Algèbres de polynômes bornés sur ensembles semi-algébriques non bornés

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Algebras of bounded polynomials on unbounded semialgebraic sets

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Contents

Introduction

The main topic of the thesis is a study of algebras of polynomials which are bounded on a given unbounded semialgebraic set. In particular to determine when a polynomial is bounded on an unbounded semi-algebraic set.

At the origin of this thesis the motivation was to attempt a generalisation to the case of unbounded sets of a celebrated theorem of Schmüdgen [Sm, 1991]. It states that every positive polynomial on a compact basic semialgebraic set can be written as a sum squares of polynomials multiplied by products of polynomials defining the semialgebraic set. In the proof of Schmüdgen the assumption of compactness is essential. He obtained this result when solving K-moment problem for compact semialgebraic sets and the proof makes essential use of functional analysis methods (spectral measures). Schmüdgen's Positivstellensatz gives a possibility to construct an algorithm to compute lower bound of a polynomial on a compact semialgebraic set. The condition that f belongs to an appropriate preordering is used in programming. Since we obtained a version of the theorem for bounded polynomials on an unbounded semialgebraic set S, we felt that from the point of view of applications it was vital to decide efficiently whether f is bounded on S.

Optimization of polynomials (i.e. finding lower or upper bounds) on semialgebraic sets is an important and challenging problem, both theoretically and practically. Nowadays there is a a very intensive activity in this direction, based mostly on sums of squares representations and more generally on Real Algebra methods. There is a number of books and survey articles devoted to various aspects of this subject, for instance [L], [Lt] and [PaS].

In order to extend the method of Schmüdgen to the case of unbounded semialgebraic sets one can consider the algebra of bounded polynomials on such sets. Actually this is partially achieved in the last chapter of this thesis, having Schweighofer's beautiful paper [Sw] as one of the inspirations for the undertaken study. To this aim it is important to understand the structure of algebras of polynomials which are bounded on a given unbounded semialgebraic set. Surprisingly this problem has been studied only recently. Actually in the PhD thesis of D. Plaumann (Konstanz 2008), supervised by Professor C. Scheiderer, among other results it was proved that for regular subsets of $\mathbb{R}^2$ (i.e. sets equal to the closure of their interior) those algebras are finitely generated. Recently Krug in [Krug] has constructed an example of a semialgebraic regular subset of $\mathbb{R}^3$ on which the algebra of bounded polynomials is not finitely generated. However, this set

is not basic closed, so the question of the finite generation remains open for this type of sets.

Let S be a subset of $\mathbb{R}^n$. Denote by

$$\mathcal{A}(S) = \{f \in \mathbb{R}[X] \mid f \textit{ is bounded on } S\}$$

the algebra of bounded polynomials on S. The set $\mathcal{A}(S)$ is a subring of $\mathbb{R}[X]$ and an algebra over $\mathbb{R}$. Note that if S is bounded, then $\mathcal{A}(S) = \mathbb{R}[X]$. Otherwise, the algebra $\mathcal{A}(S)$ is a proper subring of the ring of polynomials.

In this thesis we address several problems concerning algebras of bounded polynomials. First of all we tackle the problem of deciding the boundedness of a polynomial on a set. We achieve it for polynomials in two variables for any semialgebraic set in Section 6.1, using methods developed in Section 2. Also in the latter section we give a method of finding generators of $\mathcal{A}(S)$ for a large class of semialgebraic subsets of $\mathbb{R}^2$. In Section 3 we have established a surprising relation between complex bifurcation values of a polynomial f and the stability of the family of algebras $\mathcal{A}(S_c)$, where $S_c = \{(x, y) \in \mathbb{R}^2 |\ f(x, y) \leq c\}$.

Throughout the thesis instead of Real Algebra methods we preferred to use more geometric arguments, so we have avoided using standard language of Real Algebra. Since the problems we are dealing with can be stated quite plainly, we tried to use as simple and straightforward methods as possible and we hope that we have succeeded.

To simplify the study of algebras of bounded polynomials on a semialgebraic set S, we will consider some subsets of S which we will call tentacles. A set M is a tentacle of the set S if $M \setminus B(0, R)$ is connected for any $R > 0$ and M is one of the unbounded sets in the decomposition

$$S = K \cup M_1 \cup \ldots \cup M_l,$$

where K is compact, $l \in \mathbb{N}_0$ and $M_1, \ldots, M_l$ are closed in S and pairwise disjoint tentacle sets (see Theorem 1.19). Moreover, if $l = 0$, then $\mathcal{A}(S) = \mathbb{R}[X]$. Otherwise, we have

$$\mathcal{A}(S) = \bigcap_{i=1}^{l} \mathcal{A}(M_i).$$

A starting point for the results in Section 2 is the observation that if we consider semialgebraic subsets of $\mathbb{R}^2$, we can assume that a tentacle M is of the form

$$\{(x, y) \in \mathbb{R}^2 |\quad \beta_1(y) \leq x \leq \beta_2(y),\ y \geq R\},$$

where R is a positive real number and $\beta_1(1/Y), \beta_2(1/Y)$ are Puiseux series which parametrise semialgebraic curves. If a tentacle M of the set S is not of the above form (up to a linear change of coordinates), then $\mathcal{A}(M) = \mathbb{R}$, which implies that the algebra $\mathcal{A}(S)$ is trivial. Thus throughout Section 2 we consider a semialgebraic set M of the above form.

In Theorem 2.4 we prove that if $\mathrm{ord}_\infty\beta_1 \neq \mathrm{ord}_\infty\beta_2$, then

$$\mathcal{A}(M) = \mathbb{R}[X^iY^d|\ d \leq i\alpha],$$

where we put $\mathrm{ord}_\infty\beta = \mathrm{ord}\beta(1/Y)$ and $\alpha = \min\{\mathrm{ord}_\infty\beta_1, \mathrm{ord}_\infty\beta_2\}$. Hence the algebra of bounded polynomials in this instance is generated by monomials. The main point of the proof is the comparison of the supremum of a polynomial with its appropriate coefficients. This works whenever the width of the tentacle is essentially more than its distance from the axis.

In the second part of Section 2 we consider the case when $\mathrm{ord}_\infty\beta_1 = \mathrm{ord}_\infty\beta_2$ and $\beta_1 \neq \beta_2$. We introduce a Puiseux series β with a finite expansion, which can be computed in a finite number of steps from β_1 and β_2 (see Proposition 2.15). By identification of β with the y axis we prove Theorem 2.16 whence it follows that

$$\mathcal{A}(M) = \mathbb{R}[X,Y] \cap \mathbb{R}\left[\frac{1}{Y^{1/q}}, (X-\beta)^iY^d \mid d \leq i\eta\right],$$

where $\eta = \min\{\mathrm{ord}_\infty(\beta_1 - \beta), \mathrm{ord}_\infty(\beta_2 - \beta)\}$. We would like to note that it gives a straightforward way of checking whether a polynomial f is bounded on M. Indeed, it is a simple task to write any polynomial in terms of the above ring of bounded polynomials with Puiseux coefficients (see Proposition 2.13), afterwards it suffices to check the exponents of Y in such a representation. Note that the algebra $\mathcal{A}(S)$ need not be generated by monomials (or isomorphic to such an algebra). We would like to add that the introduction of polynomials with Puiseux coefficients lets us treat all algebras of bounded polynomials on tentacle sets as if they were generated by monomials, which facilitates their study (compare Section 5). Moreover, usually it is quite difficult to determine whether a polynomial belongs to a subring given by fixed polynomials, whereas in the case of this extended ring the representations of f are obtained after simple symbolic computation (note that β has a finite expansion).

In Section 3 we consider semialgebraic sets of the form

$$S_c = \{(x,y) \in \mathbb{R}^2|\ f(x,y) \leq c\},$$

where f is a polynomial and c is a real number. The main result of this section is Theorem 3.5 on stability of algebras $\mathcal{A}(S_c)$. Namely, we prove that the algebras of bounded polynomials on S_c are, up to some point, insensitive to the change of the parameter c. More precisely, for any $c < \tilde{c}$ we have

$$\mathcal{A}(S_c) = \mathcal{A}(S_{\tilde{c}})$$

as long as $[c,\tilde{c}] \cap B_{\mathbb{C}}(f) = \emptyset$. The set of bifurcation values $B_{\mathbb{C}}(f)$ is defined on page 44. Its most notable feature is that it is finite and can be computed for any polynomial f in two variables. The main tool in the proof of theorem on stability is a parametric version of Puiseux theorem and results of Section 2. We hope that this approach illustrates the connection between the bounded polynomials and the geometry of the fibres of f,

and might shed a new light on properties of bifurcation values as well as properness of polynomial mappings. In $\mathbb{R}^2$ it would be also interesting to study the stability of sets described by more than one polynomial inequality. This still leaves the case of higher dimensions as an open problem. Although in simple cases (for example for sets described by monomial inequalities as in Theorem 5.9) it is easy to see that they are insensitive to the change of parameters, in general the problem does not seem easy to solve.

Section 4 is devoted to the study of algebras of polynomials bounded on sets in $\mathbb{R}^n$ for arbitrary n. The first part deals with a special type of sets, which we call weighted tentacles and can be viewed as a "uniform deformation" of a lower-dimensional set along the y axis. Namely, suppose the set $S \subset \mathbb{R}^n$ has a nonempty interior. Consider a set

$$M = \{(\beta_1(y)x_1, \ldots, \beta_n(y)x_n, y) \in \mathbb{R}^{n+1} |\ x \in S,\ y \geq R\}$$

where $R > 0$ and $\beta_1(1/Y), \ldots, \beta_n(1/Y)$ are Puiseux series such that $\beta_i(y)$ are convergent and have constant positive or negative sign for $y \in [R, \infty)$.

Theorem 4.4 states that if we assume that $\mathcal{A}(S)$ is generated by monomials, then

$$\mathcal{A}(M) = \mathcal{A}(S)[Y] \cap \mathbb{R}[X^\alpha Y^d |\ \sum_{i=1}^{n} \alpha_i \lambda_i \geq d],$$

where $\lambda \in \mathbb{Q}^n$ is the tuple of weighs of the set M and we write for short $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Let us note that any semialgebraic subset of the real line has either a trivial algebra or the algebra is equal to $\mathbb{R}[X]$ (hence generated by the monomial X). Therefore the above statement generalizes Theorem 2.1 from Section 2. Moreover, in some instances in higher dimensions it gives us a practical possibility of deciding the generators of the algebra of bounded polynomials (as is the case with some examples in Section 6). The method of proof is essentially the same as in Section 2.1 i.e. uses equivalence of norms in finite dimensional spaces. Theorem 4.4 generalises and extends results of T. Netzer in [Net] who deals only with projections of such sets on the first n coordinates and have been attained using completely different arguments. Hence in [Net] the dimension of S and and the tentacle is the same, moreover S is assumed compact, of which assumptions he makes essential use.

In the last part of Section 4 we give an alternative proof (based on the nonproperness set of Jelonek) of a fact already shown by D. Plaumann and C. Scheiderer in [PlSd] that if S is an unbounded subset of a proper semialgebraic set in $\mathbb{R}^n$, then the algebra $\mathcal{A}(S)$ cannot be finitely generated. An important ingredient of the proof is Lemma 4.20 which gives a nice correlation between an algebraic property of an ideal and the geometric property of a mapping. Also in Section 4 we give an example (Example 4.2 of a semi-analytic set in $\mathbb{R}^2$ with a nonempty interior which has an infinitely generated algebra of bounded polynomials), which is interesting from the point of view of the results of Plaumann and Scheiderer on the finite generation of $\mathcal{A}(S)$ for regular semialgebraic sets in $\mathbb{R}^2$.

Because of the fact that algebras generated by monomials arise naturally in our study, we turn to their properties in Section 5. Using classical methods of convex geometry we

show the duality between convex cones of exponents of monomials and the algebras of bounded polynomials on intersections of sets given by appropriate monomial inequalities (Theorem 5.9). In particular, we would like to remark that algebras from Theorems 2.4 and 4.4 are generated by monomials. At the end of the section we compute the minimal number of monomial generators for tentacle sets on the real plane. In this context one easily sees that computation of the number of elements of the basis for an intersection and a union of a finite collection of algebras generated by monomials is also only a combinatorial task.

Suppose again that $S \subset \mathbb{R}^2$. From Theorem 6.2 on testing curves we get as a corollary that a polynomial f is bounded on S if and only if it is bounded on a finite number of generic representatives of a family of curves which depends only on the set S. The proof of this fact based on the results from Section 2 constitutes the first part of Section 6.

In Section 6.2 we present a version of Schmüdgen's Positivstellensatz for bounded polynomials. Take a basic closed semialgebraic unbounded set

$$S = \{x \in \mathbb{R}^n |\ g_1(x) \geq 0,\ \ldots,\ g_w(x) \geq 0\}$$

where $g_1, \ldots, g_w$ are polynomials. A preordering associated with S is the set

$$T = \left\{ \sum_{\sigma \in \{0,1\}^w} s_\sigma g^\sigma |\quad s_\sigma \in \sum \mathbb{R}[X]^2 \right\}.$$

Take a polynomial f. Suppose that the polynomials g_i which describe S are bounded on S. Moreover, suppose that $S_\zeta \cap \overline{\zeta(S)}^{Zar} = \overline{\zeta(S)}$ (the definition of these sets can be found on page 83). Theorem 6.13 states in particular that if f is bounded and greater than some positive constant on S, then $f \in T$. In the less likely case when $S_\zeta = \zeta(S)$, it suffices that f is bounded and positive on S (see Theorem 6.10). In the proofs of both theorems we used a simple observation that a mapping with generators (or the basis) of an algebra of bounded polynomials as coordinates is, in some way, the "canonical" bounded mapping. Hence in Schmüdgen's Postivstellensatz we can try to substitute compactness of the set by the assumption of boundedness of the polynomials. Nevertheless, note that even in the compact case Theorems 6.10 and 6.13 introduce a property that if a given polynomial f and polynomials g_i which describe the set S lie in a certain subalgebra of $\mathbb{R}[X]$, then the representation of f can be expressed with sums of squares from this subalgebra (compare Example 6.11).

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1. Preliminaries

In Preliminaries we will present some basic notions which will be used throughout the thesis. As the main theme of this work lies in the scope of semialgebraic geometry, we will start with some facts about semialgebraic sets and Puiseux series with a special emphasis on those which parametrise semialgebraic curves. Afterwards we introduce the notion of generators of an algebra, which will be essential in the study of bounded polynomials on a set. Since a bounded polynomial on an unbounded set is necessarily nonproper, we will make some remarks in Section 1.4 on the properties of nonproper mappings. At the end of the Preliminaries we turn to crucial ideas of this thesis: algebras of bounded polynomials and tentacle sets.

Notations

By $\mathbb{R}$ we denote the field of real numbers, by $\mathbb{C}$ the field of complex numbers, by $\mathbb{N}$ the set of natural numbers (not including zero) and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. The set of real polynomials in variables $X_1, \ldots, X_n$ we denote by $\mathbb{R}[X_1, \ldots, X_n]$. For brevity we will often write X for the system of variables $X_1, \ldots, X_n$ and $\mathbb{R}[X]$ for $\mathbb{R}[X_1, \ldots, X_n]$. By convention, we will write uppercase X for variables and lowercase x for points. Every polynomial of the form $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ for $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_0$ will be called a monomial.

For any set S and $i \in \mathbb{N}$ by S^i we mean the Cartesian product $S \times \cdots \times S$ of i terms, whereas we put $S^0 = \emptyset$. Furthermore, if S is a subset of a topological space denote by $IntS$ the interior of the set S, by $\overline{S}$ its closure and by FrS its boundary. For any set $S \subset \mathbb{R}^n$ there exists a smallest algebraic set (with respect to the relation of inclusion) which contains S. We will call it the Zariski closure of S and we will denote it by $\overline{S}^{Zar}$.

We will consider $\mathbb{R}^n$ equipped with the standard euclidean norm $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we will write $dist(x, S) = \inf_{y \in S} \|x - y\|$. Let us denote by $S(x, r)$ and $B(x, r)$ a sphere and an open ball in $\mathbb{R}^n$, respectively, with a centre in $x \in \mathbb{R}^n$ and radius $r > 0$. By $[a, b]$, where $a, b \in \mathbb{R}$, we denote the closed segment $\{t \in \mathbb{R} |\ a \leq t \leq b\}$ if $a < b$, the set $\{a\}$ if $a = b$ and the empty set if $a > b$.

Let $\mathcal{I}$ be an ideal in $\mathbb{R}[X]$. We will denote by $V(\mathcal{I})$ the set $\{x \in \mathbb{R}^n |\ \forall_{f \in \mathcal{I}}\ f(x) = 0\}$. If $\mathcal{I} = (f_1, \ldots, f_k)$ then we will write $V(f_1, \ldots, f_k)$ instead of $V(\mathcal{I})$. For any set $V \subset \mathbb{R}^n$

we will also denote by $\mathcal{I}(V)$ the ideal $\{f \in \mathbb{R}[X] \mid \forall_{x\in V}\ f(x) = 0\}$. We will write $(f_1, \ldots, f_k)\mathbb{R}[X]$ when we want to make clear that $(f_1, \ldots, f_k)$ is an ideal in $\mathbb{R}[X]$.

Last but not least, we say that f has constant sign on U if it is positive, negative or constantly equal zero on U. When we write $f \neq 0$, we mean that the function is not constantly equal zero. If for $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$, the partial derivatives $\frac{\partial f}{\partial x_i}$ for $i = 1, \ldots, n$ are well-defined on U, then we write $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) : U \to \mathbb{R}^n$ and call it the gradient of the function f.

1.1. Semialgebraic sets

We will recall here some basic notions and properties concerning semialgebraic sets. More details and further information can be found for example in [BCR], [BR] or [PD].

We call a set $S \subset \mathbb{R}^n$ *semialgebraic*, if it is a finite union of intersections a finite number of sets of the forms

$$\{x \in \mathbb{R}^n |\ g(x) = 0\} \qquad or \qquad \{x \in \mathbb{R}^n |\ h(x) > 0\},$$

where g and h are arbitrary real polynomials in n variables. The class of semialgebraic sets is the smallest class of sets containing the class of algebraic sets which is closed under union, intersection, complement and projection.

We have the following well-known basic property of semialgebraic sets.

Property 1.1. *If S is semialgebraic, then $IntS$, $\overline{S}$, FrS and all connected components of S are semialgebraic.*

Note that for a semialgebraic set S we have $IntS$ is empty if and only if $\overline{S}^{Zar}$ is a proper algebraic subset.

We will say that a nonempty semialgebraic set S is *of dimension n* if n is the maximal natural number such that there exists a homeomorphism $\phi : \mathbb{R}^n \supset B(0,1) \to U$, where $U \subset S$ is open in S.

A set S is called *basic semialgebraic closed* if it is of the form

$$\{x \in \mathbb{R}^n |\ g_1(x) \geq 0, \ldots, g_k(x) \geq 0\}$$

for some $g_1, \ldots, g_k \in \mathbb{R}[X]$.

A mapping $F : \mathbb{R}^n \to \mathbb{R}^k$ is called *semialgebraic* if its graph is a semialgebraic set. If $k = 1$ then the semialgebraic mapping will be called a *semialgebraic function*.

We give below a formulation of a well-known statement that we will often use (see [BCR, Theorem 2.5.5] and [KOS]).

Theorem 1.2. (Curve Selection Lemma) *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and $x \in \mathbb{R}^n$ such that there exists a sequence in S converging to x. There exists a continuous semialgebraic mapping $\gamma : [0,1) \to \mathbb{R}^n$ such that $\gamma([0,1)) \subset S$ and $\lim_{t\to 1^-} \gamma(t) = x$. Moreover, if S is unbounded, then there exists a continuous semialgebraic mapping $\gamma : [0,1) \to \mathbb{R}^n$ such that $\gamma([0,1)) \subset S$ and $\lim_{t\to 1^-} \|\gamma(t)\| = \infty$.*

Semialgebraic functions in one variable have a finite number of points where they are not continuous and a finite number of intervals where they are monotone. Let us turn now to *semialgebraic curves* i.e. sets which are images of some continuous semialgebraic mapping in one variable. As a consequence

Property 1.3. *An intersection of two semialgebraic curves is a finite union of points and images of intervals by continuous semialgebraic mappings.*

1.2. Puiseux parametrizations

Let us introduce a symbol $T^{\frac{j}{q}}$ by following relations

$$T^{\frac{1}{1}} = T, \qquad (T^{\frac{1}{kq}})^k = T^{\frac{1}{q}}, \qquad (T^{\frac{p}{q}})^k = T^{\frac{kp}{q}}.$$

By a *Puiseux series* we mean a formal series β of the form

$$\beta = \sum_{j=m}^{\infty} b_j T^{\frac{j}{q}} \tag{1.1}$$

where $m \in \mathbb{Z}$, $q \in \mathbb{N}$, $b_j \in \mathbb{R}$ for $j \geq m$. If $\beta \neq 0$, we can assume that $b_m \neq 0$. Then we put $\mathrm{ord}\beta = m/q$ and call the order of β. Additionally we put $\mathrm{ord}0 = +\infty$. Sometimes the above series are called Puiseux-Laurent series, since we allow $m < 0$.

Definition 1.4. *By a Puiseux series at infinity we will mean a Puiseux series in variable $1/Y$ i.e. a series of the form*

$$\beta = \sum_{j=m}^{\infty} b_j \left(\frac{1}{Y}\right)^{\frac{j}{q}} \tag{1.2}$$

where $m \in \mathbb{Z}$, $q \in \mathbb{N}$, $b_j \in \mathbb{R}$ for $j \geq m$.

The numbers b_j will be called the coefficients of β. If $\beta \neq 0$ is a Puiseux series at infinity of the form (1.2), we can assume that $b_m \neq 0$. By analogy we put $\mathrm{ord}_\infty\beta = m/q$ and call it the *order at infinity of* β. Additionally we put $\mathrm{ord}_\infty 0 = +\infty$. We will denote by $\mathrm{supp}(\beta)$ the set $\{i/q \in \mathbb{Q} |\ b_i \neq 0\}$.

By a *complex Puiseux series* and a *complex Puiseux series at infinity* we will mean the series of the forms (1.1) and (1.2) respectively with complex coefficients.

The set of all Puiseux series considered with addition and multiplication forms a field. The same is true for the set of all Puiseux series at infinity, complex Puiseux series as

well as complex Puiseux series at infinity. By the standard properties of order (cf. [W, Chapter 4]) we get

Property 1.5. *Let β, γ be Puiseux series at infinity. Then*
(1) $\mathrm{ord}_\infty \beta\gamma = \mathrm{ord}_\infty \beta + \mathrm{ord}_\infty \gamma$.
(2) $\mathrm{ord}_\infty(\beta + \gamma) \geq \min\{\mathrm{ord}_\infty \beta, \mathrm{ord}_\infty \gamma\}$.

We have the standard Puiseux theorem (see [W, Theorem 3.2, Chapter 4]).

Theorem 1.6. (Puiseux Theorem) *For any polynomial*

$$f = a_0(Y) + \ldots + a_d(Y)X^d \in \mathbb{C}[X, Y]$$

such that $a_d \neq 0$ there exist Puiseux series $\beta_1, \ldots, \beta_d$ with complex coefficients such that

$$f = a_d(Y)\Pi_{i=1}^d (X - \beta_i(Y)).$$

Note that the above equality is a formal equality in the ring of complex Puiseux series. From Puiseux Theorem (cf. [W, Theorem 4.1, Chapter 4]) it follows that any algebraic set in $\mathbb{C}^2$ can be parametrised locally by Puisuex series. In a suitable coordinate system one can always choose the parametrization (see [W, Theorem 2.2, Chapter 4]) to be of the special form $x = \beta(t^q)$, $y = t^q$, where $q \in \mathbb{N}$, $t \in U$, U is a neighbourhood of the origin and β is a complex Puiseux series such that $\beta(t^q)$ converges for every $t \in U$.

Consider the projective closure in $\mathbb{P}^2(\mathbb{C})$ of a nonempty algebraic set $f^{-1}(0) \subset \mathbb{C}^2$. After a choice of appropriate affine coordinates we conclude that the set $f^{-1}(0) \setminus (\mathbb{C} \times \overline{B(0,R)})$ for a sufficiently big real number R is parametrised at infinity by

$$\begin{aligned} x &= \beta(t^q) \\ y &= t^q \end{aligned} \tag{1.3}$$

where $q \in \mathbb{N}$, $|t|^q > R$ and β is a complex Puiseux series at infinity such that $\beta(t^q)$ converges for every $|t|^q > R$.

If for a Puiseux series at infinity β there exists a closed half-line $I \subset \mathbb{R}$ such that $\beta(y)$ is convergent for $y \in I$ we will say that β is *a convergent Puiseux series at infinity*. If this be the case, we will write $\beta : I \to \mathbb{R}$ and treat β both as a Puiseux series and a real function.

Take an unbounded semialgebraic curve $\Gamma \subset \mathbb{R}^{n+1}$. We will say that the tuple of convergent Puiseux series at infinity $\beta = (\beta_1, \ldots, \beta_n)$ is a *special Puiseux parametrization of the semialgebraic curve at infinity* if there exists a closed half-line $H \subset \mathbb{R}$ and a compact set $K \subset \mathbb{R}^{n+1}$ such that

$$\Gamma = \{(\beta_1(y), \ldots, \beta_n(y), y) \in \mathbb{R}^{n+1} |\ y \in H\}$$

outside $IntK$. Since we will use only the above parametrizations, we will call such a tuple simply a *Puiseux parametrization*.

Note that if $f \in \mathbb{R}[Y][X]$ for $y \in U$, where the set $U \subset \mathbb{R}$ is connected, has a constant number of distinct complex and real roots, then it has a constant number of real roots (see for example [BR, Corollary 1.5.10]). Moreover, possibly after a change of coordinates, there exists a real R sufficiently big such that every connected component of the set $f^{-1}(0) \setminus \overline{B(0,R)} \subset \mathbb{C}^2$ is parametrised by distinct parametrizations of the form (1.3).

Thus if β is a Puiseux series at infinity such that $f(\beta(y^q), y^q) = 0$ and $\beta(y_0^q) \in \mathbb{R}$ for some real number $y_0^q > R$, then $\beta(y^q) \in \mathbb{R}$ for every real $y^q > R$, provided R is sufficiently big. Therefore, β is a Puiseux series at infinity with real coefficients. In particular we get that any unbounded semialgebraic curve in $\mathbb{R}^2$ has a special Puiseux parametrization at infinity. This observation enables us to show that

Proposition 1.7. *Every unbounded semialgebraic curve in $\mathbb{R}^{n+1}$ after some change of coordinates has a special Puiseux parametrization at infinity.*

Proof: Indeed, take an unbounded semialgebraic curve Γ. There exists a permutation of coordinates such that the last coordinate of the curve is unbounded. Consider the projection $\pi_{i,n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^2$ onto the ith and $(n+1)$st coordinate. From previous considerations the curve $\pi_{i,n+1}(\Gamma) \subset \mathbb{R}^2$ has a special Puiseux parametrization at infinity β_i. It is easy to see that the tuple $(\beta_1, \ldots, \beta_n)$ is a special Puiseux parametrization at infinity of Γ. ■

From simple properties of analytic functions combined with properties of order and semialgebraic curves (Properties 1.5 and 1.3) we get

Property 1.8. *Take a closed half-line $I \subset \mathbb{R}$ and Puiseux parametrizations of semialgebraic curves at infinity $\beta, \gamma : I \to \mathbb{R}$. The following hold:*

(1) *there exists a half-line $H \subset I$ such that $(\beta - \gamma)(y)$ has a constant sign for all $y \in H$;*
(2) *either $\beta = \gamma$ or the intersection of their graphs consists of at most finite number of points;*
(3) $\operatorname{ord}_\infty \beta \geq 0$ *if and only if β is bounded on some half-line $H \subset I$;*
(4) $\operatorname{ord}_\infty \beta > 0$ *if and only if β converges to 0 at infinity;*
(5) *if* $\operatorname{ord}_\infty \beta > \operatorname{ord}_\infty \gamma$ *then there exists a half-line $H \subset I$ such that $|\beta(y)| < |\gamma(y)|$ for $y \in H$ and on the other hand, if there exists a half-line $H \subset I$ such that $|\beta(y)| < |\gamma(y)|$ for $y \in H$ then* $\operatorname{ord}_\infty \beta \geq \operatorname{ord}_\infty \gamma$.

Proof: Indeed, (1) and (2) follow from properties of analytic functions and semialgebraic curves. To prove (3)-(5) without loss of generality we can assume that $\beta, \gamma \neq 0$.

Property (3) follows from properties of analytic functions. Indeed, for some $q \in \mathbb{N}$ we can write $\beta(Y^q) = h(1/Y)$, where $h = \sum_{i=p}^{\infty} b_i Z^p$. If $\operatorname{ord}_\infty \beta \geq 0$, then $p \geq 0$ and in some neighbourhood of 0 function h is analytic, hence it is bounded on some (possibly smaller) neighbourhood of 0. In consequence β is bounded on some unbounded set in $\mathbb{R}$. On the other hand, if β is bounded on a half-line H, then h is bounded on a set either of the form $(0, \epsilon)$ or $(-\epsilon, 0)$ for some $\epsilon > 0$. Since h is the sum of a power series, it is also bounded on a neighbourood of 0. Hence $p \geq 0$ and $\operatorname{ord}_\infty \beta = p/q \geq 0$. Analogously we prove (4).

To prove (5) observe that from (2) there exists a half-line $H \subset I$ such that for $y \in H$ we have

$$\left|\frac{\beta(y)}{\gamma(y)}\right| = \left|\frac{(\frac{1}{y})^r(b_0+\tilde{\beta}(y))}{(\frac{1}{y})^s(c_0+\tilde{\gamma}(y))}\right| = \left|\frac{1}{y}\right|^{r-s}\left|\frac{b_0+\tilde{\beta}(y)}{c_0+\tilde{\gamma}(y)}\right|,$$

where $b_0, c_0 \neq 0$, $r = \mathrm{ord}_\infty\beta$, $s = \mathrm{ord}_\infty\gamma$ and $\tilde{\beta}, \tilde{\gamma}$ are Puiseux parametrizations such that $\mathrm{ord}_\infty\tilde{\beta}, \mathrm{ord}_\infty\tilde{\gamma} > 0$. Hence if $r - s > 0$, then by (4) we have $|\frac{\beta}{\gamma}(y)| < 1$ for y belonging to some half-line. The second implication in (5) follows immediately from the first one. ■

1.3. Generators of an algebra

Let $\mathcal{A}$ be a commutative algebra with a unit over $\mathbb{R}$ i.e. a linear space over $\mathbb{R}$ with a bilinear mapping $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $(\mathcal{A}, +, \cdot)$ is a commutative ring with a unit $1_\mathcal{A}$ (see [Lang]). Throughout this paper we will call such $\mathcal{A}$ an algebra.

Let $\zeta \in \mathcal{A}$. We put $\zeta^0 = 1_\mathcal{A}$ and $\zeta^{n+1} = \zeta \cdot \zeta^n$ for $n \in \mathbb{N}_0$. For $\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathcal{A}^k$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k$, where $k \in \mathbb{N}$, we will write $\zeta^\alpha = \zeta_1^{\alpha_1} \ldots \zeta_k^{\alpha_k}$.

Note that for any $\mathcal{Z} \subset \mathcal{A}$, $\mathcal{Z} \neq \emptyset$ the set

$$\{g(\zeta_1, \ldots, \zeta_k) | \quad k \in \mathbb{N},\ g \in \mathbb{R}[X_1, \ldots, X_k],\ \zeta_1, \ldots, \zeta_k \in \mathcal{Z}\}$$

is a subalgebra of $\mathcal{A}$. We will denote it by $\mathbb{R}[\mathcal{Z}]$.

Definition 1.9. *We say that an algebra $\mathcal{A}$ is generated by a set $\mathcal{Z} \subset \mathcal{A}$ if $\mathcal{A} = \mathbb{R}[\mathcal{Z}]$. By convention $\mathbb{R}[\emptyset] = \mathbb{R}$.*

In other words, an algebra $\mathcal{A}$ is generated by a set $\mathcal{Z} \subset \mathcal{A}$ if for any $f \in \mathcal{A}$ there exist some $k \in \mathbb{N}$, a finite set $A \subset \mathbb{N}_0^k$, elements $\zeta_1, \ldots, \zeta_k \in \mathcal{Z}$ and real numbers a_α for $\alpha \in A$ such that

$$f = \sum_{\alpha \in A} a_\alpha \zeta^\alpha,$$

where $\zeta = (\zeta_1, \ldots, \zeta_k)$. Note that by convention $\zeta^0 = 1_\mathcal{A}$. If $\mathcal{Z}$ generates $\mathcal{A}$, then we will talk about the elements of $\mathcal{Z}$ as *generators* of $\mathcal{A}$. If a set is defined by a formula ϕ, i.e. it is of the form $\{\zeta | \ \phi(\zeta)\}$, and the union $\{\zeta_1, \ldots, \zeta_k\} \cup \{\zeta | \ \phi(\zeta)\}$ generates an algebra $\mathcal{A}$ we will write simply $\mathcal{A} = \mathbb{R}[\zeta_1, \ldots, \zeta_k, \zeta | \ \phi(\zeta)]$.

Definition 1.10. *We will say that $\mathcal{Z}$ is a basis of $\mathcal{A}$ if it generates $\mathcal{A}$ and*

$$\forall_{\zeta \in \mathcal{Z}} \quad \zeta \notin \mathbb{R}[\mathcal{Z} \setminus \{\zeta\}].$$

For brevity we will often say that $\mathcal{Z}$ is a basis if it is a basis of $\mathbb{R}[\mathcal{Z}]$, meaning that no element of $\mathcal{Z}$ can be expressed by a polynomial in other elements of $\mathcal{Z}$. A linear space spanned over $\mathbb{R}$ by some set $\mathcal{Z}$ of polynomials usually is not an algebra. For instance $X^2 \notin lin\{1, X, Y\}$. Even if a linear space happens to be an algebra its linear basis need not be its basis as an algebra. For example $lin\{X^i | \ i \in \mathbb{N}_0\}$ is an algebra generated only by X but the set $\{X^i | \ i \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{R}$.

Property 1.11. *The following properties hold.*
(1) *If* $\mathcal{Z} \subset \mathcal{Z}'$, *then* $\mathbb{R}[\mathcal{Z}] \subset \mathbb{R}[\mathcal{Z}']$.
(2) *If* $\zeta \in \mathbb{R}[\mathcal{Z}]$, *then* $\mathbb{R}[\mathcal{Z} \cup \{\zeta\}] = \mathbb{R}[\mathcal{Z}]$.
(3) *If an algebra* $\mathcal{A}$ *is generated by a finite number of elements, then it has a basis.*

Proof: The first and second statements are obvious.

Let $\mathcal{A} = \mathbb{R}[\mathcal{Z}]$, where $\mathcal{Z}$ is a finite set. We will prove property (3) by induction with respect to the number of elements of $\mathcal{Z}$. If $\mathcal{Z} = \emptyset$ then $\mathcal{A} = \mathbb{R}$ and $\emptyset$ a basis of $\mathcal{A}$. Assume that any algebra generated by $k-1$ elements has a basis. Take $\mathcal{Z} = \{\zeta_1, \dots \zeta_k\}$. If $\zeta_i \notin \mathbb{R}[\mathcal{Z} \setminus \{\zeta_i\}]$ for $i = 1, \dots, k$, then $\mathcal{Z}$ is a basis of $\mathcal{A}$. If $\zeta_{i_0} \in \mathbb{R}[\mathcal{Z} \setminus \{\zeta_{i_0}\}]$ for some $i_0 = 1, \dots, k$, then by property (2) we get $\mathcal{A} = \mathbb{R}[\mathcal{Z} \setminus \{\zeta_{i_0}\}]$ and by the induction hypothesis $\mathbb{R}[\mathcal{Z} \setminus \{\zeta_{i_0}\}]$ has a basis. This ends the proof. ■

We should note that if $\mathbb{R}[Z] \subset \mathbb{R}[Z']$, then we cannot say anything about the relation between Z and Z'. For instance $\mathbb{R}[X-Y, X+Y] = \mathbb{R}[X, Y]$, but the sets $\{X-Y, X+Y\}$ and $\{X, Y\}$ are disjoint. This shows us also that a basis need not be unique. Moreover, an algebra can have two distinct bases which need not have the same number of elements. Take for example $\mathcal{A} = \mathbb{R}[X^2+X, X^2]$. Since $X = (X^2+X) - X^2$ then $\mathcal{A} = \mathbb{R}[X]$, so $\{X\}$ is a basis of $\mathcal{A}$. Observe that $\{X^2+X, X^2\}$ forms a basis of $\mathcal{A}$. Indeed, $X^2+X \notin \mathbb{R}[X^2]$ because $\operatorname{ord} X^2+X = 1$ and $\operatorname{ord} f(X^2) \neq 1$ for any polynomial $f \in \mathbb{R}[X]$. On the other hand $X^2 \notin \mathbb{R}[X^2+X]$ because if $X^2 = f(X^2+X)$ for some polynomial $f \in \mathbb{R}[X]$, then $(-1)^2 = f((-1)^2-1) = f(0) = 0^2$. Note that elements of a basis need not be algebraically independent, since in this example $[(X^2+X) - X^2]^2 - X^2 = 0$.

From now on we will consider only subalgebras of the algebra of polynomials in n variables with real coefficients. We will identify $1_{\mathcal{A}}$ with 1 i.e. with the constant polynomial. Note that a subalgebra of the ring of polynomials which is generated by a finite number of elements can be viewed as a subsemigroup of monomials (cf. Section 5.1).

1.4. Nonproper mappings

Take a set $S \subset \mathbb{R}^n$. We say that a mapping $F : S \to \mathbb{R}^m$ is *not proper at* $y \in \mathbb{R}^m$ if there is no neighbourhood U of y such that $F^{-1}(\overline{U})$ is compact. Put

$$J_F = \{y \in \mathbb{R}^m \mid F \text{ is not proper at } y\}.$$

We will call J_F *the set of nonproperness of* F or *Jelonek set of* F. We will say that $F : S \to \mathbb{R}^m$ is *proper on* S if the set J_F is empty. If F is proper on $S = \mathbb{R}^n$, then we will simply say that F is proper. Obviously, if a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ restricted to a set $S \subset \mathbb{R}^n$ is nonproper on S, then it is nonproper.

Jelonek in [J] has shown that

Theorem 1.12. *Let $n \geq 2$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. The Jelonek set J_F is closed, semialgebraic and for every nonempty connected component U of J_F we have*

$$1 \leq \dim U \leq n-1.$$

Moreover, U is unbounded.

In fact, for every point $x \in J_F$ there exists an image of the real line by a polynomial mapping contained in J_F and passing through x.

In the case when S is unbounded and closed we have that F is proper on S if and only if for every sequence $(a_n)_{n\in\mathbb{N}}$ of elements of S we have

$$\textit{if} \quad \lim_{n\to\infty} \|a_n\| = \infty, \quad \textit{then} \quad \lim_{n\to\infty} \|F(a_n)\| = \infty.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that c is *an asymptotic value of f on the set* $S \subset \mathbb{K}^n$ if there exists a sequence $(a_n) \subset S$ such that

$$\lim_{n\to\infty} \|a_n\| = \infty, \quad \textit{and} \quad \lim_{n\to\infty} \|f(a_n)\| = c.$$

We can see that f has asymptotic values on a closed unbounded set S if and only if it is not proper on S. It is easy to see that in this case

$$J_f = \{y \in \mathbb{R} \mid y \textit{ is an asymptotic value of } f\}.$$

We will call asymptotic values of f on $\mathbb{R}^n$ simply asymptotic values of f.

Now we will show a proposition used further in the thesis that if a polynomial has compact fibres then it is proper. Indeed,

Proposition 1.13. *For any polynomial $f \in \mathbb{R}[X]$ if a fibre $f^{-1}(a)$ is compact, then either $f^{-1}([a,b])$ is compact for any $b \geq a$ or $f^{-1}([b,a])$ is compact for any $a \geq b$.*

Proof: For polynomials in one variable the claim is obvious, whereas constant polynomials do not meet the assumptions. Let us suppose that f is a nonconstant polynomial in $n > 1$ variables. From the assumption, the set $\mathbb{R}^n \setminus f^{-1}(a)$ has exactly one unbounded connected component U. We have that either $f(U) = (a, \infty)$ or $f(U) = (-\infty, a)$, because otherwise f would be bounded on $\mathbb{R}^n$. Without loss of generality assume the first case. Hence for every $b \leq a$ we have $f^{-1}([b,a])$ is closed and bounded, since $f^{-1}([b,a]) \cap U = \emptyset$. Therefore, we get the claim. ■

In particular Proposition 1.13 implies that a polynomial is proper if and only if all its fibres are compact. Indeed, if $f \in \mathbb{R}[X]$ is proper then all fibres of f are compact. Assume that all fibres are compact. Take any $c \in \mathbb{R}$ and apply Proposition 1.13 to the polynomial $(f-c)^2$. We get that the set $((f-c)^2)^{-1}([0,1])$ is compact. Hence c cannot be an asymptotic value of f and so f is proper.

1.5. Properties of polynomials bounded on a set

In the two following subsections we will give a motivation of studying algebras of bounded polynomials from the viewpoint of tentacle sets (see Definition 1.18). First we give properties of polynomials bounded on a given set. Afterwards, we will focus on decomposition of any semialgebraic set and the relations of algebras of bounded polynomials on this set.

Let $S \subset \mathbb{R}^n$ be a given set. By $\mathcal{A}(S)$ (or by $\mathcal{A}$ if it does not lead to confusion) we denote the set of all polynomials bounded on the set S. It forms an algebra since any linear combination as well as multiplication of a finite number of polynomials bounded on S is a polynomial bounded on S.

For convenience, throughout this work we will set $\mathcal{A}(\emptyset) = \mathbb{R}[X]$ and treat the empty set as if it were compact. We will also quite often say that $\mathcal{A}$ is trivial if $\mathcal{A} = \mathbb{R}$.

Let us give some basic properties of algebras of bounded polynomials

Property 1.14. *Let $S, M \subset \mathbb{R}^n$. The following hold*

(1) *if $S \subset M$ then $\mathcal{A}(M) \subset \mathcal{A}(S)$;*
(2) *S is bounded if and only if $\mathcal{A}(S) = \mathbb{R}[X]$;*
(3) *$\mathcal{A}(S) = \mathcal{A}(\overline{S})$;*
(4) *$\mathcal{A}(S \cup M) = \mathcal{A}(S) \cap \mathcal{A}(M)$;*
(5) *if $(S \setminus M) \cup (M \setminus S)$ is bounded then $\mathcal{A}(S) = \mathcal{A}(M)$.*

Proof: Properties (1) and (4) are obvious.

To prove (2) it suffices to take under consideration the square of the euclidean norm $\|X\|^2$ which is polynomial. If S is bounded then $\|X\|^2$ is bounded on S, by definition. Since $x_i^2 \leq \|x\|^2$ for any $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, we obtain that $X_i \in \mathcal{A}(S)$ for $i = 1, \ldots, n$. Hence $\mathbb{R}[X] \subset \mathcal{A}(S)$. On the other hand, if $\mathbb{R}[X] = \mathcal{A}(S)$, then $\|X\|^2$ is bounded on S. Therefore S is bounded.

Let us note that any polynomial g is continuous and well defined on $\mathbb{R}^n$. Hence if g is bounded on S then it is bounded on $\overline{S}$. This gives us (3).

Since $S \setminus M$ and $M \setminus S$ are bounded sets, by (2) and (4) we get $\mathcal{A}(S) = \mathcal{A}(M \cap S) \cap \mathcal{A}(M \setminus S) = \mathcal{A}(M \cap S)$ and $\mathcal{A}(M) = \mathcal{A}(M \cap S)$ by analogy. This gives us property (4). ■

The opposite implication in point (5) of Property 1.14 is not true. For example take sets $S = \{x^2y^2 \leq 1\}$ and $M = \{x^2y^2 \leq 4\}$. It is easy to show that $\mathcal{A}(S) = \mathbb{R}[XY] = \mathcal{A}(M)$ (cf. Theorem 2.1 in Section 2.1). But $S \setminus M$ is empty and $M \setminus S$ is not bounded.

Note that $\mathcal{A}(S \cap M) \supset \mathbb{R}[\mathcal{A}(S) \cup \mathcal{A}(M)]$ but equality need not hold. Take for instance $S = \{(xy^2)^2 \leq 1\}$ and $M = \{(x^2y)^2 \leq 1\}$. We get that $XY \in \mathcal{A}(S \cap M) \setminus \mathbb{R}[\mathcal{A}(S) \cup \mathcal{A}(M)]$, though $(XY)^3$ does belong to both. Of course, we cannot also demand an equality in the relation $\mathbb{R}[\mathcal{A}(S) \cup \mathcal{A}(M)] \supset \mathcal{A}(S) \cup \mathcal{A}(M)$.

We will state a trivial but useful remark which we will cite quite often. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be some normed spaces.

Remark 1.15. *A function $f : \mathcal{X} \to \mathcal{Y}$ is bounded on $M \subset \mathcal{X}$ if and only if there exists a mapping $\phi : \mathcal{Z} \to \mathcal{X}$ such that $\phi(\mathcal{Z}) \supset M$ and $f \circ \phi$ is bounded on $\phi^{-1}(M)$. Moreover, f is bounded on $M \subset \mathcal{X}$ if and only if for any mapping $\phi : \mathcal{Z} \to \mathcal{X}$ the superposition $f \circ \phi$ is bounded on $\phi^{-1}(M)$.*

Indeed, to show the necessity of the condition in the first part of the remark it is sufficient to take $\phi = id_{\mathcal{X}}$. The sufficiency can be shown by noticing that if one takes a ϕ such that $\phi(Z) \supset M$ then for any $x \in M$ there exists $z \in \phi^{-1}(M)$. Hence for some constant R and any $x \in M$ we have $\|f(x)\| = \|f(\phi(z))\| \leq R$. The second part follows from a similar argument.

We say that a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an automorphism of some type (eg. polynomial, rational) if the mapping itself, as well as its inverse are of the given type. We have

Property 1.16. *Let ϕ be a polynomial automorphism of $\mathbb{R}^n$ and S some subset of $\mathbb{R}^n$. Then*

$$\mathcal{A}(S) = \mathbb{R}[\mathcal{Z}] \iff \mathcal{A}(\phi(S)) = \mathbb{R}[\phi_*^{-1}\mathcal{Z}], \tag{1.4}$$

where $\phi_*^{-1}\mathcal{Z} = \{\zeta \circ \phi^{-1} |\ \zeta \in \mathcal{Z}\}$. In other words, ϕ induces an isomorphism of the algebras $\mathcal{A}(S)$ and $\mathcal{A}(\phi(S))$.

Proof: If we put $\phi_*^{-1}(f) = f \circ \phi^{-1}$ and $\phi_*(f) = f \circ \phi$ for any $f \in \mathbb{R}[X]$, then we see that ϕ_* and ϕ_*^{-1} are homomorphisms. Moreover, ϕ_* is the inverse of ϕ_*^{-1}. Hence ϕ_* is an isomorphism of $\mathbb{R}[X]$. From Remark 1.15 we get that ϕ_*^{-1} restricted to $\mathcal{A}(S)$ is an isomorphism of $\mathcal{A}(S)$ and $\mathcal{A}(\phi(S))$. ■

In particular, a linear change of coordinates induces an isomorphism of corresponding algebras. We would like to also note that to study if a polynomial is bounded on a set it does not suffice to study its leading term. Indeed, look at the example below.

Example 1.17. Take a set $S_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x^3 + y \leq 1\}$. The polynomial $f_1 = X^3 + Y$ is obviously bounded on S_1. But its leading term X^3 is not bounded on S_1. Take a set $S_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq xy \leq 1\}$ and a polynomial $f_2 = XY + Y$. Its leading term is bounded on S_2, but the polynomial itself is not.

To finish this section let us remark that we can assume that the set of generators of any algebra $\mathcal{A} \subset \mathbb{R}[X]$ is at most countable. Indeed, $\mathcal{A}$ as a linear subspace of $\mathbb{R}[X]$ has an at most countable linear basis. In particular, the linear basis is a set of generators of the algebra $\mathcal{A}$. Therefore, without loss of generality we will only consider at most countable sets of generators of any given algebra $\mathcal{A} \subset \mathbb{R}[X]$.

1.6. Tentacle sets

We will start with a notion frequently used throughout this thesis. Right afterwards we will give its motivation in Theorem 1.19.

Definition 1.18. *A nonempty unbounded semialgebraic set $M \subset \mathbb{R}^n$ is called a tentacle set if for any $r > 0$ the set $M \setminus B(0,r)$ is connected.*

Due to the classical properties of semialgebraic sets we have the following

Theorem 1.19. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then there exists $l \in \mathbb{N}_0$ such that*

$$S = K \cup M_1 \cup ... \cup M_l, \tag{1.5}$$

where K is a bounded semialgebraic set and M_i are pairwise disjoint tentacle sets which are closed in S, i.e. $\overline{M_i} \cap S = M_i$. Moreover, if S is basic closed, then K is basic closed.

Proof: If S is bounded then put $K = S$ and $l = 0$. Assume that S is unbounded.

Consider a rational automorphism of $\mathbb{R}^n \setminus \{0\}$ given by

$$\phi(x) = \frac{x}{\|x\|^2}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. We have that $\phi(\mathbb{R}^n \setminus B(0,R)) = B(0, \frac{1}{R}) \setminus \{0\}$ for any $R > 0$. Since ϕ is semialgebraic, then $\phi(S) \subset \mathbb{R}^n$ is semialgebraic. The origin lies in $\overline{\phi(S)} \setminus \phi(S)$. Hence, by the local conic structure of semialgebraic sets ([BCR, Theorem 9.3.6]), we get that the number of connected components of $\phi(S)$ is equal to the number of connected components of $\phi(S) \cap S(0, \frac{1}{R})$ for a sufficiently small fixed real number $\frac{1}{R}$. Hence there exists $R > 0$ such that the number of connected components of $\phi(S) \cap B(0, \frac{1}{r})$ for every $\frac{1}{r} \leq \frac{1}{R}$ is constant. This gives us precisely that the number of connected components of $S \setminus B(0,r)$ for $r \geq R$ is equal to some $l \in \mathbb{N}_0$.

Take $K = S \cap \overline{B(0,R)}$ and for $i = 1, \ldots, l$ denote by M_i the connected components of $S \setminus B(0,R)$. It is easy to see that if S is a basic closed semialgeraic subset then K is also basic. ■

The above proof follows also for example from adapted cell decomposition (see [BCR, Theorem 2.3.1]) and is in fact its special restatement, but we feel that this approach gives a shorter and less technical proof. Note that in the above theorem we rule out tentacles which sum up to a tentacle.

Remark 1.20. *Tentacles M_i in Theorem 1.19 are unique in the following sense. If $S = K' \cup M'_1 \cup \ldots \cup M'_{l'}$, where the conditions of Theorem 1.19 hold for M'_i, $i \leq l'$, then $l = l'$ and there exists $R > 0$ such that, possibly after some permutation of indices,*

$$M_i \setminus B(0,R) = M'_i \setminus B(0,R).$$

In the decomposition (1.5) we can imagine that S is an octopus with K as its trunk and M_i as its tentacles. Hence we will say that M_i are *the tentacles of the set S*. The

author was inspired to call unbounded "parts" of sets in such a way by the paper [Sw].

Let us remark that using notation from Theorem 1.19 we have

$$\mathcal{A}(S) = \bigcap_{i=1}^{l} \mathcal{A}(M_i). \tag{1.6}$$

The equality follows immediately from Theorem 1.19 and Property 1.14. Hence we see that to study the algebra of bounded polynomials on a semialgebraic set it suffices to consider the algebras of bounded polynomials on its tentacles. Moreover, since $\mathcal{A}(S) = \mathcal{A}(\overline{S})$, it suffices to study closed sets. And this is exactly what we are going to do in the Sections 2 and 5.

Example 1.21. Take

$$S = \{(x,y) \in \mathbb{R}^2 |\ |x| < 1\} \cup \{(x,y) \in \mathbb{R}^2 |\ |y| < 1\}.$$

Then $\overline{S} = K \cup M_1 \cup M_2 \cup M_3 \cup M_4$, where $K = \{(x,y) \in \mathbb{R}^2 |\ |x| \leq 1,\ |y| \leq 1\}$ and

$$M_1 = \{(x,y) \in \mathbb{R}^2 |\ |x| \leq 1,\ y \leq -1\}, \quad M_2 = \{(x,y) \in \mathbb{R}^2 |\ |x| \leq 1,\ y \geq 1\},$$
$$M_3 = \{(x,y) \in \mathbb{R}^2 |\ x \leq -1,\ |y| \leq 1\}, \quad M_4 = \{(x,y) \in \mathbb{R}^2 |\ x \geq 1,\ |y| \leq 1\}.$$

It is easy to see that $\mathcal{A}(M_1) = \mathcal{A}(M_2) = \mathbb{R}[X]$ and $\mathcal{A}(M_3) = \mathcal{A}(M_4) = \mathbb{R}[Y]$. Hence $\mathcal{A}(S) = \mathcal{A}(\overline{S}) = \mathbb{R}[X] \cap \mathbb{R}[Y] = \mathbb{R}$. Note that the algebras of bounded polynomials on the tentacles are nontrivial whence the algebra $\mathcal{A}(S)$ is.

Let us turn to the case of subsets of the real plane $\mathbb{R}^2$. Now we can state a remark that explains the notation that will be used throughout Section 2.

Proposition 1.22. *Take a semialgebraic set $S \subset \mathbb{R}^2$. There exists $R > 0$ such that after a linear change of coordinates the closure $\overline{S}$ has a decomposition (1.5) such that for $i = 1, \ldots, l$ if*

$$\mathcal{A}(M_i) \neq \mathbb{R}$$

then M_i is of one of the forms

$$\{(x,y) \in \mathbb{R}^2 | \beta_1(y) \leq x \leq \beta_2(y), y \geq R\} \text{ or } \{(x,-y) \in \mathbb{R}^2 | \beta_1(y) \leq x \leq \beta_2(y), y \geq R\}, \tag{1.7}$$

where β_1, β_2 are Puiseux parametrizations at infinity of continuous semialgebraic curves well-defined on the half-line $[R, \infty)$.

Proof: Indeed, assume that S is unbounded. If FrS is bounded, then for a sufficiently big $R > 0$ we have $S \setminus B(0,R) = \mathbb{R}^2 \setminus B(0,R)$. Obviously, $\mathcal{A}(S \setminus B(0,R)) = \mathbb{R}$.

Assume that FrS is unbounded. We can write that $FrS \subset f^{-1}(0)$ for some nonconstant polynomial $f : \mathbb{R}^2 \to \mathbb{R}$. Change the coordinates so that up to the sign f is of the form

$$f = X^d + \sum_{i=1}^{d} a_i(Y) X^{d-i},$$

where $\deg a_i \leq i$. In such case, all tentacles of $f^{-1}(0)$ have special Puiseux parametrizations up to a sign change. Hence a tentacle of $\overline{S}$ that is not of the form (1.7) must for some $r, R > 0$ and Puiseux parametrizations $\beta_1, \beta_2 : [R, \infty) \to \mathbb{R}$ contain a set

$$\{(x,y) \in \mathbb{R}^2 |\ x \geq r,\ |y| \leq R\} \cup \{(x,y) \in \mathbb{R}^2 |\ x \geq r,\ y \geq R,\ x \geq \beta_1(y)\} \cup \\ \cup \{(x,-y) \in \mathbb{R}^2 |\ x \geq r,\ y \geq R,\ x \geq \beta_2(y)\} \tag{1.8}$$

or a mirror image of this set by symmetry with respect to the y axis. Denote the above tentacle by M. Take $g \in \mathcal{A}(M)$. We have $\mathcal{A}(M) \subset \mathbb{R}[Y]$, because for each y_0 the set $\{(x, y_0) |\ x \in \mathbb{R}\} \cap M$ is a half-line, hence the polynomial $g(\cdot, y_0)$ is bounded if and only if g does not depend on X. Therefore we have that $g(X,Y) = g(1,Y)$. In particular, g is bounded on the border of M. We have that there exists $C > 0$ such that $|g(\beta_1(y), y)| = |g(1,y)| \leq C$ for $y \geq R$. Hence g cannot depend on Y, otherwise it would not be bounded. Therefore, the polynomial g is constant. ■

Note that if M_i is of the form (1.7), then it still may happen that $\mathcal{A}(M_i) = \mathbb{R}$ (for example see Corollary 2.9 in Section 2.1).

Under the notation of the above proposition due to Property 1.8 we can demand that $\beta_1 - \beta_2$ has a constant sign on the appropriate half-line. In this case we will call β_1 and β_2 the *parametrizations of the borders of a tentacle set*. Hence if $Int(M) \neq \emptyset$ for a tentacle M, then we get $\beta_1 < \beta_2$ on the appropriate half-line. We have that $Int(M) = \emptyset$ if and only if $\beta_1 = \beta_2$.

Any tentacle of the form (1.7) after a change of coordinates is simply of the form

$$\{(x,y) \in \mathbb{R}^2 |\quad \beta_1(y) \leq x \leq \beta_2(y),\ y \geq R\}$$

where $R \in \mathbb{R}$ and β_1, β_2 are Puiseux parametrizations at infinity of continuous semi-algebraic curves. Therefore on the real plane it suffices to study algebras of bounded polynomials on nonempty sets of the above form with $\beta_1 - \beta_2$ of constant sign. For simplicity of notation we will demand that all tentacle sets in $\mathbb{R}^2$ are such sets.

2. Polynomials bounded on subsets of the plane

In this section we will give a characterisation of algebras of polynomials bounded on tentacle sets in the real plane. In the first section we will consider the easier case of tentacles with parametrizations of their borders having different orders at infinity. For such tentacles the algebras of bounded polynomials are easily characterised (see Theorems 2.4 and Section 5).

In the second section we will tackle the case of equal orders. We give an efficient method of checking if a polynomial is bounded on such a tentacle (Proposition 2.13 combined with Theorem 2.16) by giving a characterisation of an algebra of polynomials with Puiseux coefficients. We leave out the case of tentacles with empty interior until its study in any dimension undertaken in Section 4.3.

As to notation, in the cases when we feel that it does not lead to confusion in expressions of the form $\mathcal{A}(\{(x,y) \in \mathbb{R}^2 |\ a \leq f \leq b\})$ or $\mathcal{B}_q(\{(x,y) \in \mathbb{R}^2 |\ a \leq f \leq b\})$ we will often leave out the brackets and write simply $\mathcal{A}(a \leq f \leq b)$ and $\mathcal{B}_q(a \leq f \leq b)$.

2.1. Characterisation of polynomials bounded on tentacles with different orders of the parametrizations of their borders

Throughout this section we will consider a tentacle in $\mathbb{R}^2$ of the form

$$M = \{(x,y) \in \mathbb{R}^2 |\ \beta_1(y) \leq x \leq \beta_2(y),\ y \geq R\}, \tag{2.1}$$

where $R > 0$ and β_1, β_2 are Puiseux parametrizations at infinity of two semialgebraic curves such that the sign of $\beta_1 - \beta_2$ is constant on $[R, \infty)$. It is crucial to note that throughout this paper such sets are meant to be unbounded, that is $\beta_1(y) \leq \beta_2(y)$ for every y big enough. For the purposes of this thesis we assume $0 \cdot \infty = 0$.

Put $\alpha = \min\{\mathrm{ord}_\infty \beta_1, \mathrm{ord}_\infty \beta_2\}$.

In this section we will give an elementary characterisation of the algebra of bounded polynomials on the above type of set with $\mathrm{ord}_\infty \beta_1 \neq \mathrm{ord}_\infty \beta_2$ (Theorem 2.4). We will also, due to Theorem 2.7, attribute the same characterisation to sets M with "trivial"

asymptotics, that is the sets containing a half-line. The key to prove these results is Theorem 2.1.

We start with some remarks needed to prove the crucial Theorem 2.1. Denote

$$\|F\|_{u,v} = \sup_{x\in[u,v]} |F(x)|$$

for $u, v \in \mathbb{R}$, $F \in \mathbb{R}[X]$. For $u < v$ it is a norm in $\mathbb{R}[X]$.

The mapping

$$\mathbb{R}[X] \ni \sum_{i=0}^{d} A_i X^i \to \max_{i=0}^{d} |A_i|$$

denoted as $\|\cdot\|_{\max}$ is also a standard norm in $\mathbb{R}[X]$.

For a function $f = \sum_{i=0}^{d} a_i(Y)X^i$, defined on $\mathbb{R} \times I \subset \mathbb{R}^2$ and polynomial with respect to X we will write $\deg_X f = d$ if a_d is not constant zero function and $\deg_X 0 = -\infty$. For the above f, a given $y \in I$ and any $\lambda \in \mathbb{R}$ we have

$$f(\lambda x, y) = \sum_{i=0}^{d} a_i(y)(\lambda x)^i = \sum_{i=0}^{d} a_i(y)\lambda^i x^i.$$

Given $\lambda > 0$ and $v < 1$ we see that $x\lambda \in [v\lambda, \lambda]$ if and only if $x \in [v, 1]$. Therefore, $\sup_{x\in[v,1]} |F(x\lambda)| = \sup_{x\lambda\in[v\lambda,\lambda]} |F(x\lambda)| = \sup_{x\in[v\lambda,\lambda]} |F(x)|$ for any function $F : \mathbb{R} \to \mathbb{R}$. Hence for any fixed y we have $\|f(X, y)\|_{v\lambda,\lambda} = \|f(\lambda X, y)\|_{v,1}$ and

$$\|f(\lambda X, y)\|_{\max} = \max_{i=0,\dots d} |\lambda^i a_i(y)|.$$

Moreover, the space of polynomials in one variable of degree not bigger than d is of a finite dimension. Hence all norms on this space are equivalent. In particular, for $v < 1$ there exist positive constants $\omega(d, v) = \omega$, $\Omega(d, v) = \Omega$, depending on d and v only, such that

$$\|F\|_{v,1} \geq \omega\|F\|_{\max} \qquad and \qquad \Omega\|F\|_{v,1} \leq \|F\|_{\max}$$

for any real polynomial F of degree not greater than d.

For a function $f = \sum_{i=0}^{d} a_i(Y)X^i$ polynomial with respect to X, some $v < 1$ and any positive convergent Puiseux series at infinity β we have

$$\begin{aligned} \|f(X, y)\|_{v\beta(y),\beta(y)} &= \|f(\beta(y)X, y)\|_{v,1} \geq \\ &\geq \omega\|f(\beta(y)X, y)\|_{\max} = \quad \omega \max_{i=0,\dots,d} |\beta^i(y)a_i(y)| \end{aligned} \tag{2.2}$$

and

$$\max_{i=0,\dots,d} |\beta^i(y)a_i(y)| \geq \Omega\|f(\beta(y)X, y)\|_{v,1} = \Omega\|f(X, y)\|_{v\beta(y),\beta(y)} \tag{2.3}$$

for any $y > 0$ such that $\beta(y)$ converges to a positive value and f is well defined.

Moreover, if $\beta = b_\alpha(\frac{1}{Y})^\alpha + \textit{terms of higher degree in } 1/Y$ and $a = A_0 + A_1 Y + ... + A_k Y^k$ then

$$a \cdot \beta^i = A_k b_\alpha (\frac{1}{Y})^{i\alpha - k} + \textit{terms of higher degree in } 1/Y$$

In other words

$$a \cdot \beta^i = A_k b_\alpha Y^{k - i\alpha} + \textit{terms of lower degree in } Y$$

for any $i \in \mathbb{N}_0$. Hence

$$\operatorname{ord}_\infty(a \cdot \beta^i) = -\deg a + i \cdot \operatorname{ord}_\infty \beta. \tag{2.4}$$

Now we are prepared to prove a theorem.

Theorem 2.1. *Let $v < 1$ and β be a convergent Puiseux series at infinity with $\beta > 0$ on $[R, \infty)$, where $R > 0$. Set*

$$K_{v,\beta} = \{(x, y) \in \mathbb{R}^2 |\ v\beta(y) \leq x \leq \beta(y),\ y \geq R\}.$$

Take $f = \sum_{i=0}^{d} a_i(Y) X^i \in \mathbb{R}[X, Y]$. Then the polynomial f is bounded on $K_{v,\beta}$ if and only if

$$i \cdot \alpha \geq \deg a_i$$

for $i = 0, ..., d$, where $\alpha = \min\{\operatorname{ord}_\infty \beta, \operatorname{ord}_\infty v\beta\}$.

Proof: In this context $\alpha = \operatorname{ord}_\infty \beta$, because we have $\operatorname{ord}_\infty v\beta = \operatorname{ord}_\infty \beta$ for $v \neq 0$ and $\operatorname{ord}_\infty v\beta = \infty$ for $v = 0$. We may assume that $\deg_X f = d$.

Let us remark that since $K_{v,\beta} \cap \{(x, y) \in \mathbb{R}^2 |\ y = y_0\} = [v\beta(y_0), \beta(y_0)] \times \{y_0\}$ for any $y_0 \geq R$, the polynomial f is bounded on $K_{v,\beta}$ if and only if there exists a constant $C \in \mathbb{R}$ such that for any $y_0 \geq R$

$$C \geq \sup_{x \in M \cap \{(x,y)|\ y = y_0\}} |f(x, y_0)| = \|f(X, y_0)\|_{v\beta(y_0), \beta(y_0)} \tag{2.5}$$

Let us assume that $f \in \mathcal{A}(K_{v,\beta})$. Taking into account equation (2.5) we get that there exists a constant C such that for any $y \geq R$

$$\|f(X, y)\|_{v\beta(y), \beta(y)} \leq C.$$

Therefore, from inequality (2.2) it follows that for any $y \geq R$ and any $i = 0, ..., d$ we have

$$C \geq \|f(X, y)\|_{v\beta(y), \beta(y)} \geq \omega \max_{i=0,...,d} |\beta^i(y) a_i(y)| \geq \omega |\beta^i(y) a_i(y)|$$

for the constant $\omega = \omega(d, v)$ depending only on $\deg_X f$ and v.

Since the expressions $\beta^i(Y) a_i(Y)$ for each $i = 0, ..., d$ are bounded on the half-line $\{y \in \mathbb{R} |\ y \geq R\}$ and their order at infinity is well defined, we get

$$0 \leq \operatorname{ord}_\infty |\beta^i(Y) a_i(Y)| = -\deg a_i(Y) + i \cdot \alpha$$

by (2.4).

Now let us assume that $i\alpha \geq \deg a_i$ for all $i = 0, \ldots, d$. In such a case, from (2.4) and Property 1.8 it follows that all functions $\beta^i a_i$ are bounded outside some nonempty neighbourhood of 0. Therefore, combining it with inequality (2.3), we have that there exists a constant $\Omega = \Omega(d, v)$ depending only on $\deg_X f$ and v. Furthermore, there exists a constant $C \geq 0$ such that for each $y \geq R$ we have

$$C \geq \max_{i=0,\ldots,d} |\beta^i(y) a_i(y)| \geq \Omega \|f(X, y)\|_{v\beta(y),\beta(y)}.$$

Therefore we proved our claim. ■

Let us turn to a kind of symmetry in the behaviour of bounded polynomials in a corollary that will be often more convenient to use then Theorem 2.1. Let $v < 1$ and β be a convergent Puiseux series at infinity with $\beta > 0$ on $[R, \infty)$, where $R > 0$. Set

$$K_{v,\beta} = \{(x, y) \in \mathbb{R}^2 |\ v\beta(y) \leq x \leq \beta(y),\ y \geq R\},$$

$$S_\beta = \{(x, y) \in \mathbb{R}^2 |\ -\beta(y) \leq x \leq \beta(y),\ y \geq R\},$$

$$L_\beta = \{(x, y) \in \mathbb{R}^2 |\ 0 \leq x \leq \beta(y),\ y \geq R\}.$$

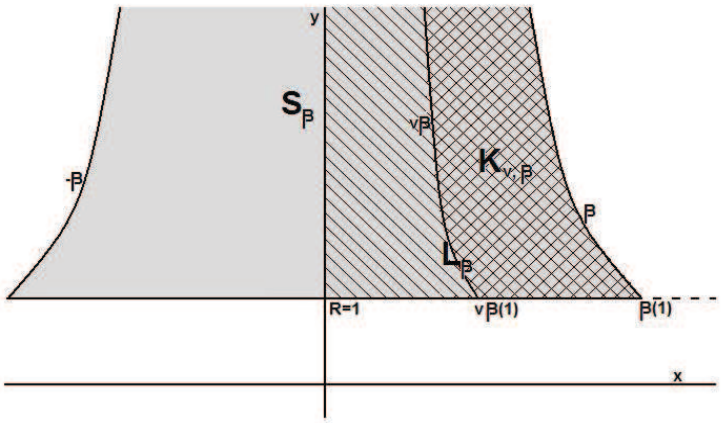

Figure 1. Sets $K_{v,\beta}$, S_β and L_β.
The set S_β is the union of L_β and its reflection with respect to the y axis, whereas $K_{v,\beta}$ need not contain a half-line.

Corollary 2.2. *Take* $f = \sum_{i=0}^{d} a_i(Y) X^i \in \mathbb{R}[X, Y]$. *The following conditions are equivalent*

(1) *the polynomial* f *is bounded on* $K_{v,\beta}$
(2) *the polynomial* f *is bounded on* S_β,
(3) *the polynomial* f *is bounded on* L_β,
(4) $i \cdot \alpha \geq \deg a_i$ *for* $i = 0, ..., d$.

In other words

$$\mathcal{A}(K_{v,\beta}) = \mathcal{A}(S_\beta) = \mathcal{A}(L_\beta) = \mathbb{R}[X^i Y^d |\ d \leq i \cdot \operatorname{ord}_\infty \beta,\ d, i \in \mathbb{N}_0]$$

Proof: The equivalence of (1) and (4) is the essence of Theorem 2.1. For equivalence of (2) and (4) we put $v = -1$. Analogously, for equivalence of (3) and (4) we put $v = 0$. ■

Since the reflection with respect to the y axis is a linear change of coordinates it is easy to conclude from Theorem 2.1 and Property 1.16 that the assertion of Theorem 2.1 holds also if β has constant negative sign and $v > -1$. Hence we conclude that

Remark 2.3. *The algebra of polynomials bounded on the tentacle*

$$\{(x, y) \in \mathbb{R}^2 |\ a\beta(y) \leq x \leq b\beta(y),\ y \geq R\},$$

where $a^2 + b^2 > 0$ *and* $\beta \neq 0$ *is a Puiseux parametrization, is of the form*

$$\mathbb{R}[X^i Y^d |\ d \leq i \cdot \mathrm{ord}_\infty \beta,\ d, i \in \mathbb{N}_0].$$

Now we will give the characterisation of algebras of bounded polynomials on a tentacle. Recall that we consider the tentacle $M = \{(x, y) \in \mathbb{R}^2 |\ \beta_1(y) \leq x \leq \beta_2(y),\ y \geq R\}$ where β_1, β_2 are Puiseux parametrizations and $\alpha = \min\{\mathrm{ord}_\infty \beta_1, \mathrm{ord}_\infty \beta_2\}$.

Theorem 2.4. *If* $\mathrm{ord}_\infty \beta_1 \neq \mathrm{ord}_\infty \beta_2$ *then*

$$\mathcal{A}(M) = \mathbb{R}[X^i Y^d |\ d \leq i\alpha],$$

where $\alpha = \min\{\mathrm{ord}_\infty \beta_1, \mathrm{ord}_\infty \beta_2\}$.

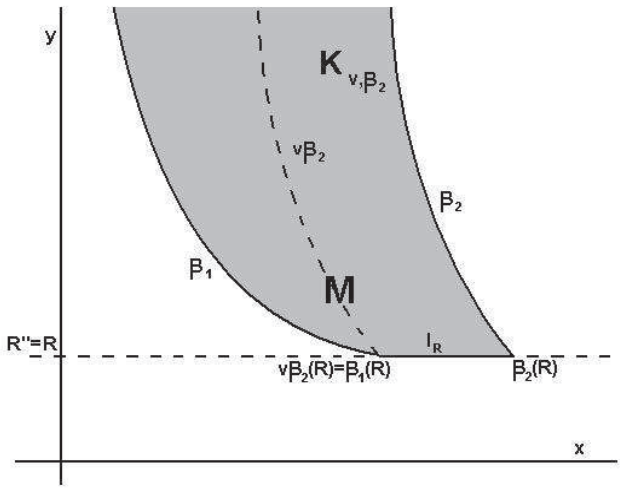

Figure 2. The set K_{v,β_2} is contained in M for R'' big enough.

Proof: Without loss of generality we can assume that $\mathrm{ord}_\infty \beta_1 > \mathrm{ord}_\infty \beta_2$. Obviously $\beta_1 \neq \beta_2$ and we have that β_1, β_2 parametrise semialgebraic curves.We have

$$M \cap \{(x, y) \in \mathbb{R}^2 |\ y = R\} = [\beta_1(R), \beta_2(R)].$$

Hence $\beta_1(R) = v\beta_2(R)$ for some $v < 1$. By Property 1.8 of the orders at infinity of Puiseux parametrizations, we can demand R to be great enough so that $|\beta_1(y)| \leq |v\beta_2(y)|$ for all $y \geq R$. Take K_{v,β_2} and S_{β_2} as on page 30. Then

$$S_{\beta_2} \supset M \supset K_{v,\beta_2}.$$

From Corollary 2.2 we get $\mathcal{A}(M) = \mathbb{R}[X^iY^d|\ d \leq i\alpha]$. ■

Let us illustrate the result with an easy example.

Example 2.5. Tentacle sets given by the following inequalities have the same algebra of bounded polynomials

$$\frac{1}{y^2} \leq x \leq \frac{2}{y}, \quad \frac{1}{y} \leq x \leq \frac{2}{y}, \quad \sum_{i=2}^{\infty} \frac{1}{y^i} \leq x \leq \sum_{i=1}^{\infty} \frac{2}{y^i} \quad \textit{and} \quad \frac{1}{y^2} \leq x \leq \frac{y-2}{y(y-1)},$$

where $y \geq 3$. The algebra of bounded polynomials in each case is precisely $\mathbb{R}[X, XY]$ by Theorems 2.1 and 2.4.

Now let us drop the assumption that $\mathrm{ord}_\infty\beta_1 \neq \mathrm{ord}_\infty\beta_2$ and assume instead that the set M contains a half-line of the y axis. First, we show

Proposition 2.6. *If $M = \{0\} \times [R, \infty)$ for some $R \in \mathbb{R}$ then*

$$\mathcal{A}(M) = \mathbb{R}[XY^k|\ k \in \mathbb{N}_0].$$

Proof: Let $f = \sum_{i=0}^{d} a_i(Y)X^i \in \mathcal{A}(M)$. Then

$$f(0, y) = const$$

for $y > r$. Therefore $a_0 = const$ and we obtain $f \in \mathbb{R}[XY^k|\ k \in \mathbb{N}_0]$. On the other hand, if $f \in \mathbb{R}[XY^k|\ k \in \mathbb{N}_0]$ then obviously $f \in \mathcal{A}(M)$. ■

Since we find it useful, we will draw this general conclusion for sets containing a half-line on the OY axis. Note that $M = \{0\} \times [R, \infty)$ for some $R > 0$ if and only if $\mathrm{ord}\beta_1 = \mathrm{ord}\beta_2 = \infty$ and recall that $0 \cdot \infty = 0$. Hence we can state that

Theorem 2.7. *Suppose that $\{0\} \times [R, \infty) \subset M$ for some $R \in \mathbb{R}$. A polynomial $f = \sum_{i=0}^{d} a_i(Y)X^i$ is bounded on M if and only if $\deg a_i \leq \alpha \cdot i$ for every $i = 0, ..., d$.*

Proof: First, in the case when $\beta_1 = \beta_2$ we see that $\alpha = \infty$, because necessarily $\beta_1 = \beta_2 = 0$. From Proposition 2.6 if f is bounded on M, then $f \in \mathbb{R}[XY^k|\ k \in \mathbb{N}_0]$. That takes place if and only if $\deg a_0 \leq 0$. If we put by our convention $0 \cdot \infty = 0$ then f is bounded on M if and only if $\deg a_i \leq \alpha \cdot i$ for every $i = 0, ..., d$.

Let $\beta_1 \neq \beta_2$. By Remark 2.3 without loss of generality we can assume that $|\beta_2| \geq |\beta_1|$ and $\beta_2 > 0$. Then by Property 1.8 of orders at infinity we have $\mathrm{ord}_\infty\beta_2 \leq \mathrm{ord}_\infty\beta_1$. Fixing a greater R if necessary, we get

$$\{(x, y) \in \mathbb{R}^2|\ 0 \leq x \leq \beta_2(y),\ y \geq R\} \subset M \subset \{(x, y) \in \mathbb{R}^2|\ -\beta_2(y) \leq x \leq \beta_2(y),\ y \geq R\}.$$

Thus, using the notation from Corollary 2.2, we obtain

$$\mathcal{A}(L_{\beta_2}) \supset \mathcal{A}(M) \supset \mathcal{A}(S_{\beta_2}),$$

which by that corollary gives us the equality

$$\mathcal{A}(L_{\beta_2}) = \mathcal{A}(S_{\beta_2}).$$

This in turn gives that f is bounded on M if and only if it is bounded on S_{β_2}. By Corollary 2.2 we conclude that f is bounded on M if and only if $\deg a_i \leq i \cdot \alpha$ for every $i = 0, ..., d$. ■

Example 2.8. Let

$$M = \{(x, y) \in \mathbb{R}^2 | -1/y \leq x \leq 1/y + 1/y^2,\ y \geq 1\}.$$

We have $\operatorname{ord}_\infty(-1/Y) = 1 = \operatorname{ord}_\infty(1/Y + 1/Y^2)$ and $\{0\} \times [R, \infty) \subset M$. From Theorem 2.7 it follows that $\mathcal{A}(M) = \mathbb{R}[X, XY]$.

Recall that $\alpha = \min(\operatorname{ord}_\infty \beta_1, \operatorname{ord}_\infty \beta_2)$. As an easy consequence of the characterisation of algebras $\mathcal{A}$ given in Theorems 2.4 and 2.7 we get

Corollary 2.9. *Suppose that* $\operatorname{ord}_\infty \beta_1 \neq \operatorname{ord}_\infty \beta_2$ *or* M *contains a half-line on the* y *axis. Then*

(1) $\mathcal{A}(M) = \mathbb{R}$ *if and only if* $\alpha < 0$,
(2) $\mathcal{A}(M) = \mathbb{R}[X]$ *if and only if* $\alpha = 0$,
(3) $\mathcal{A}(M) \supsetneq \mathbb{R}[X]$ *if and only if* $\alpha > 0$.

The essence of this corollary is that if sets get essentially more ample than a strip, they cease to admit any bounded polynomials. In particular, if a set contains a cone, then it does not admit bounded polynomials. Note that, as in the case of the set M_2 from Example 2.10 below, a set need not contain a cone to have a trivial algebra of bounded polynomials (compare [KM]). The sets which admit a nontrivial algebra of bounded polynomials can be interpreted as having their distance from the y axis essentially smaller than their width.

Example 2.10. Let

$$M_1 = \{\frac{1}{y} \leq x \leq y^2,\ y \geq 1\} \quad \textit{and} \quad M_2 = \{y^2 \leq x \leq y^3,\ y \geq 1\}.$$

By the above corollary, the only bounded polynomials on these sets are constant.

Note that all of the results in this paragraph are insensitive to x axis and y axis symmetry as well as an interchange between x and y since they are special cases of linear change of coordinates. Therefore the assertion of Theorems 2.4 and 2.7 hold for all tentacle sets of the form (1.7) as well as those with borders parameterized by x instead of y with the exponents in the characterisations appropriately interchanged. From these remarks and Theorem 2.4 it follows

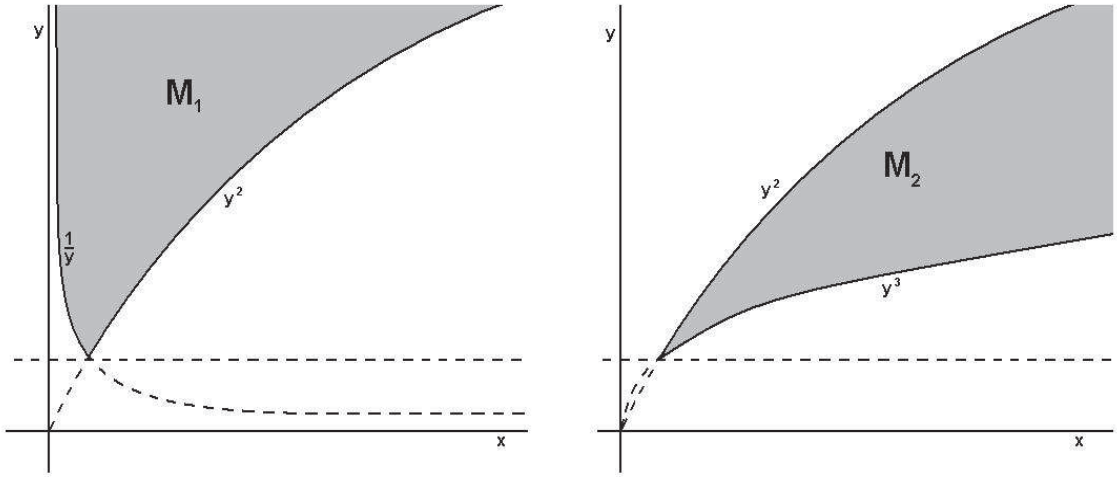

Figure 3. Typical examples of sets for which $\mathcal{A} = \mathbb{R}$

Remark 2.11. *Every algebra of bounded polynomials in the case of tentacle sets of the form (1.7) with* $\operatorname{ord}_\infty\beta_1 \neq \operatorname{ord}_\infty\beta_2$ *is an algebra of a set of the form*

$$\{(x,y) \in \mathbb{R}^2 |\ 0 \leq x \leq y^\alpha, y \geq 1\} \tag{2.6}$$

for $\alpha = \min\limits_{i=1,2} \operatorname{ord}_\infty\beta_i \in \mathbb{Q}$.

Let us emphasise that this remark implies that any two tentacle sets of the form (1.7) with parametrizations of their borders of different orders have the same algebras of bounded polynomials as long as the minimum of orders is the same. At this point the question of characterisation of bounded polynomials on tentacle sets with $\operatorname{ord}_\infty\beta_1 = \operatorname{ord}_\infty\beta_2$ is resolved fully only for sets containing a half-line. In the next sections we will turn to that problem.

2.2. Characterisation of polynomials with Puiseux coefficients bounded on tentacles with nonempty interior

In this section we turn to the problem of deciding whether a polynomial is bounded on tentacle sets which borders have parametrizations of the same order. We will achieve this through the study of polynomials with Puiseux coefficients (Definition 2.12 and Proposition 2.13) and giving a characterisation of their algebras in Theorem 2.16.

Denote by $\mathcal{R}_q[Y]$ the ring $\mathbb{R}[Y^{1/q}, \frac{1}{Y^{1/q}}]$ of Laurent polynomials in $Y^{1/q}$, where $q \in \mathbb{N}$. Of course, every $\beta \in \mathcal{R}_q[Y]$ is a convergent Puiseux series at infinity (cf. Section 1.2). If $q = 1$ these are just Laurent polynomials. By analogy

Definition 2.12. *For* $\beta \in \bigcup_{q\in\mathbb{N}} \mathcal{R}_q[Y]$ *we will say that* β *is a Puiseux polynomial.*

If $\beta = \sum_{i=p}^{m} a_i \left(\frac{1}{Y}\right)^{i/q}$, where $a_m \neq 0$, then by *the degree of β at infinity* we mean the rational number m/q and we write $\deg_\infty \beta = m/q$. We put $\deg_\infty 0 = -\infty$. Note that since $Y = (1/Y)^{-1}$, the order at infinity (the degree at infinity resp.) of β is the same as its degree (or order resp.) with the sign changed.

Note that $\beta(y)$ is convergent for every y from the positive half-line if the denominator q is even and on $\mathbb{R} \setminus \{0\}$ otherwise. Hence β can be viewed as a semialgebraic function on its domain of convergence. If $\beta \in \mathcal{R}_q[Y]$ then for any $l \in \mathbb{N}$ we have $\beta \in \mathcal{R}_{ql}[Y]$. Therefore for every two Puiseux polynomials there exists a natural number q such that they both belong to $\mathcal{R}_q[Y]$.

By $\mathcal{R}_q[Y][X]$ we will denote the ring of polynomials in X with coefficients from $\mathcal{R}_q[Y]$ and we will say that its elements are *polynomials with Puiseux coefficients*. Note that $f \in \mathcal{R}_q[Y][X]$ usually is not well defined on $\{(x, y) \in \mathbb{R}^2 |\ y = 0\}$. Of course, $\mathbb{R}[X, Y] \subset \bigcap_{q=1}^{\infty} \mathcal{R}_q[Y][X]$.

Proposition 2.13. *Any $f \in \mathcal{R}_q[Y][X]$ has unique coefficients $e_{i,k} \in \mathbb{R}$ such that*

$$f = \sum_{\substack{i = 0, \ldots, d \\ k \in D}} e_{i,k}(X - \beta)^i Y^{k/q}, \tag{2.7}$$

where $D \subset \mathbb{Z}$ is finite and $d = \deg_X f$.

Proof: Let $f \in \mathcal{R}_q[Y][X]$ be of the form $f = \sum_{j=0}^{d} a_j X^j$. Then

$$f = \sum_{j=0}^{d} a_j((X - \beta) + \beta)^j = \sum_{j=0}^{d} \sum_{i=0}^{j} a_j \binom{j}{i} \beta^{j-i}(X - \beta)^i = \\ = \sum_{i=0}^{d} \left(\sum_{j=i}^{d} a_j \binom{j}{i} \beta^{j-i} \right) (X - \beta)^i$$

Denote $c_i = \sum_{j=i}^{d} a_j \binom{j}{i} \beta^{j-i}$. Note that $c_d = a_d$. Since for every $i = 0, \ldots, d$ we have $c_i \in \mathcal{R}_q[Y]$, we can write

$$c_i = \sum_{k \in D_i} e_{i,k} Y^{k/q},$$

where $e_{i,k} \in \mathbb{R}$ and a finite set $D_i \subset \mathbb{Z}$. Put $D = \bigcup_{i=0}^{d} D_i$.

The uniqueness (up to coefficients equal zero) follows immediately from the observation that $f(X, Y^q)$ is a polynomial in X with Laurent polynomial coefficients. ■

Note that even if $f \in \mathcal{R}_s[Y][X]$ and $\beta \in \mathcal{R}_t[Y]$ for $s \neq t$, then we can consider $q = ts$ and obtain coefficients of f in the fashion of the above theorem. We want to underline that one can apply the above proposition to polynomials. In this case the

leading coefficient of f with respect to $X - \beta$ is a polynomial in Y.

Let us consider a mapping

$$L(x, y) = (x - \beta(y), y),$$

for (x, y) for which β is well defined, where $\beta \in \mathcal{R}_q[Y]$.

The mapping L is a homeomorphism of $\mathbb{R} \times [R, \infty)$ onto itself for any $R > 0$. Indeed, it is well defined for any positive R, continuous, surjective and has a continuous inverse $L^{-1}(x, y) = (x + \beta(y), y)$ for $(x, y) \in \mathbb{R} \times [R, \infty)$.

Denote by $L_* : \mathcal{R}_q[Y][X] \to \mathcal{R}_q[Y][X]$ the mapping $f \to f \circ L$. We have that L_* is an automorphism. Indeed, it is a homomorphism and L_*^{-1} defined as $L_*^{-1}(f) = f \circ L^{-1}$ for $f \in \mathcal{R}_q[Y][X]$ is its inverse.

Throughout this section β, L and L_* will be as above.

Let us consider a tentacle set

$$M = \{(x, y) \in \mathbb{R}^2 |\ \beta_1(y) \leq x \leq \beta_2(y), y \geq R\},$$

where β_1, β_2 are Puiseux parametrizations and $R > 0$. Put $\alpha = \min\{\mathrm{ord}_\infty \beta_1, \mathrm{ord}_\infty \beta_2\}$. Let us now focus on the case not yet solved for polynomials, meaning the case of $\mathrm{ord}_\infty \beta_1 = \mathrm{ord}_\infty \beta_2$.

We will denote by $\mathcal{B}_q(M)$ the subring of $\mathcal{R}_q[Y][X]$ of polynomials with Puiseux coefficients which are bounded on M. Obviously,

$$\mathcal{A}(M) = \mathcal{B}_q(M) \cap \mathbb{R}[X, Y]$$

for any $q \in \mathbb{N}$. Note that all the properties analogous to Property 1.14 for sets $S, M \subset \mathbb{R} \times [R, \infty)$ where $R > 0$ hold for algebras of bounded polynomials with Puiseux coefficients. The correspondence between $\mathcal{B}_q(M)$ and $\mathcal{A}(M)$ will be studied closer later (see Section 5 and Remark 2.19). We would like to point out that $(\frac{1}{y})^{1/q}$ is bounded for $y \geq R > 0$. Hence $\mathbb{R}[\frac{1}{Y^{1/q}}] \subset \mathcal{B}_q(M)$ for $q \in \mathbb{N}$.

Denote $\gamma_1 = \beta_1 - \beta$, $\gamma_2 = \beta_2 - \beta$. Then

$$M = \{(x, y) \in \mathbb{R}^2 |\ \gamma_1(y) \leq x - \beta \leq \gamma_2(y), y \geq R\}.$$

Due to the fact that L is a homeomorphism, we have

$$\begin{aligned} L(M) = \{(w, z) \in \mathbb{R}^2 |\ w = x - \beta(y),\ y = z,\ \gamma_1(y) \leq x - \beta(y) \leq \gamma_2(y),\ y \geq R\} = \\ = \{(w, z) \in \mathbb{R}^2 |\ \gamma_1(z) \leq w \leq \gamma_2(z),\ z \geq R\}. \end{aligned}$$

These observations allow us to make the following observation which is needed to prove the form of algebras of bounded polynomials with Puiseux coefficients, namely

Proposition 2.14. *We have*

$$\mathcal{B}_s(L(M)) = L_*^{-1}(\mathcal{B}_s(M))$$

for any $s \in q \cdot \mathbb{N}$. *In particular*

$$\mathcal{B}_s(L(M)) = \mathbb{R}[\mathcal{Z}] \quad \Longleftrightarrow \quad \mathcal{B}_s(M) = \mathbb{R}[L_*\mathcal{Z}]$$

where $L_*\mathcal{Z} = \{L_*\zeta |\ \zeta \in \mathcal{Z}\}$.

Proof: From Remark 1.15 we get as a straightforward conclusion that a function $f : D \to \mathbb{R}$ with $D \supset M$ is bounded on M if and only if $f \circ L^{-1}$ is bounded on $L(M)$ as well as the inverse. Since L_* is an isomorphism, we get the claim. In particular, for $f \in \mathcal{B}_q(M)$ we

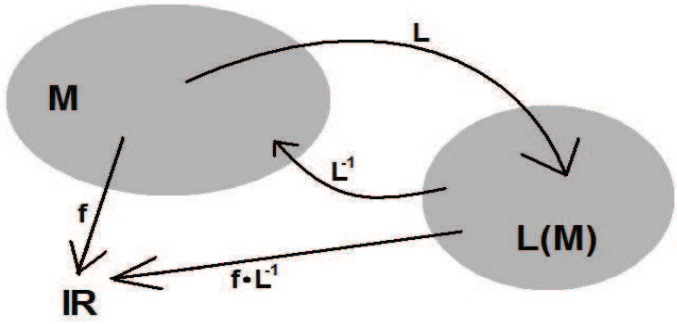

Figure 4. A function f is bounded on a set M if and only if $f \circ L^{-1}$ is bounded on $L(M)$

get that

$$L_*^{-1}f = L_*^{-1}(\sum_{\alpha \in A} a_\alpha \zeta^\alpha) = \sum a_\alpha (L_*^{-1}\zeta)^\alpha$$

for some finite collection of $\zeta_1, \ldots, \zeta_m \in Z$, a finite set $A \subset \mathbb{N}_0^m$ and $a_\alpha \in \mathbb{R}$ for $\alpha \in A$. For $g \in \mathcal{B}_q(L(M))$ it is also obvious that L_*g is generated by elements of $\mathcal{Z}$. ■

Now we will prove a proposition which gives us effectively (through a finite search of coefficients in Puiseux series) a Puiseux polynomial that will be used in the subsequent theorem.

Proposition 2.15. *Let us suppose that* $\beta_1 \neq \beta_2$ *while* $\operatorname{ord}_\infty \beta_1 = \operatorname{ord}_\infty \beta_2$. *Then there exists a Puiseux polynomial* β *such that all of the following conditions hold*

(1) $deg_\infty \beta \leq \min_{i=1,2} \operatorname{ord}_\infty(\beta_i - \beta)$
(2) $\operatorname{ord}_\infty \beta = \operatorname{ord}_\infty \beta_1$
(3) $\operatorname{ord}_\infty(\beta_1 - \beta) \neq \operatorname{ord}_\infty(\beta_2 - \beta)$.

The choice of β is illustrated on Figure 5.

Proof: We can always rewrite two Puiseux series as having the same denominator q,

so we can write $\beta_1 = \sum_{i=p}^{\infty} b_i^{(1)} (\frac{1}{Y})^{i/q}$ and $\beta_2 = \sum_{i=p}^{\infty} b_i^{(2)} (\frac{1}{Y})^{i/q}$ with $b_i^{(1)} = b_i^{(2)}$ for $i = p, \ldots, m-1$, $b_m^{(1)} \neq b_m^{(2)}$ for some $m \geq p$ (if $m = p$ then the leading coefficients of β_1, β_2 are not equal). Without loss of generality we can suppose that $b_m^{(2)} \neq 0$. Then it suffices

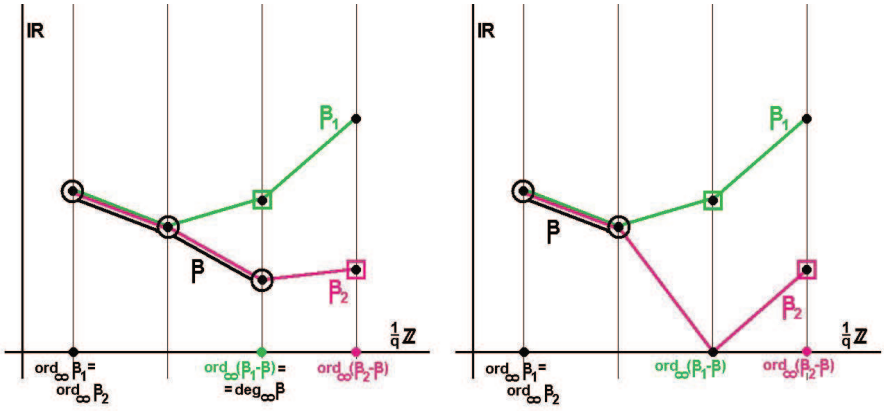

Figure 5. On this figure on the horizontal axis we mark the points of $\mathrm{supp}(\beta_1) \cup \mathrm{supp}(\beta_2)$ whereas on the vertical axis we mark the corresponding coefficients of β_1 and β_2. An exemplary Puiseux polynomial β from Proposition 2.15 is marked black. We can see that in the case on the left, when $b_m^{(1)}, b_m^{(2)} \neq 0$ we have $\deg_\infty \beta = \mathrm{ord}_\infty(\beta_1 - \beta)$, whereas in the case on the right we have $b_m^{(2)} = 0$ and $\deg_\infty \beta < \min(\mathrm{ord}_\infty(\beta_1 - \beta), \mathrm{ord}_\infty(\beta_2 - \beta))$. Moreover, in such a case β satisfying conditions of Proposition 2.15 is unique. In particular, one cannot choose β such that there is an equality in the first condition of Proposition 2.15.

to take

$$\beta = \sum_{i=p}^{m} b_i^{(1)} (\frac{1}{Y})^{i/q}.$$

We have that $\mathrm{ord}_\infty \beta_2 - \beta = m/q$ and $\mathrm{ord}_\infty \beta_1 - \beta > m/q$. Since $\deg_\infty \beta \leq m/q$, the conditions (1), (2) and (3) are satisfied. ■

Note that we do not need to know the whole formula on β_1, β_2. It suffices to indicate a finite number of initiating coefficients of β_1, β_2. So in particular β might belong to a $\mathcal{R}_q[Y]$ where q is not divisible by any of the denominators in full representations of β_1, β_2. We can view β as a Puiseux polynomial deformation from a set asymptotically close to the y axis which were studied in Section 2.1.

Theorem 2.16. *If $\beta_1 \neq \beta_2$, then there exists a Puiseux polynomial $\beta \in \mathcal{R}_q[Y]$ such that*

$$\mathcal{B}_s(M) = \mathbb{R}\left[\frac{1}{Y^{1/s}}, (X-\beta)^i Y^d \mid d \leq i \cdot \min_{i=1,2} \mathrm{ord}_\infty(\beta_i - \beta), i \in \mathbb{N}_0, d \in 1/s \cdot \mathbb{N}_0\right]$$

for any $s \in q \cdot \mathbb{N}$.

Proof: First, assume that $\mathrm{ord}_\infty \beta_1 \neq \mathrm{ord}_\infty \beta_2$. Take $f \in \mathcal{R}_q[Y][X]$. In particular, f is polynomial with respect to X and the degrees and orders of their coefficients are well

defined. By (2.2), (2.3) and (2.4) we conclude that in the proof of Theorem 2.1 one can substitute a polynomial from $\mathbb{R}[X,Y]$ by a polynomial in X with Puiseux polynomial coefficients. We obtain

$$\mathcal{B}_q(M) = \mathbb{R}\left[\frac{1}{Y^{1/q}}, X^iY^d \mid d \leq i\alpha, i \in \mathbb{N}_0, d \in 1/q \cdot \mathbb{N}_0\right]$$

for any $q \in \mathbb{N}$, where $\alpha = \min(\mathrm{ord}_\infty\beta_1, \mathrm{ord}_\infty\beta_2)$. Hence here it suffices to take $\beta = 0$.

Assume now that $\mathrm{ord}_\infty\beta_1 = \mathrm{ord}_\infty\beta_2$. Take β as in Proposition 2.15. We have that $\beta \in \mathcal{R}_q[Y]$ for some natural number q. Take $L : \mathbb{R} \times [R,\infty) \to \mathbb{R} \times [R,\infty)$, where $L(x,y) = (x - \beta(y), y)$. After putting $\gamma_1 = \beta_1 - \beta$, $\gamma_2 = \beta_2 - \beta$ we get

$$L(M) = \{(w,z) \in \mathbb{R}^2 |\ \gamma_1(z) \leq w \leq \gamma_2(z), z \geq R\},$$

where $\mathrm{ord}\gamma_1 \neq \mathrm{ord}\gamma_2$ by Proposition 2.15.

Put $\eta = \min_{i=1,2}(\mathrm{ord}_\infty\gamma_i)$. Therefore by the first part of this proof we have

$$\mathcal{B}_s(L(M)) = \mathbb{R}[\frac{1}{Z^{1/s}}, W^iZ^d \mid d \leq i\eta, i \in \mathbb{N}_0, d \in 1/s \cdot \mathbb{N}_0].$$

By Proposition 2.14 we have $\mathcal{B}_s(M) = \mathbb{R}[L_*\frac{1}{Z^{1/s}}, L_*(W^iZ^d) \mid d \leq i\eta]$. Since

$$L_*\frac{1}{Z^{1/s}} = \frac{1}{Y^{1/s}}, L_*(W^iZ^d) = (X-\beta)^iY^d,$$

we get our claim. ■

We give a slight refinement of the above theorem.

Proposition 2.17. *If $\beta_1 \neq \beta_2$ and $\mathrm{ord}_\infty\beta_1 = \mathrm{ord}_\infty\beta_2$, then the Puiseux polynomial β in Theorem 2.16 can be chosen such that it meets the assumptions of Proposition 2.15 and*

$$deg_\infty\beta < \min_{i=1,2} \mathrm{ord}_\infty(\beta_i - \beta).$$

Proof: Indeed, take β from Theorem 2.16. If $\deg_\infty \beta = \min_{i=1,2} \mathrm{ord}_\infty(\beta_i - \beta) = \eta$, then $\beta = \sum_{i=p}^m b_i(\frac{1}{Y})^{i/q}$ with $b_\eta \neq 0$. Without loss of generality we can suppose that $b_\eta > 0$. Put $\gamma = \beta - b_\eta(1/y)^\eta$. Hence from Theorem 2.4 and Proposition 2.14 we get

$$\begin{aligned}\mathcal{B}_s(M) &= \mathcal{B}_s(\beta_1 - \beta \leq x - \beta \leq \beta_2 - \beta,\ y \geq R) = \\ &= \mathcal{B}_s(-b_\eta(1/y)^\eta \leq x - \beta \leq 0,\ y \geq 1) = \\ &= \mathcal{B}_s(0 \leq x - \gamma \leq b_\eta(1/y)^\eta,\ y \geq 1)\end{aligned}$$

for any $s \in q \cdot \mathbb{N}$. Therefore by Theorem 2.16 we have

$$\mathcal{B}_s(M) = \mathbb{R}\left[\frac{1}{Y^{1/s}}, (X-\gamma)^iY^d \mid d \leq i \cdot \eta, i \in \mathbb{N}_0, d \in 1/s \cdot \mathbb{N}_0\right]$$

and $\deg_\infty \gamma < \eta \leq \min_{i=1,2} \mathrm{ord}_\infty(\beta_i - \gamma)$. Moreover, $\mathrm{ord}_\infty(\beta_1 - \gamma) = \mathrm{ord}_\infty(\beta_2 - \gamma) = \min_{i=1,2} \mathrm{ord}_\infty(\beta_i - \beta)$, because $\gamma = \beta - b_\eta(1/y)^\eta$ and b_η is the first coefficient where β_1

and β_2 differ. Note that γ equals zero if and only if β_1 and β_2 do not have common leading coefficients. ■

We see that if β meets all the assertions of the above proposition, then it is unique, because it must be equal to all initiating terms in β_1 and β_2 which coincide. Nevertheless in general, one can easily find different Puiseux polynomials which give the same algebra $\mathcal{B}_q(M)$ eg. in Example 2.8 we have $\mathcal{B}_1(M) = \mathbb{R}[\frac{1}{Y}, X, XY] = \mathbb{R}[\frac{1}{Y}, X - \frac{1}{Y}, (X - \frac{1}{Y})Y]$.

From presented results it follows that to know how the set of bounded Puiseux polynomials looks like, we have to study the common asymptotics of the borders of the tentacle. If the parametrizations of the borders have different orders or different leading coefficients, then by Theorem 2.16 and Proposition 2.17 the algebras $\mathcal{B}_q(M)$ are generated by monomials and the element $1/Y^{1/q}$. Hence the algebras of bounded polynomials are generated by monomials. But if they have common leading coefficients then we cannot tell as easily the generators of the intersections $\mathcal{B}_q \cap \mathbb{R}[X, Y]$.

Example 2.18. Take

$$M_1 = \{0 \leq y \leq \frac{1}{x^2},\ x \geq 1\} \quad \textit{and} \quad M_2 = \{0 \leq x + \frac{1}{y} \leq \frac{1}{y^2},\ y \geq 1\}.$$

We have $\mathcal{A}(M_1) = \mathbb{R}[Y, XY, X^2Y]$ and

$$\mathcal{B}_q(M_2) = \mathbb{R}[\frac{1}{Y^{1/q}}, (X + \frac{1}{Y})^i Y^d |\ d \leq 2i] = \mathbb{R}[\frac{1}{Y^{1/q}}, X + \frac{1}{Y^{1/q}}, XY + 1, XY^2 + Y]$$

for any $q \in \mathbb{N}$. From Proposition 2.13 we have

$$X^2Y = (X + \frac{1}{Y})^2 Y + 2(X + \frac{1}{Y}) + \frac{1}{Y}.$$

This gives us in particular that the set of bounded polynomials on the union of tentacles $M_1 \cup M_2$ is nonempty, because $X^2Y \in \mathcal{A}(M_1) \cap \mathcal{A}(M_2)$. Knowing this one can easily see that $\mathbb{R}[XY, X^2Y, X^2Y + Y] \subset \mathcal{A}(M_1 \cup M_2)$. The author does not know whether an equality holds.

As an easy consequence of equivalence (1.4) combined with Theorem 2.4 or Theorem 2.16 in a special case when β is polynomial we get

Remark 2.19. *Under notation and assumptions of the previous theorem if $\beta \in \mathbb{R}[Y]$, then*

$$\mathcal{A}(M) = \mathbb{R}\left[(X - \beta)^i Y^d \mid d \leq i\eta, \quad i, d \in \mathbb{N}_0\right].$$

This remark gives us that if M is an image by a polynomial automorphism $L = (X - \beta, Y)$ of some set with an algebra generated by monomials $X^{i_1}Y^{d_1}, \ldots, X^{i_k}Y^{d_k}$ for some $k \in \mathbb{N}$, then $\mathcal{A}(M) = \mathbb{R}[(X - \beta)^{i_1}Y^{d_1}, \ldots, (X - \beta)^{i_k}Y^{d_k}]$.

To end this paragraph note that the above theorems are insensitive to symmetry with respect to x axis and y axis as well as an interchange between x and y, since they are

linear change of coordinates (see Proposition 1.16). Therefore, similarly as in the previous section, the results of Theorem 2.16 and its consequences hold for tentacle sets of the form (1.7) as well as those with borders parameterized by x instead of y with the exponents in the characterisations appropriately interchanged.

Remark 2.20. *Every algebra of bounded polynomials in the case of tentacle sets of the form (1.7) with* $\operatorname{ord}_\infty\beta_1 = \operatorname{ord}_\infty\beta_2$ *and* $\beta_1 \neq \beta_2$ *is an algebra of a set of the form*

$$\{(x,y) \in \mathbb{R}^2 |\ 0 \leq x - \beta \leq y^\eta,\ y \geq 1\} \tag{2.8}$$

for $\eta = \min_{i=1,2} \operatorname{ord}_\infty(\beta_i - \beta)$ *and a Puiseux polynomial* β *satisfying conditions of Proposition 2.15 with* $\deg_\infty \beta < \eta \leq \min_{i=1,2} \operatorname{ord}_\infty(\beta_i - \beta)$.

Note that this representation is not unique (for example in the case of trivial algebras). The equality of algebras follows directly from Theorem 2.16.

To end this section let us note that we do not tackle in this thesis the problem of how the intersections of bounded polynomials with Puiseux coefficients with $\mathbb{R}[X,Y]$ look like. As long as we stay in the scope of algebras generated by monomials, everything is easy (as we will see in Section 5). However, in general, algebras of bounded polynomials are not generated by monomials (or compositions of monomials with coordinates of polynomial automorphisms) as we can see in Example 2.18, where $\mathbb{R}[X, XY, XY^2+Y] \subset \mathbb{R}[1/Y, X+1/Y, XY+1, XY^2+Y] \cap \mathbb{R}[X,Y]$ and Y is not bounded. Nevertheless, finding a representation of a polynomial in terms of other polynomials tends to be difficult, whereas symbolic division of f in the wider ring of polynomials with Puiseux (polynomial) coefficients (as in Proposition 2.13) is always easy to execute. Therefore, we content ourselves with giving an efficient method of verifying whether a polynomial is bounded on a set (as will be explained more explicitly in Section 6.1).

3. Stability of algebras of bounded polynomials in two variables

The main goal of this section will be to prove in Theorem 3.5 that if we have a family of sets in $\mathbb{R}^2$ defined by a given polynomial inequality where we let the constant terms vary, then we get only a finite collection of algebras of bounded polynomials i.e. for any $f \in \mathbb{R}[X, Y]$ there exists a finite collection of algebras $\mathcal{A}_1, \ldots, \mathcal{A}_k$ such that $\mathcal{A}(f \leq c) = \mathcal{A}_i$ for some $i = 1, \ldots, k$ and every $c \in \mathbb{R}$.

Before we prove the theorem we will prepare the tools in Sections 3.1 and 3.2. As to notation, recall that in expressions of the form $\mathcal{A}(\{(x, y) \in \mathbb{R}^n |\ a \leq f(x, y) \leq b\})$ we often leave out the brackets and write simply $\mathcal{A}(a \leq f \leq b)$ for a polynomial $f : \mathbb{R}^n \to \mathbb{R}$.

3.1. Bifurcation values

Throughout this section let $\mathbb{K}$ be the field of either real or complex numbers. Moreover, we will assume that the empty set is compact and in consequence that $\mathcal{A}(\emptyset)$ is equal to the ring of polynomials.

We say that a function $f : \mathbb{K}^n \to \mathbb{K}$ is *a trivial $\mathcal{C}^\infty$-fibration* over an open set $U \subset \mathbb{K}$ if for any $c \in U$ there exists a $\mathcal{C}^\infty$-diffeomorphism

$$\phi : f^{-1}(U) \to U \times f^{-1}(c)$$

such that

$$\pi \circ \phi = f,$$

where π is the projection on the first coordinate.

In particular, if f is a trivial $\mathcal{C}^\infty$-fibration over U, then for any $c_0 \in U$ we get that $f^{-1}(c_0)$ is diffeomorphic to $f^{-1}(c)$ for $c \in U$. Hence the fibre $f^{-1}(c_0)$ is compact if and only if for every $c \in U$ the fibre $f^{-1}(c)$ is compact as well.

We say that a number $c \in \mathbb{K}$ is *a typical value of* $f : \mathbb{K}^n \to \mathbb{K}$ if there exists a neighbourhood U of c such that f is a trivial $\mathcal{C}^\infty$-fibration over U. If c is not typical

then we will say that it is *a bifurcation value of* f. By $B_{\mathbb{K}}(f)$ we will denote the set of bifurcation values of f.

From the definition it is easy to see that

Property 3.1. *Let* $f \in \mathbb{K}[X]$. *We have*

$$B_{\mathbb{K}}(f+c) = \{\lambda + c |\ \lambda \in B_{\mathbb{K}}(f)\}$$

and

$$B_{\mathbb{K}}(cf) = \{c\lambda |\ \lambda \in B_{\mathbb{K}}(f)\}$$

for any $c \in \mathbb{K} \setminus \{0\}$.

We will make use of the above property in the next section. Now let us turn to critical and asymptotic critical values, which are of practical use.

Let $f : \mathbb{K}^n \to \mathbb{K}$. We say that $c \in \mathbb{K}$ is *a critical value of* f if there exists $a \in \mathbb{R}^n$ such that

$$f(a) = c \quad \textit{and} \quad \nabla f(a) = 0.$$

We say that $c \in \mathbb{K}$ is *an asymptotic critical value of* f if there exists a sequence $(a_k) \subset \mathbb{R}^n$ which tends to infinity such that

$$f(a_k) \to c \quad \textit{and} \quad \|a_k\|\,|\nabla f(a_k)| \to 0$$

as $a_k \to \infty$. We say that $c \in \mathbb{K}$ is *a generalized critical value of* f if it is a critical value or an asymptotic critical value. We will denote the set of generalized critical values of $f : \mathbb{K}^n \to \mathbb{K}$ as $K_{\mathbb{K}}(f)$.

Every bifurcation value which is not a critical value is necessarily an asymptotic critical value. If $f : \mathbb{K}^n \to \mathbb{K}$ is a polynomial then $B_{\mathbb{K}}(f) \subset K_{\mathbb{K}}(f)$ and $K_{\mathbb{K}}(f)$ is finite (see [Ha], [KOS], [JK]). Moreover, the set $K_{\mathbb{K}}(f)$ has at most $(2\deg f + 1)(4\deg f - 3)^n$ points more then the set of critical values. Note that if we consider a real polynomial f as a complex polynomial we have $B_{\mathbb{R}}(f) \subset B_{\mathbb{C}}(f) \cap \mathbb{R}$. For further information on bifurcation values and generalised critical values see for example [Rab], [KOS], [Par], [JK], [KS] and [NZ]. We would like also to underline that there exist explicit algorithms to compute the set of generalized critical values (eg. using Gröbner basis, see [JK]).

Example 3.2. Consider Motzkin's polynomial

$$\mathfrak{m} = 1 + X^2Y^2(Y^2 + X^2 - 3).$$

We have

$$\nabla\mathfrak{m}(x,y) = 2xy\left(y(y^2 + 2x^2 - 3), x(x^2 + 2y^2 - 3)\right)$$

for $x, y \in \mathbb{K}$. Hence after some computation we easily get that the generalized critical values are

$$K_{\mathbb{C}}(\mathfrak{m}) = \{0, 1\} = K_{\mathbb{R}}(\mathfrak{m}),$$

whereas the sets of critical values and asymptotic critical values are equal. It is known that the set of complex bifurcation values, real bifurcation values and generalized critical values of $\mathfrak{m}$ are equal.

In the case of $\mathbb{K} = \mathbb{R}$, i.e. $\mathfrak{m} : \mathbb{R}^2 \to \mathbb{R}$, it is easy to see that $\mathfrak{m}(x, y) \neq -3$ for $x, y \in \mathbb{R}$. Since $\mathfrak{m}$ is a trivial fibration over $(-\infty, 0)$, we get that $\mathfrak{m}^{-1}(c) = \mathfrak{m}^{-1}(-3) = \emptyset$ for every $c < 0$. We have that $\mathfrak{m}^{-1}(\frac{1}{2})$ is diffeomorphic to four disjoint ovals. Hence $\mathfrak{m}^{-1}(c)$ is bounded for every $c \in (0, 1)$, because $\mathfrak{m}$ is a trivial fibration over $(0, 1)$. Analogously we get that $\mathfrak{m}^{-1}(c)$ is unbounded for every $c > 1$. As to the special fibres, we have that $\mathfrak{m}^{-1}(0)$ consists of four points and $\mathfrak{m}^{-1}(1)$ is the union of a sphere $S(0, \sqrt{3})$, the x and y axis.

3.2. Fibres and bifurcation values of polynomials in two variables

In this section we will focus on polynomials in two variables and the set of complex bifurcation values.

Let $f \in \mathbb{C}[X, Y]$ be nonconstant. There exists a linear change of coordinates such that it is of the form

$$f = X^d + \sum_{i=1}^{d} a_i(Y) X^{d-i}, \tag{3.1}$$

where $\deg a_i \leq i$. Note that if f is a polynomial with real coefficients there exists a linear change of coordinates L such that $L(\mathbb{R}^2) = \mathbb{R}^2$ and f is of the form (3.1) up to the sign.

We will start with a characterisation of the set of bifurcation values by generalized critical values. Namely, in the case of two variables we have

$$B_{\mathbb{C}}(f) = K_{\mathbb{C}}(f)$$

for a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ (see [Ha], [Par]) and the number of bifurcation values of f is less or equal $\deg f(\deg f - 1)$. One can see that it is much easier to compute generalized critical values than decide what are the bifurcation values.

We will recall another characterisation of the set of bifurcation values of a polynomial f in the case of two variables due to Krasinski [Kr]. Assume that f is of the form (3.1) and let

$$D(c, Y) = \delta_0(c) Y^k + \ldots + \delta_k(c)$$

be the discriminant of the polynomial $\mathbb{C}^3 \ni (c, x, y) \to f(x, y) - c \in \mathbb{C}$ with respect to X with $\delta_0 \neq 0$ (see [Kr, Chapter 1]). The set of bifurcation values of f is exactly the union of the set of critical values of f and the set of roots of δ_0 (see [Kr, Theorem 18.1]).

The following theorem is due to Krasinski [Kr2, Lemma 2] and we will sometimes refer to it as the Parametric Puiseux Theorem (at infinity). The version given below is a slight reformulation in order to fit it in our setting.

Theorem 3.3. (T. Krasinski, 1991) *Let $f \in \mathbb{C}[X,Y]$ be of the form (3.1). Take $c_0 \notin B_{\mathbb{C}}(f)$. There exist $r, R > 0$ and holomorphic mappings*

$$\Theta_i : B(c_0, r) \times \{y \in \mathbb{C} |\ |y| > R^{1/q}\} \to \mathbb{C}^2, \quad i = 1, \ldots, m,$$

of the form

$$\Theta_i(c, y) = (\theta_i(c, y), y^q),$$

where each θ_i is holomorphic with respect to c and $\theta_i(c, Y^{1/q})$ is a complex Puiseux series at infinity for every c. Moreover,

$$f(\Theta_i(c, \cdot)) = c$$

for every $c \in B(c_0, r)$. Furthermore, for any $c \in B(c_0, r)$ the set $f^{-1}(c) \setminus (\mathbb{C} \times \overline{B(0, R^{1/q})})$ has exactly m connected components and

$$f^{-1}(c) \setminus (\mathbb{C} \times \overline{B(0, R^{1/q})}) = \bigcup_{i=1}^{m} \Theta_i(\{(c, y) \in \mathbb{C}^2 |\ |y| > R^{1/q}\}).$$

Hence if we choose a complex Puiseux series at infinity γ such that $f(\gamma(y^q), y^q) = c_0$ for $|y|^q > R$, then there exists exactly one i_0 such that

$$\{(\gamma(y^q), y^q) \in \mathbb{C}^2 |\ |y|^q > R\} = \{(\theta_{i_0}(c_0, y), y^q) \in \mathbb{C}^2 |\ |y|^q > R\}.$$

Let us remind the fact that for a real polynomial $f \in \mathbb{R}[X,Y]$ if for $y \in U$ the polynomial $f(X, y)$ has a constant number of distinct complex roots, then $f(X, y)$ has a constant number of real roots for $y \in U$, provided $U \subset \mathbb{R}$ is connected. Suppose that f is of the form (3.1). Then from Parametric Puiseux Theorem it follows that on any half-line $I \subset (-\infty, -R) \cup (R, \infty)$ the polynomial $f(X, \cdot)$ has a constant number of complex roots. Take one of the m families of Puiseux parametrizations at infinity $\{\theta(c, \cdot)\}_{c \in B(c_0, r)}$ of fibres of the real polynomial f as in the Parametric Puiseux Theorem. From the above considerations we get that if the set $graph(\theta(c_0, \cdot)) \cap \mathbb{R}^2$ is unbounded, then $graph(\theta(c, \cdot)) \cap \mathbb{R}^2$ is unbounded for all $c \in B(c_0, r) \cap \mathbb{R}$. Moreover, $\theta(c, \cdot) : I \to \mathbb{R}$ is a convergent Puiseux series at infinity with real coefficients for every $c \in (c_0 - r, c_0 + r)$ on a half-line $I \subset \mathbb{R}$. Hence

Property 3.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of the form (3.1). Under the notation of Theorem 3.3 if $[c, \tilde{c}] \subset B(c_0, r) \cap \mathbb{R}$, then every tentacle M of the set $f^{-1}([c, \tilde{c}])$ is either of the form*

$$\{(x, y) \in \mathbb{R}^2 |\ \theta_i(c_1, y^{1/q}) \leq x \leq \theta_i(\tilde{c_1}, y^{1/q}),\ y \geq \tilde{R}\}$$

or

$$\{(x, -y) \in \mathbb{R}^2 |\ \theta_j(c_2, y^{1/q}) \leq x \leq \theta_j(\tilde{c_2}, y^{1/q}),\ y \geq \tilde{R}\}$$

where $i, j \in \{1, \ldots, m\}$, $\tilde{R} > R$ is sufficiently big, $\theta_i(c_0, Y^{1/q})$ is a Puiseux parametrization at infinity and $c_1 \neq \tilde{c_1}, c_2 \neq \tilde{c_2}$, $c_1, \tilde{c_1}, c_2, \tilde{c_2} \in \{c, \tilde{c}\}$ are chosen so that the sets above are nonempty.

Therefore we have that under the above assumptions the tentacles of $f^{-1}([c,\tilde{c}])$ are of the form (1.7) from Section 1.6 on tentacle sets.

3.3. Stability

The goal of this section is to prove the following

Theorem 3.5. (Stability of algebras of bounded polynomials) *Given any polynomial $f \in \mathbb{R}[X,Y]$ and numbers $0 < c < \tilde{c} \in \mathbb{R}$ such that*

$$[c,\tilde{c}] \cap B_{\mathbb{C}}(f) = \emptyset$$

we have

$$\mathcal{A}(0 \leq f \leq c) = \mathcal{A}(0 \leq f \leq \tilde{c}).$$

The proof of this theorem follows easily from the following lemma.

Lemma 3.6. (Local stability) *Let $f \in \mathbb{R}[X,Y]$ be such that $0 \notin B_{\mathbb{C}}(f)$. There exists an $r > 0$ such that for every $\epsilon, \delta \in (0,r)$ we have*

$$\mathcal{A}(-\epsilon \leq f \leq \epsilon) = \mathcal{A}(0 \leq f \leq \delta).$$

Proof: First, consider the case when the fibre $f^{-1}(0)$ is compact. Then from the fact that 0 is a typical value it follows that for all c from some neighbourhood of 0, in particular of the form $(-r,r)$ for some $r > 0$, the fibres $f^{-1}(c)$ are also compact. From Proposition 1.13 it follows that $f^{-1}([-\epsilon,\epsilon])$ and $f^{-1}([0,\delta])$ are compact. Hence

$$\mathcal{A}(-\epsilon \leq f \leq \epsilon) = \mathbb{R}[X,Y] = \mathcal{A}(0 \leq f \leq \delta)$$

for any $0 < \epsilon, \delta < r$.

From now on we will assume that $f^{-1}(0)$ is not compact. If f is constant, then necessarily $f = 0$ and $0 \in B_{\mathbb{C}}(f)$. That contradicts the assumption. Hence we will assume that f is nonconstant. We can change the coordinates with a real linear automorphism such that f is up to the sign a monic polynomial of the form (3.1). Without loss of generality we can assume that f is exactly of the form (3.1).

Assume that the fibre $f^{-1}(0)$ has exactly l tentacles. From the Parametric Puiseux Theorem and Property 3.4 it follows that there exists an $r > 0$ such that $f^{-1}((-r,r))$ has exactly l tentacles. Moreover, each tentacle of the fibre $f^{-1}(0)$ has an unbounded intersection with exactly one tentacle of $f^{-1}((-r,r))$ i.e. they are in a one-to-one correspondence.

Choose a Puiseux parametrization at infinity $\gamma \in \mathcal{R}_q[Y]$ of a tentacle of the fibre $f^{-1}(0)$ and a tentacle M of the set $f^{-1}((-r,r))$ such that $\{(\gamma(y),y) \in \mathbb{R}^2 |\ y \geq R^{1/q}\} \cap M$ is unbounded.

Let us consider f as a complex polynomial and choose a family $\{\theta(c,\cdot)\}_{c \in B(0,r)}$ from Parametric Puiseux Theorem such that $graph(\theta(0,Y)) \supset graph(\gamma(Y^q))$. We can write

$$\theta(c, Y) = \sum_{i=p}^{\infty} b_i(c) 1/Y^i,$$

where $b_i : (-r, r) \to \mathbb{R}$ due to Theorem 3.3 are analytic for $i \geq p$ and $b_p \neq 0$.

We will now show that the claim of the lemma holds for the tentacle M. In order to do this consider analytic functions

$$b_k(0) - b_k(\epsilon)$$

defined on $(-r, r)$ for $k \geq p$. There exists an $m \geq p$ such that $b_k(0) - b_k(\cdot) = 0$ for $p \leq k < m$ and $b_m(0) - b_m(\cdot) \neq 0$. Take a set

$$A = \{\epsilon \in (0, r) |\ (b_m(0) - b_m(\epsilon))\,(b_m(0) - b_m(-\epsilon)) = 0\}\,.$$

From the choice of m the set A is a proper analytic subset of $(0, r)$. For every $\epsilon \in (0, r) \setminus A$ we have

$$m = \operatorname{ord}_\infty(\theta(\epsilon, Y) - \theta(0, Y)) = \operatorname{ord}_\infty(\theta(-\epsilon, Y) - \theta(0, Y)). \tag{3.2}$$

Hence for every $\epsilon \in (0, r) \setminus A$ the series $\theta(0, Y)$, $\theta(-\epsilon, Y)$ and $\theta(\epsilon, Y)$ have $m - p$ common initiating terms.

Take the Puiseux polynomial

$$\beta = \sum_{i=p}^{m} b_i(0) \frac{1}{Y^{i/q}}$$

as in Proposition 2.15.

For every $\epsilon \in (0, r) \setminus A$ we have

$$\operatorname{ord}_\infty(\beta - \theta(0, Y^{\frac{1}{q}})) > \operatorname{ord}_\infty(\beta - \theta(-\epsilon, Y^{\frac{1}{q}})) = \operatorname{ord}_\infty(\beta - \theta(\epsilon, Y^{\frac{1}{q}})) = \frac{m}{q}.$$

Hence from Property 3.4, Proposition 2.15 and Theorem 2.16 it follows that for any $\epsilon \in (0, r) \setminus A$ the following holds

$$\begin{aligned}\mathcal{A}(M \cap f^{-1}([0, \epsilon])) = \mathbb{R}[X, Y] \cap \mathbb{R}\left[\frac{1}{Y^{1/q}}, (X - \beta)^i Y^d \mid d \leq i \cdot \frac{m}{q}\right] = \\ = \mathcal{A}(M \cap f^{-1}([-\epsilon, 0])).\end{aligned} \tag{3.3}$$

The above algebra depends on the choice of the tentacle M, but does not depend on the choice of ϵ as long as $\epsilon \notin A$. For the purposes of the proof for the tentacle M denote the above algebra as $\mathcal{A}_M$. From the fact that 0 is a typical value of f and the above observation for $\delta, \epsilon \in (0, r) \setminus A$ we have

$$\begin{aligned}\mathcal{A}(M \cap f^{-1}([-\epsilon, \epsilon])) = \mathcal{A}(M \cap f^{-1}([-\epsilon, 0])) \cap \mathcal{A}(M \cap f^{-1}([0, \epsilon])) = \\ = \mathcal{A}_M \cap \mathcal{A}_M = \mathcal{A}_M = \mathcal{A}(M \cap f^{-1}([0, \delta])).\end{aligned} \tag{3.4}$$

Now we will show that $\mathcal{A}_M$ does not depend on the choice of ϵ even if $\epsilon \in A$. Take any $\epsilon \in (0, r)$. Since A is proper analytic there exist $\delta, \rho \in (0, r) \setminus A$ such that $\delta < \epsilon < \rho$. Hence

$$f^{-1}([0, \delta]) \cap M \ \subset \ f^{-1}([0, \epsilon]) \cap M \ \subset \ f^{-1}([0, \rho]) \cap M.$$

From (3.4) it follows that $\mathcal{A}(f^{-1}([0, \epsilon])) = \mathcal{A}_M$. By analogy we get $\mathcal{A}(f^{-1}([-\epsilon, \epsilon])) = \mathcal{A}_M$. Hence the claim of the lemma holds for the tentacle M.

Denote by $M_1, \ldots, M_l$ all tentacles of the set $f^{-1}((-r, r))$ and by $\mathcal{A}_{M_1}, \ldots, \mathcal{A}_{M_l}$ the algebras from equality (3.3). Hence for $\epsilon, \delta \in (0, r)$ we have

$$\mathcal{A}(f^{-1}([-\epsilon, \epsilon])) = \bigcap_{i=1}^{l} \mathcal{A}(M_i \cap f^{-1}([-\epsilon, \epsilon])) = \bigcap_{i=1}^{n} \mathcal{A}_{M_i}$$

and

$$\mathcal{A}(f^{-1}([0, \delta])) = \bigcap_{i=1}^{n} \mathcal{A}_{M_i}$$

where $\mathcal{A}_{M_i}$ are independent on the choice of ϵ or δ. This ends the proof. ■

Note that we cannot jump to conclusions regarding the algebra of bounded polynomials on the set $f^{-1}([-r, r])$, where r is as in the Parametric Puiseux Theorem. Indeed, take the Motzkin polynomial $\mathfrak{m} : \mathbb{R}^2 \to \mathbb{R}$ from Example 3.2. The closure of the set $\mathfrak{m}^{-1}((-1, 1)) \subset \mathbb{R}^2$ is compact, but the algebra of bounded polynomials on $\mathfrak{m}^{-1}([-1, 1])$ is not $\mathbb{R}[X, Y]$ (see Example 3.9).

Remark 3.7. *Under the assumptions of Lemma 3.6 for every $\epsilon, \delta, \rho \in (0, r)$ we have*

$$\mathcal{A}(\epsilon \leq f \leq \rho) = \mathcal{A}(-\delta \leq f \leq 0).$$

Indeed, from the above lemma and Proposition 3.1 it follows immediately that the above claim is equivalent to the claim of the lemma.

Now we are ready to prove the Stability Theorem.

Proof of Theorem 3.5: Take $0 < c < \tilde{c}$ as in the assumptions of the theorem. Since $f^{-1}([0, c]) \subset f^{-1}([0, \tilde{c}])$, we need only prove that $\mathcal{A}(0 \leq f \leq c) \subset \mathcal{A}(0 \leq f \leq \tilde{c})$. Suppose to the contrary. Then there exists a polynomial $g \in \mathbb{R}[X, Y]$ and a certain number $c_0 \in (c, \tilde{c}]$ such that $g \in \mathcal{A}(0 \leq f \leq c_0 - \epsilon)$ and $g \notin \mathcal{A}(0 \leq f \leq c_0 + \epsilon)$ for every positive number ϵ (see Figure 6). Take $\delta > 2\epsilon > 0$ such that both ϵ and δ are sufficiently small. Hence from the lemma on local stability if we put $\rho = \delta - 2\epsilon$, we get

$$\begin{aligned} \mathcal{A}(c_0 - \delta \leq f \leq c_0 - \epsilon) &= \mathcal{A}(-\delta + \epsilon \leq f - (c_0 + \epsilon) \leq 0) = \\ &= \mathcal{A}(-\delta + \epsilon \leq f - (c_0 + \epsilon) \leq \delta - \epsilon) = \\ &= \mathcal{A}(c_0 - \delta \leq f \leq c_0 + \rho). \end{aligned} \tag{3.5}$$

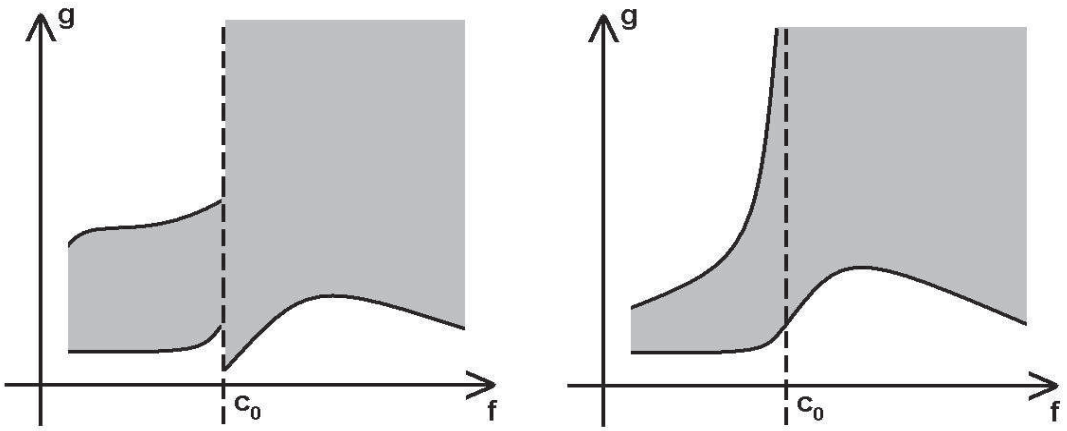

Figure 6. On this figure we have the values of a mapping with f and g for coordinates. In the proof of Theorem 3.5 there are in fact two cases. But if c_0 is a typical value of f, both are impossible. In the first case, on the left, g is bounded on $\{f < c_0\}$. Since c_0 is typical, we have $\{f \leq c\} = \overline{\{f < c\}}$. We see from the picture that it is impossible, because g is continuous. In the second case, g attains a maximum on each fibre of f, but it is impossible that its maximum tends to infinity due to Lemma 3.6.

Hence if g is bounded on $f^{-1}([0, c_0 - \epsilon])$ by assumption, then by the equality (3.5) and properties of typical values we get

$$\begin{aligned} g \ \in \ & \mathcal{A}(0 \leq f < c_0 - \delta) \cap \mathcal{A}(c_0 - \delta \leq f \leq c_0 - \epsilon) = \\ & = \mathcal{A}(0 \leq f < c_0 - \delta) \cap \mathcal{A}(c_0 - \delta \leq f \leq c_0 + \rho) = \quad \mathcal{A}(0 \leq f \leq c_0 + \rho). \end{aligned}$$

This gives a contradiction and ends the proof. ■

Remark 3.8. *Note that from the proof of Theorem 3.5 and Lemma 3.6 it follows that the algebra of bounded polynomials is stable in particular on each of the tentacles of the set* $\{(x, y) \in \mathbb{R}^2 |\ 0 \leq f(x, y) \leq c\}$.

The essence of the proof of the Theorem 3.5 lies in Lemma 3.6 on local stability. We chose to show it using the graph of a mapping. This method of proof also indicates possible applications in the study of properness of polynomial mappings.

Example 3.9. Take the Motzkin polynomial $\mathfrak{m}$ (see Example 3.2). We have that

$$\mathcal{A}(0 \leq \mathfrak{m} \leq c) = \mathbb{R}[X, Y]$$

for $c \in (0, 1)$, because of Proposition 1.13. Moreover,

$$\mathcal{A}(0 \leq \mathfrak{m} \leq 1) = \mathbb{R}[X^k Y |\ k \in \mathbb{N}_0] \cap \mathbb{R}[XY^k |\ k \in \mathbb{N}_0] = \mathbb{R}[X^i Y^j |\ i, j \in \mathbb{N}]$$

due to Proposition 2.6 and properties of algebras of bounded polynomials. It is easy to see that this algebra is infinitely generated (compare Sections 4.3 and 5).

Whereas the set $\mathfrak{m}^{-1}([0,c])$ for $c>1$ has four tentacles M_1, M_2, M_3 and M_4 such that $\mathcal{A}(M_1)=\mathcal{A}(M_2)=\mathbb{R}[X,XY,XY^2]$ and $\mathcal{A}(M_3)=\mathcal{A}(M_4)=\mathbb{R}[Y,XY,X^2Y]$. Hence

$$\mathcal{A}(0\leq\mathfrak{m}\leq c)=\mathbb{R}[XY,X^2Y,XY^2]$$

for $c>1$.

As an easy consequence of the Stability Theorem we get

Corollary 3.10. *Let $f\in\mathbb{R}[X,Y]$. If $c\neq\tilde{c}$ and $[c,\tilde{c}]\cap B_{\mathbb{C}}(f)=\emptyset$, then*

$$\mathcal{A}(f\leq c)=\mathcal{A}(f\leq\tilde{c}).$$

Proof: Indeed, it suffices to observe that for $C=|c|+|\tilde{c}|+1$ we have $c+C>0$ and $\tilde{c}+C>0$. Hence from Property 3.1 and Theorem 3.5 we get

$$\begin{aligned}\mathcal{A}(f\leq c)=\mathcal{A}(f+C\leq c+C))&=\mathcal{A}(f+C<0)\cap\mathcal{A}(0\leq f+C\leq c+C)=\\&=\mathcal{A}(f+C<0)\cap\mathcal{A}(0\leq f+C\leq\tilde{c}+C)=\mathcal{A}(f\leq\tilde{c}).\end{aligned}$$

This ends the proof. ■

Another easy consequence is

Corollary 3.11. *Take $b<c$ and $\tilde{b}<\tilde{c}$ such that*

$$[b,\tilde{b}]\cap B_{\mathbb{C}}(f)=\emptyset\quad\textit{and}\quad[c,\tilde{c}]\cap B_{\mathbb{C}}(f)=\emptyset.$$

We have that

$$\mathcal{A}(b\leq f\leq c)=\mathcal{A}(\tilde{b}\leq f\leq\tilde{c})$$

as long as $b<c$ and $\tilde{b}<\tilde{c}$.

Proof: We can suppose that $c\leq\tilde{c}$. By Property 3.1 we have that $[c-b,\tilde{c}-b]\cap B_{\mathbb{C}}(f-b)=\emptyset=[b-\tilde{c},\tilde{b}-\tilde{c}]\cap B_{\mathbb{C}}(f-\tilde{c})$. Hence by Theorem 3.5 we get

$$\mathcal{A}(b\leq f\leq c)=\mathcal{A}(b\leq f\leq\tilde{c})=\mathcal{A}(\tilde{b}\leq f\leq\tilde{c}).$$

This ends the proof. ■

Hence for the Motzkin polynomial $\mathfrak{m}:\mathbb{R}^2\to\mathbb{R}$ by Examples 3.2 and 3.9 we have

$$\mathcal{A}(b\leq\mathfrak{m}\leq c)=\begin{cases}\mathbb{R}[X,Y] & b\leq c<1,\\ \mathbb{R}[XY,X^2Y,XY^2] & b<c,\ 1<c\end{cases}$$

and the special cases are $\mathcal{A}(b\leq\mathfrak{m}\leq 1)$ and $\mathcal{A}(c\leq\mathfrak{m}\leq c)$ where $b<1\leq c$, which do not admit a finite set of generators.

In [PlSd] it is proved that if $S\subset\mathbb{R}^2$ admits only tentacles with nonempty interior, then $\mathcal{A}(S)$ is finitely generated. As a consequence we get

Proposition 3.12. *Let $f \in \mathbb{R}[X,Y]$. The algebra $\mathcal{A}(f \leq c)$ is finitely generated for $c \in \mathbb{R} \setminus B_{\mathbb{R}}(f)$.*

Proof: Take $c \in \mathbb{R} \setminus B_{\mathbb{R}}(f)$. From the properties of typical values and fibrations we get that either $f^{-1}((-\infty, c])$ is compact or each tentacle of the set $f^{-1}((-\infty, c])$ has a nonempty interior. This, combined with the result from [PlSd], gives the claim. ■

Note that in the above proposition it suffices to take $c \notin B_{\mathbb{R}}(f)$ and the set $B_{\mathbb{R}}(f)$ may be smaller then $B_{\mathbb{C}}(f) \cap \mathbb{R}$. It is not clear for the author whether Theroem 3.5 holds also if we substitute $B_{\mathbb{C}}(f)$ by $B_{\mathbb{R}}(f)$. Consider the following example given in [KS].

Example 3.13. Consider the polynomial

$$f(X,Y) = Y^4 + Y^2.$$

We have $B_{\mathbb{R}}(f) = \{0\}$ and $B_{\mathbb{C}}(f) = \{-\frac{1}{4}, 0\}$. We see that $B_{\mathbb{R}}(f) \subsetneq B_{\mathbb{C}}(f)$. Moreover,

$$\mathcal{A}(f \leq c) = \begin{cases} \mathbb{R}[X,Y] & c < 0, \\ \mathbb{R}[X^n Y |\ n \in \mathbb{N}_0] & c = 0, \\ \mathbb{R}[Y] & c > 0. \end{cases}$$

There is no change in the algebras at the complex bifurcation value $-\frac{1}{4}$.

To end this section let us remark that under the assumptions of Theorem 3.5 considered sets have only tentacles with nonempty interior. This assumption is vital as it will be later shown in Proposition 4.14 in Section 4.3.

4. Algebras of polynomials bounded on subsets of $\mathbb{R}^n$

In this section we will study algebras of bounded polynomials on special kinds of sets of arbitrary dimension n. In the first section we introduce some helpful sets given by inequalities on generators of the algebra of bounded polynomials (sets of the form (4.1) and (4.2) below) and study their properties. In the second section we turn to algebras of bounded polynomials on a special kind of tentacle sets of the form (4.6), which we will call weighted tentacles relative to a given set. Similar sets have been studied by Netzer in [Net] and their description presented in this work was inspired by his paper. The essence of these sets is their structure as "weighted deformation" of a set whose algebra of bounded polynomials is already known. In the third section we consider subsets of algebraic sets and show that their algebras are never finitely generated. This was already shown by Plaumann and Scheiderer in [PlSd], but we present a different proof based more on geometrical properties and using the set of nonproper values of a mapping.

4.1. Preliminary remarks

For the rest of this section let $S \subset \mathbb{R}^n$ and assume that $\mathcal{A}(S)$ is generated by a set $\{\zeta_i\}_{i\in I} = \mathcal{Z} \subset \mathbb{R}[X]$, where $I \subset \mathbb{N}$. Note that we do not assume that S is semialgebraic. If $\mathcal{Z} \neq \emptyset$, put

$$C_i = \sup_S |\zeta_i| \quad \textit{and} \quad C = \sum_{i\in I} C_i^2,$$

where we let $C \in \mathbb{R} \cup \{\infty\}$. Set

$$S_\cap(\mathcal{Z}) = \bigcap_{i\in I} \{x \in \mathbb{R}^n |\ |\zeta_i(x)| \leq C_i\} \tag{4.1}$$

and

$$S^2(\mathcal{Z}) = \{x \in \mathbb{R}^n |\ \sum_{i\in I} \zeta_i^2(x) \leq C\}. \tag{4.2}$$

In the case $\mathcal{Z} = \emptyset$ put $S_\cap(\emptyset) = S^2(\emptyset) = \mathbb{R}^n$. Note that sets $S_\cap(\mathcal{Z})$ and $S^2(\mathcal{Z})$ do depend on the choice of generators. In cases when the set of generators is given or it does not lead

to misunderstanding, we will simply write $S_\cap$ and S^2. It is easy to observe that

$$S \subset S_\cap \subset S^2. \tag{4.3}$$

Of course, this implies that

$$\mathcal{A}(S) \supset \mathcal{A}(S_\cap) \supset \mathcal{A}(S^2). \tag{4.4}$$

Note that if $\mathcal{Z}$ is infinite, the number C may be infinite. Then we have $S^2 = \mathbb{R}^n$ and $\mathcal{A}(S^2) = \mathbb{R}$. Nevertheless,

Property 4.1. *For the set $\mathcal{Z}$ of generators of $\mathcal{A}(S)$ we have*

$$\mathcal{A}(S) = \mathcal{A}(S_\cap(\mathcal{Z})).$$

If the set $\mathcal{Z}$ is finite, then

$$\mathcal{A}(S) = \mathcal{A}(S^2(\mathcal{Z})).$$

If the set $\mathcal{Z}$ is infinite, then there exists a function $\sigma : \mathcal{Z} \to \mathcal{A}(S)$ such that the set $\sigma(\mathcal{Z})$ generates $\mathcal{A}(S)$ and

$$\mathcal{A}(S) = \mathcal{A}\left(S^2\left(\sigma(\mathcal{Z})\right)\right).$$

Proof: By (4.4) we need to show only the inclusions $\mathcal{A}(S) \subset \mathcal{A}(S_\cap)$ and $\mathcal{A}(S) \subset \mathcal{A}(S^2)$. To prove the first two statements of the property we need to note that obviously $\mathcal{Z} \subset \mathcal{A}(S_\cap)$ and if $\mathcal{Z}$ is finite also $\mathcal{Z} \subset \mathcal{A}(S^2)$. Since $\mathcal{A}(S) = \mathbb{R}[\mathcal{Z}]$, from easy properties of generators (see Property 1.11) we get the first two equalities.

In the case when $\mathcal{Z}$ is infinite we can assume that $I = \mathbb{N}$. To choose a set of generators so that $\mathcal{A}(S) = \mathcal{A}(S^2)$, it suffices to rescale elements of $\mathcal{Z}$ conveniently. For $i \in \mathbb{N}$ put

$$\sigma(\zeta_i) = \begin{cases} \frac{\zeta_i}{2^{i/2} C_i} & if \quad C_i \neq 0 \\ \frac{\zeta_i + 1}{2^{i/2}} & if \quad C_i = 0. \end{cases} \tag{4.5}$$

Then $\sup_S |\sigma(\zeta_i)| = \frac{1}{2^{i/2}}$ and $\sigma(\mathcal{Z}) = \{\sigma(\zeta_i)\}_{i \in \mathbb{N}}$ is still a set of generators of $\mathcal{A}(S)$. We have

$$S^2(\sigma(\mathcal{Z})) = \{x \in \mathbb{R}^n |\ \sum_{i=1}^{\infty} (\sigma(\zeta_i)(x))^2 \leq 1\}$$

and $\sigma(\mathcal{Z}) \subset \mathcal{A}(S^2(\sigma(\mathcal{Z}))$. Hence $\mathcal{A}(S) = \mathcal{A}(S^2(\sigma(\mathcal{Z})))$ for this choice of generators. ■

If $\mathcal{A}(S)$ is finitely generated, then $S_\cap$ and S^2 are semialgebraic. However, if $\mathcal{A}(S)$ is not finitely generated, this need not be the case. Indeed, we have

Example 4.2. Take $S = \{(x, y) \in \mathbb{R}^2 |\ y = 0\}$. Then from Proposition 2.6 we have $\mathcal{A}(S) = \mathbb{R}[X^k Y |\ k \in \mathbb{N}_0]$. After some scaling of this basis similar to (4.5), we have also

$$\mathcal{A}(S) = \mathbb{R}\left[\frac{X^k Y + 1}{\sqrt{k!}} |\ k \in \mathbb{N}_0\right].$$

For this choice of generators we get

$$S^2 = \left\{(x,y) \in \mathbb{R}^2 |\ \sum_{k=0}^{\infty} \frac{(X^k Y + 1)^2}{k!} \leq e \right\} = \left\{(x,y) \in \mathbb{R}^2 |\ 0 \geq y \geq -2e^{x-x^2} \right\}.$$

And $\mathcal{A}(S) = \mathcal{A}(S^2)$, but S^2 is not semialgebraic. (Moreover, S^2 is not semi-analytic at infinity.)

4.2. Weighted tentacles

In this section we will turn to sets of a special form and characterisations of their algebras. There exists a natural class of sets in $\mathbb{R}^n$ to which one can easily apply methods used in our proofs of characterisations of algebras of tentacle sets in $\mathbb{R}^2$ in Section 2. The neat description (4.6) of their special cases studied in this section was inspired by the one given by Netzer in [Net] and Theorem 4.4 gives a generalisation of one of his results on algebras of bounded polynomials. We have adapted the form (4.6) of weighted tentacles since we feel that it is consequent to the form of tentacles we have studied in Section 2.

We will work in the ring $\mathbb{R}[X,Y]$, where $X = (X_1, \ldots, X_n)$ and $X_1, \ldots, X_n, Y$ are $n+1$ independent variables. Put $|\alpha| = \alpha_1 + \ldots + \alpha_n$. For $t = (t_1, \ldots, t_n), x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $t \cdot x = (t_1 x_1, \ldots, t_n x_n)$. We will denote by $\langle \cdot | \cdot \rangle$ the standard scalar product $\langle t|x\rangle = \langle (t_1, \ldots, t_n)|(x_1, \ldots, x_n)\rangle = \sum_i t_i x_i$.

For the rest of this section assume that $S \subset \mathbb{R}^n$ has a nonempty interior. Let us fix a Puiseux parametrization $\beta = (\beta_1, \cdots, \beta_n) : [R, \infty) \to \mathbb{R}^n$ of a semialgebraic curve at infinity such that β_i have constant signs on $[R, \infty)$, $R > 0$. Throughout this section set

$$\lambda = (\lambda_1, \ldots, \lambda_n) = (\mathrm{ord}_\infty \beta_1, \ldots \mathrm{ord}_\infty \beta_n) \in (\mathbb{Q}_0 \cup \{\infty\})^n.$$

Consider a set

$$M = \{(\beta(y) \cdot x, y) \in \mathbb{R}^{n+1} |\ x \in S,\ y \geq R\}. \tag{4.6}$$

We will call such a set *a weighted tentacle relative to S with the weigh λ*.

If S is semialgebraic, then M is also semialgebraic. If S has s connected components, then M has at most s tentacles. Hence if S is semialgebraic and connected, then M has exactly one tentacle. Note that $\mathcal{A}(M) = \mathcal{A}(\overline{M})$ and M is closed if S is closed.

For any $y \geq R$ put

$$M_y = \{(\beta(y) \cdot x, y) \in \mathbb{R}^{n+1} |\ x \in S\}.$$

Needless to say that M_y is semialgebraic if S is semialgebraic. If for some $i = 1, \ldots, n$ we have $\beta_i = 0$, then M is a subset of the hyperplane $\{x \in \mathbb{R}^{n+1} |\ x_i = 0\}$.

Suppose that $\beta : [R, \infty) \to \mathbb{R}^n$ from formula (4.6) is a Puiseux parametrization such that $\beta_i \neq 0$ for $i = 1, \ldots, n$. In this case M_y is homeomorphic to S for every $y \geq R$. Hence $Int(M) \neq \emptyset$ and so $\overline{M}^{Zar} = \mathbb{R}^n$. Put

$$\beta^\bullet(x, y) = \beta(y) \cdot x$$

for $y \geq R$ and $x \in \mathbb{R}$. It is a semialgebraic mapping. For a given $y \geq R$ we will denote by $\beta(y)^\bullet$ the linear mapping $\mathbb{R}^n \ni x \to \beta(y) \cdot x$. Note that this gives us

$$\beta(y)^\bullet(S) \ \times \ \{y\} = M_y.$$

For every $y \geq R$ the mapping $\beta(y)^\bullet$ is a linear automorphism of $\mathbb{R}^n$ with the inverse

$$\mathbb{R}^n \ni x \to \left(\frac{1}{\beta_1(y)} x_1, \ldots, \frac{1}{\beta_n(y)} x_n \right) \in \mathbb{R}^n.$$

Hence we will denote the inverse as $(1/\beta(y))^\bullet$ instead of $(\beta(y)^\bullet)^{-1}$. Similarly as in Section 2.2 we denote $\beta(y)^\bullet_*$ the automorhism of $\mathbb{R}[X]$ given by $g \to g \circ \beta(y)^\bullet$. Hence from Property 1.16 we get

$$\mathcal{A}(S) = \mathbb{R}[\mathcal{Z}] \iff \mathcal{A}(\beta(y)^\bullet(S)) = \mathbb{R}\left[\left(\frac{1}{\beta(y)} \right)^\bullet_* \mathcal{Z} \right] \tag{4.7}$$

for $y \geq R$. In the particular case when $\mathcal{Z}$ is a set of monomials we have

$$\mathbb{R}[\mathcal{Z}] = \mathbb{R}\left[\left(\frac{1}{\beta(y)} \right)^\bullet_* \mathcal{Z} \right].$$

Hence if $\mathcal{A}(S)$ admits a monomial basis, we have $\mathcal{A}(S) = \mathcal{A}((1/\beta(y))^\bullet(S))$.

To show how the algebra of bounded polynomials looks like on a weighted tentacle M relative to S we will show first that for a polynomial f in $n+1$ variables there exists a finite collection of values of the parameter y which check if f as a polynomial in $\mathbb{R}[X][Y]$ has bounded coefficients on the set S. Put

$$f = \sum_{\alpha \in A} a_\alpha(Y) X^\alpha \in \mathbb{R}[X, Y], \tag{4.8}$$

where A is a finite subset of $\mathbb{N}_0^n$ and $a_\alpha \neq 0$ are polynomials in variable Y for $\alpha \in A$. Set $\deg a_\alpha = d_\alpha$. Let us remind that $\deg_X f = \max_{\alpha \in A} |\alpha|$ and $\deg_Y f = \max_{\alpha \in A} d_\alpha$.

Property 4.3. *Put $d = \deg_Y f$. A polynomial f is an element of $\mathcal{A}(S)[Y]$ if and only if there exist $d+1$ distinct real numbers $y_0, \ldots, y_d$ such that*

$$f(X, y_i) \in \mathcal{A}(S)$$

for $i = 0, \ldots, d$.

Proof: If $d = 0$, then the claim is obvious. Assume that $d > 0$. The necessity is evident. We will show the sufficiency. We can write $f = \sum_{i=0}^{d} b_i(X)Y^i$, where $b_d \neq 0$. Let $y_0, \ldots, y_d$ be $d+1$ distinct points. Take the determinant v of the Vandermonde matrix

$$\begin{bmatrix} 1 & y_0 & \cdots & y_0^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_d & \cdots & y_d^d \end{bmatrix}.$$

We have that $v \in \mathbb{R} \setminus \{0\}$. For $i = 0, \ldots, d$ denote by v_i the determinant of the above matrix with the ith column replaced by the vertical vector, which is the transposition of

$$[f(X, y_0), \ldots, f(X, y_d)].$$

Then we have that

$$b_i = \frac{v_i}{v} \in \mathbb{R}[X].$$

Since by assumption $f(X, y_i) \in \mathcal{A}(S)$, we get that $v_i \in \mathcal{A}(S)$ for every $i = 0, \ldots, d$. Hence $b_i \in \mathcal{A}(S)$ and $f \in \mathcal{A}(S)[Y]$. ■

From the above and formula (4.7) it easily follows that if there exists a countable collection of points $\{y_i\}_{i\in\mathbb{N}}$ such that $\mathcal{A}(S) = \mathcal{A}(\beta(y_i)^\bullet(S))$, then to decide whether $g \in \mathcal{A}(S)[Y]$ it suffices to check if $g(X, y_i) \in \mathcal{A}(S)$ for $i \in \mathbb{N}$.

Similarly as in Section 2.1 take the nonnegative functions

$$\mathbb{R}[X] \ni \sum b_\alpha X^\alpha \to \max_\alpha |b_\alpha| \in \mathbb{R}$$

and

$$\mathbb{R}[X] \ni g \to \sup_{x\in S} |g(x)| \in \mathbb{R} \cup \{\infty\}.$$

The first function is a norm on $\mathbb{R}[X]$. The second one is a norm, when restricted to the set $\mathcal{A}(S)$. From the assumption that S has a nonempty interior, it follows that in fact the set $\mathcal{A}(S)$ is the biggest subset of $\mathbb{R}[X]$, with respect to inclusion, such that the second function restricted to this set is a norm.

Suppose that $\mathcal{A}(S)$ has a monomial basis $\{\zeta_i\}_{i\in I}$. Then the set $\{\zeta_i|\ \deg\zeta_i \leq \deg_X f\}$ is finite. Assume it is equal to $\{\zeta_i\}_{i=1,\ldots,k}$. Then the set

$$\left\{\zeta_1^{\alpha_1} \cdot \ldots \cdot \zeta_k^{\alpha_k}|\ \alpha_i \in \mathbb{N}_0,\ \langle(\deg\zeta_1, \ldots, \deg\zeta_k)|(\alpha_1, \ldots, \alpha_k)\rangle \leq \deg_X f\right\}$$

is linearly independent and finite. Moreover, it is a linear basis of the vector space of polynomials bounded on S and of degree less or equal to $\deg_X f$. Hence all norms are equivalent when restricted to the linear space $\{g \in \mathcal{A}(S)|\ \deg g \leq \deg_X f\}$. In particular, there exist positive constants ω, Ω which depend only on the set S and $\deg_X f$ such that for all polynomials $g = \sum b_\alpha X^\alpha \in \mathcal{A}(S)$ of degree less or equal $\deg_X f$ we have

$$\sup_{x\in S} |g(x)| \geq \omega \max_\alpha |b_\alpha| \tag{4.9}$$

and

$$\max_{\alpha} |b_\alpha| \geq \Omega \sup_{x \in S} |g(x)|. \tag{4.10}$$

Hence we are ready to prove

Theorem 4.4. *Suppose that $\mathcal{A}(S)$ is generated by monomials. If M is a weighted tentacle of the form (4.6) relative to the set S with the weigh $\lambda \in \mathbb{Q}^n$ and $\beta_i \neq 0$ for $i = 1, \ldots, n$, then we have*

$$\mathcal{A}(M) = \mathcal{A}(S)[Y] \ \cap \ \mathbb{R}[X^\alpha Y^d \mid \langle \alpha | \lambda \rangle \geq d, \ \alpha \in \mathbb{N}_0^n, d \in \mathbb{N}_0].$$

Compare the following proof with the proof of Theorem 2.1.

Proof: Take a polynomial f of the form (4.8). Note that f is bounded on M if and only if there exists $C > 0$ such that for all $y \geq R$ we have

$$\sup_{(x,y) \in M_y} |f(x,y)| \leq C.$$

Suppose that f is bounded on M. Hence there exists a constant $C > 0$ such that for all $y \geq R$ we have

$$C \geq \sup_{(x,y) \in M_y} |f(x,y)| = \sup_{x \in \beta(y)^\bullet(S)} |f(x,y)| = \sup_{x \in S} |f(\beta(y) \cdot x, y)|.$$

It implies that $f(X,y)$ is bounded on $\beta(y)^\bullet(S)$ for every $y \geq R$. From Property 4.3 it follows that $f \in \mathcal{A}(S)[Y]$. Moreover, from inequality (4.9) take the constant ω that does not depend on y and combine it with the above inequality. For $y \geq R$ we get that

$$C \geq \sup_{x \in S} |f(\beta(y) \cdot x, y)| \geq \omega \max_{\alpha \in A} |\beta^\alpha(y) a_\alpha(y)|.$$

Therefore for every $\alpha \in A$ the functions $y \to \beta^\alpha(y) a_\alpha(y)$ are bounded on the half-line $[R, \infty)$. This by Property 1.8 gives us that

$$0 \leq \operatorname{ord}_\infty(\beta^\alpha a_\alpha) = \operatorname{ord}_\infty a_\alpha + \operatorname{ord}_\infty \beta^\alpha = -d_\alpha + \sum_i \alpha_i \lambda_i.$$

Suppose that for every $\alpha \in A$ we have $d_\alpha \leq \langle \lambda | \alpha \rangle$ and f is an element of $\mathcal{A}(S)[Y]$. Then we can find a constant $C > 0$ such that for all $y \geq R$ and every $\alpha \in A$ we have

$$C \geq |\beta^\alpha(y) a_\alpha(y)|.$$

Hence from Property 1.8 for $y \geq R$ we have

$$\begin{aligned} C \geq \max_{\alpha \in A} |\beta^\alpha(y) a_\alpha(y)| \geq \Omega \sup_{x \in S} |f(\beta(y) \cdot x, y)| = \\ = \Omega \sup_{x \in \beta(y)^\bullet(S)} |f(x,y)| = \Omega \sup_{(x,y) \in M_y} |f(x,y)|, \end{aligned}$$

where Ω is the constant from inequality (4.10) for polynomials bounded on S and does not depend on y. Therefore f is bounded on M. ■

Let us look at an easy example below.

Example 4.5. Consider a set

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ (xy(1+z^2))^2 \leq z^2,\ x^2y^4(1+z^2)^4 \leq z^2,\ z \geq 1\}.$$

We have that

$$M = \left\{ \left(zx, \frac{y}{1+z^2}, z \right) \in \mathbb{R}^3 |\ (x, y) \in S,\ z \geq 1 \right\},$$

where

$$S = \{(x, y) \in \mathbb{R}^2 |\ (xy)^2 \leq 1,\ (xy^2)^2 \leq 1\}$$

is an unbounded semialgebraic set. From the decomposition of S into tentacles and Theorem 2.4 we get $\mathcal{A}(S) = \mathbb{R}[X, XY, XY^2] \cap \mathbb{R}[Y, XY] = \mathbb{R}[XY, XY^2]$. Moreover, we have $\operatorname{ord}_\infty Z = -1$ and $\operatorname{ord}_\infty \frac{1}{1+Z^2} = 2$. Hence by Theorem 4.4 we get

$$\begin{aligned} \mathcal{A}(M) &= \mathbb{R}[XY, XY^2][Z] \cap \mathbb{R}[X^iY^jZ^d |\ d \leq -i + 2j] = \\ &= \mathbb{R}[XY, XY^2, XYZ, XY^2Z, XY^2Z^2, XY^2Z^3]. \end{aligned}$$

In particular, $\mathcal{A}(M)$ has a six-element basis.

Observe that if $\mathcal{A}(S) = \mathbb{R}$, then $\mathcal{A}(M) = \mathbb{R}$. Indeed, from Theorem 4.4 it follows that

$$\mathcal{A}(M) = \mathbb{R}[Y] \cap \mathbb{R}[X^{(0,\ldots,0)}Y^d |\ 0 = \langle (0, \ldots, 0) | \lambda \rangle \geq d,\ d \in \mathbb{N}_0] = \mathbb{R}.$$

Let us remind that we put $\mathbb{R}[\emptyset] = \mathbb{R}$. As a corollary to Theorem 4.4 we get

Corollary 4.6. *Take a weighted tentacle M of the form (4.6) relative to the set S with the weigh $\lambda \in \mathbb{Q}^n$ and $\beta_i \neq 0$ for $i = 1, \ldots, n$. Suppose that $\mathcal{A}(S) \neq \mathbb{R}$ is generated by a set of monomials $\mathcal{Z} = \{X^\alpha | \alpha \in H\}$, where $H \subset \mathbb{N}_0^n$. Then*

$$\mathcal{A}(M) = \mathbb{R}$$

if and only if

$$\langle \lambda | \alpha \rangle < 0$$

for every $\alpha \in H$.

Proof: Take the set $D = \{\alpha \in \mathbb{N}_0^n |\ X^\alpha \in \mathcal{A}(S)\}$ of all powers of monomials bounded on S. It is easy to see that

$$D = \{m_1\alpha_1 + \ldots + m_k\alpha_k |\ k \in \mathbb{N},\ m_1, \ldots, m_k \in \mathbb{N},\ \alpha_1, \ldots, \alpha_k \in H\}.$$

First, note that

$$\langle\ \lambda\ | m_1\alpha_1 + \ldots + m_k\alpha_k \rangle = \sum_{i=1}^{k} m_i \langle \lambda | \alpha_i \rangle.$$

Therefore, we get $\langle \lambda | \alpha \rangle < 0$ for every $\alpha \in H$ if and only if $\langle \lambda | \alpha \rangle < 0$ for every $\alpha \in D$.

We have $\mathbb{R}[Y, X^\alpha |\ \alpha \in H] = \mathcal{A}(S)[Y]$ and

$$\mathcal{A}(S)[Y] \cap \mathbb{R}[X^\alpha Y^d |\ \langle \alpha | \lambda \rangle \geq d,\ \alpha \in \mathbb{N}_0^n, d \in \mathbb{N}_0] = \mathbb{R}[X^\alpha Y^d |\ \langle \alpha | \lambda \rangle \geq d,\ \alpha \in D, d \in \mathbb{N}_0].$$

Hence from Theorem 4.4 it follows that $\mathcal{A}(M) = \mathbb{R}$ if and only if

$$\mathbb{R} = \mathbb{R}[X^\alpha Y^d |\ \langle \alpha | \lambda \rangle \geq d,\ \alpha \in D, d \in \mathbb{N}_0].$$

Suppose that $\mathcal{A}(M) = \mathbb{R}$. Hence the set $\{(\alpha, d) \in D \times \mathbb{N}_0 |\ \langle \lambda | \alpha \rangle \geq d\}$ is empty. Therefore, for every $\alpha \in D$ we must have $\langle \lambda | \alpha \rangle < 0$. On the other hand, if $\langle \lambda | \alpha \rangle < 0$ for every $\alpha \in H$, then $\mathbb{R}[X^\alpha Y^d |\ \langle \alpha | \lambda \rangle \geq d,\ \alpha \in D, d \in \mathbb{N}_0] = \mathbb{R}[\emptyset] = \mathbb{R}$. This ends the proof. ■

If we suppose that S is compact, then $\mathcal{A}(S) = \mathbb{R}[X]$ and as an easy consequence of Theorem 4.4 we get

Corollary 4.7. *For the set M of the form (4.6) where S is compact and $\beta_i \neq 0$ for $i = 1, \ldots, n$ we have*

$$\mathcal{A}(M) = \mathbb{R}[X^\alpha Y^d |\ \langle \alpha | \lambda \rangle \geq d,\ \alpha \in \mathbb{N}_0^n, d \in \mathbb{N}_0]$$

where $\lambda_i = \operatorname{ord}_\infty \beta_i$ for $i = 1, \ldots, n$.

Corollary 4.8. *Suppose that M is a tentacle set of the form (4.6) with S compact. Then*

(1) $\mathcal{A}(M) = \mathbb{R}$ *if and only if* $\operatorname{ord}_\infty \beta_i < 0$ *for every* $i = 1, \ldots, n$,
(2) $\mathcal{A}(M) = \mathbb{R}[X]$ *if and only if* $\operatorname{ord}_\infty \beta_i = 0$ *for every* $i = 1, \ldots, n$,
(3) $\mathcal{A}(M) \supsetneq \mathbb{R}[X]$ *if and only if there exists* $i \in \{1, \ldots, n\}$ *such that* $\operatorname{ord}_\infty \beta_i > 0$.

Proof: Using Corollary 4.7 we easily deduce that $\mathcal{A}(M) = \mathbb{R}$ if and only if $\sum \lambda_i \alpha_i < 0$ for every $\alpha \in \mathbb{N}_0^n$. It follows that $\alpha_i \leq 0$ for $i = 1, \ldots, n$. Since α are arbitrary, we conclude that $\lambda_i < 0$ for $i = 1, \ldots, n$.

To prove the second point assume that $\mathcal{A}(M) = \mathbb{R}[X]$. Hence if $X^\alpha Y^d \in \mathcal{A}(M)$, then $d = 0$. Therefore $\sum_i \lambda_i \alpha_i = 0$ for any $\alpha \in \mathbb{N}_0^n$. In particular we can take $\alpha = (0, \ldots, 0, 1, 0 \ldots, 0)$. Hence we get that necessarily $\lambda_i = 0$ for $i = 1, \ldots, n$. The sufficiency is obvious.

Property (3) follows immediately from (2) and (1). ■

From the above it follows easily

Example 4.9. In $\mathbb{R}^n$ one can define a cone as a union of half-lines $\mathcal{C} = \{t(x_1, \ldots, x_n) |\ x \in S, t \geq 0\}$, where S is connected and bounded. Of course, as in this whole section, S has nonempty interior. Then we can see that from the above corollary it follows that $\mathcal{A}(\mathcal{C}) = \mathbb{R}$.

Now we will consider the algebras of bounded polynomials on projections of weighted tentacles. Let $\pi_n : \mathbb{R}^{n+m} \to \mathbb{R}^n$, where $m \geq 1$, be a projection on the first n coordinates and denote by $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection that leaves out only the last coordinate. We easily get

Property 4.10. *Let $K \subset \mathbb{R}^{n+m}$ be a set. Then for the projection π_n we have*

$$\mathcal{A}(\pi_n(K)) = \mathcal{A}(K) \cap \mathbb{R}[X_1, \ldots, X_n].$$

Note that from the above property it follows that if $\mathcal{A}(S) \subset \mathbb{R}[X_1, \ldots, X_k]$, then $\mathcal{A}(S) = \mathcal{A}(\pi_k(S))$. In such a case for a set M of the form (4.6) we have

$$\mathcal{A}(M) = \mathcal{A}\left(\{ \ ((\beta_1(y), \ldots, \beta_k(y)) \cdot (x_1, \ldots, x_k) \, , \ y) \ | \ (x_1, \ldots, x_k) \in \pi_k(S), \ y \geq R\}\right).$$

For a weighted tentacle M of the form (4.6) we have

$$\pi(M) = \{\beta(y) \cdot x | \ x \in S, \ y \geq R\}.$$

Let us remind that $IntS \neq \emptyset$. If S is bounded and $\mathrm{ord}_\infty \beta_1 \geq 0, \ldots, \mathrm{ord}_\infty \beta_n \geq 0$, then $\pi(M)$ is bounded. It is unbounded otherwise. Hence of course, if S is semialgebraic, $\pi(M)$ has a decomposition (1.19) into tentacle sets. But for S semialgebraic and connected $\pi(M)$ need not be a tentacle set as can be easily seen for the set $\{(z^{-1}x, zy) \in \mathbb{R}^2 | \ (x, y) \in B(0, 1)\}$ which has two tentacles.

Proposition 4.11. *Suppose that $\mathcal{A}(S)$ is generated by monomials. For a set K of the form*

$$\{\beta(y) \cdot x | \ x \in S, \ y \geq R\},$$

where β is a Puiseux parametrization such that $\beta_i \neq 0$ for each $i = 1, \ldots, n$, we have

$$\mathcal{A}(K) = \mathcal{A}(S) \cap \mathbb{R}[X^\alpha | \ \langle \lambda | \alpha \rangle \geq 0, \ \alpha \in \mathbb{N}_0^n].$$

Proof: The proof follows from Theorem 4.4 and Property 4.10. Indeed, $K = \pi(M)$ and

$$\begin{aligned} \mathcal{A}(K) = \mathcal{A}(\pi(M)) = \mathcal{A}(M) \cap \mathbb{R}[X] = \mathcal{A}(S)[Y] \cap \mathbb{R}[X^\alpha Y^d | \ d \leq \langle \lambda | \alpha \rangle] \cap \mathbb{R}[X] = \\ = \mathcal{A}(S) \cap \mathbb{R}[X^\alpha | \ \exists_{d \in \mathbb{N}_0} \ d \leq \langle \lambda | \alpha \rangle] = \mathcal{A}(S) \cap \mathbb{R}[X^\alpha | \ 0 \leq \langle \lambda | \alpha \rangle]. \end{aligned}$$

This ends the proof. ■

This proposition can be proved alternatively by an argument similar to the one presented in the proof of Theorem 4.4. Observe that if every $\mathrm{ord}_\infty \beta_i \geq 0$, then $\mathbb{R}[X^\alpha | \ \langle \lambda | \alpha \rangle \geq 0] = \mathbb{R}[X]$.

In [Net] it is shown that projections of sets of the form (4.6) (necessarily subsets if S is not compact) have stable preorderings if there holds a condition equivalent to saying that the algebra of bounded polynomials admits only constant polynomials (compare [Net, Theorem 5.4] with Proposition 4.10 and Corollary 4.8). The study of stability of preorderings (as the existence of bounds of degrees of polynomials in these representations) is also an interesting field from the viewpoint of applications, tightly connected with the K-moment problem and presently discussed in the context of unbounded set (see for instance [Sm2], [PS] or [KM]).

We will end this section with remarks on "model" sets which have the same algebra as any given set of the form (4.6).

Take a multi-index $\lambda \in \mathbb{Q}^n$ and a set $\mathcal{Z} = \{\zeta_i | \ i \in I\} \subset \mathbb{R}[X]$ of monomials that forms a basis of $\mathcal{A}(S)$ i.e. the condition 1.10 from Preliminaries holds. For this basis (possibly

after rescaling, see Property 4.1 and formula (4.5)) take the set S^2 of the form (4.2) such that $\mathcal{A}(S) = \mathcal{A}(S^2)$. Set

$$B_{\lambda,\mathcal{Z}} = \left\{ \left((1/y)^{\lambda_1} x_1 \ldots, (1/y)^{\lambda_n} x_n, y \right) \in \mathbb{R}^{n+1} |\ x \in S^2,\ y \geq 1 \right\}.$$

Note that $B_{\lambda,\mathcal{Z}}$ need not be semialgebraic, since S^2 needs not be. Directly from the characterisation of the algebras in Theorem 4.4 and Property 4.1 we get

Remark 4.12. *Suppose $\mathcal{A}(S)$ has a monomial basis $\mathcal{Z}$. For a weighted tentacle M relative to a set S with weigh $\lambda \in \mathbb{Q}^n$ and $\beta_i \neq 0$ for $i = 1, \ldots, n$ we have*

$$\mathcal{A}(M) = \mathcal{A}(B_{\lambda,\mathcal{Z}}),$$

where $\lambda_i = \mathrm{ord}_\infty \beta_i$ for $i = 1, \ldots, n$.

The choice is not unique, because there may exist $\tilde{\lambda} \neq \lambda$ such that $\mathcal{A}(B_{\lambda,\mathcal{Z}}) = \mathcal{A}(B_{\tilde{\lambda},\mathcal{Z}})$, for example in the case when the algebra admits only constants.

In the case when S is compact we can simplify $B_{\lambda,\mathcal{Z}}$. Put

$$B_\lambda = \left\{ \left((1/y)^{\lambda_1} x_1 \ldots, (1/y)^{\lambda_n} x_n, y \right) \in \mathbb{R}^{n+1} |\ x \in \overline{B(0,1)},\ y \geq 1 \right\}$$

for any $\lambda \in \mathbb{Q}^n$. Hence from Remark 4.12 follows

Remark 4.13. *If S is compact and $\beta_i \neq 0$ for $i = 1, \ldots, n$, then*

$$\mathcal{A}(M) = \mathcal{A}(B_\lambda),$$

where $\lambda_i = \mathrm{ord}_\infty \beta_i$ for $i = 1, \ldots, n$.

To end the section let us take a look at the two-dimensional case. The sets of the type (2.6) considered throughout Section 2.1 can be rewritten as either of the form

$$\{(t^\alpha x, ty) |\ (x, y) \in [0,1] \times \{1\}, t \geq 1\} \quad or \quad \{(y^\alpha x, y) |\ x \in [0,1], y \geq 1\}.$$

Whereas sets of the type (2.8) can be rewritten as $\{(y^\alpha(x - \beta), y) |\ x - \beta \in [0,1], y \geq 1\}$ where β is a Puiseux polynomial. Hence they are exactly of the type considered above. As the sets $[0,1] \times \{1\}$ and $[0,1]$ are compact, these forms can be interpreted as a kind of an "uniform deformation" of a compact set. Though every algebra of bounded polynomials is an algebra on this type of sets, tentacles of the form (1.7) are usually not equal to a set of this type.

4.3. Subsets of algebraic sets

We will begin with an observation on stability (in the sense of Theorem 3.5 in Section 3) of the algebras of bounded polynomials on subsets of algebraic sets. Afterwards we will show that these algebras do not admit finite bases. Let us note that we will use a

shortened notation $\mathcal{A}(f = c)$ to denote $\mathcal{A}(\{(x, y \in \mathbb{R}^2 |\ f(x, y) = c)\})$.

Let us make a remark that the Stability Theorem from Section 3.3 does not hold if we substitute the sets $f^{-1}([c, \tilde{c}])$ with simply the fibres $f^{-1}(c)$, unless they are bounded. To be more precise, we have the following

Proposition 4.14. *Take a nonconstant polynomial $f : \mathbb{R}^n \to \mathbb{R}$ and $c \neq \tilde{c} \in \mathbb{R}$. If $f^{-1}(c)$ or $f^{-1}(\tilde{c})$ is unbounded, then*

$$\mathcal{A}(f = c) \neq \mathcal{A}(f = \tilde{c}).$$

Proof: Without loss of generality suppose that $f^{-1}(\tilde{c})$ is unbounded. We have that $\|X\|^2 \cdot (f - c)$ is bounded on $f^{-1}(c)$. But

$$\|X\|^2 \cdot (f - c)_{|f^{-1}(\tilde{c})} = \|X\|^2 \cdot (\tilde{c} - c)_{|f^{-1}(\tilde{c})}.$$

Since $\tilde{c} - c \neq 0$ and $f^{-1}(\tilde{c})$ is unbounded we have that $\|X\|^2 \cdot (f - c) \notin \mathcal{A}(f = \tilde{c})$. ■

To find the generators of $\mathcal{A}(S)$ for an unbounded $S \subset f^{-1}(0)$ it is not sufficient to consider polynomials neither divisible by f nor generated by f i.e. $\mathcal{A}(S) \not\subseteq (f)\mathbb{R}[X] \cup \mathbb{R}[f]$. Look at the example below

Example 4.15. *Take $f = (1 + X^2)Y - 1$ and*

$$S = f^{-1}(0) = \left\{(x, y) \in \mathbb{R}^2 |\ y = \frac{1}{1 + x^2}\right\}.$$

The polynomial f is irreducible. If one takes $g = f + y$, one gets that g is bounded on S, but not constant. Hence $g \notin \mathbb{R}[f]$ and $g \notin (f)$.

Now we will give an alternative proof that an algebra of polynomials bounded on an unbounded subset of a proper algebraic set does not admit a finite basis. This fact has been already shown by Plaumann and Scheiderer in [PlSd] for a more general class of sets i.e. subsets of proper algebraic subsets of any affine $\mathbb{R}$-variety, but we feel that the proof below fits better in our setting by making an interesting use of properties of nonproper mappings and sets of the forms (4.1) and (4.2).

Theorem 4.16. *If $S \subset \mathbb{R}^n$ is an unbounded semialgebraic subset of a proper algebraic set, then $\mathcal{A}(S)$ does not admit a finite set of generators.*

The case $n = 1$ being trivial we will suppose that $n \geq 2$. Let us outline the proof of the above theorem. We will suppose that $\mathcal{A}(S)$ is finitely generated. From Lemmas 4.17 and 4.19 we will conclude that $S \subset S^2 = V(\zeta_1, \ldots, \zeta_k)$. Hence $(\zeta_1, \ldots, \zeta_k)\mathbb{R}[X] \subset \mathbb{R}[\zeta_1, \ldots, \zeta_k]$ and we will use Lemma 4.20 to show that S is bounded. This will give a contradiction.

First, note that for every $S \subset \mathbb{R}^n$ the zero ideal is always contained in $\mathcal{A}(S)$ and $\mathbb{R}^n = \{x \in \mathbb{R}^n |\ 0 = 0\}$. Moreover, for any bounded set K we have $\mathcal{A}(S) = \mathcal{A}(S \setminus K) \cap \mathcal{A}(K) = \mathcal{A}(S \setminus K)$.

Lemma 4.17. *Let $S \subset \mathbb{R}^n$ be an unbounded semialgebraic set. For any nonzero ideal $\mathcal{I} \subset \mathbb{R}[X]$ there exists a compact set K such that*

$$S \setminus K \subset V(\mathcal{I}) \iff \mathcal{I} \subset \mathcal{A}(S).$$

Proof: Indeed, if $S \setminus K \subset V(\mathcal{I})$ for some nonzero ideal $\mathcal{I} \subset \mathbb{R}[X]$ and a compact set K then obviously $f = 0$ on $S \setminus K$ for any polynomial $f \in \mathcal{I}$. Hence $f \in \mathcal{A}(S \setminus K) = \mathcal{A}(S)$. Now we will show the sufficiency. Suppose $\mathcal{I} \subset \mathcal{A}(S)$ for some nonzero ideal $\mathcal{I} = (f_1, \ldots, f_k)$. Set $K = S \setminus V(\mathcal{I})$. If $K = \emptyset$, the claim holds. Assume that $K \neq \emptyset$. Suppose first that K is unbounded. Take $f = f_1^2 + \ldots + f_k^2 \in \mathcal{I}$. Of course, $f(x) \neq 0$ for $x \notin V(f_1, \ldots, f_k)$. From Curve Selection Lemma and properties of semialgebraic curves there exists a tuple of Puiseux series at infinity $\beta : [R, \infty) \to K$ such that $f \circ \beta \neq 0$ and we have $\|\beta(t)\|^2 \geq t^p$ for $t \geq R$ and some rational $p > 0$. Hence $(f \circ \beta)(t) = a_0 \frac{1}{t^{r/q}} + a_1 \frac{1}{t^{r+1/q}} + \ldots$ for some $r \in \mathbb{N}_0$, $q \in \mathbb{Z}$ and $a_i \in \mathbb{R}$ such that $a_0 \neq 0$. Therefore there exists a natural number N such that $s = pN - r/q > 0$. Hence for some positive constant C we have

$$\big(\|X\|^{2N} \cdot |f(X)|\big) \circ \beta(t) \geq C t^{pN} \cdot \frac{1}{t^{r/q}} = c t^s \to \infty$$

as $t \to \infty$. Therefore, $\|X\|^{2N} f$ is not bounded on S. But $\|X\|^{2N} f \in \mathcal{I} \subset \mathcal{A}(S)$. This gives a contradiction. Hence $K = S \setminus V(\mathcal{I})$ is bounded. We get $S \setminus \overline{K} \subset V(\mathcal{I})$. ■

Let us illustrate the lemma above.

Example 4.18. *Take the Motzkin polynomial $\mathfrak{m}$ from Example 3.2 and the set $S = \{(x, y) \in \mathbb{R}^2 |\ \mathfrak{m}(x, y) \leq 1\}$, which is the union of the closure of the ball $B(0, \sqrt{3})$ and the locus of XY. By Example 3.9 we see that $(XY) \subset \mathcal{A}(S)$, but $\overline{S}^{Zar} = \mathbb{R}^2$. Nevertheless, it suffices to take any compact set $K \supset B(0, \sqrt{3})$, so that we get $\overline{S \setminus K}^{Zar} = V(XY)$.*

Note that if S is unbounded, then there exists a compact set K_0 such that

$$\forall_{K \supset K_0} \quad \overline{S \setminus K_0}^{Zar} = \overline{S \setminus K}^{Zar}. \tag{4.11}$$

Indeed, if the above were not true, then there would have existed an infinite strictly ascending sequence of ideals in $\mathbb{R}[X]$. This is impossible, because the ring $\mathbb{R}[X]$ is noetherian.

It follows that

Lemma 4.19. *Let S be an unbounded semialgebraic subset of $\mathbb{R}^n$ such that the set $\mathcal{Z}$ of generators of $\mathcal{A}(S)$ is finite. There exists a compact set K such that*

$$\overline{S \setminus K}^{Zar} = \overline{S_\cap \setminus K}^{Zar} = \overline{S^2 \setminus K}^{Zar}.$$

Proof: Let us first prove the existence of a compact set K such that the Zariski closures of $S \setminus K$ and $S^2 \setminus K$ are the same. If $\overline{S}^{Zar} = \mathbb{R}^n$, then $\overline{S^2}^{Zar} = \mathbb{R}^n$ due to the inclusion $S \subset S^2$. Let us suppose that $\overline{S}^{Zar} \neq \mathbb{R}^n$. For any compact set K we have

$$\overline{S \setminus K}^{Zar} \subset \overline{S^2 \setminus K}^{Zar}.$$

Take K_0 which satisfies the condition (4.11). Since the Zariski closure of the set $S \setminus K_0$ is equal to $V(\mathcal{I})$ for some nonzero ideal $\mathcal{I} \subsetneq \mathbb{R}[X]$, from Lemma 4.17 we get $\mathcal{I} \subset \mathcal{A}(S)$. Moreover, since $\mathcal{Z}$ is finite from Property 4.1 we have $\mathcal{A}(S) = \mathcal{A}(S^2) = \mathcal{A}(S^2 \setminus K_0)$ and again from Lemma 4.17 it follows that there exists a compact set Z such that $(S^2 \setminus K_0) \setminus Z \subset V(\mathcal{I})$. Hence from the choice of K_0 if we put $K = K_0 \cup Z$, we get

$$\overline{S^2 \setminus K}^{Zar} = \overline{S^2 \setminus (K_0 \cup Z)}^{Zar} \subset \overline{S \setminus K_0}^{Zar} = \overline{S \setminus K}^{Zar}.$$

This gives the first equality. We prove the second equality analogously. ■

Note that we will prove in Theorem 4.16, that with these assumptions, the above lemma is trivial, since then for any compact set we will have $\overline{S \setminus K}^{Zar}$. It will follow also that if there exists a compact set K such that $\overline{S \setminus K}^{Zar} = \overline{S_\cap \setminus K}^{Zar} \neq \mathbb{R}^n$, then every bounded polynomial on S is constant. This is not generally true (see page 63).

Now we will prove the crucial lemma suggested by Z. Jelonek.

Lemma 4.20. *Let $\zeta_1, \ldots, \zeta_k \in \mathbb{R}[X_1, \ldots, X_n]$, $n \geq 2$. If*

$$(\zeta_1, \ldots, \zeta_k)\mathbb{R}[X] \subset \mathbb{R}[\zeta_1, \ldots, \zeta_k], \tag{4.12}$$

then the mapping $(\zeta_1, \ldots, \zeta_k) : \mathbb{R}^n \to \mathbb{R}^k$ is proper.

Proof: Let us abbreviate $\zeta = (\zeta_1, \ldots, \zeta_k)$. From the assumption (4.12) there follows existence of polynomials $f_{ij} \in \mathbb{R}[Y_1, \ldots, Y_k]$ such that

$$X_i \zeta_j(X) = f_{ij}(\zeta(X)) \tag{4.13}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. We get $X_i = f_{ij}(\zeta)/\zeta_j$. Consequently it follows that the fields of fractions $\mathbb{R}(\zeta_1, \ldots, \zeta_k) = \mathbb{R}(X_1, \ldots, X_n)$ are equal. Hence $k \geq n \geq 2$.

We will show that the mapping ζ is proper on the set $\mathbb{R}^n \setminus \zeta^{-1}(0)$. Take a compact set $K \subset \mathbb{R}^k$ such that $0 \notin K$. There exists a constant C such that $\sup_K |f_{ij}| \leq C$ for every i, j. Since K and $\{0\}$ are compact and disjoint, there exists $\epsilon > 0$ such that $dist(K, 0) = \epsilon$. Hence $dist(\zeta(\zeta^{-1}(K)), 0) \geq \epsilon$. For every $x \in \zeta^{-1}(K)$ from $\|\zeta(x)\|^2 \geq \epsilon^2$ it follows that there exists $j = j(x) \in \{1, \ldots, k\}$ such that $\zeta_j^2(x) \geq \epsilon^2/k$. Hence from the above and (4.13) it follows that for the point x and every $i = 1, \ldots, n$ we have

$$|x_i| = \frac{\|f_{ij}(\zeta(x))\|}{|\zeta_j(x)|} \leq \frac{C\sqrt{k}}{\epsilon}.$$

Therefore $\zeta^{-1}(K)$ is contained in the closed ball $\overline{B(0, C\sqrt{nk}/\epsilon)}$. Hence the mapping $\zeta : \mathbb{R}^n \setminus \zeta^{-1}(0) \to \mathbb{R}^k \setminus \{0\}$ is proper. Therefore the set of nonproperness $J_\zeta \subset \{0\}$ for the mapping $\zeta : \mathbb{R}^n \to \mathbb{R}^k$. However, by Theorem 1.12 of Jelonek, the set J_ζ, if nonempty, is semialgebraic of dimension at least 1. Hence 0 cannot be an asymptotic value of ζ and ζ is proper on $\mathbb{R}^n$. ■

In fact, using the set of nonproperness for complex mappings (see [J2]) one could prove that the set of zeroes of the mapping ζ satisfying assumptions of the above theorem is finite.

Proof of Theorem 4.16: Let us assume that $\mathcal{A}(S)$ is finitely generated. If $\zeta_1, \ldots, \zeta_k$ are generators of $\mathcal{A}(S)$, then $\zeta_1 + c_1, \ldots, \zeta_k + c_k$ for any constants $c_1, \ldots, c_k \in \mathbb{R}$ are generators of $\mathcal{A}(S)$ as well. Hence we can assume that the generators $\zeta_1, \ldots, \zeta_k$ are such that for some unbounded curve $\gamma : [R, \infty) \to S$ we have $\zeta_i(\gamma(t)) \to 0$ as $t \to \infty$ for every $i = 1, \ldots, k$.

Since from assumptions $\overline{S}^{Zar} = V(\mathcal{I}) \neq \mathbb{R}^n$ for some ideal $\mathcal{I} \subset \mathbb{R}[X]$ we get $\mathcal{I} \subset \mathcal{A}(S) \neq \mathbb{R}[X]$ from Lemma 4.17. Hence from Lemma 4.19 there exists a compact set K such that

$$\overline{\{x \in \mathbb{R}^n |\ \sum \zeta_i^2(x) \leq C\} \setminus K}^{Zar} = \overline{S^2 \setminus K}^{Zar} = V(\mathcal{I}).$$

This gives us $\sum \zeta_i^2 \geq C$ on $\mathbb{R}^n \setminus K$. Should it be otherwise, then the open set

$$Z = \{x \in \mathbb{R}^n |\ \sum \zeta_i^2(x) < C\} \setminus K$$

would be nonempty. Therefore

$$\mathbb{R}^n = \overline{Z}^{Zar} \subset \overline{S^2 \setminus K}^{Zar} \neq \mathbb{R}^n$$

which gives us a contradiction. Hence $\sum \zeta_i^2 = C$ on $S \setminus K$.

On the other hand, from the choice of ζ_i it follows that $C = \sum \zeta_i^2(\gamma(t)) \to 0$ as t tends to infinity. Hence $C = 0$. From the definition of C we get $\zeta_i = 0$ on $S \setminus K$ for every i. Hence

$$S \setminus K \subset S^2 = V(\zeta_1, \ldots, \zeta_k).$$

Since the ideal $(\zeta_1, \ldots, \zeta_k)$ in $\mathbb{R}[X]$ is contained in $\mathcal{A}(S^2) = \mathbb{R}[\zeta_1, \ldots, \zeta_k]$, we can use Lemma 4.20 and get that $V(\zeta_1, \ldots, \zeta_k)$ is compact. This gives a contradiction. ■

Due to Plaumann and Scheiderer (see [PlSd]) we know that the only semialgebraic sets in $\mathbb{R}^2$ which do not admit a finite basis of the algebra of bounded polynomials are the ones whose every tentacle is a subset of a proper algebraic set i.e. sets S such that there exists a compact K such that the Zariski closure of $S \setminus K$ is proper. Nevertheless, starting from $\mathbb{R}^3$, unbounded subsets of algebraic sets are not the only ones that do not admit a finite basis. Recently, an example was given by S. Krug in [Krug] of a semialgebraic set in $\mathbb{R}^3$ whose tentacles have a nonempty interior at infinity (i.e. outside any compact set) such that its algebra cannot be finitely generated. It can be viewed as a geometric interpretation of a recent result of S. Kuroda in [Ku] on the existence of counterexamples to Hilbert's XIVth Problem in low dimension and transcendence degree. More precisely, in [Ku] it is shown that there exist infinitely generated subalgebras of the ring of polynomials in four variables arising from intersections of this ring with a subfield L of the ring of fractions $\mathbb{R}(X)$ of transcendence degree three. Nevertheless, the example of Krug is a semialgebraic set which is not basic and we feel that in context of Schmüdgen's Positivstellensatz (compare Section 6.2) it is an interesting question to ask whether there exist basic sets that do admit an infinitely generated algebra of bounded polynomials.

5. Monomial generators

In this section we will focus on algebras of polynomials generated by monomials. First we will start with the notion of semigroup and some of its basic properties. Afterwards, we will study the geometric properties of cones of monomials bounded on a given set. As an example, we will point out the generators of the algebras of polynomials bounded on tentacle sets with borders of different orders (see Section 2.1). In Theorem 5.9 we show that any closed convex cone spanned by a finite set of points with rational coefficients is in fact a convex cone spanned by the generators of an algebra of bounded polynomials on a suitable unbounded basic semialgebraic set. Hence there is a duality between convex cones in $\mathbb{R}^n$ and sets given by monomial inequalities.

5.1. Generators of semigroups

A semigroup is a set $\mathcal{M}$ with a binary operation $\cdot : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that $(\zeta \cdot \eta) \cdot \theta = \zeta \cdot (\eta \cdot \theta)$ for every $\zeta, \eta, \theta \in \mathcal{M}$. A semigroup with a neutral element $1_{\mathcal{M}}$ is called *a monoid* (see [Kin]). Any nonempty subset S of $\mathcal{M}$ which is closed under the operation $\cdot$ is called *a subsemigroup*, any subsemigroup containing the neutral element is called a *submonoid*. A monoid where all elements have their inverse is a group.

Example 5.1. Take the set $\mathbb{N}_0^m$, where $m \in \mathbb{N}$, with the operation $+$ of addition on coordinates

$$(n_1, \dots, n_m) + (k_1, \dots, k_m) = (n_1 + k_1, \dots, n_m + k_m).$$

Hence it is easily seen that $\mathbb{N}_0^m$ with the above operation is a semigroup with the neutral element $(0, \dots, 0)$. The set $\mathbb{Z}^m$ with the above operation forms a group for any $m \in \mathbb{N}$.

We will say that an element ζ of a semigroup $\mathcal{M}$ is *generated by a set* $\mathcal{Z}$ if there exist $\zeta_i \in \mathcal{Z}$, $\gamma_i \in \mathbb{N}_0$ such that $\zeta = \zeta_1^{\gamma_1} \cdots \zeta_k^{\gamma_k}$. A submonoid $A \subset \mathcal{M}$ is said to be *generated by a set* $\mathcal{Z}$ if $\mathcal{Z} \subset A$ and each element of A is generated by $\mathcal{Z}$. A set $\mathcal{Z}$ is called *a basis of* A if it generates A and every $\zeta \in \mathcal{Z}$ is not generated by the set $\{1\} \cup (\mathcal{Z} \setminus \{\zeta\})$. For example $X_1, \dots, X_n$ form a basis of the monoid of all monomials in n variables and the basis of the one-element monoid $\{1\}$ is empty.

Note that the above notion of generating a semigroup has a counterpart in generators of algebras (as in Section 1.3). And we will see in Section 5.3 that sometimes they coincide.

From now on $\mathcal{M}$ will stand for the multiplicative semigroup of monomials in n variables. We will write X^α for $X_1^{\alpha_1}\dots X_n^{\alpha_n}$, where $\alpha=(\alpha_1\dots,\alpha_n)\in\mathbb{N}_0^n$ and $X_i^0=1$. Monomials together with multiplication form a monoid. For a set $A\subset\mathcal{M}$ we put

$$\mathrm{supp}A=\{\alpha\in\mathbb{N}_0^n\mid X^\alpha\in A\}$$

and we will call it *the support of* A. The natural identification $\imath:\mathcal{M}\ni X^\alpha\to\alpha\in\mathbb{N}_0^n$ is an isomorphism of the monoid $(\mathcal{M},\cdot)$ and the monoid $\mathbb{N}_0^n$ regarded with addition on coordinates (as in Example 5.1). Indeed, $\imath(1)=\imath(X_1^0\dots X_n^0)=(0,\dots,0)$ and $\imath(X^\alpha\cdot X^\gamma)=\imath(X^{\alpha+\gamma})=\alpha+\gamma=\imath(X^\alpha)+\imath(X^\gamma)$. For a set $A\subset\mathcal{M}$ we have $\imath(A)=\mathrm{supp}A$. This approach will be useful when talking about points in $\mathbb{N}_0^n$ corresponding to monomials in n variables.

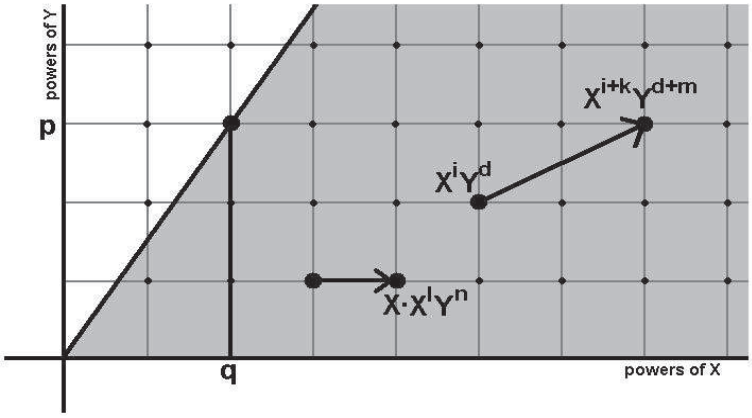

Figure 7. Multiplication of monomials $X^iY^d\cdot X^kY^m$ represented as points in $\mathbb{N}_0^2$.

If A is a subsemigroup of $\mathcal{M}$, then either $A=\{1\}$ or A is infinite. Indeed, if $1\neq\zeta\in A$, then $\zeta^i\in A$ for all $i\in\mathbb{N}$. Moreover, it is crucial to note that a monomial can be generated only by monomials of lower degrees. Indeed, for any $\zeta\in\mathcal{M}$ if $\zeta=\zeta_1^{\alpha_1}\dots\zeta_k^{\alpha_k}$ for $k>1$, $\alpha_i\in\mathbb{N}$ and $\zeta_i\neq1$ for all i then $\deg\zeta_i<\deg\zeta$.

Let us establish in $\mathcal{M}$ *the deglex ordering* (as in e.g. [DW]), that is for every $\zeta,\tilde\zeta\in\mathcal{M}$ we put

$$\zeta\preccurlyeq\tilde\zeta\iff\deg\zeta<\deg\tilde\zeta\ \ or\ \ (\deg\zeta=\deg\tilde\zeta\ \ and\ \ \zeta\preccurlyeq_{lex}\tilde\zeta),$$

where $X_1^{\alpha_1}\cdot\ldots\cdot X_n^{\alpha_n}\preccurlyeq_{lex}X_1^{\gamma_1}\cdot\ldots\cdot X_n^{\gamma_n}$ if and only if either $X_1^{\alpha_1}\cdots X_n^{\alpha_n}=X_1^{\gamma_1}\cdots X_n^{\gamma_n}$ or for the first l, for which $\alpha_l\neq\gamma_l$, we have $\alpha_l<\gamma_l$. This ordering is a well-ordering in $\mathcal{M}$. That means it is linear and each subset of $\mathcal{M}$ has the first element. Hence every infinite subset of $\mathcal{M}$ can be written as a string numbered by natural numbers.

Proposition 5.2. *Every semigroup $A\subset\mathcal{M}$ has a unique basis.*

Proof: Let us show that a basis of A always exists. We give a procedure of finding it. If $A=\{1\}$ then the basis is the empty set. Let us suppose that A is infinite. Write $A\setminus\{1\}$ as a sequence $(\zeta_i)_{i\in\mathbb{N}}$ such that $\zeta_i\preccurlyeq\zeta_j$ for $i\leq j$.

Put $\mathcal{Z}_1 = \{\zeta_1\}$. It is a basis of the monoid generated by ζ_1. Suppose we have sets $\mathcal{Z}_i$ for all $i \leq n$, where $n \geq 0$. And that each $\mathcal{Z}_i$ is the basis of a monoid generated by its elements. Take ζ_{n+1}. Put

$$\mathcal{Z}_{n+1} = \begin{cases} \mathcal{Z}_n \cup \{\zeta_{n+1}\} & if\ \zeta_{n+1} \notin \mathbb{R}[\mathcal{Z}_n], \\ \mathcal{Z}_n & otherwise. \end{cases}$$

Hence the set $\mathcal{Z}_{n+1}$ is a basis of a monoid generated by its elements. Take $\mathcal{Z} = \bigcup_{n=0}^{\infty} \mathcal{Z}_n$. It is easy to show that $\mathcal{Z}$ is a basis of A.

The basis of A is unique. Indeed, take two bases $\mathcal{Z} = \{\zeta_1, \ldots, \zeta_k, \ldots\}$ and $\tilde{\mathcal{Z}} = \{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_k, \ldots\}$ of A ordered by deglex. Choose the first k such that $\zeta_k \neq \tilde{\zeta}_k$. Without loss of generality we can assume that $\zeta_k \preccurlyeq \tilde{\zeta}_k$. Since $\tilde{\mathcal{Z}}$ is a basis of A, we get $\zeta_k = \tilde{\zeta}_1^{\alpha_1} \cdots \tilde{\zeta}_l^{\alpha_l}$ for some $l < k$, where $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_l$ are of lower degree than ζ_k. But from the choice of ζ_k, we get that it is generated by elements of $\mathcal{Z}$. This contradicts the assumption that $\mathcal{Z}$ is a basis. ■

5.2. Semigroups of monomials bounded on a set

Take the coordinate system of $\mathbb{N}_0^n$. Recall that a point $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ corresponds naturally to a monomial $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Two monomials have the same degree d if the corresponding points in $\mathbb{N}_0^n$ lie on the same hyperplane $\{x \in \mathbb{R}^n \mid \sum_i x_i = d\}$. Multiplication of two monomials $X^\alpha \cdot X^\beta = X^{\alpha+\beta}$ in this coordinate system can be interpreted as translation of the point α by the vector $[\beta]$. This is exactly a geometrical interpretation of the isomorphism between the monoid of monomials with multiplication and $\mathbb{N}_0^n$ with addition on coordinates. Moreover, since $\mathbb{N}_0^n$ is a monoid, we can speak about points generating other points (according to the definition of generating in any monoid).

Take a semialgebraic set S. Denote by $\mathcal{M}(S)$ the set of all monomials bounded on the set S. Since a product of two polynomials bounded on S remains bounded on S, the set $\mathcal{M}(S)$ with multiplication is a semigroup with the constant monomial 1 of degree 0 as the unit. It is easy to see that

$$\mathcal{A}(S) \supseteq \mathbb{R}[\mathcal{M}(S)]$$

and $\mathcal{M}(S \cup M) = \mathcal{M}(S) \cap \mathcal{M}(M)$ for sets $S, M \subset \mathbb{R}^n$.

For any set $A \subset \mathbb{R}^n$ we will denote by $\mathrm{conv}(A)$ the smallest convex subset of $\mathbb{R}^n$ containing A. In other words

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{N} t_i \alpha^{(i)} \in \mathbb{R}^n \mid N \in \mathbb{N},\ \sum_{i=1}^{N} t_i = 1,\ \alpha^{(i)} \in A,\ t_i \geq 0 \right\}.$$

We will say that a cone

$$\mathcal{C}_A = \{t\alpha \mid t \geq 0,\ \alpha \in A\}$$

is *spanned by a set* A. Note that if A is convex, then $\{X^\alpha \mid \alpha \in \mathcal{C}_A \cap \mathbb{N}_0^n\}$ is a semigroup of monomials, because $\alpha + \beta = 2(\frac{1}{2}\alpha + \frac{1}{2}\beta) \in \mathcal{C}_A$ for any $\alpha, \beta \in \mathcal{C}_A$.

We will now give some properties of the semigroup $\mathcal{M}(S)$. First, note that when taking rational exponents of coordinates X_i and treating them as functions we have to use the absolute value to make sure that appropriate powers are well-defined. Since a function is bounded on a set if and only if its absolute value is bounded, we can use the absolute value instead without loss of generality.

Property 5.3. *If the monomials* $\zeta_1 = X^{\alpha^{(1)}}, \ldots, \zeta_k = X^{\alpha^{(k)}}$ *are bounded on a set* S*, then every element of the semigroup*

$$\mathcal{C} = \mathbb{N}_0^n \cap \{t\alpha \mid t \geq 0,\ \alpha \in \operatorname{conv}(\alpha^{(1)}, \ldots, \alpha^{(k)})\}$$

corresponds to a monomial bounded on S*.*

Indeed, take $\beta \in \mathcal{C}$. Then $|X_1|^{\beta_1} \cdots |X_n|^{\beta_n} = (|\zeta_1|^{t_1} \cdots |\zeta_k|^{t_k})^t$ for some rational $t, t_1, \ldots, t_k \geq 0$ and $\sum t_i = 1$. Since $tt_i \geq 0$, we get that the function $|X_1|^{\beta_1} \cdots |X_n|^{\beta_n}$ is bounded on S. In particular, any monomial X^β is bounded on S if $\beta \in \mathcal{C}$.

From the above property it follows that the semigroup $\operatorname{supp}(\mathcal{M}(S))$ of all exponents of monomials bounded on a given set S always induces a convex cone in $\mathbb{R}^n$ i.e. $\operatorname{conv}(\operatorname{supp}\mathcal{M}(S))$ is a convex cone. Note that if $A = \{\alpha^{(1)}, \ldots, \alpha^{(k)}\}$, then

$$\mathcal{C}_{\operatorname{conv}(A)} = \left\{ \sum_{i=1}^{k} t_i \alpha^{(i)} \mid t_i \geq 0,\ i = 1, \ldots, k \right\}$$

is a so-called conical hull of points of A.

Property 5.4. *If* $\alpha^{(1)}, \ldots, \alpha^{(k)} \in \mathbb{N}_0^n$*, then the semigroup*

$$\mathcal{C} = \mathbb{N}_0^n \cap \{t\alpha \mid t \geq 0,\ \alpha \in \operatorname{conv}(\alpha^{(1)}, \ldots, \alpha^{(k)})\}$$

is finitely generated (as a subsemigroup of $\mathbb{N}_0^n$*).*

Proof: Indeed, $\mathcal{C}$ is generated by the finite set

$$\mathbb{N}_0^n \cap \{t\alpha \mid 1 \geq t \geq 0,\ \alpha \in \operatorname{conv}(\alpha^{(1)}, \ldots, \alpha^{(k)})\}.$$

Since the above set is equal to the set $\mathbb{N}_0^n \cap \{t\alpha \mid t \in \mathbb{Q},\ 1 \geq t \geq 0,\ \alpha \in \operatorname{conv}(\alpha^{(1)}, \ldots, \alpha^{(k)})\}$, we get that for any $t = p/q \geq 1$, where p, q are coprime natural numbers,

$$\frac{p}{q}(t_1\alpha^{(1)} + \ldots + t_k\alpha^{(k)}) \in \mathbb{N}_0^n \iff \frac{1}{q}(t_1\alpha^{(1)} + \ldots + t_k\alpha^{(k)}) \in \mathbb{N}_0^n$$

for any rationals $t_1, \ldots, t_k \geq 0$ such that $\sum t_i = 1$. ■

Hence $\{X^\alpha \mid \alpha \in \mathcal{C}\}$ in the above case is also a finitely generated semigroup.

Denote $\mathbb{R}_{\geq} = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{Q}_{\geq} = \mathbb{Q} \cap \mathbb{R}_{\geq}$. Note that any set H in $\mathbb{R}^n_{\geq}$ given as an intersection of a finite number of half-spaces (given by hyperplanes passing through zero) is a cone $\mathcal{C}_{\mathrm{conv}(A)}$ in $\mathbb{R}^n$ for some finite set A. Indeed, it suffices to consider the intersection of H with the hyperplane given by the equation $\sum_i x_i = 1$. It will be a convex polytope. Take its vertexes $l_1, \ldots, l_k$, which correspond to half-lines in $\mathbb{R}^n_{\geq}$ and choose $\alpha^{(1)}, \ldots, \alpha^{(k)} \neq 0$ such that $\alpha^{(i)} \in \{tl_i \mid t > 0\}$. We get $H = \mathcal{C}_{\mathrm{conv}(A)}$ for $A = \{\alpha^{(1)}, \ldots, \alpha^{(k)}\}$. Of course, if $\mathbb{N}^n \cap \{tl_i \mid t > 0\} \neq 0$, we can demand that $\alpha^{(i)}$ have natural numbers as coordinates. Hence if $H \subset \mathbb{Q}_{\geq}$ is an intersection of a finite number of half-spaces, then $\mathrm{conv}H \subset \mathbb{R}^n$ is a cone $\mathcal{C}_{\mathrm{conv}(A)}$ spanned by a finite set $A \subset \mathbb{N}^n_0$.

Remark 5.5. *Let $A \subset \mathbb{R}^n_{\geq}$. The cone $\mathcal{C}_A$ is closed if and only if there exists a compact set B such that*

$$\mathcal{C}_A = \mathcal{C}_B.$$

Indeed, if such a compact set exists, then the cone is closed. If the cone is closed, then its intersection with a hyperplane given by the equality $\sum x_i = 1$ is compact.

Hence as a consequence of Property 5.4 and the above remark we get

Theorem 5.6. *Let $S \subset \mathbb{R}^n$. The semigroup $\mathcal{M}(S)$ of monomials bounded on the set S is finitely generated if and only if there exists a finite set $A \subset \mathbb{N}^n_0$ such that it spans the cone* $\mathrm{conv}(\mathrm{supp}(\mathcal{M}(S)))$.

Proof: If $\mathcal{M}(S)$ is generated by a set $\mathcal{Z}$, then from Property 5.3 and properties of bounded polynomials we get

$$\mathrm{conv}(\mathrm{supp}\mathcal{M}(\mathrm{S})) = \mathcal{C}_{\mathrm{conv}(\mathrm{supp}\mathcal{Z})}.$$

Hence we put $A = \mathrm{supp}\mathcal{Z}$. Moreover, by Property 5.4 the above cone is closed if $\mathcal{Z}$ is finite.

On the other hand, assume that $\mathrm{conv}(\mathrm{supp}\mathcal{M}(\mathrm{S}))$ is spanned by a finite set $A \subset \mathbb{N}^n_0$. Hence the cone is closed. Take the polytope

$$\mathrm{conv}(\mathrm{supp}(\mathcal{M}(S))) \cap \{x \in \mathbb{R}^n | \ \sum x_i = 1\}.$$

Take its vertices $\gamma_1, \ldots, \gamma_l$. Then take $q_i \in \mathbb{R}$ such that $q_i \cdot \gamma_i \in \mathbb{N}^n_0$. Hence $\mathcal{M}(S)$ is generated by monomials X^α, where α are elements of the finite set

$$\mathbb{N}^n_0 \cap \{t\alpha \mid t \in \mathbb{Q}, 1 \geq t \geq 0, \ \alpha \in \mathrm{conv}(q_1\gamma_1, \ldots, q_l\gamma_l)\}.$$

This ends the proof. ■

Let us note that from the above it follows that if $\mathcal{M}(S)$ is finitely generated, then the intersection of $\mathrm{conv}(\mathrm{supp}(\mathcal{M}(S)))$ with the hyperplane given by the equality $\sum_i x_i = 1$ is a polytope.

Example 5.7. Take the set $S = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$. Then $\mathcal{M}(S) = \{XY^n \mid n \in \mathbb{N}_0\}$ and the cone $\mathrm{conv}(\mathrm{supp}\mathcal{M}(S))$ is not closed, because it is equal to the first quarter of the plane without the y axis.

We would like to underline also the constructive aspect of the above considerations in the following remark.

Remark 5.8. *Given the cone* $\mathrm{conv}(\mathrm{supp}\mathcal{M}(\mathrm{S}))$ *there exists a procedure of finding elements of the basis. If the cone is finitely generated, then this procedure stops after a finite number of steps.*

Indeed, in the case of a cone infinitely generated one of the possible methods is presented in the proof of Proposition 5.2. In the case of finite generation it suffices to find a finite set A such that $\mathrm{conv}(\mathrm{supp}\mathcal{M}(S)) = \mathcal{C}_{\mathrm{conv}(A)}$. To do that follow the proof of Theorem 5.6. Having a finite set of generators it suffices to combinatorially exclude the ones that are not elements of the basis.

5.3. Monomial bases of algebras of bounded polynomials

In this section we will show in Theorem 5.9 the duality between the points with natural coordinates from closed convex cones and semialgebraic sets described by appropriate monomial inequalities. Afterwards, we consider the algebras we have already studied in Sections 2.1 and 4.2.

Given a convex cone $\mathcal{C} \subset \mathbb{R}^n$ we can define $\mathcal{C}^\circ$ *a polar cone of* $\mathcal{C}$ by

$$\mathcal{C}^\circ = \{\eta \in \mathbb{R}^n \mid \langle \eta | \alpha \rangle \leq 0 \ for\ all\ \alpha \in \mathcal{C}\}.$$

We have that $(\mathcal{C}^\circ)^\circ = \overline{\mathcal{C}}$ (see [HUL]).

We say that a point $\alpha \in \mathcal{C}$, where $\mathcal{C}$ is a convex set, is an *extreme point of* $\mathcal{C}$ if there do not exist two distinct points $\alpha_1, \alpha_2 \in \mathcal{C}$ such that $\alpha = (\alpha_1 + \alpha_2)/2$. If $\alpha^{(1)}, \ldots, \alpha^{(l)}$ are extreme points of $\mathrm{conv}(\alpha^{(1)}, \ldots, \alpha^{(l)})$, then for any i we have

$$\mathrm{conv}\left(\alpha^{(1)}, \ldots, \alpha^{(l)}\right) \neq \mathrm{conv}\left(\{\alpha^{(1)}, \ldots, \alpha^{(l)}\} \setminus \{\alpha^{(i)}\}\right). \tag{5.1}$$

For other properties of polar cones and extreme points see [HUL].

Theorem 5.9. *Take any closed convex cone*

$$\mathcal{C} = \{t\alpha \in \mathbb{R}^n \mid t \geq 0,\ \alpha \in \mathrm{conv}(A)\},$$

where $A \subset \mathbb{N}_0^n$. *Then there exists a semialgebraic set* $S \subset \mathbb{R}^n$ *such that*

$$\mathcal{A}(S) = \mathbb{R}[X^\alpha \mid \alpha \in \mathcal{C} \cap \mathbb{N}_0^n].$$

Proof: By Theorem 5.6 we can take a finite set $\mathcal{Z}$ of generators of the semigroup $\mathcal{C} \cap \mathbb{N}_0^n$. Put

$$S = \bigcap_{\alpha \in \mathcal{Z}} \{x \in \mathbb{R}^n \mid |x^\alpha| \leq 1\}.$$

Of course, S is an unbounded semialgebraic set. Moreover, the inclusion $\mathcal{C} \cap \mathbb{N}_0^n \subset \mathrm{supp}\mathcal{M}(S)$ is obvious from the form of the set S. Hence we need to show the inverse inclusion.

Take any polynomial $f = \sum_{\alpha \in B} a_\alpha X^\alpha$, where $B = \mathrm{supp} f \subset \mathbb{N}_0^n$ is finite and $a_\alpha \neq 0$ for $\alpha \in B$. We will show that if $\mathrm{supp} f \nsubseteq \mathcal{C}$, then f is not bounded on S.

Take the affine hyperplane H given by the equality $\sum_i x_i = 1$. We can consider intersections of cones in $\mathbb{R}^n_{\geq}$ with the hyperplane H as images by a mapping $\pi : \mathbb{R}^n_{\geq} \setminus \{0\} \to H$ such that $\pi(x)$ is the intersection point of the line $\{tx \in \mathbb{R}^n \mid t \in \mathbb{R}\}$ with H.

We have that the intersection of H with the cone spanned by the set $\mathrm{conv}(\mathcal{C} \cup B)$ is a compact convex set. From Property 5.4 and the fact that the cone spanned by $\mathrm{conv}(\mathcal{C} \cup B)$ is closed it follows that it is finitely generated as a semigroup. Therefore the intersection with H must be a polytope. Hence the above intersection has extreme points $\eta^{(1)}, \ldots, \eta^{(l)}$ such that

$$\mathcal{C}_{\mathrm{conv}(\mathcal{C} \cup B)} \cap H = \mathrm{conv}(\eta^{(1)}, \ldots, \eta^{(l)}).$$

It is easy to see that $\eta^{(i)} \in \pi(\mathcal{Z} \cup B)$ for $i = 1, \ldots, l$. Moreover, if $\eta^{(i)} \in \pi(\mathcal{Z})$ for every i, then $\mathrm{supp} f \subset \mathcal{C}$, which is impossible by assumption. Hence there exists m such that $\eta^{(m)} \in \pi(B \setminus \mathcal{C})$. Denote by B' the nonempty finite set $\pi^{-1}(\eta^{(m)}) \cap B$. Take the closed convex cone

$$\mathcal{C}' = \{t\alpha \mid \alpha \in \mathrm{conv}(\{\eta^{(1)}, \ldots, \eta^{(l)}\} \setminus \{\eta^{(m)}\})\ \}.$$

Of course, $\mathcal{C} \subset \mathcal{C}'$, $B \setminus B' \subset \mathcal{C}'$ and $B' \cap \mathcal{C}' = \emptyset$ from (5.1). Since $\eta^{(m)} \notin \mathcal{C}' = \overline{\mathcal{C}'}$ and $(\mathcal{C}'^\circ)^\circ = \overline{\mathcal{C}'}$, there exists $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathcal{C}')^\circ$ such that $\langle \eta^{(m)} | \lambda \rangle > 0$. Since $\pi^{-1}(\eta^{(m)})$ is a half-line and the scalar product is linear, we have $\langle \eta | \lambda \rangle > 0$ for every $\eta \in B'$. Note that for $\eta, \eta' \in B'$ if $\eta \neq \eta'$, then

$$\langle \eta | \lambda \rangle \neq \langle \eta' | \lambda \rangle. \tag{5.2}$$

Indeed, $\langle \eta | \lambda \rangle = \langle \eta' | \lambda \rangle$ if and only if $\eta - \eta'$ is orthogonal to λ. But there exists $t \neq 1$ such that $t\eta = \eta'$, so $\langle \eta - \eta' | \lambda \rangle = (1 - t)\langle \eta | \lambda \rangle \neq 0$.

Take a curve $\gamma(t) = (t^{\lambda_1}, \ldots, t^{\lambda_n})$ for $t \geq 1$. We have for $\alpha \in \mathcal{C} \cap \mathbb{N}_0^n$ that

$$|X^\alpha \circ \gamma(t)| = |t^{\langle \alpha | \lambda \rangle}| \leq 1$$

for $t \geq 1$, because $\langle \alpha | \lambda \rangle \leq 0$. In particular the above is true for $\alpha \in \mathcal{Z}$. Hence $\gamma([1, \infty)) \subset S$. We have

$$(f \circ \gamma)(t) = \sum_{\alpha \in B \setminus B'} a_\alpha t^{\langle \lambda | \alpha \rangle} + \sum_{\alpha \in B'} a_\alpha t^{\langle \lambda | \alpha \rangle}$$

for $t \geq 1$. We see that

$$\sum_{\alpha \in B \setminus B'} a_\alpha t^{\langle \lambda | \alpha \rangle}$$

is bounded for $t \geq 1$, because $\langle \lambda | \alpha \rangle < 0$ for $\alpha \in B \setminus B' \subset \mathcal{C}'$. On the other hand in the second term

$$\sum_{\alpha \in B'} a_\alpha t^{\langle \lambda | \alpha \rangle}$$

the powers of t do not cancel out, because of (5.2). Hence it is unbounded for $t \geq 1$, because $\langle \lambda | \alpha \rangle > 0$ for $\alpha \in B'$. Therefore $f \circ \gamma$ is not bounded on $[1, \infty)$, unless $\mathrm{supp} f \subset \mathcal{C}$. This gives us that $\mathcal{A}(S) \subset \mathbb{R}[X^\alpha \mid \alpha \in \mathcal{C} \cap \mathbb{N}_0^n]$ and ends the proof. ■

Note that an intersection of two algebras generated by monomials is generated by monomials. Hence, if on each tentacle of a given set the algebra of bounded polynomials is generated by monomials, then it is easy to determine the algebra of polynomials on this given set. It is important here to remark that usually algebras of bounded polynomials are not generated by monomials. The semigroup $\mathcal{A}(M) \cap \mathcal{M}(M)$ may be trivial i.e. equal to $\{1\}$, even if $\mathcal{A}(M)$ is not. Let's look at an example

Example 5.10. Take the set $S = \{(x, y \in \mathbb{R}^2 \mid 0 \leq x - y \leq 1)\}$. We have $\mathcal{A}(S) = \mathbb{R}[X - Y]$ and of course $\mathcal{M}(S) = \{1\}$. Hence $\mathcal{A}(M)$ is not generated by monomials. (Nevertheless, in this case $\mathcal{A}(S)$ is isomorphic to an algebra $\mathbb{R}[X]$ generated by monomials.)

Even worse, there are algebras of bounded polynomials that are not isomorphic to any algebra generated by monomials. Take for example the algebra of bounded polynomials on the set M from Example 2.18, where $\mathbb{R}[XY, X^2Y, X^2Y + Y] \subset \mathcal{A}(M)$ and $Y \notin \mathcal{A}(M)$.

Nevertheless, let us now consider some explicit examples of algebras of bounded polynomials generated by monomials, which have appeared earlier in this thesis.

Let us look at sets from Section 2.1. Take an unbounded semialgebraic set

$$M = \{(x, y) \in \mathbb{R}^2 \mid \beta_1(y) \leq x \leq \beta_2(y),\ y \geq R\}, \tag{5.3}$$

for some $R > 0$ and β_1, β_2 Puiseux parametrizations of continuous semialgebraic curves with $\mathrm{ord}_\infty \beta_1 \neq \mathrm{ord}_\infty \beta_2$. Denote $\min\{\mathrm{ord}_\infty \beta_1, \mathrm{ord}_\infty \beta_2\} = \frac{p}{q}$ for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$ with no common divisors. For the moment we assume that $\min\{\mathrm{ord}_\infty \beta_1, \mathrm{ord}_\infty \beta_2\} \neq \infty$. From Theorem 2.4 it follows that

$$\mathcal{A}(M) = \mathbb{R}\left[X^i Y^d \;\middle|\; d \leq i \cdot \frac{p}{q} \right]. \tag{5.4}$$

Note that if $\frac{p}{q} < 0$ then $\mathcal{A}(M) = \mathbb{R}$ and the basis of $\mathcal{A}(M)$ is the empty set. If $\frac{p}{q} = 0$ then $\mathcal{A}(M) = \mathbb{R}[X]$. In this case we get that the set $\{X\}$ is a basis. Therefore, we suppose that $p, q \in \mathbb{N}$.

From equality (5.4) it follows

Corollary 5.11. *For a set M of the form (5.3) with $\mathrm{ord}_\infty \beta_1 \neq \mathrm{ord}_\infty \beta_2$ and $\min\{\mathrm{ord}_\infty \beta_1, \mathrm{ord}_\infty \beta_2\} \neq \infty$ the basis of the semigroup $\mathcal{M}(M)$ of monomials bounded on M is the basis of the algebra $\mathcal{A}(M)$ of polynomials bounded on M. In particular, $\mathcal{A}(M)$ is finitely generated.*

Denote by $[\,\cdot\,]$ the integer part of a real number i.e. for any real number α the value of $[\alpha]$ is the biggest integer smaller or equal to α. Then

Remark 5.12. *One can compute that the number of elements of the basis of the algebra* $\mathcal{A}(M)$ *is equal to*

$$1 + \left[\frac{p}{q}\right] + \sum_{k=2}^{q} r(k, \frac{p}{q}),$$

where

$$r(k, \alpha) = [k\alpha] - \max_{j=1,\ldots,k-1} \left[\frac{k}{j}[j\alpha]\right]$$

for $\alpha > 0$ *and* $k \geq 2$.

Indeed, consider monomials X^iY^d, $i, d \in \mathbb{N}_0$. Theorem 2.4 gives us that

$$\mathcal{M}(M) = \left\{ X^iY^d \,\middle|\, d \leq i \cdot \frac{p}{q} \right\}.$$

Hence the monomial X^iY^d is bounded on M if and only if $d \in \{0, \ldots, [i\frac{p}{q}]\}$.

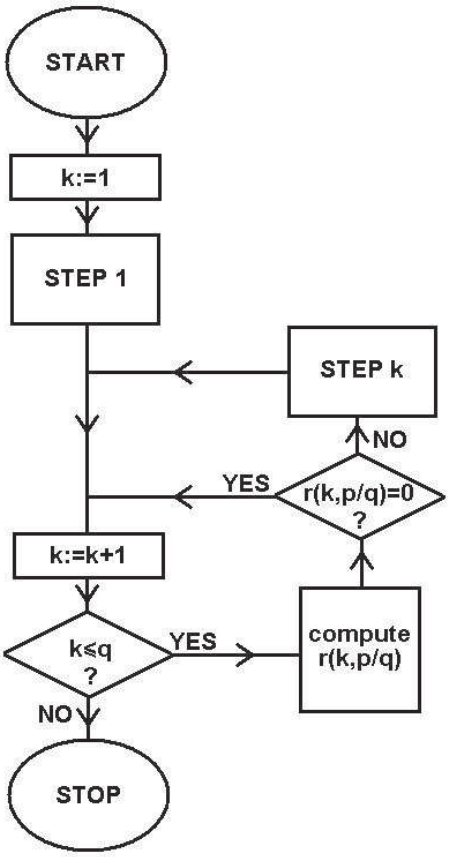

Figure 8. A procedure of choosing elements of the basis of $\mathcal{A}(M)$.

Let us consider a triangle

$$\Delta = \mathrm{conv}((0,0),(q,0),(q,p)) \cap \mathbb{N}_0^2.$$

From Properties 5.3 and 5.4 we get

$$\mathcal{M}(M) = \mathcal{C}_{\mathrm{conv}\Delta} \cap \mathbb{N}_0^n.$$

Let us consider

$$\Delta^{(i)} = \mathrm{conv}\left((0,0),(i,0),\left(i,\left[i\frac{p}{q}\right]\right)\right) \cap \mathbb{N}_0^2$$

for $i \in \mathbb{N}$. Each $\Delta^{(i)}$ is either a segment or a triangle contained in Δ. The function r in Remark 5.12 counts how many points more there are in subsequent triangles i.e. for a fixed slope p/q the value $r(k, p/q)$ is the quantity of points with integer coefficients which lie in $\Delta^{(k)}$, but do not lie in any cone generated by $\Delta^{(j)}$ for $j < k$ (the component $\max_{j=1,\ldots,k-1}\left[\frac{k}{j}[j\alpha]\right]$ of function r corresponds to this condition).

We will only outline the computation by describing the steps (see also Figure 8).

STEP 1: Take $1 + [p/q]$ points of the form $(1, d)$ from $\Delta^{(1)}$ i.e. $d = 0, \ldots, [p/q]$. Put

$$Z := \{(1,d)|\ d = 0, \ldots, [p/q]\}.$$

STEP k: If $r(k, p/q) > 0$, take $r(k, p/q)$ points $(k, d) \in \Delta^{(k)}$ such that $d = [kp/q] - r(k, p/q) + 1, \ldots, [kp/q]$. Set

$$Z := Z \cup \{(k,d)|\ d = [kp/q] - r(k,p/q) + 1, \ldots, [kp/q]\}.$$

Procedure stops when k gets bigger then q.

The outcome of this procedure is the set Z such that $\{X^iY^d|\ (i,d) \in Z\}$ is the basis of $\mathcal{M}(M)$. From the construction it follows that the set Z contains $1 + [p/q] + \sum_{k=2}^{q} r(k, p/q)$ points.

Now let us take a set $S \subset \mathbb{R}^n$ such that $\mathcal{A}(S)$ is generated by monomials and a set

$$\tilde{M} = \{(\beta(y) \cdot x, y) \in \mathbb{R}^{n+1}|\ x \in S,\ y \geq R\} \tag{5.5}$$

where $R > 0$ and $\beta = (\beta_1, \cdots, \beta_n) : [R, \infty) \to \mathbb{R}^n$ is a Puiseux parametrization of a semialgebraic curve at infinity such that $\beta_i \neq 0$. From Theorem 4.4 we get that

$$\mathcal{A}(\tilde{M}) = \mathcal{A}(S)[Y] \cap \mathbb{R}[X^\alpha Y^d|\ \langle\alpha|\lambda\rangle \geq d,\ \alpha \in \mathbb{N}_0^n, d \in \mathbb{N}_0].$$

Remark 5.13. *It is easy to see that $\mathcal{A}(\tilde{M})$ is generated by monomials and if $\mathcal{A}(S)$ is finitely generated, then so is $\mathcal{A}(\tilde{M})$.*

To end this section we would like to note that the methods presented here work also in the case when coordinates $X_1, \ldots, X_n$ are replaced by coordinates of a polynomial automorphism (this follows from the fact that polynomial automorphisms induce isomorphisms of algebras of polynomials bounded on appropriate sets cf. Property 1.16). Moreover, we can apply the methods given here to algebras from Section 2.2 of the form $\mathbb{R}[\frac{1}{Y^{1/q}}, (X-\beta)^iY^d \mid d \leq i\eta]$, which can be treated as generated by "monomials" with Puiseux coefficients and rational exponents (for which we can fix the denominator).

6. Some applications of the results

Here we give some consequences of the results presented in this thesis. First, we will show that for a semialgebraic set in $\mathbb{R}^2$ there exists a finite number of families which "test" boundedness of polynomials on the set (see Corollary 6.5). More precisely, each polynomial is bounded on the given set if and only if he is bounded on a finite number of members of these families. In the second part we will turn to a version of Schmüdgen's Positivstellensatz for unbounded sets. This version deals with representations of bounded polynomials as elements of a preordering on a given set. Since for a wide class of sets we are able to check which polynomials are bounded, we feel that this gives some constructivity to checking the assumptions of Theorems 6.10 and 6.13.

6.1. Testing curves for bounded polynomials

In this section we show that for a semialgebraic set S we can choose a finite number of families of curves such that for any polynomial f it is bounded on S if and only if it is bounded on a finite number of generic members of these families. These curves can be computed explicitly (using constructive Puiseux theorem and tools developed in Section 2, most notably Theorem 2.16).

To give a background we start with a general fact.

Proposition 6.1. *Take any closed semialgebraic set $S \subset \mathbb{R}^n$ and a polynomial $f \in \mathbb{R}[X]$. The polynomial f is bounded on S if and only if it is bounded on any continuous semialgebraic curve included in S.*

Proof: The necessity is obvious. Hence to show sufficiency we will prove that if a polynomial f is not bounded on S, then there exists a continuous semialgebraic curve $\beta \subset S$ such that f is not bounded on β.

Take any polynomial f which is not bounded on S. Without loss of generality we will assume that S is an unbounded set and f is unbounded on S from above. The set

$$Z = \bigcup_{R>0} \{x \in S |\ f(x) = \max_{S(0,R)\cap S} f\}$$

is an unbounded and semialgebraic subset of S. Hence from the Curve Selection Lemma there exists a continuous unbounded semialgebraic curve $\beta \subset Z$. Moreover, it is obvious

from the definition of Z that f is not bounded on β. ■

We will say that *a condition* Φ *holds for generic* $c \in \mathbb{R}^n$, if there exists a proper algebraic set $V \subset \mathbb{R}^n$ such that the condition Φ holds for any $c \notin V$. Note that, in the case $n = 1$, every proper algebraic set is finite.

From Theorem 2.16 we can derive an interesting consequence on deciding the boundedness of polynomials on subsets of the real plane. Let

$$M = \{(x, y) \in \mathbb{R}^2 |\ \beta_1(y) \leq x \leq \beta_2(y), y \geq R\}, \tag{6.1}$$

where $\beta_1 \neq \beta_2$ parametrise semialgebraic curves, $R > 0$ and the set M is unbounded. Put $p \in \mathbb{Z}$, $q \in \mathbb{N}$such that $\frac{p}{q} = \mathrm{ord}_\infty(\beta_1 - \beta_2)$. Although it does not influence the proof, we will suppose for simplicity that p and q are coprime. Take β_M as in Propositions 2.15 and 2.17 i.e.

$$\beta_M = \begin{cases} \sum\limits_{i=m}^{p-1} b_i^{(1)} (\frac{1}{Y})^{i/q}, & \mathrm{ord}_\infty\beta_1 = \mathrm{ord}_\infty\beta_2 = \frac{m}{q}, \\ 0, & \mathrm{ord}_\infty\beta_1 \neq \mathrm{ord}_\infty\beta_2, \end{cases}$$

where $b_i^{(1)} \in \mathbb{R}$ are the first $p - m + 1$ coefficients of β_1. Let us remind that we suppose that summing up over an empty set gives zero.

Under the above conditions and conventions we have

Theorem 6.2. *Let $f \in \mathbb{R}[X, Y]$. For a tentacle M of the form (6.1) with $\beta_1 \neq \beta_2$ take the Puiseux polynomial β_M. Then for generic $c \in \mathbb{R}$ we have that f is bounded on M if and only if it is bounded on the curve*

$$\Gamma_c = \{(x, y) \in \mathbb{R}^2 |\ c = (x - \beta_M(y))^q y^p,\ y \geq 1\}.$$

In other words, there exists a finite set $G = G(f)$ such that f is bounded on M if and only if f is bounded on some curve Γ_c, $c \notin G$.

Before proving the theorem let us make a remark.

Remark 6.3. *Note that Γ_c need not lie in M (see Example 6.4). Moreover, for $c < 0$ we have that if q is even, we get $\Gamma_c = \emptyset$. Whereas if q is odd, $c^{1/q}$ is well-defined.*

Proof of Theorem 6.2: We will write for brevity β instead of β_M. First we will expand f in a convenient form. By Proposition 2.13 the polynomial f can be represented uniquely (up to zero coefficients) as $\sum_{k\in A}\sum_{j=0}^{d} a_{j,k}(X - \beta)^j Y^{\frac{k}{q}}$, where $A \subset \mathbb{Z}$ is finite and $a_{j,k} \in \mathbb{R}$ for all $k \in A$, $j = 0, \ldots, d$. Therefore

$$f = \sum_{k\leq jp} a_{j,k}(X - \beta)^j Y^{k/q} + \sum_{k>jp} a_{j,k}(X - \beta)^j Y^{k/q}.$$

Due to Theorem 2.16 on characterisation of algebras of bounded polynomials on tentacles and its slight refinement in Proposition 2.17 we have that f is bounded on M if and only if the second sum is equal zero. Denote by $f_b = \sum_{k\leq jp} a_{j,k}(X - \beta)^j Y^{k/q}$ the bounded part.

Let us look at f on the curve Γ_c for any $c \in \mathbb{R}$. We have

$$f_{|\Gamma_c} = \sum_{k \leq jp} a_{j,k} c^{\frac{j}{q}} Y^{\frac{k-pj}{q}} + \sum_{k > jp} a_{j,k} c^{\frac{j}{q}} Y^{\frac{k-pj}{q}},$$

so the bounded part f_b is bounded on Γ_c.

Hence we see that if f is bounded on M we can write $f = f_b$, so f is bounded on Γ_c for any $c \in \mathbb{R}$. On the other hand, if f is not bounded on M then

$$\left|f_{|\Gamma_c}\right| \geq \left|\sum_{k > jp} a_{j,k} c^{\frac{j}{q}} Y^{\frac{k-pj}{q}}\right| - \left|f_{b|\Gamma_c}\right| \geq \left|\sum_{k > jp} a_{j,k} c^{\frac{j}{q}} Y^{\frac{k-pj}{q}}\right| - C,$$

for some $C \geq 0$ and any $c \in \mathbb{R}$. Therefore to show that f is unbounded on Γ_c we have to make sure that the unbounded part of f does not vanish on Γ_c. Let us look at the coefficients of any $Y^{l/q}$. So the unbounded part vanishes only for c from an algebraic set given by finitely many equalities of the form

$$G = \left\{ c \in \mathbb{R} : \sum_{\substack{k = jp + l \\ j = 0, \ldots, d}} a_{j,k} c^{j/q} = 0,\ l \in D \right\},$$

where $D \subset \{k - jp \mid k \in A, j = 0, \ldots, d\}$ is finite. If the set G had a nonempty interior, then the coefficients $a_{j,k}$ for $k > jp$ would be zeroes, so $f = f_b$, which contradicts the assumption that f is not bounded on M. Therefore for generic c, if the polynomial f is unbounded on M, then it is unbounded on the curve Γ_c. This ends the proof. ■

Example 6.4. Take

$$\begin{aligned} M &= \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{y} \leq xy^5 + y^2 \leq 1, y \geq 1\} = \\ &= \{(x,y) \in \mathbb{R}^2 \mid -\frac{1}{y^3} + \frac{1}{y^6} \leq x \leq \frac{1}{y^5} - \frac{1}{y^3},\ \ y \geq 1\} \end{aligned}$$

and a polynomial $f = X^2Y^4 + 2XY + 1$.

We have that $\beta_M = -1/Y^3$. It is easy to compute that G is empty, hence we can take any real c. In particular, take $c = 0$ and the curve

$$\Gamma_0 = \{(x,y) \in \mathbb{R}^2 |\ 0 = (x + \frac{1}{y^3})y^5,\ y \geq 1\}.$$

Note that $\Gamma_0 \cap M = \emptyset$ (in fact $\Gamma_c \setminus M$ is unbounded for every $c \in \mathbb{R}$). For $y \geq 1$ we have $-\frac{1}{y} = xy^2$ and $-\frac{1}{y^2} = xy$, so $f_{|\Gamma_0} = (-\frac{1}{y})^2 - 2\frac{1}{y^2} + 1 = 1 - \frac{1}{y^2}$. Hence $\|f_{|\Gamma_0}\| \leq 1$. From Proposition 6.2 it follows that $f \in \mathcal{A}(M)$.

From Theorem 6.2 it will follow that for any semialgebraic set there exist corresponding families of Puiseux polynomials which "test" boundedness. Take any semialgebraic set $S \subset \mathbb{R}^2$. Take all polynomials $g_1, \ldots, g_w$ used to describe it and set $g = g_1 \cdots g_w$. After a

linear change of coordinates, up to the sign, g is of the form $g = X^d + \sum_{i=1}^{d} a_i(Y)X^{d-i}$, where $\deg a_i \leq i$. Since $FrS \subset g^{-1}(0)$, we can easily decompose S as in Theorem 1.19 into l tentacles M_i. Note that the number l of tentacles cannot be greater than $\sum_{i=1}^{k} r_i$, where k is the number of polynomials g_i describing S and r_i is the number of distinct Puiseux parametrizations at infinity of $V(g_i)$. After a permutation of indices S has tentacles $M_1, \ldots, M_r$ with nonempty interior and $M_{r+1}, \ldots, M_l$ which Zariski closures are proper subsets in $\mathbb{R}^2$. In the following corollary we will deal also with them. Note that if we want to treat all tentacles at once, we have to also distinguish tentacles which borders are parametrised with positive and negative y.

As a consequence of Theorem 6.2 we get

Corollary 6.5. *For an unbounded semialgebraic set $S \subset \mathbb{R}^2$ and any $f \in \mathbb{R}[X,Y]$, after some linear change of coordinates, there exists a generic $c \in \mathbb{R}$, a finite number of Puiseux polynomials $\beta_1, \ldots, \beta_r, \gamma_{r+1}, \ldots, \gamma_l$ and integers $p_1, \ldots, p_r$ such that f is bounded on S if and only if f is bounded on curves*

$$\{(x,y) \in \mathbb{R}^2 \mid c = (x - \beta_i(y))^q y^{p_i},\ y \geq 1\} \tag{6.2}$$

for $i = 1, \ldots, k$,

$$\{(x,-y) \in \mathbb{R}^2 \mid c = (x - \beta_i(y))^q y^{p_i},\ y \geq 1\} \tag{6.3}$$

for $i = k+1, \ldots, r$,

$$\{(x,y) \in \mathbb{R}^2 \mid 0 = x - \gamma_i(y),\ y \geq 1\}$$

for $i = r+1, \ldots, s$ and

$$\{(x,-y) \in \mathbb{R}^2 \mid 0 = x - \gamma_i(y),\ y \geq 1\}$$

for $i = s+1, \ldots, l$.

Proof: If $\overline{S}$ after a linear change of coordinates and the decomposition as in Theorem 1.19 has an ample tentacle of the form (1.8), then $\mathcal{A}(S) = \mathbb{R}$ and it suffices to take $2(d+1)$ curves $\gamma_i = Y + i$ and $\gamma_{d+1+i} = -Y + i$ for $i = 0, \ldots, d$. Since $X+Y, X-Y$ is a system of coordinates, we see that f is bounded on each γ_i for $i = 0, \ldots, d$ if and only if $\deg_{X+Y} f = 0$ (compare with Property 4.3). By analogy $\deg_{X-Y} f = 0$, so f is bounded on curves γ_i for $i = 0, \ldots, 2d+1$ if and only if $\deg f = 0$. Hence f is bounded on curves γ_i for $i = 0, \ldots, 2d+1$ if and only f is constant.

Let us suppose that S does not have ample tentacles of the form (1.8). Enumerate tentacles of the set $\overline{S}$ such that $M_1, \ldots, M_r$ have nonempty interior (outside any compact set) and $M_{r+1}, \ldots, M_l$ have Zariski closures which are proper subsets of $\mathbb{R}^2$. Moreover, let $M_1, \ldots, M_k$ and $M_{r+1}, \ldots, M_s$ be the tentacles contained in $\mathbb{R} \times [0, \infty)$ and $M_{k+1}, \ldots, M_r$ and $M_{s+1}, \ldots, M_l$ be the tentacles contained in $\mathbb{R} \times (-\infty, 0]$.

For each tentacle M_i with nonempty interior and borders parametrised by positive y, $i = 1, \ldots, k$, take $\beta_i = \beta_{M_i}$ and a family of curves $\{(x,y) \in \mathbb{R}^2 \mid c = (x - \beta_i(y))^{q_i} y^{\tilde{p}_i},\ y \geq 1\}$ as in Theorem 6.2. Analogously, for each tentacle M_i with nonempty interior and borders parametrised by negative y, $i = k+1, \ldots, r$, consider $\phi(M_i)$, where $\phi = (X, -Y)$.

Take $\beta_i = \beta_{\phi(M_i)}$. It is easy to see that the family of curves $\{(x,-y) \in \mathbb{R}^2 \mid c = (x - \beta_i(y))^{q_i} y^{\tilde{p}_i},\ y \geq 1\}$ tests boundedness on M_i. We can take q as the least common multiple of the $q_1, \ldots, q_l$ and p_i such that $p_i/q = \tilde{p}_i/q_i$. From the fact that c is generic we conclude that f is bounded on $\bigcup_{i=1}^{r} M_i$ if and only if for a generic c it is bounded on the curves of the form (6.2) and (6.3).

For the tentacles $M_{r+1}, \ldots, M_l$ with proper Zariski closures it suffices to notice that each M_i is of the form

$$\{(x,y) \in \mathbb{R}^2 \mid 0 = x - \tilde{\gamma}_i(y),\ y \geq R\} \quad or \quad \{(x,-y) \in \mathbb{R}^2 \mid 0 = x - \tilde{\gamma}_i(y),\ y \geq R\}$$

for some Puiseux series at infinity $\tilde{\gamma}_i = \sum_{j=m_i}^{\infty} b_j^i (1/Y)^{j/s_i}$.

After simple calculations (see [RS]) we see that there exists D_i, which depends only on the degree of f and $\mathrm{ord}_\infty \tilde{\gamma}_i$, such that f is bounded on M_i for $i = r+1, \ldots, l$ if and only if it is bounded on a curve appropriately of the form

$$\{(x,y) \in \mathbb{R}^2 \mid 0 = x - \gamma_i(y),\ y \geq 1\} \quad or \quad \{(x,-y) \in \mathbb{R}^2 \mid 0 = x - \gamma_i(y),\ y \geq 1\},$$

where $\gamma_i = \sum_{j=m_i}^{D_i} b_j (1/Y)^{j/s_i}$ is a Puiseux polynomial. So we have the Puiseux polynomials γ_i that test the boundedness of the polynomial f on the tentacles with empty interior. Since $\mathcal{A}(S) = \bigcap_{i=1}^{l} \mathcal{A}(M_i)$, we end the proof. ■

We can see that the above corollary gives an effective method of deciding boundedness of a polynomial on a subset of $\mathbb{R}^2$. The only ingredient lacking is the ability to a priori determine what are the tentacles of the set. More precisely, if S is described by polynomials $g_1, \ldots, g_w$, then between which roots at infinity of $g = g_1 \cdots g_w$ do the tentacles of the set S lie. The question of efficient bound on a radius of a ball whose complement separates the tentacles (more precisely, a ball B such that the connected components of $S \setminus B$ are tentacles of S) is closely related to results recently settled for example in [BaK] and [BaR].

6.2. Schmüdgen's Positivstellensatz for bounded polynomials on unbounded sets

The purpose of this section is to present some versions of celebrated Schmüdgen's Positivstellensatz in the case when the set S is unbounded (Theorems 6.10 and 6.13). To this aim we will use mappings with generators of the algebra $\mathcal{A}(S)$ as their coordinates.

First, we will present elementary facts concerning preorderings and Schmüdgen's Positivstellensatz. Afterwards, we will introduce some helpful notions used in proofs of Theorems 6.10 and 6.13. Throughout this section we will meet the assumption that some functions are bounded. The previous sections give effective methods of verifying these condition in some cases.

Throughout this section let

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_w(x) \geq 0\},$$

where $g_1, \ldots, g_w \in \mathbb{R}[X]$. We assume that among the polynomials describing S there are only nonzero polynomials. We will write X for $(X_1, \ldots, X_n)$ unless stated otherwise.

For any ring A denote by $\sum A^2$ the set of all finite sums of squares of elements of A. We say that a set T is *a preordering in a ring* A if $T \subset A$ and

$$T + T \subset T, \quad T \cdot T \subset T \quad \textit{and} \quad \sum A^2 \subset T,$$

where $T + T = \{t + s |\ t, s \in T\}$ and $T \cdot T = \{t \cdot s |\ t, s \in T\}$. So in other words, T is a preordering in the ring A if it contains $\sum A^2$ and is closed with respect to multiplication and addition. Note that $\sum\ A^2$ is the smallest preordering in the ring of polynomials.

Set

$$T = \left\{ \sum_{\sigma \in \{0,1\}^w} s_\sigma g^\sigma |\quad s_\sigma \in \sum \mathbb{R}[X]^2 \right\}. \tag{6.4}$$

Such a set is *a preordering* in the ring of polynomials. It corresponds to the description of the set S. Note that for two different descriptions of the same basic semialgebraic set S there two corresponding preorderings may be different.

The following theorem was a milestone in the study of polynomial representations and alternative solutions of Hilbert's XVII problem. It was proved by Schmüdgen in [Sm] as a corollary to a result on the so called K-moment problem which lies in the scope of functional analysis and operator theory.

Theorem 6.6. (Schmüdgen's Positivstellensatz) *Let the set S be compact and f a polynomial in n variables. If $f > 0$ on S, then $f \in T$.*

Schmüdgen used methods from the field of functional analysis but the pivot of the proof was the Positivstellensatz of Krivine and Stengle (see [Kri], [St]) which is a classical result in real algebraic geometry. There have been successful attempts to prove this Theorem using only real algebraic tools (for example see [PD, Chapter 4.2], [M]).

Now let us prepare to prove some corollaries (Theorems 6.10 and 6.13) of Schmüdgen's Positivstellensatz in the case of unbounded sets. Suppose that a set $\mathcal{Z}$ generates the algebra $\mathcal{A}(S)$ (or $\mathcal{A}$ for short). For any finite number $\zeta_1, \ldots, \zeta_m$ of elements of $\mathcal{Z}$ take a mapping $\zeta = (\zeta_1, \ldots, \zeta_m) : \mathbb{R}^n \to \mathbb{R}^m$. For any unbounded set $S \subset \mathbb{R}^n$ such that $\mathcal{A}(S) = \mathbb{R}$ put $\zeta = 1$. Of course the mapping ζ is polynomial. Now we can state a following property

Proposition 6.7. *The following conditions are equivalent*

(1) *The set S is not bounded.*
(2) *Some mapping ζ is not proper on S.*
(3) *Every mapping ζ is not proper on S.*

By "some mapping ζ is not proper" we mean that for some choice of elements ζ_i of $\mathcal{Z}$ the corresponding mapping is nonproper, "every mapping ζ" alike.

Proof: The implication (3) $\Rightarrow$ (2) is obvious. To show (2) $\Rightarrow$ (1) it suffices to restate the implication equivalently: if S is bounded then every ζ is proper on S. This is evidently true since S is closed.

That leaves us the implication (1) $\Rightarrow$ (3). Choose any elements $\zeta_1, \ldots, \zeta_m \in \mathcal{Z}$. Let us suppose that S is not bounded. Then there exists a sequence (a_n) of elements of S such that $\lim_{n\to\infty} \|a_n\| = \infty$. The set $\{\zeta(a_n) \mid n \in \mathbb{N}\}$ is bounded since $\mathcal{Z} \subset \mathcal{A}(S)$. Hence $\zeta = (\zeta_1, \ldots, \zeta_m)$ is not proper. ■

Take any polynomial f bounded on S. Hence $f = \sum \varphi_\alpha \zeta^\alpha$ for some $\varphi_\alpha \in \mathbb{R}$ and $\zeta_1, \ldots, \zeta_j, \ldots, \zeta_m \in \mathcal{Z}$. Denote by f_ζ the polynomial $\sum \varphi_\alpha Y^\alpha$ in m variables i.e. f_ζ is a polynomial such that

$$f = f_\zeta \circ \zeta.$$

Let us remind that the representations of polynomials using generators are not unique. We have for example $X^2Y^2 = X \cdot XY^2 = (XY)^2$ in the algebra $\mathbb{R}[X, XY, XY^2]$ (see Example 6.16). Hence we have to specify each time which representation of f we are using, since f_ζ depends on such a choice.

Let us assume that $g_i \in \mathcal{A}$ for $i = 1, \ldots, w$. There exists $j \in \mathbb{N}$ such that for each i we have

$$g_i = \sum_{\alpha \in \{0,\ldots,j\}^l} a^i_\alpha \zeta^\alpha \tag{6.5}$$

for some basis $\zeta_1, \ldots, \zeta_l$ contained in $\mathcal{A}$ such that $\mathbb{R}[\zeta_1, \ldots, \zeta_l] \supset \mathbb{R}[g_1, \ldots, g_w]$. This need not be a basis of $\mathcal{A}$. In this paragraph we will use the representation $g_{i,\zeta} = \sum a^i_\alpha Y^\alpha$ of polynomials g_i i.e. $g_i = g_{i,\zeta} \circ \zeta$.

For a set $\mathcal{Z}$ of generators of $\mathcal{A}$ such that $\zeta_1, \ldots, \zeta_l \in \mathcal{Z}$, denote by $\zeta = (\zeta_1, \ldots, \zeta_l, \ldots, \zeta_m)$ a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ with arbitrary coordinates $\zeta_{l+1}, \ldots, \zeta_m \in \mathcal{Z}$. We want all elements $\zeta_1, \ldots, \zeta_l$ which generate polynomials $g_1, \ldots, g_w$ to be always coordinates of ζ (not necessarily all of them).

We get

$$S = \{x \in \mathbb{R}^n | \quad \sum a^1_\alpha \zeta^\alpha(x) \geq 0, \ldots, \sum a^w_\alpha \zeta^\alpha(x) \geq 0\}.$$

Moreover

$$\begin{aligned}\zeta(S) = \{y \in \mathbb{R}^m | \quad \exists_{x\in S}\ y = \zeta(x), \sum a^1_\alpha y^\alpha \geq 0, \ldots, \sum a^w_\alpha y^\alpha \geq 0\} = \\ = \{y \in \mathbb{R}^m | \quad \sum a^1_\alpha y^\alpha \geq 0, \ldots, \sum a^w_\alpha y^\alpha \geq 0\} \ \cap \ \zeta(S).\end{aligned}$$

Define

$$S_\zeta = \{y \in \mathbb{R}^m | \quad \sum a^1_\alpha y^\alpha \geq 0, \ldots, \sum a^w_\alpha y^\alpha \geq 0\}.$$

Under the introduced notation we have $\zeta(S) = S_\zeta \cap \zeta(S)$ for any representations $g_{1,\zeta}, \ldots, g_{w,\zeta}$ of the polynomials describing S.

Property 6.8. *The following properties hold*

(1) $S_\zeta \cap \zeta(S) = S_\zeta \cap \zeta(\mathbb{R}^n)$

(2) $S_\zeta \subset \zeta(S) \iff S_\zeta \subset \zeta(\mathbb{R}^n)$

(3) $S_\zeta \setminus \zeta(S) = S_\zeta \setminus \zeta(\mathbb{R}^n)$

Proof: We will show only property (1), since (2) and (3) only paraphraze it. The inclusion $S_\zeta \cap \zeta(S) \subset S_\zeta \cap \zeta(\mathbb{R}^n)$ is obvious. To show the opposite inclusion take $y \in S_\zeta \cap \zeta(\mathbb{R}^n)$ and suppose $y \notin S_\zeta \cap \zeta(S)$. For this choice of y there exists $x \in \mathbb{R}^n \setminus S$ such that $y = \zeta(x)$. Therefore for some i we have $\sum a^i_\alpha \zeta^\alpha(x) < 0$. Hence $\sum a^\alpha_i y^\alpha < 0$, so $y \notin S_\zeta$. This gives us a contradiction. ■

For simplicity of notation we will write $\mathbb{R}[\zeta]$ instead of $\mathbb{R}[\zeta_1, \ldots, \zeta_m]$ for the tuple $\zeta = (\zeta_1, \ldots, \zeta_m)$. For the preordering T given by (6.4) put

$$T_\zeta = \left\{ \sum_{\sigma \in \{0,1\}^w} s_\sigma g^\sigma \;\middle|\; s_\sigma \in \sum \mathbb{R}[\zeta]^2 \right\}.$$

Hence $T_\zeta \subset T$. Usually T_ζ is not a preordering in $\mathbb{R}[X]$ since all of its elements are bounded on the set S. Hence if S is unbounded, then $\|X\|^2 \notin T_\zeta$. On the other hand, for a bounded set S if $g_1, \ldots, g_w$ do not generate $\mathbb{R}[X]$ then it is possible to choose a basis $\mathcal{Z}$ such that T_ζ does not contain all elements of $\sum \mathbb{R}[X]^2$. In this case again, T_ζ is not a preordering in $\mathbb{R}[X]$. Nevertheless, T_ζ is a preordering in the ring $\mathbb{R}[\zeta] \subset \mathbb{R}[X]$.

First we will consider a less common situation and assume that $S_\zeta = \zeta(S)$.

Property 6.9. *Under the above assumptions i.e. the polynomials describing S are bounded on S and $S_\zeta \subset \zeta(S)$, the set S_ζ is bounded.*

The above property allows us to state the following

Theorem 6.10. *Take $f \in \mathcal{A}$. Assume that $g_1, \ldots, g_w \in \mathcal{A}$ and choose $\zeta_1, \ldots, \zeta_m \in \mathcal{A}$ such that they generate $g_1, \ldots, g_w$ and f. Suppose that $S_\zeta = \zeta(S)$. If $f > 0$ on S, then $f \in T_\zeta$.*

Proof: We have that $f = \sum \varphi_\alpha \zeta^\alpha(x) > 0$ for $x \in S$. Denote $f_\zeta = \sum \varphi_\alpha y^\alpha$. Then $f_\zeta > 0$ on $S_\zeta = \zeta(S)$.

If for the representations (6.5) we set $g^\sigma_\zeta = g^{\sigma_1}_{1,\zeta} \cdots g^{\sigma_w}_{w,\zeta}$ and

$$\zeta_*(T) = \left\{ \sum_{\sigma \in \{0,1\}^w} s_\sigma g^\sigma_\zeta \mid \quad s_\sigma \in \sum \mathbb{R}[Y_1, \ldots, Y_m]^2 \right\},$$

then $\zeta_*(T)$ is a preordering which corresponds to the description of the set S_ζ. From Schmüdgen's Positivstellensatz we get that $f_\zeta \in \zeta_*(T)$. Hence

$$f_\zeta(y) = \sum_{\sigma \in \{0,1\}^w} s_\sigma(y) \left(\sum a_\alpha y^\alpha \right)^\sigma,$$

where s_σ are sums of squares of polynomials in m variables. Substitute $y = \zeta(x)$. We get

$$f(x) = \sum_{\sigma \in \{0,1\}^w} s_\sigma(\zeta(x)) \left(\sum a_\alpha \zeta(x)^\alpha \right)^\sigma$$

for $x \in \mathbb{R}^n$. Terms $s_\sigma(\zeta(X))$ are sums of squares of polynomials in n variables, whereas $\sum a^i_\alpha \zeta(X)^\alpha = g_i$. Hence f is an element of T_ζ. ■

In particular, since $T_\zeta \subset T$, under the assumption of the above theorem we get $f \in T$.

Note that if the set S is bounded then $\mathcal{A} = \mathbb{R}[X]$. Hence we can take the monomial basis $X_1, \ldots, X_n$ of $\mathcal{A}$ and get $S_\zeta = S$. Of course, the functions describing a compact set belong to $\mathcal{A}$. Hence the above theorem in the case when S is compact and we choose the basis $X_1, \ldots, X_n$ is simply a restating of Schmüdgen's Positivstellensatz.

On the other hand, even in the case of S bounded, we can obtain representations with special features.

Example 6.11. If we take a bounded set

$$S = \{(x, y) \in \mathbb{R}^2 | \quad x^2y^4 - (x^2y^2 + y^4)^2 \geq 0,\ 1 - y^2 \geq 0,\ x^2y^2 - \frac{1}{16} \geq 0\},$$

then $\mathcal{A}(S) = \mathbb{R}[X, Y]$. In particular Y and XY are bounded on it. Take $\zeta = (Y, XY)$. We get a bounded set

$$S_{Y,XY} = \{(a, b) \in \mathbb{R}^2 | \quad b^2a^2 \geq (b^2 + a^2)^2, 1 \geq a^2,\ b^2 \geq \frac{1}{16}\}.$$

Moreover, one can show that $\zeta(S) = S_\zeta$. Hence as a straightforeward conclusion we get that any polynomial from the ring $\mathbb{R}[Y, XY]$ which is positive on S has a representation in terms of sums of squares of polynomials from this very ring.

If for the set S the equality $S_\zeta = \zeta(S)$ holds for S_ζ given by some representation of polynomials $g_1, \ldots, g_w$, Theorem 6.10 settles the problem of belonging to the preordering T, or more specifically to T_ζ, for all polynomials positive and bounded on S. Let us give an example of an unbounded set for which Theorem 6.10 works.

Example 6.12. Consider a set

$$S = \{(x, y, z) \in \mathbb{R}^3 | \quad x \geq x^2 + y^2 + x^2z^2\}.$$

Does $f = X^3Z^2 - XY - X + 2Y + 2$ have a representation in T?

From the description of S we get immediately that $\mathbb{R}[X, Y, XZ] \subset \mathcal{A}$. Put $\zeta_1 = X, \zeta_2 = Y, \zeta_3 = XZ$ and

$$S_{X,Y,XZ} = \{(a, b, c) \in \mathbb{R}^3 | \quad (a - 1/2)^2 + b^2 + c^2 \leq (1/2)^2\}.$$

Since $\zeta(\mathbb{R}^2) = (\mathbb{R}^3 \setminus V(a)) \cup V(a^2 + c^2)$, we get $\zeta(S) = S_\zeta$. Therefore we can apply Theorem 6.10 for any polynomial from the ring $\mathbb{R}[X, Y, XZ]$ which is positive on S. This is the case for f. Indeed, after regrouping the terms $f = (X^3Z^2 + 1) + (Y + 1)(2 - X)$. Since for any $(x, y, z) \in S$ we have $|y| \leq \frac{1}{2}$ and $0 \leq x \leq 1$, then $Y + 1, 2 - X$ and $X^3Z^2 + 1$ are positive on S. It follows that $f \in T_{X,Y,XZ}$.

Note that $\mathcal{A} \supsetneq \mathbb{R}[X, Y, XZ]$. It is easy to see that $Y^2 \leq \frac{1}{4(1+Z^2)}$ on S. This gives us that YZ is bounded on S. Hence if we put $\tilde{\zeta}$ a mapping with all elements of the monomial basis of $\mathcal{A}(S)$ as its coordinates, we get $\tilde{\zeta}(S) \neq S_{\tilde{\zeta}}$.

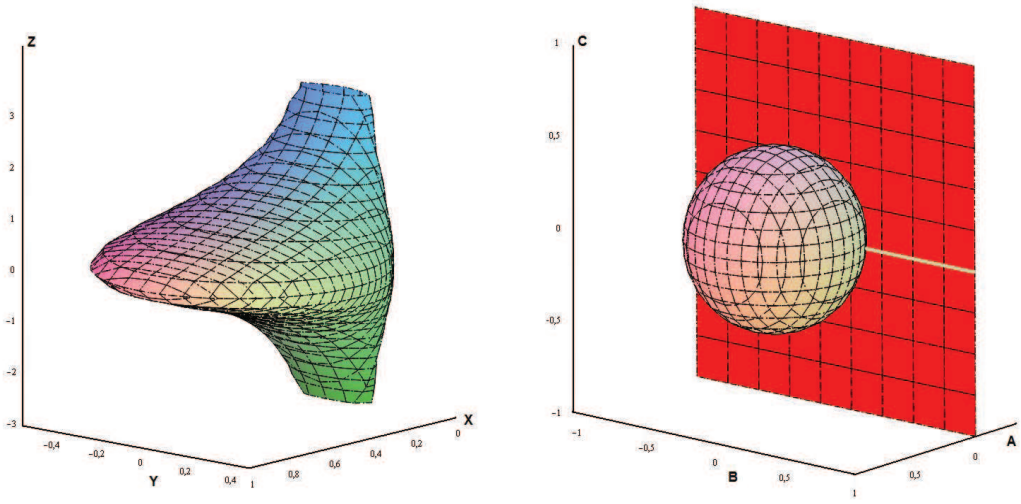

Figure 9. Sets $S = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq x^2 + y^2 + x^2 z^2\}$ and $S_{X,Y,XZ}$ with $\mathbb{R}^3 \setminus \zeta(\mathbb{R}^3)$ denoted by red.

Unfortunately, as we can see in the above example Theorem 6.10 works only when one studies a system of polynomials $g_1, \ldots, g_w, f$ generated in total by at most n generators. In general S_ζ has a nonempty interior if S has a nonempty interior whereas the set $\zeta(S)$ is at most n-dimensional. Thus if $m > n$, we get $S_\zeta \neq \zeta(S)$. Hence we must try to settle also the case when $\zeta : \mathbb{R}^n \to \mathbb{R}^m$ with $m > n$.

Using the notation of Theorem 6.10 we get

Theorem 6.13. *Take $f \in \mathcal{A}$. Assume that $g_1, \ldots, g_w \in \mathcal{A}$ and choose $\zeta_1, \ldots, \zeta_m \in \mathcal{A}$ such that they generate $g_1, \ldots, g_w$ and f. Suppose that $S_\zeta \cap \overline{\zeta(S)}^{Zar} = \overline{\zeta(S)}$. If $f > 0$ on S and all asymptotic values of f are positive with respect to S, then $f \in T_\zeta$.*

Proof: Denote $\overline{\zeta(S)}^{Zar} = V(u)$ for some polynomial u. Note that $S_\zeta \cap V(u)$ is a basic closed semialgebraic set. Since $f \in \mathcal{A}$, we have that $f = \sum a_\alpha \zeta^\alpha$.

We have that $f_\zeta > 0$ on $\overline{\zeta(S)}$ since f is supposed to have positive asymptotic values with respect to S. Indeed, for any $y \in \overline{\zeta(S)}$ there exists a sequence $\zeta(x_n) \in \zeta(S)$ converging to y. For any subsequence (x_{n_k}) which has a limit we get $f(x_{n_k}) = f_\zeta(\zeta(x_{n_k})) \to f_\zeta(y)$ since f_ζ is continuous as a polynomial. Either $x_{n_k} \to x \in S$ and $0 < f(x) = f_\zeta(y)$ or $x_{n_k} \to \infty$ and $f_\zeta(y)$ is an asymptotic value of f on S. Thus from Schmüdgen's Positivstellensatz and representation (6.5) of g_i we get

$$f_\zeta = \sum_{\sigma \in I} s_\sigma g_\zeta^\sigma + su$$

for some sums of squares $s, s_\sigma \in \sum \mathbb{R}[Y_1, \ldots, Y_m]^2$ and a finite subset I of $\mathbb{N}_0^w$.

Replace $y = \zeta(x)$ for any $x \in S$. We get

$$f(x) = \sum_{\sigma \in I} s_\sigma(\zeta(x)) g_\zeta^\sigma(\zeta(x)) + s(\zeta(x))u(\zeta(x)) =$$
$$= \sum_{\sigma \in I} s_\sigma(\zeta(x)) g^\sigma(x) + 0 \in T.$$

This ends the proof. ■

An example of application of Theorem 6.13.

Example 6.14. Let $g = -X^2 - (Z^2+1)Y^2 + 1$ and

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) \geq 0\}.$$

Does $Y + 2$ have a representation in T?

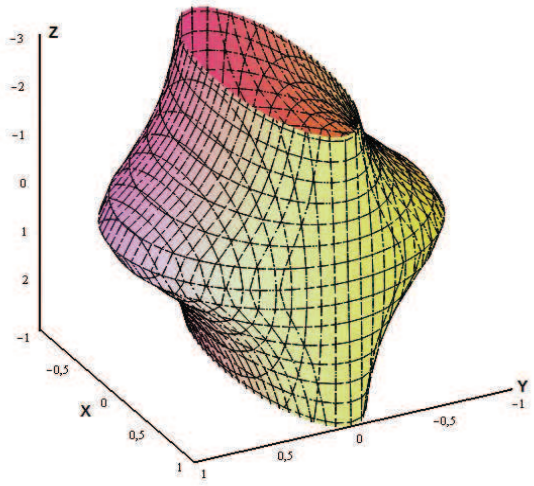

Figure 10. The set $\{(x, y, z) \in \mathbb{R}^3 | \quad g(x, y, z) = -x^2 - (z^2+1)y^2 + 1 \geq 0\}$.

The set S is a basic closed semialgebraic set which is not compact. Up to a bounded set, S is equal to

$$\left\{(x, \frac{1}{\sqrt{z^2+1}}y, z) \mid \quad x^2 + y^2 \leq 1, \ |z| \geq 1\right\}.$$

Hence, from Theorem 4.4 we have that $\mathcal{A}(S) = \mathbb{R}[X, Y, YZ]$. Put $(\zeta_1, \zeta_2, \zeta_3) = (X, Y, YZ)$. Of course, $g \in \mathcal{A}$ since $g = -\zeta_1^2 - \zeta_2^2 - \zeta_3^2 + 1$. Moreover

$$S_\zeta = \{(a, b, c) \in \mathbb{R}^3 | \quad a^2 + b^2 + c^2 \leq 1\}.$$

One can easily see that $\zeta(\mathbb{R}^3) = (\mathbb{R}^3 \setminus V(Y)) \cup \{(0, 0, 0)\}$. Hence $\zeta(S) \neq S_\zeta$. Nevertheless, the equality $\overline{\zeta(S)}^{Zar} \cap S_\zeta = \overline{\zeta(S)}$ holds. Thus from Corollary 6.15 we conclude

that any polynomial which is bounded on S from above and below by some positive constants is of the form $s_1 + s_2 g$ for some sums of squares s_1, s_2 which are bounded on S. In particular there exist sums of squares of polynomials in $\mathbb{R}[X, Y, YZ]$ such that $Y + 2 = s_1 + s_2(1 - X^2 - Y^2(Z^2 + 1))$.

It is obvious that if $\zeta_1, \ldots, \zeta_m$ generate $\mathcal{A}$ then one does not have to check in Theorems 6.10 and 6.13 whether $f \in \mathcal{A}$ is generated by $\zeta_1, \ldots, \zeta_m$. Hence we get the following corollary as a straightforward conclusion.

Corollary 6.15. *Suppose that $\mathcal{A}$ has a finite basis $\zeta_1, \ldots, \zeta_m$ and put $\zeta = (\zeta_1, \ldots, \zeta_m)$. Assume that $g_1, \ldots, g_w \in \mathcal{A}$. The following hold*

(1) *Assume that $\zeta(S) = S_\zeta$. If f is positive and bounded on S, then $f \in T_\zeta$.*

(2) *Assume that $S_\zeta \cap \overline{\zeta(S)}^{Zar} = \overline{\zeta(S)}$. If f is bounded, positive and has only positive asymptotic values on S, then $f \in T_\zeta$.*

Therefore, if $\mathcal{A}$ is finitely generated, then any polynomial bounded on S from below and above by positive constants has a representation in terms of the polynomials from $\mathbb{R}[\mathcal{Z}]$. Thus, as Theorem 6.13 shows more explicitly, finding the degree bounds for representations of bounded polynomials on sets described by polynomials bounded on them is similar to finding degree bounds of such representations in the case of compact sets (perhaps augmented by the degrees of generators of the algebra $\mathcal{A}$). Hence they depend solely on the degree and infimuum of the polynomial and the description of the set S (for effective Positivstellensätze and degree bounds see [PD], [Sw2] or [PR]).

When the algebra of bounded polynomials has a finite basis, it seems natural to take for ζ a mapping which has all the elements of this basis as coordinates. A following example shows that it makes a great difference if we take two different subsets of a basis.

Example 6.16. Take

$$S = \{(x, y) \in \mathbb{R}^2 | \quad x^2 - x^4(1 + y^2)^2 \geq 0\}.$$

We have that $S = K \cup M$ where K is bounded and

$$M = \left\{(x, y) \in \mathbb{R}^2 | \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{y^{2k}} \leq x \leq \sum_{k=1}^{\infty} (-1)^k \frac{1}{y^{2k}}, \quad y^2 \geq 4\right\}.$$

Hence from Theorem 2.7 it follows that for both connected components of M the algebra of bounded polynomials is $\mathbb{R}[X, XY, XY^2]$. Since K was bounded, we get $\mathcal{A}(S) = \mathbb{R}[X, XY, XY^2]$. In particular the unbounded set S is described by a polynomial which is bounded on S.

Take mappings $\zeta = (X, XY)$ and $\tilde{\zeta} = (X, XY^2)$. Consider the sets

$$S_{X,XY} = \{(a, b) \in \mathbb{R}^2 | \quad (a^2 + b^2)^2 - a^2 \leq 0\}$$

and

$$S_{X,XY^2} = \{(a, c) \in \mathbb{R}^2 | \quad (a^2 + ac)^2 - a^2 \leq 0\}.$$

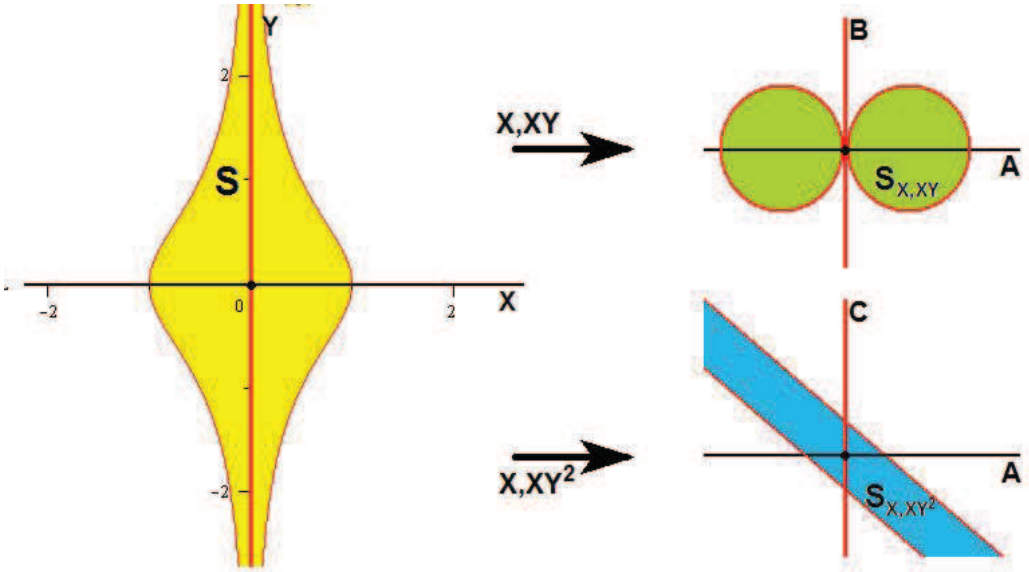

Figure 11. The sets $S_{X,XY}$ and S_{X,XY^2} from Example 6.16. The set $\mathbb{R}^3 \setminus \zeta(\mathbb{R}^3)$ is denoted by red. Take note that $ac = b^2$. Hence if we treat the sets $S_{X,XY}$ and S_{X,XY^2} as subsets of $\mathbb{R}^3$, the image of S will be the intersection of either of them with the cone $ac = b^2$.

After some easy calculation we get $\zeta(S) = S_{X,XY}$. Therefore we can apply Theorem 6.10 to the set $S_{X,XY}$.

But concerning the second set we have $\tilde{\zeta}(S) \subsetneq S_{X,XY^2}$. Furthermore, $V(X) \subset S_{X,XY^2}$ and $\overline{\tilde{\zeta}(S)}^{Zar} = \mathbb{R}^2$ while $(0, 10) \notin \overline{\tilde{\zeta}(S)}$. Hence $\overline{\tilde{\zeta}(S)}^{Zar} \cap S_{X,XY^2} \neq \overline{\tilde{\zeta}(S)}$ and we cannot apply neither Theorem 6.10 nor Theorem 6.13. (Note that if we take a different description of S, eg. given by the inequality $1 - x^2(1+y^2)^2 \geq 0$, then the equation $\overline{\tilde{\zeta}(S)}^{Zar} \cap S_{X,XY^2} = \overline{\tilde{\zeta}(S)}$ holds.)

Streszczenie

Niniejsza rozprawa doktorska poświęcona jest badaniu i wyznaczaniu algebr wielomianów ograniczonych na nieograniczonych zbiorach semialgebraicznych. W szczególności określeniu kiedy wielomian jest ograniczony na danym nieograniczonym zbiorze.

U źródła tej rozprawy leży próba uogólnienia do przypadku zbiorów nieograniczonych słynnego twierdzenia Schmüdgena [Sm, 1991]. Mówi ono, że każdy wielomian dodatni na zwartym bazowym zbiorze semialgebraicznym wyraża się za pomocą sum kwadratów wielomianów pomnożonych przez kombinacje wielomianów opisujących dany zbiór. W dowodzie Schmüdgena założenie zwartości odgrywa istotną rolę. Wynik ten uzyskał rozwiązując problem K-momentów dla zwartych zbiorów semialgebraicznych a dowód w istotny sposób używa metod analizy funkcjonalnej (miary spektralne). Positivstellensatz Schmüdgena daje możliwość skonstruowania algorytmu do obliczania kresu dolnego wielomianu na danym zwartym zbiorze semialgebraicznym. Warunek by wielomian f należał do preporządku jest używany w programowaniu. Skoro uzyskaliśmy wersję tego twierdzenia dla wielomianów ograniczonych na nieograniczonym zbiorze semialgebraicznym S, uznaliśmy, że z punktu widzenia zastosowań istotne jest efektywne decydowanie czy f jest ograniczony na S.

Optymizacja wielomianów (tj. znajdowanie kresu dolnego i górnego) na zbiorach semialgebraicznych jest ważnym i wyzywającym zadaniem, zarówno w praktyce jak i w teorii. Obecnie prowadzone są intensywne prace w tym kierunku, oparte głównie na przedstawieniach za pomocą sum kwadratów i, bardziej ogólnie, metodach Algebry Rzeczywistej. Istnieje wiele książek oraz opracowań dedykowanych różnym zagadnieniom tego tematu, na przykład [L], [Lt] oraz [PaS].

Aby rozszerzyć metodę Schmüdgena do przypadku nieograniczonych zbiorów semialgebraicznych, można rozważać algebry wielomianów ograniczonych na tych zbiorach. W rzeczy samej, pozytywne rezultaty w pewnych przypadkach zostały przedstawione w ostatnim rozdziale rozprawy, mając za inspirację interesującą pracę Schweighofera [Sw]. Aby je uzyskać, potrzeba było zrozumieć strukturę algebry wielomianów ograniczonych na danym zbiorze semialgebraicznym. Zaskakującym się wydaje, że ten problem zaczął być badany dopiero niedawno. W rozprawie doktorskiej D. Plaumanna (Konstanz, 2008), której promotorem był C. Scheiderer, pośród innych rezultatów, udowodnione zostało, że dla zbiorów regularnych w $\mathbb{R}^2$ (tj. będących domknięciem swojego wnętrza) algebry te są skończenie generowane. Niedawno S. Krug w [Krug] skonstruował przykład

semialgebraicznego zbioru regularnego w $\mathbb{R}^3$, którego algebra wielomianów ograniczonych nie jest skończenie generowana. Jednakże zbiór ten nie jest zbiorem bazowym, więc pytanie o skończoną generowalność pozostaje otwarte dla takiego typu zbiorów.

Niech S będzie podzbiorem $\mathbb{R}^n$. Oznaczmy przez

$$\mathcal{A}(S) = \{f \in \mathbb{R}[X_1, \ldots, X_n] \mid f \textit{ jest ograniczony na } S\}$$

algebrę wielomianów ograniczonych na S. Zbiór $\mathcal{A}(S)$ jest podpierścieniem pierścienia wielomianów oraz algebrą nad $\mathbb{R}$. Zauważmy, że jeśli S jest ograniczony, to $\mathcal{A}(S) = \mathbb{R}[X]$. W przeciwnym wypadku $\mathcal{A}(S)$ jest podzbiorem właściwym pierścienia wielomianów.

W niniejszej rozprawie rozpatrujemy kilka problemów dotyczących algebr wielomianów ograniczonych. Po pierwsze, zajmujemy się problemem decydowania o ograniczoności wielomianu na zbiorze. Rozwiązujemy go w Rozdziale 6.1 dla wielomianów dwóch zmiennych na dowolnym zbiorze semialgebraicznym, używając metod rozwiniętych w Rozdziale 2. W tymże rozdziale podajemy również metodę znajdowania generatorów algebry wielomianów ograniczonych dla szerokiej klasy podzbiorów semialgebraicznych płaszczyzny. W Rozdziale 3 pokazujemy zaskakującą zależność między zespolonymi wartościami bifurkacyjnymi wielomianu f a stabilnością rodziny algebr $\mathcal{A}(S_c)$, gdzie $S_c = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \leq c\}$.

W rozprawie zamiast metod Algebry Rzeczywistej woleliśmy używać raczej argumentów geometrycznych, unikaliśmy zatem używania standardowego języka Algebry Rzeczywistej. Skoro problemy, które badamy, mogą być wyrażone prosto, staraliśmy się również używać możliwie najprostszych metod i mamy nadzieję, że to się nam udało.

Aby uprościć badanie algebr wielomianów ograniczonych na zbiorze semialgebraicznym S, będziemy rozważali pewne podzbiory danego zbioru zwane mackami. Zbiór M nazywamy macką S, jeśli zbiór $M \setminus B(0, R)$ jest spójny dla dowolnego $R > 0$ oraz M jest jednym z nieograniczonych zbiorów w rozkładzie

$$S = K \cup M_1 \cup \ldots \cup M_l,$$

gdzie zbiór K jest zwarty, $l \in \mathbb{N}_0$ można wyznaczyć jednoznacznie i zbiory $M_1, \ldots, M_l$ są domkniętymi w S, parami rozłącznymi mackami (zob. Twierdzenie 1.19). Co więcej, jeśli $l = 0$, to $\mathcal{A}(S) = \mathbb{R}[X]$. W przeciwnym wypadku mamy

$$\mathcal{A}(S) = \bigcap_{i=1}^{l} \mathcal{A}(M_i).$$

Wyjściem do wyników Sekcji 2 jest obserwacja, że jeśli rozważamy podzbiory semialgebraiczne w $\mathbb{R}^2$, to możemy założyć, że macka M jest postaci

$$\{(x, y) \in \mathbb{R}^2 | \quad \beta_1(y) \leq x \leq \beta_2(y),\ y \geq R\},$$

gdzie R jest dodatnią liczbą rzeczywistą oraz $\beta_1(1/Y)$, $\beta_2(1/Y)$ są szeregami Puiseux, które parametryzują krzywe semialgebraiczne. Jeśli macka M zbioru S nie jest powyższej

postaci (z dokładnością do liniowej zamiany zmiennych), to $\mathcal{A}(M) = \mathbb{R}$, co pociąga za sobą również trywialność algebry $\mathcal{A}(S)$. Zatem w Rozdziale 2 zajmiemy się zbiorami M powyższej postaci.

W Twierdzeniu 2.4 dowodzimy, że jeśli $\mathrm{ord}_\infty\beta_1 \neq \mathrm{ord}_\infty\beta_2$, to

$$\mathcal{A}(M) = \mathbb{R}[X^iY^d |\ d \leq i\alpha],$$

gdzie $\mathrm{ord}_\infty\beta = \mathrm{ord}\,(\beta(1/Y))$ oraz $\alpha = \min\{\mathrm{ord}_\infty\beta_1, \mathrm{ord}_\infty\beta_2\}$. Zatem algebra wielomianów ograniczonych jest generowana przez jednomiany. Głównym punktem dowodu jest porównanie sumpremuum wielomianu z jego, odpowiednio wyrażonymi, współczynnikami. Udaje się to zrobić, gdy odległość zbioru od osi jest nie większa niż jego szerokość.

W drugiej części Rozdziału 2 rozważamy przypadek, gdy $\mathrm{ord}_\infty\beta_1 = \mathrm{ord}_\infty\beta_2$ oraz $\beta_1 \neq \beta_2$. Wprowadzamy szereg Puiseux o skończonym rozwinięciu β, który można wyznaczyć w skończenie wielu krokach z β_1 i β_2 (por. Stwierdzenie 2.15). Poprzez identyfikację β z osią y udawadniamy Twierdzenie 2.16, z którego wynika, że

$$\mathcal{A}(M) = \mathbb{R}[X,Y] \cap \mathbb{R}\left[\frac{1}{Y^{1/q}}, (X-\beta)^iY^d \mid d \leq i\eta\right],$$

gdzie $\eta = \min\{\mathrm{ord}_\infty(\beta_1 - \beta), \mathrm{ord}_\infty(\beta_2 - \beta)\}$. Pragniemy podkreślić, że daje to prosty sposób na sprawdzenie czy wielomian f jest ograniczony na M. Istotnie, nietrudno jest przestawić jakikolwiek wielomian w terminach powyższego pierścienia wielomianów z wielomianami Puiseux jako współczynnikami (zob. Stwierdzenie 2.13), potem wystarcza sprawdzić w jakich potęgach występuje Y. Zauważmy, że algebra $\mathcal{A}(S)$ nie musi być generowana przez jednomiany (ani też izomorficzna z żadną algebrą generowaną przez jednomiany). Dodajmy też, że wprowadzenie wielomianów o współczynnikach Puiseux pozwala nam traktować wszystkie algebry wielomianów ograniczonych na mackach jakby były generowane przez jednomiany, co ułatwia ich badanie (por. Rozdział 5). Co więcej, zwykle trudno jest określić czy wielomian należy do podpierścienia danego przez ustalone wielomiany, podczas gdy w przypadku tego rozszerzonego pierścienia przedstawienie f można uzyskać przez proste symboliczne przekształcenia (zauważmy, że β ma skończone rozwinięcie).

W Sekcji 3 rozważamy zbiory semialgebraiczne postaci

$$S_c = \{(x,y) \in \mathbb{R}^2 |\ f(x,y) \leq c\},$$

gdzie f jest wielomianem, zaś c jest liczbą rzeczywistą. Głównym wynikiem rozdziału jest Twierdzenie 3.5 o stabilności algebr $\mathcal{A}(S_c)$. Mianowicie dowodzimy, że algebry wielomianów ograniczonych na S_c są, do pewnego stopnia, nieczułe na zmianę parametru c. Ujmując to bardziej precyzyjnie, dla dowolnych $c < \tilde{c}$ mamy

$$\mathcal{A}(S_c) = \mathcal{A}(S_{\tilde{c}})$$

o ile $[c,\tilde{c}] \cap B_{\mathbb{C}}(f) = \emptyset$. Zbiór $B_{\mathbb{C}}(f)$ wartości bifurkacyjnych jest zdefiniowany na stronie 44. Należy podkreślić, że zbiór ten jest skończony i może być efektywnie

wyznaczony dla dowolnego wielomianu f dwóch zmiennych. Głównym narzędziem w dowodzie twierdzenia jest parametryczna wersja twierdzenia Puiseux oraz rezultaty Rozdziału 2. Mamy nadzieję, że to podejście ilustruje związek pomiędzy wielomianami ograniczonymi a geometrią włókien f oraz że może rzucić nowe światło na własności wartości bifurkacyjnych jak również na właściwość odwzorowań wielomianowych. Równie interesujące w $\mathbb{R}^2$ byłoby zbadanie stabilności zbiorów opisanych więcej niż jedną nierównością wielomianową. To wciąż pozostawiłoby przypadek wyższych wymiarów jako otwarty problem. Mimo że w prostych przypadkach (na przykład dla zbiorów opisanych nierównościami jednomianowymi jak w Twierdzeniu 5.9) łatwo widać, że zbiory są nieczułe na zmianę parametrów, ogólnie problem nie wydaje się łatwy do rozwiązania.

Rozdział 4 poświęcony jest badaniu algebr wielomianów ograniczonych na zbiorach w $\mathbb{R}^n$, gdzie n jest dowolne. Pierwsza część poświęcona jest specjalnemu typowi zbiorów, które nazywamy mackami z wagami i które można uznać za "jednostajne zniekształcenie" zbioru niższego wymiaru wzdłuż osi ostatniej współrzędnej. Dokładniej, załóżmy, że $S \subset \mathbb{R}^n$ ma niepuste wnętrze. Rozważmy zbiór

$$M = \{(\beta_1(y)x_1, \ldots, \beta_n(y)x_n, y) \in \mathbb{R}^{n+1} |\ x \in S,\ y \geq R\}$$

gdzie $R > 0$ oraz $\beta_1(1/Y), \ldots, \beta_n(1/Y)$ są szeregami Puiseux takimi, że $\beta_i(y)$ są zbieżne oraz mają stały dodatni lub ujemny znak dla $y \in [R, \infty)$.

W myśl Twierdzenia 4.4, jeśli algebra $\mathcal{A}(S)$ jest generowana przez jednomiany, to

$$\mathcal{A}(M) = \mathcal{A}(S)[Y] \cap \mathbb{R}\left[X^\alpha Y^d |\ \sum_{i=1}^{n} \alpha_i\beta_i \geq d\right],$$

gdzie $\lambda \in \mathbb{Q}^n$ jest ciągiem wag zbioru M oraz dla skrótu piszemy $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Zauważmy, że dowolny zbiór semialgebraiczny na prostej bądź ma algebrę wielomianów ograniczonych równą $\mathbb{R}[X]$ (więc generowaną przez jednomian X), bądź jest ona trywialna. Zatem powyższe twierdzenie uogólnia Twierdzenie 2.1 z Rozdziału 2. Co więcej, w pewnych przypadkach wyższych wymiarów daje nam to praktyczną możliwość wyznaczania generatorów algebry wielomianów ograniczonych (por. przykłady z Rozdziału 6). Twierdzenie powyższe uogólnia i rozszerza wyniki T. Netzera z [Net], który rozważa jedynie projekcje takich zbiorów na pierwsze n współrzędnych i uzyskane zostały zupełnie odmiennymi metodami. Zatem w [Net] wymiary zbioru S oraz macki są takie same, co więcej S ma być zwarty, które to założenia są istotnie wykorzystywane.

W ostatniej części Rozdziału 4 podajemy alternatywny dowód (oparty na zbiorze niewłaściwości Jelonka) faktu pokazanego już wcześniej przez D. Plaumanna i C. Scheiderera w [PlSd]. Mianowicie, jeśli S jest nieograniczonym podzbiorem właściwego zbioru algebraicznego w $\mathbb{R}^n$, to algebra $\mathcal{A}(S)$ nie może być skończenie generowana. Również w tym rozdziale pojawia się przykład (Przykład 4.2 zbioru semi-analitycznego w $\mathbb{R}^2$ o niepustym wnętrzu, którego algebra wielomianów ograniczonych jest nieskończenie generowana), który jest ciekawy ze względu na wyniki Plaumanna i Scheiderera o skończonej generowalności dla regularnych zbiorów semialgebraicznych w $\mathbb{R}^2$.

W związku z tym, że algebry generowane przez jednomiany pojawiają się w pracy w sposób naturalny, omawiamy ich własności w Rozdziale 5. Używając klasycznych metod geometrii wypukłej ustalamy relację pomiędzy wypukłymi stożkami wykładników jednomianów a zbiorami opisanymi przez odpowiednie nierówności jednomianowe (Twierdzenie 5.9). W szczególności chcielibyśmy zwrócić uwagę, że algebry z Twierdzeń 2.4 i 4.4 są generowane przez jednomiany. Na sam koniec obliczamy najmniejszą ilość generatorów jednomianowych dla algebry wielomianów ograniczonych na macce na płaszczyźnie. Na tym tle widać jasno, że obliczanie ilości elementów bazy dla przecięcia jak i sumy skończonej ilości algebr generowanych przez jednomiany również jest zadaniem czysto kombinatorycznym.

Załóżmy ponownie, że S jest semialgebraicznym podzbiorem płaszczyzny. Z Twierdzenia 2.16 o krzywych testujących wynika, że wielomian f jest ograniczony na S wtedy i tylko wtedy gdy jest ograniczony na skończonej ilości krzywych generycznie wybranych z rodziny krzywych, która zależy jedynie od zbioru S. Dowód, opierający się na wynikach przedstawionych w Rozdziale 2, jest zawarty w części pierwszej Rozdziału 6.

W Sekcji 6.2 przedstawiamy wersje Positivstellensatz Schmüdgena dla wielomianów ograniczonych. Weźmy bazowy semialgebraiczny zbiór domknięty

$$S = \{x \in \mathbb{R}^n |\ g_1(x) \geq 0,\ \ldots,\ g_w(x) \geq 0\}$$

gdzie $g_1, \ldots, g_w$ są wielomianami. Preporządkiem związanym z S jest zbiór

$$T = \left\{ \sum_{\sigma \in \{0,1\}^w} s_\sigma g^\sigma |\quad s_\sigma \in \sum \mathbb{R}[X]^2 \right\}.$$

Weźmy wielomian f. Załóżmy, że wielomiany g_i opisujące zbiór S są ograniczone na S. Co więcej, załóżmy, że $S_\zeta \cap \overline{\zeta(S)}^{Zar} = \overline{\zeta(S)}$ (definicje tych zbiorów znajdują się na stronie 83). Twierdzenie 6.13 mówi, że jeśli f na S jest ograniczony oraz większy od pewnej dodatniej stałej, to $f \in T$. W rzadszym przypadku, gdy $S_\zeta = \zeta(S)$, wystarcza, by f był na S ograniczony i dodatni (por. Twierdzenie 6.10). W dowodach obu twierdzeń użyliśmy prostej obserwacji, że odwzorowanie mające za swoje współrzędne generatory (lub bazę) algebry wielomianów ograniczonych jest, w pewnym sensie, "kanonicznym" odwzorowaniem ograniczonym. Zatem w Positivstellensatz Schmüdgena można spróbować zastąpić zwartość zbioru przez założenie o ograniczoności wielomianów. Jednakże zwróćmy uwagę, że nawet w przypadku zwartym Twierdzenia 6.10 i 6.13 pokazują, że jeśli dany wielomian f oraz wielomiany g_i opisujące zbiór S należą do pewnej podalgebry $\mathbb{R}[X]$, to f można wyrazić za pomocą sum kwadratów z tej właśnie podalgebry (por. Przykład 6.11).

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Résumé

Dans cette thèse nous étudions les algèbres de polynômes qui sont bornés sur un ensemble semi-algébrique non borné. En particulier nous déterminons si un polynôme est borné sur un ensemble semi-algébrique borné.

La motivation initiale de cette thèse est une généralisation au cas des ensembles non bornés d'un théorème célèbre de Schmüdgen [Sm, 1991] qui affirme qu'un polynôme positif sur un ensemble semi-algébrique basique et compact est une somme de carrés de polynômes multipliés par les polynômes définissant l'ensemble semi-algébrique. Dans la preuve de Schmüdgen l'hypothèse de compacité est essentielle. Le résultat est obtenu en résolvant le problème du K-moment pour les semi-algébriques compacts et la preuve utilise des méthodes d'analyse fonctionnelle (mesures spectrales). Le théorème de Schmüdgen permet de construire un algorithme donnant la borne inférieure d'un polynôme sur un ensemble semi-algèbrique compact. Comme la condition que f appartient à un préordre est utilisée dans le programme et comme nous obtenons une version du théorème pour les polynômes bornés, il nous a paru naturel de pouvoir décider effectivement quand f est borné sur S.

L'optimisation des polynômes (i.e. la détermination des bornes de ceux-ci) sur des ensembles semi-algébriques est un problème important et délicat, du point de vue théorique et pratique. Aujourd'hui il y a une grande activité dans ce domaine, essentiellement basée sur la représentation en sommes de carrés et plus généralement sur des méthodes d'algèbre réelle. On trouve un grand nombre de livres et d'articles d'introduction aux différents aspects du sujet, par exemple [L], [Lt] et [PaS].

Afin d'étendre la méthode de Schmüdgen au cas des ensembles semi-algébriques non bornés on peut considérer l'algèbre des polynômes bornés sur de tels ensembles. C'est ce qui est fait dans certains cas particuliers dans le dernier chapitre de la thèse, sous l'inspiration du bel article [Sw] de Schweighofer. Dans cette perspective il est important de comprendre la structure de l'algèbre des polynômes bornés sur un ensemble semi-algébrique non borné donné. De façon surprenante ce problème n'a été étudié que récemment. En effet dans la thèse de D. Plaumann (Konstanz, 2008), dirigée par C. Scheiderer, il est prouvé que pour les ensembles réguliers de $\mathbb{R}^2$ (i.e. les ensembles qui sont l'adhérence de leur intérieur) ces algèbres sont finiment engendrées. Récemment Krug [Krug] a construit un exemple d'un ensemble régulier non borné de $\mathbb{R}^3$ pour lequel l'algèbre des polynômes non bornés n'est pas finiment engendrée. Cependant cet ensemble n'est pas un ensemble fermé basique de sorte que la question reste ouverte pour de tels

ensembles.

Soit S un sous-ensemble de $\mathbb{R}^n$. Notons

$$\mathcal{A}(S) = \{f \in \mathbb{R}[X] \mid f \textit{ est borne sur } S\}$$

l'algèbre des polynômes bornés sur S. L'ensemble $\mathcal{A}(S)$ est un sous-anneau de $\mathbb{R}[X]$ et une algèbre sur $\mathbb{R}$. Remarquons que si S est borné, $\mathcal{A}(S) = \mathbb{R}[X]$. Dans le cas contraire, l'algèbre $\mathcal{A}(S)$ est un sous anneau propre de l'anneau des polynômes.

Dans cette thèse nous traitons différents problèmes concernant les algèbres des polynômes bornés. Tout d'abord nous abordons le problème consistant à déterminer si un polynôme est borné sur un ensemble. Nous résolvons ce problème dans la Section 6.1 pour les polynômes à deux variables définis sur des ensembles semi-algébriques quelconques, en utilisant les méthodes de la Section 2. Dans la section suivante nous donnons une méthode pour déterminer des générateurs de $\mathcal{A}(S)$ et ce pour une large classe de semi-algébriques de $\mathbb{R}^2$. Dans la Section 3 nous établissons une relation surprenante entre les valeurs de bifurcation complexes d'un polynôme et la stabilité de la famille d'algèbres $\mathcal{A}(S_c)$, où $S_c = \{(x, y) \in \mathbb{R}^2 |\ f(x, y) \leq c\}$.

Tout au long de la thèse, plutôt que des méthodes d'algèbre réelle, nous préférons utiliser des arguments géométriques, ainsi nous avons évité d'employer le langage de l'algèbre réelle. Comme les problèmes que nous attaquons sont simplement formulés, nous avons tenté d'utiliser les méthodes les plus simples et directes et nous espérons y être parvenu.

Pour simplifier l'étude des algèbres de polynômes bornés sur un ensemble S, nous considérons des sous-ensembles de S que nous appelons des tentacules. Un ensemble M est une tentacule de S si $M \setminus B(0, R)$ est connexe pour tout $R > 0$ et M est un des ensembles dans la décomposition

$$S = K \cup M_1 \cup \ldots \cup M_l,$$

avec K compact, $l \in \mathbb{N}_0$ et $M_1, \ldots, M_l$ sont des fermés de S et des tentacules deux à deux distinctes (voir le Théorème 1.19). De plus si $l = 0$, alors $\mathcal{A}(S) = \mathbb{R}[X]$, sinon

$$\mathcal{A}(S) = \bigcap_{i=1}^{l} \mathcal{A}(M_i).$$

Un point de départ pour les résultats de la Section 2 est l'observation selon laquelle pour un sous-ensemble semi-algébrique de $\mathbb{R}^2$, nous pouvons supposer que M_i est de la forme

$$\{(x, y) \in \mathbb{R}^2 |\quad \beta_1(y) \leq x \leq \beta_2(y),\ y \geq R\},$$

où R est un réel positif et $\beta_1(1/Y), \beta_2(1/Y)$ des séries de Puiseux qui paramt́rent courbes semi-algébriques. Si une tentacule M de l'ensemble S n'est pas de la forme ci-dessus (à un

changement linéaire de coordonnées près), alors $\mathcal{A}(M) = \mathbb{R}$, ce qui implique que l'algèbre $\mathcal{A}(S)$ est triviale. Dans toute la Section 2 nous considérons un semi-algébrique M de la forme ci-dessus.

Dans le Théorème 2.4 nous prouvons que $\text{ord}_\infty\beta_1 \neq \text{ord}_\infty\beta_2$, alors

$$\mathcal{A}(M) = \mathbb{R}[X^iY^d |\ d \leq i\alpha],$$

où $\alpha = \min\{\text{ord}_\infty\beta_1, \text{ord}_\infty\beta_2\}$ et $\text{ord}_\infty\beta = \text{ord}\beta(1/Y)$. Ainsi l'lagèbre des polynômes bornés est engendrée par des monômes. Le point le plus important est la comparaison du supremum d'un polynôme avec ses coefficients. Cela peut être effectué chaque fois que la distance entre la tenntacule et l'axe est plus petit que sa largeur.

Dans la deuxième partie de la Section 2 on considère le cas où $\text{ord}_\infty\beta_1 = \text{ord}_\infty\beta_2$ et $\beta_1 \neq \beta_2$. Nous introduisons une série de Puiseux β ayant un développement fini, qui peut être calculée en un nombre fini d'étapes à partir de β_1 et β_2 (voir Proposition 2.15). Par identification de β à l'axe des ordonnées nous montrons le Théorème 2.16, nous avons alors

$$\mathcal{A}(M) = \mathbb{R}[X, Y] \cap \mathbb{R}\left[\frac{1}{Y^{1/q}}, (X-\beta)^iY^d \mid d \leq i\eta\right],$$

où $\eta = \min\{\text{ord}_\infty(\beta_1 - \beta), \text{ord}_\infty(\beta_2 - \beta)\}$. Nous insistons sur le fait que cela fournit une méthode directe pour déterminer si un polynôme est borné sur M. En effet, il est aisé d'écrire un polynôme grâce aux polynômes de notre anneau de polynômes bornés à coefficients de Puiseux (voir Proposition 2.13). Il suffit ensuite d'examiner les exposants de Y dans une telle présentation. Remarquons qu'il n'est pas nécessaire que l'algèbre $\mathcal{A}(S)$ soit engendrée par des monômes (ou soit isomorphe à une telle algèbre). Ajoutons enfin que l'introduction de polynômes avec des coefficients de Puiseux permet de traiter toutes les algèbres de polynômes bornés sur des tentacules comme si elles étaient engendrées par des monômes, ce qui facilite leur étude (voir Section 5). De plus, il est en général difficile de déterminer si un polynôme appartient à un sous-anneau engendré par des polynômes donnés, tandis que dans le cas de cette extension d'anneau, les représentations de f sont obtenues par des calculs formels simples (β possédant un développement fini).

Dans la Section 3 nous considérons des ensembles semi-algébriques de la forme

$$S_c = \{(x, y) \in \mathbb{R}^2 |\ f(x, y) \leq c\},$$

où f est un polynôme et c un nombre réel. Le résultat principal de cette section est le Théorème 3.5 de stabilité des algèbres $\mathcal{A}(S_c)$. Précisément, nous obtenons que les algèbres de polynômes bornés sur S_c sont, à un nombre fini de points près, insensibles au changement de paramètres c : pour tout $c < \tilde{c}$ nous avons

$$\mathcal{A}(S_c) = \mathcal{A}(S_{\tilde{c}})$$

tant que $[c, \tilde{c}] \cap B_{\mathbb{C}}(f) = \emptyset$. L'ensemble des valeurs de bifurcation $B_{\mathbb{C}}(f)$ est défini page 44. Sa principale propriété est qu'il est fini et peut être calculé pour tout polynôme f de deux variables. L'outil principal est une version à paramètres du théorème de Puiseux

et les résultats de la section précédente. Nous espérons que cette approche illustre la relation entre les polynômes bornés et la géométrie des fibres de f, et pourra éclairer de façon nouvelle les propriétés des valeurs de bifurcation aussi bien que de propreté des polynômes. Dans $\mathbb{R}^2$ il serait intéressant d'étudier la stabilité des ensembles décrits par plus d'une inégalité polynomiale. Le cas des dimensions supérieures demeurant ouvert. Bien que dans des cas simples (par exemple pour les ensembles décrits par des inégalités monomiales comme dans le Théorème 5.9) il est facile de constater leur stabilité, en général le probème ne semble pas évident du tout.

La Section 4 est consacrée à l'étude des algèbres de polynômes bornés sur des sous-ensembles de $\mathbb{R}^n$ pour n arbitraire. La première partie s'attache à un type particulier de tels ensembles, que nous appelons tentacules pondérées et qui peuvent être vus comme une "déformation uniforme" d'un ensemble de plus petite dimension le long de l'axe y. Supposons que l'ensemble $S \subset \mathbb{R}^n$ ait un intérieur non vide. Considérons l'ensemble

$$M = \{(\beta_1(y)x_1, \ldots, \beta_n(y)x_n, y) \in \mathbb{R}^{n+1} |\ x \in S,\ y \geq R\}$$

où $R > 0$ et $\beta_1(1/Y), \ldots, \beta_n(1/Y)$ sont des séries de Puiseux telles que les $\beta_i(y)$ sont convergents et ont un signe constant, positif ou négatif, pour $y \in [R, \infty)$.

Le Théorème 4.4 affirme que si nous supposons que $\mathcal{A}(S)$ est engendré par des monômes alors

$$\mathcal{A}(M) = \mathcal{A}(S)[Y] \cap \mathbb{R}[X^\alpha Y^d |\ \sum_{i=1}^{n} \alpha_i \lambda_i \geq d],$$

où $\lambda \in \mathbb{Q}^n$ est le tuplet des poids sur l'ensemble M et $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Notons que tout semi-algébrique de la droite réelle a soit une algèbre triviale, soit une algèbre qui est égale à $\mathbb{R}[X]$ (et donc est engendrée par le monôme X). Par conséquent l'énoncé ci-dessus généralise le Théorème 2.1 de la Section 2. De plus, dans certains cas en dimensions supérieures, il donne un procédé pratique de calcul des générateurs de l'algèbre des polynômes bornés (comme dans le cas de certains exemples de la Section 6). La méthode de preuve est essentiellement la même que dans la Section 2.1 i.e. utilise une équivalence de normes dans un espace de dimension finie. Le Théorème 4.4 généralise et étend les résultats de T. Netzer dans [Net] qui utilisent seulement des projections de tels ensembles sur les n premières coordonnées et des arguments radicalements différents. Ainsi dans [Net] la dimension de S et de la tentacule est la même, et de plus S est supposé compact, ce qui est une hypothèse essentielle pour Netzer.

Dans la dernière partie de la Section 4 nous donnons une autre preuve (basée sur l'ensemble de non-propreté de Jelonek) du fait déjà obtenu par D. Plaumann et C. Scheiderer dans [PlSd], que si S est un sous-ensemble non borné d'un semi-algébrique propre, alors l'algèbre $\mathcal{A}(S)$ ne peut pas être finiment engendrée. Pour l'achever, on montre le Lemme 2 qui établit une correspondance entre une propriété algébrique d'un idéal et une propriété géométrique d'une application. Dans la même Section 4 nous produisons un exemple (Example 4.2 d'un ensemble semi-analytique de $\mathbb{R}^2$ d'intérieur non vide qui a une algèbre de polynômes bornés non finiment engendrée) qui est intéressant du point de vue des resultats de Plaumann et Scheiderer que les algèbres des polynômes

bornés sur un semi-algébrique régulier de $\mathbb{R}^2$ sont finissement engendrés.

Du fait que les algèbres engendrées par des monômes apparaissent naturellement dans notre étude, nous étudions leurs propriétés dans la Section 5. En utilisant des méthodes classiques de géométrie convexe nous montrons la dualité entre les cônes convexes des exposants des monômes et les algèbres de polynômes bornés sur les ensembles donnés par l'intersection des ensembles obtenus par des inegalités sur monômes appropriés (Theorem 5.9). Dans la dernière partie de cette section nous montrons aussi que les algèbres des Théorèmes 2.4 et 4.4 sont engendrées par des monômes et nous calculons le nombre minimal de monômes générateurs pour les tentacules dans le plan réel. Dans ce contexte on voit que le calcul du nombre des éléments d'une base de l'intersection ou bien d'une union du nombre fini des algèbres engendrées par monômes n'est qu'une tâche combinatoire.

Supposons à nouveau que $S \subset \mathbb{R}^2$. D'après le Théorème 2.16 sur les "testing" courbes le polynôme f est borné sur S si et seulement s'il est borné sur un nombre fini de représentants génériques d'une famille de courbes qui ne dépend que de l'ensemble S. La preuve basée sur les résultats de la Section 2 de cette observation constitue la première partie de la Section 6.

Dans la Section 6.2 nous présentons une version du Positivstellensatz de Schmüdgen pour les polynômes bornés. Prenons un semi-algébrique basique et non borné

$$S = \{x \in \mathbb{R}^n |\ g_1(x) \geq 0,\ \ldots,\ g_w(x) \geq 0\}$$

où $g_1, \ldots, g_w$ sont des polynômes. Un préordre associé à S est l'ensemble

$$T = \left\{ \sum_{\sigma \in \{0,1\}^w} s_\sigma g^\sigma |\quad s_\sigma \in \sum \mathbb{R}[X]^2 \right\}.$$

Considérons un polynôme f et supposons que les polynômes g_i qui définissent S sont bornés sur S. Supposons de plus que $S_\zeta \cap \overline{\zeta(S)}^{Zar} = \overline{\zeta(S)}$ (la définition de ces ensembles peut être trouvée page 83). D'après le Théorème 6.13, si f est borné et plus grand qu'une constante positive sur S, alors $f \in T$. Dans le cas moins probable où $S_\zeta = \zeta(S)$, il suffit que f soit borné et positif sur S (voir Théorème 6.10). Dans les preuves des deux théorèmes nous utilisons l'observation simple qu'une application avec générateurs (ou la base) d'une algèbre de polynômes bornés est , en tant que coordonnées, l'application bornée canonique. Ainsi dans le Postivstellensatz de Schmüdgen nous pouvons essayer de remplacer la compacité de l'ensemble par l'hypothèse que les polynômes sont bornés. Cependant, remarquons que même dans le cas compact, les théorèmes 6.10 et 6.13 introduisent une propriété que si un polynôme f et des polynômes g_i qui décrivent l'ensemble S sont dans une certaine sous-algèbre de $\mathbb{R}[X]$, alors la représentation de f peut être exprimée par des sommes de carrés dans cette algèbre (voir 6.11).

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References

[BaK] S. Basu, M. Kettner, *Bounding the number of stable homotopy types of a parametrized family of semi-algebraic sets defined by quadratic inequalities*, Proc. Lond. Math. Soc. (3) 98 (2009), no. 2, 298–324;

[BaR] S. Basu, M.-F. Roy, *Bounding the radii of balls meeting every connected component of semi-algebraic sets*, J. Symbolic Comput. 45 (2010), no. 12, 1270–1279;

[BR] R. Benedetti, J.-J. Risler, *Real algebraic and semi-algebraic sets*, Actualités Mathématiques. [Current Mathematical Topics] Hermann, Paris, 1990;

[BCR] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Springer-Verlag, Berlin, 1998;

[DW] M. Dumnicki, T. Winiarski, *Bazy Gröbnera. Efektywne metody w układach równań wielomianowych*, Wydawnictwo Naukowe Akademii Pedagogicznej, Kraków, 2007;

[DSw] N. Dunford, J. T. Schwartz, *Linear Operators, part III Spectral Operators*, Wiley, New York, 1988;

[Ha] H. V. Ha, *Nombres de Łojasiewicz et singularités a l'infini des polynômes de deux variables complexes*, C. R. Acad. Sci. Paris 311 (1990), 429-432;

[HUL] J.-B. Hiriart-Urruty, C. Lemaréchal, *Fundamentals of convex analysis*, Springer, Berlin, 2001;

[J] Z. Jelonek, *Geometry of real polynomial mappings*, Math. Z. 239 (2002), no. 2, 321–333;

[J2] Z. Jelonek, *The set of points at which a polynomial map is not proper*, Ann. Polon. Math. 58 (1993), no. 3, 259–266;

[JK] Z. Jelonek, K. Kurdyka, *On asymptotic critical values of a complex polynomial*, J. reine angew. Math. 565 (2003), 1-11;

[Kin] A.I. Kostrykin, *Wstęp do algebry, vol. 1-3*, PWN, Warszawa, 2008;

[KM] S. Kuhlmann, M. Marshall, *Positivity, sums of squares and the multi-dimentional moment problem*, Transactions of the AMS, vol. 354, no. 11, 4285-4301 (2002);

[KOS] K. Kurdyka, P. Orro, S. Simone, *Semialgebraic Sard theorem for generalized critical values*, J. Differential Geometry 56 (2000) 67-92;

[Kr] T. Krasiński, *Poziomice wielomianów dwóch zmiennych a hipoteza jakobianowa*, Acta Univ. Lodz., (1991) 1-17;

[Kr2] T. Krasiński, *On branches at infinity of a pencil of polynomials in two complex variables*, Annales Polonici Matematici 55 (1991) 213-220;

[KS] T. Krasiński, S. Spodzieja, *Bifurcation points of real polynomials*, Bull. Polish Acad. Sci. Math. 44 (1996), no. 3, 333–339;

[Kri] J.-L. Krivine, *Anneaux préordonnés*, J. Analyse Math. 12, (1964) 307–326;

[Krug] S. Krug, *Geometric interpretations of a counterexample to Hilbert's 14th problem, and rings of bounded polynomials on semialgebraic sets*, preprint arXiv:1105.2029v2;

[Ku] S. Kuroda, *A counterexample to the fourteenth problem of Hilbert in dimension three*, Michigan Math. J. 53 (2005), no. 1, 123–132;

[Lang] S. Lang, *Algebra*, PWN, Warszawa, 1973;

[L] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. 11(2000:01), no. 3 796 - 817 (electronic);

[Lt] M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, Emerging Applications of Algebraic Geometry, IMA Vol. Math. Appl. vol. 149, Springer, New -York, 2009, 157 -270;

[M] M. Marshall, *Positive polynomials and sums of squares*, Mathematical Surveys and Monographs, 146. American Mathematical Society, Providence, RI, 2008;

[Mo] T. S. Motzkin, *The arithmetic-geometric inequality*, In: Inequalities (Ed. O. Shisha) Academic Press (1967), 205-224;

[NZ] A. Némethi, A. Zaharia, *On the Bifurcation Set of a Polynomial Function and Newton Boundary*, Publ. RIMS. Kyoto Univ. 26 (1990), 681-689;

[Net] T. Netzer, *Stability of quadratic modules*, Manuscripta Math. 129 (2009), no. 2, 251–271;

[PaS] P. Parrilo, B. Sturmfels, *Minimizing polynomial functions*, Algorithmic and Quantitative Real Algebraic Geometry (S. Basu, L. Gonzales-Vega eds.), DIMACS series vol. 60 AMS, 83 - 99;

[Par] A. Parusiński, *On the bifurcation set of complex polynomial with isolated singularities at infinity*, Compositio Math. 97 (1995), 369-384;

[Pl] D. Plaumann, *Bounded polynomials, sums of squares, and the moment problem*, Ph.D. Thesis, Universität Konstanz, 2008;

[PlSd] D. Plaumann, C. Scheiderer, *The ring of bounded polynomials on a semi-algebraic set*, preprint arXiv:1002.1870v2 (to be published in Trans. Amer. Math. Soc. (2010));

[PD] A. Prestel, C.N. Delzell, *Positive Polynomials*, Springer, Berlin, 2001;

[PS] V. Powers, C. Scheiderer, *The moment problem for non-compact semialgebraic sets*, Adv. Geom. 1 (2001), no. 1, 71–88;

[PR] V. Powers, B. Reznick, *A new bound for Pólya's theorem with applications to polynomials positive on polyhedra*, Effective methods in algebraic geometry (Bath, 2000). J. Pure Appl. Algebra 164 (2001), no. 1-2, 221–229

[Rab] P. T. Rabier, *Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds*, Annals of Mathematics, 146 (1997), 647-691;

[RS] T. Rodak, S. Spodzieja, *On some characterization of proper polynomial mappings*, Bull. Polish Acad. Sci. Math. 48 (2000), no.2, 157-164;

[Sm] K. Schmüdgen, *The K-moment problem for semi-algebraic sets*, Mathematische Annalen 289 (1991) 203-206;

[Sm2] K. Schmüdgen, *On the moment problem of closed semi-algebraic sets*, J. Reine Angew. Math. 558 (2003), 225–234;

[Sw] M. Schweighofer, *Global optimization of polynomials using gradient tentacles and sums of squares*, SIAM J. Optim. 17 (2006), no. 3, 920–942;

[Sw2] M. Schweighofer, *On the complexity of Schmüdgen's positivstellensatz*, J. Complexity 20 (2004), no. 4, 529–543;

[S] S. Spodzieja, *The Łojasiewicz exponent at infinity for overdetermined polynomial mappings*, Ann. Polon. Math. 78 (2002), no. 1, 1–10;

[St] G. Stengle, *A nullstellensatz and a positivstellensatz in semialgebraic geometry*, Math. Ann. 207 (1974), 87–97;

[W] R.J. Walker, *Algebraic curves*, Springer, New York-Heidelberg, 1978;

Résumé de la thèse

Dans cette thèse nous étudions les algèbres des polynômes qui sont bornés sur un ensemble semi-algébrique non borné. Tout d'abord nous abordons le problème consistant á déterminer si un polynôme est borné sur un ensemble. Nous résolvons ce problème pour les polynômes á deux variables définis sur des ensembles semi-algébriques quelconques. Dans la section suivante nous donnons une méthode pour déterminer des générateurs de l'algèbre des polynômes bornés et ce pour une large classe de semi-algébriques du plan réel. Dans la section 3 nous établissons une relation entre les valeurs de bifurcation du complexifié d'un polynôme f á deux variables et la stabilité de la famille d'algèbres des polynômes bornés sur les ensembles $f \leq c$. Dans la section 4 nous décrivons la structure de l'algèbre des polynômes bornés sur un certain type de sous-ensembles de $\mathbb{R}^n$ avec n arbitraire, que nous appelons tentacules pondérées. Nous donnons aussi une preuve géométrique du fait que l'algèbre d'un sous-ensemble non borné d'un ensemble algébrique propre n'est pas de type fini. Dans la section suivante nous établissons une correspondance entre les cônes convexes et les algèbres des ensembles obtenus par des inégalités sur des monômes appropriés. Enfin, nous démontrons une version du Positivstellensatz de Schmudgen pour les polynômes bornés sur un ensemble non compact.

The main topic of the thesis is a study of algebras of polynomials which are bounded on a given unbounded semialgebraic set. First we tackle the problem of deciding the boundedness of a polynomial on a set. We achieve it for polynomials in two variables for any semialgebraic set. We give also a method of finding generators of the algebra of bounded polynomials for a large class of semialgebraic subsets of the real plane. In Section 3 we have established a relation between bifurcation values of a complexification of polynomial f in two variables and the family of algebras of bounded polynomials on the sets $f \leq c$. In section 4 we describe the algebras of bounded polynomials for subsets of $\mathbb{R}^n$, where n is arbitrary, which we call weighted tentacles. We also provide a geometric proof of the fact that for a unbounded subset of a proper algebraic set its algebra cannot be finitely generated. In the next section we establish a correspondence between convex cones and algebras of bounded polynomials on the sets described by monomial inequalities. At the end of this thesis we prove a version of Schmudgen's Positivstellensatz for bounded polynomials.