Un schéma de dualité pour les problèmes d'inéquations variationnelles

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RESUMÉ

Dans cette thèse, nous avont preésenté un schéma général de dualité pour des problèmes d'inéquations variationelles monotones. Cet schéma est analogue le schéma classique de dualité dans la programmation convexe en ajoutant des variables de perturbation.

Afin d'arriver á cet objetif, avant nous avons approfondi quelques propriétés et caractérisations des multi-applicationes (sous-ensemble) monotones et maximal monotones sur un point de vue global et local. En particulier, nous donnons un algorithme pour construire une extension maximal monotone d'une multi-application mnotone (sous-ensemble) arbitraire.

Nous avons spécifiquement étudié les sous-espaces affine monotone. Dans ce cas particulier, la construction d'une extension maximal monotone peut être construit par un nombre fini d'étapes.

Finalement, des applications de notre schéma de dualité quelques classes des problèmes d'inéquations variationnels sont discutées.

ABSTRACT

In this thesis, we construct a general duality scheme for monotone variational inequality problems. This scheme is analogous to the classical duality scheme in convex programming in the sense that the duality is obtained by adding perturbation variables.

In order to reach this goal, we have before deepened some properties and characterizations of monotone and maximal monotone multi-valued maps (subsets) on a global and a local point of view. In particular, we give an algorithm for constructing a maximal monotone extension of an arbitrary monotone map.

We have specifically studied monotone affine subspaces. In this particular case, the construction of a maximal monotone extension can be processed within a finite number of steps.

Finally, applications of our duality scheme to some classes of variational inequality problems are discussed.

Contents

Introduction

The finite-dimensional variational inequality problem (VIP)

$$\text{Find } \bar{x} \in C \text{ such that } \exists\, \bar{x}^* \in \Gamma(\bar{x}) \text{ with } \langle \bar{x}^*, x - \bar{x} \rangle \geq 0 \quad \forall\, x \in C,$$

where C is a non-empty closed convex subset of $\mathbb{R}^n$ and $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ a multi-valued map, provides a broad unifying setting of the study of optimization and equilibrium problems and serves as the main computational framework for the practical solution of a host of continuum problems in mathematical sciences.

The subject of variational inequalities has its origin in the calculus of variations associated with the minimization of infinite-dimensional functions. The systematic study of the subject began in the early 1960s with the seminal work of the Italian mathematician Guido Stampacchia and his collaborators, who used the variational inequality as an analytic tool for studying free boundary problems defined by non-linear partial diferential operators arising from unilateral problems in elasticity and plasticity theory and in mechanics. Some of the earliest papers on variational inequalities are [16, 21, 22, 40, 41]. In particular, the first theorem of existence and uniqueness of the solution of VIs was proved in [40].

The development of the finite-dimensional variational inequality and non-linear inequality problem also began in the early 1960 but followed a different path. Indeed, the non-linear complementarity problem was first identified in the 1964 Ph.D. thesis of Richard W. Cottle [8], who studied under the supervision of the eminent George B. Dantzig, "father of linear programming".

A brief account of the history prior to 1990 can be found in the introduction in the survey paper [15].

Related to monotone maps, Kachurovskii [17] was apparently the first to note that the gradients of differentiable convex functions are monotone maps, and he coined the term "monotonicity" for this property. Really, though, the theory of monotone mapping began with papers of Minty [25], [26], where the concept was studied directly in its full scope and the significance of maximality was brought to light.

His discovery of maximal monotonicity as a powerful tool was one of the main impulses, however, along with the introduction of sub-gradients of convex functions, that led to the resurgence of multivalued mapping as acceptable objects of discourse, especially in variational analysis.

The need for enlarging the graph of a monotone mapping in order to achieve maximal monotonicity, even if this meant that the graph would no longer be function-like, was clear to him from his previous work with optimization problems in networks, Minty [24], which revolved around the one dimensional case of this phenomenon; cf. 12.9 [39].

Much of the early research on monotone mapping was centered on infinite dimensional applications to integral equations and differential equations. The survey of Kachurovskii [18] and the book of Brézis [4] present this aspect well. But finite-dimensional applications to numerical optimization has also come to be widespread particularly in schemes of decomposition, see for example [7] and references therein.

Duality framework related to variational inequality problems has been established by many researchers [1],[2],[11],[14],[27],[37],[38]. For example, in [27] Mosco studied problems of the form

$$\text{Find } \bar{x} \in \mathbb{R}^n \text{ such that } 0 \in \Gamma(\bar{x}) + \partial g(\bar{x}), \tag{1}$$

where Γ is considered maximal monotone and g a proper lsc convex function and show that one can always associate a dual problem with (1) defined by

$$\text{Find } \bar{u}^* \in \mathbb{R}^n \text{ such that } 0 \in -\Gamma^{-}(-\bar{u}^*) + \partial g^*(\bar{u}^*), \tag{2}$$

where g^* denotes the Fenchel conjugate of g.

In this article, he shows that x solves (1) if and only if $u^* \in \Gamma(x)$ solve (2).

The above dual formulation and dual terminology is justified by the fact that this scheme is akin to the Fenchel duality scheme used in convex optimization problems. Indeed, if Γ is the subdifferential of some proper lsc convex function f, the formulations (under appropiate regularity conditions [35]) (1) and (2) are nothing else but the optimality conditions for the Fenchel primal-dual pair of convex optimization problems

$$\min\{\, f(x) + g(x) : \, x \in \mathbb{R}^n \,\}, \quad \min\{\, f^*(-u^*) + g^*(u^*) : \, u^* \in \mathbb{R}^n \,\}.$$

In contrast with this author, Auslender and Teboulle [2] established a dual framework related to Lagrangian duality (formally equivalent to Mosco's scheme), in order to produce two methods of multipliers with interior multiplier updates based on the dual and primal-dual formulations of VIP.

This duality formulation takes its inspiration in the classical Lagrangian duality framework for constrained optimization problems. In this case the closed convex subset C is explicitly defined by

$$C := \{\, x \in \mathbb{R}^n : \, f_i(x) \le 0, \, i = 1, \cdots, m \,\},$$

where $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \cdots, m$, are given proper lsc convex functions.

In this context, the dual framework established by Auslender and Teboulle is defined as

$$\text{Find } \bar{u}^* \in \mathbb{R}^m \text{ such that } \left\{ \begin{array}{c} \exists\, x \in \mathbb{R}^n \text{ with } \bar{u}^* \ge 0 \text{ and} \\ 0 \in \Gamma(x) + \sum_{i=1}^m \bar{u}_i^* \partial f_i(x) \\ 0 \in -F(x) + N_{\mathbb{R}^m_+}(\bar{u}^*), \end{array} \right. \qquad (DVP)$$

where $F(x) = (f_1(x), \cdots, f_m(x))^t$. Associated to VIP and DVP, they also introduce a primal-dual formulation defined by

$$\text{Find } (\bar{x}, \bar{u}^*) \in \mathbb{R}^n \times \mathbb{R}^m \text{ such that } (0,0) \in S(\bar{x}, \bar{u}^*), \qquad (SP)$$

where $S : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ is a multivalued map defined by

$$S(x, u^*) = \{\, (x^*, u) : \, x^* \in \Gamma(x) + \sum_{i=1}^m u_i^* \partial f_i(x), \; u \in -F(x) + N_{\mathbb{R}^m_+}(u^*) \,\}.$$

These three formulations (under Slater's condition for the constraint set C) has the following relations: x solves VIP if and only if there exists $u^* \in \mathbb{R}^m$ such that (x, u^*) solves SP. In this case u^* solves DVP.

The duality scheme we have introduced in this work (Chapter 4) takes also its inspiration in the duality in convex optimization but by adding perturbation variables. But before we observed that the monotonicity of a multivalued map $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ is a property lying on its graph which is a subset of $\mathbb{R}^n \times \mathbb{R}^n$. In particular, the inverse map Γ^{-1}, which shares the same graph than Γ, is monotone when Γ is so. Consequently we say that the graph is monotone. Thus, given the variational inequality problem VIP (primal problem)

$$\text{Find } x \in \mathbb{R}^n \text{ such that } (x, 0) \in F_p, \tag{V_p}$$

where F_p is a monotone subset of $\mathbb{R}^n \times \mathbb{R}^n$, we introduce a perturbed subset $\Phi \subset (\mathbb{R}^n \times \mathbb{R}^m) \times (\mathbb{R}^n \times \mathbb{R}^m)$ which is such that

$$(x, x^*) \in F_p \iff \exists\, u^* \in \mathbb{R}^m \text{ such that } ((x, 0), (x^*, u^*)) \in \Phi.$$

Then (V_p) is equivalent to

$$\text{Find } x \in \mathbb{R}^n \text{ such that } \exists\, u^* \in \mathbb{R}^m \text{ with } ((x, 0), (0, u^*)) \in \Phi. \tag{V_p}$$

This last formulation leads to consider the following problem

$$\text{Find } (x, u^*) \in \mathbb{R}^n \times \mathbb{R}^m \text{ such that } \exists\, u^* \in \mathbb{R}^m \text{ with } ((x, 0), (0, u^*)) \in \Phi$$

and next to define the subset $F_d \subset \mathbb{R}^m \times \mathbb{R}^m$ defined by

$$(u, u^*) \in F_d \iff \exists\, x \in \mathbb{R}^n \text{ such that } ((x, 0), (x^*, u^*)) \in \Phi.$$

Our dual variational inequality problem is defined as

$$\text{Find } u^* \in \mathbb{R}^n \text{ such that } (0, u^*) \in F_d. \tag{V_d}$$

Perturbed problems associated to (V_p) and (V_d) are also discussed. These problems are formulated as follows: For any $u \in \mathbb{R}^m$ and $x^* \in \mathbb{R}^n$ the primal perturbed problem (V_p^u) and the dual perturbed problem $(V_d^{x^*})$ are

$$\text{Find } x \in \mathbb{R}^n \text{ such that } \exists\, u^* \in \mathbb{R}^m \text{ with } ((x, u), (0, u^*)) \in \Phi$$

and

$$\text{Find } u^* \in \mathbb{R}^m \text{ such that } \exists\, x \in \mathbb{R}^n \text{ with } ((x, 0), (x^*, u^*)) \in \Phi,$$

respectively. If Φ is monotone, all the variational problems considered above are monotone.

As particular examples (Chapter 5) we recover the dual frameworks studied by Mosco [27] and Auslender and Teboulle [2].

The thesis is divided into 5 chapters:

The first chapter is devoted to set up notations and review some facts of convex analysis; we describe in details the duality scheme in convex programming. Some facts on multivalued maps are also reviewed.

In Chapter 2, we develop some new general theoretical results on monotone and maximal monotone multi-valued maps (subsets) which will be needed in our further analysis but are also of interest by themselves. Indeed, in contrast to the existing literature, new general tools of multi-valued maps (subsets) are considered in order to characterize and/or study the behaviour of monotone and maximal monotone from a global and a local point of view. In this sense, many important results related to monotone and maximal monotone are recovered as direct consequences of this new approach. In section 2.4, we present an algorithm to construct a maximal monotone extension of an arbitrary monotone map (subset).

Chapter 3, is devoted to the study of monotone and maximal monotone affine subspaces. Monotonicity and maximal monotonicity of affine subspaces are explicitly characterized by means of the eigenvalues of bordered matrices associated to these subspaces. From this characterization, we prove that any maximal monotone affine subspace can be written (under permutations of variables) as the graph of an affine map associated to a positive semi-definite matrix. The algorithm developed in the previous chapter for constructing a maximal monotone extension is significantly refined. For such subsets, the maximal monotone affine extension thus constructed is obtained in a finite number of steps.

In chapter 4, we develop our duality scheme. Primal perturbed problems, the dual problem, perturbed dual problems and the Lagrangian problem are formulated. A natural condition related to the stability for these problems is given.

In Chapter 5 we apply our duality scheme to some classes of variational inequality problems: complementarity problems, non-linear complementarity problems, etc.

Chapter 1

Notation and background of convex analysis

1.1 Preliminaries and notation

In this thesis, we assume that X and U are two finite dimensional linear spaces. Among other reasons for limiting our study to finite dimensional spaces is the fact that we shall make use frequently the concept of relative interior, we know that this concept is well adapted in a finite dimensional setting (the relative interior of a convex $C \subset \mathbb{R}^n$ is convex and non-empty when C is non-empty), but, it does not work in the infinite dimensional setting. Of course variational inequality problems appear in the infinite dimensional setting, but their treatment needs complex technical requirements that we want avoid in this first approach of duality for variational inequality problems.

We denote by X^* and U^* the dual spaces of X and U. Of course $X = X^*$ and $U = U^*$ but in order to put in evidence the distinct roles played by these spaces, we use the four symbols X, U, X^* and U^*. In this spirit $\langle \cdot, \cdot \rangle$ denotes both the duality product between the space and its dual and the classical inner product on the space.

Given $C \subset X$, we denote by $\mathrm{cl}\,(\mathrm{C})$, $\mathrm{int}\,(\mathrm{C})$, $\mathrm{ri}\,(\mathrm{C})$, $\mathrm{bd}\,(\mathrm{C})$, $\mathrm{rbd}\,(\mathrm{C})$, $\mathrm{co}\,(\mathrm{C})$, $\overline{\mathrm{co}}\,(C)$ and $\mathrm{aff}\,(\mathrm{C})$ *the closure*, *the interior*, *the relative interior*, *the boundary*, *the relative boundary*, *the convex hull*, *the closure of the convex hull* and *the affine hull* of a set C, respectively.

The *orthogonal subspace* to $C \subset X$ is defined by

$$C^{\perp} := \{\, y \in X : \langle y, x \rangle = 0 \;\text{ for all }\; x \in C \,\}.$$

Given $a \in X$, we denote by $\mathcal{N}(a)$ the family of neighborhoods of a.

Given a closed set $C \subset X$, we denote by $\mathrm{proj}_C(x)$ *the projection* of x onto C, which is the set of all points in C that are the closest to x for a given norm, that is

$$\mathrm{proj}_{\mathrm{C}}(\mathrm{x}) := \{\, \bar{\mathrm{y}} \in \mathrm{C} : \|\mathrm{x} - \bar{\mathrm{y}}\| = \inf_{\mathrm{y} \in \mathrm{C}} \|\mathrm{x} - \mathrm{y}\| \,\}.$$

Unless otherwise specified, the norm used is the Euclidean norm. For this norm, when C is closed and convex, $\mathrm{proj}_C(x)$ is reduced to a singleton. In fact, this property can be used to characterize the closed convex subsets of X. Indeed C is closed and convex if and only if the projection operator $\mathrm{proj}_C(\cdot)$, is single-valued on X [5], [28].

Given a closed convex set C, the *normal cone* and the *tangent cone* to C at x, denoted respectively by $N_C(x)$ and $T_C(x)$ are defined by

$$N_C(x) := \begin{cases} \{\, x^* : \langle x^*, y - x \rangle \leq 0, \;\; \forall\, y \in C \,\} & \text{if } \; x \in C, \\ \\ \emptyset & \text{if not} \end{cases}$$

and

$$T_C(x) = \{\, v : \exists\, \{x_k\} \subset C,\, \{t_k\} \subset \mathbb{R},\, x_k \to x,\, t_k \to 0^+ \text{ and } \frac{x_k - x}{t_k} \to v \,\}.$$

We shall use the following convention:

$$A + \emptyset = \emptyset + A = \emptyset, \;\; \text{for any set } A.$$

1.2 Convex analysis

Given $f : X \to (-\infty, +\infty]$, we say that f is convex if its epigraph

$$\mathrm{epi}\,(\mathrm{f}) = \{\, (\mathrm{x}, \alpha) \in \mathrm{X} \times \mathbb{R} : \mathrm{f}(\mathrm{x}) \leq \alpha \,\}$$

is convex in $X \times \mathbb{R}$. We say that f is concave if $(-f)$ is convex. We say that f is lower semi-continuous (lsc in short) at $\bar{x}$ if for every $\lambda \in \mathbb{R}$ such that $f(\bar{x}) > \lambda$ there exists a neighborhood $V \in \mathcal{N}(\bar{x})$ such that $x \in V$, implies $f(x) > \lambda$. f is said to be lsc if it is lsc at every point of X. This is equivalent to saying that its epigraph is closed in $X \times \mathbb{R}$. The function f is said to be upper semi-continuous (usc in short) if $(-f)$ is lsc. A convex function f is said to be proper if $f(x) > -\infty$ for every $x \in X$ and its domain

$$\mathrm{dom}\,(\mathrm{f}) = \{\, \mathrm{x} \in \mathrm{X} : \mathrm{f}(\mathrm{x}) < +\infty \,\}$$

is nonempty. Note that if f is a convex function, then $\mathrm{dom}\,(\mathrm{f})$ is a convex set.

Given $f : X \to (-\infty, +\infty]$, its *Fenchel-conjugate* is

$$f^*(x^*) = \sup_{x \in X} [\langle x^*, x \rangle - f(x)],$$

and its *biconjugate* is

$$f^{**}(x) = \sup_{x^* \in X^*} [\langle x^*, x \rangle - f^*(x^*)] = \sup_{x^* \in X^*} \inf_{z \in X} [\langle x^*, x - z \rangle + f(z)].$$

By construction f^* and f^{**} are two convex and lsc functions, and $f^{**}(x) \leq f(x)$ for all $x \in X$. A crucial property is the following (see for instance [3],[10],[35], etc.).

Proposition 1.2.1 *Assume that f is a proper convex function on X and lsc at $\bar{x}$. Then $f^{**}(\bar{x}) = f(\bar{x})$.*

Assume that f is a proper convex function and $x \in X$. The *subdifferential* of f at x is the set $\partial f(x)$ defined by

$$\partial f(x) = \{\, x^* \in X^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \text{ for all } \; y \in X \,\}$$

or equivalently, using the definition of the conjugate,

$$\partial f(x) = \{\, x^* \in X^* : f(x) + f^*(x^*) \leq \langle x^*, x \rangle \,\}.$$

Clearly, $\partial f(x) = \emptyset$ if $x \notin \mathrm{dom}\,(\mathrm{f})$ or if f is not lsc at x.

By construction, the set $\partial f(x)$ is closed and convex for all $x \in X$. Also $\partial f(x)$ is bounded and nonempty on the interior of dom (f).

The domain of ∂f and its graph are the sets

$$\mathrm{dom}\,(\partial \mathrm{f}) = \{\, \mathrm{x} \in \mathrm{X} : \partial \mathrm{f}(\mathrm{x}) \neq \emptyset \,\},$$

$$\mathrm{graph}\,(\partial \mathrm{f}) = \{\, (\mathrm{x}, \mathrm{x}^*) \in \mathrm{X} \times \mathrm{X}^* : \mathrm{x}^* \in \partial \mathrm{f}(\mathrm{x}) \,\}.$$

Clearly dom $(\partial \mathrm{f}) \subset$ dom (f) but in general these sets do not coincide, moreover dom $(\partial \mathrm{f})$, unlike dom (f), may be not convex when f is convex as seen from the following example taken from [35]

Example 1.2.1 *Let us define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f(x_1, x_2) = \begin{cases} max\{|x_1|,\ 1 - \sqrt{x_2}\} & \textit{if} \ \ x_2 \geq 0, \\ \\ +\infty & \textit{if not} . \end{cases}$$

It is easily seen that f is convex proper and lsc,

$$\mathrm{dom}\,(\partial \mathrm{f}) = (\mathbb{R} \times [0, +\infty[\ )\ \setminus\ (\ ]-1, +1[\times\{0\})$$

which is not convex and do not coincide with dom (f).

However it is known that, for a convex function f, the interior (the relative interior) of dom $(\partial \mathrm{f})$ is convex and coincides with the interior (the relative interior) of $dom(f)$.

Another very important property of the sub-differential of a convex function f is the property called *cyclic-monotonicity*, i.e., for any finite family $\{(x_i, x_i^*),\ i = i_0, i_1, \cdots, i_{k+1}\}$ contained in the graph of ∂f such that $i_0 = i_{k+1}$, the following inequality holds:

$$\sum_{i=0}^{k} \langle x_{j_i}^*, x_{j_{i+1}} - x_{j_i} \rangle \leq 0.$$

In particular, for every $x_1^* \in \partial f(x_1)$ and $x_2^* \in \partial f(x_2)$, we have

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0,$$

which corresponds to the classical *monotonicity* of ∂f.

Now, a few words on the continuity properties of the subdifferential. Recall that a multivalued map $\Gamma : X \overset{\rightarrow}{\longrightarrow} X^*$ is said to be closed if its graph

$$\mathrm{graph}\,(\Gamma) = \{\, (\mathrm{x}, \mathrm{x}^*) : \mathrm{x}^* \in \Gamma(\mathrm{x}) \,\}$$

is a closed subset of $X \times X^*$. The map Γ is said to be usc at $\bar{x}$ if for all open subset Ω of X such that $\Omega \supset \Gamma(\bar{x})$ there exists a neighborhood $V \in \mathcal{N}(\bar{x})$ such that $\Gamma(V) \subset \Omega$. It is known that the subdifferential of a proper convex function f is usc at any $x \in \mathrm{int}\,(\mathrm{dom(f)})$. Furthermore if in addition f is lsc, then the map ∂f is closed.

1.3 The duality scheme in convex programming

Because our duality scheme for monotone variational inequality problems takes its inspiration in the duality scheme for convex optimization problems, we describe this scheme in detail.

An optimization problem in X is of the form:

$$m = \min[\tilde{f}(x) \;:\; x \in C], \qquad (P_C)$$

where $\tilde{f} : X \to (-\infty, +\infty]$ and C is a nonempty subset of X. If C is convex and $\tilde{f}$ is convex, then we are faced with a convex optimization problem.

Step 1. ***The primal problem*** **:**

It consists to replace the constrained problem (P_C) by an equivalent, apparently unconstrained, problem:

$$m = \min[f(x) \;:\; x \in X], \qquad (P)$$

with $f(x) = \tilde{f}(x) + \delta_C(x)$, where δ_C, is the indicator function of C, i.e.,

$$\delta_C(x) = \begin{cases} \quad 0 & \text{if } \; x \in C, \\ \\ +\infty & \text{if } \; x \notin C. \end{cases}$$

If $\tilde{f}$ is convex and C is convex, then f is convex. Also, if $\tilde{f}$ is lsc and C is closed, then f is lsc. More details on these properties can be found, for instance, in [3],[10],[35], etc. Of course (P_C) and (P) have the same set of optimal solutions. (P) is called the *primal problem.*

Step 2. ***The perturbations*** **:**

In this step, we introduce a *perturbation function* $\varphi : X \times U \to (-\infty, +\infty]$ such that

$$\varphi(x, 0) = f(x), \quad \text{for all } x \in X.$$

Then, we consider the associated perturbed problems

$$h(u) = \min[\varphi(x, u) : \ x \in X]. \tag{P_u}$$

The problems (P_u) are called the *primal perturbed problems.*

If φ is convex on $X \times U$ then the problems (P_u) are convex and the function h is convex on U. Unfortunately h may be not lsc when φ is lsc.

Denote by $S(u)$ the set of optimal solutions of (P_u). Then $S(0)$ is nothing else but the set of optimal solutions of (P). If φ is convex on $X \times U$, then, for all $u \in U$, $S(u)$ is a convex (may be empty) subset of X. If φ is lsc on $X \times U$, then $S(u)$ is closed.

Step 3. ***The dual problem*** **:**

Let us consider the *Fenchel-conjugate* function h^* of h.

$$\begin{aligned} h^*(u^*) &= \sup_u[\langle u, u^* \rangle - h(u)] \\ & \quad \sup_{x,u}[\langle 0, x \rangle + \langle u^*, u \rangle - \varphi(x, u)] = \varphi^*(0, u^*). \end{aligned}$$

Then, the biconjugate is

$$h^{**}(u) = \sup_{u^*}[\langle u^*, u \rangle - \varphi^*(0, u^*)].$$

By construction, h^{**} is convex and lsc on U. Furthermore $h^{**}(u) \leq h(u)$ for all u. In particular

$$m_d = h^{**}(0) \leq h(0) = m.$$

Let us define the function $d : U \to [-\infty, +\infty]$ by

$$d(u^*) = \varphi^*(0, u^*), \quad \forall\, u^* \in U.$$

Then the *dual problem* is

$$-m_d = -h^{**}(0) = \inf_{u^*} d(u^*). \tag{D}$$

By construction, the function d is convex and lsc. Therefore (D) is a convex optimization problem.

There is no duality gap ($m_d = m$), if h is a proper convex function which is lsc at 0.

By analogy with the construction of the *primal perturbed problems*, we introduce the *dual perturbed problems* as

$$k(x^*) = \inf_{u^*} \varphi^*(x^*, u^*) \tag{D_{x^*}}$$

and we denote by $T(x^*)$ the sets of optimal solutions of these problems. The function k is convex but not necessarily lsc. The sets $T(x^*)$ are closed and convex but they may be empty. In particular $T(0)$ is the set of optimal solutions of (D). Furthermore,

$$k^*(x) = \varphi^{**}(x, 0).$$

In the particular case where φ is a proper lsc convex function on $X \times U$ (this implies that f is convex and lsc on X), $\varphi^{**} = \varphi$. It results that (P) is the dual of (D) and the duality scheme we have described is thoroughly symmetric.

Step 4. ***The Lagrangian function*** **:**

Let us define on $X \times U^*$ the function

$$L(x, u^*) = \inf_u [\langle -u, u^* \rangle + \varphi_x(u)]$$

where the function φ_x is defined by

$$\varphi_x(u) = \varphi(x, u), \quad \text{for all} \quad (x, u) \in X \times U.$$

By construction, for any fixed x, the function $u^* \to L(x, u^*)$ is concave and usc because it is an infimum of affine functions. On the other hand, if φ is convex on $X \times U$, then, for any fixed u^*, the function $x \to L(x, u^*)$ is convex on X.

Because the classical *sup-inf* inequality, we have

$$\sup_{u^*} \inf_x L(x, u^*) \leq \inf_x \sup_{u^*} L(x, u^*).$$

Let us compute these two terms.

We begin with the term on the right hand side.

$$\inf_x \sup_{u^*} L(x, u^*) = \inf_x \sup_{u^*} \inf_u [\langle 0 - u, u^* \rangle + \varphi_x(u)] = \inf_x (\varphi_x)^{**}(0).$$

We know that $(\varphi_x)^{**} \leq \varphi_x$. Hence we have the following relation

$$\inf_x \sup_{u^*} L(x, u^*) = \inf_x (\varphi_x)^{**}(0) \leq \inf_x \varphi_x(0) = \inf_x \varphi(x, 0) = m.$$

Next, we deal with the term on the left.

$$\begin{aligned} \sup_{u^*} \inf_x L(x, u^*) &= \sup_{u^*} \inf_{(x,u)} [\langle x, 0 \rangle - \langle u, u^* \rangle + \varphi_x(u)], \\ &= \sup_{u^*} [-\varphi^*(0, u^*)] = -\inf_{u^*} d(u^*) = m_d. \end{aligned}$$

Thus the *sup-inf* inequality becomes

$$m_d = \sup_{u^*} \inf_x L(x, u^*) \leq \inf_x \sup_{u^*} L(x, u^*) = \inf_x (\varphi_x)^{**}(0) \leq m.$$

Furthermore, if for each x the function φ_x is proper and convex on U and lsc at 0, then

$$(\varphi_x)^{**}(0) = \varphi_x(0) = \varphi(x, 0) = f(x).$$

In this case

$$\inf_x \sup_{u^*} L(x,u^*) = \inf_x f(x) = m.$$

Step 5. ***Optimal solutions and saddle points*** **:**

By definition, $(\bar{x}, \bar{u}^*)$ is said to be a saddle point of L if

$$L(\bar{x}, u^*) \leq L(\bar{x}, \bar{u}^*) \leq L(x, \bar{u}^*), \quad \text{for all} \quad (x,u^*) \in X \times U^*.$$

The fundamental property of saddle points is that $(\bar{x}, \bar{u}^*)$ is a saddle point of L if and only if

$$\sup_{u^*} \inf_x L(x,u^*) = \inf_x \sup_{u^*} L(x,u^*),$$

$\bar{x}$ is an optimal solution of

$$\inf_x [\, \sup_{u^*} L(x,u^*) \,]$$

and $\bar{u}^*$ is an optimal solution of

$$\sup_{u^*} [\, \inf_x L(x,u^*) \,].$$

In the case where φ_x is proper convex and lsc on U for all x (this is true in particular when φ proper convex and lsc on $X \times U$), $(\bar{x}, \bar{u}^*)$ is a saddle point of L if and only if $m = m_d$ (there is no duality gap), $\bar{x}$ is an optimal solution of (P) and $\bar{u}^*$ is an optimal solution of (D). In this case, if $\mathcal{SP}$ denotes the saddle points set of L, then

$$\mathcal{SP} \subset S(0) \times T(0).$$

The equality holds if $0 \in \mathrm{ri}\,(\mathrm{proj}_{\,\mathrm{U}}(\mathrm{dom}\,(\varphi)))$.

The following example shows that the previous inclusion can be strict when $0 \in \mathrm{bd}\,(\mathrm{proj}_{\,\mathrm{U}}(\mathrm{dom}\,(\varphi)))$.

Example 1.3.1 *Take $X = U = \mathbb{R}$. Define $f : X \to \mathbb{R} \cup +\infty$ by*

$$f(x) = \begin{cases} 1 & \text{if } \ x \geq 0, \\ \\ +\infty & \text{if not }. \end{cases}$$

The function $\varphi : X \times U \to \mathbb{R} \cup +\infty$ *defined by*

$$\varphi(x,u) = \begin{cases} e^{-\sqrt{xu}} & \text{if } x,\, u \geq 0, \\ \\ +\infty & \text{if not} \end{cases}$$

is a perturbation of f *with* $0 \in \mathrm{bd}\,(\mathrm{proj}_{\,\mathrm{U}}(\mathrm{dom}\,(\varphi)))$. *By definition*

$$\varphi^*(0,u^*) = \begin{cases} 0 & \text{if } u^* \leq 0, \\ \\ +\infty & \text{if } u^* > 0. \end{cases}$$

This imply that $h(0) = 1 > h^{**}(0) = 0$ *and therefore*

$$(0,0) \in S(0) \times T(0) \quad \text{but} \quad (0,0) \notin \mathcal{SP}.$$

<u>**Step 6.**</u> <u>***Sensitivity analysis***</u> **:**

If the perturbation function φ is convex, $m > -\infty$ and h is bounded from above on an open convex neighborhood V of 0, then h is convex and continuous on V. It follows that

$$h^{**}(u) = h(u), \quad \text{for all } u \in V.$$

Then, the set of optimal solutions of the *dual problem* (Q^C) is nothing else but $\partial h(0)$ which is convex, compact, not empty. Furthermore, the multivalued map $u \overset{\rightarrow}{\longrightarrow} \partial h(u)$ is usc on V. On the other hand, if φ is convex, proper and lsc on $\mathbb{R}^n \times \mathbb{R}^m$, then $\varphi^{**} = \varphi$ and $\bar{x}$ is a solution of (P^C) if and only if $\bar{x} \in \partial k^{**}(0)$. Next, if $m_d < \infty$ and k is bounded from above on an open convex neighborhood W of 0, then k is convex and continuous on W. It follows that

$$k^{**}(x^*) = k(x^*), \quad \text{for all } x^* \in W.$$

Then, the optimal solution set of (P^C) is $\partial k(0)$ which is convex, compact, not empty and the multivalued map $x^* \overset{\rightarrow}{\longrightarrow} \partial k(x^*)$ is usc on W.

Next, we analyse the behavior of the solution set $S(u)$ of the perturbed problem (P_u) under small perturbations u. In all what follows we assume again that φ is convex, proper and lsc on $\mathbb{R}^n \times \mathbb{R}^m$ and h is bounded from above on an open convex neighborhood V of 0. Since

$$S(u) = \{x \,:\, \varphi(x,u) - h(u) \leq 0\}$$

and h is continuous on V, the map S is closed on V, this means that if we have a sequence $\{(x_k, u_k)\}$ converging to $(\bar{x}, 0))$ with $x_k \in S(u_k)$, then $\bar{x} \in S(0)$. Let $u \in V$ and $u^* \in \partial h(u)$, then $x \in S(u)$ if and only if

$$\varphi(x,u) = h(u) = \langle u^*, u \rangle - h^*(u^*) = \langle u^*, u \rangle - \varphi^*(0, u^*).$$

Thus,

$$x \in S(u) \Leftrightarrow (x,u) \in \partial\varphi^*(0,u^*) \Leftrightarrow (0,u^*) \in \partial\varphi(x,u).$$

Without no additional assumptions, the sets $S(u)$ may be empty or unbounded.

In addition to the previous assumptions, we suppose now that the set $\{0\} \times \partial h(0)$ is contained in the interior of $\mathrm{dom}\,(\varphi^*)$ and we consider the set

$$K_1 = \{v^* \,:\, \exists u^* \in \partial h(0) \text{ such that } \|v^* - u^*\| \leq 1\},$$

K_1 is bounded and not empty. There exists $W \subset V$ neighborhood of 0 in U such that

$$\partial h(u) \subset K_1 \;\text{ and }\; \{0\} \times \partial h(u) \subset \mathrm{int}\,(\mathrm{dom}\,\varphi^*) \qquad \text{for all u} \in \text{W}.$$

Then a bounded set $K_2 \subset X \times U$ exists so that

$$\emptyset \neq \partial\varphi^*(0,u^*) \subset K_2, \qquad \text{for all } u \in W \text{ and } \text{ for all } u^* \in \partial h(u).$$

One deduces that some bounded set $K_3 \subset X$ exists so that

$$\emptyset \neq S(u) \subset K_3, \qquad \text{for all } u \in W.$$

It follows that the multivalued map S is usc in a neighborhood of 0.

Chapter 2

Monotonicity and maximal monotonicity

If one asks someone to define the convexity of a function, one generally obtains an analytical definition. But the true essence of convexity is of a geometrical nature: a function is convex if its epigraph is convex. In the same manner, the monotonicity of a multivalued map $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ is in fact a property on its graph $F = \{(x, x^*) : x^* \in \Gamma(x)\} \subset X \times X^*$. Thus a good and rich approach to monotonicity is geometric and consists in working on sets (the graphs of maps) instead of the classical analytical approach where mappings are favoured. As an illustration, the inverse map Γ^- is monotone if and only if Γ is so because the two maps share the same graph. Another illustration is maximal monotonicity, a subset F of $X \times X^*$ is maximal monotone if any monotone subset G containing F coincides with F. The definition of maximal monotonicity of multivalued maps follows. Monotonicity and maximal monotonicity of subsets are preserved when appropriate permutations of variables are done, this will be an essential trick for the duality scheme for variational inequality problems introduced in Chapter 4 and applied in Chapter 5.

Convexity and monotonicity are intimately related. The sub-differential of a convex function is maximal monotone, but all maximal monotone maps are not issued from convex functions, a necessary condition is cyclic monotonicity. This condition cannot be translated in terms of graphs. This illustrates the fact that monotonicity is a larger concept than convexity.

In order to study the maximal monotonicity of a subset F, it is useful to introduce the subset $\widetilde{F} \subset X \times X^*$ defined by

$$\widetilde{F} = \{\, (x, x^*) : \, \langle x^* - y^*, x - y \rangle \geq 0 \;\text{ for all } (y, y^*) \in F \,\}.$$

If G is a monotone subset containing F, then G is contained in $\widetilde{F}$, i.e., $\widetilde{F}$ contains all monotone extensions of F (it results that a subset F is maximal monotone if and only if $\widetilde{F}$ and F coincide). With $\widetilde{F}$ is associated the map $\widetilde{\Gamma} : X \overset{\rightarrow}{\longrightarrow} X^*$, Γ is maximal monotone if and only if $\widetilde{\Gamma}$ and Γ coincide. The properties of $\widetilde{\Gamma}$ are studied in section 2.3. Another essential tool introduced in this subsection in order to study the maximal monotonicity of a monotone map Γ is the map $\Gamma_S : X \overset{\rightarrow}{\longrightarrow} X^*$ defined by

$$\mathrm{graph}\,(\Gamma_\mathrm{S}) = \mathrm{cl}\,[\mathrm{graph}\,(\Gamma) \cap (\mathrm{S} \times \mathrm{X}^*)]$$

where S is a subset of $\mathrm{dom}\,(\Gamma)$. A fundamental property is: if V is an open convex set contained in the convex hull of domain of Γ such that $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{S}) = \mathrm{cl}\,(\mathrm{V})$, then

$$\widetilde{\Gamma}(x) = \mathrm{co}\,(\Gamma_\mathrm{S}(\mathrm{x})) \;\; \text{for all } \mathrm{x} \in \mathrm{V}.$$

We shall use this property in section 2.7 to construct a maximal extension of a monotone map.

In section 2.4, given $\Gamma : X \times U \overset{\rightarrow}{\longrightarrow} X^* \times U^*$ and a fixed point $\bar{u} \in U$, we study the map $\Sigma_{\bar{u}} : X \overset{\rightarrow}{\longrightarrow} X^*$ defined by

$$\Sigma_{\bar{u}}(x) = \{\, x^* : \exists\, u^* \in U^* \text{ such that } (x^*, u^*) \in \Gamma(x, \bar{u}) \,\}.$$

The geometric meaning of this map is that its graph is the projection onto $X \times X^*$ of a restriction of the graph of Γ. The introduction of this restriction is a major tool in the construction of the duality scheme given in chapter 4.

2.1 Definitions and notation

Definition 2.1.1 *A set $F \subset X \times X^*$ is said to be* monotone *if*

$$\langle x^* - y^*, x - y \rangle \geq 0 \;\; \textit{for all} \;\; (x, x^*),\, (y, y^*) \in F$$

and it is said maximal monotone *if for any monotone subset G of $X \times X^*$ such that $F \subset G$ we have $F = G$.*

Given $F \subset X \times X^*$, we denote by $\widetilde{F}$ the subset

$$\widetilde{F} := \{ (x, x^*) \in X \times X^* : \langle x^* - y^*, x - y \rangle \geq 0 \ \text{ for all } \ (y, y^*) \in F \}.$$

This set $\widetilde{F}$ is closed since it is an intersection of closed sets.

We have the following fundamental result.

Proposition 2.1.1 *Assume that F and G are two subsets of $X \times X^*$.*

a) If $F \subset G$ then $\widetilde{G} \subset \widetilde{F}$.

b) F is monotone if and only if $F \subset \widetilde{F}$.

c) If $F \subset G$ and G is monotone then F is monotone and $G \subset \widetilde{F}$.

d) F is maximal monotone if and only if $F = \widetilde{F}$.

Proof.

a) Either $\widetilde{G} = \emptyset$ or there exists $(x, x^*) \in \widetilde{G}$, then $\langle x^* - y^*, x - y \rangle \geq 0$, for all $(y, y^*) \in G$. Since $F \subset G$, $\langle x^* - y^*, x - y \rangle \geq 0$ for all $(y, y^*) \in F$. This implies that $(x, x^*) \in \widetilde{F}$.

b) $\Rightarrow$) Let $(x, x^*) \in F$. Since F is monotone, then $\langle x^* - y^*, x - y \rangle \geq 0$ for all $(y, y^*) \in F$. This implies that $(x, x^*) \in \widetilde{F}$.

$\Leftarrow$) Let (x, x^*), $(y, y^*) \in F$. Since $F \subset \widetilde{F}$, then $\langle x^* - y^*, x - y \rangle \geq 0$. Thus, F is monotone.

c) Monotonicity of F follows clearly from the monotonicity of G, and therefore $G \subset \widetilde{F}$ follows from *a*) and *b*).

d) $\Rightarrow$) If F is maximal monotone, it is monotone and $F \subset \widetilde{F}$ by *b*). In order to prove the reverse inclusion, assume, for contradiction, that there exists $(x, x^*) \in \widetilde{F} \setminus F$. Then $G = F \cup \{(x, x^*)\}$ is a monotone set, a contradiction with F maximal monotone.

$\Leftarrow$) Assume that $F = \widetilde{F}$. Then F is monotone. If G is a monotone set such that $G \supset F$, then using part c), one has $G \subset \widetilde{F} = F$. ■

Proposition 2.1.2 *Let $F \subset X \times X^*$, then:*

a) F is monotone if and only if $\mathrm{cl}\,(\mathrm{F})$ *is monotone.*

b) If F is maximal monotone then it is closed.

Proof.

a) Let $(x, x^*), (y, y^*) \in \mathrm{cl}\,(\mathrm{F})$. Then, there exist two sequences in F, $\{(x_n, x_n^*)\}$ and $\{(y_n, y_n^*)\}$ such that $(x_n, x_n^*) \to (x, x^*)$ and $(y_n, y_n^*) \to (y, y^*)$. For all n, $\langle y_n^* - x_n^*, y_n - x_n\rangle \geq 0$. Passing to the limit one obtains that $\langle y^* - x^*, y - x\rangle \geq 0$, which implies that $\mathrm{cl}\,(\mathrm{F})$ is monotone. Conversely, the inclusion $F \subset \mathrm{cl}\,(\mathrm{F})$ implies the monotonicity of F when $\mathrm{cl}\,(\mathrm{F})$ is monotone.

b) Follows from the fact that $F = \widetilde{F}$. ■

Given $G \subset X \times X^*$, we denote by Γ and Γ^- the multivalued maps defined respectively on X and X^* by

$$\begin{aligned} \Gamma(x) &:= \{x^* \in X^* : (x, x^*) \in G\}, \\ \Gamma^-(x^*) &:= \{x \in X : (x, x^*) \in G\}. \end{aligned}$$

Thus G can be considered as the graph of both Γ and Γ^-.

Remark. The maps Γ and Γ^- are said to be *monotone* when G is monotone, *maximal monotone* when G is maximal monotone.

The domains of Γ and Γ^- are the sets

$$\begin{aligned} \mathrm{dom}\,(\Gamma) &:= \{x : \Gamma(x) \neq \emptyset\} = \mathrm{proj}_{\,\mathrm{X}}(\mathrm{G}), \\ \mathrm{dom}\,(\Gamma^-) &:= \{x^* : \Gamma^-(x^*) \neq \emptyset\} = \mathrm{proj}_{\,\mathrm{X}^*}(\mathrm{G}). \end{aligned}$$

Similarly, we define the multivalued maps

$$\begin{aligned} \widetilde{\Gamma}(x) &:= \{x^* \in X^* : (x, x^*) \in \widetilde{G}\}, \\ (\widetilde{\Gamma^-})(x^*) &:= \{x \in X : (x, x^*) \in \widetilde{G}\}. \end{aligned}$$

It is clear that $(\widetilde{\Gamma})^- = (\widetilde{\Gamma^-})$.

In view of Proposition 2.1.1, the equality $\widetilde{\Gamma} = \Gamma$ holds if and only if the multivalued map Γ is maximal monotone. Same fact for Γ^- and $(\widetilde{\Gamma^-})$.

Proposition 2.1.3 *Let $G \subset X \times X^*$. Then, for any x and x^* the sets $\widetilde{\Gamma}(x)$ and $(\widetilde{\Gamma^-})(x^*)$ are two closed convex sets. In particular, $\Gamma(x)$ and $\Gamma^-(x^*)$ are closed convex sets when G is maximal monotone.*

Proof. By definition,

$$\widetilde{\Gamma}(x) = \bigcap_{(y,y^*)\in\widetilde{G}} \{\, x^* : \langle x^* - y^*, x - y\rangle \geq 0 \,\}$$

and

$$\widetilde{\Gamma}^-(x^*) = \bigcap_{(y,y^*)\in\widetilde{G}} \{\, x : \langle x^* - y^*, x - y\rangle \geq 0 \,\}.$$

These sets are closed and convex as intersections of half spaces. ■

Subdifferentials of proper convex lower semi-continuous functions are maximal monotone maps (see for instance [35]). But a maximal monotone map is not necessarily associated with a convex function as shown in the following example.

Example 2.1.1 *The map $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\Gamma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

is maximal monotone. Indeed, by definition,

$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} \in \widetilde{\Gamma}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ if and only if}$$

$$x_1^*x_1 + x_2^*x_2 - (x_1^* + x_2)y_1 + (x_1 - x_2^*)y_2 \geq 0, \quad \forall\ (y_1, y_2) \in \mathbb{R}^2,$$

which implies that

$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \Gamma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By Proposition 2.1.1, Γ *is maximal monotone. But, there is no convex function* f *such that* $\partial f(x) = \Gamma(x)$ $\forall x \in \mathbb{R}^2$*. Indeed, a necessary and sufficient condition for the existence of a convex function such that* $\partial f(x) = \Gamma(x)$ *for all* x*, is* Γ *cyclically-monotone. Here consider the points:* $(1,0)$, $(0,1)$ *and* $(1,1)$ *in* $\operatorname{graph}(\Gamma)$*. Then*

$$\langle x_0^*, x_1 - x_0 \rangle = 0, \quad \langle x_1^*, x_2 - x_1 \rangle = 1 \quad \textit{and} \quad \langle x_2^*, x_3 - x_2 \rangle = 0.$$

Hence, Γ *is not cyclically-monotone.*

2.2 Monotonicity on a product space

In this section, we do an immediate, but fundamental observation which will be essential for the construction of the duality scheme for variational inequality problems.

Assume that Φ is a subset of the space $(X \times U) \times (X^* \times U^*)$. Φ can be considered also as a subset of any of the following spaces

- $(X \times U^*) \times (X^* \times U)$,
- $(X^* \times U) \times (X \times U^*)$,
- $(X^* \times U^*) \times (X \times U)$,
- $(U \times X) \times (U^* \times X^*)$,
- $(U \times X^*) \times (U^* \times X)$,
- $(U^* \times X) \times (U \times X^*)$,
- $(U^* \times X^*) \times (U \times X)$.

If Φ considered as a subset of one of these spaces is (maximal) monotone, it is so for each of them.

This is no more true for cyclic monotonicity. To see that, consider the function $\varphi : \mathbb{R} \times \mathbb{R} \to (-\infty, +\infty]$ defined by

$$\varphi(x,u) = \begin{cases} 0 & \text{if } x = u, \\ \\ +\infty & \text{if not .} \end{cases}$$

Next, consider for Φ the graph of the subdifferential of φ.

$$\Phi = \mathrm{graph}\,(\partial\varphi) = \{\, ((\mathrm{x},\mathrm{u}),(\mathrm{x}^*,\mathrm{u}^*)) \in \mathbb{R}^4 : \mathrm{x} = \mathrm{u},\ \mathrm{u}^* = -\mathrm{x}^* \,\}$$

and

$$\Psi = \{\, ((x,u^*),(x^*,u)) \in \mathbb{R}^4 : x = u,\ u^* = -x^* \,\}.$$

Φ is maximal cyclically monotone, Ψ is maximal monotone but not cyclically monotone. Indeed Ψ corresponds to the map Γ defined in example 2.1.1.

However it is easy to see that $F \subset X \times X^*$ is cyclically monotone if and only if the set $F^- \subset X^* \times X$ defined by $(x^*,x) \in F^- \iff (x,x^*) \in F$ is cyclically monotone.

It is very important to know if the monotonicity or maximal monotonicity holds when we analyze only some projections over appropriate subspaces. In this sense we present the following proposition.

Proposition 2.2.1 *Assume that Φ is a given subset of $(X \times U) \times (X^* \times U^*)$. Define $E = \mathrm{proj}_{\mathrm{X}\times\mathrm{X}^*}(\Phi)$ and $F = \mathrm{proj}_{\mathrm{U}\times\mathrm{U}^*}(\Phi)$. If E and F are (maximal) monotone, then Φ is (maximal) monotone.*

Proof. It is clear that monotonicity of Φ follows from the monotonicity of E and F. Next, assume that E and F are maximal monotone. In view of Proposition 2.1.1, it is suffices to show that the inclusion $\widetilde{\Phi} \subset \Phi$ is verified. Let $(x,u,x^*,u^*) \in \widetilde{\Phi}$. By definition

$$\langle x^* - y^*, x - y\rangle + \langle u^* - v^*, u - v\rangle \geq 0 \text{ for all } (y,y^*) \in E,\ (v,v^*) \in F.$$

Assume, for contradiction, that $(x,x^*) \notin E$, then a vector $(\bar{y},\bar{y}^*) \in E$ exists so that $\langle x^* - \bar{y}^*, x - \bar{y}\rangle < 0$ and consequently $\langle u^* - v^*, u - v\rangle > 0$, for all $(v,v^*) \in F$. One deduce that $(u,u^*) \in \widetilde{F} = F$. Take $(y,v,y^*,v^*) = (\bar{y},u,\bar{y}^*,u^*) \in \Phi$, then one has

$$\langle x^* - \bar{y}^*, x - \bar{y}\rangle = \langle x^* - \bar{y}^*, x - \bar{y}\rangle + \langle u^* - u^*, u - u\rangle \geq 0,$$

in contradiction with the inequality above. One obtains that Φ is maximal monotone. ■

2.3 More on $\widetilde{\Gamma}$

Proposition 2.3.1 *Let $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ be a multivalued map. Let Σ and $\bar{\Sigma}$ be defined by*

$$\Sigma(x) = \mathrm{co}\,(\Gamma(\mathrm{x})) \quad \textit{and} \quad \bar{\Sigma}(\mathrm{x}) = \overline{\mathrm{co}}\,(\Gamma(\mathrm{x})) \quad \textit{for any} \quad \mathrm{x} \in \mathrm{X}.$$

Then Σ is monotone if and only if Γ is so, the same result holds for $\bar{\Sigma}$.

Proof. Assume that $\Gamma : \mathbb{R}^n \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^n$ is monotone. Take any $(x,x^*), (y,y^*) \in$ graph (Σ). Then there exist $x_i^* \in \Gamma(x)$, $y_i^* \in \Gamma(y)$, $t_i \geq 0$, $s_i \geq 0$, $i = 0, \cdots, n$ such that

$$1 = \sum_{i=0}^{n} t_i = \sum_{i=0}^{n} s_i, \quad x^* = \sum_{i=0}^{n} t_i x_i^* \text{ and } y^* = \sum_{i=0}^{n} s_i y_i^*.$$

The monotonicity of Γ implies

$$\langle x_i^* - y_j^*, x - y\rangle \geq 0 \text{ for all } i,j = 0, \cdots, n.$$

Thus,

$$\langle x^* - y^*, x - y\rangle = \sum_{i=0}^{n} t_i \sum_{j=0}^{n} s_j \langle x_i^* - y_j^*, x - y\rangle \geq 0,$$

and the monotonicity of Σ follows. Conversely, since the graph of Σ contains the graph of Γ, monotonicity of Σ implies monotonicity of Γ.

If Σ is monotone, the graph of Γ is a monotone subset, this is also the case of the closure of the graph which contains the graph of $\bar{\Sigma}$. The converse is immediate. ■

Proposition 2.3.2 *Let $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ be a monotone multivalued map. Denote $C = \overline{\text{co}}\,\text{dom}\,(\Gamma)$. Assume that $\bar{x} \in \text{dom}\,(\widetilde{\Gamma}) \cap \text{C}$. Then, $N_C(\bar{x})$ coincides with the recession cone of $\widetilde{\Gamma}(\bar{x})$, i.e.,*

$$N_C(\bar{x}) = (\widetilde{\Gamma}(\bar{x}))_\infty.$$

It follows that

$$\widetilde{\Gamma}(x) = \widetilde{\Gamma}(x) + N_C(x) \text{ for all } x \in C.$$

Proof. Clearly, $w^* \in N_C(\bar{x})$ if and only if

$$\langle x^* + tw^* - y^*, \bar{x} - y\rangle \geq 0 \text{ for all } (y, y^*) \in \text{graph}\,(\Gamma),\ \text{t} \geq 0,\ \text{x}^* \in \widetilde{\Gamma}(\bar{\text{x}}).$$

Thus, $N_C(\bar{x})$ consists of the vectors w^* such that

$$x^* + tw^* \in \widetilde{\Gamma}(\bar{x}) \text{ for all } x^* \in \widetilde{\Gamma}(\bar{x}),\ t \geq 0.$$

In other words

$$N_C(\bar{x}) = (\widetilde{\Gamma}(\bar{x}))_\infty,$$

because $\widetilde{\Gamma}(\bar{x})$ is closed and convex. ■

In the next results, it is assumed that Γ is monotone and the interior of the convex hull of the domain of Γ is not empty. In a forthcoming section, these results will be generalized by considering the relative interior.

Theorem 2.3.1 *Assume that the multivalued map $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ is monotone and $D = \text{co}\,(\text{dom}\,(\Gamma))$ has nonempty interior. Then for all $\bar{x} \in \text{int}\,(\text{D})$ there exists a compact $K \subset X^*$ and a neighbourhood V of $\bar{x}$ such that $\emptyset \neq \widetilde{\Gamma}(x) \subset K$ for all $x \in V$.*

Proof. Assume that $X = \mathbb{R}^n$. Let G be the graph of Γ. Since $\bar{x}$ belongs to the interior of the convex hull of the domain of Γ, there exist $\bar{t}_i > 0$ and $(x_i, x_i^*) \in G$ for $i = 0, 1, \cdots, n$ such that

$$\bar{x} = \sum_{i=0}^{n} \bar{t}_i x_i\ , \quad 1 = \sum_{i=0}^{n} \bar{t}_i$$

and the n vectors $(x_i - x_0),\ i = 1, 2, \cdots, n,$ are linearly independent .

Let some $\epsilon > 0$ be such that $\epsilon < \bar{t}_i$ for $i = 0, 1, \cdots, n$. Let V be defined by

$$V = \{x = \sum_{i=0}^{n} t_i x_i \ : \ 1 = \sum_{i=0}^{n} t_i \quad \text{and} \quad \epsilon \leq t_i \text{ for all } i\}.$$

Then V is a neighbourhood of $\bar{x}$. Given $c \in \mathbb{R}^n$ and $x \in V$, let us define

$$\alpha(c, x) = \sup\,[\langle c, x^* \rangle \ : \ x^* \in \widetilde{\Gamma}(x)\,].$$

Then,

$$-\infty \leq \alpha(c, x) \leq \beta(c, x)$$

where

$$\beta(c, x) = \sup_{x^*}\,[\langle c, x^* \rangle \ : \ \langle x^*, x_i - x \rangle \leq \langle x_i^*, x_i - x \rangle, i = 0, 1, 2, \cdots, n\,]. \quad (P_e)$$

The dual of the linear program (P_e) is the problem

$$\widetilde{\beta}(c, x) = \inf_{u}\,[\sum_{i=0}^{n} u_i \langle x_i^*, x_i - x \rangle \ : \ u_i \geq 0 \text{ and } \sum_{i=0}^{n} u_i (x_i - x) = c\,]. \qquad (D_e)$$

Because x belongs to V and V is contained in the interior of the convex hull of the $(n+1)$ points x_i, (D_e) is feasible and therefore $\widetilde{\beta}(c, x) = \beta(c, x)$.

Next, let us consider the linear program

$$\min\,[\sum_{i=0}^{n} u_i : \ u_i \geq 0 \text{ and } \sum_{i=0}^{n} u_i (x_i - x) = c\,]. \tag{2.1}$$

As (D_e), this problem is feasible. We shall show that this problem has one unique optimal solution that we will denote by $u(c, x)$. Furthermore we shall prove that the function $(c, x) \to u(c, x)$ is continuous on $X \times V$. Indeed, the n vectors $(x_i - x)$, $i = 1, 2, \cdots, n$ are linearly independent. Thus, there exist uniquely defined $\lambda_i(c, x) \in \mathbb{R}$, $i = 1, 2, \cdots, n$ such that

$$c = \sum_{i=1}^{n} \lambda_i(c, x)(x_i - x). \tag{2.2}$$

Also, there are uniquely defined $\gamma_i(c, x) > 0$, $i = 1, 2, \cdots, n$ such that

$$(x - x_0) = \sum_{i=1}^{n} \gamma_i(c, x)(x_i - x). \tag{2.3}$$

Thus, u is feasible for problem (2.1) if and only if

$$u \geq 0 \text{ and } \lambda_i(c,x) = u_i - u_0\gamma_i(c,x), \quad i = 1,2,\cdots,n \tag{2.4}$$

and therefore

$$u_0 \geq 0 \text{ and } \lambda_i(c,x) + u_0\gamma_i(c,x) \geq 0, \quad i = 1,2,\cdots,n.$$

Problem (2.1) becomes

$$\inf_{u_0} [\, u_0(1 + \sum_{i=1}^{n} \gamma_i(c,x)) : \; u_0 \geq \max\{0, \; \max_i [\, -\frac{\lambda_i(c,x)}{\gamma_i(c,x)} : \; i = 1,2,\cdots,n\,]\} \,].$$

Since $\sum_{i=1}^{n} \gamma_i(c,x)) > 0$, the previous problem has one unique optimal solution,

$$u_0(c,x) = \max\{0, \; \max_i [\, -\frac{\lambda_i(c,x)}{\gamma_i(c,x)} : \; i = 1,2,\cdots,n\,]\}. \tag{2.5}$$

Hence problem (2.1) has one unique optimal solution denoted by $u(c,x)$.

In order to prove the continuity of $u(c,x)$, define the $n \times n$ matrix

$$A(x) = [\, x_1 - x, x_2 - x, \cdots, x_n - x\,].$$

By definition, $A(x)$ is nonsingular for all $x \in V$ and the function $x \to A(x)$ is continuous on X. Thus from equations (2.2) and (2.3), the functions

$$(c,x) \to \lambda(c,x) = [A(x)]^{-1}c \quad \text{and} \quad (c,x) \to \gamma(c,x) = [A(x)]^{-1}(x - x_0)$$

are continuous on $X \times V$. Hence, from (2.4) and (2.5), the function $(c,x) \to u(c,x)$ is continuous on $X \times V$.

Next, define

$$\rho(c,x) = \sum_{i=0}^{n} u_i(c,x)\langle x_i^*, x_i - x\rangle,$$

$$M = \sup_{x,c} [\, \rho(c,x) \, : \, x \in V, \; \|c\| \leq 1\,] \quad \text{and} \quad K = \{x^* \, : \, \|x^*\| \leq M\}.$$

Then for all c such that $\|c\| \leq 1$ and for all $x \in V$, one has

$$\begin{aligned} \alpha(c,x) &= \sup_{x^*}[\, \langle c, x^*\rangle : \; x^* \in \widetilde{\Gamma}(x)\,] \\ &\leq \sup_{x^*} [\langle c, x^*\rangle \, : \, \langle x^*, x_i - x\rangle \leq \langle x_i^*, x_i - x\rangle, \quad i = 0,1,2,\cdots,n\,] \\ &= \beta(c,x) = \widetilde{\beta}(c,x) \leq \rho(c,x) \leq M. \end{aligned}$$

Thus,

$$\sup_{x^*\in\widetilde{\Gamma}(x)} \|x^*\| \leq \sup_{x^*} [\ \|x^*\| :\ \langle x^*, x_i - x\rangle \leq \langle x_i^*, x_i - x\rangle,\ \ i = 0, 1, 2, \cdots, n\,] \leq M.$$

Then, for all $x \in V$

$$\tilde{\Gamma}(x) \subset \{x^* \ :\ \langle x^*, x_i - x\rangle \leq \langle x_i^*, x_i - x\rangle,\ \ i = 0, 1, 2, \cdots, n\} \subset K. \tag{2.6}$$

Therefore the boundedness of $\widetilde{\Gamma}$ on V follows.

Next, we shall prove that $\widetilde{\Gamma}(\bar{x})$ is not empty. Assume, for contradiction, that $\tilde{\Gamma}(\bar{x})$ is empty. Since

$$\emptyset = \tilde{\Gamma}(\bar{x}) = \bigcap_{(x,x^*)\in F} \{\bar{x}^* \in \mathbb{R}^n \ :\ \langle \bar{x}^* - x^*, \bar{x} - x\rangle \geq 0\} \subset K$$

and K is compact, there exist $(x_j, x_j^*) \in F$, $j = n+1, n+2, \cdots, n+q$ such that

$$\emptyset = \Big(\bigcap_{j=n+1,\cdots,n+q} \{\bar{x}^* \ :\ \langle \bar{x}^* - x_j^*, \bar{x} - x_j\rangle \geq 0\}\Big) \cap K.$$

Next, in view of (2.6),

$$\emptyset = \bigcap_{j=0,\cdots,n+q} \{\bar{x}^* \ :\ \langle \bar{x}^* - x_j^*, \bar{x} - x_j\rangle \geq 0\}. \tag{2.7}$$

Consider the $(n+q+1)\times n$ matrix $A = (\bar{x} - x_0, \bar{x} - x_1, \cdots, \bar{x} - x_{n+q})$ and the $(n+q+1)$ vector a with components $a_j = \langle x_j^*, \bar{x} - x_j\rangle$, then (2.7) is equivalent to

$$\nexists\ x^* \in \mathbb{R}^n \text{ such that } A^t x^* \geq a.$$

This condition is equivalent to (theorem on alternatives, see for instance [35], Section 22)

$$\exists\ u \in \mathbb{R}^{n+q+1} \text{ such that } u \geq 0,\ \ Au = 0 \text{ and } \langle a, u\rangle > 0.$$

Without loss of generality, we assume that $\sum u_i = 1$. Then $Au = 0$ implies $\bar{x} = \sum u_i x_i$. Next, $\langle a, u\rangle > 0$ implies

$$0 < \sum_{j=0}^{n+q} u_j \langle x_j^*, \sum_{i=0}^{n+q} u_i x_i - x_j\rangle = \sum_{i,j}^{n+q} u_i u_j \langle x_j^*, x_i - x_j\rangle.$$

Hence

$$0 < -\sum_{i,j}^{n+q} u_i u_j \langle x_i^* - x_j^*, x_i - x_j \rangle.$$

In contradiction with

$$\langle x_i^* - x_j^*, x_i - x_j \rangle \geq 0, \quad \forall\, i, j$$

which is implied by F monotone. ■

As an immediate consequence of this result we have the convexity of the interior and of the closure of the domain of a maximal monotone map.

Corollary 2.3.1 *Assume that* $\Gamma : X \overset{\rightarrow}{\longrightarrow} X^*$ *is a maximal monotone map and that* $\mathrm{aff}\,(\mathrm{dom}\,(\Gamma)) = \mathrm{X}$*. then the interior and the closure of* $\mathrm{dom}\,(\Gamma)$ *are convex subsets. Moreover*

$$\mathrm{cl}\,(\mathrm{int}\,(\mathrm{dom}\,(\Gamma))) = \mathrm{cl}\,(\mathrm{dom}\,(\Gamma)) \quad \textit{and} \quad \mathrm{int}\,(\mathrm{cl}\,(\mathrm{dom}\,(\Gamma))) = \mathrm{int}\,(\mathrm{dom}\,(\Gamma)).$$

Proof. Since Γ is maximal monotone, $\widetilde{\Gamma} = \Gamma$. By Theorem 2.3.1,

$$\mathrm{int}\,(\mathrm{co}\,(\mathrm{dom}\,(\Gamma))) \subset \mathrm{dom}\,(\Gamma) \subset \mathrm{co}\,(\mathrm{dom}\,(\Gamma)).$$

This implies that

$$\mathrm{int}\,(\mathrm{co}\,(\mathrm{dom}\,(\Gamma))) = \mathrm{int}\,(\mathrm{dom}\,(\Gamma))$$

and

$$\mathrm{cl}\,(\mathrm{int}\,(\mathrm{dom}\,(\Gamma))) = \overline{\mathrm{co}}(\mathrm{dom}\,(\Gamma)) = \mathrm{cl}\,(\mathrm{dom}\,(\Gamma)).$$

On the other hand, since

$$\mathrm{int}\,(\overline{\mathrm{co}}(\mathrm{dom}\,(\Gamma))) = \mathrm{int}\,(\mathrm{co}\,(\mathrm{dom}\,(\Gamma)))$$

we obtain

$$\mathrm{int}\,(\mathrm{cl}\,(\mathrm{dom}\,(\Gamma))) = \mathrm{int}\,(\mathrm{dom}(\Gamma)).$$

The result is proved. ■

The next result is more general than Corollary 2.3.1 since the map Γ is not assumed to be maximal.

Theorem 2.3.2 *With the assumptions and notations of Theorem 2.3.1, the interior and the closure of* $\operatorname{dom}(\widetilde{\Gamma})$ *are convex sets. Moreover*

$$\operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))) = \operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma})) \quad \textit{and} \quad \operatorname{int}(\operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma}))) = \operatorname{int}(\operatorname{dom}(\widetilde{\Gamma})).$$

Proof. Let $\bar{x}, \bar{y} \in \operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma}))$ and $\alpha \in (0,1)$. We will prove that

$$\bar{z} = \alpha\bar{x} + (1-\alpha)\bar{y} \in \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))).$$

Let $\{x_k\}$ and $\{y_k\}$ be two sequences in $\operatorname{dom}(\widetilde{\Gamma})$ converging to $\bar{x}$ and $\bar{y}$ respectively. Consider $z_k = \alpha x_k + (1-\alpha)y_k$, then $z_k \in \operatorname{co}(\operatorname{dom}(\widetilde{\Gamma}))$. We distinguish the two following cases.

i) $\exists\, k_0 \in \mathbb{N}$ such that $\forall\, k \geq k_0$, there exists $x_k^* \in \widetilde{\Gamma}(x_k)$ and $y_k^* \in \widetilde{\Gamma}(y_k)$ with

$$\langle x_k^* - y_k^*, x_k - y_k \rangle \geq 0.$$

Define $\Sigma_k : X \overset{\longrightarrow}{\longrightarrow} X^*$ such that

$$\operatorname{graph}(\Sigma_k) = \operatorname{graph}(\Gamma) \cup \{(x_k, x_k^*)\} \cup \{(y_k, y_k^*)\}.$$

By definition, Σ_k is monotone, then Theorem 2.3.1 and Proposition 2.1.1 imply

$$z_k \in \operatorname{cl}(\operatorname{int}(\operatorname{co}(\operatorname{dom}(\Sigma_k)))) \subset \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Sigma}_k))) \subset \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))).$$

Passing to the limit, we obtain $\bar{z} \in \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma})))$.

ii) Otherwise, since $\widetilde{\Gamma}(x_k)$ and $\widetilde{\Gamma}(y_k)$ are nonempty, $\forall\, k_0 \in \mathbb{N}$, $\exists\, k \geq k_0$ such that there exist $x_k^* \in \widetilde{\Gamma}(x_k)$ and $y_k^* \in \widetilde{\Gamma}(y_k)$ with

$$\langle x_k^* - y_k^*, x_k - y_k \rangle < 0.$$

Take $z_k^* = \alpha x_k^* + (1-\alpha)y_k^*$. Let us prove that $z_k^* \in \widetilde{\Gamma}(z_k)$. For that, consider any $(w, w^*) \in \operatorname{graph}(\Gamma)$ and prove that $A = \langle z_k^* - w^*, z_k - w \rangle \geq 0$.

$$\begin{aligned}
A &= \langle \alpha x_k^* - \alpha y_k^* + y_k^* - w^*, (1-\alpha)y_k - (1-\alpha)x_k + x_k - w \rangle \\
&= \alpha(1-\alpha)\langle x_k^* - y_k^*, y_k - x_k \rangle + \alpha\langle x_k^* - w^* + w^* - y_k^*, x_k - w \rangle \\
&\quad + (1-\alpha)\langle y_k^* - w^*, y_k - w + w - x_k \rangle + \langle y_k^* - w^*, x_k - w \rangle \\
&= \alpha(1-\alpha)\langle x_k^* - y_k^*, y_k - x_k \rangle + \alpha\langle x_k^* - w^*, x_k - w \rangle \\
&\quad + (1-\alpha)\langle y_k^* - w^*, y_k - w \rangle.
\end{aligned}$$

Thus, for any $(w, w^*) \in \operatorname{graph}(\Gamma)$, $\langle z_k^* - w^*, z_k - w \rangle \geq 0$, and therefore $(z_k, z_k^*) \in \operatorname{graph}(\widetilde{\Gamma})$. Define $\Sigma_k : X \rightrightarrows X^*$ by

$$\operatorname{graph}(\Sigma_\mathrm{k}) = \operatorname{graph}(\Gamma) \cup \{(\mathrm{z_k}, \mathrm{z_k^*})\}.$$

By definition, Σ_k is monotone, then Theorem 2.3.1 and Proposition 2.1.1 imply

$$z_k \in \operatorname{cl}(\operatorname{int}(\operatorname{co}(\operatorname{dom}(\Sigma_\mathrm{k})))) \subset \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Sigma}_\mathrm{k}))) \subset \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))).$$

Passing to the limit we obtain that $\bar{z} \in \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma})))$.

Summarizing, we have proved that for any $\bar{x}, \bar{y} \in \operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma}))$ and $\alpha \in (0,1)$, $\bar{z} = \alpha\bar{x} + (1-\alpha)\bar{y} \in \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))) \subset \operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma}))$. We deduce that $\operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma}))$ is convex. Furthermore, taking $\bar{x} = \bar{y} \in \operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma}))$, we obtain that

$$\operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))) = \operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma})). \tag{2.8}$$

iii) We now prove that $\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))$ is convex. Let $\bar{x}, \bar{y} \in \operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))$ and $\alpha \in (0,1)$. Let us prove that $\bar{z} = \alpha\bar{x} + (1-\alpha)\bar{y} \in \operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))$. Since $\bar{x}, \bar{y} \in \operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))$ and $\operatorname{int}(\operatorname{co}(\operatorname{dom}(\Gamma))) \neq \emptyset$, there exists $\bar{t} > 0$ such that $x = \bar{x} + \bar{t}(\bar{x} - a)$ and $y = \bar{y} + \bar{t}(\bar{y} - a)$ belong to $\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))$ with $a \in \operatorname{int}(\operatorname{co}(\operatorname{dom}(\Gamma)))$. Repeating the two previous cases with x and y, we obtain that $z = \alpha x + (1-\alpha)y \in \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma})))$. This implies that there exists $\hat{z} \in \operatorname{dom}(\widetilde{\Gamma})$ such that $\bar{z} \in \operatorname{int}(\operatorname{co}(\operatorname{dom}(\Gamma) \cup \{\hat{z}\}))$. Next, define $\Gamma_1 : X \rightrightarrows X^*$ such that

$$\operatorname{graph}(\Gamma_1) = \operatorname{graph}(\Gamma) \cup (\{\hat{\mathrm{z}}\} \times \widetilde{\Gamma}(\hat{\mathrm{z}})).$$

Then $\bar{z} \in \operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}_1))$. Therefore $\bar{z} \in \operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))$. Hence the convexity of $\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma}))$ follows, and therefore from (2.8), one has

$$\operatorname{int}(\operatorname{dom}(\widetilde{\Gamma})) = \operatorname{int}(\operatorname{cl}(\operatorname{dom}(\widetilde{\Gamma}))).$$

From i), ii) and iii), the result follows. ∎

Now, we are turn our interest in some genericity properties of monotone maps.

Definition 2.3.1 *Let* $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ *be a monotone multivalued map and* $\mathrm{S} \subset \mathrm{dom}\,(\Gamma)$. *We associate with* Γ *and* S *the map* $\Gamma_{\mathrm{S}} : X \overset{\longrightarrow}{\longrightarrow} X^*$ *defined by*

$$\mathrm{graph}\,(\Gamma_{\mathrm{S}}) = \mathrm{cl}\,[\mathrm{graph}\,(\Gamma) \cap (\mathrm{S} \times \mathrm{X}^*)].$$

Since Γ *is monotone and the closedness of monotone subsets are monotone, it follows that* Γ_{S} *is also monotone.*

Next, given $\bar{x} \in C = \overline{\mathrm{co}}\,(\mathrm{dom}\,\Gamma)$ *and* $\bar{d} \in T_C(\bar{x})$, *we define*

$$\widetilde{\gamma}(\bar{x}, \bar{d}) = \liminf_{t \to 0_+} \inf_{x^*} [\, \langle x^*, \bar{d} \rangle \, : \, x^* \in \widetilde{\Gamma}(\bar{x} + t\bar{d}) \,]$$

and

$$\gamma_{\mathrm{S}}\,(\bar{x}, \bar{d}) = \liminf_{(d,t) \to (\bar{d}, 0_+)} \inf_{x^*} [\, \langle x^*, d \rangle \, : \, x^* \in \Gamma(\bar{x} + td), \quad \bar{x} + td \in \mathrm{S} \,].$$

Then, we have the following results.

Theorem 2.3.3 *Let* $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ *be a monotone multivalued map. Denote* $D = \mathrm{co}\,(\mathrm{dom}\,(\Gamma))$. *Assume that* $\mathrm{int}\,(\mathrm{D}) \neq \emptyset$ *and we are given* $S \subset \mathrm{dom}\,(\Gamma)$ *and an open convex subset* V *of* D *such that* $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{S}) = \mathrm{cl}\,(\mathrm{V})$. *Then,*

a) $\widetilde{\Gamma}(x) = \mathrm{co}\,(\Gamma_{\mathrm{S}}(\mathrm{x})), \ \forall\, \mathrm{x} \in \mathrm{V}$.

b) $\widetilde{\Gamma}$ *is monotone on* V.

It follows that any maximal monotone map containing Γ *coincides with* $\widetilde{\Gamma}$ *on* V.

Proof. i) It is clear that $\mathrm{graph}\,(\Gamma_{\mathrm{S}}) \subset \mathrm{cl}\,(\mathrm{graph}\,(\Gamma)) \subset \mathrm{graph}\,(\widetilde{\Gamma})$. Let $\bar{x} \in V \subset \mathrm{int}\,(\mathrm{D})$. By Theorem 2.3.1, there exist a compact K and a neighborhood $V_{\bar{x}}$ of $\bar{x}$, $V_{\bar{x}} \subset V$ such that for all $x \in V_{\bar{x}}$, $\Gamma_{\mathrm{S}}\,(x) \subset \widetilde{\Gamma}(x) \subset K$. For such x, the set $\Gamma_{\mathrm{S}}\,(x)$ is bounded, it is closed since $\mathrm{graph}\,(\Gamma_{\mathrm{S}})$ is closed. Therefore, one has

$$\mathrm{co}\,(\Gamma_{\mathrm{S}}(\mathrm{x})) = \overline{\mathrm{co}}\,(\Gamma_{\mathrm{S}}(\mathrm{x})) \subset \widetilde{\Gamma}(\mathrm{x}), \quad \forall\, \mathrm{x} \in \mathrm{V}_{\bar{\mathrm{x}}}.$$

ii) Next, we prove that $\widetilde{\Gamma}(\bar{x}) \subset \mathrm{co}\,(\Gamma_{\mathrm{S}}(\bar{\mathrm{x}}))$. Assume, for contradiction, that there exists $a^* \in \widetilde{\Gamma}(\bar{x})$ such that $a^* \notin \mathrm{co}\,(\Gamma_{\mathrm{S}}(\bar{\mathrm{x}}))$. In view of separation theorems (see for instance [35], section 11), there exists a vector d, $\|d\| = 1$, such that

$$\sup[\ \langle d, \xi^* - a^* \rangle : \xi^* \in \mathrm{co}\,(\Gamma_{\mathrm{S}}(\bar{\mathrm{x}}))\ ] < 0. \tag{2.9}$$

Since $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{S}) = \mathrm{cl}\,(\mathrm{V})$, there exist a sequence of vectors $\{d_k\} \in X$ and a sequence of positive real numbers $\{t_k\}$ such that

$$x_k = \bar{x} + t_k d_k \in S \cap V_{\bar{x}}\ ,\quad d_k \to d \ \text{ and } \ t_k \to 0 \text{ as } k \to +\infty.$$

Let $x_k^* \in \Gamma(x_k)$, then $x_k^* \in K$. Without loss of generality, we assume that the whole sequence $\{x_k^*\}$ converges to some $\bar{x}^*$. Then $\bar{x}^* \in \Gamma_{\mathrm{S}}\,(\bar{x})$.

On the other hand, since

$$a^* \in \widetilde{\Gamma}(\bar{x}) = \{\ \bar{x}^* :\ \langle x^* - \bar{x}^*, x - \bar{x} \rangle \geq 0,\ \forall (x, x^*) \in \mathrm{graph}\,(\Gamma)\ \}$$

and for all k, $(x_k, x_k^*) \in \mathrm{graph}\,(\Gamma)$, then

$$\langle x_k^* - a^*, d_k \rangle = \frac{1}{t_k} \langle x_k^* - a^*, x_k - \bar{x} \rangle \geq 0.$$

Thus, for every k

$$\langle x_k^* - a^*, d_k \rangle \geq 0.$$

Passing to the limit, we obtain $\langle \bar{x}^* - a^*, d \rangle \geq 0$, in contradiction with (2.9).

iii) It remains to prove that $\widetilde{\Gamma}$ is monotone on V. It suffices to show that Γ_{S} is monotone on V. Let $x^* \in \Gamma_{\mathrm{S}}\,(x)$ and $y^* \in \Gamma_{\mathrm{S}}\,(y)$. Then, there exist two sequences $\{(x_k, x_k^*)\}$ and $\{(y_k, y_k^*)\}$ in $\mathrm{graph}\,(\Gamma)$ that converge to (x, x^*) and (y, y^*) respectively. Since Γ is monotone, one has $\langle x_k^* - y_k^*, x_k - y_k \rangle \geq 0$. Passing to the limit, we obtain

$$\langle x^* - y^*, x - y \rangle \geq 0$$

as required. ■

Corollary 2.3.2 *Let* $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ *be a monotone multivalued map and* $\bar{x} \in \mathrm{int}\,(\mathrm{dom}(\Gamma))$*. Assume that* $\Gamma(\bar{x})$ *is a convex subset of* X^* *and the map* Γ *is closed on a neighborhood* W *of* $\bar{x}$*. Then* $\widetilde{\Gamma}(\bar{x}) = \Gamma(\bar{x})$*.*

Proof. Choose for W a convex open neighborhood, next set $\mathrm{S} = \mathrm{W}$ in the theorem. The closedness of Γ implies that $\Gamma_{\mathrm{S}}(\bar{x}) = \Gamma(\bar{x})$, and therefore $\widetilde{\Gamma}(\bar{x}) = \Gamma(\bar{x})$ as required. ∎

Theorem 2.3.4 *Under the assumptions of Theorem 2.3.3, it holds, for all $\bar{x} \in V$ and $\bar{d} \in X$:*

$$\begin{array}{llcl} a) & \widetilde{\gamma}(\bar{x}, \bar{d}) & = & \lim\limits_{t \to 0_+} \inf\limits_{x^*} [\, \langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x} + t\bar{d})\,]. \\ b) & \widetilde{\gamma}(\bar{x}, \bar{d}) & = & \lim\limits_{t \to 0_+} \sup\limits_{x^*} [\, \langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x} + t\bar{d})\,]. \\ c) & \widetilde{\gamma}(\bar{x}, \bar{d}) & = & \sup [\, \langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x})\,]. \\ d) & \widetilde{\gamma}(\bar{x}, \bar{d}) & = & \gamma_{\mathrm{S}}(\bar{x}, \bar{d}). \end{array}$$

Proof.

a,b) Let t_1, t_2 be such that $0 < t_1 < t_2$ and $\bar{x} + t_1\bar{d},\ \bar{x} + t_2\bar{d} \in V$. Since $\widetilde{\Gamma}$ is monotone on V, for all $x_1^* \in \widetilde{\Gamma}(\bar{x} + t_1\bar{d})$, $x_2^* \in \widetilde{\Gamma}(\bar{x} + t_2\bar{d})$,

$$(t_2 - t_1)\langle x_2^* - x_1^*, \bar{d} \rangle \geq 0$$

and therefore

$$\langle x_2^*, \bar{d} \rangle \geq \langle x_1^*, \bar{d} \rangle, \quad \forall\, x_i^* \in \widetilde{\Gamma}(\bar{x} + t_i\bar{d}),\ i = 1, 2.$$

It follows that

$$\sup[\, \langle x_2^*, \bar{d} \rangle : x_2^* \in \widetilde{\Gamma}(\bar{x} + t_2\bar{d})\,] \geq \inf[\, \langle x_2^*, \bar{d} \rangle : x_2^* \in \widetilde{\Gamma}(\bar{x} + t_2\bar{d})\,] \geq$$

$$\geq \sup[\, \langle x_1^*, \bar{d} \rangle : x_1^* \in \widetilde{\Gamma}(\bar{x} + t_1\bar{d})\,] \geq \inf[\, \langle x_1^*, \bar{d} \rangle : x_1^* \in \widetilde{\Gamma}(\bar{x} + t_1\bar{d})\,].$$

Hence

$$\begin{array}{rcl} \widetilde{\gamma}(\bar{x}, \bar{d}) & = & \lim\limits_{t \to 0_+} \inf\limits_{x^*} [\, \langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x} + t\bar{d})\,] \\ & = & \lim\limits_{t \to 0_+} \sup\limits_{x^*} [\, \langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x} + t\bar{d})\,]. \end{array}$$

c) For any $t > 0$ such that $\bar{x} + t\bar{d} \in V$ and $x_t^* \in \Gamma(\bar{x} + t\bar{d})$ one has

$$\langle x_t^* - x^*, \bar{d} \rangle = \frac{1}{t} \langle x_t^* - x^*, \bar{x} + t\bar{d} - \bar{x} \rangle \geq 0 \quad \text{for all } x^* \in \widetilde{\Gamma}(\bar{x}),$$

and therefore

$$\widetilde{\gamma}(\bar{x}, \bar{d}) \geq \sup \left[\langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x}) \right].$$

Now suppose, for contradiction, that the converse inequality does not hold. Then there exists λ such that

$$\widetilde{\gamma}(\bar{x}, \bar{d}) > \lambda > \sup \left[\langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x}) \right].$$

Since $\widetilde{\Gamma}$ is usc in $\bar{x}$, there exists $\bar{t} > 0$ such that for all $t \in]0, \bar{t}[$, $\bar{x} + t\bar{d} \in V$ and

$$\lambda > \sup_{x^*} \left[\langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x} + t\bar{d}) \right].$$

Then

$$\lambda \geq \liminf_{t \to 0_+} \sup_{x^*} \left[\langle x^*, \bar{d} \rangle : x^* \in \widetilde{\Gamma}(\bar{x} + t\bar{d}) \right] \geq \widetilde{\gamma}(\bar{x}, \bar{d}).$$

In contradiction with the assumption on λ.

d) We shall prove that

$$\begin{aligned} \widetilde{\gamma}(\bar{x}, \bar{d}) &= \liminf_{(d,t) \to (\bar{d}, 0_+)} \inf_{x^*} \left[\langle x^*, d \rangle : x^* \in \Gamma(\bar{x} + td), \quad \bar{x} + td \in \bar{x}^* \in \mathrm{S} \right] \\ &= \limsup_{(d,t) \to (\bar{d}, 0_+)} \sup_{x^*} \left[\langle x^*, d \rangle : x^* \in \Gamma(\bar{x} + td), \quad \bar{x} + td \in \mathrm{S} \right], \end{aligned}$$

from what the result will follow. By definition, for all $t > 0$

$$\langle x^* - \bar{x}^*, d \rangle = \frac{1}{t} \langle x^* - \bar{x}^*, (\bar{x} + td) - \bar{x} \rangle \geq 0 \; \forall x^* \in \Gamma(\bar{x} + td), \, \bar{x}^* \in \widetilde{\Gamma}(\bar{x}).$$

Then

$$\inf_{x^*} \left[\langle x^*, d \rangle : x^* \in \Gamma(\bar{x} + td), \quad \bar{x} + td \in \mathrm{S} \right] \geq \langle \bar{\mathrm{x}}^*, \mathrm{d} \rangle \quad \text{for all } \bar{\mathrm{x}}^* \in \widetilde{\Gamma}(\bar{\mathrm{x}}),$$

and therefore, for all $\bar{x}^* \in \widetilde{\Gamma}(\bar{x})$,

$$\liminf_{(d,t) \to (\bar{d}, 0_+)} \inf_{x^*} \left[\langle x^*, d \rangle : x^* \in \Gamma(\bar{x} + td), \quad \bar{x} + td \in \mathrm{S} \right] \geq \langle \bar{\mathrm{x}}^*, \bar{\mathrm{d}} \rangle.$$

Then, it follows from c),

$$\liminf_{(d,t)\to(\bar{d},0_+)} \inf_{x^*} [\, \langle x^*, d\rangle \,:\, x^* \in \Gamma(\bar{x}+td), \quad \bar{x}+td \in \mathrm{S}\,] \geq \tilde{\gamma}(\bar{\mathrm{x}}, \bar{\mathrm{d}}). \quad (2.10)$$

On the other hand, since $\widetilde{\Gamma}$ is usc in $\bar{x}$, given $\epsilon > 0$, there exist $\delta > 0$ and an open bounded convex neighborhood $W_{\bar{d}}$ of $\bar{d}$, such that $t \in (0, \delta)$ and $d \in W_{\bar{d}}$, implies $\bar{x} + td \in V$ and

$$\Gamma(\bar{x}+td) \subset \widetilde{\Gamma}(\bar{x}) + \epsilon B_1(0), \quad \forall\, \bar{x}+td \in V \cap \mathrm{S}\,,$$

where $B_1(0)$ is the Euclidean unit ball of X^*. It follows from c),

$$\limsup_{(d,t)\to(\bar{d},0_+)} \sup_{x^*} [\, \langle x^*, d\rangle \,:\, x^* \in \Gamma(\bar{x}+td), \; \bar{x}+td \in \mathrm{S}\,] \leq \gamma(\bar{\mathrm{x}}, \bar{\mathrm{d}}) + \epsilon \mathrm{M},$$

for some $M > 0$. Taking $\epsilon \to 0_+$,

$$\limsup_{(d,t)\to(\bar{d},0_+)} \sup [\, \langle x_t^*, d\rangle \,:\, x_t^* \in \Gamma(\bar{x}+td), \quad \bar{x}+td \in \mathrm{S}\,] \leq \tilde{\gamma}(\bar{\mathrm{x}}, \bar{\mathrm{d}}). \quad (2.11)$$

The result follow from (2.10) and (2.11). ■

The two latter results were concerned with points in the interior of $C = \overline{\mathrm{co}}(\mathrm{dom}\,\Gamma)$. Next, we consider points belonging to the boundary of C. We begin with the following proposition.

Proposition 2.3.3 *Let* $\Gamma : X \longrightarrow X^*$ *be a monotone multivalued map. Assume that* $\mathrm{int}\,(\mathrm{C}) \neq \emptyset$, $\bar{x} \in \mathrm{bd}\,(\mathrm{C})$ *and there exist a subset* $S \subset \mathrm{dom}\,(\Gamma)$ *and a neighborhood* V *of* $\bar{x}$ *satisfying* $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{S}) = \mathrm{cl}\,(\mathrm{V} \cap \mathrm{C})$.

Assume also that there exists a sequence $\{(x_k, x_k^*)\}_{k\in\mathbb{N}} \subset (\mathrm{S} \times \mathrm{X}^*) \cap \mathrm{graph}\,(\Gamma)$ *such that* $x_k = \bar{x} + t_k d_k$, $t_k \to 0_+$, $d_k \to \bar{d} \in \mathrm{int}\,(\mathrm{T_C}(\bar{\mathrm{x}}))$ *and* $\|x_k^*\| \to +\infty$. *Then,*

a) $\Gamma_{\mathrm{S}}\,(\bar{x}) = \widetilde{\Gamma}(\bar{x}) = \emptyset$,

b) $\gamma_{\mathrm{S}}\,(\bar{x}, d) = \tilde{\gamma}(\bar{x}, d) = -\infty, \quad \forall\, d \in \mathrm{int}\,(\mathrm{T_C}(\bar{\mathrm{x}}))$,

c) For all $(x_k, x_k^*) \in \mathrm{graph}\,(\Gamma)$ *with* $\{x_k\}$ *converging to* $\bar{x}$ *one has* $\|x_k^*\| \to +\infty$.

Proof.

a) Since $\Gamma_S(\bar{x}) \subset \widetilde{\Gamma}(\bar{x})$, we shall prove that $\widetilde{\Gamma}(\bar{x}) = \emptyset$. Assume for contradiction that there exists $\bar{x}^* \in \widetilde{\Gamma}(\bar{x})$. Without loss of generality, we assume that the whole sequence $\{\frac{x_k^*}{\|x_k^*\|}\}_{k\in\mathbb{N}}$ converges to some w^*. Since $x_k \in S \subset \mathrm{dom}\,(\Gamma)$ and $x_k^* \in \Gamma(x_k)$

$$\langle \frac{x_k^*}{\|x_k^*\|} - \frac{z^*}{\|x_k^*\|}, x_k - z\rangle \geq 0, \quad \forall z \in \mathrm{int}\,(\mathrm{C}),\ \mathrm{z}^* \in \widetilde{\Gamma}(\mathrm{z}).$$

Passing to the limit we obtain,

$$\langle w^*, \bar{x} - z\rangle \geq 0, \quad \forall z \in C.$$

It follows that $w^* \in N_C(\bar{x})$. Since $\|w^*\| = 1$ and $\bar{d} \in \mathrm{int}\,(\mathrm{T_C}(\bar{\mathrm{x}}))$ one has

$$\langle w^*, \bar{d}\rangle < 0. \tag{2.12}$$

On the other hand, $(x_k, x_k^*) \in \mathrm{graph}\,(\Gamma)$ and $(\bar{x}, \bar{x}^*) \in \mathrm{graph}\,(\widetilde{\Gamma})$. Then

$$\langle \frac{x_k^*}{\|x_k^*\|} - \frac{\bar{x}^*}{\|x_k^*\|}, d_k\rangle = \frac{1}{t_k\|x_k^*\|}\langle x_k^* - \bar{x}^*, x_k - \bar{x}\rangle \geq 0.$$

Passing to the limit, we obtain $\langle w^*, \bar{d}\rangle \geq 0$, in contradiction with (2.12).

b) Let $d \in \mathrm{int}\,(\mathrm{T_C}(\bar{\mathrm{x}}))$. Consider a sequence $\{(x_k, x_k^*)\}_{k\in\mathbb{N}} \subset \mathrm{graph}\,(\widetilde{\Gamma})$ such that $x_k = \bar{x} + t_k d_k$, $t_k \to 0_+$ and $d_k \to d$. We shall prove that

$$\beta = \limsup_{k\to+\infty}\langle x_k^*, d_k\rangle = -\infty,$$

from what the result will follow. We assume, for contradiction, that $-\infty < \beta$. Since $\widetilde{\Gamma}(\bar{x}) = \emptyset$ and, by definition, the graph of $\widetilde{\Gamma}$ is closed, the whole sequence $\|x_k^*\|$ converges to $+\infty$ as $k \to +\infty$. Without loss of generality, assume that the whole sequence $\{\frac{x_k^*}{\|x_k^*\|}\}_{k\in\mathbb{N}}$ converges to some w^*. Since $\widetilde{\Gamma}$ is monotone on $\mathrm{int}\,(\mathrm{C}) \cap \mathrm{S} \cap \mathrm{V}$, $w^* \in N_C(\bar{x})$, and therefore $\langle w^*, d\rangle < 0$. Thus, there exists $k_0 \in \mathbb{N}$, such that $k \geq k_0$, implies

$$\frac{\beta - 1}{\|x_k^*\|} \leq \langle \frac{x_k^*}{\|x_k^*\|}, d_k\rangle < 0.$$

Passing to the limit, obtain $0 \leq \langle w^*, d\rangle < 0$, a contradiction.

c) Follows from a) and from the fact that graph $(\widetilde{\Gamma})$ is closed. ■

The following result establishes a formulation of $\widetilde{\Gamma}$ on the boundary of the convex hull of dom (Γ).

Theorem 2.3.5 *Let* $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ *be a monotone multivalued map. Denote by C the closure of the convex hull of* dom (Γ)*. Assume that* int $(\mathrm{C}) \neq \emptyset$*, that* $\bar{x} \in \mathrm{bd}\,(\mathrm{C})$ *and that there exist a subset* $\mathrm{S} \subset \mathrm{dom}\,(\Gamma)$ *and a neighborhood* $V \in \mathcal{N}(\bar{x})$ *satisfying* $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{S}) = \mathrm{cl}\,(\mathrm{V} \cap \mathrm{C})$*. Then,*

$$\widetilde{\Gamma}(\bar{x}) = \overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x}))) + N_C(\bar{x}).$$

Proof. i) We prove that $\overline{\mathrm{co}}(\Gamma_S(\bar{x})) + N_C(\bar{x}) \subset \widetilde{\Gamma}(\bar{x})$. Indeed, by definition,

$$\mathrm{graph}\,(\Gamma_{\mathrm{S}}) \subset \mathrm{cl}\,(\mathrm{graph}\,(\Gamma)) \subset \mathrm{graph}\,(\widetilde{\Gamma}).$$

Thus, $\Gamma_{\mathrm{S}}(\bar{x}) \subset \widetilde{\Gamma}(\bar{x})$ and therefore $\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x})) \subset \widetilde{\Gamma}(\bar{x})$ because $\widetilde{\Gamma}(\bar{x})$ is closed and convex. On the other hand, by Proposition 2.3.2, $\widetilde{\Gamma}(\bar{x}) = \widetilde{\Gamma}(\bar{x}) + N_C(\bar{x})$. It follows that

$$\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x})) + N_C(\bar{x}) \subset \widetilde{\Gamma}(\bar{x}).$$

ii) We prove that $\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x})) + N_C(\bar{x})$ is closed. In view of the result on the closure of the sum of two closed convex sets, it is enough to prove that

$$-N_C(\bar{x}) \cap [\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x}))]_\infty = \{0\}.$$

Since $\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x})) \subset \widetilde{\Gamma}(\bar{x})$, Proposition 2.3.2 implies

$$[\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x}))]_\infty \subset (\widetilde{\Gamma}(\bar{x}))_\infty \subset N_C(\bar{x}).$$

Next, $-N_C(\bar{x}) \cap N_C(\bar{x}) = \{0\}$, because int $(\mathrm{C}) \neq \emptyset$. Therefore,

$$-N_C(\bar{x}) \cap [\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x}))]_\infty \subset -N_C(\bar{x}) \cap N_C(\bar{x}) = \{0\},$$

as required.
iii) We prove that $\widetilde{\Gamma}(\bar{x}) \subset \overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x})) + N_C(\bar{x})$. Assume, for contradiction, that there exists $\bar{x}^* \in \widetilde{\Gamma}(\bar{x})$ such that $\bar{x}^* \notin \overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x})) + N_C(\bar{x})$. Since

$\overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(\bar{x})) + N_C(\bar{x})$ is closed and convex, applying separation theorems (see for instance [35], section 11), there exists a vector $\bar{d}$, $\|\bar{d}\| = 1$ such that

$$\sup[\,\langle \bar{d}, x_1^* + x_2^*\rangle : \; x_1^* \in \Gamma_{\mathrm{S}}(\bar{x}), \; x_2^* \in N_C(\bar{x})\,] < \langle \bar{d}, \bar{x}^*\rangle. \tag{2.13}$$

Since $N_C(\bar{x})$ is a cone,

$$0 = \sup\,[\,\langle \bar{d}, x_2^*\rangle : x_2^* \in N_C(\bar{x})\,]. \tag{2.14}$$

Hence $\bar{d} \in T_C(\bar{x})$. On the other hand, combining (2.14) and (2.13) one obtains

$$\sup\,[\,\langle \bar{d}, x^* - \bar{x}^*\rangle : x_1^* \in \Gamma_{\mathrm{S}}(\bar{x})\,] < 0.$$

We shall show that there exists some $d \in \mathrm{int}\,(\mathrm{T_C(\bar{x})})$ such that

$$\sup\,[\,\langle d, x^* - \bar{x}^*\rangle : x_2^* \in \Gamma_{\mathrm{S}}(\bar{x})\,] < 0.$$

Take some $v \in \mathrm{int}\,(\mathrm{T_C(\bar{x})})$. For all positive integer k, set

$$d_k = \bar{d} + \frac{1}{k} v \in \mathrm{int}\,(\mathrm{T_C(\bar{x})}).$$

We shall prove that for k large enough

$$\sup\,[\,\langle d_k, x^* - \bar{x}^*\rangle : x^* \in \Gamma_{\mathrm{S}}(\bar{x})\,] < 0.$$

If not, for all k, there exists $x_k^* \in \Gamma_S(\bar{x})$ such that

$$\frac{1}{k} + \langle d_k, x_k^* - \bar{x}^*\rangle \geq 0 > \langle \bar{d}, x_k^* - \bar{x}^*\rangle. \tag{2.15}$$

Since $\Gamma_S(\bar{x})$ is closed, it follows that $\|x_k^*\| \to +\infty$ as $k \to +\infty$. Without loss of generality, we assume that $\frac{x_k^*}{\|x_k^*\|}$ converges to w^*. Then, proceeding as in the proof Proposition 2.3.3 a), $w^* \in N_C(\bar{x})$. The relations in (2.15) imply

$$\frac{1}{\|x_k^*\|} + \langle v, \frac{x_k^*}{\|x_k^*\|} - \frac{\bar{x}^*}{\|x_k^*\|}\rangle = k[\frac{1}{k}\frac{1}{\|x_k^*\|} + \frac{1}{k}\langle v, \frac{x_k^*}{\|x_k^*\|} - \frac{\bar{x}^*}{\|x_k^*\|}\rangle] > 0.$$

Passing to the limit,

$$\langle v, w^*\rangle \geq 0,$$

a contradiction. Hence there exists $d \in \operatorname{int}(\mathrm{T_C}(\bar{\mathrm{x}}))$ such that

$$\sup\left[\,\langle d, x_2^* \rangle : x_2^* \in \Gamma_{\mathrm{S}}(\bar{x})\,\right] < \langle d, \bar{x}^* \rangle. \tag{2.16}$$

Next, let us consider a sequence $\{(x_k, x_k^*)\} \subset (\mathrm{S} \times \mathrm{X}^*) \cap \operatorname{graph}(\Gamma)$ such that $x_k = \bar{x} + t_k d_k$, $t_k \to 0_+$, $d_k \to d$. Since $d \in \operatorname{int}(\mathrm{T_C}(\bar{\mathrm{x}}))$ and, by assumption, $\widetilde{\Gamma}(\bar{x})$ is not empty, Proposition 2.3.3 implies that the sequence $\{x_k^*\}$ is bounded. Without loss of generality it can be assumed that x_k^* converges to some x^*. Then $x^* \in \Gamma_{\mathrm{S}}(\bar{x})$.

On the other hand, since $\bar{x}^* \in \widetilde{\Gamma}(\bar{x})$,

$$\langle \bar{x}^* - x_k^*, d_k \rangle = \frac{1}{t_k} \langle \bar{x}^* - x_k^*, x_k - \bar{x} \rangle \leq 0, \quad \forall\, k \in \mathbb{N}.$$

Passing to the limit

$$\langle \bar{x}^* - x^*, d \rangle \leq 0, \quad \text{with} \;\; x^* \in \Gamma_S(\bar{x}),$$

a contradiction with (2.16). Hence

$$\widetilde{\Gamma}(\bar{x}) \subset \overline{\mathrm{co}}(\Gamma_S(\bar{x})) + N_C(\bar{x}),$$

as required. ∎

We summarize the different results above in the following theorem.

Theorem 2.3.6 *Let $\Gamma : X \rightrightarrows X^*$ be a monotone multivalued map. Denote by C the closure of the convex hull of* $\operatorname{dom}(\Gamma)$. *Assume that* $\operatorname{int}(\mathrm{C}) \neq \emptyset$ *and that there exists* $\mathrm{S} \subset \operatorname{dom}(\Gamma)$ *such that* $\operatorname{cl}(\mathrm{S}) = \mathrm{C}$. *Then, the multivalued map $\Lambda : X \rightrightarrows X^*$ defined by*

$$\Lambda(x) = \begin{cases} \overline{\mathrm{co}}(\Gamma_{\mathrm{S}}(x)) + N_C(x) & \textit{if } \; x \in C, \\[2ex] \emptyset & \textit{if } \; x \notin C, \end{cases}$$

is the unique maximal monotone map containing Γ with domain contained in C.

Proof. By construction Λ is monotone on X. Theorems 2.3.3 and 2.3.5 imply

$$\Lambda(x) = \widetilde{\Gamma}(x) + N_C(x) \text{ for all } x \in X. \tag{2.17}$$

In order to prove the maximality of Λ, by Proposition 2.1.1 c) we must prove that

$$\widetilde{\Lambda}(x) = \Lambda(x) \text{ for all } x \in X.$$

We first show that

$$\widetilde{\Lambda}(x) = \Lambda(x) \text{ for all } x \in C. \tag{2.18}$$

Indeed, since, by construction, $\mathrm{graph}\,(\Gamma) \subset \mathrm{graph}\,(\Lambda)$, (2.17) and Proposition 2.1.1 c) imply

$$\Lambda(x) \subset \widetilde{\Lambda}(x) \subset \widetilde{\Gamma}(x) \subset \Lambda(x) \text{ for all } x \in C.$$

Thus,

$$\widetilde{\Lambda}(x) = \Lambda(x) = \widetilde{\Gamma}(x) \text{ for all } x \in C. \tag{2.19}$$

Next, we prove that

$$\widetilde{\Lambda}(x) = \emptyset \text{ for all } x \notin C.$$

Assume, for contradiction, that $\widetilde{\Lambda}(\tilde{x}) \neq \emptyset$, for some $\tilde{x} \notin C$. Define $H : X \overset{\longrightarrow}{\longrightarrow} X^*$ by

$$\mathrm{graph\,(H) = graph\,(\Lambda) \cup (\{\tilde{x}\} \times \widetilde{\Lambda}(\tilde{x}))}.$$

By definition, H is monotone. Denote by D the convex hull of $C \cup \{\tilde{x}\}$. In view of Theorem 2.3.1,

$$\widetilde{H}(z) \neq \emptyset \text{ for all } z \in \mathrm{int\,(D)}. \tag{2.20}$$

Consider $\bar{x} \in \mathrm{bd\,(C) \cap int\,(D)}$ and $0 \neq v \in N_C(\bar{x})$, such that

$$x_e = \bar{x} + v \in \mathrm{int\,(D)}.$$

Take $x_e^* \in \widetilde{H}(x_e)$. By definition,

$$\langle y^* - x_e^*, y - x_e \rangle \geq 0 \text{ for all } (y, y^*) \in \mathrm{graph\,(H)},$$

which implies in particular,

$$\langle y^* - x_e^*, y - x_e\rangle \geq 0 \;\text{ for all } (y, y^*) \in \operatorname{graph}(\Lambda). \tag{2.21}$$

On the other hand, since $\operatorname{graph}(\Lambda) \subset \operatorname{graph}(\mathrm{H})$, relation (2.20) and Proposition 2.1.1 a), imply that $\widetilde{\Lambda}(\bar{x}) \neq \emptyset$. Thus, from (2.19),

$$\emptyset \neq \Lambda(\bar{x}) = \widetilde{\Lambda}(\bar{x}).$$

From (2.21), we have

$$\langle x^* + tv - x_e^*, \bar{x} - x_e\rangle \geq 0 \;\text{ for all } x^* \in \widetilde{\Gamma}(\bar{x}),\, t \geq 0.$$

Then

$$\langle x^* - x_e^*, \bar{x} - x_e\rangle - t\|v\|^2 \geq 0 \;\text{ for all } x^* \in \widetilde{\Gamma}(\bar{x}),\, t \geq 0.$$

Letting $t \to +\infty$, we have a contradiction. Thus

$$\widetilde{\Lambda}(x) = \emptyset = \Lambda(x) \;\text{ for all } x \notin C.$$

Consequently, from (2.18)

$$\widetilde{\Lambda}(x) = \Lambda(x) \;\text{ for all } x \in X,$$

as required.

It remains to prove the uniqueness of Λ. Assume that Λ_1 is a maximal monotone map satisfying the above property. Since $\operatorname{graph}(\Gamma) \subset \operatorname{graph}(\Lambda_1)$, Proposition 2.1.1, implies that

$$\operatorname{graph}(\Lambda_1) = \operatorname{graph}(\widetilde{\Lambda}_1) \subset \operatorname{graph}(\widetilde{\Gamma}) \tag{2.22}$$

and

$$\operatorname{graph}(\widetilde{\widetilde{\Gamma}}) \subset \operatorname{graph}(\widetilde{\Lambda}_1) = \operatorname{graph}(\Lambda_1). \tag{2.23}$$

Thus, from (2.22), (2.23) and monotonicity of $\widetilde{\Gamma}$ on C, imply that

$$\Lambda_1(x) \subset \widetilde{\Gamma}(x) = \widetilde{\widetilde{\Gamma}}(x) \subset \Lambda_1(x) \;\text{ for all } x \in C.$$

Hence, the maximality of Λ_1, implies that

$$\Lambda_1(x) = \Lambda_1(x) + N_C(x) = \widetilde{\Gamma}(x) + N_C(x) = \Lambda(x) \text{ for all } x \in X$$

and the uniqueness follows. ■

Remark. The above theorem is an extension of the well known result on convex functions (see for instance [35]): assume that f is proper, convex, lower semicontinuous and such that $\operatorname{int}(\operatorname{dom}(\mathrm{f})) \neq \emptyset$. Then

$$\partial f(x) = \overline{\mathrm{co}}(S(x)) + K(x) \text{ for all } x,$$

where $K(x)$ is the normal cone to $\operatorname{dom}(\mathrm{f})$ at x (empty if $x \notin \operatorname{dom}(\mathrm{f})$) and $S(x)$ is the set of all limits of sequences of the form $\{\nabla f(x_k)\}$ such that f is differentiable at x_k and $\{x_k\}$ tends to x. We will see later, that maximal monotone maps are reduced to a point almost everywhere on the interior of their domain.

As a direct consequence of this theorem we have the well known result on the sum of two maximal monotone maps.

Proposition 2.3.4 *Let* $\Gamma^i : X \overset{\longrightarrow}{\longrightarrow} X^*$, $i = 1, 2$, *be two maximal monotone maps. Assume that* $\operatorname{int}(\operatorname{dom}(\Gamma^1)) \cap \operatorname{int}(\operatorname{dom}(\Gamma^2)) \neq \emptyset$. *Then the multivalued map* $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ *defined by*

$$\Gamma(x) = \Gamma^1(x) + \Gamma^2(x)$$

is also maximal monotone.

Proof. Denote by C the closure of $\operatorname{dom}(\Gamma) = \operatorname{dom}(\Gamma^1) \cap \operatorname{dom}(\Gamma^2)$. It is clear C is convex with nonempty interior. Consider $S \subset \operatorname{int}(\mathrm{C})$ such that $\operatorname{cl}(\mathrm{S}) = \mathrm{C}$. In view of Theorem 2.3.6, we shall prove that

$$\overline{\mathrm{co}}(\Gamma_S(x)) + N_C(x) \subset \Gamma(x) \text{ for all } x \in X.$$

We first prove that Γ is closed. Let $\{(x_k, x_k^*)\} \subset \operatorname{graph}(\Gamma)$ such that $(x_k, x_k^*) \to (x, x^*)$. Then there exist two sequences $\{x_{1k}^*\}$ and $\{x_{2k}^*\}$ with $x_{1k}^* \in \Gamma^1(x_k)$, $x_{2k}^* \in \Gamma^2(x_k)$ and $x_{1k}^* + x_{2k}^* = x^*$.

We claim that $\{x_{1k}^*\}$ (and therefore also $\{x_{2k}^*\}$) is bounded. Assume, for contradiction, that the sequence of $\{x_{1k}^*\}$ is unbounded. Without loss of generality we assume that the sequence $\{\|x_{1k}^*\|\}$ converges to $+\infty$ and $\{\frac{x_{1k}^*}{\|x_{1k}^*\|}\}$ converges to w^*. Then $\{\|x_{2k}^*\|\}$ converges to $+\infty$ and $\{\frac{x_{2k}^*}{\|x_{2k}^*\|}\}$ converges to $-w^*$. Then w^* belongs to the normal cone of $\mathrm{dom}\,(\Gamma_1)$ at x and $-w^*$ belongs to the normal cone of $\mathrm{dom}\,(\Gamma_2)$. This is a contradiction with $\mathrm{int}\,(\mathrm{dom}\,(\Gamma_1)) \cap \mathrm{int}\,(\mathrm{dom}\,(\Gamma_2)) \neq \emptyset$. Since the two sequences $\{x_{1k}^*\}$ and $\{x_{2k}^*\}$ are bounded we obtain that Γ is closed.

It is clear that, for all $x \in X$, $\Gamma(x)$ is convex. Thus the closedness of Γ and the definition of Γ_S imply the inclusion above. ■

2.4 Restriction of a maximal monotone map

In this section, we are given $\Gamma : X \times U \overset{\longrightarrow}{\longrightarrow} X^* \times U^*$ and a fixed point $\bar{u} \in U$. Define $\Sigma_{\bar{u}} : X \overset{\longrightarrow}{\longrightarrow} X^*$ by

$$\Sigma_{\bar{u}}(x) = \{\, x^* : \exists\, u^* \in U^* \text{ such that } (x^*, u^*) \in \Gamma(x, \bar{u})\,\}.$$

The graph of $\Sigma_{\bar{u}}$ is nothing else that the projection on $X \times X^*$ of the set

$$\mathrm{graph}\,(\Gamma) \cap (\mathrm{X} \times \{\bar{\mathrm{u}}\} \times \mathrm{X}^* \times \mathrm{U}^*).$$

The domain of $\Sigma_{\bar{u}}$ is such that

$$\mathrm{dom}\,(\Sigma_{\bar{\mathrm{u}}}) \times \{\bar{\mathrm{u}}\} = \mathrm{dom}\,(\Gamma) \cap (\mathrm{X} \times \{\bar{\mathrm{u}}\}). \tag{2.24}$$

We shall study some properties of the map $\Sigma_{\bar{u}}$.

It is clear that $\Sigma_{\bar{u}}$ is monotone if Γ is monotone. If $\bar{u} \notin \mathrm{proj}_{\,\mathrm{U}}(\mathrm{dom}\,(\Gamma))$ then $\Sigma_{\bar{u}}(x) = \emptyset$ for all $x \in X$. In the next theorem we assume that Γ is maximal monotone.

Theorem 2.4.1 *Let $\Gamma : X \times U \overset{\longrightarrow}{\longrightarrow} X^* \times U^*$ be a maximal monotone multi-valued map and let $(\bar{x}, \bar{u}) \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma))$. Then, there exists a neighborhood V of $\bar{x}$ such that $\widetilde{\Sigma}_{\bar{u}}(x) = \Sigma_{\bar{u}}(x)$ for every $x \in V$.*

Proof. Let $V \times W$ be an open convex neighborhood of $(\bar{x}, \bar{u})$ contained in the interior of dom (Γ) and a compact $K \subset X^* \times U^*$ such that

$$\Gamma(x,u) \subset K \quad \text{for all } (x,u) \in V \times W. \tag{2.25}$$

Such V, W and K exist in view of Theorem 2.3.1. For any $x \in V$, $\Sigma_{\bar{u}}(x)$ is a convex subset of U^* because $\Gamma(x, \bar{u})$ is a convex subset of $X^* \times U^*$. We shall prove that that $\Sigma_{\bar{u}}$ is closed on V. Assume that $x \in V$ and the sequence $\{(x_k, x_k^*)\} \subset (V \times X^*) \cap \operatorname{graph}(\Sigma_{\bar{u}})$ converges to (x, x^*). We must prove that (x, x^*) belongs to the graph of $\Sigma_{\bar{u}}$. By definition of $\Sigma_{\bar{u}}$, a sequence $\{u_k^*\} \subset U^*$ exists such that $(x_k^*, u_k^*) \in \Gamma(x_k, \bar{u})$. The sequence $\{u_k^*\}$ is bounded because of (2.25). Let u^* be a cluster point of the sequence. Without loss of generality, we assume that the whole sequence $((x_k, \bar{u}), (x_k^*, u_k^*))$ converges to $((x, \bar{u}), (x^*, u^*))$. Then, the maximal monotonicity of Γ implies that $(x^*, u^*) \in \Gamma(x, \bar{u})$. Next, the definition of $\Sigma_{\bar{u}}$ implies that (x, x^*) belongs to graph $(\Sigma_{\bar{u}})$, and therefore $\Sigma_{\bar{u}}$ is closed on V. Next apply Corollary 2.3.2 to $\Sigma_{\bar{u}}$. ■

The next result gives some information on the domain of $\Sigma_{\bar{u}}$.

Lemma 2.4.1 *Assume that* $\Gamma : X \times U \overset{\longrightarrow}{\longrightarrow} X^* \times U^*$ *is maximal monotone and the interior of its domain is nonempty. Assume also that* $\bar{u} \in \operatorname{proj}_{\mathrm{U}}(\operatorname{int}(\operatorname{dom}(\Gamma)))$. *Then the sets* $\operatorname{int}(\operatorname{dom}(\Sigma_{\bar{u}}))$ *and* $\operatorname{cl}(\operatorname{dom}(\Sigma_{\bar{u}}))$ *are convex and*

$$\begin{aligned} \operatorname{cl}(\operatorname{int}(\operatorname{dom}(\Sigma_{\bar{u}}))) &= \operatorname{cl}(\operatorname{dom}(\Sigma_{\bar{u}})), \\ \operatorname{int}(\operatorname{dom}(\Sigma_{\bar{u}})) &= \operatorname{int}(\operatorname{cl}(\operatorname{dom}(\Sigma_{\bar{u}}))). \end{aligned}$$

Furthermore, $x \in \operatorname{bd}(\operatorname{dom}(\Sigma_{\bar{u}}))$ *if and only if* $(x, \bar{u}) \in \operatorname{bd}(\operatorname{dom}(\Gamma))$.

Proof. Since Γ is maximal monotone, $\operatorname{cl}(\operatorname{int}(\operatorname{dom}(\Gamma)) = \operatorname{cl}(\operatorname{dom}(\Gamma))$ and $\operatorname{int}(\operatorname{cl}(\operatorname{dom}(\Gamma))) = \operatorname{int}(\operatorname{dom}(\Gamma))$. Combine with relation (2.24). ■

For the main result of this section, one needs the following lemma.

Lemma 2.4.2 *Let* K *be a closed convex subset of* $\mathbb{R}^n \times \mathbb{R}^p$ *and let* $C = \{ x \in \mathbb{R}^n : (x, 0) \in K \}$. *Assume that* $(0,0) \in \operatorname{bd}(\mathrm{K})$ *and there exists* $\widetilde{x}$ *such that*

$(\widetilde{x}, 0) \in \mathrm{int}\,(\mathrm{K})$. *Then* $0 \in \mathrm{bd}\,(\mathrm{C})$ *and the following relation holds between the normal cones at* $(0,0) \in K$ *and* 0 *to* C

$$x^* \in N_C(0) \iff \exists\, u^* \in \mathbb{R}^p \;\; \textit{such that} \;\; (x^*, u^*) \in N_K(0,0).$$

Proof. It is clear that if $(x^*, u^*) \in N_K(0,0)$ then $x^* \in N_C(0)$. To show the converse statement, assume that $x^* \in N_C(0)$, $x^* \neq 0$. Then, because $0 \in \mathrm{bd}\,(\mathrm{C})$,

$$0 = \inf_x [\, \langle -x^*, x \rangle : x \in C \,]. \qquad (P)$$

Let us consider a convex function $g : \mathbb{R}^n \times \mathbb{R}^p \to]-\infty, +\infty[$ such that

$$g(\widetilde{x}, 0) < 0, \;\; K = \{\, (x,u) : g(x,u) \leq 0 \,\} \subset \mathrm{int}\,(\mathrm{dom}\,(\mathrm{g})).$$

Such a function is easily constructed. Then (P) can be written as

$$0 = \inf_{x,u} [\, \langle -x^*, x \rangle + \langle 0, u \rangle : g(x,u) \leq 0, \; u = 0 \,]. \qquad (P)$$

Because the Slater condition holds for this convex problem and $(0,0)$ is solution, there is $\lambda \geq 0$, $(z^*, v^*) \in \partial g(0,0)$ and $w^* \in \mathbb{R}^p$ such that

$$-x^* + \lambda z^* = 0, \;\; \lambda v^* + w^* = 0.$$

Then, because $x^* \neq 0$, $\lambda > 0$ and $\lambda^{-1}(x^*, -w^*) \in N_K(0,0)$. It follows that $(x^*, -w^*) \in N_K(0,0)$. ■

Now, we can prove the following basic result.

Theorem 2.4.2 *Assume that* $\Gamma : X \times U \rightrightarrows X^* \times U^*$ *is maximal monotone and the interior of its domain is nonempty. Let* $\bar{u} \in \mathrm{proj}_{\,\mathrm{U}}(\mathrm{int}\,(\mathrm{dom}\,(\Gamma)))$. *Then,* $\Sigma_{\bar{u}}$ *is maximal monotone on* X.

Proof. i) Let $x \in \mathrm{int}\,(\mathrm{dom}(\Sigma_{\bar{u}}))$, then $(x, \bar{u}) \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma))$. Theorem 2.4.1 implies that $\widetilde{\Sigma}_{\bar{u}}(x) = \Sigma_{\bar{u}}(x)$. We have proved that $\Sigma_{\bar{u}}$ is maximal monotone on the interior of its domain.

ii) Next, consider some x in the boundary of the domain of $\Sigma_{\bar{u}}$. Proceeding as in Theorem 2.4.1 we see that $\Sigma_{\bar{u}}(x)$ is convex. We shall prove that $\Sigma_{\bar{u}}$ is closed in x. Consider any sequence $\{(x_k, x_k^*)\} \subset \mathrm{graph}\,(\Sigma_{\bar{u}})$ converging to (x, x^*). Then a sequence $\{u_k^*\} \subset U^*$ exists such that $(x_k^*, u_k^*) \in \Gamma(x_k, \bar{u})$. We claim that the sequence $\{u_k^*\}$ is bounded. Otherwise, there exists a subsequence with $\|u_{k_l}^*\| \to +\infty$. Let w^* be a cluster point of the sequence $\{\|u_{k_l}^*\|^{-1} u_{k_l}^*\}$. For simplification and without loss of generality, we assume that the whole sequence $\{\|u_k^*\|^{-1} u_k^*\}$ converges to w^*. Proceeding as in the proof of Proposition 2.3.3 a), we deduce that $(0, w^*)$ belongs to the normal cone at $(x, \bar{u})$ to $\mathrm{dom}\,(\Gamma)$.

Choose $\bar{x}$ such that $(\bar{x}, \bar{u}) \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma))$ and $\epsilon > 0$ small enough in order that $(\bar{x}, \bar{u} + \epsilon w^*)$ still belongs to $\mathrm{dom}\,(\Gamma)$. Since $(0, w^*)$ belongs to the normal cone

$$0 < \epsilon = \langle (0, w^*), (\bar{x} - x, \epsilon w^*) \rangle \leq 0,$$

which is not possible.

Thus, the sequence $\{u_k^*\}$ is bounded, without loss of generality we assume that the whole sequence $\{((x_k, \bar{u}), (x_k^*, u_k^*))\}$ converges to $((x, \bar{u}), (x^*, u^*))$. Then, the maximal monotonicity of Γ implies that $(x^*, u^*) \in \Gamma(x, \bar{u})$ and therefore $x^* \in \Sigma(x)$. We deduce that $\Sigma_{\bar{u}}$ is closed on the whole space.

Combining with Lemma 2.4.2 and Theorem 2.3.6, we obtain the maximal monotonicity. ∎

2.5 Dealing with relative interiors

Until now, we have assumed that the affine hull of $\mathrm{dom}\,(\Gamma)$ is the whole space. Next, we shall generalize the previous results to the case where Γ is still monotone, $\mathrm{dom}\,(\Gamma) \neq \emptyset$ but $\mathrm{aff}\,(\mathrm{dom}\,(\Gamma)) \neq \mathrm{X}$. From now, we assume that

$$\mathrm{aff}\,(\mathrm{dom}\,(\Gamma)) = \mathrm{a} + \mathrm{L} \text{ with } \mathrm{a} \in \mathrm{dom}\,(\Gamma) \text{ and L is a linear subspace of X.}$$

Proposition 2.5.1 *Assume that Γ is maximal monotone, then*

a) $\Gamma(x) = \Gamma(x) + L^\perp$ *for all* $x \in \mathrm{dom}\,(\Gamma)$

b) there exists $\Sigma : L \longrightarrow L$ *maximal monotone such that*

$$\Sigma(l) = \{l^* \in L : l^* \in \Gamma(a+l)\} \;\; \textit{and} \;\; \Gamma(a+l) = \Sigma(l) + L^\perp \;\; \textit{for all} \;\; l \in L.$$

c) *the relative interior and the closure of* $\operatorname{dom}(\Gamma)$ *are convex sets. Moreover*

$$\operatorname{cl}(\operatorname{ri}(\operatorname{dom}(\Gamma))) = \operatorname{cl}(\operatorname{dom}(\Gamma)) \quad \textit{and} \quad \operatorname{ri}(\operatorname{cl}(\operatorname{dom}(\Gamma))) = \operatorname{ri}(\operatorname{dom}(\Gamma)).$$

It follows that Σ *can be also written as*

$$\Sigma(l) = \{l^* \in L : \exists\, l_1^* \in L^\perp \ \textit{with}\ l^* + l_1^* \in \Gamma(a+l)\}.$$

Proof.

a) Let $y^* \in L^\perp$. Then $\langle y^*, x - \bar{x}\rangle = 0$ for all $x, \bar{x} \in \operatorname{dom}(\Gamma)$.

Let $\bar{x}^* \in \Gamma(\bar{x})$, then for all $(x, x^*) \in \operatorname{graph}(\Gamma)$,

$$0 \leq \langle \bar{x}^* - x^*, \bar{x} - x\rangle = \langle \bar{x}^* + y^* - x^*, \bar{x} - x\rangle.$$

It follows that $\bar{x}^* + y^* \in \widetilde{\Gamma}(\bar{x}) = \Gamma(\bar{x})$.

b) i) Since $L + L^\perp = X = X^*$, it follows from definition of Σ that

$$\Gamma(a+l) = \Sigma(l) + L^\perp \quad \text{for all} \quad l \in L.$$

ii) By definition, Σ is monotone. Assume that $z^* \in \widetilde{\Sigma}(z)$. Then

$$\langle z^* - l^*, z - l\rangle \geq 0 \quad \text{for all} \quad (l, l^*) \in \operatorname{graph}(\Sigma).$$

Next, since $\Gamma(a+l) = \Sigma(l) + L^\perp$, for all $(a+l, x^*) \in \operatorname{graph}(\Sigma)$ and $w^* \in L^\perp$

$$\langle z^* - (x^* - w^*), (a+z) - (a+l)\rangle \geq 0.$$

This implies that $z \in L$ and therefore

$$\langle z^* - x^*, (a+z) - (a+l)\rangle \geq 0 \quad \text{for all} \quad (a+l, x^*) \in \operatorname{graph}(\Sigma).$$

Thus $z^* \in \widetilde{\Gamma}(z+a) = \Gamma(a+z)$. Therefore $z^* \in \Sigma(z)$.

c) By definition, $\operatorname{ri}(\operatorname{dom}(\Gamma)) = \mathrm{a} + \operatorname{int}(\operatorname{dom}(\Sigma))$ and $\operatorname{cl}(\operatorname{dom}(\Gamma)) = \mathrm{a} + \operatorname{cl}(\operatorname{dom}(\Sigma))$. The result follows from the maximality monotonicity of Σ and Corollary 2.3.1. ■

Next, we shall construct a maximal monotone extension of a monotone map. Recall that there are several ways to construct such extensions and there is no uniqueness of these extensions. We illustrate this plurality in the following example.

Example 2.5.1 *Let us consider* $\Gamma : \mathbb{R}^2 \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^2$ *be such that*

$$\Gamma(x_1, x_2) = \begin{cases} \{(0,0)\} & \textit{if } \ x_1 = x_2, \\ \emptyset & \textit{otherwise} \ . \end{cases}$$

Γ *is monotone. Next, consider* Γ_1, $\Gamma_2 : \mathbb{R}^2 \overset{\longrightarrow}{\longrightarrow} \mathbb{R}^2$ *be defined by*

$$\Gamma_1(x_1, x_2) = \begin{cases} \{\, t(1,-1) : \, t \in \mathbb{R} \,\} & \textit{if } \ x_1 = x_2, \\ \emptyset & \textit{otherwise} \end{cases}$$

and

$$\Gamma_2(x_1, x_2) = \{\, (0,0) \,\} \ \textit{ for all } \ (x_1, x_2) \in \mathbb{R}^2.$$

Γ_1 *and* Γ_2 *are two maximal monotone extensions of* Γ. $\operatorname{aff}(\operatorname{dom}(\Gamma_1) = \operatorname{aff}(\operatorname{dom}(\Gamma))$ *and* $\operatorname{aff}(\operatorname{dom}(\Gamma_2) = \mathbb{R}^2 \neq \operatorname{aff}(\operatorname{dom}(\Gamma))$.

In this work we shall construct maximal monotone extensions preserving the affine hulls of the domain.

We begin with the following proposition.

Proposition 2.5.2 *Let* $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ *monotone. Define* $\Sigma : L \overset{\longrightarrow}{\longrightarrow} L^* = L$ *by*

$$l^* \in \Sigma(l) \iff \exists \, l^*_\perp \in L^\perp \ \textit{such that} \ l^* + l^*_\perp \in \Gamma(a + l).$$

Then

a) The multivalued map Σ *is monotone*

b) We have the following relation between $\widetilde{\Gamma}$ *and* $\widetilde{\Sigma}$

$$\widetilde{\Gamma}(a + l) = \widetilde{\Sigma}(l) + L^\perp \ \textit{ for all } \ l \in L.$$

c) If indeed Σ is such that

$$\Gamma(a+l) = \Sigma(l) + L^{\perp} \quad \textit{for all} \quad l \in L.$$

Then Σ is maximal monotone if and only if Γ it so.

Proof. Monotonicity of Σ is easily shown. Let us prove b). Let $\bar{l}^* \in \widetilde{\Gamma}(a+\bar{l})$. Then, by definition of $\widetilde{\Gamma}$,

$$\langle \bar{l}^* - l^*, (a+\bar{l}) - (a+l) \rangle \geq 0 \quad \text{for all} \quad l \in \text{dom}\,(\Sigma),\ \mathrm{l}^* \in \Gamma(\mathrm{a}+\mathrm{l}).$$

By definition of Σ, last inequality implies that

$$\langle \bar{l}^* - (l^* - l^*_{\perp}), \bar{l} - l \rangle \geq 0 \ \forall \ (l, l^*) \in \text{graph}\,(\Sigma),\ \mathrm{l}^*_{\perp} \in \mathrm{L}^{\perp} \text{ s.t. } \mathrm{l}^* + \mathrm{l}^*_{\perp} \in \Gamma(\mathrm{a}+\mathrm{l}).$$

Then

$$\langle \bar{l}^* - l^*, \bar{l} - l \rangle \geq 0 \quad \text{for all} \quad (l, l^*) \in \text{graph}\,(\Sigma).$$

Thus $\bar{l}^* \in \widetilde{\Sigma}(\bar{l})$, and therefore $\widetilde{\Gamma}(a+\bar{l}) \subset \widetilde{\Sigma}(\bar{l}) = \widetilde{\Sigma}(\bar{l}) + L^{\perp}$.

Next, assume that $\bar{l}^* \in \widetilde{\Sigma}(\bar{l})$ and $\bar{l}^*_{\perp} \in L^{\perp}$. Then

$$\langle \bar{l}^* - l^*, \bar{l} - l \rangle \geq 0 \quad \text{for all} \quad (l, l^*) \in \text{graph}\,(\Sigma)$$

and

$$\langle \bar{l}^*_{\perp}, \bar{l} - l \rangle \geq 0 \quad \text{for all} \quad (l, l^*) \in \text{graph}\,(\Sigma).$$

Thus, for all $(a+l, l^*) \in \text{graph}\,(\Gamma)$, $\mathrm{l}^*_{\perp} \in \mathrm{L}^{\perp}$ such that $l^* - l^*_{\perp} \in \Sigma(l)$,

$$\langle \bar{l}^* + \bar{l}^*_{\perp} - (l^* - l^*_{\perp}), \bar{l} - l \geq 0.$$

Then

$$\langle \bar{l}^* + \bar{l}^*_{\perp} - l^*, \bar{l} - l \geq 0 \quad \text{for all} \quad (a+l, l^*) \in \text{graph}\,(\Gamma).$$

Therefore $\bar{l}^* + \bar{l}^*_{\perp} \in \widetilde{\Gamma}(a+l)$, and the reverse inclusion follows.

We now prove c). The maximal monotonicity of Γ implies the maximal monotonicity of Σ, see Proposition 2.5.1. In order to prove the converse, we first prove that $\text{dom}\,(\widetilde{\Gamma}) \subset \text{aff}\,(\text{dom}\,(\Gamma))$. Indeed, given $(\bar{x}, \bar{x}^*) \in \text{graph}\,(\widetilde{\Gamma})$, one has

$$\langle \bar{x}^* - l^*, \bar{x} - (a+l) \rangle \geq 0 \quad \text{for all} \quad (a+l, l^*) \in \text{graph}\,(\Gamma).$$

Hence

$$\langle \bar{x}^* - (l^* + l^*_\perp), \bar{x} - (a + l) \rangle \geq 0 \quad \text{for all} \quad (l, l^*) \in \operatorname{graph}(\Sigma),\ \mathrm{l}^*_\perp \in \mathrm{L}^\perp.$$

Therefore the inclusion follows. Thus, for all $l \in L$ so that $a + l \in \operatorname{dom}(\widetilde{\Gamma})$,

$$\widetilde{\Gamma}(a + l) = \widetilde{\Sigma}(l) + L^\perp = \Sigma(l) + L^\perp = \Gamma(a + l),$$

and the maximality of Γ follows. ■

The above proposition implies in particular that if Σ_{max} is a maximal extension of Σ, the multivalued map $\Gamma_{max} : X \overset{\longrightarrow}{\longrightarrow} X^*$ defined by $\Gamma_{max}(x) = \Sigma_{max}(x - a) + L^\perp$ is also a maximal extension of Γ.

As an application of Theorem 2.4.2 and Proposition 2.5.2, we have the following extension of Proposition 2.3.4.

Proposition 2.5.3 *Let $\Gamma_i : X \overset{\longrightarrow}{\longrightarrow} X^*$, $i = 1, 2$, be two maximal monotone maps. Assume that* $\operatorname{ri}(\operatorname{dom}(\Gamma_1)) \cap \operatorname{ri}(\operatorname{dom}(\Gamma_2)) \neq \emptyset$. *Then the multivalued map $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ defined by*

$$\Gamma(x) = \Gamma_1(x) + \Gamma_2(x) \quad \textit{for all } x \in \operatorname{dom}(\Gamma_1) \cap \operatorname{dom}(\Gamma_2)$$

is also maximal monotone.

Proof. Fix $a \in \operatorname{ri}(\operatorname{dom}(\Gamma_1)) \cap \operatorname{ri}(\operatorname{dom}(\Gamma_2))$. Then, there exists two linear subspaces L_1 and L_2 such that $\operatorname{aff}(\operatorname{dom}(\Gamma_1)) = \mathrm{a} + \mathrm{L}_1$ and $\operatorname{aff}(\operatorname{dom}(\Gamma_2)) = \mathrm{L}_2$.

Take $L = E_0 = L_1 \cap L_2$, $E_1 = L^\perp \cap L_1$, $E_2 = L^\perp \cap L_2$ and $E_3 = L_1^\perp \cap L_2^\perp$. Then $\operatorname{aff}(\operatorname{dom}(\Gamma)) = \mathrm{a} + \mathrm{L}$. Any $x \in X$ can be uniquely expressed as

$$x = a + l_0 + l_1 + l_2 + l_3 \quad \text{with} \ \ l_i \in E_i, \ \ i = 0, 1, 2, 3.$$

For $i = 1, 2$ we consider the maximal monotone map $\Sigma_i : L_i \overset{\longrightarrow}{\longrightarrow} L_i$ which is such that $\Gamma_i(a + l) = \Sigma_i(l) + L_i^\perp$ for any $l \in L_i$, next we consider the map $\widehat{\Sigma}_i : L \overset{\longrightarrow}{\longrightarrow} L$ which is such that

$$\widehat{\Sigma}_i(l) = \{\, l^* : \exists\, \xi^* \in E_i \ \text{ such that } \ (l^*, \xi^*) \in \Sigma_i(l, 0) \,\}.$$

These map are maximal monotone in view of Theorem 2.4.2. Next, Proposition 2.3.4 implies that the sum $\widehat{\Sigma} = \widehat{\Sigma}_1 + \widehat{\Sigma}_2$ is maximal monotone on L.

Let $x \in \mathrm{dom}\,(\Gamma)$. Let $l \in L$ be such that $x = a + l$. Then,

$$\Gamma_1(x) = \{x^* = l_0^* + l_1^* + l_2^* + l_3^* \,:\, (l_0^*, l_1^*) \in \Sigma_1(l, 0),\, l_2^* \in E_2,\, l_3^* \in E_3\},$$

and

$$\Gamma_2(x) = \{x^* = t_0^* + t_1^* + t_2^* + t_3^* \,:\, (t_0^*, t_2^*) \in \Sigma_2(l, 0),\, t_1^* \in E_1,\, t_3^* \in E_3\}.$$

We deduce that

$$\Gamma(x) = \{x^* = a^* + b^* \,:\, a^* \in \widehat{\Sigma}(l),\, b^* \in L^{\perp}\}.$$

From what we deduce that Γ is maximal monotone. ■

The following result is well known.

Corollary 2.5.1 *Let* $F : X \overset{\longrightarrow}{\longrightarrow} X^*$ *be a monotone multivalued map and* $\alpha > 0$. *F is maximal monotone if and only if* $\Gamma = F + \alpha I$ *is maximal monotone.*

Proof. $\Rightarrow$) Follows from Proposition 2.5.3, since αI is maximal monotone.
$\Leftarrow$) Since F is monotone, it is enough to prove that $\mathrm{graph}\,(\widetilde{\mathrm{F}}) \subset \mathrm{graph}\,(\mathrm{F})$. Let $x^* \in \widetilde{F}(x)$. By definition,

$$\langle x^* - y^*, x - y\rangle \geq 0 \quad \text{for all} \quad (y, y^*) \in \mathrm{graph}\,(\mathrm{F}),$$

and therefore, for every $(z, z^*) \in \mathrm{graph}\,(\Gamma)$,

$$\langle (x^* + \alpha x) - z^*, x - z\rangle = \alpha\|x - z\|^2 + \langle x^* - (z^* - \alpha z), x - z\rangle \geq 0.$$

The maximality of Γ implies that $x^* + \alpha x \in \Gamma(x)$, and therefore $x^* \in F(x)$, as required. ■

The next result is an extension of Theorem 2.3.1.

Proposition 2.5.4 *Assume in Theorem 2.3.1 that* $\mathrm{dom}\,(\Gamma) \neq \emptyset$ *but* $\mathrm{int}\,(\mathrm{D})$ *not necessarily not empty. Then* $\widetilde{\Gamma}$ *is nonempty on* $\mathrm{ri}\,(\mathrm{D})$*. Actually, for all* $\bar{x} \in \mathrm{ri}\,(\mathrm{D})$ *there exist a compact* $K \subset X^*$ *and a neighborhood* V *of* $\bar{x}$ *such that* $\emptyset \neq \widetilde{\Gamma}(x) \subset K + L^{\perp}$ *for all* $x \in V \cap D$*, where* L *is a linear subspace such that* $\mathrm{aff}\,(\mathrm{dom}\,(\Gamma)) = \mathrm{a} + \mathrm{L}$*,* $a \in \mathrm{dom}\,(\Gamma)$*.*

Proof. Define Σ as in Proposition 2.5.2. Then the convex hull $\hat{D}$ of $\mathrm{dom}\,(\Sigma)$ has not empty interior. It is clear that $\bar{x} \in \mathrm{ri}\,(\mathrm{D})$ if and only if $\bar{x} - a \in \mathrm{int}\,(\hat{\mathrm{D}})$. Thus by Theorem 2.3.1, a compact $\hat{K} \subset L^*$ and a neighborhood $\hat{V}$ exist such that $\widetilde{\Sigma}(x) \subset \hat{K}$ for all $x \in \hat{V}$. The result follows from Proposition 2.5.2. ∎

Next, we generalize Theorem 2.3.3.

Proposition 2.5.5 *With the same notations in Proposition 2.5.2, consider* $\Gamma : X \stackrel{\longrightarrow}{\longrightarrow} X^*$ *be a proper monotone map. Assume that there exist* $S \subset \mathrm{dom}\,(\Gamma)$ *and a relative open subset* $V \subset C$ *such that* $\mathrm{cl}\,(\mathrm{V} \cap \mathrm{S}) = \mathrm{cl}\,(\mathrm{V})$*. Denote* $\hat{S}$*,* $\hat{V} \subset L$ *such that* $a + \hat{S} = S$ *and* $a + \hat{V} = V$*. Then*

$$\mathrm{co}\,(\Gamma_{\mathrm{S}}(\mathrm{a}+\mathrm{l})) + \mathrm{L}^{\perp} \subset \widetilde{\Gamma}(\mathrm{a}+\mathrm{l}) = \widetilde{\Sigma}(\mathrm{l}) + \mathrm{L}^{\perp} = \mathrm{co}\,(\Sigma_{\hat{\mathrm{S}}}(\mathrm{l})) + \mathrm{L}^{\perp} \quad \textit{for all} \quad \mathrm{l} \in \hat{\mathrm{V}}.$$

It follows that $\widetilde{\Gamma}$ *is monotone on* V *and therefore any maximal monotone map containing* $\Gamma + L^{\perp}$ *coincides with* $\widetilde{\Gamma}$ *on* V*.*

Proof. Since $\mathrm{graph}\,(\widetilde{\Gamma})$ is a closed subset, the inclusion

$$\mathrm{co}\,(\Gamma_{\mathrm{S}}(\mathrm{a}+\mathrm{l})) + \mathrm{L}^{\perp} \subset \widetilde{\Gamma}(\mathrm{a}+\mathrm{l})$$

is clearly verified. On the other hand, Proposition 2.5.2 and Theorem 2.3.3 imply that

$$\widetilde{\Gamma}(a+l) = \widetilde{\Sigma}(l) + L^{\perp} = \mathrm{co}\,(\Sigma_{\hat{\mathrm{S}}}(\mathrm{l})) + \mathrm{L}^{\perp} \quad \text{for all} \quad \mathrm{l} \in \hat{\mathrm{V}}.$$

Therefore, the result follows. ∎

The inclusion in Proposition 2.5.5 can be strict. Indeed, consider $\Gamma : \mathbb{R}^2 \stackrel{\longrightarrow}{\longrightarrow} \mathbb{R}^2$ defined by

$$\Gamma(x,y) \begin{cases} \{(\frac{1}{x}(-1,1)\} & \text{if} \quad x = y,\, 0 < |x| \leq 1, \\ \\ \emptyset & \text{otherwise}\ . \end{cases}$$

Taking any dense subset S of $\mathrm{co}\,(\mathrm{dom}\,(\Gamma))$ and $a=(0,0)$, we obtain

$$\widetilde{\Gamma}(0,0)=\{\,t(-1,1):t\in\mathbb{R}\,\},\quad \text{but}\quad \Gamma_S(0,0)=\emptyset.$$

The next proposition generalizes Theorem 2.3.5 when the interior of C is not assumed to be not empty.

Proposition 2.5.6 *With the same notations of Proposition 2.5.5, assume in Theorem 2.3.5 that* Γ *is a proper monotone map with* $\mathrm{int}\,(\mathrm{C})$ *not necessarily not empty. Then*

$$\widetilde{\Gamma}(a+\bar{l})=\overline{\mathrm{co}}\,\Sigma_{\hat{S}}(\bar{l})+N_C(a+\bar{l}),$$

where $a+\bar{l}\in\mathrm{rbd}\,(\mathrm{C})$.

Proof. Denote $\hat{C}=C-a$. Then $\bar{l}\in\mathrm{bd}\,(\hat{\mathrm{C}})$. By Theorem 2.3.5,

$$\widetilde{\Sigma}(\bar{l})=\overline{\mathrm{co}}\,\Sigma_{\hat{S}}(\bar{l})+N_{\hat{C}}(\bar{l}).$$

The result follows from Proposition 2.5.2 and from the fact that $N_C(a+\bar{l})=N_{\hat{C}}(\bar{l})+L^{\perp}$. ■

Similarly to Theorem 2.3.6, we have the following result on maximal monotone extensions of monotone maps when the interior of the convex hull of their respective domains is not assumed to be not empty.

This result is a direct consequence of Theorem 2.3.6 and Proposition 2.5.2.

Theorem 2.5.1 *With the same notation of Proposition 2.5.5, assume that* $\Gamma:X\overset{\longrightarrow}{\longrightarrow}X^*$ *is a monotone multivalued map. Denote by* C *the closure of the convex hull of* $\mathrm{dom}\,(\Gamma)$ *and* $\hat{C}=C-a$. *Let* $S\subset\mathrm{dom}\,(\Gamma)$ *be such that* $\mathrm{cl}\,(\mathrm{S})=\mathrm{C}$. *Then, the multivalued map* $\Lambda:X\overset{\longrightarrow}{\longrightarrow}X^*$ *defined by*

$$\Lambda(a+l)=\begin{cases}\overline{\mathrm{co}}\,(\Sigma_{\hat{S}}(l))+N_C(a+l) & \text{if } l\in\hat{C},\\ \\ \emptyset & \text{if } l\notin\hat{C},\end{cases}$$

is the unique maximal monotone map containing Γ *with domain contained in* C.

Finally, the following proposition is an extension of Theorem 2.4.2.

Proposition 2.5.7 *Assume that* $\Gamma : X \times U \overset{\longrightarrow}{\longrightarrow} X^* \times U^*$ *is maximal monotone and* $\bar{u} \in \mathrm{proj}_{\,\mathrm{U}}(\mathrm{ri}\,(\mathrm{dom}\,(\Gamma)))$*. Then the multivalued map* $\Sigma_{\bar{u}} : X \overset{\longrightarrow}{\longrightarrow} X^*$ *defined by*

$$\Sigma_{\bar{u}}(x) = \{\, x^* : \exists\, u^* \in U^* \text{ such that } (x^*, u^*) \in \Gamma(x, \bar{u}) \,\}$$

is maximal monotone.

It follows that the relative interior and the closure of $\mathrm{dom}\,(\Sigma_{\bar{\mathrm{u}}})$ *are convex and satisfy the following relations*

$$\mathrm{cl}\,(\mathrm{ri}\,(\mathrm{dom}\,(\Sigma_{\bar{\mathrm{u}}}))) = \mathrm{cl}\,(\mathrm{dom}\,(\Sigma_{\bar{\mathrm{u}}})) \quad \text{and} \quad \mathrm{ri}\,(\mathrm{cl}\,(\mathrm{dom}\,(\Sigma_{\bar{\mathrm{u}}}))) = \mathrm{ri}\,(\mathrm{dom}\,(\Sigma_{\bar{\mathrm{u}}})).$$

Furthermore,

$$x \in \mathrm{rbd}\,(\mathrm{dom}\,(\Sigma_{\bar{\mathrm{u}}})) \Longleftrightarrow (\mathrm{x}, \bar{\mathrm{u}}) \in \mathrm{rbd}\,(\mathrm{dom}\,(\Gamma)).$$

Proof. In order to simplify the notations, assume, without loss of generality, that $(0,0) \in \mathrm{ri}\,(\mathrm{dom}\,(\Gamma))$ and $\bar{u} = 0$.

The linear subspace $L = \mathrm{aff}\,(\mathrm{dom}\,(\Gamma))$ can be written as

$$L = \{\, (x, u) \in X \times U : Ax + Bu = 0 \,\},$$

where A and B are two matrices of appropriate order. Then

$$L^{\perp} = \mathrm{img}\,([\mathrm{A}, \mathrm{B}]^{\mathrm{t}})$$

and $\mathrm{aff}\,(\mathrm{dom}\,(\Sigma_0)) = \ker(\mathrm{A})$.

Define $\Sigma : L \overset{\longrightarrow}{\longrightarrow} L^* = L$ by

$$\Gamma(x, u) = \Sigma(x, u) + L^{\perp},$$

then Σ is maximal monotone, by Proposition 2.5.1 b). This implies that the multivalued map $\Sigma^0 : \ker(A) \overset{\longrightarrow}{\longrightarrow} \ker(A)$ defined by

$$\Sigma^0(x) = \{\, x^* : \exists\, u^* \in \mathrm{proj}_{\,\mathrm{U}^*}(\mathrm{L}^*) \text{ such that } (\mathrm{x}^*, \mathrm{u}^*) \in \Sigma(\mathrm{x}, 0) \,\}$$

is also maximal monotone.

By definition,

$$\Sigma_0 = \Sigma^0 + \mathrm{img}\,(\mathrm{A}^{\mathrm{t}}) = \Sigma^0 + (\ker(\mathrm{A}))^{\perp},$$

from what we deduce that Σ_0 is maximal monotone. ■

2.6 Strict and strong monotonicity

Proposition 2.6.1 *Let* $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ *be a monotone multivalued map*

a) If Γ *is strictly monotone, then* Γ^- *is single valued on its domain,*

b) If Γ *is strongly monotone with modulus* $\alpha > 0$*, then* Γ^- *is* $\frac{1}{\alpha}-$ *Lipschitz. Furthermore,*

$$\widetilde{\Gamma}(C) = \bigcup_{x \in C} \widetilde{\Gamma}(x) = X^*,$$

where $C = \overline{\text{co}}\,(\text{dom}\,(\Gamma))$.

In particular maximal strongly monotone maps are surjective.

Proof.

a) Let (x^*, x), $(y^*, y) \in \text{graph}\,(\Gamma^-)$. By definition,

$$\langle x^* - y^*, x - y \rangle > 0, \quad \text{if } x \neq y.$$

Hence Γ^- is single valued on its domain.

b) Let (x^*, x), $(y^*, y) \in \text{graph}\,(\Gamma^-)$. By definition,

$$\alpha \|x - y\|^2 \leq \langle x^* - y^*, x - y \rangle \leq \|x^* - y^*\| \|x - y\|,$$

and therefore

$$\|x - y\| \leq \frac{1}{\alpha} \|x^* - y^*\|.$$

Hence Γ^- is $\frac{1}{\alpha}-$ *Lipschitz.*

Now, we shall prove that $\widetilde{\Gamma}(C) = X^*$. For that, define $F : X \overset{\longrightarrow}{\longrightarrow} X^*$ by $F(x) = \Gamma(x) - \alpha x$. Since Γ is strongly monotone with modulus α, F is monotone. By Theorem 2.5.1 a maximal monotone $\widehat{F}$ containing F exists such that $\text{dom}\,(\widehat{\text{F}}) \subset \overline{\text{co}}(\text{dom}\,(\text{F})) = \text{C}$. Since $\hat{F}$ is maximal monotone, Corollary 2.5.1 implies that the multivalued map $\hat{\Gamma} = \hat{F} + \alpha I$ is strongly maximal monotone and is contained in $\widetilde{\Gamma}$. Clearly, $\text{dom}\,(\hat{\Gamma}) = \text{dom}\,(\hat{\text{F}})$. Thus, in order to prove that $\widetilde{\Gamma}^(C) = X^*$, we shall prove that $\mathrm{dom}\,(\hat{\Gamma}^-) = \mathrm{X}^*$. To do that, it is enough to prove that $\hat{C} = \mathrm{dom}\,(\hat{\Gamma}^-)$ is both open and closed, indeed a subset of $\mathbb{R}^n$ which is both closed and open is either the empty set or the whole set.

i) Proof that $\hat{C}$ is open: Assume, for contradiction, that $\hat{C}$ is not open. Then there exists some $x^* \in \hat{C} \cap \mathrm{bd}\,(\hat{\mathrm{C}})$. Thus $\hat{\Gamma}^-(x^*)$ is not empty and by Proposition 2.3.2, its recession cone is $N_{\overline{\mathrm{co}}\,(\hat{C})}(x^*)$. It follows that $\hat{\Gamma}^-(x^*)$ is unbounded, contradicting the fact that $\hat{\Gamma}^-$ is single-valued on $\hat{C}$.

ii) Proof that $\hat{C}$ is closed. Consider a sequence $\{(x_k^*, x_k)\} \subset \mathrm{graph}\,(\hat{\Gamma}^-)$ such that $\{x_k^*\}$ converges to some x^*. Let $y^* \in \hat{C}$. Then,

$$\|\hat{\Gamma}^-(x_k) - \hat{\Gamma}^-(y^*)\| \le \frac{1}{\alpha}\|x_k^* - y^*\|,$$

and therefore the sequence $\{\hat{\Gamma}^-(x_k)\}$ has cluster points. Let x be such a point. The closedness of $\mathrm{graph}\,(\hat{\Gamma}^-)$ implies $(x^*, x) \in \mathrm{graph}\,(\hat{\Gamma}^-)$. Hence $x^* \in \hat{C}$. ■

Remark. For proper convex lower semicontinuous functions there exists an (almost) equivalence between their strict convexity and the differentiability of their conjugates: if f is strictly convex then ∂f^* is single-valued on its domain; conversely, if ∂f^* is single-valued on its domain, then f is strictly convex on the relative interior of its domain.

Example 2.1.1 shows that the equivalence does not hold for maximal monotone maps, the strict monotonicity corresponding to the strict monotonicity and the maps to the subdifferentials. In this example Γ and Γ^- are maximal monotone and single-valued but they are not strictly monotone.

The following result gives a characterization of maximal strictly monotone maps.

Proposition 2.6.2 *Let $\Gamma : X \rightrightarrows X^*$ be a strictly monotone map. It is maximal monotone if and only if it is closed and its* $\mathrm{dom}\,(\Gamma^-)$ *is open and convex.*

Proof. Assume that Γ is maximal monotone. Then Γ is closed and from Proposition 2.6.1 $\mathrm{dom}\,(\Gamma^-)$ is open and convex. Conversely, in view of Proposition 2.6.1 and the closedness of Γ, one has, for $\mathrm{S} = \mathrm{dom}\,(\Gamma^-)$, that

$$(\Gamma^-)_{\mathrm{S}}\,(x^*) = \begin{cases} \Gamma^-(x^*) & \text{if } x \in \mathrm{dom}\,(\Gamma^-), \\ \\ \emptyset & \text{if } x \notin \mathrm{dom}\,(\Gamma^-). \end{cases}$$

Thus, by Theorem 2.3.6, Γ^- and, hence, Γ are maximal monotone. ■

Next, we shall state a well known result which provides a very important characterization of maximal monotone maps. This result is a direct consequence of Proposition 2.6.1.

Proposition 2.6.3 *(Minty Theorem) Let $F : X \xrightarrow{\rightarrow} X^*$ be a monotone map. F is maximal monotone if and only if* $\mathrm{dom}\,(\Gamma^-) = \mathrm{X}^*$, *where* $\Gamma = F + I$.

Proof. $\Rightarrow$) Γ is strongly monotone with modulus 1. Since F is maximal monotone, Γ is also maximal monotone, and therefore, from Proposition 2.6.1, $\mathrm{dom}\,(\Gamma^-) = \mathrm{X}^*$.
$\Leftarrow$) We shall prove that any $(x, x^*) \in \mathrm{graph}\,(\widetilde{\mathrm{F}})$ belongs to $\mathrm{graph}\,(\mathrm{F})$ too. Since $\mathrm{dom}\,(\Gamma^-) = \mathrm{X}^*$, there exists $(\xi, \xi^*) \in \mathrm{graph}\,(\mathrm{F})$ such that $x^*+x = \xi^*+\xi$. By definition of $\widetilde{F}$,

$$0 \leq \langle x^* - \xi^*, x - \xi \rangle = -\langle x - \xi, x - \xi \rangle,$$

and therefore $(x, x^*) = (\xi, \xi^*) \in \mathrm{graph}\,(\mathrm{F})$. ■

Proposition 2.6.4 *Let $\Gamma : X \xrightarrow{\rightarrow} X^*$ be a monotone map with bounded domain. Then $\widetilde{\Gamma}(C) = X^*$, where $C = \overline{\mathrm{co}}(\mathrm{dom}\,(\Gamma))$.*

Proof. By same argument of Proposition 2.6.1 b), there exists a maximal monotone map $\hat{\Gamma}$ containing Γ such that $\mathrm{dom}\,(\hat{\Gamma}) \subset \mathrm{C}$. Clearly $\mathrm{dom}\,(\hat{\Gamma})$ is bounded and that $\hat{\Gamma}$ is contained in $\widetilde{\Gamma}$. Thus, in order to prove that $\widetilde{\Gamma}(C) = X^*$, we shall prove that $\mathrm{dom}\,(\hat{\Gamma}^-) = \mathrm{X}^*$. As in the proof of Proposition 2.6.1,

we shall show that $\hat{C} = \mathrm{dom}\,(\hat{\Gamma}^-)$ is both open and closed.
i) Assume, for contradiction, that $\hat{C}$ is not open. Take some $x^* \in \hat{C} \cap \mathrm{bd}\,(\hat{\mathrm{C}})$. Then $\hat{\Gamma}^-(x^*)$ is not empty and, by Proposition 2.3.2, its recession cone is $N_{\overline{\mathrm{co}}\,(\hat{C})}(x^*)$. This implies that $\hat{\Gamma}^-(x^*)$ is unbounded, contradicting the fact that $\mathrm{dom}\,(\hat{\Gamma})$ is bounded.
ii) Proof that $\hat{C}$ is closed. Consider a sequence $\{(x_k^*, x_k)\} \subset \mathrm{graph}\,(\hat{\Gamma}^-)$ such that $\{x_k^*\}$ converges to some x^*. Since $\mathrm{dom}\,(\hat{\Gamma})$ is bounded, the sequence $\{x_k\}$ has cluster points. Let x be such a point. The closedness of $\mathrm{graph}\,(\hat{\Gamma}^-)$ implies $(x^*, x) \in \mathrm{graph}\,(\hat{\Gamma}^-)$. Hence $x^* \in \hat{C}$. ■

Proposition 2.6.5 *Let* $\Gamma : X \rightrightarrows X^*$ *be a maximal monotone map. Assume that for all sequence* $\{(x_k, x_k^*)\} \subset \mathrm{graph}\,(\Gamma)$ *such that* $\|x_k\| \to +\infty$ *we have* $\|x_k^*\| \to +\infty$*. Then* $\mathrm{dom}\,(\Gamma^-) = \mathrm{X}^*$.

Proof. We first recall that for all $x^* \in X^*$, $\Gamma^-(x^*)$ is bounded. As in the previous proposition, we shall show that $\hat{C} = \mathrm{dom}\,(\Gamma^-)$ is both open and closed.
i) Assume, for contradiction, that $\hat{C}$ is not open. Take some $x^* \in \hat{C} \cap \mathrm{bd}\,(\hat{\mathrm{C}})$. Thus $\Gamma^-(x^*)$ is not empty and, by Proposition 2.3.2, its recession cone is $N_{\overline{\mathrm{co}}\,\hat{\mathrm{C}})}(x^*)$. It follows that $\Gamma^-(x^*)$ is unbounded, contradicting the fact that $\mathrm{dom}\,(\Gamma)$ is bounded.
ii) Proof that $\hat{C}$ is closed. Consider a sequence $\{(x_k^*, x_k)\} \subset \mathrm{graph}\,(\Gamma^-)$ such that $\{x_k^*\}_{k \in \mathbb{N}}$ converges to some x^*. By assumption, the sequence $\{x_k\}$ has cluster points. Let x be such a point. The closedness of $\mathrm{graph}\,(\Gamma^-)$ implies $(x^*, x) \in \mathrm{graph}\,(\Gamma^-)$. Hence $x^* \in \hat{C}$. ■

Proposition 2.6.6 *Let* $\Gamma : X \rightrightarrows X^*$ *be a maximal monotone map such that* $\hat{x} \in \mathrm{ri}\,(\mathrm{dom}\,(\Gamma))$*. Assume that for all sequence* $\{(x_k, x_k^*)\} \subset \mathrm{graph}\,(\Gamma)$ *with* $\|x_k\| \to +\infty$ *we have*

$$\limsup_{k \to +\infty} \langle x_k^*, x_k - \hat{x} \rangle > 0.$$

Then $\Gamma^-(0)$ *is a nonempty compact set.*

Proof. Define the set

$$A = \{ x \in \mathrm{dom}\,(\Gamma) : \exists \mathrm{x}^* \in \Gamma(\mathrm{x}),\ \langle \mathrm{x}^*, \mathrm{x} - \hat{\mathrm{x}} \rangle \le 0 \}.$$

Clearly, A is bounded and $\Gamma^{-}(0) \subset A$. Consider $r > 0$ such that the Euclidean ball $B_r(\hat{x})$ contains A. Define $F : X \rightrightarrows X^*$ by $F(x) = \Gamma(x) + N_{B_{2r}(\hat{x})}(x)$. Since $\hat{x} \in \mathrm{ri}\,(\mathrm{dom}\,(\Gamma)) \cap \mathrm{B}_{2r}(\hat{\mathrm{x}})$, F is maximal monotone and therefore, since $\mathrm{dom}\,(\mathrm{F})$ is bounded, there exists $\bar{x} \in B_{2r}(\hat{x})$ such that

$$0 \in F(\bar{x}) = \Gamma(\bar{x}) + N_{B_{2r}(\hat{x})}(\bar{x}).$$

Let $\bar{x}^* \in N_B(\bar{x})$ such that $-\bar{x}^* \in \Gamma(\bar{x})$. Let us prove that $\bar{x} \in A$. For that, assume, for contradiction, that $\bar{x} \notin A$. Since $\bar{x}^* \in N_B(\bar{x})$, $\langle \bar{x}^*, \bar{x} - x \rangle \ge 0$, for all $x \in B$. In particular, for $x = \hat{x}$, we have $\langle \bar{x}^*, \bar{x} - \hat{x} \rangle \ge 0$. On the other hand, since $\bar{x} \notin A$ and $-\bar{x}^* \in \Gamma(\bar{x})$, $\langle \bar{x}^*, \bar{x} - \hat{x} \rangle < 0$, which is not possible. Hence $\bar{x} \in A$, and therefore $\bar{x}^* = 0$. Thus, $0 \in \Gamma(\bar{x})$, as required. ■

Theorem 2.6.1 *Let $\Gamma : X \rightrightarrows X^*$ be a maximal monotone multivalued map such that $C = \mathrm{int}\,(\mathrm{dom}(\Gamma)) \ne \emptyset$. Then Γ is single-valued almost everywhere (in the Lebesgue sense) on C.*

Proof. The result, when $\dim(X) = 1$, is a well known result on monotonic functions of one real variable, see for instance Natanson [29]. Assume that $\dim(X) = n > 1$. For every $i = 1, \cdots, n$, let us define, for every $x \in C$,

$$\theta_i(x) = \max\,[\, \langle x^* - y^*, e_i \rangle : x^*, y^* \in \Gamma(x) \,],$$

where for $i = 1, \cdots, n$, e_i denotes the ith canonical vector in X. Since $\Gamma(x)$ is compact for all $x \in C$ and the map Γ is usc, the function θ_i is upper semicontinuous and therefore measurable on C. Thus the set $D_i = \{x \in C : \theta_i(x) \le 0\}$ is measurable because C is convex and open.

Let $x \in D = \cap_{i=1}^{n} D_i$. If $x^*, y^* \in \Gamma(x)$, then $|x_i^* - y_i^*| \le 0$ for all i. Hence $x^* = y^*$. Thus D is the set of $x \in C$ such that $\Gamma(x)$ is reduced to a singleton. D and their complement $D^c = \cup_{i=1}^{n} D_i^c$ are measurable. We shall prove that for all i, $\mathrm{meas}\,(\mathrm{D_i^c}) = 0$, from what it is deduced that $\mathrm{meas}\,(\mathrm{D^c}) = 0$. We give the proof for $i = n$. In fact, we shall prove that $\mathrm{meas}\,(\mathrm{D^c} \cap \mathrm{P}) = 0$, for all P of the type $P = \prod_{i=1}^{n} [\bar{x}_i, \bar{x}_i + \epsilon]$, with $\epsilon > 0$, from what the result follows. Denotes by $1_{D_n^c}$ the characteristic function of D_n^c defined by

$$1_{D_n^c}(x) = \begin{cases} 1 & \text{if } \; x \in D_n^c, \\ 0 & \text{if } \; x \notin D_n^c. \end{cases}$$

By Fubini's theorem,

$$\mathrm{meas}\,(\mathrm{D_n^c} \cap \mathrm{P}) = \int_{\mathrm{P}} 1_{\mathrm{D_n^c}}(\mathrm{x})\mathrm{dx} = \int_{\mathrm{Q}} [\int_{\bar{\mathrm{x}}_\mathrm{n}}^{\bar{\mathrm{x}}_\mathrm{n}+\epsilon} 1_{\mathrm{D_n^c}}(\mathrm{x})\mathrm{dx_n}]\mathrm{dx_1} \cdots \mathrm{dx_{n-1}}, \tag{2.26}$$

where $Q = \{y = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} : \exists\, x_n \text{ with } (x_1, \cdots, x_{n-1}, x_n) \in P \cap C\}$. For $y \in Q$, let us define

$$D(y) = \{x_n \in \mathbb{R} : (y, x_n) \in D_n \cap P\}.$$

By definition, $D(y)$ is the set of points where the multivalued map $h_y : \mathbb{R} \overset{\longrightarrow}{\longrightarrow} \mathbb{R}$ defined by

$$h_y(t) = \langle \Gamma(y, \bar{x}_n + t), e_n \rangle$$

is reduced to a singleton. This map h_y is monotone. Applying again the result on monotonic functions of one real variable, we obtain that $\mathrm{meas}\,([\mathrm{D(y)}]^\mathrm{c}) = 0$. Report in (2.26), we deduce that $\mathrm{meas}\,(\mathrm{D_n^c} \cap \mathrm{P}) = 0$. ∎

2.7 Maximal monotone extensions

Let G be a subset of $X \times X^*$. If G is not monotone, there is no $\bar{G}$ monotone containing G. If G is monotone, with an argument based on the axiom of choice, it is possible to prove that there exists a maximal monotone extension of G. This extension is not unique as seen in the following example:

Example 2.7.1 $G = \{(-1,-1), (1,1)\} \subset \mathbb{R}^2$, *G is monotone. The two following sets*

$$\begin{aligned} G_1 &= \{(x, x^*) \in \mathbb{R}^2 \,:\, x^* = x\}\,, \\ G_2 &= \,]-\infty, 1] \times \{-1\} \cup \{1\} \times [-1, \infty[\end{aligned}$$

are maximal monotone, and they both contain G.

The axiom of choice is not constructive. We shall show how to construct a maximal monotone extension.

Let $F : X \rightrightarrows X^*$ be a monotone map. Denote by C, the closure of the convex hull of dom (F). We assume that int (C) is nonempty and we are given a countable set $S = \{x_0, x_1, \cdots, x_n, \cdots\} \subset \mathrm{int}\,(\mathrm{C})$ such that $\mathrm{cl}\,(\mathrm{S}) = \mathrm{C}$. In the construction, by convention, $A + \emptyset = \emptyset$.

Algorithm:

Step 0 Define $F_0 : X \rightrightarrows X^*$ by

$$F_0(x) = \begin{cases} F(x) + N_C(x) & \text{if } x \in C, \\ \\ \emptyset & \text{if not }. \end{cases}$$

By construction F_0 is monotone and $\mathrm{dom}\,(\mathrm{F_0}) = \mathrm{dom}\,(\mathrm{F})$.

Step k In the previous steps a monotone map F_k has been obtained with $\mathrm{graph}\,(\mathrm{F_k}) \supset \mathrm{graph}\,(\mathrm{F})$.

- If $\mathrm{dom}\,(\mathrm{F_k}) \supset \mathrm{S}$, by Theorem 2.3.6, the multivalued map

$$x \rightrightarrows \widetilde{F}_k(x) + N_C(x)$$

is maximal monotone. STOP.

- If not, take

$$p(k) = \min[\, p \in \mathbb{N} : x_p \in S \cap (\mathrm{dom}\,(\mathrm{F_k}))^{\mathrm{c}}\,],$$

and define

$$F_{k+1}(x) = \begin{cases} F_k(x) & \text{if } x \in \mathrm{dom}\,(\mathrm{F_k}), \\ \\ \widetilde{F}_k(x_{p(k)}) & \text{if } x = x_{p(k)}, \\ \\ \emptyset & \text{otherwise }. \end{cases}$$

By construction, F_{k+1} is monotone,

$$\mathrm{graph}\,(\mathrm{F}) \subset \mathrm{graph}\,(\mathrm{F_k}) \subset \mathrm{graph}\,(\mathrm{F_{k+1}})$$

and

$$\mathrm{dom}\,(\mathrm{F_{k+1}}) = \mathrm{dom}\,(\mathrm{F_k}) \cup \{\mathrm{x_{p(k)}}\} \subset \mathrm{C}.$$

Do $k = k+1$ and go back to step k.

End of algorithm

Take $D = \cup_k \mathrm{dom}\,(\mathrm{F_k})$. Define $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ by

$$\Gamma(x) = \begin{cases} F_k(x) & \text{if } \; x \in \mathrm{dom}\,(\mathrm{F_k}), \\ \\ \emptyset & \text{if } x \notin D. \end{cases}$$

Γ is monotone and $C \supset \mathrm{dom}\,(\Gamma) = \mathrm{D} \supset \mathrm{S}$. Thus, in view of Theorem 2.3.6, the multivalued map $\Sigma : X \overset{\longrightarrow}{\longrightarrow} X^*$ defined by $\Sigma(x) = \widetilde{\Gamma}(x) + N_C(x)$ is maximal monotone. Its graph contains the graph of F and $\mathrm{cl}\,(\mathrm{dom}\,(\Sigma)) = \mathrm{C}$. ■

Chapter 3

Monotonicity and maximal monotonicity of affine subspaces

Monotone linear variational inequality problems constitute an important class of variational inequality problems. They are of the form:

$$\text{Find } \bar{x} \in C \quad \text{such that} \quad \langle A\bar{x} - c, y - \bar{x} \rangle \geq 0 \ \ \forall\, y \in C,$$

where $c \in \mathbb{R}^n$, A is a $n \times n$ positive semidefinite matrix and $C \subset \mathbb{R}^n$ is a polyhedral convex subset. Linear and quadratic optimization programs can be formulated in this way.

The map $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\Gamma(x) = Ax - c$ is maximal monotone, its graph

$$E = \{\, (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : \ Ax + (-I)x^* = c \,\},$$

where I denotes the identity matrix of order n, is a maximal monotone affine subspace of $\mathbb{R}^n \times \mathbb{R}^n$.

In this chapter, we generalize the above representation to subspaces of the form

$$E = \{\, (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : \ Ax + Bx^* = c \,\},$$

where A and B are two $p \times n$ matrices and $c \in \mathbb{R}^p$.

It is clear that any affine subspace of $\mathbb{R}^n \times \mathbb{R}^n$ can be written in this way.

In sections 3.1 and 3.2 we characterize the monotonicity and the maximal monotonicity of these subspaces. An important result shows that at any maximal monotone subset is associated a permutation of the variables and a positive subdefinite matrix. Based on this result, we give a finite algorithm to obtain a maximal monotone affine extension of a monotone map subspace. The linearity structure allows a quite simpler construction than the one given in Chapter 2 for non affine subspaces.

The last section is concerned with the restriction of an affine monotone subspace.

3.1 Monotone affine subspaces

In this section we consider subsets of $\mathbb{R}^n \times \mathbb{R}^n$ of the form

$$E = \{ (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : Ax + Bx^* = c \},$$

where $c \in \mathbb{R}^p$, A and B are two $p \times n$ matrices.

Without loss of generality, we assume that there is no redundance in the linear system, i.e., the $p \times 2n$ matrix $C = [A, B]$ is of rank p.

By definition, the set E is monotone if and only if

$$\inf[\, \langle x_2^* - x_1^*, x_2 - x_1 \rangle : (x_i, x_i^*) \in E,\ i = 1, 2\,] \geq 0.$$

Denote P the $2n \times 2n$ matrix defined by

$$P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where I is the identity matrix of order n; then the monotonicity of E is equivalent to

$$\inf_u [\, \langle Pu, u \rangle : Cu = 0\,] = 0,$$

which is also equivalent to the following condition:

$$Cu = 0 \quad \Longrightarrow \quad \langle Pu, u \rangle \geq 0. \tag{PSD}$$

Then we have the following characterization of monotone affine subspaces.

Theorem 3.1.1 *1. The subset E is monotone if and only if $p \geq n$ and the $p \times p$ matrix $M = AB^t + BA^t$ has exactly $p - n$ positive eigenvalues.*

2. The subset E is maximal monotone if and only if $p = n$ and E is monotone.

Proof.

1. Let us consider the inertia of the $(2n+p) \times (2n+p)$ *bordered* matrix

$$T = \begin{pmatrix} P & C^t \\ C & 0 \end{pmatrix}.$$

This inertia $\mathrm{In}\,(\mathrm{T})$ is the triple

$$\mathrm{In}\,(\mathrm{T}) = (\nu_+, \nu_-, \nu_0),$$

where ν_+, ν_- and ν_0 denote respectively the numbers of positive, negative and zero eigenvalues of T ($\nu_+ + \nu_- + \nu_0 = 2n+p$). By construction, since $\mathrm{rank}\,(\mathrm{C}) = \mathrm{p}$, we have $\mu_- \geq p$. Then condition (PSD) (see [6], [9]) is equivalent to say that ν_- is exactly p. Moreover, in view of a result on the *Schur's Complement* (see [6], [9]),

$$\begin{aligned} \mathrm{In}\,(\mathrm{T}) &= \mathrm{In}\,(\mathrm{P}) + \mathrm{In}\,(0 - \mathrm{CP}^{-1}\mathrm{C}^t) \\ &= (n, n, 0) + \mathrm{In}\,(-\mathrm{AB}^t - \mathrm{BA}^t). \end{aligned}$$

Thus, F is monotone if and only if $p \geq n$ and the matrix M has exactly $p - n$ positive eigenvalues.

2. By Proposition 2.1.1, E monotone is maximal monotone if and only if

$$(\bar{x}, \bar{x}^*) \in \widetilde{E} \quad \Longrightarrow \quad A\bar{x} + B\bar{x}^* = c,$$

where

$$\widetilde{E} = \{\, (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : \langle x^* - \xi^*, x - \xi \rangle \geq 0, \quad A\xi + B\xi^* = c \,\}.$$

Set $\bar{c} = c - A\bar{x} - B\bar{x}^*$, then E monotone is maximal monotone if and only if it satisfies the following condition

$$\inf_u [\, \langle Pu, u\rangle : Cu = \bar{c}\,] \geq 0 \quad \Longrightarrow \quad \bar{c} = 0. \tag{3.1}$$

It is known that a quadratic function which is bounded from below on a convex polyhedral set reaches its minimum at some feasible point $\bar{u}$ (see [13]). Then according to the KKT optimality condition, there exists a vector $\bar{v}$ such that $P\bar{u} = C^t\bar{v}$. Then

$$\bar{u} = P^{-1}C^t\bar{v} = PC^t\bar{v} \quad \text{and} \quad \bar{c} = C\bar{u} = CPC^t\bar{v} = M\bar{v}.$$

Note that

$$\langle P\bar{u}, \bar{u}\rangle = \langle C^t\bar{v}, PC^t\bar{v}\rangle = \langle M\bar{v}, \bar{v}\rangle. \tag{3.2}$$

If $p = n$ and the subset E is monotone and the symmetric matrix M is negative semidefinite. The left hand side relation in (3.1) and relation (3.2) imply that $0 \leq \langle M\bar{v}, \bar{v}\rangle \leq 0$ and therefore $\bar{c} = M\bar{v} = 0$. The sufficient condition follows.

Next, assume that E is monotone and $p > n$. Since M is symmetric, there exist a $p \times p$ orthogonal matrix Q ($QQ^t = I$), $n \times n$ negative semidefinite diagonal matrix D_1 and a $(p-n)\times(p-n)$ positive definite diagonal matrix D_2, such that

$$QMQ^t = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Define

$$\widehat{C} = QC = \begin{pmatrix} \widehat{A}_1 & \widehat{B}_1 \\ \widehat{A}_2 & \widehat{B}_2 \end{pmatrix}, \quad \widehat{M} = \widehat{C}P\widehat{C}^t \quad \text{and} \quad \widehat{c} = Qc = \begin{pmatrix} \widehat{c}_1 \\ \widehat{c}_2 \end{pmatrix}.$$

Then

$$\widehat{M} = \begin{pmatrix} \widehat{A}_1\widehat{B}_1^t + \widehat{B}_1\widehat{A}_1^t & \widehat{A}_1\widehat{B}_2^t + \widehat{B}_1\widehat{A}_2^t \\ \\ \widehat{A}_2\widehat{B}_1^t + \widehat{B}_2\widehat{A}_1^t & \widehat{A}_2\widehat{B}_2^t + \widehat{B}_2\widehat{A}_2^t \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ \\ 0 & D_2 \end{pmatrix}.$$

This implies that the $n \times n$ matrix

$$\widehat{A}_1 \widehat{B}_1^t + \widehat{B}_1 \widehat{A}_1^t = D_1$$

has no positive eigenvalues and the $n \times 2n$ matrix $[\widehat{A}_1, \widehat{B}_1]$ has rank n. It follows from part 2 of the proof that the subset

$$\widehat{E}_1 = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : \widehat{A}_1 x + \widehat{B}_1 x^* = \widehat{c}_1\}$$

is maximal monotone. Since $p > n$ and $[A, B]$ is of rank p, this set strictly contains the set

$$\widehat{E} = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : \widehat{A}_1 x + \widehat{B}_1 x^* = \widehat{c}_1, \quad \widehat{A}_2 x + \widehat{B}_2 x^* = \widehat{c}_2\},$$

which is obviously equal to E. The theorem follows. ■

As immediate consequences of this theorem, we deduce the following results.

Corollary 3.1.1 *Assume that $C = [A, B]$ has rank p.*

i) If $p < n$, then E is not monotone.

ii) If $p > n$, then E is not maximal monotone.

iii) If E is monotone, $\dim(E) \leq n$.

iv) E is maximal monotone if and only if E is monotone and $\dim(E) = n$.

3.2 A characterization of maximal mononotonicity for affine subspaces

The following result says that any affine maximal monotone subspace can be written, under appropriate permutation of its variables, as the graph of an affine map. Before, the following notations are useful: For a subset $\mathcal{I} \subset \{1, 2, \cdots, n\}$, denotes $\mathcal{I}^c = \{\, i \in \{1, 2, \cdots, n\} : i \notin \mathcal{I}\,\}$ and for $x \in \mathbb{R}^n$, denotes $x_{\mathcal{I}} = (x_{i_1}, x_{i_2}, \cdots, x_{i_r})^t$, where $\{i_1 < i_2 < \cdots < i_r\} = \mathcal{I}$. Finally, for a matrix C, denotes by c_{ij} the element in the ith line and jth column of C.

Theorem 3.2.1 *Let E be an affine subspace of* $\mathbb{R}^n \times \mathbb{R}^n$. *This subset is maximal monotone if and only if there exist a subset $\mathcal{I} \subset \{1, 2, \cdots, n\}$, a* $\operatorname{card}(\mathcal{I}) \times \operatorname{card}(\mathcal{I})$ *positive semidefinite matrix M, a* $\operatorname{card}(\mathcal{I}) \times \operatorname{card}(\mathcal{I}^c)$ *matrix P and two vectors $q_{\mathcal{I}} \in \mathbb{R}^{\operatorname{card}(\mathcal{I})}$ and $q_{\mathcal{I}^c} \in \mathbb{R}^{\operatorname{card}(\mathcal{I}^c)}$ such that*

$$\begin{pmatrix} x^*_{\mathcal{I}} \\ x_{\mathcal{I}^c} \end{pmatrix} = \begin{pmatrix} M & P \\ -P^t & 0 \end{pmatrix} \begin{pmatrix} x_{\mathcal{I}} \\ x^*_{\mathcal{I}^c} \end{pmatrix} + \begin{pmatrix} q_{\mathcal{I}} \\ q_{\mathcal{I}^c} \end{pmatrix}.$$

Proof. The part *"if"* of the condition is clearly sufficient. Conversely, assume that E is maximal monotone. Since $\dim(E) = n$, there exist $c \in \mathbb{R}^n$ and C a $n \times 2n$ matrix of rank n such that

$$E = \{\, w = (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : Cw = c \,\}.$$

Since the maximal monotonicity is preserved under translation, we can assume without loss of generality that E is a linear subspace, i.e., $c = 0$. Then $q_{\mathcal{I}}$ and $q_{\mathcal{I}^c}$ are null vectors. We construct the subset $\mathcal{I}$ and the matrices M and P with the help of the following algorithm.

I.- **Initialization**

For $i = 1, 2, \cdots, n$ do $\delta(i) = 0$.

II.- **Construction**

Do $i = 1$.

(A) Find $k \geq i$ such that $c_{k\,n+i} \neq 0$.

- If such a k exists, permute lines i and k of C and **go to Processing**.
- If not, find $k \geq i$ such that $c_{ki} \neq 0$.
  - if such a k exists, do $\delta(i) = 1$, permute lines i and k, columns i and $n + i$ of C and **go to Processing**.
  - otherwise, **STOP**: E is not maximal monotone.

(B) **Processing:**

- divide line i of C by $c_{i\,n+i}$.

– for $k \neq i$ and $j = 1, 2, \cdots, 2n$ do $c_{kj} = c_{kj} - c_{k\,n+i} c_{ij}$.

If $i < n$, increase i by one and go to (A).

If not, the algorithm is finished.

Justification of the algorithm

a) The algorithm cannot stop at step $i \leq n$.

We shall prove that we have a contradiction with E monotone if $i \leq n$. Because in previous steps, $j = 1, \cdots, i-1$, some permutation between columns j and $n+j$ may have occurred, and for simplicity, we construct the vectors z and z^* as follows

$$(z_j, z_j^*) = \begin{cases} (x_j, x_j^*) & \text{if } \delta(j) = 0, \\ \\ (x_j^*, x_j) & \text{if } \delta(j) = 1. \end{cases}$$

Let us define

$$\widehat{E} = \{\, w = (z, z^*) \in \mathbb{R}^n \times \mathbb{R}^n : \widehat{C} w = 0 \,\},$$

where

$$\widehat{C} = \left(\begin{array}{ccc|ccc} D & a & F_1 & I & b & F_2 \\ e^t & 0 & f^t & 0 & 0 & g^t \\ P & 0 & Q_1 & 0 & 0 & Q_2 \end{array} \right),$$

D, P, F_k and Q_k, $k = 1, 2$, are respectively $(i-1) \times (i-1)$, $(n-i) \times (i-1)$, $(i-1) \times (n-i)$ and $(n-i) \times (n-i)$ matrices, the vectors a, b and e are in $\mathbb{R}^{i-1}$, f and g in $\mathbb{R}^{n-i}$. Finally, I denotes the identity matrix of order $i-1$. Because E is maximal monotone, $\widehat{E}$ is so.

For $k = 1, 2, \cdots, n$, define

$$z_k = \begin{cases} 0 & \text{if } k \neq i, \\ 1 & \text{if } k = i. \end{cases} \quad \text{and} \quad z_k^* = \begin{cases} b_k - a_k & \text{if } k < i, \\ -1 & \text{if } k = i, \\ 0 & \text{if } k > i. \end{cases}$$

It is clear that the vector (z, z^*) belongs to $\widehat{E}$ and that $\langle z, z^* \rangle = -1$. This is in contradiction with $\widehat{E}$ monotone.

b) The algorithm goes to its end.
Define $\mathcal{I} \subset \{1, 2, \cdots, n\}$ by

$$i \in \mathcal{I} \iff \delta(i) = 0.$$

Then the linear subspace E is of form

$$E = \{\, w = ((x_{\mathcal{I}}, x^*_{\mathcal{I}^c}), (x^*_{\mathcal{I}}, x_{\mathcal{I}^c})) \in \mathbb{R}^n \times \mathbb{R}^n : \widehat{C}w = 0 \,\},$$

where

$$\widehat{C} = \left(\begin{array}{cc|cc} -M & -P & I & 0 \\ -Q^t & 0 & 0 & I \end{array} \right),$$

M is a $\mathrm{card}\,(\mathcal{I}) \times \mathrm{card}\,(\mathcal{I})$ matrix, P and Q are two $\mathrm{card}\,(\mathcal{I}) \times \mathrm{card}\,(\mathcal{I}^{\mathrm{c}})$ matrices. No confusion being possible, I stands both the identity matrices of order $\mathrm{card}\,(\mathcal{I})$ and $\mathrm{card}\,(\mathcal{I}^{\mathrm{c}})$.

Thus, $(x, x^*) \in E$ if and only if we have

$$\begin{pmatrix} x^*_{\mathcal{I}} \\ x_{\mathcal{I}^c} \end{pmatrix} = \begin{pmatrix} M & P \\ Q^t & 0 \end{pmatrix} \begin{pmatrix} x_{\mathcal{I}} \\ x^*_{\mathcal{I}^c} \end{pmatrix}.$$

Since E is monotone,

$$\langle M x_{\mathcal{I}}, x_{\mathcal{I}} \rangle + \langle (P^t + Q^t) x_{\mathcal{I}}, x^*_{\mathcal{I}^c} \rangle \geq 0 \ \text{ for all } (x_{\mathcal{I}}, x^*_{\mathcal{I}^c}) \in \mathbb{R}^n.$$

Set $x^*_{\mathcal{I}^c} = 0$, we deduce that M is positive semidefinite. Next, given any $x_{\mathcal{I}} \in \mathbb{R}^{\mathrm{card}\,(\mathcal{I})}$ set $x^*_{\mathcal{I}^c} = -(P^t + Q^t)x_{\mathcal{I}}$, we deduce that $P^t + Q^t = 0$. ■

Remark. Since $\mathrm{rank}\,(\mathrm{P}) \leq \min\{\mathrm{card}\,(\mathcal{I}), \mathrm{card}\,(\mathcal{I}^{\mathrm{c}})\})$, there exists a non-singular $\mathrm{card}\,(\mathcal{I}) \times \mathrm{card}\,(\mathcal{I})$ matrix Q such that

$$P^t Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where the order of the identity matrix is the rank of P.

Set

$$\begin{pmatrix} y_{\mathcal{I}} \\ y^*_{\mathcal{I}^c} \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} x_{\mathcal{I}} \\ x^*_{\mathcal{I}^c} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y^*_{\mathcal{I}} \\ y_{\mathcal{I}^c} \end{pmatrix} = \begin{pmatrix} Q^t & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x^*_{\mathcal{I}} \\ x_{\mathcal{I}^c} \end{pmatrix}.$$

Denote

$$\widehat{q} = \begin{pmatrix} \widehat{q}_{\mathcal{I}} \\ \widehat{q}_{\mathcal{I}^c} \end{pmatrix} = \begin{pmatrix} Q^t & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} q_{\mathcal{I}} \\ q_{\mathcal{I}^c} \end{pmatrix}$$

and

$$\widehat{C} = \begin{pmatrix} Q^t & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M & P \\ -P^t & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & I & 0 \\ M_{21} & M_{22} & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the affine subspace $\widehat{E}$ defined by

$$\widehat{E} = \{\, (w, w^*) = ((y_{\mathcal{I}}, y^*_{\mathcal{I}^c}), (y^*_{\mathcal{I}}, y_{\mathcal{I}^c})) \in \mathbb{R}^n \times \mathbb{R}^n : w^* = \widehat{C}w + \widehat{q} \,\}$$

is also maximal monotone. ■

3.3 Construction of an affine maximal monotone extension

Assume that E is an affine monotone, but not maximal monotone, subspace. Then $\dim(E) < n$, by Corollary 3.1.1 iii). We know that any monotone subset has maximal monotone extensions, by the axiom of choice. But a maximal extension of an affine subspace is not necessarily an affine subspace as the following example shows.

Example 3.3.1 *Consider* $E = \{\, (x, x^*) \in \mathbb{R}^2 \times \mathbb{R}^2 : Ax + Bx^* = 0 \,\}$*, where* A *and* B *are the* 3×2 *matrices*

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The matrix $C = [A, B]$ *has rank* 3*. Easy computations lead to*

$$M = AB^t + BA^t = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

whose eigenvalues are: $-\sqrt{6}$, 0 *and* $\sqrt{6}$. *In view of Theorem 3.1.1, the set* E *is monotone but not maximal monotone.*

Next, define the closed convex polyhedral $K = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \leq x_2 \}$. *By definition, the normal cone* N_K *is a maximal monotone map on* $\mathbb{R}^2$. *Its graph contains* E. *Clearly, this extension is not an affine subspace.*

In Section 2.7, we have shown how to construct a maximal monotone extension of an arbitrary monotone map (subset). Of course this construction can be used for a monotone affine subspace, but it does not lead to an affine extension. We shall show below how to construct one. Furthermore, in this particular case the construction becomes quite simpler.

Because the (maximal) monotonicity of E is preserved under translations, it is enough to work on linear subspaces, thus in the following discussion, we consider the subset

$$E = \{ (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n : Ax + Bx^* = 0 \}.$$

As usual, we denote by $\widetilde{E}$ the set

$$\widetilde{E} = \{ (\xi, \xi^*) : \langle \xi^* - x^*, \xi - x \rangle \geq 0, \ \forall (x, x^*) \in E \}.$$

Since $(0, 0)$ belongs to E,

$$(\xi, \xi^*) \in \widetilde{E} \implies \langle \xi, \xi^* \rangle \geq 0.$$

The following lemma gives a monotone linear extension of E.

Lemma 3.3.1 *Assume that* E *is monotone and take any* $(\xi, \xi^*) \in \widetilde{E}$. *Then the linear subspace*

$$\widehat{E} = \{ (x, x^*) + \lambda(\xi, \xi^*) : (x, x^*) \in E, \ \lambda \in \mathbb{R} \}$$

is monotone.

Proof. The set $\widehat{E}$ is monotone if and only if for all (x_1, x_1^*), $(x_2, x_2^*) \in E$ and for all λ_1, $\lambda_2 \in \mathbb{R}$

$$\langle (x_2^* - x_1^*) + (\lambda_2 - \lambda_1)\xi^*, (x_2 - x_1) + (\lambda_2 - \lambda_1)\xi \rangle \geq 0. \tag{3.3}$$

It is clear that inequality (3.3) holds when $\lambda_1 = \lambda_2$. If $\lambda_1 \neq \lambda_2$, this inequality is equivalent to

$$\langle \frac{(x_2^* - x_1^*)}{(\lambda_2 - \lambda_1)} - \xi^*, \frac{(x_2 - x_1)}{(\lambda_2 - \lambda_1)} - \xi \rangle \geq 0,$$

which is true, since E is a linear subspace and $(\xi, \xi^*) \in \widetilde{E}$. ■

We shall use this result to design an algorithm which constructs a maximal monotone linear subspace containing E in a finite number of steps.

Algorithm.

Step 0. Take $E_0 = E$.

Step k. In the previous steps, a monotone linear subspace E_{k-1} has been constructed with $E_{k-1} \supset E$.

- If $\dim(E_{k-1}) = n$, Corollary 3.1.1 implies that E_{k-1} is maximal monotone. STOP.

- Otherwise, by Proposition 2.1.1 there exists an element $(x_k, x_k^*) \in \widetilde{E}_{k-1} \setminus E_{k-1}$. Define

  $$E_k = E_{k-1} + F_k,$$

  where

  $$F_k = \{ \lambda(x_k, x_k^*) : \lambda \in \mathbb{R} \}.$$

  By Lemma 3.3.1, the linear subspace E_k is monotone. By construction

  $$\dim(E) \leq \dim(E_{k-1}) < \dim(E_k) = \dim(E_{k-1}) + 1.$$

- Do $k = k + 1$ and go back to Step k. ■

As a corollary of this result and Theorem 3.2.1, we have the following characterization of monotone affine subspaces.

Proposition 3.3.1 *An affine linear subspace E is monotone if and only if there exist a subset $\mathcal{I} \subset \{1, 2, \cdots, n\}$, a $\mathrm{card}\,(\mathcal{I}) \times \mathrm{card}\,(\mathcal{I})$ positive semidefinite matrix M, a $\mathrm{card}\,(\mathcal{I}) \times \mathrm{card}\,(\mathcal{I}^c)$ matrix P and two vectors $q_{\mathcal{I}} \in \mathbb{R}^{\mathrm{card}\,(\mathcal{I})}$*

and $q_{\mathcal{I}^c} \in \mathbb{R}^{\mathrm{card}\,(\mathcal{I}^c)}$ *such that*

$$(x, x^*) \in E \implies \begin{pmatrix} x^*_{\mathcal{I}} \\ x_{\mathcal{I}^c} \end{pmatrix} = \begin{pmatrix} M & P \\ -P^t & 0 \end{pmatrix} \begin{pmatrix} x_{\mathcal{I}} \\ x^*_{\mathcal{I}^c} \end{pmatrix} + \begin{pmatrix} q_{\mathcal{I}} \\ q_{\mathcal{I}^c} \end{pmatrix}.$$

Proof. By the previous construction, there exists a maximal monotone affine subspace containing E. Apply Theorem 3.2.1. ■

Remark. The difference between Proposition 3.3.1 and Theorem 3.2.1 is that the implication is one way for monotonicity and both ways for maximal monotonicity.

3.4 Restriction of an affine monotone subspace

In this section we assume that $X = X^* = \mathbb{R}^n$ and $U = U^* = \mathbb{R}^m$ and $\Phi \subset (X \times U) \times (X^* \times U^*)$ is an affine subspace. As in section 2.4, given $\bar{u} \in \mathrm{proj}_{\,\mathrm{U}}\Phi$, we consider the subspace

$$\Phi_{\bar{u}} = \{(x, x^*) \in X \times X^* : \exists\, u^* \in U^* \ \text{ such that } \ ((x, \bar{u}), (x^*, u^*)) \in \Phi\,\}.$$

Proposition 3.4.1 *Assume that* Φ *is (maximal) monotone. If we fixed* $\bar{u} \in \mathrm{proj}_{\,\mathrm{U}}\Phi$*. Then* $\Phi_{\bar{u}}$ *is a (maximal) monotone affine subspace.*

Proof. Φ can be set as

$$\Phi = \{\, ((x, u), (x^*, u^*)) : Ax + Bu + Cx^* + Du^* = c\,\},$$

where A and C are $p \times n$ matrices, B and D are $p \times m$ matrices and $c \in \mathbb{R}^p$. As usual, we assume that $c = 0$ and that the matrix $C = [A, B, C, D]$ has rank p. For simplicity, we assume that $\bar{u} = 0$. It is clear that the monotonicity of Φ_0 follows from the monotonicity of Φ. Next, assume that Φ is maximal monotone. By Theorem 3.1.1, $p = n + m$. Thanks to the remark just after Theorem 3.2.1, we can assume that the linear subspace Φ is of the following form

$$((x_{\mathcal{I}}, x_{\mathcal{I}^c}, u_{\mathcal{J}}, u_{\mathcal{J}^c}), (x^*_{\mathcal{I}}, x^*_{\mathcal{I}^c}, u^*_{\mathcal{J}}, u^*_{\mathcal{J}^c})) \in \Phi \quad \text{if and only if}$$

$$
\begin{pmatrix} x^*_{\mathcal{I}_1} \\ u^*_{\mathcal{J}_1} \\ x^*_{\mathcal{I}_2} \\ u^*_{\mathcal{J}_2} \\ x_{\mathcal{I}_1^c} \\ u_{\mathcal{J}_1^c} \\ x_{\mathcal{I}_2^c} \\ u_{\mathcal{J}_2^c} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} & I & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} & 0 & I & 0 & 0 \\ M_{31} & M_{32} & M_{33} & M_{34} & 0 & 0 & 0 & 0 \\ M_{41} & M_{42} & M_{43} & M_{44} & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{\mathcal{I}_1} \\ u_{\mathcal{J}_1} \\ x_{\mathcal{I}_2} \\ u_{\mathcal{J}_2} \\ x^*_{\mathcal{I}_1^c} \\ u^*_{\mathcal{J}_1^c} \\ x^*_{\mathcal{I}_2^c} \\ u^*_{\mathcal{J}_2^c} \end{pmatrix},
$$

where

$$
\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I} \subset \{1, 2, \cdots, n\}, \quad \mathcal{J}_1 \cup \mathcal{J}_2 = \mathcal{J} \subset \{1, 2, \cdots, m\},
$$

$\mathcal{I}_1^c \cup \mathcal{I}_2^c = \mathcal{I}^c$, $\mathcal{J}_1^c \cup \mathcal{J}_2^c = \mathcal{J}^c$ and $\mathcal{I}_1 \cap \mathcal{I}_2 \mathcal{J}_1 \cap \mathcal{J}_2 = \mathcal{I}_1^c \cap \mathcal{I}_2^c = \mathcal{J}_1^c \cap \mathcal{J}_2^c = \emptyset$.

Thus, $((x_{\mathcal{I}}, x_{\mathcal{I}^c}), (x^*_{\mathcal{I}}, x^*_{\mathcal{I}^c}))$ belongs to Φ_0 if and only if

$$
\begin{pmatrix} x^*_{\mathcal{I}_1} \\ x^*_{\mathcal{I}_2} \\ x_{\mathcal{I}_1^c} \\ x_{\mathcal{I}_2^c} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{13} & I & 0 \\ M_{31} & M_{33} & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{\mathcal{I}_1} \\ x_{\mathcal{I}_2} \\ x^*_{\mathcal{I}_1^c} \\ x^*_{\mathcal{I}_2^c} \end{pmatrix}.
$$

For this (x, x^*) a possible value for $(0, u^*)$ is

$$
\begin{pmatrix} u^*_{\mathcal{J}_1} \\ u^*_{\mathcal{J}_2} \end{pmatrix} = \begin{pmatrix} M_{21} & M_{23} & 0 & 0 \\ M_{41} & M_{43} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{\mathcal{I}_1} \\ x_{\mathcal{I}_2} \\ x^*_{\mathcal{I}_1^c} \\ x^*_{\mathcal{I}_2^c} \end{pmatrix}.
$$

Thus we have shown that $\dim(\Phi_0) \geq n$. Since Φ_0 is monotone, it is maximal monotone with $\dim(\Phi_0) = n$. ■

Remark. Since Φ is an affine subspace, the condition $\bar{u} \in \operatorname{proj}_{\mathrm{U}}(\Phi)$ is equivalent to $\bar{u} \in \operatorname{proj}_{\mathrm{U}}(\operatorname{ri}(\operatorname{proj}_{\mathrm{X}\times\mathrm{U}}(\Phi)))$. Thus, Proposition 3.4.1 can also be seen as a corollary of Proposition 2.5.7.

Chapter 4

A duality scheme for variational inequality problems

We begin this chapter formulating the convex duality scheme for optimization problems in terms of a duality scheme involving variational inequality problems. Then, we show how this scheme can be extended to general monotone variational inequality problems not issued from optimization problems.

The construction of the scheme is expressed in terms of monotone subsets and their projections. We have seen that projections preserve monotonicity but not maximality. Thus, preservation of maximality is a key question.

As in convex duality, associated to the original problem (called primal), by using perturbations, one obtains a dual, a perturbed primal, a perturbed dual, and a lagrangian-type problems. Duality is symmetric, that is to say, under a maximality type condition, the dual of the dual problem is the primal problem. Finally, we conclude with Section 4.2.5, related to a sensitivity analysis, that is to say, we study the behavior of the set of solutions of the problem under small perturbations.

4.1 The duality scheme for variational inequality problems resulting from convex optimization problems

In this section, we shall traduce the duality scheme for optimization problems in terms of variational inequality problems. As in chapter 1, one considers the problem:

$$\text{Find } \ \bar{x} \in X \ \text{ such that } \ f(\bar{x}) \leq f(x), \quad \forall\, x \in X, \tag{P}$$

where $f : X \to]-\infty, +\infty]$ is a proper lsc convex function. Next, let $\varphi : X \times U \to]-\infty, +\infty]$ be a lsc convex function such that

$$\varphi(x, 0) = f(x), \quad \forall\, x \in X.$$

By construction φ is proper since $\mathrm{dom}\,(\varphi) \supset \mathrm{dom}\,(\mathrm{f}) \times \{0\}$. Next, for each $u \in U$, define $\varphi_u : X \to]-\infty, +\infty]$ by

$$\varphi_u(x) = \varphi(x, u), \quad \forall\, x \in X.$$

These functions are convex and lsc. The function φ_u is proper if and only if $u \in \mathrm{proj}_{\,\mathrm{U}}(\mathrm{dom}\,(\varphi))$. The perturbed problems are:

$$\text{Find } \ \bar{x}_u \in X \ \text{ such that } \ \varphi_u(\bar{x}_u) \leq \varphi_u(x), \quad \forall\, x \in X. \tag{P_u}$$

Next, let F, Φ and Φ_u be the graphs of ∂f, $\partial\varphi$ and $\partial\varphi_u$, respectively. Since f, φ and φ_u are proper, convex and lsc functions, the sets F, Φ and Φ_u are *cyclically maximal monotone.*

The problems (P) and (P_u) are respectively equivalent to the following Variational Inequality Problems (VIP):

$$\text{Find } \ \bar{x} \in X \ \text{ such that } \ (\bar{x}, 0) \in F \tag{V}$$

and

$$\text{Find } \ \bar{x}_u \in X \ \text{ such that } \ (\bar{x}_u, 0) \in \Phi_u. \qquad (V_u).$$

It is natural to say that problem (P) $((P_u))$ is nondegenerate if the function f (φ_u) is proper. Thus, we say that $\mathrm{dom}\,(\mathrm{f})$ and $\mathrm{dom}\,(\varphi_\mathrm{u})$ are the *domains of nondegeneracy* of (P) and (P_u), respectively. By analogy, we say that the problems (V) and (V_u) are nondegenerate when F and Φ_u are nonempty. The sets, $\mathrm{proj}_\mathrm{X}(\mathrm{F})$ and $\mathrm{proj}_\mathrm{X}(\Phi_\mathrm{u})$ are called the *domains of nondegeneracy* of (V) and (V_u), respectively. Unfortunately, the domains of nondegeneracy of (P) and (V) $((P_u)$ and $(V_u))$ do not coincide in general as shown by the following example:

Example 4.1.1 *Take* $X = \mathbb{R}$ *and define* $f : X \to]-\infty, +\infty]$ *by*

$$f(x) = \begin{cases} -\sqrt{x} & \textit{if } \ x \geq 0, \\ \\ +\infty & \textit{otherwise.} \end{cases}$$

Then, $\quad \mathrm{dom}\,(\mathrm{f}) = [\,0, +\infty\,[\quad$ *and* $\quad \mathrm{dom}\,(\mathrm{F}) =]\,0, +\infty\,[.$

The following proposition is rather immediate.

Proposition 4.1.1 *Assume that* $\varphi : X \times U \to]-\infty, +\infty]$ *is a proper lsc convex function. Then*

a) $\mathrm{proj}_\mathrm{X}(\Phi) \subset \mathrm{proj}_\mathrm{X}(\mathrm{dom}(\varphi))$.

b) $\mathrm{ri}\,(\mathrm{proj}_\mathrm{X}\Phi) = \mathrm{ri}\,(\mathrm{proj}_\mathrm{X}(\mathrm{dom}(\varphi)))$, *this set is convex.*

c) $\mathrm{cl}\,(\mathrm{proj}_\mathrm{X}\Phi) = \mathrm{cl}\,(\mathrm{proj}_\mathrm{X}(\mathrm{dom}\,(\varphi)))$, *this set is convex.*

d) $\mathrm{proj}_\mathrm{U}(\Phi) \subset \{\mathrm{u} : \mathrm{proj}_\mathrm{X}(\Phi_\mathrm{u}) \neq \emptyset\} = \mathrm{proj}_\mathrm{U}(\mathrm{dom}\,(\varphi))$.

e) $\mathrm{ri}\,(\mathrm{proj}_\mathrm{U}\Phi) = \mathrm{ri}\,(\{\mathrm{u} : \mathrm{proj}_\mathrm{X}(\Phi_\mathrm{u}) \neq \emptyset\}) = \mathrm{ri}\,(\mathrm{proj}_\mathrm{U}(\mathrm{dom}(\varphi)))$, *this set is convex.*

f) $\mathrm{cl}\,(\mathrm{proj}_\mathrm{U}\Phi) = \mathrm{cl}\,(\{\mathrm{u} : \mathrm{proj}_\mathrm{X}(\Phi_\mathrm{u}) \neq \emptyset\}) = \mathrm{cl}\,(\mathrm{proj}_\mathrm{U}(\mathrm{dom}(\varphi)))$, *this set is convex.*

Proof.

a) By definition, $x \in \operatorname{proj}_{\mathrm{X}}(\Phi)$ if and only if there exists $u \in U$ such that $(x,u) \in \operatorname{dom}(\partial\varphi)$. This implies, in particular, that $(x,u) \in \operatorname{dom}(\varphi)$ which is equivalent to say that $x \in \operatorname{proj}_{\mathrm{X}}(\operatorname{dom}(\varphi))$.

b) By (a), $\operatorname{ri}(\operatorname{proj}_{\mathrm{X}}(\Phi)) \subset \operatorname{ri}(\operatorname{proj}_{\mathrm{X}}(\operatorname{dom}(\varphi)))$. The converse inclusion follows from the relation $\operatorname{ri}(\operatorname{proj}_{\mathrm{X}}(\operatorname{dom}(\varphi))) = \operatorname{proj}_{\mathrm{X}}(\operatorname{ri}(\operatorname{dom}(\varphi)))$, which is due to the fact that the projection on a linear space is linear.

c) By (a), $\operatorname{cl}(\operatorname{proj}_{\mathrm{X}}(\Phi)) \subset \operatorname{cl}(\operatorname{proj}_{\mathrm{X}}(\operatorname{dom}(\varphi)))$. On the other hand, since $\operatorname{cl}(\operatorname{proj}_{\mathrm{X}}(\operatorname{dom}(\varphi))) = \operatorname{cl}(\operatorname{ri}(\operatorname{proj}_{\mathrm{X}}(\operatorname{dom}(\varphi))))$, part $b)$ implies that $\operatorname{cl}(\operatorname{ri}(\operatorname{proj}_{\mathrm{X}}(\operatorname{dom}(\varphi)))) = \operatorname{cl}(\operatorname{ri}(\operatorname{proj}_{\mathrm{X}}(\Phi))) \subset \operatorname{cl}(\operatorname{proj}_{\mathrm{X}}(\Phi))$, and therefore the converse inclusion follows.

d) By definition, $u \in \operatorname{proj}_{\mathrm{U}}(\Phi)$ if and only if there exists $(x, x^*, u^*) \in X \times X^* \times U^*$ such that $(x^*, u^*) \in \partial\varphi(x,u)$. This implies that $x^* \in \partial\varphi_u(x)$, and therefore $\operatorname{proj}_{\mathrm{X}}(\Phi_{\mathrm{u}}) \neq \emptyset$. Next, assume that $\operatorname{proj}_{\mathrm{X}}(\Phi_{\mathrm{u}}) \neq \emptyset$. Then there exists $(x, x^*) \in X \times X^*$ such that $x^* \in \partial\varphi_u(x)$. Hence $(x,u) \in \operatorname{dom}(\varphi)$, and therefore $u \in \operatorname{proj}_{\mathrm{U}}(\operatorname{dom}(\varphi))$.

e) Similarly to $b)$, $\operatorname{ri}(\operatorname{proj}_{\mathrm{U}}(\Phi)) = \operatorname{ri}(\operatorname{proj}_{\mathrm{U}}(\operatorname{dom}(\varphi)))$. Thus, $e)$ follows from $d)$.

f) Similarly to $c)$, $\operatorname{cl}(\operatorname{proj}_{\mathrm{U}}(\Phi)) = \operatorname{cl}(\operatorname{proj}_{\mathrm{U}}(\operatorname{dom}(\varphi)))$. Thus, $f)$ follows from $d)$. ∎

The following proposition establishes a relation between Φ and Φ_u.

Proposition 4.1.2 *Assume that* $\varphi : X \times U \to]-\infty, +\infty]$ *is a proper lsc convex function. Then,*

a) $\Phi_u \supset \operatorname{proj}_{\mathrm{X}\times\mathrm{X}^*}[\Phi \cap ((\mathrm{X} \times \{\mathrm{u}\}) \times (\mathrm{X}^* \times \mathrm{U}^*))]$*;*

b) If $u \in \operatorname{ri}(\operatorname{proj}_{\mathrm{U}}\Phi)$*, then*

$$\Phi_u = \operatorname{proj}_{\mathrm{X}\times\mathrm{X}^*}[\Phi \cap ((\mathrm{X} \times \{\mathrm{u}\}) \times (\mathrm{X}^* \times \mathrm{U}^*))].$$

Proof. a) follows from the definitions of Φ and Φ_u. Let us prove b). Assume that $(\bar{x}, \bar{x}^*) \in \Phi_u$, then, $0 \in \partial[\varphi_u(.) - \langle \bar{x}^*, . \rangle](\bar{x})$. Next, define $\psi : X \times U \to]-\infty, +\infty]$ such that $\psi(x, u) = \varphi(x, u) - \langle \bar{x}^*, x \rangle$. ψ is proper lsc and convex function. Associate with ψ the minimization problem

$$h(u) = \inf_x \psi(x, u). \qquad (P_1)$$

Then, the dual optimization problem of (P_1) is:

$$h^{**}(u) = \sup_{u^*}[\langle u^*, u \rangle - \varphi^*(\bar{x}^*, u^*)]. \qquad (Q_1)$$

Since $u \in \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{h}))$, $h(u) = h^{**}(u)$ and $\partial h(u) \neq \emptyset$. Thus, for any $\bar{u}^* \in \partial h(u)$,

$$\varphi(\bar{x}, u) - \langle \bar{x}^*, \bar{x} \rangle = \min_x \psi(x, u) = \sup_{u^*}[\langle u^*, u \rangle - \varphi^*(\bar{x}^*, u^*)] = \langle \bar{u}^*, u \rangle - \varphi^*(\bar{x}^*, \bar{u}^*)$$

i.e.

$$(\bar{x}^*, \bar{u}^*) \in \partial\varphi(\bar{x}, u).$$

This shows that

$$\Phi_u \subset \mathrm{proj}_{\mathrm{X} \times \mathrm{X}^*}[\Phi \cap ((\mathrm{X} \times \{\mathrm{u}\}) \times (\mathrm{X}^* \times \mathrm{U}^*))],$$

and therefore the equality follows. ■

Remark. Since Φ_u is the graph of the subdifferential of a proper lsc convex function, the sets $\mathrm{ri}\,(\mathrm{proj}_{\mathrm{X}}(\Phi_{\mathrm{u}}))$ and $\mathrm{cl}\,(\mathrm{proj}_{\mathrm{X}}(\Phi_{\mathrm{u}}))$ are convex when $u \in \mathrm{ri}\,(\mathrm{proj}_{\mathrm{U}}(\Phi))$. This is also the case when $u \notin \mathrm{proj}_{\mathrm{U}}(\Phi)$ because in this case the two sets are empty. When u belongs to the boundary of the projection of Φ on U, the following set

$$\mathrm{cl}\,(\mathrm{proj}_{\mathrm{X}}[\Phi \cap ((\mathrm{X} \times \{\mathrm{u}\}) \times (\mathrm{X}^* \times \mathrm{U}^*))])$$

may be not convex. This explains why b) does not hold in general.

Let us provide an example of such a situation.

Example 4.1.2 *Take* $X = U = \mathbb{R}$. *Define* $\varphi : X \times U \to]-\infty, +\infty]$ *by*

$$\varphi(x,u) \begin{cases} max[\,x^2, 1 - \sqrt{u}\,] & \text{if } u \geq 0, \\ \\ +\infty & \text{if not.} \end{cases}$$

The function φ *is proper convex and lsc (see example 1.2.1). Take* $u = 0$. *Then,*

$$\Phi_0 = \{\,(x, 2x) : |x| > 1\,\} \cup (\{-1\} \times [\,-2, 0\,]) \cup \{\,(x, 0) : |x| < 1\,\} \cup (\{1\} \times [\,0, 2\,])$$

and

$$\mathrm{proj}_{\,X \times X^*}[\Phi \cap ((X \times \{0\}) \times (X^* \times U^*))] = \{\,(x, 2x) : |x| \geq 1\,\}.$$

Hence

$$\mathrm{ri}\,(\mathrm{proj}_{\,X}(\Phi_0)) = \mathrm{cl}\,(\mathrm{proj}_{\,X}(\Phi_0)) = X$$

and

$$\mathrm{cl}\,(\mathrm{proj}_{\,X}[\Phi \cap ((X \times \{0\}) \times (X^* \times U^*))]) = X \setminus]-1, +1[.$$

The first set is convex, but the second one is not. In this example $u = 0$ *belongs to the boundary of* $\mathrm{proj}_{\,U}(\Phi)$. ■

According to Proposition 4.1.2, when $0 \in \mathrm{ri}\,(\mathrm{proj}_{\,U}(\Phi))$ ($u \in \mathrm{ri}\,(\mathrm{proj}_{\,U}(\Phi))$), the variational inequality problem (V) ((V_u)) can be formulated as

$$\text{Find } x \in X \text{ such that } \exists\, u^* \in U^* \text{ with } (x, 0, 0, u^*) \in \Phi, \qquad (V^0)$$

$$(\text{ Find } x_u \in X \text{ such that } \exists\, u^* \in U^* \text{ with } (x_u, u, 0, u^*) \in \Phi). \qquad (V^u)$$

Next, we shall consider a dual formulation of problem (V). Here, again we refer the duality scheme in optimization problem.

The dual optimization problem associated to (P) is

$$\text{Find } \bar{u}^* \in U^* \text{ such that } d(\bar{u}^*) \leq d(u^*),\ \forall\, u^* \in U^*, \qquad (D)$$

where the function $d : U^* \to [-\infty, +\infty]$ is defined by

$$d(u^*) = \varphi^*(0, u^*) = \varphi_0^*(u^*),\ \forall\, u^* \in U^*.$$

The variational formulation of (D) is

$$\text{Find } \ \bar{u}^* \in U^* \ \text{ such that } \ (0, \bar{u}^*) \in G, \tag{DV}$$

where

$$G^- = \operatorname{graph}(\partial \mathrm{d}) = \operatorname{graph}(\partial \varphi_0^*).$$

We say that (DV) is a *dual variational inequality problem* associated to (V).

Assume that $0 \in \operatorname{ri}(\operatorname{proj}_{X^*}(\Phi))$. Then, in the same way that we have done for the primal problems (V) and (V_0), we reformulate (DV) in terms of Φ as

$$\text{Find } \ \bar{u}^* \in U^* \ \text{ such that } \exists\, x \in X \text{ with } (x, 0, 0, u^*) \in \Phi. \tag{DV^0}$$

Also, the perturbed variational inequality problems associated to (DV) are

$$\text{Find } \ \bar{u}^* \in U^* \ \text{ such that } \ (0, \bar{u}^*) \in G_{x^*}, \tag{DV_{x^*}}$$

where $(G_{x^*})^- = \operatorname{graph}(\partial \varphi^*_{x^*})$ and the function $\varphi^*_{x^*}$ is defined by

$$\varphi^*_{x^*}(u^*) = \varphi^*(x^*, u^*), \ \forall\, u^* \in U^*.$$

Here, the elements x^* belonging to $\operatorname{proj}_{X^*}(\Phi)$ are taken as the dual perturbation parameters.

The dual perturbed optimization problems associated to (D) are

$$\text{Find } \ \bar{u}^*_{x^*} \in U^* \ \text{ such that } \ \varphi^*_{x^*}(\bar{u}^*_{x^*}) \le \varphi^*_{x^*}(u^*_{x^*}), \ \forall\, u^* \in U^*, \tag{D_{x^*}}$$

which, if $x^* \in \operatorname{ri}(\operatorname{proj}_{X^*}(\Phi))$, are equivalent, in terms of Φ, to:

$$\text{Find } \ \bar{u}^*_{x^*} \in U^* \ \text{ such that } \exists\, x \in X \text{ with } (x, 0, x^*, \bar{u}^*_{x^*}) \in \Phi. \tag{DV^{x^*}}$$

Our next step consists in giving a variational inequality formulation for the lagrangian. Recall that the lagrangian function $L : X \times U^* \to [-\infty, +\infty]$ associated to the perturbation function $\varphi : X \times U \to]-\infty, +\infty]$ is defined by

$$L(x, u^*) = \inf_{u \in U} [\, \langle -u, u^* \rangle + \varphi(x, u)\,].$$

The saddle-points of L are the points $(\bar{x}, \bar{u}^*) \in X \times U^*$ such that

$$L(\bar{x}, u^*) \leq L(\bar{x}, \bar{u}^*) \leq L(x, \bar{u}^*), \ \forall\, (x, u^*) \in X \times U^*.$$

The saddle-points of L are associated to the convex optimization problems (P) and (D) as follows: $(\bar{x}, \bar{u}^*)$ is a saddle point of L if and only if $\bar{x}$ is a solution of problem (P), $\bar{u}^*$ is a solution of problem (D) and there is no duality gap. Thus, $(\bar{x}, \bar{u}^*)$ is a saddle point of L if and only if

$$\varphi(\bar{x}, 0) + \varphi^*(0, \bar{u}^*) = \langle (\bar{x}, 0), (0, \bar{u}^*) \rangle.$$

In terms of Φ, $(\bar{x}, \bar{u}^*)$ is a saddle point of L if and only if

$$((\bar{x}, 0), (0, \bar{u}^*)) \in \Phi.$$

Thus, the *variational inequality saddle point problem* is

$$\text{Find } \ (\bar{x}, \bar{u}^*) \in X \times U^* \ \text{ such that } \ ((\bar{x}, 0), (0, \bar{u}^*)) \in \Phi, \qquad (SPV)$$

which can be equivalently formulated as

$$\text{Find } \ (\bar{x}, \bar{u}^*) \in X \times U^* \ \text{ such that } \ ((\bar{x}, \bar{u}^*), (0, 0)) \in \Psi, \qquad (SV)$$

where the set Ψ is defined by

$$((x, u^*), (x^*, u)) \in \Psi \iff ((x, u)(x^*, u^*)) \in \Phi.$$

Unfortunately, the cyclic monotonicity property is not preserved in general when permutations are done on the variables (see example 2.1.1 and the discussion that follows). Therefore, (SV) is not necessarily associated to a convex optimization problem when (SPV) is so.

4.2 A duality scheme for monotone variational inequality problems

As mentioned, variational inequality problems are not necessarily associated with optimization problems. In this section, we describe a duality scheme working in the general case of variational inequality problems. This scheme is inspired by the scheme described in the previous section.

4.2.1 The primal variational problem

We start with the *variational inequality problem*

$$\text{Find } \; x \in X \text{ such that } (x, 0) \in F_p \tag{V_p}$$

where F_p is a subset of $X \times X^*$. We denote the solutions set of (V_p) by

$$S_p = \{\, x \in X : (x, 0) \in F_p \,\}.$$

Problem (V_p) is said to be monotone if F_p is monotone. If F_p is maximal monotone, then S_p is closed and convex, possibly empty.

4.2.2 Introducing perturbations

Next, we consider $\Phi \subset (X \times U) \times (X^* \times U^*)$ such that

$$(x, x^*) \in F_p \iff \exists\, u^* \in U^* \text{ with } ((x, 0), (x^*, u^*)) \in \Phi.$$

F_p is monotone if Φ is monotone, but not necessarily maximal monotone if Φ is maximal monotone. We shall give later some conditions which ensure F_p to be maximal monotone.

It is clear that (V_p) is equivalent to the problem

$$\text{Find } \; x \in X \text{ such that } \exists\, u^* \in U^* \text{ with } ((x, 0), (0, u^*)) \in \Phi,$$

which is also equivalent to the Lagrangian-type problem

$$\text{Find } \; (x, u^*) \in X \times U^* \text{ such that } ((x, 0), (0, u^*)) \in \Phi, \tag{V_L}$$

whose solution set is denoted by

$$S_l = \{\, (x, u^*) \in X \times U^* : ((x, 0), (0, u^*)) \in \Phi \,\}.$$

4.2.3 The dual problem

The Lagrangian-type problem (V_L) leads to consider the following problem

$$\text{Find } \; u^* \in U^* \text{ such that } \exists\, x \in X \text{ with } ((x, 0), (0, u^*)) \in \Phi$$

and next to define the subset $F_d \subset U \times U^*$ defined by

$$(u, u^*) \in F_d \iff \exists\, x \in X \text{ such that } ((x, u), (0, u^*)) \in \Phi.$$

Then, the *dual variational inequality problem* is defined as:

$$\text{Find } \ u^* \in U^* \text{ such that } (0, u^*) \in F_d. \tag{V_d}$$

We denote by S_d the set of solutions of (V_d),

$$S_d = \{\, u^* \in U^* : (0, u^*) \in F_d \,\}.$$

The duality scheme we have described above is thoroughly symmetric. Furthermore, if Φ is monotone, then (V_p), (V_d) and (V_L) are monotone variational inequality problems.

If this duality scheme is associated with the duality scheme in convex optimization (i.e. when Φ is the graph of the subdifferential of the perturbation function φ), then S_p, S_d are respectively the sets of optimal solutions of the primal and the dual optimization problems and S_l is the set of saddle-points of the corresponding Lagrangian function.

It is clear that

$$S_p = \mathrm{proj}_{\,X}(S_l), \quad S_d = \mathrm{proj}_{\,U^*}(S_l) \ \text{ and } \ S_l \subset S_p \times S_d.$$

It follows that the sets S_p and S_d are convex when S_l is convex, but they are not necessarily closed when S_l is closed. Recall that S_l is closed and convex when Φ is maximal monotone.

In general we have not $S_l = S_p \times S_d$ as seen from the following example

Example 4.2.1 *Let us consider*

$$F_p = \{\, (x, 0) : x \in \mathbb{R} \,\} \subset \mathbb{R}^2$$

and

$$\Phi = \{\, ((x, u), (x^*, u^*)) \ : \ x^* = u = x + u^* \} \subset \mathbb{R}^4.$$

Then F_p and Φ are maximal monotone and the relation

$$(x, x^*) \in F_p \iff \exists\, u^* \in U^* \text{ with } ((x, 0), (x^*, u^*)) \in \Phi$$

is verified. It follows that

$$F_d = \{ (0, u^*) : u^* \in \mathbb{R} \} \subset \mathbb{R}^2.$$

Hence

$$S_l = \{(x, u^*) \; : \; 0 = x + u^*\} \neq S_p \times S_d = \mathbb{R} \times \mathbb{R}.$$

In this example, $0 \in \operatorname{ri}(\operatorname{proj}_{\mathrm{U}}(\Phi)) = \mathbb{R}$*.*

Remark. This example shows that, in contrast with the convex optimization duality scheme, the condition $0 \in \operatorname{ri}(\operatorname{proj}_{\mathrm{U}}(\Phi))$ does not imply the equality $S_l = S_p \times S_d$.

4.2.4 Perturbed variational inequality problems

Given $u \in U$, we define the *primal perturbed problem* (V_p^u) as

$$\text{Find } \; x_u \in S_p(u), \tag{V_p^u}$$

where

$$S_p(u) = \{ x \in X : \exists \, u^* \in U^* \text{ with } ((x, u), (0, u^*)) \in \Phi \}.$$

Note that $S_p(0) = S_p$.

We also define, for each $u \in U$, the subsets $F_p^u \subset X \times X^*$ as

$$(x, x^*) \in F_p^u \iff \exists \, u^* \in U^* \;\text{ with }\; ((x, u), (x^*, u^*)) \in \Phi.$$

It follows that the problem (V_p^u) is equivalent to

$$\text{Find } \; x_u \in X \;\text{ such that }\; (x_u, 0) \in F_p^u.$$

Similarly, for each $x^* \in X^*$, the *dual perturbed problem* $(V_d^{x^*})$ is defined as

$$\text{Find } \; u^*_{x^*} \in S_d(x^*), \tag{$V_d^{x^*}$}$$

where

$$S_d(x^*) = \{ u^* \in U^* : \exists \, x \in X \text{ with } ((x, 0), (x^*, u^*)) \in \Phi \}.$$

Note that $S_d(0) = S_d$.

It is clear that the problem $(V_d^{x^*})$ can be equivalently formulated as

$$\text{Find } \ u^*_{x^*} \in U^* \ \text{ s.t. } \ (0, u^*_{x^*}) \in F_d^{x^*}, \qquad (V_d^{x^*})$$

where the subset $F_d^{x^*} \subset U \times U^*$ is defined by

$$(u, u^*) \in F_d^{x^*} \iff \exists\, x \in X \ \text{ with } \ ((x, u), (x^*, u^*)) \in \Phi.$$

It is convenient to introduce the following map: $\Lambda : X^* \times U \rightrightarrows X \times U^*$ defined by

$$\Lambda(x^*, u) = \{(x, u^*) \,:\, (x, u, x^*, u^*) \in \Phi\}.$$

It is clear that

$$S_p(u) = \mathrm{proj}_{\,X}(\Lambda(0, u)), \ S_d(x^*) = \mathrm{proj}_{\,U^*}(\Lambda(x^*, 0)) \ \text{ and } \ S_l = \Lambda(0, 0). \quad (4.1)$$

If Φ is monotone, then problems (V_L), (V_p^u) and $(V_d^{x^*})$ are monotone.

4.2.5 Sensitivity and stability analysis

In the same way that in the duality scheme for convex optimization, one seeks, as much as possible, to choose the perturbation function φ in the class of convex lsc functions, we shall try to choose the perturbation subset Φ in the class of maximal monotone subsets. Such a condition on Φ needs the monotonicity of F_p and, in turn, implies the monotonicity of F_d. The following proposition is rather immediate.

Proposition 4.2.1 *Assume that Φ is a maximal monotone subset. Then, the map Λ is maximal monotone, the set S_l is closed and convex, and the sets S_p and S_d are convex. Furthermore, S_l compact and nonempty if and only if $(0, 0)$ belongs to the interior of the domain of Λ. This condition is equivalent to say that both sets S_p and S_d are compact and nonempty.*

If $(0, 0)$ does not belong to the interior of the domain of Λ, then the solution sets S_p, S_d and S_l are unbounded or empty. To say more, one must look at the sets F_p and F_d. Unfortunately, unlike in convex duality, Φ maximal monotone does not imply that F_p and F_d are maximal monotone as we see in the following example.

Example 4.2.2 *(see Example 4.1.2) Consider for* Φ*, the graph of the subdifferential of the function* $\varphi : X \times U \to]-\infty, +\infty]$ *defined by*

$$\varphi(x,u) = \begin{cases} max[\,x^2, 1 - \sqrt{u}\,] & \text{if } u \geq 0, \\ \\ +\infty & \text{if not.} \end{cases}$$

Then

$$F_p = \{(x, 2x) : |x| \geq 1\},$$

which is monotone but not maximal monotone. In this particular example, $0 \in \mathrm{bd}\,(\mathrm{proj}_{\,\mathrm{U}}(\Phi))$.

Linear and convex quadratic programming are two cases where the convex optimization duality scheme works with minimal assumptions. The corresponding case in our scheme is the case where Φ is affine.

Proposition 4.2.2 *Assume that* Φ *is a maximal monotone affine subspace and* $(u, x^*) \in \mathrm{proj}_{\,\mathrm{U}}(\Phi) \times \mathrm{proj}_{\,\mathrm{X}^*}(\Phi)$*. Then, the sets* F_p^u *and* $F_d^{x^*}$ *are maximal monotone affine subspace and solutions sets* S_l*,* $S_p(u)$ *and* $S_d(x^*)$ *are also affine subspaces.*

Proof. The assumptions are equivalent to assumptions in Proposition 3.4.1, so, we deduce that F_p^u and $F_d^{x^*}$ are maximal monotone affine subspaces. On the other hand, by definition, for all $(x^*, u) \in X^* \times U$, the sets $\Lambda(x^*, u)$ are affine subspaces. Thus, the sets S_l, $S_p(u)$ and $S_d(x^*)$ are affine subspaces, because, by (4.1), are projection of affine subspaces. ■

For the treatment of the nonaffine case, the following classical result is useful, see for instance [10].

Proposition 4.2.3 *Given* $f : \mathbb{R}^n \to \mathbb{R}^p$ *linear and* $C \subset \mathbb{R}^n$ *such that* $\mathrm{ri}\,(\mathrm{C})$ *and* $\mathrm{cl}\,(\mathrm{C})$ *are convex. If* $\mathrm{ri}\,(\mathrm{cl}\,(\mathrm{C})) = \mathrm{ri}\,(\mathrm{C})$*, then* $f(\mathrm{ri}\,(\mathrm{C})) = \mathrm{ri}\,(\mathrm{f(C)})$.

The projection onto a linear space is linear. We obtain the following results on primal and dual problems.

Proposition 4.2.4 *Assume that* $\Phi \subset (X \times U) \times (X^* \times U^*)$ *is a maximal monotone subset and the interior of the domain of* Λ *is not empty. Assume in addition that* $0 \in \operatorname{int}(\operatorname{proj}_{\mathrm{U}}(\Phi))$. *Then, there exists an open neighborhood* N *of* $0 \in U$ *such that for each* $u \in N$ *the set* F_p^u *is maximal monotone and the solution set* $S_p(u)$ *is closed and convex.*

Proof. The map Λ is maximal monotone and therefore its domain verify the conditions in Proposition 4.2.3. Hence,

$$\operatorname{ri}(\operatorname{proj}_{\mathrm{U}}(\Phi)) = \operatorname{ri}(\operatorname{proj}_{\mathrm{U}}(\operatorname{dom}(\Lambda))) = \operatorname{proj}_{\mathrm{U}}(\operatorname{ri}(\operatorname{dom}(\Lambda))).$$

Since by assumption, $\operatorname{int}(\operatorname{dom}(\Lambda)) \neq \emptyset$, the relative interior above is the interior. Then, there exists an open neighborhood N of 0 such that $N \subset \operatorname{proj}_{\mathrm{U}}(\operatorname{int}(\operatorname{dom}(\Lambda)))$. Thus, for all $u \in N$, the assumptions of Theorem 2.4.2 hold, we deduce the maximal monotonicity of the multivalued map Γ_u defined by

$$x^* \in \Gamma_u(x) \iff \exists\, u^* \in U^* \text{ such that } (x, u^*) \in \Lambda(x^*, u).$$

Its graph F_p^u, is maximal monotone. Since $S_p(u) = \Gamma_u^-(0)$ this set is closed and convex ■

Because the duality scheme is symmetric, we have the following result, which no proof is required.

Proposition 4.2.5 *Assume that* $\Phi \subset (X \times U) \times (X^* \times U^*)$ *is maximal monotone and the interior of the domain of* Λ *is not empty. Assume in addition that* $0 \in \operatorname{int}(\operatorname{proj}_{\mathrm{X}^*}(\Phi))$. *Then, there exists an open neighborhood* W *of* 0 *such that for each* $x^* \in W$ *the set* $F_d^{x^*}$ *is maximal monotone and the set* $S_d(x^*)$ *is closed and convex.*

Next, define the following multivalued maps, $\Sigma_p : X^* \overset{\longrightarrow}{\longrightarrow} X$ and $\Sigma_d : U \overset{\longrightarrow}{\longrightarrow} U^*$ by

$$\mathrm{graph}\,(\Sigma_\mathrm{p}) = \{\, (\mathrm{x}^*, \mathrm{x}) : (\mathrm{x}, \mathrm{x}^*) \in \mathrm{F}_\mathrm{p}^0 \,\} \quad \text{and} \quad \mathrm{graph}\,(\Sigma_\mathrm{d}) = \mathrm{F}_\mathrm{d}^0.$$

It is clear that

$$\mathrm{dom}\,(\Sigma_{\mathrm{p}}) = \mathrm{dom}\,(\mathrm{S_d}) \quad \text{and} \quad \mathrm{dom}\,(\Sigma_{\mathrm{d}}) = \mathrm{dom}\,(\mathrm{S_p}).$$

The following two propositions give other characterizations (see Proposition 4.2.1) for the condition $(0,0) \in \mathrm{int}\,(\mathrm{dom}\,(\Lambda))$.

Proposition 4.2.6 *With the same assumptions of Proposition 4.2.4, the following three statements are equivalent:*

i) $(0,0) \in \mathrm{int}\,(\mathrm{dom}\,(\Lambda))$.

ii) $0 \in \mathrm{int}\,(\mathrm{dom}\,(\Sigma_{\mathrm{p}})) = \mathrm{int}\,(\mathrm{dom}\,(\mathrm{S_d}))$.

iii) There exists a compact $K \subset X$ such that $\emptyset \neq S_p(0) \subset K$.

Proof. a) By assumption, the multivalued map Λ is maximal monotone and $0 \in \mathrm{proj}_{\,\mathrm{U}}(\mathrm{int}\,(\mathrm{dom}\,(\Lambda)))$, which is exactly the assumption of Proposition 2.4.2, from what we deduce that the multivalued map Σ_p is maximal monotone and the equivalence between i) and ii) follows.

b) By definition,

$$S_p(0) = \Sigma_p(0).$$

Since Σ_p is maximal monotone, the condition $0 \in \mathrm{int}\,(\mathrm{dom}\,(\Sigma_{\mathrm{p}}))$ is equivalent to the existence of a compact subset $K \subset X$ such that $\emptyset \neq \Sigma_p(0) \subset K$, from what the equivalence between i) and ii) follows. ■

The dual version of this proposition is the following, for which no proof is required.

Proposition 4.2.7 *With the same assumptions of Proposition 4.2.5, the following three statements are equivalent:*

i) $(0,0) \in \mathrm{int}\,(\mathrm{dom}\,(\Lambda))$.

ii) $0 \in \mathrm{int}\,(\mathrm{dom}\,(\Sigma_{\mathrm{d}})) = \mathrm{int}\,(\mathrm{dom}\,(\mathrm{S_p}))$.

iii) There exists a compact $T \subset U^*$ such that $\emptyset \neq S_d(0) \subset T$.

As a simple consequence of these last results we have the following:

Corollary 4.2.1 *With the same assumption of Proposition 4.2.4 or Proposition 4.2.5, we have that $S_p(0)$ is convex, compact and not empty if and only if $S_d(0)$ it so. Furthermore, there exists an open neighborhood $N \times W \subset U \times X^*$ of $(0,0)$, such that, for each $(u,x^*) \in N \times W$, the subsets F_p^u and $F_d^{x^*}$ are maximal monotone.*

Next, we shall study the behavior of the primal and dual perturbed problems in a neighborhood of $u = 0$.

Theorem 4.2.1 *Assume that $\Phi \subset (X \times U) \times (X^* \times U^*)$ is maximal monotone and $(0,0) \in \operatorname{int}(\operatorname{dom}(\Lambda))$. Then, there exist K, a compact subset of X, T, a compact subset of U^*, V, an open convex neighborhood of 0 in U, and W, an open convex neighborhood of 0 in X^*, such that*

a) For all $(u,x^*) \in V \times W$, $\emptyset \neq S_p(u) \subset K$ and $\emptyset \neq S_d(x^*) \subset T$.

b) The multivalued maps S_p and S_d are usc on V and W, respectively.

Proof. By assumption, the multivalued map Λ is maximal monotone and $(0,0) \in \operatorname{int}(\operatorname{dom}(\Lambda))$, from what we deduce that a compact subset K of X, a compact subset T of U^*, an open convex neighborhood V of 0 in U, and an open convex neighborhood W of 0 in X^* exist, satisfying

$$\emptyset \neq \Lambda(x^*,u) \subset K \times T, \quad \forall\, (x^*,u) \in W \times V.$$

By (4.1),

$$S_p(u) = \mathrm{proj}_{\,\mathrm{X}}[\Lambda(0,\mathrm{u})] \quad \text{and} \quad \mathrm{S_d(x^*)} = \mathrm{proj}_{\,\mathrm{U^*}}[\Lambda(\mathrm{x^*},0)],$$

from what we obtain, for all $(x^*,u) \in W \times V$,

$$\emptyset \neq S_p(u) \subset K \quad \text{and} \quad \emptyset \neq S_d(x^*) \subset T.$$

Hence a) follows.

Next, the upper semicontinuity in b) follows from the fact that Λ is usc on $W \times V$ and the projections $\mathrm{proj}_{\,\mathrm{X}}$ and $\mathrm{proj}_{\,\mathrm{U^*}}$ are continuous functions.■

To finish this chapter, we will generalize the above results when interior is replaced by relative interior.

We first note that if the subset $\Phi \subset (X \times U) \times (U^* \times X^*)$ is maximal monotone, the following conditions are equivalent:

i) $0 \in \mathrm{proj}_{\mathrm{U}}(\mathrm{ri}\,(\mathrm{dom}\,(\Lambda)))$.

ii) $0 \in \mathrm{proj}_{\mathrm{U}}(\mathrm{ri}\,(\mathrm{proj}_{\mathrm{U}\times\mathrm{X}}(\Phi)))$.

iii) $0 \in \mathrm{ri}\,\mathrm{proj}_{\mathrm{U}}(\Phi)$.

Similar equivalences are obtained when projecting onto X^*.

We start with the generalization of Propositions 4.2.4 and 4.2.5.

Proposition 4.2.8 *Assume that* $\Phi \subset (X \times U) \times (X^* \times U^*)$ *is a maximal monotone subset and* $0 \in \mathrm{ri}\,(\mathrm{proj}_{\mathrm{U}}(\Phi))$ *(respectively* $0 \in \mathrm{ri}\,(\mathrm{proj}_{\mathrm{X}^*}(\Phi))$*). Then there exists a relative open neighborhood* N *(respectively* W*) of zero such that for each* $u \in N$ *(respectively* $x^* \in W$*) the subset* F_p^u *(respectively* $F_d^{x^*}$*) is maximal monotone and the solution subset* $S_p(u)$ *(respectively* $S_d(x^*)$*) is closed and convex.*

Proof. The assumptions imply that Λ is maximal monotone and $0 \in \mathrm{proj}_{\mathrm{U}}(\mathrm{ri}\,(\mathrm{dom}\,(\Lambda)))$ (respectively $0 \in \mathrm{proj}_{\mathrm{X}^*}(\mathrm{ri}\,(\mathrm{dom}\,(\Lambda)))$), from what deduce that a relative open neighborhood N (respectively W) of 0 exists such that $N \subset \mathrm{proj}_{\mathrm{U}}(\mathrm{ri}\,(\mathrm{dom}\,(\Lambda)))$ (respectively $N \subset \mathrm{proj}_{\mathrm{X}^*}(\mathrm{ri}\,(\mathrm{dom}\,(\Lambda)))$). The result follows from Proposition 2.5.7 applied with $\Gamma = \Lambda$. ■

Remark. A similar proof shows that if $0 \in \mathrm{int}\,(\mathrm{proj}_{\mathrm{U}}(\Phi))$ (respectively $0 \in \mathrm{int}\,(\mathrm{proj}_{\mathrm{X}^*}(\Phi))$), the relative open neighborhood N (respectively W) becomes indeed an open subset. We shall use this remark in the next chapter.

Proposition 4.2.9 *Assume that* $\Phi \subset (X \times U) \times (X^* \times U^*)$ *is maximal monotone and* $0 \in \mathrm{ri}\,(\mathrm{proj}_{\mathrm{U}}(\Phi))$. *Denote by* L_p*, the linear subspace parallel to* $\mathrm{aff}\,(\mathrm{dom}\,(\Sigma_{\mathrm{p}}))$. *Then the following three statements are equivalent*

i) $(0,0) \in \mathrm{ri}\,(\mathrm{dom}\,(\Lambda))$.

ii) $0 \in \mathrm{ri}\,(\mathrm{dom}\,(\Sigma_\mathrm{p})) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{S_d}))$.

iii) There exists a compact $K \subset X$ *such that* $\emptyset \neq S_p(0) \subset K + L_p^\perp$.

Proof. a) By assumption, the multivalued map Λ is maximal monotone and $0 \in \mathrm{proj}_{\,\mathrm{U}}(\mathrm{ri}\,(\mathrm{dom}\,(\Lambda)))$, which are exactly the assumptions of Proposition 2.5.7, from what we deduce that the multivalued map Σ_p is maximal monotone and the equivalence between i) and ii) holds.

b) By definition,

$$S_p(0) = \Sigma_p(0).$$

The maximal monotonicity of Σ_p implies that the multivalued map $\Sigma_p^0 : L_p \overset{\longrightarrow}{\longrightarrow} L_p$, defined by

$$\Sigma_p(x^*) = \Sigma_p^0(x^* - a^*) + L_p^\perp,$$

for some fixed $a^* \in \mathrm{dom}\,(\Sigma_\mathrm{p})$, is maximal monotone.

The assumption $0 \in \mathrm{ri}\,(\mathrm{dom}\,(\Sigma_\mathrm{p}))$ is equivalent to $-a^* \in \mathrm{int}\,(\mathrm{dom}\,(\Sigma_\mathrm{p}^0))$, and therefore, by Theorem 2.3.1, the existence of a compact $K \subset L_p \subset X$ such that

$$\emptyset \neq \Sigma_p^0(-a^*) \subset K.$$

Taking $a^* = 0$, we deduce the equivalence between ii) and iii). ■

Similarly to Theorem 4.2.7, the dual version of proposition 4.2.9 is the following result.

Proposition 4.2.10 *Assume that* $\Phi \subset (X \times U) \times (X^* \times U^*)$ *is maximal monotone and* $0 \in \mathrm{ri}\,(\mathrm{proj}_{\,\mathrm{X}^*}(\Phi))$. *Denote by* L_d*, the linear subspace parallel to* $\mathrm{aff}\,(\mathrm{dom}\,(\Sigma_\mathrm{d}))$. *Then the following three statements are equivalent*

i) $(0,0) \in \mathrm{ri}\,(\mathrm{dom}\,(\Lambda))$.

ii) $0 \in \mathrm{ri}\,(\mathrm{dom}\,(\Sigma_\mathrm{d})) = \mathrm{ri}\,(\mathrm{dom}\,(\mathrm{S_p}))$.

iii) There exists a compact $T \subset U^*$ *such that* $\emptyset \neq S_d(0) \subset T + L_d^\perp$.

Finally, we generalize Proposition 4.2.1.

Proposition 4.2.11 *Assume that* $\Phi \subset (X \times U) \times (X^* \times U^*)$ *is maximal monotone and* $(0,0) \in \mathrm{ri}\,(\mathrm{dom}\,(\Lambda))$. *Denotes by* L_p *and* L_d *the parallel subspace to* $\mathrm{aff}\,(\mathrm{proj}_{X^*}(F_p^0)) = \mathrm{aff}\,(\mathrm{dom}\,(\Sigma_p))$ *and* $\mathrm{aff}\,(\mathrm{proj}_U(F_d^0)) = \mathrm{aff}\,(\mathrm{dom}\,(\Sigma_d))$, *respectively. Then, there exist* K, *a compact subset of* X, T, *a compact subset of* U^*, V, *a relative open convex neighborhood of* 0 *in* U, *and* W, *a relative open convex neighborhood of* 0 *in* X^* *such that*

a) For all $(u, x^*) \in V \times W$, $\emptyset \neq S_p(u) = S_p(u) + L_p^\perp \subset K + L_p^\perp$ *and* $\emptyset \neq S_d(x^*) = S_d(x^*) + L_d^\perp \subset T + L_d^\perp$.

b) The multivalued maps S_p *and* S_d *are usc on* V *and* W, *respectively.*

Proof. By assumption, Λ is maximal monotone and $(0,0) \in \mathrm{ri}\,(\mathrm{dom}(\Lambda))$.

Denote by L, the affine hull of $\mathrm{dom}\,(\Lambda)$. Then

$$L = \{\, (x^*, u) \in X^* \times U : Ax^* + Bu = 0 \,\},$$

for some matrices A and B of appropriate order. This implies that

$$L^\perp = \mathrm{img}\,([A, B]^t), \quad \mathrm{proj}_X(L^\perp) = L_p^\perp \quad \text{and} \quad \mathrm{proj}_{U^*}(L^\perp) = L_d^\perp. \tag{4.2}$$

By Proposition 2.5.4, there exist a compact subset K of X, a compact subset T of U^*, an open convex neighborhood V of 0 in U and an open convex neighborhood W of 0 in X^* exist satisfying

$$\emptyset \neq \Lambda(x^*, u) = \Lambda(x^*, u) + L^\perp \subset K \times T + L^\perp, \quad \forall\, (x^*, u) \in W \times V. \tag{4.3}$$

By (4.1),

$$S_p(u) = \mathrm{proj}_X[\Lambda(0, u)] \quad \text{and} \quad S_d(x^*) = \mathrm{proj}_{U^*}[\Lambda(x^*, 0)].$$

Relations (4.2) and (4.3) imply that for all $(u, x^*) \in V \times W$,

$$\emptyset \neq S_p(u) = S_p(u) + L_p^\perp \subset K + L_p^\perp$$

and

$$\emptyset \neq S_d(x^*) = S_d(x^*) + L_d^\perp \subset T + L_d^\perp,$$

from the what a) is verified. On the other hand, since the multivalued map Λ is usc on $W \times V$ and the projections proj_X and proj_{U^*} are continuous functions, the upper semicontinuity in b) is verified. ∎

Chapter 5

Applications of the VIP-duality scheme

In this chapter we apply our duality scheme, described in the last chapter, to some variational inequality problems associated to monotone multivalued maps.

In Section 5.1, we deal with problems involving the sum of two monotone maps. Many problems can be set under this formulation, we shall discuss some of them in Section 5.2.

In the last section of this chapter, we also apply our duality scheme to the sum of more that two monotone maps.

5.1 Sum of two monotone maps

In this section, we are given two multivalued maps $\Gamma_1, \Gamma_2 : X \overset{\longrightarrow}{\longrightarrow} X^*$ and we consider the sum $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ which is defined as follows

$$\Gamma(x) = \Gamma_1(x) + \Gamma_2(x)$$

with, by convention, $A + \emptyset = \emptyset$ for all $A \subset X^*$.

Then, we consider the variational inequality problem:

$$\text{Find } \bar{x} \in \operatorname{dom}(\Gamma) \text{ such that } 0 \in \Gamma(\bar{\mathrm{x}}). \tag{V}$$

We assume that $\mathrm{dom}\,(\Gamma) = \mathrm{dom}\,(\Gamma_1) \cap \mathrm{dom}\,(\Gamma_2)$ is not empty.
In order to set the problem in our formulation, let us define

$$F_p = \{(x, x^*) : \exists\, x_1^*,\, x_2^* \in X^* \ \text{ s.t. } \ x^* = x_1^* + x_2^*,\ x_i^* \in \Gamma_i(x)\,,\, i = 1, 2\,\}.$$

Then (V) can be formulated as

$$\text{Find } \bar{x} \in X \ \text{ such that } \ (\bar{x}, 0) \in F_p. \tag{V_p}$$

5.1.1 Introducing the perturbation

Take $U = X \times X$, $U^* = X^* \times X^*$ and define $\Phi \subset (X \times U) \times (X^* \times U^*)$ as

$$\Phi = \{\, ((x, u_1, u_2), (x^*, u_1^*, u_2^*)) : u_1^* + u_2^* = x^* \text{ and } u_i^* \in \Gamma_i(x + u_i)\,,\, i = 1, 2\,\}.$$

It is clear that

$$(x, x^*) \in F_p \iff \exists\, u^* \in U^* \text{ such that } ((x, 0), (x^*, u^*)) \in \Phi.$$

The following proposition shows that the (maximal) monotonicity of Γ_1 and Γ_2 imply the (maximal) monotonicity of Φ.

Proposition 5.1.1 *Assume that the multivalued maps Γ_1 and Γ_2 are monotone, then the subset Φ is monotone. If Γ_1 and Γ_2 are maximal monotone, then Φ is maximal monotone.*

Proof. a) Assume that Γ_1 and Γ_2 are monotone. Let $((x, u), (x^*, u^*))$ and $((y, v), (y^*, v^*))$ be two elements belonging to Φ. We shall prove that

$$A = \langle y^* - x^*, y - x\rangle + \sum_{i=1}^{2} \langle v_i^* - u_i^*, v_i - u_i\rangle \geq 0.$$

Since $\langle y^* - x^*, y - x\rangle = \langle v_1^* - u_1^*, y - x\rangle + \langle v_2^* - u_2^*, y - x\rangle$,

$$A = \sum_{i=1}^{2} \langle v_i^* - u_i^*, (y + v_i) - (x + u_i)\rangle.$$

By construction, $(x+u_i, u_i^*)$ and $(y+v_i, v_i^*)$ belong to the graph of Γ_i, $i = 1, 2$, therefore $A \geq 0$.

b) Assume now that Γ_1 and Γ_2 are maximal monotone. Let us consider the multivalued map $\Sigma : U^* \times X \overset{\longrightarrow}{\longrightarrow} U \times X^*$ defined by

$$\Sigma(u^*, x) = \{(u, x^*) \,:\, ((x, u), (x^*, u^*) \in \Phi\}.$$

Then,

$$\Sigma(u_1^*, u_2^*, x) = [\Gamma_1^-(u_1^*) - x] \times [\Gamma_2^-(u_2^*) - x] \times \{u_1^* + u_2^*\},$$

and

$$\mathrm{dom}\,(\Sigma) = \mathrm{dom}\,(\Gamma_1^-) \times \mathrm{dom}\,(\Gamma_2^-) \times \mathrm{X}.$$

Since the graph of Σ corresponds to Φ after permutation of variables, Φ is maximal monotone if and only if Σ is maximal monotone. For this, by Proposition 2.1.1, it is enough to prove that

$$\widetilde{\Sigma}(u_1^*, u_2^*, x) = [\Gamma_1^-(u_1^*) - x] \times [\Gamma_2^-(u_2^*) - x] \times \{u_1^* + u_2^*\}.$$

Assume that $(u_1, u_2, x^*) \in \widetilde{\Sigma}(u_1^*, u_2^*, x)$. Then, for all $(v_1, v_1^*) \in \mathrm{graph}\,(\Gamma_1)$, $(v_2, v_2^*) \in \mathrm{graph}\,(\Gamma_2)$ and $y \in X$, one has $B \geq 0$, where

$$B = \langle u_1^* - v_1^*, u_1 - v_1 + y\rangle + \langle u_2^* - v_2^*, u_2 - v_2 + y\rangle + \langle x^* - v_1^* - v_2^*, x - y\rangle.$$

Easy computations give

$$B = \langle u_1^* - v_1^*, u_1 - v_1\rangle + \langle u_2^* - v_2^*, u_2 - v_2\rangle + \langle x^* - v_1^* - v_2^*, x\rangle - \langle x^* - u_1^* - u_2^*, y\rangle.$$

Let $(v_1, v_1^*) \in \mathrm{graph}\,(\Gamma_1)$ and $(v_2, v_2^*) \in \mathrm{graph}\,(\Gamma_2)$ be fixed. Since $B \geq 0$ for all $y \in X$, one obtains

$$x^* = u_1^* + u_2^*.$$

Using this identity for x^* in the last expression of B, we see that

$$\sum_{i=1,2} \langle u_i^* - v_i^*, (u_i + x) - v_i\rangle \geq 0,\ \forall (v_i, v_i^*) \in \mathrm{graph}\,(\Gamma_\mathrm{i}),\ \mathrm{i} = 1, 2.$$

Assume for contradiction that $u_1 + x \notin \Gamma_1^-(u_1^*)$, then there exists $(v_1, v_1^*) \in \mathrm{graph}\,(\Gamma_1)$ so that $\langle u_1^* - v_1^*, (u_1 + x) - v_1\rangle < 0$ and consequently $\langle u_2^* - v_2^*, (u_2 + x) - v_2\rangle > 0$ whenever $(v_2, v_2^*) \in \mathrm{graph}\,(\Gamma_2)$. One deduces that $u_2 + x \in \Gamma_2^-(u_2^*)$. Next, take $(v_2, v_2^*) = (u_2 + x, u_2^*)$, then one has $\langle u_2^* - v_2^*, (u_2 + x) - v_2\rangle = 0$ in contradiction with the inequality above.

One obtains that Σ is maximal monotone. ∎

5.1.2 The dual problem

According to the duality scheme, the subset F_d is defined by the relation

$$(u, u^*) \in F_d \iff \exists\, x \in X \text{ such that } ((x, u), (0, u^*)) \in \Phi.$$

In the present case,

$$F_d = \{\, (u, u^*) : \exists\, x \in X \text{ such that } u_1^* + u_2^* = 0 \text{ and } u_i^* \in \Gamma_i(x + u_i),\ i = 1, 2\,\}.$$

The dual VIP problem can be equivalently formulated as :

$$\text{Find } \bar{u}^* = (\bar{u}_1^*, \bar{u}_2^*) \in U^* \text{ such that } \begin{cases} \exists\, x \in X \text{ with } \bar{u}_1^* + \bar{u}_2^* = 0 \\ \text{and } \bar{u}_i^* \in \Gamma_i(x),\ i = 1, 2, \end{cases}, \qquad (V_d)$$

$$\text{Find } \ \bar{u}^* \in X^* \ \text{ such that } \ \Gamma_1^-(\bar{u}^*) \cap \Gamma_2^-(-\bar{u}^*) \neq \emptyset \qquad (V_d)$$

or again

$$\text{Find } \ \bar{u}^* \in X^* \ \text{ such that } \ 0 \in \Gamma_1^-(\bar{u}^*) - \Gamma_2^-(-\bar{u}^*). \qquad (V_d)$$

Let us introduce the maps Σ_1 and Σ_2 defined by

$$\Sigma_1(u^*) = \Gamma_1^-(u^*), \ \ \Sigma_2(u^*) = -\Gamma_2^-(-u^*) \ \ \forall\, u^*,$$

then problem (V_d) is also equivalent to

$$\text{Find } \ \bar{u}^* \in X^* \ \text{ such that } \ 0 \in \Sigma_1(\bar{u}^*) + \Sigma_2(\bar{u}^*), \qquad (V_d)$$

which is exactly of the same form that the primal problem (V_p). This imply that the duality is thoroughly symmetric.

It is easily seen that Σ_1 and Σ_2 are (maximal) monotone when Γ_1 and Γ_2 are so.

5.1.3 The perturbed problems and the lagrangian problem

Given the perturbation variable $u = (u_1, u_2) \in X \times X$, the subset F_p^u is

$$F_p^u = \{\, (x, x^*) : x^* \in \Gamma_1(x + u_1) + \Gamma_2(x + u_2)\,\}$$

and, therefore, the primal perturbed problem (V_p^u) becomes

$$\text{Find } \bar{x} \in X \text{ such that } 0 \in \Gamma_1(\bar{x}+u_1) + \Gamma_2(\bar{x}+u_2). \qquad (V_p^u)$$

Next, given the perturbation variable $x^* \in X^*$, the subset $F_d^{x^*}$ is

$$F_d^{x^*} = \left\{ (u,u^*) : \begin{array}{c} \exists\, x \in X \text{ with } u_1^* + u_2^* = x^* \\ \text{and } u_i^* \in \Gamma_i(x+u_i),\ i=1,2, \end{array} \right\}$$

and therefore the dual perturbed problem $(V_d^{x^*})$ becomes

$$\text{Find } \bar{u}^* \in U^* \text{ such that } \left\{ \begin{array}{c} \exists\, x \in X \text{ with } \bar{u}_1^* + \bar{u}_2^* = x^* \\ \text{and } \bar{u}_i^* \in \Gamma_i(x),\ i=1,2, \end{array} \right. \qquad (V_d^{x^*})$$

which can be equivalently formulated as

$$\text{Find } \bar{u}^* \in X^* \text{ such that } \Gamma_1^-(\bar{u}^*) \cap \Gamma_2^-(x^* - \bar{u}^*) \neq \emptyset \qquad (V_d^{x^*})$$

$$\text{Find } \bar{u}^* \in X^* \text{ such that } 0 \in \Gamma_1^-(\bar{u}^*) - \Gamma_2^-(x^* - \bar{u}^*). \qquad (V_d^{x^*})$$

Given x^*, the set of solutions of problem $(V_d^{x^*})$ is not empty whenever $x^* \in \operatorname{img}(\Gamma_1 + \Gamma_2) = \operatorname{img}(\Gamma)$ which, for example, is verified (see propositions 2.6.1 and 2.6.4) under any of the following conditions:

i) Γ is a strongly maximal monotone map,

ii) Γ is maximal monotone and $\operatorname{dom}(\Gamma)$ is bounded.

Finally, the multivalued map Λ becomes

$$\Lambda(x^*,u) = \{\, (x,u^*) : u_1^* \in \Gamma_1(x+u_1),\ u_2^* \in \Gamma_2(x+u_2) \text{ and } u_1^* + u_2^* = x^* \,\}$$

and, therefore, problem (V_L) is formulated as

$$\text{Find } (\bar{x}, \bar{u}^*) \in X \times U^* \text{ such that } \left\{ \begin{array}{c} \bar{u}_1^* + \bar{u}_2^* = 0 \text{ and} \\ \bar{u}_i^* \in \Gamma_i(\bar{x}),\ i=1,2. \end{array} \right. \qquad (V_L)$$

The solutions sets of problems (V_p^u), $(V_d^{x^*})$ and (V_L) are, respectively,

$$S_p(u) = \{ x \in X : 0 \in \Gamma_1(x+u_1) + \Gamma_2(x+u_2) \},$$

$$S_d(x^*) = \{ (v^*, x^* - v^*) \in U^* : 0 \in \Gamma_1^-(v^*) - \Gamma_2^-(x^* - v^*) \}$$

and

$$S_l = \{ (x, (v^*, -v^*)) \in X \times U^* : v^* \in \Gamma_1(x) \text{ and } -v^* \in \Gamma_2(x) \}.$$

5.1.4 Sensitivity and stability analysis

Let us consider the following conditions:

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are maximal monotone} \tag{Mm}$$

$$\operatorname{int}(\operatorname{dom}(\Gamma_1)) \cap \operatorname{int}(\operatorname{dom}(\Gamma_2)) \neq \emptyset \tag{Dsp}$$

and the weaker form

$$\operatorname{ri}(\operatorname{dom}(\Gamma_1)) \cap \operatorname{ri}(\operatorname{dom}(\Gamma_2)) \neq \emptyset. \tag{Dwp}$$

We have already seen that condition (Mm) implies that Φ is maximal monotone, by Proposition 2.5.3, hence S_l is closed and convex. We turn our interest on the solutions sets of the (perturbed) primal problems. For that we prove the following proposition.

Proposition 5.1.2 *Assume that conditions* (Mm) *and* (Dsp) *hold. Then* $0 \in \operatorname{proj}_{\mathrm{U}}(\operatorname{int}(\operatorname{proj}_{\mathrm{X}\times\mathrm{U}}(\Phi)))$.

Proof. Let $\tilde{x} \in \operatorname{int}(\operatorname{dom}(\Gamma_1))) \cap \operatorname{int}(\operatorname{dom}(\Gamma_2)))$, then there exist N neighborhood of $\tilde{x}$ and V neighborhood of 0 such that $x \in N$ and $u_1, u_2 \in V$ imply $x + u_1, x + u_2 \in \operatorname{int}(\operatorname{dom}(\Gamma_1))) \cap \operatorname{int}(\operatorname{dom}(\Gamma_2)))$. Hence $N \times V \times V$ is contained in $\operatorname{int}(\operatorname{proj}_{\mathrm{X}\times\mathrm{U}}(\Phi)))$. The result follows. ■

Applying Proposition 4.2.4, one obtains that the subset F_p^u is maximal monotone and the solution set $S_p(u)$ is closed and convex (eventually, empty or unbounded) for u in an **open neighborhood** of $0 \in U$. It is worth noticing that F_p^0 is nothing else but the graph of the map Γ which is consequently maximal monotone. We recover in this way the result already quoted in Proposition 2.3.4. This way appears more elegant.

A similar result is obtained when condition (Dsp) is replaced by (Dwp). No proof is required.

Proposition 5.1.3 *Assume that conditions* (Mm) *and* (Dwp) *hold. Then* $0 \in \operatorname{proj}_{\mathrm{U}}(\operatorname{ri}(\operatorname{proj}_{\mathrm{X}\times\mathrm{U}}(\Phi)))$.

Next, we deal with the (perturbed) dual problems. We have seen that the dual has the same form of the primal. Conditions (Dsp) and (Dwp) are transformed in

$$\operatorname{dom}(\Gamma_1^-) \cap -\operatorname{int}(\operatorname{dom}(\Gamma_2^-)) \neq \emptyset \tag{Dsd}$$

and the weaker form

$$\operatorname{dom}(\Gamma_1^-) \cap -\operatorname{ri}(\operatorname{dom}(\Gamma_2^-)) \neq \emptyset, \tag{Dwd}$$

then we have the following result.

Proposition 5.1.4 *Assume that conditions* (Mm) *and* (Dsd) *hold. Then* $0 \in \operatorname{int}(\operatorname{proj}_{\mathrm{X}^*}(\Phi))$.

Proof. Let $v^* \in \operatorname{dom}(\Gamma_1^-) \cap -\operatorname{int}(\operatorname{dom}(\Gamma_2^-))$, then there exists an open convex neighborhood W of 0 in X^* such that $x^* \in W$ implies $x^* - v^* \in \operatorname{dom}(\Gamma_2^-)$, from what $W \subset \operatorname{proj}_{\mathrm{X}^*}(\Phi)$. The result follows. ■

Note, that from Propositions 5.1.4 and 4.2.8, that the subset $F_d^{x^*}$ is maximal monotone and the solution set $S_d(x^*)$ is closed and convex for x^* in an **open neighborhood** of $0 \in X^*$.

A similar result is obtaining when interior is replaced by relative interior, but, in this case, in view of Proposition 4.2.8, the subsets $F_d^{x^*}$ are maximal monotone and the subsets $S_d(x^*)$ are closed and convex, for x^* belonging only in a **relative open neighborhood** of $0 \in X^*$.

Proposition 5.1.5 *Assume that conditions* (Mm) *and* (Dwd) *hold. Then* $0 \in \operatorname{ri}(\operatorname{proj}_{\mathrm{X}^*}(\Phi))$.

Remark.
i) In general, by definition of Φ, $\operatorname{int}(\operatorname{proj}_{\mathrm{X}^* \times \mathrm{U}^*}(\Phi)) = \emptyset$.
ii) The condition (Dsd) does not imply, in general, that $\operatorname{int}(\operatorname{proj}_{\mathrm{X}^* \times \mathrm{U}}(\Phi)) \neq \emptyset$. For that, consider, for example, that Γ_1^- and Γ_2^- are constant (say $\Gamma_1^- \equiv \bar{u}_1$ and $\Gamma_2^- \equiv \bar{u}_2$). In this case,

$$\operatorname{dom}(\Gamma_1^-) = -\operatorname{int}(\operatorname{dom}(\Gamma_2^-)) = \mathrm{X}^*$$

and

$$\mathrm{proj}_{\,\mathrm{X^*\times U}}(\Phi) = \{\, (\bar{\mathrm{u}}_1 - \mathrm{x}, \bar{\mathrm{u}}_2 - \mathrm{x}, \mathrm{u}_1^* + \mathrm{u}_2^*) : \ \mathrm{x} \in \mathrm{X}, \ \mathrm{u}_\mathrm{i}^* \in \mathrm{dom}\,(\Gamma_\mathrm{i}^-), \ \mathrm{i} = 1, 2 \,\},$$

which, obviously, has empty interior.

Next, we will specialize the conditions in Theorem 4.2.1 in order to establish the stability property of primal and dual perturbed problems.

In the following proposition, which is rather immediate, one assumes that the multivalued maps $\Gamma = \Gamma_1 + \Gamma_2$ and $\Sigma = \Sigma_1 + \Sigma_2$ are maximal monotone, where Σ_1 and Σ_2 are defined by $\Sigma_1 \equiv \Gamma_1^-$ and $\Sigma_2(v^*) = -\Gamma_2^-(-v^*), \quad \forall\, v^* \in X^*$.

Proposition 5.1.6 *In the following six statements, we have the following equivalences: $a \Leftrightarrow b \Leftrightarrow c$ and $a' \Leftrightarrow b' \Leftrightarrow c'$.*

a) *The solution set $S_p(0)$ is bounded and not empty.*

b) $0 \in \mathrm{int}\,(\mathrm{img}\,(\Gamma))$.

c) $\exists\, W \in \mathcal{N}(0)$ *such that* $\bigcup_{v^* \in X^*} [\Gamma_1^-(v^*) \cap \Gamma_2^-(x^* - v^*)] \neq \emptyset, \quad \forall\, x^* \in W$.

a') *The solution set $S_d(0)$ is bounded and not empty.*

b') $0 \in \mathrm{int}\,(\mathrm{img}\,(\Sigma))$.

c') $\exists\, V \in \mathcal{N}(0)$ *such that* $\bigcup_{z \in X} [\Gamma_1(z) \cap -\Gamma_2(z - x)] \neq \emptyset, \quad \forall\, x \in V$.

Applying Proposition 4.2.1 and Theorem 4.2.1, one obtains that, under conditions b) (or c) and b') (or c'), there exist an open neighborhood $\hat{V} \times \hat{W} \subset U \times X^*$ of $(0, 0)$ and a compact $K \times L \subset X \times U^*$ such that:

- $\emptyset \neq S_p(u) \subset K$ and $\emptyset \neq S_d(x^*) \subset L, \quad \forall\, (u, x^*) \in \hat{V} \times \hat{W}$
- The multivalued maps S_p and S_d are usc on $\hat{V}$ and $\hat{W}$, respectively.

5.2 The constrained VIP

In this section we shall consider some particular examples that can be formulated as sum of two maximal monotone multivalued maps. Indeed, every variational inequality problem constrained to some closed convex subset, and, associated to a maximal monotone map, can be formulated in this framework.

We start with the general version.

5.2.1 The general case

In this part we consider the following variational inequality problem VIP.

$$\text{Find } \bar{x} \in C \ \text{ such that } \ \exists\, \bar{x}^* \in \Gamma_1(\bar{x}) \ \text{ with } \ \langle \bar{x}^*, x - \bar{x} \rangle \geq 0,\ \forall\, x \in C, \quad (V)$$

where C is a closed convex subset of X, and Γ_1 a given maximal monotone multivalued map, such that, $\mathrm{dom}\,(\Gamma_1) \cap \mathrm{C} \neq \emptyset$.

Consider $\Gamma_2 = N_C$, the normal cone map associated to C, which, by definition, is maximal monotone, and $\Gamma = \Gamma_1 + \Gamma_2$.

Then (V) can be formulated as:

$$\text{Find } \bar{x} \in C \ \text{ such that } \ 0 \in \Gamma(\bar{x}).$$

Thus, we are faced with the problem considered in the last section.

It is easily seen that $\Gamma_2^-(0) = N_C^-(0) = C$ and for $x^* \neq 0$, $N_C^-(x^*)$ is (eventually empty or unbounded) a closed convex subset of $\mathrm{bd}\,(\mathrm{C})$. Moreover

$$\bigcup_{x^* \neq 0} N_C^-(x^*) = \mathrm{bd}\,(\mathrm{C}).$$

In this setting, the set F_p becomes

$$F_p = \{(x, x^*) : \exists\, z^* \in \Gamma_1(x) \ \text{ with } \ x^* - z^* \in N_C(x)\,\}.$$

Then (V) can also be formulated as

$$\text{Find } \bar{x} \in X \ \text{ such that } \ (\bar{x}, 0) \in F_p. \qquad (V_p)$$

Introducing the perturbation.

Define $\Phi \subset (X \times U) \times (X^* \times U^*)$ as:

$$\Phi = \{((x, u_1, u_2), (x^*, u_1^*, u_2^*)) : u_1^* \in \Gamma_1(x+u_1), \;\; u_2^* = x^* - u_1^* \in N_C(x+u_2)\}.$$

Since, by assumption, Γ_1 and Γ_2 are maximal monotone, Proposition 5.1.1 implies that Φ is maximal monotone and, therefore, the monotonicity of F_p.

The dual problem.

According to our duality scheme, the subset F_d, in this case becomes

$$F_d = \{\, (u, u^*) : \exists\, x \in X \text{ s.t. } u_1^* \in \Gamma_1(x + u_1) \text{ and } u_2^* = -u_1^* \in N_C(x + u_2)\,\},$$

which, by the monotonicity of Φ, is also monotone.

The dual VIP problem can be equivalently formulated as:

$$\text{Find } \bar{u}^* = (\bar{u}_1^*, \bar{u}_2^*) \in U^* \text{ such that } \begin{cases} \exists\, x \in X \text{ with } \bar{u}_1^* \in \Gamma_1(x) \\ \text{and } \bar{u}_2^* = -u_1^* \in N_C(x), \end{cases} \qquad (V_d)$$

$$\text{Find } \bar{u}^* \in X^* \text{ such that } \Gamma_1^-(\bar{u}^*) \cap N_C^-(-\bar{u}^*) \neq \emptyset \qquad (V_d)$$

or again

$$\text{Find } \bar{u}^* \in X^* \text{ such that } 0 \in \Gamma_1^-(\bar{u}^*) - N_C^-(-\bar{u}^*). \qquad (V_d)$$

Since $N_C^-(0) = C$, the second one of these formulations of (V_d) implies in particular that

$$0 \in \Gamma_1(C) \iff 0 \text{ is a solution of problem } (V_d).$$

The third one of these formulations is a dual framework studied by Mosco, in [27].

The perturbed problems and the lagrangian problem.

Given the perturbation variable $u \in X \times X$, the subset F_p^u is

$$F_p^u = \{\, (x, x^*) : x^* \in \Gamma_1(x + u_1) + N_C(x + u_2)\,\},$$

and, therefore, the primal perturbed problem (V_p^u) becomes

$$\text{Find } \bar{x} \in X \text{ such that } 0 \in \Gamma_1(\bar{x} + u_1) + N_C(\bar{x} + u_2). \tag{V_p^u}$$

Next, given the perturbation variable $x^* \in X^*$, the subset $F_d^{x^*}$ is

$$F_d^{x^*} = \left\{ (u, u^*) : \begin{array}{c} \exists\, x \in X \text{ with } u_1^* + u_2^* = x^*, \\ u_1^* \in \Gamma_1(x + u_1) \text{ and } u_2^* \in N_C(x + u_2) \end{array} \right\},$$

and, therefore, the dual perturbed problem $(V_d^{x^*})$ becomes

$$\text{Find } \bar{u}^* = (\bar{u}_1^*, \bar{u}_2^*) \in U^* \text{ such that } \left\{ \begin{array}{c} \exists\, x \in X \text{ with } \bar{u}_1^* \in \Gamma_1(x) \\ \text{and } \bar{u}_2^* = x^* - u_1^* \in N_C(x), \end{array} \right. \tag{$V_d^{x^*}$}$$

which can be equivalently formulated as

$$\text{Find } \bar{u}^* \in X^* \text{ such that } \Gamma_1^-(\bar{u}^*) \cap N_C^-(x^* - \bar{u}^*) \neq \emptyset \tag{$V_d^{x^*}$}$$

$$\text{Find } \bar{u}^* \in X^* \text{ such that } 0 \in \Gamma_1^-(\bar{u}^*) - N_C^-(x^* - \bar{u}^*). \tag{$V_d^{x^*}$}$$

Again, the fact that $N_C^-(0) = C$, the second one of these formulations of $(V_d^{x^*})$ implies in particular that

$$w^* \in \Gamma_1(C) \iff w^* \text{ is a solution of problem } (V_d^{w^*}).$$

Finally, the multivalued map Λ becomes

$$\Lambda(x^*, u) = \{\, (x, u^*) : u_1^* \in \Gamma_1(x + u_1),\, u_2^* \in N_C(x + u_2) \text{ and } u_1^* + u_2^* = x^* \,\},$$

and therefore, the problem (V_L) is formulated as:

$$\text{Find } (\bar{x}, \bar{u}^*) \in X \times U^* \text{ such that } \left\{ \begin{array}{c} \bar{u}_1^* \in \Gamma_1(\bar{x}) \text{ and} \\ \bar{u}_2^* = -\bar{u}_1^* \in N_C(\bar{x}). \end{array} \right. \tag{V_L}$$

The solutions sets of problems (V_p^u), $(V_d^{x^*})$ and (V_L) are:

$$S_p(u) = \{ x \in X : 0 \in \Gamma_1(x + u_1) + N_C(x + u_2) \},$$

$$S_d(x^*) = \{(v^*, x^* - v^*) \in U^* : \, 0 \in \Gamma_1^-(v^*) - N_C^-(x^* - v^*)\,\}$$

and

$$S_l = \{(x, (v^*, -v^*)) \in X \times U^* : \, v^* \in \Gamma_1(x) \text{ and } -v^* \in N_C(x)\,\}.$$

Since Φ is maximal monotone, the subset S_l is (eventually, empty or unbounded) closed and convex.

Sensitivity and stability analysis.

In the present case, conditions (Dsp), (Dwp), (Dsd) and (Dwd) become:

$$\mathrm{int}\,(\mathrm{dom}\,(\Gamma_1)) \cap \mathrm{int}\,(\mathrm{C}) \neq \emptyset, \tag{Dsp}$$

$$\mathrm{ri}\,(\mathrm{dom}\,(\Gamma_1)) \cap \mathrm{ri}\,(\mathrm{C}) \neq \emptyset, \tag{Dwp}$$

$$\mathrm{dom}\,(\Gamma_1^-) \cap -\mathrm{int}\,(\mathrm{img}\,(\mathrm{N_C})) \neq \emptyset, \tag{Dsd}$$

$$\mathrm{dom}\,(\Gamma_1^-) \cap -\mathrm{ri}\,(\mathrm{img}\,(\mathrm{N_C})) \neq \emptyset. \tag{Dwd}$$

For example, if the subset C is bounded, the condition (Dsd) is clearly verified, in this case, $\mathrm{img}\,(\mathrm{N_C}) = \mathrm{X}^*$; see Proposition 2.6.4.

5.2.2 Complementarity problems

In this example assume that C is a closed convex cone. Then,

$$N_C(0) = C^* \text{ and for } x \neq 0,\; N_C(x) \text{ is contained in } \mathrm{bd}\,(\mathrm{C}^*),$$

where C^* is the polar cone of C defined by

$$C^* := \{\, x^* \in X^* : \langle x^*, x\rangle \leq 0,\, \forall\, x \in C\,\}.$$

By definition,

$$N_C^-(x^*) = N_{C^*}(x^*) \quad \forall\, x^* \in X^*.$$

The cone $-C^*$ is called the positive cone of C and is denoted by C^+. In this setting, problem (V) from the previous section becomes

$$\text{Find } \bar{x} \in C \text{ such that } \exists\, \bar{x}^* \in \Gamma_1(\bar{x}) \cap C^+ \text{ with } \langle \bar{x}^*, \bar{x}\rangle = 0. \tag{CP}$$

(CP) is called a general *complementarity problem.* As usual, we assume that Γ_1 is maximal monotone and $\mathrm{dom}\,(\Gamma_1) \cap \mathrm{C} \neq \emptyset$.

In the present case, the subset F_p becomes

$$F_p = \{(x, x^*) : \exists\, z^* \in \Gamma_1(x) \;\text{ with }\; x^* - z^* \in N_C(x)\,\}.$$

Then (V) can also be formulated as

$$\text{Find } \bar{x} \in X \;\text{ such that }\; (\bar{x}, 0) \in F_p. \tag{V_p}$$

Introducing the perturbation.

In this case the perturbed set Φ becomes:

$$\Phi = \{((x, u_1, u_2), (x^*, u_1^*, u_2^*)) : u_1^* \in \Gamma_1(x + u_1),\;\; u_2^* = x^* - u_1^* \in N_C(x + u_2)\}.$$

Since Γ_1 and $\Gamma_2 = N_C$ are maximal monotone, Proposition 5.1.1 implies that Φ is maximal monotone and, therefore ensures the monotonicity of F_p.

The dual problem.

According to the duality scheme, the subset F_d becomes

$$F_d = \{\,(u, u^*) : \exists\, x \in X \text{ s.t. } u_1^* \in \Gamma_1(x + u_1) \text{ and } u_2^* = -u_1^* \in N_C(x + u_2)\,\},$$

which, by the monotonicity of Φ, is also monotone.

The dual VIP problem can be equivalently formulated as:

$$\text{Find } \bar{u}^* = (\bar{u}_1^*, \bar{u}_2^*) \in U^* \text{ such that } \begin{cases} \exists\, x \in X \text{ with } \bar{u}_1^* \in \Gamma_1(x) \\ \text{and } \bar{u}_2^* = -u_1^* \in N_C(x), \end{cases} \tag{V_d}$$

$$\text{Find } \bar{u}^* \in X^* \;\text{ such that }\; \Gamma_1^-(\bar{u}^*) \cap N_{C^*}(-\bar{u}^*) \neq \emptyset \tag{V_d}$$

$$\text{Find } \bar{u}^* \in X^* \;\text{ such that }\; 0 \in \Gamma_1^-(\bar{u}^*) + N_{C^+}(\bar{u}^*) \tag{V_d}$$

or again

$$\text{Find } \bar{u}^* \in C^+ \;\text{ such that }\; \exists\, \bar{u} \in \Gamma_1^-(\bar{u}^*) \cap C \;\text{ with }\; \langle \bar{u}^*, \bar{u} \rangle = 0. \tag{V_d}$$

Since $N_{C^*}(0) = C$, the second one of these formulations of (V_d) implies in particular that

$$0 \in \Gamma_1(C) \iff 0 \;\text{ is a solution of problem } (V_d).$$

The perturbed problems and the lagrangian problem.

Given the perturbation variable $u \in X \times X$, the subset F_p^u is

$$F_p^u = \{ (x, x^*) : x^* \in \Gamma_1(x + u_1) + N_C(x + u_2) \},$$

and therefore, the primal perturbed problem (V_p^u) becomes

$$\text{Find } \bar{x} \in X \text{ such that } 0 \in \Gamma_1(\bar{x} + u_1) + N_C(\bar{x} + u_2). \qquad (V_p^u)$$

Next, given the perturbation variable $x^* \in X^*$, the subset $F_d^{x^*}$ is

$$F_d^{x^*} = \left\{ (u, u^*) : \begin{array}{c} \exists\, x \in X \text{ with } u_1^* + u_2^* = x^*, \\ u_1^* \in \Gamma_1(x + u_1) \text{ and } u_2^* \in N_C(x + u_2) \end{array} \right\}$$

or, equivalently,

$$F_d^{x^*} = \{ (u, u^*) : [\Gamma_1^-(u_1^*) - u_1] \cap [N_{C^*}(u_2^*) - u_2] \neq \emptyset \text{ and } u_1^* + u_2^* = x^* \},$$

and therefore, the dual perturbed problem $(V_d^{x^*})$ becomes

$$\text{Find } \bar{u}^* = (\bar{u}_1^*, \bar{u}_2^*) \in U^* \text{ such that } \left\{ \begin{array}{c} \exists\, x \in X \text{ with } \bar{u}_1^* \in \Gamma_1(x) \\ \text{and } \bar{u}_2^* = x^* - u_1^* \in N_C(x), \end{array} \right. \qquad (V_d^{x^*})$$

which can be equivalently formulated as

$$\text{Find } \; \bar{u}^* \in X^* \; \text{ such that } \; \Gamma_1^-(\bar{u}^*) \cap N_{C^*}(x^* - \bar{u}^*) \neq \emptyset \qquad (V_d^{x^*})$$

$$\text{Find } \; \bar{u}^* \in X^* \; \text{ such that } \; 0 \in \Gamma_1^-(\bar{u}^*) + N_{C^+}(\bar{u}^* - x^*) \qquad (V_d^{x^*})$$

$$\text{Find } \; \bar{u}^* \in C^+ \; \text{ such that } \; \exists\, \bar{u} \in \Gamma_1^-(\bar{u}_{x^*} + x^*) \cap C \; \text{ with } \; \langle \bar{u}_{x^*}, \bar{u} \rangle = 0. \; (V_d^{x^*})$$

Again, by the fact that $N_{C^*}(0) = C$, the second one of these formulations of $(V_d^{x^*})$ implies in particular that

$$w^* \in \Gamma_1(C) \iff w^* \;\text{ is a solution of problem } (V_d^{w^*}).$$

Finally, the multivalued map Λ becomes

$$\Lambda(x^*, u) = \{ (x, u^*) : u_1^* \in \Gamma_1(x + u_1),\, u_2^* \in N_C(x + u_2) \text{ and } u_1^* + u_2^* = x^* \},$$

and, therefore, problem (V_L) is formulated as:

$$\text{Find } (\bar{x}, \bar{u}^*) \in X \times U^* \text{ such that } \quad \begin{cases} \bar{u}_1^* \in \Gamma_1(\bar{x}) \text{ and} \\ \bar{u}_2^* = -\bar{u}_1^* \in N_C(\bar{x}). \end{cases} \tag{V_L}$$

The solutions sets of problems (V_p^u), $(V_d^{x^*})$ and (V_L), become:

$$S_p(u) = \{x \in X : 0 \in \Gamma_1(x + u_1) + N_C(x + u_2) \},$$

$$S_d(x^*) = \{(v^*, x^* - v^*) \in U^* : 0 \in \Gamma_1^-(v^*) + N_{C^+}(v^* - x^*) \}$$

and

$$S_l = \{(x, (v^*, -v^*)) \in X \times U^* : v^* \in \Gamma_1(x) \text{ and } -v^* \in N_C(x) \}.$$

Sensitivity and stability analysis.

In the present case, conditions (Dsp), (Dwp), (Dsd) and (Dwd) are:

$$\operatorname{int}(\operatorname{dom}(\Gamma_1)) \cap \operatorname{int}(\mathrm{C}) \neq \emptyset, \tag{Dsp}$$

$$\operatorname{ri}(\operatorname{dom}(\Gamma_1)) \cap \operatorname{ri}(\mathrm{C}) \neq \emptyset, \tag{Dwp}$$

$$\operatorname{dom}(\Gamma_1^-) \cap \operatorname{int}(\mathrm{C}^+) \neq \emptyset, \tag{Dsd}$$

$$\operatorname{dom}(\Gamma_1^-) \cap \operatorname{ri}(\mathrm{C}^+) \neq \emptyset. \tag{Dwd}$$

5.2.3 Nonlinear complementarity problems

Let us assume in this case that $C = \mathbb{R}_+^n$, the nonnegative orthant in $\mathbb{R}^n$. Then $C^* = \mathbb{R}_-^n = -\mathbb{R}_+^n$. Also, we assume that Γ_1 is maximal monotone and $\operatorname{dom}(\Gamma_1) \cap \mathbb{R}_+^n \neq \emptyset$.

In this setting problem (V) is

$$\text{Find } \bar{x} \in C \text{ such that } \exists\, \bar{x}^* \in \Gamma_1(\bar{x}) \cap \mathbb{R}_+^n \text{ with } \langle \bar{x}^*, \bar{x} \rangle = 0. \tag{CP}$$

The set F_p is

$$F_p = \{(x, x^*) : \exists\, z^* \in \Gamma_1(x) \ \text{ with } \ x^* - z^* \in N_{\mathbb{R}^n_+}(x)\,\}.$$

Then (V) can also be formulated as

$$\text{Find } \bar{x} \in X \ \text{ such that } \ (\bar{x}, 0) \in F_p. \tag{V_p}$$

<u>Introducing the perturbation</u>.

In this case the perturbed subset Φ becomes

$$\Phi = \{((x, u_1, u_2), (x^*, u_1^*, u_2^*)) : u_1^* \in \Gamma_1(x+u_1), \ \ u_2^* = x^* - u_1^* \in N_{\mathbb{R}^n_+}(x+u_2)\}.$$

Again, in view of Proposition 5.1.1, the maximality of Γ_1 and $\Gamma_2 = N_{\mathbb{R}^n_+}$ implies the maximality of Φ and therefore the monotonicity of F_p.

<u>The dual problem</u>.

According to the duality scheme, the subset F_d becomes

$$F_d = \{\,(u, u^*) : \exists\, x \in X \text{ s.t. } u_1^* \in \Gamma_1(x+u_1) \text{ and } u_2^* = -u_1^* \in N_{\mathbb{R}^n_+}(x+u_2)\,\},$$

which, by the monotonicity of Φ, is also monotone.

The dual VIP problem can be equivalently formulated as:

$$\text{Find } \bar{u}^* = (\bar{u}_1^*, \bar{u}_2^*) \in U^* \text{ such that } \begin{cases} \exists\, x \in X \text{ with } \bar{u}_1^* \in \Gamma_1(x) \\ \text{and } \bar{u}_2^* = -u_1^* \in N_{\mathbb{R}^n_+}(x), \end{cases} \tag{V_d}$$

$$\text{Find } \ \bar{u}^* \in X^* \ \text{ such that } \ \Gamma_1^-(\bar{u}^*) \cap -N_{\mathbb{R}^n_+}(-\bar{u}^*) \neq \emptyset \tag{V_d}$$

$$\text{Find } \ \bar{u}^* \in X^* \ \text{ such that } \ 0 \in \Gamma_1^-(\bar{u}^*) + N_{\mathbb{R}^n_+}(\bar{u}^*), \tag{V_d}$$

or, again,

$$\text{Find } \bar{u}^* \in \mathbb{R}^n_+ \ \text{ such that } \ \exists\, \bar{u} \in \Gamma_1^-(\bar{u}^*) \cap \mathbb{R}^n_+ \ \text{ with } \ \langle \bar{u}^*, \bar{u} \rangle = 0. \tag{V_d}$$

The second one of these formulations of (V_d) implies in particular that

$$0 \in \Gamma_1(\mathbb{R}^n_+) \iff 0 \ \text{ is a solution of problem } (V_d).$$

The perturbed problems and the lagrangian problem.

Given the perturbation variable $u \in X \times X$, the subset F_p^u is

$$F_p^u = \{\, (x, x^*) : x^* \in \Gamma_1(x + u_1) + N_{\mathbb{R}^n_+}(x + u_2) \,\},$$

and therefore, the primal perturbed problem (V_p^u) becomes

$$\text{Find } \bar{x} \in X \text{ such that } 0 \in \Gamma_1(\bar{x} + u_1) + N_{\mathbb{R}^n_+}(\bar{x} + u_2). \qquad (V_p^u)$$

Next, given the perturbation variable $x^* \in X^*$, the subset $F_d^{x^*}$ is

$$F_d^{x^*} = \left\{ (u, u^*) : \begin{array}{c} \exists\, x \in X \text{ with } u_1^* + u_2^* = x^*, \\ u_1^* \in \Gamma_1(x + u_1) \text{ and } u_2^* \in N_{\mathbb{R}^n_+}(x + u_2) \end{array} \right\}$$

or equivalently

$$F_d^{x^*} = \{\, (u, u^*) : [\Gamma_1^-(u_1^*) - u_1] \cap [N_{\mathbb{R}^n_-}(u_2^*) - u_2] \neq \emptyset \text{ and } u_1^* + u_2^* = x^* \,\},$$

and therefore, the dual perturbed problem $(V_d^{x^*})$ becomes

$$\text{Find } \bar{u}^* = (\bar{u}_1^*, \bar{u}_2^*) \in U^* \text{ s.t. } \left\{ \begin{array}{c} \exists\, x \in X \text{ with } \bar{u}_1^* \in \Gamma_1(x) \\ \text{and } \bar{u}_2^* = x^* - u_1^* \in N_{\mathbb{R}^n_+}(x), \end{array} \right. \qquad (V_d^{x^*})$$

which can be equivalently formulated as

$$\text{Find } \bar{u}^* \in X^* \text{ such that } \Gamma_1^-(\bar{u}^*) \cap -N_{\mathbb{R}^n_+}(\bar{u}^* - x^*) \neq \emptyset \qquad (V_d^{x^*})$$

$$\text{Find } \bar{u}^* \in X^* \text{ such that } 0 \in \Gamma_1^-(\bar{u}^*) + N_{\mathbb{R}^n_+}(\bar{u}^* - x^*) \qquad (V_d^{x^*})$$

$$\text{Find } \bar{u}^* \in \mathbb{R}^n_+ \text{ such that } \exists\, \bar{u} \in \Gamma_1^-(\bar{u}_{x^*} + x^*) \cap \mathbb{R}^n_+ \text{ with } \langle \bar{u}_{x^*}, \bar{u} \rangle = 0. \qquad (V_d^{x^*})$$

Again, the second one of these formulations of $(V_d^{x^*})$ implies in particular that

$$w^* \in \Gamma_1(\mathbb{R}^n_+) \iff w^* \text{ is a solution of problem } (V_d^{w^*}).$$

Finally, the multivalued map Λ becomes

$$\Lambda(x^*, u) = \{\, (x, u^*) : u_1^* \in \Gamma_1(x + u_1),\ u_2^* \in N_{\mathbb{R}^n_+}(x + u_2) \text{ and } u_1^* + u_2^* = x^* \,\},$$

and therefore, the problem (V_L) is formulated as:

$$\text{Find } (\bar{x}, \bar{u}^*) \in X \times U^* \text{ such that } \quad \begin{cases} \bar{u}_1^* \in \Gamma_1(\bar{x}) \text{ and} \\ \bar{u}_2^* = -\bar{u}_1^* \in N_{\mathbb{R}_+^n}(\bar{x}). \end{cases} \tag{V_L}$$

In this particular example, the solutions sets of problems (V_p^u), $(V_d^{x^*})$ and (V_L), become:

$$S_p(u) = \{x \in X :\ 0 \in \Gamma_1(x + u_1) + N_{\mathbb{R}_+^n}(x + u_2)\},$$

$$S_d(x^*) = \{(v^*, x^* - v^*) \in U^* :\ 0 \in \Gamma_1^-(v^*) + N_{\mathbb{R}_+^n}(v^* - x^*)\}$$

and

$$S_l = \{(x, (v^*, -v^*)) \in X \times U^* :\ v^* \in \Gamma_1(x) \text{ and } -v^* \in N_{\mathbb{R}_+^n}(x)\}.$$

Sensitivity and stability analysis.

In the present case, conditions (Dsp), (Dwp), (Dsd) and (Dwd) are:

$$\operatorname{int}(\operatorname{dom}(\Gamma_1))) \cap \operatorname{int}(\mathbb{R}_+^n) \neq \emptyset, \tag{Dsp}$$

$$\operatorname{ri}(\operatorname{dom}(\Gamma_1))) \cap \operatorname{int}(\mathbb{R}_+^n) \neq \emptyset, \tag{Dwp}$$

$$\operatorname{dom}(\Gamma_1^-) \cap \operatorname{int}(\mathbb{R}_+^n) \neq \emptyset. \tag{Dsd) = (Dwd}$$

5.2.4 The Auslender-Teboulle lagrangian duality

In this example consider $C = \cap_{k=1}^r C_k$, where C_k is the closed convex set defined by $C_k = \{x \in X : g_k(x) \leq 0\}$, with g_k given proper lsc convex function. Denote N_{C_k} the normal cone to C_k.

Again, assume that the multivalued map Γ_1 is maximal monotone and $\operatorname{dom}(\Gamma_1) \cap \mathrm{C} \neq \emptyset$.

By duality in linear programming, a vector x^* belongs to $N_{C_k}(x)$ if and only if $g_k(x) \leq 0$ and there exists a real number $\mu_k^* \geq 0$ for which $x^* = \mu_k^* \partial g_k(x)$ and $\mu_k^* g_k(x) = 0$.

Then (see [35], Corollary 28.2.1) under Slater's condition

$$\exists\, \tilde{x} \in X \text{ such that } g_k(\tilde{x}) < 0,\ k = 1, 2, \cdots, r, \tag{5.1}$$

$N_C(x) = \sum_{k=1}^{r} N_{C_k}(x)$, and, therefore, the following relation

$$z^* \in N_C(x) \Leftrightarrow \left\{ \begin{array}{c} \exists\, w^* \in \mathbb{R}^r \text{ such that} \\ z^* \in \sum_{j=1}^{r} w_j^* \partial g_j(x), \\ w^* \geq 0,\ \ g(x) \leq 0, \\ \langle w^*, g(x) \rangle = 0, \end{array} \right. \tag{5.2}$$

holds, where $g(x) = (g_1(x), g_2(x), \cdots, g_r(x))^t$.

The above relation, implying, that existence of z^* belonging to normal cone $N_C(x)$, is equivalent to the existence of $w^* \in \mathbb{R}^r$ satisfying expression (5.2).

The set F_p is defined as:

$$F_p = \{\, (x, x^*) : \, x^* \in \Gamma_1(x) + N_C(x) \,\}$$

or, under assumption (5.1), as

$$F_p = \left\{ (x, x^*) : \quad \begin{array}{c} \exists\, w^* \in \mathbb{R}^r \text{ such that} \\ x^* \in \Gamma_1(x) + \sum_{j=1}^{r} w_j^* \partial g_j(x), \\ g(x) \in N_{\mathbb{R}^n_+}(w^*) \end{array} \right\}.$$

In this setting, under assumption (5.1), problem (V_p) becomes:

$$\text{Find } x \in X \text{ such that} \quad \left\{ \begin{array}{c} \exists\, w^* \in \mathbb{R}^r \text{ for which} \\ 0 \in \Gamma_1(x) + \sum_{j=1}^{r} w_j^* \partial g_j(x), \\ g(x) \in N_{\mathbb{R}^n_+}(w^*). \end{array} \right. \tag{$\hat{V}_p$}$$

Introducing the perturbation.

In this case, the perturbed subset Φ becomes

$$\Phi = \{((x, u_1, u_2), (x^*, u_1^*, u_2^*)) : u_1^* \in \Gamma_1(x+u_1),\ \ u_2^* = x^* - u_1^* \in N_{\mathbb{R}^n_+}(x+u_2)\}$$

or, under assumption (5.1),

$$\Phi = \left\{ ((x, u_1, u_2), (x^*, u_1^*, u_2^*)) : \quad \begin{array}{c} \exists\, w^* \in \mathbb{R}^r \text{ such that} \\ u_1^* \in \Gamma_1(x + u_1), \\ u_2^* \in \sum_{j=1}^{r} w_j^* \partial g_j(x + u_2), \\ g(x) \in N_{\mathbb{R}^n_+}(w^*),\ \ u_1^* + u_2^* = x^*. \end{array} \right\}.$$

In view of Proposition 5.1.1, the maximality of Γ_1 and $\Gamma_2 = N_C$ implies the maximality of Φ and therefore the monotonicity of F_p.

The dual problem.

According to the duality scheme, the subset F_d becomes

$$F_d = \{ (u, u^*) : \exists\, x \in X \text{ s.t. } u_1^* \in \Gamma_1(x + u_1) \text{ and } u_2^* = -u_1^* \in N_C(x + u_2) \},$$

which, by the monotonicity of Φ, it is also monotone. Thus, under assumption (5.1), the dual problem (V_d) can be equivalently formulated as:

$$\text{Find } w^* \in \mathbb{R}_+^r \text{ such that } \begin{cases} \exists\, x \in X \text{ for which} \\ 0 \in \Gamma_1(x) + \sum_{j=1}^r w_j^* \partial g_j(x), \\ g(x) \in N_{\mathbb{R}_+^n}(w^*). \end{cases} \qquad (\hat{V}_d)$$

In this case, the vectors of the form $(v^*, -v^*) \in X^* \times X^*$ for which

$$v^* \in \Gamma_1(x) \cap -\sum_{j=1}^r w_j^* \partial g_j(x),$$

are solutions of our original dual problem, i.e., $((0,0),(v^*,-v^*)) \in F_d$.

The perturbed problems and the lagrangian problem.

Given the perturbation variable $u \in X \times X$, the subset F_p^u is

$$F_p^u = \{ (x, x^*) : x^* \in \Gamma_1(\bar{x} + u_1) + N_C(\bar{x} + u_2) \}$$

and therefore, the primal perturbed problem becomes:

$$\text{Find } \bar{x} \in X \text{ such that } 0 \in \Gamma_1(\bar{x} + u_1) + N_C(\bar{x} + u_2), \qquad (V_p^u)$$

which, under assumption (5.1), is equivalent to:

$$\text{Find } x \in X \text{ such that } \begin{cases} \exists\, w^* \in \mathbb{R}^r \text{ for which} \\ 0 \in \Gamma_1(x + u_1) + \sum_{j=1}^r w_j^* \partial g_j(x + u_2), \\ g(x + u_2) \in N_{\mathbb{R}_+^r}(w^*). \end{cases} \qquad (\hat{V}_p^u)$$

Next, given the perturbation variable $x^* \in X^*$, the subset $F_d^{x^*}$ is

$$F_d^{x^*} = \{ (u, u^*) : [\Gamma_1^-(u_1^*) - u_1] \cap [N_C^-(u_2^*) - u_2] \neq \emptyset \text{ and } u_1^* + u_2^* = x^* \}$$

and, therefore, the dual perturbed problem becomes:

$$\text{Find } \bar{u}^* \in U^* \text{ such that } [\Gamma_1^-(u_1^*)] \cap [N_C^-(u_2^*)] \neq \emptyset \text{ and } u_1^* + u_2^* = x^*, \quad (V_d^{x^*})$$

which, under assumption (5.1), is equivalent to:

$$\text{Find } w^* \in \mathbb{R}_+^r \text{ such that } \quad \begin{cases} \exists\, x \in X \text{ for which} \\ x^* \in \Gamma_1(x) + \sum_{j=1}^r w_j^* \partial g_j(x), \\ g(x) \in N_{\mathbb{R}_+^r}(w^*). \end{cases} \qquad (\hat{V}_d^{x^*})$$

In this case, vectors of the form $(v^*, -v^*) \in X^* \times X^*$ for which

$$v^* \in [\Gamma_1(x) - x^*] \cap -\sum_{j=1}^r w_j^* \partial g_j(x),$$

are such that, $((0,0),(v^*,-v^*)) \in F_d^{x^*}$.

Finally, the multifunction $\Lambda : X^* \times U \overset{\longrightarrow}{\longrightarrow} X \times U^*$ is defined by

$$\Lambda(x^*, u) =$$

$$\{ (x, u^*) \in X \times U^* :\ u_1^* \in \Gamma_1(x + u_1),\ u_2^* \in N_C(x + u_2) \text{ and } u_1^* + u_2^* = x^* \}$$

and therefore, the lagrangian problem becomes

$$\text{Find } (\bar{x}, \bar{u}^*) \in X \times U^* \text{ such that } \quad \begin{cases} \bar{u}_1^* \in \Gamma_1(\bar{x}) \ \text{ and} \\ \bar{u}_2^* = -\bar{u}_1^* \in N_C(\bar{x}), \end{cases} \qquad (V_L)$$

which, under assumption (5.1), is equivalent to:

$$\text{Find } (x, w^*) \in X \times \mathbb{R}_+^r \text{ such that } \begin{cases} 0 \in \Gamma_1(x) + \sum_{j=1}^r w_j^* \partial g_j(x), \\ g(x) \in N_{\mathbb{R}_+^r}(w^*). \end{cases} \qquad (\hat{V}_L)$$

In this case, the vectors of the form $(x, (v^*, -v^*)) \in X \times U^*$, for which

$$v^* \in [\Gamma_1(x)] \cap -\sum_{j=1}^r w_j^* \partial g_j(x),$$

belong to $\Lambda(0,0)$.

The formulations of problems $(\hat{V}_p)$, $(\hat{V}_d)$ and $(\hat{V}_L)$ are, respectively, the well known formulations of primal problem, dual problem, and primal-dual problem developed by Auslender and Teboulle, in [2].

5.3 Sum of more that two monotone maps

In this section we shall apply our duality scheme to the sum of more that two monotone maps. Thus, this section can be seen as a generalization of Section 5.1.

Consider $X = X^* = \mathbb{R}^n$ and for $i = 1, 2, \cdots, q$, $Y_i = Y_i^* = \mathbb{R}^{r_i}$ and the multivalued maps $\Gamma_i : Y_i \overset{\longrightarrow}{\longrightarrow} Y_i^*$.

Define $\Gamma : X \overset{\longrightarrow}{\longrightarrow} X^*$ by

$$\Gamma(x) = \sum_{i=1}^{q} A_i^t \Gamma_i(A_i x + a_i),$$

where for $i = 1, 2, \cdots, q$, A_i is a $r_i \times n$ surjective matrix and $a_i \in \mathbb{R}^{r_i}$.

It is clear that if the multivalued maps Γ_i, $i = 1, 2, \cdots, q$, are monotone, then the multivalued map Γ is also monotone.

We shall apply the duality scheme to the following problem

$$\text{Find } \bar{x} \in X \text{ such that } 0 \in \Gamma(\bar{x}). \tag{V}$$

Assume that $\operatorname{dom}(\Gamma)$ is not empty. Denote $F_p = \operatorname{graph}(\Gamma)$, then

$$F_p = \left\{ (x, x^*) : \begin{array}{c} \text{for } i = 1, 2, \cdots, q, \ \exists\, y_i^* \in Y_i^* \text{ with} \\ y_i^* \in \Gamma_i(A_i x + a_i) \text{ and } \sum_{i=1}^{q} A_i^t y_i^* = x^* \end{array} \right\}$$

and therefore the problem (V) can be written as

$$\text{Find } \bar{x} \in X \text{ such that } (\bar{x}, 0) \in F_p. \tag{V_p}$$

5.3.1 Introducing the perturbation

Take

$$U = U_1 \times U_2 \times \cdots \times U_q \quad \text{and} \quad U^* = U_1^* \times U_2^* \times \cdots \times U_q^*,$$

where for $i = 1, 2, \cdots, q$, $U_i = U_i^* = \mathbb{R}^{r_i}$.

Let us introduce the perturbed subset $\Psi \subset (X \times U) \times (X^* \times U^*)$,

$$\Psi = \left\{ ((x,u),(x^*,u^*)) : \begin{array}{c} \sum_{i=1}^q A_i^t u_i^* = x^* \text{ and} \\ u_i^* \in \Gamma_i(A_i x + a_i + u_i),\ i = 1, 2, \cdots, q \end{array} \right\}.$$

By definition,

$$(x, x^*) \in F_p \iff \exists\, u^* \in U^* \text{ such that } ((x,0),(x^*,u^*)) \in \Psi.$$

With the same techniques used to prove Proposition 5.1.1 we can prove the following proposition

Proposition 5.3.1 *Assume that the multivalued maps Γ_i for $i = 1, 2, \cdots, q$, are monotone, then the subset Ψ is monotone. If the Γ_i's are maximal monotone, then Ψ is also maximal monotone.*

Proof. a) Assume that the multivalued maps Γ_i, $i = 1, 2, \cdots, q$ are monotone. Let $((x,u),(x^*,u^*))$ and $((y,v),(y^*,v^*))$ be two elements belonging to Φ. We shall prove that

$$B = \langle y^* - x^*, y - x\rangle + \sum_{i=1}^q \langle v_i^* - u_i^*, v_i - u_i\rangle \geq 0.$$

Since $\langle y^* - x^*, y - x\rangle = \sum_{i=1}^q \langle v_i^* - u_i^*, A_i y - A_i x\rangle$,

$$B = \sum_{i=1}^q \langle v_i^* - u_i^*, (A_i y + a_i + v_i) - (A_i x + a_i + u_i)\rangle.$$

By construction, $(A_i x + a_i + u_i, u_i^*)$ and $(A_i y + a_i + v_i, v_i^*)$ belong to E_i for $i = 1, 2, \cdots, q$, therefore $B \geq 0$.

b) Assume now that the Γ_i's are maximal monotone. Let us consider the multivalued map $\Sigma : U^* \times X \stackrel{\longrightarrow}{\longrightarrow} U \times X^*$ defined by

$$\Sigma(u^*, x) = \{(u, x^*) \,:\, ((x,u),(x^*,u^*) \in \Psi\}.$$

Then,

$$\Sigma(u_1^*, \cdots, u_q^*, x) = [\Gamma_1^-(u_1^*) - A_1x - a_1] \times \cdots \times [\Gamma_q^-(u_q^*) - A_qx - a_q] \times \{\sum_{i=1}^{q} A_i^t u_i^*\}.$$

Since the graph of Σ corresponds to Φ after permutation of variables, Φ is maximal monotone if and only if Σ is maximal monotone. For this, it is enough to prove that

$$\widetilde{\Sigma}(u_1^*, \cdots, u_q^*, x) = [\Gamma_1^-(u_1^*) - A_1x - a_1] \times \cdots \times [\Gamma_q^-(u_q^*) - A_qx - a_q] \times \{\sum_{i=1}^{q} A_i^t u_i^*\}.$$

Assume that $(u_1, u_2, \cdots, u_q, x^*) \in \widetilde{\Sigma}(u_1^*, u_2^*, \cdots, u_q^*, x)$. Then, for all $(v_i, v_i^*) \in \text{graph}\,(\Gamma_\text{i})$, $i = 1, 2, \cdots, q$ and $y \in X$ one has, $B \geq 0$, where

$$B = \sum_{i=1}^{q} \langle u_i^* - v_i^*, u_i + A_iy + a_i - v_i\rangle + \langle x^* - \sum_{i=1}^{q} A_i^t v_i^*, x - y\rangle.$$

Easy computations give

$$B = \sum_{i=1}^{q} \langle u_i^* - v_i^*, u_i + a_i - v_i\rangle + \langle x^* - \sum_{i=1}^{q} A_i^t v_i^*, x\rangle + \langle \sum_{i=1}^{q} A_i^t u_i^* - x^*, y\rangle,$$

from what

$$x^* = \sum_{i=1}^{q} A_i^t u_i^*.$$

Using this relation in the last expression of B, we see that

$$\sum_{i=1}^{q} \langle u_i^* - v_i^*, (u_i + A_ix + a_i) - v_i\rangle \geq 0, \ \forall (v_i, v_i^*) \in \text{graph}\,(\Gamma_\text{i}), \text{ i} = 1, 2 \cdots, \text{q}.$$

Assume for contradiction that $u_1 + x \notin \Gamma_1^-(u_1^*)$, then there exists $(v_1, v_1^*) \in \text{graph}\,(\Gamma_1)$ satisfying $\langle u_1^* - v_1^*, (u_1 + x) - v_1\rangle < 0$ and consequently, for some index $i \neq 1$, $\langle u_i^* - v_i^*, (u_i + A_ix + a_i) - v_i\rangle > 0$ whenever $(v_i, v_i^*) \in \text{graph}\,(\Gamma_\text{i})$. One deduces that $u_i + A_ix + a_i \in \Gamma_i^-(u_i^*)$. Next, take $(v_i, v_i^*) = (u_i + A_ix + a_i, u_i^*)$, then one has $\langle u_i^* - v_i^*, (u_i + A_ix + a_i) - v_i\rangle = 0$ in contradiction with the inequality above.

One obtains that Σ is maximal monotone. ■

5.3.2 The dual problem

According to the duality scheme, the subset F_d is defined by the relation

$$(u,u^*) \in F_d \iff \exists\, x \in X \text{ such that } ((x,u),(0,u^*)) \in \Phi,$$

in the present case,

$$F_d = \left\{ (u,u^*) : \begin{array}{c} \exists\, x \in X \text{ such that } \sum_{i=1}^q A_i^t u_i^* = 0 \text{ and} \\ u_i^* \in \Gamma_i(A_i x + a_i + u_i),\ i = 1,2,\cdots,q \end{array} \right\}.$$

The dual problem is formulated as:

$$\text{Find } \bar{u}^* \in U^* \text{ such that } \left\{ \begin{array}{l} \exists\, x \in X \text{ with } \sum_{i=0}^q A_i^t \bar{u}_i^* = 0 \text{ and} \\ \bar{u}_i^* \in \Gamma_i(A_i x + a_i),\ i = 0,1,\cdots,q, \end{array} \right. \qquad (V_d)$$

which can also be formulated as

$$\text{Find } \bar{u}^* \in U^* \text{ s.t } \sum_{i=1}^q A_i^t \bar{u}_i^* = 0 \text{ and } \bigcap_{i=1}^q [A_i^\dagger(\Gamma_i^-(\bar{u}_i^*) - a_i) + \ker(A_i)] \neq \emptyset,$$

where $A^\dagger$ denotes the Moore-Penrose pseudoinverse of matrix A, which, if A is surjective, has the expression $A^\dagger = A^t(AA^t)^{-1}$.

5.3.3 The perturbed problems and the lagrangian problem

Given the perturbation variable $u \in X \times X$, the subset F_p^u is

$$F_p^u = \left\{ (x,x^*) : \begin{array}{c} \text{for } i = 1,2,\cdots,q,\ \exists\, y_i^* \in Y_i^* \text{ with} \\ y_i^* \in \Gamma_i(A_i x + a_i + u_i) \text{ and } \sum_{i=1}^q A_i^t y_i^* = x^* \end{array} \right\}$$

and therefore the primal perturbed problem (V_p^u) can be written as

$$\text{Find } \bar{x}_u \in X \text{ s.t } \left\{ \begin{array}{c} \text{for } i = 0,1,\cdots,q,\ \ \exists\, y_i^* \in Y^* \text{ with} \\ y_i^* \in \Gamma_i(A_i \bar{x}_u + a_i + u_i) \text{ and } \sum_{i=0}^q A_i^t y_i^* = 0. \end{array} \right. \qquad (V_p^u)$$

Next, given the perturbation variable $x^* \in X^*$, the subset $F_d^{x^*}$ is

$$F_d^{x^*} = \left\{ (u, u^*) : \begin{array}{c} \exists\, x \in X \text{ such that } \sum_{i=1}^q A_i^t u_i^* = x^* \text{ and} \\ u_i^* \in \Gamma_i(A_i x + a_i + u_i),\ i = 1, 2, \cdots, q \end{array} \right\}$$

and therefore, the dual perturbed problem becomes:

$$\text{Find } \bar{u}^* \in U^* \text{ such that } \left\{ \begin{array}{c} \exists\, x \in X \text{ with } \sum_{i=0}^q A_i^t \bar{u}_i^* = x^* \text{ and} \\ \bar{u}_i^* \in \Gamma_i(A_i x + a_i),\ i = 0, 1, \cdots, q, \end{array} \right. \qquad (V_d^{x^*})$$

which is equivalent to

$$\text{Find } \bar{x}_u \in X \text{ s.t. } \sum_{i=1}^q A_i^t \bar{u}_i^* = x^* \text{ and } \bigcap_{i=1}^q [A_i^\dagger(\Gamma_i^-(\bar{u}_i^*) - a_i) + \ker(A_i)] \neq \emptyset.$$

Finally, the multifunction $\Lambda : X^* \times U \xrightarrow{\longrightarrow} X \times U^*$ is defined by

$$\Lambda(x^*, u) = \left\{ (x, u^*) : \begin{array}{c} \sum_{i=1}^q A_i^t u_i^* = x^* \text{ and} \\ u_i^* \in \Gamma_i(A_i x + a_i + u_i),\ i = 1, 2, \cdots, q \end{array} \right\}$$

and therefore, the Lagrangian problem becomes

$$\text{Find } (\bar{x}, \bar{u}^*) \in X \times U^* \text{ s.t. } \left\{ \begin{array}{c} \sum_{i=1}^q A_i^t \bar{u}_i^* = 0 \text{ and} \\ \bar{u}_i^* \in \Gamma_i(A_i \bar{x} + a_i),\ i = 1, 2, \cdots, q. \end{array} \right. \qquad (V_L)$$

5.3.4 Sensitivity and stability analysis

In this general case, let us consider the following conditions:

$$\Gamma_1, \Gamma_2 \cdots, \Gamma_q \text{ are maximal monotone} \qquad (Mm)$$

$$\bigcap_{i=1}^q \{\, x \in X : \ A_i x + a_i \in \operatorname{int}(\operatorname{dom}(\Gamma_\mathrm{i}))\,\} \neq \emptyset \qquad (Dsp)$$

and the weaker form

$$\bigcap_{i=1}^q \{\, x \in X : \ A_i x + a_i \in \operatorname{ri}(\operatorname{dom}(\Gamma_\mathrm{i}))\,\} \neq \emptyset. \qquad (Dwp)$$

We have already seen that condition (Mm) implies that Φ is maximal monotone, hence S_l is closed and convex. Similarly to Proposition 5.1.2, we obtain the following sufficient condition in order to establish the maximality of F_p^u, for u belonging in some open neighborhood of 0.

Proposition 5.3.2 *Assume that conditions* (Mm) *and* (Dsp) *hold. Then* $0 \in \mathrm{proj}_{\mathrm{U}}(\mathrm{int}\,(\mathrm{proj}_{\mathrm{X\times U}}(\Phi)))$.

Proof. Let $\tilde{x} \in \bigcap_{i=1}^{q} \{\, x \in X :\, A_i x + a_i \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma_{\mathrm{i}}))\,\}$, then there exist N neighborhood of $\tilde{x}$ and V neighborhood of 0 such that for all $x \in N$ and $i = 1, 2\cdots, q$, $u_i \in V$, imply that $A_i x + u_i + a_i \in \mathrm{int}\,(\mathrm{dom}\,(\Gamma_{\mathrm{i}})))$. Hence $N \times [\prod_{i=1}^{q} V]$ is contained in $\mathrm{int}\,(\mathrm{proj}_{\mathrm{X\times U}}(\Phi)))$. The result follows. ■

If condition (Dsp) is replaced by (Dwp), we obtain that the maximality of F_p^u is only verified for u belonging in some relative open neighborhood of $0 \in U$. Indeed, we have the following proposition.

Proposition 5.3.3 *Assume that conditions* (Mm) *and* (Dwp) *hold. Then* $0 \in \mathrm{proj}_{\mathrm{U}}(\mathrm{ri}\,(\mathrm{proj}_{\mathrm{X\times U}}(\Phi)))$.

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