

Finite covers of a hyperbolic 3-manifold and virtual fibers.

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Some conjectures.

Let M be a hyperbolic 3-manifold, connected, closed and oriented.

Theorem (Kahn, Markovic)

The fundamental group $\pi_1 M$ contains a surface subgroup.

Conjectures

- (1) (**Virtually Haken.**) *There exists a finite cover $M' \rightarrow M$ containing an **incompressible surface**, i.e. an embedded surface T in M' such that the map induced by the embedding $\iota : T \rightarrow M'$ on fundamental groups $\iota_* : \pi_1 T \rightarrow \pi_1 M'$ is injective.*
- (2) (**Virtually positive first Betti number.**) *There exists a finite cover $M' \rightarrow M$ with $b_1(M') > 0$.*
- (3) (**Virtually infinite first Betti number.**) *For each $n \in \mathbb{N}$, there exists a finite cover $M_n \rightarrow M$ with $b_1(M_n) \geq n$.*
- (4) (**Thurston.**) *There exists a finite cover $M' \rightarrow M$ which fibers over the circle \mathbb{S}^1 .*

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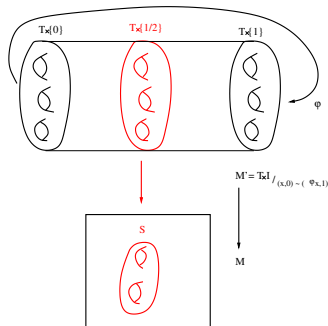
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Definition

A 3-manifold M is said to be **virtually fibered** if it admits a finite cover which fibers over the circle.

An **embedded** surface S in M is a **virtual fiber** if there is a finite cover of M in which the preimage of S is a fiber.

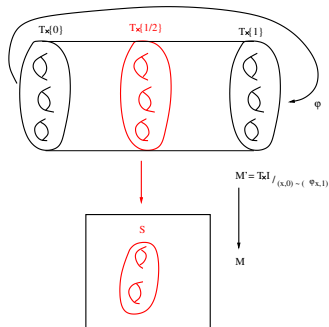


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Main theorem.

Theorem (1, main theorem.)

Fix $\epsilon \leq \text{Inj}(M)/2$. There exists an explicit constant $k = k(\epsilon, \text{Vol}(M))$ such that:

If $M' \rightarrow M$ is a cover of finite degree d ,

with an embedded, closed, orientable, pseudo-minimal surface F , which splits M' into q compression bodies C_1, \dots, C_q with the following properties:

- ① Every simple closed curve embedded in C_j of length $\leq \epsilon$ is nul-homotopic in C_j .
- ② $k g \ln g < \ln \ln d / q$, with $g = \max_j \{g(C_j)\}$.

Then the finite cover M' contains an embedded surface T of genus $g(T) \leq g$ which is a virtual fiber.

In particular, M virtually fibers over the circle and M' is Haken.

Conjecture (*)

The technical assumption (1) is not necessary.

Remark

If $\text{Vol}(M)$ is fixed, $\lim_{\epsilon \rightarrow 0} k(\epsilon, \text{Vol}(M)) \rightarrow +\infty$.

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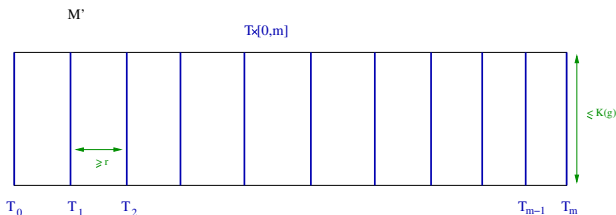
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Ideas of the proof of the main theorem.

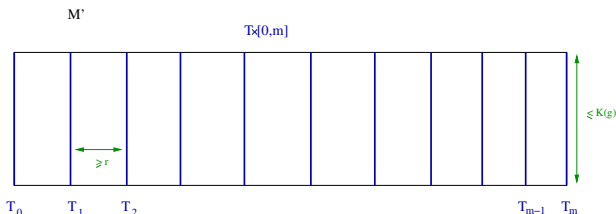
Suppose that the ratio $g \ln g / \ln \ln(d/q)$ is “small enough”.
Proof in two steps.

First step: Construct an embedded “long and thin” product $T \times [0, m]$ in M' , satisfying the following properties.



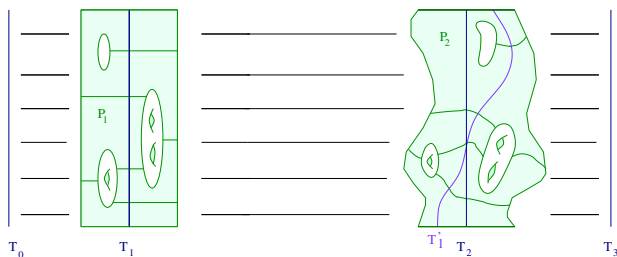
- The surface T is orientable and closed, with genus $g(T) \leq g$.
- The number $m = m(\frac{d}{q}, g)$ is “large”.
- The surfaces $T_j := T \times \{j\}$ have their diameters uniformly bounded from above by $K = K(g)$.
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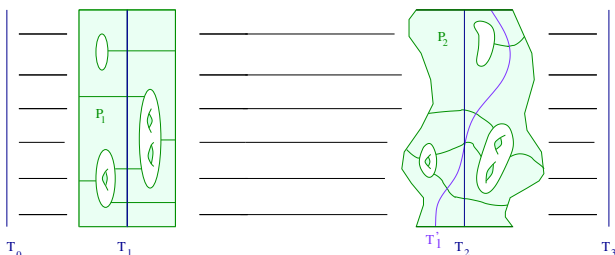
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Second step: Use this product to construct a virtual fibration of M' .



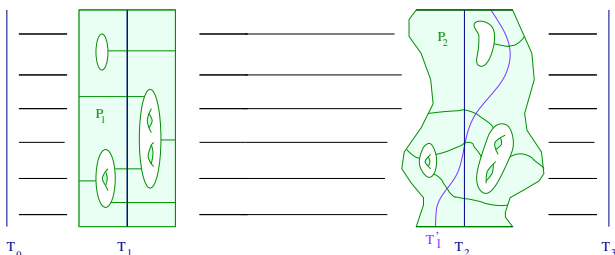
- Choose \mathcal{D} , a Dirichlet fundamental polyhedron for M in \mathbb{H}^3 .
- For each surface T_j , consider the **pattern of fundamental domains** P_j that is the union of the fundamental domains meeting T_j .
- Find two patterns:
 - disjoint from each other,
 - isometric to the same "model" pattern P ,
 - containing parallel surfaces T_1 and T'_1 .
- Cut along T_1 and T'_1 , glue them together to get a finite fibered cover N of M .

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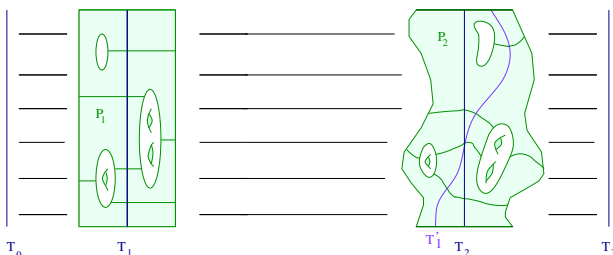
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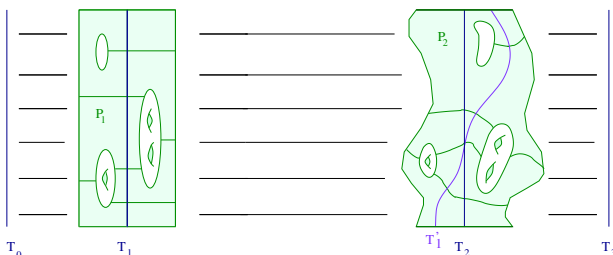
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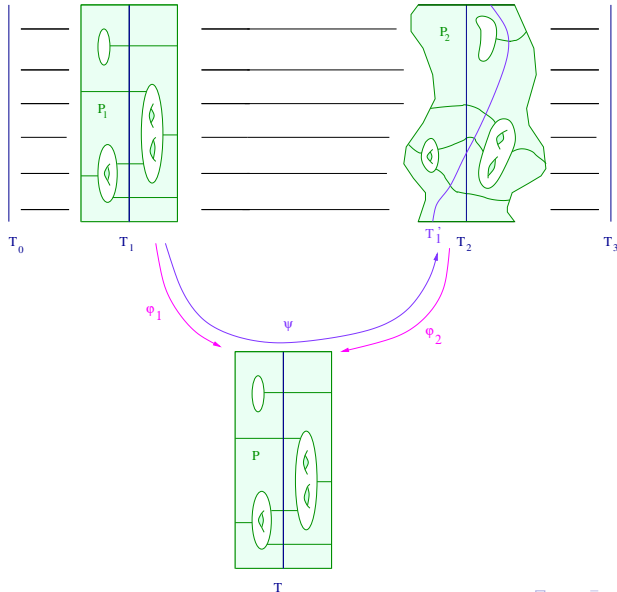


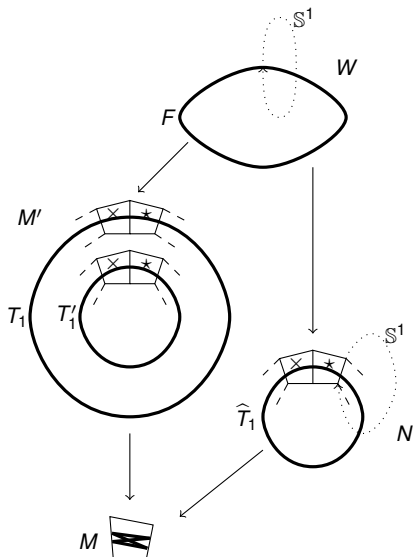
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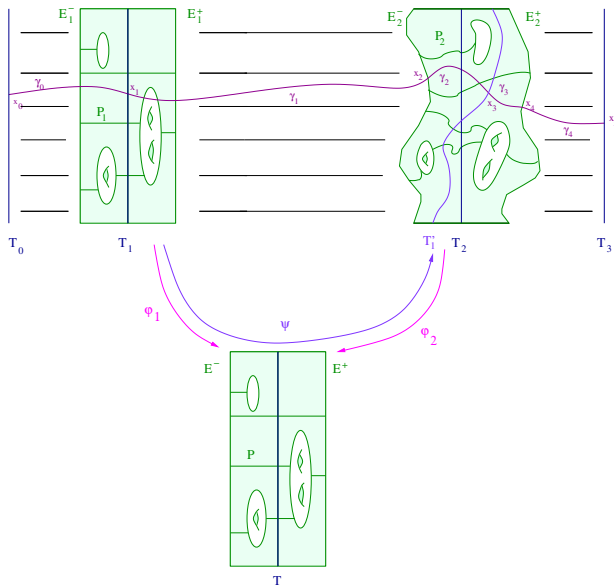
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The regular case.

Theorem (2, regular case.)

Fix $\epsilon \leq \text{Inj}(M)/2$. There exists an explicit constant $k' = k'(\epsilon, \text{Vol}(M))$ such that:

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Then the finite cover M' is a fiber bundle over the circle, and a fiber can be obtained from a component of F , possibly after some surgeries.

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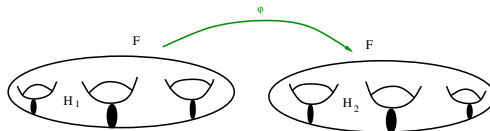
Theorem (2, regular case.)

Fix $\epsilon \leq \text{Inj}(M)/2$. There exists an explicit constant $k' = k'(\epsilon, \text{Vol}(M))$ such that:
 If $M' \rightarrow M$ is a **regular** cover of finite degree d ,
 with an embedded, closed, orientable, pseudo-minimal surface F , which splits M' into
 q compression bodies C_1, \dots, C_q with the following properties:

- ① Every simple closed curve embedded in C_j of length $\leq \epsilon$ is nul-homotopic in C_j .
- ② $k' g^2 < \ln \ln d / q$, with $g = \max_j \{g(C_j)\}$.

Then the finite cover M' is a fiber bundle over the circle, and a fiber can be obtained from a component of F , possibly after some surgeries.

Application to Heegaard splittings.



If F is a surface of genus at least 1, $\chi_-(F) = 2g(F) - 2$.

Definition

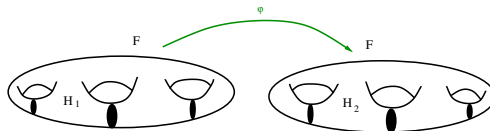
The Heegaard characteristic: $\chi_-^h(M) = 2g(M) - 2$.

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Remark

If $M' \rightarrow M$ is a cover of finite degree d , $\chi_-^h(M') \leq d\chi_-^h(M)$.

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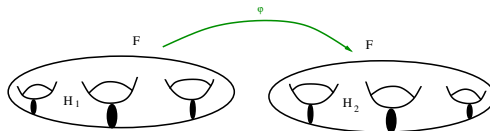
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Heegaard gradient and conjectures of Lackenby.

Definition (Lackenby)

Heegaard gradient:

$$\nabla^h(M) = \inf_i \left\{ \frac{\chi_-^h(M_i)}{d_i} \right\}.$$

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- (1) *The Heegaard gradient of M is zero if and only if M virtually fibers over the circle.*
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The sub-logarithmic version is true.

Definition

Let $\eta \in (0, 1)$.

η -sub-logarithmic Heegaard gradient:

$$\nabla_{\log, \eta}^h(M) = \inf_i \left\{ \frac{\chi_-^h(M_i)}{(\ln \ln d_i)^\eta} \right\}.$$

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Proposition (3, Sub-logarithmic version of Lackenby's conjectures.)

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The homological viewpoint.

Let $\alpha \in H^1(M, \mathbb{Z})$ be a non-trivial element.

Definition

A $\|\alpha\|$ -**minimizing** surface R is an embedded surface with homology class Poincaré-dual to α , and minimizing Thurston's norm: $\chi_-(R) = \|\alpha\|$.

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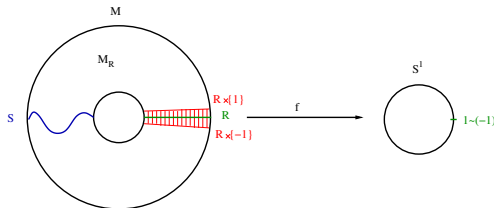
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Definition

Let M_R be the 3-manifold obtained from M by removing a regular neighborhood of R diffeomorphic to $R \times (-1, 1)$. The **circular characteristic** of α , denoted by $\chi_-^c(\alpha)$, is the minimum over all $\|\alpha\|$ -minimizing surfaces R of the Heegaard characteristic of the cobordism $(M_R, R \times \{1\}, R \times \{-1\})$.



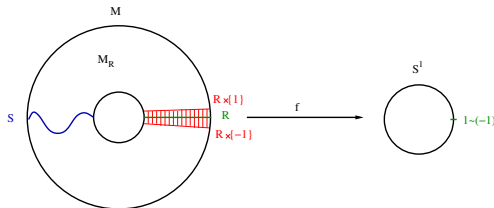
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$\chi_-^c(\alpha) = \|\alpha\| + h(\alpha)$, where $h(\alpha)$ is the minimum over all $\|\alpha\|$ -minimizing surfaces R of the minimal number of critical points of index 1 and 2 of a Morse function $M_R \rightarrow [-1, 1]$.

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Applications to circular decompositions.

Theorem (4, Adapted from a result of Lackenby)

There exists an explicit constant $\ell' = \ell'(\epsilon, \text{Vol}(M))$ such that:

*Fix $\alpha \in H^1(M)$ a non-trivial cohomology class and R a $\|\alpha\|$ -minimizing surface. Let $M' \rightarrow M$ be a d -sheeted **regular** cover and $\alpha' \in H^1(M', \mathbb{Z})$ the Poincaré-dual class associated to a connected component R' of the preimage of R in M' .*

If $\ell' \chi_-^c(\alpha') \leq \sqrt[4]{d}$, then the manifold M fibers over the circle and the surface R is a fiber.

Virtually infinite first Betti number.

Question: Find a tower of finite covers $\dots \rightarrow M_{i+1} \rightarrow M_i \rightarrow \dots \rightarrow M$ such that $\lim_{i \rightarrow +\infty} b_1(M_i) = +\infty$?

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Let $\dots \rightarrow M_i \rightarrow M_{i-1} \rightarrow \dots \rightarrow M_1 \rightarrow M$ a tower of finite covers of M such that for all $i \geq 1$, $M_i \rightarrow M_{i-1}$ is regular, of group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r_i}$ (with the convention that $M_0 = M$). Set $R_i = r_1 + r_2 + \dots + r_i$.

Suppose that one of the following assumptions is satisfied:

(a) $\pi_1 M$ does not have property (τ) with respect to the family $\{\pi_1 M_i\}_{i \in \mathbb{N}}$ (for example if $\lim_{i \rightarrow +\infty} h(M_i) = 0$) and $\inf_{i \in \mathbb{N}} \frac{r_{i+1}}{2^{R_i}} > 0$, or

(b) each cover $M_i \rightarrow M$ is regular and $\lim_{i \rightarrow +\infty} \frac{r_{i+1}}{2^{R_i}} = +\infty$.

Then $\pi_1 M$ is virtually large. In particular, the first Betti number of M is virtually infinite.

Corollary (7)

Let $\dots \rightarrow M_i \rightarrow M_{i-1} \rightarrow \dots \rightarrow M_1 \rightarrow M$ be an infinite tower of finite covers of M as in the previous theorem.

Then the first Betti number of M is virtually infinite if neither of the following properties is satisfied.

① $\inf_{i \in \mathbb{N}} h(M_i) > 0$ and the sequence $(\frac{r_{i+1}}{2^{R_i}})_{i \in \mathbb{N}}$ admits a bounded subsequence.

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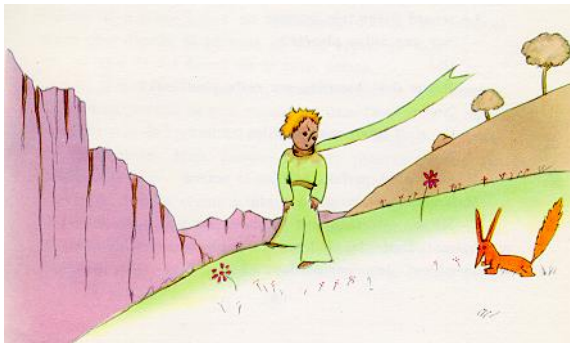
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Thank you for your attention !!!!



Merci pour votre attention !!!!