. Then $\varphi_{t,u}(y) := \varphi_{r,u}(z)$. As $u > \tau_{s,x}$, let $s_1 \in [s, g_{s,u}(x)[, x_1 \in G_{\mathbb{Q}}$ such that $Z_{s,u}(x) = Z_{s_1,u}(x_1)$. Then $\varphi_{s,u}(x) := \varphi_{s_1,u}(x_1)$. Now $Z_{s,u}(x) = Z_{s_1,u}(x_1)$ together with $s_1 < g_{s,u}(x)$, yield $Z_{s,h}(x) = Z_{s_1,h}(x_1)$ for all $h \in [g_{s,u}(x), u]$ by the previous remark. Similarly from $Z_{s,u}(x) = Z_{t,u}(y) = Z_{r,u}(z)$ and $r < g_{t,u}(y)(= g_{s,u}(x))$, we get $Z_{s,h}(x) = Z_{r,h}(z)$ for all $h \in [g_{s,u}(x), u]$ and $Z_{s,h}(x) = Z_{r,h}(z) = 0$ if $h = g_{s,u}(x)$. This shows that $Z_{r,h}(z) = Z_{s_1,h}(x_1)$ for all $h \in [g_{s,u}(x), u]$ and $Z_{r,h}(z) = Z_{s_1,h}(x_1) = 0$ if $h = g_{s,u}(x)$.

By construction, we have $\varphi_{s_1,u}(x_1) = \varphi_{r,u}(z)$ and consequently $\varphi_{s,u}(x) = \varphi_{t,u}(y)$.

• On the event $\{\tau_{s,x} < t, \tau_{t,y} < u\}$, since $\tau_{t,y}$ is a common zero of $(Z_{s,r}(x))_{r\geq s}$ and $(Z_{t,r}(y))_{r\geq t}$ before u, it comes that $g_{t,u}(y) = g_{s,u}(x)$. Define (r,z) and (s_1,x_1) similarly to the previous case. Then $Z_{s,h}(x) = Z_{r,h}(z) = Z_{s_1,h}(x_1)$ for all $h \in [g_{s,u}(x), u]$ and a fortiori $\varphi_{s,u}(x) = \varphi_{s_1,u}(x_1) = \varphi_{r,u}(z) = \varphi_{t,u}(y)$.

• On the event $\{\tau_{s,x} < t, u < \tau_{t,y}\}$, choose $s_1 \in [s, g_{s,t}(x)[, x_1 \in G_{\mathbb{Q}}$ such that $Z_{s,t}(x) = Z_{s_1,t}(x_1)$, then $\varphi_{s,t}(x) = \varphi_{s_1,t}(x_1)$ and $Z_{s,r}(x) = Z_{s_1,r}(x_1)$ for all $r \geq t$ (by Lemma 4.4 (iv)). Since $u > \tau_{s,x}$ and $Z_{s,u}(x) = Z_{s_1,u}(x_1)$, we get $\varphi_{s,u}(x) := \varphi_{s_1,u}(x_1)$. Now

$$\tau_{t,y} = \inf\{r \geq t : Z_{t,r}(y) = 0\} = \inf\{r \geq t : Z_{s,r}(x) = 0\}.$$

As $Z_{s,r}(x) = Z_{s_1,r}(x_1)$ for all $r \geq t$, we deduce that $\tau_{t,y} = d_{s_1,t}(x_1)$. Hence, it remains to show that $\varphi_{s_1,u}(x_1) = \varphi_{t,u}(\varphi_{s_1,t}(x_1))$ if $u < d_{s_1,t}(x_1)$ which can easily be checked from the construction. $\square$

Proposition 4.4. *φ is a coalescing solution of* (E).

Proof. We use the notations: $Y_u := Y_{0,u}(0),\ \varphi_u := \varphi_{0,u}(0)$. We first show that φ is a $W(\alpha_1, \cdots, \alpha_N)$ process on G. Define for all $n \geq 1$: $T_0^n(Y) = 0$,

$$\begin{aligned} T_{k+1}^n(Y) &= \inf\{r \geq T_k^n(Y), d(\varphi_r, \varphi_{T_k^n}) = \frac{1}{2^n}\} = \inf\{r \geq T_k^n(Y), |Y_r - Y_{T_k^n}| = \frac{1}{2^n}\} \\ &= \inf\{r \geq T_k^n(Y), ||Y_r| - |Y_{T_k^n}|| = \frac{1}{2^n}\}, k \geq 0. \end{aligned}$$

Remark that $|Y|$ is a reflected Brownian motion and denote $T_k^n(Y)$ simply by T_k^n. From the proof of Proposition 2.2, for all $K > 0$ a.s.

$$\lim_{n\to+\infty} \sup_{t\leq K} |T^n_{\lfloor 2^{2n}t \rfloor} - t| = 0.$$

Set $\varphi_k^n = 2^n \varphi_{T_k^n}$. Then, since almost surely $t \longmapsto \varphi_t$ is continuous, a.s. $\forall t \geq 0$, $\lim\limits_{n\to+\infty} \frac{1}{2^n}\varphi^n_{\lfloor 2^{2n}t \rfloor} = \varphi_t$. By Proposition 2.2, it remains to show that for all $n \geq 0$, $(\varphi_k^n, k \geq 0)$ is a Markov chain (started at 0) whose transition mechanism is described by (2.2). If $Y_k^n = 2^n Y_{T_k^n}$, then, by the proof of Proposition 2.2 (since SBM is a special case of $W(\alpha_1, \cdots, \alpha_N)$), for all $n \geq 0$, $(Y^n_k)_{k\geq 1}$ is a Markov chain on $\mathbb{Z}$ started at 0 whose law is described by

$$Q(0,1) = 1 - Q(0,-1) = \alpha^+, \quad Q(m,m+1) = Q(m,m-1) = \frac{1}{2} \ \forall m \neq 0.$$

Let $k \geq 1$ and $x_0, .., x_k \in G$ such that $x_0 = x_k = 0$ and $|x_{h+1} - x_h| = 1$ if $h \in [0, k-1]$. We write

$$\{x_h, x_h = 0, h \in [1,k]\} = \{x_{i_0}, .., x_{i_q}\}, \ i_0 = 0 < i_1 < \cdots < i_q = k$$

and

$$\{x_h, x_h \neq 0, h \in [1,k]\} = \{x_h\}_{h\in[i_0+1, i_1-1]} \cup \cdots \cup \{x_h\}_{h\in[i_{q-1}+1, i_{k-1}]}.$$

Assume that

$$\{x_h\}_{h\in[i_0+1, i_1-1]} \subset D_{j_0}, \cdots, \{x_h\}_{h\in[i_{q-1}+1, i_{k-1}]} \subset D_{j_{q-1}}$$

and define

$$A^n_h = (Y^n_h = \varepsilon(x_h)|x_h|), \quad E = (\vec{e}(\varphi^n_{i_0+1}) = \vec{e}_{j_0}, \cdots, \vec{e}(\varphi^n_{i_{q-1}+1}) = \vec{e}_{j_{q-1}}).$$

If $i \in [1,p]$, we have

$$(\varphi^n_{k+1} = \vec{e}_i, \varphi^n_k = x_k, \cdots, \varphi^n_0 = x_0) = \bigcap_{h=0}^{k} A^n_h \bigcap (Y^n_{k+1} - Y^n_k = 1) \bigcap E \bigcap (\vec{e}(\varphi^n_{k+1}) = \vec{e}_i)$$

and $(\varphi^n_k = x_k, \cdots, \varphi^n_0 = x_0) = \bigcap_{h=0}^{k} A^n_h \bigcap E$. Now

$$\mathbb{P}(\varphi^n_{k+1} = \vec{e}_i | \varphi^n_0 = x_0, \cdots, \varphi^n_k = 0) = \frac{\alpha_i}{\alpha^+} \mathbb{P}(Y^n_{k+1} - Y^n_k = 1 | Y^n_k = 0) = \alpha_i.$$

Obviously, the previous argument can be applied to show that the transition probabilities of $(\varphi^n_k, k \geq 0)$ are given by (2.2) and so φ is a $W(\alpha_1, \cdots, \alpha_N)$ process on G started at 0. Using (2.4) for φ, it follows that $\forall f \in D(\alpha_1, \cdots, \alpha_N)$,

$$f(\varphi_t) = f(0) + \int_0^t f'(\varphi_s) dB_s + \frac{1}{2} \int_0^t f''(\varphi_s) ds$$

where

$$B_t = |\varphi_t| - \tilde{L}_t(|\varphi|) = |Y_t| - \tilde{L}_t(|Y|) = \int_0^t \widetilde{sgn}(Y_s)dY_s$$

by Tanaka's formula for symmetric local time. But Y solves (4.1) and therefore $\int_0^t \widetilde{sgn}(Y_s)dY_s = \int_0^t \widetilde{sgn}(Y_s)W(ds)$. Since a.s. $\widetilde{sgn}(Y_s) = \varepsilon(\varphi_s)$ for all $s \geq 0$, it comes that $\forall f \in D(\alpha_1, \cdots, \alpha_N)$,

$$f(\varphi_{0,t}(x)) = f(x) + \int_0^t f'(\varphi_{0,s}(x))\varepsilon(\varphi_{0,s}(x))W(ds) + \frac{1}{2}\int_0^t f''(\varphi_{0,s}(x))ds$$

when $x = 0$. Finally, by distinguishing the cases $t \leqslant \tau_{0,x}$ and $t > \tau_{0,x}$, we see that the previous equation is also satisfied for $x \neq 0$. $\square$

Corollary 4.1. *K^{m^+,m^-} is a stochastic flow of kernels solution of (E).*

Proof. By Proposition 4.2 (i) and Jensen inequality, K^{m^+,m^-} is a stochastic flow of kernels. The fact that K^{m^+,m^-} is a solution of (E) is a consequence of the previous proposition and is similar to Lemma 4.6 [36] using Proposition 4.2 (i). $\square$

Remarks 4.3. *(i) Define $\hat{K}_{s,t}(x,y) = K^{m^+,m^-}_{s,t}(x)\otimes\delta_{\varphi_{s,t}}(y)$. Then $\hat{K}$ is a stochastic flow of kernels on G^2.*

(ii) If $(m^+,m^-) = (\delta_{(\frac{\alpha_1}{\alpha^+},\cdots,\frac{\alpha_p}{\alpha^+})}, \delta_{(\frac{\alpha_{p+1}}{\alpha^-},\cdots,\frac{\alpha_N}{\alpha^-})})$, then

$$\begin{aligned} K^W_{s,t}(x) &= \delta_{x+\vec{e}(x)\varepsilon(x)W_{s,t}}1_{\{t\leq\tau_{s,x}\}} \\ &+ \Big(\sum_{i=1}^{p}\frac{\alpha_i}{\alpha^+}\delta_{\vec{e}_i|Z_{s,t}(x)|}1_{\{Z_{s,t}(x)>0\}} + \sum_{i=p+1}^{N}\frac{\alpha_i}{\alpha^-}\delta_{\vec{e}_i|Z_{s,t}(x)|}1_{\{Z_{s,t}(x)\leq 0\}}\Big)1_{\{t>\tau_{s,x}\}} \end{aligned} \quad (4.15)$$

is a Wiener solution of (E).

(iii) If $(m^+,m^-) = (\sum_{i=1}^{p}\frac{\alpha_i}{\alpha^+}\delta_{(0,..,0,1,0,..,0)}, \sum_{i=p+1}^{N}\frac{\alpha_i}{\alpha^-}\delta_{(0,..,0,1,0,..,0)})$, then $K^{m^+,m^-} = \delta_\varphi$.

4.3 Unicity of flows associated to (E)

Let K be a solution of (E) and fix $s \in \mathbb{R}, x \in G$. Then $(K_{s,t}(x))_{t\geq s}$ can be modified such that, a.s., the mapping $t \longmapsto K_{s,t}(x)$ is continuous from $[s, +\infty[$ into $\mathcal{P}(G)$. To prove this, let $(g_n)_{n\geq 1}$ be a sequence of functions dense in $C_0(G)$. Recall the definition of $D'(\alpha_1, \cdots, \alpha_N)$ from (2.5) and that $h_n := P_{\frac{1}{n}} g_n \in D'(\alpha_1, \cdots, \alpha_N)$. Then $\{h_n, n \geq 1\}$ is dense in $C_0(G)$. Let λ_t be the unique linear form on $H := \text{Vect}\{h_n, n \geq 1\}$ defined by

$$\lambda_t h_n = h_n(x) + \int_s^t K_{s,u}(h_n'\varepsilon)(x) dW_u + \frac{1}{2}\int_s^t K_{s,u} h_n''(x) du.$$

For a fixed $t \geq s$, a.s., $\lambda_t f = K_{s,t} f(x)$ for all $f \in H$. Using the continuity of λ_t with respect to t, we have a.s.

$$\forall t \geq s, f \in H, \ |\lambda_t f| \leq ||f||_\infty.$$

Therefore, we can extend λ_t to a continuous linear form on $C_0(G)$. For a fixed $t \geq s$, a.s., $\forall f \in C_0(G), \lambda_t f = K_{s,t} f(x)$. Consequently, a.s., for all $t \in [s, \infty[\cap\mathbb{Q}, \lambda_t$ is a positive linear form on $C_0(G)$. Let $f \in C_0(G)$ and $(f_n)_n \subset H$, with $\lim\limits_{n\to\infty} f_n = f$. Then

$$|\lambda_t f - \lambda_{\frac{\lfloor qt \rfloor}{q}} f| \leq 2||f_n - f||_\infty + |\lambda_t f_n - \lambda_{\frac{\lfloor qt \rfloor}{q}} f_n|.$$

By letting $q \to \infty$ and then $n \to \infty$, we get $\lambda_t f = \lim_{q\to\infty} \lambda_{\frac{\lfloor qt \rfloor}{q}} f$. In other words

$$\lim_{q\to\infty} \lambda_{\frac{\lfloor qt \rfloor}{q}} = \lambda_t \text{ weakly.} \tag{4.16}$$

As a result a.s., for all $t \geq s, \lambda_t$ is positive linear form on $C_0(G)$. By Riesz's theorem, there exists a measure μ_t such that

$$\lambda_t f = \int f(y)\mu_t(dy) \text{ for all } f \in C_0(G).$$

With probability 1, for all $t \in [s, \infty[\cap\mathbb{Q}, \mu_t$ is a probability measure on G. Take a uniformly bounded sequence $(f)_k \subset C_0(G)$ which converges pointwise to 1. Then

$$|\mu_t(G) - 1| \leq |\lambda_t(1) - \lambda_t(f_k)| + |\lambda_t f_k - \lambda_{\frac{\lfloor qt \rfloor}{q}} f_k| + |\lambda_{\frac{\lfloor qt \rfloor}{q}} f_k - 1|.$$

From (4.16), $\lim_{q\to\infty} \lambda_{\frac{\lfloor qt \rfloor}{q}} f_k = \lambda_t f_k$ and

$$\limsup_{q\to\infty} |\lambda_{\frac{\lfloor qt \rfloor}{q}} f_k - 1| \le \limsup_{q\to\infty} \int |f_k - 1| d\mu_{\frac{\lfloor qt \rfloor}{q}} = \int |f_k - 1| d\mu_t.$$

Finally $|\mu_t(G) - 1| \le 2\int |f_k - 1| d\mu_t$ which converges to 0 as $k \to \infty$ by dominated convergence.

We will always consider this modification for $(K_{s,t}(x))_{t\ge s}$.

Lemma 4.6. *Let (K, W) be a solution of (E). Then $\forall x \in G, s \in \mathbb{R}, a.s.$*

$$K_{s,t}(x) = \delta_{x+\vec{e}(x)\varepsilon(x)W_{s,t}}, \text{ if } s \le t \le \tau_{s,x} \text{ where } \tau_{s,x} = \inf\{r \ge s,\ \varepsilon(x)|x| + W_{s,r} = 0\}.$$

Proof. We follow [36] (Lemma 3.1). Suppose $x \ne 0, x \in D_i$, $1 \le i \le p$ and take $s = 0$. Let $\beta_i = 1$ and consider a set of numbers $(\beta_j)_{1\le j\le N, j\ne i}$ such that $\sum_{j=1}^{N} \beta_j \alpha_j = 0$. If $f(h\vec{e}_j) = \beta_j h$ for all $1 \le j \le N$, then $f \in D(\alpha_1, \cdots, \alpha_N)$. Set $\tilde{\tau}_x = \inf\{r; K_{0,r}(x)(\cup_{j\ne i} D_j) > 0\}$ and apply f in (E) to get

$$\int_{D_i\setminus\{0\}} |y| K_{0,t}(x, dy) = |x| + W_t \text{ for all } t \le \tilde{\tau}_x. \tag{4.17}$$

By applying $f_k(y) = |y|^2 e^{\frac{-|y|}{k}}$, $k \ge 1$ in (E), we have for all $t \ge 0$,

$$K_{0,t\wedge\tilde{\tau}_x} f_k(x) = f_k(x) + \int_0^t 1_{[0,\tilde{\tau}_x]}(u) K_{0,u}(\varepsilon f_k')(x) dW_u + \frac{1}{2}\int_0^{t\wedge\tilde{\tau}_x} K_{0,u} f_k''(x) du.$$

As $k \to \infty$, $K_{0,t\wedge\tilde{\tau}_x} f_k(x) \longrightarrow \int_0^t |y|^2 K_{0,t\wedge\tilde{\tau}_x}(x, dy)$ by monotone convergence. Now

$$\int_0^t 1_{[0,\tilde{\tau}_x]}(u) K_{0,u}(\varepsilon f_k')(x) dW_u - \int_0^t 1_{[0,\tilde{\tau}_x]}(u) \int_G 2|y| K_{0,u}(x, dy) dW_u$$

$$= \int_0^t 1_{[0,\tilde{\tau}_x]}(u) \int_G (f_k'(y) - 2|y|) K_{0,u}(x, dy) dW_u.$$

Let $A > 0, xe^{-x} \le A$ for all $x \ge 0$. Then $|f_k'(y) - 2|y|| \le (4 + A)|y|$. Using dominated convergence for stochastic integrals (see [45] page 142) and (4.17), we see that

$$\int_0^t 1_{[0,\tilde{\tau}_x]}(u) K_{0,u}(\varepsilon f_k')(x) dW_u \longrightarrow \int_0^t 1_{[0,\tilde{\tau}_x]}(u) \int_G 2|y| K_{0,u}(x, dy) dW_u$$

as $k \to \infty$. We easily check that $|f_k''(y)| \leq 2e^{\frac{-1}{k}|y|} + \frac{4+A}{k}|y|$ and so $\int_0^{t\wedge\tilde{\tau}_x} K_{0,u} f_k''(x)du \longrightarrow 0$ as $k \to \infty$. By identifying the limits, we get

$$\int_{D_i\setminus\{0\}} (|y| - |x| - W_t)^2 K_{0,t}(x, dy) = 0 \quad \forall\, t \leq \tilde{\tau}_x.$$

This proves that for $t \leq \tilde{\tau}_x$, $K_{0,t}(x) = \delta_{x+\vec{e}(x)W_t}$. The fact that $\tau_{0,x} = \tilde{\tau}_x$ easily follows. $\square$

The previous lemma entails the following

Corollary 4.2. *If (K, W) is a solution of (E), then $\sigma(W) \subset \sigma(K)$.*

Proof. For all $x \in D_1$, we have $K_{0,t}(x) = \delta_{\vec{e_1}(|x|+W_t)}$ if $t \leq \tau_{0,x}$. If f is a positive function on G such that $f_1(h) = h$, then $W_t = K_{0,t}f(x) - |x|$ for all $t \leq \tau_{0,x}, x \in D_1$. By considering a sequence $(x_k)_{k\geq 0}$ converging to ∞, this shows that $\sigma(W_t) \subset \sigma(K_{0,t}(y), y \in D_1)$. $\square$

4.3.1 Unicity of the Wiener solution

In order to complete the proof of Theorem 4.1, we will prove the following

Proposition 4.5. *Equation (E) has at most one Wiener solution. This means that: if K and K' are two Wiener solutions, then for all $s \leq t, x \in G, K_{s,t}(x) = K'_{s,t}(x)$ a.s.*

Proof. Denote by P be the semigroup of the $W(\alpha_1, \cdots, \alpha_N)$ process, A and $D(A)$ being respectively its generator and its domain on $C_0(G)$. Recall the definition of $D'(\alpha_1, \cdots, \alpha_N)$ from (2.5) and that

$$\forall t > 0 \quad P_t(C_0(G)) \subset D'(\alpha_1, \cdots, \alpha_N) \subset D(A)$$

(see Proposition 2.5). Define
$\mathcal{S} = \{f : G \longrightarrow \mathbb{R} : f, f', f'' \in C_b(G^*)$ and are prolongeable by continuity at 0 on each ray, $\lim_{x\to\infty} f(x) = 0\}$.

For $t > 0, h$ a measurable bounded function on G^*, let: $\lambda_t h(x) = 2p_t h_j(|x|)$, if $x \in D_j$, where h_j is the extension of h_j that equals 0 on $]-\infty, 0]$. Then, we first check the following identity:

$$(P_t f)' = -P_t f' + \lambda_t f' \quad \text{on } G^* \quad \text{for all } f \in \mathcal{S}. \tag{4.18}$$

Fix $f \in \mathcal{S}, x = h\vec{e_j}, t > 0$. We have

$$P_t f(x) = 2\sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}} f_i(y-h)p_t(0,y)dy + \int_{\mathbb{R}} f_j(y+h)p_t(0,y)dy - \int_{\mathbb{R}} f_j(y-h)p_t(0,y)dy.$$

and so

$$(P_t f)'(x) = -2\sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}} f_i'(y-h)p_t(0,y)dy + \int_{\mathbb{R}} f_j'(y+h)p_t(0,y)dy + \int_{\mathbb{R}} f_j'(y-h)p_t(0,y)dy$$

$$= -P_t f'(x) + 2\int_{\mathbb{R}} f_j'(y+h)p_t(0,y)dy = -P_t f'(x) + \lambda_t f'(x).$$

Now, we will verify that $(P_t f)' \in \mathcal{S}$. Clearly $(P_t f)' \in C_b(G^*)$ and is prolongeable by continuity at 0 on each ray. Furthermore, a simple integration by parts yields

$$\int_{\mathbb{R}} f_j'(y+h)p_t(0,y)dy = C\int_{\mathbb{R}} f_j(y+h)yp_t(0,y)dy \text{ for some } C \in \mathbb{R}$$

and since $\lim_{x\to\infty} f(x) = 0$, we get $\lim_{x\to\infty}(P_t f)'(x) = 0$. On the other hand

$$(P_t f)''(x) = 2\sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}} f_i''(y-h)p_t(0,y)dy + \int_{\mathbb{R}} f_j''(y+h)p_t(0,y)dy - \int_{\mathbb{R}} f_j''(y-h)p_t(0,y)dy$$

$$= 2\sum_{i=1}^{N} \alpha_i \int_{\mathbb{R}} f_i''(y)p_t(0,y+h)dy + \int_{\mathbb{R}} f_j''(y)p_t(0,y-h)dy - \int_{\mathbb{R}} f_j''(y)p_t(0,y+h)dy.$$

The first line in the equality above shows that $(P_t f)'' \in C_b(G^*)$ and is prolongeable by continuity at 0 on each ray. The second line and (2.6) show that the same holds for $(P_t f)'''$.

Let (K, W) be a stochastic flow that solves (E) (not necessarily a Wiener flow). Our first aim is to establish the following identity

$$K_{0,t} f(x) = P_t f(x) + \int_0^t K_{0,u}(D(P_{t-u} f))(x)dW_u \tag{4.19}$$

where $Dg(x) = \varepsilon(x).g'(x)$. Note that $\int_0^t K_{0,u}(D(P_{t-u}f))(x)dW_u$ is well defined. In fact

$$\int_0^t E[K_{0,u}(D(P_{t-u}f))(x)]^2 du \leq \int_0^t P_u((D(P_{t-u}f))^2)(x)du \leq \int_0^t ||(P_{t-u}f)'||_\infty^2 du$$

and the right hand side is bounded since (4.18) is satisfied and f' is bounded. Set $g = P_\epsilon f = P_{\frac{\epsilon}{2}} P_{\frac{\epsilon}{2}} f$. Then, since $P_{\frac{\epsilon}{2}} f \in C_0(G)$ ($\lim_{x\to\infty} P_{\frac{\epsilon}{2}} f(x) = 0$ comes from $\lim_{x\to\infty} f(x) = 0$), we have $g \in D'(\alpha_1, \cdots, \alpha_N)$. Now

$$K_{0,t}g(x) - P_t g(x) - \int_0^t K_{0,u}(D(P_{t-u}g))(x)dW_u = \sum_{p=0}^{n-1}(K_{0,\frac{(p+1)t}{n}} P_{t-\frac{(p+1)t}{n}} g - K_{0,\frac{pt}{n}} P_{t-\frac{pt}{n}} g)(x)$$

$$-\sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u} D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})g)(x)dW_u - \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u} D(P_{t-\frac{(p+1)t}{n}} g)(x)dW_u.$$

For all $p \in \{0, .., n-1\}$, $g_{p,n} = P_{t-\frac{(p+1)t}{n}} g \in D'(\alpha_1, \cdots, \alpha_N)$ and so by replacing in (E), we get

$$\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u} D g_{p,n}(x)dW_u = K_{0,\frac{(p+1)t}{n}} g_{p,n}(x) - K_{0,\frac{pt}{n}} g_{p,n}(x) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u} A g_{p,n}(x)du$$

$$= K_{0,\frac{(p+1)t}{n}} g_{p,n}(x) - K_{0,\frac{pt}{n}} g_{p,n}(x) - \frac{t}{n} K_{0,\frac{pt}{n}} A g_{p,n}(x) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_{0,u} - K_{0,\frac{pt}{n}}) A g_{p,n}(x)du$$

Then we can write

$$K_{0,t}g(x) - P_t g(x) - \int_0^t K_{0,u}(D(P_{t-u}g))(x)dW_u = A_1(n) + A_2(n) + A_3(n)$$

where:

$$A_1(n) = -\sum_{p=0}^{n-1} K_{0,\frac{pt}{n}} [P_{t-\frac{pt}{n}} g - P_{t-\frac{(p+1)t}{n}} g - \frac{t}{n}.AP_{t-\frac{(p+1)t}{n}} g](x)$$

$$A_2(n) = -\sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u} D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})g)(x)dW_u$$

$$A_3(n) = \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_{0,u} - K_{0,\frac{pt}{n}}) AP_{t-\frac{(p+1)t}{n}} g(x)du.$$

Using $||K_{0,u}f||_\infty \leq ||f||_\infty$ if f is a bounded measurable function, we obtain:

$$\begin{aligned}
|A_1(n)| &\leq \sum_{p=0}^{n-1} ||P_{t-\frac{pt}{n}}g - P_{t-\frac{(p+1)t}{n}}g - \frac{t}{n}.AP_{t-\frac{(p+1)t}{n}}g||_\infty \\
&\leq \sum_{p=0}^{n-1} ||P_{t-\frac{(p+1)t}{n}}[P_{\frac{t}{n}}g - g - \frac{t}{n}.Ag]||_\infty \\
&\leq n||P_{\frac{t}{n}}g - g - \frac{t}{n}.Ag||_\infty \xrightarrow[n\to+\infty]{} 0
\end{aligned}$$

Note that $A_2(n)$ is the sum of orthogonal terms in $L^2(\Omega)$. Consequently

$$||A_2(n)||^2_{L^2(\Omega)} = \sum_{p=0}^{n-1} || \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_{0,u}D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})g)(x)dW_u||^2_{L^2(\Omega)}$$

By applying Jensen inequality, we get

$$||A_2(n)||^2_{L^2(\Omega)} \leq \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_uV_u^2(x)du$$

where $V_u = (P_{t-u}g)' - (P_{t-\frac{(p+1)t}{n}}g)'$. By (4.18), one can decompose V_u as follows:

$$V_u = X_u + Y_u; \quad X_u = -P_{t-u}g' + P_{t-\frac{(p+1)t}{n}}g', \quad Y_u = \lambda_{t-u}g' - \lambda_{t-\frac{(p+1)t}{n}}g'$$

Using the trivial inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain: $P_uV_u^2(x) \leq 2P_uX_u^2(x) + 2P_uY_u^2(x)$ and so

$$||A_2(n)||^2_{L^2(\Omega)} \leq 2B_1(n) + 2B_2(n)$$

where $B_1(n) = \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_uX_u^2(x)du \quad B_2(n) = \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_uY_u^2(x)du$

If $p \in [0, n-1]$ and $u \in [\frac{pt}{n}, \frac{(p+1)t}{n}]$, $P_uX_u^2(x) \leq P_{u+t-\frac{p+1}{n}t}(g' - P_{\frac{p+1}{n}t-u}g')^2(x)$. The change of variable $v = (p+1)t - nu$, yields:

$$\begin{aligned}
B_1(n) &\leq \int_0^t P_{t-\frac{v}{n}}(P_{\frac{v}{n}}g' - g')^2(x)dv \\
&\leq \int_0^t (P_tg'^2(x) - 2P_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')(x) + P_{t-\frac{v}{n}}g'^2(x))dv
\end{aligned}$$

Since g' is bounded, by dominated convergence to show that $B_1(n)$ tends to 0 as $n \to +\infty$, it sufficies to check that $\lim_{n\to\infty} P_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')(x) = P_tg'^2(x)$. Write

$$P_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')(x) = 2\sum_{i=1}^{N} \alpha_i p_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')_i(-h) + p_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')_j(h) - p_{t-\frac{v}{n}}(g'P_{\frac{v}{n}}g')_j(-h)$$

Its is enough to prove that

$$\lim_{n\to\infty} p_{t-\frac{v}{n}}(g' P_{\frac{v}{n}} g'_i)(y) = p_t(g'^2)_i(y) \text{ for all } \ y \in \mathbb{R}^*, i \in [1,N]. \tag{4.20}$$

We have

$$p_{t-\frac{v}{n}}(g' P_{\frac{v}{n}} g')_i(y) = p_{t-\frac{v}{n}}(g'_i (P_{\frac{v}{n}} g')_i)(y) = \int_{\mathbb{R}} g'_i(z)(P_{\frac{v}{n}} g')_i(z) p_{t-\frac{v}{n}}(y,z)dz$$

$$= 2\sum_{k=1}^{N} \alpha_k \int_{\mathbb{R}_+} g'_i(z) p_{\frac{v}{n}} g'_k(-z) p_{t-\frac{v}{n}}(y,z)dz + \int_{\mathbb{R}} g'_i(z)(p_{\frac{v}{n}} g'_i(z) - p_{\frac{v}{n}} g'_i(-z)) p_{t-\frac{v}{n}}(y,z)dz$$

which converges clearly by dominated convergence towards $\int_{\mathbb{R}} g'_i(z)^2 p_t(y,z)dz = p_t(g'^2)_i(y)$.

Now (4.20) is proved and as a result $B_1(n) \longrightarrow 0$ as $n \to \infty$. For $B_2(n)$, we have

$$P_u Y_u^2(x) = 2\sum_{i=1}^{N} \alpha_i p_u((Y_u^2)_i)(-|x|) + p_u((Y_u^2)_j)(|x|) - p_u((Y_u^2)_j)(-|x|) \text{ if } x \in \Delta_j$$

where $(Y_u)_i = 2p_{t-u} g'_i - 2p_{t-\frac{(p+1)t}{n}} g'_i$, defined on $\mathbb{R}_+^*$. For all $i \in [1,N], y \in \mathbb{R}^*$, we have

$$\sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} p_u((Y_u^2)_i)(y) = 4\sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} p_u(p_{t-u} g'_i - 2p_{t-\frac{(p+1)t}{n}} g'_i)^2(y)du$$

In what preceded, it was shown that this quantity tends to 0 as $n \to +\infty$ when p is replaced by P in general and consequently $B_2(n)$ tends to 0 as $n \to +\infty$. Now

$$||A_3(n)||_{L^2(\Omega)} \leq \sum_{p=0}^{n-1} || \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_{0,u} - K_{0,\frac{pt}{n}}) A P_{t-\frac{(p+1)t}{n}} g(x)du||_{L^2(\Omega)}$$

Set $h_{p,n} = AP_{t-\frac{(p+1)t}{n}} g$. Then, $h_{p,n} \in D'(\alpha_1, \cdots, \alpha_N)$ for all $p \in [0, n-1]$ (if $p = n-1$ remark that $h_{p,n} = P_{\frac{\epsilon}{2}} A P_{\frac{\epsilon}{2}} f$). By the Cauchy-Schwarz inequality

$$||A_3(n)||_{L^2(\Omega)} \leq \sqrt{t} \left\{ \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} E[((K_{0,u} - K_{0,\frac{pt}{n}}) h_{p,n}(x))^2]du \right\}^{\frac{1}{2}}$$

If $u \in [\frac{pt}{n}, \frac{(p+1)t}{n}]$:

$$\begin{aligned}
E[((K_{0,u} - K_{0,\frac{pt}{n}})h_{p,n}(x))^2] &\leq E[K_{0,\frac{pt}{n}}(K_{\frac{pt}{n},u}h_{p,n} - h_{p,n})^2(x)] \\
&\leq E[K_{0,\frac{pt}{n}}(K_{\frac{pt}{n},u}h_{p,n}^2 - 2h_{p,n}K_{\frac{pt}{n},u}h_{p,n} + h_{p,n}^2)(x)] \\
&\leq ||P_{u-\frac{pt}{n}}h_{p,n}^2 - 2h_{p,n}P_{u-\frac{pt}{n}}h_{p,n} + h_{p,n}^2||_\infty \\
&\leq 2||h_{p,n}||_\infty ||P_{u-\frac{pt}{n}}h_{p,n} - h_{p,n}||_\infty + ||P_{u-\frac{pt}{n}}h_{p,n}^2 - h_{p,n}^2||_\infty
\end{aligned}$$

Therefore $||A_3(n)||_{L^2(\Omega)} \leq \sqrt{t}(2C_1(n) + C_2(n))^{\frac{1}{2}}$, where:

$$C_1(n) = \sum_{p=0}^{n-1} ||h_{p,n}||_\infty \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}}h_{p,n} - h_{p,n}||_\infty du$$
$$C_2(n) = \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}}h_{p,n}^2 - h_{p,n}^2||_\infty du$$

Since $||h_{p,n}||_\infty \leq ||Ag||_\infty$ and $||P_{u-\frac{pt}{n}}h_{p,n} - h_{p,n}||_\infty \leq ||P_{u-\frac{pt}{n}}Ag - Ag||_\infty$, it comes that:

$$\begin{aligned}
C_1(n) &\leq ||Ag||_\infty \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}}Ag - Ag||_\infty du \\
&\leq ||Ag||_\infty \int_0^t ||P_{\frac{z}{n}}Ag - Ag||_\infty dz
\end{aligned}$$

As $Ag \in C_0(G)$, $C_1(n)$ tends to 0 clearly. On the other hand, $h_{p,n}^2 \in D(\alpha_1, \cdots, \alpha_N)$. In fact since $h_{p,n}$ is continuous

$$\sum_{i=0}^{N} \alpha_i (h_{p,n}^2)_i'(0+) = \sum_{i=0}^{N} 2\alpha_i (h_{p,n})_i(0+)(h_{p,n})_i'(0+) = h_{p,n}(0) \sum_{i=0}^{N} 2\alpha_i (h_{p,n})_i'(0+) = 0$$

We may apply (2.4) to get:

$$C_2(n) = \frac{1}{n} \sum_{p=0}^{n-1} \int_0^t ||P_{\frac{z}{n}}h_{p,n}^2 - h_{p,n}^2||_\infty dz \leq \frac{1}{2n} \sum_{p=0}^{n-1} \int_0^t \int_0^{\frac{z}{n}} ||(h_{p,n}^2)''||_\infty du dz$$

Now we verify that $h_{p,n}', h_{p,n}''$ are uniformly bounded with respect to n and $0 \leq p \leq n-1$. In fact $||h_{p,n}''||_\infty = ||2Ah_{p,n}||_\infty \leq 2||AP_{\frac{\epsilon}{2}}f||_\infty$. Write $h_{p,n} = P_{t-\frac{p+1}{n}t+\frac{\epsilon}{2}} P_{\frac{\epsilon}{4}} AP_{\frac{\epsilon}{4}} f$ where $P_{\frac{\epsilon}{4}} AP_{\frac{\epsilon}{4}} f \in D'(\alpha_1, \cdots, \alpha_N)$. By (4.18), this shows that $||h_{p,n}'||_\infty$ is uniformly bounded with respect to $n, p \in [0, n-1]$ and the same holds for $||(h_{p,n}^2)''||_\infty$. Consequently $C_2(n)$ tends to 0 as $n \to \infty$. As a result:

$$K_{0,t}g(x) = P_t g(x) + \int_0^t K_{0,u}(D(P_{t-u}g))(x)dW_u$$

Letting ϵ tend to 0, then $K_{0,t}g(x)$ tends to $K_{0,t}f(x)$ in $L^2(\Omega)$. Furthermore
$$|| \int_0^t K_{0,u}(D(P_{t-u}g))(x)dW_u - \int_0^t K_{0,u}(D(P_{t-u}f))(x)dW_u||^2_{\mathbb{L}^2(\Omega)}$$

$$\leq \int_0^t P_u((P_{t-u}g)' - (P_{t-u}f)')^2(x)du$$

Using the derivation formula (4.18), the right side may be decomposed as $I_\epsilon + J_\epsilon$, where

$$I_\epsilon = \int_0^t P_u(P_{t-u}g' - P_{t-u}f')^2(x)du \leq tP_t(g' - f')^2(x)$$

$$J_\epsilon = \int_0^t P_u(\lambda_{t-u}g' - \lambda_{t-u}f')^2(x)du$$

We have $g'(y) = -P_\epsilon f'(y) + 2\lambda_\epsilon f'(y) \longrightarrow f'(y)$ as $\epsilon \to 0, P_t(x, dy)$ $a.s.$ and so by dominated convergence $I_\epsilon \longrightarrow 0$ as $\epsilon \to 0$. Similarly J_ϵ tends to 0 as $\epsilon \to 0$. This establishes (5.5). Now suppose that (K, W) is a wiener solution of (E) and let $f \in \mathcal{S}$. Since $K_{0,t}f(x) \in L^2(\mathcal{F}_\infty^{W_{0,\cdot}})$, let $K_{0,t}f(x) = P_t f(x) + \sum_{n=1}^\infty J_t^n f(x)$ be the decomposition in Wiener chaos of $K_{0,t}f(x)$ in L^2 sense (see [45] page 202). By iterating (5.5) (recall that $(P_t f)' \in \mathcal{S}$), we see that for all $n \geq 1$

$$J_t^n f(x) = \int_{0<s_1<\cdots<s_n<t} P_{s_1}(D(P_{s_2-s_1} \cdots D(P_{t-s_n}f)))(x)dW_{0,s_1} \cdots dW_{0,s_n}.$$

If K' is another Wiener flow satisfying (5.5), then $K_{0,t}f(x)$ and $K'_{0,t}f(x)$ must have the same Wiener chaos decomposition for all $f \in \mathcal{S}$, that is $K_{0,t}f(x) = K'_{0,t}f(x)$ a.s. Consequently $K_{0,t}f(x) = K'_{0,t}f(x)$ a.s. for all $f \in D'(\alpha_1, \cdots, \alpha_N)$ since this last set is included in $\mathcal{S}$ and the result extends for all $f \in C_0(G)$ by a density argument. This ends the proof when $x \neq 0$. The case $x = 0$ can be deduced from property (4) in the definition of a stochastic flow of kernels. $\square$

Consequence: We already know that K^W given by (4.15) is a Wiener solution of (E). Since $\sigma(W) \subset \sigma(K)$, we can define K^* the stochastic flow obtained by filtering K with respect to $\sigma(W)$ (Section 3.3.1). Then $\forall s \leq t, x \in G, \quad K^*_{s,t}(x) = E[K_{s,t}(x)|\sigma(W)]$ $a.s.$ As a result, (K^*, W) solves also (E) and by the last proposition,

$$\forall s \leq t, x \in G, \quad E[K_{s,t}(x)|\sigma(W)] = K^W_{s,t}(x) \ a.s. \tag{4.21}$$

From now on, (K, W) **is a solution of** (E) defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Let $P^n_t = E[K^{\otimes n}_{0,t}]$ be the compatible family of Feller semigroups associated to K. We retain the notations introduced in Section 4.2 for all functions of W $(Y_{s,t}(x), Z_{s,t}(x), g_{s,t}(x) \cdots)$. In the next section, starting from K, we construct a flow of mappings φ^c which is a solution of (E). This flow will play an important role to characterize the law of K.

4.3.2 Construction of a stochastic flow of mappings from K

Let $x \in G$, $t > 0$. By (6.14), on $\{t > \tau_{0,x}\}$, $K_{0,t}(x)$ is supported on

$$\{|Z_{0,t}(x)|\vec{e}_i, \quad 1 \leq i \leq p\} \ \text{ if } Z_{0,t}(x) > 0$$

and is supported on

$$\{|Z_{0,t}(x)|\vec{e}_i, \quad p+1 \leq i \leq N\} \ \text{ if } Z_{0,t}(x) \leq 0.$$

In [35] (Section 2.6), the n point motion X^n started at $(x_1, \cdots, x_n) \in G^n$ and associated with P^n has been constructed on an extension $\Omega \times \Omega'$ of Ω such that the law of $\omega' \longmapsto X^n_t(\omega, \omega')$ is given by $K_{0,t}(x_1, dy_1) \cdots K_{0,t}(x_n, dy_n)$. For each $(x, y) \in G^2$, let $(X^x_t, Y^y_t)_{t \geq 0}$ be the two point motion started at (x, y) associated with P^2 as in Section 2.6 [35]. Then $|X^x_t| = |Z_{0,t}(x)|, |Y^y_t| = |Z_{0,t}(y)|$ for all $t \geq 0$ and so

$$T^{x,y} := \inf\{r \geq 0, X^x_r = Y^y_r\} < +\infty \ \text{ a.s.}$$

To $(P^n)_{n \geq 1}$, we associate the compatible family of Markovian coalescent semigroups $(P^{n,c})_{n \geq 1}$ as described in Theorem 3.3. Then we have the following

Lemma 4.7. *$(P^{n,c})_{n \geq 1}$ is a compatible family of Feller semigroups associated with a coalescing flow of mappings φ^c.*

Proof. By Theorem 3.3, we only need to check that: $\forall t > 0, \varepsilon > 0, x \in G$,

$$\lim_{y \to x} \mathbb{P}(\{T^{x,y} > t\} \cap \{d(X_t^x, Y_t^y) > \varepsilon\}) = 0 \ \ (C).$$

As $|X_u^x| = |Z_{0,u}(x)|, |Y_u^y| = |Z_{0,u}(y)|$ for all $u \geq 0$, we have $\{t < T^{x,y}\} \subset \{t < T_{\varepsilon(x)|x|,\varepsilon(y)|y|}\}$. For y close to x, $\{d(X_t^x, Y_t^y) > \varepsilon\} \subset \{\inf(\tau_{0,x}, \tau_{0,y}) < t\}$. Now (C) holds from Lemma 4.2. $\square$

Consequence: Let ν (respectively ν^c) be the Feller convolution semigroup associated with $(P^n)_{n\geq 1}$ (respectively $(P^{n,c})_{n\geq 1}$). By Theorem 3.4, there exists a joint realization (K^1, K^2) where K^1 and K^2 are two stochastic flows of kernels satisfying $K^1 \overset{law}{=} \delta_{\varphi^c}$, $K^2 \overset{law}{=} K$ and such that:

(i) $\hat{K}_{s,t}(x, y) = K^1_{s,t}(x) \otimes K^2_{s,t}(y)$ is a stochastic flow of kernels on G^2,

(ii) For all $s \leq t, x \in G$, $K^2_{s,t}(x) = E[K^1_{s,t}(x)|K^2]$ a.s.

For $s \leq t$, let

$$\hat{\mathcal{F}}_{s,t} = \sigma(\hat{K}_{u,v}, s \leq u \leq v \leq t), \ \ \mathcal{F}^i_{s,t} = \sigma(K^i_{u,v}, s \leq u \leq v \leq t), \ i = 1, 2.$$

Then $\hat{\mathcal{F}}_{s,t} = \mathcal{F}^1_{s,t} \vee \mathcal{F}^2_{s,t}$. To simplify notations, we shall assume that φ^c is defined on the original space $(\Omega, \mathcal{A}, \mathbb{P})$ and that (i) and (ii) are satisfied if we replace (K^1, K^2) by (δ_{φ^c}, K). Recall that (i) and (ii) are also satisfied by the pair $(\delta_\varphi, K^{m^+,m^-})$ constructed in Section 4.2. Now

$$K_{s,t}(x) = E[\delta_{\varphi^c_{s,t}(x)}|K] \ \ a.s. \text{ for all } \ s \leq t, x \in G, \tag{4.22}$$

and using (6.14), we obtain

$$K^W_{s,t}(x) = E[\delta_{\varphi^c_{s,t}(x)}|\sigma(W)] \ a.s. \text{ for all } \ s \leq t, x \in G, \tag{4.23}$$

with K^W being the Wiener flow given by (4.15).

Proposition 4.6. *The stochastic flow φ^c solves (E).*

Proof. Fix $t > 0, x \in G$. By (6.16), $\delta_{\varphi^c_{0,t}(x)}$ is supported on $\{|Z_{0,t}(x)|\vec{e}_j, 1 \leq j \leq N\}$ a.s. and so $|\varphi^c_{0,t}(x)| = |Z_{0,t}(x)|$. Similarly, using (6.16), we have

$$\varphi^c_{0,t}(x) \in G^+ \Leftrightarrow Z_{0,t}(x) \geq 0 \text{ and } \varphi^c_{0,t}(x) \in G^- \Leftrightarrow Z_{0,t}(x) \leq 0. \tag{4.24}$$

Consequently $\varepsilon(\varphi^c_{0,t}(x)) = \widetilde{sgn}(Z_{0,t}(x))$ a.s. Since $\varphi^c_{0,\cdot}(x)$ is a $W(\alpha_1, \cdots, \alpha_N)$ process started at x, it satisfies Theorem 2.3; $\forall f \in D(\alpha_1, \cdots, \alpha_N)$,

$$f(\varphi^c_{0,t}(x)) = f(x) + \int_0^t f'(\varphi^c_{0,u}(x))dB_u + \frac{1}{2}\int_0^t f''(\varphi^c_{0,u}(x))du \ \ a.s.$$

with $B_t = |\varphi_{0,t}(x)| - \tilde{L}_t(|\varphi_{0,\cdot}(x)|) - |x| = |Z_{0,t}(x)| - \tilde{L}_t(|Z_{0,\cdot}(x)|) - |x|$. Tanaka's formula and (4.24) yield

$$B_t = \int_0^t \widetilde{sgn}(Z_{0,u}(x))dZ_{0,u}(x) = \int_0^t \widetilde{sgn}(Z_{0,u}(x))dW_u = \int_0^t \varepsilon(\varphi^c_{0,u}(x))dW_u.$$

Likewise for all $s \leq t, x \in G, f \in D(\alpha_1, \cdots, \alpha_N)$,

$$f(\varphi^c_{s,t}(x)) = f(x) + \int_s^t f'(\varphi^c_{s,u}(x))\varepsilon(\varphi^c_{s,u}(x))dW_u + \frac{1}{2}\int_s^t f''(\varphi^c_{s,u}(x))du \ \ a.s.$$

□

We will see later (Remark 4.3) that $\varphi^c \overset{law}{=} \varphi$ where φ is the stochastic flow of mappings constructed in Section 4.2.

4.3.3 Two probability measures associated to K

For all $t \geq \tau_{s,x}$, set:

$$V^{+,i}_{s,t}(x) = K_{s,t}(x)(D_i \setminus \{0\}) \ \forall 1 \leq i \leq p$$

and

$$V^{-,N}_{s,t}(x) = K_{s,t}(x)(D_N), \quad V^{-,i}_{s,t}(x) = K_{s,t}(x)(D_i \setminus \{0\}) \ \ \forall p+1 \leq i \leq N-1,$$

$$V^+_{s,t}(x) = (V^{+,i}_{s,t}(x))_{1 \leq i \leq p}, V^-_{s,t}(x) = (V^{-,i}_{s,t}(x))_{p+1 \leq i \leq N}, \ V_{s,t}(x) = (V^+_{s,t}(x), V^-_{s,t}(x)).$$

For $s=0$, we use the abbreviated notations:

$$Z_t(x)=Z_{0,t}(x),\quad V_t^+(x)=V_{0,t}^+(x),\quad V_t^-(x)=V_{0,t}^-(x),\; V_t(x)=(V_t^+(x),V_t^-(x))$$

and if $x=0$, $Z_t=Z_{0,t}(0)$, $\; V_t^+=V_{0,t}^+(0)$, $\; V_t^-=V_{0,t}^-(0)$, $V_t=(V_t^+,V_t^-)$.
By (6.14), $\forall x\in G, s\le t$, with probability 1,

$$\begin{aligned}K_{s,t}(x) &= \delta_{\vec{e}(x)|Z_{s,t}(x)|}1_{\{t\le\tau_{s,x}\}}\\ &+ \Big(\sum_{i=1}^{p}V_{s,t}^{+,i}(x)\delta_{\vec{e}_i|Z_{s,t}(x)|}1_{\{Z_{s,t}(x)>0\}}+\sum_{i=p+1}^{N}V_{s,t}^{-,i}(x)\delta_{\vec{e}_i|Z_{s,t}(x)|}1_{\{Z_{s,t}(x)\le 0\}}\Big)1_{\{t>\tau_{s,x}\}}.\end{aligned}$$

Define

$$\mathcal{F}_{s,t}^K=\sigma(K_{v,u}, s\le v\le u\le t),\quad \mathcal{F}_{s,t}^W=\sigma(W_{v,u}, s\le v\le u\le t)$$

and assume that all these σ-fields are right-continuous and include all $\mathbb{P}$-negligible sets. When $s=0$, we denote $\mathcal{F}_{0,t}^K,\mathcal{F}_{0,t}^W$ simply by $\mathcal{F}_t^K,\mathcal{F}_t^W$. Recall that for all $s\in\mathbb{R}, x\in G$, the mapping $t\longmapsto K_{s,t}(x)$ defined from $[s,+\infty[$ into $\mathcal{P}(G)$ is continuous. Then the following Markov property holds.

Lemma 4.8. *Let $x,y\in G$ and T be an $(\mathcal{F}_t^K)_{t\ge0}$-stopping time such that $K_{0,T}(x)=\delta_y$ a.s. Then $K_{0,\cdot+T}(x)$ is independent of $\mathcal{F}_T^K$ and has the same law as $K_{0,\cdot}(y)$.*

Proof. Let $p\ge1, 0\le t_1<\cdots<t_p$ and $g_1,\cdots,g_p$ be p bounded Lipschitzs functions from $\mathcal{P}(G)$ into $\mathbb{R}$. Let $A\in\mathcal{F}_T^K$ and $[T]_n=\inf\{\frac{j}{2^n}:\frac{j}{2^n}>T\}$. Since K is a flow, we may write

$$\begin{aligned}E\Big[\prod_{j=1}^{p}g_j(K_{0,t_j+T}(x))1_A\Big] &= \lim_{n\to\infty}\sum_{i=1}^{\infty}E\Big[\prod_{j=1}^{p}g_j(K_{0,t_j+\frac{i}{2^n}}(x))1_{A\cap\{\frac{i-1}{2^n}\le T<\frac{i}{2^n}\}}\Big]\\ &= \lim_{n\to\infty}E[1_A G(K_{0,[T]_n}(x))]\end{aligned}$$

where $G(\mu)=E\big[\prod_{j=1}^{p}g_j(\mu K_{0,t_j})\big]$. To complete the proof, it remains to check that G is continuous on $\mathcal{P}(G)$. We can also suppose $p=1$ (see the proof of Proposition 3.3). Let $\mu,\mu_k\in\mathcal{P}(G)$ such that $\mu=\lim_{k\to\infty}\mu_k$ weakly. Recall the definition of the distance d on $\mathcal{P}(G)$ from Section 3.2.3. As g_1 is Lipschitz, it sufficies to prove

$$\lim_{k\to\infty}E\left[d(\mu_k K_{0,t},\mu K_{0,t})\right]=0\ (t:=t_1). \tag{4.25}$$

Recall the definition $P_t^n = E[K_{0,t}^{\otimes n}]$ and let $f \in C_0(G)$. Then

$$E\left[\left(\int K_{0,t}f(x)\mu_k(dx) - \int K_{0,t}f(x)\mu(dx)\right)^2\right] = \int P_t^2(f\otimes f)(x,y)\mu_k(dx)\mu_k(dy)$$

$$-2\int P_t^2(f\otimes f)(x,y)\mu_k(dx)\mu(dy) + \int P_t^2(f\otimes f)(x,y)\mu(dx)\mu(dy).$$

As P^2 is Feller, it is easy to deduce (6.17). $\square$

The previous lemma shows that for all x, $K_{0,\tau_{0,x}+\cdot}(x)$ is independent of $\mathcal{F}^K_{\tau_{0,x}}$ and is equal in law to $K_{0,\cdot}(0)$.

Let T and L be the random times defined by:

$$T = \inf\{r \geq 0 : Z_r = 1\}, \quad L = \sup\{r \in [0,T] : Z_r = 0\}.$$

Consider the following σ-fields

$$\mathcal{F}_{L-} = \sigma(X_L, X \text{ is bounded } \ (\mathcal{F}_t^W)_{t\geq 0} - \text{previsible process}),$$

$$\mathcal{F}_{L+} = \sigma(X_L, X \text{ is bounded } \ (\mathcal{F}_t^W)_{t\geq 0} - \text{progressive process}).$$

Then $\mathcal{F}_{L+} = \mathcal{F}_{L-}$ (see the Appendix). Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a bounded continuous function and set $X_t = E[f(V_t)|\sigma(W)]$. By (6.15), the process $r \longmapsto V_r$ is constant on the excursions of $r \longmapsto Z_r$ out of 0.

Lemma 4.9. *There exists an $\mathcal{F}^W$-progressive version of X denoted Y that is constant on the excursions of Z out of 0 and satisfies $Y_L = Y_T$ a.s.*

Proof. By induction, for all integers k and n, define the sequence of stopping times $S_{k,n}$ and $T_{k,n}$ by the relations: $T_{0,n} = 0$ and for $k \geq 1$,

$$S_{k,n} = \inf\{t \geq T_{k-1,n} : |Z_t| = 2^{-n}\}, \quad T_{k,n} = \inf\{t \geq S_{k,n} : Z_t = 0\}.$$

In the following $U_{k,n}$ will denote $U_{S_{k,n}}$. Then for $t \in [S_{k,n}, T_{k,n}[$, $U_t = U_{k,n}$ a.s. For all bounded continuous functions $g_1, \cdots, g_p$, $A \in \mathcal{F}^K_{S_{k,n}}$ and $(u_1, \cdots, u_p) \in (\mathbb{R}_+^*)^p$, we have

$$\prod_{i=1}^{p} g_i(W_{S_{k,n},u_i+S_{k,n}})1_A = \lim_{j\to\infty}\sum_{h=0}^{\infty}\prod_{i=1}^{p} g_i(W_{\frac{h+1}{2^j},u_i+\frac{h+1}{2^j}})1_{A\cap\{S_{k,n}\le\frac{h+1}{2^j}\}\cap\{\frac{h}{2^j}<S_{k,n}\}}.$$

Remark that

$$\prod_{i=1}^{p} g_i(W_{\frac{h+1}{2^j},u_i+\frac{h+1}{2^j}}) \text{ is } \sigma(K_{\frac{h+1}{2^j},u}, u \ge \frac{h+1}{2^j}) \text{ measurable}$$

and

$$A \cap \{S_{k,n} \le \frac{h+1}{2^j}\} \cap \{\frac{h}{2^j} < S_{k,n}\} \in \mathcal{F}^K_{\frac{h+1}{2^j}}.$$

Consequently $\sigma(W_{S_{k,n},u+S_{k,n}}, u \ge 0)$ is independent of $\mathcal{F}^K_{S_{k,n}}$. Therefore $X_{k,n} := E[f(U_{k,n})|W] = E[f(U_{k,n})|\mathcal{F}^W_{S_{k,n}}]$ is $\mathcal{F}^W_{S_{k,n}}$ measurable. We define inductively a sequence X^n of $\mathcal{F}^W$-progressive processes. Let $I_n = \bigcup_{k\ge1}[S_{k,n}, T_{k,n}[$ and $X^0_t = E[f(U_{k,0})|W]$ if $t \in [S_{k,0}, T_{k,0}[$, $X^0_t = f(0)$ if $t \notin I_0$. Then X^0 is $\mathcal{F}^W$-progressive. Suppose a $\mathcal{F}^W$-progressive process X^n is defined such that $X^n_t = E[f(U_{k,n})|W]$ if $t \in [S_{k,n}, T_{k,n}[$ and $X^n_t = f(0)$ if $t \notin I_n$. Define X^{n+1} by setting:

$$X^{n+1}_t = \begin{cases} f(0) & \text{if } t \notin I_{n+1} \\ X^n_{S_{k,n}} & \text{if } t \in I_{n+1} \bigcap [S_{k,n}, T_{k,n}[\text{ (for some } k) \\ E[f(U_{l,n+1})|W] & \text{if } t \in [S_{l,n+1}, T_{l,n+1}[\bigcap I^c_n \text{ (for some } l) \end{cases}$$

The process X^{n+1} is $\mathcal{F}^W$-progressive and for all $t \in [S_{l,n+1}, T_{l,n+1}[$, $X^{n+1}_t = E[f(U_{l,n+1})|W]$ a.s. For all t, X^n_t is a stationary sequence. Set $\tilde{X}_t = \limsup_{n\to\infty} X^n_t$. For $t > 0$, a.s., there exists integers k and n such that $t \in [S_{k,n}, T_{k,n}[$. Thus a.s., $\tilde{X} = X_t$. Now set $Y_0 = f(0)$, $Y_t = \limsup_{n\to\infty} \tilde{X}_{t+\frac{1}{n}}$, $t > 0$. Then Y is a modification of X which is $\mathcal{F}^W$-progressive (see Lemma 3.2 [31]) and constant on the excursions of Z out of 0. Moreover $Y_L = Y_T$ a.s. □

We take for X this version. Then X_T is $\mathcal{F}_{L+}$ measurable. Notice that $Z_T \neq 0$ and from

$$X_T = \lim_{n\to\infty}\sum_{i=1}^{\infty} X_{\frac{i}{2^n}} 1_{\{\frac{i-1}{2^n}\le T<\frac{i}{2^n}\}}$$

we have $X_T = E[f(V_T)|\sigma(W)]$ a.s.

Lemma 4.10. $E[X_T|\mathcal{F}_{L-}] = E[f(V_T)]$

Proof. Let S be an $\mathcal{F}^W$-stopping time and $d_S = \inf\{t \geq S : Z_t = 0\}$. We have $\{S < L\} = \{d_S < T\}$ (up to some negligible set) and so $1_{\{S<L\}}$ is $\mathcal{F}^W_{d_S}$-measurable. Using (6.15) and the continuity of $t \mapsto K_{0,t}(0)$, we have a.s.,

$$f(Z_t) = \int f(\varepsilon(y)|y|)K_{0,t}(0,dy) \text{ for all } \ f \in C_0(G), t \geq 0. \tag{4.26}$$

By an approximation argument $\int |y|K_{0,t}(0,dy) = |Z_t|$ which entails that $K_{0,t}(0,dy) = \delta_0(dy)$ if $Z_t = 0$ and in particular $K_{0,d_S}(0) = \delta_0$ a.s. Moreover, $\sigma(Z_{d_S+\cdot}) \subset \sigma(K_{0,d_S+\cdot}(0))$ by (4.26). Let $H = \inf\{r \geq 0 : Z_{r+d_S} = 1\}$, then by Lemma 6.10,

$$\begin{aligned} E[X_T 1_{\{d_s<T\}}] &= E[f(V_{d_S+H})1_{\{d_S<T\}}] \\ &= E[f(V_T)]\mathbb{P}(d_S < T) \end{aligned}$$

Since the σ-field $\mathcal{F}_{L-}$ is generated by the random variables $1_{\{S<L\}}$ (see [47] page 344), this implies the lemma. $\square$

The fact that $\mathcal{F}_{L-} = \mathcal{F}_{L+}$ and the fact that X_T is $\mathcal{F}_{L+}$ measurable imply that $X_T = E[f(V_T)|W] = E[f(V_T)]$ a.s. Since this holds for all bounded continuous function f, this proves that V_T is independent of $\sigma(W)$ and the same holds if we replace T by $\inf\{t \geq 0 : Z_t = a\}$ where $a > 0$.

Define inductively: $T^+_{0,n} = 0$ and for $k \geq 1$:

$$S^+_{k,n} = \inf\{t \geq T^+_{k-1,n} : Z_t = 2^{-n}\}, \quad T^+_{k,n} = \inf\{t \geq S^+_{k,n} : Z_t = 0\}$$

Set $V^+_{k,n} = V^+_{S^+_{k,n}}$. Then, we have the following lemma.

Lemma 4.11. *For all n, $(V^+_{k,n})_{k\geq 1}$ is a sequence of i.i.d. random variables. Furthermore, this sequence is independent of W.*

Proof. For all $k \geq 2$, $V^+_{k,n}$ is $\sigma(K_{0,T^+_{k-1,n}+t}(0), t \geq 0)$ measurable and $V^+_{k-1,n}$ is $\mathcal{F}^K_{0,T^+_{k-1,n}}$ measurable. This proves the first claim by Lemma 6.10. Now, we show by induction on q that $(V^+_{1,n}, \cdots, V^+_{q,n})$ is independent of $\sigma(W)$. For $q = 1$, this has been justified. Suppose $(V^+_{1,n}, \cdots, V^+_{q-1,n})$ is independent of $\sigma(W)$. Write

$$\sigma(W_{0,u}, u \geq 0) = \sigma(Z_{u\wedge T^+_{q-1,n}}, u \geq 0) \vee \sigma(Z_{u+T^+_{q-1,n}}, u > 0)$$

Since $(V^+_{1,n}, \cdots, V^+_{q-1,n})$ is $\mathcal{F}^K_{0,T^+_{q-1,n}}$ measurable, $\sigma(Z_{u+T^+_{q-1,n}}, u > 0) \subset \sigma(K_{0,T^+_{q-1,n}+t}(0), t \geq 0)$, we conclude that $(V^+_{1,n}, \cdots, V^+_{q,n})$ and $\sigma(W)$ are independent. $\square$

Let m^+_n to be the common law of $(V^+_{k,n})_{k\geq 1}$ for each $n \geq 1$ and define m^+ to be the law of V^+_1 conditionally to $(Z_1 > 0)$. Then, the following lemma holds.

Lemma 4.12. *The sequence $(m^+_n)_{n\geq 1}$ converges weakly towards m^+. For all $t > 0$, under $\mathbb{P}(\cdot|Z_t > 0)$, V^+_t and W are independent and the law of V^+_t is given by m^+.*

Proof. For each bounded continuous function $f : \mathbb{R}^p \longrightarrow \mathbb{R}$,

$$\begin{aligned} E[f(V^+_t)|W]1_{\{Z_t>0\}} &= \lim_{n\to\infty} \sum_k E\big[1_{t\in[S^+_{k,n},T^+_{k,n}[} f(V^+_{k,n})|W\big] \\ &= \lim_{n\to\infty} \sum_k 1_{t\in[S^+_{k,n},T^+_{k,n}[} \left(\int f dm^+_n\right). \\ &= \big[1_{\{Z_t>0\}} \lim_{n\to\infty} \int f dm^+_n + \varepsilon_n(t)\big]. \end{aligned}$$

with $\lim_{n\to\infty} \varepsilon_n(t) = 0$ a.s. Consequently

$$\lim_{n\to\infty} \int f dm^+_n = \frac{1}{\mathbb{P}(Z_t > 0)} E[f(V^+_t)1_{Z_t>0}]$$

As the left hand side no longer depends on t, we obtain the desired result. $\square$

We define analogously the measure m^- by considering the following stopping times: $T^-_{0,n} = 0$ and for $k \geq 1$:

$$S^-_{k,n} = \inf\{t \geq T^-_{k-1,n} : Z_t = -2^{-n}\}, \quad T^-_{k,n} = \inf\{t \geq S^-_{k,n} : Z_t = 0\}$$

Set $V^-_{k,n} = V^-_{S^-_{k,n}}$ and let m^-_n be the common law of $(V^-_{k,n})_{k\geq 1}$. Define m^- to be the conditional law of V^-_1 knowing $(Z_1 < 0)$. Then, the sequence $(m^-_n)_{n\geq 1}$ converges weakly towards m^-. Furthermore, for all $t > 0$, the conditional law of V^-_t knowing $(Z_t < 0)$ is given by m^-. As a result, we have:

$$E[f(V^-_t)|W]1_{\{Z_t<0\}} = 1_{\{Z_t<0\}} \int f dm^-$$

for each measurable bounded $f : \mathbb{R}^{N-p} \longrightarrow \mathbb{R}$. Follow the same steps as before but consider $(Z_{u+\tau_{0,x}}(x), u \geq 0)$ for all x, we show that the conditional law of $V^+_t(x)$

knowing $(Z_t(x) > 0, t > \tau_{0,x})$ no longer depends on $t > 0$. Denote by m_x^+ such a law. Then Lemma 6.10 states that m_x^+ does not depend on $x \in G$. Consequently $m_x^+ = m^+$ for all x and we get

$$E[f(V_t^+(x))|W]1_{\{Z_t(x)>0,t>\tau_{0,x}\}} = 1_{\{Z_t(x)>0,t>\tau_{0,x}\}} \int f dm^+ \tag{4.27}$$

for each measurable bounded $f : \mathbb{R}^p \longrightarrow \mathbb{R}$. Similarly

$$E[h(V_t^-(x))|W]1_{\{Z_t(x)<0,t>\tau_{0,x}\}} = 1_{\{Z_t(x)<0,t>\tau_{0,x}\}} \int h dm^- \tag{4.28}$$

for each measurable bounded $h : \mathbb{R}^{N-p} \longrightarrow \mathbb{R}$.

4.3.4 Unicity in law of K

Define

$$p(x) = |x|\vec{e}_1 1_{\{x\in G^+\}} + |x|\vec{e}_{p+1} 1_{\{x\in G^-, x\neq 0\}}, \quad x \in G.$$

Fix $x \in G$, $0 < s < t$ and let $x_s = p(\varphi_{0,s}^c(x))$. Then:

(i) $\varphi_{s,r}^c(x) = x + \vec{e}(x)\varepsilon(x)W_{s,r}$ for all $r \leq \tau_{s,x}$ (from Lemma 4.6).

(ii) $\tau_{s,x} = \tau_{s,p(x)}$ and $\varphi_{s,r}^c(x) = \varphi_{s,r}^c(p(x))$ for all $r \geq \tau_{s,x}$ since φ^c is a coalescing flow.

(iii) $\tau_{s,\varphi_{0,s}^c(x)} = \tau_{s,x_s}$ and $\varphi_{s,r}^c(\varphi_{0,s}^c(x)) = \varphi_{s,r}^c(x_s)$ for all $r \geq \tau_{s,x_s}$ by (ii) and the independence of increments of φ^c.

(iv) On $\{t > \tau_{s,x_s}\}$, $\varphi_{0,t}^c(x) = \varphi_{s,t}^c(\varphi_{0,s}^c(x)) = \varphi_{s,t}^c(x_s)$ by the flow property of φ^c and (iii).

(v) Clearly $\tau_{s,x_s} = \inf\{r \geq s,\ Z_{0,r}(x) = 0\}$ a.s. Since $\{\tau_{0,x} < s < g_{0,t}(x)\} \subset \{t > \tau_{s,x_s}\}$ a.s., we deduce that

$$\mathbb{P}(\varphi_{0,t}^c(x) = \varphi_{s,t}^c(x_s)|\tau_{0,x} < s < g_{0,t}(x)) = 1$$

(vi) Recall that $\hat{\mathcal{F}}_{0,s}$ and $\hat{\mathcal{F}}_{s,t}$ are independent ($\hat{K}$ is a flow) and $\hat{\mathcal{F}}_{0,t} = \hat{\mathcal{F}}_{0,s} \vee \hat{\mathcal{F}}_{s,t}$. By (6.15), we have $K_{s,t}(x_s) = E[\delta_{\varphi^c_{s,t}(x_s)} | \mathcal{F}^K_{0,t}]$. As a result of (v),

$$\mathbb{P}(K_{s,t}(x_s) = K_{0,t}(x) | \tau_{0,x} < s < g_{0,t}(x)) = 1. \tag{4.29}$$

Lemma 4.13. *Let $\mathbb{P}_{t,x_1,\cdots,x_n}$ be the law of $(K_{0,t}(x_1), \cdots, K_{0,t}(x_n), W)$ where $t \geq 0$ and $x_1, \cdots, x_n \in G$. Then, $\mathbb{P}_{t,x_1,\cdots,x_n}$ is uniquely determined by $\{\mathbb{P}_{u,x}, u \geq 0, x \in G\}$.*

Proof. Recall the definition of $T^{x_0,\cdots,x_q}_{s_0,\cdots,s_q}$ from Section 4.2.2. We will prove the lemma by induction on n. For $n = 1$, this is clear. Notice that if $t < \tau_{0,z}$, then $K_{0,t}(z)$ is $\sigma(W)$ measurable and if $t > T^{z_1,z_2}_{0,0}$, then $K_{0,t}(z_1) = K_{0,t}(z_2)$. Suppose the result holds for $n \geq 1$ and let $x_{n+1} \in G$. Then by the previous remark, we only need to check that the law of $(K_{0,t}(x_1), \cdots, K_{0,t}(x_{n+1}), W)$ conditionally to $A = \{\sup_{1 \leq i \leq n+1} \tau_{0,x_i} < t < T^{x_1,\cdots,x_{n+1}}_{0,\cdots,0}\}$ only depends on $\{\mathbb{P}_{u,x}, u \geq 0, x \in G\}$. Remark that on A, $\{g_{0,t}(x_i), 1 \leq i \leq n+1\}$ are distinct and so by summing over all possible cases, we may replace A by

$$E = \{\sup_{1 \leq i \leq n+1} \tau_{0,x_i} < t < T^{x_1,\cdots,x_{n+1}}_{0,\cdots,0}, g_{0,t}(x_1) < \cdots < g_{0,t}(x_n) < g_{0,t}(x_{n+1})\}$$

Recall the definition of f from Section 4.2.2 and let $S = f(g_{0,t}(x_n), g_{0,t}(x_{n+1}))$, $E_s = E \cap \{S = s\}$ for $s \in \mathbb{D}$. Then it will be sufficient to show that the law of $(K_{0,t}(x_1), \cdots, K_{0,t}(x_{n+1}), W)$ conditionally to E_s only depends on $\{\mathbb{P}_{u,x}, u \geq 0, x \in G\}$ where $s \in \mathbb{D}$ is fixed such that $s < t$. On E_s,

(i) $(K_{0,t}(x_1), \cdots, K_{0,t}(x_n), W)$ is a measurable function of $(V_s(x_1), \cdots, V_s(x_n), W)$ as
$(V_r(x_i), r \geq \tau_{0,x_i})$ is constant on the excursions of $(Z_r(x_i), r \geq \tau_{0,x_i})$.

(ii) There exits a random variable X_{n+1} which is $\mathcal{F}^W_{0,s}$ measurable and satisfies $K_{0,t}(x_{n+1}) = K_{s,t}(X_{n+1})$ (from (4.29)).

Clearly, the law of $(V_s(x_1), \cdots, V_s(x_n), K_{s,t}(X_{n+1}), W)$ is uniquely determined by $\{\mathbb{P}_{s,x_1,\cdots,x_n}, \mathbb{P}_{t-s,y}, y \in G\}$. This completes the proof. ∎

Proposition 4.7. *Let (K^{m^+,m^-}, W') be the solution constructed in Section 4.2.2 associated with (m^+, m^-). Then $K \stackrel{law}{=} K^{m^+,m^-}$.*

Proof. From (4.27) and (4.28), $(K_{0,t}(x), W) \stackrel{law}{=} (K_{0,t}^{m^+,m^-}(x), W')$ for all $t > 0$ and $x \in G$. Notice that all the properties (i)-(v) mentioned just above are satisfied by the flow φ constructed in Section 4.2.2 and consequently K^{m^+,m^-} satisfies also (4.29) using the same arguments. By following the same steps as in the proof of Lemma 4.13, we show by induction on n that

$$(K_{0,t}(x_1), \cdots, K_{0,t}(x_n), W) \stackrel{law}{=} (K_{0,t}^{m^+,m^-}(x_1), \cdots, K_{0,t}^{m^+,m^-}(x_n), W')$$

for all $t > 0, x_1, \cdots, x_n \in G$. This proves the proposition. □

Remark 4.3. *When K is a stochastic flow of mappings, then by definition*

$$(m^+, m^-) = \left(\sum_{i=1}^{p} \frac{\alpha_i}{\alpha^+} \delta_{(0,..,0,1,0,..,0)}, \sum_{i=p+1}^{N} \frac{\alpha_i}{\alpha^-} \delta_{(0,..,0,1,0,..,0)}\right).$$

This shows that there is only one flow of mappings solving (E).

4.3.5 The case $\alpha^+ = \frac{1}{2}, N > 2$

Let K^W be the flow given by (4.15), where $Z_{s,t}(x) = \varepsilon(x)|x| + W_t - W_s$. It is easy to verify that K^W is a Wiener flow. **Fix** $s \in \mathbb{R}, x \in G$. Then, by following ideas of Section 4.2.2, one can construct a Brownian motion on the real line W and a process $(X_{s,t}^x, t \geq s)$ which is a $W(\alpha_1, \cdots, \alpha_N)$ process started at x such that

- (i) for all $t \geq s, f \in D(\alpha_1, \cdots, \alpha_N)$,

$$f(X_{s,t}^x) = f(x) + \int_s^t (\varepsilon f')(X_{s,u}^x) dW_u + \frac{1}{2}\int_s^t f''(X_{s,u}^x) du \quad a.s.$$

- (ii) for all $t \geq s$, $K_{s,t}^W(x) = E[\delta_{X_{s,t}^x} | \sigma(W)]$ a.s.

By conditioning with respect to $\sigma(W)$ in (i), this shows that K^W solves (E).

Now, let (K, W) be any other solution of (E) and set $P_t^n = E[K_{0,t}^{\otimes n}]$. From the hypothesis $\alpha^+ = \frac{1}{2}$, we see that $h(x) = \varepsilon(x)|x|$ belongs to $D(\alpha_1, \cdots, \alpha_N)$ and by applying h in (E), we get $K_{0,t}h(x) = h(x) + W_t$. Denote by (X^{x_1}, X^{x_2}) the two-point motion started at $(x_1, x_2) \in G^2$ associated to P^2. Since $|X^{x_i}|$ is a reflected Brownian motion started at $|x_i|$ (Theorem 2.3), we have $E[|X_t^{x_i}|^2] = t + |x_i|^2$. From the preceding observation $E[h(X_t^{x_1})h(X_t^{x_2})] = E[K_{0,t}h(x_1)K_{0,t}h(x_2)] = h(x_1)h(x_2)+$ t and therefore

$$E[(h(X_t^{x_1}) - h(X_t^{x_2}) - h(x_1) + h(x_2))^2] = 0.$$

This shows that $h(X_t^{x_1}) - h(X_t^{x_2}) = h(x_1) - h(x_2)$. Now we check inductively that P^n does not depend on K. For $n = 1$, this follows from Proposition 2.5. Suppose the result holds for n and let $(x_1, \cdots, x_{n+1}) \in G^{n+1}$ such that $h(x_i) \neq h(x_j), i \neq j$. Let $\tau_{x_i} = \inf\{r \geq 0 : X_r^{x_i} = 0\} = \inf\{r \geq 0 : h(X_r^{x_i}) = 0\}$ and $(x_i, x_j) \in G^+ \times G^-$ such that $h(x_i) < h(x_k), h(x_h) < h(x_j)$ for all $(x_k, x_h) \in G^+ \times G^-$ (when (x_i, x_j) does not exist the proof is simpler). Clearly τ_{x_k} is a function of X^{x_h} for all $h, k \in [1, n+1]$ and so for all measurable bounded $f : G^{n+1} \longrightarrow \mathbb{R}$,

$$f(X_t^{x_1}, \cdots, X_t^{x_{n+1}})1_{\{t<\tau_{x_i}, \inf_{1\leq k\leq n+1} \tau_{x_k} = \tau_{x_i}\}} \text{ is a function of } X^{x_i}$$

and

$$f(X_t^{x_1}, \cdots, X_t^{x_{n+1}})1_{\{t<\tau_{x_j}, \inf_{1\leq k\leq n+1} \tau_{x_k} = \tau_{x_j}\}} \text{ is a function of } X^{x_j}.$$

where $t > 0$ is fixed. This shows that $E[f(X_t^{x_1}, \cdots, X_t^{x_{n+1}})1_{\{t<\inf_{1\leq k\leq n+1} \tau_{x_k}\}}]$ only depends on P^1. Consider the following stopping times

$$S_0 = \inf_{1\leq i\leq n+1} \tau_{x_i}, S_{k+1} = \inf\{r \geq S_k : \exists j \in [1, n+1], X_r^{x_j} = 0, X_{S_k}^{x_j} \neq 0\}, k \geq 0.$$

Remark that $(S_k)_{k\geq 0}$ is a function of X^{x_h} for all $h \in [1, n+1]$. By summing over all possible cases we need only check the unicity in law of $(X_t^{x_1}, \cdots, X_t^{x_{n+1}})$ conditionally to $A = \{S_k < t < S_{k+1}, X_{S_k}^{x_h} = 0\}$ where $k \geq 0, h \in [1, n+1]$ are fixed. Write $A = B \cap \{t - S_k < T\}$ where $B = \{S_k < t, X_{S_k}^{x_h} = 0\} = \{S_k < t, X_{S_k}^{x_i} \neq 0 \text{ if } i \neq h\}$ and $T = \inf\{r \geq 0, \exists j \neq h : X_{r+S_k}^{x_j} = 0\}$. On A, $X_t^{x_i}$ is a function of $(X_{S_k}^{x_i}, X_t^{x_h})$ and therefore for all measurable bounded $f : G^{n+1} \longrightarrow \mathbb{R}$,

$$f(X_t^{x_1}, \cdots, X_t^{x_{n+1}})1_A \text{ may be written as } g((X_{S_k}^{x_i})_{i\neq h}, X_t^{x_h})1_A$$

where g is measurable bounded from G^{n+1} into $\mathbb{R}$. By the strong Markov property for $X = (X^{x_1}, \cdots, X^{x_{n+1}})$, we have

$$1_B E[1_{\{t-S_k<T\}} g((X^{x_i}_{S_k})_{i\neq h}, X^{x_h}_t)|\mathcal{F}^X_{S_k}] = 1_B \psi(t - S_k, (X^{x_i}_{S_k})_{i\neq h})$$

where

$$\psi(u, y_1, \cdots, y_n) = E[1_{\{u<\inf\{r\geq 0: \exists j\in[1,n], X^{y_j}_r = 0\}\}} g(y_1, \cdots, y_n, X^0_u)].$$

This shows that $E[f(X^{x_1}_t, \cdots, X^{x_{n+1}}_t)1_A]$ only depends on the law of $(X^{x_i})_{i\neq h}$. As a result, $P^{n+1}_t((x_1, \cdots, x_{n+1}), dy)$ is unique whenever $h(x_i) \neq h(x_j), i \neq j$ and by an approximation argument for all $(x_1, \cdots, x_{n+1}) \in G^{n+1}$. Since a stochastic flow of kernels is uniquely determined by the compatible system of its n-point motions, this proves (2) of Theorem 4.2.

4.4 Appendix $\mathcal{F}_{L+} = \mathcal{F}_{L-}$

Recall the definitions of the random times T and L. For any random time S define the following σ-fields

$$\mathcal{F}_{S-} = \sigma(X_S, X \text{ is bounded } (\mathcal{F}^W_t)_{t\geq 0} - \text{previsible process}),$$

$$\mathcal{F}_S = \sigma(X_S, X \text{ is bounded } (\mathcal{F}^W_t)_{t\geq 0} - \text{optional process}),$$

$$\mathcal{F}_{S+} = \sigma(X_S, X \text{ is bounded } (\mathcal{F}^W_t)_{t\geq 0} - \text{progressive process}).$$

We follow Lemma 4.11 [36] with more details. Denote by $(\mathcal{F}^L_t)_{t\geq 0}$ the natural augmentation of $(\mathcal{F}^W_t \vee \sigma(L))_{t\geq 0}$. From the theory of enlargement of filtration, we have $\mathcal{F}_{L+} = \mathcal{F}^L_L$ (see [27] page 77). For $\varepsilon > 0$, define $T_\varepsilon = \inf\{r \geq 0 : \quad Z_{r+L} = \varepsilon\}$. Since the filtration $(\mathcal{F}^L_t)$ satisfies the usual conditions, we have:

$$\mathcal{F}_{L+} = \mathcal{F}^L_L = \mathcal{F}^L_{L+} = \bigcap_{\epsilon>0} \mathcal{F}^L_{L+T_\epsilon}.$$

We will prove the following assertions:

(i) $\mathcal{F}_{L+T_\epsilon} = \mathcal{F}^L_{L+T_\epsilon} = \sigma(Z_{u\wedge(T_\epsilon+L)}, u \geq 0) \vee \sigma(T_\epsilon + L) = \sigma(Z_{u\wedge(T_\epsilon+L)}, u \geq 0)$,

(ii) $\sigma(Z_{u\wedge(T_\epsilon+L)}, u \geq 0) = \sigma(Z_{u\wedge L}, u \geq 0) \vee \sigma(Z_{(L+u)\wedge(L+T_\varepsilon)}, u \geq 0) \vee \sigma(L)$,

We follow [40] and begin by:

(i) Since $L+T_\varepsilon > L$, we have $\mathcal{F}_{L+T_\epsilon} = \mathcal{F}^L_{L+T_\epsilon}$ (see [27] Lemma 5.7 (c) page 78). Now $\mathcal{F}^W$ is the Brownian filtration and so $\mathcal{F}_{L+T_\epsilon} = \mathcal{F}_{(L+T_\epsilon)-}$ (see [42] Lemma 8.9). The σ- field $\mathcal{F}_{L+T_\epsilon}$ is generated by the class $\mathcal{C}$ of processes

$$f(Z_{t_1}, \cdots, Z_{t_n}) 1_{]t_n,+\infty[}(L+T_\epsilon)$$

for all increasing positive sequences $(t_i)_{1\leq i\leq n}$ and measurable bounded f (see [13]) which writes as

$$f(Z_{t_1\wedge(L+T_\epsilon)}, \cdots, Z_{t_n\wedge(L+T_\epsilon)}) 1_{]t_n,+\infty[}(L+T_\epsilon).$$

Consequently $\mathcal{F}_{L+T_\varepsilon} \subset \sigma(Z_{u\wedge(T_\epsilon+L)}, u \geq 0) \vee \sigma(T_\epsilon + L)$. The other inclusion is trivial and (i) is proved. (ii) is easy. As in (i), we have $\mathcal{F}_L = \sigma(Z_{u\wedge L}, u \geq 0) \vee \sigma(L)$. By combining (i) and (ii), we get

$$\mathcal{F}_{L+} = \bigcap_{\epsilon>0} \mathcal{F}_L \vee \sigma(Z_{(L+u)\wedge(L+T_\varepsilon)}, u \geq 0).$$

On the other hand, $(Z_{L+u\wedge T_\varepsilon})_{u\geq 0}$ is a $BES(3)$ killed at time T_ε independent of $\mathcal{F}_L$ by the decomposition of David Williams [27]. The Lindvall-Rogers lemma [4], yields

$$\mathcal{F}_{L+} = \mathcal{F}_L \vee \bigcap_{\epsilon>0} \sigma(Z_{L+u\wedge T_\varepsilon}, u \geq 0) = \mathcal{F}_L$$

since the filtration of $BES(3)$ is Brownian.

Chapter 5

Discrete approximations to flows of Tanaka's SDE related to WBM

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5.1 Introduction and main results

In the foregoing chapter, we have defined a Tanaka's SDE related to WBM which depends on kernels. It was shown that there is only one Wiener solution and only one flow of mappings solving this equation. In the terminology of Le Jan and Raimond, these are respectively the stronger and the weaker among all solutions. In this chapter, we provide discrete approximations to these flows. Among recent papers on approximating flows, let us mention [43] where the author construct an approximation for the Harris flow and the Arratia flow. Let us first recall Tanaka's SDE related to Walsh BM

Definition 5.1. *(Equation (T)).*

Fix $N \in \mathbb{N}^*$, $\alpha_1, \cdots, \alpha_N > 0$ such that $\sum_{i=1}^{N} \alpha_i = 1$ and consider the graph G defined in Section 2.3.1. On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let W be a Brownian motion on the real line and K be a stochastic flow of kernels on G. We say that (K, W) solves (T) if for all $s \le t, f \in D(\alpha_1, \cdots, \alpha_N), x \in G$,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}f'(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du \quad a.s.$$

If $K = \delta_\varphi$ is a solution of (T) where φ is a flow of mappings, we just say that (φ, W) solves (T).

It was shown in the previous chapter that if (K, W) solves (T), then $\sigma(W) \subset \sigma(K)$ and therefore one can just say that K solves (T). We also recall the following

Theorem 5.1. *There exists a unique Wiener flow K^W (resp. flow of mappings φ) which solves (T).*

For all $z \in G$, recall the definition $\vec{e}(z) = \vec{e}_i$ if $z \in D_i, z \neq 0$ (convention $\vec{e}(0) = \vec{e}_N$). As described in Theorem 4.1, the unique Wiener solution of (T) is simply

$$K^W_{s,t}(x) = \delta_{x+\vec{e}(x)W_{s,t}} 1_{\{t \le \tau_{s,x}\}} + \sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i W^+_{s,t}} 1_{\{t > \tau_{s,x}\}}. \tag{5.1}$$

where

$$\tau_{s,x} = \inf\{r \ge s : x + \vec{e}(x)W_{s,r} = 0\} = \inf\{r \ge s : W_{s,r} = -|x|\}. \tag{5.2}$$

However, the construction of the unique (see Remark 4.3) flow of mappings φ relies on flipping Brownian excursions and is more complicated. Another construction of φ using Kolmogorov extension theorem can be derived from Section 3.4 similarly to

Tanaka's equation. Here, we restrict our attention to discrete models.

Note that the one point motion associated to any solution of (T) is the $W(\alpha_1, \cdots, \alpha_N)$ process on G. Let $G_{\mathbb{N}} = \{x \in G; |x| \in \mathbb{N}\}$ and $\mathcal{P}(G)$ (resp. $\mathcal{P}(G_{\mathbb{N}})$) be the space of all probability measures on G (resp. $G_{\mathbb{N}}$). We now come to the discrete description of (φ, K^W) and first introduce

Definition 5.2. *(Discrete flows) We say that a process $\psi_{p,q}(x)$ (resp. $N_{p,q}(x)$) indexed by $\{p \le q \in \mathbb{Z}, x \in G_{\mathbb{N}}\}$ with values in $G_{\mathbb{N}}$ (resp. $\mathcal{P}(G_{\mathbb{N}})$) is a discrete flow of mappings (resp. kernels) on $G_{\mathbb{N}}$ if:*

(i) The family $\{\psi_{i,i+1}; i \in \mathbb{Z}\}$ (resp. $\{N_{i,i+1}; i \in \mathbb{Z}\}$) is independent.

(ii)$\forall p \in \mathbb{Z}, x \in G_{\mathbb{N}}$, a.s.

$$\psi_{p,p+2}(x) = \psi_{p+1,p+2}(\psi_{p,p+1}(x)) \ (\textit{resp. } N_{p,p+2}(x) = N_{p,p+1}N_{p+1,p+2}(x))$$

where

$$N_{p,p+1}N_{p+1,p+2}(x, A) := \sum_{y \in G_{\mathbb{N}}} N_{p+1,p+2}(y, A) N_{p,p+1}(x, \{y\}) \textit{ for all } A \subset G_{\mathbb{N}}.$$

We call (ii), the cocycle or flow property.

The main difficulty in the construction of the flow φ associated to (3.4) [36] is that it has to keep the consistency of the flow. This problem does not arise in discrete time. Starting from the following two remarks,

(i) $\varphi_{s,t}(x) = x + \text{sgn}(x) W_{s,t}$ if $s \le t \le \tau_{s,x}$,

(ii) $|\varphi_{s,t}(0)| = W^+_{s,t}$ and $\text{sgn}(\varphi_{s,t}(0))$ is independent of W for all $s \le t$,

one can easily expect the discrete analogous of φ as follows: consider an original random walk S and a family of signs (η_i) which are independent. Then

(i) a particle at time k and position $n \neq 0$, just follows what the $S_{k+1} - S_k$ tells him (goes to $n+1$ if $S_{k+1} - S_k = 1$ and to $n-1$ if $S_{k+1} - S_k = -1$),

(ii) a particle at 0 at time k does not move if $S_{k+1} - S_k = -1$, and moves according to η_k if $S_{k+1} - S_k = 1$.

The situation on a finite half-lines is very close. Let $S = (S_n)_{n\in\mathbb{Z}}$ be a simple random walk on $\mathbb{Z}$, that is $(S_n)_{n\in\mathbb{N}}$ and $(S_{-n})_{n\in\mathbb{N}}$ are two independent simple random walks on $\mathbb{Z}$ and $(\vec{\eta}_i)_{i\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with law $\sum_{i=1}^{N}\alpha_i\delta_{\vec{e}_i}$ which is independent of S. For $p \leq n$, set

$$S_{p,n} = S_n - S_p,\ \ S^+_{p,n} = S_n - \min_{h\in[p,n]} S_h = S_{p,n} - \min_{h\in[p,n]} S_{p,h}.$$

and for $p \in \mathbb{Z}, x \in G_{\mathbb{N}}$, define

$$\Psi_{p,p+1}(x) = x + \vec{e}(x)S_{p,p+1} \text{ if } \ x \neq 0, \Psi_{p,p+1}(0) = \vec{\eta}_p S^+_{p,p+1}.$$

$$K_{p,p+1}(x) = \delta_{x+\vec{e}(x)S_{p,p+1}} \text{ if } \ x \neq 0, K_{p,p+1}(0) = \sum_{i=1}^{N}\alpha_i\delta_{S^+_{p,p+1}\vec{e}_i}.$$

In particular, we have $K_{p,p+1}(x) = E[\delta_{\Psi_{p,p+1}(x)}|\sigma(S)]$. Now, we extend this definition for all $p \leq n \in \mathbb{Z}, x \in G_{\mathbb{N}}$ by setting

$$\Psi_{p,n}(x) = x1_{\{p=n\}} + \Psi_{n-1,n} \circ \Psi_{n-2,n-1} \circ \cdots \circ \Psi_{p,p+1}(x)1_{\{p>n\}},$$

$$K_{p,n}(x) = \delta_x 1_{\{p=n\}} + K_{p,p+1}\cdots K_{n-2,n-1}K_{n-1,n}(x)1_{\{p>n\}}.$$

We equip $\mathcal{P}(G)$ with the following topology of weak convergence:

$$\beta(P,Q) = \sup\left\{|\int gdP - \int gdQ|, \|g\|_\infty + \sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|} \leq 1, g(0) = 0\right\}.$$

In this chapter, starting from (Ψ, K), we construct (φ, K^W) and in particular show the following

Theorem 5.2. *(1) Ψ (resp. K) is a discrete flow of mappings (resp. kernels) on $G_{\mathbb{N}}$.*
(2) There exists a joint realization (ψ, N, φ, K^W) on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that
(i) $(\psi, N) \overset{law}{=} (\Psi, K)$.
(ii) (φ, W) (resp. (K^W, W)) is the unique flow of mappings (resp. Wiener flow)

which solves (T).

(iii) For all $s \in \mathbb{R}, T > 0, x \in G, x_n \in \frac{1}{\sqrt{n}} G_{\mathbb{N}}$ *such that* $\lim_{n\to\infty} x_n = x$ *we have*

$$\lim_{n\to\infty} \sup_{s\le t\le s+T} |\frac{1}{\sqrt{n}}\psi_{\lfloor ns\rfloor,\lfloor nt\rfloor}(\sqrt{n}x_n) - \varphi_{s,t}(x)| = 0 \quad a.s.$$

and

$$\lim_{n\to\infty} \sup_{s\le t\le s+T} \beta(K_{\lfloor ns\rfloor,\lfloor nt\rfloor}(\sqrt{n}x_n)(\sqrt{n}.), K^W_{s,t}(x)) = 0 \quad a.s. \tag{5.3}$$

This theorem implies also the following

Corollary 5.1. *For all* $s \in \mathbb{R}, x \in G_{\mathbb{N}}$*, let* $t \longmapsto \Psi(t)$ *be the linear interpolation of* $\left(\Psi_{\lfloor ns\rfloor,k}(x), k \ge \lfloor ns\rfloor\right)$ *and* $\Psi^n_{s,t}(x) := \frac{1}{\sqrt{n}}\Psi(nt)$*,* $K^n_{s,t}(x) = E[\delta_{\Psi^n_{s,t}(x)}|\sigma(S)], t \ge s, n \ge 1$*. For all* $1 \le p \le q$*,* $(x_i)_{1\le i\le q} \subset G$*, let* $x^n_i \in \frac{1}{\sqrt{n}} G_{\mathbb{N}}$ *such that* $\lim_{n\to\infty} x^n_i = x_i$*. Define*

$$Y^n = \left(\Psi^n_{s_1,.}(\sqrt{n}x^n_1), \cdots, \Psi^n_{s_p,.}(\sqrt{n}x^n_p), K^n_{s_{p+1},.}(\sqrt{n}x^n_{p+1}), \cdots, K^n_{s_q,.}(\sqrt{n}x^n_q)\right).$$

Then

$$Y^n \xrightarrow[n\to+\infty]{law} Y \quad in \quad \prod_{i=1}^{p} C([s_i, +\infty[, G) \times \prod_{j=p+1}^{q} C([s_j, +\infty[, \mathcal{P}(G))$$

where

$$Y = \left(\varphi_{s_1,.}(x_1), \cdots, \varphi_{s_p,.}(x_p), K^W_{s_{p+1},.}(x_{p+1}), \cdots, K^W_{s_q,.}(x_q)\right).$$

Our proof of Theorem 5.2 is based on a remarkable transformation introduced by Csaki and Vincze [44] which is strongly linked with Tanaka's SDE. Let S be a simple random walk on $\mathbb{Z}$ (SRW) and ε be a Bernoulli random variable independent of S (just one!). Then there exists a SRW M such that

$$\sigma(M) = \sigma(\varepsilon, S)$$

and moreover

$$(\frac{1}{\sqrt{n}}S(nt), \frac{1}{\sqrt{n}}M(nt))_{t\ge 0} \xrightarrow[n\to+\infty]{\text{law}} (B_t, W_t)_{t\ge 0} \ \text{ in } C([0,\infty[, \mathbb{R}^2).$$

where $t \longmapsto S(t)$ (resp. $M(t)$) is the linear interpolation of S (resp. M) and B, W are two Brownian motions satisfying Tanaka's equation

$$dW_t = \operatorname{sgn}(W_t) dB_t.$$

We will carry a thorough study of this transformation in Section 5.2.1 and then extend the result of Csaki and Vincze to Walsh Brownian motion (Proposition 5.1); Let $S = (S_n)_{n \in \mathbb{N}}$ be a SRW and associate to S the process $Y_n := S_n - \min_{k \le n} S_k$, flip independently every "excursion "of Y to each ray D_i with probability α_i, then the resulting process is not far from a random walk on G whose law is given by (2.2). In Section 5.3, we study the scaling limits of Ψ, K.

5.2 Csaki-Vincze transformation and consequences

In this section, we review a relevant result of Csaki and Vincze and then derive some useful consequences offering a better understanding of Tanaka's equation.

5.2.1 Csaki-Vincze transformation

Theorem 5.3. *([44] page 109) Let $S = (S_n)_{n \ge 0}$ be a SRW. Then, there exists a SRW $\overline{S} = (\overline{S}_n)_{n \ge 0}$ such that:*

$$\overline{Y}_n := \max_{k \le n} \overline{S}_k - \overline{S}_n \Rightarrow |\overline{Y}_n - |S_n|| \le 2 \quad \forall n \in \mathbb{N}.$$

Proof. Let $X_i = S_i - S_{i-1}, i \ge 1$ and define

$$\tau_1 = \min\{i > 0 : S_{i-1}S_{i+1} < 0\}, \ \tau_{l+1} = \min\{i > \tau_l : S_{i-1}S_{i+1} < 0\} \ \forall l \ge 1.$$

For $j \ge 1$, set

$$\overline{X}_j = \sum_{l \ge 0} (-1)^{l+1} X_1 X_{j+1} 1_{\{\tau_l + 1 \le j \le \tau_{l+1}\}}.$$

Then $(\overline{X}_j)_{j \ge 1}$ is a sequence of i.i.d. Bernoulli random variables. Let

$$\overline{S}_0 = 0, \ \overline{S}_j = \overline{X}_1 + \cdots + \overline{X}_j, \ j \ge 1.$$

Then the following properties are easy to check

(i) For $k \in [\tau_l + 1, \tau_{l+1}]$, we have

$$\overline{S}_k - \overline{S}_{\tau_l} = \sum_{j=\tau_l+1}^{k} \overline{X}_j = (-1)^{l+1} X_1 \sum_{j=\tau_l+1}^{k} X_{j+1} = (-1)^{l+1} X_1 (S_{k+1} - S_{\tau_l+1}).$$

So

$$\overline{S}_k - \overline{S}_{\tau_l} \begin{cases} \leq & \text{if } \tau_l + 1 \leq k \leq \tau_{l+1} - 2 \\ = 1 & \text{if } k = \tau_{l+1} - 1 \\ = 2 & \text{if } k = \tau_{l+1}. \end{cases}$$

(ii) Let $\overline{M}_n = \max_{k \leq n} \overline{S}_k$, then $\overline{M}_{\tau_l} = \overline{S}_{\tau_l} = 2l$, $l \geq 1$.

(iii) For any $\tau_l \leq n < \tau_{l+1}$, we have $2l \leq \overline{M}_n \leq 2l + 1$.

(iv)

$$\overline{S}_k = \begin{cases} 2l + 1 - |S_{k+1}| \leq & \text{if } \tau_l + 1 \leq k \leq \tau_{l+1} - 1 \\ 2l + 2 - |S_k| & \text{if } k = \tau_{l+1} \end{cases}$$

Therefore

$$\overline{Y}_k = \overline{M}_k - \overline{S}_k \leq |S_{k+1}| \leq |S_k| + 1$$

and

$$\overline{Y}_k = \overline{M}_k - \overline{S}_k \geq |S_{k+1}| - 1 \geq |S_k| - 2$$

This shows that the theorem holds for $\overline{S}$. We call $T(S) = \overline{S}$ the Csaki-Vincze transformation of S.

$\square$

Note that T is an even function, in other words $T(S) = T(-S)$. As a consequence of (ii) and (iii) in the proof of the last theorem, we have

$$\tau_l = \min \{n \geq 0, \overline{S}_n = 2l\} \ \forall l \geq 1. \tag{5.4}$$

This entails the following

Corollary 5.2. *(1) Let S be a SRW and define $\overline{S} = T(S)$. Then*

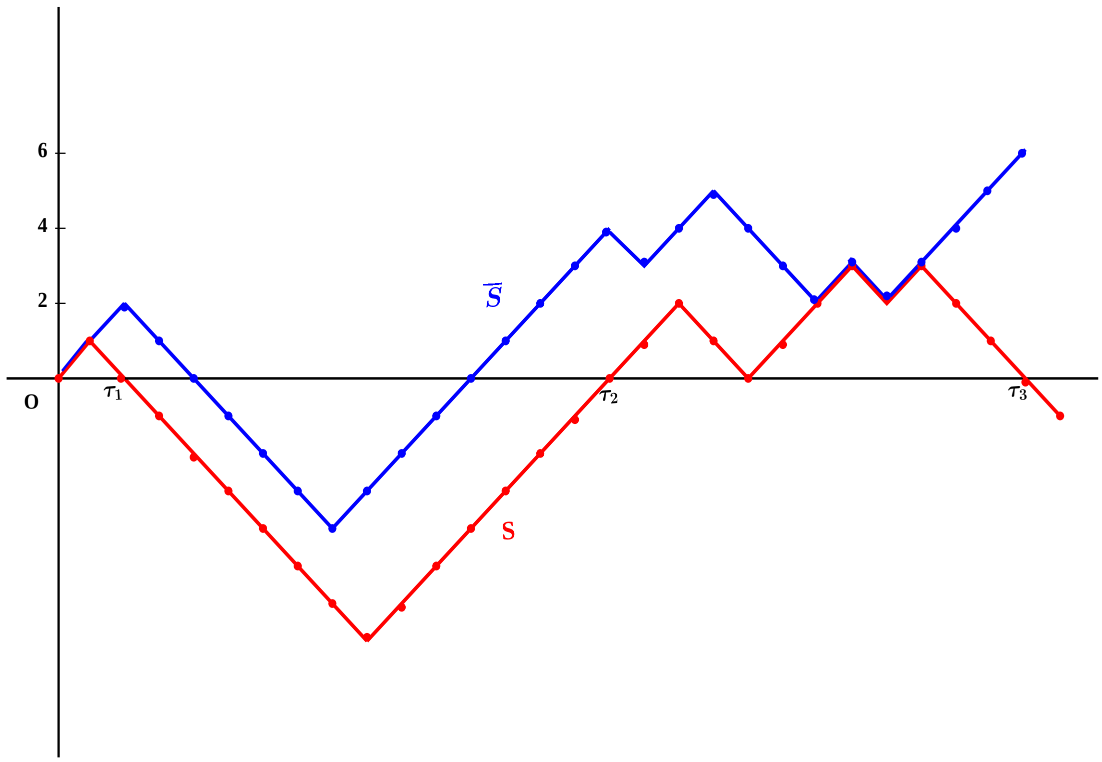

Figure 5.1: S and $\overline{S}$.

(i) For all $n \geq 0$, we have $\sigma(\overline{S}_j, j \leq n) \vee \sigma(S_1) = \sigma(S_j, j \leq n+1)$.

(ii) S_1 is independent of $\sigma(\overline{S})$.

(2) Let $\overline{S} = (\overline{S}_k)_{k\geq 0}$ be a SRW. Then

(i) There exists a SRW S such that:

$$\overline{Y}_n := \max_{k\leq n}\overline{S}_k - \overline{S}_n \Rightarrow |\overline{Y}_n - |S_n|| \leq 2 \quad \forall n \in \mathbb{N}.$$

(ii) $T^{-1}\{\overline{S}\}$ is reduced to exactly two elements S and $-S$ where S is obtained by adding information to $\overline{S}$.

Proof. (1) We retain the notations just before the corollary. (i) To prove the inclusion $\subset$, we only need to check that $\{\tau_l + 1 \leq j \leq \tau_{l+1}\} \in \sigma(S_h, h \leq n+1)$ for a fixed $j \leq n$. But this is clear since $\{\tau_l = m\} \in \sigma(S_h, h \leq m+1)$ for all $l, m \in \mathbb{N}$.

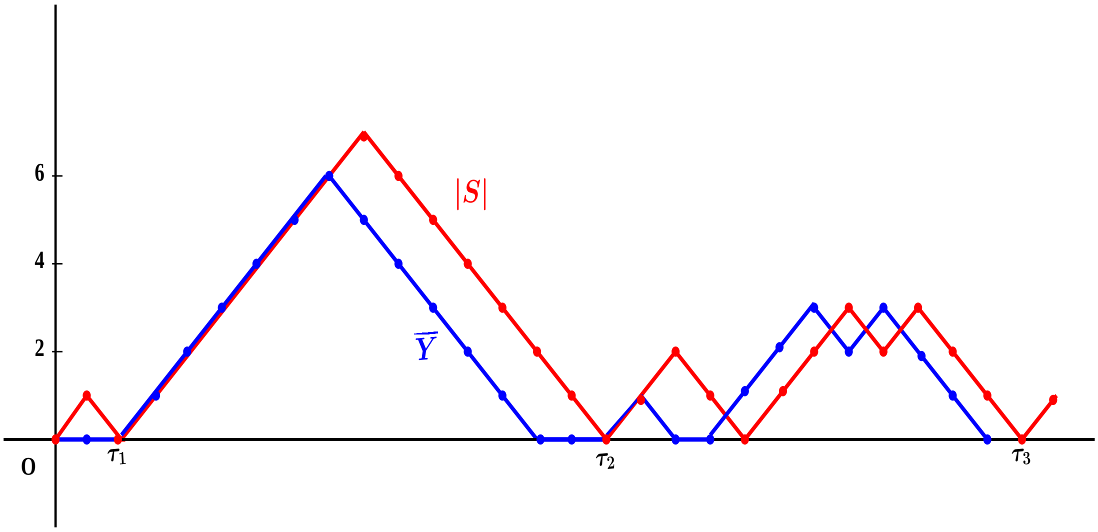

Figure 5.2: $|S|$ and $\overline{Y}$.

For all $1 \leq j \leq n$, we have $X_{j+1} = \sum_{l\geq 0}(-1)^{l+1}X_1\overline{X}_j 1_{\{\tau_l+1\leq j\leq \tau_{l+1}\}}$. By (5.4), $\{\tau_l + 1 \leq j \leq \tau_{l+1}\} \in \sigma(\overline{S}_h, h \leq j-1)$ and so the inclusion $\supset$ holds. (ii) We may write

$$\tau_1 = \min\{i > 1 : X_1 S_{i-1} X_1 S_{i+1} < 0\},\ \tau_{l+1} = \min\{i > \tau_l : X_1 S_{i-1} X_1 S_{i+1} < 0\}\ \forall l \geq 1.$$

This shows that $\overline{S}$ is $\sigma(X_1 X_{j+1}, j \geq 0)$-measurable and (ii) is proved.
(2) (i) Set $\overline{X}_j = \overline{S}_j - \overline{S}_{j-1}, j \geq 1$ and $\tau_l = \min\{n \geq 0, \overline{S}_n = 2l\}$ for all $l \geq 1$. Let ε be a random variable independent of $\overline{S}$ such that:

$$\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2}.$$

Define

$$X_{j+1} = \varepsilon 1_{\{j=0\}} + \left(\sum_{l\geq 0}(-1)^{l+1}\varepsilon\overline{X}_j 1_{\{\tau_l+1\leq j\leq \tau_{l+1}\}}\right) 1_{\{j\geq 1\}}.$$

Setting $S_0 = 0, S_j = X_1 + \cdots + X_j, j \geq 1$, its is not hard to see that the sequence of the random times $\tau_i(S), i \geq 1$ defined from S as in Theorem 5.3 is exactly $\tau_i, i \geq 1$ so that $T(S) = \overline{S}$. (ii) Let S such that $T(S) = \overline{S}$. By (1), $\sigma(\overline{S}) \vee \sigma(S_1) = \sigma(S)$ and S_1 is independent of $\overline{S}$ which proves (ii). $\square$

5.2.2 The link with Tanaka's equation

Let S be a SRW, $\overline{S} = -T(S)$ and $t \longmapsto S(t)$ (resp. $\overline{S}(t)$) be the linear interpolation of S (resp. $\overline{S}$) on $\mathbb{R}$. Define for all $n \geq 1$,

$$S_t^{(n)} = \frac{1}{\sqrt{n}} S(nt), \overline{S}_t^{(n)} = \frac{1}{\sqrt{n}} \overline{S}(nt).$$

Then, it can be easily checked (see Proposition 2.4 in [16] page 107) that

$$(\overline{S}_t^{(n)}, S_t^{(n)})_{t \geq 0} \xrightarrow[n \to +\infty]{\text{law}} (B_t, W_t)_{t \geq 0} \text{ in } C([0, \infty[, \mathbb{R}^2).$$

In particular B and W are two standard Brownian motions. On the other hand, $|Y_n^+ - |S_n|| \leq 2 \ \forall n \in \mathbb{N}$ with $Y_n^+ := \overline{S}_n - \min_{k \leq n} \overline{S}_k$ by Theorem 5.3 which implies

$$|W_t| = B_t - \min_{0 \leq u \leq t} B_u.$$

Tanaka's formula for local time gives

$$|W_t| = \int_0^t sgn(W_u) dW_u + L_t(W) = B_t - \min_{0 \leq u \leq t} B_u,$$

where $L_t(W)$ is the local time at 0 of W and so

$$dW_u = sgn(W_u) dB_u. \tag{5.5}$$

We deduce that for each SRW S the pair $(-T(S), S)$, suitably normalized and time scaled converges in law towards (B, W) satisfying (5.5). Finally, remark that

$$-T(S) = \overline{S} \Rightarrow -T(-S) = \overline{S}$$

is the analogue of

$$W \text{solves } (5.5) \Rightarrow -W \text{solves } (5.5).$$

We have seen how to construct solutions to (5.5) by means of T. In the sequel, we will use this approach to construct a stochastic flow of mappings which solves equation (T) in general.

5.2.3 Extensions

Let $S = (S_n)_{n\geq 0}$ be a SRW and set $Y_n := \max\limits_{k\leq n} S_k - S_n$. For $0 \leq p < q$, we say that $E = [p, q]$ is an excursion for Y if the following conditions are satisfied (with the convention $Y_{-1} = 0$):

- $Y_p = Y_{p-1} = Y_q = Y_{q+1} = 0$.
- $\forall\ p \leq j < q, Y_j = 0 \Rightarrow Y_{j+1} = 1$.

For example in Figure 5.2, $[2, 14], [16, 18]$ are excursions for $\overline{Y}$. If $E = [p, q]$ is an excursion for Y, define $e(E) := p, \quad f(E) := q$.

Let $(E_i)_{i\geq 1}$ be the random set of all excursions of Y ordered such that: $e(E_i) < e(E_j)\ \forall i < j$. From now on, we call E_i the ith excursion of Y. Then, we have

Proposition 5.1. *On a probability space $(\Omega, \mathcal{A}, P)$, consider the following jointly independent processes:*

- $\vec{\eta} = (\vec{\eta}_i)_{i\geq 1}$*, a sequence of i.i.d. random variables distributed according to* $\sum\limits_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$.
- $(\overline{S}_n)_{n\in\mathbb{N}}$ *a SRW.*

Then, there exists, on an extension of $(\Omega, \mathcal{A}, P)$ a Markov chain $(M_n)_{n\in\mathbb{N}}$ started at 0 *with stochastic matrix given by (2.2) such that:*

$$\overline{Y}_n := \max_{k\leq n} \overline{S}_k - \overline{S}_n \Rightarrow |M_n - \vec{\eta}_i \overline{Y}_n| \leq 2$$

on the ith excursion of $\overline{Y}$.

Proof. Fix $S \in T^{-1}\{\overline{S}\}$. Then, by Corollary 5.2, we have $|\overline{Y}_n - |S_n|| \leq 2 \quad \forall n \in \mathbb{N}$. Consider a sequence $(\vec{\beta}_i)_{i\geq 1}$ of i.i.d. random variables distributed according to $\sum\limits_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$ which is independent of $(\overline{S}, \vec{\eta})$. Denote by $(\tau_l)_{l\geq 1}$ the sequence of random times constructed in the proof of Theorem 5.3 from S. It is sufficient to look to what happens at each interval $[\tau_l, \tau_{l+1}]$ (with the convention $\tau_0 = 0$).

Using (5.4), we see that in $[\tau_l, \tau_{l+1}]$ there are two jumps of $\max\limits_{k\leq n} \overline{S}_k$; from $2l$ to $2l+1$ (J_1) and from $2l+1$ to $2l+2$ (J_2). The last jump (J_2) occurs always at τ_{l+1} by (5.4). Consequently there are only 3 possible cases:

(i) There is no excursion of $\overline{Y}$ (J_1 and J_2 occur respectively at $\tau_l + 1$ and $\tau_l + 2$, see $[0, \tau_1]$ in Figure 5.2).
(ii) There is just one excursion of $\overline{Y}$ (see $[\tau_1, \tau_2]$ in Figure 5.2).
(iii) There are 2 excursions of $\overline{Y}$ (see $[\tau_2, \tau_3]$ in Figure 5.2).
Note that: $\overline{Y}_{\tau_l} = \overline{Y}_{\tau_{l+1}} = S_{\tau_l} = S_{\tau_{l+1}} = 0$. In the case (i), we necessarily have $\tau_{l+1} = \tau_l + 2$. Set $M_n = \vec{\beta}_l.|S_n| \quad \forall n \in [\tau_l, \tau_{l+1}]$.
To treat other cases, the following remarks may be useful: from the expression of $\overline{S}$, we have $\forall l \geq 0$

(a) If $k \in [\tau_l + 2, \tau_{l+1}]$, $\overline{S}_{k-1} = 2l + 1 \Longleftrightarrow S_k = 0$.

(b) If $k \in [\tau_l, \tau_{l+1}]$, $\overline{Y}_k = 0 \Rightarrow |S_{k+1}| \in \{0, 1\}$ and $S_{k+1} = 0 \Rightarrow \overline{Y}_k = 0$.

In the case (ii), let E_l^1 be the unique excursion of $\overline{Y}$ in the interval $[\tau_l, \tau_{l+1}]$. Then, we have two subcases:
(ii1) $f(E_l^1) = \tau_{l+1} - 2$ (J_1 occurs at $\tau_{l+1} - 1$).
If $\tau_l + 2 \leq k \leqslant f(E_l^1) + 1$, then $k - 1 \leqslant f(E_l^1)$ and so $\overline{S}_{k-1} \neq 2l + 1$. Using (a), we get: $S_k \neq 0$. Thus, in this case the first zero of S after τ_l is τ_{l+1}. Set: $M_n = \vec{\eta}_{N(E_l^1)}.|S_n|$, where $N(E)$ is the number of the excursion E.
(ii2)$f(E_l^1) = \tau_{l+1} - 1$ (J_1 occurs at $\tau_l + 1$ and so $\overline{Y}_{\tau_l+1} = 0$). In this case, using (b) and the figure below we observe that the first zero τ_l^* of S after τ_l is $e(E_l^1) + 1 = \tau_l + 2$.

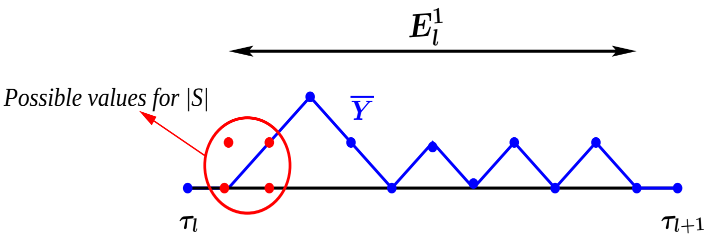

Figure 5.3: The case (ii2).

Set

$$M_n = \begin{cases} \vec{\beta}_l.|S_n| & \text{if } \ n \in [\tau_l, \tau_l^* - 1] \\ \vec{\eta}_{N(E_l^1)}.|S_n| & \text{if } \ n \in [\tau_l^*, \tau_{l+1}] \end{cases}$$

In the case (iii), let E_l^1 and E_l^2 denote respectively the first and 2nd excursion of $\overline{Y}$ in $[\tau_l, \tau_{l+1}]$. We have $\tau_l + 2 \leq k \leq e(E_l^2) \Rightarrow k - 1 \leq e(E_l^2) - 1 = f(E_l^1) \Rightarrow \overline{S}_{k-1} \neq 2l + 1 \Rightarrow S_k \neq 0$ by (a). Hence, the first zero of S after τ_l is $\tau_l^* := e(E_l^2) + 1$ using $\overline{Y}_k = 0 \Rightarrow |S_{k+1}| \in \{0, 1\}$ in (b). Set

$$M_n = \begin{cases} \vec{\eta}_{N(E_l^1)}.|S_n| & \text{if } n \in [\tau_l, \tau_l^* - 1] \\ \vec{\eta}_{N(E_l^2)}.|S_n| & \text{if } n \in [\tau_l^*, \tau_{l+1}] \end{cases}$$

Let $(M_n)_{n\in\mathbb{N}}$ be the process constructed above. Then clearly $|M_n - \vec{\eta}_i \overline{Y}_n| \leq 2$ on the ith excursion of $\overline{Y}$.

To complete the proof, it remains to show that the law of $(M_n)_{n\in\mathbb{N}}$ is given by (2.2). The only point to verify is $\mathbb{P}(M_{n+1} = \vec{e}_i | M_n = 0) = \alpha_i$. For this, consider on another probability space the jointly independent processes $(S, \vec{\gamma}, \vec{\lambda})$ such that S is a SRW and $\vec{\gamma}, \vec{\lambda}$ have the same law as $\vec{\eta}$. Let $(\tau_l)_{l\geq 1}$ be the sequence of random times defined from S as in Theorem 5.3. For all $l \in \mathbb{N}$, denote by τ_l^* the first zero of S after τ_l and set

$$V_n = \begin{cases} \vec{\gamma}_l.|S_n| & \text{if } \ n \in [\tau_l, \tau_l^* - 1] \\ \vec{\lambda}_l.|S_n| & \text{if } \ n \in [\tau_l^*, \tau_{l+1}] \end{cases}$$

It is clear, by construction, that $M \overset{law}{=} V$. We can write:

$$\{\tau_0, \tau_0^*, \tau_1, \tau_1^*, \tau_2, \cdots\} = \{T_0, T_1, T_2, \cdots\} \ \text{ with } T_0 = 0 < T_1 < T_2 < \cdots$$

For all $k \geq 0$, let $\vec{\zeta}_k := \sum_{j=0}^{N} \vec{e}_j 1_{\{V|_{[T_k, T_{k+1}]} \in D_j\}}$. Obviously, S and $\vec{\zeta}_k$ are independent and $\vec{\zeta}_k \overset{law}{=} \sum_{i=1}^{N} \alpha_i \delta_{\vec{e}_i}$. Furthermore

$$\begin{aligned} \mathbb{P}(V_{n+1} = \vec{e}_i | V_n = 0) &= \frac{1}{\mathbb{P}(S_n = 0)} \sum_{k=0}^{+\infty} \mathbb{P}(V_{n+1} = \vec{e}_i, S_n = 0, n \in [T_k, T_{k+1}[) \\ &= \frac{1}{\mathbb{P}(S_n = 0)} \sum_{k=0}^{+\infty} \mathbb{P}(\vec{\zeta}_k = \vec{e}_i, S_n = 0, n \in [T_k, T_{k+1}[) \\ &= \alpha_i \end{aligned}$$

This ends the proof of the proposition. $\square$

Remark 5.1. *With the notations of Proposition 5.1, let $(\vec{\eta}.\overline{Y})$ be the Markov chain defined by $(\vec{\eta}.\overline{Y})_n = \vec{\eta}_i\overline{Y}_n$ on the ith excursion of $\overline{Y}$ and $(\vec{\eta}.\overline{Y})_n = 0$ if $\overline{Y}_n = 0$. Then the stochastic Matrix of $(\vec{\eta}.\overline{Y})$ is given by*

$$M(0,\vec{e}_i) = \frac{\alpha_i}{2}, \quad M(n\vec{e}_i,(n\pm 1)\vec{e}_i) = \frac{1}{2}\ \forall i \in [1,N],\ n \in \mathbb{N}^*, M(0,0) = \frac{1}{2}. \tag{5.6}$$

As a consequence of Proposition 5.1, $(\vec{\eta}.\overline{Y})$ rescales as Walsh Brownian motion. This might be proved without having to resort to Proposition 5.1, by showing that the family of laws is tight and that any limit process along a subsequence is the Walsh process.

5.3 Proof of main results

5.3.1 Scaling limits of (Ψ, K)

Set $\vec{\eta}_{p,n} = \vec{e}(\Psi_{p,n})$ for all $p \le n$ where $\Psi_{p,n} = \Psi_{p,n}(0)$.

Proposition 5.2. *(i) For all $p \le n$, $|\Psi_{p,n}| = S^+_{p,n}$.*

(ii) For all $p < n < q$,

$$\mathbb{P}(\vec{\eta}_{p,q} = \vec{\eta}_{n,q}|min_{h\in[p,q]}S_h = min_{h\in[n,q]}S_h) = 1$$

and

$$\mathbb{P}(\vec{\eta}_{p,n} = \vec{\eta}_{p,q}|min_{h\in[p,n]}S_h = min_{h\in[p,q]}S_h, S^+_{p,j} > 0\ \forall j \in [n,q]) = 1.$$

(iii) Set $T_{p,x} = \inf\{q \ge p : S_q - S_p = -|x|\}$. Then for all $p \le n$, $x \in G_{\mathbb{N}}$,

$$\Psi_{p,n}(x) = (x + \vec{e}(x)S_{p,n})1_{\{n\le\ T_{p,x}\}} + \Psi_{p,n}1_{\{n>T_{p,x}\}};$$

$$K_{p,n}(x) = E[\delta_{\Psi_{p,n}(x)}|\sigma(S)] = \delta_{x+\vec{e}(x)S_{p,n}}1_{\{n\le T_{p,x}\}} + \sum_{i=1}^{N}\alpha_i\delta_{S^+_{p,n}\vec{e}_i}1_{\{n>T_{p,x}\}}.$$

Proof. (i) We take $p = 0$ and prove the result by induction on n. For $n = 0$, this is clear. Suppose the result holds for n. If $\Psi_{0,n} \in G^*$, then $S^+_{0,n} > 0$ and so $\min_{h\in[0,n]} S_h = \min_{h\in[0,n+1]} S_h$. Moreover

$$\Psi_{0,n+1} = \Psi_{0,n} + \vec{\eta}_{0,n} S_{n,n+1} = (S_{n+1} - \min_{h\in[0,n]} S_h)\vec{\eta}_{0,n} = S^+_{0,n+1}\vec{\eta}_{0,n}.$$

If $\Psi_{0,n} = 0$, then $S^+_{0,n} = 0$ and $|\Psi_{0,n+1}| = S^+_{n,n+1}$. But $\min_{h\in[0,n+1]} S_h = \min(\min_{h\in[0,n]} S_h, S_{n+1}) = \min(S_n, S_{n+1})$ since $S^+_{0,n} = 0$ which proves (i).

(ii) Let $p < n < q$. If $\min_{h\in[p,q]} S_h = \min_{h\in[n,q]} S_h$, then $S^+_{p,q} = S^+_{n,q}$. When $S^+_{p,q} = 0$, we have $\vec{\eta}_{p,q} = \vec{\eta}_{n,q} = \vec{e}_N$ by convention. Suppose that $S^+_{p,q} > 0$, then clearly

$$J := \sup\{j < q : S^+_{p,j} = 0\} = \sup\{j < q : S^+_{n,j} = 0\}.$$

By the flow property of Ψ, we have $\Psi_{p,q} = \Psi_{n,q} = \Psi_{J,q}$. The second assertion of (ii) is also clear.

(iii) By (i), we have $\Psi_{p,n} = \Psi_{p,n}(x) = 0$ if $n = T_{p,x}$ and so $\Psi_{p,\cdot}(x)$ is given by $\Psi_{p,\cdot}$ after $T_{p,x}$ using the cocyle property. The last claim is easy to establish. □

For all $s \in \mathbb{R}$, let d_s (resp. d_∞) be the distance of uniform convergence on every compact subset of $C([s, +\infty[, G)$ (resp. $C(\mathbb{R}, \mathbb{R})$). Denote by $\mathbb{D} = \{s_n, n \in \mathbb{N}\}$ the set of all dyadic numbers of $\mathbb{R}$ and define

$$\widetilde{C} = C(\mathbb{R}, \mathbb{R}) \times \prod_{n=0}^{+\infty} C([s_n, +\infty[, G)$$

equipped with the metric:

$$d(x, y) = d_\infty(x', y') + \sum_{n=0}^{+\infty} \frac{1}{2^n} \inf(1, d_{s_n}(x_n, y_n)) \text{ where } x = (x', x_{s_0}, \cdots), y = (y', y_{s_0}, \cdots).$$

Let $t \longmapsto S(t)$ be the linear interpolation of S on $\mathbb{R}$ and define $S^{(n)}_t = \frac{1}{\sqrt{n}} S(nt), n \geq 1$. If $u \leq 0$, we define $\lfloor u \rfloor = -\lfloor -u \rfloor$. Then, we have

$$S^{(n)}_t = S^n_t + o(\frac{1}{\sqrt{n}}), \text{ with } S^n_t := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}.$$

Let $\Psi^n_{s,t} = \Psi^n_{s,t}(0)$ (defined in Corollary 5.1). Then $\Psi^{(n)}_{s,t} := \frac{1}{\sqrt{n}} \Psi_{\lfloor ns \rfloor, \lfloor nt \rfloor} + o(\frac{1}{\sqrt{n}})$ and we have the following

Lemma 5.1. *Let $\mathbb{P}_n$ be the law of $Z^n = (S_\cdot^{(n)}, (\Psi_{s_i,\cdot}^{(n)})_{i\in\mathbb{N}})$ in $\widetilde{C}$. Then $(\mathbb{P}_n, n \geq 1)$ is tight.*

Proof. By Donsker theorem $\mathbb{P}_{S^{(n)}} \longrightarrow \mathbb{P}_W$ in $C(\mathbb{R},\mathbb{R})$ as $n \to \infty$ where $\mathbb{P}_W$ is the law of any Brownian motion on the real line. Let $\mathbb{P}_{Z_{s_i}}$ be the law of any $W(\alpha_1, \cdots, \alpha_N)$ process started at 0 at time s_i. Plainly, the law of $\Psi_{p,p+\cdot}$ is given by (5.6). By the above propositions 2.3 and 5.1, for all $i \in \mathbb{N}$, $\mathbb{P}_{\Psi_{s_i,\cdot}^{(n)}} \longrightarrow \mathbb{P}_{Z_{s_i}}$ in $C([s_i, +\infty[, G)$ as $n \to \infty$. Now the lemma holds using Proposition 2.4 [16] (page 107). □

Fix a sequence $(n_k, k \in \mathbb{N})$ such that $Z^{n_k} \xrightarrow[k\to+\infty]{\text{law}} Z$ in $\widetilde{C}$. In the next paragraph, we will describe the law of Z. Notice that $(\Psi_{p,n})_{p\leq n}$ and S can be recovered from $(Z^{n_k})_{k\in\mathbb{N}}$. Using Skorokhod representation theorem, we may assume that Z is defined on the original probability space and the preceding convergence holds almost surely. Write $Z = (W, \psi_{s_1,\cdot}, \psi_{s_2,\cdot}, \cdots)$. Then, $(W_t)_{t\in\mathbb{R}}$ is a Brownian motion on the real line and $(\psi_{s,t})_{t\geq s}$ is a $W(\alpha_1, \cdots, \alpha_N)$ process started at 0 for all $s \in \mathbb{D}$.

Description of the limit process

Set $\vec{\gamma}_{s,t} = \vec{e}(\psi_{s,t}), s \in \mathbb{D}, s < t$ and define $\min_{u,v} = \min_{r\in[u,v]} W_r$, $u \leq v \in \mathbb{R}$. Then, we have

Proposition 5.3. *(i) For all $s \leq t, s \in \mathbb{D}$, $|\psi_{s,t}| = W_{s,t}^{+}$.*

(ii) For all $s < t, u < v$, $s, u \in \mathbb{D}$,

$$\mathbb{P}(\vec{\gamma}_{s,t} = \vec{\gamma}_{u,v} | min_{s,t} = min_{u,v}) = 1 \ \textit{if} \ \mathbb{P}(min_{s,t} = min_{u,v}) > 0.$$

Proof. (i) is immediate from the convergence of Z^{n_k} towards Z and Proposition 5.2 (i). (ii) We first prove that for all $s < t < u$,

$$\mathbb{P}(\vec{\gamma}_{s,u} = \vec{\gamma}_{t,u} | \min_{s,u} = \min_{t,u}) = 1 \text{ if } s, t \in \mathbb{D} \tag{5.7}$$

and

$$\mathbb{P}(\vec{\gamma}_{s,t} = \vec{\gamma}_{s,u} | \min_{s,t} = \min_{s,u}) = 1 \text{ if } s \in \mathbb{D}. \tag{5.8}$$

Fix $s < t < u$ with $s, t \in \mathbb{D}$ and let show that a.s.

$$\{\min_{s,u} = \min_{t,u}\} \subset \{\exists k_0, \quad \vec{\eta}_{\lfloor n_k s\rfloor, \lfloor n_k u\rfloor} = \vec{\eta}_{\lfloor n_k t\rfloor, \lfloor n_k u\rfloor} \text{ for all } k \geq k_0\}. \tag{5.9}$$

We have $\{\min_{s,u} = \min_{t,u}\} = \{\min_{s,t} < \min_{t,u}\}$ *a.s.* By uniform convergence the last set is contained in

$$\{\exists k_0, \min_{\lfloor n_k s\rfloor \leq j \leq \lfloor n_k t\rfloor} S_j < \min_{\lfloor n_k t\rfloor \leq j \leq \lfloor n_k u\rfloor} S_j \text{ for all } \; k \geq k_0\}$$

which is a subset of

$$\{\exists k_0, \min_{\lfloor n_k s\rfloor \leq j \leq \lfloor n_k u\rfloor} S_j = \min_{\lfloor n_k t\rfloor \leq j \leq \lfloor n_k u\rfloor} S_j \text{ for all } k \geq k_0\}.$$

This gives (5.9) using Proposition 5.2 (ii). Since $x \longrightarrow \vec{e}(x)$ is continuous on G^*, on $\{\min_{s,u} = \min_{t,u}\}$, we have

$$\vec{\gamma}_{s,u} = \lim_{k\to\infty} \vec{e}(\frac{1}{\sqrt{n_k}} \Psi_{\lfloor n_k s\rfloor, \lfloor n_k u\rfloor}) = \lim_{k\to\infty} \vec{e}(\frac{1}{\sqrt{n_k}} \Psi_{\lfloor n_k t\rfloor, \lfloor n_k u\rfloor}) = \vec{\gamma}_{t,u} \quad a.s.$$

which proves (5.7). If $s \in \mathbb{D}, t > s$ and $\min_{s,t} = \min_{s,u}$, then s and t are in the same excursion interval of $W^+_{s,}$ and so $W^+_{s,r} > 0$ for all $r \in [t, u]$. As preceded,$\{\min_{s,t} = \min_{s,u}\}$ is a.s. containted in

$$\{\exists k_0, \min_{\lfloor n_k s\rfloor \leq j \leq \lfloor n_k t\rfloor} S_j = \min_{\lfloor n_k s\rfloor \leq j \leq \lfloor n_k u\rfloor} S_j \;, S^+_{\lfloor n_k s\rfloor, j} > 0 \; \forall j \in [\lfloor n_k t\rfloor, \lfloor n_k u\rfloor], k \geq k_0\}.$$

Now (5.8) can be deduced from Proposition 5.2 (ii). To prove (ii), suppose that $s \leq u, \min_{s,t} = \min_{u,v}$. There are two cases to discuss, (a) $s \leq u \leq v \leq t$, (b) $s \leq u \leq t \leq v$ (in any other case $\mathbb{P}(\min_{s,t} = \min_{u,v}) = 0$). In case (a), we have $\min_{s,t} = \min_{u,v} = \min_{u,t}$ and so $\vec{\gamma}_{s,t} = \vec{\gamma}_{u,t} = \vec{\gamma}_{u,v}$ by (5.7) and (5.8). Similarly in case (b), we have $\vec{\gamma}_{s,t} = \vec{\gamma}_{u,t} = \vec{\gamma}_{u,v}$. □

Proposition 5.4. *Fix $s < t, s \in \mathbb{D}, n \geq 1$ and $\{(s_i, t_i); 1 \leq i \leq n\}$ with $s_i < t_i, s_i \in \mathbb{D}$. Then*

(i) $\vec{\gamma}_{s,t}$ is independent of $\sigma(W)$.

(ii) For all $i \in [1,N]$, $h \in [1,n]$, we have

$$E\big[1_{\{\vec{\gamma}_{s,t}=\vec{e}_i\}}|(\vec{\gamma}_{s_i,t_i})_{1\le i\le n}, W\big] = 1_{\{\vec{\gamma}_{s_h,t_h}=\vec{e}_i\}} \text{ on } \{min_{s,t} = min_{s_h,t_h}\}.$$

(iii) The law of $\vec{\gamma}_{s,t}$ knowing $(\vec{\gamma}_{s_i,t_i})_{1\le i\le n}$ and W is given by $\sum_{i=1}^{N}\alpha_i\delta_{\vec{e}_i}$ when $min_{s,t} \notin \{min_{s_i,t_i}; 1 \le i \le n\}$.

This entirely describes the law of $(W, \psi_{s,\cdot}, s \in \mathbb{D})$ in $\widetilde{C}$ independently of $(n_k, k \in \mathbb{N})$ and consequently $Z^n \xrightarrow[n\to+\infty]{law} Z$ in $\widetilde{C}$.

Proof. (i) is clear. (ii) is a consequence of Proposition 5.3 (ii). (iii) Write $\{s, t, s_i, t_i, 1 \le i \le n\} = \{r_k, 1 \le k \le m\}$ with $r_j < r_{j+1}$ for all $1 \le j \le m-1$. Suppose that $s = r_i, t = r_h$ with $i < h$. Then a.s. $\{\min_{r_j,r_{j+1}}, i \le j \le h-1\}$ are distinct and it will be sufficient to show that $\vec{\gamma}_{s,t}$ is independent of $\sigma((\vec{\gamma}_{s_i,t_i})_{1\le i\le n}, W)$ conditionally to $A = \{\min_{s,t} = \min_{r_j,r_{j+1}},\ \min_{s,t} \ne \min_{s_i,t_i}$ for all $1 \le i \le n\}$ for $j \in [i, h-1]$. On A, we have $\vec{\gamma}_{s,t} = \vec{\gamma}_{r_j,r_{j+1}}$, $\{\min_{s_i,t_i}, 1 \le i \le n\} \subset \{\min_{r_k,r_{k+1}}, k \ne j\}$ and so $\{\vec{\gamma}_{s_i,t_i}, 1 \le i \le n\} \subset \{\vec{\gamma}_{r_k,r_{k+1}}, k \ne j\}$. Since $\vec{\gamma}_{r_1,r_2}, \cdots, \vec{\gamma}_{r_{m-1},r_m}, W$ are independent, it is now easy to conclude. $\square$

In the sequel, we still assume that all processes are defined on the same probability space and that $Z^n \xrightarrow[n\to+\infty]{\text{a.s.}} Z$ in $\widetilde{C}$. In particular $\forall s \in \mathbb{D}, T > 0$,

$$\lim_{k\to+\infty} \sup_{s\le t\le s+T} \Big|\frac{1}{\sqrt{k}}\Psi_{\lfloor ks\rfloor,\lfloor kt\rfloor} - \psi_{s,t}\Big| = 0 \ \ a.s. \tag{5.10}$$

Extension of the limit process

For a fixed $s < t$, $\min_{s,t}$ is attained in $]s, t[$ a.s. By Proposition 5.3 (ii), on a measurable set $\Omega_{s,t}$ with probability 1, $\lim_{s'\to s+, s'\in\mathbb{D}} \vec{\gamma}_{s',t}$ exists. Define $\vec{\varepsilon}_{s,t} = \lim_{s'\to s+, s'\in\mathbb{D}} \vec{\gamma}_{s',t}$ on $\Omega_{s,t}$ and give an arbitrary value to $\vec{\varepsilon}_{s,t}$ on $\Omega^c_{s,t}$. Now, let $\varphi_{s,t} = \vec{\varepsilon}_{s,t}W^+_{s,t}$. Then for all $s \in \mathbb{D}, t > s$, $(\vec{\varepsilon}_{s,t}, \varphi_{s,t})$ is a modification of $(\vec{\gamma}_{s,t}, \psi_{s,t})$. For all $s \in \mathbb{R}$, $t > s$, $\varphi_{s,t} = \lim_{n\to\infty} \varphi_{s_n,t}$ *a.s.* where $s_n = \frac{\lfloor 2^n s\rfloor+1}{2^n}$ and in particular $(\varphi_{s,t})_{t\ge s}$ is a $W(\alpha_1, \cdots, \alpha_N)$ process started at 0. Again, Proposition 5.3 (ii) yields

$$\forall s < t, u < v,\ \mathbb{P}(\vec{\varepsilon}_{s,t} = \vec{\varepsilon}_{u,v}|\min_{s,t} = \min_{u,v}) = 1 \text{ if } \mathbb{P}(\min_{s,t} = \min_{u,v}) > 0. \tag{5.11}$$

Define:

$$\varphi_{s,t}(x) = (x + \vec{e}(x)W_{s,t})1_{\{t \le \tau_{s,x}\}} + \varphi_{s,t}1_{\{t > \tau_{s,x}\}}, s \le t, x \in G,$$

where $W_{s,t} = W_t - W_s$ and $\tau_{s,x}$ is given by (5.2).

Proposition 5.5. *Let $x \in G, x_n \in \frac{1}{\sqrt{n}}G_{\mathbb{N}}, \lim_{n\to\infty} x_n = x,\ s \in \mathbb{R}, T > 0$. Then a.s.*

$$\lim_{n\to+\infty} \sup_{s\le t\le s+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns\rfloor,\lfloor nt\rfloor}(\sqrt{n}x_n) - \varphi_{s,t}(x)| = 0.$$

Proof. Let s' be a dyadic number such that $s < s' < s+T$. By (5.11), for $t > s'$, a.s.

$$\{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\}.$$

and so, a.s. $\forall t > s', t \in \mathbb{D}$;

$$\{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\}.$$

If $t > s', \min_{s,t} = \min_{s',t}$ and $t_n \in \mathbb{D}, t_n \downarrow t$ as $n \to \infty$, then $\min_{s,t_n} = \min_{s',t_n}$ which entails that $\varphi_{s,t_n} = \varphi_{s',t_n}$ and a fortiori $\varphi_{s,t} = \varphi_{s',t}$ by letting $n \to \infty$. Thus a.s. $\forall t > s'$;

$$\{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\}.$$

Finally a.s.

$$\forall s' \in \mathbb{D}\cap]s, s+T[, \forall t > s';\ \{\min_{s,t} = \min_{s',t}\} \subset \{\varphi_{s,t} = \varphi_{s',t}\}. \tag{5.12}$$

By standard properties of Brownian paths, a.s. $\min_{s,s+T} \notin \{W_s, W_{s+T}\}$ and

$$\forall p \in \mathbb{N}^*; \min_{s,s+\frac{1}{p}} < W_s,\ \min_{s,s+\frac{1}{p}} \ne W_{s+\frac{1}{p}}, \exists! u_p \in]s, s+\frac{1}{p}[: \min_{s,s+\frac{1}{p}} = W_{u_p}.$$

The reasoning below holds almost surely: Take $p \ge 1, \min_{s,s+\frac{1}{p}} > \min_{s,s+T}$. Let $\mathcal{S}_p \in]s, s+\frac{1}{p}[: \min_{s,s+\frac{1}{p}} = W_{\mathcal{S}_p}$ and s' be a (random) dyadic number in $]s, \mathcal{S}_p[$. Then $\min_{s,s'} > \min_{s',t}$ for all $t \in [\mathcal{S}_p, s+T]$. By uniform convergence, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \ge n_0,\ \forall \mathcal{S}_p \le t \le s+T,\ \min_{u\in[s,s']} S_{\lfloor nu\rfloor} > \min_{u\in[s',t]} S_{\lfloor nu\rfloor} \text{ and so } \Psi_{\lfloor ns'\rfloor,\lfloor nt\rfloor} = \Psi_{\lfloor ns\rfloor,\lfloor nt\rfloor}.$$

Hence for $n \geq n_0$, we have

$$\sup_{\mathcal{S}_p \leq t \leq s+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns \rfloor, \lfloor nt \rfloor} - \varphi_{s,t}| = \sup_{\mathcal{S}_p \leq t \leq s+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns' \rfloor, \lfloor nt \rfloor} - \varphi_{s',t}| \text{ (using (5.12))}$$

and so

$$\begin{aligned}
\sup_{s \leq t \leq s+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns \rfloor, \lfloor nt \rfloor} - \varphi_{s,t}| &\leq \sup_{s \leq t \leq \mathcal{S}_p} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns \rfloor, \lfloor nt \rfloor} - \varphi_{s,t}| + \sup_{\mathcal{S}_p \leq t \leq s+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns \rfloor, \lfloor nt \rfloor} - \varphi_{s,t}| \\
&\leq \sup_{s \leq t \leq s+\frac{1}{p}} (\frac{1}{\sqrt{n}}S^+_{\lfloor ns \rfloor, \lfloor nt \rfloor} + W^+_{s,t}) + \sup_{\mathcal{S}_p \leq t \leq s+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns' \rfloor, \lfloor nt \rfloor} - \varphi_{s',t}| \\
&\leq \sup_{s \leq t \leq s+\frac{1}{p}} (\frac{1}{\sqrt{n}}S^+_{\lfloor ns \rfloor, \lfloor nt \rfloor} + W^+_{s,t}) + \sup_{s' \leq t \leq s'+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns' \rfloor, \lfloor nt \rfloor} - \varphi_{s',t}|.
\end{aligned}$$

From (5.10), a.s. $\forall u \in \mathbb{D}$, $\lim_{n \to +\infty} \sup_{u \leq t \leq u+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor nu \rfloor, \lfloor nt \rfloor} - \varphi_{u,t}| = 0$. By letting n go to $+\infty$ and then p go to $+\infty$, we obtain

$$\lim_{n \to \infty} \sup_{s \leq t \leq s+T} |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns \rfloor, \lfloor nt \rfloor} - \varphi_{s,t}| = 0 \ a.s. \tag{5.13}$$

We now show that

$$\lim_{n \to +\infty} \frac{1}{n}T_{\lfloor ns \rfloor, \sqrt{n}x_n} = \tau_{s,x} \ a.s. \tag{5.14}$$

We have

$$\frac{1}{n}T_{\lfloor ns \rfloor, \sqrt{n}x_n} = \inf\{r \geq \frac{\lfloor ns \rfloor}{n} : S^n_r - S^n_s = -|x_n|\}.$$

For $\epsilon > 0$, from

$$\lim_{n \to \infty} \sup_{u \in [\tau_{s,x}, \tau_{s,x}+\epsilon]} |(S^n_u - S^n_s + |x_n|) - (W_{s,u} + |x|)| = 0,$$

we get

$$\lim_{n \to \infty} \inf_{u \in [\tau_{s,x}, \tau_{s,x}+\epsilon]} (S^n_u - S^n_s + |x_n|) = \inf_{u \in [\tau_{s,x}, \tau_{s,x}+\epsilon]} (W_{s,u} + |x|) < 0$$

which implies that $\frac{1}{n}T_{\lfloor ns \rfloor, \sqrt{n}x_n} < \tau_{s,x} + \epsilon$ for n large. If $x = 0$, $\frac{1}{n}T_{\lfloor ns \rfloor, \sqrt{n}x_n} \geq \frac{\lfloor ns \rfloor}{n}$ entails obviously (5.14). If $x \neq 0$, then working in $[s, \tau_{s,x} - \epsilon]$ as before and using $\inf_{u \in [s, \tau_{s,x}-\epsilon]}(W_u - W_s + |x|) > 0$, we prove that $\frac{1}{n}T_{\lfloor ns \rfloor, \sqrt{n}x_n} \leq \tau_{s,x} - \epsilon$ for n large which establishes (5.14).

Now

$$\sup_{s\le t\le s+T} \left|\frac{1}{\sqrt{n}}\Psi_{\lfloor ns\rfloor,\lfloor nt\rfloor}(\sqrt{n}x_n) - \varphi_{s,t}(x)\right| \le \sup_{s\le t\le s+T} Q^{1,n}_{s,t} + \sup_{s\le t\le s+T} Q^{2,n}_{s,t} \tag{5.15}$$

where

$$Q^{1,n}_{s,t} = |(x_n + \vec{e}(x_n)(S^n_t - S^n_s))1_{\{\lfloor nt\rfloor \le T_{\lfloor ns\rfloor,\sqrt{n}x_n}\}} - (x+\vec{e}(x)W_{s,t})1_{\{t\le\tau_{s,x}\}}|,$$

$$Q^{2,n}_{s,t} = |\frac{1}{\sqrt{n}}\Psi_{\lfloor ns\rfloor,\lfloor nt\rfloor}1_{\{\lfloor nt\rfloor > T_{\lfloor ns\rfloor,\sqrt{n}x_n}\}} - \varphi_{s,t}1_{\{t>\tau_{s,x}\}}|.$$

By (5.13), (5.14) and the convergence of $\frac{1}{\sqrt{n}}S_{\lfloor n.\rfloor}$ towards W on compact sets, the right-hand side of (5.15) converges to 0 when $n\to+\infty$. $\square$

Remark 5.2. *From the definition of $\vec{\varepsilon}_{s,t}$ (or Proposition 5.5), it is obvious that $\vec{\varepsilon}_{r_1,r_2},\cdots,\vec{\varepsilon}_{r_{m-1},r_m}, W$ are independent for all $r_1<\cdots<r_m$. Using (5.11), we easily check that (i), (ii) and (iii) of Proposition 5.4 are satisfied for all $s<t, n\ge 1, \{(s_i,t_i); 1\le i\le n\}$ with $s_i<t_i$ (the proof remains the same as Proposition 5.4).*

Proposition 5.6. *φ is the unique stochastic flow of mappings solution of (T).*

Proof. Fix $s<t<u, x\in G$ and let prove that $\varphi_{s,u}(x)=\varphi_{t,u}\circ\varphi_{s,t}(x)$ a.s. We follow Lemma 4.3 [36] and denote $\tau_{s,x}$ by $\tau_s(x)$. All the equalities below hold a.s.

On the event $\{u<\tau_s(x)\}, \varphi_{s,t}(x) = x+\vec{e}(x)W_{s,t}, \tau_t(\varphi_{s,t}(x)) = \tau_s(x) < u$ and

$$\varphi_{t,u}\circ\varphi_{s,t}(x) = x+\vec{e}(x)(W_{s,t}+W_{t,u}) = x+\vec{e}(x)W_{s,u} = \varphi_{s,u}(x).$$

On the event $\{\tau_s(x)\in]t,u]\}$, we still have $\varphi_{s,t}(x) = x+\vec{e}(x)W_{s,t}$ and $\tau_t(\varphi_{s,t}(x)) = \tau_s(x)\le u$, thus

$$\varphi_{t,u}\circ\varphi_{s,t}(x) = \vec{\varepsilon}_{t,u}W^+_{t,u} = \vec{\varepsilon}_{s,u}W^+_{s,u} = \varphi_{s,u}(x).$$

since on the event $\{\tau_s(x)\in]t,u]\}, \min_{s,u} = \min_{t,u}$ and $W^+_{s,u} = W_u - \min_{s,u} = W^+_{t,u}$.

On the event $\{\tau_s(x)\le t\}\cap\{\tau_t(\varphi_{s,t}(x))\le u\}, \varphi_{s,t}(x) = \vec{\varepsilon}_{s,t}W^+_{s,t}$ and

$$\varphi_{t,u}\circ\varphi_{s,t}(x) = \varphi_{t,u}(\vec{\varepsilon}_{s,t}W^+_{s,t}) = \vec{\varepsilon}_{t,u}W^+_{t,u} = \vec{\varepsilon}_{s,u}W^+_{s,u} = \varphi_{s,u}(x)$$

since $W^+_{s,\tau_t(\varphi_{s,t}(x))} = 0$ and thus $\min_{s,u} = \min_{t,u}$ which implies $\vec{\varepsilon}_{s,u} = \vec{\varepsilon}_{t,u}$ and $W^+_{s,u} = W^+_{t,u}$.

On the event $\{\tau_s(x) \leq t\} \cap \{\tau_t(\varphi_{s,t}(x)) > u\}, \varphi_{s,t}(x) = \vec{\varepsilon}_{s,t}W^+_{s,t}$ and

$$\varphi_{t,u} \circ \varphi_{s,t}(x) = \varphi_{t,u}(\vec{\varepsilon}_{s,t}W^+_{s,t}) = \vec{\varepsilon}_{s,t}(W^+_{s,t} + W_{t,u}) = \vec{\varepsilon}_{s,u}W^+_{s,u} = \varphi_{s,u}(x).$$

since in this case $\min_{s,u} = \min_{s,t}$ which implies $\vec{\varepsilon}_{s,u} = \vec{\varepsilon}_{s,t}$ and

$$\begin{aligned} W^+_{s,u} &= W_u - \min_{s,u} \\ &= W_u - W_s + W_s - \min_{s,t} \\ &= W^+_{s,t} + W_{t,u}. \end{aligned}$$

Thus we have, a.s. $\varphi_{s,u}(x) = \varphi_{t,u} \circ \varphi_{s,t}(x)$ which proves the cocyle property for φ. It is now easy to check that φ is a stochastic flow of mappings in the sense of Definition 3.1.

Note that $(\varphi_{0,t}, t \geq 0)$ is a $W(\alpha_1, \cdots, \alpha_N)$ process started at 0 and therefore satisfies Freidlin-Sheu formula (Theorem 2.3). Let $f \in D(\alpha_1, \cdots, \alpha_N)$, then for all $t \geq 0$,

$$f(\varphi_{0,t}) = f(0) + \int_0^t f'(\varphi_{0,u})dB_u + \frac{1}{2}\int_0^t f''(\varphi_{0,u})du \quad a.s.$$

where $B_t = |\varphi_{0,t}| - \tilde{L}_t(|\varphi_{0,\cdot}|)$ and $\tilde{L}_t(|\varphi_{0,\cdot}|)$ is the symmetric local time at 0 of $|\varphi_{0,\cdot}|$. Since $|\varphi_{0,t}| = W_t - \min_{0,t}$, we get $B_t = W_t$. Let $x \in D_i \setminus \{0\}$ and $f_i(r) = f(r\vec{e}_i), r \geq 0$. Since $\lim_{z\to 0, z\in D_i, z\neq 0} f'(z)$ and $\lim_{z\to 0, z\in D_i, z\neq 0} f''(z)$ exist, we can construct g which is C^2 on $\mathbb{R}$ and coincides with f_i on $\mathbb{R}_+$. By Itô's formula,

$$g(|x| + W_t) = g(|x|) + \int_0^t g'(|x| + W_u)dW_u + \frac{1}{2}\int_0^t g''(|x| + W_u)du \quad a.s.$$

and so for $t \leqslant \tau_0(x)$, we have

$$f(\varphi_{0,t}(x)) = f(x) + \int_0^t f'(\varphi_{0,u}(x))dW_u + \frac{1}{2}\int_0^t f''(\varphi_{0,u}(x))du \quad a.s.$$

Set $\alpha = f(0) + \int_0^{\tau_0(x)} f'(\varphi_{0,u})dW_u + \frac{1}{2}\int_0^{\tau_0(x)} f''(\varphi_{0,u})du = f(\varphi_{0,\tau_0(x)}) = f(0)$ since $W^+_{0,\tau_0(x)} = 0$. Then for $t > \tau_0(x)$, write

$$\begin{aligned} f(\varphi_{0,t}(x)) = f(\varphi_{0,t}) &= \alpha + \int_{\tau_0(x)}^t f'(\varphi_{0,u})dW_u + \frac{1}{2}\int_{\tau_0(x)}^t f''(\varphi_{0,u})du \\ &= f(0) + \int_{\tau_0(x)}^t f'(\varphi_{0,u}(x))dW_u + \frac{1}{2}\int_{\tau_0(x)}^t f''(\varphi_{0,u}(x))du. \end{aligned}$$

But $f(x)+\int_0^{\tau_0(x)} f'(\varphi_{0,u}(x))dW_u+\frac{1}{2}\int_0^{\tau_0(x)} f''(\varphi_{0,u}(x))du = f(\varphi_{0,\tau_0(x)}(x)) = f(0)$ and so, for all $t\geq 0, f\in D(\alpha_1,\cdots,\alpha_N), x\in G$,

$$f(\varphi_{0,t}(x)) = f(x)+\int_0^t f'(\varphi_{0,u}(x))dW_u+\frac{1}{2}\int_0^t f''(\varphi_{0,u}(x))du \quad a.s. \tag{5.16}$$

Thus φ is a flow of mappings solution of (T) and so Proposition 5.6 is proved (see Remark 4.3). Let give another simple proof of the unicity of φ and for this consider any flow of mappings (ψ, W) solution of (T). By lemma 4.3,

$$\psi_{0,t}(x) = x+\vec{e}(x)W_{0,t} \text{ for } 0\leq t\leq \tau_{0,x} \text{ with } \tau_{0,x} \text{ given by (5.2).} \tag{5.17}$$

As $\sigma(W_t)\subset\sigma(\psi_{0,t}(y), y\in G)$, we can define a Wiener stochastic flow K^* obtained by filtering δ_ψ with respect to $\sigma(W)$ satisfying: $\forall s\leq t, x\in G, K^*_{s,t}(x) = E[\delta_{\psi_{s,t}(x)}|\sigma(W)]$ $a.s.$ In particular K^* solves (T) and since K^W given by (5.1) is the unique Wiener solution of (T), we get: $\forall s\leq t, x\in G,\ K^W_{s,t}(x) = E[\delta_{\varphi_{s,t}(x)}|\sigma(W)]$ $a.s.$ (see Proposition 4.5). As $K^W_{0,t}(0)$ is supported on $\{W^+_{0,t}\vec{e}_i,\quad 1\leq i\leq N\}$, we deduce that $|\psi_{0,t}(0)| = W^+_{0,t}$. Combining this with (5.17), we see that

$$\inf\{r\geq 0 : \psi_{0,r}(x) = \psi_{0,r}(0)\} = \tau_{0,x}.$$

This implies that $\psi_{0,r}(x) = \psi_{0,r}(0)$ for all $r\geq\tau_{0,x}$ by applying the strong Markov property (see Lemma 3.2). Note that $W_{0,\cdot}$ can be recovered out from $W^+_{0,\cdot}$ and consequently $\psi_{0,\cdot}(x)$ is a measurable function of $\psi_{0,\cdot}(0)$ for all $x\in G$. Therefore, for all $(x_1,\cdots,x_n)\in G^n$, $(\psi_{0,\cdot}(x_1),\cdots,\psi_{0,\cdot}(x_n))$ is unique in law since $\psi_{0,\cdot}(0)$ is a Walsh Brownian motion. This completes the proof. $\square$

The Wiener flow

In order to finish the proof of Theorem 5.2 and Corollary 5.1, we need only check the following lemma (the proof of (5.3) is similar)

Lemma 5.2. *Under the hypothesis of Proposition 5.5, we have*

$$\sup_{t\in[s,s+T]}\beta(K^W_{s,t}(x), K^n_{s,t}(\sqrt{n}x_n)) \xrightarrow[n\to+\infty]{} 0 \quad a.s.$$

Proof. Let $g : G \longrightarrow \mathbb{R}$ such that $\|g\|_\infty + \sup_{x\neq y} \frac{|g(x)-g(y)|}{|x-y|} \leq 1, g(0) = 0$. Then,

$$\left|\int_G g(y)K^W_{s,t}(x)(dy) - \int_G g(y)K^n_{s,t}(\sqrt{n}x_n)(dy)\right| \leq V^{1,n}_{s,t} + V^{2,n}_{s,t}$$

where

$$V^{1,n}_{s,t} = \left|g(x_n + \vec{e}(x_n)S^n_{s,t})1_{\{\lfloor nt\rfloor \leq T_{\lfloor ns\rfloor,\sqrt{n}x_n}\}} - g(x + \vec{e}(x)W_{s,t})1_{\{t\leq \tau_{s,x}\}}\right|,$$

$$V^{2,n}_{s,t} = \sum_{j=1}^N \alpha_j \left|g(\vec{e}_j W^+_{s,t})1_{\{t>\tau_{s,x}\}} - g(\vec{e}_j S^+_{n,s,t} + o_n)1_{\{\lfloor nt\rfloor > T_{\lfloor ns\rfloor,\sqrt{n}x_n}\}}\right|$$

and $o_n \in G$ is a $\sigma(S)$ measurable random variable such that $|o_n| \leq \frac{1}{\sqrt{n}}$, $S^n_{s,t} = S^n_t - S^n_s$, $S^+_{n,s,t} = \frac{1}{\sqrt{n}}S^+_{\lfloor ns\rfloor,\lfloor nt\rfloor}$. As $\lfloor x\rfloor - 1 \leq x \leq \lfloor x\rfloor + 1$ for all $x \in \mathbb{R}$, we get

$$V^{1,n}_{s,t} \leq \sup_{t\in I_{n,s,x}} |x_n+\vec{e}(x_n)S^{(n)}_{s,t} - x - \vec{e}(x)W_{s,t}| + \sup_{t\in J_{n,s,x}} |g(x_n+\vec{e}(x_n)S^{(n)}_{s,t})| + \sup_{t\in K_{n,s,x}} |g(x+\vec{e}(x)W_{s,t})|$$

with
$$I_{n,s,x} = [s, \tau_{s,x} \vee (\tfrac{1}{n} + \tfrac{1}{n}T_{\lfloor ns\rfloor,\sqrt{n}x_n})],$$

$$J_{n,s,x} = [\tau_{s,x}, (\frac{1}{n}T_{\lfloor ns\rfloor,\sqrt{n}x_n} + \frac{1}{n}) \vee \tau_{s,x}], \quad K_{n,s,x} = [\tau_{s,x} \wedge (\frac{1}{n}T_{\lfloor ns\rfloor,\sqrt{n}x_n} - \frac{1}{n}), \tau_{s,x}].$$

Using $|g(y)| \leq |y|$, we obtain

$$\sup_{t\in J_{n,s,x}} |g(x_n+\vec{e}(x_n)S^{(n)}_{s,t})| + \sup_{t\in K_{n,s,x}} |g(x+\vec{e}(x)W_{s,t})| \leq \sup_{t\in J_{n,s,x}} ||x_n|+S^{(n)}_{s,t}| + \sup_{t\in K_{n,s,x}} ||x|+W_{s,t}|.$$

Since $\lim_{n\to+\infty} \frac{1}{n}T_{\lfloor ns\rfloor,\sqrt{n}x_n} = \tau_{s,x}$ a.s., the right-hand side converges to 0. By discussing the cases $x = 0, x \neq 0$, we easily see that

$$\lim_{n\to\infty} \sup_{t\in I_{n,s,x}} |x_n + \vec{e}(x_n)S^{(n)}_{s,t} - x - \vec{e}(x)W_{s,t}| = 0$$

and a fortiori $\lim_{n\to\infty} \sup_{t\in[s,s+T]} V^{1,n}_{s,t} = 0$. By the same manner, we arrive at $\lim_{n\to\infty} \sup_{t\in[s,s+T]} V^{2,n}_{s,t} = 0$ which proves the lemma. $\square$

Chapter 6

Stochastic flows associated to Tanaka's SDE on the circle

Joint work with Olivier Raimond

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6.1 Introduction and main results

Consider Tanaka's equation

$$\varphi_{s,t}(x) = x + \int_s^t \operatorname{sgn}(\varphi_{s,u}(x))dW_u, \quad s \le t, x \in \mathbb{R}, \tag{6.1}$$

where $(W_t)_{t\in\mathbb{R}}$ is a Brownian motion on $\mathbb{R}$ (that is $(W_t)_{t\ge 0}$ and $(W_{-t})_{t\ge 0}$ are two independent standard Brownian motions) and φ is a stochastic flow of mappings, both defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. In [36], Le Jan and Raimond have extended (6.1) to kernels and then classified all the laws of solutions by means of probability measures on $[0,1]$. A stochastic flow of kernels K is said to solve Tanaka's equation if and only if for all $s \le t, x \in \mathbb{R}$, $f \in C_b^2(\mathbb{R})$ (f is C^2 on $\mathbb{R}$ and f', f'' are bounded), a.s.

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(f'\operatorname{sgn})(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du. \tag{6.2}$$

The main result of [36] is a one-to-one correspondence between probability measures m on $[0,1]$ and solutions associated to (6.2). The case $m = \delta_{\frac{1}{2}}$ corresponds to the unique $\sigma(W)$-adapted solution (Wiener flow) of (6.2) given by

$$K^W_{s,t}(x) = \delta_{x+\operatorname{sgn}(x)(W_t - W_s)} 1_{\{t \le \tau_{s,x}\}} + \frac{1}{2}(\delta_{W^+_{s,t}} + \delta_{-W^+_{s,t}}) 1_{\{t > \tau_{s,x}\}}$$

where

$$\tau_{s,x} = \inf\{r \ge s : W_r - W_s = -|x|\}, \quad W^+_{s,t} := W_t - \inf_{u\in[s,t]} W_u.$$

For $m = \frac{1}{2}(\delta_0 + \delta_1)$, we recover the unique flow of mappings solving (6.1) which was firstly introduced in [51]. In [20], a more general Tanaka's equation has been defined on a graph related to Walsh's Brownian motion. In this work, we deal with another simple oriented graph with two edges and two vertices that will be embedded in the unit circle $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$.
A function f defined on $\mathscr{C}$ is said to be derivable in $z_0 \in \mathscr{C}$ if

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 e^{ih}) - f(z_0)}{h}$$

exists. Let $C^2(\mathscr{C})$ be the space of all functions f defined on $\mathscr{C}$ having first and second continuous derivatives f' and f''. Let $\mathcal{P}(\mathscr{C})$ be the space of probability measures on $\mathscr{C}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions dense in $\{f \in C(\mathscr{C}), ||f||_\infty \leq 1\}$. We equip $\mathcal{P}(\mathscr{C})$ with the distance

$$d(\mu, \nu) = \left(\sum_n 2^{-n} (\int f_n d\mu - \int f_n d\nu)^2 \right)^{\frac{1}{2}}, \mu, \nu \in \mathcal{P}(\mathscr{C}). \tag{6.3}$$

In the following, $\arg(z) \in [0, 2\pi[$ denotes the argument of $z \in \mathbb{C}$.

Definition 6.1. *Fix $l \in]0, \pi]$ and define for $z \in \mathscr{C}$,*

$$\epsilon(z) = 1_{\{arg(z) \in [0,l]\}} - 1_{\{arg(z) \in]l, 2\pi[\}}.$$

On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let W be a Brownian motion on $\mathbb{R}$ and K be a stochastic flow of kernels on $\mathscr{C}$. We say that (K, W) solves Tanaka's equation on $\mathscr{C}$ denoted $(T_{\mathscr{C}})$ if for all $s \leq t, f \in C^2(\mathscr{C}), x \in \mathscr{C}$, a.s.

$$K_{s,t} f(x) = f(x) + \int_s^t K_{s,u}(\epsilon f')(x) dW_u + \frac{1}{2} \int_s^t K_{s,u} f''(x) du. \tag{6.4}$$

If K is a solution of $(T_{\mathscr{C}})$ and $K = \delta_\varphi$ with φ is a stochastic flow of mappings, we simply say that (φ, W) solves $(T_{\mathscr{C}})$.

If (K, W) is a solution of $(T_{\mathscr{C}})$, it was shown in [35] (Section 6) that $\sigma(W) \subset \sigma(K)$ (see also Lemma 6.7 (ii) in this paper). So we will simply say that K solves $(T_{\mathscr{C}})$.
In this paper, given two probability measures on $[0, 1]$, m^+ and m^- satisfying some

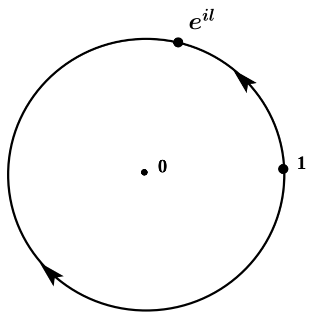

Figure 6.1: Tanaka's equation on $\mathscr{C}$.

moment conditions, we construct a flow K^{m^+,m^-} solution of $(T_{\mathscr{C}})$. Using the main result of [36], we will attach to the particular points 1 and e^{il} two flows K^+ and K^- associated respectively to (6.4) and to (6.4) driven by $-W$. The respective laws of K^+ and K^- are uniquely determined by the measures m^+ and m^-. This will require a suitable description of the law of (K^+, K^-). The flows K^+ and K^- provide the additional randomness when K^{m^+,m^-} passes through 1 or e^{il}. Away from these two points, K^{m^+,m^-} just follows Brownian motion. We now state our main result and postpone the details to the next section.

Theorem 6.1. *Let m^+ and m^- be two probability measures on $[0,1]$ satisfying*

$$\int_0^1 u\ m^+(du) = \int_0^1 u\ m^-(du) = \frac{1}{2}. \tag{6.5}$$

(1) To (m^+, m^-) is associated a stochastic flow of kernels K^{m^+,m^-} solution of $(T_{\mathscr{C}})$.

- *To $m^+ = m^- = \delta_{\frac{1}{2}}$ is associated a Wiener solution K^W.*
- *To $m^+ = m^- = \frac{1}{2}(\delta_0 + \delta_1)$ is associated a flow of mappings φ.*

(2) For all stochastic flow of kernels K solution of $(T_{\mathscr{C}})$, there exists a unique pair of measures (m^+, m^-) satisfying (6.5) such that $K \overset{law}{=} K^{m^+,m^-}$.

For all $s \le t$, let

$$\mathcal{F}^W_{s,t} = \sigma(W_u - W_v, s \le u \le v \le t).$$

Then (K^{m^+,m^-}, φ) will be constructed on the same probability space such that:

Proposition 6.1. *(1) There exists an increasing sequence* $(T_k)_{k\geq 1}$ *of* $(\mathcal{F}^W_{0,t})_{t\geq 0}$*-stopping times such that a.s.* $\lim_{k\to\infty} T_k = +\infty$ *and* $\varphi_{0,T_k}(x) = e^{il}, K^{m^+,m^-}_{0,T_k}(x) = \delta_{e^{il}}$ *for all* $x \in \mathscr{C},\ k \geq 1$.

(2) There exists an increasing sequence $(S_k)_{k\geq 0}$ *of* $(\mathcal{F}^W_{0,t})_{t\geq 0}$*-stopping times such that a.s.* $\lim_{k\to\infty} S_k = +\infty$ *and* $\varphi_{0,S_k}(x) = 1, K^{m^+,m^-}_{0,S_k}(x) = \delta_1$ *for all* $x \in \mathscr{C},\ k \geq 1$.

6.2 Construction of flows associated to $(T_{\mathscr{C}})$

Fix two probability measures m^+ and m^- on $[0,1]$ with mean $\frac{1}{2}$.

6.2.1 Coupling flows associated with two Tanaka's equations on $\mathbb{R}$

In this section, we follow [36]. By Kolmogorov extension theorem, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which one can construct a process $(\varepsilon^+_{s,t}, \varepsilon^-_{s,t}, U^+_{s,t}, U^-_{s,t}, W_{s,t})_{-\infty<s\leq t<\infty}$ taking values in $\{-1,1\}^2 \times [0,1]^2 \times \mathbb{R}$ such that:

(i) $W_{s,t} := W_t - W_s$ for all $s \leq t$ and W is a Brownian motion on $\mathbb{R}$.

(ii) Given W, $(\varepsilon^+_{s,t}, U^+_{s,t})_{s\leq t}$ and $(\varepsilon^-_{s,t}, U^-_{s,t})_{s\leq t}$ are independent.

(iii) For fixed $s < t$, $(\varepsilon^\pm_{s,t}, U^\pm_{s,t})$ is independent of W and

$$(\varepsilon^\pm_{s,t}, U^\pm_{s,t}) \overset{law}{=} (u\delta_1(dx) + (1-u)\delta_{-1}(dx))m^\pm(du).$$

In particular

$$\mathbb{P}(\varepsilon^\pm_{s,t} = 1 | U^\pm_{s,t}) = U^\pm_{s,t}. \tag{6.6}$$

Define for all $s < t$

$$\mathrm{m}^+_{s,t} = \inf\{W_u; u \in [s,t]\}, \quad \mathrm{m}^-_{s,t} = \sup\{W_u; u \in [s,t]\}.$$

Then

(iv) For all $s<t$ and $\{(s_i,t_i); 1\le i\le n\}$ with $s_i<t_i$, the law of $(\varepsilon^{\pm}_{s,t}, U^{\pm}_{s,t})$ knowing $(\varepsilon^{\pm}_{s_i,t_i}, U^{\pm}_{s_i,t_i})_{1\le i\le n}$ and W is given by

$$(u\delta_1(dx)+(1-u)\delta_{-1}(dx))m^{\pm}(du)$$

when $\mathrm{m}^{\pm}_{s,t}\notin\{\mathrm{m}^{\pm}_{s_i,t_i}; 1\le i\le n\}$ and is given by

$$\sum_{i=1}^{n}\delta_{\varepsilon^{\pm}_{s_i,t_i},U^{\pm}_{s_i,t_i}}\times\frac{1_{\{\mathrm{m}^{\pm}_{s,t}=\mathrm{m}^{\pm}_{s_i,t_i}\}}}{\mathrm{Card}\{i;\mathrm{m}^{\pm}_{s_i,t_i}=\mathrm{m}^{\pm}_{s,t}\}}$$

otherwise.

Note that (i)-(iv) uniquely define the law of

$$(\varepsilon^{+}_{s_1,t_1}, U^{+}_{s_1,t_1}, \varepsilon^{-}_{s_1,t_1}, U^{-}_{s_1,t_1},\cdots,\varepsilon^{+}_{s_n,t_n}, U^{+}_{s_n,t_n}, \varepsilon^{-}_{s_n,t_n}, U^{-}_{s_n,t_n}, W)$$

for all $s_i<t_i, 1\le i\le n$.
By construction, for all $s<t, u<v$, if $\mathbb{P}(\mathrm{m}^{\pm}_{s,t}=\mathrm{m}^{\pm}_{u,v})>0$, then

$$\mathbb{P}(\varepsilon^{\pm}_{s,t}=\varepsilon^{\pm}_{u,v}, U^{\pm}_{s,t}=U^{\pm}_{u,v}|\mathrm{m}^{\pm}_{s,t}=\mathrm{m}^{\pm}_{u,v})=1. \tag{6.7}$$

For $s\le t, x\in\mathbb{R}$, define

$$\tau^{\pm}_s(x)=\inf\{r\ge s: W_{s,r}=\mp|x|\}$$

and

$$W^{+}_{s,t}=W_t-\mathrm{m}^{+}_{s,t}=W_{s,t}-\inf_{s\le u\le t}W_{s,u},$$

$$W^{-}_{s,t}=\mathrm{m}^{-}_{s,t}-W_t=\sup_{s\le u\le t}W_{s,u}-W_{s,t}.$$

Finally for all $s\le t, x\in\mathbb{R}$, set

$$\varphi^{\pm}_{s,t}(x)=(x\pm\mathrm{sgn}(x)W_{s,t})1_{\{t\le\tau^{\pm}_s(x)\}}+\varepsilon^{\pm}_{s,t}W^{\pm}_{s,t}1_{\{t>\tau^{\pm}_s(x)\}},$$

$$K^{\pm}_{s,t}(x)=\delta_{x\pm\mathrm{sgn}(x)W_{s,t}}1_{\{t\le\tau^{\pm}_s(x)\}}+(U^{\pm}_{s,t}\delta_{W^{\pm}_{s,t}}+(1-U^{\pm}_{s,t})\delta_{-W^{\pm}_{s,t}})1_{\{t>\tau^{\pm}_s(x)\}}.$$

Recall the following

Theorem 6.2. *[36] (i) $\varphi^{\pm}$ is a stochastic flow of mappings which satisfies: for all $x \in \mathbb{R}, s \leq t,$ a.s.*

$$\varphi^{\pm}_{s,t}(x) = x \pm \int_s^t sgn(\varphi^{\pm}_{s,u}(x))dW_u.$$

(ii) $K^{\pm}$ is a stochastic flow of kernels which satisfies: for all $x \in \mathbb{R}, s \leq t, f \in C^2_b(\mathbb{R})$, a.s.

$$K^{\pm}_{s,t}f(x) = f(x) \pm \int_s^t K^{\pm}_{s,u}(sgnf')(x)dW_u + \frac{1}{2}\int_s^t K^{\pm}_{s,u}f''(x)du.$$

(iii) For all $x \in \mathbb{R}$, all $s \leq t$ and all bounded continuous function f, a.s.

$$K^{\pm}_{s,t}f(x) = E[f(\varphi^{\pm}_{s,t}(x))|K^{\pm}].$$

6.2.2 Modification of flows

For our later need, we will construct modifications of $\varphi^{\pm}, K^{\pm}$ which are measurable with respect to (s,t,x,ω). On a set of probability 1, define for all $s < t$, $(s_n, t_n) = (\frac{\lfloor ns \rfloor + 1}{n}, \frac{\lfloor nt \rfloor - 1}{n})$ and

$$(\widetilde{\varepsilon}^{\pm}_{s,t}, \widetilde{U}^{\pm}_{s,t}) = (\limsup_{n\to\infty} \varepsilon^{\pm}_{s_n,t_n}, \limsup_{n\to\infty} U^{\pm}_{s_n,t_n}).$$

Then, we have the following

Lemma 6.1. *(i) For all $s < t$, a.s. $\widetilde{\varepsilon}^{\pm}_{s,t} = \varepsilon^{\pm}_{s,t}$, $\widetilde{U}^{\pm}_{s,t} = U^{\pm}_{s,t}$.*
(ii) Consider the random sets

$$\mathscr{D}^{+} = \{(s,t) \in \mathbb{R}^2; s < t, m^{+}_{s,t} < min(W_s, W_t)\},$$

$$\mathscr{D}^{-} = \{(s,t) \in \mathbb{R}^2; s < t, m^{-}_{s,t} > max(W_s, W_t)\}.$$

Then a.s. for all (s,t) and (u,v) in $\mathscr{D}^{\pm}$, we have

$$\{m^{\pm}_{s,t} = m^{\pm}_{u,v}\} \subset \{\widetilde{\varepsilon}^{\pm}_{s,t} = \widetilde{\varepsilon}^{\pm}_{u,v}, \widetilde{U}^{\pm}_{s,t} = \widetilde{U}^{\pm}_{u,v}\}.$$

Proof. (i) By (6.7), for all $s < t, u < v$ such that $(s,t,u,v) \in \mathbb{Q}^4$, we have

$$m^{\pm}_{s,t} = m^{\pm}_{u,v} \Longrightarrow (\varepsilon^{\pm}_{s,t}, U^{\pm}_{s,t}) = (\varepsilon^{\pm}_{u,v}, U^{\pm}_{u,v}).$$

Fix $s < t$. With probability 1, $\mathrm{m}^{\pm}_{s,t}$ is attained in $]s,t[$ and thus a.s. there exists n_0 such that

$$m^{\pm}_{s,t} = m^{\pm}_{s_n,t_n} = m^{\pm}_{s_{n_0},t_{n_0}} \text{ for all } n \geq n_0. \tag{6.8}$$

Taking the limit, we get $(\widetilde{\varepsilon}^{\pm}_{s,t}, \widetilde{U}^{\pm}_{s,t}) = (\varepsilon^{+}_{s_{n_0},t_{n_0}}, U^{\pm}_{s_{n_0},t_{n_0}})$ a.s. From (6.7) and (6.8), we also draw that $(\varepsilon^{\pm}_{s,t}, U^{\pm}_{s,t}) = (\varepsilon^{+}_{s_{n_0},t_{n_0}}, U^{\pm}_{s_{n_0},t_{n_0}})$ a.s. and (i) is proved.

(ii) With probability 1, for all (s,t) and (u,v) in $\mathscr{D}^{\pm}$,

$$\begin{aligned}
\{\mathrm{m}^{\pm}_{s,t} = \mathrm{m}^{\pm}_{u,v}\} \;&\subset\; \{\exists n_0 : \mathrm{m}^{\pm}_{s_n,t_n} = \mathrm{m}^{\pm}_{u_n,v_n} \text{ for all } n \geq n_0\} \\
&\subset\; \{\exists n_0 : (\varepsilon^{+}_{s_n,t_n}, U^{\pm}_{s_n,t_n}) = (\varepsilon^{+}_{u_n,v_n}, U^{\pm}_{u_n,v_n}) \text{ for all } n \geq n_0\} \\
&\subset\; \{\widetilde{\varepsilon}^{\pm}_{s,t} = \widetilde{\varepsilon}^{\pm}_{u,v}, \widetilde{U}^{\pm}_{s,t} = \widetilde{U}^{\pm}_{u,v}\}.
\end{aligned}$$

□

We may now consider the following modifications of $\varphi^{\pm}$ and $K^{\pm}$ defined for all $s \leq t, x \in \mathbb{R}$ by

$$\widetilde{\varphi}^{\pm}_{s,t}(x) = (x \pm \mathrm{sgn}(x) W_{s,t}) 1_{\{t \leq \tau^{\pm}_s(x)\}} + \widetilde{\varepsilon}^{\pm}_{s,t} W^{\pm}_{s,t} 1_{\{t > \tau^{\pm}_s(x)\}},$$

$$\widetilde{K}^{\pm}_{s,t}(x) = \delta_{x \pm \mathrm{sgn}(x) W_{s,t}} 1_{\{t \leq \tau^{\pm}_s(x)\}} + (\widetilde{U}^{\pm}_{s,t} \delta_{W^{\pm}_{s,t}} + (1 - \widetilde{U}^{\pm}_{s,t}) \delta_{-W^{\pm}_{s,t}}) 1_{\{t > \tau^{\pm}_s(x)\}}.$$

Then, we have the following

Lemma 6.2. *(i) The mapping*

$$(s,t,x,\omega) \longmapsto (\widetilde{\varphi}^{\pm}_{s,t}(x,\omega), \widetilde{K}^{\pm}_{s,t}(x,\omega))$$

is measurable from $\{(s,t,x,\omega), s \leq t, x \in \mathbb{R}, \omega \in \Omega\}$ *into* $\mathbb{R} \times \mathcal{P}(\mathbb{R})$.

(ii) For all s,t,x, *a.s.*

$$\varphi^{\pm}_{s,t}(x) = \widetilde{\varphi}^{\pm}_{s,t}(x), \quad K^{\pm}_{s,t}(x) = \widetilde{K}^{\pm}_{s,t}(x).$$

In particular, Theorem 6.2 holds also for the flows $\widetilde{\varphi}^{\pm}, \widetilde{K}^{\pm}$.

Proof. (i) Clearly

$$(s,t,\omega) \longmapsto (\widetilde{\varepsilon}^{\pm}_{s,t}(\omega), \widetilde{U}^{\pm}_{s,t}(\omega), W_{s,t}(\omega))$$

is measurable. For all $t \geq s$, we have

$$\{\tau_s^+(x) > t\} = \{\inf_{s \leq r \leq t} W_{s,r} + |x| > 0\}$$

which shows that $(s, x, \omega) \longmapsto \tau_s^+(x, \omega)$ is measurable and a fortiori $(s, x, \omega) \longmapsto \tau_s^-(x, \omega)$ is also measurable. (ii) is a consequence of Lemma 6.1 (i). $\square$

To simplify notations, throughout the rest of the paper, we will denote $\widetilde{\varepsilon}^{\pm}_{s,t}, \widetilde{U}^{\pm}_{s,t}, \widetilde{\varphi}^{\pm}_{s,t}, \widetilde{K}^{\pm}_{s,t}$ simply by $\varepsilon^{\pm}_{s,t}, U^{\pm}_{s,t}, \varphi^{\pm}_{s,t}, K^{\pm}_{s,t}$.

6.2.3 The construction

In this paragraph, we construct the flows K^{m^+,m^-} and φ as in Theorem 6.1 respectively from (K^+, K^-) and (φ^+, φ^-). Let

$$\rho_s = \inf\{r \geq s, \ \sup(W^+_{s,r}, W^-_{s,r}) = l\}.$$

For $t \in [s, \rho_s]$, define

$$\varphi_{s,t}(1) = \exp(i\varphi^+_{s,t}(0))$$

and

$$\varphi_{s,t}(e^{il}) = \exp(i(l + \varphi^-_{s,t}(0))).$$

For $z \in \mathscr{C} \setminus \{1, e^{il}\}$, define $(\varphi_{s,t}(z))_{s \leq t \leq \rho_s}$ by

$$\begin{aligned} \varphi_{s,t}(z) &= ze^{i\epsilon(z)W_{s,t}} 1_{\{s \leq t \leq \rho_s \wedge \tau_s(z)\}} \\ &+ \left(\varphi_{s,t}(1) 1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}} = 1\}} + \varphi_{s,t}(e^{il}) 1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}} = e^{il}\}}\right) 1_{\{\tau_s(z) < t \leq \rho_s\}}. \end{aligned}$$

where

$$\tau_s(z) = \inf\{r \geq s, \ ze^{i\epsilon(z)W_{s,r}} = 1 \text{ or } e^{il}\}.$$

Note that on $\{\tau_s(z) < \rho_s\} \cap \{ze^{i\epsilon(z)W_{s,\tau_s(z)}} = 1\}$, we have $W^+_{s,\tau_s(z)} = 0$ and consequently $\varphi_{s,\tau_s(z)}(1) = 1$. By analogy, on $\{\tau_s(z) < \rho_s\} \cap \{ze^{i\epsilon(z)W_{s,\tau_s(z)}} = e^{il}\}$, we have $W^-_{s,\tau_s(z)} = 0$ and so $\varphi_{s,\tau_s(z)}(e^{il}) = e^{il}$.

Since $(s,\omega) \longmapsto \rho_s(\omega)$ and $(s,z,\omega) \longmapsto \tau_s(z,\omega)$ are measurabe, it follows from Lemma 6.2 that

$$(s,t,z,\omega) \longmapsto \varphi_{s,t}(z,\omega)1_{\{s\le t\le \rho_s(\omega)\}}$$

is measurable from $\{(s,t,z,\omega), s\le t, z\in\mathscr{C}, \omega\in\Omega\}$ into $\mathscr{C}$ (we set by convention $z\times 0 = 1$ for all $z\in\mathscr{C}$). Now we consider the sequence

$$\rho_s^0 = s,\ \ \rho_s^{k+1} = \rho_{\rho_s^k}, k\ge 0$$

and extend our definition for all $s\le t, z\in\mathscr{C}$ by setting

$$\varphi_{s,t}(z) = \sum_{k\ge 0} 1_{\{\rho_s^k\le t<\rho_s^{k+1}\}}\varphi_{\rho_s^k,t}\circ\varphi_{\rho_s^{k-1},\rho_s^k}\circ\cdots\circ\varphi_{s,\rho_s}(z).$$

Then $(s,t,z,\omega)\longmapsto\varphi_{s,t}(z,\omega)$ is measurable from $\{(s,t,z,\omega), s\le t, z\in\mathscr{C},\omega\in\Omega\}$ into $\mathscr{C}$. By the same way, for $t\in[s,\rho_s]$, set

$$K_{s,t}^{m^+,m^-}(1) = U_{s,t}^+\delta_{\exp(iW_{s,t}^+)} + (1-U_{s,t}^+)\delta_{\exp(-iW_{s,t}^+)}$$

and

$$K_{s,t}^{m^+,m^-}(e^{il}) = U_{s,t}^-\delta_{\exp(i(l+W_{s,t}^-))} + (1-U_{s,t}^-)\delta_{\exp(i(l-W_{s,t}^-))}.$$

Then define

$$\begin{aligned} K_{s,t}^{m^+,m^-}(z) &= \delta_{ze^{i\epsilon(z)W_{s,t}}}1_{\{s\le t\le\rho_s\wedge\tau_s(z)\}} \\ &+ \left(K_{s,t}^{m^+,m^-}(1)1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}}=1\}} + K_{s,t}^{m^+,m^-}(e^{il})1_{\{ze^{i\epsilon(z)W_{s,\tau_s(z)}}=e^{il}\}}\right)1_{\{\tau_s(z)<t\le\rho_s\}}. \end{aligned}$$

We extend this definition for all $s\le t, z\in\mathscr{C}$ by setting

$$K_{s,t}^{m^+,m^-}(z) = \sum_{k\ge 0} 1_{\{\rho_s^k\le t<\rho_s^{k+1}\}}K_{s,\rho_s}^{m^+,m^-}\cdots K_{\rho_s^{k-1},\rho_s^k}^{m^+,m^-}K_{\rho_s^k,t}^{m^+,m^-}(z).$$

Then $(s,t,z,\omega)\longmapsto K_{s,t}^{m^+,m^-}(z,\omega)$ is measurable from $\{(s,t,z,\omega), s\le t, z\in\mathscr{C},\omega\in\Omega\}$ into $\mathcal{P}(\mathscr{C})$.

For every choice $s_1<t_1<\cdots<s_n<t_n$, φ_{s_i,t_i} is $\sigma(\varepsilon_{u,v}^+,\varepsilon_{u,v}^-,W_{u,v}, s_i\le u\le v\le t_i)$ measurable and the σ-fields $\sigma(\varepsilon_{u,v}^+,\varepsilon_{u,v}^-,W_{u,v}, s_i\le u\le v\le t_i)$ for $1\le i\le n$ are independent by construction. This implies the independence of the family $\{\varphi_{s_i,t_i},\ 1\le i\le n\}$ and a fortiori the family $\{K_{s_i,t_i}^{m^+,m^-}, 1\le i\le n\}$ is independent. It is also evident that the laws of $\varphi_{s,t}$ and $K_{s,t}^{m^+,m^-}$ only depend on $t-s$.

6.2.4 The flow property for K^{m^+,m^-} and φ.

To prove the flow property for both φ and K^{m^+,m^-}, we start by the following

Proposition 6.2. *Let $s \in \mathbb{R}$ and S, T be two $(\mathcal{F}^W_{s,r})_{r\geq s}$-stopping times such that $s \leq S \leq T \leq \rho_S$. Then a.s. for all $u \in [T, \rho_S], z \in \mathscr{C}$, we have*

$$\varphi_{S,u}(z) = \varphi_{T,u} \circ \varphi_{S,T}(z)$$

and

$$K^{m^+,m^-}_{S,u}(z) = K^{m^+,m^-}_{S,T} K^{m^+,m^-}_{T,u}(z).$$

Proof. First we prove the result for φ a.s. on the set of probability 1 (see Lemma 6.1 (ii)):

$$\tilde{\Omega} = \{\omega \in \Omega : \forall\ (s_1, t_1), (s_2, t_2) \in \mathscr{D}^{\pm}, m^{\pm}_{s_1,t_1} = m^{\pm}_{s_2,t_2} \Rightarrow \varepsilon^{\pm}_{s_1,t_1} = \varepsilon^{\pm}_{s_2,t_2}\}$$

and simultaneously for all (u, z) in the following sets

$$\begin{aligned}
E_{(i)} &= \{(u, z) : u < \tau_S(z), T \leq u \leq \rho_S\},\\
E_{(ii)} &= \{(u, z) : \tau_S(z) \leq u, T \leq u \leq \rho_S, T < \tau_S(z)\},\\
E_{(iii)} &= \{(u, z) : \tau_S(z) \leq u, T \leq u \leq \rho_S, T \geq \tau_S(z), u < \tau_T(\varphi_{S,T}(z))\},\\
E_{(iv)} &= \{(u, z) : \tau_S(z) \leq u, T \leq u \leq \rho_S, T \geq \tau_S(z), u \geq \tau_T(\varphi_{S,T}(z))\}.
\end{aligned}$$

All the equalities below hold a.s. on $\tilde{\Omega}$ simultaneously for all (u, z). For all $z \in \mathscr{C}$, set $Z = \varphi_{S,T}(z)$.

(i) Let $(z, u) \in E_{(i)}, \theta = \arg(z)$. Then as $T < \tau_S(z)$, we have $\theta \notin \{0, l\}$ and

$$\begin{aligned}
\tau_T(Z) &= \inf\{r \geq T,\ Ze^{i(\epsilon(Z)W_{T,r})} = 1 \text{ or } e^{il}\}\\
&= \inf\{r \geq T,\ e^{i(\theta+\epsilon(z)W_{S,T}+\epsilon(Z)W_{T,r})} = 1 \text{ or } e^{il}\} = \tau_S(z)
\end{aligned}$$

since $\epsilon(z) = \epsilon(Z)$. Consequently $\varphi_{S,u}(z) = \varphi_{T,u} \circ \varphi_{S,T}(z)$.

(ii) Let $(z, u) \in E_{(ii)}$. Then, we still have $\tau_T(Z) = \tau_S(z)$ and $\varphi_{T,\tau_T(Z)}(Z) = \varphi_{S,\tau_S(z)}(z)$. Recall that

$$\varphi_{S,u}(z) = \varphi_{S,u}(1)1_{\{\varphi_{S,\tau_S(z)}(z)=1\}} + \varphi_{S,u}(e^{il})1_{\{\varphi_{S,\tau_S(z)}(z)=e^{il}\}}$$

and

$$\varphi_{T,u}(Z) = \varphi_{T,u}(1)1_{\{\varphi_{T,\tau_T(Z)}(Z)=1\}} + \varphi_{T,u}(e^{il})1_{\{\varphi_{T,\tau_T(Z)}(Z)=e^{il}\}}.$$

Suppose for example $\varphi_{S,\tau_S(z)}(z) = \varphi_{T,\tau_T(Z)}(Z) = 1$, then $W^+_{T,\tau_T(Z)} = W^+_{S,\tau_S(z)} = 0$ and so $W^+_{T,r} = W^+_{S,r}$ for all $r \geq \tau_T(Z)(= \tau_S(z))$. From the definition,

$$\varphi_{S,u}(z) = \varphi_{S,u}(1) = \exp(i\varphi^+_{S,u}(0)) \text{ and } \varphi_{T,u}(Z) = \varphi_{T,u}(1) = \exp(i\varphi^+_{T,u}(0)).$$

If $W^+_{T,u} = W^+_{S,u} = 0$, then $\varphi_{S,u}(z) = \varphi_{T,u}(Z) = 1$. Suppose that $W^+_{T,u} = W^+_{S,u} > 0$, then $W_u > \mathrm{m}^+_{T,u}$ and $W_u > \mathrm{m}^+_{S,u}$. Since S and T are two $(\mathcal{F}^W_{s,r})_{r\geq s}$-stopping times, we have

$$W_T > \mathrm{m}^+_{T,u} \text{ and } W_S > \mathrm{m}^+_{S,u}.$$

In other words, (T,u) and (S,u) are in $\mathscr{D}^+$ so that $\varepsilon^+_{S,u} = \varepsilon^+_{T,u}$ and $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$.
(iii) Let $(z,u) \in E_{(iii)}$. Assume for example that $\varphi_{S,\tau_S(z)}(z) = 1$, then $Z = \varphi_{S,T}(1)$ and

$$\begin{aligned}
\varphi_{T,u}(Z) &= \exp(i(\arg(Z) + \epsilon(Z)W_{T,u})) \\
&= \exp(i(\varphi^+_{S,T}(0) + \epsilon(Z)W_{T,u})) = \exp(i(\varepsilon^+_{S,T}W^+_{S,T} + \epsilon(Z)W_{T,u})).
\end{aligned}$$

As $T \leq u < \tau_T(Z)$, it follows that $Z \notin \{1, e^{il}\}$ (if $Z \in \{1, e^{il}\}$, then $\tau_T(Z) = T$), $\epsilon(Z) = \varepsilon^+_{S,T}$ and so $\varphi_{T,u}(Z) = \exp(i\varepsilon^+_{S,T}(W_u - \mathrm{m}^+_{S,T}))$. As $Z \neq 1$, we necessarily have $W^+_{S,T} > 0$. Thus

$$\tau_T(Z) = \inf\{r \geq T : \; W_r - \mathrm{m}^+_{S,T} = 0 \text{ or } \; l\} \text{ if } \varepsilon^+_{S,T} = 1$$

and

$$\tau_T(Z) = \inf\{r \geq T : \; W_r - \mathrm{m}^+_{S,T} = 0 \text{ or } \; 2\pi - l\} \text{ if } \varepsilon^+_{S,T} = -1.$$

The assumption $u < \tau_T(Z)$ implies $\mathrm{m}^+_{S,u} = \mathrm{m}^+_{S,T}$ and a fortiori $\varphi_{T,u}(Z) = \exp(i\varepsilon^+_{S,T}W^+_{S,u})$. On the other hand,

$$\varphi_{S,u}(z) = \exp(i\varphi^+_{S,u}(0)) = \exp(i\varepsilon^+_{S,u}W^+_{S,u}).$$

But $(S,T) \in \mathscr{D}^+$ (from $W^+_{S,T} > 0$), $(S,u) \in \mathscr{D}^+$ (from $u < \tau_T(Z)$ which entails that $W^+_{S,u} > 0$). Consequently $\varepsilon^+_{S,u} = \varepsilon^+_{S,T}$ and so $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$.

(iv) Let $(z,u) \in E_{(iv)}$. Assume for example that $\varphi_{S,\tau_S(z)}(z) = 1$ and first that $\varepsilon^+_{S,T} = 1$. Then

$$\tau_T(Z) = \inf\{r \geq T : \ W_r - \mathrm{m}^+_{S,T} \in \{0, l\}\}.$$

If $W_{\tau_T(Z)} - \mathrm{m}^+_{S,T} = l$, then $u = \tau_T(Z) = \rho_S$ and $\varphi_{S,u}(z) = \varphi_{T,u}(Z) = e^{il}$.
If $W_{\tau_T(Z)} - \mathrm{m}^+_{S,T} = 0$, then $\varphi_{T,\tau_T(Z)}(Z) = 1$ and $\varphi_{T,u}(Z) = \varphi_{T,u}(1)$.
Since $\varphi_{S,\tau_S(z)}(z) = 1$, we have $\varphi_{S,u}(z) = \varphi_{S,u}(1)$. Moreover $W^+_{T,\tau_T(Z)} = W^+_{S,\tau_T(Z)} = 0$ implies $W^+_{T,r} = W^+_{S,r}$ for all $r \geq \tau_T(Z)$.
Now, if u satisfies $W^+_{T,u} = W^+_{S,u} = 0$, then $\varphi_{T,u}(Z) = \varphi_{S,u}(z) = 1$. If not, we have $\varepsilon^+_{T,u} = \varepsilon^+_{S,u}$ and $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$ exactly as in (ii).
Assume that $\varepsilon^+_{S,T} = -1$, then $\tau_T(Z)$ satisfies $W_{\tau_T(Z)} - \mathrm{m}^+_{S,T} = 0$ (recall that $\tau_T(Z) \leq \rho_S$) and $\varphi_{T,u}(Z) = \varphi_{S,u}(z)$ as before.
The result for K^{m^+,m^-} can be proved by considering

$$\tilde{\Omega} = \{\omega \in \Omega : \forall\ (s_1,t_1),(s_2,t_2) \in \mathscr{D}^{\pm}, m^{\pm}_{s_1,t_1} = m^{\pm}_{s_2,t_2} \Rightarrow U^{\pm}_{s_1,t_1} = U^{\pm}_{s_2,t_2}\}$$

and the sets $E_{(i)}, \cdots, E_{(iv)}$ by replacing $\varphi_{S,T}(z)$ by $e^{iW^+_{S,T}}$ in $E_{(iii)}$ and $E_{(iv)}$. However, the proof remains similar. $\square$

Corollary 6.1. *Let $s \in \mathbb{R}$ and $s \leq S \leq T$ be two $(\mathcal{F}^W_{s,r})_{r\geq s}$-stopping times. Then, with probability 1, for all $u \geq T, z \in \mathscr{C}$, we have*

$$\varphi_{S,u}(z) = \varphi_{T,u} \circ \varphi_{S,T}(z)$$

and

$$K^{m^+,m^-}_{S,u}(z) = K^{m^+,m^-}_{S,T} K^{m^+,m^-}_{T,u}(z).$$

Proof. Fix $k \in \mathbb{N}$ and define the family of stopping times $T^0 = (T \vee \rho^k_S) \wedge \rho^{k+1}_S, \ \ T^i = \rho_{T^{i-1}}, i \geq 1$. Then T^i is an $(\mathcal{F}^W_{s,r})_{r\geq s}$-stopping time for all $i \geq 0$. Furthermore as $r \longmapsto \rho_r$ is increasing, we have $\rho^{k+i}_S \leq T^i \leq \rho^{k+i+1}_S$ for all $i \geq 0$. Applying successively Proposition 6.2, we have a.s. $\forall z \in \mathscr{C}, i \geq 0$,

$$\varphi_{S,u}(z) = \varphi_{\rho^{k+i}_S,u} \circ \varphi_{T^{i-1},\rho^{k+i}_S} \circ \cdots \circ \varphi_{T^0,\rho^{k+1}_S} \circ \varphi_{\rho^k_S,T^0} \circ \varphi_{S,\rho^k_S}(z) \text{ for all } u \in [\rho^{k+i}_S, T^i]$$

and

$$\varphi_{S,u}(z) = \varphi_{T^i,u} \circ \varphi_{\rho_S^{k+i},T^i} \circ \cdots \circ \varphi_{T^0,\rho_S^{k+1}} \circ \varphi_{\rho_S^k,T^0} \circ \varphi_{S,\rho_S^k}(z) \text{ for all } u \in [T^i, \rho_S^{k+i+1}].$$

On $\{\rho_S^k \leq T < \rho_S^{k+1}\}$, we have $T^i = \rho_T^i$ for all $i \geq 0$ whence a.s. on $\{\rho_S^k \leq T < \rho_S^{k+1}\}$, $\forall z \in \mathscr{C}$,

$$\varphi_{S,u}(z) = \varphi_{\rho_S^{k+i},u} \circ \varphi_{\rho_T^{i-1},\rho_S^{k+i}} \circ \cdots \circ \varphi_{T,\rho_S^{k+1}} \circ \varphi_{S,T}(z) \text{ for all } u \in [\rho_S^{k+i}, \rho_T^i], i \geq 0$$

and

$$\varphi_{S,u}(z) = \varphi_{\rho_T^i,u} \circ \varphi_{\rho_S^{k+i},\rho_T^i} \circ \cdots \circ \varphi_{T,\rho_S^{k+1}} \circ \varphi_{S,T}(z) \text{ for all } u \in [\rho_T^i, \rho_S^{k+i+1}], i \geq 0.$$

Now define $S^1 = (T \vee \rho_S^{k+1}) \wedge \rho_T^1$ and $S^{i+1} = \rho_{S^i}, i \geq 1$. Then $(S^i)_{i \geq 1}$ is a family of $(\mathcal{F}_{s,r}^W)_{r \geq s}$-stopping times satisfying $\rho_T^i \leq S^{i+1} \leq \rho_T^{i+1}$ for all $i \geq 0$. Applying successively Proposition 6.2, we get a.s. $\forall z \in \mathscr{C}$,

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{\rho_T^i,u} \circ \varphi_{S^i,\rho_T^i} \circ \cdots \circ \varphi_{S^1,\rho_T^1} \circ \varphi_{T,S^1}(\varphi_{S,T}(z)) \text{ for all } u \in [\rho_T^i, S^{i+1}], i \geq 0$$

and

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{S^{i+1},u} \circ \varphi_{\rho_T^i,S^{i+1}} \circ \cdots \circ \varphi_{S^1,\rho_T^1} \circ \varphi_{T,S^1}(\varphi_{S,T}(z)) \text{ for all } u \in [S^{i+1}, \rho_T^{i+1}], i \geq 0.$$

On $\{\rho_S^k \leq T < \rho_S^{k+1}\}$, we have $S^i = \rho_S^{k+i}$ for all $i \geq 1$ and consequently a.s. on $\{\rho_S^k \leq T < \rho_S^{k+1}\}$, $\forall z \in \mathscr{C}$,

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{\rho_T^i,u} \circ \varphi_{\rho_S^{k+i},\rho_T^i} \circ \cdots \circ \varphi_{\rho_S^{k+1},\rho_T^1} \circ \varphi_{T,\rho_S^{k+1}}(\varphi_{S,T}(z)) \text{ for all } u \in [\rho_T^i, \rho_S^{k+i+1}], i \geq 0,$$

and

$$\varphi_{T,u}(\varphi_{S,T}(z)) = \varphi_{\rho_S^{k+i+1},u} \circ \varphi_{\rho_T^i,\rho_S^{k+i+1}} \circ \cdots \circ \varphi_{\rho_S^{k+1},\rho_T^1} \circ \varphi_{T,\rho_S^{k+1}}(\varphi_{S,T}(z)) \text{ for all } u \in [\rho_S^{k+i+1}, \rho_T^{i+1}], i \geq 0.$$

We have shown that a.s. $\forall z \in \mathscr{C}$,

$$1_{\{\rho_S^k \leq T < \rho_S^{k+1}\}} \varphi_{T,u} \circ \varphi_{S,T}(z) = 1_{\{\rho_S^k \leq T < \rho_S^{k+1}\}} \varphi_{S,u}(z) \text{ for all } u \geq T.$$

By summing over k, we get that a.s. $\forall z \in \mathscr{C}, u \geq T$, $\varphi_{T,u} \circ \varphi_{S,T}(z) = \varphi_{S,u}(z)$. The flow property for K^{m^+,m^-} holds by the same reasoning. $\square$

6.2.5 K^{m^+,m^-} can be obtained by filtering φ

For all $-\infty \le s \le t \le +\infty$, let

$$\mathcal{F}_{s,t}^{U^+,U^-,W} = \sigma(U^+_{u,v}, U^-_{u,v}, W_{u,v}, s \le u \le v \le t) = \sigma(K^+_{u,v}, K^-_{u,v}, s \le u \le v \le t).$$

Then we have the following

Corollary 6.2. *For all $z \in \mathscr{C}$, all $s < t$ and all continuous function f,*

$$K_{s,t}^{m^+,m^-} f(z) = E[f(\varphi_{s,t}(z))|\mathcal{F}_{s,t}^{U^+,U^-,W}] \ a.s.$$

Proof. Fix $s \le t, z \in \mathscr{C}$. We have by (6.6) that a.s.

$$K_{s,t}^{m^+,m^-} f(z) 1_{\{s \le t \le \rho_s\}} = E[f(\varphi_{s,t}(z))|\mathcal{F}_{s,t}^{U^+,U^-,W}] 1_{\{s \le t \le \rho_s\}} \ .$$

A fortiori, if Z is a random variable independent of $\mathcal{F}_{s,t}^{U^+,U^-,W}$, then a.s.

$$K_{s,t}^{m^+,m^-} f(Z) 1_{\{s \le t \le \rho_s\}} = E[f(\varphi_{s,t}(Z))|\mathcal{F}_{s,t}^{U^+,U^-,W}] 1_{\{s \le t \le \rho_s\}} \ . \tag{6.9}$$

Let $t_i^n = s + \frac{(t-s)i}{n}, n \ge 1, i \in [0,n]$ and for $i \in [1,n]$ define $A_{n,i} = \{t_i^n \le \rho_{t_{i-1}^n}\}$, $A_n = \cap_{i=1}^n A_{n,i}$. Then $A_{n,i} \in \mathcal{F}_{t_{i-1}^n,t_i^n}^{U^+,U^-,W}$. Note that since K^+ and K^- are two stochastic flows, we have $\mathcal{F}_{s,t}^{U^+,U^-,W} = \bigvee_{i=1}^n \mathcal{F}_{t_{i-1}^n,t_i^n}^{U^+,U^-,W}$. By Corollary 6.1, a.s.

$$K_{s,t}^{m^+,m^-}(z) = K_{s,t_1^n}^{m^+,m^-} \cdots K_{t_{n-1}^n,t}^{m^+,m^-}(z)$$

and

$$\varphi_{s,t}(z) = \varphi_{t_{n-1}^n,t} \circ \cdots \circ \varphi_{s,t_1^n}(z).$$

Recall that the σ-fields $\mathcal{F}_{t_{i-1}^n,t_i^n}^{U^+,U^-,W}$ for $1 \le i \le n$ are independent. Then, using (6.9), we get that a.s.

$$K_{s,t}^{m^+,m^-} f(z) 1_{A_n} = E[f(\varphi_{s,t}(z))|\mathcal{F}_{s,t}^{U^+,U^-,W}] 1_{A_n},$$

and therefore a.s.

$$K_{s,t}^{m^+,m^-} f(z) = E[f(\varphi_{s,t}(z))|\mathcal{F}_{s,t}^{U^+,U^-,W}] + \big(K_{s,t}^{m^+,m^-} f(z) - E[f(\varphi_{s,t}(z))|\mathcal{F}_{s,t}^{U^+,U^-,W}]\big) 1_{A_n^c}.$$

To finish the proof, it remains to prove that $\mathbb{P}(A_n^c) \to 0$ as $n \to \infty$. Write

$$\mathbb{P}(A_n^c) = \sum_{i=1}^{n} \mathbb{P}(A_{n,i}^c) = \sum_{i=1}^{n} \mathbb{P}(t_i^n - t_{i-1}^n > \rho_{t_{i-1}^n} - t_{i-1}^n) = n\mathbb{P}(\frac{t-s}{n} > \rho_0).$$

Let $\rho^{\pm} = \inf\{r \geq 0 : W_{0,r}^{\pm} = l\}$. Then

$$\mathbb{P}(A_n^c) \leq n\left(\mathbb{P}\left(\frac{t-s}{n} > \rho^+\right) + \mathbb{P}\left(\frac{t-s}{n} > \rho^-\right)\right) = 2n\mathbb{P}\left(\frac{t-s}{n} > \rho^+\right).$$

We have $\rho^+ \overset{law}{=} \inf\{r \geq 0 : |W_r| = l\}$. Let $T_l = \inf\{r \geq 0 : W_r = l\}$, then

$$\mathbb{P}(A_n^c) \leq 4n\mathbb{P}\left(\frac{t-s}{n} > T_l\right) = 4n \int_{\frac{t-s}{n}}^{+\infty} \frac{l}{\sqrt{2\pi x^3}} \exp(\frac{-l^2}{2x})dx$$

(see [45] page 107). By the change of variable $v = nx$, the right hand side converges to 0 as $n \to \infty$ which finishes the proof. $\square$

6.2.6 The L^2 continuity

To conclude that K^{m^+,m^-} and φ are two stochastic flows, it remains to prove the following

Proposition 6.3. *For all $t \geq 0$, $x \in \mathscr{C}$ and $f \in C(\mathscr{C})$, we have*

$$\lim_{y\to x} E\big[\,(f(\varphi_{0,t}(x)) - f(\varphi_{0,t}(y)))^2\,\big] = \lim_{y\to x} E\big[(K_{0,t}^{m^+,m^-}f(x) - K_{0,t}^{m^+,m^-}f(y))^2\big] = 0.$$

Proof. By Jensen's inequality and the preceding corollary, it suffices to prove the result for φ and by the proof of Lemma 1.11 [35] (see also Lemma 1 [20]), this amounts to showing that

$$\lim_{y\to x} \mathbb{P}(d(\varphi_{0,t}(x), \varphi_{0,t}(y)) > \eta) = 0 \text{ for all } t > 0, \eta > 0 \text{ and } x \in \mathscr{C}. \tag{6.10}$$

Fix $\eta > 0, t > 0$ and for $x \in \mathscr{C}, \theta \in [0, 2\pi[$, set

$$A_{x,\theta} = \{d(\varphi_{0,t}(x), \varphi_{0,t}(e^{i\theta})) > \eta\}.$$

For simplicity, we will write $\tau(x)$ instead of $\tau_0(x)$. For $\theta \in]0, l[$, we have

$$\mathbb{P}(A_{1,\theta}) \leq \mathbb{P}(t < \tau(e^{i\theta})) + \mathbb{P}(A_{1,\theta} \cap \{\varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = 1, t \geq \tau(e^{i\theta})\}) + \mathbb{P}(\varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = e^{il}).$$

If $t \geq \tau(e^{i\theta})$ and $\varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = 1$, then $\varphi_{0,t}(e^{i\theta}) := \varphi_{0,t}(1)$. Thus

$$\mathbb{P}(A_{1,\theta} \cap \{\varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = 1, t \geq \tau(e^{i\theta})\}) = 0.$$

From $\lim_{\theta\to 0+} \tau(e^{i\theta}) = 0$ a.s. and

$$\mathbb{P}(\varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = e^{il}) = \mathbb{P}(\theta + W_{0,\tau(e^{i\theta})} = l),$$

we get $\lim_{\theta\to 0+} \mathbb{P}(A_{1,\theta}) = 0$ and similarly, we can prove that $\lim_{\theta\to(2\pi)-} \mathbb{P}(A_{1,\theta}) = 0$. Thus (6.10) holds for $x = 1$ and by the same way for $x = e^{il}$.

Fix $x = e^{i\theta_0} \in \mathscr{C}$ such that $\theta_0 \in]l, 2\pi[$. For all $\theta \in]l, 2\pi[$, we have

$$\mathbb{P}(A_{x,\theta}) \leq \sum_{z\in\{1,e^{il}\}} \mathbb{P}(A_{x,\theta} \cap \{\varphi_{0,\tau(x)}(x) = \varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = z\}) + \epsilon_\theta$$

where ϵ_θ is given by

$$\mathbb{P}(\varphi_{0,\tau(x)}(x) = 1, \varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = e^{il}) + \mathbb{P}(\varphi_{0,\tau(x)}(x) = e^{il}, \varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = 1)$$

and tends to 0 as $\theta \to 0$. Let prove for example that

$$\lim_{\theta\to\theta_0} \mathbb{P}(B_\theta) = 0 \text{ where } \ B_\theta = A_{x,\theta} \cap \{\varphi_{0,\tau(x)}(x) = \varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = 1\}.$$

For $l < \theta < \theta_0$, write

$$\mathbb{P}(B_\theta) = \mathbb{P}(B_\theta \cap \{t \leq \tau(x)\}) + \mathbb{P}(B_\theta \cap \{\tau(x) < t < \tau(e^{i\theta})\}) + \mathbb{P}(B_\theta \cap \{t \geq \tau(e^{i\theta})\}).$$

It is easy to see that

$$\lim_{\theta\to\theta_0-} \Big(\mathbb{P}(B_\theta \cap \{t \leq \tau(x)\}) + \mathbb{P}(B_\theta \cap \{\tau(x) < t < \tau(e^{i\theta})\})\Big) = 0.$$

Now

$$\begin{aligned}
\mathbb{P}(B_\theta \cap \{t \geq \tau(e^{i\theta})\}) &= \mathbb{P}(B_\theta \cap \{\tau(e^{i\theta}) \leq t \wedge \rho_{\tau(x)}\}) + \mathbb{P}(B_\theta \cap \{\rho_{\tau(x)} < \tau(e^{i\theta}) \leq t\}) \\
&\leq \mathbb{P}(B_\theta \cap \{\tau(e^{i\theta}) \leq t \wedge \rho_{\tau(x)}\}) + \mathbb{P}(\rho_{\tau(x)} < \tau(e^{i\theta})).
\end{aligned}$$

Obviously $\lim_{\theta\to\theta_0} \mathbb{P}(\rho_{\tau(x)} < \tau(e^{i\theta})) = 0$. Set $Y = \varphi_{0,\tau(x)}(e^{i\theta})$, then a.s. on $B_\theta \cap \{\tau(e^{i\theta}) \leq t \wedge \rho_{\tau(x)}\}$, we have, $\varphi_{0,t}(e^{i\theta}) = \varphi_{\tau(x),t}(Y)$ by Corollary 6.1 and $\tau_{\tau(x)}(Y) =$

$\tau(e^{i\theta}) \leq \rho_{\tau(x)}$. We recall that $\varphi_{\tau(x),s}(Y) := \varphi_{\tau(x),s}(1)$ for all $s \in [\tau_{\tau(x)}(Y), \rho_{\tau(x)}]$ and a fortiori $\varphi_{\tau(x),s}(Y) = \varphi_{\tau(x),s}(1)$ for all $s \geq \tau_{\tau(x)}(Y)$ (by the definition of φ). This shows that a.s. on $B_\theta \cap \{\tau(e^{i\theta}) \leq t \wedge \rho_{\tau(x)}\}$, we have

$$d(\varphi_{0,t}(x), \varphi_{0,t}(e^{i\theta})) = d(\varphi_{\tau(x),t}(1), \varphi_{\tau(x),t}(1)) = 0$$

Finally $\lim_{\theta\to\theta_0-} \mathbb{P}(B_\theta) = 0$ and by interchanging the roles of θ_0 and θ, we have $\lim_{\theta\to\theta_0+} \mathbb{P}(B_\theta) = 0$. Similarly

$$\lim_{\theta\to\theta_0} \mathbb{P}(A_{x,\theta} \cap \{\varphi_{0,\tau(x)}(x) = \varphi_{0,\tau(e^{i\theta})}(e^{i\theta}) = e^{il}\}) = 0$$

so that (6.10) is satisfied for all x such that $\arg(x) \in]l, 2\pi[$ and a fortiori for all $x \in \mathscr{C}$. ☐

6.2.7 The flows φ and K^{m^+,m^-} solve $(T_{\mathscr{C}})$

In this paragraph we prove the following

Proposition 6.4. *Both φ and K^{m^+,m^-} solve $(T_{\mathscr{C}})$.*

Proof. First we check the result for φ.

First step. Let S be an $(\mathcal{F}^W_{0,\cdot})$-stopping time. Then for all $x \in \mathscr{C}$, $f \in C^2(\mathscr{C})$, a.s. $\forall t \in [0, \rho_S - S]$,

$$f(\varphi_{S,S+t}(x)) = f(x) + \int_0^t (f'\epsilon)(\varphi_{S,S+u}(x))dW_{S,S+u} + \frac{1}{2}\int_0^t f''(\varphi_{S,S+u}(x))du.$$

We begin by $x = 1$ and first show that $(\varphi^+_{S,S+t}(0), t \geq 0)$ is a Brownian motion. We will check the following two points:

(i) For all $0 < s < t$, we have

$$\mathbb{P}(\varepsilon^+_{S,S+t} = \varepsilon^+_{S,S+s} | \mathrm{m}^+_{S,S+t} = \mathrm{m}^+_{S,S+s}) = 1.$$

(ii) For all $0 < s < t$, the conditional law of $\varepsilon^+_{S,S+t}$ knowing $(W_{S,S+u}, u \geq 0)$ and $\sigma(\varepsilon^+_{S,S+r}, 0 \leq r \leq s)$ is $\frac{1}{2}(\delta_1 + \delta_{-1})$ on the event $\{\mathrm{m}^+_{S,S+t} < \mathrm{m}^+_{S,S+s}\}$.

(i) Pick $0 < s < t$. Since $(W_{S,S+u}, u \geq 0)$ is a Brownian motion, a.s. $(S, S+s), (S, S+$

t) are in $\mathscr{D}^+$. Now (i) follows at once from Lemma 6.1 (ii). To prove (ii), recall that for all $s < s' < t$, the conditional law of $\varepsilon^+_{s,t}$ knowing $\sigma(\varepsilon^+_{u,v}, s \le u \le v \le s') \vee \sigma(W)$ is $\frac{1}{2}(\delta_1 + \delta_{-1})$ on $\{\mathrm{m}^+_{s,t} < \mathrm{m}^+_{s,s'}\}$. Note also that a.s. for all $(u,v) \in \mathscr{D}^+$,

$$\varepsilon^+_{u,v} = \lim_{u' \to u+, v' \to v-} \varepsilon^+_{u',v'}.$$

Let $n \ge 1$ and $0 < r_1 < \cdots < r_n \le s$. Take a family $\{f, g_1, \cdots, g_n\}$ of bounded continuous functions from $\mathbb{R}$ into $\mathbb{R}$ and a bounded continuous function $h : C(\mathbb{R}_+, \mathbb{R}) \longrightarrow \mathbb{R}$. Note that a.s. $(S, S+t), (S, S+r_i), 1 \le i \le n$ are in $\mathscr{D}^+$ and consequently

$$E\Big[f(\varepsilon^+_{S,S+t}) \prod_{i=1}^n g_i(\varepsilon^+_{S,S+r_i}) h(W_{S,S+\cdot}) 1_{\{\mathrm{m}^+_{S,S+t} < \mathrm{m}^+_{S,S+s}\}}\Big]$$

$$= \lim_{q \to \infty} E\Big[f(\varepsilon^+_{\frac{\lfloor qS \rfloor + 1}{q}, \frac{\lfloor qS \rfloor - 1}{q} + t}) \prod_{i=1}^n g_i(\varepsilon^+_{\frac{\lfloor qS \rfloor + 1}{q}, \frac{\lfloor qS \rfloor - 1}{q} + r_i}) h(W_{S,S+\cdot}) 1_{\{\mathrm{m}^+_{S,S+t} < \mathrm{m}^+_{S,S+s}\}}\Big].$$

Using our previous remark, (ii) can easily be completed. Now (i) and (ii) entail that $(\varphi^+_{S,S+t}(0), t \ge 0)$ is a Brownian motion (see Remark 4.4 [36]). By Itô's formula, we have for all $f \in C^2(\mathscr{C})$ a.s. $\forall t \ge 0$,

$$f(\exp(i\varphi^+_{S,S+t}(0))) = f(1) + \int_0^t f'(\exp(i\varphi^+_{S,S+u}(0))) d\varphi^+_{S,S+u}(0) + \frac{1}{2} \int_0^t f''(\exp(i\varphi^+_{S,S+u}(0))) du.$$

Tanaka's formula for local time yields a.s. $\forall t \in [0, \rho_S - S]$,

$$\begin{aligned} |\varphi^+_{S,S+t}(0)| &= \int_0^t \mathrm{sgn}(\varphi^+_{S,S+u}(0)) d\varphi^+_{S,S+u}(0) + L_t \\ &= W^+_{S,S+t} \end{aligned}$$

where L_t is the local time in 0 of $\varphi^+_{S,S+\cdot}(0)$ and the last equality is satisfied by the definition of φ^+. Hence a.s. $\forall t \in [0, \rho_S - S]$,

$$\varphi^+_{S,S+t}(0) = \int_0^t \mathrm{sgn}(\varphi^+_{S,S+u}(0)) dW_{S,S+u} = \int_0^t \epsilon(\varphi_{S,S+u}(0)) dW_{S,S+u}.$$

Recall that $\varphi_{S,S+t}(1) = e^{i\varphi^+_{S,S+t}(0)}$ for all $t \in [0, \rho_S - S]$, thus the first step holds for $x = 1$. The first step is similarly satisfied for $x = e^{il}$ and for all $x \in \mathscr{C} \setminus \{1, e^{il}\}$ by distinguishing the cases $t \le \tau_S(x) - S$ and $t > \tau_S(x) - S$.

Second step. Let S be an $(\mathcal{F}^W_{0,.})$-stopping time, $\mathcal{G}_t = \sigma(\varphi_{0,u}(x), x \in \mathscr{C}, 0 \le u \le t)$, $t \ge 0$. Then $\sigma(\varphi_{S,(S+u)\wedge\rho_S}(x), x \in \mathscr{C}, u \ge 0)$ is independent of $\mathcal{G}_S$.
Clearly

$$\sigma(\varphi_{S,(S+u)\wedge\rho_S}(x), x \in \mathscr{C}, u \ge 0) \subset \sigma(\varphi^+_{S,S+u}(0), u \ge 0) \vee \sigma(\varphi^-_{S,S+u}(0), u \ge 0).$$

Fix $0 < u_1 < \cdots < u_n$, then a.s. $(S, S+u_1), \cdots, (S, S+u_n)$ are in $\mathscr{D}^+ \cap \mathscr{D}^-$. Take a family $\{f_1, g_1, \cdots, f_n, g_n\}$ of bounded continuous functions from $\mathbb{R}$ into $\mathbb{R}$ and let $A \in \mathcal{G}_S$. Then

$$E\Big[\prod_{i=1}^n f_i(\varphi^+_{S,S+u_i}(0))g_i(\varphi^-_{S,S+u_i}(0))1_A\Big]$$
$$= \lim_{q\to\infty} E\Big[\prod_{i=1}^n f_i(\varphi^+_{\frac{\lfloor qS\rfloor+1}{q},\frac{\lfloor qS\rfloor-1}{q}+u_i}(0))g_i(\varphi^-_{\frac{\lfloor qS\rfloor+1}{q},\frac{\lfloor qS\rfloor-1}{q}+u_i}(0))1_A\Big].$$

For q large enough ($\frac{2}{q} < u_1$), we have

$$E\Big[\prod_{i=1}^n f_i(\varphi^+_{\frac{\lfloor qS\rfloor+1}{q},\frac{\lfloor qS\rfloor-1}{q}+u_i}(0))g_i(\varphi^-_{\frac{\lfloor qS\rfloor+1}{q},\frac{\lfloor qS\rfloor-1}{q}+u_i}(0))1_A\Big]$$
$$= \sum_{m\ge 0} E\Big[\prod_{i=1}^n f_i(\varphi^+_{\frac{m+1}{q},\frac{m-1}{q}+u_i}(0))g_i(\varphi^-_{\frac{m+1}{q},\frac{m-1}{q}+u_i}(0))1_{A\cap\{\frac{m}{q}\le S<\frac{m+1}{q}\}}\Big]$$

with $A \cap \{\frac{m}{q} \le S < \frac{m+1}{q}\} \in \mathcal{G}_{\frac{m+1}{q}} \subset \sigma(\varphi^+_{u,v}(x), \varphi^-_{u,v}(x), x \in \mathscr{C}, 0 \le u \le v \le \frac{m+1}{q})$. Now using the independence of increments and the stationarity of (φ^+, φ^-), the second step easily holds.

Third step. φ solves $(T_{\mathscr{C}})$.
Denote ρ^k_0 simply by ρ^k. For all $k \in \mathbb{N}$, a.s. $u \longmapsto \varphi_{\rho^k,u}(x)$ is continuous on $[\rho^k, \rho^{k+1}]$ for all $x \in \mathscr{C}$. Consequently for all $x \in \mathscr{C}$, a.s. $u \longmapsto \varphi_{0,u}(x)$ is continuous on $[0, +\infty[$ and in particular, $\varphi_{0,\rho^k}(x)$ is $\mathcal{G}_{\rho^k}$ measurable. Now fix $f \in C^2(\mathscr{C}), t \ge 0, x \in \mathscr{C}$ and define for $z \in \mathscr{C}$,

$$\begin{aligned} H_{(f,t)}(z) &= f(\varphi_{\rho^1,\rho^1+t\wedge(\rho^2-\rho^1)}(z)) - f(z) - \int_0^{t\wedge(\rho^2-\rho^1)} (f'\epsilon)(\varphi_{\rho^1,\rho^1+u}(z))dW_{\rho^1,\rho^1+u} \\ &- \frac{1}{2}\int_0^{t\wedge(\rho^2-\rho^1)} f''(\varphi_{\rho^1,\rho^1+u}(z))du. \end{aligned}$$

Then a.s. $z \longmapsto H_{(f,t)}(z)$ is measurable from $\mathscr{C}$ into $\mathbb{R}$. Moreover $H_{(f,t)}$ is $\sigma(\varphi_{\rho^1,(\rho^1+u)\wedge\rho^2}, u \geq 0)$ measurable and $H_{(f,t)}(z) = 0$ a.s. for all $z \in \mathscr{C}$ by the first step. The second step yields $H_{(f,t)}(\varphi_{0,\rho^1}(x)) = 0$ a.s. and we may replace z by $\varphi_{0,\rho^1}(x)$ directly in the stochastic integral so that, using the flow property, we get

$$\begin{aligned} f(\varphi_{0,\rho^1+t\wedge(\rho^2-\rho^1)}(x)) &= f(\varphi_{0,\rho^1}(x)) + \int_0^{t\wedge(\rho^2-\rho^1)} (f'\epsilon)(\varphi_{0,\rho^1+u}(x))dW_{\rho^1,\rho^1+u} \\ &+ \frac{1}{2}\int_0^{t\wedge(\rho^2-\rho^1)} f''(\varphi_{0,\rho^1+u}(x))du \\ &= f(x) + \int_0^{\rho^1+t\wedge(\rho^2-\rho^1)} (f'\epsilon)(\varphi_{0,u}(x))dW_u + \frac{1}{2}\int_0^{\rho^1+t\wedge(\rho^2-\rho^1)} f''(\varphi_{0,u}(x))du. \end{aligned}$$

By induction, we have a.s. $\forall k \in \mathbb{N}$,

$$\begin{aligned} f(\varphi_{0,\rho^k+t\wedge(\rho^{k+1}-\rho^k)}(x)) &= f(x) + \int_0^{\rho^k+t\wedge(\rho^{k+1}-\rho^k)} (f'\epsilon)(\varphi_{0,u}(x))dW_u \\ &+ \frac{1}{2}\int_0^{\rho^k+t\wedge(\rho^{k+1}-\rho^k)} f''(\varphi_{0,u}(x))du. \end{aligned}$$

This implies that φ solves $(T_{\mathscr{C}})$. The fact that K^{m^+,m^-} solves $(T_{\mathscr{C}})$ is similar to Proposition 4.1 (ii) in [36] using Corollary 6.2. □

6.3 Coalescence (Proof of Proposition 6.1)

For $r \geq 0$, we denote $W^{\pm}_{0,r}$ simply by $W^{\pm}_r$. For all $a \in \mathbb{R}, b \geq 0$ define

$$T_a = \inf\{r \geq 0 : W_r = a\}$$

and

$$\rho^{\pm}_b = \inf\{r \geq 0 : W^{\pm}_r = b\}.$$

We will further need the following

Lemma 6.3. *For all $a > 0, b > 0, c < 0$, we have $\mathbb{P}(T_a < \rho^-_b \wedge T_c) > 0$.*

Proof. Fix $\eta \in]0, \frac{b}{2} \wedge (-c)[$ and let $k \geq 1$ such that $k\eta \geq a$. Now define the sequence of stopping times $(R_i)_{i\geq 0}$ such that $R_0 = 0$ and for $i \geq 0$,

$$R_{i+1} = \inf\{r \geq R_i : |W_r - W_{R_i}| = \eta\}.$$

Let $A = \cap_{i=1}^k \{W_{R_i} = W_{R_{i-1}} + \eta\}$. Then on A, $\sup_{r\le R_k} W_r = k\eta \ge a$ and for all $i \in [0, k-1], u \in [R_i, R_{i+1}]$,

$$W_u^- = \sup_{r\le u} W_r - W_u = \sup_{R_i \le s\le u} (W_s - W_u) \le 2\eta < b.$$

Moreover $\inf_{0\le r\le R_k} W_r > -\eta \ge c$. Since $A \subset \{T_a < \rho_b^- \wedge T_c\}$ and $\mathbb{P}(A) = \frac{1}{2^k}$, this proves the lemma. □

Let $a > 0$. Since $\{T_a < \rho_a^- \wedge T_{-a}\} \subset \{T_a < \rho_a^-\}$, we deduce that $\mathbb{P}(T_a < \rho_a^-) > 0$. Obviously $\rho_a^+ \le T_a$. Since $W \overset{law}{=} -W$, we have $\mathbb{P}(\rho_a^+ < \rho_a^-) = \mathbb{P}(\rho_a^- < \rho_a^+) = \frac{1}{2}$. Remark also that

$$\rho_a^+ \wedge \rho_a^- = \inf\{r \ge 0 : W_r^+ + W_r^- = a\}.$$

This shows that on $\{\rho_a^+ < \rho_a^-\}$, we have $W_{\rho_a^+}^- = 0$ and similarly on $\{\rho_a^- < \rho_a^+\}$, we have $W_{\rho_a^-}^+ = 0$.

6.3.1 The case $l = \pi$

The case $l = \pi$ is the easier one.

Lemma 6.4. *With probability* 1*, for all $x \in \mathscr{C}$, we have*

$$\varphi_{0,\rho_\pi^+}(x) = -1, \quad K_{0,\rho_\pi^+}^{m^+,m^-}(x) = \delta_{-1}$$

and

$$\varphi_{0,\rho_\pi^-}(x) = 1, \quad K_{0,\rho_\pi^-}^{m^+,m^-}(x) = \delta_1.$$

Proof. The proof is obvious. □

To prove Proposition 6.1, consider the sequence of stopping times given by $\sigma_0 = 0$ and for $k \ge 0$,

$$\sigma_{2k+1} = \inf\{u \ge \sigma_{2k} : W_{\sigma_{2k},u}^+ = \pi\}, \sigma_{2k+2} = \inf\{u \ge \sigma_{2k+1} : W_{\sigma_{2k+1},u}^- = \pi\}.$$

Then $(\sigma_{2k+1})_{k\ge 0}$ (resp. $(\sigma_{2k})_{k\ge 0}$) satisfies (1) (resp. (2)) of Proposition 6.1.

6.3.2 The case $l \neq \pi$

We fix $\delta > 0$ such that $0 < l - \delta < l + \delta < \pi$. For $s, a \in \mathbb{R}$ define

$$T_{s,a} = \inf\{r \geq s : W_{s,r} = a\}$$

and

$$\rho^-_{s,\delta} = \inf\{r \geq s : W^-_{s,r} = \delta\}.$$

For any $(\mathcal{F}^W_{0,\cdot})$-stopping time S, let

$$A_S = \{T_{S,2(\pi-l)} < T_{S,-l} \wedge \rho^-_{S,\delta}\}.$$

Then, we have

$$A_S = \left\{\varphi_{S,\cdot}(e^{-il}) \text{ reaches } e^{il} \text{ before } 1 \text{ and before that } \varphi_{S,\cdot}(e^{il}) \text{ arrives in } e^{i(l+\delta)} \text{ or } e^{i(l-\delta)}\right\}.$$

Define the sequence $(\sigma_k)_{k\geq 0}$ of $(\mathcal{F}^W_{0,t})_{t\geq 0}$-stopping time by $\sigma_0 = 0$ and for $k \geq 0, \sigma_{k+1} = T_{\rho_{\sigma_k},2(\pi-l)} (= T_{\rho_{\sigma_k},\arg(e^{-il})-\arg(e^{il})})$. Then set, for $k \geq 0$,

$$C_k = \{W^+_{\sigma_k,\rho_{\sigma_k}} = l\} \cap A_{\rho_{\sigma_k}}.$$

Note that the events $\{W^+_{\sigma_k,\rho_{\sigma_k}} = l\}$ and $A_{\rho_{\sigma_k}}$ are independent. The following proposition describes what happens on C_k.

Proposition 6.5. *With probability* 1, *for all* $k \geq 0$, *on* C_k, *we have*

(i) $\forall x \in \mathscr{C}$, $\arg(\varphi_{\sigma_k,\rho_{\sigma_k}}(x)) \in [l, 2\pi - l]$.

(ii) $\forall x \in \mathscr{C}$ *with* $\arg(x) \in [l, 2\pi - l]$, *we have* $\varphi_{\rho_{\sigma_k},\sigma_{k+1}}(x) = e^{il}$.

(iii) $\forall x \in \mathscr{C}$, $\varphi_{\sigma_k,\sigma_{k+1}}(x) = e^{il}$.

(iv) $\forall x \in \mathscr{C}$, $\varphi_{0,\sigma_{k+1}}(x) = e^{il}, K^{m^+,m^-}_{0,\sigma_{k+1}}(x) = \delta_{e^{il}}$.

Proof. We take $k = 0$ (the proof is similar for all k) and denote ρ_0 simply by ρ.
(i) Fix $x \in \mathscr{C}$. If $\tau_0(x) \leq \rho$, then $\varphi_{0,\rho}(x) \in \{\varphi_{0,\rho}(1), \varphi_{0,\rho}(e^{il})\}$. On C_0, we have

$W_\rho^+ = l$ and so $W_\rho^- = 0$ (see the lines after Lemma 6.3). Consequently $\varphi_{0,\rho}(e^{il}) = 0$ and $\varphi_{0,\rho}(1) \in \{e^{il}, e^{-il}\}$.

Suppose $\rho < \tau_0(x)$, then necessarily $\arg(x) \in]l, 2\pi[$ and

$$\varphi_{0,\rho}(x) = \exp(i(\arg(x) - W_\rho)) = \exp(i(\arg(x) - l - \inf_{0\le u\le\rho} W_u)).$$

Since $\rho < \tau_0(x)$, we have $\arg(x) - \inf_{0\le u\le\rho} W_u < 2\pi$ and therefore $\arg(\varphi_{0,\rho}(x)) < 2\pi - l$. It is also clear that $\arg(\varphi_{0,\rho}(x)) \ge l$ which proves the first statement.

(ii) Let $x \in \mathscr{C}$ with $\arg(x) \in [l, 2\pi - l]$. Then as $\varphi_{\rho,\cdot}(e^{-il})$ arrives to e^{il} before 1, $\varphi_{\rho,\cdot}(x)$ reaches e^{il} before σ_1. Let h be the greatest integer such that $\rho_\rho^h (= \rho^{h+1}) \le \sigma_1$. Then $\varphi_{\rho,\sigma_1}(x) = \varphi_{\rho^{h+1},\sigma_1}(y)$ where $y = \varphi_{\rho,\rho^{h+1}}(x)$. Clearly $\tau_{\rho^{h+1}}(y) = \tau_\rho(x) \le \sigma_1$. Therefore $\varphi_{\rho,\sigma_1}(x) = \varphi_{\rho^{h+1},\sigma_1}(e^{il})$. But $-W_{\rho,u} + 2(\pi - l) \ge W_{\rho,u}^-$ for all $u \in [\rho, \sigma_1]$ and so $W_{\rho,\sigma_1}^- = 0$. As $\rho^{h+1} \ge \rho$, we get $W_{\rho^{h+1},\sigma_1}^- = 0$. That is $\varphi_{\rho,\sigma_1}(x) = e^{il}$.

(iii) and (iv) are immediate from the flow property (Corollary 6.1) and (i), (ii). The result for K^{m^+,m^-} can be proved by following the same steps with minor modifications. $\square$

Since σ_k is an $(\mathcal{F}_{0,t}^W)_{t\ge0}$-stopping time, the sequence $(C_k)_{k\ge0}$ is independent and satisfies $\mathbb{P}(C_k) = \mathbb{P}(C_0) = \mathbb{P}(A_0) \times \mathbb{P}(W_\rho^+ = l)$ for all $k \ge 0$. By Lemma 6.3, $\sum_{k\ge0} \mathbb{P}(C_k) = \infty$ and the Borel-Cantelli lemma yields $\mathbb{P}(\overline{\lim} C_k) = 1$. We deduce that with probability 1,

$$\varphi_{0,\sigma_k}(\mathscr{C}) = e^{il} \text{ and } K_{0,\sigma_k}^{m^+,m^-}(\mathscr{C}) = \delta_{e^{il}} \text{ for infintely many } k.$$

Lemma 6.5. *Let*

$$k_0(\omega) = 0, \; k_{n+1}(\omega) = \inf\{k > k_n(\omega) : \omega \in C_k\}.$$

Set $\sigma'_n = \sigma_{k_n}, n \ge 1$. *Then* $(\sigma'_n)_{n\ge1}$ *is a sequence of* $(\mathcal{F}_{0,t}^W)_{t\ge0}$*-stopping times such that a.s.* $\lim_{n\to\infty} \sigma'_n = +\infty$ *and* $\varphi_{0,\sigma'_n}(x) = e^{il}, K_{0,\sigma'_n}^{m^+,m^-}(x) = \delta_{e^{il}}$ *for all* $x \in \mathscr{C}, \; n \ge 1$.

Proof. Remark that $C_k \in \mathcal{F}_{\sigma_{k+1}}^W$ for all $k \ge 0$. For all $n \ge 1, t \ge 0$, we have

$$\{\sigma_{k_n} \le t\} = \cup_{k\ge1}\{\sigma_k \le t, k_n = k\}.$$

It remains to prove that $\{k_n = k\} \in \mathcal{F}^W_{\sigma_{k+1}}$. We will prove this by induction on n. For $n = 1$, this is clear since $\{k_1 = 1\} = C_1$ and for $k \geq 2$,

$$\{k_1 = k\} = C_1^c \cap \cdots \cap C_{k-1}^c \cap C_k.$$

Suppose the result holds for n. Then for all $k \geq 2$,

$$\{k_{n+1} = k\} = \cup_{1\leq i\leq k-1} \big(\{k_n = i\} \cap C_{i+1}^c \cap \cdots C_{k-1}^c \cap C_k\big)$$

and the desired result holds for $n+1$ using the induction hypothesis.

$\square$

We have proved Part (1) of Proposition 6.1. Part (2) can be deduced by analogy.

6.4 The support of K^{m^+,m^-}

In this section ρ_0^k and K^{m^+,m^-} will be denoted simply by ρ^k and K.

6.4.1 The case $l = \pi$

When m^+ and m^- are both different from $\frac{1}{2}(\delta_0 + \delta_1)$, a precise description of $\mathrm{supp}(K_{0,t}(1))$ can be given as follows. Recall the definition of the sequence $(\sigma_k)_{k\geq 0}$ from Section 6.3.1. Then

$$\mathrm{supp}(K_{0,t}(1)) = \{e^{iW^+_{\sigma_{2k},t}}, e^{-iW^+_{\sigma_{2k},t}}\} \text{ for all } \ \sigma_{2k} \leq t \leq \sigma_{2k+1}$$

and

$$\mathrm{supp}(K_{0,t}(1)) = \{e^{i(\pi+W^-_{\sigma_{2k+1},t})}, e^{i(\pi-W^-_{\sigma_{2k+1},t})}\} \text{ for all } \ \sigma_{2k+1} \leq t \leq \sigma_{2k+2}.$$

In fact for all $s \leq t$,

$$\mathrm{supp}(K_{s,t}(1)) = \{e^{iX_{s,t}}, e^{-iX_{s,t}}\},$$

with $X_{s,t}$ being the unique reflecting Brownian motion on $[0, \pi]$ (see [6]) solution of

$$X_{s,t} = W_{s,t} + L^0_{s,t} - L^\pi_{s,t}, \quad t \geq s,$$

and

$$L^x_{s,t} = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_s^t 1_{\{|X_{s,u}-x|\le\varepsilon\}}du, \quad x = 0, \pi.$$

If $m^+ = m^- = \delta_{\frac{1}{2}}$, then K is a Wiener flow such that $K_{s,t}(1) = \frac{1}{2}(\delta_{e^{iX_{s,t}}} + \delta_{e^{-iX_{s,t}}})$ for all $s \le t$.

6.4.2 The case $l \neq \pi$

From the definition of K, $K_{\rho^k,t}(x)$ is carried by at most two points for all $k \ge 0$, $t \in [\rho^k, \rho^{k+1}], x \in \mathscr{C}$. It is therefore clear that a.s.

$$\forall t \ge 0,\ x \in \mathscr{C},\ \text{Card supp } K_{0,t}(x) < \infty.$$

We assume in this section that m^+ and m^- are both distinct from $\frac{1}{2}(\delta_0 + \delta_1)$ (for the other case, see Remark 6.1 below).

Fix a decreasing positive sequence $(\alpha_k)_{k\ge1}$ such that $\alpha_1 < \inf(l, 2(\pi-l))$. Now define $A_1 = \{\ W^+_{0,\rho^1} = l\}$ and for $k \ge 1$,

$$\begin{aligned} A_{2k} &= \{W^-_{\rho^{2k-1},\rho^{2k}} = l,\ \alpha_{2k} < \sup_{\rho^{2k-1}\le u\le\rho^{2k}} W_{\rho^{2k-1},u} < \alpha_{2k-1}\} \\ &= \{W^-_{\rho^{2k-1},\ \rho^{2k}} = l, -l+\alpha_{2k} < W_{\rho^{2k-1},\rho^{2k}} < -l+\alpha_{2k-1}\}, \\ A_{2k+1} &= \{W^+_{\rho^{2k},\rho^{2k+1}} = l,\ -\alpha_{2k} < \inf_{\rho^{2k}\le u\le\rho^{2k+1}} W_{\rho^{2k},u} < -\alpha_{2k+1}\} \\ &= \{W^+_{\rho^{2k},\rho^{2k+1}} = l,\ l-\alpha_{2k} < W_{\rho^{2k},\rho^{2k+1}} < l-\alpha_{2k+1}\}. \end{aligned}$$

We are going to prove the following

Proposition 6.6. *Let $C_n = \cap_{i=1}^n A_i$. Then for all $n \ge 1$,*

(i) $\mathbb{P}(C_n) > 0$,

(ii) Card supp $(K_{0,\rho^n}(1)) = n+1$ *a.s. on* C_n.

Moreover a.s. for all $k \ge 0$,

(ii1) On C_{2k},

$$supp\left(K_{0,\rho^{2k}}(1)\right) = \{P^{2k}_i, 1 \le i \le 2k+1\},$$

with $\arg(P_i^{2k}) < \arg(P_{i+1}^{2k})$ *for all* $i \in [1, 2k]$,

$$P_1^{2k} = 1,\ P_2^{2k} = e^{2il} \textit{and } P_{2k+1}^{2k} = e^{i(-l-W_{\rho^{2k-1},\rho^{2k}})}.$$

(ii2) On C_{2k+1}*, we have*

$$supp\left(K_{0,\rho^{2k+1}}(1)\right) = \{P_i^{2k+1},\ 1 \le i \le 2k+2\},$$

with $\arg(P_i^{2k+1}) < \arg(P_{i+1}^{2k+1})$ *for all* $i \in [1, 2k+1]$,

$$P_1^{2k+1} = e^{il}, P_2^{2k+1} = e^{i(2l-W_{\rho^{2k},\rho^{2k+1}})} \textit{and } P_{2k+2}^{2k+1} = e^{-il}.$$

To prove this proposition, let first establish the following

Lemma 6.6. *Fix* $0 < \alpha < \beta < l$ *and define*

$$E = \{W_\rho^- = l, \alpha < \sup_{0\le u\le\rho} W_u < \beta\}$$

where $\rho = \inf\{r \ge 0 : \sup(W_r^+, W_r^-) = l\}$*. Then* $\mathbb{P}(E) > 0$.

Proof. Recall the definition of T_a from the begining of Section 6.3. Consider the event

$$F = \{T_\alpha < T_{\beta-l} < T_\beta\} \cap \{\text{after } T_{\beta-l}, W \text{ reaches } \alpha - l \text{ before } \beta - l + \alpha\}.$$

Clearly $\mathbb{P}(F) > 0$. Note that ρ can be expressed as

$$\rho = \inf\{t \ge 0 : \sup_{0\le u\le t} W_u - \inf_{0\le u\le t} W_u = l\}.$$

On F, we have $T_{\beta-l} < \rho \le T_{\alpha-l}$ and so $\alpha < \sup_{0\le u\le\rho} W_u < \beta$. Moreover, on F

$$W_\rho^+ = W_\rho - \inf_{0\le u\le t} W_u < \beta - l + \alpha - (\alpha - l) < \beta < l.$$

In other words $W_\rho^- = l$ which proves the inclusion $F \subset E$ and allows to deduce the lemma. $\square$

Proof of Proposition 6.6 (i) The sequence $(A_i)_{i\geq 1}$ is independent and therefore we only need to check that $\mathbb{P}(A_n) > 0$ for all $n \geq 1$. But this is immediate from Lemma 6.6 for n even. By replacing W with $-W$, it is also immediate for n odd.
(ii) We denote the properties (ii1) and (ii2) respectively by $\mathcal{P}_{2k}$ and $\mathcal{P}_{2k+1}$. Let prove all the $(\mathcal{P}_i)_{i\geq 0}$ by induction. First $\mathcal{P}_0$ and $\mathcal{P}_1$ are clearly satisfied since $K_{0,0}(1) = \delta_1$ and supp $K_{0,\rho^1}(1) = \{e^{il}, e^{-il}\}$ on C_1. Suppose $(\mathcal{P}_i)_{0\leq i\leq 2k}$ hold for $k \geq 0$. On C_{2k+1}, $K_{\rho^{2k},\cdot}(e^{2il})$ cannot reach $\delta_{e^{il}}$ before ρ^{2k+1} since

$$W_{\rho^{2k},\cdot} < W^+_{\rho^{2k},\cdot} \leq l \text{ on }]\rho^{2k}, \rho^{2k+1}].$$

Moreover, on C_{2k+1},

$$\sup_{\rho^{2k}\leq u\leq \rho^{2k+1}} (2\pi - l - W_{\rho^{2k-1},\rho^{2k}} - W_{\rho^{2k},u}) = 2\pi - (W_{\rho^{2k-1},\rho^{2k}} + W_{\rho^{2k},\rho^{2k+1}}) < 2\pi.$$

Thus, on C_{2k+1}, $K_{\rho^{2k},\cdot}(P^{2k}_{2k+1})$ cannot reach δ_1 before ρ^{2k+1} and $\mathcal{P}_{2k+1}$ easily holds. Similarly, on C_{2k+2}, $K_{\rho^{2k+1},\cdot}(e^{-il})$ cannot reach δ_1 before ρ^{2k+2} since

$$W^-_{\rho^{2k+1},\cdot} > -W_{\rho^{2k+1},\cdot} \text{ on }]\rho^{2k+1}, \rho^{2k+2}].$$

Moreover, on C_{2k+2}, we have $W_{\rho^{2k},\rho^{2k+1}} + W_{\rho^{2k+1},\rho^{2k+2}} < 0$ and therefore $K_{\rho^{2k+1},\cdot}(P^{2k+1}_2)$ cannot reach e^{il} before ρ^{2k+2} so that $\mathcal{P}_{2k+2}$ holds.

Remark 6.1. *When $m^+ \neq m^-, m^- = \frac{1}{2}(\delta_0 + \delta_1)$, by considering*

$$E_{2i+1} = A_{2i+1}, i \geq 0, E_{2i} = A_{2i} \cap \{K_{\rho^{2i-1},\rho^{2i}}(e^{il}) = \delta_1\}, i \geq 1,$$

and then $F_n = \cap_{1\leq i\leq n} E_i$, we similarly show that supp$(K_{0,t}(1))$ may be sufficiently large with positive probability.

6.5 Unicity of flows associated to $(T_{\mathscr{C}})$

Let K be a solution of $(T_{\mathscr{C}})$. Fix $s \in \mathbb{R}, x \in \mathscr{C}$, then $(K_{s,t}(x))_{t\geq s}$ can be modified such that, a.s. the mapping $t \longmapsto K_{s,t}(x)$ is continuous from $[s, +\infty[$ into $\mathcal{P}(\mathscr{C})$. We will always consider this modification for $(K_{s,t}(x))_{t\geq s}$. Moreover we will assume that all the σ-fields which will be considered contain all $\mathbb{P}$ negligible sets.

Lemma 6.7. *Let (K, W) be a solution of $(T_{\mathscr{C}})$. Then*

(i) $\forall x \in \mathscr{C}, s \in \mathbb{R}$ denote $\tau_s(x) = \inf\{r \geq s,\ xe^{i\epsilon(x)W_{s,r}} = 1$ or $e^{il}\}$. Then a.s.

$$K_{s,t}(x) = \delta_{xe^{i\epsilon(x)W_{s,t}}},\ \textit{if}\ \ s \leq t \leq \tau_s(x).$$

(ii) $\sigma(W) \subset \sigma(K)$.

Proof. (i) We follow Lemma 3.1 [36]. Fix $x \in \mathscr{C}$ and suppose for example that $\arg(x) \in]0, l[$. Define

$$\mathscr{C}^+ = \{z \in \mathscr{C} : \arg(z) \in]0, l[\},\ \mathscr{C}^- = \mathscr{C} \setminus \mathscr{C}^+ \tag{6.11}$$

and

$$\tilde{\tau}_x = \inf\left\{t \geq 0 : K_{0,t}(x, \mathscr{C}^-) > 0\right\}.$$

Let $f \in C^2(\mathscr{C})$ such that $f(z) = \arg(z)$ if $z \in \mathscr{C}^+$. By applying f in $(T_{\mathscr{C}})$, we have for $t < \tilde{\tau}_x$,

$$K_{0,t}f(x) = f(x) + W_t$$

and a fortiori, for $t < \tilde{\tau}_x$,

$$\int_{\mathscr{C}} \arg(y) K_{0,t}(x, dy) = \arg(x) + W_t. \tag{6.12}$$

By applying f^2 in $(T_{\mathscr{C}})$, we also have for $t < \tilde{\tau}_x$,

$$K_{0,t}f^2(x) = f^2(x) + 2\int_0^t \int_{\mathscr{C}} \arg(y) K_{0,u}(x, dy) dW_u + t.$$

Using (6.12), we obtain that for $t < \tilde{\tau}_x$,

$$\int_{\mathscr{C}} (\arg(y) - \arg(x) - W_t)^2 K_{0,t}(x, dy) = 0.$$

By continuity a.s.

$$K_{0,t}(x) = \delta_{xe^{i\epsilon(x)W_t}} \text{ for all } t \in [0, \tilde{\tau}_x].$$

The fact that $\tau_0(x) = \tilde{\tau}_x$ easily follows.

(ii) Let $(f_n)_{n\geq 1}$ be a sequence in $C^2(\mathscr{C})$ such that $f_n'(z) \to \epsilon(z)$ as $n \to \infty$ for all $z \in \mathscr{C} \setminus \{1, e^{il}\}$. Applying f_n in $(T_{\mathscr{C}})$, we get

$$\int_0^t K_{0,u}(\epsilon f_n')(1) dW_u = K_{0,t}f_n(1) - f_n(1) - \frac{1}{2}\int_0^t K_{0,u}f_n''(1) du.$$

It is easy to check that $\int_0^t K_{0,u}(\epsilon f_n')(1)dW_u$ converges towards W_t in $L^2(\mathbb{P})$ as $n \to \infty$ whence

$$W_t = \lim_{n\to\infty}\left(K_{0,t}f_n(1) - f_n(1) - \frac{1}{2}\int_0^t K_{0,u}f_n''(1)du\right) \text{ in } L^2(\mathbb{P})$$

which proves (ii). □

6.5.1 Unicity of the Wiener solution.

Our aim in this section is to prove that $(T_{\mathscr{C}})$ admits only one Wiener solution (such that $\sigma(W) \subset \sigma(K)$). This solution corresponds of course to $m^+ = m^- = \delta_{\frac{1}{2}}$. For this, we will essentially follow the general idea of [32], namely the Wiener chaos decomposition. Let p be semigroup of the standard Brownian motion on $\mathbb{R}$. Then the semigroup of Brownian motion on $\mathscr{C}$ writes

$$P_t(e^{ix}, e^{iy}) = \sum_{k\in\mathbb{Z}} p_t(x, y + 2k\pi), \quad x, y \in [0, 2\pi[.$$

For all $f \in C^1(\mathscr{C})$, we easily check that $P_t f \in C^1(\mathscr{C})$ and $(P_t f)' = P_t f'$. Let $Af = \frac{1}{2}f''$, $f \in C^2(\mathscr{C})$ be the generator of P.

Proposition 6.7. *Equation $(T_{\mathscr{C}})$ has at most one Wiener solution: If (K, W) is a solution such that $\sigma(W) \subset \sigma(K)$, then $\forall t \geq 0, f \in C^\infty(\mathscr{C}), x \in \mathscr{C}$,*

$$K_{0,t}f(x) = P_t f(x) + \sum_{n=1}^{\infty} J_t^n f(x) \ \textit{in}\ L^2(\mathbb{P})$$

where

$$J_t^n f(x) = \int_{0<s_1<\cdots<s_n<t} P_{s_1}(D(P_{s_2-s_1}\cdots D(P_{t-s_n}f)))(x)dW_{0,s_1}\cdots dW_{0,s_n} \tag{6.13}$$

no longer depends on K and $Df(x) = \epsilon(x).f'(x)$.

Proof. Let (K, W) be a stochastic flow that solves $(T_{\mathscr{C}})$ (not necessarily a Wiener flow). Our first aim now is to establish the following

Lemma 6.8. *Fix $f \in C^\infty(\mathscr{C}), x \in \mathscr{C}$. Then*

$$K_{0,t}f(x) = P_t f(x) + \int_0^t K_{0,u}(D(P_{t-u}f))(x)dW_u.$$

Let $f \in C^\infty(\mathscr{C}), x \in \mathscr{C}$ and denote $K_{0,t}$ simply by K_t. Note that $\int_0^t K_u(D(P_{t-u}f))(x)dW_u$ is well defined. In fact

$$\int_0^t E[K_u(D(P_{t-u}f))(x)]^2 du \leq \int_0^t P_u((D(P_{t-u}f))^2)(x)du \leq \int_0^t ||(P_{t-u}f)'||_\infty^2 du$$

and the right-hand side is smaller than $t||f'||_\infty^2$. Now

$$K_t f(x) - P_t f(x) - \int_0^t K_u(D(P_{t-u}f))(x)dW_u = \sum_{p=0}^{n-1}(K_{\frac{(p+1)t}{n}} P_{t-\frac{(p+1)t}{n}} f - K_{\frac{pt}{n}} P_{t-\frac{pt}{n}} f)(x)$$

$$-\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})f)(x)dW_u - \sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D(P_{t-\frac{(p+1)t}{n}} f)(x)dW_u.$$

For all $p \in \{0, .., n-1\}$, set $f_{p,n} = P_{t-\frac{(p+1)t}{n}} f \in C^\infty(\mathscr{C})$ and so by replacing f by $f_{p,n}$ in $(T_{\mathscr{C}})$, we get

$$\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u(Df_{p,n})(x)dW_u = K_{\frac{(p+1)t}{n}} f_{p,n}(x) - K_{\frac{pt}{n}} f_{p,n}(x) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u(Af_{p,n})(x)du$$

$$= K_{\frac{(p+1)t}{n}} f_{p,n}(x) - K_{\frac{pt}{n}} f_{p,n}(x) - \frac{t}{n} K_{\frac{pt}{n}}(Af_{p,n})(x) - \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_u - K_{\frac{pt}{n}})(Af_{p,n})(x)du.$$

Then we can write

$$K_t f(x) - P_t f(x) - \int_0^t K_u(D(P_{t-u}f))(x)dW_u = A_1(n) + A_2(n) + A_3(n),$$

where

$$\begin{aligned}
A_1(n) &= -\sum_{p=0}^{n-1} K_{\frac{pt}{n}}[P_{t-\frac{pt}{n}} f - P_{t-\frac{(p+1)t}{n}} f - \frac{t}{n} A P_{t-\frac{(p+1)t}{n}} f](x), \\
A_2(n) &= -\sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})f)(x)dW_u, \\
A_3(n) &= \sum_{p=0}^{n-1}\int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_u - K_{\frac{pt}{n}}) A P_{t-\frac{(p+1)t}{n}} f(x)du.
\end{aligned}$$

Using $||K_u g||_\infty \leq ||g||_\infty$ if g is a bounded measurable function, we obtain

$$|A_1(n)| \leq \sum_{p=0}^{n-1} \left|\left| P_{t-\frac{(p+1)t}{n}}[P_{\frac{t}{n}} f - f - \frac{t}{n}.Af] \right|\right|_\infty \leq n \left|\left| P_{\frac{t}{n}} f - f - \frac{t}{n} Af \right|\right|_\infty .$$

Since $f \in C^\infty(\mathscr{C})$, this shows that $A_1(n)$ converges to 0 as $n \to \infty$. Note that $A_2(n)$ is the sum of orthogonal terms in $L^2(\mathbb{P})$. Consequently

$$||A_2(n)||^2_{L^2(\mathbb{P})} = \sum_{p=0}^{n-1} \left|\left| \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} K_u D((P_{t-u} - P_{t-\frac{(p+1)t}{n}})f)(x) dW_u \right|\right|^2_{L^2(\mathbb{P})} .$$

By applying Jensen's inequality, we arrive at

$$||A_2(n)||^2_{L^2(\mathbb{P})} \leq \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} P_u V_u^2(x) du$$

where $V_u = (P_{t-u} f)' - (P_{t-\frac{(p+1)t}{n}} f)' = P_{t-u} f' - P_{t-\frac{(p+1)t}{n}} f'$. For all $u \in [\frac{pt}{n}, \frac{(p+1)t}{n}]$, we have

$$P_u V_u^2(x) \leq ||V_u||^2_\infty = \left|\left| P_{t-\frac{(p+1)t}{n}} \left(P_{\frac{(p+1)t}{n}-u} f' - f' \right) \right|\right|^2_\infty \leq ||P_{\frac{(p+1)t}{n}-u} f' - f'||^2_\infty .$$

Consequently

$$||A_2(n)||^2_{L^2(\mathbb{P})} \leq \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{\frac{(p+1)t}{n}-u} f' - f'||^2_\infty du = n \int_0^{\frac{t}{n}} ||P_u f' - f'||^2_\infty du,$$

and one can deduce that $A_2(n)$ tends to 0 as $n \to +\infty$ in $L^2(\mathbb{P})$. Now

$$||A_3(n)||_{L^2(\mathbb{P})} \leq \sum_{p=0}^{n-1} \left|\left| \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} (K_u - K_{\frac{pt}{n}}) A P_{t-\frac{(p+1)t}{n}} f(x) du \right|\right|_{L^2(\mathbb{P})} .$$

Set $h_{p,n} = AP_{t-\frac{(p+1)t}{n}} f$. Then $h_{p,n} \in C^\infty(\mathscr{C})$ for all $p \in [0, n-1]$. By the Cauchy-Schwarz inequality

$$||A_3(n)||_{L^2(\mathbb{P})} \leq \sqrt{t} \left\{ \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} E[((K_u - K_{\frac{pt}{n}}) h_{p,n}(x))^2] du \right\}^{\frac{1}{2}} .$$

If $u \in [\frac{pt}{n}, \frac{(p+1)t}{n}]$:

$$\begin{aligned}
E[((K_u - K_{\frac{pt}{n}})h_{p,n}(x))^2] &\leq E[K_{\frac{pt}{n}}(K_{\frac{pt}{n},u}h_{p,n} - h_{p,n})^2(x)] \\
&\leq E[K_{\frac{pt}{n}}(K_{\frac{pt}{n},u}h_{p,n}^2 - 2h_{p,n}K_{\frac{pt}{n},u}h_{p,n} + h_{p,n}^2)(x)] \\
&\leq P_{\frac{pt}{n}}\left(P_{u-\frac{pt}{n}}h_{p,n}^2 - 2h_{p,n}P_{u-\frac{pt}{n}}h_{p,n} + h_{p,n}^2\right)(x) \\
&\leq ||P_{u-\frac{pt}{n}}h_{p,n}^2 - 2h_{p,n}P_{u-\frac{pt}{n}}h_{p,n} + h_{p,n}^2||_\infty \\
&\leq 2||h_{p,n}||_\infty ||P_{u-\frac{pt}{n}}h_{p,n} - h_{p,n}||_\infty + ||P_{u-\frac{pt}{n}}h_{p,n}^2 - h_{p,n}^2||_\infty.
\end{aligned}$$

Therefore $||A_3(n)||_{L^2(\mathbb{P})} \leq \sqrt{t}(2C_1(n) + C_2(n))^{\frac{1}{2}}$, where

$$C_1(n) = \sum_{p=0}^{n-1} ||h_{p,n}||_\infty \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}}h_{p,n} - h_{p,n}||_\infty du$$

and

$$C_2(n) = \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}}h_{p,n}^2 - h_{p,n}^2||_\infty du.$$

From $||h_{p,n}||_\infty \leq ||Af||_\infty$ and $||P_{u-\frac{pt}{n}}h_{p,n} - h_{p,n}||_\infty \leq ||P_{u-\frac{pt}{n}}Af - Af||_\infty$, we get

$$C_1(n) \leq ||Af||_\infty \sum_{p=0}^{n-1} \int_{\frac{pt}{n}}^{\frac{(p+1)t}{n}} ||P_{u-\frac{pt}{n}}Af - Af||_\infty du \leq ||Af||_\infty \int_0^t ||P_{\frac{s}{n}}Af - Af||_\infty ds.$$

As $Af \in C^\infty(\mathscr{C})$, $C_1(n)$ tends to 0 obviously. On the other hand, $h_{p,n}^2 \in C^\infty(\mathscr{C})$ and so

$$C_2(n) = \frac{1}{n}\sum_{p=0}^{n-1} \int_0^t ||P_{\frac{s}{n}}h_{p,n}^2 - h_{p,n}^2||_\infty ds \leq \frac{1}{2n}\sum_{p=0}^{n-1} \int_0^t \int_0^{\frac{s}{n}} ||(h_{p,n}^2)''||_\infty du ds.$$

Now we easily verify that $h_{p,n}, h'_{p,n}, h''_{p,n}$ are uniformly bounded with respect to n and $0 \leq p \leq n-1$. As a result $C_2(n)$ tends to 0 as $n \to \infty$. This establishes Lemma 6.8.

Assume that (K, W) is a Wiener solution of $(T_{\mathscr{C}})$ and for $t \geq 0, f \in C^\infty(\mathscr{C}), x \in \mathscr{C}$, let $K_{0,t}f(x) = P_tf(x) + \sum_{n=1}^\infty J_t^n f(x)$ be the decomposition in Wiener chaos of $K_{0,t}f(x)$ in L^2 sense (recall that $K_{0,t}f(x) \in L^2(\mathcal{F}_\infty^{W_{0,\cdot}})$). By iterating the identity of Lemma 6.8, we see that for all $n \geq 1$, $J_t^n f(x)$ is given by (6.13). $\square$

Consequence: Let (K, W) be a solution of $(T_{\mathscr{C}})$ and K^W be the unique Wiener solution of $(T_{\mathscr{C}})$. Since $\sigma(W) \subset \sigma(K)$, we can define K^* the stochastic flow obtained by filtering K with respect to $\sigma(W)$ (Lemma 3-2 (ii) in [35]). Then, $\forall s \leq t, x \in \mathscr{C}$, a.s.

$$K^*_{s,t}(x) = E[K_{s,t}(x)|\sigma(W)].$$

As a result, (K^*, W) solves also $(T_{\mathscr{C}})$ and by the last proposition, $\forall s \leq t, x \in \mathscr{C}$, a.s.

$$E[K_{s,t}(x)|\sigma(W)] = K^W_{s,t}(x). \tag{6.14}$$

6.5.2 Proof of Theorem 6.1 (2)

From now on, (K, W) **is a solution of** $(T_{\mathscr{C}})$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Let $P^n_t = E[K^{\otimes n}_{0,t}]$ be the compatible family of Feller semigroups associated to K.

A stochastic flow of mappings associated to K.

Let $(P^{n,c})_{n\geq 1}$ be the family of compatible Markov semigroups associated to $(P^n)_{n\geq 1}$ by Theorem 4.1 [35]. Then we have

Lemma 6.9. *$(P^{n,c})_{n\geq 1}$ is a compatible family of Feller semigroups associated with a flow of mappings φ^c.*

Proof. For each $(x, y) \in \mathscr{C}^2$, let $(X^x_t, Y^y_t)_{t\geq 0}$ be the two point motion started at (x, y) associated with P^2 constructed as in Section 2.6 [35] on an extension $\Omega \times \Omega'$ of Ω such that the law of (X^x_t, Y^y_t) given $\omega \in \Omega$ is $K_{0,t}(x) \otimes K_{0,t}(y)$. Define

$$T^{x,y} := \inf\{t \geq 0 : X^x_t = Y^y_t\}.$$

By Theorem 3.3, we only need to check that: $\forall t > 0, \varepsilon > 0, x \in \mathscr{C}$,

$$\lim_{y \to x} \mathbb{P}(\{T^{x,y} > t\} \cap \{d(X^x_t, Y^y_t) > \varepsilon\}) = 0 \qquad (C).$$

Recall that for all $0 \leq t \leq \rho$ $(:= \rho_0)$,

$$K^W_{0,t}(1) = \frac{1}{2}(\delta_{e^{iW^+_t}} + \delta_{e^{-iW^+_t}}).$$

This shows that $K_{0,t}(1)$ is supported on $\{\delta_{e^{iW_t^+}}, \delta_{e^{-iW_t^+}}\}$ and so $X_t^1 = e^{iW_t^+}$ or $e^{-iW_t^+}$ for all $0 \leq t \leq \rho$. Moreover, if $y \notin \{1, e^{il}\}$, then $X_t^y = ye^{i\varepsilon(y)W_t}$ for all $0 \leq t \leq \tau(y)(:= \tau_0(y))$ by Lemma 6.7 (i).

Fix $t > 0, \varepsilon > 0$ and let $A = \{T^{1,y} > t\} \cap \{d(X_t^1, Y_t^y) > \varepsilon\}$ where y is close to 1 and $y \neq 1$. Write

$$\mathbb{P}(A) = \mathbb{P}(A \cap \{t \leq \tau(y)\}) + \mathbb{P}(A \cap \{t > \tau(y)\}).$$

Since $\tau(y)$ tends to 0 as y tends to 1, we have $\lim_{y\to 1} \mathbb{P}(A \cap \{t \leq \tau(y)\}) = 0$. Moreover

$$\mathbb{P}(A \cap \{t > \tau(y)\}) \leq \mathbb{P}(B) + \mathbb{P}(X_{\tau(y)}^y = e^{il}).$$

where $B = A \cap \{t > \tau(y), X_{\tau(y)}^y = 1\}$. Obviously

$$\mathbb{P}(B) \leq \mathbb{P}(B \cap \{\tau(y) < \rho\}) + \mathbb{P}(\tau(y) \geq \rho)$$

with $\lim_{y\to 1} \mathbb{P}(\tau(y) \geq \rho) = 0$. On $B \cap \{\tau(y) < \rho\}$, we have $X_{\tau(y)}^1 = X_{\tau(y)}^y = 1$ and a fortiori $T^{1,y} \leq \tau(y)$. As a result

$$\mathbb{P}(B \cap \{\tau(y) < \rho\}) \leq \mathbb{P}(t < T^{1,y} \leq \tau(y)).$$

Since the right-hand side converges to 0 as $y \to 1$, (C) is satisfied for $x = 1$ and by analogy for $x = e^{il}$. Let $x \notin \{1, e^{il}\}$ and y be close to x, then X^x and X^y move parallely until one of the two processes reach 1 or e^{il} say at time T. Since P^2 is Feller, the strong Markov property at time T and the established result for $x \in \{1, e^{il}\}$ allows to deduce (C) for x. $\square$

Consequence: Let ν (respectively ν^c) be the Feller convolution semigroup associated with $(P^n)_{n\geq 1}$ (respectively $(P^{n,c})_{n\geq 1}$). By Theorem 3.4, there exists a joint realization (K^1, K^2) where K^1 and K^2 are two stochastic flows of kernels satisfying $K^1 \overset{law}{=} \delta_{\varphi^c}$, $K^2 \overset{law}{=} K$ and such that:

(i) $\hat{K}_{s,t}(x, y) = K_{s,t}^1(x) \otimes K_{s,t}^2(y)$ is a stochastic flow of kernels on $\mathscr{C}^2$,

(ii) For all $s \leq t, x \in \mathscr{C}$, a.s. $K_{s,t}^2(x) = E[K_{s,t}^1(x)|K^2]$.

For all $s \le t$, let

$$\hat{\mathcal{F}}_{s,t} = \sigma(\hat{K}_{u,v}, s \le u \le v \le t), \quad \mathcal{F}^i_{s,t} = \sigma(K^i_{u,v}, s \le u \le v \le t), \ i = 1, 2.$$

Then $\hat{\mathcal{F}}_{s,t} = \mathcal{F}^1_{s,t} \vee \mathcal{F}^2_{s,t}$. To simplify notations, we shall assume that φ^c is defined on the original space $(\Omega, \mathcal{A}, \mathbb{P})$ and that (i) and (ii) are satisfied if we replace (K^1, K^2) by (δ_{φ^c}, K). Recall that (i) and (ii) are also satisfied by the pair $(\delta_\varphi, K^{m^+,m^-})$ constructed in Section 6.2.3. Now (ii) rewrites, for all $s \le t, x \in \mathscr{C}$,

$$K_{s,t}(x) = E[\delta_{\varphi^c_{s,t}(x)}|K] \ \ a.s. \tag{6.15}$$

and using (6.14), we obtain, for all $s \le t, x \in \mathscr{C}$,

$$K^W_{s,t}(x) = E[\delta_{\varphi^c_{s,t}(x)}|\sigma(W)] \ a.s. \tag{6.16}$$

with K^W being the Wiener flow associated with $m^+ = m^- = \frac{1}{2}$.

The law of (K, φ^c).

For all $s \le t$, define $\mathcal{F}^K_{s,t} = \sigma(K_{v,u}, s \le v \le u \le t)$ and recall the definition $\mathcal{F}^W_{s,t} = \sigma(W_{v,u}, s \le v \le u \le t)$. Assume that these σ-fields are right-continuous and include all $\mathbb{P}$-negligible sets. When $s = 0$, we denote $\mathcal{F}^K_{0,t}, \mathcal{F}^W_{0,t}$ simply by $\mathcal{F}^K_t, \mathcal{F}^W_t$. For each each $x \in \mathscr{C}$, recall that $t \longmapsto K_{0,t}(x)$ is continuous from $[0, +\infty[$ into $\mathcal{P}(\mathscr{C})$. We denote by $\mathbb{P}_x$, the law of this process which is a probability measure on $C(\mathbb{R}_+, \mathcal{P}(\mathscr{C}))$. We begin this section by the following Markov property

Lemma 6.10. *Let $x, y \in \mathscr{C}$ and T be an $(\mathcal{F}^K_t)_{t\ge0}$-stopping time. On $\{K_{0,T}(x) = \delta_y\}$, the law of $K_{0,T+\cdot}(x)$ knowing $\mathcal{F}^K_T$ is given by $\mathbb{P}_y$.*

Proof. Let $p \ge 1, 0 \le t_1 < \cdots < t_p$ and $g, g_1, \cdots, g_p$ be $p+1$ Lipschitzs functions from $\mathcal{P}(\mathscr{C})$ into $\mathbb{R}$. Let $A \in \mathcal{F}^K_T$ and $[T]_n = \inf\{\frac{j}{2^n} : \frac{j}{2^n} > T\}$. Since K is a flow, we may write

$$\begin{aligned} E\Big[\prod_{j=1}^p g_j(K_{0,T+t_j}(x))g(K_{0,T}(x))1_A\Big] &= \lim_{n\to\infty} \sum_{i=1}^\infty E\left[\prod_{j=1}^p g_j(K_{0,\frac{i}{2^n}+t_j}(x))g(K_{0,\frac{i}{2^n}}(x))1_{A\cap\{\frac{i-1}{2^n}\le T<\frac{i}{2^n}\}}\right] \\ &= \lim_{n\to\infty} E[G(K_{0,[T]_n}(x))g(K_{0,[T]_n}(x))1_A]. \end{aligned}$$

where $G(\mu) = E\big[\prod_{j=1}^{p} g_j(\mu K_{0,t_j})\big]$ and μK is the probability measure $\mu K(\cdot) = \int_{\mathscr{C}} \mu(dx)K(x,\cdot)$ for all kernel K and $\mu \in \mathcal{P}(\mathscr{C})$. Suppose for the moment that G is continuous on $\mathcal{P}(\mathscr{C})$, then

$$E\big[\prod_{j=1}^{p} g_j(K_{0,T+t_j}(x))g(K_{0,T}(x))1_A\big] = E\big[G(K_{0,T}(x))g(K_{0,T}(x))1_A\big].$$

By an approximation argument,

$$E\big[\prod_{j=1}^{p} g_j(K_{0,T+t_j}(x))1_{\{K_{0,T}(x)=\delta_y\}}1_A\big] = E[G(\delta_y)1_{\{K_{0,T}(x)=\delta_y\}}1_A]$$

which proves the lemma. To check the continuity of G, suppose for simplicity that $p = 1$. Let $\mu, \mu_k \in \mathcal{P}(\mathscr{C})$ such that $\lim_{k\to\infty} d(\mu_k, \mu) = 0$ where d is the distance of weak convergence defined by (6.3). As g_1 is Lipschitz, it sufficies to prove

$$\lim_{k\to\infty} E\left[d(\mu_k K_{0,t}, \mu K_{0,t})\right] = 0 \ (t := t_1). \tag{6.17}$$

Recall the definition $P_t^n = E[K_{0,t}^{\otimes n}]$ and let $f \in C(\mathscr{C})$. Then

$$E\left[\left(\int K_{0,t}f(x)\mu_k(dx) - \int K_{0,t}f(x)\mu(dx)\right)^2\right] = \int P_t^2(f\otimes f)(x,y)\mu_k(dx)\mu_k(dy)$$

$$-2\int P_t^2(f\otimes f)(x,y)\mu_k(dx)\mu(dy) + \int P_t^2(f\otimes f)(x,y)\mu(dx)\mu(dy).$$

As P^2 is Feller, it is easy to deduce (6.17). $\square$

Recall the definitions of $\mathscr{C}^+$ and $\mathscr{C}^-$ from (6.11) and set for all $s \le t$,

$$U_{s,t}^+ = K_{s,t}(1, \mathscr{C}^+), \ U_{s,t}^- = K_{s,t}(e^{il}, \mathscr{C}^-).$$

For $s = 0$, we denote $U_{0,t}^+, U_{0,t}^-$ simply by U_t^+, U_t^-. Let

$$\rho^+ = \inf\{r \ge 0 : W_r^+ = l\}, \quad L = \sup\{r \in [0, \rho^+] : W_r^+ = 0\}.$$

Thanks to (6.16), on the event $\{0 \le t \le \rho^+\}$, a.s.

$$E[\delta_{\varphi_{0,t}^c(1)}|\sigma(W)] = \frac{1}{2}(e^{iW_t^+} + e^{-iW_t^+}).$$

This shows that $\varphi^c_{0,t}(1) \in \{e^{iW^+_t}, e^{-iW^+_t}\}$ for all $0 \le t \le \rho^+$. Let $h \in C(\mathscr{C})$ such that $\forall x \in [-l, l]$, $h(e^{ix}) = |x|$. Using (6.15) and the continuity of $t \longmapsto K_{0,t}(1)$, we have a.s.

$$K_{0,t}(g \circ h)(1) = g(W^+_t) \text{ for all } g \in C_0(\mathbb{R}), t \in [0, \rho^+].$$

Thus a.s. $\forall t \in [0, \rho^+]$, $K_{0,t}h(1) = W^+_t$ and ρ^+ can be expressed as

$$\rho^+ = \inf\{t \ge 0 : K_{0,t}h(1) = l\}.$$

Define the σ-fields:

$$\mathcal{F}_{L-} = \sigma(X_L, X \text{ is a bounded } \mathcal{F}^W_{0,\cdot} - \text{previsible process}),$$

$$\mathcal{F}_{L+} = \sigma(X_L, X \text{ is a bounded } \mathcal{F}^W_{0,\cdot} - \text{progressive process}).$$

Then $\mathcal{F}_{L+} = \mathcal{F}_{L-}$ (see Section 4.4). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded continuous function and set

$$X_t = E[f(U^+_t)|\sigma(W)]1_{\{0 \le t \le \rho^+\}}.$$

By (6.15), the process U^+ is constant on the excursions of W^+ out of 0 before ρ^+.

Lemma 6.11. *There exists an $\mathcal{F}^W$-progressive version of X denoted Y that is constant on the excursions of W^+ out of 0 before ρ^+ and satisfies $Y_L = Y_{\rho^+}$ a.s.*

Proof. We closely follow Lemma 4.12 [36] and correct an error at the end of the proof there. By induction, for all integers k and n, define the sequence of stopping times $S_{k,n}$ and $T_{k,n}$ by the relations: $T_{0,n} = 0$ and for $k \ge 1$,

$$\begin{aligned} S_{k,n} &= \inf\{t \ge T_{k-1,n} : W^+_t = 2^{-n}\}, \\ T_{k,n} &= \inf\{t \ge S_{k,n} : W^+_t = 0\}. \end{aligned}$$

In the following $U^+_{k,n}$ will denote $U^+_{S_{k,n}}$. On $\{t \in [S_{k,n}, T_{k,n}[, t \le \rho^+\}$, we have $U^+_t = U^+_{k,n}$ a.s. Let $X_{k,n} := E[f(U^+_{k,n})|W]1_{\{S_{k,n} \le \rho^+\}}$. Since $\sigma(W_{S_{k,n}, u+S_{k,n}}, u \ge 0)$ is independent of $\mathcal{F}^K_{S_{k,n}}$, we have $X_{k,n} = E[f(U_{k,n})|\mathcal{F}^W_{S_{k,n}}]1_{\{S_{k,n} \le \rho^+\}}$ which is $\mathcal{F}^W_{S_{k,n}}$ measurable. Set $I_n = \bigcup_{k \ge 1}[S_{k,n}, T_{k,n}[$ and define

$$X_t^n = \begin{cases} X_{k,n} & \text{if } t \in [S_{k,n}, T_{k,n}[\text{ (for some } k) \text{ and } t \leq \rho^+, \\ f(0) & \text{if } t \in I_n^c \cap [0, \rho^+], \\ 0 & \text{if } t > \rho^+. \end{cases}$$

Then X^n is $\mathcal{F}^W$-progressive. For all $t \geq 0$, set $\tilde{X}_t = \limsup_{n\to\infty} X_t^n$. The process $\tilde{X}$ is $\mathcal{F}^W$-progressive and for all $t \geq 0$, $\tilde{X}_t = X_t$ a.s. In fact, on $\{0 < t < \rho^+\}$, choose k_0 and n_0 such that $t \in [S_{k_0,n_0}, T_{k_0,n_0}[$, then $X_t^{n_0} = X_{k_0,n_0}$. For all $n \geq n_0$, there exists an integer l_n such that $t \in [S_{l_n,n}, T_{l_n,n}[$. Thus $X_t^n = X_{l_n,n} = X_{k_0,n_0}$ since S_{k_0,n_0} and $S_{l_n,n}$ belong to the same excursion interval of W^+ containing also t. Now set $Y_0 = f(0), Y_t = \limsup_{n\to\infty} \tilde{X}_{t+\frac{1}{n}}$ for all $t > 0$. Then Y is a modification of X which is $\mathcal{F}^W$-progressive and constant on the excursions of W^+ out of 0 before ρ^+. Moreover $Y_L = Y_{\rho^+}$ a.s. □

We take for X this $\mathcal{F}^W$-progressive version. Then $X_{\rho^+} = E[f(U_{\rho^+}^+)|\sigma(W)]$ is $\mathcal{F}_{L+}$ measurable.

Lemma 6.12. $E[X_{\rho^+}|\mathcal{F}_{L-}] = E[f(U_{\rho^+}^+)]$.

Proof. Let S be an $\mathcal{F}^W$-stopping time and $d_S = \inf\{t \geq S : W_t^+ = 0\}$. We have $\{S < L\} = \{d_S < \rho^+\}$ (up to some negligible set) and so $1_{\{S<L\}}$ is $\mathcal{F}_{0,d_S}^W$-measurable. Let $H = d_S \wedge \rho^+$, then

$$E[X_{\rho^+} 1_{\{d_S<\rho^+\}}] = E[f(U_{H+K}^+) 1_{\{d_S<\rho^+, K_{0,H}(1)=\delta_1\}}].$$

where $K = \inf\{r \geq 0 : K_{0,H+r}h(1) = l\}$. Applying Lemma 6.10 at time H, we get

$$E[X_{\rho^+} 1_{\{d_s<\rho^+\}}] = E[f(U_{\rho^+}^+)] E[1_{\{d_S<\rho^+, K_{0,H}(1)=\delta_1\}}] = E[f(U_{\rho^+}^+)] \mathbb{P}(d_S < \rho^+).$$

Since the σ-field $\mathcal{F}_{L-}$ is generated by the random variables $1_{\{S<L\}}$ (see [47] page 344), the lemma holds. □

The previous lemma implies that $U_{\rho^+}^+$ is independent of $\sigma(W)$ (Lemma 4.14 [36]) and the same holds if we replace ρ^+ by $\inf\{t \geq 0 : W_t^+ = a\}$ where $0 < a \leq l$. For n

such that $2^{-n} < l$, define inductively $T^+_{0,n} = 0$ and for $k \geq 1$:

$$\begin{aligned} S^+_{k,n} &= \inf\{t \geq T^+_{k-1,n} : W^+_t = 2^{-n}\}, \\ T^+_{k,n} &= \inf\{t \geq S^+_{k,n} : W^+_t = 0\}. \end{aligned}$$

Set $V^+_{k,n} = U^+_{S^+_{k,n}}$. Then, we have the following

Lemma 6.13. *For all $q \geq 1$, conditionally to $\{S^+_{q,n} \leq \rho^+\}$, $V^+_{1,n}, \cdots, V^+_{q,n}, W$ are independent and $V^+_{1,n}, \cdots, V^+_{q,n}$ have the same law (which depends on n but no longer depends on q).*

Proof. We prove the result by induction on q. For $q = 1$, this has been justified. Suppose the result holds for $q-1$ and let (f_j) be an approximation of ϵ as in the proof of Lemma 6.7 (ii). For a fixed $t \geq 0$,

$$W_{T^+_{q-1,n}, t+T^+_{q-1,n}} = \lim_{j \to \infty} \left(K_{0,t+T^+_{q-1,n}} f_j(1) - K_{0,T^+_{q-1,n}} f_j(1) - \frac{1}{2} \int_0^t K_{0,u+T^+_{q-1,n}} f''_j(1) du \right) \text{ in } L^2(\mathbb{P}).$$

On $\{S^+_{q,n} \leq \rho^+\}$, we have $K_{0,T^+_{q-1,n}}(1) = \delta_1$ and therefore,

$$W_{T^+_{q-1,n}, t+T^+_{q-1,n}} = \lim_{j \to \infty} \left(K_{0,t+T^+_{q-1,n}} f_j(1) - f_j(1) - \frac{1}{2} \int_0^t K_{0,u+T^+_{q-1,n}} f''_j(1) du \right) \quad (6.18)$$

in $L^2(\mathbb{P}(.|S^+_{q,n} \leq \rho^+))$. As $2^{-n} < l$, $\{S^+_{q,n} \leq \rho^+\} = \{T^+_{q-1,n} \leq \rho^+\}$ a.s. Choose a family $\{g_1, \cdots, g_q, g, h\}$ of bounded continuous functions on $\mathbb{R}$. Using (6.18), an application of Lemma 6.10 at time $T^+_{q-1,n}$ shows that

$$\begin{aligned} & E_{(S^+_{q,n} \leq \rho^+)} \left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}}) g(W_{t \wedge T^+_{q-1,n}}) h(W_{T^+_{q-1,n}, t+T^+_{q-1,n}}) \right] \\ &= E_{(S^+_{q,n} \leq \rho^+)} \left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}}) g(W_{t \wedge T^+_{q-1,n}}) \right] E[h(W_t)] E[g_q(U^+_{S^+_{1,n}})]. \end{aligned}$$

Since $\{S^+_{q-1,n} \leq \rho^+\} \subset \{S^+_{q,n} \leq \rho^+\}$, we have by the induction hypothesis

$$E_{(S^+_{q,n} \leq \rho^+)} \left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}}) g(W_{t \wedge T^+_{q-1,n}}) \right] = E_{(S^+_{q-1,n} \leq \rho^+)} \left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}}) \right] E_{(S^+_{q,n} \leq \rho^+)} [g(W_{t \wedge T^+_{q-1,n}})].$$

In conclusion

$$E_{(S^+_{q,n}\le\rho^+)}\left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}})g(W_{t\wedge T^+_{q-1,n}})h(W_{T^+_{q-1,n},t+T^+_{q-1,n}})\right]$$

$$= E_{(S^+_{q-1,n}\le\rho^+)}\left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}})\right] E_{(S^+_{q,n}\le\rho^+)}\left[g(W_{t\wedge T^+_{q-1,n}})h(W_{T^+_{q-1,n},t+T^+_{q-1,n}})\right] E[g_q(U^+_{S^+_{1,n}})].$$

The last identity remains satisfied if we replace $g(W_{t\wedge T^+_{q-1,n}})h(W_{T^+_{q-1,n},t+T^+_{q-1,n}})$ by a finite product $\prod_{i=1}^{k} g^i(W_{t_i\wedge T^+_{q-1,n}})h^i(W_{T^+_{q-1,n},t_i+T^+_{q-1,n}})$. As a result, for all bounded continuous $g : C(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}$,

$$E_{(S^+_{q,n}\le\rho^+)}\left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}})g(W)\right] = E_{(S^+_{q-1,n}\le\rho^+)}\left[\prod_{i=1}^{q-1} g_i(U^+_{S^+_{i,n}})\right] E_{(S^+_{q,n}\le\rho^+)}[g(W)]E[g_q(U^+_{S^+_{1,n}})].$$

Iterating this relation, we get

$$E_{(S^+_{q,n}\le\rho^+)}\left[\prod_{i=1}^{q} g_i(U^+_{S^+_{i,n}})g(W)\right] = \prod_{i=1}^{q} E[g_i(U^+_{S^+_{1,n}})]E_{(S^+_{q,n}\le\rho^+)}[g(W)].$$

In particular, for all $i \in [1,q]$,

$$E_{(S^+_{q,n}\le\rho^+)}[g_i(U^+_{S^+_{i,n}})] = E[g_i(U^+_{S^+_{1,n}})].$$

This completes the proof. □

Let m^+_n be the law of $U^+_{1,n}$ and m^+ be the law of U^+_1 under $\mathbb{P}(.|\rho^+ > 1)$. Then, we have the

Lemma 6.14. *The sequence $(m^+_n)_{n\ge1}$ converges weakly towards m^+. For all $t > 0$, under $\mathbb{P}(\cdot|\rho^+ > t)$, U^+_t and W are independent and the law of U^+_t is given by m^+.*

Proof. For each bounded continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\begin{aligned}
E[f(U^+_t)|W]1_{\{0<t<\rho^+\}} &= \lim_{n\to\infty}\sum_k E\left[1_{\{t\in[S^+_{k,n},T^+_{k,n}[\}}f(V^+_{k,n})|W\right]1_{\{0<t<\rho^+\}} \\
&= \lim_{n\to\infty}\sum_k 1_{\{t\in[S^+_{k,n},T^+_{k,n}\wedge\rho^+[\}}\left(\int f dm^+_n\right) \\
&= \left[1_{\{0<t<\rho^+\}}\lim_{n\to\infty}\int f dm^+_n + \varepsilon_n(t)\right]
\end{aligned}$$

with $\lim_{n\to\infty} \varepsilon_n(t) = 0$ a.s. Consequently

$$\lim_{n\to\infty} \int f dm_n^+ = \frac{1}{\mathbb{P}(\rho^+ > t)} E[f(U_t^+) 1_{\{\rho^+ > t\}}].$$

The left-hand side no longer depends on t, which completes the proof. $\square$

We define the measure m^- by analogy. Let $\rho^- = \inf\{t \geq 0 : W_t^- = l\}$, then for all $t > 0$, under $\mathbb{P}(\cdot|\rho^- > t)$, U_t^- and W are independent and the law of U_t^- is m^-. Recall the definition $\rho = \inf(\rho^+, \rho^-)$. Then the law of U_t^+ (respectively U_t^-) knowing $\{0 < t < \rho\}$ is given by m^+ (respectively m^-). Note that φ^c is constructed such that for all $x, y \in \mathscr{C}$ a.s. $\varphi^c_{0,\cdot}(x)$ and $\varphi^c_{0,\cdot}(y)$ collide whenever they meet. By (6.15), if $x \in \mathscr{C}$ and $\tau_0(x) < \rho$, then $K_{0,t}(x) \in \{K_{0,t}(1), K_{0,t}(e^{il})\}$ if $t \in [\tau_0(x), \rho]$. This entails the following

Lemma 6.15. *Let $\mathbb{P}_{t,x_1,\dots,x_n}$ be the law of $(K_{0,t}(x_1), \dots, K_{0,t}(x_n), W)$ conditionally to $A = \{0 < t < \rho\}$ where $t > 0$ and $x_1, \dots, x_n \in \mathscr{C}$. Then $\mathbb{P}_{t,x_1,\dots,x_n}$ only depends on $\{\mathbb{P}_{u,1}, \mathbb{P}_{u,e^{il}}, u \geq 0\}$.*

Proof. As already observed we only need to show that the law of $(K_{0,t}(1), K_{0,t}(e^{il}), W)$ conditionally to A only depends on $\{\mathbb{P}_{u,1}, \mathbb{P}_{u,e^{il}}, u \geq 0\}$. Let

$$g_t^\pm = \sup\{u \in [0,t] : W_u^\pm = 0\}$$

and write

$$A = (A \cap \{g_t^+ < g_t^-\}) \cup (A \cap \{g_t^- < g_t^+\}).$$

Then we may replace A by $A \cap \{g_t^+ < g_t^-\}$. For all $n \geq 0$, let $\mathbb{D}_n = \{\frac{k}{2^n},\ k \in \mathbb{N}\}$ and $\mathbb{D} = \cup_{n\in\mathbb{N}} \mathbb{D}_n$. Define for $0 \leq u < v$,

$$n(u,v) = \inf\{n \in \mathbb{N} : \mathbb{D}_n \cap]u, v[\neq \emptyset\}$$

and

$$f(u,v) = \inf(\mathbb{D}_{n(u,v)} \cap]u,v[), \quad S = f(g_t^-, g_t^+).$$

By summing over all possible values of S, we only need to show that the law of $(K_{0,t}(1), K_{0,t}(e^{il}), W)$ conditionally to $E_s = A \cap \{g_t^- < s < g_t^+\}$ where $s \in \mathbb{D}$ is fixed only depends on $\{\mathbb{P}_{u,1}, \mathbb{P}_{u,e^{il}}, u \geq 0\}$. On E_s, we have $y := \varphi^c_{0,s}(1) \in \{e^{iW^+_s}, e^{-iW^+_s}\}$ by (6.16) and

$$\tau_s(y) = \inf\{r \geq s : W_r - \mathrm{m}^+_{0,s} = 0\} = \inf\{r \geq s : W^+_r = 0\} = g^+_t.$$

Clearly $\varphi^c_{s,\tau_s(y)}(y) = \varphi^c_{s,\tau_s(y)}(1) = 1$ and a fortiori $\varphi^c_{s,r}(y) = \varphi^c_{s,r}(1)$ for all $r \geq \tau_s(y)$. The flow property of φ^c, yields, on E_s a.s.

$$\varphi^c_{0,t}(1) = \varphi^c_{s,t}(y) = \varphi^c_{s,t}(1).$$

Using (6.15), we get $K_{0,t}(1) = K_{s,t}(1)$ a.s. on E_s. This shows that

$$(K_{0,t}(1), K_{0,t}(e^{il}), W)1_{E_s} = (K_{s,t}(1), K_{0,t}(e^{il}), W)1_{E_s}.$$

On E_s, $K_{0,t}(e^{il})$ is a measurable function of (U^-_s, W) because $r \longmapsto U^-_r$ is constant on the excursions of W^- on $[0, \rho]$. Now the law of $(U^-_s, K_{s,t}(1), W)$ on $E_s (= E_s \cap \{t \leq \rho_s\})$ only depends on $\{\mathbb{P}_{t-s,1}, \mathbb{P}_{s,e^{il}}\}$ which finishes the proof. $\square$

Proposition 6.8. *Let (K^{m^+,m^-}, W') be the solution constructed in Section 6.2.3 associated with (m^+, m^-). Then $K \overset{law}{=} K^{m^+,m^-}$.*

Proof. For simplicity, we will note W' also by W. Then $(K_{0,t}(x), W) \overset{law}{=} (K^{m^+,m^-}_{0,t}(x), W)$ conditionally to $\{0 < t < \rho\}$ for $x = 1, e^{il}$ and consequently for all $x \in \mathscr{C}$. By following the same steps as in the proof of Lemma 6.15, we show that

$$(K_{0,t}(1), K_{0,t}(e^{il}), W) \overset{law}{=} (K^{m^+,m^-}_{0,t}(1), K^{m^+,m^-}_{0,t}(e^{il}), W)$$

conditionally to $\{0 < t < \rho\}$ and so $K_{0,t} \overset{law}{=} K^{m^+,m^-}_{0,t}$ conditionally to $\{0 < t < \rho\}$. Of course for all $s \in \mathbb{R}$, we still have $K_{s,t} \overset{law}{=} K^{m^+,m^-}_{s,t}$ conditionally to $\{s < t < \rho_s\}$. Let $t > 0, t^n_i = \frac{it}{n}, n \geq 1, i \in [0, n]$ and define $A_{n,i} = \{t^n_i \leq \rho_{t^n_{i-1}}\} \in \mathcal{F}^W_{t^n_{i-1}, t^n_i}$, $A_n = \cap^n_{i=1} A_{n,i}$. Then

$$(K_{0,t^n_1}, \cdots, K_{t^n_{n-1},t}) \overset{law}{=} (K^{m^+,m^-}_{0,t^n_1}, \cdots, K^{m^+,m^-}_{t^n_{n-1},t}) \text{ on } A_n.$$

Recall that $\mathbb{P}(A^c_n) \to 0$ as $n \to \infty$ (see the proof of Corollary 6.2). Letting $n \to \infty$ and using the flow property for both K and K^{m^+,m^-}, we deduce that $K_{0,t} \overset{law}{=} K^{m^+,m^-}_{0,t}$ $\square$

Open problems

(1) Consider the oriented graph of Figure 6.2 and classify the solutions of

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}f'(x)dW_u + \frac{1}{2}\int_s^t K_{s,u}f''(x)du, f \in D \qquad (6.19)$$

where D is the space of all functions f whih are C^2 on the graph excepted the set of vertexes and which satisfy a boundary condition at each vertex similarly to Figure 4.4.

We conjecture that the solutions of 6.19 are classified by a family of pairs of measures

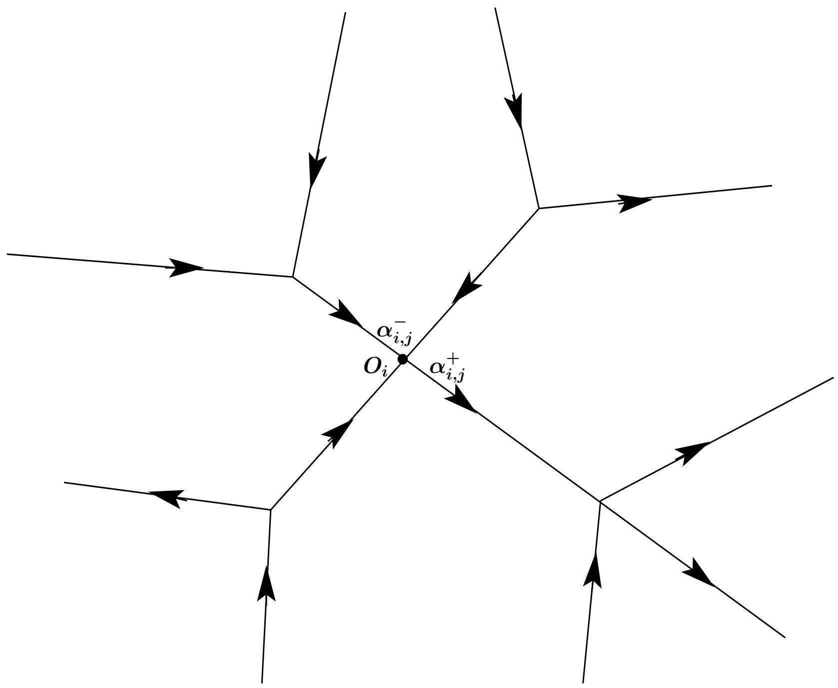

Figure 6.2: A general oriented graph

$\{(m_O^+, m_O^-), O \in V\}$ where V is the vertex set of the graph.

(2) Study $\mathrm{supp}(K_{0,\cdot}^{m^+,m^-}(1))$ constructed in the last chapter.

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