

Les langues de Arnold de la famille standard
double

Explosion des cycles dans la famille $z^2 + \lambda$

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Plan

1. Arnold tongues of the double standard map family
2. Growth of the coefficients of the Laurent series of the inverse of the Böttcher map in the escaping case
3. Levin inequality on the position of the critical values of the multiplier map
4. Non local connectedness criterium for the Julia set of an infinite satellite renormalizable quadratic polynomial
5. Geometrical model of the explosion of cycles

Double standard map

The double standard maps family has been defined by Michał Misiurewicz and Ana Rodrigues as the following family of maps of $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ ($x \in \mathbb{T}^1$)

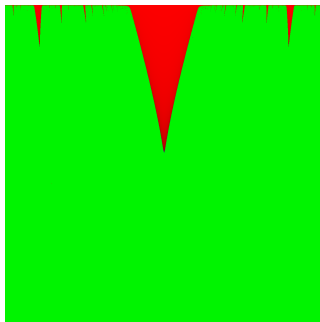
$$f_{a,b}(x) = 2x + a - \frac{b}{\pi} \sin(2\pi x) \pmod{1}.$$

It is parametrized by the two real parameters $a \in \mathbb{T}^1$ and $b \in [0, 1]$. This is a real family of perturbations of the family of double covering of the circle $x \mapsto 2x + a$.

Double standard map

Definition (Misiurewicz-Rodrigues)

A **tongue** of the double standard map family is a connected component of the set Ω of $(a, b) \in \mathbb{T}^1 \times [0, 1]$ such that $f_{a,b}$ has an attracting cycle.



Double standard map

In order to study these sets, one needs to introduce the notion of **type**.

- ▶ There exists a unique map $\varphi_{a,b} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ which is continuous, of degree 1 and preserving the orientation such that

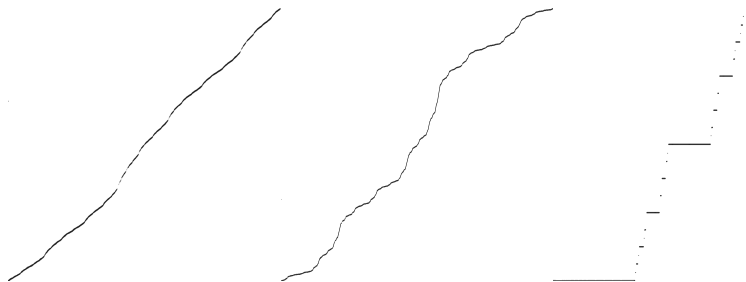
$$\varphi_{a,b} \circ f_{a,b}(x) = 2\varphi_{a,b}(x) \pmod{1}.$$

This semiconjugacy depends continuously on (a, b) .

- ▶ Using the exponential map, the mappings $f_{a,b}$ can be extended into the holomorphic maps $g_{a,b} : \mathbb{C}^* \rightarrow \mathbb{C}^*$ defined by

$$g_{a,b}(z) = e^{2a\sqrt{-1}\pi} z^2 e^{-b(z-1/z)}.$$

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Exemples de graphes de $\varphi_{a,b}$ avec
 $(a, b) = (0.64, 0.22)$, $(0.33, 0.59)$ et $(0, 0.99)$

Double standard map

- ▶ The mappings $g_{a,b}$ are symmetric with respect to the unit circle in the following sense : $g_{a,b}(1/\bar{z}) = 1/\overline{g_{a,b}(z)}$.
- ▶ Thanks to this symmetry and to the structure of the singular set, the mapping $g_{a,b}$ has at most one attracting or parabolic cycle on the circle.

Definition (Misiurewicz-Rodrigues)

*If there is such a cycle, the mapping $f_{a,b}$ is said to be of **type** τ where τ is the image of the unique critical component of the basin by the semiconjugacy $\varphi_{a,b}$.*

The only possible types belongs to the set of periodic cycles of the doubling map $(\text{ mod } 1) x \mapsto 2x$.

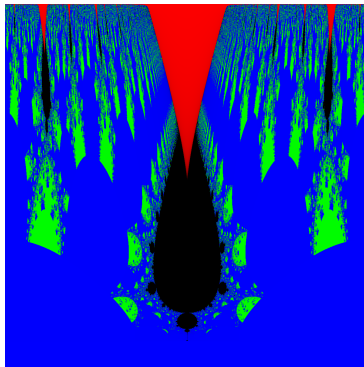
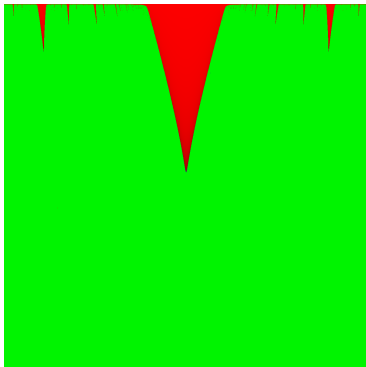
Double standard map

Theorem (D.)

For any possible type τ , there exists one and only one connected component of the set Ω for which the corresponding mappings are all of type τ .

As a consequence, the tongues are uniquely labelled with type. The proof consists in making a path inside the tongue connecting the starting parameter to a fixed parameter and keeping the type constant. This is produced by quasiconformal deformation.

Les langues de Arnold de la famille standard double



Infinitely satellite renormalizable quadratic polynomials

Theorem (Douady-Sorensen)

There exist parameters $\lambda \in \mathbb{C}$ such that the quadratic polynomial $P_\lambda(z) = z^2 + \lambda$ is infinitely renormalizable and such that its Julia set J_λ is not locally connected.

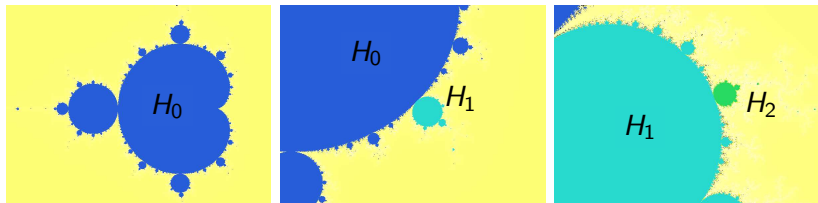
Remark

Here, the renormalizations are satellite renormalizations.

Infinitely renormalizable quadratic polynomials

- ▶ Start with a hyperbolic component H_0 of the Mandelbrot set M . It is the main component of a copy of M . Then proceed by induction, that is, given a hyperbolic component H_n , choose an adjacent hyperbolic component H_{n+1} which is not H_{n-1} (it is then the main component of a copy of a smaller copy of M).
- ▶ It yields a decreasing sequence of copies of M and parameters λ (or corresponding maps P_λ) which belongs to the intersection of such a sequence are said to be infinitely satellite renormalizable (ISR).

Infinitely renormalizable quadratic polynomials



Encoding ISR quadratic polynomials

Such a sequence of renormalization can be encoded by the starting component H_0 and a sequence of rotation numbers $\left(\frac{p_n}{q_n}\right)_n \in \mathbb{Q}^{\mathbb{N}}$ with $\left|\frac{p_n}{q_n}\right| \leq \frac{1}{2}$. The numbers p_n/q_n are determined as follows.

- ▶ For all $\lambda \in H_n$, the polynomial P_λ has an attracting periodic cycle of period m_n .
- ▶ The multiplier map $\rho_n : H_n \rightarrow \mathbb{D}$ is a holomorphic isomorphism and it extends injectively to the boundary.
- ▶ The closures of the components H_n and H_{n+1} intersect at a parameter λ_n such that $\rho_n(\lambda_n) = e^{2\pi i p_n/q_n}$.

One can try to quantify the previous theorem using this encoding. That is, try to find the most general condition on the sequence of rotation numbers $\left(\frac{p_n}{q_n}\right)_n$ that implies non local connectedness of the Julia set of the corresponding polynomial.

Thanks to renormalization techniques, this condition will depend only on the tail of the sequence $\left(\frac{p_n}{q_n}\right)_n$.

An explicit condition for non local connectedness

Theorem (G. Levin, 2007)

Let H_0 be a hyperbolic component of M and $\left(\frac{p_n}{q_n}\right)_n$ a sequence of rational numbers such that $p_n/q_n \rightarrow 0$.

If

$$\limsup \left| \frac{p_{n+1}}{q_{n+1}} \right|^{1/q_n} < 1,$$

then

- ▶ the sequence of parameters λ_n (defined by the data $\left(H_0, \left(\frac{p_n}{q_n}\right)_n\right)$) converges to a parameter λ_* such that P_{λ_*} is ISR,
- ▶ the Julia set of P_{λ_*} is not locally connected.

General view of the proof

1. Show that the sequence $(\lambda_n)_n$ converges,
2. apply Douady-Sullivan criterium.

Douady-Sullivan criterium

Theorem (Douady-Sullivan criterium)

Let X be a compact subset of the Julia set of some quadratic polynomial P such that :

- ▶ $X \cap \text{crit}(P) = \emptyset$,
- ▶ $P : X \rightarrow X$ is bijective.

Then, if the set X is infinite, the Julia set of P is not locally connected.

The set X we will use in the proof of Levin's theorem will be the closure of the union of a sequence of periodic cycles of P_{λ_*} .

Some main ideas of the proof

We find a decreasing sequence of topological disks D_n in the parameter space such that :

- ▶ $\lambda_n \in D_n$,
- ▶ $\text{diam}(D_n) \rightarrow 0$,
- ▶ D_n is the preimage of a round disk $D\left(2\pi i \frac{p_n}{q_n}, r_n\right)$ by an inverse branch of the mapping $\log \rho_n$.

Some main ideas of the proof

The radius r_n is chosen so that :

- ▶ the function $\log \rho_n : D_n \rightarrow D\left(2\pi i \frac{p_n}{q_n}, r_n\right)$ is univalent,
- ▶ explosions functions can be defined on the disk $D(0, r_n^{1/q_n})$.

The latter "means" the following. We call $C_n(\lambda)$ the cycle of the polynomial P_λ which is attracting for $\lambda \in H_n$.

Then, on $D_n \setminus \{\lambda_n\}$ the cycle $C_{n+1}(\lambda)$ can be followed locally holomorphically (while it collapses into $C_n(\lambda)$ for $\lambda = \lambda_n$).

Some main ideas of the proof

We set

$$X = \overline{\bigcup_n C_n(\lambda_*)},$$

where $\lambda_* = \lim \lambda_n$.

In order to apply Douady-Sullivan criterium we have to show that $C_n(\lambda)$ avoids a fixed neighbourhood of the critical point.

Recall that we suppose $\left| \frac{\rho_{n+1}}{q_{n+1}} \right|^{1/q_n} \leq \alpha$ for some $\alpha < 1$.

Let $d_n = |\rho_n(\lambda_*) - \rho_n(\lambda_n)|$ and suppose $\left(\frac{d_n}{r_n} \right)^{1/q_n} \leq \beta$ for some fixed $\beta < 1$.

- ▶ By a compactness argument : $\frac{\max \text{dist}(C_{n+1}, C_n)}{\min \text{dist inside } C_n} \leq \text{Const.}$
- ▶

$$\begin{aligned} \min \text{dist inside } C_n &= \text{dist between two "consecutive" points} \\ &= |C_n^+ - C_n| \end{aligned}$$

as a holomorphic function defined on $D(0, r_{n-1}^{1/q_{n-1}})$ (explosion functions).

The latter can be estimated via the hyperbolic metric on disks.

Then we have :

$$|C_{n+1} - C_n| \leq \frac{Const}{q_{n-1}} \left| \frac{d_{n-1}}{r_{n-1}} \right|^{1/q_{n-1}} .$$

We obtain the following inequality.

$$\sum_n |C_{n+1} - C_n| \leq \sum_n \frac{Const}{q_n} < \infty .$$

What remains to be shown is that $\left(\frac{d_n}{r_n}\right)^{1/q_n} \leq \beta$ (for some fixed $\beta < 1$).

► Thanks to good knowledge (Yoccoz inequalities) $r_n^{1/q_n} \rightarrow 1$,

► $d_n = |\rho_n(\lambda_*) - \rho_n(\lambda_n)| \asymp \frac{1}{q_n^2} \left| \frac{p_{n+1}}{q_{n+1}} \right|$.

As a consequence,

$$\begin{aligned} \left(\frac{d_n}{r_n}\right)^{1/q_n} &\leq (1 + o(1)) \left| \frac{p_{n+1}}{q_{n+1}} \right|^{1/q_n} \\ &\leq (1 + o(1))\alpha. \end{aligned}$$

Take $\alpha < \beta < 1$.

Perspectives

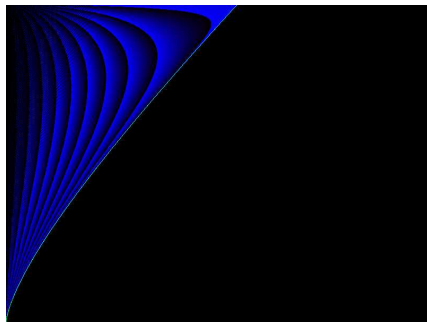
- ▶ Order of contact at the tip of the tongues (transversality)
- ▶ Extension of Levin's result to the general unicritical families $z^d + \lambda$
- ▶ Find an optimal condition for non local connectedness (model)
- ▶ ...

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Étapes de la démonstration :

1. Existence d'un unique (a_τ, b_τ) tel que f_{a_τ, b_τ} est de type τ avec un cycle superattractif (on a alors $b_\tau = 1$).
2. Étant donné g_{a_0, b_0} de type τ non superattractif, construction d'une famille d'applications g_t holomorphes sur \mathbb{C}^* par déformation quasiconforme de g_{a_0, b_0} , paramétrée par le multiplicateur du cycle $t \in]0, 1[$.
3. Grâce à une caractérisation topologique de la famille $(g_{a, b})_{a, b}$, on peut voir cette famille comme un chemin dans la famille standard double.
4. Ce chemin est continu (réel analytique) et son extrémité quand $t \rightarrow 0$ est le paramètre (a_τ, b_τ) .

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Tracé des chemins construits dans la démonstration du théorème de connexité.