Second order analysis of optimal control problems with singular arcs. Optimality conditions and shooting algorithm.

Maria Soledad Aronna

Thèse pour l'obtention du titre de

DOCTEUR DE L'ÉCOLE POLYTECHNIQUE

Spécialité: **Mathématiques Appliquées**

par

María Soledad Aronna

Second order analysis of optimal control problems with singular arcs.

Optimality conditions and shooting algorithm.

J. Frédéric Bonnans	INRIA & Ecole Polytechnique	Directeur
Jean-Baptiste Caillau	Université de Bourgogne & CNRS	Président du jury
Bean-San Goh	University of Malaya, Malaysia	Examinateur
Urszula Ledzewicz	Southern Illinois University, USA	Rapporteur
Pablo A. Lotito	Pladema CONICET, Tandil, Argentina	Directeur
Helmut Maurer	WWU Münster, Germany	Examinateur
Emmanuel Trélat	Université Pierre et Marie Curie	Rapporteur

December 15th., 2011.

Contents

Abstract

This thesis deals with optimal control problems for systems that are affine in one part of the control variable. First, we state necessary and sufficient second order conditions when all control variables enter linearly. We have bound control constraints and a bang-singular solution. The sufficient condition is restricted to the scalar control case. We propose a shooting algorithm and provide a sufficient condition for its local quadratic convergence. This condition guarantees the stability of the optimal solution and the local quadratic convergence of the algorithm for the perturbed problem in some cases. We present numerical tests that validate our method. Afterwards, we investigate an optimal control problems with systems that are affine in one part of the control variable. We obtain second order necessary and sufficient conditions for optimality. We propose a shooting algorithm, and we show that the sufficient condition just mentioned is also sufficient for the local quadratic convergence. Finally, we study a model of optimal hydrothermal scheduling. We investigate, by means of necessary conditions due to Goh, the possible occurrence of a singular arc.

Résumé

Dans cette thèse on s'intéresse aux problèmes de commande optimale pour des systèmes affines dans une partie de la commande. Premièrement, on donne une condition nécessaire du second ordre pour le cas ou le système est affine dans toutes les commandes. On a des bornes sur les contrôles et une solution bang-singulière. Une condition suffisante est donnée pour le cas d'une commande scalaire. On propose après un algorithme de tir et une condition suffisante pour sa convergence quadratique locale. Cette condition garantit la stabilité de la solution optimale et implique que l'algorithme converge quadratiquement localement pour le problème perturbé, dans certains cas. On présente des essais numériques qui valident notre méthode. Ensuite, on étudie un système affine dans une partie des commandes. On obtient des conditions nécessaire et suffisante du second ordre. Ensuite, on propose un algorithme de tir et on montre que la condition suffisante mentionnée garantit que cet algorithme converge quadratiquement localement. Enfin, on étudie un problème de planification d'une centrale hydrothermal. On analyse au moyen des conditions nécessaires obtenues par Goh, la possible apparition d'arcs singuliers.

Acknowledgements

This thesis is the fruit of years of research I spent between the COMMANDS-INRIA team of the Center of Applied Mathematics (CMAP) at Ecole Polytechnique, France and the 'Control y Optimización' team of the National University of Rosario, in Argentina. During this time, I have had the pleasure to work with several people whose various contributions where invaluable to the accomplishment of my research and to the production of this manuscript. It is my pleasure to convey my sincere gratitude to them all.

First of all, I want to thank my supervisor J. Frédéric Bonnans for his constant support and full-time commitment. The work presented here is the result of our many discussions. I have learnt enormously by his side. His guidance was crucial for the achievement of this thesis.

My special thanks also goes to Pablo Lotito, my advisor in Argentina. I have appreciated working with him, and I am very grateful for his invitation to work within his team. He has given me the chance to be part of this international cooperation.

I acknowlegde the referees, Urszula Ledzewicz and Emmanuel Trélat for their useful remarks and comments that contributed to this final version of my thesis. Many thanks are sent to all members of the jury: Jean Baptiste Caillau, Bean San Goh and Helmut Maurer. I thank also Andrei Dmitruk for accepting the invitation to work with us. It was an enriching experience.

My heartfelt gratitude goes to the 'Control y Optimización' team and my friends of the Mathematics Department in Rosario. I thank Laura, Pepa, Elina, Lis, Jor, Eduardo, Vivi, Fran, Eva, Sabri, Iso and Ale for their support. Special thanks go to my French teacher Carmen, and to the Olimpíada Matemática Argentina and all its organisers.

My time at CMAP, Ecole Polytechnique was made enjoyable in large part due to the many colleagues: Francisco, Camille, Xavier, Laurent, Emilie, Sylvie, Khalid, Zhihao, Florent, Clément, Zixian, Maxime and Michael.

Last but not least, I would like to thank my family for all their love and encouragement. For my parents and who raised me with a deep love and support me in all my pursuits. And my thanks go to all my close friends in Argentina for giving me their support during this period.

Thank you.

0

Introduction

0.1 Brief introduction to optimal control theory

In this thesis we study an optimal control problem

$$\varphi_0(x(T)) \to \min, \tag{1}$$
$$\dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. on } [0,T], \tag{2}$$
$$x(0) = x_0, \tag{3}$$
$$u \in U, \tag{4}$$

where $u \in \mathcal{U} := L_\infty(0,T;\mathbb{R}^m)$ is termed *control variable,* $x \in \mathcal{X} := W^1_\infty(0,T;\mathbb{R}^n)$, is called *state variable*, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, and $U \subset \mathbb{R}^m$ is a closed convex set. When needed, put $w = (x,u)$ for a point in the product space $\mathcal{W} := \mathcal{X} \times \mathcal{U}$.

A *trajectory* is an element $w \in \mathcal{W}$ that satisfies the state equation (2). If in addition constraint (4) holds, say that w is a *feasible point* of the problem (1)-(4). Denote by $\mathcal{A}$ the *set of feasible points.* Call *feasible variation* for $\hat{w} \in \mathcal{A}$ an element $\delta w \in \mathcal{W}$ such that $\hat{w} + \delta w \in \mathcal{A}$. Hence, our optimal control problem consists of finding a feasible trajectory w that minimizes the cost function φ_0.

Later on a problem having finitely many initial-final state constraints is considered. However, we found it more illustrative to present the basics of optimal control theory for the simplified framework (1)-(4).

The investigation of optimal control problems dates from the late 1950s, when two important advances were made. The first one is the Pontryagin Maximum Principle [111] and consists of a set of necessary conditions for optimality. The second one was the Dynamic Programming Principle [13] that allows us to transform the minimization problem (1)-(4) into a problem of finding the solution to a partial differential equation, known as the Hamilton-Jacobi-Bellman equation. Both approaches make use of the *pre-Hamiltonian function*

$$\begin{array}{rcl} H : \mathbb{R}^n \times \mathbb{R}^{n,*} \times \mathbb{R}^m & \to & \mathbb{R}, \\ (x,p,u) & \mapsto & pf(x,u), \end{array} \tag{5}$$

where with $\mathbb{R}^{n,*}$ we indicate the $n-$dimensional space of row-vector with real components.

0.1.1 Pontryagin Maximum Principle

The Pontryagin Maximum Principle (PMP) provides a necessary condition for optimality by means of the existence of a certain dual variable called *costate variable.* Let $(x,u) \in \mathcal{W}$ be an optimal solution of (1)-(4). Then the PMP states that there exists a Lipschitz continuous function $p : [0,T] \to \mathbb{R}^{n,*}$ that is solution of the *costate equation*

$$-\dot{p}(t) = H_x(x(t), p(t), u(t)), \quad p(T) = \varphi_0'(x(T)), \tag{6}$$

and such that $(x(t), p(t), u(t))$ satisfies the *minimum condition*

$$H(x(t), p(t), u(t)) = \min_{v \in U} H(x(t), p(t), v), \quad \text{a.e. on } [0,T]. \tag{7}$$

When initial-final state constraints are considered, the PMP has a slightly more complex statement that includes a multiplier associated to the cost function and the initial-final constraints.

0.1.2 Hamilton-Jacobi-Bellman equation

The Hamilton-Jacobi-Bellman (HJB) equation is a partial differential equation resulting from the Dynamic Programming Principle. It provides a necessary and sufficient condition for optimality in terms of the *value function*

$$V(\zeta, t) := \inf_{(x,u)} \{\varphi_0(x(T)) : \dot{x}(s) = f(x(s), u(s)) \text{ and } u(s) \in U \text{ a.e. on } [t,T],\ x(t) = \zeta\}.$$

The pillar of the dynamic programming approach is the following statement: *the value function V is unique viscosity solution of the HJB equation*

$$\begin{cases} W_t(\zeta,t) + \inf_{a \in U} H(\zeta, a, W_\zeta(\zeta,t)) = 0, & \text{for } (\zeta,t) \in \mathbb{R}^n \times (0,T), \\ W(\zeta,T) = \varphi_0(\zeta). \end{cases} \tag{HJB}$$

The concept of *viscosity solution* was introduced by Crandall and Lions in [37], and in their later work with Evans [36] they provided a more refined set of definitions and properties. The classical references concerning viscosity solutions and their connection with optimal control theory are the books of Fleming-Soner [58] and Bardi-Capuzzo Dolcetta [11].

In this thesis we do not investigate this approach.

0.1.3 Second order conditions

Let $(x(t), p(t), u(t))$ be a feasible solution of (1)-(4) satisfying the PMP, and such that

$$u(t) \in \operatorname{int} U, \quad \text{a.e. on } [0,T]. \tag{8}$$

Hence, the minimum condition (7) implies the *stationarity* of the pre-Hamiltonian

$$H_u(x(t), p(t), u(t)) = 0, \quad \text{a.e. on } [0,T], \tag{9}$$

and the positive semidefiniteness of its Hessian matrix

$$H_{uu}(x(t), p(t), u(t)) \succeq 0, \quad \text{a.e. on } [0, T]. \tag{10}$$

The latter inequality is termed *Legendre-Clebsch condition* and it is a second order necessary condition for optimality. On the other hand, observe that if the *strengthened Legendre-Clebsch condition*

$$H_{uu}(x(t), p(t), u(t)) \succ \alpha I, \quad \text{a.e. on } [0, T] \tag{11}$$

holds for some positive α, it guarantees that $u(t)$ is a local minimizer of $H(x(t), p(t), v)$ almost everywhere on $[0, T]$.

Set $J(u) := \varphi_0(x(T))$ the cost associated to a given control $u \in \mathcal{U}$, i.e. x is the solution of (2)-(3) corresponding to u. It can be shown that if there exists $\alpha > 0$ such that

$$D^2_{uu} J(u)(v, v) \geq \alpha \|v\|_2^2, \quad \text{for every } v \in L_2(0, T; \mathbb{R}^m), \tag{12}$$

then u is a strict local minimum satisfying *quadratic growth,* i.e. for some $\varepsilon, \rho > 0$,

$$J(u + v) \geq J(u) + c\|v\|_2^2, \quad \text{for every } \|v\|_\infty < \rho. \tag{13}$$

Hence, (12) is a *sufficient condition for local optimality.*

Concerning second order conditions for optimality, many references can be cited and several different approaches are encountered in the literature. Next we list a set of works related to second order conditions that is far from being complete, but that intends to give an outlook on the research in this area. The books Bryson-Ho [30] and Agrachev-Sachkov [3] provide a set of second order conditions for different kinds of problems. The articles Jacobson [75], Agrachev [1], Sarychev [116], Zeidan [128, 129], Zeidan-Zezza [131, 130], Pales-Zeidan [105] and the book by Bonnard and Chyba [24] treated this subject as well. The reader is then referred to these works and the references therein for further information on the topic of second order analysis.

0.1.4 The shooting method

Concerning numerical methods for optimal control, two approaches can be encountered in practice: direct and indirect methods. Direct methods consist of solving the nonlinear programming problem obtained by discretizing the state and control variables. On the other hand, indirect methods use the shooting algorithm to solve the two-point boundary value problem resulting from the Pontryagin Maximum Principle. In this thesis we study only the indirect approach. Betts in [14] gave a meticulous survey on direct and indirect techniques, including 100 references. This article together with the works by von Stryk-Bulirsch [126], Biegler-Cervantes [17], Pesch [106], Trélat [125] and Biegler [15] provide a detailed panorama of the subject.

Observe that the minimization condition (7) of the PMP allows, under hypothesis (8) and (11), to express the control as a function of the state and costate variables:

$$u(t) = \Gamma(x(t), p(t)), \quad t \in [0, T]. \tag{14}$$

Eliminating u by means of the latter expression from the set of conditions provided by the PMP yields the two-point boundary value problem

$$\begin{aligned} \dot{x}(t) &= f(x(t), \Gamma(x(t), p(t))), \\ \dot{p}(t) &= -H_x(x(t), p(t), \Gamma(x(t), p(t))), \end{aligned} \tag{15}$$

with boundary conditions

$$x(0) = x_0, \quad p(T) = \varphi_0'(x(T)). \tag{16}$$

The latter is a system of ordinary differential equations having boundary conditions both in the initial and final times. We define the shooting function

$$\begin{array}{rcl} \mathcal{S} : \mathbb{R}^{n,*} & \rightarrow & \mathbb{R}^{n,*}, \\ p_0 & \mapsto & \mathcal{S}(p_0) := \varphi_0'(x(T)) - p(T), \end{array} \tag{17}$$

where x and p are the solution of (15) with initial conditions $x(0) = x_0$ and $p(0) = p_0$. Hence, $\mathcal{S}$ assigns to each estimate of the initial values, the value of the final condition of the corresponding solution. The shooting algorithm consists of approximating a zero of this function. In other words, the method finds suitable initial values for which the corresponding solution of the differential equation system satisfies the final conditions.

0.2 Singular arcs

In some problems, the candidate extremal or some subarc of it cannot be directly determined by the minimum condition (7). This occurs when the strengthened Legendre-Clebsch condition (11) does not hold, and hence

$$H_{uu} \text{ is singular}, \tag{18}$$

and only semidefinite. Such arcs are called *singular arcs.*

Singular arcs appear, for example, when the system is affine in one or more control variable.

This topic has been extensively studied and many references can be cited. See Bell-Jacobson [12], Bryson-Ho [30] and Bonnard-Chyba [24], among others.

0.2.1 Affine systems

When systems are affine in all the control variables, there is a classical technique that provides second order necessary conditions. It is based on the fact that the control u does not appear in the stationarity condition (9). Therefore, the expression (9) can be differentiated with respect to the time variable. Once this derivation is performed, $\dot{x}$ and $\dot{p}$ are replaced using (2) and (6). This yields a new algebraic equation in the variables (x, p, u) that can contain the control with a non-zero coefficient. If it does not, another time differentiation is allowed. This is done until an expression depending explicitly on u is obtained.

By means of the technique just described a set of necessary conditions was derived in the 60s. For the scalar control case, Kelley in [79] proved that

$$-\frac{\partial}{\partial u}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} H_u(x(t), p(t), u(t))\right] \geq 0. \tag{19}$$

The latter inequality is known as the *generalized Legendre-Clebsch condition.* The result was extended by Kopp and Moyer [82] for higher order derivatives. In Kelley-Kopp-Moyer [81] it is shown that the smallest order of derivative of H_u where u appears without an identically zero coefficient is even. This says that if M is the smallest positive integer such that $\frac{\partial}{\partial u}(\mathrm{d}^M/\mathrm{d}t^M) H_u(x(t), p(t), u(t)) \neq 0$, then M is even. The integer $N := M/2$ is called *order of the singular arc.* Summing up all these results together yields: *if N is the order of the singular arc then the coefficient of u in $(\mathrm{d}^M/\mathrm{d}t^M) H_u(x(t), p(t), u(t))$ is nonnegative.* Goh in [67] extended this necessary condition for the vector control problem by proving that

$$(-1)^N \frac{\partial}{\partial u}\left[\frac{\mathrm{d}^{2N}}{\mathrm{d}t^{2N}} H_u(x(t), p(t), u(t))\right] \succeq 0, \tag{20}$$

along an optimal trajectory. He used a transformation of variables introduced by himself in [68].

Regarding sufficient conditions for this kind of problem, the first results came out a decade later. Dmitruk in [40] gave necessary and sufficient conditions for optimality in terms of the second variation of the Lagrangian function. He used Goh's Transformation above-mentioned. In Dmitruk [41, 43] he extended previous results to a more general type of minimum. These results have a strong connection with the first two chapters of this thesis and therefore, we present them in more detail later on.

Many other works can be cited concerning second order conditions for totally singular extremals. The reader is referred to the introduction of Chapter 1 to a survey of the literature on this topic.

0.2.2 Partially affine systems

Consider now a problem having a pre-Hamiltonian that is linear in one part of the control variables and nonlinear in the remaining part. The necessary condition Goh gave in [67] actually applies to this kind of problem. It extends both the Legendre-Clebsch for the nonlinear case in equation (10) and the 'vector generalized Legendre-Clebsch condition' in (20). Assume that the control can be written in the form (u, v), with v appearing linearly in the pre-Hamiltonian. Then the necessary condition in Goh [67] for a singular arc of order 1 says that *along an optimal trajectory,*

$$\begin{pmatrix} H_{uu} & -\dfrac{\partial}{\partial u}\dot{H}_v^\top \\ -\dfrac{\partial}{\partial u}\dot{H}_v & -\dfrac{\partial}{\partial v}\ddot{H}_v \end{pmatrix} \succeq 0. \tag{21}$$

In this text we refer to (21) as *Goh-Legendre-Clebsch condition.* In the recent article [69], Goh gave a concise presentation of this result and some other related necessary conditions.

Concerning sufficient conditions, in [95] Maurer and Osmolovskii derive one for a problem with this 'mixed' structure, but where the optimal trajectory has a bang-bang affine control. Our framework is different since the components that appear linearly in the system are assumed to be singular. No derivation of sufficient conditions for partially affine systems was found in the literature.

Remark 0.2.1. *The terminology just introduced for the conditions* (10) *and* (20) *is the one we use throughout all this thesis, but some variants can be encountered in the literature. Actually, the Legendre-Clebsch condition* (10) *is found under the name of Legendre condition, while what we call here generalized Legendre-Clebsch condition* (20) *is sometimes known as Legendre-Clebsch condition (as in Bonnard-Chyba [24]) or as Kelley's condition when* $N = 1$ *(as in Robbins [112]).*

0.3 Structure of the thesis

The thesis consists of four chapters and each of them corresponds to an article (either published, submission or preprint).

Chapter 1 is entitled *Second order conditions for bang-singular extremals* and corresponds to a joint work with J.F. Bonnans, A.V. Dmitruk and P.A. Lotito. It will appear in Numerical Algebra, Control and Optimization, in a special issue in honor of Professor Helmut Maurer, under the title *Quadratic conditions for bang-*

singular extremals. It was also published as the INRIA Research Report Nr. 6674 [8].

Chapter 2 is entitled *A shooting algorithm for problems with singular arcs* and corresponds to a joint work with J.F. Bonnans and P. Martinon. It was published as the INRIA Research Report Nr. 7763 [10] under the title *A well-posed shooting algorithm for problems with singular arcs.*

Chapter 3 is entitled *Partially affine control problems: second order conditions and a shooting algorithm* and M.S. Aronna is the sole author. It was published as the INRIA Research Report Nr. 7764 [7] under the title *Partially affine control problems: second order conditions and a well-posed shooting algorithm.*

Chapter 4 is entitled *Continuous Time Optimal Hydrothermal Scheduling* and corresponds to a joint work with J.F. Bonnans and P.A. Lotito. It was published in Proceedings of the International Conference on Engineering Optimization, under the title *Continuous Time Optimal Hydrothermal Scheduling.* See reference [9].

Next we give a short summary of the results of each chapter.

0.4 Summary of the results of the thesis

0.4.1 Second order conditions for bang-singular extremals

In the first chapter we give second order necessary and sufficient conditions for the problem

$$J := \varphi_0(x(T)) \to \min, \tag{22}$$

$$\dot{x}(t) = \sum_{i=0}^{m} u_i(t) f_i(x(t)), \quad \text{a.e. on } [0,T], \tag{23}$$

$$x(0) = x_0, \tag{24}$$

$$u(t) \geq 0, \ \text{for a.a. } t \in (0,T), \tag{25}$$

$$\varphi_i(x(T)) \leq 0, \quad \text{for } i = 1, \dots, d_\varphi, \tag{26}$$

$$\eta_j(x(T)) = 0, \quad \text{for } j = 1 \dots, d_\eta, \tag{27}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 0, \dots, m$, $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 0, \dots, d_\varphi$, $\eta_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \dots, d_\eta$ and $u_0 \equiv 1$. The sufficient condition is restricted to the scalar control case.

We shall start by noticing that all the second order conditions that we mentioned in Section 0.2.1 do not apply when the control variable touches the boundary of the admissible set since condition (8) was needed. This means that those results hold only for *totally singular extremals.* The novelty of Chapter 1 is to provide second order conditions for solutions that eventually touch the boundary of U. In other words, we allow the constraint (25) to be *active.*

For a given control component i, an arc is termed *singular* if $H_{u_i} = 0$ almost everywhere on the arc, otherwise it is called *bang.* Under the strict complementary assumption to be introduced afterwards, the singular arcs occur when the control is positive, and the bang arcs when the control is null. The studied trajectory is a finite concatenation of bang and singular arcs. We call this kind of solution a *bang-singular extremal.*

Basic concepts.

We shall introduce some definitions needed for the statement of the principal theorems of Chapter 1. We study the two types of minimum defined as follows.

Definition 0.4.1. *A pair $w^0 = (x^0, u^0) \in \mathcal{W}$ is said to be a weak minimum of problem* (22)-(27) *if there exists $\varepsilon > 0$ such that the cost function attains at w^0 its minimum on the set of feasible trajectories $w = (x, u)$ verifying*

$$\|x - x^0\|_\infty < \varepsilon, \quad \|u - u^0\|_\infty < \varepsilon.$$

A trajectory w^0 is a Pontryagin minimum if for any positive N there exists $\varepsilon_N > 0$ such that w^0 is a minimum point on the set of feasible trajectories w verifying

$$\|x - x^0\|_\infty < \varepsilon_N, \quad \|u - u^0\|_\infty \leq N, \quad \|u - u^0\|_1 < \varepsilon_N.$$

Let $\lambda = (\alpha, \beta, p) \in \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0, T; \mathbb{R}^{n,*})$. Define the *pre-Hamiltonian*[1]

$$H[\lambda](x, u, t) := p(t) \sum_{i=0}^{m} u_i f_i(x),$$

the *terminal Lagrangian*

$$\ell[\lambda](q) := \sum_{i=0}^{d_\varphi} \alpha_i \varphi_i(q) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(q),$$

and the *Lagrangian*

$$\Phi[\lambda](w) := \ell[\lambda](x(T)) + \int_0^T p(t) \left(\sum_{i=0}^{m} u_i(t) f_i(x(t)) - \dot{x}(t) \right) \mathrm{d}t. \tag{28}$$

The optimality of a given feasible trajectory $\hat{w} = (\hat{x}, \hat{u})$ is investigated.

[1] We use the notation with the multiplier indicated between square brackets like in Dmitruk [43].

Definition 0.4.2. *Denote by* $\Lambda \subset \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0,T;\mathbb{R}^{n,*})$ *the set of Pontryagin multipliers associated to* $\hat{w}$. *It consists of the elements* $\lambda = (\alpha, \beta, p)$ *satisfying the Pontryagin Maximum Principle, i.e. such that*

$$|\alpha| + |\beta| = 1, \tag{29}$$

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d_\varphi}) \geq 0, \tag{30}$$

the function p *is solution of the costate equation and satisfies the transversality condition in the end-point* T

$$-\dot{p}(t) = H_x[\lambda](\hat{x}(t), \hat{u}(t), t), \;\; p(T) = \ell'[\lambda](\hat{x}(T)), \tag{31}$$

and the following minimum condition holds

$$H[\lambda](\hat{x}(t), \hat{u}(t), t) = \min_{v \geq 0} H[\lambda](\hat{x}(t), v, t), \;\text{ a.e. on } [0,T]. \tag{32}$$

Remark 0.4.3. Λ *can be identified with a finite dimensional compact set.*

The second order condition that we establish first involves the critical cone and second variation of the Lagrangian function to be defined below. Consider the *linearized state equation:*

$$\begin{cases} \dot{z}(t) = A(t)z(t) + B(t)v(t), & \text{a.e. on } [0,T], \\ z(0) = 0, \end{cases} \tag{33}$$

with $A(t) := \sum_{i=0}^m \hat{u}_i f_i'(\hat{x}(t))$ and $B(t)v(t) := \sum_{i=1}^m v_i(t) f_i(\hat{u}(t))$. The solution z of (33) is called *linearized state variable.* The *linearization of the cost and final constraints* is given by

$$\begin{cases} \varphi_i'(\hat{x}(T))z(T) \leq 0, \;\; i = 0, \ldots, d_\varphi, \\ \eta_j'(\hat{x}(T))z(T) = 0, \;\; j = 1, \ldots, d_\eta. \end{cases} \tag{34}$$

To each index $i = 1, \ldots, m$ we associate the *active and free sets*

$$I_0^i := \left\{ t \in [0,T] : \max_{\lambda \in \Lambda} H_{u_i}[\lambda](t) > 0 \right\}, \quad I_+^i := [0,T] \backslash I_0^i, \tag{35}$$

We assume *strict complementarity for the control constraint:* $\hat{u}_i = 0$ a.e. on I_0^i and $\hat{u}_i > 0$ a.e. on I_+^i. Consider now the *critical cone*

$$\mathcal{C}_2 := \left\{ (z,v) \in W_2^1(0,T;\mathbb{R}^n) \times L_2(0,T;\mathbb{R}^m) : v_i = 0 \text{ on } I_0^i, \text{ (33) and (34) hold} \right\}, \tag{36}$$

and the *second variation of the Lagrangian function*

$$\begin{aligned} \Omega[\lambda](\delta x, \delta u) &:= \tfrac{1}{2} D^2 \Phi[\lambda](\hat{w})(\delta x, \delta u)^2 \\ &\tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\delta x(T))^2 + \int_0^T \left(\tfrac{1}{2} \delta x^\top Q[\lambda] \delta x + \delta u^\top C[\lambda] \delta x \right) \mathrm{d}t, \end{aligned} \tag{37}$$

where $Q := H_{xx}$ and $C := H_{ux}$.

Results

First we provide a detailed derivation of the necessary condition below. Another proof can be found in Levitin-Milyutin-Osmolovskii [86].

Theorem 0.4.4. *If $\hat{w}$ is a weak minimum then*

$$\max_{\lambda \in \Lambda} \Omega[\lambda](z, v) \geq 0, \quad \text{for all } (z, v) \in \mathcal{C}_2. \tag{38}$$

Afterwards, we use this result, the Goh's transformation of variables introduced in [68] and some techniques given by Dmitruk in [40, 43] to show a necessary condition in another space of perturbations. Many details are omitted in order to achieve a concise presentation.

Let us begin by recalling the Goh's transformation: given (z, v) satisfying (33), set

$$y(t) := \int_0^t v(s)\mathrm{d}s, \quad \xi := z - By. \tag{39}$$

Notice that ξ satisfies the linear differential equation

$$\dot{\xi} = A\xi + B_1 y, \quad \xi(0) = 0, \tag{40}$$

where $B_1 := AB - \dot{B}$. Performing Goh's transformation in Ω yields

$$\begin{aligned} \Omega_{\mathcal{P}}[\lambda](\xi, y, v) :=& g[\lambda](\xi(T), y(T)) \\ &+ \int_0^T (\tfrac{1}{2}\xi^\top Q[\lambda]\xi + y^\top M[\lambda]\xi + \tfrac{1}{2} y^\top R[\lambda] y + v^\top V[\lambda] y)\mathrm{d}t. \end{aligned} \tag{41}$$

By transforming the critical cone $\mathcal{C}_2$ we obtain a cone $\mathcal{P}$ in the new space of perturbations. The objective is providing conditions that do not involve variable v. With this aim we define, for a compact and convex set $M \subset \mathbb{R}^s$,

$$G(M) := \{\lambda \in M : V_{ij}[\lambda](t) = 0 \text{ on } I_+^i \cap I_+^j, \text{ for any pair } 1 \leq i < j \leq m\}.$$

By Remark 0.4.3, the convex hull of Λ, denoted by $\operatorname{co}\Lambda$, can be identified with a finite dimensional compact and convex set. The following necessary condition is established.

Theorem 0.4.5. *If $\hat{w}$ is a weak minimum then*

$$\max_{\lambda \in G(\operatorname{co}\Lambda)} \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) \geq 0, \quad \text{on } \mathcal{P}. \tag{42}$$

Denote by $\mathcal{P}_2$ the closure of the cone $\mathcal{P}$ in the $W_2^1(0, T; \mathbb{R}^n) \times L_2(0, T; \mathbb{R}_+^m) \times \mathbb{R}^m$−topology and consider the following quadratic mapping that does not depend on v.

Definition 0.4.6. *For $(\xi, y, h) \in \mathcal{P}_2$ and $\lambda \in G(\Lambda)$ define*

$$\begin{aligned}\Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) :=& g[\lambda](\xi(T), h) + \Xi[\lambda](\xi, y, h) \\ &+ \int_0^T (\tfrac{1}{2}\xi^\top Q[\lambda]\xi + y^\top M[\lambda]\xi + \tfrac{1}{2}y^\top R[\lambda]y)\mathrm{d}t,\end{aligned} \tag{43}$$

where Ξ is a quadratic mapping (that we define later).

The following necessary condition is established.

Theorem 0.4.7. *Let $\hat{w}$ be a weak minimum, then*

$$\max_{\lambda \in G(\mathrm{co}\,\Lambda)} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq 0, \quad \text{for all } (\xi, y, h) \in \mathcal{P}_2. \tag{44}$$

Finally, we provide a sufficient condition for Pontryagin optimality for the case of a scalar control. We make use of the concepts below

Definition 0.4.8.

- *A sequence $\{v_k\} \subset \mathcal{U}$ converges to 0 in the Pontryagin sense if $\|v_k\|_1 \to 0$ and there exists N such that $\|v_k\|_\infty < N$.*
- *For $(y, h) \in \mathcal{U}_2 \times \mathbb{R}$, consider the order $\gamma(y, h) := \int_0^T y(t)^2 \mathrm{d}t + h^2$.*
- *The extremal $\hat{w}$ satisfies $\gamma-$quadratic growth condition in the Pontryagin sense if there exists $\rho > 0$ such that, for every sequence of feasible variations $\{(\delta x_k, v_k)\}$ with $\{v_k\}$ converging to 0 in the Pontryagin sense,*

$$J(\hat{u} + v_k) - J(\hat{u}) \geq \rho\gamma(y_k, y_k(T)), \tag{45}$$

holds for large enough k, where y_k is defined by (39). Equivalently, for all $N > 0$, there exists $\varepsilon > 0$ such that if $\|v\|_\infty < N$ and $\|v\|_1 < \varepsilon$, then (45) holds.

- *The trajectory $\hat{w}$ is normal if $\alpha_0 > 0$ for every $\lambda \in \Lambda$.*

The sufficient condition is as follows.

Theorem 0.4.9 (Sufficient condition for scalar control)**.** *Suppose that there exists $\rho > 0$ such that*

$$\max_{\lambda \in \Lambda} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq \rho\gamma(y, h), \quad \text{on } (\xi, y, h) \in \mathcal{P}_2. \tag{46}$$

Then $\hat{w}$ is a Pontryagin minimum satisfying $\gamma-$ quadratic growth. Furthermore, if $\hat{w}$ is normal, the converse holds.

Remark 0.4.10. *In case the bang arcs are absent, i.e. the control is totally singular, this theorem reduces to one proved in Dmitruk [41, 43].*

0.4.2 A shooting method

We begin Chapter 2 by studying the problem

$$\varphi_0(x(0),x(T)) \to \min, \tag{47}$$

$$\dot{x}(t) = \sum_{i=0}^{m} u_i(t) f_i(x(t)), \quad \text{a.e. on } [0,T], \tag{48}$$

$$\eta_j(x(0),x(T)) = 0, \quad \text{for } j = 1,\dots,d_\eta. \tag{49}$$

Notice that no inequality constraints are considered. We assume that the studied trajectory $\hat{w}$ satisfies the following *qualification hypothesis.* Consider the mapping

$$\begin{array}{rcl} G\colon \mathbb{R}^n \times \mathcal{U} & \to & \mathbb{R}^{d_\eta} \\ (x_0,u) & \mapsto & \eta(x_0,x_T), \end{array} \tag{50}$$

where x_T is the solution of (2.2) associated to (x_0,u).

Assumption 0.4.11. *The derivative of G at $(\hat{x}_0,\hat{u})$ is onto.*

It is a known fact that the Assumption 0.4.11 implies the uniqueness of multiplier. We denote this unique multiplier by $\hat{\lambda} = (\hat{\beta},\hat{p})$.

For this problem, we propose a shooting algorithm and we show a sufficient condition that guarantees its local quadratic convergence. This condition coincides with the second order sufficient condition for weak optimality proved by Dmitruk in [40], and already mentioned earlier in this introduction. Furthermore, in some cases, we prove that this sufficient condition ensures the stability of the optimal solution under small data perturbation.

Afterwards we deal with a problem having control bounds of the type

$$0 \leq u(t) \leq 1, \quad \text{a.e. on } [0,T]. \tag{51}$$

By means of a certain transformation we obtain similar results that those established for (47)-(49).

Finally we present numerical tests that validate our method.

The algorithm

The PMP consists of the following conditions: the costate equation

$$-\dot{\hat{p}}(t) = H_x[\hat{\lambda}](\hat{x}(t),\hat{u}(t),t), \quad \text{on } [0,T], \tag{52}$$

with the transversality conditions

$$\hat{p}(0) = -D_{x_0}\ell[\hat{\lambda}](\hat{x}(0), \hat{x}(T)), \tag{53}$$
$$\hat{p}(T) = D_{x_T}\ell[\hat{\lambda}](\hat{x}(0), \hat{x}(T)), \tag{54}$$

and the minimum condition

$$H[\hat{\lambda}](\hat{x}(t), \hat{u}(t), t) = \min_{v \in \mathbb{R}^m} H[\hat{\lambda}](\hat{x}(t), v, t), \quad \text{a.e. on } [0, T]. \tag{55}$$

Let the *switching function* $\Phi : [0, T] \to \mathbb{R}^{m,*}$ be defined by

$$\Phi(t) := H_u[\hat{\lambda}](\hat{x}(t), \hat{u}(t), t). \tag{56}$$

Observe that the minimum condition (55) is equivalent to

$$\Phi(t) = 0, \quad \text{a.e. on } [0, T]. \tag{57}$$

Assuming that the *strengthened generalized Legendre-Clebsch condition* $-\frac{\partial}{\partial u}\ddot{\Phi} \succ 0$ holds, we can write $\hat{u}$ in terms of $\hat{x}$ and $\hat{p}$ from equation

$$\ddot{\Phi} = 0, \quad \text{a.e. on } [0, T]. \tag{58}$$

Observe that (58) together with

$$\Phi_T = 0, \;\; \dot{\Phi}_0 = 0, \tag{59}$$

imply the stationarity condition (57).

Notation: Denote by (OS) the set of equations composed by (48)-(49), (52)-(54), (58), (59).

Define the *shooting function*

$$\begin{aligned} \mathcal{S} : \mathrm{D}(\mathcal{S}) := \mathbb{R}^n \times \mathbb{R}^{n+d_\eta,*} \; &\to \mathbb{R}^{d_\eta} \times \mathbb{R}^{2n+2m,*}, \\ \big(x_0, p_0, \beta\big) =: \nu \; &\mapsto \mathcal{S}(\nu) := \begin{pmatrix} \eta(x(0), x(T)) \\ p_0 + D_{x_0}\ell[\lambda](x(0), x(T)) \\ p_T - D_{x_T}\ell[\lambda](x(0), x(T)) \\ \Phi(T) \\ \dot{\Phi}(0) \end{pmatrix}, \end{aligned} \tag{60}$$

where (x, u, p) is a solution of (48), (52), (58) with initial conditions x_0 and p_0, and $\lambda := (\beta, p)$. Note that solving (OS) consists of finding $\nu \in \mathrm{D}(\mathcal{S})$ such that

$$\mathcal{S}(\nu) = 0. \tag{61}$$

Since the number of equations in (2.23) is greater than the number of unknowns, the Gauss-Newton method is a suitable approach to solve it. The *shooting method* suggested here consists of solving (2.23) by the Gauss-Newton method. The latter is applicable provided that $\mathcal{S}'(\hat{\nu})$ is one-to-one, where $\hat{\nu}$ is the solution of (61), and in this case the algorithm converges locally quadratically.

The sufficient condition

We shall briefly present the second order sufficient condition due to Dmitruk [40] that we use in the main statements of Chapter 2. The concepts that we need here where already introduced in the presentantion of Chapter 1 in Paragraph 0.4.1 above. Recall the critical cone $\mathcal{P}_2$ and the quadratic mappings $\Omega_{\mathcal{P}}$ and $\Omega_{\mathcal{P}_2}$.

Theorem 0.4.12. *The trajectory $\hat{w}$ is a weak minimum of (P) satisfying $\gamma-$quadratic growth condition in the weak sense if and only if $V \equiv 0$ and $\Omega_{\mathcal{P}_2}$ is coercive on $\mathcal{P}_2$.*

Remark 0.4.13. *Actually, the nullity of V is a necessary condition for weak optimality due to Goh [67] and it implies that the function Ξ in $\Omega_{\mathcal{P}_2}$ is null and $\Omega_{\mathcal{P}} = \Omega_{\mathcal{P}_2}$.*

The results

We give first a sufficient condition for the local quadratic convergence of the shooting algorithm:

Theorem 0.4.14. *Let $\hat{w}$ be a weak minimum of* (47)-(49) *such that $\Omega_{\mathcal{P}_2}$ is coercive on $\mathcal{P}_2$. Then the shooting algorithm is locally quadratically convergent.*

Afterwards, we study the family of problems depending on the real parameter μ

$$\begin{aligned}
&\varphi_0^\mu(x(0), x(T)) \to \min,\\
&\dot{x}(t) = \sum_{i=0}^m u_i(t) f_i^\mu(x(t)), \quad \text{a.e. on } [0,T],\\
&\eta^\mu(x(0), x(T)) = 0.
\end{aligned} \tag{P$_\mu$}$$

All the data functions are continuously differentiable with respect to the parameter μ, and problem (P_0) coincides with (47)-(49). We establish the stability under small perturbation.

Theorem 0.4.15 (Stability of the optimal solution)**.** *Assume that the shooting system generated by problem* (47)-(49) *is square (i.e. it can be reduced to a square system) and let $\hat{w}$ be such that $\Omega_{\mathcal{P}_2}$ is coercive on $\mathcal{P}_2$. Then there exists a neighborhood $\mathcal{J} \subset \mathbb{R}$ of 0, and a continuous differentiable mapping $\mu \mapsto w^\mu = (x^\mu, u^\mu)$, from $\mathcal{J}$ to $\mathcal{W}$, where w^μ is a weak solution for* (P_μ). *Furthermore, w^μ verifies the coercivity condition for $\Omega_{\mathcal{P}_2}^\mu$. Therefore, the quadratic growth in the weak sense holds and the shooting algorithm for* (P_μ) *converges locally quadratically.*

For a problem having control bounds like in (51) we prove similar results to the two just given. We proceed as follows. Denote by (CP) the problem defined by (47)-(49), (51). Assuming that we know the structure of the optimal solution $\hat{u}$ of (CP) and an approximation of its switching times we propose a shooting system by means of a transformed problem (TP). Afterwards we prove that we can transform $\hat{w}$ into a solution $\hat{W}$ of (TP). Finally we conclude that if $\hat{W}$ satisfies the coercivity condition for its corresponding $\Omega_{\mathcal{P}_2}$, the Theorem 0.4.14 can be applied. Then the shooting method converges locally quadratically. Furthermore, if the system is square the stability result holds for $\hat{W}$, in view of Theorem 0.4.15. The latter yields the stability of $\hat{w}$ in a certain sense that we define later on and that is very close to weak optimality.

0.4.3 Partially affine problems

In Chapter 3 we investigate the problem

$$J := \varphi_0(x(0), x(T)) \to \min, \tag{62}$$

$$\dot{x}(t) = \sum_{i=0}^{m} v_i(t) f_i(x(t), u(t)), \quad \text{a.e. on } [0, T], \tag{63}$$

$$\eta_j(x(0), x(T)) = 0, \text{ for } j = 1\dots, d_\eta, \tag{64}$$

$$\varphi_i(x(0), x(T)) \leq 0, \text{ for } i = 1, \dots, d_\varphi. \tag{65}$$

Observe that v enters linearly in the pre-Hamiltonian. We derive second order necessary and sufficient conditions for weak optimality in terms of the second variation of the Lagrangian function. Afterwards, we propose a shooting algorithm and show that the provided sufficient condition guarantees the local quadratic convergence of this algorithm.

There is a difference that we encounter in the derivation of second order conditions for this 'mixed' framework that we would like to mention. The Goh's transformation used in this case does not change the entire control variable, but only the controls that are affine in the pre-Hamiltonian. More precisely, if we denote by $(\bar{x}, \bar{u}, \bar{v})$ the perturbation for a trajectory $(\hat{x}, \hat{u}, \hat{v})$, the transformation is given by

$$\bar{y}(t) := \int_0^t \bar{v}(s)\mathrm{d}s, \quad \bar{\xi}(t) := \bar{x}(t) - B(t)\bar{y}(t). \tag{66}$$

Hence, the new perturbation is $(\bar{\xi}, \bar{u}, \bar{y})$.

Basic concepts and properties

As usual, we study a nominal feasible trajectory $(\hat{x}, \hat{u}, \hat{v})$. We consider the associated set of Lagrange multipliers Λ_L consisting of the elements $\lambda = (\alpha, \beta, p)$ that verify the normalization for (α, β), the non-negativity for α, the costate equation and the following *stationarity* conditions

$$\begin{cases} H_u[\lambda](\hat{x}(t), \hat{u}(t), \hat{v}(t), t) = 0, \\ H_v[\lambda](\hat{x}(t), \hat{u}(t), \hat{v}(t), t) = 0, \end{cases} \quad \text{a.e. on } [0,T]. \tag{67}$$

The second variation associated to problem (62)-(65) is given by

$$\begin{aligned} \Omega[\lambda](\bar{x}, \bar{u}, \bar{v}) := &\ \tfrac{1}{2}\ell''[\lambda](\hat{x}_0, \hat{x}_T)(\bar{x}_0, \bar{x}_T)^2 \\ &+ \int_0^T [\tfrac{1}{2}\bar{x}^\top Q[\lambda]\bar{x} + \bar{u}^\top E[\lambda]\bar{x} + \bar{v}^\top C[\lambda]\bar{x} + \tfrac{1}{2}\bar{u}^\top R_0[\lambda]\bar{u} + \bar{v}^\top K[\lambda]\bar{u}]\mathrm{d}t. \end{aligned} \tag{68}$$

The main statements involve the subset of multipliers given by

$$\Lambda_L^\# := \{\lambda \in \Lambda_L : R_0[\lambda] \succeq 0 \text{ and } K[\lambda] \equiv 0\}. \tag{69}$$

The critical cone $\mathcal{C}_2$, the quadratic mapping $\Omega_{\mathcal{P}_2}$ and the transformed cone $\mathcal{P}_2$ are the natural extensions for the mixed case of the ones defined before.

Results

In the first part of the chapter we establish the following second order conditions.

Theorem 0.4.16 (Second order necessary condition). *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in \Lambda_L^\#} \Omega[\lambda](\bar{x}, \bar{u}, \bar{v}) \geq 0, \quad \text{on } \mathcal{C}_2. \tag{70}$$

Theorem 0.4.17 (Transformed second order necessary condition). *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in G(\mathrm{co}\,\Lambda_L^\#)} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \geq 0, \quad \text{on } \mathcal{P}_2. \tag{71}$$

Theorem 0.4.18 (Sufficient condition). *Assume that there exists $\rho > 0$ such that*

$$\max_{\lambda \in G(\mathrm{co}\,\Lambda_L^\#)} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \geq \rho\gamma(\bar{\xi}_0, \bar{u}, \bar{y}, \bar{h}), \quad \text{on } \mathcal{P}_2. \tag{72}$$

Then $\hat{w}$ is a weak minimum satisfying $\gamma-$quadratic growth in the weak sense.

In the second part of the chapter we propose a shooting algorithm that uses the system

$$\begin{pmatrix} H_u \\ -\ddot{H}_v \end{pmatrix} = 0 \tag{73}$$

to eliminate the control (u, v) in terms of the state and costate variables. We give a sufficient condition for its local quadratic convergence that coincides with the sufficient condition stated above.

Theorem 0.4.19. *If $\hat{w}$ is a weak minimum satisfying* (72) *then the shooting algorithm converges locally quadratically.*

0.4.4 Optimal hydrothermal scheduling

In Chapter 4 we study the model of hydrothermal scheduling described next. We know (somehow) the complete trajectory of the electricity demand $d(t)$ and the thermal production cost P which is an increasing, positive function of the load. If we call $\pi(t)$ the total amount of electricity produced with the hydropower plants, the remainder $d(t) - \pi(t)$ will be produced with the thermal units. For the sake of planning, the water has an economic value that has also to be considered in the cost definition. More precisely, we consider the problem on $[0, T]$ given by

$$\begin{aligned}
&\int_0^T P[d(t) - \pi(t)]\mathrm{d}t - a\sum_{i=1}^m y_i(T) \to \min, \\
&\dot{y}(t) = b(t) - s(t) - q(t), \\
&\pi(t) = \sum_{i=1}^m \rho_i(y_i)q_i \leq d(t), \\
&0 \leq y_i(t) \leq y^M, \quad \text{for } i = 1, \ldots, m, \\
&0 \leq q_i(t) \leq q^M, \quad \text{for } i = 1, \ldots, m, \\
&s(t) \geq 0,
\end{aligned} \tag{P}$$

where q_i is the outflow of plant i, y_i is the water volume in the valley i, b_i is its water inflow and s_i is the spilled out water at plant i. The control variables are q and s. We assume that the efficiency of each turbine is a positive and increasing function of the volumen $\rho_i(y_i)$, the maximum volume of each valley is given by y^M and the maximum allowed flow in each turbine is q^M.

Results

By means of a change of variables on the controls we convert problem (P) into a problem having the partially affine structure studied in Chapter 3. We focus on the occurrence of singular arcs in this transformed problem. Our main result is an analysis of the Goh-Legendre-Clebsch condition. We show that for some choices of the efficiency functions $\rho_i(y_i)$, the Goh-Legendre-Clebsch condition always holds (respectively does not hold).

1

Second order conditions for bang-singular extremals

Joint work with J.F. Bonnans, A.V. Dmitruk and P.A. Lotito. Accepted for publication in Numerical Algebra, Control and Optimization, special issue in honor of Helmut Maurer [to appear in 2012], under the title *Quadratic conditions for bang- singular extremals.* Published as the INRIA Research Report Nr. 6674 [June 2011]

Abstract

This paper deals with optimal control problems for systems affine in the control variable. We consider nonnegativity constraints on the control, and finitely many equality and inequality constraints on the final state. For the vector control case we obtain second order necessary optimality conditions, and for the scalar control we get second order sufficient conditions as well.

1.1 Introduction

In this article we obtain second order conditions for an optimal control problem affine in the control. First we consider a pointwise nonnegativity constraint on the control, end-point state constraints and a fixed time interval. Then we extend the result to bound constraints on the control, initial-final state constraints and problems involving parameters. We do not assume that the multipliers are unique. We study weak and Pontryagin minima.

There is already an important literature on this subject. The case without control constraints, i.e. when the extremal is totally singular, has been extensively studied since the mid 1960s. Kelley in [79] treated the scalar control case and presented a necessary condition involving the second order derivative of the switching function. The result was extended by Kopp and Moyer [82] for higher order derivatives, and in [81] it was shown that the order had to be even. Goh in [68] proposed a special change of variables obtained via a linear ODE and in [67] used this transformation to derive a necessary condition for the vector control problem. An extensive survey of these articles can be found in Gabasov and Kirillova [62]. Jacobson and Speyer in [76], and together with Lele in [77] obtained necessary conditions by adding a penalization term to the cost functional. Gabasov and Kirillova [62], Krener [83], Agrachev and Gamkrelidze [2] obtained a countable series of necessary conditions that in fact use the idea behind the Goh transformation. Milyutin in [96] discovered an abstract essence of this approach and obtained even stronger necessary conditions. More references can be found in Dmitruk [44]. The main feature of this kind of problem, where the control enters linearly, is that the corresponding second variation does not contain the Legendre term, so the methods of the classical calculus of variations are not applicable for obtaining sufficient conditions. This is why the literature was mostly devoted to necessary conditions, which are actually a consequence of the nonnegativity of the second variation. For a long time, the proposed sufficient conditions were not of second order since they required not only properties along the reference extremal, but also those in a neighborhood of it. Those conditions were based either on field theory or on the existence of a so-called Krotov function. Hence, they were rather far from being necessary. See e.g. works of Moyer [99], Gurman [71] and Dykhta [51]. On the other hand, the above-mentioned Goh's transformation allows one to convert the second variation into another functional that hopefully turns out to have a positive Legendre term with respect to some state variable that can be taken as a new control. Dmitruk in [40] proposed a quadratic order involving only state variations, and proved that the corresponding coercivity of the second variation is sufficient for the weak optimality (whereas its nonnegativity is a necessary condition). He used the abstract approach developed by Levitin, Milyutin and Osmolovskii in [86], and considered finitely many inequality and equality constraints on the endpoints and the possible nonuniqueness of multipliers. In [41, 43] he also obtained necessary and sufficient conditions of this quadratic order, again closely related, for Pontryagin minimality. More recently, Bonnard et al. in [21] provided second order sufficient conditions for the minimum time problem of a single-input system in terms of the existence of a conjugate time.

The case with linear control constraints and a “purely” bang-bang control without singular subarcs has been extensively investigated over the past 15 years. Milyutin and Osmolovskii in [97] provided second order necessary and sufficient conditions based on the general theory of [86]. Agrachev, Stefani and Zezza [4] reduced the problem to a finite dimensional problem with the switching instants as variables and obtained a sufficient condition for strong optimality. Osmolovskii and Maurer [104] showed that this condition is equivalent to that in [97]. Some other results were obtained by Sarychev [115], Poggiolini and Spadini [107], Maurer and Osmolovskii [94, 93]. Felgenhauer in [52, 53, 54] studied both second order optimality conditions and sensitivity of the optimal solution.

The mixed case, where the control is partly bang-bang, partly singular was studied in [108] by Poggiolini and Stefani. They obtained a second order sufficient condition with an additional geometrical hypothesis (which is not needed here) and claimed that it is not clear whether this hypothesis is ‘almost necessary’, in the sense that it is not obtained straightforward from a necessary condition by strengthening an inequality. In [109, 110] they derived a second order sufficient condition for the special case of a time-optimal problem. The single-input time-optimal problem was also studied by means of synthesis-like methods. See, among others, Sussmann [123, 122, 121], Schättler [117] and Schättler-Jankovic [118]. Both bang-bang and bang-singular structures were analyzed in these works, but they did not give second order conditions. The main goal of the present article is to provide a second order sufficient condition for bang-singular extremals in a general Mayer problem, that is ‘almost necessary’. Our approach is quite different from that in Poggiolini and Stefani [108, 109, 110].

The article is organized as follows. In the section 1.2 we present the problem and give basic definitions. In section 1.3 we perform a second order analysis. More precisely, we derive the second variation of the Lagrangian functions and obtain a necessary condition. Afterwards, in the section 1.4, we present the Goh transformation and a new necessary condition in the transformed variables. In section 1.5 we obtain a sufficient condition for scalar control. Finally, we give an example with a scalar control where the second order sufficient condition can be verified in section 1.6. The appendix is devoted to a series of technical properties that are used to prove the main results.

1.2 Statement of the problem and assumptions

1.2.1 Statement of the problem

Consider the spaces $\mathcal{U} := L_\infty(0,T;\mathbb{R}^m)$ and $\mathcal{X} := W^1_\infty(0,T;\mathbb{R}^n)$ as control and state spaces, respectively. Denote with u and x their elements, respectively. When needed, put $w = (x,u)$ for a point in $\mathcal{W} := \mathcal{X} \times \mathcal{U}$. In this paper we investigate the optimal control problem

$$J := \varphi_0(x(T)) \to \min, \tag{1.1}$$

$$\dot{x}(t) = \sum_{i=0}^{m} u_i f_i(x), \quad x(0) = x_0, \tag{1.2}$$

$$u(t) \geq 0, \text{ a.e. on } t \in [0,T], \tag{1.3}$$

$$\varphi_i(x(T)) \leq 0, \text{ for } i = 1,\dots,d_\varphi, \quad \eta_j(x(T)) = 0, \text{ for } j = 1\dots,d_\eta. \tag{1.4}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 0,\dots,m$, $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 0,\dots,d_\varphi$, $\eta_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1,\dots,d_\eta$ and $u_0 \equiv 1$. Assume that data functions f_i are twice continuously differentiable. Functions φ_i and η_j are assumed to be twice differentiable.

A *trajectory* is an element $w \in \mathcal{W}$ that satisfies the state equation (1.2). If, in addition, constraints (1.3) and (1.4) hold, we say that w is a *feasible point* of the problem (1.1)-(1.4). Denote by $\mathcal{A}$ the *set of feasible points.* A *feasible variation* for $\hat{w} \in \mathcal{A}$ is an element $\delta w \in \mathcal{W}$ such that $\hat{w} + \delta w \in \mathcal{A}$.

Definition 1.2.1. *A pair* $w^0 = (x^0, u^0) \in \mathcal{W}$ *is said to be a weak minimum of problem* (1.1)-(1.4) *if there exists an* $\varepsilon > 0$ *such that the cost function attains at* w^0 *its minimum on the set*

$$\left\{ w = (x,u) \in \mathcal{A} : \|x - x^0\|_\infty < \varepsilon, \|u - u^0\|_\infty < \varepsilon \right\}.$$

We say w^0 *is a Pontryagin minimum of problem* (1.1)-(1.4) *if, for any positive* N*, there exists an* $\varepsilon_N > 0$ *such that* w^0 *is a minimum point on the set*

$$\left\{ w = (x,u) \in \mathcal{A} : \|x - x^0\|_\infty < \varepsilon_N, \|u - u^0\|_\infty \leq N, \|u - u^0\|_1 < \varepsilon_N \right\}.$$

Consider $\lambda = (\alpha, \beta, \psi) \in \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0,T;\mathbb{R}^{n,*})$, i.e. ψ is a Lipschitz-continuous function with values in the $n-$dimensional space of row-vectors with real components $\mathbb{R}^{n,*}$. Define the *pre-Hamiltonian* (or Pontryagin) function

$$H[\lambda](x,u,t) := \psi(t) \sum_{i=0}^m u_i f_i(x),$$

the *terminal Lagrangian* function

$$\ell[\lambda](q) := \sum_{i=0}^{d_\varphi} \alpha_i \varphi_i(q) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(q),$$

and the *Lagrangian* function

$$\Phi[\lambda](w) := \ell[\lambda](x(T)) + \int_0^T \psi(t) \left(\sum_{i=0}^m u_i(t) f_i(x(t)) - \dot{x}(t) \right) \mathrm{d}t. \tag{1.5}$$

In this article the optimality of a given feasible trajectory $\hat{w} = (\hat{x}, \hat{u})$ is studied. Whenever some argument of f_i, H, ℓ, Φ or their derivatives is omitted, assume that they are evaluated over this trajectory. Without loss of generality suppose that

$$\varphi_i(\hat{x}(T)) = 0, \text{ for all } i = 0, 1, \ldots, d_\varphi. \tag{1.6}$$

1.2.2 First order analysis

Definition 1.2.2. *Denote by* $\Lambda \subset \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0,T;\mathbb{R}^{n,*})$ *the set of Pontryagin multipliers associated with* $\hat{w}$ *consisting of the elements* $\lambda = (\alpha, \beta, \psi)$ *satisfying the Pontryagin Maximum Principle, i.e. having the following properties:*

$$|\alpha| + |\beta| = 1, \tag{1.7}$$

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{d_\varphi}) \geq 0, \tag{1.8}$$

function ψ is solution of the costate equation and satisfies the transversality condition at the endpoint T, i.e.

$$-\dot{\psi}(t) = H_x[\lambda](\hat{x}(t), \hat{u}(t), t), \ \psi(T) = \ell'[\lambda](\hat{x}(T)), \tag{1.9}$$

and the following minimum condition holds

$$H[\lambda](\hat{x}(t), \hat{u}(t), t) = \min_{v \geq 0} H[\lambda](\hat{x}(t), v, t), \ \text{a.e. on } [0, T]. \tag{1.10}$$

Remark 1.2.3. *For every $\lambda \in \Lambda$, the following two conditions hold.*

(i) $H_{u_i}[\lambda]$ *is continuous in time,*

(ii) $H_{u_i}[\lambda](t) \geq 0$, *a.e. on* $[0, T]$.

Recall the following well known result for which a proof can be found e.g. in Alekseev, Tikhomirov and Fomin [5], Kurcyusz and Zowe [84].

Theorem 1.2.4. *The set Λ is not empty.*

Remark 1.2.5. *Since ψ may be expressed as a linear continuous mapping of (α, β) and since* (1.7) *holds, Λ is a finite-dimensional compact set. Thus, it can be identified with a compact subset of $\mathbb{R}^s$, where $s := d_\varphi + d_\eta + 1$.*

The following expression for the derivative of the Lagrangian function holds

$$\Phi_u[\lambda](\hat{w})v = \int_0^T H_u[\lambda](\hat{x}(t), \hat{u}(t), t)v(t)\mathrm{d}t. \tag{1.11}$$

Consider $v \in \mathcal{U}$ and the *linearized state equation:*

$$\begin{cases} \dot{z}(t) = \sum_{i=0}^m \hat{u}_i(t) f_i'(\hat{x}(t))z(t) + \sum_{i=1}^m v_i(t) f_i(\hat{u}(t)), & \text{a.e. on } [0, T], \\ z(0) = 0. \end{cases} \tag{1.12}$$

Its solution z is called the *linearized state variable.*

With each index $i = 1, \dots, m$, we associate the sets

$$I_0^i := \left\{ t \in [0, T] : \max_{\lambda \in \Lambda} H_{u_i}[\lambda](t) > 0 \right\}, \quad I_+^i := [0, T] \backslash I_0^i, \tag{1.13}$$

and the *active set*

$$\tilde{I}_0^i := \{t \in [0, T] : \hat{u}_i(t) = 0\}. \tag{1.14}$$

Notice that $I_0^i \subset \tilde{I}_0^i$, and that I_0^i is relatively open in $[0, T]$ as each $H_{u_i}[\lambda]$ is continuous.

Assumption 1. Assume *strict complementarity for the control constraint,* i.e. for every $i = 1, \dots, m$,

$$I_0^i = \tilde{I}_0^i, \text{ up to a set of null measure.} \tag{1.15}$$

Observe then that for any index $i = 1, \dots, m$, the control $\hat{u}_i(t) > 0$ a.e. on I_+^i, and given $\lambda \in \Lambda$,

$$H_{u_i}[\lambda](t) = 0, \text{ a.e. on } I_+^i.$$

Assumption 2. For every $i = 1, \dots, m$, the active set I_0^i is a finite union of intervals, i.e.

$$I_0^i = \bigcup_{j=1}^{N_i} I_j^i,$$

for I_j^i subintervals of $[0, T]$ of the form $[0, d)$, $(c, T]$; or (c, d) if $c \neq 0$ and $d \neq T$. Denote by $c_1^i < d_1^i < c_2^i < \ldots < c_{N_i}^i < d_{N_i}^i$ the endpoints of these intervals. Consequently, I_+^i is a finite union of intervals as well.

Remark 1.2.6 (On the multi-dimensional control case)**.** *We would like to make a comment concerning solutions with more than one control component being singular at the same time. In [32, 33], Chitour et al. proved that generic systems with three or more control variables, or with two controls and drift did not admit singular optimal trajectories (by means of Goh's necessary condition [67]). Consequently, the study of generic properties of control-affine systems is restricted to problems having either one dimensional control or two control variables and no drift. Nevertheless, there are motivations for investigating problems with an arbitrary number of inputs that we point out next. In [85], Ledzewicz and Schättler worked on a model of cancer treatment having two control variables entering linearly in the pre-Hamiltonian and nonzero drift. They provided necessary optimality conditions for solutions with both controls being singular at the same time. Even if they were not able to give a proof of optimality they claimed to have strong expectations that this structure is part of the solution. Other examples can be found in the literature. Maurer in [92] analyzed a resource allocation problem (taken from Bryson-Ho [30]). The model had two controls and drift, and numerical computations yielded a candidate solution containing two simultaneous singular arcs. For a system with a similar structure, Gajardo et al. in [63] discussed the optimality of an extremal with two singular control components at the same time. Another motivation that we would like to point out is the technique used in Aronna et al. [10] to study the shooting algorithm for bang-singular solutions. In order to treat this kind of extremals, they perform a transformation that yields a new system and an associated totally singular solution. This new system involves as* *many control variables as singular arcs of the original solution. Hence, even a one-dimensional problem can lead to a multi-dimensional totally singular solution. These facts give a motivation for the investigation of multi-input control-affine problems.*

1.2.3 Critical cones

Let $1 \leq p \leq \infty$, and call $\mathcal{U}_p := L_p(0,T;\mathbb{R}^m)$, $\mathcal{U}_p^+ := L_p(0,T;\mathbb{R}_+^m)$ and $\mathcal{X}_p := W_p^1(0,T;\mathbb{R}^n)$. Recall that given a topological vector space E, a subset $D \subset E$ and $x \in E$, a *tangent direction* to D at x is an element $d \in E$ such that there exists sequences $(\sigma_k) \subset \mathbb{R}_+$ and $(x_k) \subset D$ with

$$\frac{x_k - x}{\sigma_k} \to d.$$

It is a well known result, see e.g. [35], that the tangent cone to $\mathcal{U}_2^+$ at $\hat{u}$ is

$$\{v \in \mathcal{U}_2 : \ v_i \geq 0 \text{ on } I_0^i, \ \text{for}\, i = 1, \ldots, m\}.$$

Given $v \in \mathcal{U}_p$ and z the solution of (1.12), consider the *linearization of the cost and final constraints*

$$\begin{cases} \varphi_i'(\hat{x}(T))z(T) \leq 0, \ i = 0, \ldots, d_\varphi, \\ \eta_j'(\hat{x}(T))z(T) = 0, \ j = 1, \ldots, d_\eta. \end{cases} \tag{1.16}$$

For $p \in \{2, \infty\}$, define the L_p-*critical cone* as

$$\mathcal{C}_p := \left\{(z,v) \in \mathcal{X}_p \times \mathcal{U}_p : v \text{ tangent to } \mathcal{U}_p^+, \ (1.12) \text{ and } (1.16) \text{ hold}\right\}.$$

Certain relations of inclusion and density between some approximate critical cones are needed. Given $\varepsilon \geq 0$ and $i = 1, \ldots, m$, define the $\varepsilon-$*active sets,* up to a set of null measure

$$I_\varepsilon^i := \{t \in (0,T) : \hat{u}_i(t) \leq \varepsilon\},$$

and the sets

$$\mathcal{W}_{p,\varepsilon} := \{(z,v) \in \mathcal{X}_p \times \mathcal{U}_p : v_i = 0 \text{ on } I_\varepsilon^i, \ (1.12) \text{ holds}\}.$$

By Assumption 1, the following explicit expression for $\mathcal{C}_2$ holds

$$\mathcal{C}_2 = \{(z,v) \in \mathcal{W}_{2,0} : (1.16) \text{ holds}\}. \tag{1.17}$$

Consider the $\varepsilon-$*critical cones*

$$\mathcal{C}_{p,\varepsilon} := \{(z,v) \in \mathcal{W}_{p,\varepsilon} : (1.16) \text{ holds}\}. \tag{1.18}$$

Let $\varepsilon > 0$. Note that by (1.17), $\mathcal{C}_{2,\varepsilon} \subset \mathcal{C}_2$. On the other hand, given $(z, v) \in \mathcal{C}_{\infty,\varepsilon}$, it easily follows that $\hat{u} + \sigma v \in \mathcal{U}^+$ for small positive σ. Thus v is tangent to $\mathcal{U}^+$ at $\hat{u}$, and this yields $\mathcal{C}_{\infty,\varepsilon} \subset \mathcal{C}_\infty$.

Recall the following technical result, see Dmitruk [45].

Lemma 1.2.7 (on density). *Consider a locally convex topological space X, a finite-faced cone $C \subset X$, and a linear manifold L dense in X. Then the cone $C \cap L$ is dense in C.*

Lemma 1.2.8. *Given $\varepsilon > 0$ the following properties hold.*

(a) $\mathcal{C}_{\infty,\varepsilon} \subset \mathcal{C}_{2,\varepsilon}$ with dense inclusion.

(b) $\bigcup_{\varepsilon>0} \mathcal{C}_{2,\varepsilon} \subset \mathcal{C}_2$ with dense inclusion.

Proof. **(a)** The inclusion is immediate. As $\mathcal{U}$ is dense in $\mathcal{U}_2$, $\mathcal{W}_{\infty,\varepsilon}$ is a dense subspace of $\mathcal{W}_{2,\varepsilon}$. By Lemma 1.2.7, $\mathcal{C}_{2,\varepsilon} \cap \mathcal{W}_{\infty,\varepsilon}$ is dense in $\mathcal{C}_{2,\varepsilon}$, as desired.
(b) The inclusion is immediate. In order to prove density, consider the following dense subspace of $\mathcal{W}_{2,0}$:

$$\mathcal{W}_{2,\bigcup} := \bigcup_{\varepsilon>0} \mathcal{W}_{2,\varepsilon},$$

and the finite-faced cone in $\mathcal{C}_2 \subset \mathcal{W}_{2,0}$. By Lemma 1.2.7, $\mathcal{C}_2 \cap \mathcal{W}_{2,\bigcup}$ is dense in $\mathcal{C}_2$, which is what we needed to prove. $\square$

1.3 Second order analysis

1.3.1 Second variation

Consider the following quadratic mapping on $\mathcal{W}$;

$$\begin{aligned}\Omega[\lambda](\delta x, \delta u) :=& \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\delta x(T))^2 \\ &+ \tfrac{1}{2}\int_0^T [(H_{xx}[\lambda]\delta x, \delta x) + 2(H_{ux}[\lambda]\delta x, \delta u)]\,\mathrm{d}t.\end{aligned}$$

The next lemma provides a second order expansion for the Lagrangian function involving operator Ω. Recall the following notation: given two functions $h : \mathbb{R}^n \to \mathbb{R}^{n_h}$ and $k : \mathbb{R}^n \to \mathbb{R}^{n_k}$, we say that h is a *big-O* of k around 0 and denote it by

$$h(x) = \mathcal{O}(k(x)),$$

if there exists positive constants δ and M such that $|h(x)| \leq M|k(x)|$ for $|x| < \delta$. It is a *small-o* if M goes to 0 as $|x|$ goes to 0. Denote this by

$$h(x) = o(k(x)).$$

Lemma 1.3.1. *Let $\delta w = (\delta x, \delta u) \in \mathcal{W}$. Then for every multiplier $\lambda \in \Lambda$, the function Φ has the following expansion (omitting time arguments):*

$$\Phi[\lambda](\hat{w}+\delta w) = \int_0^T H_u[\lambda]\delta u \mathrm{d}t + \Omega[\lambda](\delta x, \delta u) + \tfrac{1}{2}\int_0^T (H_{uxx}[\lambda]\delta x, \delta x, \delta u)\mathrm{d}t$$
$$+ \mathcal{O}(|\delta x(T)|^3) + \int_0^T |(\hat{u}+\delta u)(t)|\mathcal{O}(|\delta x(t)|^3)\,\mathrm{d}t.$$

Proof. Omit the dependence on λ for the sake of simplicity. Use the Taylor expansions

$$\ell(\hat{x}(T)+\delta x(T)) = \ell(\hat{x}(T)) + \ell'(\hat{x}(T))\delta x(T) + \tfrac{1}{2}\ell''(\hat{x}(T))(\delta x(T))^2 + \mathcal{O}(|\delta x(T)|^3),$$

$$f_i(\hat{x}(t)+\delta x(t)) = f_i(\hat{x}(t)) + f_i'(\hat{x}(t))\delta x(t) + \tfrac{1}{2}f_i''(\hat{x}(t))(\delta x(t))^2 + \mathcal{O}(|\delta x(t)|^3),$$

in the expression

$$\Phi(\hat{w}+\delta w) = \ell(\hat{x}+\delta x(T)) + \int_0^T \psi\left[\sum_{i=0}^m (\hat{u}_i+\delta u_i)f_i(\hat{x}+\delta x) - \dot{\hat{x}} - \dot{\delta x}\right]\mathrm{d}t.$$

Afterwards, use the identity

$$\int_0^T \psi\sum_{i=0}^m \hat{u}_i f_i'(\hat{x})\delta x \mathrm{d}t = -\ell'(\hat{x}(T))\delta x(T) + \int_0^T \psi\dot{\delta x}\mathrm{d}t,$$

obtained by integration by parts and equation (1.2) to get the desired result. $\square$

The previous lemma yields the following identity for every $(\delta x, \delta u) \in \mathcal{W}$:

$$\Omega[\lambda](\delta x, \delta u) = \tfrac{1}{2}D^2\Phi[\lambda](\hat{w})(\delta x, \delta u)^2.$$

1.3.2 Necessary condition

This section provides the following second order necessary condition in terms of Ω and the critical cone $\mathcal{C}_2$.

Theorem 1.3.2. *If $\hat{w}$ is a weak minimum then*

$$\max_{\lambda\in\Lambda}\Omega[\lambda](z,v) \geq 0, \quad \text{for all } (z,v)\in\mathcal{C}_2. \tag{1.19}$$

For the sake of simplicity, define $\bar{\varphi} : \mathcal{U} \to \mathbb{R}^{d_\varphi+1}$, and $\bar{\eta} : \mathcal{U} \to \mathbb{R}^{d_\eta}$ as

$$\begin{aligned}\bar{\varphi}_i(u) &:= \varphi_i(x(T)), \text{ for } i = 0,1,\dots,d_\varphi,\\ \bar{\eta}_j(u) &:= \eta_j(x(T)), \text{ for } j = 1,\dots,d_\eta,\end{aligned} \tag{1.20}$$

where x is the solution of (1.2) corresponding to u.

Definition 1.3.3. *We say that the equality constraints are nondegenerate if*

$$\bar{\eta}'(\hat{u}) \text{ is onto from } \mathcal{U} \text{ to } \mathbb{R}^{d_\eta}. \tag{1.21}$$

If (1.21) *does not hold, we call them degenerate.*

Write the problem in the following way

$$\bar{\varphi}_0(u) \to \min; \quad \bar{\varphi}_i(u) \leq 0,\ i = 1, \dots, d_\varphi,\ \bar{\eta}(u) = 0,\ u \in \mathcal{U}_+. \tag{P}$$

Suppose that $\hat{u}$ is a local weak solution of (P)**.** Next we prove Theorem 1.3.2. Its proof is divided into two cases: degenerate and nondegenerate equality constraints. For the first case the result is immediate and is tackled in the next Lemma. In order to show Theorem 1.3.2 for the latter case we introduce an auxiliary problem parameterized by certain critical directions (z, v), denoted by (QP$_v$). We prove that val(QP$_v$) ≥ 0 and, by a result on duality, the desired second order condition will be derived.

Lemma 1.3.4. *If equality constraints are degenerate, then* (1.19) *holds.*

Proof. Notice that there exists $\beta \neq 0$ such that $\sum_{j=1}^{d_\eta} \beta_j \eta_j'(\hat{x}(T)) = 0$, since $\bar{\eta}'(\hat{u})$ is not onto. Consider $\alpha = 0$ and $\psi = 0$. Take $\lambda := (\alpha, \beta, \psi)$ and notice that both λ and $-\lambda$ are in Λ. Observe that

$$\Omega[\lambda](z, v) = \tfrac{1}{2} \sum_{j=1}^{d_\eta} \beta_j \eta_j''(\hat{x}(T))(z(T))^2.$$

Thus $\Omega[\lambda](z, v) \geq 0$ either for λ or $-\lambda$. The required result follows. □

Take $\varepsilon > 0$, $(z, v) \in \mathcal{C}_{\infty,\varepsilon}$, and rewrite (1.18) using the notation in (1.20),

$$\begin{aligned} \mathcal{C}_{\infty,\varepsilon} = \{(z, v) \in \mathcal{X} \times \mathcal{U} :\ & v_i(t) = 0 \text{ on } I_\varepsilon^i,\quad i = 1, \dots, m, \\ & (1.12) \text{ holds},\ \bar{\varphi}_i'(\hat{u})v \leq 0,\ i = 0, \dots, d_\varphi,\ \bar{\eta}'(\hat{u})v = 0\}. \end{aligned}$$

Consider the problem

$$\begin{aligned} & \delta\zeta \to \min \\ & \bar{\varphi}_i'(\hat{u})r + \bar{\varphi}_i''(\hat{u})(v, v) \leq \delta\zeta, \text{ for } i = 0, \dots, d_\varphi, \\ & \bar{\eta}'(\hat{u})r + \bar{\eta}''(\hat{u})(v, v) = 0, \\ & -r_i(t) \leq \delta\zeta,\ \text{ on } I_0^i,\ \text{ for } i = 1, \dots, m. \end{aligned} \tag{QP$_v$}$$

Proposition 1.3.5. *Let $(z, v) \in \mathcal{C}_{\infty,\varepsilon}$. If the equality constraints are nondegenerate, problem* (QP$_v$) *is feasible and* val (QP$_v$) ≥ 0.

Proof. Let us first prove feasibility. As $\bar{\eta}'(\hat{u})$ is onto, there exists $r \in \mathcal{U}$ such that the equality constraint in (QP_v) is satisfied. Take

$$\delta\zeta := \max(\|r\|_\infty, \bar{\varphi}_i'(\hat{u})r + \bar{\varphi}''(\hat{u})(v,v)).$$

Thus the pair $(r, \delta\zeta)$ is feasible for (QP_v).

Let us now prove that $\mathrm{val}\,(\mathrm{QP}_v) \geq 0$. On the contrary suppose that there exists a feasible solution $(r, \delta\zeta)$ with $\delta\zeta < 0$. The last constraint in (QP_v) implies $\|r\|_\infty \neq 0$. Set, for $\sigma > 0$,

$$\tilde{u}(\sigma) := \hat{u} + \sigma v + \tfrac{1}{2}\sigma^2 r, \quad \tilde{\zeta}(\sigma) := \tfrac{1}{2}\sigma^2 \delta\zeta. \tag{1.22}$$

The goal is finding $u(\sigma)$ feasible for (P) such that for small σ,

$$u(\sigma) \xrightarrow{\mathcal{U}} \hat{u}, \text{ and } \bar{\varphi}_0(u(\sigma)) < \bar{\varphi}_0(\hat{u}),$$

contradicting the weak optimality of $\hat{u}$.

Notice that $\hat{u}_i(t) > \varepsilon$ a.e. on $[0,T]\backslash I_\varepsilon^i$, and then $\tilde{u}(\sigma)_i(t) > -\tilde{\zeta}(\sigma)$ for sufficiently small σ. On I_ε^i, if $\tilde{u}(\sigma)_i(t) < -\tilde{\zeta}(\sigma)$ then necessarily

$$\hat{u}_i(t) < \tfrac{1}{2}\sigma^2(\|r\|_\infty + |\delta\zeta|),$$

as $v_i(t) = 0$. Thus, defining the set

$$J_\sigma^i := \{t : 0 < \hat{u}_i(t) < \tfrac{1}{2}\sigma^2(\|r\|_\infty + |\delta\zeta|)\},$$

we get $\{t \in [0,T] : \tilde{u}(\sigma)_i(t) < -\tilde{\zeta}(\sigma)\} \subset J_\sigma^i$. Observe that on J_σ^i, the function $|\tilde{u}(\sigma)_i(t) + \tilde{\zeta}(\sigma)|/\sigma^2$ is dominated by $\|r\|_\infty + |\delta\zeta|$. Since $\mathrm{meas}(J_\sigma^i)$ goes to 0 by the Dominated Convergence Theorem, we obtain

$$\int_{J_\sigma^i} |\tilde{u}(\sigma)_i(t) + \tilde{\zeta}(\sigma)|\mathrm{d}t = o(\sigma^2).$$

Take

$$\tilde{\tilde{u}}(\sigma) := \begin{cases} \tilde{u}(\sigma) & \text{on } [0,T]\backslash J_\sigma^i, \\ -\tilde{\zeta}(\sigma) & \text{on } J_\sigma^i. \end{cases}$$

Thus, $\tilde{\tilde{u}}$ satisfies

$$\tilde{\tilde{u}}(\sigma)(t) \geq -\tilde{\zeta}(\sigma), \quad \text{a.e. on } [0,T], \tag{1.23}$$

$$\|\tilde{\tilde{u}}(\sigma) - \hat{u}\|_1 = o(\sigma^2), \quad \|\tilde{\tilde{u}}(\sigma) - \hat{u}\|_\infty = O(\sigma^2),$$

and the following estimates hold

$$\begin{aligned} \bar{\varphi}_i(\tilde{\tilde{u}}(\sigma)) &= \bar{\varphi}_i(\hat{u}) + \sigma\bar{\varphi}_i'(\hat{u})v + \tfrac{1}{2}\sigma^2[\bar{\varphi}_i'(\hat{u})r + \bar{\varphi}_i''(\hat{u})(v,v)] + o(\sigma^2) \\ &< \bar{\varphi}_i(\hat{u}) + \tilde{\zeta}(\sigma) + o(\sigma^2), \end{aligned} \tag{1.24}$$

$$\bar{\eta}(\tilde{\tilde{u}}(\sigma)) = \sigma\bar{\eta}'(\hat{u})v + \tfrac{1}{2}\sigma^2[\bar{\eta}'(\hat{u})r + \bar{\eta}''(\hat{u})(v,v)] + o(\sigma^2) = o(\sigma^2).$$

As $\bar{\eta}'(\hat{u})$ is onto on $\mathcal{U}$ we can find a corrected control $u(\sigma)$ satisfying the equality constraint and such that $\|u(\sigma) - \tilde{\tilde{u}}(\sigma)\|_\infty = o(\sigma^2)$. Deduce by (1.23) that $u(\sigma) \geq 0$ a.e. on $[0,T]$, and by (1.24) that it satisfies the terminal inequality constraints. Thus $u(\sigma)$ is feasible for (P) and it satisfies (1.22). This contradicts the weak optimality of $\hat{u}$. □

Recall that a *Lagrange multiplier* associated with $\hat{w}$ is a pair (λ,μ) in $\mathbb{R}^{d_\varphi+1} \times \mathbb{R}^{d_\eta} \times W^1_\infty(0,T;\mathbb{R}^{n,*}) \times \mathcal{U}^*$ with $\lambda = (\alpha,\beta,\psi)$ satisfying (1.7), (1.8), $\mu \geq 0$ and the *stationarity condition*

$$\int_0^T H_u[\lambda](t)v(t)\mathrm{d}t + \int_0^T v(t)\mathrm{d}\mu(t) = 0, \quad \text{for every } v \in \mathcal{U}.$$

Here $\mathcal{U}^*$ denotes the dual space of $\mathcal{U}$. Simple computations show that (λ,μ) is a Lagrange multiplier if and only if λ is a Pontryagin multiplier and $\mu = H_u[\lambda]$. Thus $\mu \in L_\infty(0,T;\mathbb{R}^{m,*})$.

Let us come back to Theorem 1.3.2.

Proof. [of Theorem 1.3.2] Lemma 1.3.4 covers the degenerate case. Assume thus that $\bar{\eta}'(\hat{u})$ is onto. Take $\varepsilon > 0$ and $(z,v) \in \mathcal{C}_{\infty,\varepsilon}$. Applying Proposition 1.3.5, we see that there cannot exist r and $\delta\zeta < 0$ such that

$$\begin{aligned}
&\bar{\varphi}_i'(\hat{u})r + \bar{\varphi}_i''(\hat{u})(v,v) \leq \delta\zeta,\ i = 0,\dots,d_\varphi,\\
&\bar{\eta}'(\hat{u})r + \bar{\eta}''(\hat{u})(v,v) = 0,\\
&-r_i(t) \leq \delta\zeta,\ \text{on } I_0^i,\ \text{for } i = 1,\dots,m.
\end{aligned}$$

By the Dubovitskii-Milyutin Theorem (see [49]) we obtain the existence of $(\alpha,\beta) \in \mathbb{R}^s$ and $\mu \in \mathcal{U}^*$ with supp $\mu_i \subset I_0^i$, and $(\alpha,\beta,\mu) \neq 0$ such that

$$\sum_{i=0}^{d_\varphi} \alpha_i\bar{\varphi}_i'(\hat{u}) + \sum_{i=1}^{d_\eta} \beta_j\bar{\eta}_j'(\hat{u}) - \mu = 0, \tag{1.25}$$

and denoting $\lambda := (\alpha,\beta,\psi)$, with ψ being solution of (1.9), the following holds:

$$\sum_{i=0}^{d_\varphi} \alpha_i\bar{\varphi}_i''(\hat{u})(v,v) + \sum_{i=1}^{d_\eta} \beta_j\bar{\eta}_j''(\hat{u})(v,v) \geq 0.$$

By Lemma 1.8.2 we obtain

$$\Omega[\lambda](z,v) \geq 0. \tag{1.26}$$

Observe that (1.25) implies that $\lambda \in \Lambda$. Consider now $(\bar{z}, \bar{v}) \in \mathcal{C}_2$, and note that Lemma 1.2.8 guarantees the existence of a sequence $\{(z_\varepsilon, v_\varepsilon)\} \subset \mathcal{C}_{\infty,\varepsilon}$ converging to $(\bar{z}, \bar{v})$ in $\mathcal{X}_2 \times \mathcal{U}_2$. Recall Remark 1.2.5. Let $\lambda_\varepsilon \in \Lambda$ be such that (1.26) holds for $(\lambda_\varepsilon, z_\varepsilon, v_\varepsilon)$. Since (λ_ε) is bounded, it contains a limit point $\bar{\lambda} \in \Lambda$. Thus (1.26) holds for $(\bar{\lambda}, \bar{z}, \bar{v})$, as required. $\square$

1.4 Goh Transformation

Consider an arbitrary linear system:

$$\begin{cases} \dot{z}(t) = A(t)z(t) + B(t)v(t), & \text{a.e. on } [0,T], \\ z(0) = 0, \end{cases} \tag{1.27}$$

where $A(t) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ is an essentially bounded function of t, and $B(t) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ is a Lipschitz-continuous function of t. With each $v \in \mathcal{U}$ associate the state variable $z \in \mathcal{X}$ solution of (1.12). Let us present a transformation of the variables $(z, v) \in \mathcal{W}$, first introduced by Goh in [68]. Define two new state variables as follows:

$$\begin{cases} y(t) := \displaystyle\int_0^t v(s)\mathrm{d}s, \\ \xi(t) := z(t) - B(t)y(t). \end{cases} \tag{1.28}$$

Thus $y \in \mathcal{Y} := W^1_\infty(0, T; \mathbb{R}^m)$, $y(0) = 0$ and ξ is an element of space $\mathcal{X}$. It easily follows that ξ is a solution of the linear differential equation

$$\dot{\xi}(t) = A(t)\xi(t) + B_1(t)y(t), \quad \xi(0) = 0, \tag{1.29}$$

where

$$B_1(t) := A(t)B(t) - \dot{B}(t). \tag{1.30}$$

For the purposes of this article take

$$A(t) := \sum_{i=0}^m \hat{u}_i f_i'(\hat{x}(t)), \text{ and } B(t)v(t) := \sum_{i=1}^m v_i(t) f_i(\hat{u}(t)). \tag{1.31}$$

Then (1.27) coincides with the linearized equation (1.12).

1.4.1 Transformed critical directions

As optimality conditions on the variables obtained by the Goh Transformation will be derived, a new set of critical directions is needed. Take a point (z, v) in $\mathcal{C}_\infty$, and define ξ and y by the transformation (1.28). Let $h := y(T)$ and notice that since (1.16) is satisfied, the following inequalities hold,

$$\begin{aligned} &\varphi_i'(\hat{x}(T))(\xi(T)+B(T)h) \leq 0, \text{ for } i=0,\dots,d_\varphi,\\ &\eta_j'(\hat{x}(T))(\xi(T)+B(T)h) = 0, \text{ for } j=1,\dots,d_\eta. \end{aligned} \tag{1.32}$$

Define the set of *transformed critical directions*

$$\mathcal{P} := \left\{ \begin{array}{l} (\xi, y, h) \in \mathcal{X}\times\mathcal{Y}\times\mathbb{R}^m : \dot{y}_i = 0 \text{ over } I_0^i,\ y(0)=0,\ h := y(T),\\ (1.29) \text{ and } (1.32) \text{ hold} \end{array} \right\}.$$

Observe that for every $(\xi, y, h) \in \mathcal{P}$ and $1 \leq i \leq m$,

$$y_i \text{ is constant over each connected component of } I_0^i, \tag{1.33}$$

and at the endpoints the following conditions hold

$$\begin{aligned} &y_i = 0 \text{ on } [0, d_1^i),\ \text{ if } 0 \in I_0^i,\ \text{ and}\\ &y_i = h_i \text{ on } (c_{N_i}^i, T],\ \text{ if } T \in I_0^i, \end{aligned} \tag{1.34}$$

where c_1^i and d_1^i were introduced in Assumption 2. Define the set

$$\mathcal{P}_2 := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m : (1.29),\ (1.32),\ (1.33) \text{ and } (1.34) \text{ hold}\}.$$

Lemma 1.4.1. *$\mathcal{P}$ is a dense subset of $\mathcal{P}_2$ in the $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m-$topology.*

Proof. The inclusion is immediate. In order to prove the density, consider the following sets.

$$X := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m : (1.29),\ (1.33) \text{ and } (1.34) \text{ hold}\},$$

$$L := \{(\xi, y, y(T)) \in \mathcal{X}\times\mathcal{Y}\times\mathbb{R}^m :\ y(0)=0,\ (1.29) \text{ and } (1.33) \text{ hold}\},$$

$$C := \{(\xi, y, h) \in X :\ (1.32) \text{ holds}\}.$$

By Lemma 1.8.1, L is a dense subset of X. The conclusion follows with Lemma 1.2.7. $\square$

1.4.2 Transformed second variation

We are interested in writing Ω in terms of variables y and ξ defined in (1.28). Introduce the following notation for the sake of simplifying the presentation.

Definition 1.4.2. *Consider the following matrices of sizes $n \times n, m \times n$ and $m \times n$, respectively.*

$$Q[\lambda] := H_{xx}[\lambda], \quad C[\lambda] := H_{ux}[\lambda], \quad M[\lambda] := B^\top Q[\lambda] - \dot{C}[\lambda] - C[\lambda]A, \tag{1.35}$$

where A and B were defined in (1.31). *Notice that M is well-defined as C is Lipschitz-continuous on t. Decompose matrix $C[\lambda]B$ into its symmetric and skew-symmetric parts, i.e. consider*

$$S[\lambda] := \tfrac{1}{2}(C[\lambda]B + (C[\lambda]B)^\top), \quad V[\lambda] := \tfrac{1}{2}(C[\lambda]B - (C[\lambda]B)^\top). \tag{1.36}$$

Remark 1.4.3. *Observe that, since $C[\lambda]$ and B are Lipschitz-continuous, $S[\lambda]$ and $V[\lambda]$ are Lipschitz-continuous as well. In fact, simple computations yield*

$$S_{ij}[\lambda] = \tfrac{1}{2}\psi(f_i' f_j + f_j' f_i), \quad V_{ij}[\lambda] = \tfrac{1}{2}\psi[f_i, f_j], \quad \text{for } i,j = 1,\dots,m, \tag{1.37}$$

where

$$[f_i, f_j] := f_i' f_j - f_j' f_i. \tag{1.38}$$

With this notation, Ω takes the form

$$\Omega[\lambda](\delta x, v) = \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\delta x(T))^2 + \tfrac{1}{2}\int_0^T [(Q[\lambda]\delta x, \delta x) + 2(C[\lambda]\delta x, v)]\mathrm{d}t.$$

Define the $m \times m$ matrix

$$R[\lambda] := B^\top Q[\lambda]B - C[\lambda]B_1 - (C[\lambda]B_1)^\top - \dot{S}[\lambda], \tag{1.39}$$

where B_1 was introduced in equation (1.30). Consider the function $g[\lambda]$ from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$ defined by:

$$g[\lambda](\zeta, h) := \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\zeta + B(T)h)^2 + \tfrac{1}{2}(C[\lambda](T)(2\zeta + B(T)h), h). \tag{1.40}$$

Remark 1.4.4. *(i) We use the same notation for the matrices $Q[\lambda]$, $C[\lambda]$, $M[\lambda]$, $\ell''[\lambda](\hat{x}(T))$ and for the bilinear mapping they define.*

(ii) Observe that when $m = 1$, the function $V[\lambda] \equiv 0$ since it becomes a skew-symmetric scalar.

Definition 1.4.5. *Define the mapping over $\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ given by*

$$\begin{aligned}\Omega_{\mathcal{P}}[\lambda](\xi, y, v) &:= g[\lambda](\xi(T), y(T)) \\ &\quad + \int_0^T \{\tfrac{1}{2}(Q[\lambda]\xi, \xi) + 2(M[\lambda]\xi, y) + \tfrac{1}{2}(R[\lambda]y, y) + (V[\lambda]y, v)\}\mathrm{d}t,\end{aligned} \tag{1.41}$$

with $g[\lambda]$, $Q[\lambda]$, $M[\lambda]$, $R[\lambda]$ and $V[\lambda]$ defined in (1.35)-(1.40).

The following theorem shows that $\Omega_{\mathcal{P}}$ coincides with Ω. See e.g. [43].

Theorem 1.4.6. *Let* $(z,v) \in \mathcal{W}$ *satisfying* (1.12) *and* (ξ, y) *be defined by* (1.28). *Then*

$$\Omega[\lambda](z,v) = \Omega_{\mathcal{P}}[\lambda](\xi,y,v).$$

Proof. We omit the dependence on λ for the sake of simplicity. Replace z by its expression in (1.28) and obtain

$$\begin{aligned}\Omega(z,v) &= \tfrac{1}{2}\ell''(\hat{x}(T))(\xi(T)+B(T)y(T))^2 \\ &+ \tfrac{1}{2}\int_0^T [(Q(\xi+By),\xi+By) + (C(\xi+By),v) + (C^\top v, \xi+By)]\mathrm{d}t.\end{aligned} \tag{1.42}$$

Integrating by parts yields

$$\int_0^T (C\xi,v)\mathrm{d}t = [(C\xi,y)]_0^T - \int_0^T (\dot{C}\xi + C(A\xi+B_1y),y)\mathrm{d}t, \tag{1.43}$$

and

$$\begin{aligned}\int_0^T (CBy,v)\mathrm{d}t &= \int_0^T ((S+V)y,v)\mathrm{d}t \\ &= \tfrac{1}{2}[(Sy,y)]_0^T + \int_0^T (-\tfrac{1}{2}(\dot{S}y,y) + (Vy,v))\mathrm{d}t.\end{aligned} \tag{1.44}$$

Combining (1.42), (1.43) and (1.44) we get the desired result. □

Corollary 1.4.7. *If* $V[\lambda] \equiv 0$ *then* Ω *does not involve* v *explicitly, and it can be expressed in terms of* $(\xi, y, y(T))$.

In view of (1.37), the previous corollary holds in particular if $[f_i, f_j] = 0$ on the reference trajectory for each pair $1 \leq i < j \leq m$.

Corollary 1.4.8. *If* $\hat{w}$ *is a weak minimum, then*

$$\max_{\lambda\in\Lambda} \Omega_{\mathcal{P}}[\lambda](\xi,y,v) \geq 0,$$

for every $(z,v) \in \mathcal{C}_2$ *and* (ξ,y) *defined by* (1.28).

1.4.3 New second order condition

In this section we present a necessary condition involving the variable (ξ, y, h) in $\mathcal{P}_2$. To achieve this we remove the explicit dependence on v from the second variation, for certain subset of multipliers. Recall that we consider $\lambda = (\alpha, \beta)$ as elements of $\mathbb{R}^s$.

Definition 1.4.9. *Given $M \subset \mathbb{R}^s$, define*

$$G(M) := \{\lambda \in M : V_{ij}[\lambda](t) = 0 \text{ on } I_+^i \cap I_+^j, \text{ for any pair } 1 \le i < j \le m\}.$$

Theorem 1.4.10. *Let $M \subset \mathbb{R}^s$ be convex and compact, and assume that*

$$\max_{\lambda \in M} \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) \ge 0, \quad \text{for all } \ (\xi, y, h) \in \mathcal{P}. \tag{1.45}$$

Then

$$\max_{\lambda \in G(M)} \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) \ge 0, \quad \text{for all } \ (\xi, y, h) \in \mathcal{P}.$$

The proof is based on some techniques introduced in Dmitruk [40, 43] for the proof of similar theorems.

Let $1 \le i < j \le m$ and $t^* \in \operatorname{int} I_+^i \cap I_+^j$. Take $y \in \mathcal{Y}$ satisfying

$$y(0) = y(T) = 0, \quad y_k = 0, \text{ for } \ k \ne i, k \ne j. \tag{1.46}$$

Such functions define a linear continuous mapping $r : \mathbb{R}^{s,*} \to \mathbb{R}$ by

$$\lambda \mapsto r[\lambda] := \int_0^T (V[\lambda](t^*)y, \dot{y})\mathrm{d}t. \tag{1.47}$$

By condition (1.46), and since $V[\lambda]$ is skew-symmetric,

$$\int_0^T (V[\lambda](t^*)y, \dot{y})\mathrm{d}t = V_{ij}[\lambda](t^*) \int_0^T (y_i\dot{y}_j - y_j\dot{y}_i)\mathrm{d}t.$$

Each r is an element of the dual space of $\mathbb{R}^{s,*}$, and it can thus be identified with an element of $\mathbb{R}^s$. Consequently, the subset of $\mathbb{R}^s$ defined by

$$R_{ij}(t^*) := \{r \in \mathbb{R}^s : y \in \mathcal{Y} \text{ satisfies (1.46)}, \ r \text{ is defined by (1.47)}\},$$

is a linear subspace of $\mathbb{R}^s$. Now, consider all the finite collections

$$\Theta_{ij} := \left\{\theta = \{t^1 < \cdots < t^{N_\theta}\} : t^k \in \operatorname{int} I_+^i \cap I_+^j \text{ for } k = 1, \dots, N_\theta\right\}.$$

Define

$$\mathcal{R} := \sum_{i<j} \bigcup_{\theta \in \Theta_{ij}} \sum_{k=1}^{N_\theta} R_{ij}(t^k).$$

Note that $\mathcal{R}$ is a linear subspace of $\mathbb{R}^s$. Given $(\xi, y, y(T)) \in \mathcal{P}$, let the mapping $p_y : \mathbb{R}^{s,*} \to \mathbb{R}$ be given by

$$\lambda \mapsto p_y[\lambda] := \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}). \tag{1.48}$$

Thus, p_y is an element of $\mathbb{R}^s$.

Lemma 1.4.11. *Let* $(\bar{\xi}, \bar{y}, \bar{y}(T)) \in \mathcal{P}$ *and* $r \in \mathcal{R}$. *Then there exists a sequence* $\{(\xi^\nu, y^\nu, y^\nu(T))\}$ *in* $\mathcal{P}$ *such that*

$$\Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) \longrightarrow p_{\bar{y}}[\lambda] + r[\lambda]. \tag{1.49}$$

Proof. Take $(\bar{\xi}, \bar{y}, \bar{y}(T)) \in \mathcal{P}$, its corresponding critical direction $(\bar{z}, \bar{v}) \in \mathcal{C}$ related via (1.28) and $p_{\bar{y}}$ defined in (1.48). Assume that $r \in R_{ij}(t^*)$ for some $1 \leq i < j \leq m$ and $t^* \in \text{int } I_+^i \cap I_+^j$, i.e. r is associated via (1.47) to some function $\tilde{y}$ verifying (1.46). Take $\tilde{y}(t) = 0$ when $t \notin [0, T]$. Consider

$$\tilde{y}^\nu(t) := \tilde{y}(\nu(t - t^*)), \quad \breve{y}^\nu := \bar{y} + \tilde{y}^\nu. \tag{1.50}$$

Let $\breve{\xi}^\nu$ be the solution of (1.29) corresponding to $\breve{y}^\nu$. Observe that for large enough ν, as $t^* \in \text{int } I_+^i \cap I_+^j$,

$$\dot{\breve{y}}_k^\nu = 0, \text{ a.e. on } I_0^k, \text{ for } k = 1, \dots, m. \tag{1.51}$$

Let $(\tilde{z}^\nu, \tilde{v}^\nu)$ and $(\breve{z}^\nu, \breve{v}^\nu)$ be the points associated by transformation (1.28) with $(\tilde{\xi}^\nu, \tilde{y}^\nu, \tilde{y}^\nu(T))$ and $(\breve{\xi}^\nu, \breve{y}^\nu, \breve{y}^\nu(T))$, respectively. By (1.51), we get

$$\breve{v}_k^\nu = 0, \text{ a.e. on } I_0^k, \text{ for } k = 1, \dots, m.$$

Note, however, that $(\breve{z}^\nu, \breve{v}^\nu)$ can violate the terminal constraints defining $\mathcal{C}_\infty$, i.e. the constraints defined in (1.16). Let us look for an estimate of the magnitude of this violation. Since

$$\|\tilde{y}^\nu\|_1 = \mathcal{O}(1/\nu), \tag{1.52}$$

and $(\tilde{\xi}^\nu, \tilde{y}^\nu)$ is solution of (1.29), Gronwall's Lemma implies

$$|\tilde{\xi}^\nu(T)| = \mathcal{O}(1/\nu).$$

On the other hand, notice that $\breve{z}^\nu(T) = \bar{z}(T) + \tilde{\xi}^\nu(T)$, and thus

$$|\breve{z}^\nu(T) - \bar{z}(T)| = \mathcal{O}(1/\nu).$$

By Hoffman's Lemma (see [73]), there exists $(\Delta z^\nu, \Delta v^\nu) \in \mathcal{W}$ satisfying $\|\Delta v^\nu\|_\infty + \|\Delta z^\nu\|_\infty = \mathcal{O}(1/\nu)$, and such that $(z^\nu, v^\nu) := (\breve{z}^\nu, \breve{v}^\nu) + (\Delta z^\nu, \Delta v^\nu)$ belongs to $\mathcal{C}_\infty$. Let $(\xi^\nu, y^\nu, y^\nu(T)) \in \mathcal{P}$ be defined by (1.28). Let us show that for each $\lambda \in M$,

$$\lim_{\nu \to \infty} \Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) = p_{\bar{y}}[\lambda] + r[\lambda].$$

Observe that

$$\lim_{\nu\to\infty} \Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) - p_{\bar{y}}[\lambda] = \lim_{\nu\to\infty} \int_0^T \{(V[\lambda]\bar{y}, \dot{\tilde{y}}^\nu) + (V[\lambda]\tilde{y}^\nu, \dot{\tilde{y}}^\nu)\}\mathrm{d}t, \tag{1.53}$$

since the terms involving $\xi^\nu - \bar{\xi}$, $y^\nu - \bar{y}$ or Δv^ν vanish as $\|\xi^\nu - \bar{\xi}\|_\infty \to 0$ and $\|y^\nu - \bar{y}\|_1 \to 0$. Integrating by parts the first term in the right hand-side of (1.53) we obtain

$$\int_0^T (V[\lambda]\bar{y}, \dot{\tilde{y}}^\nu)\mathrm{d}t = [(V[\lambda]\bar{y}, \tilde{y}^\nu)]_0^T - \int_0^T \{(\dot{V}[\lambda]\bar{y}, \tilde{y}^\nu) + (V[\lambda]\dot{\bar{y}}, \tilde{y}^\nu)\}\mathrm{d}t \overset{\nu\to\infty}{\to} 0,$$

by (1.52) and since $\tilde{y}^\nu(0) = \tilde{y}^\nu(T) = 0$. Coming back to (1.53) we have

$$\begin{aligned}
\lim_{\nu\to\infty} \Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) - p_{\bar{y}}[\lambda] &= \lim_{\nu\to\infty} \int_0^T (V[\lambda]\tilde{y}^\nu, \dot{\tilde{y}}^\nu)\mathrm{d}t \\
&= \lim_{\nu\to\infty} \int_0^T (V[\lambda](t)\tilde{y}(\nu(t-t^*)), \dot{\tilde{y}}(\nu(t-t^*)))\mathrm{d}\nu t \\
&= \lim_{\nu\to\infty} \int_{-\nu t^*}^{\nu(T-t^*)} (V[\lambda](t^* + s/\nu)\tilde{y}(s), \dot{\tilde{y}}(s))\mathrm{d}s = r[\lambda],
\end{aligned}$$

and thus (1.49) holds when $r \in R_{ij}(t^*)$.

Consider the general case when $r \in \mathcal{R}$, i.e. $r = \sum_{i<j}\sum_{k=1}^{N_{ij}} r_{ij}^k$, with each r_{ij}^k in $R_{ij}(t_{ij}^k)$. Let $\tilde{y}_{ij}^k$ be associated with r_{ij}^k by (1.47). Define $\tilde{y}_{ij}^{k,\nu}$ as in (1.50), and follow the previous procedure for $\bar{y} + \sum_{i<j}\sum_{k=1}^{N_{ij}} \tilde{y}_{ij}^{k,\nu}$ to get the desired result.

□

Proof. **[of Theorem 1.4.10]** Take $(\bar{\xi}, \bar{y}, \bar{y}(T)) \in \mathcal{P}$ and $r \in \mathcal{R}$. By Lemma 1.4.11 there exists a sequence $\{(\xi^\nu, y^\nu, y^\nu(T))\}$ in $\mathcal{P}$ such that for each $\lambda \in M$,

$$\Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) \to \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda].$$

Since this convergence is uniform over M, from (1.45) we get that

$$\max_{\lambda\in M}(\Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda]) \geq 0, \quad \text{for all } r \in \mathcal{R}.$$

Hence

$$\inf_{r\in\mathcal{R}}\max_{\lambda\in M}(\Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda]) \geq 0, \tag{1.54}$$

where the expression in brackets is linear both in λ and r. Furthermore, note that M and $\mathcal{R}$ are convex, and M is compact. In light of MinMax Theorem [113, Corollary 37.3.2, page 39] we can invert the order of inf and max in (1.54) and obtain

$$\max_{\lambda\in M}\inf_{r\in\mathcal{R}}(\Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{y}, \dot{\bar{y}}) + r[\lambda]) \geq 0. \tag{1.55}$$

Suppose that, for certain $\lambda \in M$, there exists $r \in \mathcal{R}$ with $r[\lambda] \neq 0$. Then the infimum in (1.55) is $-\infty$ since $\mathcal{R}$ is a linear subspace. Hence, this λ does not provide the maximal value of the infima, and so, we can restrict the maximization to the set of $\lambda \in M$ for which $r[\lambda] = 0$ for every $r \in \mathcal{R}$. Note that this set is $G(M)$, and thus the conclusion follows. $\square$

Consider for $i, j = 1, \dots, m$:

$$I_{ij} := \{t \in (0,T) : \hat{u}_i(t) = 0,\ \hat{u}_j(t) > 0\}.$$

By Assumption 2, I_{ij} can be expressed as a finite union of intervals, i.e.

$$I_{ij} = \bigcup_{k=1}^{K_{ij}} I_{ij}^k, \text{ where } I_{ij}^k := (c_{ij}^k, d_{ij}^k).$$

Let $(z, v) \in \mathcal{C}_\infty$, $i \neq j$, and y be defined by (1.28). Notice that y_i is constant on each (c_{ij}^k, d_{ij}^k). Denote with $y_{i,j}^k$ its value on this interval.

Proposition 1.4.12. *Let* $(z, v) \in \mathcal{C}_\infty$, y *be defined by* (1.28) *and* $\lambda \in G(\Lambda)$. *Then*

$$\int_0^T (V[\lambda]y, v)\mathrm{d}t = \sum_{\substack{i \neq j \\ i,j=1}}^{m} \sum_{k=1}^{K_{ij}} y_{i,j}^k \left\{ [V_{ij}[\lambda]y_j]_{c_{ij}^k}^{d_{ij}^k} - \int_{c_{ij}^k}^{d_{ij}^k} \dot{V}_{ij}[\lambda]y_j \mathrm{d}t \right\}.$$

Proof. Observe that

$$\int_0^T (V[\lambda]y, v)\mathrm{d}t = \sum_{\substack{i \neq j \\ i,j=1}}^{m} \int_0^T V_{ij}[\lambda]y_i v_j \mathrm{d}t, \tag{1.56}$$

since $V_{ii}[\lambda] \equiv 0$. Fix $i \neq j$, and recall that that $V_{ij}[\lambda]$ is differentiable in time (see expression (1.37)). Since $(z, v) \in \mathcal{C}_\infty$ and $\lambda \in G(\Lambda)$,

$$\begin{aligned} \int_0^T V_{ij}[\lambda]y_i v_j \mathrm{d}t &= \int_{I_{ij}} V_{ij}[\lambda]y_i v_j \mathrm{d}t = \sum_{k=1}^{K_{ij}} \int_{c_{ij}^k}^{d_{ij}^k} V_{ij}[\lambda]y_i v_j \mathrm{d}t \\ &= \sum_{k=1}^{K_{ij}} y_{i,j}^k \left\{ [V_{ij}[\lambda]y_j]_{c_{ij}^k}^{d_{ij}^k} - \int_{c_{ij}^k}^{d_{ij}^k} \dot{V}_{ij}[\lambda]y_j \mathrm{d}t \right\}, \end{aligned} \tag{1.57}$$

where the last equality was obtained by integrating by parts and knowing that y_i is constant on I_{ij}. The desired result follows from (1.56) and (1.57). $\square$

Given a real function h and $c \in \mathbb{R}$, define

$$h(c+) := \lim_{t \to c+} h(t), \text{ and } h(c-) := \lim_{t \to c-} h(t).$$

Definition 1.4.13. *Let $(\xi, y, h) \in \mathcal{P}_2$ and $\lambda \in G(\Lambda)$. Define*

$$\Xi[\lambda](\xi, y, h) :=$$
$$2\sum_{\substack{i\neq j\\ i,j=1}}^{m}\sum_{\substack{k=1\\ c_{ij}^k\neq 0}}^{K_{ij}} y_{i,j}^k\left\{V_{ij}[\lambda](d_{ij}^k)y_j(d_{ij}^k+) - V_{ij}[\lambda](c_{ij}^k)y_j(c_{ij}^k-) - \int_{c_{ij}^k}^{d_{ij}^k}\dot{V}_{ij}[\lambda]y_j\mathrm{d}t\right\},$$

where the above expression is interpreted as follows:

(i) $y_j(d_{ij}^k+) := h_j$, *if* $d_{ij}^k = T$,

(ii) $V_{ij}[\lambda](c_{ij}^k)y_j(c_{ij}^k-) := 0$, *if* $\hat{u}_i > 0$ *and* $\hat{u}_j > 0$ *for* $t < c_{ij}^k$,

(iii) $V_{ij}[\lambda](d_{ij}^k)y_j(d_{ij}^k+) := 0$, *if* $\hat{u}_i > 0$ *and* $\hat{u}_j > 0$ *for* $t > d_{ij}^k$.

Proposition 1.4.14. *The following properties for Ξ hold.*

(i) $\Xi[\lambda](\xi, y, h)$ *is well-defined for each* $(\xi, y, h) \in \mathcal{P}_2$, *and* $\lambda \in G(\Lambda)$.

(ii) If $\{(\xi^\nu, y^\nu, y^\nu(T))\} \subset \mathcal{P}$ *converges in the* $\mathcal{X}_2\times\mathcal{U}_2\times\mathbb{R}^m-$ *topology to* $(\xi, y, h) \in \mathcal{P}_2$ *, then*
$$\int_0^T (V[\lambda]y^\nu, \dot{y}^\nu)\mathrm{d}t \overset{\nu\to\infty}{\longrightarrow} \Xi[\lambda](\xi, y, h).$$

Proof. **(i)** Take $(\xi, y, h) \in \mathcal{P}_2$. First observe that $y_i \equiv y_{i,j}^k$ over (c_{ij}^k, d_{ij}^k). As $c_{ij}^k \neq 0$, two possible situations can arise,

(a) for $t < c_{ij}^k : \hat{u}_j = 0$, thus y_j is constant, and consequently $y_j(c_{ij}^k-)$ is well-defined,

(b) for $t < c_{ij}^k : \hat{u}_i > 0$ and $\hat{u}_j > 0$, thus $V_{ij}[\lambda](c_{ij}^k) = 0$ since $\lambda \in G(\Lambda)$.

The same analysis can be done for $t > d_{ij}^k$ when $d_{ij}^k \neq T$. We conclude that Ξ is correctly defined.
(ii) Observe that since y^ν converges to y in the $\mathcal{U}_2-$topology and since y_i^ν is constant over I_{ij}, then y_i is constant as well, and y_i^ν goes to y_i pointwise on I_{ij}. Thus, $y_i^\nu(c_{ij}^k) \longrightarrow y_{i,j}^k$, and $y_i^\nu(d_{ij}^k) \longrightarrow y_{i,j}^k$. Now, for the terms on y_j, the same analysis can be made, which yields either $y_j^\nu(c_{ij}^k) \longrightarrow y_j(c_{ij}^k-)$ or $V_{ij}[\lambda](c_{ij}^k) = 0$; and, either $y_j^\nu(d_{ij}^k) \longrightarrow y_j(d_{ij}^k+)$ or $V_{ij}[\lambda](d_{ij}^k) = 0$, when $d_{ij}^k < T$. For $d_{ij}^k = T$, $y_j^\nu(T) \longrightarrow h_j$ holds. ☐

Definition 1.4.15. *For $(\xi, y, h) \in \mathcal{P}_2$ and $\lambda \in G(\Lambda)$ define*

$$\Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) := g[\lambda](\xi(T), h) + \Xi[\lambda](\xi, y, h) + \int_0^T ((Q[\lambda]\xi, \xi) + 2(M[\lambda]\xi, y) + (R[\lambda]y, y))\mathrm{d}t.$$

Remark 1.4.16. *Observe that when $m = 1$, the mapping $\Xi \equiv 0$ since $V \equiv 0$. Thus, in this case, $\Omega_{\mathcal{P}_2}$ can be defined for any element $(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}$ and any $\lambda \in \Lambda$. If we take $(z, v) \in \mathcal{W}$ satisfying* (1.12)*, and define (ξ, y) by* (1.28)*, then*

$$\Omega[\lambda](z, v) = \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) = \Omega_{\mathcal{P}_2}[\lambda](\xi, y, y(T)).$$

For $m > 1$, the previous equality holds for $(z, v) \in \mathcal{C}_\infty$.

Lemma 1.4.17. *Let $\{(\xi^\nu, y^\nu, y^\nu(T)\} \subset \mathcal{P}$ be a sequence converging to $(\xi, y, h) \in \mathcal{P}_2$ in the $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m-$topology. Then*

$$\lim_{\nu \to \infty} \Omega_{\mathcal{P}}[\lambda](\xi^\nu, y^\nu, \dot{y}^\nu) = \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h).$$

Denote with $\operatorname{co}\Lambda$ the convex hull of Λ.

Theorem 1.4.18. *Let $\hat{w}$ be a weak minimum, then*

$$\max_{\lambda \in G(\operatorname{co}\Lambda)} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq 0, \quad \text{for all } (\xi, y, h) \in \mathcal{P}_2. \tag{1.58}$$

Proof. Corollary 1.4.8 together with Theorem 1.4.10 applied to $M := \operatorname{co}\Lambda$ yield

$$\max_{\lambda \in G(\operatorname{co}\Lambda)} \Omega_{\mathcal{P}}[\lambda](\xi, y, \dot{y}) \geq 0, \quad \text{for all } (\xi, y, y(T)) \in \mathcal{P}.$$

The result follows from Lemma 1.4.1 and Lemma 1.4.17. □

Remark 1.4.19. *Notice that in case* (1.21) *is not satisfied, condition* (1.58) *does not provide any useful information as $0 \in \operatorname{co}\Lambda$. On the other hand, if* (1.21) *holds, every $\lambda = (\alpha, \beta, \psi) \in \Lambda$ necessarily has $\alpha \neq 0$, and thus $0 \notin \operatorname{co}\Lambda$.*

1.5 Sufficient condition

Consider the problem for a scalar control, i.e. let $m = 1$. This section provides a sufficient condition for Pontryagin optimality.

In view of the notion of Pontryagin optimality, one can consider the following kind of convergence.

Definition 1.5.1. *A sequence $\{v_k\} \subset \mathcal{U}$ converges to 0 in the Pontryagin sense if $\|v_k\|_1 \to 0$ and there exists N such that $\|v_k\|_\infty < N$.*

Definition 1.5.2. *Given $(y, h) \in \mathcal{U}_2 \times \mathbb{R}$, let*

$$\gamma(y, h) := \int_0^T y(t)^2 \mathrm{d}t + |h|^2.$$

Definition 1.5.3. *We say that $\hat{w}$ satisfies $\gamma-$quadratic growth condition in the Pontryagin sense if there exists $\rho > 0$ such that, for every sequence of feasible variations $\{(\delta x_k, v_k)\}$ with $\{v_k\}$ converging to 0 in the Pontryagin sense,*

$$J(\hat{u} + v_k) - J(\hat{u}) \geq \rho\gamma(y_k, y_k(T)), \tag{1.59}$$

holds for a large enough k, where y_k is defined by (1.28). *Equivalently, for all $N > 0$, there exists $\varepsilon > 0$ such that if $\|v\|_\infty < N$ and $\|v\|_1 < \varepsilon$, then* (1.59) *holds.*

Definition 1.5.4. *We say that $\hat{w}$ is normal if $\alpha_0 > 0$ for every $\lambda \in \Lambda$.*

Theorem 1.5.5. *Suppose that there exists $\rho > 0$ such that*

$$\max_{\lambda \in \Lambda} \Omega_{\mathcal{P}_2}[\lambda](\xi, y, h) \geq \rho\gamma(y, h), \quad \text{for all } (\xi, y, h) \in \mathcal{P}_2. \tag{1.60}$$

Then $\hat{w}$ is a Pontryagin minimum satisfying $\gamma-$ quadratic growth. Furthermore, if $\hat{w}$ is normal, the converse holds.

Remark 1.5.6. *In case the bang arcs are absent, i.e. the control is totally singular, this theorem follows from one proved in Dmitruk [41, 43].*

Recall that Φ is defined in (1.5). We will use the following technical result.

Lemma 1.5.7. *Consider $\{v_k\} \subset \mathcal{U}$ converging to 0 in the Pontryagin sense. Let $u_k := \hat{u} + v_k$ and let x_k be the corresponding solution of equation* (1.2). *Then for every $\lambda \in \Lambda$,*

$$\Phi[\lambda](x_k, u_k) = \Phi[\lambda](\hat{x}, \hat{u}) + \int_0^T H_u[\lambda](t)v_k(t)\mathrm{d}t + \Omega[\lambda](z_k, v_k) + o(\gamma_k), \tag{1.61}$$

where z_k is defined by (1.12), *$\gamma_k := \gamma(y_k, y_k(T))$, and y_k is defined by* (1.28).

Proof. By Lemma 1.3.1 we can write

$$\Phi[\lambda](x_k, u_k) = \Phi[\lambda](\hat{x}, \hat{u}) + \int_0^T H_u[\lambda](t)v_k(t)\mathrm{d}t + \Omega[\lambda](z_k, v_k) + R_k,$$

where, in view of Lemma 1.8.5,

$$R_k := \Delta_k\Omega[\lambda] + \int_0^T (H_{uxx}[\lambda](t)\delta x_k(t), \delta x_k(t), v_k(t))\mathrm{d}t + o(\gamma_k), \tag{1.62}$$

with $\delta x_k := x_k - \hat{x}$, and

$$\Delta_k\Omega[\lambda] := \Omega[\lambda](\delta x_k, v_k) - \Omega[\lambda](z_k, v_k). \tag{1.63}$$

Next, we prove that

$$R_k = o(\gamma_k). \tag{1.64}$$

Note that $\mathcal{Q}(a,a) - \mathcal{Q}(b,b) = \mathcal{Q}(a+b, a-b)$, for any bilinear mapping $\mathcal{Q}$, and any pair a, b. Put $\eta_k := \delta x_k - z_k$. Hence, from (1.63), we get

$$\begin{aligned}\Delta_k\Omega[\lambda] =& \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(\delta x_k(T) + z_k(T), \eta_k(T)) \\ &+ \tfrac{1}{2}\int_0^T (H_{xx}[\lambda](\delta x_k + z_k), \eta_k)\mathrm{d}t + \int_0^T (H_{ux}[\lambda]\eta_k, v_k)\mathrm{d}t.\end{aligned}$$

By Lemmas 1.8.5 and 1.8.13 in the Appendix, the first and the second terms are of order $o(\gamma_k)$. Integrate by parts the last term to obtain

$$\int_0^T (H_{ux}[\lambda]\eta_k, v_k)\mathrm{d}t \tag{1.65}$$

$$= [(H_{ux}[\lambda]\eta_k, y_k)]_0^T - \int_0^T \{(\dot{H}_{ux}[\lambda]\eta_k, y_k) + (H_{ux}[\lambda]\dot{\eta}_k, y_k)\}\mathrm{d}t. \tag{1.66}$$

Thus, by Lemma 1.8.13 we deduce that the first two terms in (1.66) are of order $o(\gamma_k)$. It remains to deal with last term in the integral. Replace $\dot{\eta}_k$ by its expression in equation (1.125) of Lemma 1.8.13:

$$\begin{aligned}\int_0^T (H_{ux}[\lambda]\dot{\eta}_k, y_k)\mathrm{d}t &= \int_0^T (H_{ux}[\lambda]\left(\sum_{i=0}^{1} \hat{u}_i f_i'(\hat{x})\eta_k + v_k f_1'(\hat{x})\delta x_k + \zeta_k\right), y_k)\mathrm{d}t \\ &= o(\gamma_k) + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_k^2}{2}\right) H_{ux}[\lambda] f_1'(\hat{x})\delta x_k \mathrm{d}t,\end{aligned} \tag{1.67}$$

where the second equality follows from Lemmas 1.8.5 and 1.8.13. Integrating the last term by parts, we obtain

$$\begin{aligned}&\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_k^2}{2}\right) H_{ux}[\lambda] f_1'(\hat{x})\delta x_k \mathrm{d}t = \left[\frac{y_k^2}{2} H_{ux}[\lambda] f_1'(\hat{x})\delta x_k\right]_0^T \\ &- \int_0^T \frac{y_k^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(H_{ux}[\lambda] f_1'(\hat{x})\right)\delta x_k \mathrm{d}t - \int_0^T \frac{y_k^2}{2} H_{ux}[\lambda] f_1'(\hat{x})\delta \dot{x}_k \mathrm{d}t \\ &= o(\gamma_k) - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_k^3}{6}\right) H_{ux}[\lambda] f_1'(\hat{x}) f_1(\hat{x})\mathrm{d}t \\ &= o(\gamma_k) - \left[\frac{y_k^3}{6} H_{ux}[\lambda] f_1'(\hat{x}) f_1(\hat{x})\right]_0^T + \int_0^T \frac{y_k^3}{6}\frac{\mathrm{d}}{\mathrm{d}t}\left(H_{ux}[\lambda] f_1'(\hat{x}) f_1(\hat{x})\right)\mathrm{d}t \\ &= o(\gamma_k),\end{aligned} \tag{1.68}$$

where we used Lemma 1.8.13 and, in particular, equation (1.126). From (1.67) and (1.68), it follows that the term in (1.65) is of order $o(\gamma_k)$. Thus,

$$\Delta_k\Omega[\lambda] \leq o(\gamma_k). \tag{1.69}$$

Consider now the third order term in (1.62):

$$\begin{aligned}
&\int_0^T (H_{uxx}[\lambda]\delta x_k, \delta x_k, v_k)\mathrm{d}t = [y_k \delta x_k^\top H_{uxx}[\lambda]\delta x_k]_0^T \\
&- \int_0^T y_k \delta x_k^\top \dot{H}_{uxx}[\lambda]\delta x_k \mathrm{d}t - 2\int_0^T y_k \delta x_k^\top H_{uxx}[\lambda]\dot{\delta x}_k \mathrm{d}t \\
&= o(\gamma_k) - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(y_k^2)\delta x_k^\top H_{uxx}[\lambda] f_1(\hat{x})\mathrm{d}t \\
&= o(\gamma_k) - \left[y_k^2 \delta x_k^\top H_{uxx}[\lambda] f_1(\hat{x})\right]_0^T - \int_0^T y_k^2 v_k f_1(\hat{x})^\top H_{uxx}[\lambda] f_1(\hat{x})\mathrm{d}t = o(\gamma_k),
\end{aligned} \tag{1.70}$$

by Lemmas 1.8.5 and 1.8.13. The last inequality follows from integrating by parts one more time as it was done in (1.68). Consider expression (1.62). By inequality (1.69) and equation (1.70), equality (1.64) is obtained and thus, the desired result follows.

□

Proof. [of Theorem 1.5.5] *Part 1.* First we prove that if $\hat{w}$ is a normal Pontryagin minimum satisfying the $\gamma-$quadratic growth condition in the Pontryagin sense then (1.60) holds for some $\rho > 0$. Here the necessary condition of Theorem 1.3.2 is used. Define $\hat{y}(t) := \int_0^t \hat{u}(s)\mathrm{d}s$, and note that $(\hat{w}, \hat{y})$ is, for some $\rho' > 0$, a Pontryagin minimum of

$$\begin{aligned}
&\tilde{J} := J - \rho'\gamma(y - \hat{y}, y(T) - \hat{y}(T)) \to \min, \\
&(1.2)\text{-}(1.4), \dot{y} = u,\ y(0) = 0.
\end{aligned} \tag{1.71}$$

Observe that the critical cone $\tilde{\mathcal{C}}_2$ for (1.71) consists of the points $(z, v, \delta y)$ in $\mathcal{X}_2 \times \mathcal{U}_2 \times W_2^1(0, T; \mathbb{R})$ verifying $(z, v) \in \mathcal{C}_2$, $\dot{\delta y} = v$ and $\delta y(0) = 0$. Since the pre-Hamiltonian at point $(\hat{w}, \hat{y})$ coincides with the original pre-Hamiltonian, the set of multipliers for (1.71) consists of the points (λ, ψ_y) with $\lambda \in \Lambda$.

Applying the second order necessary condition of Theorem 1.3.2 at the point $(\hat{w}, \hat{y})$ we see that, for every $(z, v) \in \mathcal{C}_2$ and $\delta y(t) := \int_0^t v(s)\mathrm{d}s$, there exists $\lambda \in \Lambda$ such that

$$\Omega[\lambda](z, v) - \alpha_0\rho'(\|\delta y\|_2^2 + \delta y^2(T)) \geq 0, \tag{1.72}$$

where $\alpha_0 > 0$ since $\hat{w}$ is normal. Take $\rho := \min_{\lambda\in\Lambda} \alpha_0\rho' > 0$. Applying the Goh transformation in (1.72), condition (1.60) for the constant ρ follows.

Part 2. We shall prove that if (1.60) holds for some $\rho > 0$, then $\hat{w}$ satisfies $\gamma-$quadratic growth in the Pontryagin sense. On the contrary, assume that the quadratic growth condition (1.59) is not valid. Consequently, there exists a sequence $\{v_k\} \subset \mathcal{U}$ converging to 0 in the Pontryagin sense such that, denoting $u_k := \hat{u} + v_k$,

$$J(\hat{u} + v_k) \leq J(\hat{u}) + o(\gamma_k), \tag{1.73}$$

where $y_k(t) := \int_0^t v_k(s)ds$ and $\gamma_k := \gamma(y_k, y_k(T))$. Denote by x_k the solution of equation (1.2) corresponding to u_k, define $w_k := (x_k, u_k)$ and let z_k be the solution of (1.12) associated with v_k. Take any $\lambda \in \Lambda$. Multiply inequality (1.73) by α_0, add the nonpositive term $\sum_{i=0}^{d_\varphi} \alpha_i \varphi_i(x_k(T)) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(x_k(T))$ to its left-hand side, and obtain the inequality

$$\Phi[\lambda](x_k, u_k) \leq \Phi[\lambda](\hat{x}, \hat{u}) + o(\gamma_k). \tag{1.74}$$

Recall expansion (1.61). Let $(\bar{y}_k, \bar{h}_k) := (y_k, y_k(T))/\sqrt{\gamma_k}$. Note that the elements of this sequence have unit norm in $\mathcal{U}_2 \times \mathbb{R}$. By the Banach-Alaoglu Theorem, extracting if necessary a sequence, we may assume that there exists $(\bar{y}, \bar{h}) \in \mathcal{U}_2 \times \mathbb{R}$ such that

$$\bar{y}_k \rightharpoonup \bar{y}, \text{ and } \bar{h}_k \to \bar{h}, \tag{1.75}$$

where the first limit is taken in the weak topology of $\mathcal{U}_2$. The remainder of the proof is split into two parts.

(a) Using equations (1.61) and (1.74) we prove that $(\bar{\xi}, \bar{y}, \bar{h}) \in \mathcal{P}_2$, where $\bar{\xi}$ is a solution of (1.29).

(b) We prove that $(\bar{y}, \bar{h}) = 0$ and that it is the limit of $\{(\bar{y}_k, \bar{h}_k)\}$ in the strong sense. This leads to a contradiction since each $(\bar{y}_k, \bar{h}_k)$ has unit norm.

(a) We shall prove that $(\bar{\xi}, \bar{y}, \bar{h}) \in \mathcal{P}_2$. From (1.61) and (1.74) it follows that

$$0 \leq \int_0^T H_u[\lambda](t) v_k(t) \mathrm{d}t \leq -\Omega_{\mathcal{P}_2}[\lambda](\xi_k, y_k, h_k) + o(\gamma_k),$$

where ξ_k is solution of (1.29) corresponding to y_k. The first inequality holds as $H_u[\lambda] v_k \geq 0$ almost everywhere on $[0, T]$ and we replaced $\Omega_{\mathcal{P}}$ by $\Omega_{\mathcal{P}_2}$ in view of Remark 1.4.16. By the continuity of mapping $\Omega_{\mathcal{P}_2}[\lambda]$ over $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}$ deduce that

$$0 \leq \int_0^T H_u[\lambda](t) v_k(t) \mathrm{d}t \leq O(\gamma_k),$$

and thus, for each composing interval (c, d) of I_0,

$$\lim_{k \to \infty} \int_c^d H_u[\lambda](t) \varphi(t) \frac{v_k(t)}{\sqrt{\gamma_k}} dt = 0, \tag{1.76}$$

for every nonnegative Lipschitz continuous function φ with $\operatorname{supp}\varphi \subset (c,d)$. The latter expression means that the support of φ is included in (c,d). Integrating by parts in (1.76) and by (1.75) we obtain

$$0 = \lim_{k\to\infty} \int_c^d \frac{\mathrm{d}}{\mathrm{d}t} \left(H_u[\lambda](t)\varphi(t)\right) \bar{y}_k(t)\mathrm{d}t = \int_c^d \frac{\mathrm{d}}{\mathrm{d}t} \left(H_u[\lambda](t)\varphi(t)\right) \bar{y}(t)\mathrm{d}t.$$

By Lemma 1.8.6, $\bar{y}$ is nondecreasing over (c,d). Hence, in view of Lemma 1.8.8, we can integrate by parts in the previous equation to get

$$\int_c^d H_u[\lambda](t)\varphi(t)\mathrm{d}\bar{y}(t) = 0. \tag{1.77}$$

Take $t_0 \in (c,d)$. By the strict complementary in Assumption 1, there exists $\lambda_0 \in \Lambda$ such that $H_u[\lambda_0](t_0) > 0$. Hence, in view of the continuity of $H_u[\lambda_0]$, there exists $\varepsilon > 0$ such that $H_u[\lambda_0] > 0$ on $(t_0 - 2\varepsilon, t_0 + 2\varepsilon) \subset (c,d)$. Choose φ such that $\operatorname{supp}\varphi \subset (t_0 - 2\varepsilon, t_0 + 2\varepsilon)$, and $H_u[\lambda_0](t)\varphi(t) = 1$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$. Since $\mathrm{d}\bar{y} \geq 0$, equation (1.77) yields

$$\begin{aligned} 0 &= \int_c^d H_u[\lambda](t)\varphi(t)\mathrm{d}\bar{y}(t) \geq \int_{t_0-\varepsilon}^{t_0+\varepsilon} H_u[\lambda](t)\varphi(t)\mathrm{d}\bar{y}(t) \\ &= \int_{t_0-\varepsilon}^{t_0+\varepsilon} \mathrm{d}\bar{y}(t) = \bar{y}(t_0+\varepsilon) - \bar{y}(t_0-\varepsilon). \end{aligned}$$

As ε and $t_0 \in (c,d)$ are arbitrary we find that

$$\mathrm{d}\bar{y}(t) = 0, \quad \text{on } I_0, \tag{1.78}$$

and thus (1.33) holds. Let us prove condition (1.34) for $(\bar{\xi}, \bar{y}, \bar{h})$. Suppose that $0 \in I_0$. Take $\varepsilon > 0$, and notice that by Assumption 1 there exists $\lambda' \in \Lambda$ and $\delta > 0$ such that $H_u[\lambda'](t) > \delta$ for $t \in [0, d_1 - \varepsilon]$, and thus by (1.76) we obtain $\int_0^{d_1-\varepsilon} v_k(t)/\sqrt{\gamma_k}\mathrm{d}t \to 0$, as $v_k \geq 0$. Then for all $s \in [0, d_1)$, we have

$$\bar{y}_k(s) \to 0,$$

and thus

$$\bar{y} = 0, \text{ on } [0, d_1), \text{ if } 0 \in I_0. \tag{1.79}$$

Suppose that $T \in I_0$. Then, we can derive $\int_{a_N+\varepsilon}^T \bar{v}_k(t)\mathrm{d}t \to 0$ by an analogous argument. Thus, the pointwise convergence

$$\bar{h}_k - \bar{y}_k(s) \to 0,$$

holds for every $s \in (a_N, T]$, and then,

$$\bar{y} = \bar{h}, \text{ on } (a_N, T], \text{ if } T \in I_0. \tag{1.80}$$

It remains to check the final conditions (1.32) for $\bar{h}$. Let $0 \leq i \leq d_\varphi$,

$$\begin{aligned} \varphi_i'(\hat{x}(T))(\bar{\xi}(T) + B(T)\bar{h}) &= \lim_{k\to\infty} \varphi_i'(\hat{x}(T)) \left(\frac{\xi_k(T) + B(T)h_k}{\sqrt{\gamma_k}} \right) \\ &= \lim_{k\to\infty} \varphi_i'(\hat{x}(T)) \frac{z_k(T)}{\sqrt{\gamma_k}}. \end{aligned} \tag{1.81}$$

A first order Taylor expansion of the function φ_i around $\hat{x}(T)$ gives

$$\varphi_i(x_k(T)) = \varphi_i(\hat{x}(T)) + \varphi_i'(\hat{x}(T))\delta x_k(T) + O(|\delta x_k(T)|^2).$$

By Lemmas 1.8.5 and 1.8.13 in the Appendix, we can write

$$\varphi_i(x_k(T)) = \varphi_i(\hat{x}(T)) + \varphi_i'(\hat{x}(T))z_k(T) + o(\sqrt{\gamma_k}).$$

Thus

$$\varphi_i'(\hat{x}(T))\frac{z_k(T)}{\sqrt{\gamma_k}} = \frac{\varphi_i(x_k(T)) - \varphi_i(\hat{x}(T))}{\sqrt{\gamma_k}} + o(1). \tag{1.82}$$

Since x_k satisfies (1.4), equations (1.81) and (1.82) yield, for $1 \leq i \leq d_\varphi$: $\varphi_i'(\hat{x}(T))(\bar{\xi}(T) + B(T)\bar{h}) \leq 0$. For $i = 0$ use inequality (1.73). Analogously,

$$\eta_j'(\hat{x}(T))(\bar{\xi}(T) + B(T)\bar{h}) = 0, \quad \text{for } j = 1, \dots, d_\eta.$$

Thus $(\bar{\xi}, \bar{y}, \bar{h})$ satisfies (1.32), and by (1.78), (1.79) and (1.80), we obtain

$$(\bar{\xi}, \bar{y}, \bar{h}) \in \mathcal{P}_2.$$

(b) Return to the expansion (1.61). Equation (1.74) and $H_u[\lambda] \geq 0$ imply

$$\begin{aligned} &\Omega_{\mathcal{P}_2}[\lambda](\xi_k, y_k, y_k(T)) = \\ &\qquad \Phi[\lambda](x_k, u_k) - \Phi[\lambda](\hat{x}, \hat{u}) - \int_0^T H_u[\lambda]v_k \mathrm{d}t - o(\gamma_k) \leq o(\gamma_k). \end{aligned}$$

Thus

$$\liminf_{k\to\infty} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}_k, \bar{y}_k, \bar{h}_k) \leq \limsup_{k\to\infty} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}_k, \bar{y}_k, \bar{h}_k) \leq 0. \tag{1.83}$$

Recall Assumption 2, and let N be the number of connected components of I_0. Set $\varepsilon > 0$, and for each composing interval (c, d) of I_0, consider a smaller interval of the form $(c+\varepsilon/2N, d-\varepsilon/2N)$. Denote their union as I_0^ε. Notice that $I_0\backslash I_0^\varepsilon$ is of measure ε. Put $I_+^\varepsilon := [0,T]\backslash I_0^\varepsilon$. Split $\Omega_{\mathcal{P}_2}$ as follows

$$\Omega^\varepsilon_{\mathcal{P}_2,0}[\lambda](\xi,y,h) := \int_0^T \{(Q[\lambda]\xi,\xi) + (M[\lambda]\xi,y)\}\mathrm{d}t + \int_{I_0^\varepsilon}(R[\lambda]y,y)\mathrm{d}t + g[\lambda](\xi(T),h),$$

and

$$\Omega^\varepsilon_{\mathcal{P}_2,+}[\lambda](y) := \int_{I_+^\varepsilon}(R[\lambda]y,y)\mathrm{d}t.$$

Consider the Hilbert space

$$\Gamma_2 := \{(\xi,y,h)\in\mathcal{X}_2\times\mathcal{U}_2\times\mathbb{R} : (1.29),\ (1.33)\,\text{and}\,(1.34)\ \text{hold}\}.$$

Let $\rho>0$ be the constant in (1.60) and define

$$\Lambda^\rho := \{\lambda\in\operatorname{co}\Lambda : \Omega_{\mathcal{P}_2}[\lambda]-\rho\gamma \text{ is weakly l.s.c. on } \Gamma_2\}.$$

Equation (1.60) and Lemma 1.8.12 imply that

$$\max_{\lambda\in\Lambda^\rho}\Omega_{\mathcal{P}_2}[\lambda](\bar\xi,\bar y,\bar h)\geq\rho\gamma(\bar y,\bar h). \tag{1.84}$$

Denote by $\bar\lambda$ the element in Λ^ρ that reaches the maximum in (1.84). Observe that $\Omega^\varepsilon_{\mathcal{P}_2,0}[\bar\lambda]$ is weakly continuous. Thus, $\Omega_{\mathcal{P}_2}[\bar\lambda]-\rho\gamma$ is weakly l.s.c. if and only if the quadratic mapping

$$y\mapsto\Omega^\varepsilon_{\mathcal{P}_2,+}[\bar\lambda](y)-\rho\int_{I_+^\varepsilon}|y(t)|^2\mathrm{d}t, \tag{1.85}$$

is weakly l.s.c. over the subspace $\mathcal{U}_{2,I_+^\varepsilon}$ of $\mathcal{U}_2$ consisting of the functions that are constant over each connected component of $[0,T]\backslash I_+^\varepsilon$. Applying Lemma 1.8.11 to the mapping in (1.85) we see that $R[\bar\lambda](t)\succeq\rho$ on I_+^ε. Consequently, $\Omega^\varepsilon_{\mathcal{P}_2,+}[\bar\lambda]$ is a Legendre form on $\mathcal{U}_{2,I_+^\varepsilon}$, and thus

$$\Omega^\varepsilon_{\mathcal{P}_2,+}[\bar\lambda](\bar y)\leq\liminf_{k\to\infty}\Omega^\varepsilon_{\mathcal{P}_2,+}[\bar\lambda](\bar y_k). \tag{1.86}$$

Hence,

$$\begin{aligned}\rho\gamma(\bar y,\bar h)\leq&\Omega_{\mathcal{P}_2}[\bar\lambda](\bar\xi,\bar y,\bar h)\leq\lim_{k\to\infty}\Omega^\varepsilon_{\mathcal{P}_2,0}[\bar\lambda](\xi_k,\bar y_k,\bar h_k)\\&+\liminf_{k\to\infty}\Omega^\varepsilon_{\mathcal{P}_2,+}[\bar\lambda](\bar y_k)=\liminf_{k\to\infty}\Omega_{\mathcal{P}_2}[\bar\lambda](\bar\xi_k,\bar y_k,\bar h_k),\end{aligned}$$

where first inequality is due to condition (1.60), and the second inequality is an immediate consequence of (1.86). Inequality (1.83) implies that the right-hand side of the last expression is nonpositive. Hence,

$$(\bar y,\bar h)=0,\ \text{and}\ \lim_{k\to\infty}\Omega_{\mathcal{P}_2}[\bar\lambda](\bar\xi_k,\bar y_k,\bar h_k)=0.$$

As $\Omega^{\varepsilon}_{\mathcal{P}_2,0}[\bar{\lambda}]$ is weakly continuous, $\lim_{k\to\infty}\Omega^{\varepsilon}_{\mathcal{P}_2,0}[\bar{\lambda}](\bar{\xi}_k,\bar{y}_k,\bar{h}_k)=0$ and thus

$$\lim_{k\to\infty}\Omega^{\varepsilon}_{\mathcal{P}_2,+}[\bar{\lambda}](\bar{y}_k)=0. \tag{1.87}$$

We have: $\Omega^{\varepsilon}_{\mathcal{P}_2,+}[\bar{\lambda}]$ is a Legendre form and $\bar{y}_k \rightharpoonup 0$ on I^{ε}_+. Thus, by (1.87),

$$\bar{y}_k \to 0, \quad \text{on } \mathcal{U}_{2,I^{\varepsilon}_+}.$$

As $\{\bar{y}_k\}$ converges uniformly on I^{ε}_0, the strong convergence holds on $[0,T]$, and thus

$$(\bar{y}_k,\bar{h}_k) \longrightarrow (0,0), \quad \text{on } \mathcal{U}_2\times\mathbb{R}. \tag{1.88}$$

This leads to a contradiction since $(\bar{y}_k,\bar{h}_k)$ has unit norm for every $k\in I\!N$. Thus, $\hat{w}$ is a Pontryagin minimum satisfying quadratic growth.

□

1.6 Extensions and an example

1.6.1 Including parameters

Consider the following optimal control problem where the initial state is not determined, some parameters are included and a more general control constraint is considered.

$$J := \varphi_0(x(0),x(T),r(0)) \to \min, \tag{1.89}$$

$$\dot{x}(t)=\sum_{i=0}^{m} u_i(t)f_i(x(t),r(t)), \tag{1.90}$$

$$\dot{r}(t)=0, \tag{1.91}$$

$$a_i \le u_i(t)\le b_i,\ \text{for a.a. } t\in(0,T),\ i=1,\dots,m \tag{1.92}$$

$$\varphi_i(x(0),x(T),r(0))\le 0,\ \text{for } i=1,\dots,d_{\varphi}, \tag{1.93}$$

$$\eta_j(x(0),x(T),r(0))=0,\ \text{for } j=1\dots,d_{\eta}, \tag{1.94}$$

where $u\in\mathcal{U}$, $x\in\mathcal{X}$, $r\in\mathbb{R}^{n_r}$ is a parameter considered as a state variable with zero-dynamics, $a,b\in\mathbb{R}^m$, functions $f_i:\mathbb{R}^{n+n_r}\to\mathbb{R}^n$, $\varphi_i:\mathbb{R}^{2n+n_r}\to\mathbb{R}$, and $\eta:\mathbb{R}^{2n+n_r}\to\mathbb{R}^{d_\eta}$ are twice continuously differentiable. As r has zero dynamics, the costate variable ψ_r corresponding to equation (1.91) does not appear in the pre-Hamiltonian. Denote with ψ the costate variable associated with (1.90). The pre-Hamiltonian function for problem (1.89)-(1.94) is given by

$$H[\lambda](x,r,u,t)=\psi(t)\sum_{i=0}^{m}u_if_i(x,r).$$

Let $(\hat{x}, \hat{r}, \hat{u})$ be a feasible solution for (1.90)-(1.94). Since $\hat{r}(\cdot)$ is constant, we can denote it by $\hat{r}$. Assume that

$$\varphi_i(\hat{x}(0), \hat{x}(T), \hat{r}) = 0, \quad \text{for } i = 0, \ldots, d_\varphi.$$

An element $\lambda = (\alpha, \beta, \psi_x, \psi_r) \in \mathbb{R}^{d_\varphi + d_\eta + 1} \times W^1_\infty(0, T; \mathbb{R}^{n,*}) \times W^1_\infty(0, T; \mathbb{R}^{n_r,*})$ is a Pontryagin multiplier for $(\hat{x}, \hat{r}, \hat{u})$ if it satisfies (1.7), (1.8), the costate equation for ψ

$$\begin{cases} -\dot{\psi}_x(t) = H_x[\lambda](\hat{x}(t), \hat{r}, \hat{u}(t), t), \text{ a.e. on } [0, T] \\ \psi_x(0) = -\ell_{x_0}[\lambda](\hat{x}(0), \hat{x}(T), \hat{r}), \\ \psi_x(T) = \ell_{x_T}[\lambda](\hat{x}(0), \hat{x}(T), \hat{r}), \end{cases}$$

and for ψ_r

$$\begin{cases} -\dot{\psi}_r(t) = H_r[\lambda](\hat{x}(t), \hat{r}, \hat{u}(t), t), \text{ a.e. on } [0, T] \\ \psi_r(0) = -\ell_r[\lambda](\hat{x}(0), \hat{x}(T), \hat{r}), \ \psi_r(T) = 0. \end{cases} \tag{1.95}$$

Observe that (1.95) implies the stationarity condition

$$\ell_r(\hat{x}(0), \hat{x}(T), \hat{r}) + \int_0^T H_r[\lambda](t)\mathrm{d}t = 0.$$

Take $v \in \mathcal{U}$ and consider the linearized state equation

$$\begin{cases} \dot{z}(t) = \displaystyle\sum_{i=0}^m \hat{u}_i(t)[f_{i,x}(\hat{x}(t), \hat{r})z(t) + f_{i,r}(\hat{x}(t), \hat{r})\delta r(t)] + \sum_{i=1}^m v_i(t) f_i(\hat{x}(t), \hat{r}), \\ \dot{\delta r}(t) = 0, \end{cases} \tag{1.96}$$

where we can see that $\delta r(\cdot)$ is constant and thus we denote it by δr. Let the linearized initial-final constraints be

$$\begin{aligned} \varphi_i'(\hat{x}(0), \hat{x}(T), \hat{r})(z(0), z(T), \delta r) \leq 0, \quad \text{for } i = 1, \ldots, d_\varphi, \\ \eta_j'(\hat{x}(0), \hat{x}(T), \hat{r})(z(0), z(T), \delta r) = 0, \quad \text{for } j = 1, \ldots, d_\eta. \end{aligned} \tag{1.97}$$

Define for each $i = 1, \ldots, m$ the sets

$$\begin{aligned} I_a^i &:= \{t \in [0, T] : \max_{\lambda \in \Lambda} H_{u_i}[\lambda](t) > 0\}, \\ I_b^i &:= \{t \in [0, T] : \max_{\lambda \in \Lambda} H_{u_i}[\lambda](t) < 0\}, \\ I_{\text{sing}}^i &:= [0, T] \backslash (I_a^i \cup I_b^i). \end{aligned}$$

Assumption 3. Consider the natural extension of Assumption 2, i.e. for each $i = 1, \ldots, m$, the sets I_a^i and I_b^i are finite unions of intervals, i.e.

$$I_a^i = \bigcup_{j=1}^{N_a^i} I_{j,a}^i, \quad I_b^i = \bigcup_{j=1}^{N_b^i} I_{j,b}^i,$$

for $I^i_{j,a}$ and $I^i_{j,b}$ being subintervals of $[0,T]$ of the form $[0,c)$, $(d,T]$; or (c,d) if $c \neq 0$ and $d \neq T$. Notice that $I^i_a \cap I^i_b = \emptyset$. Call $c^i_{1,a} < d^i_{1,a} < c^i_{2,a} < \ldots < c^i_{N^i_a,a} < d^i_{N^i_a,a}$ the endpoints of these intervals corresponding to bound a, and define them analogously for b. Consequently, I^i_{sing} is a finite union of intervals as well. Assume that a concatenation of a bang arc followed by another bang arc is forbidden.

Assumption 4. Strict complementarity assumption for control constraints:

$$\begin{cases} I^i_a = \{t \in [0,T] : \hat{u}_i(t) = a_i\}, \text{ up to a set of null measure,} \\ I^i_b = \{t \in [0,T] : \hat{u}_i(t) = b_i\}, \text{ up to a set of null measure.} \end{cases}$$

Consider

$$\mathcal{C}_2 := \left\{ \begin{array}{l} (z, \delta r, v) \in \mathcal{X}_2 \times \mathbb{R}^{n_r} \times \mathcal{U}_2 : (1.96)\text{-}(1.97) \text{ hold,} \\ v_i = 0 \text{ on } I^i_{\rm a} \cup I^i_{\rm b}, \text{ for } i = 1, \ldots, m \end{array} \right\}.$$

The Goh transformation allows us to obtain variables (ξ, y) defined by

$$y(t) := \int_0^t v(s) \mathrm{d}s, \ \xi := z - \sum_{i=1}^m y_i f_i.$$

Notice that ξ satisfies the equation

$$\begin{aligned} \dot{\xi} &= A^x \xi + A^r \delta r + B^x_1 y, \\ \xi(0) &= z(0), \end{aligned} \tag{1.98}$$

where, denoting $[f_i, f_j]^x := f_{i,x} f_j - f_{j,x} f_i$,

$$A^x := \sum_{i=0}^m \hat{u}_i f_{i,x}, \quad A^r := \sum_{i=0}^m \hat{u}_i f_{i,r}, \quad B^x_1 y := \sum_{j=1}^m y_j \sum_{i=0}^m \hat{u}_i [f_i, f_j]^x.$$

Consider the transformed version of (1.97),

$$\begin{aligned} &\varphi'_i(\hat{x}(0), \hat{x}(T), \hat{r})(\xi(0), \xi(T) + B(T)h, \delta r) \leq 0, \ i = 1, \ldots, d_\varphi, \\ &\eta'_j(\hat{x}(0), \hat{x}(T), \hat{r})(\xi(0), \xi(T) + B(T)h, \delta r) = 0, \ j = 1, \ldots, d_\eta, \end{aligned} \tag{1.99}$$

and let the cone $\mathcal{P}$ be given by

$$\mathcal{P} := \left\{ \begin{array}{l} (\xi, \delta r, y, h) \in \mathcal{X} \times \mathbb{R}^{n_r} \times \mathcal{Y} \times \mathbb{R}^m : y(0) = 0, \ h = y(T), \\ (1.98) \text{ and } (1.99) \text{ hold, } y'_i = 0 \text{ on } I^i_{\rm a} \cup I^i_{\rm b}, \text{ for } i = 1, \ldots, m \end{array} \right\}.$$

Observe that each $(\xi, \delta r, y, h) \in \mathcal{P}$ satisfies

$$y_i \text{ constant over each composing interval of } I^i_{\rm a} \cup I^i_{\rm b}, \tag{1.100}$$

and at the endpoints,

$$\begin{cases} y_i = 0 \text{ on } [0,d], \text{ if } 0 \in I_{\mathrm{a}}^i \cup I_{\mathrm{b}}^i, \text{ and,} \\ y_i = h_i \text{ on } [c,T], \text{ if } T \in I_{\mathrm{a}}^i \cup I_{\mathrm{b}}^i, \end{cases} \tag{1.101}$$

where $[0,d)$ is the first maximal composing interval of $I_{\mathrm{a}}^i \cup I_{\mathrm{b}}^d$ when $0 \in I_{\mathrm{a}}^d \cup I_{\mathrm{b}}^d$, and $(c,T]$ is its last composing interval when $T \in I_{\mathrm{a}}^i \cup I_{\mathrm{b}}^i$. Define

$$\mathcal{P}_2 := \left\{ \begin{array}{l} (\xi, \delta r, y, h) \in \mathcal{X}_2 \times \mathbb{R}^{n_r} \times \mathcal{U}_2 \times \mathbb{R}^m : \\ (1.98),\ (1.99),\ (1.100) \text{ and } (1.101) \text{ hold for } i = 1, \dots, m \end{array} \right\}.$$

Recall definitions in equations (1.35), (1.36), (1.39), (1.40), (1.41). Minor simplifications appear in the computations of these functions as the dynamics of r are null and δr is constant. We outline these calculations in an example.

Consider $M \subset \mathbb{R}^s$ and the subset of $M \subset \mathbb{R}^s$ defined by

$$G(M) := \{\lambda \in M : V_{ij}[\lambda] = 0 \text{ on } I_{\mathrm{sing}}^i \cap I_{\mathrm{sing}}^j, \text{ for every pair } 1 < i \neq j \leq m\}.$$

Using the same techniques, we obtain the equivalent of Theorem 1.4.18:

Corollary 1.6.1. *Suppose that $(\hat{x}, \hat{r}, \hat{u})$ is a weak minimum for problem* (1.89)-(1.94)*. Then*

$$\max_{\lambda \in G(\mathrm{co}\,\Lambda)} \Omega_{\mathcal{P}_2}[\lambda](\xi, \delta r, y, h) \geq 0, \quad \text{for all } (\xi, \delta r, y, h) \in \mathcal{P}_2.$$

By a simple adaptation of the proof of Theorem 1.5.5 we get the equivalent result.

Corollary 1.6.2. *Let $m = 1$. Suppose that there exists $\rho > 0$ such that*

$$\max_{\lambda \in \Lambda} \Omega_{\mathcal{P}_2}[\lambda](\xi, \delta r, y, h) \geq \rho\gamma(y,h), \quad \text{for all } (\xi, \delta r, y, h) \in \mathcal{P}_2. \tag{1.102}$$

Then $(\hat{x}, \hat{r}, \hat{u})$ is a Pontryagin minimum that satisfies $\gamma-$quadratic growth.

1.6.2 Application to minimum-time problems

Consider the problem

$$\begin{aligned} &J := T \to \min, \\ &\text{s.t. } (1.90) - (1.94). \end{aligned}$$

Observe that by the change of variables:

$$x(s) \leftarrow x(Ts), \quad u(s) \leftarrow u(Ts), \tag{1.103}$$

we can transform the problem into the following formulation.

$$
\begin{aligned}
&J := T(0) \to \min, \\
&\dot{x}(s) = T(s) \sum_{i=0}^{m} u_i(s) f_i(x(s), r(s)), \quad \text{a.e. on } [0,1], \\
&\dot{r}(s) = 0, \quad \text{a.e. on } [0,1], \\
&\dot{T}(s) = 0, \quad \text{a.e. on } [0,1], \\
&a_i \leq u_i(s) \leq b_i, \quad \text{a.e. on } [0,1],\ i = 1, \dots, m, \\
&\varphi_i(x(0), x(1), r(0)) \leq 0,\ \text{for } i = 1, \dots, d_\varphi, \\
&\eta_j(x(0), x(T), r(0)) = 0,\ \text{for } j = 1 \dots, d_\eta.
\end{aligned}
$$

We can apply Corollaries 1.6.1 and 1.6.2 to the problem written in this form. We outline the calculations in the following example.

1.6.2.1 Example: Markov-Dubins problem

Consider a problem over the interval $[0, T]$ with free final time T :

$$
\begin{aligned}
&J := T \to \min, \\
&\dot{x}_1 = -\sin x_3,\ x_1(0) = 0,\ x_1(T) = b_1, \\
&\dot{x}_2 = \cos x_3,\ x_2(0) = 0,\ x_2(T) = b_2, \\
&\dot{x}_3 = u,\ x_3(0) = 0,\ x_3(T) = \theta, \\
&-1 \leq u \leq 1,
\end{aligned}
\tag{1.104}
$$

with $0 < \theta < \pi$, b_1 and b_2 fixed.

This problem was originally introduced and studied by Markov in [88], and much later, it was investigated by Dubins in [48]. More recently, the problem was investigated by Sussmann and Tang [124], Soueres and Laumond [120], Boscain and Piccoli [28], among others.

Here we will study the optimality of the extremal

$$
\hat{u}(t) := \begin{cases} 1 & \text{on } [0, \theta], \\ 0 & \text{on } (\theta, \hat{T}]. \end{cases}
\tag{1.105}
$$

Observe that by the change of variables (1.103) we can transform (1.104) into the following problem on the interval $[0, 1]$.

$$\begin{aligned}
&J := T(0) \to \min, \\
&\dot{x}_1(s) = -T(s)\sin x_3(s),\ x_1(0) = 0,\ x_1(1) = b_1, \\
&\dot{x}_2(s) = T(s)\cos x_3(s),\ x_2(0) = 0,\ x_2(1) = b_2, \\
&\dot{x}_3(s) = T(s)u(s),\ x_3(0) = 0,\ x_3(1) = \theta, \\
&\dot{T}(s) = 0, \\
&-1 \le u(s) \le 1.
\end{aligned} \tag{1.106}$$

We obtain for state variables:

$$\hat{x}_3(s) = \begin{cases} \hat{T}s & \text{on } [0, \theta/\hat{T}], \\ \theta & \text{on } (\theta/\hat{T}, 1], \end{cases} \tag{1.107}$$

$$\hat{x}_1(s) = \begin{cases} \cos(\hat{T}s) - 1 & \text{on } [0, \theta/\hat{T}], \\ \hat{T}\sin\theta(\theta/\hat{T} - s) + \cos\theta - 1 & \text{on } (\theta/\hat{T}, 1], \end{cases}$$

$$\hat{x}_2(s) = \begin{cases} \sin\hat{T}s & \text{on } [0, \theta/\hat{T}], \\ \hat{T}\cos\theta(s - \theta/\hat{T}) + \sin\theta & \text{on } (\theta, \hat{T}]. \end{cases}$$

Since the terminal values for x_1 and x_2 are fixed, the final time $\hat{T}$ is determined by the previous equalities. The pre-Hamiltonian for problem (1.106) is

$$H[\lambda](s) := T(s)(-\psi_1(s)\sin x_3(s) + \psi_2(s)\cos x_3(s) + \psi_3(s)u(s)). \tag{1.108}$$

The final Lagrangian is

$$\ell := \alpha_0 T(1) + \sum_{j=1}^{3} (\beta^j x_j(0) + \beta_j x_j(1)).$$

As $\dot{\psi}_1 \equiv 0$, and $\dot{\psi}_2 \equiv 0$, we get

$$\psi_1 \equiv \beta_1, \quad \psi_2 \equiv \beta_2, \quad \text{on } [0, 1].$$

Since the candidate control $\hat{u}$ is singular on $[\theta/\hat{T}, 1]$, we have $H_u[\lambda] \equiv 0$. By (1.108), we obtain

$$\psi_3(s) = 0, \quad \text{on } [\theta/\hat{T}, 1]. \tag{1.109}$$

Thus $\beta_3 = 0$. In addition, as the costate equation for ψ_3 is

$$-\dot{\psi}_3 = \hat{T}(-\beta_1\cos\hat{x}_3 - \beta_2\sin\hat{x}_3),$$

by (1.107) and (1.109), we get

$$\beta_1\cos\theta + \beta_2\sin\theta = 0. \tag{1.110}$$

From (1.107) and (1.109) and since H is constant and equal to $-\alpha_0$, we get

$$H = \hat{T}(-\beta_1 \sin\theta + \beta_2 \cos\theta) \equiv -\alpha_0. \tag{1.111}$$

Proposition 1.6.3. *The following properties hold*

(i) $\alpha_0 > 0$,

(ii) $H_u[\lambda](s) < 0$ *on* $[0, \theta/\hat{T})$ *for all* $\lambda \in \Lambda$.

Proof. **Item (i)** Suppose that $\alpha_0 = 0$. By (1.110) and (1.111), we obtain

$$\beta_1 \cos\theta + \beta_2 \sin\theta = 0, \text{ and } -\beta_1 \sin\theta + \beta_2 \cos\theta = 0.$$

Suppose, w.l.g., that $\cos\theta \neq 0$. Then $\beta_1 = -\beta_2 \dfrac{\sin\theta}{\cos\theta}$ and thus

$$\beta_2 \frac{\sin^2\theta}{\cos\theta} + \beta_2 \cos\theta = 0.$$

We conclude that $\beta_2 = 0$ as well. This implies $(\alpha_0, \beta_1, \beta_2, \beta_3) = 0$, which contradicts the non-triviality condition (1.7). So, $\alpha_0 > 0$, as required.
Item (ii) Observe that

$$H_u[\lambda](s) \leq 0, \quad \text{on } [0, \theta/\hat{T}),$$

and $H_u[\lambda] = \psi_3$. Let us prove that ψ_3 is never 0 on $[0, \theta/\hat{T})$. Suppose there exists $s_1 \in [0, \theta/\hat{T})$ such that $\psi_3(s_1) = 0$. Thus, since $\psi_3(\theta/\hat{T}) = 0$ as indicated in (1.109), there exists $s_2 \in (s_1, \theta/\hat{T})$ such that $\dot{\psi}_3(s_2) = 0$, i.e.

$$\beta_1 \cos(\hat{T}s_2) + \beta_2 \sin(\hat{T}s_2) = 0. \tag{1.112}$$

Equations (1.110) and (1.112) imply that $\tan(\theta/\hat{T}) = \tan(s_2/\hat{T})$. This contradicts $\theta < \pi$. Thus $\psi_3(s) \neq 0$ for every $s \in [0, \theta/\hat{T})$, and consequently,

$$H_u[\lambda](s) < 0, \quad \text{for } s \in [0, \theta/\hat{T}).$$

□

Since $\alpha_0 > 0$, then $\delta T = 0$ for each element of the critical cone, where δT is the linearized state variable T. Observe that as $\hat{u} = 1$ on $[0, \theta/\hat{T}]$, then

$$y = 0 \text{ and } \xi = 0, \text{ on } [0, \theta/\hat{T}], \text{ for all } (\xi, \delta T, y, h) \in \mathcal{P}_2.$$

We look for the second variation in the interval $[\theta/\hat{T}, 1]$. The Goh transformation gives

$$\xi_3 = z_3 - \hat{T}y,$$

and since $\dot{z}_3 = \hat{T}v$, we get $z_3 = \hat{T}y$ and thus $\xi_3 = 0$. Then, as $H_{ux} = 0$ and $\ell'' = 0$, we get

$$\Omega[\lambda] = \int_{\theta/\hat{T}}^{1} (\beta_1 \sin\theta - \beta_2 \cos\theta) y^2 dt = \alpha_0 \int_0^1 y^2 dt.$$

Notice that if $(\xi, \delta T, y, h) \in \mathcal{P}_2$, then h satisfies $\xi_3(T) + \hat{T}h = 0$, and, as $\xi_3(T) = 0$, we get $h = 0$. Thus

$$\Omega[\lambda](\xi, y, h) = \alpha_0 \int_0^T y^2 dt = \alpha_0 \gamma(y, h), \quad \text{on } \mathcal{P}_2.$$

Since Assumptions 3 and 4 hold, we conclude by Corollary 1.6.2 that $(\hat{x}, \hat{T}, \hat{u})$ is a Pontryagin minimum satisfying quadratic growth.

1.7 Conclusion

We provided a set of necessary and sufficient conditions for a bang-singular extremal. The sufficient condition is restricted to the scalar control case. These necessary and sufficient conditions are close in the sense that, to pass from one to the other, one has to strengthen a non-negativity inequality transforming it into a coercivity condition.

This is the first time that a sufficient condition that is 'almost necessary' is established for a bang-singular extremal for the general Mayer problem. In some cases the condition can be easily checked as it can be seen in the example.

1.8 Appendix

Lemma 1.8.1. *Let*

$$X := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m : \ (1.29), (1.33)\text{-}(1.34) \text{ hold}\},$$

$$L := \{(\xi, y, y(T)) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m : y(0) = 0, \ (1.29) \text{ and } (1.33)\}.$$

Then L is a dense subset of X in the $\mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m-$topology.

Proof. (See Lemma 6 in [47].) Let us prove the result for $m = 1$. The general case is a trivial extension. Let $(\bar{\xi}, \bar{y}, \bar{h}) \in X$ and $\varepsilon, \delta > 0$. Consider $\phi \in \mathcal{Y}$ such that $\|\bar{y} - \phi\|_2 < \varepsilon/2$. In order to satisfy condition (1.34) take

$$\begin{cases} y_\delta(t) := 0, & \text{for } t \in [0, d_1], \quad \text{if } c_1 = 0, \\ y_\delta(t) := h, & \text{for } t \in [c_N, T], \quad \text{if } d_N = T, \end{cases}$$

where c_j, d_j were introduced in Assumption 2. Since $\bar{y}$ is constant on each I_j, define y_δ constant over these intervals with the same constant value as $\bar{y}$. It remains to define

y_δ over I_+. Over each maximal composing interval (a,b) of I_+, define y_δ as described below. Take $c := \bar{y}(a-)$ if $a > 0$, or $c := 0$ if $a = 0$; and let $d := \bar{y}(b+)$ if $b < T$, or $d := h$ when $b = T$. Define two affine functions $\ell_{1,\delta}$ and $\ell_{2,\delta}$ satisfying

$$\begin{aligned} &\ell_{1,\delta}(a) = c,\ \ \ell_{1,\delta}(a+\delta) = \phi(a+\delta), \\ &\ell_{2,\delta}(b) = d,\ \ \ell_{2,\delta}(b-\delta) = \phi(b-\delta). \end{aligned} \tag{1.113}$$

Take

$$y_\delta(t) := \begin{cases} \ell_{1,\delta}(t), & \text{for } t \in [a, a+\delta], \\ \phi(t), & \text{for } t \in (a+\delta, b-\delta), \\ \ell_{2,\delta}(t), & \text{for } t \in [b-\delta, b], \end{cases} \tag{1.114}$$

and notice that $\|\phi - y_\delta\|_{2,[a,b]} \leq \frac{1}{k}\max(|c|,|d|,M)$, where $M := \sup_{t\in[a,b]}|\phi(t)|$. Finally, observe that $y_\delta(T) = h$, and, for sufficiently small δ,

$$\|\bar{y} - y_\delta\|_2 \leq \|\bar{y} - \phi\|_2 + \|\phi - y_\delta\|_2 < \varepsilon.$$

Thus, the result follows. $\square$

Lemma 1.8.2. *Let $\lambda \in \Lambda$ and $(z,v) \in \mathcal{C}_2$. Then*

$$\sum_{i=0}^{d_\varphi} \alpha_i \bar{\varphi}_i''(\hat{u})(v,v) + \sum_{i=1}^{d_\eta} \beta_j \bar{\eta}_j''(\hat{u})(v,v) = \Omega[\lambda](z,v). \tag{1.115}$$

Proof. Let us compute the left-hand side of (1.115). Notice that

$$\sum_{i=0}^{d_\varphi} \alpha_i \bar{\varphi}_i(\hat{u}) + \sum_{i=1}^{d_\eta} \beta_j \bar{\eta}_j(\hat{u}) = \ell[\lambda](\hat{x}(T)). \tag{1.116}$$

Let us look for a second order expansion for ℓ. Consider first a second order expansion of the state variable:

$$x = \hat{x} + z + \tfrac{1}{2} z_{vv} + o(\|v\|_\infty^2),$$

where z_{vv} satisfies

$$\dot{z}_{vv} = A z_{vv} + D^2_{(x,u)^2} F(\hat{x},\hat{u})(z,v)^2, \quad z_{vv}(0) = 0, \tag{1.117}$$

with $F(x,u) := \sum_{i=0}^m u_i f_i(x)$. Consider the second order expansion for ℓ :

$$\begin{aligned} \ell[\lambda](x(T)) =& \ell[\lambda]((\hat{x} + z + \tfrac{1}{2} z_{vv})(T)) + o(\|v\|_1^2) \\ =& \ell[\lambda](\hat{x}(T)) + \ell'[\lambda](\hat{x}(T))(z(T) + \tfrac{1}{2} z_{vv}(T)) \\ &+ \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))(z(T) + \tfrac{1}{2} z_{vv}(T))^2 + o(\|v\|_1^2). \end{aligned} \tag{1.118}$$

Step 1. Compute

$$\begin{aligned}
\ell'[\lambda](\hat{x}(T))z_{vv}(T) &= \psi(T)z_{vv}(T) - \psi(0)z_{vv}(0)\\
&= \int_0^T [\dot{\psi}z_{vv} + \psi\dot{z}_{vv}]\mathrm{d}t = \int_0^T \{-\psi A z_{vv} + \psi(Az_{vv} + D^2F_{(x,u)^2}(z,v)^2)\}\mathrm{d}t\\
&= \int_0^T D^2H[\lambda](z,v)^2\mathrm{d}t.
\end{aligned}$$

Step 2. Compute $\ell''[\lambda](\hat{x}(T))(z(T), z_{vv}(T))$. Applying Gronwall's Lemma, we obtain $\|z\|_\infty = O(\|v\|_1)$, and $\|z_{vv}\|_\infty = O(\|v^2\|_1)$. Thus

$$|(z(T), z_{vv}(T))| = O(\|v\|_1^3),$$

and we conclude that

$$|\ell''[\lambda](\hat{x}(T))(z(T), z_{vv}(T))| = O(\|v\|_1^3).$$

Step 3. See that $\ell''[\lambda](\hat{x}(T))(z_{vv}(T))^2 = O(\|v\|_1^4)$. Then by (1.118) we get,

$$\begin{aligned}
\ell[\lambda](x(T)) =& \ell[\lambda](\hat{x}(T)) + \ell'[\lambda](\hat{x}(T))z(T)\\
&+ \tfrac{1}{2}\ell''[\lambda](\hat{x}(T))z^2(T) + \tfrac{1}{2}\int_0^T D^2_{(x,u)^2}H[\lambda](z,v)^2\mathrm{d}t + o(\|v\|_1^2)\\
=& \ell[\lambda](\hat{x}(T)) + \ell'[\lambda](\hat{x}(T))z(T) + \Omega[\lambda](z,v) + o(\|v\|_1^2).
\end{aligned}$$

The conclusion follows by (1.116). □

Lemma 1.8.3. *Given $(z,v) \in \mathcal{W}$ satisfying* (1.12)*, the following estimation holds for some $\rho > 0$:*

$$\|z\|_2^2 + |z(T)|^2 \leq \rho\gamma(y, y(T)),$$

where y is defined by (1.28)*.*

Remark 1.8.4. *ρ depends on $\hat{w}$, i.e. it does not vary with (z,v).*

Proof. Every time we mention ρ_i we are referring to a constant depending on $\|A\|_\infty$, $\|B\|_\infty$ or both. Consider ξ, the solution of equation (1.29) corresponding to y. Gronwall's Lemma and the Cauchy-Schwartz inequality imply

$$\|\xi\|_\infty \leq \rho_1\|y\|_2. \tag{1.119}$$

This last inequality, together with expression (1.28), implies

$$\|z\|_2 \leq \|\xi\|_2 + \|B\|_\infty\|y\|_2 \leq \rho_2\|y\|_2. \tag{1.120}$$

On the other hand, equations (1.28) and (1.119) lead to

$$|z(T)| \leq |\xi(T)| + \|B\|_\infty |y(T)| \leq \rho_1 \|y\|_2 + \|B\|_\infty |y(T)|.$$

Then, by the inequality $ab \leq \frac{a^2+b^2}{2}$, we get

$$|z(T)|^2 \leq \rho_3(\|y\|_2^2 + |y(T)|^2). \tag{1.121}$$

The conclusion follows from equations (1.120) and (1.121). □

The next lemma is a generalization of the previous result to the nonlinear case. See Lemma 6.1 in Dmitruk [43].

Lemma 1.8.5. *Let $w = (x, u)$ be the solution of* (1.2) *with $\|u\|_2 \leq c$ for some constant c. Put $(\delta x, v) := w - \hat{w}$. Then*

$$|\delta x(T)|^2 + \|\delta x\|_2^2 \leq \rho \gamma(y, y(T)),$$

where y is defined by (1.28) *and ρ depends on c.*

Lemma 1.8.6. *Let $\{y_k\} \subset L_2(a, b)$ be a sequence of continuous non-decreasing functions that converges weakly to $y \in L_2(a, b)$. Then y is non-decreasing.*

Proof. Let $s, t \in (a, b)$ be such that $s < t$, and $\varepsilon_0 > 0$ such that $s + \varepsilon_0 < t - \varepsilon_0$. For every $k \in I\!N$, and every $0 < \varepsilon < \varepsilon_0$, the following inequality holds

$$\int_{s-\varepsilon}^{s+\varepsilon} y_k(\nu) d\nu \leq \int_{t-\varepsilon}^{t+\varepsilon} y_k(\nu) d\nu.$$

Taking the limit as k goes to infinity and multiplying by $\frac{1}{2\varepsilon}$, we deduce that

$$\frac{1}{2\varepsilon} \int_{s-\varepsilon}^{s+\varepsilon} y(\nu) d\nu \leq \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} y(\nu) d\nu.$$

As (a, b) is a finite measure space, y is a function of $L_1(a, b)$ and almost all points in (a, b) are Lebesgue points (see Rudin [114, Theorem 7.7]). Thus, by taking ε to 0, it follows from the previous inequality that

$$y(s) \leq y(t),$$

which is what we wanted to prove. □

Lemma 1.8.7. *Consider a sequence $\{y_k\}$ of non-decreasing continuous functions in a compact real interval I and assume that $\{y_k\}$ converges weakly to 0 in $\mathcal{U}_2$. Then it converges uniformly to 0 on any interval $(a, b) \subset I$.*

Proof. Take an arbitrary interval $(a,b) \subset I$. First prove the pointwise convergence of $\{y_k\}$ to 0. On the contrary, suppose that there exists $c \in (a,b)$ such that $\{y_k(c)\}$ does not converge to 0. Thus there exist $\varepsilon > 0$ and a subsequence $\{y_{k_j}\}$ such that $y_{k_j}(c) > \varepsilon$ for each $j \in I\!N$, or $y_{k_j}(c) < -\varepsilon$ for each $j \in I\!N$. Suppose, without loss of generality, that the first statement is true. Thus

$$0 < \varepsilon(b-c) < y_{k_j}(c)(b-c) \leq \int_c^b y_{k_j}(t)\mathrm{d}t, \tag{1.122}$$

where the last inequality holds since y_{k_j} is nondecreasing. But the right-hand side of (1.122) goes to 0 as j goes to infinity. This contradicts the hypothesis and thus the pointwise convergence of $\{y_k\}$ to 0 follows. The uniform convergence is a direct consequence of the monotonicity of the functions y_k. □

Lemma 1.8.8. *[50, Theorem 22, Page 154 - Volume I] Let a and b be two functions of bounded variation in $[0,T]$. Suppose that one is continuous and the other is right-continuous. Then*

$$\int_0^T a(t)db(t) + \int_0^T b(t)da(t) = [ab]_{0-}^{T+}.$$

Lemma 1.8.9. *Let $m = 1$, i.e. consider a scalar control variable. Then, for any $\lambda \in \Lambda$, the function $R[\lambda](t)$ defined in (1.39) is continuous in t.*

Proof. Consider definition (1.36). Condition $V[\lambda] \equiv 0$ yields $S[\lambda] = C[\lambda]B$, and since $R[\lambda]$ is scalar, we can write

$$R[\lambda] = B^\top Q[\lambda]B - 2C[\lambda]B_1 - \dot{C}[\lambda]B - C[\lambda]\dot{B}.$$

Note that $B = f_1$, $B_1 = [f_0, f_1]$, $C[\lambda] = -\psi f_1'$, and $Q[\lambda] = -\psi(f_0'' + \hat{u}f_1''$. Thus

$$\begin{aligned} R[\lambda] =& \psi(f_0'' + \hat{u}f_1'')(f_1, f_1) - 2\psi f_1'(f_0'f_1 - f_1'f_0) \\ &+ \psi(f_0' + \hat{u}f_1')f_1'f_1 - \psi f_1''(f_0 + \hat{u}f_1)f_1 - \psi f_1'f_1'(f_0 + \hat{u}f_1) \\ =& \psi[f_1, [f_1, f_0]]. \end{aligned}$$

Since f_0 and f_1 are twice continuously differentiable, we conclude that $R[\lambda]$ is continuous in time. □

Lemma 1.8.10. *[72] Consider a quadratic form $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ where $\mathcal{Q}_1$ is a Legendre form and $\mathcal{Q}_2$ is weakly continuous over some Hilbert space. Then $\mathcal{Q}$ is a Legendre form.*

Lemma 1.8.11. *[72, Theorem 3.2] Consider a real interval I and a quadratic form $\mathcal{Q}$ over the Hilbert space $L_2(I)$, given by*

$$\mathcal{Q}(y) := \int_I y^\top(t)R(t)y(t)\mathrm{d}t.$$

Then $\mathcal{Q}$ is weakly l.s.c. over $L_2(I)$ iff

$$R(t) \succeq 0, \quad \text{a.e. on } I. \tag{1.123}$$

Lemma 1.8.12. *[42, Theorem 5] Given a Hilbert space H, and $a_1, a_2, \dots, a_p \in H$, set*

$$K := \{x \in H : (a_i, x) \leq 0, \text{ for } i = 1, \dots, p\}.$$

Let M be a convex and compact subset of $\mathbb{R}^s$, and let $\{Q^\psi : \psi \in M\}$ be a family of continuous quadratic forms over H with the mapping $\psi \to Q^\psi$ being affine. Set $M_\# := \{\psi \in M : \ Q^\psi \text{ is weakly l.s.c.}\}$ and assume that

$$\max_{\psi \in M} Q^\psi(x) \geq 0, \text{ for all } x \in K.$$

Then

$$\max_{\psi \in M_\#} Q^\psi(x) \geq 0, \text{ for all } x \in K.$$

The following result is an adaptation of Lemma 6.5 in [43].

Lemma 1.8.13. *Consider a sequence $\{v_k\} \subset \mathcal{U}$ and $\{y_k\}$ their primitives defined by (1.28). Call $u_k := \hat{u} + v_k$, x_k its corresponding solution of (1.2), and let z_k denote the linearized state corresponding to v_k, i.e. the solution of (1.12). Define, for each $k \in I\!N$,*

$$\delta x_k := x_k - \hat{x}, \quad \eta_k := \delta x_k - z_k, \quad \gamma_k := \gamma(y_k, y_k(T)). \tag{1.124}$$

Suppose that $\{v_k\}$ converges to 0 in the Pontryagin sense. Then

(i)

$$\dot{\eta}_k = \sum_{i=0}^m \hat{u}_i f_i'(\hat{x})\eta_k + \sum_{i=1}^m v_{i,k} f_i'(\hat{x})\delta x_k + \zeta_k, \tag{1.125}$$

$$\dot{\delta x}_k = \sum_{i=0}^m u_{i,k} f_i'(\hat{x})\delta x_k + \sum_{i=1}^m v_{i,k} f_i(\hat{x}) + \zeta_k, \tag{1.126}$$

where $\|\zeta_k\|_2 \leq o(\sqrt{\gamma_k})$ and $\|\zeta_k\|_\infty \to 0$,

(ii) $\|\eta_k\|_\infty \leq o(\sqrt{\gamma_k})$.

Proof. **(i,ii)** Consider the second order Taylor expansions of f_i,

$$f_i(x_k) = f_i(\hat{x}) + f_i'(\hat{x})\delta x_k + \tfrac{1}{2} f_i''(\hat{x})(\delta x_k, \delta x_k) + o(|\delta x_k(t)|^2).$$

We can write

$$\dot{\delta x}_k = \sum_{i=0}^{m} u_{i,k} f_i'(\hat{x})\delta x_k + \sum_{i=1}^{m} v_{i,k} f_i(\hat{x}) + \zeta_k, \tag{1.127}$$

with

$$\zeta_k := \tfrac{1}{2} \sum_{i=0}^{m} u_{i,k} f_i''(\hat{x})(\delta x_k, \delta x_k) + o(|\delta x_k(t)|^2) \sum_{i=0}^{m} u_{i,k}. \tag{1.128}$$

As $\{u_k\}$ is bounded in L_∞ and $\|\delta x_k\|_\infty \to 0$, we get $\|\zeta_k\|_\infty \to 0$ and the following L_2-norm bound:

$$\begin{aligned} \|\zeta_k\|_2 &\leq const. \sum_{i=0}^{m} \|u_{i,k}(\delta x_k, \delta x_k)\|_2 + o(\gamma_k) \| \sum_{i=0}^{m} u_{i,k}\|_1 \\ &\leq const. \|u_k\|_\infty \|\delta x_k\|_2^2 = O(\gamma_k) \leq o(\sqrt{\gamma_k}). \end{aligned} \tag{1.129}$$

Let us look for the differential equation of η_k defined in (1.124). By (1.127), and adding and substracting the term $\sum_{i=1}^{m} \hat{u}_i f_i'(\hat{x})\delta x_k$ we obtain

$$\dot{\eta}_k = \sum_{i=0}^{m} \hat{u}_i f_i'(\hat{x})\eta_k + \sum_{i=1}^{m} v_{i,k} f_i'(\hat{x})\delta x_k + \zeta_k.$$

Thus we obtain **(i).** Applying Gronwall's Lemma to this last differential equation we get

$$\|\eta_k\|_\infty \leq \| \sum_{i=1}^{m} v_{i,k} f_i'(\hat{x})\delta x_k + \zeta_k\|_1. \tag{1.130}$$

Since $\|v_k\|_\infty < N$ and $\|v_k\|_1 \to 0$, we also find that $\|v_k\|_2 \to 0$. Applying the Cauchy-Schwartz inequality to (1.130), from (1.129) we get **(ii).**

$\square$

2

A shooting algorithm for problems with singular arcs

Joint work with J.F. Bonnans and P. Martinon. Published as the INRIA Research Report Nr. 7763 [October 2011], under the title *A well-posed shooting algorithm for problems with singular arcs.*

Abstract

In this article we propose a shooting algorithm for a class of optimal control problems for which all control variables appear linearly. The shooting system has, in the general case, more equations than unknowns and the Gauss-Newton method is used to compute a zero of the shooting function. This shooting algorithm is locally quadratically convergent if the derivative of the shooting function is one-to-one at the solution. The main result of this paper is to show that the latter holds whenever a sufficient condition for weak optimality is satisfied. We note that this condition is very close to a second order necessary condition. For the case when the shooting system can be reduced to one having the same number of unknowns and equations (square system) we prove that the mentioned sufficient condition guarantees the stability of the optimal solution under small perturbations and the invertibility of the Jacobian matrix of the shooting function associated to the perturbed problem. We present numerical tests that validate our method.

2.1 Introduction

The classical shooting method is used to solve boundary value problems. Hence, it is used to compute the solution of optimal control problems by solving the boundary value problem derived from the Pontryagin Maximum Principle.

Some references can be mentioned regarding the shooting method. The first two works we can find in the literature, dating from years 1956 and 1962 respectively, are Goodman-Lance [70] and Morrison et al. [98]. Both present the same method for solving two-point boundary value problems in a general setting, not necessarily related to an optimal control problem. The latter article applies to more general formulations. The method was studied in detail in Keller's book [78], and later on Bulirsch [31] applied it to the resolution of optimal control problems.

The case we deal with in this paper where the shooting method is used to solve optimal control problems with control-affine systems is treated in, e.g., Maurer [92], Oberle [101, 102], Fraser-Andrews [60], Martinon [89] and Vossen [127]. These works provided a series of algorithms and numerical examples with different control structures, but no theoretical foundation is supplied. In particular, Vossen [127] dealt with a problem in which the control can be written as a function of the state variable, i.e. the control has a feedback representation. He proposed an algorithm that involved a finite dimensional optimization problem induced by the switching times. The main difference between Vossen's work and the study here presented is that we treat the general problem (no feedback law is necessary). Furthermore we justify the well-definition and the convergence of our algorithm via second order sufficient conditions of the original control problem. In some of the just mentioned papers the control variable had only some of its components entering linearly. This particular structure is studied in more detailed in Aronna [7], and in the present article we study problems having all affine inputs.

In [25] Bonnard and Kupka studied the optimal time problem of a generic single-input affine system without control constraints, with fixed initial point and terminal point constrained to a given manifold. For this class of problems they established a link between the injectivity of the shooting function and the optimality of the trajectory by means of the conjugate and focal points theory. Bonnard et al. [23] provides a survey on a series of algorithms for the numerical computation of these points, which can be employed to test the injectivity of the shooting function in some cases. The reader is referred to [23], Bonnard-Chyba [24] and references therein for further information about this topic.

In addition, Malanowski-Maurer [87] and Bonnans-Hermant [20] dealt with a problem having mixed control-state and pure state running constraints and satisfying the strong Legendre-Clebsch condition (which is not verified in our affine-input case). They all established a link between the invertibility of the Jacobian of the shooting function and some second order sufficient condition for optimality. They provided stability analysis as well.

We start this article by presenting an optimal control problem affine in the control, with terminal constraints and free control variables. For this kind of problem we state a set of optimality conditions which is equivalent to the Pontryagin Maximum Principle. Afterwards, the second order strengthened generalized Legendre-Clebsch condition is used to eliminate the control variable from the stationarity condition. The resulting set of conditions turns out to be a two-point boundary value problem, i.e. a system of ordinary differential equations having boundary conditions both in the initial and final times. We define the shooting function as the mapping that assigns to each estimate of the initial values, the value of the final condition of the corresponding solution. The shooting algorithm consists of approximating a zero of this function. In other words, the method finds suitable initial values for which the corresponding solution of the differential equation system satisfies the final conditions.

Since the number of equations happens to be, in general, greater than the number of unknowns, the Gauss-Newton method is a suitable approach for solving this overdetermined system of equations. The reader is referred to Dennis [38], Fletcher [59] and Dennis et al. [39] for details and implementations of Gauss-Newton technique. This method is applicable when the derivative of the shooting function is one-to-one at the solution, and in this case it converges locally quadratically.

The main result of this paper is to provide a sufficient condition for the injectivity of this derivative, and to notice that this condition is quite weak since, for qualified problems, it characterizes quadratic growth in the weak sense (see Dmitruk [40, 43]). Once the unconstrained case is investigated, we pass to a problem having bounded controls. To treat this case, we perform a transformation yielding a new problem without bounds, we prove that an optimal solution of the original problem is also optimal for the transformed one and we apply our above-mentioned result to this modified formulation.

It is interesting to mention that by means of the latter result we can justify, in particular, the invertibility of the Jacobian of the shooting function proposed by Maurer [92]. In this work, Maurer suggested a method to treat problems having scalar bang-singular-bang solutions and provided a square system of equations (i.e. a system having as many equations as unknowns) meant to be solved by Newton's algorithm. However, the systems that can be encountered in practice may not be square and hence our approach is suitable.

We provide a deeper analysis in the case when the shooting system can be reduced to one having equal number of equations and unknowns. In this framework, we investigate the stability of the optimal solution. It is shown that the above-mentioned sufficient condition guarantees the stability of the optimal solution under small perturbation of the data and the invertibility of the Jacobian of the shooting function associated to the perturbed problem. Felgenhauer in [56, 55] provided sufficient conditions for the stability of the structure of the optimal control, but assuming that the perturbed problem had an optimal solution.

Our article is organized as follows. In section 2.2 we present the optimal control problem without bound constraints, for which we provide an optimality system in section 2.3. We give a description of the shooting method in section 2.4. In section 2.5 we present a set of second order necessary and sufficient conditions, and the statement of the main result. We introduce a linear quadratic optimal control problem in section 2.6. In section 2.7 we present a variable transformation relating the shooting system and the optimality system of the linear quadratic problem mentioned above. In section 2.8 we deal with the control constrained case. A stability analysis for both unconstrained and constrained control cases is provided in section 2.9. Finally we present some numerical tests in section 2.10, and we devote section 2.11 to the conclusions of the article.

2.2 Statement of the Problem

Consider the spaces $\mathcal{U} := L_\infty(0,T;\mathbb{R}^m)$ and $\mathcal{X} := W^1_\infty(0,T;\mathbb{R}^n)$, as control and state spaces, respectively. Denote by u and x their elements, respectively. When needed, put $w = (x,u)$ for a point in the product space $\mathcal{W} := \mathcal{X} \times \mathcal{U}$. In this paper we investigate the optimal control problem

$$J := \varphi_0(x_0, x_T) \to \min, \tag{2.1}$$

$$\dot{x}_t = \sum_{i=0}^m u_{i,t} f_i(x_t), \qquad \text{a.e. on } [0,T], \tag{2.2}$$

$$\eta_j(x_0, x_T) = 0, \quad \text{for } j = 1, \dots, d_\eta, \tag{2.3}$$

where final time T is fixed, $u_0 \equiv 1$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i = 0, \dots, m$ and $\eta_j : \mathbb{R}^{2n} \to \mathbb{R}$ for $j = 1, \dots, d_\eta$. Assume that data functions φ_0, f_i and η_j have Lipschitz-continuous second derivatives. Denote by (P) the problem defined by (2.1)-(2.3). An element $w \in \mathcal{W}$ satisfying (2.2)-(2.3) is called a *feasible trajectory.*

Set $\mathcal{X}_* := W^1_\infty(0,T;\mathbb{R}^{n,*})$ the space of Lipschitz-continuous functions with values in the $n-$dimensional space of row-vectors with real components $\mathbb{R}^{n,*}$. Consider an element $\lambda := (\beta, p) \in \mathbb{R}^{d_\eta,*} \times \mathcal{X}_*$ and define the *pre-Hamiltonian* function

$$H[\lambda](x,u,t) := p_t \sum_{i=0}^m u_i f_i(x), \tag{2.4}$$

the *initial-final Lagrangian* function

$$\ell[\lambda](\zeta_0, \zeta_T) := \varphi_0(\zeta_0, \zeta_T) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(\zeta_0, \zeta_T), \tag{2.5}$$

and the *Lagrangian* function

$$\mathcal{L}[\lambda](w) := \ell[\lambda](x_0, x_T) + \int_0^T p_t \left(\sum_{i=0}^m u_{i,t} f_i(x_t) - \dot{x}_t \right) \mathrm{d}t. \tag{2.6}$$

We study a nominal feasible trajectory $\hat{w} = (\hat{x}, \hat{u})$. Next we present a *qualification hypothesis* that is assumed throughout the article. Consider the mapping

$$\begin{array}{rcl} G\colon \mathbb{R}^n \times \mathcal{U} & \to & \mathbb{R}^{d_\eta} \\ (x_0, u) & \mapsto & \eta(x_0, x_T), \end{array} \tag{2.7}$$

where x_T is the solution of (2.2) associated to (x_0, u).

Assumption 2.2.1. *The derivative of G at $(\hat{x}_0, \hat{u})$ is onto.*

Assumption 2.2.1 is usually known as *qualification of equality constraints.*

Definition 2.2.2. *It is said that the trajectory $\hat{w}$ is a weak minimum of problem (P) if there exists $\varepsilon > 0$ such that $\hat{w}$ is a minimum in the set of feasible trajectories $w = (x, u) \in \mathcal{W}$ satisfying*

$$\|x - \hat{x}\|_\infty < \varepsilon, \quad \|u - \hat{u}\|_\infty < \varepsilon.$$

The following first order necessary condition holds for $\hat{w}$.

Theorem 2.2.3. *If $\hat{w}$ is a weak solution then there exists $\lambda = (\beta, p) \in \mathbb{R}^{d_\eta,*} \times \mathcal{X}_*$ such that p is solution of the costate equation*

$$-\dot{p}_t = D_x H[\lambda](\hat{x}_t, \hat{u}_t, t), \quad \text{a.e. on } [0, T], \tag{2.8}$$

with transversality conditions

$$p_0 = -D_{x_0}\ell[\lambda](\hat{x}_0, \hat{x}_T), \tag{2.9}$$

$$p_T = D_{x_T}\ell[\lambda](\hat{x}_0, \hat{x}_T), \tag{2.10}$$

and the stationarity condition

$$D_u H[\lambda](\hat{x}_t, \hat{u}_t, t) = 0, \quad \text{a.e. on } [0, T], \tag{2.11}$$

is verified.

It follows easily that since the pre-Hamiltonian H is affine in all the control variables, (2.11) is equivalent to the *minimum condition*

$$H[\lambda](\hat{x}_t, \hat{u}_t, t) = \min_{v \in \mathbb{R}^m} H[\lambda](\hat{x}_t, v, t), \quad \text{a.e. on } [0, T]. \tag{2.12}$$

In order words, the element $(\hat{w}, \lambda)$ in Theorem 2.2.3 satisfies the *qualified Pontryagin Maximum Principle* and λ is a *Pontryagin multiplier.* It is known that the Assumption 2.2.1 implies the existence and uniqueness of multiplier. We denote this unique multiplier by $\hat{\lambda} = (\hat{\beta}, \hat{p})$.

Let the *switching function* $\Phi : [0, T] \to \mathbb{R}^{m,*}$ be defined by

$$\Phi_t := D_u H[\hat{\lambda}](\hat{x}_t, \hat{u}_t, t) = (\hat{p}_t f_i(\hat{x}_t))_{i=1}^m. \tag{2.13}$$

Observe that the stationarity condition (2.11) can be written as

$$\Phi_t = 0, \quad \text{a.e. on } [0, T]. \tag{2.14}$$

2.3 Optimality System

In this section we present an optimality system, i.e. a set of equations that are necessary for optimality. We obtain this system from the conditions in Theorem 2.2.3 above and assuming that the *strengthened generalized Legendre-Clebsch condition* (to be defined below) holds.

Observe that, since H is affine in the control, the switching function Φ introduced in (2.13) does not depend explicitly on u. Let an index $i = 1, \dots, m$, and $(\mathrm{d}^{M_i}\Phi/\mathrm{d}t^{M_i})$ be the lowest order derivative of Φ in which u_i appears with a coefficient that is not identically zero on $(0, T)$. In Kelley et al. [81] it is stated that M_i is even, assuming that the extremal is normal (as it is the case here since $\hat{w}$ satisfies the PMP in its qualified form). The integer $N_i := M_i/2$ is called *order of the singular arc.* As we have just said, the control u cannot be retrieved from equation (2.11). In order to be able to express $\hat{u}$ in terms of $(\hat{p}, \hat{x})$ from equation

$$\ddot{\Phi}_t = 0, \quad \text{a.e. on } [0, T], \tag{2.15}$$

we make the following hypothesis.

Assumption 2.3.1. *The strengthened generalized Legendre-Clebsch condition (see e.g.*
Kelley [79] and Goh [67]) holds, i.e.

$$-\frac{\partial}{\partial u}\ddot{\Phi}_t \succ 0, \quad \text{on } [0, T]. \tag{2.16}$$

Here, by $X \succ 0$ we mean that the matrix X is positive definite. Notice that function $\ddot{\Phi}$ is affine in u, and thus $\hat{u}$ can be written in terms of $(\hat{p}, \hat{x})$ from (2.15) by inverting the matrix in (2.16). Furthermore, due to the regularity hypothesis imposed on the data functions, $\hat{u}$ turns out to be a continuous function of time.

Hence, the condition (2.15) is included in our optimality system and we can use it to compute $\hat{u}$ in view of Assumption 2.3.1. In order to guarantee the stationarity condition (2.14) we consider the endpoint conditions

$$\Phi_T = 0, \ \dot{\Phi}_0 = 0. \tag{2.17}$$

Remark 2.3.2. *We could choose another pair of endpoint conditions among the four possible ones:* $\Phi_0 = 0$, $\Phi_T = 0$, $\dot{\Phi}_0 = 0$ *and* $\dot{\Phi}_T = 0$, *always including at least one of order zero. The choice we made will simplify the presentation of the result afterwards.*

Notation: Denote by (OS) the set of equations composed by (2.2)-(2.3),(2.8)-(2.10), (2.15), (2.17), i.e. the system

$$\begin{cases} \dot{x}_t = \sum_{i=0}^{m} u_{i,t} f_i(x_t), \quad \text{a.e. on } [0,T], \\ \eta_j(x_0, x_T) = 0, \quad \text{for } j = 1, \dots, d_\eta, \\ -\dot{p}_t = D_x H[\lambda](\hat{x}_t, \hat{u}_t, t), \quad \text{a.e. on } [0,T], \\ p_0 = -D_{x_0} \ell[\lambda](\hat{x}_0, \hat{x}_T), \\ p_T = D_{x_T} \ell[\lambda](\hat{x}_0, \hat{x}_T), \\ \ddot{\Phi}_t = 0, \quad \text{a.e. on } [0,T], \\ \Phi_T = 0, \ \dot{\Phi}_0 = 0. \end{cases} \tag{OS}$$

Let us give explicit expressions for $\dot{\Phi}$ and $\ddot{\Phi}$. Denote the *Lie bracket* of two smooth vector fields $g, h\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$[g,h](x) := g'(x)h(x) - h'(x)g(x). \tag{2.18}$$

Define $A\colon \mathbb{R}^{n+m} \to \mathcal{M}_{n\times n}(\mathbb{R})$ and $B\colon \mathbb{R}^n \to \mathcal{M}_{n\times m}(\mathbb{R})$ by

$$A(x,u) := \sum_{i=0}^{m} u_i f_i'(x), \quad B(x)v := \sum_{i=1}^{m} v_i f_i(x), \tag{2.19}$$

for every $v \in \mathbb{R}^m$. Notice that the ith. column of $B(x)$ is $f_i(x)$. For $(x,u) \in \mathcal{W}$ satisfying (2.2), let $B_1(x_t,u_t) \in \mathcal{M}_{n\times m}(\mathbb{R})$ given by

$$B_1(x_t,u_t) := A(x_t,u_t)B(x_t) - \frac{\mathrm{d}}{\mathrm{d}t}B(x_t). \tag{2.20}$$

In view of (2.19) and (2.20), the expressions in (2.17) can be rewritten as

$$\Phi_t = p_t B(x_t), \quad \dot{\Phi}_t = -p_t B_1(x_t,u_t). \tag{2.21}$$

2.4 Shooting Algorithm

The aim of this section is to present an appropriated numerical scheme to solve system (OS). For this purpose define the *shooting function*

$$\begin{aligned} \mathcal{S}\colon \mathrm{D}(\mathcal{S}) := \mathbb{R}^n \times \mathbb{R}^{n+d_\eta,*} &\to \mathbb{R}^{d_\eta} \times \mathbb{R}^{2n+2m,*}, \\ \big(x_0, p_0, \beta\big) =: \nu &\mapsto \mathcal{S}(\nu) := \begin{pmatrix} \eta(x_0,x_T) \\ p_0 + D_{x_0}\ell[\lambda](x_0,x_T) \\ p_T - D_{x_T}\ell[\lambda](x_0,x_T) \\ p_T B(x_T) \\ p_0 B_1(x_0,u_0) \end{pmatrix}, \end{aligned} \tag{2.22}$$

where (x, u, p) is a solution of (2.2),(2.8),(2.15) corresponding to the initial conditions (x_0, p_0), and with $\lambda := (\beta, p)$. Here we denote either by (a_1, a_2) or $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ an element of the product space $A_1 \times A_2$. Notice that the control u retrieved from (2.15) is continuous in time, as we have already pointed out after Assumption 2.3.1. Hence, we can refer to the value u_0, as it is done in the right hand-side of (2.22). Observe that in a simpler framework having fixed initial state and no final constraints, the shooting function would depend only on p_0. In our case, since the initial state is not fixed and a multiplier associated with the initial-final constraints must be considered, $\mathcal{S}$ has more independent variables. Note that solving (OS) consists of finding $\nu \in \mathrm{D}(\mathcal{S})$ such that

$$\mathcal{S}(\nu) = 0. \tag{2.23}$$

Since the number of equations in (2.23) is greater than the number of unknowns, the Gauss-Newton method is a suitable approach to solve it. This algorithm will solve the equivalent least squares problem

$$\min_{\nu \in \mathrm{D}(\mathcal{S})} \left| \mathcal{S}\left(\nu\right) \right|^2. \tag{2.24}$$

At each iteration k, given the approximate values ν^k, it looks for Δ^k that gives the minimum of the linear approximation of problem

$$\min_{\Delta \in \mathrm{D}(\mathcal{S})} \left| \mathcal{S}(\nu^k) + \mathcal{S}'(\nu^k)\Delta \right|^2. \tag{2.25}$$

Afterwards it updates

$$\nu^{k+1} \leftarrow \nu^k + \Delta^k. \tag{2.26}$$

In order to solve the linear approximation of problem (2.25) at each iteration k, we look for Δ^k in the kernel of the derivative of the objective function, i.e. Δ^k satisfying

$$\mathcal{S}'(\nu^k)^\top \mathcal{S}'(\nu^k)\Delta^k + \mathcal{S}'(\nu^k)^\top \mathcal{S}(\nu^k) = 0. \tag{2.27}$$

Hence, to compute direction Δ^k the matrix $\mathcal{S}'(\nu^k)^\top \mathcal{S}'(\nu^k)$ must be nonsingular. Thus, Gauss-Newton method will be applicable provided that $\mathcal{S}'(\hat{\nu})^\top \mathcal{S}'(\hat{\nu})$ is invertible, where $\hat{\nu} := (\hat{x}_0, \hat{p}_0, \hat{\beta})$. Easily follows that $\mathcal{S}'(\hat{\nu})^\top \mathcal{S}'(\hat{\nu})$ is nonsingular if and only if $\mathcal{S}'(\hat{\nu})$ is one-to-one. Summarizing, the *shooting algorithm* we propose here consists of solving the equation (2.23) by the Gauss-Newton method defined by (2.26)-(2.27).

Since the right hand-side of system (2.23) is zero, the Gauss-Newton method converges locally quadratically if the function $\mathcal{S}$ has Lipschitz-continuous derivative. The latter holds here given the regularity assumptions on the data functions. This convergence result is stated in the proposition below. See, e.g., Fletcher [59] or Bonnans [19] for a proof.

Proposition 2.4.1. *If $\mathcal{S}'(\hat{\nu})$ is one-to-one then the shooting algorithm is locally quadratically convergent.*

The main result of this article is to present a condition that guarantees the quadratic convergence of the shooting method near the optimal local solution $(\hat{w}, \hat{\lambda})$. This condition involves the second variation studied in Dmitruk [40, 43], more precisely, the sufficient optimality conditions therein presented.

2.4.1 Linearization of a Differential Algebraic System

For the aim of finding an expression of $\mathcal{S}'(\hat{\nu})$, we make use of the linearization of (OS) and thus we introduce the following concept.

Definition 2.4.2 (Linearization of a Differential Algebraic System)**.** *Consider a system of differential algebraic equations (DAE) with endpoint conditions*

$$\dot{\zeta}_t = \mathcal{F}(\zeta_t, \alpha_t), \tag{2.28}$$

$$0 = \mathcal{G}(\zeta_t, \alpha_t), \tag{2.29}$$

$$0 = \mathcal{I}(\zeta_0, \zeta_T), \tag{2.30}$$

where $\mathcal{F} : \mathbb{R}^{m+n} \to \mathbb{R}^n$, $\mathcal{G} : \mathbb{R}^{m+n} \to \mathbb{R}^{d_\mathcal{G}}$ and $\mathcal{I} : \mathbb{R}^{2n} \to \mathbb{R}^{d_\mathcal{I}}$ are C^1 functions. Let (ζ^0, α^0) be a C^1 solution. We call linearized system at point (ζ^0, α^0) the following DAE in the variables $\bar{\zeta}$ and $\bar{\alpha}$,

$$\dot{\bar{\zeta}}_t = \mathrm{Lin}\, \mathcal{F} \mid_{(\zeta^0_t, \alpha^0_t)} (\bar{\zeta}_t, \bar{\alpha}_t), \tag{2.31}$$

$$0 = \mathrm{Lin}\, \mathcal{G} \mid_{(\zeta^0_t, \alpha^0_t)} (\bar{\zeta}_t, \bar{\alpha}_t), \tag{2.32}$$

$$0 = \mathrm{Lin}\, \mathcal{I} \mid_{(\zeta^0_0, \zeta^0_T)} (\bar{\zeta}_0, \bar{\zeta}_T), \tag{2.33}$$

where

$$\mathrm{Lin}\, \mathcal{F} \mid_{(\zeta^0_t, \alpha^0_t)} (\bar{\zeta}_t, \bar{\alpha}_t) := \mathcal{F}'(\zeta^0_t, \alpha^0_t)(\bar{\zeta}_t, \bar{\alpha}_t), \tag{2.34}$$

and the analogous definitions hold for $\mathrm{Lin}\, \mathcal{G}$ *and* $\mathrm{Lin}\, \mathcal{H}$*.*

The technical result below will simplify the computation of the linearization of (OS). Its proof is immediate.

Lemma 2.4.3 (Commutation of linearization and differentiation)**.** *Given $\mathcal{G}$ and $\mathcal{F}$ as in the previous definition, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Lin}\, \mathcal{G} = \mathrm{Lin}\, \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{G}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Lin}\, \mathcal{F} = \mathrm{Lin}\, \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}. \tag{2.35}$$

2.4.2 Linearized optimality system

In the sequel, whenever the argument of functions A, B, B_1, etc. is omitted, assume that they are evaluated at the reference extremal $(\hat{w}, \hat{\lambda})$. Define the $m \times n-$matrix C, the $n \times n-$matrix Q and the $m \times n-$matrix M by

$$C := H_{ux}, \quad Q := H_{xx}, \quad M := B^\top Q - \dot{C} - CA. \tag{2.36}$$

Notice that the ith. row of matrix C is the function pf_i', for $i = 1, \ldots, m$. Denote with $(z, v, \bar{\lambda} := (\bar{\beta}, q))$ the linearized variable $(x, u, \lambda = (\beta, p))$. In view of equations (2.21) and (2.36) we can write

$$\text{Lin } \Phi_t = q_t B_t + z_t^\top C_t^\top. \tag{2.37}$$

The linearization of system (OS) at point $(\hat{x}, \hat{u}, \hat{\lambda})$ consists of the *linearized state equation*

$$\dot{z}_t = A_t z_t + B_t v_t, \quad \text{a.e. on } [0, T], \tag{2.38}$$

with endpoint conditions

$$0 = D\eta(\hat{x}_0, \hat{x}_T)(z_0, z_T), \tag{2.39}$$

the linearized costate equation

$$-\dot{q}_t = q_t A_t + z_t^\top Q_t + v_t^\top C_t, \quad \text{a.e. on } [0, T], \tag{2.40}$$

with endpoint conditions

$$q_0 = -\left[z_0^\top D^2_{x_0^2} \ell + z_T^\top D^2_{x_0 x_T} \ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_0} \eta_j \right]_{(\hat{x}_0, \hat{x}_T)}, \tag{2.41}$$

$$q_T = \left[z_T^\top D^2_{x_T^2} \ell + z_0^\top D^2_{x_0 x_T} \ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T} \eta_j \right]_{(\hat{x}_0, \hat{x}_T)}, \tag{2.42}$$

and the algebraic equations

$$0 = \text{Lin } \ddot{\Phi} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}(qB + Cz), \quad \text{a.e. on } [0, T], \tag{2.43}$$

$$0 = \text{Lin } \Phi_T = q_T B_T + C_T z_T, \tag{2.44}$$

$$0 = \text{Lin } \dot{\Phi}_0 = -\frac{\mathrm{d}}{\mathrm{d}t}(qB + Cz)_{t=0}. \tag{2.45}$$

Here we used equation (2.37) and commutation property of Lemma 2.4.3 to write (2.43) and (2.47). Observe that (2.43)-(2.47) and Lemma 2.4.3 yield

$$0 = \text{Lin } \Phi_t = q_t B_t + z_t^\top C_t^\top, \quad \text{on } [0, T], \tag{2.46}$$

and

$$0 = \text{Lin } \dot{\Phi}_t = -qB_1 - z^\top M^\top + v^\top(-CB + B^\top C^\top), \quad \text{a.e. on } [0,T].$$

By means of Theorem 2.5.2 to be stated in Section 2.5 afterwards we can see that the coefficient of v in previous expression vanishes, and hence,

$$0 = \text{Lin } \dot{\Phi}_t = -qB_1 - z^\top M^\top, \quad \text{on } [0,T]. \tag{2.47}$$

Note that both equations (2.46) and (2.47) hold everywhere on $[0,T]$ since all the involved functions are continuous in time.

Notation: denote by (LS) the set of equations (2.38)-(2.47).

Once we have computed the linearized system (LS), we can write the derivative of $\mathcal{S}$ in the direction $\bar{\nu} := \big(z_0, q_0, \bar{\beta}\big)$ as follows.

$$\mathcal{S}'(\hat{\nu})\bar{\nu} = \begin{pmatrix} D\eta(\hat{x}_0, \hat{x}_T)(z_0, z_T) \\ q_0 + \Big[z_0^\top D^2_{x_0^2}\ell + z_T^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_0}\eta_j\Big]_{(\hat{x}_0,\hat{x}_T)} \\ q_T - \Big[z_T^\top D^2_{x_T^2}\ell + z_0^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T}\eta_j\Big]_{(\hat{x}_0,\hat{x}_T)} \\ q_T B_T + z_T^\top C_T^\top \\ q_0 B_{1,0} + z_0^\top M_0^\top \end{pmatrix}, \tag{2.48}$$

where (v,z,q) is the solution of (2.38),(2.40),(2.43) associated with the initial condition (z_0,q_0) and the multiplier $\bar{\beta}$. Thus, we get the property below.

Proposition 2.4.4. *$\mathcal{S}'(\hat{\nu})$ is one-to-one if the only solution of* (2.38)-(2.40),(2.43) *with the initial conditions $z_0 = 0,\ q_0 = 0$ and with $\bar{\beta} = 0$ is $(v,z,q) = 0$.*

2.5 Second Order Optimality Conditions

In this section we summarize a set of second order necessary and sufficient conditions. At the end of the section we state a sufficient condition for the local quadratic convergence of the shooting algorithm presented in Section 2.4. The latter is the main result of this article.

Recall the matrices C and Q defined in (2.36), and the space $\mathcal{W}$ given at the beginning of Section 2.2. Consider the quadratic mapping on $\mathcal{W}$,

$$\Omega(z,v) := \tfrac{1}{2}D^2\ell\,(z_0,z_T)^2 + \tfrac{1}{2}\int_0^T \big[z^\top Q z + 2v^\top C z\big]\,\mathrm{d}t. \tag{2.49}$$

It is a well-known result that for each $(z,v) \in \mathcal{W}$,

$$D^2\mathcal{L}\,(z,v)^2 = \Omega(z,v). \tag{2.50}$$

We next recall the classical second order necessary condition for optimality that states that the second variation of the Lagrangian function is nonnegative on the critical cone. In our case, the *critical cone* is given by

$$\mathcal{C} := \{(z,v) \in \mathcal{W} : \text{(2.38)-(2.39) hold}\}, \tag{2.51}$$

and the second order optimality condition is as follows.

Theorem 2.5.1 (Second order necessary optimality condition). *If $\hat{w}$ is a weak minimum of (P) then*

$$\Omega(z,v) \geq 0, \quad \text{for all } (z,v) \in \mathcal{C}. \tag{2.52}$$

A proof of previous theorem can be found in, e.g., Levitin, Milyutin and Osmolovskii [86]. The following necessary condition is due to Goh [67] and it is a nontrivial consequence (not immediate) of Theorem 2.5.1. Define first the $m \times m-$matrix

$$R := B^\top QB - CB_1 - (CB_1)^\top - \frac{\mathrm{d}}{\mathrm{d}t}(CB). \tag{2.53}$$

Theorem 2.5.2 (Goh's Necessary Condition). *If $\hat{w}$ is a weak minimum of (P), then*

$$CB \text{ is symmetric}, \tag{2.54}$$

and

$$R \succeq 0. \tag{2.55}$$

Remark 2.5.3. *Observe that* (2.54) *is equivalent to $pf_i'f_j = pf_j'f_i$, for every pair $i,j = 1,\dots,m$. These identities can be written in terms of Lie brackets as*

$$p[f_i, f_j] = 0, \quad \text{for } i,j = 1,\dots,m. \tag{2.56}$$

Notice that (2.54) *implies, in view of* (2.53)*, that R is symmetric. The components of matrix R can be written as*

$$R_{ij} = p[f_i, [f_j, f_0]], \tag{2.57}$$

and hence, its symmetry implies

$$p[f_i, [f_j, f_0]] = p[f_j, [f_i, f_0]], \quad \text{for } i,j = 1,\dots,m. \tag{2.58}$$

The latter expressions involving Lie brackets can be often found in the literature.

The result that we present next is due to Dmitruk [40] and is stated in terms of the coercivity of Ω in a transformed space of variables. Let us give the details of the involved transformation and the transformed second variation. Given $(z, v) \in \mathcal{W}$, define

$$y_t := \int_0^t v_s \mathrm{d}s, \tag{2.59}$$

$$\xi_t := z_t - B(\hat{x}_t) y_t. \tag{2.60}$$

This change of variables, first introduced by Goh in [68], can be perform in any linear system of differential equations, and it is known as *Goh's transformation*.

We aim to perform Goh's transformation in (2.49). To this end consider the spaces $\mathcal{U}_2 := L_2(0, T; \mathbb{R}^m)$ and $\mathcal{X}_2 := W_2^1(0, T; \mathbb{R}^n)$, the function $g \colon \mathbb{R}^{2n+m} \to \mathbb{R}$ defined by

$$g(\zeta_0, \zeta_T, h) := D^2\ell\,(\zeta_0, \zeta_T + B_T h)^2 + h^\top C_T (2\zeta_T + B_T h), \tag{2.61}$$

and the quadratic mapping

$$\begin{aligned} \bar{\Omega} \colon \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m &\to \mathbb{R} \\ (\xi, y, h) &\mapsto \tfrac{1}{2} g(\xi_0, \xi_T, h) + \tfrac{1}{2} \int_0^T \{\xi^\top Q \xi + 2 y^\top M \xi + y^\top R y\} \mathrm{d}t, \end{aligned} \tag{2.62}$$

where the involved matrices where introduced in (2.19), (2.36) and (2.53).

Proposition 2.5.4. *If $\hat{w}$ is a weak minimum of (P), then*

$$\Omega(z, v) = \bar{\Omega}(\xi, y, y_T), \tag{2.63}$$

whenever $(z, v) \in \mathcal{W}$ and $(\xi, y, y_T) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m$ satisfy (2.59)-(2.60).

The latter result follows by integrating by parts the terms containing v in (2.49), and by replacing z by its expression in (2.60). See, e.g., Aronna et al. [8] for the detailed calculations that lead to (2.63).

Define the *order function* $\gamma \colon \mathbb{R}^n \times \mathcal{U}_2 \times \mathbb{R}^m \to \mathbb{R}$ as

$$\gamma(\xi_0, y, h) := |\xi_0|^2 + \int_0^T y_t^2 \mathrm{d}t + |h|^2. \tag{2.64}$$

We call $(\delta x, v) \in \mathcal{W}$ a *feasible variation* for $\hat{w}$ if $(\hat{x} + \delta x, \hat{u} + v)$ satisfies (2.2)-(2.3).

Definition 2.5.5. *We say that $\hat{w}$ satisfies the $\gamma-$growth condition in the weak sense if there exists $\rho > 0$ such that, for every sequence of feasible variations $\{(\delta x^k, v^k)\}$ converging to 0 in $\mathcal{W}$,*

$$J(\hat{u} + v^k) - J(\hat{u}) \geq \rho \gamma(\xi_0^k, y^k, y_T^k), \tag{2.65}$$

holds for big enough k, where $y_t^k := \int_0^t v_s^k \mathrm{d}s$, and ξ^k is given by (2.60).

In previous definition, given that $(\delta x^k, v^k)$ is a feasible variation for each k, the sequence $\{(\delta x^k, v^k)\}$ goes to 0 in $\mathcal{W}$ if and only if $\{v^k\}$ goes to 0 in $\mathcal{U}$.

Observe that if $(z, v) \in \mathcal{W}$ satisfies (2.38)-(2.39), then $(\xi, y, h := y_T)$ given by transformation (2.59)-(2.60) verifies

$$\dot{\xi} = A\xi + B_1 y, \tag{2.66}$$

$$D\eta(\hat{x}_0, \hat{x}_T)(\xi_0, \xi_T + B_T h) = 0. \tag{2.67}$$

Set the *transformed critical cone*

$$\mathcal{P}_2 := \{(\xi, y, h) \in \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^m : \text{(2.66)-(2.67) hold}\}. \tag{2.68}$$

The following is an immediate consequence of the sufficient condition established in Dmitruk [40] (or [43, Theorem 3.1]).

Theorem 2.5.6. *The trajectory $\hat{w}$ is a weak minimum of (P) satisfying $\gamma-$ growth condition in the weak sense if and only if* (2.54) *holds and there exists $\rho > 0$ such that*

$$\bar{\Omega}(\xi, y, h) \geq \rho\gamma(\xi_0, y, h), \quad \text{on } \mathcal{P}_2. \tag{2.69}$$

The result presented in [40] applies to a more general case having finitely many equalities and inequalities constraints on the initial and final state, and a set of multipliers consisting possibly of more than one element.

Remark 2.5.7. *If* (2.69) *holds then necessarily*

$$R \succeq \rho I, \tag{2.70}$$

where I represents the identity matrix.

Theorem 2.5.8. *If $\hat{w}$ is a weak minimum of (P) satisfying* (2.69)*, then the shooting algorithm is locally quadratically convergent.*

We present the proof of previous theorem at the end of Section 2.7.

Remark 2.5.9. *It is interesting to observe that condition* (2.69) *is a quite weak assumption in the sense that it is necessary for $\gamma-$growth and its corresponding relaxed condition* (2.52) *holds necessarily for every weak minimum.*

Remark 2.5.10 (Verification of (2.69))**.** *The sufficient condition in* (2.69) *can be sometimes checked analytically. On the other hand, when the initial point ξ_0 is fixed, it can be characterized by a Riccati-type equation and/or the nonexistence of a focal point as it was established in Zeidan [129]. Furthermore, under certain hypotheses, the condition* (2.69) *can be verified numerically as proposed in [22] by Bonnard, Caillau and Trélat (see also the survey in [23]).*

2.6 Corresponding LQ Problem

In this section we study the linear-quadratic problem (LQ) given by

$$\bar{\Omega}(\xi, y, h_T) \to \min, \tag{2.71}$$

$$(2.66)\text{-}(2.67), \tag{2.72}$$

$$\dot{h} = 0, \quad h_0 \text{ free.} \tag{2.73}$$

Here y is the control, ξ and h are the state variables. Note that if condition (2.69) holds then (LQ) has a unique optimal solution $(\xi, y, h) = 0$. Furthermore, recall that (2.69) yields (2.70) as it was said in Remark 2.5.7. In other words, (2.69) implies that the strengthened Legendre-Clebsch condition holds at $(\xi, y, h) = 0$. Hence, the unique local optimal solution of (LQ) is characterized by the first optimality system, that we denote afterwards by (LQS). In Section 2.7 we present a one-to-one linear mapping that transforms each solution of (LS) (introduced in section 2.4.2) into a solution of this new optimality system (LQS). Theorem 2.5.8 will follow.

Denote by χ and χ_h the costate variables corresponding to ξ and h, respectively; and by β^{LQ} the multiplier associated to the initial-final linearized state constraint (2.67). Note that the qualification hypothesis in Assumption 2.2.1 implies that $\{D\eta_j(\hat{x}_0, \hat{x}_T)\}_{j=1}^{d_\eta}$ are linearly independent. Hence any weak solution (ξ, y, h) of (LQ) has a unique associated multiplier $\lambda^{LQ} := (\chi, \chi_h, \beta^{LQ})$ solution of the system that we describe next. The pre-Hamiltonian of (LQ) is

$$\mathcal{H}[\lambda^{LQ}](\xi, y) := \chi(A\xi + B_1 y) + \tfrac{1}{2}(\xi^\top Q\xi + 2y^\top M\xi + y^\top Ry). \tag{2.74}$$

Observe that $\mathcal{H}$ does not depend on h since the latter has zero dynamics and does not appear in the running cost. The endpoint Lagrangian is given by

$$\ell^{LQ}[\lambda^{LQ}](\xi_0, \xi_T, h_T) := \tfrac{1}{2}g(\xi_0, \xi_T, h_T) + \sum_{j=1}^{d_\eta} \beta_j^{LQ} D\eta_j(\xi_0, \xi_T + B_T h_T). \tag{2.75}$$

The costate equation for χ is

$$-\dot{\chi} = D_\xi \mathcal{H}[\lambda^{LQ}] = \chi A + \xi^\top Q + y^\top M, \tag{2.76}$$

with endpoint conditions

$$\begin{aligned} \chi_0 = &\ -D_{\xi_0}\ell^{LQ}[\lambda^{LQ}] \\ = &\ -\left[\xi_0^\top D^2_{x_0^2}\ell + (\xi_T + B_T h)^\top D^2_{x_0 x_T}\ell + \textstyle\sum_{j=1}^{d_\eta} \beta_j^{LQ} D_{x_0}\eta_j\right], \end{aligned} \tag{2.77}$$

$$\begin{aligned}\chi_T = &\ D_{\xi_T}\ell^{LQ}[\lambda^{LQ}] \\ = &\ \xi_0^\top D^2_{x_0x_T}\ell + (\xi_T + B_Th)^\top D^2_{x_T^2}\ell + h^\top C_T + \textstyle\sum_{j=1}^{d_\eta}\beta_j^{LQ}D_{x_T}\eta_j.\end{aligned} \tag{2.78}$$

For costate variable χ_h we get the equation

$$\dot{\chi}_h = 0, \tag{2.79}$$

$$\chi_{h,0} = 0, \tag{2.80}$$

$$\chi_{h,T} = D_h\ell^{LQ}[\lambda^{LQ}]. \tag{2.81}$$

Hence, $\chi_h \equiv 0$ and thus (2.81) yields

$$0 = \xi_0^\top D^2_{x_0x_T}\ell B_T + (\xi_T + B_Th)^\top (D^2_{x_T^2}\ell B_T + C_T^\top) + \sum_{j=1}^{d_\eta}\beta_j^{LQ}D_{x_T}\eta_jB_T. \tag{2.82}$$

The stationarity with respect to the new control y implies

$$0 = D_y\mathcal{H} = \chi B_1 + \xi^\top M^\top + y^\top R. \tag{2.83}$$

Notation: Denote by (LQS) the set of equations consisting of (2.66)-(2.67), (2.73), (2.76)-(2.78),(2.82) and (2.83), i.e. (LQS) is the system

$$\begin{cases} \dot{\xi} = A\xi + B_1y, \\ D\eta(\hat{x}_0,\hat{x}_T)(\xi_0, \xi_T + B_Th) = 0, \\ \dot{h} = 0, \\ -\dot{\chi} = D_\xi\mathcal{H}[\lambda^{LQ}] = \chi A + \xi^\top Q + y^\top M, \\ \chi_0 = -\left[\xi_0^\top D^2_{x_0^2}\ell + (\xi_T + B_Th)^\top D^2_{x_0x_T}\ell + \displaystyle\sum_{j=1}^{d_\eta}\beta_j^{LQ}D_{x_0}\eta_j\right], \\ \chi_T = \xi_0^\top D^2_{x_0x_T}\ell + (\xi_T + B_Th)^\top D^2_{x_T^2}\ell + h^\top C_T + \displaystyle\sum_{j=1}^{d_\eta}\beta_j^{LQ}D_{x_T}\eta_j, \\ 0 = \xi_0^\top D^2_{x_0x_T}\ell B_T + (\xi_T + B_Th)^\top (D^2_{x_T^2}\ell B_T + C_T^\top) + \displaystyle\sum_{j=1}^{d_\eta}\beta_j^{LQ}D_{x_T}\eta_jB_T, \\ 0 = \chi B_1 + \xi^\top M^\top + y^\top R. \end{cases} \tag{LQS}$$

Notice that (LQS) is a first order optimality system for problem (2.71)-(2.73).

2.7 The Transformation

In this section we show how to transform a solution of (LS) into a solution of (LQS) via a one-to-one linear mapping. Given $(z, v, q, \bar{\beta}) \in \mathcal{X} \times \mathcal{U} \times \mathcal{X}_* \times \mathbb{R}^{d_\eta,*}$, define

$$y_t := \int_0^t v_s ds,\ \xi := z - By,\ \chi := q + y^\top C,\ \chi_h := 0,\ h := y_T,\ \beta_j^{LQ} := \bar{\beta}_j. \tag{2.84}$$

The next lemma shows that the point $(\xi, y, h, \chi, \chi_h, \beta^{LQ})$ is solution of (LQS) provided that $(z, v, q, \bar{\beta})$ is solution of (LS).

Lemma 2.7.1. *The one-to-one linear mapping defined by* (2.84) *converts each solution of (LS) into a solution of (LQS).*

Proof. Let $(z, v, q, \bar{\beta})$ be a solution of (LS), and set $(\xi, y, \chi, \beta^{LQ})$ by (2.84).
Part I. We shall prove that $(\xi, y, \chi, \beta^{LQ})$ satisfies conditions (2.66) and (2.67). Equation (2.66) follows by differentiating expression of ξ in (2.84), and equation (2.67) follows from (2.39).
Part II. We shall prove that $(\xi, y, \chi, \beta^{LQ})$ verifies (2.76)-(2.78) and (2.82). Differentiate χ in (2.84), use equations (2.40) and (2.84), recall definition of M in (2.36) and obtain

$$\begin{aligned} -\dot{\chi} = &\; -\dot{q} - v^\top C - y^\top \dot{C} \\ = &\; qA + z^\top Q - y^\top \dot{C} \\ = &\; \chi A + \xi^\top Q + y^\top(-CA + B^\top Q - \dot{C}) \\ = &\; \chi A + \xi^\top Q + y^\top M. \end{aligned} \tag{2.85}$$

Hence (2.76) holds. Equations (2.77) and (2.78) follow from (2.41) and (2.42). Combine (2.42) and (2.44) to get

$$\begin{aligned} 0 = &\; q_T B_T + z_T^\top C_T^\top \\ = &\; \left[z_T^\top D^2_{x_T^2} \ell + z_0^\top D^2_{x_0 x_T} \ell + \textstyle\sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T} \eta_j \right]_{(\hat{x}_0, \hat{x}_T)} B_T + z_T^\top C_T^\top. \end{aligned} \tag{2.86}$$

Performing transformation (2.84) in the previous equation yields (2.82).
Part III. We shall prove that (2.83) holds. Differentiating (2.46) we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Lin}\ \Phi = \frac{\mathrm{d}}{\mathrm{d}t}(qB + z^\top C^\top). \tag{2.87}$$

Consequently, by (2.38) and (2.40),

$$0 = -(qA + z^\top Q + v^\top C)B + q\dot{B} + (z^\top A^\top + v^\top B^\top)C^\top + z^\top \dot{C}^\top, \tag{2.88}$$

where the coefficient of v vanishes in view of (2.54). Recall (2.20) and (2.36). Performing transformation (2.84) and regrouping the terms we get from (2.88),

$$0 = -\chi B_1 - \xi^\top M^\top + y^\top (CB_1 - B^\top Q B + B^\top A^\top C^\top + B^\top \dot{C}^\top). \tag{2.89}$$

Equation (2.83) follows from (2.53) and condition (2.54).

Parts I, II and III show that $(\xi, y, \chi, \beta^{LQ})$ is a solution of (LQS), and hence the result follows.

□

Remark 2.7.2. *Observe that the unique assumption we needed in previous proof was Goh's condition* (2.54) *that follows from the weak optimality of* $\hat{w}$.

Proof. [of Theorem 2.5.8] We shall prove that (2.69) implies that $\mathcal{S}'(\hat{\nu})$ is one-to-one. Take $(z, v, q, \bar{\beta})$ a solution of (LS), and let $(\xi, y, \chi, \chi_h, \beta^{LQ})$ be defined by (2.84), that we know by Lemma 2.7.1 is solution of (LQS). As it has been already pointed out at the beginning of Section 2.6, condition (2.69) implies that the unique solution of (LQS) is 0. Hence $(\xi, y, \chi, \chi_h, \beta^{LQ}) = 0$ and thus $(z, v, q, \bar{\beta}) = 0$. Conclude that the unique solution of (LS) is 0. The latter assertion implies, in view of Proposition 2.4.4, that $\mathcal{S}'(\hat{\nu})$ is one-to-one. The result follows from Proposition 2.4.1. $\square$

2.8 Control Constrained Case

In this section we add the following bounds to the control variables

$$0 \leq u_{i,t} \leq 1, \quad \text{for a.a. } t \in [0,T], \text{ for } i = 1, \dots, m. \tag{2.90}$$

Denote with (CP) the problem given by (2.1)-(2.3) and (2.90).

Definition 2.8.1. *A feasible trajectory* $\hat{w} \in \mathcal{W}$ *is a Pontryagin minimum of (CP) if for any positive* N *there exists* $\varepsilon_N > 0$ *such that* $\hat{w}$ *is a minimum in the set of feasible trajectories* $w = (x, u) \in \mathcal{W}$ *satisfying*

$$\|x - \hat{x}\|_\infty < \varepsilon_N, \; \|u - \hat{u}\|_1 < \varepsilon_N, \; \|u - \hat{u}\|_\infty < N.$$

Given $i = 1, \dots, m$, we say that $\hat{u}_i$ has a *bang arc* in $(a, b) \subset (0, T)$ if $\hat{u}_{i,t} = 0$ a.e. on (a, b) or $\hat{u}_{i,t} = 1$ a.e. on (a, b), and it has a *singular arc* if $0 < \hat{u}_{i,t} < 1$ a.e. on (a, b).

Assumption 2.8.2. *Each component* $\hat{u}_i$ *is a finite concatenation of bang and singular arcs.*

A time $t \in (0, T)$ is called *switching time* if there exists an index $1 \leq i \leq m$ such that $\hat{u}_i$ switches at time t from singular to bang, or vice versa, or from one bound in (2.90) to the other.

Remark 2.8.3. *Assumption 2.8.2 rules out the solutions having an infinite number of swit- chings in a bounded interval. This behavior is usually known as Fuller's phenomenon (see Fuller [61]). Many examples can be encountered satisfying Assumption 2.8.2 as is the case of the three problems presented in Section 2.10.*

With the purpose of solving (CP) numerically we assume that the structure of the concatenation of bang and singular arcs of the optimal solution $\hat{w}$ and an approximation of its switching times are known. This initial guess can be obtained, for instance, by solving the nonlinear problem resulting from the discretization of the optimality conditions or by a continuation method. See Betts [14] or Biegler [16] for a detailed survey and description of numerical methods for nonlinear programming problems. For the continuation method the reader is referred to Martinon [89].

This section is organized as follows. From (CP) and the known structure of $\hat{u}$ and its switching times we create a new problem that we denote by (TP). Afterwards we prove that we can transform $\hat{w}$ into a weak solution $\hat{W}$ of (TP). Finally we conclude that if $\hat{W}$ satisfies the coercivity condition (2.69), then the shooting method for problem (TP) converges locally quadratically. In practice, the procedure will be as follows: obtain somehow the structure of the optimal solution of (CP), create problem (TP), solve (TP) numerically obtaining $\hat{W}$, and finally transform $\hat{W}$ to find $\hat{w}$.

Next we present the transformed problem.

Assumption 2.8.4. *Assume that each time a control $\hat{u}_i$ switches from bang to singular or vice versa, there is a discontinuity of first kind.*

Here, by *discontinuity of first kind* we mean that each component of $\hat{u}$ has a finite nonzero jump at the switching times, and the left and right limits exist.

By Assumption 2.8.2 the set of switching times is finite. Consider the partition of $[0,T]$ induced by the switching times:

$$\{0 =: \hat{T}_0 < \hat{T}_1 < \ldots < \hat{T}_{N-1} < \hat{T}_N := T\}. \tag{2.91}$$

Set $\hat{I}_k := [\hat{T}_{k-1}, \hat{T}_k]$, and define for $k = 1, \ldots, N$,

$$S_k := \{1 \leq i \leq m : \ \hat{u}_i \text{ is singular on } \hat{I}_k\}, \tag{2.92}$$

$$E_k := \{1 \leq i \leq m : \ \hat{u}_i = 0 \text{ a.e. on } \hat{I}_k\}, \tag{2.93}$$

$$N_k := \{1 \leq i \leq m : \ \hat{u}_i = 1 \text{ a.e. on } \hat{I}_k\}. \tag{2.94}$$

Clearly $S_k \cup E_k \cup N_k = \{1, \ldots, m\}$.

Assumption 2.8.5. *For each $k = 1, \ldots, N$, denote by u_{S_k} the vector with components u_i with $i \in S_k$. Assume that the strengthened generalized Legendre-Clebsch condition holds on $\hat{I}_k$, i.e.*

$$-\frac{\partial}{\partial u_{S_k}} \ddot{H}_{u_{S_k}} \succ 0, \quad \text{on } \hat{I}_k. \tag{2.95}$$

Hence, u_{S_k} can be retrieved from equation

$$\ddot{H}_{u_{S_k}} = 0, \tag{2.96}$$

since the latter is affine on u_{S_k} as it has been already pointed out in Section 2.3. Observe that the expression obtained from (2.96) involves only the state variable $\hat{x}$ and the corresponding adjoint state $\hat{p}$. Hence, it results that $\hat{u}_{S_k}$ is continuous on $\hat{I}_k$ with finite limits at the endpoints of this interval. As the components $\hat{u}_i$ with $i \notin S_k$ are either identically 1 or 0, we conclude that

$$\hat{u} \text{ is continuous on } \hat{I}_k. \tag{2.97}$$

By Assumption 2.8.4 and condition (2.97) (derived from Assumption 2.8.5) we get that there exists $\rho > 0$ such that

$$\rho < \hat{u}_{i,t} < 1 - \rho, \quad \text{a.e. on } \hat{I}_k, \text{ for } k = 1, \dots, N, \ i \in S_k. \tag{2.98}$$

Next we present a new control problem obtained in the following way. For each $k = 1, \dots, N$, we perform the change of time variable that converts the interval $\hat{I}_k$ into $[0, 1]$, afterwards we fix the bang control variables to their bounds and finally, we associate a free control variable to each index in S_k. More precisely, consider for $k = 1, \dots, N$ the control variables $u_i^k \in L_\infty(0, 1; \mathbb{R})$, with $i \in S_k$, and the state variables $x^k \in W_\infty^1(0, 1; \mathbb{R}^n)$. Let the constants $T_k \in \mathbb{R}$, for $k = 1, \dots, N-1$, which will be considered as state variables of zero-dynamics. Set $T_0 := 0$, $T_N := T$ and define the problem on the interval $[0, 1]$

$$\varphi_0(x_0^1, x_1^N) \to \min, \tag{2.99}$$

$$\dot{x}^k = (T_k - T_{k-1}) \left(\sum_{i \in N_k \cup \{0\}} f_i(x^k) + \sum_{i \in S_k} u_i^k f_i(x^k) \right), \qquad k = 1, \dots, N, \tag{2.100}$$

$$\dot{T}_k = 0, \qquad k = 1, \dots, N-1, \tag{2.101}$$

$$\eta(x_0^1, x_1^N) = 0, \tag{2.102}$$

$$x_1^k = x_0^{k+1}, \quad k = 1, \dots, N-1. \tag{2.103}$$

Denote by (TP) the problem consisting of equations (2.99)-(2.103). The link between the original problem (CP) and the transformed one (TP) is given in Lemma 2.8.6 below. Set for each $k = 1, \dots, N$:

$$\hat{x}_s^k := \hat{x}(\hat{T}_{k-1} + (\hat{T}_k - \hat{T}_{k-1})s), \quad \text{for } s \in [0, 1], \tag{2.104}$$

$$\hat{u}_{i,s}^k := \hat{u}_i(\hat{T}_{k-1} + (\hat{T}_k - \hat{T}_{k-1})s), \quad \text{for } i \in S_k, \text{ a.a. } s \in [0, 1]. \tag{2.105}$$

Set

$$\hat{W} := ((\hat{x}^k)_{k=1}^N, (\hat{u}_i^k)_{k=1,i\in S_k}^N, (\hat{T}_k)_{k=1}^{N-1}). \tag{2.106}$$

Lemma 2.8.6. *If $\hat{w}$ is a Pontryagin minimum of (CP), then $\hat{W}$ is a weak solution of (TP).*

Proof. The idea of the proof is to derive the weak optimality of $\hat{W}$ from the Pontryagin optimality of $\hat{w}$ and condition (2.98). Since $\hat{w}$ is a Pontryagin minimum for (CP), there exists $\varepsilon > 0$ such that $\hat{w}$ is a minimum in the set of feasible trajectories $w = (x, u)$ satisfying

$$\|x - \hat{x}\|_\infty < \varepsilon, \quad \|u - \hat{u}\|_1 < \varepsilon, \quad \|u - \hat{u}\|_\infty < 1. \tag{2.107}$$

Consider $\bar{\delta}, \bar{\varepsilon} > 0$, and a feasible solution $((x^k), (u_i^k), (T_k))$ for (TP) such that

$$|T_k - \hat{T}_k| \leq \bar{\delta}, \quad \|u_i^k - \hat{u}_i^k\|_\infty < \bar{\varepsilon}, \quad \text{for all } k = 1, \dots, N. \tag{2.108}$$

We shall relate ε in (2.107) with $\bar{\delta}$ and $\bar{\varepsilon}$ in (2.108). Let $k = 1, \dots, N$. Denote $I_k := (T_{k-1}, T_k)$, and define for each $i = 1, \dots, m$:

$$u_{i,t} := \begin{cases} 0, & \text{if } t \in I_k \text{ and } i \in E_k, \\ u_i^k\left(\frac{t-T_{k-1}}{T_k - T_{k-1}}\right), & \text{if } t \in I_k \text{ and } i \in S_k, \\ 1, & \text{if } t \in I_k \text{ and } i \in N_k. \end{cases} \tag{2.109}$$

Let x be the solution of (2.2) associated to u and having $x_0 = x_0^1$. We shall prove that (x, u) is feasible for the original problem (CP). Observe that condition (2.103) implies that $x_t = x^k\left(\frac{t-T_{k-1}}{T_k - T_{k-1}}\right)$ when $t \in I_k$, and thus $x_1 = x_1^N$. It follows that (2.3) holds. We shall check condition (2.90). For $i \in E_k \cup N_k$, it follows from the definition in (2.109). Consider now $i \in S_k$. Since (2.98) holds, by (2.105) we get

$$\rho < \hat{u}_{i,s}^k < 1 - \rho, \quad \text{a.e. on } (0, 1). \tag{2.110}$$

Thus, by (2.108) and if $\bar{\varepsilon} < \rho$, we get $0 < u_{i,s}^k < 1$ a.e. on $(0, 1)$. This yields

$$0 < u_{i,t} < 1, \quad \text{a.e. on } I_k, \tag{2.111}$$

and thus the feasibility of (x, u) for (CP).

We now estimate $\|u - \hat{u}\|_1$. For $k = 1, \dots, N$ and $i \in S_k$,

$$\begin{aligned} \int_{I_k\cap\hat{I}_k} |u_{i,t} - \hat{u}_{i,t}|\mathrm{d}t \leq & \int_{I_k\cap\hat{I}_k} \left|u_i^k\left(\tfrac{t-T_{k-1}}{T_k-T_{k-1}}\right) - \hat{u}_i^k\left(\tfrac{t-T_{k-1}}{T_k-T_{k-1}}\right)\right|\mathrm{d}t \\ & + \int_{I_k\cap\hat{I}_k} \left|\hat{u}_i^k\left(\tfrac{t-T_{k-1}}{T_k-T_{k-1}}\right) - \hat{u}_i^k\left(\tfrac{t-\hat{T}_{k-1}}{\hat{T}_k-\hat{T}_{k-1}}\right)\right|\mathrm{d}t. \end{aligned} \tag{2.112}$$

Note that by Assumption 2.8.4 and condition (2.97), each $\hat{u}_i^k$ is uniformly continuous on $\hat{I}_k$, and thus there exists $\theta_{ki} > 0$ such that if $|s - s'| < \theta_{ki}$ then $|\hat{u}_{i,s}^k - \hat{u}_{i,s'}^k| < \bar{\varepsilon}$. Set $\bar{\theta} := \min \theta_{ki} > 0$. Consider then $\bar{\delta}$ such that if $|T_k - \hat{T}_k| < \bar{\delta}$, then $\left| \frac{t-T_{k-1}}{T_k - T_{k-1}} - \frac{t-\hat{T}_{k-1}}{\hat{T}_k - \hat{T}_{k-1}} \right| < \bar{\theta}$. From (2.108) and (2.112) we get

$$\int_{I_k \cap \hat{I}_k} |u_{i,t} - \hat{u}_{i,t}| \mathrm{d}t < 2\bar{\varepsilon}\, \mathrm{meas}\,(I_k \cap \hat{I}_k). \tag{2.113}$$

Assume, w.l.o.g., that $T_k < \hat{T}_k$ and note that

$$\int_{T_k}^{\hat{T}_k} |u_{i,t} - \hat{u}_{i,t}| \mathrm{d}t \leq \int_{T_k}^{\hat{T}_k} \left| u_i^k \left(\frac{t - T_{k-1}}{T_k - T_{k-1}} \right) - \hat{u}_i^k \left(\frac{t - \hat{T}_{k-1}}{\hat{T}_k - \hat{T}_{k-1}} \right) \right| \mathrm{d}t < \bar{\delta}\, \bar{\varepsilon}, \tag{2.114}$$

where we used (2.108) in the last inequality. From (2.113) and (2.114) we get $\|u_i - \hat{u}_i\|_1 < \bar{\varepsilon}(2T + (N-1)\bar{\delta})$. Thus $\|u - \hat{u}\|_1 < \varepsilon$ if

$$\bar{\varepsilon}(2T + (N-1)\bar{\delta}) < \varepsilon / m. \tag{2.115}$$

We conclude from (2.107) that $((x^k), (u_i^k), (T_k))$ is a minimum on the set of feasible points satisfying (2.108) and (2.115). Thus $\hat{W}$ is a weak solution of (TP), as it was to be proved. $\square$

We shall next propose a shooting function associated to (TP). The pre-Hamiltonian of the latter is

$$\tilde{H} := \sum_{k=1}^{N} (T_k - T_{k-1}) H^k, \tag{2.116}$$

where, denoting by p^k the costate variable associated to x^k,

$$H^k := p^k \left(\sum_{i \in N_k \cup \{0\}} f_i(x^k) + \sum_{i \in S_k} u_i^k f_i(x^k) \right). \tag{2.117}$$

Observe that Assumption 2.8.5 made on $\hat{u}$ yields

$$-\frac{\partial}{\partial u} \ddot{\tilde{H}}_u \succ 0, \quad \text{on } [0,1], \tag{2.118}$$

i.e. the strengthened generalized Legendre-Clebsch condition holds in problem (TP) at $\hat{w}$. Hence we can define the shooting function for (TP) as it was done in Section 2.4 for (P).

The endpoint Lagrangian is

$$\tilde{\ell} := \varphi_0(x_0^1, x_1^N) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(x_0^1, x_1^N) + \sum_{k=1}^{N-1} \theta_k (x_1^k - x_0^{k+1}). \tag{2.119}$$

The costate equation for p^k is given by

$$\dot{p}^k = -(T_k - T_{k-1}) D_{x^k} H^k, \tag{2.120}$$

with endpoint conditions

$$p_0^1 = -D_{x_0^1} \tilde{\ell} = -D_{x_0^1} \varphi_0 - \sum_{j=1}^{d_\eta} \beta_j D_{x_0^1} \eta_j, \tag{2.121}$$

$$\begin{aligned} p_1^k &= \theta^k, \quad \text{for } k = 1, \dots, N-1, \\ p_0^k &= \theta^{k-1}, \quad \text{for } k = 2, \dots, N, \end{aligned} \tag{2.122}$$

$$p_1^N = D_{x_1^N} \tilde{\ell} = D_{x_1^N} \varphi_0 + \sum_{j=1}^{d_\eta} \beta_j D_{x_1^N} \eta_j. \tag{2.123}$$

For the costate variables p^{T_k} associated with T_k we get the equations

$$\dot{p}^{T_k} = -H^k + H^{k+1}, \quad p_0^{T_k} = 0, \quad p_1^{T_k} = 0, \quad \text{for } k = 1, \dots, N-1, . \tag{2.124}$$

Remark 2.8.7. *We can sum up the conditions in* (2.124) *integrating the first one and obtaining* $\int_0^1 (H^{k+1} - H^k) \mathrm{dt} = 0$, *and hence, since* H^k *is constant on the optimal trajectory, we get the equivalent condition*

$$H_1^k = H_0^{k+1}, \quad \text{for } k = 1, \dots, N-1. \tag{2.125}$$

So we can remove the shooting variable p^{T_k} *and keep the continuity condition on the pre-Hamiltonian.*

Observe that (2.103) and (2.122) imply the continuity of the two functions obtained by concatenating the states and the costates, i.e. the continuity of X and P defined by

$$X_0 := x_0^1,\ X_s := x^k(s - (k-1)), \quad \text{for } s \in (k-1, k],\ k = 1, \dots, N, \tag{2.126}$$

$$P_0 := p_0^1,\ P_s := p^k(s - (k-1)), \quad \text{for } s \in (k-1, k],\ k = 1, \dots, N. \tag{2.127}$$

Thus, while iterating the shooting method, we can either include the conditions (2.103) and (2.122) in the definition of the shooting function or integrate the differential equations for x^k and p^k from the values x_1^{k-1} and p_1^{k-1} previously obtained. The latter option reduces the number of variables and hence the size of the problem, but is less stable. We shall present below the shooting function for the more stable case. For this end define the $n \times n-$matrix

$$A^k := \sum_{i \in N_k \cup \{0\}} f_i'(\hat{x}^k) + \sum_{i \in S_k} \hat{u}_i^k f_i'(\hat{x}^k), \tag{2.128}$$

the $n \times |S_k|-$matrix B^k with columns $f_i(\hat{x}^k)$ with $i \in S_k$, and

$$B_1^k := A^k B^k - \frac{\mathrm{d}}{\mathrm{d}t} B^k. \tag{2.129}$$

We shall denote by $g_i(x^k, u^k)$ the ith. column of B_1^k for each i in S_k. Here u^k is the $|S_k|-$dimensional vector of components u_i^k. The resulting shooting function for (TP) is given by

$$\begin{aligned}
\mathcal{S}\colon \mathbb{R}^{Nn+N-1} \times \mathbb{R}^{Nn+d_\eta,*} &\to \mathbb{R}^{d_\eta+(N-1)n} \times \mathbb{R}^{(N+1)n+N-1+2\sum|S_k|,*}, \\
\big((x_0^k),(T_k),(p_0^k),\beta\big) =: \nu &\mapsto \mathcal{S}(\nu) := \begin{pmatrix} \eta(x_0^1, x_1^N) \\ (x_1^k - x_0^{k+1})_{k=1,\dots,N-1} \\ p_0^1 + D_{x_0^1}\tilde{\ell}[\lambda](x_0^1, x_1^N) \\ (p_1^k - p_0^{k+1})_{k=1,\dots,N-1} \\ p_1^N - D_{x_1^N}\tilde{\ell}[\lambda](x_0^1, x_1^N) \\ (H_1^k - H_0^{k+1})_{k=1,\dots,N-1} \\ (p_0^k f_i(x_0^k))_{k=1,\dots,N,\ i\in S_k} \\ (p_0^k g_i(x_0^k, u_0^k))_{k=1,\dots,N,\ i\in S_k} \end{pmatrix}.
\end{aligned} \tag{2.130}$$

Here we put both conditions $\tilde{H}_u = 0$ and $\dot{\tilde{H}}_u = 0$ at the beginning of the interval since we have already pointed out in Remark 2.3.2 that all the possible choices were equivalent.

Since problem (TP) has the same structure than problem (P) in section 2.2, i.e. they both have free control variable (initial-final constraints), we can apply Theorem 2.5.8 and obtain the analogous result below.

Theorem 2.8.8. *Assume that $\hat{w}$ is a Pontryagin minimum of (CP) such that $\hat{W}$ defined in* (2.106) *satisfies condition* (2.69) *for problem (TP). Then the shooting algorithm for (TP) is locally quadratically convergent.*

Remark 2.8.9. *Once system* (2.130) *is obtained, observe that two numerical implementations can be done: one integrating each variable on the interval* $[0, 1]$ *and the other one, going back to the original interval* $[0, T]$, *and using implicitly the continuity conditions* (2.103), (2.122) *and* (2.125) *at each switching time. The latter implementation is done in the numerical tests of Section 2.10 below. In this case, the sensibility with respect to the switching times is obtained from the derivative of the shooting function.*

2.8.1 Reduced Systems

In some cases we can show that some of the conditions imposed to the shooting function in (2.130) are redundant. Hence, they can be removed from the formulation yielding a smaller system that we will refer as *reduced system* and which is associated to a *reduced shooting function.*

Recall that when defining $\mathcal{S}$ we are implicitly imposing that $\ddot{\tilde{H}}_u \equiv 0$. The latter condition together with $\dot{\tilde{H}}_{u,0} = \tilde{H}_{u,1} = 0$, both included in the definition of $\mathcal{S}$, imply that $\dot{\tilde{H}}_u \equiv \tilde{H}_u \equiv 0$. Hence,

$$p_1^k f_i(x_1^k) = p_1^k g_i(x_1^k, u_1^k) = 0, \quad \text{for } k = 1, \dots, N, \; i \in S_k, \tag{2.131}$$

and, in view of the continuity conditions (2.103) and (2.122),

$$p_0^{k+1} f_i(x_0^{k+1}) = p_0^{k+1} g_i(x_0^{k+1}, u_0^{k+1}) = 0, \quad \text{for } k = 1, \dots, N-1, \; i \in S_k. \tag{2.132}$$

Therefore, if a component of the control is singular on I_k and remains being singular on I_{k+1}, then there is no need to impose the boundary conditions on $\tilde{H}_u$ and $\dot{\tilde{H}}_u$ since they are a consequence of the continuity conditions and the implicit equation $\ddot{\tilde{H}}_u \equiv 0$.

Observe now that from (2.117), (2.130) and previous two equations (2.131) and (2.132) we obtain,

$$H_1^k = p_1^k \sum_{N_k \cup \{0\}} f_i(x_1^k) = p_0^{k+1} \sum_{N_k \cup \{0\} \backslash S_{k+1}} f_i(x_0^{k+1}). \tag{2.133}$$

On the other hand,

$$H_0^{k+1} = p_0^{k+1} \sum_{N_{k+1} \cup \{0\} \backslash S_k} f_i(x_0^{k+1}). \tag{2.134}$$

Thus, $H_1^k = H_0^{k+1}$ if $N_k \cup \{0\} \backslash S_{k+1} = N_{k+1} \cup \{0\} \backslash S_k$. The latter equality holds if and only if at instant T_k all the switchings are either bang-to-singular or singular-to-bang.

Definition 2.8.10 (Reduced shooting function)**.** *We call reduced shooting function and we denote it by* $\mathcal{S}^r$ *the function obtained from* $\mathcal{S}$ *defined in* (2.130) *by removing the condition* $H_1^k = H_0^{k+1}$ *whenever all the switchings occurring at* T_k *are either bang-to-singular or singular-to-bang, and removing*

$$p_0^k f_i(x_0^k) = 0, \quad p_0^k g_i(x_0^k, u_0^k) = 0, \tag{2.135}$$

for $k = 2, \dots, N$ *and* $i \in S_{k-1} \cap S_k$.

2.8.2 Square Systems

The reduced system above-presented can occasionally result *square*, in the sense that the reduced function $\mathcal{S}^r$ has as many variables as outputs. This situation occurs, e.g., in problems 1 and 3 of Section 2.10. The fact that the reduced system turns out to be square is a consequence of the structure of the optimal solution. In general, the optimal solution $\hat{u}$ yields a square reduced system if and only if each singular arc is in the interior of $[0,T]$ and at each switching time only one control component switches. This can be interpreted as follows: each singular arc contributes to the formulation with two inputs that are its entry and exit times, and with two outputs that correspond to $p_0^k f_i(x_0^k) = g_i(x_0^k, u_0^k) = 0$, being I_k the first interval where the component is singular and i the index of the analyzed component. On the other hand, whenever a bang-to-bang transition occurs, it contributes to the formulation with one input for the switching time and one output associated to the continuity of the pre-Hamiltonian (which is sometimes expressed as a zero of the switching function).

2.9 Stability under Data Perturbation

In this section we investigate the stability of the optimal solution under data perturbation. We shall prove that, under condition (2.69), the solution is stable under small perturbations of the data functions φ_0, f_i and η. Assume for this stability analysis that the shooting system of the studied problem can be reduced to a square one. We gave a description of this situation in Subsection 2.8.2. Even if the above-mentioned square systems appear in control constrained problems, we start this section by establishing a stability result of the optimal solution for an unconstrained problem. Afterwards, in Subsection 2.9.2, we apply the latter result to problem (TP) and this way we obtain a stability result for the control constrained problem (CP).

2.9.1 Unconstrained control case

Consider then problem (P) presented in Section 2.2, and the family of problems depending on the real parameter μ given by:

$$\begin{aligned}
&\varphi_0^\mu(x_0, x_T) \to \min, \\
&\dot{x}_t = \sum_{i=0}^{m} u_{i,t} f_i^\mu(x_t), \quad \text{for } t \in (0,T), \\
&\eta^\mu(x_0, x_T) = 0.
\end{aligned} \tag{P$_\mu$}$$

Assume that $\varphi_0^\mu : \mathbb{R}^{2n+1} \to \mathbb{R}$ and $\eta^\mu : \mathbb{R}^{2n+1} \to \mathbb{R}^{d_\eta}$ have Lipschitz-continuous second derivatives in the variable (x_0, x_T) and continuously differentiable with respect to μ, and $f_i^\mu : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is twice continuously differentiable with respect to x and continuously differentiable with respect to the parameter μ. In this formulation, the problem (P_0) associated to $\mu = 0$ coincides with (P), i.e. $\varphi_0^0 = \varphi_0$, $f_i^0 = f_i$ for $i = 0, \ldots, m$ and $\eta^0 = \eta$. Recall (2.69) in Theorem 2.5.6, and write the analogous condition for (P_μ) as follows:

$$\bar{\Omega}^\mu(\xi, y, h) \geq \rho\gamma(\xi_0, y, h), \quad \text{on } \mathcal{P}_2^\mu, \tag{2.136}$$

where $\bar{\Omega}^\mu$ and $\mathcal{P}_2^\mu$ are the second variation and critical cone associated to (P_μ), respectively. Let $\mathcal{S}^\mu$ be the shooting function for (P_μ). Thus, we can write

$$\begin{aligned} \mathcal{S}^\mu : \ \ & \mathbb{R}^M \times \mathbb{R} && \to \mathbb{R}^M, \\ & (\,\nu\,,\,\mu\,) && \mapsto \mathcal{S}^\mu(\nu), \end{aligned} \tag{2.137}$$

where we indicate with M the dimension of the domain of $\mathcal{S}$. The following stability result will be established.

Theorem 2.9.1 (Stability of the optimal solution)**.** *Assume that the shooting system generated by problem (P) is square and let $\hat{w}$ be a solution satisfying the uniform positivity condition* (2.69)*. Then there exists a neighborhood $\mathcal{J} \subset \mathbb{R}$ of 0, and a continuous differentiable mapping $\mu \mapsto w^\mu = (x^\mu, u^\mu)$, from $\mathcal{J}$ to $\mathcal{W}$, where w^μ is a weak solution for* (P_μ)*. Furthermore, w^μ verifies the uniform positivity* (2.136)*. Therefore, in view of Theorems 2.5.6 and 2.5.8, the $\gamma-$ growth holds, and the shooting algorithm for (P^μ) is locally quadratically convergent.*

Let us start showing the following stability result for the family of shooting functions $\{\mathcal{S}^\mu\}$.

Lemma 2.9.2. *Under the hypotheses of Theorem 2.9.1, there exists a neighborhood $\mathcal{I} \subset \mathbb{R}$ of 0 and a continuous differentiable mapping $\mu \mapsto \nu^\mu = (x_0^\mu, p_0^\mu, \beta^\mu)$, from $\mathcal{I}$ to $\mathbb{R}^M$, such that $\mathcal{S}^\mu(\nu^\mu) = 0$. Furthermore, the solutions (x^μ, u^μ, p^μ) of* (2.2)-(2.8)-(2.15) *with initial condition (x_0^μ, p_0^μ) and associated multiplier β^μ provide a family of feasible trajectories $w^\mu := (x^\mu, u^\mu)$ verifying*

$$\|x^\mu - \hat{x}\|_\infty + \|u^\mu - \hat{u}\|_\infty + \|p^\mu - \hat{p}\|_\infty + |\beta^\mu - \hat{\beta}| = \mathcal{O}(\mu). \tag{2.138}$$

Proof. Since (2.69) holds, the result in Theorem 2.5.8 yields the non-singularity of the square matrix $D_\nu S^0(\hat{\nu})$. Hence, the Implicit Function Theorem is applicable and we can then guarantee the existence of a neighborhood $\mathcal{B} \subset \mathbb{R}^M$ of $\hat{\nu}$, a neighborhood $\mathcal{I} \subset \mathbb{R}$ of 0, and a continuously differentiable function $\Gamma : \mathcal{I} \to \mathcal{B}$ such that

$$\mathcal{S}^\mu(\Gamma(\mu)) = 0, \quad \text{for all } \mu \in \mathcal{I}. \tag{2.139}$$

Finally, write $\nu^\mu := \Gamma(\mu)$ and use the continuity of $D\Gamma$ on $\mathcal{I}$ to get the first part of the statement.

The feasibility of w^μ holds since equation (2.139) is verified. Finally, the estimation (2.138) follows from the stability of the system of differential equation provided by the shooting method. $\square$

Once we obtained the existence of this w^μ feasible for (P^μ), we may wonder whether it is locally optimal. For this aim, we shall investigate the stability of the sufficient condition (2.69). Denote by $\bar{\Omega}^\mu$ and $\mathcal{P}_2^\mu$ the quadratic mapping and critical cone related to (P^μ), respectively. Given that all the functions involved in $\bar{\Omega}^\mu$ are continuously differentiable with respect to μ, the mapping $\bar{\Omega}^\mu$ itself is continuously differentiable with respect to μ. For the perturbed cone we get the following approximation result.

Lemma 2.9.3. *Assume the same hypotheses as in Theorem 2.9.1. Take $\mu \in \mathcal{I}$ and $(\xi^\mu, y^\mu, h^\mu) \in \mathcal{P}_2^\mu$. Then there exists $(\xi, y, h) \in \mathcal{P}_2$ such that*

$$|\xi_0^\mu - \xi_0| + \|y^\mu - y\|_2 + |h^\mu - h| = \mathcal{O}(\mu). \tag{2.140}$$

The definition below will be useful in the proof of previous Lemma.

Definition 2.9.4. *Define the function $\bar{\eta} : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}^{d_\eta}$, given by*

$$\bar{\eta}(u, x_0) := \eta(x_0, x_T), \tag{2.141}$$

where x is the solution of (2.2) associated to (u, x_0).

Proof. [of Lemma 2.9.3] Recall that $D\bar{\eta}(\hat{u}, \hat{x}_0)$ is onto by Assumption 2.2.1. Call back the definition of the critical cone $\mathcal{C}$ given in (2.51), and notice that we can rewrite it as $\mathcal{C} = \{(z, v) \in \mathcal{W} : \mathcal{G}(z, v) = 0\} = \operatorname{Ker} \mathcal{G}$, with $\mathcal{G}(z, v) := D\eta(\hat{x}_0, \hat{x}_T)(z_0, z_T)$ being an onto linear application from $\mathcal{W}$ to $\mathbb{R}^{d_\eta}$. In view of Goh's Transformation (2.59)-(2.60),

$$D\eta(\hat{x}_0, \hat{x}_T)(z_0, z_T) = D\eta(\hat{x}_0, \hat{x}_T)(\xi_0, \xi_T + B_T y_T), \tag{2.142}$$

for $(z, v) \in \mathcal{W}$ and (ξ, y) being its corresponding transformed direction. Thus, the cone $\mathcal{P}_2$ can be written as $\mathcal{P}_2 = \{\zeta \in \mathcal{H} : \mathcal{K}(\zeta) = 0\} = \operatorname{Ker} \mathcal{K}$, with $\zeta := (\xi, y, h)$,

$\mathcal{H} := \mathcal{X}_2 \times \mathcal{U}_2 \times \mathbb{R}^n$, and $\mathcal{K}(\zeta) := D\eta(\hat{x}_0, \hat{x}_T)(\xi_0, \xi_T + B_T h)$. Then $\mathcal{K} \in \mathcal{L}(\mathcal{H}, \mathbb{R}^{d_\eta})$ and it is surjective. Analogously, $\mathcal{P}_2^\mu = \{\zeta \in \mathcal{H} : \mathcal{K}^\mu(\zeta) = 0\} = \operatorname{Ker} \mathcal{K}^\mu$, with

$$\|\mathcal{K}^\mu - \mathcal{K}\|_{\mathcal{L}(\mathcal{H}, \mathbb{R}^{d_\eta})} = \mathcal{O}(\mu). \tag{2.143}$$

Let us now prove the desired stability property. Take $\zeta^\mu \in \mathcal{P}_2^\mu = \operatorname{Ker} \mathcal{K}^\mu$ having $\|\zeta\|_{\mathcal{H}}^\mu = 1$. Hence $\mathcal{K}(\zeta^\mu) = \mathcal{K}^\mu(\zeta^\mu) + (\mathcal{K} - \mathcal{K}^\mu)(\zeta^\mu)$, and by estimation (2.143),

$$|\mathcal{K}(\zeta^\mu)| = \mathcal{O}(\mu). \tag{2.144}$$

Observe that, since $\mathcal{H} = \operatorname{Ker} \mathcal{K} \oplus \operatorname{Im} \mathcal{K}^\top$, there exists $\zeta^{\mu,*} \in \mathcal{H}^*$ such that

$$\zeta := \zeta^\mu + \mathcal{K}^\top(\zeta^{\mu,*}) \in \operatorname{Ker} \mathcal{K}. \tag{2.145}$$

This yields $0 = \mathcal{K}(\zeta) = \mathcal{K}(\zeta^\mu) + \mathcal{K}\mathcal{K}^\top(\zeta^{\mu,*}) = (\mathcal{K} - \mathcal{K}^\mu)(\zeta^\mu) + \mathcal{K}\mathcal{K}^\top(\zeta^{\mu,*})$. Given that $\mathcal{K}$ is onto, the operator $\mathcal{K}\mathcal{K}^\top$ is invertible and thus

$$\zeta^{\mu,*} = -(\mathcal{K}\mathcal{K}^\top)^{-1}(\mathcal{K} - \mathcal{K}^\mu)(\zeta^\mu). \tag{2.146}$$

The estimation (2.144) above implies $\|\zeta^{\mu,*}\|_{\mathcal{H}^*} = \mathcal{O}(\mu)$. It follows then from (2.145) that $\|\zeta^\mu - \zeta\|_{\mathcal{H}} = \mathcal{O}(\mu)$, and therefore, the desired result holds. □

Proof. [of Theorem 2.9.1] We shall begin by observing that Lemma 2.9.2 provides a neighborhood $\mathcal{I}$ and a class of solutions $\{(x^\mu, u^\mu, p^\mu, \beta^\mu)\}_{\mu \in \mathcal{I}}$ satisfying (2.138). We shall prove that $w^\mu = (x^\mu, u^\mu)$ satisfies the sufficient condition (2.136) close to 0.

Suppose on the contrary that there exists a sequence of parameters $\mu_k \to 0$ and critical directions $(\xi^{\mu_k}, y^{\mu_k}, h^{\mu_k}) \in \mathcal{P}_2^{\mu_k}$ with $\gamma(\xi_0^{\mu_k}, y^{\mu_k}, h^{\mu_k}) = 1$, such that

$$\bar{\Omega}^{\mu_k}(\xi^{\mu_k}, y^{\mu_k}, h^{\mu_k}) \leq o(1). \tag{2.147}$$

Since $\bar{\Omega}^\mu$ is Lipschitz-continuous in μ, from previous inequality we get

$$\bar{\Omega}(\xi^{\mu_k}, y^{\mu_k}, h^{\mu_k}) \leq o(1). \tag{2.148}$$

In view of Lemma 2.9.3, there exists for each k, a direction $(\xi^k, y^k, h^k) \in \mathcal{P}_2$ satisfying

$$|\xi_0^k - \xi_0^{\mu_k}| + \|y^k - y^{\mu_k}\|_2 + |h^k - h^{\mu_k}| = \mathcal{O}(\mu_k). \tag{2.149}$$

Hence, by inequality (2.148) and given that $\hat{w}$ satisfies (2.69),

$$\rho\gamma(\xi_0^k, y^k, h^k) \leq \bar{\Omega}(\xi^k, y^k, h^k) \leq o(1). \tag{2.150}$$

However, the left hand-side of this last inequality cannot go to 0 since (ξ_0^k, y^k, h^k) is close to $(\xi_0^{\mu_k}, y^{\mu_k}, h^{\mu_k})$ by estimation (2.149), and the elements of the latter sequence have unit norm. This leads to a contradiction. Hence, the result follows. □

2.9.2 Control constrained case

In this paragraph we aim to investigate the stability of the shooting algorithm applied to the problem with control bounds (CP) studied in Section 2.8. Observe that previous Theorem 2.9.1 guarantees the weak optimality for the perturbed problem when the control constraints are absent. In case we have control constraints, this stability result is applied to the transformed problem (TP) (given by equations (2.99)-(2.103) of Section 2.8) yielding a similar stability property, but for which the nominal point and the perturbed ones are weak optimal for (TP). This means that they are optimal in the class of extremals having the same control structure, and switching times and singular arcs sufficiently close in L_∞. An extremal satisfying optimality in this sense will be called *weak-structural optimal,* and a formal definition would be as follows.

Definition 2.9.5 (Weak-structural optimality). *A feasible solution $\hat{w}$ for problem (CP) is called a weak-structural solution if its transformed extremal $\hat{W}$ given by* (2.104)-(2.106) *is a weak solution of (TP).*

Theorem 2.9.6 (Sufficient condition for the extended weak minimum in the control constrained case). *Let $\hat{w}$ be a feasible solution for (CP) satisfying Assumptions 2.8.2 and 2.8.4. Consider the transformed problem (TP) and the corresponding transformed solution $\hat{W}$ given by* (2.104)-(2.106). *If $\hat{w}$ satisfies* (2.69) *for (TP), then $\hat{w}$ is an extended weak solution for (CP).*

Proof. It follows from the sufficient condition in Theorem 2.5.6 applied to (TP). □

Consider the family of perturbed problems given by:

$$
\begin{aligned}
&\varphi_0^\mu(x_0, x_T) \rightarrow \min,\\
&\dot{x}_t = \sum_{i=0}^{m} u_{i,t} f_i^\mu(x_t), \quad \text{for } t \in (0,T),\\
&\eta^\mu(x_0, x_T) = 0,\\
&0 \leq u_t \leq 1, \quad \text{a.e on } (0,T).
\end{aligned}
\tag{CP$_\mu$}
$$

The following stability result follows from Theorem 2.9.1.

Theorem 2.9.7 (Stability in the control constrained case). *Assume that the reduced shooting system generated by problem (CP) is square. Let $\hat{w}$ be the solution of (CP) and $\{\hat{T}_k\}_{k=1}^N$ its switching times. Denote by $\hat{W}$ its transformation via equation* (2.106). *Suppose that $\hat{W}$ satisfies uniform positivity condition* (2.69) *for problem (TP). Then there exists a neighborhood $\mathcal{J} \subset \mathbb{R}$ of 0 such that for every parameter $\mu \in \mathcal{J}$ there* *exists a weak-structural optimal extremal w^μ of (CP^μ) with switching times $\{T_k^\mu\}_{k=1}^N$ satisfying the estimation*

$$\sum_{k=1}^N |T_k^\mu - \hat{T}_k| + \sum_{k=1}^N \sum_{i\in S_k} \|u_i^\mu - \hat{u}_i\|_{\infty, I_k^\mu \cap \hat{I}_k} + \|x^\mu - \hat{x}\|_\infty = \mathcal{O}(\mu), \tag{2.151}$$

where $I_k^\mu := (T_{k-1}^\mu, T_k)$. Furthermore, the transformed perturbed solution W^μ verifies uniform positivity (2.136) *and hence quadratic growth in the weak sense for problem (TP) holds, and the shooting algorithm for* (CP_μ) *is locally quadratically convergent.*

2.9.3 Additional analysis for the scalar control case

Consider a particular case where the control $\hat{u}$ is scalar. The lemma below shows that the perturbed solutions are Pontryagin extremals for (CP_μ) provided that the following assumption holds.

Assumption 2.9.8. *(a) The switching function H_u is never zero in the interior of a bang arc. Hence if $\hat{u} = 1$ on (t_1, t_2) then $H_u < 0$ on (t_1, t_2), and if $\hat{u} = -1$ on (t_1, t_2) then $H_u > 0$ on (t_1, t_2).*

(b) If $\hat{T}_k$ is a bang-to-bang switching time then $\dot{H}_u(\hat{T}_k) \neq 0$.

The property (a) is called *strict complementarity for the control constraint.*

Lemma 2.9.9. *Suppose that $\hat{u}$ satisfies Assumption 2.9.8. Let w^μ as in Theorem 2.9.7 above. Then w^μ is a Pontryagin extremal for* (CP_μ)*.*

Proof. We intend to prove that w^μ satisfies the minimum condition (2.12) given by the Pontryagin Maximum Principle. Observe that on the singular arcs, $H_u^\mu = 0$ since w^μ is the solution associated to a zero of the shooting function. It suffices then to study the stability of the sign of H_u^μ on the bang arcs around a switching time. First suppose that $\hat{u}$ has a bang-to-singular switching at $\hat{T}_k$. Assume, without loss of generality, that $\hat{u} \equiv 1$ on $\hat{I}_k$ and $\hat{u}$ is singular on $[\hat{T}_k, \hat{T}_{k+1}]$. Let us write

$$\ddot{H}_u^\mu = a^\mu + u^\mu b^\mu, \tag{2.152}$$

where a^μ and $b^\mu := \frac{\partial}{\partial u} \ddot{H}_u^\mu$ are continuous functions on $[0, T]$, and continuously differentiable with respect to μ since they depend on x^μ and p^μ. Assumption 2.8.5 yields $b^0 < 0$ on $[\hat{T}_k, \hat{T}_{k+1}]$, and therefore

$$b^\mu < 0, \quad \text{on } [T_k^\mu, T_{k+1}^\mu]. \tag{2.153}$$

Due to (2.152), the sign of $\ddot{H}_u^\mu$ around T_k^μ depends on $u^\mu(T_k^\mu+) - u^\mu(T_k^\mu-)$. But this quantity is negative since u^μ passes from its upper bound to a singular arc. From the latter assertion and (2.153) follows

$$\ddot{H}_u^\mu(T_k^\mu-) < 0, \tag{2.154}$$

and thus H_u^μ is concave at the junction time T_k^μ. Since H_u^μ is null on $[T_k^\mu, T_{k+1}^\mu]$, its concavity implies that it has to be negative before entering this arc. Hence, w^μ respects the minimum condition on the interval $\hat{I}_k$.

Consider now the case when $\hat{u}$ has a bang-to-bang switching at $\hat{T}_k$. Let us begin by showing that $H_u^\mu(T_k^\mu) = 0$. Suppose on the contrary that $H_u^\mu(T_k^\mu) \neq 0$. Then $H^\mu(T_k^\mu+) - H^\mu(T_k^\mu-) \neq 0$, contradicting the continuity condition imposed on H in the shooting system. Hence $H_u^\mu(T_k^\mu) = 0$. On the other hand, since $\dot{H}_u(\hat{T}_k) \neq 0$ by Assumption 2.9.8, the value $\dot{H}_u^\mu(T_k^\mu)$ has the same sign for small μ. This implies that H_u^μ has the same sign before and after T_k^μ that H_u (before and after $\hat{T}_k$), respectively. The result follows. $\square$

Remark 2.9.10. *We end this analysis by mentioning that if the transformed solution $\hat{W}$ satisfies the uniform positivity* (2.69) *for (TP), then $\hat{w}$ verifies the sufficient condition established in Aronna et al. [8] and hence it is actually a Pontryagin minimum. This follows from the fact that in condition* (2.69) *we are allowed to perturb the switching times, and hence* (2.69) *is more restrictive (or demanding) that the condition in [8].*

2.10 Numerical Simulations

Now we aim to check numerically the extended shooting method described above. More precisely, we want to compare the classical $n \times n$ shooting formulation to an extended formulation with the additional conditions on the pre-Hamiltonian continuity. We test three problems with singular arcs: a fishing and a regulator problem and the well-known Goddard problem, which we have already studied in [65, 90]. For each problem, we perform a batch of shootings on a large grid around the solution. We then check the convergence and the solution found, as well as the singular values and condition number of the Jacobian matrix of the shooting function.

2.10.1 Test problems

2.10.1.1 Fishing problem

The first example we consider is a fishing problem described in [34]. The state $x_t \in \mathbb{R}$ represents the fish population (halibut), the control $u_t \in \mathbb{R}$ is the fishing activity, and the objective is to maximize the net revenue of fishing over a fixed time interval. The coefficient $(E - c/x)$ takes into account the greater fishing cost for a low fish population. The problem is

$$\begin{cases} \max \displaystyle\int_0^T (E - c/x_t)\; u_t\, U_{\max} \mathrm{d}t, \\ \dot{x}_t = r\, x_t\; (1 - x_t/k) \;-\; u_t\, U_{\max}, \\ 0 \le u_t \le 1, \quad \forall t \in [0, T], \\ x_0 = 70, \quad x_T \text{ free}, \end{cases} \tag{P$_1$}$$

with $T = 10$, $E = 1$, $c = 17.5$, $r = 0.71$, $k = 80.5$ and $U_{\max} = 20$.

Remark 2.10.1. *The state and control were rescaled by a factor* 10^6 *compared to the original data for a better numerical behavior.*

Remark 2.10.2. *Since we have an integral cost, we add a state variable to adapt* (P$_1$) *to the initial-final cost formulation. It is well-known that its corresponding costate variable is constantly equal to 1.*

The pre-Hamiltonian for this problem is

$$H := (c/x - E)\, u\, U_{\max} + p[r\, x\, (1 - x/k) - u\, U_{\max}], \tag{2.155}$$

and hence the switching function

$$\Phi_t = D_u H_t = U_{\max}(c/x_t - E - p_t), \quad \forall t \in [0, T]. \tag{2.156}$$

The optimal control follows the bang-bang law

$$\begin{cases} u_t^* = 0 & \text{if } \Phi_t > 0, \\ u_t^* = 1 & \text{if } \Phi_t < 0. \end{cases} \tag{2.157}$$

Over a singular arc where $\Phi = 0$, we assume that the relation $\ddot{\Phi} = 0$ gives the expression of the singular control *(t is omitted for clarity)*

$$u^*_{\text{singular}} = \frac{k\; r}{2(c/x - p)U_{max}} \left(\frac{c}{x} - \frac{c}{k} - p + \frac{2px}{k} - \frac{2px^2}{k^2} \right). \tag{2.158}$$

The solution obtained for (P$_1$) has the structure **bang-singular-bang**, as shown on Figure 2.1.

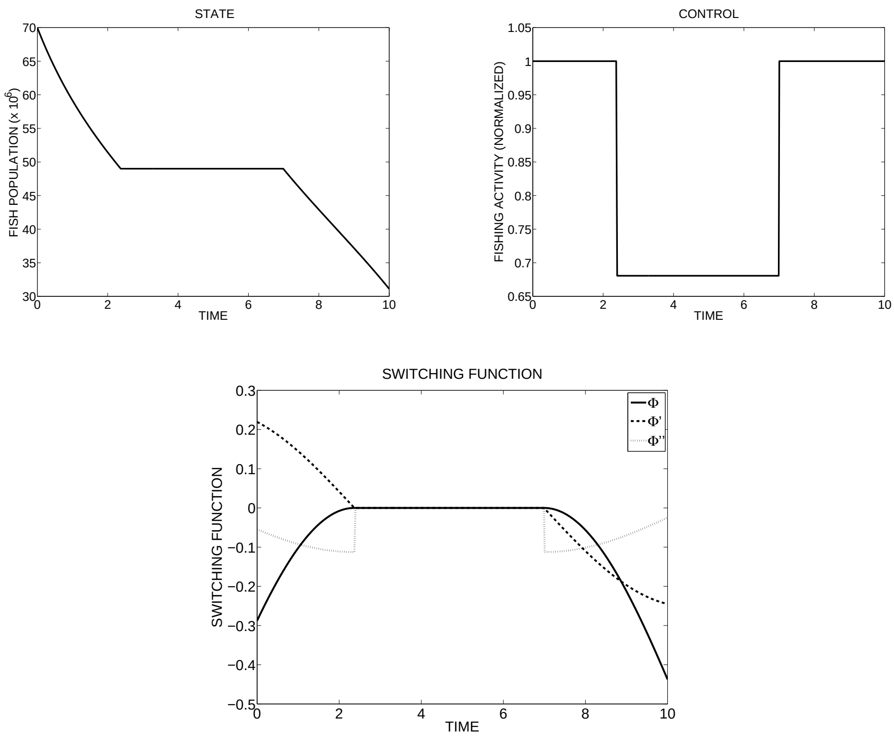

Figure 2.1: Fishing Problem

Shooting formulations. Assuming the control structure, the shooting unknowns are the initial costate and the limits of the singular arc,

$$\nu := (p_0, t_1, t_2) \in \mathbb{R}^3.$$

The classical shooting formulation uses the entry conditions on t_1

$$\mathcal{S}_1(\nu) := (p_T, \Phi_{t_1}, \dot{\Phi}_{t_1}).$$

Solving $S_1(\nu) = 0$ is a square nonlinear system, for which a quasi-Newton method can be used. Note that even if there is no explicit condition on t_2 in S, the value of p_T does depend on t_2 via the control switch.

The extended shooting formulation adds two conditions corresponding to the continuity of the pre-Hamiltonian at the junctions between bang and singular arcs. We denote $[H]_t := H_{t+} - H_{t-}$ the pre-Hamiltonian jump, and define

$$\tilde{\mathcal{S}}_1(\nu) = (p_{10}, \Phi_{t_1}, \dot{\Phi}_{t_1}, [H]_{t_1}, [H]_{t_2}). \tag{2.159}$$

To solve $\tilde{S}_1(\nu) = 0$ we use a nonlinear least-square algorithm (see paragraph 2.10.2 below for more details).

2.10.1.2 Regulator problem

The second example is the quadratic regulator problem described in Aly [6]. We want to minimize the integral of the sum of the squares of the position and speed of a mobile over a fixed time interval, the control being the acceleration.

$$\left\{\begin{array}{l} \min \frac{1}{2} \int_0^T \left(x_{1,t}^2 + x_{2,t}^2\right) \mathrm{d}t, \\ \dot{x}_{1,t} = x_{2,t}, \\ \dot{x}_{2,t} = u_t, \\ -1 \leq u_t \leq 1, \quad \text{a.e. on } [0,T], \\ x_0 = (0,1), \quad x_T \text{ free}, \\ T = 5. \end{array}\right. \tag{P$_2$}$$

The corresponding pre-Hamiltonian

$$H := \tfrac{1}{2}(x_1^2 + x_2^2) + p_1 x_2 + p_2 u, \tag{2.160}$$

and hence we have the switching function

$$\Phi_t := D_u H_t = p_{2,t}. \tag{2.161}$$

The bang-bang optimal control satisfies

$$u_t^* = -\text{sign } p_{2,t} \quad \text{if } \Phi_t \neq 0. \tag{2.162}$$

The singular control is again obtained from $\ddot{\Phi} = 0$ and verifies

$$u_{\text{singular},t}^* = x_{1,t}. \tag{2.163}$$

The solution for this problem has the structure **bang-singular**, as shown on Figure 2.2.

Shooting formulations. Assuming the control structure, the shooting unknowns are

$$\nu := (p_{1,0}, p_{2,0}, t_1) \in \mathbb{R}^3. \tag{2.164}$$

For the classical shooting formulation, in order to have a square system, we can for instance combine the two entry conditions on Φ and $\dot{\Phi}$, since we only have one additional unknown which is the entry time t_1. Thus we define

$$\mathcal{S}_2(\nu) := (p_{1,T}, p_{2,T}, \Phi_{t_1}^2 + \dot{\Phi}_{t_2}^2). \tag{2.165}$$

The extended formulation does not require such a trick, we simply have

$$\tilde{\mathcal{S}}_2(\nu) := (p_{1,T}, p_{2,T}, \Phi_{t_1}, \dot{\Phi}_{t_1}, [H]_{t_1}). \tag{2.166}$$

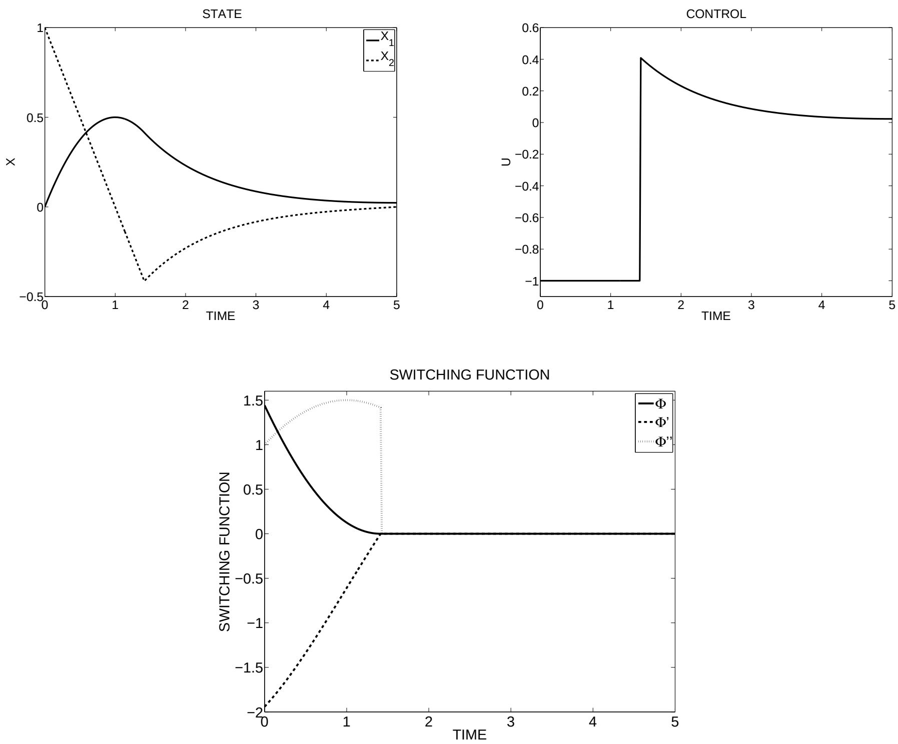

Figure 2.2: Regulator Problem

2.10.1.3 Goddard problem

The third example is the well-known Goddard problem, introduced in Goddard [66] and studied for instance in Seywald-Cliff [119]. This problem models the ascent of a rocket through the atmosphere, and we restrict here ourselves to vertical (unidimensional) trajectories. The state variables are the altitude, speed and mass of the rocket during the flight, for a total dimension of 3. The rocket is subject to gravity, thrust and drag forces. The final time is free, and the objective is to reach a certain altitude with a minimal fuel consumption, i.e. a maximal final mass.

$$\left\{\begin{array}{l} \max\ m_T, \\ \dot{r} = v, \\ \dot{v} = -1/r^2 + 1/m(T_{\max}u - D(r,v)) \\ \dot{m} = -bT_{\max}u, \\ 0 \le u_t \le 1, \quad \text{a.e. on } (0,1), \\ r_0 = 1,\ v_0 = 0,\ m_0 = 1, \\ r_t = 1.01, \\ T \text{ free}, \end{array}\right. \tag{P$_3$}$$

with the parameters $b = 7$, $T_{\max} = 3.5$ and the drag given by

$$D(r,v) := 310v^2e^{-500(r-1)}.$$

The pre-Hamiltonian function here is

$$H := p_r v + p_v(-1/r^2 + 1/m(T_{\max}u - D(r,v))) - p_m b T_{\max} u, \tag{2.167}$$

where p_r, p_v and p_m are the costate variables associated to r, v and m, respectively. The switching function is

$$\Phi := D_u H = T_{\max}((1-p_m)b + p_v/m). \tag{2.168}$$

Hence, the bang-bang optimal control is given by

$$\begin{cases} u_t^* = 0 & \text{if } \Phi_t > 0, \\ u_t^* = 1 & \text{if } \Phi_t < 0, \end{cases} \tag{2.169}$$

and the singular control can be obtained by formally solving $\ddot{\Phi} = 0$. The expression of u^*_{singular}, however, is quite complicated and is not recalled here. The solution for this problem has the well-known typical structure **1-singular-0**, as shown on Figures 2.3 and 2.4.

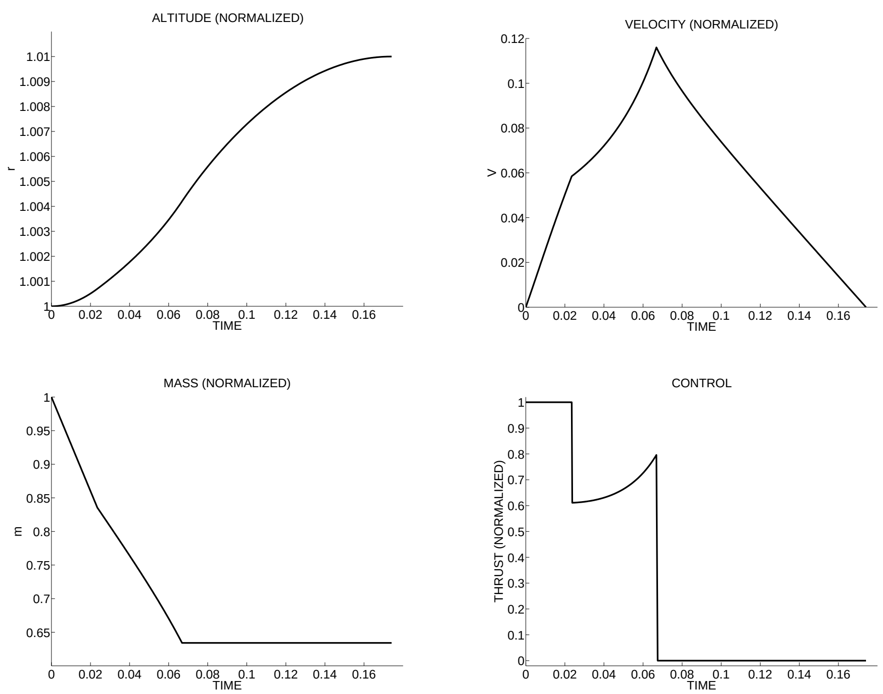

Figure 2.3: Goddard Problem

Shooting formulations. Once again fixing the control structure, the shooting unknowns are

$$\nu = (p_{1,0}, p_{2,0}, p_{3,0}, t_1, t_2, T) \in \mathbb{R}^6. \tag{2.170}$$

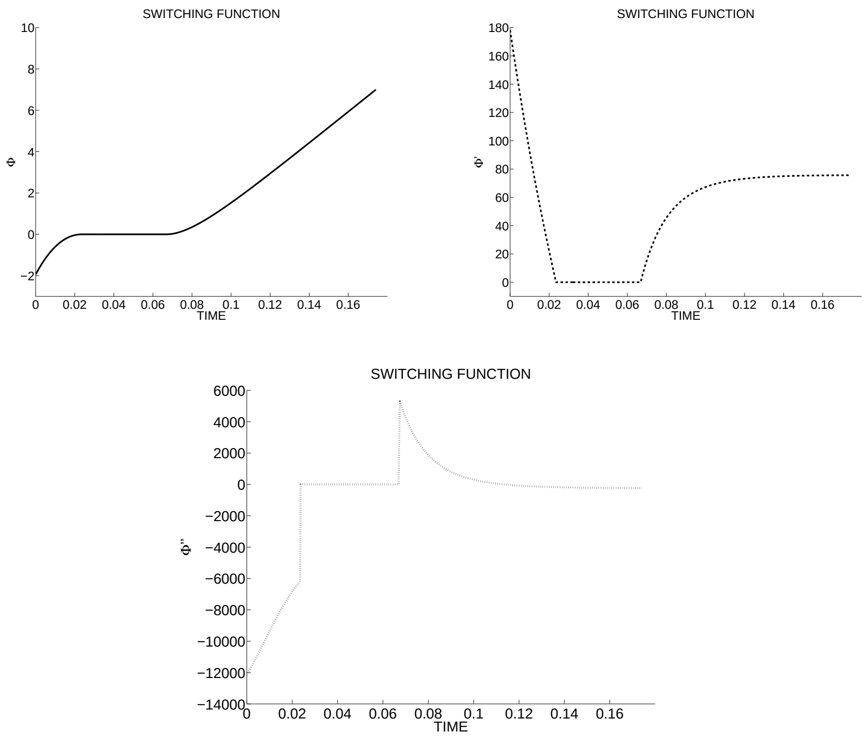

Figure 2.4: Goddard Problem

Here is the classical shooting formulation with the entry conditions on t_1

$$\mathcal{S}_3(\nu) := (x_{1,T} - 1.01, p_{2,T}, p_{3,T} + 1, \Phi_{t_1}, \dot{\Phi}_{t_1}, H_T), \tag{2.171}$$

while the extended formulation is

$$\tilde{\mathcal{S}}_3(\nu) := (x_{1,T} - 1.01, p_{2,T}, p_{3,T} + 1, \Phi_{t_1}, \dot{\Phi}_{t_1}, H_T, [H]_{t_1}, [H]_{t_2}). \tag{2.172}$$

2.10.2 Results

All tests were run on a 12-core platform, with the parallelized (OPENMP) version of the SHOOT ([91]) package. The ODE solver is a fixed step 4th. order Runge Kutta method with 500 steps. The classical shooting is solved with a basic Newton method, and the extended shooting with a basic Gauss-Newton method. Both algorithms use a fixed step length of 1 and a maximum of 1000 iterations. In addition to the singular/bang structure, the value of the control on the bang arcs is also fixed according to the expected solution.

The values for the initial costates are taken in $[-10, 10]$, and the values for the entry/exit times in $[0, T]$ for (P_1) and (P_2). For (P_3), the entry, exit and final times are taken in $[0, 0.2]$. The number of grid points is set around to 10 000 for the three problems. These grids for the starting points are quite large and rough, which explains the low success rate for (P_1) and (P_3). However, the solution was found for all three problems.

For each problem, the results are summarized in 3 tables. The first table indicates the total CPU time for all shootings over the grid, the success rate of convergence to the solution, the norm of the shooting function at the solution, and the objective value. The second table recalls the solution found by both formulations: initial costate and junction times, as well as final time for (P_3). The third table gives the singular values for the Jacobian matrix at the solution, as well as its condition number $\kappa := \sigma_1/\sigma_n$.

We observe that for all three problems (P_1), (P_2) and (P_3), both formulations converge to the same solution, ν^* and the objective being identical to more than 6 digits. The success rate over the grid, total CPU time and norm of the shooting function at the solution are close for both formulations. Concerning the singular values and condition number of the Jacobian matrix, we note that for (P_2) the extended formulation has the smallest singular value going from 10^{-8} to 1, thus improving the condition number by a factor 10^8. This is caused by the combination of the two entry conditions into a single one that we used in the classical formulation for this problem: as the singular arc lasts until t_f, there is only one additional unknown, the entry time.

Overall, these results validate the extended shooting formulation, which perform at least as well as the classical formulation and has a theoretical foundation.

Remark 2.10.3. *Several additional tests runs were made using the HYBRD ([64]) and NL2SNO ([39]) solvers for the classical and extended shootings instead of the basic Newton and Gauss-Newton method. The results were similar, apart from a higher success rate for the HYBRD solver compared to NL2SNO.*

Remark 2.10.4. *We also tested both formulations using the sign of the switching function to determine the control value over the bang arcs, instead of forcing the value. However, this causes a numerical instability at the exit of a singular arc, where the switching function is supposed to be 0 but whose sign determines the control at the beginning of the following bang arc. This instability leads to much more erratic results for both shooting formulations, but with the same general tendencies.*

Problem 1:

Shooting grid: $[-10, 10] \times [0, T]^2$, 21^3 gridpoints, 9261 shootings.

Shooting	CPU	Success	Convergence	Objective
Classical	74 s	21.28 %	1.43E-16	-106.9059979
Extended	86 s	22.52 %	6.51E-16	-106.9059979

Table 1.1: (P_1) CPU times, success rate, convergence and objective

Shooting	p_0	t_1	t_2
Classical	-0.462254744307241	2.37041478456004	6.98877992494185
Extended	-0.462254744307242	2.37041478456004	6.98877992494185

Table 1.2: (P_1) solution ν^* found

Shooting	σ_1	σ_2	σ_3	κ
Classical	3.61	0.43	5.63E-02	64.12
Extended	27.2	1.71	3.53E-01	77.05

Table 1.3: (P_1) singular values and condition number for the Jacobian

Problem 2

Shooting grid: $[-10, 10]^2 \times [0, T]$, 21^3 gridpoints, 9261 shootings.

Shooting	CPU	Success	Convergence	Objective
Classical	468 s	94.14 %	1.17E-16	0.37699193037
Extended	419 s	99.36 %	1.22E-13	0.37699193037

Table 2.1: (P_2) CPU times, success rate, convergence and objective

Shooting	$p_{1,0}$	$p_{2,0}$	t_1
Classical	0.942173346483640	1.44191017584598	1.41376408762863
Extended	0.942173346476773	1.44191017581021	1.41376408762893

Table 2.2: (P_2) solution ν^* found

Shooting	σ_1	σ_2	σ_3	κ
Classical	24.66	5.19	1.96E-08	1.26E+09
Extended	24.70	5.97	1.13	21.86

Table 2.3: (P_2) singular values and condition number for the Jacobian

Problem 3

Shooting grid: $[-10,10]^3 \times [0,0.2]^3$, $4^3 \times 5^3$ gridpoints, 8000 shootings.

Shooting	CPU	Success	Convergence	Objective
Classical	42 s	0.82 %	5.27E-13	-0.634130666
Extended	52 s	0.85 %	1.29E-10	-0.634130666

Table 3.1: (P_3) CPU times, success rate, convergence and objective

Shoot.	$p_{r,0}$	$p_{v,0}$	$p_{m,0}$
Class.	-50.9280055899288	-1.94115676279896	-0.693270270795148
Exten.	-50.9280055901093	-1.94115676280611	-0.693270270787320
	t_1	t_2	t_f
Class.	0.02350968417421373	0.06684546924474312	0.174129456729642
Exten.	0.02350968417420884	0.06684546924565564	0.174129456733106

Table 3.2: (P_3) solution ν^* found

Shooting	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	κ
Classical	6182	9.44	8.13	2.46	0.86	1.09E-03	5.67E+06
Extended	6189	12.30	8.23	2.49	0.86	1.09E-03	5.67E+06

Table 3.3: (P_3) singular values and condition number for the Jacobian

2.11 Conclusion

Theorems 2.5.8 and 2.8.8 provide a theoretical support for an extension of the shooting algorithm for problems with all the control variables entering linearly and having singular arcs. The shooting functions here presented are not the ones usually implemented in numerical methods as we have already pointed out in previous section. They come from systems having more equations than unknowns in the general case, while before in practice only square systems have been used. Anyway, we are not able to prove the injectivity of the derivative of the shooting function when we remove some equations, i.e. we are not able to determine which equations are redundant, and we suspect that it can vary for different problems.

The proposed algorithm was tested in three simple problems, where we compared its performance with the classical shooting method for square systems. The percentages of convergence are similar in both approaches, the singular values and condition number of the Jacobian matrix of the shooting function coincide in two problems, and are better for our formulation in one of the problems. Summarizing, we can observe that the proposed method works as well as the one currently used in practice and has a theoretical foundation.

In the bang-singular-bang case, as in the fishing and Goddard's problems, our formulation coincides with the algorithm proposed by Maurer [92].

Whenever the system can be reduced to a square one, given that the sufficient condition for the non-singularity of the Jacobian of the shooting function coincides with a sufficient condition for optimality, we could established the stability of the optimal local solution under small perturbations of the data.

3

Partially affine control problems: second order conditions and a shooting algorithm

Published as the INRIA Research Report Nr. 7764 [October 2011], under the title *Partially affine control problems: second order conditions and a well-posed shooting algorithm.*

Abstract

This paper deals with optimal control problems for systems that are affine in one part of the control variables and nonlinear in the rest of the control variables. We have finitely many equality and inequality constraints on the initial and final states. First we obtain second order necessary and sufficient conditions for weak optimality. Afterwards, we propose a shooting algorithm. We show that the sufficient condition above-mentioned is also sufficient for the local quadratic convergence of this algorithm.

3.1 Introduction

In this article we investigate an optimal control that is affine in one part of the control variables and nonlinear in the rest of the control variables. We consider a finite quantity of initial-final state constraints. First we provide second order necessary and sufficient conditions for weak optimality. We do not assume uniqueness of the multipliers. Then we prove that the stated sufficient condition is also sufficient for local quadratic convergence of the shooting algorithm. Some of the techniques used in Aronna et al. [8, 10] (or chapters 1 and 2) are employed.

The investigation of this particular framework is motivated by some models encountered in practice. Among these we can mention: 1. an hydro-thermal electricity production problem studied in Bortolossi et al. [26] and Aronna et al. [9] or (Chapter 4), 2. the Goddard's problem in 3 dimensions introduced in Goddard [66] and analysed in, e.g., Martinon et al. [18], 3. the problem of atmospheric flight considered by Oberle in [103].

The subject of optimality conditions for these partially affine problems have been studied by Goh in [68, 69], Dmitruk in [46], Dmitruk and Shishov in [47], and Maurer and Osmolovskii [95]. In the papers by Goh, second order necessary conditions are derived assuming uniqueness of the multiplier. They are consequence of the classical Legendre-Clebsch condition applied to a transformed problem, and are currently known as Goh-Legendre-Clebsch conditions. Dmitruk and Shishov [47] analysed the quadratic functional associated to the second variation of the Lagrangian function. They provided a set of necessary conditions for the nonnegativity of this quadratic functional. In [46] Dmitruk proposed, without proof, necessary and sufficient conditions for a problem having a particular structure: the affine control variable applies to a term depending only on the state variable, i.e. the affine and nonlinear controls are 'uncoupled'. This hypothesis is not used in the present work. The conditions established here coincide with those suggested in Dmitruk [46] when the latter are applicable. All of these four articles use Goh's Transformation, first introduced in [67], to derive their conditions. We use this transformation as well. On the other hand, in [95] Maurer and Osmolovskii gave a sufficient condition for a class of problems having one affine control subject to bounds and such that it is bang-bang at the optimal solution. This structure is not studied here since no control constraints are considered. Our affine control is suppose to be totally singular.

Regarding the shooting method applied to the numerical solution of partially affine problems we can mention the articles Bonnans et al. [90, 18], Oberle [101, 102, 103] and Oberle-Taubert [100]. All these works present interesting implementations of a shooting-like method to solve partially affine control problems having bang-singular or bang-bang solutions and, in some cases, running-state constraints are considered.

The article is organised as follows. In section 3.2 we present the problem, the basic definitions and properties. A necessary condition is established in section 3.3. In section 3.4 we introduce Goh's Transformation, that is an essential tool for the rest of the article. In Section 3.5 we show a new necessary condition, and in Section 3.6 we give a sufficient for weak optimality. A shooting algorithm is proposed in Section 3.7, and in Section 3.8 we give a sufficient condition for this algorithm to converge locally quadratically.

3.2 Statement of the problem and assumptions

3.2.1 Statement of the problem.

In this paper we investigate the optimal control problem (P) given by

$$J := \varphi_0(x_0, x_T) \to \min, \tag{3.1}$$

$$\dot{x}_t = \sum_{i=0}^{m} v_{i,t} f_i(x_t, u_t), , \quad \text{a.e. on } [0,T], \tag{3.2}$$

$$\eta_j(x_0, x_T) = 0, \text{ for } j = 1 \dots, d_\eta, \tag{3.3}$$

$$\varphi_i(x_0, x_T) \leq 0, \text{ for } i = 1, \dots, d_\varphi, \tag{3.4}$$

where $f_i : \mathbb{R}^{n+l} \to \mathbb{R}^n$ for $i = 0, \dots, m$, $\varphi_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 0, \dots, d_\varphi$, $\eta_j : \mathbb{R}^{2n} \to \mathbb{R}$ for $j = 1, \dots, d_\eta$ and $v_0 \equiv 1$ (it is not a variable). The nonlinear control u belongs to $\mathcal{U} := L_\infty(0,T;\mathbb{R}^l)$, while by $\mathcal{V} := L_\infty(0,T;\mathbb{R}^m)$ we denote the space of affine controls, and $\mathcal{X} := W^1_\infty(0,T;\mathbb{R}^n)$ refers to the state space. When needed, put $w = (x,u,v)$ for a point in $\mathcal{W} := \mathcal{X} \times \mathcal{U} \times \mathcal{V}$. Assume that all data functions f_i have Lipschitz-continuous second derivatives.

A *trajectory* is an element $w \in \mathcal{W}$ that satisfies the state equation (3.2). If in addition constraints (3.3) and (3.4) hold, say that w is a *feasible trajectory* of problem (P). Denote by $\mathcal{A}$ the *set of feasible trajectories.*

The following type of minimum is considered.

Definition 3.2.1. *A feasible trajectory $\hat{w} = (\hat{x}, \hat{u}, \hat{v}) \in \mathcal{W}$ is said to be a weak minimum of (P) if there exists $\varepsilon > 0$ such that the cost function attains at $\hat{w}$ its minimum in the set of feasible trajectories satisfying*

$$\|x - x^0\|_\infty < \varepsilon, \quad \|u - \hat{u}\|_\infty < \varepsilon, \quad \|v - \hat{v}\|_\infty < \varepsilon.$$

In the sequel, we study a nominal feasible trajectory $\hat{w} = (\hat{x}, \hat{u}, \hat{v}) \in \mathcal{W}$. An element $\delta w \in \mathcal{W}$ is termed *feasible variation* for $\hat{w}$ if $\hat{w} + \delta w \in \mathcal{A}$. We write $\mathbb{R}^{d,*}$ for the $d-$dimensional space of row vectors with real components. Take $\lambda = (\alpha, \beta, p)$ in $\mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0, T; \mathbb{R}^{n,*})$, i.e. p is a Lip-continuous function with values in $\mathbb{R}^{n,*}$. Define the *pre-Hamiltonian* function

$$H[\lambda](x, u, v, t) := p_t \sum_{i=0}^{m} v_i f_i(x, u),$$

the *terminal Lagrangian* function

$$\ell[\lambda](q) := \sum_{i=0}^{d_\varphi} \alpha_i \varphi_i(q) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(q),$$

and the *Lagrangian* function

$$\mathcal{L}[\lambda](w) := \ell[\lambda](\hat{x}_0, \hat{x}_T) + \int_0^T p_t \left(\sum_{i=0}^{m} v_{i,t} f_i(x_t, u_t) - \dot{x}_t \right) \mathrm{d}t. \tag{3.5}$$

In the sequel, whenever some argument of f_i, H, ℓ, $\mathcal{L}$ or their derivatives is omitted, assume that they are evaluated at $\hat{w}$. Without loss of generality suppose that

$$\varphi_i(\hat{x}_0, \hat{x}_T) = 0, \text{ for all } i = 0, 1, \dots, d_\varphi. \tag{3.6}$$

3.2.2 Lagrange and Pontryagin multipliers

Definition 3.2.2. *An element $\lambda = (\alpha, \beta, p) \in \mathbb{R}^{d_\varphi+1,*} \times \mathbb{R}^{d_\eta,*} \times W^1_\infty(0, T; \mathbb{R}^{n,*})$ is a Lagrange multiplier associated to $\hat{w}$ if it satisfies the following conditions:*

$$|\alpha| + |\beta| = 1, \tag{3.7}$$

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{d_\varphi}) \geq 0, \tag{3.8}$$

the function p is solution of the costate equation

$$-\dot{p}_t = H_x[\lambda](\hat{x}_t, \hat{u}_t, \hat{v}_t, t), \tag{3.9}$$

and satisfies the transversality conditions

$$\begin{aligned} p_0 &= -D_{x_0}\ell[\lambda](\hat{x}_0, \hat{x}_T), \\ p_T &= D_{x_T}\ell[\lambda](\hat{x}_0, \hat{x}_T), \end{aligned} \tag{3.10}$$

and the following stationarity conditions hold

$$\begin{cases} H_u[\lambda](\hat{x}(t), \hat{u}(t), \hat{v}(t), t) = 0, \\ H_v[\lambda](\hat{x}(t), \hat{u}(t), \hat{v}(t), t) = 0, \end{cases} \quad \text{a.e. on } [0, T]. \tag{3.11}$$

We say that λ is a Pontryagin multiplier if it satisfies (3.7)-(3.10) *and the following minimum condition*

$$H[\lambda](\hat{x}_t, \hat{u}_t, \hat{v}_t, t) = \min_{(u,v)\in\mathbb{R}^{l+m}} H[\lambda](\hat{x}_t, u, v, t), \quad \text{a.e. on } [0, T]. \tag{3.12}$$

Denote by Λ_L and Λ_P the sets of Lagrange and Pontryagin multipliers, respectively.

It easily follows from the previous definitions that

$$\Lambda_P \subset \Lambda_L, \tag{3.13}$$

and

$$H_{v_j}[\lambda] = pf_j(\hat{x}, \hat{u}) \equiv 0, \quad \text{for } j = 1, \dots, m. \tag{3.14}$$

Recall the following well-known result.

Theorem 3.2.3. *The set Λ_L and Λ_P are non empty and compact.*

Proof. Regarding the existence of a Pontryagin multiplier, the reader is referred to Ioffe-Tihomirov [74].

In order to prove the compactness, observe that p may be expressed as a linear continuous mapping of (α, β). Thus, since the normalization (3.7) holds, both Λ_L and Λ_P are finite-dimensional compact sets. □

In view of previous result, note that Λ_L and Λ_P can be identified with compact subsets of $\mathbb{R}^s$, where $s := d_\varphi + d_\eta + 1$.

Given a square symmetric real matrix X, we write $X \succeq 0$ to indicate that it is positive semidefinite and $X \succ 0$ when it is positive definite. The minimum condition (3.12) yields following set of properties.

Lemma 3.2.4. *For every Pontryagin multiplier $\lambda \in \Lambda_P$,*

(i)
$$H_{uu}[\lambda] = p \sum_{i=0}^{m} \hat{v}_i D^2_{uu} f_i(\hat{x}, \hat{u}) \succeq 0, \tag{3.15}$$

(ii)
$$\begin{gathered} H_{uv}[\lambda] \equiv 0, \\ pD_u f_j(\hat{x}, \hat{u}) \equiv 0, \quad \text{for } j = 0, \dots, m. \end{gathered} \tag{3.16}$$

Proof. Observe that (3.12) implies that the matrix

$$\begin{pmatrix} H_{uu}[\lambda] & H_{vu}[\lambda]^\top \\ H_{vu}[\lambda] & H_{vv}[\lambda] \end{pmatrix} \tag{3.17}$$

is positive semidefinite. Since $H_{vv}[\lambda] \equiv 0$, the positive semidefiniteness in (i) and the first identity in (ii) follow. The latter equation yields the second identity in (ii) for $j = 1, \dots, m$. Finally, use (3.11) to get the analogous condition for $j = 0$. $\square$

For $(\bar{x}_0, \bar{u}, \bar{v}) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$, consider the *linearized state equation*

$$\dot{\bar{x}}_t = A_t \bar{x}_t + D_t \bar{u}_t + B_t \bar{v}_t, \quad \text{a.e. on } [0, T], \tag{3.18}$$
$$\bar{x}(0) = \bar{x}_0, \tag{3.19}$$

where
$$A_t := \sum_{i=0}^{m} \hat{v}_i f_{i,x}(\hat{x}, \hat{u}), \quad D_t := \sum_{i=0}^{m} \hat{v}_i f_{i,u}(\hat{x}, \hat{u}), \tag{3.20}$$

and $B : [0, T] \to \mathcal{M}_{n\times m}(\mathbb{R})$ such that for every $v \in \mathbb{R}^m$,

$$B_t v := \sum_{i=0}^{m} v_i f_i(\hat{x}_t, \hat{u}_t), \tag{3.21}$$

i.e. the ith. column of B is $f_i(\hat{x}, \hat{u})$. The variable $\bar{x}$ in (3.18)-(3.19) is called *linearized state variable.*

3.2.3 Critical cones

Set $\mathcal{X}_2 := W^1_2(0, T; \mathbb{R}^n)$, $\mathcal{U}_2 := L_2(0, T; \mathbb{R}^l)$ and $\mathcal{V}_2 := L_2(0, T; \mathbb{R}^m)$. Put $\mathcal{W}_2 := \mathcal{X}_2 \times \mathcal{U}_2 \times \mathcal{V}_2$ for the corresponding product space. Define,

$$\mathcal{H}_2 := \{\bar{w} \in \mathcal{W}_2 : (3.18)\text{-}(3.19) \text{ hold}\}, \ \mathcal{H}_\infty := \{\bar{w} \in \mathcal{W} : (3.18)\text{-}(3.19) \text{ hold}\}. \tag{3.22}$$

$\mathcal{M}_{n\times m}(\mathbb{R})$: the space of $n \times m-$real matrices

Let $p \in \{2, \infty\}$. Given $\bar{w} \in \mathcal{H}_p$, consider the *linearization of the initial-final constraints and cost function:*

$$D\eta_j(\hat{x}_0, \hat{x}_T)(\bar{x}_0, \bar{x}_T) = 0, \text{ for } j = 1, \dots, d_\eta, \tag{3.23}$$

$$D\varphi_i(\hat{x}_0, \hat{x}_T)(\bar{x}_0, \bar{x}_T) \leq 0, \text{ for } i = 0, \dots, d_\varphi. \tag{3.24}$$

Define the L_p-*critical cone*

$$\mathcal{C}_p := \{\bar{w} \in \mathcal{H}_p : \ (3.23) - (3.24) \text{ hold}\}. \tag{3.25}$$

Lemma 3.2.5. *The critical cone $\mathcal{C}_\infty$ is a dense subset of $\mathcal{C}_2$.*

In order to prove previous lemma, recall the following technical result, (see e.g. Dmitruk [45] for a proof).

Lemma 3.2.6 (on density)**.** *Consider a locally convex topological space X, a finite-faced cone $C \subset X$, and a linear manifold L dense in X. Then the cone $C \cap L$ is dense in C.*

Proof. [of Lemma 3.2.5] Set $X := \mathcal{H}_2$, $L := \mathcal{H}_\infty$, $C := \mathcal{C}_2$ and apply Lemma 3.2.6. □

3.3 Second order analysis

We begin this section by presenting the second variation of the Lagrangian function the we denote by Ω. All the second order conditions in this paper are established in terms of either Ω or some transformed form of Ω. The main result of the current section is the second order necessary condition provided by Theorem 3.3.6.

3.3.1 Second variation

Let us consider the quadratic mapping:

$$\begin{aligned}\Omega[\lambda](\bar{x}, \bar{u}, \bar{v}) := {} & \tfrac{1}{2}\ell''[\lambda](\hat{x}_0, \hat{x}_T)(\bar{x}_0, \bar{x}_T)^2 \\ & + \int_0^T [\tfrac{1}{2}\bar{x}^\top Q[\lambda]\bar{x} + \bar{u}^\top E[\lambda]\bar{x} + \bar{v}^\top C[\lambda]\bar{x} + \tfrac{1}{2}\bar{u}^\top R_0[\lambda]\bar{u} + \bar{v}^\top K[\lambda]\bar{u}]\mathrm{d}t,\end{aligned} \tag{3.26}$$

where the involved matrices are, omitting arguments,

$$Q := H_{xx}, \quad E := H_{ux}, \quad C := H_{vx}, \quad R_0 := H_{uu}, \quad K := H_{vu}. \tag{3.27}$$

Lemma 3.3.1 (Lagrangian expansion)**.** *Let $w \in \mathcal{W}$ be a solution of* (3.2)*, and set $\delta w := w - \hat{w}$. Then for every multiplier $\lambda \in \Lambda_L$,*

$$\begin{aligned}\mathcal{L}[\lambda](w) = {} & \mathcal{L}[\lambda](\hat{w}) + \Omega[\lambda](\delta x, \delta u, \delta v) + \int_0^T [H_{vxx}[\lambda](\delta x, \delta x, \delta v) \\ & + 2H_{vux}[\lambda](\delta x, \delta u, \delta v) + H_{vuu}[\lambda](\delta u, \delta u, \delta v)]\,\mathrm{d}t \\ & + \mathcal{O}(|(\delta x_0, \delta x_T)|^3) + (1 + \|v\|_1)\|(\delta x, \delta u)\|_\infty \mathcal{O}(\|(\delta x, \delta u)\|_2^2),\end{aligned} \tag{3.28}$$

where the time variable was omitted for the sake of simplicity.

Proof. Omit the dependence on λ for the sake of simplicity. In order to achieve the expression (3.28) consider the second order Taylor representations below, written in a compact form,

$$\ell(x_0, x_T) = \ell + D\ell(\delta x_0, \delta x_T) + \tfrac{1}{2}D^2\ell(\delta x_0, \delta x_T)^2 + \mathcal{O}(|(\delta x_0, \delta x_T)|^3), \tag{3.29}$$

$$f_i(x_t, u_t) = f_{i,t} + Df_{i,t}(\delta x_t, \delta u_t) + \tfrac{1}{2}D^2 f_{i,t}(\delta x_t, \delta u_t)^2 + \mathcal{O}(\|(\delta x, \delta u)\|_3^3), \tag{3.30}$$

where, whenever the argument is missing the corresponding function is evaluated on the reference trajectory $\hat{w}$. Observe that the costate equation (3.9) and the transversality conditions (3.10) yield

$$D\ell\,(\delta x_0, \delta x_T) = -p_0\,\delta x_0 + p_T\,\delta x_T = \int_0^T p\left[-\sum_{i=0}^m \hat{v}_i D_x f_i\,\delta x + \dot{\delta x}\right]\mathrm{d}t, \tag{3.31}$$

Recall the expression of the Lagrangian given in (3.5). Replacing $\ell(x_0, x_T)$ and $f_i(x, u)$ in (3.5) by their Taylor expansions (3.29)-(3.30) and using the identity (3.31) we get

$$\begin{aligned}\mathcal{L}(w) = {} & \mathcal{L}(\hat{w}) + \int_0^T [H_u\delta u + H_v\delta v]\mathrm{d}t + \Omega(\delta x, \delta u, \delta v) \\ & + \int_0^T [H_{vxx}(\delta x, \delta x, \delta v) + 2H_{vux}(\delta x, \delta u, \delta v) + H_{vuu}(\delta u, \delta u, \delta v)]\,\mathrm{d}t \\ & + \mathcal{O}(|(\delta x_0, \delta x_T)|^3) + \|(\delta x, \delta u)\|_\infty \int_0^T p\sum_{i=0}^m v_i\,\mathcal{O}(\|(\delta x, \delta u)\|_2^2)\mathrm{d}t.\end{aligned}$$

Finally, to obtain (3.28) use stationarity condition (3.11). □

Remark 3.3.2. *The last lemma gives the identity:*

$$\Omega[\lambda](\bar{w}) = \tfrac{1}{2}D^2\mathcal{L}[\lambda](\hat{w})\,\bar{w}^2 \tag{3.32}$$

3.3.2 Second order necessary condition

Recall the classical second order condition below, a proof of which can be found in [86].

Theorem 3.3.3 (Classical second order necessary condition)**.** *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in \Lambda_L} \Omega[\lambda](\bar{x}, \bar{u}, \bar{v}) \geq 0, \quad \text{on } \mathcal{C}_\infty. \tag{3.33}$$

In this paragraph we aim to strengthen previous necessary condition by proving that the maximum in (3.33) can be taken in a smaller set of multipliers. We shall first give a description of the subset of Lagrange multipliers we work with.

Remark 3.3.4. *Condition* (3.33) *can be extended to the cone $\mathcal{C}_2$ by the continuity of* $\Omega[\lambda]$ *and the compactness of* Λ_L.

Recall the definition of the Hilbert space $\mathcal{H}_2$ introduced in (3.22), and consider the subset of Λ_L given by

$$\Lambda_L^\# := \{\lambda \in \Lambda_L : \Omega[\lambda] \text{ is weakly-l.s.c. on } \mathcal{H}_2\}. \tag{3.34}$$

The two results below are established in this section. Lemma 3.3.5 provides a characterization of $\Lambda_L^\#$ and Theorem 3.3.6 gives a strengthened second order necessary condition.

Lemma 3.3.5.

$$\Lambda_L^\# = \{\lambda \in \Lambda_L : R_0[\lambda] \succeq 0 \text{ and } K[\lambda] \equiv 0\}. \tag{3.35}$$

Theorem 3.3.6 (Second order necessary condition)**.** *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in \Lambda_L^\#} \Omega[\lambda](\bar{x}, \bar{u}, \bar{v}) \geq 0, \quad \text{on } \mathcal{C}_2. \tag{3.36}$$

In order to prove Lemma 3.3.5 consider the matrix

$$\tfrac{1}{2}\begin{pmatrix} R_0[\lambda] & K[\lambda]^\top \\ K[\lambda] & 0 \end{pmatrix}, \tag{3.37}$$

and note that it is the coefficient of the quadratic term on $\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$ in $\Omega[\lambda]$. Hence $\Omega[\lambda]$ can be written as the sum of a weakly-continuous mapping on the space $\mathcal{H}_2$ and the quadratic operator given by

$$\int_0^T (\tfrac{1}{2}\bar{u}^\top R_0[\lambda]\bar{u} + \bar{v}^\top K[\lambda]\bar{u})\mathrm{d}t. \tag{3.38}$$

Recall next a characterization of weakly-l.s.c. forms.

Lemma 3.3.7. *[72, Theorem 3.2] Consider a real interval I and a quadratic form $\mathcal{Q}$ over the Hilbert space $L_2(I)$, given by*

$$\mathcal{Q}(y) := \int_I y_t^\top R_t y_t \mathrm{d}t.$$

Then $\mathcal{Q}$ is weakly-l.s.c. over $L_2(I)$ iff

$$R_t \succeq 0, \quad \text{a.e. on } I. \tag{3.39}$$

Lemma 3.3.5 follows from last result and it yields, owing to Lemma 3.2.4:

$$\Lambda_P \subset \Lambda_L^\#. \tag{3.40}$$

Theorem 3.3.6 is a consequence of Remark 3.3.4, Lemma 3.3.5 and the following result on quadratic forms:

Lemma 3.3.8. *[42, Theorem 5] Given a Hilbert space H, and $a_1, a_2, \ldots, a_p$ in H, set*

$$K := \{x \in H : (a_i, x) \leq 0, \text{ for } i = 1, \ldots, p\}. \tag{3.41}$$

Let M be a convex and compact subset of $\mathbb{R}^s$, and let $\{Q^\psi : \psi \in M\}$ be a family of continuous quadratic forms over H, the mapping $\psi \to Q^\psi$ being affine. Set $M^\# := \{\psi \in M : \ Q^\psi$ is weakly-l.s.c. on $H\}$ and assume that

$$\max_{\psi \in M} Q^\psi(x) \geq 0, \text{ for all } x \in K. \tag{3.42}$$

Then

$$\max_{\psi \in M^\#} Q^\psi(x) \geq 0, \text{ for all } x \in K. \tag{3.43}$$

3.4 Goh Transformation

In this section we introduce a linear transformation of variables $\bar{x}$, $\bar{u}$ and $\bar{v}$. Afterwards we define the critical cone in the new space of variables denoted by $\mathcal{P}_2$, and we show that performing the mentioned transformation in Ω yields a new quadratic operator called Ω_{P_2} on the transformed space.

Consider hence the linear system in (3.18) and the change of variables

$$\begin{cases} \bar{y}_t := \int_0^t \bar{v}_s \mathrm{d}s, \\ \bar{\xi}_t := \bar{x}_t - B_t \bar{y}_t, \end{cases} \quad \text{for } t \in [0, T]. \tag{3.44}$$

This change of variables, first introduced by Goh [68], can be done in any linear system of differential equations, and it is often called *Goh's transformation.* Observe that $\bar{\xi}$ defined in that way satisfies the linear equation

$$\dot{\bar{\xi}} = A\bar{\xi} + D\bar{u} + B_1\bar{y}, \quad \bar{\xi}_0 = \bar{x}_0, \tag{3.45}$$

where A and D were given in (3.20), and

$$B_{1,t} := A_t B_t - \frac{\mathrm{d}}{\mathrm{d}t} B_t. \tag{3.46}$$

3.4.1 Tranformed critical cones

In this paragraph we present the critical cones obtained after Goh's transformation, that we use later on to formulate the optimality conditions. Recall the linearized constraints in (3.23)-(3.24) and the critical cone given by (3.25) in paragraph 3.2.3. Let $(\bar{x}, \bar{u}, \bar{v}) \in \mathcal{C}_\infty$ be a critical direction. Define $(\bar{\xi}, \bar{y})$ by transformation (3.44) and set $\bar{h} := \bar{y}_T$. Note that (3.23)-(3.24) yields

$$D\eta_j(\hat{x}_0, \hat{x}_T)(\bar{\xi}_0, \bar{\xi}_T + B_T\bar{h}) = 0, \quad \text{for } j = 1, \ldots, d_\eta, \tag{3.47}$$

$$D\varphi_i(\hat{x}_0, \hat{x}_T)(\bar{\xi}_0, \bar{\xi}_T + B_T\bar{h}) \leq 0, \quad \text{for } i = 0, \ldots, d_\varphi. \tag{3.48}$$

Recall the definition of $\mathcal{W}_2$ in paragraph 3.2.3. Denote by $\mathcal{Y}$ the space $W^1_\infty(0, T; \mathbb{R}^m)$, and consider the cones

$$\mathcal{P} := \{(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \in \mathcal{W} \times \mathbb{R}^m : \bar{y}_0 = 0,\, \bar{y}_T = \bar{h},\, (3.45),\, (3.47)\text{-}(3.48) \text{ hold}\}, \tag{3.49}$$

$$\mathcal{P}_2 := \{(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \in \mathcal{W}_2 \times \mathbb{R}^m : (3.45),\, (3.47)\text{-}(3.48) \text{ hold}\}. \tag{3.50}$$

Remark 3.4.1. *Notice that $\mathcal{P}$ consists of the directions obtained by transformating the elements of $\mathcal{C}_\infty$ via Goh's Transformation* (3.44).

Lemma 3.4.2. *$\mathcal{P}$ is a dense subspace of $\mathcal{P}_2$ in the $\mathcal{W}_2 \times \mathbb{R}^m-$topology.*

Proof. Notice that the inclusion is immediate. In order to prove the density, consider the linear spaces

$$X := \{(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \in \mathcal{W}_2 \times \mathbb{R}^m : (3.45) \text{ holds}\}, \tag{3.51}$$

$$L := \{(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \in \mathcal{W}_\infty \times \mathbb{R}^m : \bar{y}(0) = 0,\, \bar{y}(T) = \bar{h}, \text{ and } (3.45) \text{ holds}\}, \tag{3.52}$$

and the cone

$$C := \{(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \in X : (3.47) - (3.48) \text{ holds}\}. \tag{3.53}$$

Since L is a dense linear subspace of X (by Lemma 6 in [47] or Lemma 7.1 in [8]), and C is a finite-faced cone of X, we get the desired density by Lemma 3.2.6. $\square$

3.4.2 Transformed second variation

In Theorem 3.4.3 below we prove that performing the Goh transformation in Ω yields the new quadratic operator $\Omega_{\mathcal{P}}$ in variables $(\bar{\xi},\bar{u},\bar{y},\bar{v},\bar{h})$ defined below. Recall the definitions in (3.27) and set for $\lambda\in\Lambda_L^{\#}$,

$$\begin{aligned}\Omega_{\mathcal{P}}[\lambda](\bar{\xi},\bar{u},\bar{y},\bar{v},\bar{h}) := g[\lambda](\bar{\xi}_0,\bar{\xi}_T,\bar{h}) + \int_0^T \big(\tfrac{1}{2}\bar{\xi}^\top Q[\lambda]\bar{\xi} + \bar{u}^\top E[\lambda]\bar{\xi}\\ + \bar{y}^\top M[\lambda]\bar{\xi} + \tfrac{1}{2}\bar{u}^\top R_0[\lambda]\bar{u} + \bar{y}^\top J[\lambda]\bar{u} + \tfrac{1}{2}\bar{y}^\top R_1[\lambda]\bar{y} + \bar{v}^\top V[\lambda]\bar{y}\big)\,\mathrm{d}t,\end{aligned} \tag{3.54}$$

where

$$M := B^\top Q - \dot{C} - CA,\quad J := B^\top E^\top - CD, \tag{3.55}$$

$$S := \tfrac{1}{2}(CB + (CB)^\top),\quad V := \tfrac{1}{2}(CB - (CB)^\top), \tag{3.56}$$

$$R_1 := B^\top QB - \tfrac{1}{2}(CB_1 + (CB_1)^\top) - \dot{S}, \tag{3.57}$$

$$g[\lambda](\zeta_0,\zeta_T,h) := \tfrac{1}{2}\ell''(\zeta_0,\zeta_T + B_T h)^2 + h^\top(C_T\zeta_T + \tfrac{1}{2}S_T h). \tag{3.58}$$

Theorem 3.4.3. *Let $(\bar{x},\bar{u},\bar{v})\in\mathcal{H}_2$ (given in (3.22)) and $(\bar{\xi},\bar{y})$ defined by the transformation (3.44). Then*

$$\Omega[\lambda](\bar{x},\bar{u},\bar{v}) = \Omega_{\mathcal{P}}[\lambda](\bar{\xi},\bar{u},\bar{y},\bar{v},\bar{y}_T). \tag{3.59}$$

Proof. We omit the dependence on λ for the sake of simplicity. Replacing $\bar{x}$ in the definition (3.26) of Ω by its expression in (3.44) yields

$$\begin{aligned}\Omega(\bar{x},\bar{u},\bar{v}) = \;& \tfrac{1}{2}\ell''(\hat{x}_0,\hat{x}_T)(\bar{\xi}_0,\bar{\xi}_T + B_T\bar{y}_T)^2 + \int_0^T \big[\tfrac{1}{2}(\bar{\xi}+B\bar{y})^\top Q(\bar{\xi}+B\bar{y})\\ &+ \bar{u}^\top E(\bar{\xi}+B\bar{y}) + \bar{v}^\top C(\bar{\xi}+B\bar{y}) + \tfrac{1}{2}\bar{u}^\top R_0\bar{u}\big]\,\mathrm{d}t.\end{aligned} \tag{3.60}$$

Integrating by parts the first term containing $\bar{v}$ yields, owing to (3.45),

$$\int_0^T \bar{v}^\top C\bar{\xi}\mathrm{d}t = [\bar{y}^\top C\bar{\xi}]_0^T - \int_0^T \bar{y}^\top\{\dot{C}\bar{\xi} + C(A\bar{\xi} + D\bar{u} + B_1\bar{y})\}\mathrm{d}t. \tag{3.61}$$

Using the descomposition of CB introduced in (3.56) we get

$$\begin{aligned}\int_0^T \bar{v}^\top CB\bar{y}\mathrm{d}t &= \int_0^T \bar{v}^\top(S+V)\bar{y}\mathrm{d}t\\ &= \tfrac{1}{2}[\bar{y}^\top S\bar{y}]_0^T + \int_0^T(-\tfrac{1}{2}\bar{y}^\top\dot{S}\bar{y} + \bar{v}^\top V\bar{y})\mathrm{d}t.\end{aligned} \tag{3.62}$$

Combining (3.60), (3.61) and (3.62), the identity (3.59) follows. □

Recall Theorem 3.3.6. Observe that by performing Goh's transformation (3.44) in (3.36) and in view of Remark 3.4.1, we obtain the following form for the second order necessary condition.

Corollary 3.4.4. *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in \Lambda_L^\#} \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \dot{\bar{y}}, \bar{h}) \geq 0, \quad \text{on } \mathcal{P}. \tag{3.63}$$

3.5 New second order necessary condition

We aim to remove the dependence of $\bar{v}$ from the formulation of the second order necessary condition. Note that in (3.63), given that the considered multipliers are in $\Lambda_L^\#$, the matrix $K[\lambda]$ vanishes and that is why we do not include it in $\Omega_{\mathcal{P}}$. However, there still remains the term $\bar{v}^\top V[\lambda]\bar{y}$. Next we prove that we can restrict the maximum in (3.63) to the subset of $\Lambda_L^\#$ consisting of the multipliers for which $V[\lambda]$ vanishes. We use, in an essential way, some techniques introduced by Dmitruk [40, 43] for the proof of similar results.

Definition 3.5.1. *Given $M \subset \mathbb{R}^s$, define*

$$G(M) := \{\lambda \in M : V_{ij}[\lambda] \equiv 0 \text{ on } [0,T]\}.$$

Theorem 3.5.2. *Let $M \subset \mathbb{R}^s$ be convex and compact, and assume that*

$$\max_{\lambda \in M} \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \dot{\bar{y}}, \bar{y}_T) \geq 0, \quad \text{on } \mathcal{P}. \tag{3.64}$$

Then

$$\max_{\lambda \in G(M)} \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \dot{\bar{y}}, \bar{y}_T) \geq 0, \quad \text{on } \mathcal{P}. \tag{3.65}$$

The proof of Theorem 3.5.2 is based on some techniques introduced in Dmitruk [40, 43], and was given in detail in Aronna et al. [8] (or Theorem 1.4.10 of Chapter 1) for a system that is affine in all the control variables. For the case treated here, the same proof holds with minor modifications and hence there is no point in writing it again.

When $\hat{w}$ has a unique associated Lagrange multiplier, as a consequence of Theorem 3.5.2 we get the corollary below. This corollary is one of the necessary conditions stated by Goh in [67].

Corollary 3.5.3. *When $\hat{w}$ is a weak minimum having a unique associated multiplier, $V \equiv 0$ or, equivalently, CB is symmetric.*

From theorems 3.3.6 and 3.5.2 we get

Theorem 3.5.4 (New necessary condition)**.** *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in G(\operatorname{co}\Lambda_L^{\#})} \Omega_{\mathcal{P}}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \dot{\bar{y}}, \bar{y}_T) \geq 0, \quad \text{on } \mathcal{P}. \tag{3.66}$$

Observe that for $\lambda \in G(\operatorname{co}\Lambda_L^{\#})$, the quadratic form $\Omega[\lambda]$ does not depend on $\bar{v}$ since its coefficients vanish. We can then consider its continuous extension to $\mathcal{P}_2$, given by

$$\begin{aligned} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) := g[\lambda](\bar{\xi}_0, \bar{\xi}_T, \bar{h}) + \int_0^T \big(\tfrac{1}{2}\bar{\xi}^\top Q[\lambda]\bar{\xi} + \bar{u}^\top E[\lambda]\bar{\xi} \\ + \bar{y}^\top M[\lambda]\bar{\xi} + \tfrac{1}{2}\bar{u}^\top R_0[\lambda]\bar{u} + \bar{y}^\top J[\lambda]\bar{u} + \tfrac{1}{2}\bar{y}^\top R_1[\lambda]\bar{y} \big)\, \mathrm{d}t. \end{aligned} \tag{3.67}$$

Applying Theorem 3.5.4 we obtain

Theorem 3.5.5. *If $\hat{w}$ is a weak minimum of problem (P), then*

$$\max_{\lambda \in G(\operatorname{co}\Lambda_L^{\#})} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \geq 0, \quad \text{on } \mathcal{P}_2. \tag{3.68}$$

Remark 3.5.6. *The latter optimality condition does not involve variable $\bar{v}$. It is stated in the space of variables $(\bar{\xi}, \bar{u}, \hat{y}, \bar{h})$.*

3.6 Second order sufficient condition for weak minimum

This section provides a second order sufficient condition for strict weak optimality with quadratic growth. The proof is an adaptation of the proof of Theorem 5.5 in [8] (or Theorem 1.5.5 in Chapter 1), with important simplifications due to the absence of control constraints.

The quadratic growth above mentioned will be established with respect to the *order*

$$\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{h}) := |\bar{x}_0|^2 + |\bar{h}|^2 + \int_0^T (|\bar{u}_t| + |\bar{y}_t|)^2 \mathrm{d}t. \tag{3.69}$$

for $(\bar{x}_0, \bar{u}, \bar{y}, \bar{h}) \in \mathcal{U}_2 \times \mathcal{V}_2 \times \mathbb{R}^{n+m}$. It can also be considered as a function of $(\bar{x}_0, \bar{u}, \bar{v}) \in \mathbb{R}^n \times \mathcal{U}_2 \times \mathcal{V}_2$ by means of the identity

$$\tilde{\gamma}(\bar{x}_0, \bar{u}, \bar{v}) := \gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T), \tag{3.70}$$

with $\bar{y}$ being the primitive of $\bar{v}$ defined in (3.44).
Notation: We write γ to refer to either γ or $\tilde{\gamma}$.

Definition 3.6.1. *[Quadratic Growth] We say that $\hat{w}$ satisfies $\gamma-$quadratic growth condition in the weak sense if there exists $\rho > 0$ such that*

$$J(w) \geq J(\hat{w}) + \rho\gamma(x_0 - \hat{x}_0, u - \hat{u}, v - \hat{v}), \tag{3.71}$$

for every feasible trajectory w satisfying $\|w - \hat{w}\|_\infty < \varepsilon$.

Theorem 3.6.2 (Sufficient condition)**.** *Assume that there exists $\rho > 0$ such that*

$$\max_{\lambda \in G(\operatorname{co}\Lambda_L^\#)} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi}, \bar{u}, \bar{y}, \bar{h}) \geq \rho\gamma(\bar{\xi}_0, \bar{u}, \bar{y}, \bar{h}), \quad \text{on } \mathcal{P}_2. \tag{3.72}$$

Then $\hat{w}$ is a weak minimum satisfying $\gamma-$quadratic growth in the weak sense.

The remainder of this section is devoted to the proof of Theorem 3.6.2. We shall start by establishing some technical results that will be needed for the main result.

For the lemma below recall the definition of the space $\mathcal{H}_2$ in (3.22).

Lemma 3.6.3. *There exists $\rho > 0$ such that*

$$|\bar{x}_0|^2 + \|\bar{x}\|_2^2 + |\bar{x}_T|^2 \leq \rho\gamma(\bar{x}_0, \bar{u}, \bar{v}), \tag{3.73}$$

for every linearized trajectory $(\bar{x}, \bar{u}, \bar{v}) \in \mathcal{H}_2$. The constant ρ depends on $\|A\|_\infty$, $\|B\|_\infty$, $\|D\|_\infty$ and $\|B_2\|_\infty$.

Proof. Every time we put ρ_i we refer to a positive constant depending on $\|A\|_\infty$, $\|B\|_\infty$, $\|D\|_\infty$, and/or $\|B_2\|_\infty$. Let $(\bar{x}, \bar{u}, \bar{v}) \in \mathcal{H}_2$ and $(\bar{\xi}, \bar{y})$ be defined by Goh's Transformation (3.44). Thus $(\bar{\xi}, \bar{u}, \bar{y})$ is solution of (3.45) having $\bar{\xi}_0 = \bar{x}_0$. Gronwall's Lemma and Cauchy-Schwartz inequality yield

$$\|\bar{\xi}\|_\infty \leq \rho_1(|\bar{\xi}_0|^2 + \|\bar{u}\|_2^2 + \|\bar{y}\|_2^2)^{1/2} \leq \rho_1\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T)^{1/2}, \tag{3.74}$$

with $\rho_1 = \rho_1(\|A\|_1, \|D\|_\infty, \|B_1\|_\infty)$. This last inequality together with the relation between $\bar{\xi}$ and xb provided by (3.44) imply

$$\|\bar{x}\|_2 \leq \|\bar{\xi}\|_2 + \|B\|_\infty\|\bar{y}\|_2 \leq \rho_2\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T)^{1/2}, \tag{3.75}$$

for $\rho_2 = \rho_2(\rho_1, \|B\|_\infty)$. On the other hand, (3.44) and estimation (3.74) lead to

$$|\bar{x}_T| \leq |\bar{\xi}_T| + \|B\|_\infty|\bar{y}_T| \leq \rho_1\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T)^{1/2} + \|B\|_\infty|\bar{y}_T|. \tag{3.76}$$

Then, in view of inequality '$ab \leq \frac{a^2+b^2}{2}$,'

$$|\bar{x}_T|^2 \leq \rho_3\gamma(\bar{x}_0, \bar{u}, \bar{y}, \bar{y}_T), \tag{3.77}$$

for some $\rho_3 = \rho_3(\rho_1, \|B\|_\infty)$. The desired estimation follows from (3.75) and (3.77). $\square$

Notice that Lemma 3.6.3 above gives an estimation of the linearized state in terms of γ. The following result shows that the analogous property holds for the variation of the state variable as well. Recall the state dynamics (3.2).

Lemma 3.6.4. *For every $C > 0$ there exists $\rho > 0$ such that*

$$|\delta x_0|^2 + \|\delta x\|_2^2 + |\delta x_T|^2 \leq \rho\gamma(\delta x_0, \delta u, \delta v). \tag{3.78}$$

for every (x, u, v) solution of (3.2) having $\|v\|_2 \leq C$, and where we set $\delta w := w - \hat{w}$. The constant ρ depends on C, $\|B\|_\infty$, $\|\dot{B}\|_\infty$ and the Lipschitz constants of f_i.

Proof. Consider (x, u, v) solution of (3.2) with $\|v\|_2 \leq C$. Let $\delta w := w - \hat{w}$, and $\xi := \delta x - B\delta y$, with B given in (3.21) and $y_t := \int_0^t v_s \mathrm{d}s$. Note that

$$\begin{aligned} \dot{\xi} &= \sum_{i=0}^m [v_i f_i(x, u) - \hat{v}_i f_i(\hat{x}, \hat{u})] - \dot{B}\delta y - \sum_{i=1}^m \delta v_i f_i(\hat{x}, \hat{u}) \\ &= \sum_{i=0}^m v_i [f_i(\hat{x} + \xi + B\delta y, \hat{u} + \delta u) - f_i(\hat{x}, \hat{u})] - \dot{B}\delta y, \end{aligned} \tag{3.79}$$

where $v_0 \equiv 1$. In view of the Lipschitz-continuity of f_i,

$$|f_i(\hat{x} + \xi + B\delta y, \hat{u} + \delta u) - f_i(\hat{x}, \hat{u})| \leq L(|\xi| + \|B\|_\infty |\delta y| + |\delta u|), \tag{3.80}$$

for some $L > 0$. Thus, from (3.79) it follows

$$\begin{aligned} |\dot{\xi}| &\leq L(|\xi| + \|B\|_\infty|\delta y| + |\delta u|)(1 + |v|) + \|\dot{B}\|_\infty|\delta y| \\ &= L(|\xi|(1 + |v|) + \|B\|_\infty|\delta y| + |\delta u| + \|B\|_\infty|\delta y||v| + |\delta u||v|) + \|\dot{B}\|_\infty|\delta y|. \end{aligned}$$

Applying Gronwall's Lemma and Cauchy-Schwartz inequality to previous estimation yields

$$\|\xi\|_\infty \leq \rho_1(|\xi_0| + \|\delta u\|_1 + \|\delta y\|_1 + \|\delta y\|_2\|v\|_2 + \|\delta u\|_2\|v\|_2), \tag{3.81}$$

for $\rho_1 = \rho_1(L, C, \|B\|_\infty, \|\dot{B}\|_\infty)$. Hence, since $\|\delta x\|_2 \leq \|\xi\|_2 + \|B\|_\infty\|\delta y\|_2$, by Cauchy-Schwartz inequality and previous estimation, the desired result follows. $\square$

Finally, the following lemma gives an estimation for the difference between the variation of the state variable and the linearized state.

Lemma 3.6.5. *Consider $C > 0$ and $w = (x, u, v) \in \mathcal{W}$ a feasible trajectory having $\|w - \hat{w}\|_\infty < C$. Set $(\delta x, \bar{u}, \bar{v}) := w - \hat{w}$ and $\bar{x}$ its corresponding linearized state. Consider*

$$\eta := \delta x - \bar{x}. \tag{3.82}$$

Then,

$$\dot{\eta} = \sum_{i=0}^{m} \hat{v}_i D_x f_i(\hat{x},\hat{u})\eta + \sum_{i=1}^{m} \bar{v}_i Df_i(\hat{x},\hat{u})(\delta x,\bar{u}) + \zeta, \tag{3.83}$$

with

$$\|\zeta\|_\infty < \mathcal{O}(C), \quad \|\zeta\|_2 < \mathcal{O}(\gamma). \tag{3.84}$$

If in addition, $\|\bar{u}\|_2 + \|\bar{v}\|_2 \to 0$, *the following estimations for hold:*

$$\|\eta\|_\infty < o(\sqrt{\gamma}), \quad \|\dot{\eta}\|_2 < o(\sqrt{\gamma}). \tag{3.85}$$

Proof. Let us begin by observing that the variation of the state variable satisfies the differential equation:

$$\dot{\delta x} = \sum_{i=1}^{m} \bar{v}_i f_i(\hat{x},\hat{u}) + \sum_{i=0}^{m} v_i[f_i(x,u) - f_i(\hat{x},\hat{u})]. \tag{3.86}$$

Consider the following Taylor expansions for f_i :

$$f_i(x,u) = f_i(\hat{x},\hat{u}) + Df_i(\hat{x},\hat{u})(\delta x,\bar{u}) + \tfrac{1}{2}D^2 f_i(\hat{x},\hat{u})(\delta x,\bar{u})^2 + o(|(\delta x,\bar{u})|^2). \tag{3.87}$$

Combining (3.86) and (3.87) yields

$$\dot{\delta x} = \sum_{i=1}^{m} \bar{v}_i f_i(\hat{x},\hat{u}) + \sum_{i=0}^{m} v_i Df_i(\hat{x},\hat{u})(\delta x,\bar{u}) + \zeta, \tag{3.88}$$

with the remainder being

$$\zeta := \tfrac{1}{2}\sum_{i=0}^{m} v_i[D^2 f_i(\hat{x},\hat{u})(\delta x,\bar{u})^2 + o(|(\delta x,\bar{u})|^2)]. \tag{3.89}$$

The linearized equation together with (3.88) lead to (3.83), and, in view of (3.89), it can be seen that the estimations in (3.84) hold. Applying Gronwall's Lemma in (3.83), and using Cauchy-Schwartz inequality afterwards lead to

$$\|\eta\|_\infty \leq \rho_1 \left\| \sum_{i=1}^{m} \bar{v}_i Df_i(\hat{x},\hat{u})(\delta x,\bar{u}) + \zeta \right\|_1 \leq \rho_2 \left\{ \|\bar{v}\|_2(\|\delta x\|_2 + \|\bar{u}\|_2) + \|\zeta\|_2 \right\}, \tag{3.90}$$

for some positive ρ_1, ρ_2. Finally, using estimation of Lemma 3.6.4 and (3.84) just obtained, the inequalities in (3.85) follow. □

In view of Lemmas 3.3.1, 3.6.3, 3.6.4 and 3.6.5 we can justify the following technical result that is an essential point in the proof of Theorem 3.6.2.

Lemma 3.6.6. *Let $w \in \mathcal{W}$ be a feasible variation. Set $(\delta x, \bar{u}, \bar{v}) := w - \hat{w}$, and $\bar{x}$ its corresponding linearized state, i.e., the solution of* (3.18)-(3.19) *associated to $\bar{u}$, $\bar{v}$ and δx_0. Assume that $\|w - \hat{w}\|_\infty \to 0$. Then*

$$\mathcal{L}[\lambda](w) = \mathcal{L}[\lambda](\hat{w}) + \Omega[\lambda](\bar{x}, \bar{u}, \bar{v}) + o(\gamma). \tag{3.91}$$

Proof. Omit the dependence on λ for the sake of simplicity. Recall the expansion of the Lagrangian function given in Lemma 3.3.1. Notice that by Lemma 3.6.4, $\mathcal{L}(w) = \mathcal{L}(\hat{w}) + \Omega(\delta x, \bar{u}, \bar{v}) + o(\gamma)$. Hence,

$$\mathcal{L}(w) = \mathcal{L}(\hat{w}) + \Omega(\bar{x}, \bar{u}, \bar{v}) + \Delta\Omega + o(\gamma), \tag{3.92}$$

with $\Delta\Omega := \Omega(\delta x, \bar{u}, \bar{v}) - \Omega(\bar{x}, \bar{u}, \bar{v})$. The next step is using Lemmas 3.6.3, 3.6.4 and 3.6.5 to prove that

$$\Delta\Omega = o(\gamma). \tag{3.93}$$

Note that $\mathcal{Q}(a,a) - \mathcal{Q}(b,b) = \mathcal{Q}(a+b, a-b)$, for any bilinear mapping $\mathcal{Q}$, and any pair a, b. Put $\eta := \delta x - \bar{x}$ as it is done in Lemma 3.6.5. Hence,

$$\Delta\Omega = \tfrac{1}{2}\ell''((\delta x_0 + \bar{x}_0, \delta x_T + \bar{x}_T), (0, \eta_T)) + \int_0^T [\tfrac{1}{2}(\delta x + \bar{x})^\top Q\eta + \bar{u}^\top E\eta + \bar{v}^\top C\eta]\mathrm{d}t. \tag{3.94}$$

The estimations in Lemmas 3.6.3 and 3.6.4 yield $\Delta\Omega = \int_0^T \bar{v}^\top C\eta \mathrm{d}t + o(\gamma)$. Integrating by parts in the latter expression and using (3.85) lead to

$$\int_0^T \bar{v}^\top C\eta \mathrm{d}t = [\bar{y}^\top C\eta]_0^T - \int_0^T \bar{y}^\top (\dot{C}\eta + C\dot{\eta})\mathrm{d}t = o(\gamma), \tag{3.95}$$

and hence the desired result follows. □

Proof. [of Theorem 3.6.2]

We shall prove that if (3.72) holds for some $\rho > 0$, then $\hat{w}$ satisfies $\gamma-$quadratic growth in the weak sense. By the contrary assume that the quadratic growth condition (3.71) is not satisfied. Consequently, there exists a sequence of feasible trajectories $\{w_k\}$ converging to $\hat{w}$ in the weak sense, such that

$$J(w_k) \leq J(\hat{w}) + o(\gamma_k), \tag{3.96}$$

with $\delta w_k := w_k - \hat{w}$ and $\gamma_k := \gamma(\delta x_{k,0}, \bar{u}_k, \bar{v}_k)$. Let $(\bar{\xi}_k, \bar{u}_k, \bar{y}_k)$ be the transformed directions defined by (3.44). We divide the remainder of the proof in two steps.

(I) First we prove that the sequence given by

$$(\tilde{\xi}_k, \tilde{u}_k, \tilde{y}_k, \tilde{h}_k) := (\bar{\xi}_k, \bar{u}_k, \bar{y}_k, \bar{h}_k)/\sqrt{\gamma_k} \tag{3.97}$$

contains a subsequence converging to an element $(\tilde{\xi}, \tilde{u}, \tilde{y}, \tilde{h})$ of $\mathcal{P}_2$ in the weak topology, i.e. $(\tilde{u}_k, \tilde{y}_k) \rightharpoonup (\tilde{u}, \tilde{y})$ in the weak topology of $\mathcal{U}_2 \times \mathcal{V}_2$ and $(\tilde{\xi}_k, \tilde{h}_k) \to (\tilde{\xi}, \tilde{h})$ in the strong sense in $\mathcal{X}_2 \times \mathbb{R}^m$.

(II) Afterwards, employing the latter sequence and its weak limit, we show that (3.72) together with (3.96) lead to a contradiction.

We shall begin by **Part (I).** Take any Lagrange multiplier λ in $\Lambda_L^{\#}$. Multiply inequality (3.96) by α_0, then add the nonpositive term

$$\sum_{i=0}^{d_\varphi} \alpha_i \varphi_i(x_{k,0}, x_{k,T}) + \sum_{j=1}^{d_\eta} \beta_j \eta_j(x_{k,0}, x_{k,T}), \tag{3.98}$$

to its left-hand side and obtain the inequality

$$\mathcal{L}[\lambda](w_k) \leq \mathcal{L}[\lambda](\hat{w}) + o(\gamma_k). \tag{3.99}$$

Recall now expansion (3.91) given in Lemma 3.6.6. Note that the elements of the sequence $(\tilde{\xi}_{k,0}, \tilde{u}_k, \tilde{y}_k, \tilde{h}_k)$ have unit $\mathbb{R}^n \times \mathcal{U}_2 \times \mathcal{V}_2 \times \mathbb{R}^m-$norm. The Banach-Alaoglu Theorem (see e.g. [29, Theorem III.15]), implies that, extracting if necessary a subsequence, there exists $(\tilde{\xi}_0, \tilde{u}, \tilde{y}, \tilde{h}) \in \mathbb{R}^n \times \mathcal{U}_2 \times \mathcal{V}_2 \times \mathbb{R}^m$ such that

$$\tilde{\xi}_{k,0} \to \tilde{\xi}_0, \quad \tilde{u}_k \rightharpoonup \tilde{u}, \quad \tilde{y}_k \rightharpoonup \tilde{y}, \quad \tilde{h}_k \to \tilde{h}, \tag{3.100}$$

where the two limits indicated with $\rightharpoonup$ are taken in the weak topology of $\mathcal{U}_2$ and $\mathcal{V}_2$, respectively. Let $\tilde{\xi}$ be the solution of equation (3.45) associated with $(\tilde{\xi}_0, \tilde{u}, \tilde{y})$. Note that $\tilde{\xi}$ is the limit of $\tilde{\xi}_k$ in $\mathcal{X}_2$. For the aim of proving that $(\tilde{\xi}, \tilde{u}, \tilde{v}, \tilde{h})$ belongs to $\mathcal{P}_2$, we shall check that the initial-final conditions (3.47)-(3.48) are verified. For each index $0 \leq i \leq d_\varphi$,

$$D\varphi_i(\hat{x}_0, \hat{x}_T)(\tilde{\xi}_0, \tilde{\xi}_T + B_T \tilde{h}) = \lim_{k\to\infty} D\varphi_i(\hat{x}_0, \hat{x}_T) \left(\frac{\bar{x}_{k,0}, \bar{x}_{k,T}}{\sqrt{\gamma_k}} \right). \tag{3.101}$$

In order to prove that the right hand-side of (3.101) is non-positive, consider the following first order Taylor expansion of function φ_i around $(\hat{x}_0, \hat{x}_T)$:

$$\varphi_i(x_{k,0}, x_{k,T}) = \varphi_i(\hat{x}_0, \hat{x}_T) + D\varphi_i(\hat{x}_0, \hat{x}_T)(\delta x_{k,0}, \delta x_{k,T}) + o(|(\delta x_{k,0}, \delta x_{k,T})|).$$

Previous equation and Lemmas 3.6.3 and 3.6.5 imply

$$\varphi_i(x_{k,0},x_{k,T}) = \varphi_i(\hat{x}_0,\hat{x}_T) + D\varphi_i(\hat{x}_0,\hat{x}_T)(\bar{x}_{k,0},\bar{x}_{k,T}) + o(\sqrt{\gamma_k}).$$

Thus, the following approximation for the right hand-side in (3.101) holds:

$$D\varphi_i(\hat{x}_0,\hat{x}_T)\left(\frac{\bar{x}_{k,0},\bar{x}_{k,T}}{\sqrt{\gamma_k}}\right) = \frac{\varphi_i(x_{k,0},x_{k,T}) - \varphi_i(\hat{x}_0,\hat{x}_T)}{\sqrt{\gamma_k}} + o(1). \tag{3.102}$$

Since w_k is a feasible trajectory, it satisfies (3.4), and then equations (3.101) and (3.102) yield, for $1 \leq i \leq d_\varphi$, $D\varphi_i(\hat{x}_0,\hat{x}_T)(\tilde{\xi}_0,\tilde{\xi}_T + B_T\tilde{h}) \leq 0$. For $i=0$ use inequality (3.96) to get the corresponding inequality. Analogously,

$$D\eta_j(\hat{x}_0,\hat{x}_T)(\tilde{\xi}_0,\tilde{\xi}_T + B_T\tilde{h}) = 0, \quad \text{for } j = 1,\dots,d_\eta. \tag{3.103}$$

Thus $(\tilde{\xi},\tilde{u},\tilde{y},\tilde{h})$ satisfies (3.47)-(3.48), and hence it belongs to $\mathcal{P}_2$.

Let us deal with **Part (II).** Notice that from (3.91) and (3.99) we get

$$\Omega_{\mathcal{P}_2}[\lambda](\tilde{\xi}_k,\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) \leq o(1), \tag{3.104}$$

and thus

$$\liminf_{k\to\infty} \Omega_{\mathcal{P}_2}[\lambda](\tilde{\xi}_k,\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) \leq 0. \tag{3.105}$$

Consider the subset of $G(\operatorname{co}\Lambda_L^{\#})$ given by

$$\Lambda_L^{\#,\rho} := \{\lambda \in G(\operatorname{co}\Lambda_L^{\#}) : \Omega_{\mathcal{P}_2}[\lambda] - \rho\gamma \text{ is weakly l.s.c. on } \mathcal{H}_2 \times \mathbb{R}^m\}. \tag{3.106}$$

Applying Lemma 3.3.8 to the inequality (3.72) yields

$$\max_{\lambda\in\Lambda_L^{\#,\rho}} \Omega_{\mathcal{P}_2}[\lambda](\bar{\xi},\bar{u},\bar{y},\bar{h}) \geq \rho\gamma(\bar{\xi}_0,\bar{u},\bar{y},\bar{h}), \quad \text{on } \mathcal{P}_2. \tag{3.107}$$

Take $\tilde{\lambda} \in \Lambda_L^{\#,\rho}$ that attains the maximum in (3.107) for the direction $(\tilde{\xi},\tilde{u},\tilde{y},\tilde{h})$. Hence

$$\begin{aligned} 0 \;&\leq \Omega_{\mathcal{P}_2}[\tilde{\lambda}](\tilde{\xi},\tilde{u},\tilde{y},\tilde{h}) - \rho\gamma(\tilde{\xi}_0,\tilde{u},\tilde{y},\tilde{h}) \\ &\leq \liminf_{k\to\infty} \Omega_{\mathcal{P}_2}[\tilde{\lambda}](\tilde{\xi}_k,\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) - \rho\gamma(\tilde{\xi}_{k,0},\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) \leq -\rho, \end{aligned} \tag{3.108}$$

since $\Omega_{\mathcal{P}_2}[\tilde{\lambda}] - \rho\gamma$ is weakly-l.s.c., $\gamma(\tilde{\xi}_{k,0},\tilde{u}_k,\tilde{y}_k,\tilde{h}_k) = 1$ for every k and inequality (3.105) holds. This leads us to a contradiction, since $\rho > 0$, and so the desired result follows.

□

3.7 Shooting algorithm

The purpose of this section is to present an appropriated numerical scheme to solve the problem given by equations (3.1)-(3.3), that we denote with (SP). Notice that **no inequality constraints are considered.** Next we present a *qualification hypothesis* that is assumed throughout the remainder of this article. Consider the mapping

$$\begin{array}{rcl} G\colon \mathbb{R}^n \times \mathcal{U} \times \mathcal{V} & \to & \mathbb{R}^{d_\eta} \\ (x_0, u, v) & \mapsto & \eta(x_0, x_T), \end{array} \tag{3.109}$$

where x_T is the solution of (3.2) associated to (x_0, u, v).

Assumption 3.7.1. *The derivative of G at $(\hat{x}_0, \hat{u}, \hat{v})$ is onto.*

The Assumption 2.2.1 is usually known as *qualification of equality constraints.* It is a known fact that the Assumption 2.2.1 implies the uniqueness of multiplier and the normality of the extremal. Hence we can consider $\alpha_0 = 1$. We denote this unique multiplier by $\hat{\lambda} = (\hat{\beta}, \hat{p})$.

3.7.1 Optimality system

We aim to provide an optimality system for (SP) in the form of a boundary value problem. First, call back condition (3.11) given by the Pontryagin maximum principle (PMP) in Section 3.2.

We shall recall that for the case where all the control variables appear nonlinearly ($m = 0$), the classical technique is using the stationarity equation

$$H_u[\hat{\lambda}](\hat{w}) = 0, \tag{3.110}$$

to solve $\hat{u}$ as a function of $(\hat{x}, \hat{\lambda})$. One is able to do this by assuming, for instance, the *strengthened Legendre-Clebsch condition*

$$H_{uu}[\hat{\lambda}](\hat{w}) \succ 0. \tag{3.111}$$

In this case, in view of the Implicit Function Theorem, we can write $\hat{u} = U[\hat{\lambda}](\hat{x})$ with U being differentiable. Hence, replacing the occurrences of $\hat{u}$ by $U[\hat{\lambda}](\hat{x})$ in the conditions provided by the PMP yields a two point boundary value problem.

When the system is affine in all the control variables ($l = 0$), we cannot eliminate the control from the equation $H_v = 0$ and then a different technique is implemented (see [10] or Section 2.3 in Chapter 2). Let then $1 \leq i \leq m$, and $d^{M_i} H_v / dt^{M_i}$ be the lowest order derivative of H_v in which $\hat{v}_i$ appears with a coefficient that is not identically zero. Kelley et al. in [81] proved that M_i is even when the investigated extremal is normal. This implies that $\dot{H}_v$ depends only on $\hat{x}$ and $\hat{\lambda}$. Hence, it is differentiable and the expression

$$\ddot{H}_v[\hat{\lambda}](\hat{w}) = 0 \tag{3.112}$$

is well-defined. The control $\hat{v}$ can be retrieved from (3.112) provided that, for instance, the *strengthened generalized Legendre-Clebsch condition* (see Goh [67])

$$-\frac{\partial \ddot{H}_v}{\partial v}[\hat{\lambda}](\hat{w}) \succ 0 \tag{3.113}$$

holds. In this case, we can write $\hat{v} = V[\hat{\lambda}](\hat{x})$ with V being differentiable. An optimality system in the form of a boundary value problem can be then obtained by replacing $\hat{v}$ by $V[\hat{\lambda}](\hat{x})$ in the PMP.

In the problem studied here, where $l > 0$ and $m > 0$, we aim to use both equations (3.110) and (3.112) to retrieve the control $(\hat{u}, \hat{v})$ as a function of the state and costate variables. We next describe a procedure to achieve this elimination that was given in Goh [69]. Let us show that H_v can be differentiated two times in the time variable as it was done in the totally affine case. We shall start by proving that we can use (3.110) to write $\dot{\hat{u}}$ as a function of $(\hat{\lambda}, \hat{w})$. In fact, since $H_{uv} = 0$ (see Lemma 3.2.4), the coefficient of $\dot{\hat{v}}$ in $\dot{H}_u$ is zero and hence,

$$\dot{H}_u = \dot{H}_u[\hat{\lambda}](\hat{x}, \hat{u}, \hat{v}, \dot{\hat{u}}) = 0. \tag{3.114}$$

Thus, if the strong Legendre-Clebsch condition (3.111) holds, $\dot{\hat{u}}$ can be eliminated from (3.114) yielding

$$\dot{\hat{u}} = \Gamma[\hat{\lambda}](\hat{x}, \hat{u}, \hat{v}). \tag{3.115}$$

Observe that $H_v = H_v[\hat{\lambda}](\hat{x}, \hat{u})$, i.e. it does not depend on v. In view of (3.115) we can write $\dot{H}_v = \dot{H}_v[\hat{\lambda}](\hat{x}, \hat{u}, \hat{v})$. Thus, a priori, $\dot{H}_v$ is a function of $(\hat{\lambda}, \hat{x}, \hat{u}, \hat{v})$. However, since Goh in [67] stated that

$$\frac{\partial \dot{H}_v}{\partial v} = 0 \tag{3.116}$$

on the reference trajectory, $\dot{H}_v$ does not depend on $\hat{v}$, and so

$$\dot{H}_v = \dot{H}_v[\hat{\lambda}](\hat{x}, \hat{u}) = 0. \tag{3.117}$$

We can then differentiate one more time (3.117) obtaining (3.112) as it was desired.

Remark 3.7.2. *Observe that if the system has the special structure*

$$\dot{x} = f_0(x,u) + \sum_{i=1}^{m} v_i f_i(x), \tag{3.118}$$

then in the expression (3.115), $\dot{\hat{u}}$ *is affine on* $\hat{v}$. *In fact, by differentiating* H_u *we get*

$$\begin{aligned} \dot{H}_u &= H_{ux}H_p - H_x H_{up} + H_{uu}\dot{\hat{u}} \\ &= \hat{p}\left(\textstyle\sum_{i=0}^m \hat{v}_i f_{i,ux}\right)\left(\textstyle\sum_{j=0}^m \hat{v}_j f_j\right) - \hat{p}\left(\textstyle\sum_{j=0}^m \hat{v}_j f_{j,x}\right)\left(\textstyle\sum_{i=0}^m \hat{v}_i f_{i,u}\right) + H_{uu}\dot{\hat{u}} \\ &= \hat{p}\textstyle\sum_{i,j=0}^m \hat{v}_i\hat{v}_j(f_{i,ux}f_j - f_{j,x}f_{i,u}) + H_{uu}\dot{\hat{u}}. \end{aligned} \tag{3.119}$$

In our case, we have $f_{i,u} = 0$ *for* $i = 1,\dots,m$, *and hence it follows from* (3.119),

$$\dot{H}_u = \hat{p}\sum_{j=0}^{m} \hat{v}_j(f_{0,ux}f_j - f_{j,x}f_{0,u}) + H_{uu}\dot{\hat{u}}. \tag{3.120}$$

Thus, given that H_{uu} *does not depend on* $\hat{v}$, *we can deduce from* (3.120) *that* $\dot{\hat{u}}$ *is affine on* $\hat{v}$.

Observe now that the derivative of the function

$$\begin{pmatrix} H_u \\ -\ddot{H}_v \end{pmatrix} \tag{3.121}$$

with respect to (u,v) is given by

$$\mathcal{J} := \begin{pmatrix} H_{uu} & 0 \\ -\dfrac{\partial \ddot{H}_v}{\partial u} & -\dfrac{\partial \ddot{H}_v}{\partial v} \end{pmatrix}, \tag{3.122}$$

where we used (3.16). Therefore, the equations (3.110) and (3.112) can be used to solve $(\hat{u},\hat{v})$ in terms of $(\hat{\lambda},\hat{x})$ provided that $\mathcal{J}$ is invertible. Notice that if (3.111) and (3.113) are verified, $\mathcal{J}$ is definite positive and consequently, nonsingular.

Finally, see that (3.112) together with the boundary conditions

$$H_v[\hat{\lambda}](\hat{w}_T) = 0, \tag{3.123}$$

$$\dot{H}_v[\hat{\lambda}](\hat{w}_0) = 0, \tag{3.124}$$

guarantee the second identity in (3.11).

Notation: Denote by (OS) the set of equations consisting of (3.2)-(3.3), (3.7), (3.9)-(3.10), (3.110), (3.112) and the boundary conditions (3.123)-(3.124).

Remark 3.7.3. *Instead of* (3.123)-(3.124), *we could choose another pair of endpoint conditions among the four possible ones:* $H_v(0) = 0$, $H_v(T) = 0$, $\dot{H}_v(0) = 0$ *and* $\dot{H}_v(T) = 0$, *always including at least one of order zero. The choice we made will simplify the presentation of the result afterwards.*

The rest of this article is very close to what was done in Aronna el at- [10] (or Sections 2.4-2.7 of Chapter 2 of this thesis). The presentation here is then more concise, and the reader is referred to the mentioned sections for further details.

3.7.2 The algorithm

The aim of this section is to present an appropriated numerical scheme to solve system (OS). Let us define the *shooting function*

$$\begin{aligned} \mathcal{S} : \mathrm{D}(\mathcal{S}) := \mathbb{R}^n \times \mathbb{R}^{n+d_\eta,*} \quad &\to \quad \mathbb{R}^{d_\eta} \times \mathbb{R}^{2n+2m,*}, \\ \big(x_0, p_0, \beta\big) =: \nu \quad &\mapsto \quad \mathcal{S}(\nu) := \begin{pmatrix} \eta(x_0, x_T) \\ p_0 + D_{x_0}\ell[\lambda](x_0, x_T) \\ p_T - D_{x_T}\ell[\lambda](x_0, x_T) \\ H_v[\lambda](w_T) \\ \dot{H}_v(w_0) \end{pmatrix}, \end{aligned} \tag{3.125}$$

where (x, u, v, p) is a solution of (3.2),(3.9),(3.110),(3.112) with initial conditions x_0 and p_0, and $\lambda := (p, \beta)$. Note that solving (OS) consists of finding $\hat{\nu} \in \mathrm{D}(\mathcal{S})$ such that

$$\mathcal{S}(\hat{\nu}) = 0. \tag{3.126}$$

Since the number of equations in (3.126) is greater than the number of unknowns, the Gauss-Newton method is a suitable approach to solve it (see [10] or Section 2.4 of Chapter 2 of this thesis for further details of Gauss-Newton method). The *shooting algorithm* we propose here consists of solving the equation (3.126) by the Gauss-Newton method. As we already know, Gauss-Newton is applicable when the derivative of the shooting function $\mathcal{S}$ is one-to-one in a neighborhood of $\hat{\nu}$, and in this case it converges locally quadratically.

The main result of this last part of the article is presenting a condition that guarantees the local quadratic convergence of the shooting method. This condition involves the sufficient optimality condition of Theorem 3.6.2 in Section 3.6.

3.8 Convergence of the shooting algorithm

The purpose of this section is proving the following result:

Theorem 3.8.1. *If $\hat{w}$ is a weak minimum of (SP) satisfying* (3.72) *then the shooting algorithm is locally quadratically convergent.*

3.8.1 Linearized optimality system

In this paragraph we shall compute the linearization of (OS). The reader is referred to the Appendix for a definition of linearized system of differential algebraic equations and for a commutation property between the linearization and time differentiation. We denote by $\operatorname{Lin}\mathcal{F}$ the linearization of function $\mathcal{F}$. Each time the argument of a function is missing, assume that it is evaluated on $(\hat{w}, \hat{\lambda})$. Recall the definitions of B in (3.21) and of C in (3.27). Notice that, since $H_v = pB$,

$$\operatorname{Lin} H_v = \bar{p}B + \bar{x}^\top C^\top. \tag{3.127}$$

The linearization of system (OS) at point $(\hat{x}, \hat{u}, \hat{v}, \hat{\lambda})$ consists of the linearized state equation (3.18) with initial-final condition (3.23), the linearized costate equation

$$-\dot{\bar{p}}_t = \bar{p}_t A_t + \bar{x}_t^\top Q_t + \bar{u}_t^\top E_t + \bar{v}_t^\top C_t, \quad \text{a.e. on } [0,T], \tag{3.128}$$

with initial-final conditions

$$\bar{p}_0 = -\left[\bar{x}_0^\top D^2_{x_0^2}\ell + \bar{x}_T^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_0}\eta_j \right]_{(\hat{x}_0, \hat{x}_T)}, \tag{3.129}$$

$$\bar{p}_T = \left[\bar{x}_T^\top D^2_{x_T^2}\ell + \bar{x}_0^\top D^2_{x_0 x_T}\ell + \sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T}\eta_j \right]_{(\hat{x}_0, \hat{x}_T)}, \tag{3.130}$$

and the algebraic equations

$$0 = \operatorname{Lin} H_u = \bar{p}D + \bar{x}^\top E^\top + \bar{u}^\top R_0, \tag{3.131}$$

$$0 = \operatorname{Lin} \ddot{H}_v = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\bar{p}B + \bar{x}^\top C^\top), \quad \text{a.e. on } [0,T], \tag{3.132}$$

$$0 = (\operatorname{Lin} H_v)_T = \bar{p}_T B_T + \bar{x}_T^\top C_T^\top, \tag{3.133}$$

$$0 = (\operatorname{Lin} \dot{H}_v)_0 = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\bar{p}B + \bar{x}^\top C^\top), \tag{3.134}$$

where we used equation (3.127) and commutation property of Lemma 3.9.2. There is no need to detail the derivatives in (3.133) and (3.134) since we will not make use of them later. Observe that (3.132) -(3.134) and Lemma 3.9.2 yield

$$0 = \operatorname{Lin} H_v = \bar{p}B + \bar{x}^\top C^\top, \quad \text{a.e. on } [0,T]. \tag{3.135}$$

Note that equation (3.135) holds everywhere on $[0,T]$ since all the involved functions are continuous in time.

Notation: denote by (LS) the set of equations consisting of (3.18), (3.23), (3.128)-(3.134).

Once we have computed the linearized system (LS), we can write the derivative of $\mathcal{S}$ in the direction $\bar{\nu} := \left(\bar{x}_0, \bar{p}_0, \bar{\beta}\right)$ as follows:

$$\mathcal{S}'(\hat{\nu})\bar{\nu} = \begin{pmatrix} D\eta(\hat{x}_0,\hat{x}_T)(\bar{x}_0,\bar{x}_T) \\ \bar{p}_0 + \left[\bar{x}_0^\top D^2_{x_0^2}\ell + \bar{x}_T^\top D^2_{x_0x_T}\ell + \sum_{j=1}^{d_\eta}\bar{\beta}_j D_{x_0}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)} \\ \bar{p}_T - \left[\bar{x}_T^\top D^2_{x_T^2}\ell + \bar{x}_0^\top D^2_{x_0x_T}\ell + \sum_{j=1}^{d_\eta}\bar{\beta}_j D_{x_T}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)} \\ \bar{p}_T B_T + \bar{x}_T^\top C_T \\ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\bar{p}B + \bar{x}^\top C^\top\right) \end{pmatrix}, \tag{3.136}$$

where $(\bar{x},\bar{u},\bar{v},\bar{p})$ is the solution of (3.18),(3.128),(3.131),(3.132) associated to the initial condition $(\bar{x}_0,\bar{p}_0)$ and the multiplier $\bar{\beta}$.

Proposition 3.8.2. *The derivative $\mathcal{S}'(\hat{\nu})$ is one-to-one if the only solution of* (3.18), (3.128), (3.131), (3.132) *with the initial conditions $\bar{x}_0 = 0,\ \bar{p}_0 = 0$ and with $\bar{\beta} = 0$ is* $(\bar{x},\bar{u},\bar{v},\bar{p}) = 0$.

3.8.2 Additional LQ problem

In this paragraph we present a linear-quadratic control problem (LQ) in the variables $(\bar{\xi},\bar{u},\bar{y},\bar{h})$ having $\Omega_{\mathcal{P}_2}$ (defined in (3.67)) as cost functional. Note that if condition (3.72) holds then (LQ) has a unique optimal solution $(\xi,y,h) = 0$. Furthermore, (3.72) yields the strong convexity of the pre-Hamiltonian of (LQ) and hence its unique optimal solution is characterized by its first order optimality system. Here we present a one-to-one linear mapping that transforms each solution of (LS) (introduced in paragraph 3.8.1 above) into a solution of this new optimality system. Theorem 3.8.1 will follow.

Let us consider the linear-quadratic problem (LQ) given by:

$$\Omega_{\mathcal{P}_2}(\bar{\xi},\bar{u},\bar{y},\bar{h}) \to \min, \tag{3.137}$$

$$(3.45)\text{-}(3.47), \tag{3.138}$$

$$\dot{h} = 0. \tag{3.139}$$

Here $\bar{u}$ and $\bar{y}$ are the control variables, $\bar{\xi}$ and $\bar{h}$ are the state variables. Denote by $\bar{\chi}$ and $\bar{\chi}_h$ the costate variables corresponding to $\bar{\xi}$ and $\bar{h}$, respectively. Note that the qualification hypothesis in Assumption 3.7.1 implies that $\{D\eta_j(\hat{x}_0,\hat{x}_T)\}_{j=1}^{d_\eta}$ are linearly independent. Hence any weak solution $(\bar{\xi},\bar{u},\bar{y},\bar{h})$ of (LQ) has a unique associated multiplier $\lambda^{LQ} := (\bar{\chi}, \bar{\chi}_h, \beta^{LQ})$ solution of the system that we describe next. The pre-Hamiltonian for (LQ) is

$$\begin{aligned}&\mathcal{H}[\lambda^{LQ}](\bar{\xi}, \bar{u}, \bar{y}) := \bar{\chi}(A\bar{\xi} + D\bar{u} + B_1\bar{y}) \\ &+(\tfrac{1}{2}\bar{\xi}^\top Q\bar{\xi} + \bar{u}^\top E\bar{\xi} + \bar{y}^\top M\bar{\xi} + \tfrac{1}{2}\bar{u}^\top R_0\bar{u} + \bar{y}^\top J\bar{u} + \tfrac{1}{2}\bar{y}^\top R_1\bar{y}).\end{aligned} \tag{3.140}$$

Observe that $\mathcal{H}$ does not depend on $\bar{h}$ since it has zero dynamics and does not appear in the running cost. The initial-final Lagrangian is

$$\ell^{LQ}[\lambda^{LQ}](\bar{\xi}_0, \bar{\xi}_T, \bar{h}_T) := g(\bar{\xi}_0, \bar{\xi}_T, \bar{h}_T) + \sum_{j=1}^{d_\eta} \beta_j^{LQ} D\eta_j(\bar{\xi}_0, \bar{\xi}_T + B_T\bar{h}_T). \tag{3.141}$$

The costate equation for $\bar{\chi}$ is:

$$-\dot{\bar{\chi}}_t = D_{\bar{\xi}}\mathcal{H}[\lambda^{LQ}] = \bar{\chi}A + \bar{\xi}^\top Q + \bar{u}^\top E + \bar{y}^\top M, \tag{3.142}$$

with the boundary conditions:

$$\begin{aligned}\bar{\chi}_0 =\ & -D_{\bar{\xi}_0}\ell^{LQ}[\lambda^{LQ}] \\ =\ & -\rho_0^{LQ}\left[\bar{\xi}_0^\top D^2_{x_0^2}\ell + (\bar{\xi}_T + B_T\bar{h})^\top D^2_{x_0x_T}\ell\right] - \textstyle\sum_{j=1}^{d_\eta}\beta_j^{LQ} D_{x_0}\eta_j,\end{aligned} \tag{3.143}$$

$$\begin{aligned}\bar{\chi}_T =\ & D_{\xi_T}\ell^{LQ}[\lambda^{LQ}] \\ =\ & \rho_0^{LQ}\left[\bar{\xi}_0^\top D^2_{x_0x_T}\ell + (\bar{\xi}_T + B_T\bar{h})^\top D^2_{x_T^2}\ell\right] + \bar{h}^\top C_T + \textstyle\sum_{j=1}^{d_\eta}\beta_j^{LQ} D_{x_T}\eta_j.\end{aligned} \tag{3.144}$$

For costate variable $\bar{\chi}_h$ we get the equation

$$\dot{\bar{\chi}}_h = 0, \tag{3.145}$$

$$\bar{\chi}_{h,0} = 0, \tag{3.146}$$

$$\bar{\chi}_{h,T} = D_{\bar{h}}\ell^{LQ}[\lambda^{LQ}]. \tag{3.147}$$

Hence, $\bar{\chi}_h \equiv 0$ and thus (3.147) yields

$$0 = \rho_0^{LQ}\left[\bar{\xi}_0^\top D^2_{x_0x_T}\ell\, B_T + (\bar{\xi}_T + B_T\bar{h})^\top (D^2_{x_T^2}\ell\, B_T + C_T^\top)\right] + \sum_{j=1}^{d_\eta}\beta_j^{LQ} D_{x_T}\eta_j B_T. \tag{3.148}$$

The **stationarity with respect to the new control** $\bar{y}$ implies

$$0 = \mathcal{H}_{\bar{u}} = \bar{\chi}D + \xi^\top E^\top + \bar{u}^\top R_0 + \bar{y}^\top J, \tag{3.149}$$

$$0 = \mathcal{H}_{\bar{y}} = \bar{\chi}B_1 + \bar{\xi}^\top M^\top + \bar{u}^\top J^\top + \bar{y}^\top R_1. \tag{3.150}$$

Notation: Denote by (LQS) the set of equations consisting of (3.138)-(3.139), (3.142)-(3.144),(3.148) and (3.150), and observe that (LQS) is an **optimality system** of problem (3.137)-(3.139).

3.8.3 The transformation

In this paragraph we show how to transform a solution of (LS) into a solution of (LQS) via a one-to-one linear mapping. Given $(\bar{x}, \bar{u}, \bar{v}, \bar{p}, \bar{\beta}) \in \mathcal{W} \times W_{1,\infty} \times \mathbb{R}^{d_\eta,*}$, define

$$\bar{y}_t := \int_0^t \bar{v}_s ds,\ \bar{\xi} := \bar{x} - B\bar{y},\ \bar{\chi} := \bar{p} + \bar{y}^\top C,\ \bar{\chi}_h := 0,\ \bar{h} := \bar{y}_T,\ \beta_j^{LQ} := \bar{\beta}_j. \quad (3.151)$$

The next lemma shows that the point $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \bar{\chi}_h, \beta^{LQ})$ is solution of (LQS) provided that $(\bar{x}, \bar{u}, \bar{v}, \bar{p}, \bar{\beta})$ is solution of (LS).

Lemma 3.8.3. *The one-to-one linear mapping defined by* (3.151) *converts each solution of (LS) into a solution of (LQS).*

Remark 3.8.4. *Recall Corollary 3.5.3 for the proof below.*

Proof. Let $(\bar{x}, \bar{u}, \bar{v}, \bar{p}, \bar{\beta})$ be a solution of (LS), and set $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \beta^{LQ})$ by (3.151).
Part I. We shall prove that $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \beta^{LQ})$ satisfies conditions (3.138). Equation (3.45) follows differentiating expression of $\bar{\xi}$ in (3.151), and equation (3.47) follows from (3.23).
Part II. We shall prove that $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \beta^{LQ})$ verifies (3.142)-(3.144) and (3.148). Differentiate $\bar{\chi}$ in (3.151), use equations (3.128) and (3.151), recall definition of M in (3.55) and obtain:

$$\begin{aligned} -\dot{\bar{\chi}} &= -\dot{\bar{p}} - \bar{v}^\top C - \bar{y}^\top \dot{C} \\ &= \bar{p}A + \bar{x}^\top Q + \bar{u}^\top E - \bar{y}^\top \dot{C} \\ &= \bar{\chi}A + \bar{\xi}^\top Q + \bar{u}^\top E + \bar{y}^\top(-CA + B^\top Q - \dot{C}) \\ &= \bar{\chi}A + \xi^\top Q + \bar{u}^\top E + y^\top M. \end{aligned} \quad (3.152)$$

Hence (3.142) holds. Equations (3.143)-(3.144) follow from (3.129)-(3.130). Combine (3.130) and (3.133) to get

$$\begin{aligned} 0 &= \bar{p}_T B_T + \bar{x}_T^\top C_T^\top \\ &= \left[\bar{p}_T^\top D^2_{x_T^2}\ell + \bar{x}_0^\top D^2_{x_0 x_T}\ell + \textstyle\sum_{j=1}^{d_\eta} \bar{\beta}_j D_{x_T}\eta_j\right]_{(\hat{x}_0,\hat{x}_T)} B_T + \bar{x}_T^\top C_T^\top. \end{aligned} \quad (3.153)$$

Performing transformation (3.151) in the previous equation yields (3.148).
Part III. Let us show that the stationarity with respect to $\bar{y}$ in (3.149) is verified. The transformation in (3.151) together with equation (3.131) imply

$$\begin{aligned} 0 &= (\bar{\chi} - \bar{y}^\top C)D + (\bar{\xi} + B\bar{y})^\top E^\top + \bar{u}^\top R_0 \\ &= \bar{\chi}D + \bar{\xi}^\top E^\top + \bar{u}^\top R_0 + \bar{y}^\top(B^\top E^\top - CD). \end{aligned} \quad (3.154)$$

Calling back definition of J in (3.55), stationarity condition (3.149) follows.

Part IV. Finally, we shall prove that (3.150) holds. Perform the transformation (3.151) in equation (3.135) to obtain

$$0 = (\bar{\chi} - \bar{y}^\top C)B + (\bar{\xi} + B\bar{y})^\top C^\top = \bar{\chi}B + \bar{\xi}^\top C^\top, \tag{3.155}$$

since Corollary 3.5.3 holds when the multiplier in unique. Differentiating previous expression we obtain

$$\begin{aligned} 0 = \quad & -(\bar{\chi}A + \bar{\xi}^\top Q + \bar{u}^\top E + \bar{y}^\top M)\, B + +\bar{\chi}\dot{B} \\ & +(A\bar{\xi} + D\bar{u} + B_1\bar{y})^\top C^\top + \bar{\xi}^\top \dot{C}^\top. \end{aligned} \tag{3.156}$$

Recall the definitions of B_1 in (3.46) and of J in (3.55), of R_1 in (3.57), and use then in (3.156) to get (3.150).

Parts I to IV show that $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \beta^{LQ})$ is a solution of (LQS), and hence the result follows.

□

Remark 3.8.5. *Observe that the unique assumption we needed in previous proof was the symmetry condition in Corollary 3.5.3.*

Proof. [of Theorem 3.8.1] Let $(\bar{x}, \bar{u}, \bar{v}, \bar{p}, \bar{\beta})$ be a solution of (LS), and let $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \bar{\chi}_h, \beta^{LQ})$ be defined by the transformation in (3.151). Hence we know by Lemma 3.8.3 that $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \bar{\chi}_h, \beta^{LQ})$ is solution of (LQS). As it has been already pointed out, condition (3.72) implies that the unique solution of (LQS) is 0. Hence $(\bar{\xi}, \bar{u}, \bar{y}, \bar{h}, \bar{\chi}, \bar{\chi}_h, \beta^{LQ}) = 0$ and thus $(\bar{x}, \bar{u}, \bar{v}, \bar{p}, \bar{\beta}) = 0$. Conclude that the unique solution of (LS) is 0. This implies the injectivity of $\mathcal{S}'$ at $\hat{\nu}$, and hence the result follows.

□

3.9 Conclusion

We provided a set of necessary and sufficient conditions for a problem having a part of the control variable entering linearly in the pre-Hamiltonian. These conditions apply to a weak minimum and do not assume the uniqueness of multipliers.

We proposed a shooting algorithm based on the procedure described by Goh [69] to compute the control variables in terms of the state and costate variables. We proved that the sufficient condition above-mentioned guarantees the local quadratic convergence of the shooting algorithm.

Appendix

Linearization of a Differential Algebraic System

For the aim of finding an expression of $\mathcal{S}'(\hat{\nu})$, we make use of the linearization of (OS) and thus we recall the following concept:

Definition 3.9.1 (Linearization of a Differential Algebraic System)**.** *Consider a system of differential algebraic equations:*

$$\dot{\zeta}_t = \mathcal{F}(\zeta_t, \alpha_t), \tag{3.157}$$

$$0 = \mathcal{G}(\zeta_t, \alpha_t), \tag{3.158}$$

$$0 = \mathcal{I}(\zeta_0, \zeta_T), \tag{3.159}$$

with $\mathcal{F} : \mathbb{R}^{m+n} \to \mathbb{R}^n$, $\mathcal{G} : \mathbb{R}^{m+n} \to \mathbb{R}^{n_\mathcal{G}}$, *and* $\mathcal{I} : \mathbb{R}^{2n} \to \mathbb{R}^{n_\mathcal{I}}$. *Let* (ζ^0, α^0) *be a solution. We call linearized system at point* (ζ^0, α^0) *the following DAE in the variables* $\bar{\zeta}$ *and* $\bar{\alpha}$:

$$\dot{\bar{\zeta}}_t = \operatorname{Lin} \mathcal{F} \mid_{(\zeta_t^0, \alpha_t^0)} (\bar{\zeta}_t, \bar{\alpha}_t), \tag{3.160}$$

$$0 = \operatorname{Lin} \mathcal{G} \mid_{(\zeta_t^0, \alpha_t^0)} (\bar{\zeta}_t, \bar{\alpha}_t), \tag{3.161}$$

$$0 = \operatorname{Lin} \mathcal{I} \mid_{(\zeta_0^0, \zeta_T^0)} (\bar{\zeta}_0, \bar{\zeta}_T), \tag{3.162}$$

where

$$\operatorname{Lin} \mathcal{F} \mid_{(\zeta_t^0, \alpha_t^0)} (\bar{\zeta}_t, \bar{\alpha}_t) := \mathcal{F}'(\zeta_t^0, \alpha_t^0)(\bar{\zeta}_t, \bar{\alpha}_t), \tag{3.163}$$

and the analogous definitions for $\operatorname{Lin} \mathcal{G}$ *and* $\operatorname{Lin} \mathcal{H}$.

The technical result below will simplify the computation of the linearization of (OS). Its proof is immediate.

Lemma 3.9.2 (Commutation of linearization and differentiation)**.** *Given* $\mathcal{G}$ *and* $\mathcal{F}$ *as in the previous definition, it holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Lin} \mathcal{G} = \operatorname{Lin} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{G}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Lin} \mathcal{F} = \operatorname{Lin} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}. \tag{3.164}$$

4

Continuous Time Optimal Hydrothermal Scheduling

Joint work with J.F. Bonnans and P.A. Lotito. Published in Proceedings of the International Conference on Engineering Optimization [2008], under the title *Continuous Time Optimal Hydrothermal Scheduling.*

Abstract

We consider an optimal control problem of optimal hydrothermal scheduling. The model, already discussed in [27], is deterministic and takes into account the dependence of efficiency of hydroelectric energy with respect to the volume of water in dams. The thermal cost is a strongly convex and nondecreasing function of the thermal power. We study the possible occurence of a singular arc, for which necessary conditions due to Goh are known (Goh-Legendre-Clebsch condition in (21)). We are able to give conditions under which these conditions are automatically satisfied.

4.1 Introduction

On a weekly basis optimal hydrothermal scheduling is usually performed using linear programming techniques, see [57] and the references therein. In such a case limitations due to switch on-off of engines as well as the variable efficiency as function of height of water are neglected. Our proposal is to study the hydrothermal scheduling by taking into account this variable efficiency. Our model is deterministic and with continuous time, which is meaningful for a short-term horizon. In this respect we follow the model in [27]. These authors obtain existence and also uniqueness of minimizers in some special situations.

The novelty of this paper is to apply the tools of optimal control theory. We focus on the analysis of singular arcs. Our main result is a characterization of the Goh-Legendre-Clebsch (GLC) condition [67, 69]. As a consequence we show that for some choices of the efficiency coefficients, this condition always holds (resp. does not hold). When the GL condition holds, the algebraic variables (controls) can be eliminated from some algebraic expressions and expressed as functions of the differential variables (state and costate).

4.2 Background on singular arcs

Consider an optimal control problem with state equation

$$\dot{y}(t) = f(u(t), w(t), y(t)), \tag{4.1}$$

where $y(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{m_1}$ is the "nonlinear control", and $w(t) \in \mathbb{R}^{m_2}$ is such that the mapping $w \mapsto f(u, w, y)$ is affine, and therefore we call w the "affine control". For simplicity we assume that the cost is a function $\phi(y(T))$ of the terminal state (the horizon is fixed; if an integral cost is present then we can eliminate it thanks to an additional state variable). Let $(\bar{u}, \bar{w}, \bar{y})$ be a solution of this unconstrained problem. The Hamiltonian associated with the problem is

$$H(u, w, y, p) := p \cdot f(u, w, y). \tag{4.2}$$

The dynamics of the costate is

$$-\dot{p}(t) = H_y(u(t), w(t), y(t), p(t)). \tag{4.3}$$

By Pontryagin's principle we have that the Hamiltonian

$$H(u, w, y, p) := p \cdot f(u, w, y), \tag{4.4}$$

attains its minimum over the control variables along the trajectory, i.e.,

$$H(\bar{u}(t), \bar{w}(t), \bar{y}(t), \bar{p}(t)) \leq H(u, w, \bar{y}(t), \bar{p}(t)), \quad \text{for all } (u, w). \tag{4.5}$$

The first-order optimality conditions for this problem are

$$\text{(i)} \;\; H_u(\bar{u}(t), \bar{w}(t), \bar{y}(t), \bar{p}(t)) = 0; \quad \text{(ii)} \;\; H_w(\bar{u}(t), \bar{w}(t), \bar{y}(t), \bar{p}(t)) = 0. \tag{4.6}$$

The second-order optimality condition is, skipping the variables and denoting $A \succeq B$ if A, B are symmetric matrices of same size and $A - B$ is positive semidefinite:

$$\begin{pmatrix} H_{uu} & H_{uw} \\ H_{wu} & H_{ww} \end{pmatrix} \succeq 0. \tag{4.7}$$

This is known as the Legendre-Clebsch condition.

Since $w \mapsto H(u, w, y, p)$ is affine, $H_{ww} = 0$ and so the Legendre-Clebsch is equivalent to the two relations below:

$$\text{(i)} \;\; H_{uu} \succeq 0 \quad \text{(ii)} \;\; H_{uw} = 0. \tag{4.8}$$

If $H_{uu} \succ 0$ (is positive definite) then by the Implicit Function Theorem (IFT) we can locally solve Eq.(4.6)(i) by expressing $\bar{u}$ as a function say U of $(\bar{w}, \bar{y}, \bar{p})$. However since Eq.(4.6)(ii) does not depend on $\bar{w}$, we cannot eliminate $\bar{w}$ from the latter.

Further results were obtained by considering high-order time derivatives H_w along the optimal trajectory. Generalizations of the Legendre-Clebsch condition, for optimal control problems with Hamiltonian affine w.r.t. the control, were obtained by Kelley [79, 80], and then higher order conditions were obtained by [81]. Robbins [112] consider the case when only some of the components of the control enter in an affine way, then eliminating the other components using Pontryagin's principle formulates a reduced system for which similar conditions hold. However, this construction by reduction is not obvious. By a different approach Goh could obtain more general conditions called Goh-Legendre-Clebsch conditions [67, 69]. In the simplest case they are as follows:

$$\begin{pmatrix} H_{uu} & -(\dot{H}_w)_u^\top \\ -(\dot{H}_w)_u & -(\ddot{H}_w)_w \end{pmatrix} \succeq 0. \tag{4.9}$$

In the above relation the total derivatives denoted by one or two points must be understood as being applied to H_w. The expression of the total derivative is obtained by applying the chain rule, replacing the derivatives of state and costate with their expression, but without elimination of the nonlinear control and its derivatives. For a qualified constrained problem, the above relations remain valid along an arc where these constraints are not active.

We will now see how to apply these results to our model and see if this allows to eliminate the algebraic variables (an essential condition for designing shooting algorithms in the presence of singular arcs, see e.g. [90] and Aronna [7]).

4.3 Optimal Hydropower Generation Problem

We know (somehow) the complete trajectory of the electricity demand $d(t)$, the thermal production cost P which is an increasing, positive function of the load. We need to decide for each time how much energy we will produce with each kind of generation units, complying with the restrictions. If we call $\pi(t)$ the total amount of electricity produced with the hydropower plants, the rest, $d(t) - \pi(t)$ will be produced with the thermal units. For the sake of planning, the water has an economic value that has also to be considered in the cost definition. More precisely, the criterium to be minimized is

$$J(q,s) = \int_0^T P[d(t) - \pi(t)]\mathrm{d}t - a \sum_i y_i(T), \tag{4.10}$$

where $q(t) = (q_i(t))$, $i = 1, 2, \ldots, m$, is the outflow in each plant, $y(t) = (y_i(t))$, $i = 1, 2, \ldots, m$, is the water volume in the valley i at time t, $\pi(t) = \sum \rho_i(y_i)q_i$ and $s(t) = (s_i(t))$, $i = 1, 2, \ldots, m$, is the spilled out water at each plant at time t. We assume that the efficiency of each turbine is a positive and increasing function of the volumen $\rho_i(y_i)$, the maximum volume of each valley is given by y^M, and the maximum allowed flow in each turbine is q^M. Now the dynamic equation for the water volume on each valley is given by

$$\dot{y}_i(t) = b_i(t) - s_i(t) - q_i(t), \tag{4.11}$$

where $b_i(t)$ is the water inflow for each valley and we consider also that

$$0 \leq y_i(t) \leq y^M, \tag{4.12}$$
$$0 \leq q_i(t) \leq q^M, \tag{4.13}$$
$$0 \leq s_i(t), \tag{4.14}$$
$$\sum \rho_i(y_i)q_i \leq d. \tag{4.15}$$

4.4 Singular arcs study with two plants

We have the following dynamical system

$$\begin{cases} \dot{y_1} = b_1 - s_1 - q_1, \\ \dot{y_2} = b_2 - s_2 - q_2. \end{cases} \tag{4.16}$$

The Hamiltonian is, in this case

$$H(p_1, p_2, y_1, y_2, q_1, q_2, s_1, s_2) = p_1(b_1 - s_1 - q_1) + p_2(b_2 - s_2 - q_2) + P(d - \rho_1 q_1 - \rho_2 q_2). \tag{4.17}$$

By PMP we have that

$$0 = H_{q_i} = -p_i - P'(d - \sum_{k=1}^{2} \rho_k q_k)\rho_i, \tag{4.18}$$

and

$$\dot{p_i} = -H_{y_i} = -P'(d - \sum_{k=1}^{2} \rho_k q_k)\rho_i'. \tag{4.19}$$

From (4.18) we know that

$$p_i = -P'(d - \sum_{k=1}^{2} \rho_k q_k)\rho_i, \tag{4.20}$$

and then, as P' and ρ_i are strictly positive, we have that

$$p_i < 0. \tag{4.21}$$

From (4.19), we can deduce that

$$\dot{p_i} < 0, \tag{4.22}$$

since $\rho_i' > 0$.

We now make a change of variables on controls in order to highlight the presence of an affine control. Consider the following new control variables

$$\pi = \frac{\rho_1 q_1 + \rho_2 q_2}{2}, \ \eta = \frac{\rho_1 q_1 - \rho_2 q_2}{2}.$$

The resulting dynamics is

$$\begin{cases} \dot{y}_1 = b_1 - s_1 - \dfrac{\pi + \eta}{\rho_1}, \\ \dot{y}_2 = b_2 - s_2 - \dfrac{\pi - \eta}{\rho_2}, \end{cases} \tag{4.23}$$

and the new Hamiltonian reads

$$H(p, y, \pi, \eta, s) = p_1 \left(b_1 - s_1 - \frac{\pi + \eta}{\rho_1} \right) + p_2 \left(b_2 - s_2 - \frac{\pi - \eta}{\rho_2} \right) + P(d - 2\pi). \tag{4.24}$$

If the qualified form of Pontryagin's principle holds, then we have that the costate p satisfies

$$-\dot{p}_1 = H_{y_1} = \frac{\rho_1' p_1 (\pi + \eta)}{\rho_1^2}, \ -\dot{p}_2 = H_{y_2} = \frac{\rho_2' p_2 (\pi - \eta)}{\rho_2^2}. \tag{4.25}$$

In the sequel we note that we can assume that $s_1 = s_2 = 0$, since when control constraints are not active (which by the definition is the case on singular arcs we are interested in) there is no spillover. In order to check Goh's condition we have to compute the first- and second-order time derivatives of H_η along the optimal trajectory. Their expression is stated in the next lemma.

Proposition 4.4.1. *The expression of H_η and its first-order time derivative $\dot{H}_\eta$, as well as the one of the partial derivative w.r.t. η of $\ddot{H}_\eta$, are*

$$H_\eta = -\frac{p_1}{\rho_1} + \frac{p_2}{\rho_2}, \tag{4.26}$$

$$\dot{H}_\eta = \frac{p_1 \rho_1' b_1}{\rho_1^2} - \frac{p_2 \rho_2' b_2}{\rho_2^2}, \tag{4.27}$$

$$\frac{\partial \ddot{H}_\eta}{\partial \eta} = -p_1 b_1 \left(\frac{\rho_1'^2}{\rho_1^4} + \frac{1}{\rho_1} \left[\frac{\rho_1'}{\rho_1^2} \right]' \right) - p_2 b_2 \left(\frac{\rho_2'^2}{\rho_2^4} + \frac{1}{\rho_2} \left[\frac{\rho_2'}{\rho_2^2} \right]' \right). \tag{4.28}$$

Proof. The expression of H_η and of its first-order time derivative are esay consequences of the state and costate equations. The second-order time derivative is

$$\begin{aligned}\ddot{H}_\eta = & - \frac{\rho_1'^2}{\rho_1^4} p_1(\pi+\eta) b_1 + p_1 \left[\frac{\rho_1'}{\rho_1^2}\right]' (b_1 - \frac{\pi+\eta}{\rho_1}) b_1 + p_1 \frac{\rho_1'}{\rho_1^2} \dot{b}_1 \\ & + \frac{\rho_2'^2}{\rho_2^4} p_2(\pi-\eta) b_2 - p_2 \left[\frac{\rho_2'}{\rho_2^2}\right]' (b_2 - \frac{\pi-\eta}{\rho_2}) b_2 - p_2 \frac{\rho_2'}{\rho_2^2} \dot{b}_2.\end{aligned}$$

And then, differentiating with respect to η we obtain

$$\begin{aligned}\frac{\partial \ddot{H}_\eta}{\partial \eta} &= -\frac{\rho_1'^2}{\rho_1^4} p_1 b_1 - \frac{p_1 b_1}{\rho_1}\left[\frac{\rho_1'}{\rho_1^2}\right]' - \frac{\rho_2'^2}{\rho_2^4} p_2 b_2 - \frac{p_2 b_2}{\rho_2}\left[\frac{\rho_2'}{\rho_2^2}\right]' && (4.29)\\ &= -p_1 b_1 \left(\frac{\rho_1'^2}{\rho_1^4} + \frac{1}{\rho_1}\left[\frac{\rho_1'}{\rho_1^2}\right]'\right) - p_2 b_2 \left(\frac{\rho_2'^2}{\rho_2^4} + \frac{1}{\rho_2}\left[\frac{\rho_2'}{\rho_2^2}\right]'\right), && (4.30)\end{aligned}$$

as was to be proved. □

Lemma 4.4.2. *Goh-Legendre-Clebsch condition is satisfied if and only if*

$$\frac{\partial \ddot{H}_\eta}{\partial \eta} \leq 0. \tag{4.31}$$

Proof. We recall the matrix of Goh-Legendre-Clebsch condition (we use here the fact that both π and η are scalar variables):

$$\begin{pmatrix} H_{\pi\pi} & -\frac{\partial \dot{H}_\eta}{\partial \pi} \\ -\frac{\partial \dot{H}_\eta}{\partial \pi} & -\frac{\partial \ddot{H}_\eta}{\partial \eta} \end{pmatrix}. \tag{4.32}$$

Let us look for $H_{\pi\pi}$.

$$H_\pi = -\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} - 2P'(d-2\pi), \tag{4.33}$$

and then

$$H_{\pi\pi} = 4P''(d-2\pi), \tag{4.34}$$

which is strictly positive as P is strictly convex. From (4.27) we get that

$$\frac{\partial \dot{H}_\eta}{\partial \pi} = 0. \tag{4.35}$$

Then the matrix in (4.32) is as follows

$$\begin{pmatrix} 4P''(d-2\pi) & 0 \\ 0 & -\frac{\partial \ddot{H}_\eta}{\partial \eta} \end{pmatrix}. \tag{4.36}$$

We conclude then that it is semidefinite positive if and only if

$$\frac{\partial \ddot{H}_\eta}{\partial \eta} \leq 0. \tag{4.37}$$

□

Using the fact that $p_i < 0,\ i = 1, 2$, and that $b_i \geq 0$, from (4.28) we can obtain sufficient conditions for having (not having) Goh's condition. We put

$$\Phi(\rho) := \frac{\rho'^2}{\rho^4} + \frac{1}{\rho}\left[\frac{\rho'}{\rho^2}\right]'. \tag{4.38}$$

Corollary 4.4.3. *(i) If*

$$\Phi(\rho_i) \leq 0,\ i = 1, 2, \tag{4.39}$$

then Goh's condition is satified.

(ii) If

$$b_i > 0, \Phi(\rho_i) > 0,\ i = 1, 2, \tag{4.40}$$

then Goh's condition is not satified, and hence, there are no singular arcs.

Lemma 4.4.4. *If*

$$\frac{\partial \ddot{H}_\eta}{\partial \eta} \neq 0$$

then we can retrieve η, π *from the system of equations*

$$\begin{cases} H_\pi = 0, \\ \ddot{H}_\eta = 0. \end{cases} \tag{4.41}$$

Proof. The Jacobian matrix for the system in (4.41) is the following one

$$J = \begin{pmatrix} H_{\pi\pi} & H_{\pi\eta} \\ \dfrac{\partial \ddot{H}_\eta}{\partial \pi} & \dfrac{\partial \ddot{H}_\eta}{\partial \eta} \end{pmatrix}. \tag{4.42}$$

Then, as $H_{\pi\eta} = 0$ and $H_{\pi\pi} > 0$ we will have that J is nonsingular if and only if

$$\frac{\partial \ddot{H}_\eta}{\partial \eta} \neq 0.$$

□

4.5 Singular arcs study with m plants

In this paragraph we will analyse Goh-Legendre-Clebsch condition for the problem with m dams. The change of variables in this case varies from the one we made for two dams and is as follows: we consider for $i = 1, \ldots, m$ the control $\eta_i = \rho_i q_i - \pi$. The dynamical system can be written as

$$\dot{y}_i = b_i - s_i - \frac{\pi + \eta_i}{\rho_i}, \tag{4.43}$$

with the additional constraint $\sum_{i=1}^{m} \eta_i = 0$. Then we have that $\sum \rho_i q_i = m\pi$, and moreover, we can work out η_m from this constraint. Then:

$$\eta_m = -\sum_{i=1}^{m-1} \eta_i. \tag{4.44}$$

Then, the dynamic system is as follows:

$$\begin{cases} \dot{y}_i = b_i - s_i - \dfrac{\pi + \eta_i}{\rho_i}, \ i = 1, \ldots m-1, \\ \dot{y}_m = b_m - s_m - \dfrac{\pi - \sum_{i=1}^{m-1} \eta_i}{\rho_m}, \end{cases} \tag{4.45}$$

and the Hamiltonian is

$$H(\pi, \eta, s, y, p) = \sum_{i=1}^{m-1} p_i(b_i - s_i - \frac{\pi + \eta_i}{\rho_i}) + p_m(b_m - s_m - \frac{\pi - \sum_{i=1}^{m-1} \eta_i}{\rho_m}) + P(d - m\pi). \tag{4.46}$$

PMP states that:

$$\begin{cases} \dot{p}_i = -H_{y_i} = -\dfrac{p_i \rho_i'(\pi + \eta_i)}{\rho_i^2}, \ i = 1, \ldots, m-1, \\ \dot{p}_m = -H_{y_m} = -\dfrac{p_m \rho_m'(\pi - \sum_{i=1}^{m-1} \eta_i)}{\rho_m^2}, \\ p_i(T) = -a, \ i = 1, \ldots, m \end{cases}$$

where $\pi(t)$, $\eta(t)$, $s(t)$, $y(t)$, $p(t)$ are optimal.

Proposition 4.5.1. *For all $i = 1, \ldots, m-1$, we have that*

$$H_{\eta_i} = -\frac{p_i}{\rho_i} + \frac{p_m}{\rho_m}, \tag{4.47}$$

$$\dot{H}_{\eta_i} = \frac{\rho_i' b_i p_i}{\rho_i^2} - \frac{\rho_m' b_m p_m}{\rho_m^2}, \tag{4.48}$$

$$\ddot{H}_{\eta_i} = A_i - A_m, \tag{4.49}$$

where

$$A_i = -\frac{\rho_i'^2}{\rho_i^4} p_i(\pi + \eta_i) b_i + p_i \left[\frac{\rho_i'}{\rho_i^2}\right]' (b_i - \frac{\pi + \eta_i}{\rho_i}) b_i + p_i \frac{\rho_i'}{\rho_i^2} \dot{b}_i, \tag{4.50}$$

$$A_m = -\frac{\rho_m'^2}{\rho_m^4} p_m(\pi - \sum_{i=1}^{m-1} \eta_i) b_m + p_m \left[\frac{\rho_m'}{\rho_m^2}\right]' (b_m - \frac{\pi - \sum_{i=1}^{m-1} \eta_i}{\rho_m}) b_m + p_m \frac{\rho_m'}{\rho_m^2} \dot{b}_m. \tag{4.51}$$

Proof. It is easy to retrieve this expressions by making an analogy with the equations for two plants and taking into account the difference between the dynamical systems of both problems. ∎

Theorem 4.5.2. *The matrix in Goh-Legendre-Clebsch condition is the following one:*

$$\begin{pmatrix} m^2 P''(d - m\pi) & 0 \\ 0 & -\dfrac{\partial \ddot{H}_\eta}{\partial \eta} \end{pmatrix}, \tag{4.52}$$

where

$$\frac{\partial \ddot{H}_\eta}{\partial \eta} = -\text{diag}(p_i b_i \Phi(\rho_i)) - p_m b_m \Phi(\rho_m) \mathbf{1}\mathbf{1}^T. \tag{4.53}$$

With $\mathbf{1}$ *we denote the* $m-1$ *dimensional column vector with all its components equal to 1, and we have already introduced* Φ *in the last paragraph.*

Proof. From (4.46):

$$H_\pi = -mP'(d - m\pi), \tag{4.54}$$

and then

$$H_{\pi\pi} = m^2 P''(d - m\pi). \tag{4.55}$$

Now, from (4.48) we obtain that

$$\frac{\partial \dot{H}_{\eta_i}}{\partial \pi} = 0. \tag{4.56}$$

From (4.49), we know that

$$\ddot{H}_{\eta_i} = A_i - A_m, \tag{4.57}$$

then, let us look for

$$\frac{\partial A_i}{\partial \eta_j}, \ for \ i = 1, \ldots, m, \ j = 1, \ldots, m-1. \tag{4.58}$$

From (4.50) and (4.51)

$$\frac{\partial A_i}{\partial \eta_i} = -p_i b_i \left(\frac{\rho_i'^2}{\rho_i^4} + \frac{1}{\rho_i} \left[\frac{\rho_i'}{\rho_i^2} \right]' \right) = -p_i b_i \Phi(\rho_i), \ i = 1, \ldots, m-1, \tag{4.59}$$

$$\frac{\partial A_i}{\partial \eta_j} = 0, \ i, j = 1, \ldots, m-1, \ i \neq j, \tag{4.60}$$

$$\frac{\partial A_m}{\partial \eta_i} = p_m b_m \left(\frac{\rho_m'^2}{\rho_m^4} + \frac{1}{\rho_m} \left[\frac{\rho_m'}{\rho_m^2} \right]' \right) = -p_m b_m \Phi(\rho_m), \ i = 1, \ldots, m-1. \tag{4.61}$$

Then

$$\frac{\partial \ddot{H}_{\eta_i}}{\partial \eta_i} = \frac{\partial A_i}{\partial \eta_i} - \frac{\partial A_m}{\partial \eta_i} = -p_i b_i \Phi(\rho_i) - p_m b_m \Phi(\rho_m), \tag{4.62}$$

$$\frac{\partial \ddot{H}_{\eta_i}}{\partial \eta_j} = \frac{\partial A_i}{\partial \eta_j} - \frac{\partial A_m}{\partial \eta_j} = -p_m b_m \Phi(\rho_m), \text{ if } i \neq j. \tag{4.63}$$

We arrived then to the expected result.

□

Using the fact that $p_i < 0$ since it is solution of an homogeneous linear differential equation and its final value is negative, and that $b_i \geq 0$; from Theorem 4.5.2 we can obtain sufficient conditions for having (not having) Goh's condition:

Corollary 4.5.3. *(i) If*

$$\Phi(\rho_i) \leq 0, \ i = 1, \dots, m, \tag{4.64}$$

then Goh's condition is satified.

(ii) If

$$b_i > 0, \Phi(\rho_i) > 0, \ i = 1, \dots, m, \tag{4.65}$$

then Goh's condition is not satified, and hence, there are no singular arcs.

Proof. If condition (i) is satisfied we will have that the matrix in Goh-Legendre-Clebsch condition is a sum of two semidefinite positive matrix, and thus, it will be semidefinite positive. That proves item (i).

For item (ii) we use the same argument.

□

Remark 4.5.4. *When we eliminated η_m, we could have chosen any other control η_i, and then there are other sufficient conditions for having (not having) Goh-Legendre-Clebsch condition by putting another η_i in the place of η_m.*

Conclusion

In this thesis we studied two types of optimal control problems. The first class was governed by a control-affine system and the second type has a partially control-affine dynamics.

In Chapter 1 we gave a pair of second order necessary and sufficient conditions for a bang-singular solution for a control-affine problem. The sufficient condition is restricted to the scalar control case, and it is the main result of the chapter. These necessary and sufficient conditions are close in the sense that, to pass from one to the other, one has to strengthen a non-negativity inequality transforming it into a coercivity condition. We checked the sufficient condition in a simple example.

In Chapter 2 we suggested a shooting algorithm for problems having control-affine dynamics. The algorithm can be applied to compute an optimal bang-singular solution. We provided a sufficient condition for this algorithm to be convergent and we related this condition to a second order sufficient condition of optimality for totally singular extremals. The sufficient condition obtained in Chapter 1 is an extension of the latter that applies to bang-singular solutions. The convergent result is the main theorem of the chapter. The algorithm was tested in three simple problems. In some cases, we could established the stability of the optimal local solution under small perturbations of the data.

In Chapter 3 we studied a problem governed by a partially control-affine system. We provided a set of necessary and sufficient conditions for a totally singular solution. No control constraints were considered. We proposed a shooting algorithm, and we proved that the just mentioned sufficient condition guarantees the local quadratic convergence of this shooting.

Finally, in Chapter 4, we exhibited a model of electricity production that responds to a partially control-affine structure, after an appropriated change of variables. It is then a motivation of the investigation of Chapter 3. This problem has control bounds. We studied the possible occurrence of a singular arc by means of second order conditions established by Goh in [67, 69].

Bibliography

[1] A. A. Agrachev. A necessary condition for second order optimality in the general nonlinear case. *Mat. Sb. (N.S.)*, 102(144)(4):551–568, 632, 1977.

[2] A.A. Agrachev and R.V. Gamkrelidze. Second order optimality principle for a time- optimal problem. *Math. USSR, Sbornik*, 100(4), 1976.

[3] A.A. Agrachev and Y.L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[4] A.A. Agrachev, G. Stefani, and P.L. Zezza. Strong optimality for a bang-bang trajectory. *SIAM J. Control and Optimization*, 41:991–1014, 2002.

[5] V.M. Alekseev, V.M. Tikhomirov, and S.V. Fomin. *Optimal Control.* Nauka, Moscow, 1979. [in Russian].

[6] G.M. Aly. The computation of optimal singular control. *International Journal of Control*, 28(5):681–688, 1978.

[7] M.S. Aronna. Partially affine control problems: second order conditions and a well-posed shooting algorithm. *INRIA Research Rapport Nr. 7764*, October 2011.

[8] M.S. Aronna, J. F. Bonnans, A. V. Dmitruk, and P.A. Lotito. Quadratic conditions for bang-singular extremals. *Numerical Algebra, Control and Optimization, special issue in honor of Helmut Maurer, and Rapport de Recherche INRIA Number 7764.* [to appear in 2012].

[9] M.S. Aronna, J.F. Bonnans, and P.A. Lotito. Continuous time optimal hydrothermal scheduling. *Proceedings of the International Conference on Engineering Optimization*, 2008.

[10] M.S. Aronna, J.F. Bonnans, and Martinon P. A well-posed shooting algorithm for optimal control problems with singular arcs. *INRIA Research Rapport Nr. 7763*, October 2011.

[11] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems and Control: Foundations and Applications. Birkhäuser, Boston, 1997.

[12] D.J. Bell and D.H. Jacobson. *Singular Optimal Control Problems*. Academic Press, 1975.

[13] R. Bellman. *Dynamic programming*. Princeton University Press, Princeton, 1961.

[14] J.T. Betts. Survey of numerical methods for trajectory optimization. *AIAA J. of Guidance, Control and Dynamics*, 21:193–207, 1998.

[15] L.T. Biegler. An overview of simultaneous strategies for dynamic optimization. *Chemical Engineering and Processing: Process Intensification*, 46(11):1043 – 1053, 2007. Special Issue on Process Optimization and Control in Chemical Engineering and Processing.

[16] L.T. Biegler. *Nonlinear programming*, volume 10 of *MOS-SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2010. Concepts, algorithms, and applications to chemical processes.

[17] L.T. Biegler and A. Cervantes. Optimization strategies for dynamic systems. *Encyclopedia of Optimization*, 2000.

[18] F. Bonnans, J. Laurent-Varin, P. Martinon, and E. Trélat. Numerical study of optimal trajectories with singular arcs for an Ariane 5 launcher. *Journal of Guidance, Control, and Dynamics*, 32(1):51–55, 2009.

[19] J.F. Bonnans. *Optimisation continue*. Dunod, 2006.

[20] J.F. Bonnans and A. Hermant. Revisiting the analysis of optimal control problems with several state constraints. *Control Cybernet.*, 38(4A):1021–1052, 2009.

[21] B. Bonnard, J.-B. Caillau, and E. Trélat. Geometric optimal control of elliptic Keplerian orbits. *Discrete Contin. Dyn. Syst. Ser. B*, 5(4):929–956 (electronic), 2005.

[22] B. Bonnard, J.-B. Caillau, and Emmanuel Trélat. Cotcot: short reference manual. ENSEEIHT-IRIT Technical report RT/APO/05/1, 2005.

[23] B. Bonnard, J.B. Caillau, and E. Trélat. Second order optimality conditions in the smooth case and applications in optimal control. *ESAIM Control Optim. Calc. Var.*, 13(2):207–236 (electronic), 2007.

[24] B. Bonnard and M. Chyba. *Singular trajectories and their role in control theory*, volume 40 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2003.

[25] B. Bonnard and I. Kupka. Théorie des singularités de l'application entrée/sortie et optimalité des trajectoires singulières dans le problème du temps minimal. *Forum Math.*, 5(2):111–159, 1993.

[26] H. J. Bortolossi, M. V. Pereira, and C. Tomei. Optimal hydrothermal scheduling with variable production coefficient. *Math. Methods Oper. Res.*, 55(1):11–36, 2002.

[27] H. J. Bortolossi, M. V. Pereira, and C. Tomei. Optimal hydrothermal scheduling with variable production coefficient. *Mathematical Methods of Operations Research*, 55(1):11–36, 2002.

[28] U. Boscain and B. Piccoli. *Optimal syntheses for control systems on 2-D manifolds*. Springer-Verlag, Berlin, 2004.

[29] H. Brézis. *Analyse fonctionnelle*. Masson, Paris, 1983.

[30] A.E. Bryson, Jr. and Y.C. Ho. *Applied optimal control*. Hemisphere Publishing Corp. Washington, D. C., 1975. Optimization, estimation, and control, Revised printing.

[31] R. Bulirsch. Die mehrzielmethode zur numerischen lösung von nichtlinearen randwertproblemen und aufgaben der optimalen steuerung. Technical report, Carl-Cranz-Gesellschaft, Deutsches Zentrum für Luft- und Raumfahrt (DLR), Oberpfaffenhofen, Germany, 1971.

[32] Y. Chitour, F. Jean, and E. Trélat. Genericity results for singular curves. *J. Differential Geom.*, 73(1):45–73, 2006.

[33] Y. Chitour, F. Jean, and E. Trélat. Singular trajectories of control-affine systems. *SIAM J. Control Optim.*, 47(2):1078–1095, 2008.

[34] C.W. Clark. *Mathematical Bioeconomics.* John Wiley & Sons, 1976.

[35] R. Cominetti and J.-Penot. Tangent sets of order one and two to the positive cones of some functional spaces. *Applied Mathematics and Optimization*, (36):291–312, 1997.

[36] M.G. Crandall, L.C. Evans, and P.-L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc*, 282:487–502, 1984.

[37] M.G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton Jacobi equations. *Bull. American Mathematical Society*, 277:1–42, 1983.

[38] J.E. Dennis. Nonlinear least-squares. *In: D. Jacobs, Editor, The State of the Art in Numerical Analysis*, pages 269–312, 1977.

[39] J.E. Dennis, D.M. Gay, and R. E. Welsch. An adaptive nonlinear least-squares algorithm. *ACM Trans. Math. Softw.*, 7:348–368, 1981.

[40] A.V. Dmitruk. Quadratic conditions for a weak minimum for singular regimes in optimal control problems. *Soviet Math. Doklady*, 18(2), 1977.

[41] A.V. Dmitruk. Quadratic conditions for a Pontryagin minimum in an optimal control problems, linear in the control, with a constraint on the control. *Soviet Math. Doklady*, 28(2):364–368, 1983.

[42] A.V. Dmitruk. Jacobi-type conditions for the problem of Bolza with inequalities. *Math. Notes*, 35(6):427–435, 1984.

[43] A.V. Dmitruk. Quadratic order conditions for a Pontryagin minimum in an optimal control problem linear in the control. *Math. USSR Izvestiya*, 28:275–303, 1987.

[44] A.V. Dmitruk. Quadratic order conditions of a local minimum for singular extremals in a general optimal control problem. *Proceedings of Symposia in Pure Mathematics*, 64:163–198, 1999.

[45] A.V. Dmitruk. Jacobi type conditions for singular extremals. *Control & Cybernetics*, 37(2):285–306, 2008.

[46] A.V. Dmitruk. Quadratic order optimality conditions for extremals completely singular in part of controls. *Operations Research Proceedings 2010, Selected Papers of the Annual International Conference of the German Operations Research Society*, 0:341–346, 2011.

[47] A.V. Dmitruk and K.K. Shishov. Analysis of a quadratic functional with a partly singular Legendre condition. *Moscow University Comput. Math. and Cybernetics*, 34(1):16–25, 2010.

[48] L. E. Dubins. On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. *Amer. J. Math.*, 79:497–516, 1957.

[49] A.Ya. Dubovitskii and A.A. Milyutin. Extremum problems with constraints. *USSR Comp. Math. and Math. Phys.*, 5(3):1–80, 1965.

[50] N. Dunford and J. Schwartz. *Linear operators, Vol I.* Interscience, New York, 1958, 1963.

[51] V.A. Dykhta. Conditions of a local minimum for singular modes in systems with linear control. *Automatics and Remote Control*, 42(12):1583–1587, 1982.

[52] U. Felgenhauer. On stability of bang-bang type controls. *SIAM J. Control Optim.*, 41(6):1843–1867 (electronic), 2003.

[53] U. Felgenhauer. Optimality and sensitivity for semilinear bang-bang type optimal control problems. *Int. J. Appl. Math. Comput. Sci.*, 14(4):447–454, 2004.

[54] U. Felgenhauer. Optimality properties of controls with bang-bang components in problems with semilinear state equation. *Control Cybernet.*, 34(3):763–785, 2005.

[55] U. Felgenhauer. Controllability and stability for problems with bang-singular-bang optimal control. 2011. [submitted].

[56] U. Felgenhauer. Structural stability investigation of bang-singular-bang optimal controls. *Journal of Optimization Theory and Applications*, 2011. [published as 'online first'].

[57] E.C. Finardi, E.L. da Silva, and C.A. Sagastizábal. Solving the unit commitment problem of hydropower plants via Lagrangian relaxation and sequential quadratic programming. *Computational & Applied Mathematics*, 24(3):317–341, 2005.

[58] W.H. Fleming and H.M. Soner. *Controlled Markov processes and viscosity solutions*. Springer, New York, 1993.

[59] R. Fletcher. *Practical methods of optimization. Vol. 1.* John Wiley & Sons Ltd., Chichester, 1980. Unconstrained optimization, A Wiley-Interscience Publication.

[60] G. Fraser-Andrews. Finding candidate singular optimal controls: a state of the art survey. *J. Optim. Theory Appl.*, 60(2):173–190, 1989.

[61] A.T. Fuller. Study of an optimum non-linear control system. *J. of Electronics and Control*, 15:63–71, 1963.

[62] R. Gabasov and F.M. Kirillova. High-order necessary conditions for optimality. *J. SIAM Control*, 10:127–168, 1972.

[63] P. Gajardo, H. Ramírez C., and A. Rapaport. Minimal time sequential batch reactors with bounded and impulse controls for one or more species. *SIAM J. Control Optim.*, 47(6):2827–2856, 2008.

[64] B.S. Garbow, K.E. Hillstrom, and J.J. More. *User Guide for Minpack-1.* National Argonne Laboratory, Illinois, 1980.

[65] J. Gergaud and P. Martinon. An application of PL continuation methods to singular arcs problems. In A. Seeger, editor, *Recent Advances in Optimization*, volume 563 of *Lectures Notes in Economics and Mathematical Systems*, pages 163–186. Springer-Verlag, 2006.

[66] R.H. Goddard. *A Method of Reaching Extreme Altitudes*, volume 71(2) of *Smithsonian Miscellaneous Collections*. Smithsonian institution, City of Washington, 1919.

[67] B.S. Goh. Necessary conditions for singular extremals involving multiple control variables. *J. SIAM Control*, 4:716–731, 1966.

[68] B.S. Goh. The second variation for the singular Bolza problem. *J. SIAM Control*, 4(2):309–325, 1966.

[69] B.S. Goh. Optimal singular rocket and aircraft trajectories. In *Control and Decision Conference, 2008. CCDC 2008*, pages 1531 –1536, 2008.

[70] T.R. Goodman and G.N. Lance. The numerical integration of two-point boundary value problems. *Math. Tables Aids Comput.*, 10:82–86, 1956.

[71] V. I. Gurman. *Degenerate optimal control problems.* Izdat. "Nauka", Moscow, 1977. Series in Theoretical Foundations of Engineering Cybernetics.

[72] M.R. Hestenes. Applications of the theory of quadratic forms in Hilbert space to the calculus of variations. *Pacific J. Math.*, 1(4):525–581, 1951.

[73] A. Hoffman. On approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureau of Standards, Section B, Mathematical Sciences*, 49:263–265, 1952.

[74] A. D. Ioffe and V. M. Tihomirov. *Theory of extremal problems*, volume 6 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, 1979. Translated from the Russian by Karol Makowski.

[75] D. H. Jacobson. Sufficient conditions for nonnegativity of the second variation in singular and nonsingular control problems. *SIAM J. Control*, 8:403–423, 1970.

[76] D. H. Jacobson and Speyer J.L. Necessary and sufficient conditions for optimality for singular control problems: A limit approach. *J. Math. Anal. Appl.*, 34:239–266, 1971.

[77] D.H. Jacobson, M.M. Lele, and J.L. Speyer. New necessary conditions of optimality for control problems with state-variable inequality constraints. *Journal of Mathematical Analysis and Applications*, 35:255–284, 1971.

[78] H.B. Keller. *Numerical methods for two-point boundary-value problems.* Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1968.

[79] H.J. Kelley. A second variation test for singular extremals. *AIAA Journal*, 2:1380–1382, 1964.

[80] H.J. Kelley. A transformation approach to singular subarcs in optimal trajectory and control problems. *J. Soc. Indust. Appl. Math. Ser. A Control*, 2:234–240, 1964.

[81] H.J. Kelley, R.E. Kopp, and H.G. Moyer. Singular extremals. In *Topics in Optimization*, pages 63–101. Academic Press, New York, 1967.

[82] R.E. Kopp and H.G. Moyer. Necessary conditions for singular extremals. *AIAA Journal*, 3:1439–1444, 1965.

[83] A.J. Krener. The high order maximal principle and its application to singular extremals. *SIAM J. on Control*, 15(2):256–293, 1977.

[84] S. Kurcyusz and J. Zowe. Regularity and stability for the mathematical programming problem in Banach spaces. In *Applied Mathematics and Optimization*, pages 49–62. Springer, New York, 1979.

[85] U. Ledzewicz and H. Schättler. Multi-input optimal control problems for combined tumor anti-angiogenic and radiotherapy treatments. *Journal of Optimization Theory and Applications.* Published as 'online first' [to appear 2012].

[86] E. S. Levitin, A. A. Milyutin, and N. P. Osmolovskiĭ. Higher order conditions for local minima in problems with constraints. *Uspekhi Mat. Nauk*, 33(6(204)):85–148, 272, 1978.

[87] K. Malanowski and H. Maurer. Sensitivity analysis for parametric control problems with control-state constraints. *Computational Optimization and Applications*, 5:253–283, 1996.

[88] A.A. Markov. Some examples of the solution of a special kind of problem on greatest and least quantities. *Soobshch. Karkovsk. Mat. Obshch.*, 1:250–276, 1887. (in Russian).

[89] P. Martinon. Numerical resolution of optimal control problems by a piecewise linear continuation method. *PhD thesis. Institut National Polytechnique de Toulouse*, 2005. http://www.cmap.polytechnique.fr/ martinon/docs/Martinon-Thesis.pdf.

[90] P. Martinon, J.F. Bonnans, and E. Trélat. Singular arcs in the generalized Goddard's problem. *J. Optim. Theory Appl.*, 139(2):439–461, 2008.

[91] P. Martinon and J. Gergaud. Shoot2.0: An indirect grid shooting package for optimal control problems, with switching handling and embedded continuation. Technical report, Inria Saclay, 2011. RR-7380.

[92] H. Maurer. Numerical solution of singular control problems using multiple shooting techniques. *J. of Optimization theory and applications*, 18(2):235–257, 1976.

[93] H. Maurer and N.P. Osmolovskii. Second order optimality conditions for bang-bang control problems. *Control and Cybernetics*, 32:555–584, 2003.

[94] H. Maurer and N.P. Osmolovskii. Second order sufficient conditions for time-optimal bang-bang control. *SIAM J. Control Optim.*, 42:2239–2263, 2003.

[95] H. Maurer and N.P. Osmolovskii. Second order sufficient optimality conditions for a control problem with continuous and bang-bang control components: Riccati approach. In *System modeling and optimization*, volume 312 of *IFIP Adv. Inf. Commun. Technol.*, pages 411–429. Springer, Berlin, 2009.

[96] A.A. Milyutin. On quadratic conditions for an extremum in smooth problems with a finite-dimensional range. *Methods of the Theory of Extremal Problems in Economics*, pages 138–177, 1981.

[97] A.A. Milyutin and N.P. Osmolovskii. Calculus of variations and optimal control. *American Mathematical Society*, 1998.

[98] D.D. Morrison, J.D. Riley, and J.F. Zancanaro. Multiple shooting method for two-point boundary value problems. *Comm. ACM*, 5:613–614, 1962.

[99] H.G. Moyer. Sufficient conditions for a strong minimum in singular control problems. *SIAM J. Control*, 11:620–636, 1973.

[100] H. J. Oberle and K. Taubert. Existence and multiple solutions of the minimum-fuel orbit transfer problem. *J. Optim. Theory Appl.*, 95(2):243–262, 1997.

[101] H.J. Oberle. Numerische Behandlung singulärer Steuerungen mit der Mehrzielmethode am Beispiel der Klimatisierung von Sonnenhäusern. *PhD thesis. Technische Universität München*, 1977.

[102] H.J. Oberle. Numerical computation of singular control problems with application to optimal heating and cooling by solar energy. *Appl. Math. Optim.*, 5(4):297–314, 1979.

[103] H.J. Oberle. Numerical computation of singular control functions in trajectory optimizatio. *Journal of Guidance, Control and Dynamics*, 13:153–159, 1990.

[104] N. P. Osmolovskii and H. Maurer. Equivalence of second order optimality conditions for bang-bang control problems. I. Main results. *Control Cybernet.*, 34(3):927–950, 2005.

[105] Z. Páles and V. Zeidan. First- and second-order necessary conditions for control problems with constraints. *Transactions of the American Mathematical Society*, 346:421–453, 1994.

[106] H. J. Pesch. A practical guide to the solution of real-life optimal control problems. *Control and Cybernetics*, 23:7–60, 1994.

[107] L. Poggiolini and M. Spadini. Strong local optimality for a bang-bang trajectory in a Mayer problem. *SIAM J. Control Optimization*, 49:140–161, 2011.

[108] L. Poggiolini and G. Stefani. On second order sufficient conditions for a bang-singular arc. *Proceedings of science - SISSA*, 2005.

[109] L. Poggiolini and G. Stefani. Minimum time optimality of a partially singular arc: second order conditions. In *Lagrangian and Hamiltonian methods for nonlinear control 2006*, volume 366 of *Lecture Notes in Control and Inform. Sci.*, pages 281–291. Springer, Berlin, 2007.

[110] L. Poggiolini and G. Stefani. Sufficient optimality conditions for a bang-singular extremal in the minimum time problem. *Control Cybernet.*, 37(2):469–490, 2008.

[111] L. Pontryagin, V. Boltyanski, R. Gamkrelidze, and E. Michtchenko. *The Mathematical Theory of Optimal Processes*. Wiley Interscience, New York, 1962.

[112] H.M. Robbins. A generalized Legendre-Clebsch condition for the singular case of optimal control. *IBM J. of Research and Development*, 11:361–372, 1967.

[113] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1970.

[114] W. Rudin. *Real and complex analysis*. Mc Graw-Hill, New York, 1987.

[115] A. V. Sarychev. First- and second-order sufficient optimality conditions for bang-bang controls. *SIAM J. Control Optim.*, 35(1):315–340, 1997.

[116] A.V. Sarychev. Index of second variation of a control system. *Mat. Sb. (N.S.)*, 113(155)(3(11)):464–486, 496, 1980.

[117] H. Schättler. A local feedback synthesis of time-optimal stabilizing controls in dimension three. *Math. Control Signals Systems*, 4(3):293–313, 1991.

[118] H. Schättler and M. Jankovic. A synthesis of time-optimal controls in the presence of saturated singular arcs. *Forum Math.*, 5(3):203–241, 1993.

[119] H. Seywald and E.M. Cliff. Goddard problem in presence of a dynamic pressure limit. *Journal of Guidance, Control, and Dynamics*, 16(4):776–781, 1993.

[120] P. Souères and J.-P. Laumond. Shortest paths synthesis for a car-like robot. *IEEE Trans. Automat. Control*, 41(5):672–688, 1996.

[121] H. J. Sussmann. Regular synthesis for time-optimal control of single-input real analytic systems in the plane. *SIAM J. Control Optim.*, 25(5):1145–1162, 1987.

[122] H. J. Sussmann. The structure of time-optimal trajectories for single-input systems in the plane: the C^∞ nonsingular case. *SIAM J. Control Optim.*, 25(2):433–465, 1987.

[123] H. J. Sussmann. The structure of time-optimal trajectories for single-input systems in the plane: the general real analytic case. *SIAM J. Control Optim.*, 25(4):868–904, 1987.

[124] H.J. Sussmann and G. Tang. Shortest paths for the reeds-shepp car: A worked out example of the use of geometric techniques in nonlinear optimal control. Technical report, Rutgers Center for Systems and Control Technical Report 91-10, 1991.

[125] E. Trélat. *Contrôle optimal: théorie et applications.* Vuibert, Collections Mathématiques Concrètes, 2005.

[126] O. von Stryk and R. Bulirsch. Direct and indirect methods for trajectory optimization. *Ann. Oper. Res.*, 37(1-4):357–373, 1992. Nonlinear methods in economic dynamics and optimal control (Vienna, 1990).

[127] G. Vossen. Switching time optimization for bang-bang and singular controls. *J. Optim. Theory Appl.*, 144(2):409–429, 2010.

[128] V. Zeidan. First and second order sufficient conditions for optimal control and the calculus of variations. *Appl. Math. Optim.*, 11(3):209–226, 1984.

[129] V. Zeidan. Sufficiency criteria via focal points and via coupled points. *SIAM J. Control Optim.*, 30(1):82–98, 1992.

[130] V. Zeidan and P. Zezza. The conjugate point condition for smooth control sets. *J. Math. Anal. Appl.*, 132(2):572–589, 1988.

[131] V. Zeidan and P. Zezza. Necessary conditions for optimal control problems: conjugate points. *SIAM J. Control Optim.*, 26(3):592–608, 1988.