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Karine Kuyumzhiyan

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Karine Kuyumzhiyan. Actions des groupes algébriques sur les variétés affines et normalité d'adhérences d'orbites. Mathématiques générales [math.GM]. Université de Grenoble, 2011. Français. NNT : 2011GRENM015 . tel-00685202

HAL Id: tel-00685202

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THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE GRENOBLE

Spécialité : **Mathématiques**

Arrêté ministériel : 7 août 2006

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préparée au sein de **l'Institut Fourier**
et de **l'Ecole Doctorale MSTII**

Actions des groupes algébriques sur les variétés affines et normalité d'adhérences d'orbites

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Remerciements

Je remercie tout d'abord mes directeurs de thèse, Ivan V. Arzhantsev et Mikhail G. Zaidenberg, pour leur soutien et confiance pendant les années de thèse. Ils m'ont introduit ce domaine d'algèbre très riche, ils m'ont guidée, ils m'ont posé des problèmes et ont relu attentivement et soigneusement mes nombreux manuscrits. Ivan V. Arzhantsev a également été mon directeur de thèse en Master et je lui en exprime ma gratitude.

Malgré la date tardive à laquelle je leur ai remis le premier exemplaire, Frank Kutzschebauch et Nicolas Ressayre ont accepté d'être les rapporteurs de cette thèse. Je leur suis très reconnaissante de l'intérêt qu'ils ont porté à mon travail.

Je veux remercier Laurent Manivel pour sa participation au jury de soutenance, mais aussi pour toutes les discussions utiles concernant les systèmes de racines exceptionnels et pour son aide sur la partie administrative de la soutenance. Merci à Adrien Dubouloz pour l'organisation des groupes de travail "Automorphismes des Espaces Affines" ; ces rencontres régulières ont beaucoup aidé à échanger des idées dans un groupe de chercheurs s'intéressant aux mêmes sujets. Un grand merci également à Dmitry N. Akhiezer de m'avoir fait l'honneur d'assister à ma soutenance en tant qu'examinateur.

Je tiens à remercier mes collègues avec lesquels j'ai travaillé pendant ces années, en particulier Frédéric Mangolte. La rédaction de notre travail en commun m'a beaucoup appris et m'a procuré un grand plaisir.

De même je souhaite exprimer ma gratitude envers les membres permanents de l'Institut Fourier ; en particulier Michel Brion et Emmanuel Peyre. Merci à Françoise Truc, ma co-bureau pendant ma 3^{ème} année de thèse, elle m'a toujours répondu à mes questions linguistiques.

Je tiens aussi à saluer les thésards : Aline, Jorge, Ximena, Hernan, Chloe, Delphine, Mateusz, Ronan, Roland, Junyan, Kevin, Clélia, Gunnar, Sasha, Marco (et merci pour le vrai café italien!), déjà docteurs : Alvaro, Rodrigo et Alexander, et tous les autres bien sûr !

Je remercie tous le personnel administratif et technique, grâce à qui j'ai pu préparer ma thèse dans d'excellentes conditions.

Je dois beaucoup à l'Université d'État de Moscou et à l'Université Indépendante

de Moscou pour m'avoir permis d'apprendre les mathématiques et m'avoir plongée dans le monde de mathématiques contemporaines.

À Grenoble, j'ai bien profité de sports proposés par l'université, mais aussi des montagnes. Cela m'a permis régulièrement de vider ma tête pour avoir de nouvelles idées. Je remercie Christian Ramon, prof de ping-pong, et toutes les équipes de l'EGUG et de l'ESMUG.

Je n'oublie pas mes amis des autres laboratoires : Olga et Vassil, Sveta, Marina, Lena, Sergey et Tanya, Olivier, Olya et les autres, avec qui j'ai passé de bons moments.

Un grand merci à Ilya Bogdanov, à ma mère Elena et mon père Georgiy pour leur chaleur et pour leur soutien pendant ces années.

Благодарности

Хочу выразить глубокую благодарность моим научным руководителям Ивану Владимировичу Аржанцеву и Михаилу Григорьевичу Зайденбергу за ту неоценимую поддержку, которую они оказывали мне в течение всех лет обучения в университете и в аспирантуре, за помощь в выборе своего направления в математике, за постановки задач и помощь в их решении.

Огромное спасибо оппонентам Франку Кутчебауху и Николя Рессейру, взявшим на себя нелёгкий труд по прочтению данной рукописи. Благодарю также Лорана Манивеля, Дмитрия Наумовича Ахиезера и Адриена Дюбуло за присутствие здесь в качестве членов жюри.

Я очень благодарна Институту Фурье за тот уют и условия для работы, которые были созданы во время моего пребывания здесь. Спасибо всем сотрудникам за улыбки и помощь в новой для меня обстановке, мне было очень приятно ходить на лекции и участвовать в семинарах, а также просто разговаривать о математике!

Выражаю также свою признательность моим родителям Елене Александровне и Георгию Валерьевичу за тепло и моральную поддержку. Огромное спасибо Илье Богданову за моральную поддержку и ценные идеи.

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Introduction (version française)

Cette thèse est consacrée aux actions des groupes de transformations algébriques sur les variétés affines algébriques. Dans la première partie, on étudie la normalité des adhérences des orbites de tore maximal dans un module rationnel de groupe algébrique simple. La seconde partie porte sur les actions du groupe d'automorphismes d'une variété affine. Nous nous intéressons aux propriétés de transitivité et de transitivité infinie de ces actions sur le lieu lisse de la variété.

Partie I

Normalité pour des adhérences d'orbites

Nous commençons par une brève synthèse. Soit \mathbb{k} un corps algébriquement clos de caractéristique nulle et G un groupe affine algébrique de corps de base \mathbb{k} . Par exemple, on peut considérer le groupe spécial linéaire $SL(n)$. Rappelons qu'une variété algébrique affine irréductible X est dite *normale* si son algèbre des fonctions régulières $\mathbb{k}[X]$ est intégralement close dans son corps de fractions. L'étude de la normalité des adhérences d'orbites a une longue histoire. Un des premiers résultats importants est celui de B. Kostant [37] qui a démontré dans les années 1960 que le complété du cône nilpotent dans le module adjoint est normal. H. Kraft et C. Procesi [39] ont démontré que dans le module adjoint $\mathfrak{sl}(\mathfrak{n})$, les adhérences de toutes les $SL(n)$ -orbites sont normales. Autrement dit, l'adhérence, dans la topologie de Zariski, de toute classe de conjugaison de matrices est normale. Pour un corps \mathbb{k} de caractéristique positive, le même résultat pour $SL(n)$ a été établi par S. Donkin [19]. Plus tard, H. Kraft et C. Procesi [40] ont étudié la même question pour les modules adjoints des autres groupes classiques. Ils ont indiqué certaines orbites dont l'adhérence n'est pas normale en terme de diagrammes de Young. E. Sommers [64] a résolu le cas restant pour D_n , plus précisément, le cas de classe de conjugaison très paire. La normalité pour l'adhérence d'une orbite dans le module adjoint est équivalente à la normalité pour l'adhérence d'une certaine orbite nilpotente dans le même module. Dans tous les articles précédemment cités, la plupart des raisonnements sont effectués sur des orbites nilpotentes. Les cas

des groupes F_4 , G_2 , E_6 ont été résolus par A. Broer, H. Kraft et E. Sommers, respectivement dans [12], [38] et [63]. Il n'y a pas encore de réponse complète pour E_7 et E_8 .

Passons maintenant aux actions de tore algébrique T , c'est-à-dire de groupe algébrique affine isomorphe à $\mathbb{k}^\times \times \dots \times \mathbb{k}^\times$, où $\mathbb{k}^\times = \mathbb{k} \setminus \{0\}$. Une variété X algébrique irréductible est dite *torique* si X est normale et si T agit sur X avec une orbite ouverte. Les variétés toriques jouent un rôle important dans la géométrie algébrique, la topologie et la combinatoire, grâce à leur description remarquable en termes de géométrie convexe (voir [24]). Si un tore algébrique T agit sur une variété Y , l'adhérence de l'orbite $X = \overline{Ty}$ d'un point $y \in Y$ est un candidat naturel pour une variété torique. Pour démontrer que X est torique, il faut vérifier que X est normale. Donc la propriété de normalité est très intéressante.

Il y a un lien entre normalité des adhérences d'orbites de tore et normalité des adhérences de G -orbites. Effectivement, l'étude des orbites de G -action dans le produit Cartésien $\mathfrak{g} \oplus \mathfrak{g}$ du module adjoint peut être réduite à l'étude des T -orbites dans le module adjoint \mathfrak{g} lui-même, à l'aide du Théorème des slices étales de Luna.

Dans ce paragraphe, on se restreint au cas de T -actions et on introduit une terminologie et une description combinatoire. Soit V un T -module rationnel. On note $\Lambda = \Lambda(T)$ le *réseau des caractères* de T . Par rapport à l'action de T , le module V peut être diagonalisé :

$$V = \bigoplus_{\mu \in \Lambda} V_\mu, \quad \text{où } V_\mu = \{v \in V \mid tv = \mu(t)v \quad \forall t \in T\}.$$

On note $M(V) = \{\mu \in \Lambda \mid V_\mu \neq 0\}$ l'ensemble des poids de V , qui est un ensemble fini. Chaque vecteur non nul v dans V se décompose de la manière suivante : $v = v_{\mu_1} + \dots + v_{\mu_s}$, $v_{\mu_i} \in V_{\mu_i}$, $v_{\mu_i} \neq 0$. Soit $M(v)$ l'ensemble $\{\mu_1, \dots, \mu_s\}$. Cet ensemble engendre un semi-groupe $\mathbb{Z}_{\geq 0}(\mu_1, \dots, \mu_s)$. On peut aussi engendrer un sous-réseau $\mathbb{Z}(\mu_1, \dots, \mu_s)$ et un cône rationnel polyédral $\mathbb{Q}_{\geq 0}(\mu_1, \dots, \mu_s)$ dans l'espace $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

Définition. L'ensemble des points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ est dit *saturé* si

$$\mathbb{Z}_{\geq 0}(\mu_1, \dots, \mu_s) = \mathbb{Z}(\mu_1, \dots, \mu_s) \cap \mathbb{Q}_{\geq 0}(\mu_1, \dots, \mu_s).$$

L'ensemble des points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ est dit *héréditairement normal* si tout sous-ensemble en est saturé.

Dans [32, I, §1, Lemma 1], il est démontré que l'adhérence \overline{Tv} de la T -orbite d'un vecteur v est normale si et seulement si l'ensemble des caractères $\{\mu_1, \dots, \mu_s\}$ est saturé. Ce critère combinatoire joue un rôle clé dans la première partie de ma thèse.

Dans la littérature, il y a aussi des résultats concernant la propriété de normalité des adhérences des orbites pour les T -actions projectives. Soit $X(v)$ l'adhérence d'une T -orbite $T[v]$ d'un point $[v] \in \mathbb{P}(V)$ dans la projectivisation d'un T -module rationnel V . Considérons l'enveloppe convexe $P(v)$ de $M(v)$ dans $\Lambda_{\mathbb{Q}}$; alors $X(v)$ est normale si et seulement si l'ensemble $\{\mu - \mu_0 \mid \mu \in M(v)\}$ est saturé pour tout sommet μ_0 du polytope $P(v)$. Ce critère a été énoncé et développé par J.B. Carrell et A. Kurth [13].

Considérons un cadre plus général. Soit G un groupe algébrique affine semi-simple simplement connexe, $T \subset G$ un tore maximal et $B \subset G$ un sous-groupe de Borel. A.A. Klyachko [34] a démontré que l'adhérence d'une T -orbite générique sur la variété des drapeaux G/B est normale. Plus tard, R. Dabrowski [16] a prouvé que l'adhérence d'une T -orbite générique dans G/P , où $P \subset G$ est un sous-groupe parabolique, est aussi normale. On peut trouver des exemples d'adhérences non normales d'orbites toriques non génériques dans [13].

Il est bien connu que l'adhérence de toute T -orbite sur une variété torique est normale. En utilisant la méthode des U -invariants, on peut prouver la normalité de l'adhérence de toute G -orbite sur une variété sphérique pour tout groupe G connexe réductif. Dans le cas de complexité 1, I.V. Arzhantsev [1] a démontré que pour une action d'un groupe G connexe réductif sur une variété X normale avec une famille à un paramètre des G -orbites sphériques générales et possédant un bon quotient $\pi: X \rightarrow X//G$, où $X//G$ est une courbe, l'adhérence de toute G -orbite est normale.

La propriété de saturation est importante dans plusieurs problèmes algébriques et géométriques. N. White [68] a démontré que l'ensemble des vecteurs d'incidence de la base d'un matroïde réalisable est saturé. Cette propriété a pour conséquence le fait suivant : pour tout point \underline{y} dans le cône affine au-dessus de la Grassmannienne classique $\text{Gr}(k, n)$, l'adhérence $\overline{T\underline{y}}$ est normale.

Soit Γ un graphe fini avec n sommets, on peut lui associer la collection finie $M(\Gamma)$ de vecteurs dans le réseau \mathbb{Z}^n suivante :

$$M(\Gamma) = \{\varepsilon_i + \varepsilon_j : (ij) \text{ est une arête de } \Gamma\},$$

où $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ est la base standard de \mathbb{Z}^n . La propriété de saturation pour cet ensemble est équivalente au fait suivant : soient C et C' deux cycles minimaux, impairs dans Γ , alors soit C et C' ont un sommet commun, soit il existe une arête de Γ qui relie un sommet de C à un sommet de C' . Ceci a été montré par H. Ohsugi et T. Hibi [52], A. Simis, W. Vasconcelos, et R. Villarreal [62]. Soit $\mathcal{A}(\Gamma) = \mathbb{k}[x_i x_j : (ij) \text{ est une arête de } \Gamma]$. Algébriquement, la propriété de saturation de $M(\Gamma)$ est équivalente au fait que la sous-algèbre $\mathcal{A}(\Gamma)$ de l'algèbre polynomiale $\mathbb{k}[x_1, x_2, \dots, x_n]$ soit intégralement close dans son corps de fractions $Q\mathcal{A}(\Gamma)$.

C. Chindris [14] a obtenu certains résultats généraux concernant les carquois et la propriété de saturation. Il a démontré qu'un carquois Q fini et connexe sans

cycles orientés est un carquois de Dynkin ou euclidien si et seulement si tous les semi-groupes orbitaux des représentations de Q sont saturés.

La propriété de saturation dans une situation légèrement différente se révèle être importante aussi dans d'autres problèmes de la théorie des représentations. Dans les articles de N. Ressayre [60] et P.-L. Montagard, B. Pasquier, N. Ressayre [50], les auteurs parlent du calcul de certains semi-groupes dans le réseau des poids. S'il est connu *a priori* qu'un semi-groupe M est finiment engendré, alors on peut obtenir M en deux étapes. Premièrement, on trouve des inégalités qui définissent le cône $\text{cone}(M)$, engendré par le semi-groupe M . Deuxièmement, on est obligé de vérifier quels points entiers de $\text{cone}(M)$ appartiennent effectivement au semi-groupe M . Dans la plupart des cas intéressants, M coïncide avec l'ensemble des points entiers dans $\text{cone}(M)$ (c'est le *problème de saturation*).

Pour l'action adjointe $SL(n) : \mathfrak{sl}(\mathfrak{n})$, le résultat de B. Sturmfels [65, Ex. 3.7], [66] dit que l'adhérence de toute T -orbite est normale. G. Bobiński et G. Zwara [7] ont interprété ce résultat combinatoire en termes de représentations des carquois. J. Morand [51] a classifié tous les groupes affines algébriques semi-simples tels que l'adhérences de toute T -orbite dans le module adjoint est normale.

Nos méthodes ont beaucoup d'éléments en commun avec la technique utilisée dans ces articles.

Soient G un groupe algébrique semi-simple connexe simplement connexe et T un tore maximal dans G . Formulons la tâche générale qui est résolue dans la Partie I.

Trouver tous les G -modules V simples rationnels tels que, pour tout vecteur $v \in V$, l'adhérence de sa T -orbite soit une variété affine normale.

Rappelons la description d'ensemble de T -poids d'un G -module rationnel $V(\lambda)$ de plus haut poids λ . Soient Φ le système des racines qui correspond à G , Ξ le réseau des racines et W le groupe de Weyl de Φ . Le *polytope des poids* $P(\lambda)$ du module $V(\lambda)$ est l'enveloppe convexe $\text{conv}\{w\lambda \mid w \in W\}$ de la W -orbite du point λ dans $\Lambda_{\mathbb{Q}}$. Alors

$$M(\lambda) = (\lambda + \Xi) \cap P(\lambda)$$

(voir [25, Theorem 14.18] et [11, Exercices to Ch. VIII §7]). Donc, toutes les adhérences de T -orbites dans $V = V(\lambda)$ sont normales si et seulement si $M(\lambda)$ est héréditairement normal.

Résultats de la Partie I

En rassemblant les résultats de [8, 42, 43], on obtient le théorème suivant.

Théorème A. *Pour les types suivants de groupes algébriques simples et les modules correspondants, ainsi que leurs duaux, l'adhérence de toute orbite de tore*

maximal est normale. Dans tous les autres cas, le module correspondant contient une orbite de tore maximal dont l'adhérence est non normale.

<i>Système de racines</i>	<i>Plus haut poids</i>	<i>Traité dans</i>
$A_n, n \geq 1$	π_1	Cas 2.1
$A_n, n \geq 1$	$\pi_1 + \pi_n$	[51, 65, 66], Cas 2.2
A_1	$3\pi_1$	Cas 2.3
A_1	$4\pi_1$	Cas 2.4
A_2	$2\pi_1$	Cas 2.5
A_3	π_2	Cas 2.7
A_4	π_2	Cas 2.7
A_5	π_2	Cas 2.7
A_5	π_3	Cas 2.6
$B_n, n \geq 2$	π_1	Cas 3.1
B_2	π_2	Cas 3.3
B_2	$2\pi_2$	Cas 3.2
B_3	π_3	Cas 3.3
B_4	π_4	Cas 3.3
$C_n, n \geq 3$	π_1	Cas 3.4
C_3	π_2	Cas 3.5
C_4	π_2	Cas 3.5
$D_n, n \geq 4$	π_1	Cas 3.6
D_4	π_2	Cas 3.7
D_4	π_3	Cas 3.8
D_4	π_4	Cas 3.8
D_5	π_4	Cas 3.9
D_6	π_5	Cas 3.10
D_6	π_6	Cas 3.10
F_4	π_1	Cas 4.1
G_2	π_1	Cas 4.2

La première partie de cette thèse est constituée de quatre chapitres. Présentons ici le contenu de chacun de ces chapitres.

Dans le **Chapitre 1**, on reformule les problèmes évoqués ci-dessus dans un langage combinatoire. On introduit les méthodes utilisées dans toute la Partie I. Si $M(\lambda)$ est inclus dans $M(\mu)$ et si un sous-ensemble non saturé est trouvé pour λ , alors il peut être utilisé comme sous-ensemble non saturé pour μ . C'est pour cela que dans beaucoup de cas, il suffit de construire un sous-ensemble non saturé pour

répondre par la négative à la question de normalité simultanée des adhérences de toutes les T -orbites. Ainsi, les sous-ensembles non saturés vont apparaître très souvent. On note NSS un sous-ensemble non saturé. Le *sous-ensemble non saturé élargi* est un sous-ensemble non saturé $\{v_1, \dots, v_r\}$ et qui admet un vecteur v_0 tel que

$$i. \ v_0 \in (\mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geq 0}(v_1, v_2, \dots, v_r)) \setminus \mathbb{Z}_{\geq 0}(v_1, v_2, \dots, v_r),$$

ii. il existe une $\mathbb{Q}_{\geq 0}$ -représentation

$$v_0 = q_1 v_{i_1} + \dots + q_s v_{i_s}, \quad v_{i_j} \in \{v_1, v_2, \dots, v_r\}$$

telle que les vecteurs v_{i_1}, \dots, v_{i_s} sont linéairement indépendants et les coefficients q_i appartiennent à $[0, 1[$.

De tels sous-ensembles sont appelés $ENSS$; on les note $\{v_0; v_1, \dots, v_r\}$. Il est aisé de démontrer que si l'ensemble $M = \{v_1, \dots, v_r\}$ n'est pas saturé, alors il existe un vecteur v_0 tel que $\{v_0; v_1, \dots, v_r\}$ est un $ENSS$.

À partir d'un ensemble fini de points, on peut facilement construire un $ENSS$. Par contre, la vérification effective du fait que ce que l'on vient de construire est bien un $ENSS$, est moins évidente. L'une de nos méthodes est la suivante. Soient v_0, v_1, \dots, v_r des vecteurs dans un espace vectoriel rationnel \mathbb{Q}^n et soit f une fonction linéaire sur \mathbb{Q}^n . Nous disons que f est une *fonction linéaire séparante* pour la collection $\{v_0; v_1, \dots, v_r\}$ lorsque la valeur $f(v_0)$ ne peut pas être représentée comme une combinaison linéaire des valeurs $f(v_1), \dots, f(v_r)$ avec des coefficients entiers positifs ou nuls. Supposons que $v_0 \in \mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geq 0}(v_1, v_2, \dots, v_r)$ et que v_0 peut être représenté comme une $\mathbb{Q}_{\geq 0}$ -combinaison de vecteurs linéairement indépendants v_1, \dots, v_r avec des coefficients dans l'intervalle $[0, 1[$. Alors, l'existence d'une fonction séparante garantit que $\{v_0; v_1, \dots, v_r\}$ est un $ENSS$.

À présent, décrivons les méthodes qui servent pour établir qu'un ensemble M de points est héréditairement normal. Choisissons une base telle que pour tout $v \in M$, les coordonnées de v dans cette base soient entières. Représentons chaque point comme un vecteur-colonne de ces coordonnées dans cette base. Soit $K = K(M)$ une matrice entière $n \times r$ formée par tous ces vecteurs-colonnes. L'algorithme suivant concerne les algèbres monomiales.

Définition. À chaque colonne $K_i = (k_{1i}, \dots, k_{ni})^T$, on associe un monôme de Laurent $t^{K_i} = t_1^{k_{1i}} \dots t_n^{k_{ni}}$. L'idéal torique I_K associé à K est le noyau de l'homomorphisme de \mathbb{k} -algèbres

$$\mathbb{k}[x_1, x_2, \dots, x_r] \rightarrow \mathbb{k}[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}], \quad x_i \mapsto t^{K_i}.$$

Définition. Supposons que u_+ et u_- sont deux vecteurs dans \mathbb{Z}_+^n à supports disjoints. Soit $f = x^{u_+} - x^{u_-} \in I_K$. On dit que f est un *circuit* dans I_K si les deux conditions suivantes sont satisfaites :

- toutes les coordonnées de u_+ et u_- n'ont pas de diviseur commun supérieur à 1 ;
- l'ensemble des variables qui apparaissent effectivement dans f est minimal par rapport à l'inclusion entre tous les binômes de I_K .

Les deux critères suivants sont énoncés par B. Sturmfels.

Théorème B ([65, Theorem 3.8]). *Un ensemble M de points est héréditairement normal si et seulement si tout circuit dans $I_{K(M)}$ possède au moins un monôme sans facteur carré.*

Supposons que l'ensemble de vecteurs $M \subset \mathbb{Q}^n$ est de rang d , $d \leq n$. Soit $L = \langle v \mid v \in M \rangle$ l'espace vectoriel engendré par les vecteurs de M . L'ensemble M est dit *unimodulaire* si pour tous vecteurs linéairement indépendants $v_1, \dots, v_d \in M$, le volume d -dimensionnel $\text{vol}_d(v_1, v_2, \dots, v_d)$ est constant en valeur absolue. Dans une base fixée de L , cette condition signifie que les valeurs absolues de tous les déterminants non nuls $|\det(v_1, v_2, \dots, v_d)|$, $v_1, v_2, \dots, v_d \in M$, calculés dans cette base, sont égales.

Si l'ensemble M est unimodulaire, son intersection avec tout sous-espace $L_1 \subset L$ est aussi unimodulaire. En effet, on peut toujours choisir une base dans L compatible avec L_1 .

Théorème C ([65, Theorem 3.5]). *Tout ensemble unimodulaire est héréditairement normal.*

Plus généralement, on dit qu'un ensemble $M \subset \mathbb{Q}^n$ de rang d est *presque unimodulaire* si on peut choisir un sous-ensemble $\{v_1, v_2, \dots, v_d\} \subseteq M$ tel que

$$\text{vol}_d(v_1, v_2, \dots, v_d) = m$$

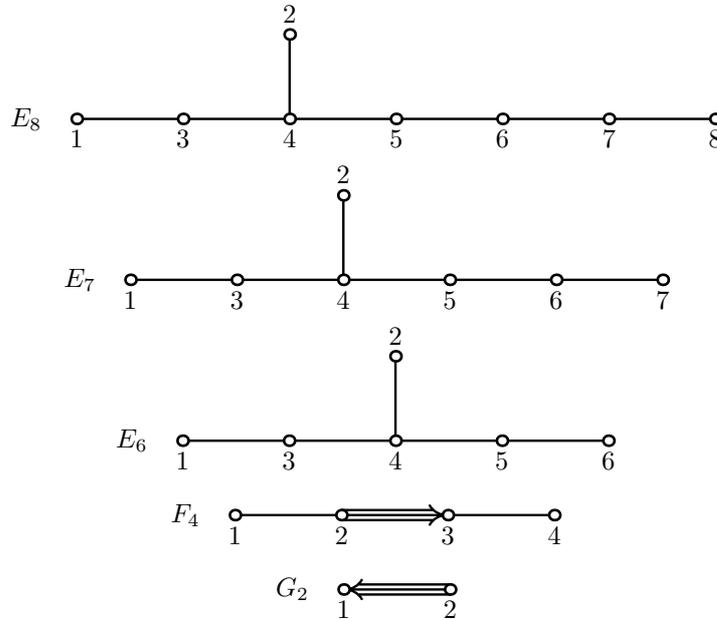
et que pour tout autre vecteur $w \in M$ et pour tout i la valeur

$$\text{vol}_d(v_1, v_2, \dots, \widehat{v}_i, \dots, v_d, w)$$

est divisible par m , c'est-à-dire est égale à km avec $k \in \mathbb{Z}$. Si on fixe une base dans l'espace vectoriel $\langle M \rangle$, cette propriété peut être vérifiée en comparant toutes les valeurs des déterminants correspondants dans cette base donnée au lieu de comparer les valeurs vol_d . La valeur $m = \det(v_1, v_2, \dots, v_d)$ est dite le *volume* d'un ensemble presque unimodulaire.

Des ensembles presque unimodulaires sont utiles dans les démonstrations où il faut montrer qu'un certain ensemble de vecteurs est héréditairement normal.

Dans le **Chapitre 4**, nous considérons les groupes exceptionnels. On utilise la même technique, y compris les ensembles unimodulaires et presque unimodulaires. On peut voir ici que l'adhérence de chaque T -orbite est normale dans le cas des premières représentations fondamentales de F_4 et de G_2 . En revanche, dans tous les autres cas, il existe au moins une T -orbite dont l'adhérence n'est pas normale.



Partie II

La seconde partie de la thèse est consacrée à la transitivité infinie du groupe d'automorphismes des variétés algébriques sur un corps de caractéristique zéro. Cette partie est constituée de deux chapitres. Dans le **Chapitre 5**, le corps de base est algébriquement clos, tandis que dans le **Chapitre 6**, le corps de base est \mathbb{R} . Ces deux cas sont complètement différents, c'est pourquoi nous présentons séparément un résumé historique bref pour chacun de ces deux cas.

Groupes d'automorphismes de variétés complexes

Définition. L'action d'un groupe G sur un ensemble A est dite m -transitive si pour deux m -uplets quelconques de points distincts (a_1, a_2, \dots, a_m) et (b_1, b_2, \dots, b_m) de A , il existe un élément $g \in G$ tel que $g(a_i) = b_i$, $i = 1, 2, \dots, m$. Par extension, les actions qui sont m -transitives pour tous $m \in \mathbb{N}$ seront dites *infiniment transitives*.

Les variétés munies d'actions de groupes multiples transitives sont assez rares. En effet, soit X une variété analytique lisse et soit G un groupe de Lie agissant sur X . Evidemment, pour $m > \dim G$, le groupe G ne peut pas agir de façon m -transitive sur X . Conformément à A. Borel, un groupe de Lie réel ne peut pas agir de façon 4-transitive sur une variété simplement connexe (voir les Théorèmes 5 et 6 dans [9]). La transitivité multiple des actions de groupes de Lie a été étudiée par J. Tits [67]. Il a classifié toutes les actions de groupes de Lie doublement et triplement transitives. Plus tard, ce résultat a été redémontré via d'autres méthodes par L. Kramer [41].

Dans le cas algébrique, F. Knop [35] a trouvé toutes les actions 2-transitives d'un groupe G algébrique connexe sur une variété algébrique X , où le corps de base est un corps algébriquement clos quelconque. Dans ce cas-là, soit $X = \mathbb{P}^m$ et $G = PGL(m+1)$, soit $X = \mathbb{A}^m$ et G est un produit semi-direct $L \ltimes \text{Transl}$, où Transl est le groupe de translations de \mathbb{A}^m et L est un groupe qui agit linéairement sur \mathbb{A}^m et transitivement sur $\mathbb{A}^m \setminus \{0\}$. Tous ces groupes linéaires sont énumérés dans [35]. En particulier, ce résultat implique que la seule action 3-transitive d'un groupe algébrique est l'action naturelle de $PGL(2)$ sur \mathbb{P}^1 , tandis que les actions 4-transitives n'existent pas. V. Popov [57] a estimé le degré de transitivité "général" pour tous les groupes algébriques simples et leurs actions sur variétés algébriques. Ici transitivité générale signifie que l'action induite sur la puissance cartésienne correspondante X^m a une orbite ouverte.

Finalement, on peut conclure qu'il n'y a pas d'espace avec une action infiniment transitive lorsqu'on se restreint aux actions de groupes de Lie ou bien aux groupes algébriques.

Dans toute la suite, on considère l'action d'un groupe complet d'automorphismes d'une variété. La transitivité infinie pour le groupe de tous les automorphismes algébriques est bien connue pour l'espace affine \mathbb{A}^n sur un corps quelconque algébriquement clos \mathbb{k} , où $n \geq 2$. La démonstration est donnée, par exemple, dans [31, Lemma 5.5].

J.-P. Rosay et W. Rudin [61] ont étudié l'action du groupe de tous les automorphismes holomorphes de \mathbb{C}^n , $n \geq 2$, sur les sous-ensembles dénombrables. Pour deux suites quelconques discrètes et infinies de points, il a été démontré que la première peut être envoyée sur la deuxième par une application holomorphe. Cependant, cette application n'est pas toujours un automorphisme. De plus, la classe de suites pouvant être envoyées sur une suite standard, par exemple l'ensemble des nombres entiers, par un automorphisme holomorphe, est assez pauvre. La situation est complètement différente pour les ensembles denses. J.-P. Rosay et W. Rudin ont démontré que, pour deux sous-ensembles quelconques A et B dénombrables denses dans \mathbb{C}^n , $n \geq 2$, il y a un automorphisme holomorphe g tel que $g(A) = B$.

Notons $\text{Aut}(X)$ le groupe de tous les automorphismes algébriques d'une variété affine X . Ce groupe est trivial en général mais il est néanmoins assez riche dans quelques cas intéressants. Evidemment, il ne peut pas être algébrique s'il est de dimension infinie. Dans ces cas, la description de ce groupe est assez compliquée. Pour les surfaces dans \mathbb{A}^3 données par des équations de forme $uv - P(x) = 0$ et $uv^n - P(x) = 0$, où $\deg P > 1$, L. Makar-Limanov [47, 48] a donné la description complète du groupe d'automorphismes. Pour les surfaces dites surfaces de Gizatullin (voir ci-après), une telle description a été obtenue par V.I. Danilov et M.H. Gizatullin [17].

Rappelons la définition de l'invariant de Makar-Limanov d'une variété, donnée dans [23, §9]

Définition. *L'invariant de Makar-Limanov* $\text{ML}(X)$ d'une variété affine X est l'intersection des noyaux de toutes les dérivations localement nilpotentes dans $\mathbb{k}[X]$ ou, autrement dit, la sous-algèbre dans $\mathbb{k}[X]$ des invariants communs pour tous les sous-groupes unipotents à un paramètre de $\text{Aut}(X)$.

L'invariant de Makar-Limanov est un outil puissant pour distinguer des variétés affines. En particulier, avec l'aide de cet invariant, L. Makar-Limanov [49] a démontré que l'hypersurface cubique de Russel d'équation $x + x^2y + z^2 + t^3 = 0$ dans \mathbb{C}^4 , n'est pas algébriquement isomorphe à \mathbb{C}^3 , tandis qu'elle est difféomorphe à \mathbb{C}^3 . Dans le problème de linéarisation, cet invariant est aussi très efficace, voir [36].

Donnons quelques définitions. On suppose que le corps de base \mathbb{k} est algébriquement clos et de caractéristique zéro. Toutes les variétés sont irréductibles et réduites. L'action effective du groupe additif $G_a(\mathbb{k})$ sur la variété algébrique X définit un sous-groupe à un paramètre unipotent dans le groupe d'automorphismes $\text{Aut}(X)$.

Définition. Par *groupe spécial d'automorphismes* $\text{SAut}(X)$, nous désignons le sous-groupe de $\text{Aut}(X)$ engendré par tous ces sous-groupes unipotents à un paramètre.

Illustrons ces notions avec des surfaces affines algébriques. Étant donnée une surface affine normale X non isomorphe à $\mathbb{k} \times \mathbb{k}^*$, les trois conditions suivantes sur X sont équivalentes :

- l'action de $\text{SAut}(X)$ sur X est infiniment transitive sur un sous-ensemble de X ayant un complément fini ;
- X admet une complétion par une chaîne de courbes lisses rationnelles ;
- l'invariant de Makar-Limanov $\text{ML}(X)$ est trivial.

Ces surfaces sont appelées *surfaces de Gizatullin*. L'équivalence de deux premières caractérisations a été démontrée par M.H. Gizatullin [26] et puis par V.I. Danilov et M.H. Gizatullin [17]. La troisième équivalence pour les surfaces lisses était implicite dans [26]. Dans le cas de variétés normales, elle a été justifiée par A. Dubouloz [20].

Il est clair qu'un automorphisme algébrique ne peut pas envoyer un point lisse sur un point singulier, ni l'inverse. C'est pourquoi nous formulons notre tâche de la façon suivante. Soient m un nombre entier et X une variété affine algébrique, pas nécessairement lisse. On cherche X telle que l'action de $\text{Aut}(X)$ sur X , restreinte au lieu lisse X_{reg} , soit m -transitive, c'est-à-dire qu'elle envoie tout m -uplet ordonné de points distincts de X_{reg} sur tout autre m -uplet de points distincts de X_{reg} .

S. Kaliman et M. Zaidenberg [31, §5] ont considéré une hypersurface X dans \mathbb{A}^{n+2} donnée par l'équation $uv - f(x_1, \dots, x_n) = 0$, où $n \geq 2$ et $f \in \mathbb{k}[x_1, \dots, x_n]$ est un polynôme arbitraire non constant. Par la suite, de telles variétés X sont appelées *suspensions* au-dessus de $Y = \mathbb{A}^n$. S. Kaliman et M. Zaidenberg ont démontré que pour ces variétés X , le groupe $\text{SAut}(X)$ agit de façon infiniment transitive sur X_{reg} .

Définition. Soit X une variété algébrique sur un corps \mathbb{k} . Le point $x \in X$ est dit *flexible* si l'espace tangent $T_x X$ est engendré, en tant qu'espace vectoriel, par les vecteurs tangents des orbites $H \cdot x$ des sous-groupes unipotents à un paramètre $H \subseteq \text{Aut}(X)$. Une variété X est dite *flexible* si tout point lisse $x \in X_{\text{reg}}$ l'est.

Cette propriété est apparue pour la première fois dans [3], où elle jouait un rôle technique. Plus tard, I.V. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch et M. Zaidenberg [2] ont démontré qu'elle était équivalente à la transitivité du groupe $\text{SAut}(X)$, justifiant une conjecture dans [3].

Théorème ([2, Theorem 0.1]). *Pour une variété affine X de dimension au moins 2, les conditions suivantes sont équivalentes.*

- (i) *Le groupe $\text{SAut}(X)$ agit transitivement sur X_{reg} .*
- (ii) *Le groupe $\text{SAut}(X)$ agit infiniment transitivement sur X_{reg} .*
- (iii) *La variété X est flexible.*

Résultats du Chapitre 5

Dans ce chapitre, nous étudions les actions infiniment transitives d'un groupe spécial d'automorphismes $\text{SAut}(X)$ sur une variété affine algébrique irréductible X sur un corps algébriquement clos de caractéristique zéro. La transitivité infinie de $\text{SAut}(X)$ est *a priori* plus forte que la transitivité infinie du groupe total d'automorphismes $\text{Aut}(X)$. Cependant, pour nous il est beaucoup plus agréable de travailler avec $\text{SAut}(X)$ puisque nos outils le décrivent d'une façon très efficace.

Définition. Par *suspension* au-dessus d'une variété affine Y on désigne une hypersurface $X \subseteq Y \times \mathbb{A}^2$ donnée par l'équation $uv - f(y) = 0$, où $\mathbb{A}^2 = \text{Spec } \mathbb{k}[u, v]$ et $f \in \mathbb{k}[Y]$ n'est pas constante.

En particulier, pour X une suspension au-dessus de Y , on a $\dim X = 1 + \dim Y$.

Les résultats principaux de [3] sont contenus dans les théorèmes suivants. Rappelons que pour un groupe semi-simple algébrique G et un sous-groupe parabolique $P \subset G$, l'espace homogène G/P est appelé *variété de drapeaux*. C'est une variété projective.

Théorème. *Soit G/P une variété de drapeaux sur \mathbb{k} . Tout cône affine normal X au-dessus de G/P est flexible et son groupe spécial d'automorphismes $\text{SAut}(X)$ agit infiniment transitivement sur le lieu lisse $X_{\text{reg}} = X \setminus \{0\}$.*

Rappelons qu'une variété torique X est dite *non dégénérée* si $\mathbb{k}[X]$ ne contient aucune fonction non constante inversible.

Théorème. *Soit X une variété affine torique non dégénérée sur \mathbb{k} de dimension au moins 2. Alors, X est flexible et son groupe spécial d'automorphismes $\text{SAut}(X)$ agit infiniment transitivement sur le lieu lisse X_{reg} .*

Théorème D. *Soit X une variété affine flexible sur \mathbb{k} . On suppose soit que $X \cong \mathbb{A}^1$, soit que $\dim X \geq 2$ et que le groupe spécial d'automorphismes $\text{SAut}(X)$ agit infiniment transitivement sur le lieu lisse X_{reg} . Alors, toutes les suspensions itérées au-dessus de X ont les mêmes propriétés. Elles sont notamment flexibles et leurs groupes spéciaux d'automorphismes agissent infiniment transitivement sur le lieu lisse.*

En conclusion de ces théorèmes, on peut déduire que l'invariant de Makar-Limanov de ces variétés est trivial.

Dans la plupart des démonstrations, on utilise le fait suivant [23, §1.5]. Pour un corps de caractéristique zéro, il y a une bijection entre les dérivations localement nilpotentes de l'algèbre $\mathbb{k}[X]$ et les sous-groupes à un paramètre de $\text{Aut}(X)$ algébriques unipotents.

Groupe d'automorphismes d'une variété réelle

Soit X une variété affine algébrique sur \mathbb{R} . Pour nous, une variété affine algébrique réelle est le lieu de zéros d'un ensemble fini de polynômes à coefficients réels dans un espace affine réel. L'application $f: X \rightarrow Y$ est dite régulière si elle peut être exprimée dans des coordonnées comme

$$f(x) = \left(\frac{p_1(x)}{q_1(x)}, \dots, \frac{p_n(x)}{q_n(x)} \right) \text{ où } p_1, \dots, p_n, q_1, \dots, q_n \in \mathbb{R}[X]$$

avec q_1, \dots, q_n qui ne s'annulent pas sur X . Remarquons que ce ne sont pas toutes les applications f de ce type qui peuvent être relevées comme des applications

régulières $X(\mathbb{C}) \rightarrow Y(\mathbb{C})$. Effectivement, s'il y a un dénominateur non trivial q_i , alors, même s'il ne s'annule pas sur X , il va avoir des zéros sur $X(\mathbb{C})$; donc cette expression dans les coordonnées ne va pas marcher pour $X(\mathbb{C})$. De plus, il peut être impossible de relever cette application régulièrement sur $X(\mathbb{C})$. L'application $f: X \rightarrow X$ est dite un *automorphisme algébrique* si f est régulière, bijective et f^{-1} est régulière. Comme précédemment, considérons le groupe $\text{Aut}(X)$ de tous les automorphismes algébriques de X . En général, ce groupe est assez pauvre, donc les variétés X telles que le groupe $\text{Aut}(X)$ soit assez riche sont intéressantes.

La transitivité infinie pour les surfaces réelles algébriques est bien étudiée. Commençons avec une surface analytique réelle lisse connexe qui est difféomorphe à au moins une variété réelle algébrique, autrement dit, ayant au moins un modèle rationnel. De telles surfaces ont été classifiées par I. Biswas et J. Huisman.

Théorème ([5, Theorem 1.2]). *Soit S une 2-variété réelle analytique compacte connexe.*

1. *Si S est orientable de genre supérieur à 1, alors S n'admet aucun modèle rationnel.*
2. *Si S est non orientable ou bien difféomorphe à S^2 ou $S^1 \times S^1$, alors il y a, à un isomorphisme près, exactement un modèle rationnel de S . Autrement dit, deux modèles rationnels de S sont isomorphes.*

I. Biswas, J. Huisman et F. Mangolte [5], [29] ont démontré la transitivité infinie pour cette classe de surfaces.

Théorème ([29, Theorem 1.4]). *Pour toute surface X rationnelle non singulière compacte connexe réelle algébrique, le groupe $\text{Aut}(X)$ agit infiniment transitivement sur X .*

Pour les surfaces réelles algébriques non connexes, il n'y a pas de 2-transitivité pour des raisons de connexité. Effectivement, deux points pris dans une même composante connexe ne peuvent pas être envoyés sur une paire de points appartenant aux composantes différentes. On va donc introduire la propriété suivante.

Définition. Par extension, l'action d'un groupe G sur un ensemble $Y = Y^1 \sqcup Y^2 \sqcup \dots \sqcup Y^s$ est dite *infiniment transitive composante par composante* si, pour tout s -uplet (m_1, \dots, m_s) , cette action est transitive sur $(m_1 + \dots + m_s)$ -uplets de forme $(P_1^1, \dots, P_{m_1}^1, P_1^2, \dots, P_{m_2}^2, \dots, P_1^s, \dots, P_{m_s}^s)$, où $P_j^i \in Y^i$ sont deux à deux distincts.

Conformément à l'article récent de J. Blanc et F. Mangolte [6], cette propriété n'est pas toujours vraie pour les surfaces réelles algébriques non connexes. Soit $\#M$ le nombre de composantes connexes d'une variété compacte M .

Théorème ([6, Theorem 1]). *Soit X une surface non singulière réelle projective. Le groupe $\text{Aut}(X)$ agit infiniment transitivement composante par composante sur X si et seulement si X est géométriquement rationnelle et $\#X \leq 3$.*

Dans le cas de trois composantes connexes, ce théorème peut être renforcé [6, Theorem 2].

Il faut remarquer que la transitivité infinie dans tous ces cas a été obtenue au moyen du groupe $\text{Aut}(X)$ et non pas de $\text{SAut}(X)$. À l'heure actuelle, la question de la transitivité infinie de l'action de $\text{SAut}(X)$ sur ces surfaces demeure une question ouverte.

Résultats du Chapitre 6

Ces résultats peuvent être trouvés dans [3, 44]. À présent, nous étudions les actions infiniment transitives du groupe $\text{SAut}(X)$ pour des variétés de dimension quelconque sur le corps \mathbb{R} . Tout d'abord, au prix de quelques restrictions légères sur la suspension, on peut généraliser le Théorème D.

Théorème. *Soit Y une variété flexible réelle algébrique. On suppose que le lieu lisse Y_{reg} est connexe et que le groupe spécial d'automorphismes $\text{SAut}(Y)$ agit de façon m -transitive sur Y_{reg} pour tout $m \in \mathbb{N}$. Considérons la suspension $X = \text{Susp}(Y, f)$, où la fonction $f \in \mathbb{R}[Y]$ vérifie la condition $f(Y_{\text{reg}}) = \mathbb{R}$. Alors, la variété X est flexible et le groupe spécial d'automorphismes $\text{SAut}(X)$ agit de façon m -transitive sur X_{reg} pour tout $m \in \mathbb{N}$.*

Dans le cas particulier d'une suspension au-dessus d'une droite affine, le résultat du Théorème D reste vrai sur un corps quelconque de caractéristique zéro.

Théorème. *Soit \mathbb{k} un corps de caractéristique zéro. Étant donné un polynôme $f \in \mathbb{k}[x]$ tel que $f(\mathbb{k}) = \mathbb{k}$, on peut considérer une surface $X \subseteq \mathbb{A}_{\mathbb{k}}^3$ donnée par l'équation $f(x) - uv = 0$. Alors, X est flexible et le groupe spécial d'automorphismes $\text{SAut}(X)$ agit de façon m -transitive sur X_{reg} pour tout $m \in \mathbb{N}$.*

Dans le cas complexe, ces surfaces ont été classifiées dans des termes abstraits par T. Bandman et L. Makar-Limanov [4].

Si X_{reg} n'est pas connexe, le groupe $\text{SAut}(X)$ n'est même pas transitif sur X_{reg} . Effectivement, l'action de $\text{SAut}(X)$ sur X laisse fixe chaque composante connexe de X : chaque automorphisme spécial g peut être décomposé sous la forme $\prod h_j(1)$, où chaque h_i est un sous-groupe à un paramètre unipotent. Pour tout $x \in X$, la courbe $t \rightarrow \prod h_i(t) \cdot x$ relie x avec $g \cdot x$. Ainsi, pour les variétés réelles non connexes, on est obligé d'étudier la transitivité infinie composante par composante.

Soit Y une variété algébrique. On peut décomposer $Y_{\text{reg}} = Y^1 \sqcup \dots \sqcup Y^s$ selon des composantes connexes. Pour la suspension $X = \text{Susp}(Y, f)$, on a une partition

$X_{\text{reg}} = X^1 \sqcup \dots \sqcup X^{s'}$. Il est facile de démontrer que si deux points x et x' sont dans la même composante connexe de X^j , leurs projections naturelles sur Y appartiennent toutes les deux à une même composante connexe Y^i de Y_{reg} . En revanche, on peut avoir plusieurs composantes connexes de X_{reg} au-dessus d'une composante connexe de Y_{reg} . Par exemple, si Y^i est compacte et f est strictement positive sur Y^i , les deux branches de l'hyperbole $uv = f(y)$ sont au-dessus de tout $y \in Y^i$. La réunion de toutes ces branches forme deux composantes connexes de $\text{Susp}(Y^i, f)$. Pour de telles variétés, on généralise le Théorème 6.1 comme ci-dessous (article en commun avec F. Mangolte [44]).

Théorème. *Soit Y une variété affine algébrique définie sur \mathbb{R} et soit $f \in \mathbb{R}[Y]$. On suppose que pour toute composante connexe Y^i de Y_{reg} , la dimension $\dim Y^i$ est au moins 2 et que f est non constante sur Y^i . Si Y est flexible et si l'action de $\text{SAut}(Y)$ sur Y_{reg} est infiniment transitive sur chaque composante connexe, alors la suspension $X = \text{Susp}(Y, f)$ est flexible et $\text{SAut}(X)$ agit sur X_{reg} infiniment transitivement composante par composante.*

Ce théorème est le résultat principal du Chapitre 6.

Introduction

This thesis is devoted to the actions of groups of algebraic transformations on affine algebraic varieties. In the first part we study normality of closures of maximal torus orbits in the rational modules of simple algebraic groups. The second part deals with actions of automorphism groups on affine varieties. We study transitivity and infinite transitivity of such an action on the smooth locus.

Part I

The normality problem for orbit closures

We start with a brief survey. Let G be an affine algebraic group, e.g. the special linear group $SL(n)$. We suppose that the ground field \mathbb{k} is algebraically closed and of characteristic zero. Recall that an irreducible affine algebraic variety X is called *normal* if its algebra of regular functions $\mathbb{k}[X]$ is integrally closed in its field of fractions. The study of normality of orbit closures has a long history. The very first result is due to B. Kostant [37] who showed that the full nilpotent cone in the adjoint module is normal. H. Kraft and C. Procesi [39] proved that in the adjoint module $\mathfrak{sl}(\mathfrak{n})$ all the $SL(n)$ -orbit closures are normal. In other words, the closure in the Zarisky topology of each matrix conjugacy class is normal. For a field \mathbb{k} of positive characteristic, the same result for $SL(n)$ was established by S. Donkin [19]. Later on H. Kraft and C. Procesi [40] studied the same question for the adjoint modules of other classical groups. They pointed out some orbits with non-normal closures in terms of Young diagrams. E. Sommers [64] solved the remaining case inside D_n , namely the case of the very even conjugacy class. The normality for an orbit closure in the adjoint module is equivalent to the normality of a certain nilpotent orbit closure in the same module. In all aforementioned articles the authors dealt mainly with nilpotent orbits. The cases F_4 , G_2 , E_6 are resolved by A. Broer, H. Kraft, and E. Sommers in [12], [38], and [63], respectively. There is no complete answer for E_7 and E_8 yet.

Now we come to actions of an algebraic torus T , i.e. of an affine algebraic group isomorphic to $\mathbb{k}^\times \times \dots \times \mathbb{k}^\times$, where $\mathbb{k}^\times = \mathbb{k} \setminus \{0\}$. An irreducible algebraic variety

is called *toric* if X is normal and T acts on X with an open orbit. Toric varieties play an important role in algebraic geometry, topology and combinatorics due to their remarkable description in terms of convex geometry, see [24]. If an algebraic torus T acts on a variety Y , then the orbit closure $X = \overline{T y}$ of a point $y \in Y$ is a natural candidate for a toric variety. To verify that it is indeed toric, it suffices to check that X is normal. So the normality property is of great interest.

There is a link between normality of closures of torus orbits and normality of closures of G -orbits. Indeed, the study of orbits of the G -action in the Cartesian square $\mathfrak{g} \oplus \mathfrak{g}$ of the adjoint module can be reduced to the study of T -orbits in the adjoint module \mathfrak{g} itself, using Luna's Étale Slice theorem.

For the time being we restrict ourselves to the case of T -actions and introduce some terminology and a combinatorial interpretation. Let V be a rational T -module. We denote by $\Lambda = \Lambda(T)$ the *character lattice* of T . With respect to the T -action, the module V can be diagonalized:

$$V = \bigoplus_{\mu \in \Lambda} V_{\mu}, \quad \text{where} \quad V_{\mu} = \{v \in V \mid tv = \mu(t)v \quad \forall t \in T\}.$$

We denote by $M(V) = \{\mu \in \Lambda \mid V_{\mu} \neq 0\}$ the set of weights of V . Each nonzero vector v in V has its *weight decomposition* $v = v_{\mu_1} + \dots + v_{\mu_s}$, $v_{\mu_i} \in V_{\mu_i}$, $v_{\mu_i} \neq 0$. Denote by $M(v)$ the set $\{\mu_1, \dots, \mu_s\}$. We can generate a semigroup $\mathbb{Z}_{\geq 0}(\mu_1, \dots, \mu_s)$ with these weights. We can also generate a sublattice $\mathbb{Z}(\mu_1, \dots, \mu_s)$ and a rational polyhedral cone $\mathbb{Q}_{\geq 0}(\mu_1, \dots, \mu_s)$ in the space $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition (Definition 1.2). The set of points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ is called *saturated* if

$$\mathbb{Z}_{\geq 0}(\mu_1, \dots, \mu_s) = \mathbb{Z}(\mu_1, \dots, \mu_s) \cap \mathbb{Q}_{\geq 0}(\mu_1, \dots, \mu_s).$$

The set of points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ is called *hereditarily normal* if each its subset is saturated.

It was proved in [32, I, §1, Lemma 1] that the closure $\overline{T v}$ of the T -orbit of a vector v is normal if and only if the set of characters $\{\mu_1, \dots, \mu_s\}$ is saturated. This combinatorial criterion plays a key role in the first part of the thesis.

There are also some results in the literature concerning normality of orbit closures for projective T -actions. Let $X(v)$ be the closure of the T -orbit $T[v]$ of a point $[v] \in \mathbb{P}(V)$ in the projectivisation of a rational T -module V . Let $P(v)$ be the convex hull of $M(v)$ in $\Lambda_{\mathbb{Q}}$. Then $X(v)$ is normal if and only if the set $\{\mu - \mu_0 \mid \mu \in M(v)\}$ is saturated for every vertex μ_0 of the polytope $P(v)$. This and other criteria are given by J.B. Carrell and A. Kurth [13].

Consider a more general setting. Let G be a simply connected semisimple affine algebraic group, $T \subset G$ be a maximal torus, $B \subset G$ be a Borel subgroup. A.A. Klyachko [34] proved that the closure of a general T -orbit on the flag variety

G/B is normal. Later on R. Dabrowski [16] showed that the closure of a general T -orbit in G/P , where $P \subset G$ is a parabolic subgroup, is also normal. Examples of non-normal closures of non-general torus orbits can be found in [13].

It is well known that the closure of each T -orbit on a toric variety is normal. Using the method of U -invariants, we can prove normality of the closure of each G -orbit on a spherical variety for any connected reductive group G . In the case of complexity one, I.V. Arzhantsev [1] showed that for an action of a connected reductive group G on a normal variety X with a one-parameter family of generic spherical G -orbits and a good quotient $\pi: X \mapsto X//G$, where $X//G$ is a curve, the closure of any G -orbit is normal.

The saturation property is important in many algebraic and geometric problems. N. White [68] proved that the set of incidence vectors of the bases of a realizable matroid is saturated. A geometric consequence of this fact is that for any point y in the affine cone over the classical Grassmannian $\text{Gr}(k, n)$ the closure \overline{Ty} is normal.

Given a finite graph Γ with n vertices, we can associate the following finite collection $M(\Gamma)$ of vectors in the lattice \mathbb{Z}^n :

$$M(\Gamma) = \{\varepsilon_i + \varepsilon_j : (ij) \text{ is an edge of } \Gamma\},$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ is the standard basis of \mathbb{Z}^n . The saturation property for this set is equivalent to the fact that for two arbitrary minimal odd cycles C and C' in Γ , either C and C' have a common vertex or there exists an edge of Γ joining a vertex of C with a vertex of C' , see H. Ohsugi and T. Hibi [52], A. Simis, W. Vasconcelos, and R. Villarreal [62]. Algebraically, the saturation property for $M(\Gamma)$ is equivalent to the integral closedness of the subalgebra $\mathcal{A}(\Gamma)$ of the polynomial algebra $\mathbb{k}[x_1, x_2, \dots, x_n]$ in its field of fractions $Q\mathcal{A}(\Gamma)$,

$$\mathcal{A}(\Gamma) = \mathbb{k}[x_i x_j : (ij) \text{ is an edge of } \Gamma].$$

C. Chindris [14] obtained some general results concerning quivers and the saturation property. It was shown that a finite, connected quiver Q without oriented cycles is a Dynkin or Euclidean quiver if and only if all orbit semigroups of representations of Q are saturated.

The saturation property in a slightly different setting occurs to be important also in some other problems of representation theory, see N. Ressayre [60] and P.-L. Montagard, B. Pasquier, N. Ressayre [50]. These papers deal with counting certain semigroups in the weight lattice. If it is known *a priori* that the semigroup M is finitely generated, then M can be found in two steps. On the first step we find inequalities defining the cone $\text{cone}(M)$, generated by the semigroup M . On the second step we have to choose those integer points of $\text{cone}(M)$, which actually

belong to the semigroup M . In many interesting cases M coincides with the set of integer points in $\text{cone}(M)$ (the *saturation problem*).

For the adjoint action $SL(n) : \mathfrak{sl}(\mathfrak{n})$ the result of B. Sturmfels [65, Ex. 3.7], [66] says that all T -orbit closures are normal. G. Bobiński and G. Zwara [7] interpreted this combinatorial result in terms of representations of quivers. J. Morand [51] classified all semisimple affine algebraic groups such that all T -orbit closures in the adjoint module are normal.

Our methods have much in common with technique used in these papers.

Let G be a connected simply connected semisimple algebraic group and T be a maximal torus in G . We formulate here the main problem being solved in Part I.

Find all simple rational G -modules V such that for each vector $v \in V$ the closure of its T -orbit is a normal affine variety.

Let us recall the description of the set of T -weights of a rational G -module $V(\lambda)$ with the highest weight λ . Denote by Φ the root system corresponding to G . Let Ξ be the root lattice, and let W be the Weyl group of Φ . The *weight polytope* $P(\lambda)$ of the module $V(\lambda)$ is the convex hull $\text{conv}\{w\lambda \mid w \in W\}$ of the W -orbit of the point λ in $\Lambda_{\mathbb{Q}}$. Then

$$M(\lambda) = (\lambda + \Xi) \cap P(\lambda),$$

see [25, Theorem 14.18] and [11, Exercices to Ch. VIII §7]. Hence, all T -orbit closures in $V = V(\lambda)$ are normal if and only if $M(\lambda)$ is hereditarily normal.

Results of Part I

Summarizing results of [8, 42, 43], we obtain the following theorem.

Theorem (Theorem 1.1). *For the following types of simple algebraic groups and the corresponding modules, and for their dual modules, the closures of all maximal torus orbits are normal. In all other cases, the module contains a maximal torus orbit with nonnormal closure.*

<i>Root system</i>	<i>Highest weight</i>	<i>Checked in</i>
$A_n, n \geq 1$	π_1	Case 2.1
$A_n, n \geq 1$	$\pi_1 + \pi_n$	[51, 65, 66], Case 2.2
A_1	$3\pi_1$	Case 2.3
A_1	$4\pi_1$	Case 2.4
A_2	$2\pi_1$	Case 2.5
A_3	π_2	Case 2.7
A_4	π_2	Case 2.7
A_5	π_2	Case 2.7
A_5	π_3	Case 2.6

<i>Root system</i>	<i>Highest weight</i>	<i>Checked in</i>
$B_n, n \geq 2$	π_1	Case 3.1
B_2	π_2	Case 3.3
B_2	$2\pi_2$	Case 3.2
B_3	π_3	Case 3.3
B_4	π_4	Case 3.3
$C_n, n \geq 3$	π_1	Case 3.4
C_3	π_2	Case 3.5
C_4	π_2	Case 3.5
$D_n, n \geq 4$	π_1	Case 3.6
D_4	π_2	Case 3.7
D_4	π_3	Case 3.8
D_4	π_4	Case 3.8
D_5	π_4	Case 3.9
D_6	π_5	Case 3.10
D_6	π_6	Case 3.10
F_4	π_1	Case 4.1
G_2	π_1	Case 4.2

The first part of the thesis consists of four chapters. Let us present the content of each chapter.

In **Chapter 1** we reformulate the above problems in combinatorial language and introduce the methods used throughout Part I. If $M(\lambda) \subseteq M(\mu)$, and a non-saturated subset for λ is known, then it can be used as a non-saturated subset for μ . Hence, in many cases it suffices to construct one non-saturated subset to answer in the negative the question of simultaneous normality of all T -orbit closures. So non-saturated subsets will occur very often. In the sequel we refer to a nonsaturated subset as an *NSS*. By an *extended nonsaturated subset* we mean a nonsaturated subset $\{v_1, \dots, v_r\}$ augmented by a vector v_0 such that

i. $v_0 \in (\mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geq 0}(v_1, v_2, \dots, v_r)) \setminus \mathbb{Z}_{\geq 0}(v_1, v_2, \dots, v_r)$,

ii. there exists a $\mathbb{Q}_{\geq 0}$ -representation

$$v_0 = q_1 v_{i_1} + \dots + q_s v_{i_s}, \quad v_{i_j} \in \{v_1, v_2, \dots, v_r\}$$

with linearly independent vectors v_{i_1}, \dots, v_{i_s} and coefficients $q_i \in [0, 1[$.

We call these subsets *ENSSs* and denote them by $\{v_0; v_1, \dots, v_r\}$. It is easy to show that if the set $M = \{v_1, \dots, v_r\}$ is not saturated, then there exists a vector v_0 such that $\{v_0; v_1, \dots, v_r\}$ is an ENSS.

Usually it is not difficult to guess an ENSS. However, it is harder to check that the guessed set is indeed an ENSS. One of our methods consists in the following. Let v_0, v_1, \dots, v_r be vectors in a rational vector space \mathbb{Q}^n , and let f be a linear function on \mathbb{Q}^n . We call f a *discriminating linear function* for the collection $\{v_0; v_1, \dots, v_r\}$ if the value $f(v_0)$ cannot be represented as a linear combination of the values $f(v_1), \dots, f(v_r)$ with nonnegative integer coefficients. Assume that $v_0 \in \mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geq 0}(v_1, v_2, \dots, v_r)$, and that v_0 can be represented as a $\mathbb{Q}_{\geq 0}$ -combination of linearly independent vectors v_1, \dots, v_r with coefficients from the interval $[0, 1[$. Then the existence of a discriminating function guarantees that $\{v_0; v_1, \dots, v_r\}$ is an ENSS.

Let us describe machinery which serves to establish that a given set of points M is hereditarily normal. Choose a basis such that for any $v \in M$, the coordinates of v with respect to this basis are all integers. Represent each point as a column vector of its coordinates in this basis. Let $K = K(M)$ be an integer $n \times r$ matrix formed by all these column vectors. The following algorithm deals with the monomial algebras.

Definition (Definition 1.17). To each column $K_i = (k_{1i}, \dots, k_{ni})^T$ we associate a Laurent monomial $t^{K_i} = t_1^{k_{1i}} \dots t_n^{k_{ni}}$. The *toric ideal* I_K associated with K is the kernel of the \mathbb{k} -algebra homomorphism

$$\mathbb{k}[x_1, x_2, \dots, x_r] \rightarrow \mathbb{k}[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}], \quad x_i \mapsto t^{K_i}.$$

Definition (Definition 1.18). Suppose that u_+ and u_- are two vectors in \mathbb{Z}_+^n with disjoint supports, and let $f = x^{u_+} - x^{u_-} \in I_K$. We say that f is a *circuit* in I_K if the following two conditions hold:

- the coordinates of both u_+ and u_- have no common divisor greater than 1;
- the set of variables which actually occur in f is minimal with respect to inclusion among all the binomials of I_K .

The following two criteria are due to B. Sturmfels.

Theorem ([65, Theorem 3.8]). *A set of points M is hereditarily normal if and only if every circuit in $I_{K(M)}$ has at least one square-free monomial.*

Assume that the set of vectors $M \subset \mathbb{Q}^n$ has rank d , $d \leq n$, and let $L = \langle v \mid v \in M \rangle$ be the linear span of vectors from M . The set M is called *unimodular* if for any linearly independent vectors $v_1, \dots, v_d \in M$ the d -dimensional volume $\text{vol}_d(v_1, v_2, \dots, v_d)$ has constant absolute value. In a fixed basis of L , the above condition is equivalent to the fact that the absolute values of all nonzero determinants $|\det(v_1, v_2, \dots, v_d)|$, $v_1, v_2, \dots, v_d \in M$, computed in this basis are equal.

If the set M is unimodular, then its intersection with any subspace $L_1 \subset L$ is also unimodular. Indeed, we can always choose in L a basis compatible with L_1 .

Theorem ([65, Theorem 3.5], Theorem 1.12). *Any unimodular set is hereditarily normal.*

More generally, we say that a subset $M \subset \mathbb{Q}^n$ of rank d is *almost unimodular* if we can choose a subset $\{v_1, v_2, \dots, v_d\} \subseteq M$ such that

$$\text{vol}_d(v_1, v_2, \dots, v_d) = m,$$

and for any other vector $w \in M$ and each i the value

$$\text{vol}_d(v_1, v_2, \dots, \widehat{v}_i, \dots, v_d, w)$$

is divisible by m , i.e. equals km for some $k \in \mathbb{Z}$. If one fixes a basis in the linear space $\langle M \rangle$, then this property can be checked by comparing the values of the corresponding determinants in the given basis instead of computing vol_d 's. The value $m = \det(v_1, v_2, \dots, v_d)$ is called the *volume* of an almost unimodular subset.

Almost unimodular sets are used to prove that certain sets of vectors are hereditarily normal. We argue by contradiction supposing that an ENSS $(v_0; v_1, \dots, v_r)$ exists in the given set M . Then, using almost unimodularity of M , we analyze coefficients of the corresponding $\mathbb{Q}_{\geq 0}$ -combination for v_0 , according to the definition of an ENSS. There is a finite number of possible values for these coefficients. Then, using this data and additional information about the weight lattice, we show that $(v_0; v_1, \dots, v_r)$ is not an ENSS. This implies that M is hereditarily normal.

It is known that the closure of a **generic** T -orbit is normal, see [8]. In **Chapter 1** we give a combinatorial counterpart of this fact.

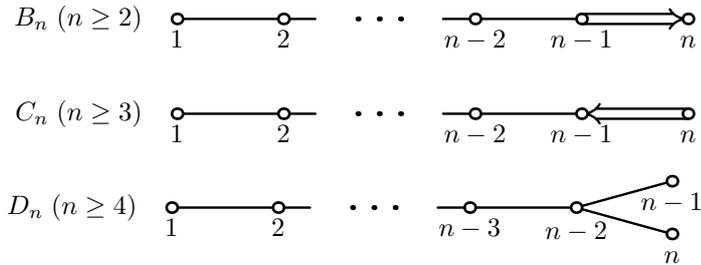
In **Chapter 2** we consider the case of the special linear group. Actually, this occurs to be the most difficult case. It corresponds to the root system A_n .

$$A_n \ (n \geq 1) \quad \circ_1 \text{---} \circ_2 \text{---} \circ_3 \quad \cdots \quad \circ_{n-1} \text{---} \circ_n$$

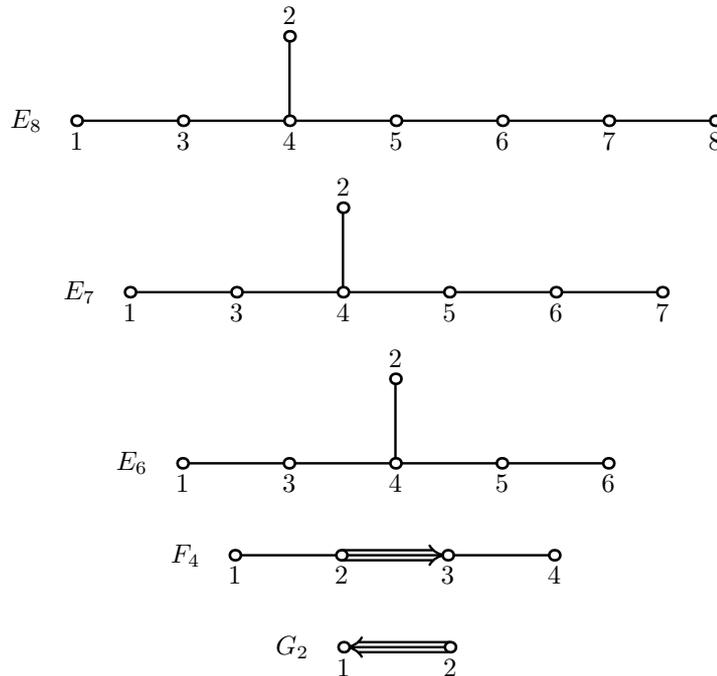
For the modules mentioned in Theorem 1.1, we apply Theorems 1.19 and 1.12 in the proofs. Some reasoning uses the graph theory language. Our reference for graph theory is [27].

Among the cases not listed in Theorem 1.1, the fundamental representations are the most difficult to treat. To work with this class, we use the following observation. If a non-saturated subset in the set of weights of the k th fundamental representation of $SL(n)$ is found, then the analogous non-saturated subset exists in the set of weights of the k th fundamental representation of $SL(n+k)$. Then a procedure similar to the Euclidean algorithm reduces all the cases to those for which an ENSS is already constructed.

In **Chapter 3** we treat the other classical groups. The most difficult cases here are spinor modules for D_5 and D_6 . We prove that the sets $M(\pi_4)$, $M(\pi_5)$ for D_5 and $M(\pi_5)$, $M(\pi_6)$ for D_6 are hereditarily normal. Methods used for A_n are not sufficient in these cases. We need to use almost unimodular sets to get some information about the sets which are candidates to be an ENSS.



In **Chapter 4** we consider exceptional groups. The same technique is used, involving unimodular and almost unimodular sets. It turns out that all T -orbit closures are normal just in two cases; namely, for the first fundamental representations of F_4 and G_2 .



Part II

The second part of the thesis is devoted to the infinite transitivity of the automorphism groups of algebraic varieties over a field of characteristic zero. It consists of two chapters. In **Chapter 5** the ground field is algebraically closed, whereas in **Chapter 6** the ground field is \mathbb{R} . These two settings are quite different. Let us give a brief historical overview of these cases independently.

Automorphism groups of complex varieties

Definition (Definition 5.1). The action of a group G on a set A is called *m-transitive*, if for any two tuples of pairwise distinct points (a_1, a_2, \dots, a_m) and (b_1, b_2, \dots, b_m) in A there exists an element $g \in G$ such that $g(a_i) = b_i$, $i = 1, 2, \dots, m$. By abuse of language, the actions which are *m-transitive* for all $m \in \mathbb{N}$ will be called *infinitely transitive*.

Manifolds with multiple transitive group actions are quite rare. Indeed, let X be a smooth manifold, and consider a Lie group G acting on X . Clearly, for $m > \dim X$ it cannot act *m-transitively* on X . According to A. Borel, a real Lie group cannot act 4-transitively on a simply connected manifold, see Theorems 5 and 6 in [9]. Multiple transitivity of the Lie group actions on homogeneous spaces was studied by J. Tits [67]. He classified all doubly and triply transitive Lie group actions. This result was later reproved, using other methods, by L. Kramer [41].

In the algebraic setting, F. Knop [35] found all 2-transitive actions of a connected algebraic group G on an algebraic variety X , where the ground field can be any algebraically closed field. In this case either $X = \mathbb{P}^m$ and $G = PGL(m+1)$, or $X = \mathbb{A}^m$ and G is a semidirect product $L \ltimes \text{Transl}$, where Transl stands for the group of translations of \mathbb{A}^m and L is a group acting linearly on \mathbb{A}^m and transitively on $\mathbb{A}^m \setminus \{0\}$. All such linear groups are listed in [35]. In particular, it follows from this result that the only 3-transitive action of an algebraic group is the natural action of $PGL(2)$ on \mathbb{P}^1 , whereas no 4-transitive action exists. V. Popov [57] estimates the degree of "general" transitivity for all simple algebraic groups G and their actions on algebraic varieties. Here general transitivity means that the induced action on the corresponding Cartesian power X^m has an open orbit.

Summarizing, we conclude that there is no space with an infinitely transitive action if we restrict ourselves to actions of Lie groups or algebraic group.

From now on, let us consider the action of the full automorphism group of a variety. The infinite transitivity of the group of all algebraic automorphisms is well known for the affine space \mathbb{A}^n over any infinite field \mathbb{k} , where $n \geq 2$. For the proof see e.g. [31, Lemma 5.5].

Rosay and Rudin [61] studied the action of the group of all holomorphic automorphisms of \mathbb{C}^n , $n \geq 2$, on countable subsets. For any two infinite discrete sequences of points they showed that the first can be sent to the second by means of a holomorphic mapping. However this mapping does not need to be an automorphism. Moreover, the class of sequences that can be mapped to a standard sequence, i.e. to the set of integers, by a holomorphic automorphism, is rather poor. The situation is quite different for dense sets. They proved that for any two countable dense subsets A and B of \mathbb{C}^n , $n \geq 2$, there exists a holomorphic automorphism g such that $g(A) = B$.

Denote by $\text{Aut}(X)$ the group of all algebraic automorphisms of an affine variety X . This group is trivial in general, but in some interesting cases it is however rich. Obviously it cannot be algebraic if it is infinite dimensional. In these cases its description is rather complicated. For the surfaces in \mathbb{A}^3 given by the equations $uv - P(x) = 0$ and $uv^n - P(x) = 0$, where $\deg P > 1$, the automorphism group was completely described by L. Makar-Limanov [47, 48]. For so called Gizatullin surfaces (see below) such a description was obtained by V.I. Danilov and M.H. Gizatullin [17].

Definition (Definition 5.40). Recall [23, §9] that the *Makar-Limanov invariant* $\text{ML}(X)$ of an affine variety X is the intersection of the kernels of all locally nilpotent derivations of $\mathbb{k}[X]$, or, in other words, the subalgebra in $\mathbb{k}[X]$ of common invariants for all one parameter unipotent subgroups of $\text{Aut}(X)$.

The Makar-Limanov invariant is a powerful tool to distinguish affine varieties. In particular, L. Makar-Limanov [49] showed with the help of this invariant that the Russel cubic hypersurface $x + x^2y + z^2 + t^3 = 0$ in \mathbb{C}^4 is not algebraically isomorphic to \mathbb{C}^3 , while it is diffeomorphic to \mathbb{C}^3 . This invariant occurred to be extremely useful in the linearization problem, see survey [36].

Let us give some definitions. We assume the ground field \mathbb{k} to be algebraically closed of characteristic zero. All the varieties are supposed to be irreducible and reduced. An effective action of the additive group $G_a(\mathbb{k})$ on the algebraic variety X defines a one-parameter unipotent subgroup in the automorphism group $\text{Aut}(X)$.

Definition (Definition 5.2). By a *special automorphism group* $\text{SAut}(X)$ we mean the subgroup of $\text{Aut}(X)$ generated by all its one-parameter unipotent subgroups.

Let us illustrate these notions on the example of affine algebraic surfaces. Given a normal affine surface X non-isomorphic to $\mathbb{k} \times \mathbb{k}^*$, the following three conditions on X are equivalent:

- the action of $\text{SAut}(X)$ on X is infinitely transitive everywhere except a finite number of points;
- X admits a completion by a chain of smooth rational curves;

- the Makar-Limanov invariant $\text{ML}(X)$ is trivial.

These surfaces are often called the *Gizatullin surfaces*. The equivalence of two first characterizations was shown by M.H. Gizatullin [26] and then by V.I. Danilov and M.H. Gizatullin [17]. The third characterization for smooth surfaces was implicit in [26]. For normal varieties it was proved by A. Dubouloz [20].

It is clear that an algebraic automorphism cannot send a smooth point to a singular one, and vice versa. So our setting is the following. Let m be an integer, and let X be an affine algebraic variety, not necessarily smooth. We are seeking for X such that the action of $\text{Aut}(X)$ on X , restricted to the smooth locus X_{reg} , is m -transitive, i.e. maps any ordered m -tuple of pairwise distinct points on X_{reg} to any other such m -tuple.

S. Kaliman and M. Zaidenberg [31, §5] considered a hypersurface X in \mathbb{A}^{n+2} given by equation $uv - f(x_1, \dots, x_n) = 0$, where $n \geq 2$ and $f \in \mathbb{k}[x_1, \dots, x_n]$ is an arbitrary non-constant polynomial. In the sequel such a variety X is called a *suspension* over $Y = \mathbb{A}^n$. They proved that the group $\text{SAut}(X)$ of such a variety X is infinitely transitive on X_{reg} .

Definition (Definition 5.3). Let X be an algebraic variety over \mathbb{k} . We say that a point $x \in X$ is *flexible* if the tangent space $T_x X$ is spanned by the tangent vectors to the orbits $H.x$ of one-parameter unipotent subgroups $H \subseteq \text{Aut}(X)$. The variety X is called *flexible* if every smooth point $x \in X_{\text{reg}}$ is.

This property appeared first in [3], where it played a technical role. Later on I.V. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch and M. Zaidenberg [2] showed that it is equivalent to the transitivity of the group $\text{SAut}(X)$, so confirming a conjecture in [3].

Theorem ([2, Theorem 0.1]). *For an affine variety X of dimension at least 2, the following conditions are equivalent.*

- (i) *The group $\text{SAut}(X)$ acts transitively on X_{reg} .*
- (ii) *The group $\text{SAut}(X)$ acts infinitely transitively on X_{reg} .*
- (iii) *The variety X is flexible.*

Results of Chapter 5

In this chapter we study infinitely transitive actions of the special automorphism group $\text{SAut}(X)$ on an irreducible affine algebraic variety X over an algebraically closed field of characteristic zero. Infinite transitivity of $\text{SAut}(X)$ is *a priori* stronger than infinite transitivity of the whole automorphism group $\text{Aut}(X)$. However, it is more convenient for us since our tools describe $\text{SAut}(X)$ in a rather comfortable manner.

Definition (Definition 5.4). We call a *suspension* over an affine variety Y the hypersurface $X \subseteq Y \times \mathbb{A}^2$ given by equation $uv - f(y) = 0$, where $\mathbb{A}^2 = \text{Spec } \mathbb{k}[u, v]$, and $f \in \mathbb{k}[Y]$ is non-constant. In particular, $\dim X = 1 + \dim Y$.

The main results of [3] are contained in the following theorems.

Recall that for a semisimple algebraic group G and a parabolic subgroup $P \subset G$, the homogeneous space G/P is called a *flag variety*. It is a projective variety.

Theorem (Theorem 5.5). *Consider a flag variety G/P over \mathbb{k} . Then every normal affine cone X over G/P is flexible and its special automorphism group $\text{SAut}(X)$ acts infinitely transitively on the smooth locus $X_{\text{reg}} = X \setminus \{0\}$.*

Recall that a toric variety X is called *non-degenerate* if $\mathbb{k}[X]$ contains no non-constant invertible functions.

Theorem (Theorem 5.11). *Let X be any non-degenerate affine toric variety over \mathbb{k} of dimension at least 2. Then X is flexible and its special automorphism group $\text{SAut}(X)$ acts infinitely transitively on the smooth locus X_{reg} .*

Theorem (Theorem 5.30). *Suppose that an affine variety X over \mathbb{k} is flexible and either $X \cong \mathbb{A}^1$, or $\dim X \geq 2$ and the special automorphism group $\text{SAut}(X)$ acts infinitely transitively on the smooth locus X_{reg} . Then all iterated suspensions over X have the same properties, namely, they are flexible and the special automorphism group acts infinitely transitively on the set of its smooth points.*

As a conclusion of these theorems, we can deduce that the Makar-Limanov invariant is trivial for the varieties under consideration.

Our main tool used throughout the proofs is the following, see [23, §1.5]. For a field of characteristic zero, there is a bijection between locally nilpotent derivations of the algebra $\mathbb{k}[X]$ and one-parameter unipotent algebraic subgroups of $\text{Aut}(X)$.

Automorphism groups of real varieties

Let X be an affine algebraic variety over \mathbb{R} . By an algebraic variety we mean the zero locus of a finite set of polynomials with real coefficients in the real affine space. The map $f: X \rightarrow Y$ is called regular if it can be written in coordinates as

$$f(x) = \left(\frac{p_1(x)}{q_1(x)}, \dots, \frac{p_n(x)}{q_n(x)} \right), \text{ where } p_1, \dots, p_n, q_1, \dots, q_n \in \mathbb{R}[X]$$

and q_1, \dots, q_n do not attain zero on X . Note that not all such maps f can be lifted to a regular map $X(\mathbb{C}) \rightarrow Y(\mathbb{C})$. Indeed, if there is a nontrivial denominator q_i , then being nonvanishing on X it can attain zero on $X(\mathbb{C})$. So this coordinate

decomposition will not work for $X(\mathbb{C})$. It can be impossible to extend it regularly to $X(\mathbb{C})$. The map $f: X \rightarrow X$ is called an *algebraic automorphism* if f is regular, bijective, and f^{-1} is regular. As before, denote by $\text{Aut}(X)$ the group of all algebraic automorphisms of X . This group is poor in general. So the class of varieties X such that the group $\text{Aut}(X)$ is rich enough is of interest.

The infinite transitivity for real algebraic surfaces is rather well understood. Let us start with a smooth connected real analytic surface diffeomorphic to at least one real algebraic variety, in other words, having at least one rational model. Such surfaces are all classified by I. Biswas and J. Huisman.

Theorem ([5, Theorem 1.2]). *Let S be a compact connected real analytic two-manifold.*

1. *If S is orientable of genus greater than 1, then S does not admit any rational model.*
2. *If S is either nonorientable, or is diffeomorphic to one of S^2 and $S^1 \times S^1$, then there is exactly one rational model of S , up to isomorphism. In other words, any two rational models of S are isomorphic.*

I. Biswas, J. Huisman and F. Mangolte [5], [29] prove infinite transitivity for this class of surfaces.

Theorem ([29, Theorem 1.4]). *Whenever X is a rational nonsingular compact connected real algebraic surface, the group $\text{Aut}(X)$ acts infinitely transitively on X .*

For non-connected real algebraic surfaces there is no two-transitivity by the reason of connectedness. Indeed, two points in the same connected component cannot be sent to a pair of points in different connected components. So it is quite natural to introduce the following notion.

Definition (Definition 6.2). By abuse of language, we say that the action of a group G on a set $Y = Y^1 \sqcup Y^2 \sqcup \dots \sqcup Y^s$ is *infinitely transitive on each connected component* if for every s -tuple (m_1, \dots, m_s) , it is transitive on $(m_1 + \dots + m_s)$ -tuples of the form $(P_1^1, \dots, P_{m_1}^1, P_1^2, \dots, P_{m_2}^2, \dots, P_1^s, \dots, P_{m_s}^s)$, where $P_j^i \in Y^i$ are pairwise distinct.

According to a recent work of J. Blanc and F. Mangolte, this property does not always hold for non-connected real algebraic surfaces. Let $\#M$ be the number of connected components of a compact manifold M .

Theorem ([6, Theorem 1]). *Let X be a nonsingular real projective surface. The group $\text{Aut}(X)$ is infinitely transitive on each connected component of X if and only if X is geometrically rational and $\#X \leq 3$.*

In the three component case, this theorem can be strengthened, see [6, Theorem 2].

Let us note that the transitivity in the above cases is obtained by means of the group $\text{Aut}(X)$ and not with $\text{SAut}(X)$. Meanwhile, it is unknown whether $\text{SAut}(X)$ acts transitively on these surfaces.

Results of Chapter 6

These results can be found in [3, 44]. We study infinitely transitive actions of the group $\text{SAut}(X)$ for varieties of arbitrary dimension over the ground field \mathbb{R} . First, under some mild restrictions on the suspension, we can generalize Theorem 5.30.

Theorem (Theorem 6.1). *Let Y be a flexible real algebraic variety. Suppose that the smooth locus Y_{reg} is connected and the special automorphism group $\text{SAut}(Y)$ acts m -transitively on Y_{reg} for every $m \in \mathbb{N}$. Consider the suspension $X = \text{Susp}(Y, f)$, where the function $f \in \mathbb{R}[Y]$ satisfies the condition $f(Y_{\text{reg}}) = \mathbb{R}$. Then the variety X is flexible and the special automorphism group $\text{SAut}(X)$ acts m -transitively on X_{reg} for every $m \in \mathbb{N}$.*

In the particular case of a suspension over an affine line, the result of Theorem 5.30 remains valid over any field of characteristic zero.

Theorem (Theorem 5.31). *Let \mathbb{k} be a field of characteristic zero. Given a polynomial $f \in \mathbb{k}[x]$ with $f(\mathbb{k}) = \mathbb{k}$, consider a surface $X \subseteq \mathbb{A}_{\mathbb{k}}^3$ with equation $f(x) - uv = 0$. Then X is flexible and the special automorphism group $\text{SAut}(X)$ acts m -transitively on X_{reg} for every $m \in \mathbb{N}$.*

These surfaces were characterized in abstract terms by T. Bandman and L. Makar-Limanov [4].

If X_{reg} is non-connected, then $\text{SAut}(X)$ is not even transitive on X_{reg} . Indeed, the action of $\text{SAut}(X)$ on X fixes each connected component of X : every special automorphism g admits a decomposition $\prod h_j(1)$, where each h_i is a one-parameter unipotent group. For any $x \in X$, the arc $t \rightarrow \prod h_i(t) \cdot x$ connects x to $g \cdot x$. So for non-connected real algebraic varieties we study infinite transitivity on each connected component, see Definition 6.2.

Let us consider an algebraic variety Y . Decompose $Y_{\text{reg}} = Y^1 \sqcup \dots \sqcup Y^s$ into connected components. For the suspension $X = \text{Susp}(Y, f)$ we have a splitting $X_{\text{reg}} = X^1 \sqcup \dots \sqcup X^{s'}$. It is easy to show that if two points x and x' are in the same connected component X^j , then their natural projections to Y are both in the same connected component Y^i of Y_{reg} . However, there can be more than one connected component of X_{reg} over one connected component of Y_{reg} . For example, if Y^i is compact and f is strictly positive on Y^i , then we have two branches of hyperbola

$uv = f(y)$ over each $y \in Y^i$. The union of these branches form two connected components of $\text{Susp}(Y^i, f)$. For such varieties we generalize Theorem 6.1 as follows (joint work with F. Mangolte [44]).

Theorem (Theorem 6.3). *Let Y be an affine algebraic variety defined over \mathbb{R} and let $f \in \mathbb{R}[Y]$. Assume that for each connected component Y^i of Y_{reg} , the dimension $\dim Y^i$ is at least 2 and f is non-constant on Y^i . If Y is flexible and the action of $\text{SAut}(Y)$ on Y_{reg} is infinitely transitive on each connected component, then the suspension $X = \text{Susp}(Y, f)$ is flexible and $\text{SAut}(X)$ acts on X_{reg} infinitely transitively on each connected component.*

This theorem is the main result of Chapter 6.

Part I

Normality of maximal torus orbits closures

Chapter 1

Hereditary normality and unimodular sets

Let, as before, G be a connected simply connected simple algebraic group which corresponds to an irreducible root system Φ . Fix a maximal torus $T \subset G$. The main goal is to classify all simple finite-dimensional rational G -modules V such that for any $v \in V$ the closure \overline{Tv} is normal. Obviously, this does not depend on choice of T . It is known that each simple module is uniquely (up to isomorphism) defined by its highest weight. In turn, every such weight can be represented as a nonnegative integer combination of fundamental weights. So we formulate the result in terms of root systems and highest weights. We enumerate fundamental weights as in [53, Chapter 4].

Theorem 1.1. *For the following types of simple algebraic groups and the corresponding modules, and for their dual modules, the closures of all maximal torus orbits are normal. In all other cases, the module contains a maximal torus orbit with nonnormal closure.*

<i>Root system</i>	<i>Highest weight</i>	<i>Checked in</i>
$A_n, n \geq 1$	π_1	Case 2.1
$A_n, n \geq 1$	$\pi_1 + \pi_n$	[51, 65, 66], Case 2.2
A_1	$3\pi_1$	Case 2.3
A_1	$4\pi_1$	Case 2.4
A_2	$2\pi_1$	Case 2.5
A_3	π_2	Case 2.7
A_4	π_2	Case 2.7
A_5	π_2	Case 2.7
A_5	π_3	Case 2.6
$B_n, n \geq 2$	π_1	Case 3.1

<i>Root system</i>	<i>Highest weight</i>	<i>Checked in</i>
B_2	π_2	Case 3.3
B_2	$2\pi_2$	Case 3.2
B_3	π_3	Case 3.3
B_4	π_4	Case 3.3
$C_n, n \geq 3$	π_1	Case 3.4
C_3	π_2	Case 3.5
C_4	π_2	Case 3.5
$D_n, n \geq 4$	π_1	Case 3.6
D_4	π_2	Case 3.7
D_4	π_3	Case 3.8
D_4	π_4	Case 3.8
D_5	π_4	Case 3.9
D_6	π_5	Case 3.10
D_6	π_6	Case 3.10
F_4	π_1	Case 4.1
G_2	π_1	Case 4.2

In this chapter we give some algebraic definitions and reformulate the problem in combinatorial terms. On one hand, we prove that the saturation property holds for each subset in the set of weights of the representations listed in Theorem 1.1. On the other hand, we produce non-saturated subsets in sets of weights for all other representations.

1.1 Normality of torus orbits closures

Let V be a rational T -module. Recall that $\Lambda = \Lambda(T)$ stands for its character lattice. With respect to the T -action, the module V can be decomposed:

$$V = \bigoplus_{\mu \in \Lambda} V_{\mu}, \quad \text{where } V_{\mu} = \{v \in V \mid tv = \mu(t)v \quad \forall t \in T\}.$$

We denote by $M(V) = \{\mu \in \Lambda \mid V_{\mu} \neq 0\}$ the set of weights of V . Each nonzero vector v in V has its *weight decomposition*

$$v = v_{\mu_1} + \cdots + v_{\mu_s}, \quad v_{\mu_i} \in V_{\mu_i}, \quad v_{\mu_i} \neq 0.$$

Let $M(v)$ stand for the set $\{\mu_1, \dots, \mu_s\}$. With these weights, we generate a semi-group $\mathbb{Z}_{\geq 0}(\mu_1, \dots, \mu_s)$, a sublattice $\mathbb{Z}(\mu_1, \dots, \mu_s)$ and a rational polyhedral cone $\mathbb{Q}_{\geq 0}(\mu_1, \dots, \mu_s)$ in the space $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 1.2. The set of points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ is called *saturated* if

$$\mathbb{Z}_{\geq 0}(\mu_1, \dots, \mu_s) = \mathbb{Z}(\mu_1, \dots, \mu_s) \cap \mathbb{Q}_{\geq 0}(\mu_1, \dots, \mu_s).$$

The set of points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ is called *hereditarily normal* if each its subset is saturated.

Recall that an irreducible affine algebraic variety X is called *normal* if its algebra of regular functions $\mathbb{k}[X]$ is integrally closed in its field of fractions. The following combinatorial criterion of normality of a T -orbit closure in a T -module [32, I, §1, Lemma 1] plays a key role.

Proposition 1.3. *Let V be a finite dimensional rational T -module and $v = v_{\mu_1} + \dots + v_{\mu_s}$ be the weight decomposition of a vector $v \in V$. The closure \overline{Tv} of the T -orbit of v is normal if and only if $M(v)$ is saturated.*

Corollary 1.4. *Let V be a finite dimensional rational T -module. The closures of all T -orbits in the module V are normal if and only if the set $M(V)$ is hereditarily normal.*

Remark 1.5. The weight system is multiplied by -1 while replacing a representation V of the torus T by its dual V^* . Hence the property of normality of all T -orbits closures is preserved.

1.2 Weight decomposition

Let G be a connected simply connected semisimple algebraic group. We choose a Borel subgroup B in G and the maximal torus $T \subset B$. Denote by Φ the root system of the Lie algebra $\text{Lie}(G)$ associated with the maximal torus T . Let Φ^+ be the set of positive roots corresponding to B , and let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be simple roots in Φ^+ . Denote by π_i the fundamental weight corresponding to the simple root α_i . It is well-known that the weights π_1, \dots, π_r form a basis of the character lattice $\Lambda = \Lambda(T)$ of the torus T . Since G is simply connected, this lattice is also the *weight lattice* of the root system Φ . The semigroup generated by fundamental weights coincides with the semigroup of dominant weights Λ_+ . The subgroup of Λ generated by the root system Φ is called the *root lattice*, we denote it by Ξ . It is known that Ξ is the sublattice of Λ of finite index, and $\alpha_1, \dots, \alpha_r$ form a basis of Ξ . For convenience, we consider weights $\mu \in \Lambda$ as points of the rational vector space $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. If the weight lattice Λ is realized in the space \mathbb{Q}^n , then we denote by $\varepsilon_1, \dots, \varepsilon_n$ the standard basis in \mathbb{Q}^n . In all the cases, except A_n , we work in this standard basis.

Let W be the Weyl group of the root system Φ . Then W can be seen as a finite group of linear isometries of $\Lambda_{\mathbb{Q}}$ generated by reflections s_{α} , where α is a

root, see [30]. Recall that the reflection s_α can be given as $s_\alpha: \beta \mapsto \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha$, here (\cdot, \cdot) denote the W -invariant scalar product. Note that for any irreducible root system Φ the group W acts transitively on the set of roots of equal length, see [10, Proposition 6.1.11]

Let $V(\lambda)$ be a simple G -module with the highest weight $\lambda \in \Lambda_+$. Recall the following description of the set of T -weights of the module $V(\lambda)$. The *weight polytope* $P(\lambda)$ of the module $V(\lambda)$ is the convex hull $\text{conv}\{w\lambda \mid w \in W\}$ of the W -orbit of the point λ in $\Lambda_{\mathbb{Q}}$. Then

$$M(\lambda) = (\lambda + \Xi) \cap P(\lambda),$$

see [25, Theorem 14.18] and [11, Exercises to Ch. VIII §7]. There is a partial order on the vector space $\Lambda_{\mathbb{Q}}$: $\lambda \succeq \mu$ if and only if $\lambda - \mu$ is a linear combination of simple roots with nonnegative integer coefficients. The highest weight λ of the representation is the maximal element of $M(\lambda)$ with respect to this order.

Remark 1.6. Actually, we could start with a non-dominant weight λ in this construction. Then the formula $M(\lambda) = (\lambda + \Xi) \cap P(\lambda)$ gives the set of weights of a representation such that its highest weight belongs to the W -orbit of the vector λ .

In our situation, Corollary 1.4 can be reformulated in the following way:

Proposition 1.7. *Let $V(\lambda)$ be a simple G -module with the highest weight λ . Then the closure of each T -orbit in $V(\lambda)$ is normal if and only if $M(\lambda)$ is hereditarily normal.*

In the proof of Theorem 1.1 we distinguish positive cases, where we have to prove that certain sets $M(\lambda)$ are hereditarily normal, and negative cases, where it suffices to construct one nonsaturated subset in $M(\lambda)$. In general, the negative cases are easier than the positive ones. The following lemma and remark show how to produce nonsaturated subsets.

Lemma 1.8. *Let $\lambda, \lambda' \in \Lambda_+$. Suppose that $\lambda \succeq \lambda'$, then $M(\lambda) \supseteq M(\lambda')$.*

Proof. Use the criterion from [11, Exercice 1 to Section VIII, §7]: the weight $\lambda' \in \lambda + \Xi$ belongs to $M(\lambda)$ if and only if for all $w \in W$ the weight $\lambda - w\lambda'$ is a linear combination of simple roots with nonnegative integer coefficients. First notice that under our assumptions λ' belongs to $M(\lambda)$. Indeed, for $w = e$ the weight $\lambda - \lambda'$ is a linear combination of simple roots with nonnegative integer coefficients due to the assumption, and for $w \neq e$ it is known that $w\lambda' = \lambda' - \mu$, where μ is a sum of positive roots. Hence $\lambda - w\lambda' = \lambda - \lambda' + \mu$ is a linear combinations of simple roots with nonnegative integer coefficients. It means that $\lambda' \in M(\lambda)$, and all points of the form $w\lambda'$, where $w \in W$, belong to $M(\lambda)$. Using convexity, we obtain that $M(\lambda') \subseteq M(\lambda)$. \square

Remark 1.9. Take $\lambda' \in \Lambda_+$ and assume that $M(\lambda')$ is not hereditarily normal. Then for all $\lambda \in \Lambda_+$ such that $\lambda \succeq \lambda'$ the set $M(\lambda)$ is not hereditarily normal.

However, there are some negative cases which present difficulties. In the case A_n we need to treat infinitely many cases of fundamental representations π_k , where $k, n \in \mathbb{Z}_{\geq 0}$, $1 \leq k \leq n$. Remark 1.9 is not applicable here. So we use a procedure which is based on the descent as in Euclidean algorithm.

1.3 Nonsaturated subsets

The proof of the following lemma can be found in [51].

Lemma 1.10. *Let M be a finite set of vectors in \mathbb{Q}^n .*

- (i) *If M is linearly independent, then M is saturated.*
- (ii) *If M is not saturated and contains both vectors v and $-v$, then either $M \setminus \{v\}$ or $M \setminus \{-v\}$ is not saturated.*
- (iii) *Let $v \in \mathbb{Q}_{\geq 0}(M)$. Then there exists a linearly independent subset $M' \subseteq M$ such that $v \in \mathbb{Q}_{\geq 0}(M')$.*

We refer to a nonsaturated subset as an *NSS*. By an *extended nonsaturated subset* we mean a nonsaturated subset $\{v_1, \dots, v_r\}$ augmented by a vector v_0 such that

- $v_0 \in (\mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geq 0}(v_1, v_2, \dots, v_r)) \setminus \mathbb{Z}_{\geq 0}(v_1, v_2, \dots, v_r)$,
- there exists a $\mathbb{Q}_{\geq 0}$ -representation

$$v_0 = q_1 v_{i_1} + \dots + q_s v_{i_s}, \quad v_{i_j} \in \{v_1, v_2, \dots, v_r\}$$

with linearly independent vectors v_{i_1}, \dots, v_{i_s} and coefficients $q_i \in [0, 1[$. These subsets will be called *ENSSs* and will be denoted by $\{v_0; v_1, \dots, v_r\}$.

The fractional part of a real value q is denoted by $\{q\}$, and the integer part by $\lfloor q \rfloor$. We often say "points" instead of "elements of M ". The following lemma is useful in proofs by contradiction.

Lemma 1.11. *Suppose that the set $M = \{v_1, \dots, v_r\}$ is not saturated. Then there exists a vector v_0 such that $\{v_0; v_1, \dots, v_r\}$ is an *ENSS*.*

Proof. Consider any vector $v_0 \in (\mathbb{Z}(M) \cap \mathbb{Q}_{\geq 0}(M)) \setminus \mathbb{Z}_{\geq 0}(M)$, and the corresponding $\mathbb{Q}_{\geq 0}$ -combination $v_0 = q_1 v_1 + \dots + q_r v_r$. By Lemma 1.10(iii) there exists a linearly independent subset $\{v_{i_1}, \dots, v_{i_s}\} \subseteq \{v_1, \dots, v_r\}$ and the collection of $\mathbb{Q}_{\geq 0}$ -coefficients q'_j such that $v_0 = q'_1 v_{i_1} + \dots + q'_s v_{i_s}$. If some $q'_j \geq 1$, consider

another vector $v'_0 = v_0 - [q'_1]v_{i_1} - \cdots - [q'_s]v_{i_s}$ instead of v_0 . It is easy to see that v'_0 also belongs to $\mathbb{Z}(v_1, \dots, v_r)$ and to $\mathbb{Q}_{\geq 0}(v_1, \dots, v_r)$, and does not belong to $\mathbb{Z}_{\geq 0}(v_1, \dots, v_r)$. However all the coefficients of the new $\mathbb{Q}_{\geq 0}$ -combination belong to the interval $[0, 1[$. This means that $\{v'_0; v_1, \dots, v_r\}$ is an ENSS. \square

To prove that a certain set is an ENSS, we often use the following tool. Let v_0, v_1, \dots, v_r be some vectors in a rational vector space L , and let f be a linear function on L . We call f a *discriminating linear function* for the collection $\{v_0; v_1, \dots, v_r\}$ if the value $f(v_0)$ cannot be represented as a linear combination of the values $f(v_1), \dots, f(v_r)$ with nonnegative integer coefficients. If v_0 belongs to $\mathbb{Z}(v_1, v_2, \dots, v_r) \cap \mathbb{Q}_{\geq 0}(v_1, v_2, \dots, v_r)$ and can be represented as a $\mathbb{Q}_{\geq 0}$ -combination of linearly independent vectors v_1, \dots, v_r with coefficients from the interval $[0, 1[$, then the existence of a discriminating function guarantees that $\{v_0; v_1, \dots, v_r\}$ is an ENSS.

1.4 Unimodular and almost unimodular sets

Assume that the set of vectors $M \subset \mathbb{Q}^n$ has rank d , $d \leq n$, and $L = \langle v \mid v \in M \rangle$ is the linear span of vectors from M . The set M is called *unimodular* if for any linearly independent vectors $v_1, \dots, v_d \in M$ the value of the d -dimensional volume $\text{vol}_d(v_1, v_2, \dots, v_d)$ has constant absolute value. If a basis in L is fixed, then the condition above is equivalent to the fact that the absolute values of all nonzero determinants $|\det(v_1, v_2, \dots, v_d)|$, $v_1, v_2, \dots, v_d \in M$, computed in this basis, are equal.

If the set M is unimodular, then its intersection with any subspace $L_1 \subset L$ is also unimodular. This is clear because we can choose in L any basis compatible with L_1 .

Theorem 1.12 ([65, Theorem 3.5]). *Any unimodular set of vectors M is hereditarily normal.*

Proof. Suppose on the contrary that there exists an ENSS $\{v_0; v_1, \dots, v_r\}$ in M . Denote by d' the dimension of the subspace $L_1 = \langle v_1, \dots, v_r \rangle$, $d' \leq d$. The set $\{v_1, \dots, v_r\}$ is unimodular. According to the definition of an ENSS, vectors in the corresponding $\mathbb{Q}_{\geq 0}$ -combination for v_0 are linearly independent. Complete them with elements of M to a basis $v_1, v_2, \dots, v_{d'}$ of the space L_1 . Due to unimodularity and Cramer's formulae, the coordinates of each v_i , $i > d'$, in the basis $v_1, v_2, \dots, v_{d'}$ are equal to 0, 1, and -1 . Then, using the initial \mathbb{Z} -combination for v_0 and substituting the decompositions for v_i in it, we deduce that v_0 has integer coordinates in the basis $v_1, \dots, v_{d'}$. Since v_0 has a unique decomposition in any fixed basis, all the coefficients of the initial $\mathbb{Q}_{\geq 0}$ -combination are integers. \square

We say that a subset $M \subset \mathbb{Q}^n$ of rank d is *almost unimodular* if we can choose a subset $\{v_1, v_2, \dots, v_d\} \subseteq M$ such that $\text{vol}_d(v_1, v_2, \dots, v_d) = m$, and for any other vector $w \in M$ and for each i the value

$$\text{vol}_d(v_1, v_2, \dots, \widehat{v}_i, \dots, v_d, w)$$

is divisible by m . If one fixes a basis in the linear space $\langle M \rangle$, then this property can be checked by comparing the values of the corresponding determinants in the given basis instead of computing vol_d s. The value $m = \det(v_1, v_2, \dots, v_d)$ is called the *volume* of an almost unimodular subset. In invariant terms, M is almost unimodular if and only if there exists a base of the lattice $\mathbb{Z}(M)$ formed by some elements of M .

Lemma 1.13. *Let M be an almost unimodular set of volume m and of rank d . Then all the determinants in M are divisible by m .*

Proof. Expand each $w \in M$ in the basis (v_1, v_2, \dots, v_d) . By Cramer's formulae, they all have integer coordinates:

$$\text{if } w = a_1v_1 + \dots + a_dv_d, \quad \text{then } a_i = \frac{\det(v_1, v_2, \dots, \widehat{v}_i, w, \dots, v_d)}{\det(v_1, v_2, \dots, v_d)},$$

where all $a_i \in \mathbb{Z}$ because of almost unimodularity. Furthermore, for any vectors $w_1, \dots, w_d \in M$ we have $\det(w_1, \dots, w_d) = \det A \cdot \det(v_1, \dots, v_d)$, where A is an integer matrix, which expresses the set of vectors w_1, \dots, w_d in the basis (v_1, v_2, \dots, v_d) . Since $\det A \in \mathbb{Z}$, the value $\det(w_1, \dots, w_d)$ is divisible by m , and the proof is completed. \square

To treat spinor modules in D_5 and D_6 cases, we need the following observations.

Lemma 1.14. *Consider an almost unimodular set M such that all determinants in M are contained in the set $m \cdot \{1, a_1, \dots, a_k\}$, and for some vectors $w_1, \dots, w_d \in M$ the value $\det(w_1, \dots, w_d)$ equals am . Then, if we decompose any vector $w \in M$ in the basis w_1, \dots, w_d , the coefficients of this decomposition belong to the set $\{\pm 1/a, \pm a_1/a, \dots, \pm a_k/a\}$.*

Proof. The proof follows directly from Cramer's formulae, see above. \square

By a *primitive subset* $\{v_1, \dots, v_d\}$ in an almost unimodular set of volume m we mean any d -element subset such that its determinant equals $\pm m$. In fact, the proof of Lemma 1.13 says that for any primitive subset $\{v_1, \dots, v_d\} \subseteq M$ the set M belongs to $\mathbb{Z}(v_1, \dots, v_d)$.

Example. Consider the set M containing the 16 points $\{(\pm 1, \pm 1, \pm 1, \pm 1)\}$. It is easy to see that determinants of all 4-tuples equal 0, 8, or 16. This means that M is almost unimodular.

Lemma 1.15. *Suppose that an almost unimodular set M of rank d and of volume m is not hereditarily normal, and $\{v_0; v_1, \dots, v_r\}$ is an ENSS. Assume that the corresponding $\mathbb{Q}_{\geq 0}$ -combination for v_0 involves only the linearly independent vectors v_1, \dots, v_d .*

- (i) *If $d' = \text{rk} \langle v_1, \dots, v_d \rangle = d$, then $|\text{vol}_d(v_1, \dots, v_d)| \neq m$.*
- (ii) *If $d' < d$, then for any vectors $w_{d'+1}, \dots, w_d \in M$ which are linearly independent with $v_1, \dots, v_{d'}$, the value $|\text{vol}_d(v_1, \dots, v_{d'}, w_{d'+1}, \dots, w_d)|$ is different from m .*

Proof. (i) If $|\text{vol}_d(v_1, \dots, v_d)| = m$, then by Lemma 1.13 the decomposition of v_0 in the basis v_1, \dots, v_d has integer coefficients. Since v_1, \dots, v_d are linearly independent, this \mathbb{Z} -combination coincides with the initial $\mathbb{Q}_{\geq 0}$ -combination. This contradicts the fact that it is an ENSS.

(ii) We may suppose that the vectors $w_{d'+1}, \dots, w_d$ enter in the initial $\mathbb{Q}_{\geq 0}$ -combination for v_0 with zero coefficients, and then use the reasoning of the previous part. \square

Lemma 1.16. *Consider an almost unimodular set M of rank d which is not hereditarily normal and such that all values of determinants in it equal 0 , $\pm m$, or $\pm 2m$. Let $\{v_0; v_1, \dots, v_r\}$ be an ENSS such that the corresponding $\mathbb{Q}_{\geq 0}$ -combination for v_0 contains exactly the vectors v_1, \dots, v_l . Then there exists a vector v'_0 and a subset $\{v_{i_1}, \dots, v_{i_s}\} \subseteq \{v_1, \dots, v_l\}$ such that $\{v'_0; v_1, \dots, v_r\}$ is an ENSS and all the coefficients in the corresponding $\mathbb{Q}_{\geq 0}$ -combination $v'_0 = q'_{i_1} v_{i_1} + \dots + q'_{i_s} v_{i_s}$ are equal to 0 or $\frac{1}{2}$.*

Proof. Assume that the set $\{v_1, \dots, v_r\}$ has rank d' . Let us first show that if $d' < d$, then in $\langle v_1, \dots, v_r \rangle$ all $\text{vol}_{d'}(w_1, \dots, w_{d'}) \in \{0, \pm m', \pm 2m'\}$, $w_i \in \{v_1, \dots, v_r\}$ for some m' (if $d = d'$, this is a tautology). Choose a basis of vectors $w_i \in M$ in $L = \langle M \rangle$ compatible with $L_1 = \langle v_1, \dots, v_r \rangle$. Let $w_{d'+1}, \dots, w_d$ be vectors of this basis belonging to $L \setminus L_1$. Consider all possible values

$$\{\text{vol}_d(u_1, \dots, u_{d'}, w_{d'+1}, \dots, w_d) \mid u_1, \dots, u_{d'} \in L_1\}.$$

In the basis constructed above the corresponding matrices have the form $\begin{pmatrix} A & B \\ 0 & E \end{pmatrix}$, where E stands for the identity matrix. This implies

$$\det A \in \{\pm m, \pm 2m\}.$$

Now we can omit the points from $M \setminus L_1$ and suppose that $d' = d$, and $m' = m$. Moreover, vectors v_1, \dots, v_l are linearly independent. If $l < d$, then

augment $\{v_1, \dots, v_l\}$ by new $(d-l)$ vectors v_{l+1}, \dots, v_d , linearly independent with $\{v_1, \dots, v_l\}$, and assume that v_{l+1}, \dots, v_d appear in the initial $\mathbb{Q}_{\geq 0}$ -combination for v_0 with zero coefficients. If $\text{vol}(v_1, \dots, v_d) = m$, then by Lemma 1.15 the set $\{v_0; v_1, \dots, v_r\}$ is not an ENSS. If $\text{vol}(v_1, \dots, v_d) = 2m$, then by Lemma 1.14 all the vectors v_i for $i > d$ can be decomposed in the basis v_1, v_2, \dots, v_d with integer or half-integer coefficients. Substituting these decompositions in the initial \mathbb{Z} -combination for v_0 , we obtain that the initial $\mathbb{Q}_{\geq 0}$ -combination has integer or half-integer coefficients (the last $d-l$ of them being zero). If some of them exceed 1, then, as in Lemma 1.11, we replace the vector v_0 by v'_0 and obtain that all coefficients are equal to 0 or $\frac{1}{2}$. \square

As it was mentioned in the Introduction, one more method to check hereditary normality can be formulated as follows. Choose a basis such that for any $v \in M$, the coordinates of v with respect to this basis are all integers. Represent each point as a column vector of its coordinates in this basis. Let $K = K(M)$ be an integer $n \times r$ matrix formed by all these column vectors. The below algorithm deals with the monomial algebras.

Definition 1.17. To each column $K_i = (k_{1i}, \dots, k_{ni})^T$ we associate a Laurent monomial $t^{K_i} = t_1^{k_{1i}} \dots t_n^{k_{ni}}$. The *toric ideal* I_K associated with K is the kernel of the \mathbb{k} -algebra homomorphism

$$\mathbb{k}[x_1, x_2, \dots, x_r] \rightarrow \mathbb{k}[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}], \quad x_i \mapsto t^{K_i}.$$

Definition 1.18. Suppose that u_+ and u_- are two vectors in \mathbb{Z}_+^n with disjoint supports, and let $f = x^{u_+} - x^{u_-} \in I_K$. We say that f is a *circuit* in I_K if the following two conditions hold:

- the coordinates of both u_+ and u_- have no common divisor greater than 1;
- the set of variables which actually occur in f is minimal with respect to inclusion among all the binomials of I_K .

Theorem 1.19 ([65, Thm. 3.8]). *A set of points M is hereditarily normal if and only if every circuit in $I_{K(M)}$ has at least one square-free monomial.*

1.5 Normality of a generic T -orbit closure

Answering a question posed by V. Batyrev, we study here the generic torus orbit closures. By a generic orbit we mean below the orbit of a vector v such that all weight components of v are nonzero.

We need the following simple lemma.

Lemma 1.20. *Let L be a vector space over \mathbb{Q} . Suppose that the set $S = \{v_1, \dots, v_n\} \in L$ generates L as a cone, i.e. $\mathbb{Q}_{\geq 0}(v_1, \dots, v_n) = L$. Then the set S is saturated.*

Proof. For each vector $v_i \in S$, there exists a $\mathbb{Q}_{\geq 0}$ -combination of some vectors from S , resulting in the vector $-v_i$. Multiplying this combination by the LCM of the denominators of the coefficients, we obtain that

$$-d_i v_i \in \mathbb{Z}_{\geq 0}(v_1, \dots, v_n)$$

for some integer d_i . Denote $D = d_1 \dots d_n$. We see that

$$-Dv_i \in \mathbb{Z}_{\geq 0}(v_1, \dots, v_n)$$

for all i , hence $(-D)\mathbb{Z}_{\geq 0}(v_1, \dots, v_n) \subseteq \mathbb{Z}_{\geq 0}(v_1, \dots, v_n)$. Consequently,

$$\mathbb{Z}(v_1, \dots, v_n) = (-D)\mathbb{Z}_{\geq 0}(v_1, \dots, v_n) + \mathbb{Z}_{\geq 0}(v_1, \dots, v_n) \subseteq \mathbb{Z}_{\geq 0}(v_1, \dots, v_n).$$

It follows immediately that S is saturated. \square

Now we can prove the following statement.

Proposition 1.21. *Let G be a semisimple algebraic group, and $T \subset G$ be a maximal torus. Then the closure of a generic T -orbit in any rational finite-dimensional G -module is normal.*

Proof. For the generic vector $v \in V(\lambda)$ all the components of its weight decomposition are nonzero, hence the question of normality of the closure for its T -orbit is equivalent to the question of saturatedness for the whole set $M(\lambda)$. First we show that for any $\lambda \in \Lambda$ the cone $\mathbb{Q}_{\geq 0}(M(\lambda))$ is a linear space. Indeed, the sum $\sum_{\mu \in M(\lambda)} \mu$ is W -invariant, hence it is orthogonal to all roots. From the fact that G is semisimple we conclude that $\sum_{\mu \in M(\lambda)} \mu = 0$, it means that $\mathbb{Q}_{\geq 0}(M(\lambda))$ is a linear space. Finally by Lemma 1.20 the set $M(\lambda)$ is saturated. \square

This fact has also a geometric explanation, as the rapporteur has kindly noticed. Indeed, given a T -action on a vector space V and a point $v \in V$, if $0 \in M(v)$, then the T -orbit Tv is closed, see Proposition 6.15 in [59]. It means that each point in $\overline{Tv} = Tv$ is smooth, hence Tv is normal.

Chapter 2

The root system A_n

Let $G = SL(n)$ and Φ of type A_{n-1} . The root system A_{n-1} , where $n \geq 2$, is formed by vectors $\{\pm(\varepsilon_i - \varepsilon_j) | 1 \leq i < j \leq n\} \cup \{0\}$.

Now we pass to another notation. Consider

$$e_i = \left(-\frac{1}{n}, -\frac{1}{n}, \dots, -\frac{1}{n}, \underset{i\text{th place}}{\frac{n-1}{n}}, -\frac{1}{n}, \dots, -\frac{1}{n} \right).$$

Notice that e_1, e_2, \dots, e_n (further referred to as a *quasi-basis*) satisfy the only linear relation

$$e_1 + e_2 + \dots + e_n = 0. \quad (*)$$

In this quasi-basis we still have $A_{n-1} = \{\pm(e_i - e_j) | 1 \leq i < j \leq n\} \cup \{0\}$. The root lattice Ξ here is a lattice generated by the vectors $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$. Due to the ambiguity of notation,

$$\Xi = \{a_1 e_1 + a_2 e_2 + \dots + a_n e_n : n \mid (a_1 + a_2 + \dots + a_n)\},$$

where \mid stands for divisibility, i.e. $n \mid m \iff \exists z \in \mathbb{Z}, m = nz$. With respect to the simple roots $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$, the fundamental weights can be written in the form

$$\pi_k = e_1 + \dots + e_k.$$

Identify Λ with the \mathbb{Z} -lattice generated by e_1, e_2, \dots, e_n . A weight $\chi_a = a_1 e_1 + \dots + a_n e_n$ is dominant if and only if $a_1 \geq a_2 \geq \dots \geq a_n$.

For a positive integer $s \mid n$, define

$$\mathbb{Z}_{\equiv 0(s)}(e_1, \dots, e_n) = \left\{ \sum_{i=1}^n x_i e_i : x_i \in \mathbb{Z}, s \mid \sum_{i=1}^n x_i \right\}.$$

In this notation the root lattice Ξ coincides with $\mathbb{Z}_{\equiv 0(n)}(e_1, \dots, e_n)$.

The role of the Weyl group W is played here by the permutation group S_n , which acts on Λ by permutations of coordinates.

2.1 Positive results

In this section we prove that certain weight systems are hereditarily normal.

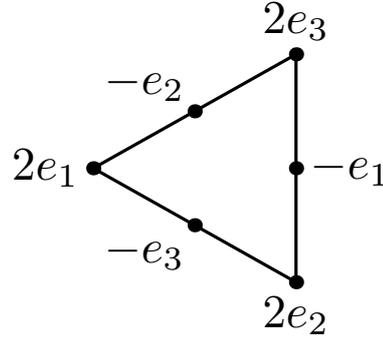
Case 2.1. The tautological representation. Its highest weight equals $\pi_1 = e_1$. The set $M(e_1)$ equals $\{e_1, \dots, e_n\}$. It is unimodular, hence hereditarily normal.

Case 2.2. The adjoint representation. Its highest weight λ is equal to $\pi_1 + \pi_{n-1} = e_1 - e_n$. Acting by $W = S_n$, we get all vectors of the form $e_i - e_j$. Taking the convex hull adds only $\bar{0}$ to this set. We get $M(\lambda) = \{0\} \cup \{e_i - e_j : 1 \leq i, j \leq n, i \neq j\}$. It was proved in [65, Ex. 3.7] or [51, Theorem 1] that this set is hereditarily normal.

Case 2.3. The representation of $SL(2)$ with the highest weight $3\pi_1$. Here we have $M(\lambda) = \{(\frac{3}{2}, -\frac{3}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{3}{2}, \frac{3}{2})\}$ (in the usual basis). It easily follows from Lemma 1.20 that the set $M(\lambda)$ is hereditarily normal.

Case 2.4. The representation of $SL(2)$ with the highest weight $4\pi_1$. One has to verify that the set $M(\lambda) = \{(2, -2), (1, -1), (0, 0), (-1, 1), (-2, 2)\}$ is hereditarily normal. According to Lemma 1.20, it can be done with a short case-by-case consideration.

Case 2.5. The representation of $SL(3)$ with the highest weight $2\pi_1 = 2e_1$. All the weights of this representation are pointed in the figure below.



After an appropriate change of basis, $K(M(\lambda)) = \begin{pmatrix} 1 & 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 0 & -2 & 2 \end{pmatrix}$.

The set of all circuits of $I_{K(M(\lambda))}$ is $\{x_1x_2x_3 - 1, x_4x_5x_6 - 1, x_1^2x_4 - 1, x_2^2x_5 - 1, x_3^2x_6 - 1, x_1^2x_2^2 - x_6, x_1^2x_3^2 - x_5, x_2^2x_3^2 - x_4, x_1^2 - x_4x_5, x_2^2 - x_4x_6, x_3^2 - x_5x_6\}$. By Theorem 1.19, it is hereditarily normal.

Case 2.6. The representation of $SL(6)$ with the highest weight π_3 . The highest weight λ equals $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$,

$$M(\lambda) = \left\{ \frac{1}{2}(\varepsilon_1, \dots, \varepsilon_6) : \varepsilon_i = \pm 1, \sum \varepsilon_i = 0 \right\}.$$

The matrix K formed by elements of $M(\lambda)$ in the standard basis is $(K_1 \mid -K_1)$, where

$$K_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This matrix is of rank 5 because $\dim \mathbb{Q}(M(\lambda)) = 5$. Unimodularity of the set of columns of K is equivalent to the same property of K_1 . To check this, we can exclude the redundant first row, and calculate the values of all 5×5 -minors of K_2 ,

$$K_2 = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

An easy check in Maple 7 shows that K_2 is unimodular, all its non-zero minors are equal to ± 16 , hence $M(\lambda)$ is hereditarily normal.

Case 2.7. The representations of $SL(4)$, $SL(5)$ and $SL(6)$ with the highest weight $\pi_2 = e_1 + e_2$. The set of weights $M(\lambda)$ is equal to $\{e_i + e_j : 1 \leq i < j \leq n\}$, where $n = 4, 5, 6$. Suppose that there exists an ENSS $\{w; v_1, \dots, v_m\}$, $v_i \in M(\lambda)$.

Let Γ be the graph on n vertices associated with $\{v_1, \dots, v_m\}$ in the following way. Each vector $v_s = e_{i_s} + e_{j_s}$ gives rise to an edge joining vertices i_s and j_s . Construct a subgraph $\Gamma' \subset \Gamma$: take all the vertices of Γ and all the edges of Γ entering into the \mathbb{Q}_+ -combination above with nonzero coefficients. Write the coefficients of the \mathbb{Q}_+ -combination at the edges of Γ' . The further proof consists of a search of all possible graphs Γ' . The following observations simplify the search.

(0.1) The number of edges in each connected component of Γ' is not greater than the number of vertices. Otherwise, the vectors corresponding to the edges of this component are linearly dependent.

(0.2) The number of edges in Γ' is less than the number of vertices (it follows from (*) that the dimension of the enveloping space equals $n - 1$).

(0.3) The graph Γ' does not contain even cycles. It follows from the fact that the edges of an even cycle are linearly dependent: their alternating sum is zero.

(0.4) It follows from (0.1) that each connected component of Γ' either is a tree or contains exactly one cycle. In the second case it follows from (0.3) that this cycle is odd.

(0.5) It follows from (0.2) that Γ' has a vertex of degree 0 or 1.

At each vertex, count the sum of all coefficients on the incident edges, and for each sum take its fractional part. All these fractional parts are equal due to the fact that all the sums in vertices (they equal the coordinates of v) become integer after subtracting (*) with an appropriate coefficient. Now we conclude that

(0.6) Γ' does not contain vertices of degree 0 and 1 simultaneously: if it does, the fractional parts of the sums in vertices are all equal to 0, but in the terminal vertex this sum has only one summand and is not an integer. We consider these two cases independently.

Case 1. Graph Γ' has a vertex of degree 0.

(1.1) As follows from (0.6), any other connected component of Γ' is either a point or has no terminal vertices. Moreover, (0.4) implies that it is an odd cycle.

(1.2) We have $n \leq 6$, consequently, the number of edges in Γ' is ≤ 5 , but any odd cycle has ≥ 3 edges, and we yield that Γ' has at most one cycle.

Fulfil an exhaustive search within all graphs Γ' having a vertex of degree 0:

- $n = 4$, graph is a cycle of length 3 and a point,
- $n = 5$, graph is a cycle of length 3 and two points,
- $n = 6$, graph is a cycle of length 3 and three points,
- $n = 6$, graph is a cycle of length 5 and a point.

The only possible \mathbb{Q}_+ -combination in these cases is $\frac{1}{2}(v_1 + \dots + v_s)$. This means that $v = e_1 + e_2 + \dots + e_s$, $s \in \{3, 5\}$, e_1, \dots, e_s correspond to the vertices of the cycle. But it does not lie in $\mathbb{Z}(v_1, \dots, v_m)$ when n is even. When $n = 5$, consider also the graph Γ . Since v is a \mathbb{Z} -combination of the edges of Γ , Γ has more than 3 edges: $\Gamma \supset \Gamma'$, $\Gamma \neq \Gamma'$ and Γ' has 3 edges. In the representation $e_1 + e_2 + \dots + e_s$ the sum of coefficients of v is odd, hence we should apply (*) to the existing \mathbb{Z} -combination to get the representation $e_1 + e_2 + \dots + e_s$. For this purpose the edges of Γ should touch all the vertices of Γ (we name this property (**)).

In Fig. 1 the graph Γ' is drawn. To satisfy (**), Γ should contain at least the following edges (up to symmetry): see Fig. 2, 3 or 4. The vertices corresponding to e_i are called V_i . But in all cases we get a contradiction, since $e_1 + e_2 + e_3$ is already a \mathbb{Z}_+ -combination:

- in Fig. 2: $e_1 + e_2 + e_3 = V_1V_2 + V_2V_3 + V_1V_3 + V_4V_5$,
- in Fig. 3: $e_1 + e_2 + e_3 = V_4V_2 + V_2V_1 + V_1V_3 + V_3V_5$,
- in Fig. 4: $e_1 + e_2 + e_3 = V_4V_2 + V_2V_5 + 2V_1V_3$.

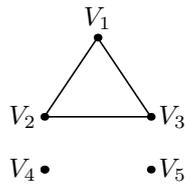


Fig. 1

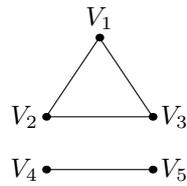


Fig. 2

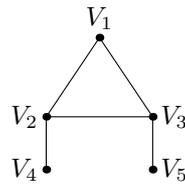


Fig. 3

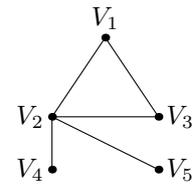


Fig. 4

We have shown that the graph Γ does not provide an NSS if Γ' has a vertex of degree 0.

Case 2. The graph Γ' has a vertex X of degree 1 and no vertex of degree 0. Let XY be an edge incident to X . We need to subtract $(*)$ with the same multiplicity as at XY . It follows that:

(2.1) Since X is a terminal vertex of Γ' , either XY is a connected component of Γ' or the degree of Y is ≥ 3 . Indeed, suppose that the degree of Y is 2. Let YZ be the second edge incident to Y , and let q be the value written at YZ . After subtracting $(*)$ the coefficient at Y becomes equal to q , but it should be integer, and we know that $q \in]0, 1[$. This is a contradiction.

Find all possible connected components of Γ' .

- On 2 vertices:
- On 3 vertices:
- On 4 vertices: and

If Γ' has a connected component on 5 or 6 vertices, then this component coincides with Γ' . Using this observation together with (0.2), we obtain that Γ' is a tree. Taking into account (2.1), it remains to consider only the following trees: , and . But the edges of are linearly dependent (when $n = 6$, one should sum all the thin edges, then subtract the thick one, and obtain $(*)$). Therefore, this graph should not be considered. The result is

- On 5 vertices:
- On 6 vertices:

Fulfil an exhaustive search within all graphs Γ' on n vertices satisfying all the conditions above. In the case when one of the connected components of Γ' is a claw (i.e., all the edges are incident to one vertex) with at least 4 vertices, its central vertex will correspond to e_1 (it is easy to see that Γ' cannot have more than one claw).

n	Splitting into connected components	Admissible graphs
4	$2 + 2$	
4	4	or
5	$2 + 3$	
5	5	
6	$2 + 2 + 2$	
6	$2 + 4$	or
6	6	

The graphs and do not satisfy our conditions: their edges are linearly dependent.

If we start with , we can obtain only e_1 as the \mathbb{Q}_+ -combination: all the three edges should appear in the \mathbb{Q}_+ -combination with the same coefficient, say a , $a \in]0, 1[$. We sum these three vectors, obtain $3ae_1 + ae_2 + ae_3 + ae_4$, and subtract (*) with an appropriate coefficient. Finally we obtain $2ae_1$. In this notation it already has integer coordinates (equal to zero), this means that all the other coordinates, $2a$ among them, should be integers, $a = \frac{1}{2}$, $v = e_1$. But v cannot be obtained as a \mathbb{Z} -combination of vectors of type $e_i + e_j$. Indeed, each v_i has an even sum of coordinates, n is even, subtracting (*) with an integer coefficient does not change parity of the sum of coordinates, and this proves that any vector from $\mathbb{Z}\{v_i\}_{i=1}^m$ has even sum of coordinates.

The edges of the graph are linearly dependent (here $n = 5$) because (2·first edge + the sum of the edges of the cycle) = 0.

The graph : using similar reasoning, $v = e_1$ or $2e_1$. But there exists an edge in $\Gamma \setminus \Gamma'$, hence e_1 is a \mathbb{Z}_+ -combination of the edges of Γ : take the sum of thick edges of

The edges of and are linearly dependent.

In the graph the vector v may be equal only to e_1 , but e_1 can not be obtained as a \mathbb{Z} -combination: 6 is even, and the sum of coordinates of e_1 is odd.

In the graph the vector v has to be proportional to e_1 , moreover, the coefficient should be even (we use the reasoning as above, from the fact that 6 is even it follows that the sum of coordinates is even for any vector from $\mathbb{Z}(v_1, \dots, v_m)$). But if we add any edge to this set, $2e_1$ will be obtained as a \mathbb{Z}_+ -combination: take the sum of thick edges of

All the cases are considered, and this completes the proof.

2.2 Negative results

Let λ be a dominant weight not listed in Theorem 1.1. One has to construct an NSS in $M(\lambda)$. There are two possibilities for λ : either the absolute values of all its usual coordinates are < 1 , or λ has a coordinate with the absolute value ≥ 1 . Speaking informally, the second case is practically always the consequence of the first one (Lemma 1.8), but the NSS in the first case is constructed recursively and its capacity increases when n increases. The construction of the second case gives an NSS of only 4 vectors for any n .

By x_i we denote the i th quasi-coordinate of a vector, if its quasi-basis representation is fixed. Note that a discriminating function $f = a_1x_1 + \dots + a_nx_n$ in quasi-basis is well-defined if and only if $a_1 + \dots + a_n = 0$ (for its definition, see Section 1.3).

2.2.1 The fundamental weights

In this case λ equals

$$\pi_k = \pi_{k,n} = \left(\frac{n-k}{n}, \dots, \frac{n-k}{n}, -\frac{k}{n}, \dots, -\frac{k}{n} \right)$$

in the usual basis, $0 < k < n$, $n \geq 3$ (if $n = 2$, the corresponding representation is mentioned in Theorem 1.1). In some proofs we consider π_k for different $SL(n)$ s simultaneously, so the second index in the notation $\pi_{k,n}$ carries this data. Here $M(\lambda) = \{w\lambda : w \in S_n\}$. The highest weight is equal to $e_1 + e_2 + \dots + e_k$ in quasi-basis, all the points of $M(\lambda)$ have form $e_{i_1} + e_{i_2} + \dots + e_{i_k}$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$.

Now we can reformulate the problem. Let $\{e_i\}$ be the quasi-basis, $k < n$, and the weight $\lambda = \pi_k$ is not listed in Theorem 1.1. One has to find a non-saturated subset in the set

$$\{e_{i_1} + e_{i_2} + \dots + e_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq n\}.$$

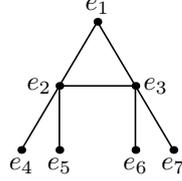
The construction uses induction on n . In the next section we produce the NSSs which are the base of induction.

Important particular cases

Example 2.1. $n = 7, k = 2$. The NSS consists of those and only those vectors which are the sums of two quasi-basis vectors connected with an edge in the graph below. We have

$$v = e_1 + e_2 + e_3 = \frac{1}{2} \left((e_1 + e_2) + (e_2 + e_3) + (e_1 + e_3) \right),$$

$$v = -(e_4 + e_5 + e_6 + e_7) = 2(e_2 + e_3) - (e_2 + e_4) - (e_2 + e_5) - (e_3 + e_6) - (e_3 + e_7).$$



Let $f = 5(x_2 + x_3) - 2(x_1 + x_4 + x_5 + x_6 + x_7)$. Then

$$f(e_1 + e_2) = f(e_1 + e_3) = f(e_2 + e_4) = f(e_2 + e_5) = f(e_3 + e_6) = f(e_3 + e_7) = 3,$$

$$f(e_2 + e_3) = 10, \quad f(v) = f(e_1 + e_2 + e_3) = 5 \cdot 2 - 2 = 8.$$

It is clear that 8 cannot be represented as a sum where each summand equals either 3 or 10.

Example 2.2. $n = 8, k = 3$. Consider the following vectors (in quasi-basis):

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Take $v = (1, 1, 1, 1, 1, 0, 0, 0) = \frac{1}{3}(v_1 + v_2 + v_3 + v_4 + v_5) = 2v_5 - v_6 - v_7 - v_8$. Let $f = x_1 + 5(x_2 + x_3 + x_4) + 2x_5 - 6(x_6 + x_7 + x_8)$. Then

$$f(v_1) = 12, \quad f(v_2) = f(v_3) = 8, \quad f(v_4) = 11, \\ f(v_5) = 15, \quad f(v_6) = f(v_7) = f(v_8) = 4, \quad f(v) = 18.$$

It is easy to see that 18 cannot be represented as a sum of 4, 8, 11, 12, or 15.

Example 2.3. $n = 2k, k \geq 4$.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{k-1} \\ v_k \\ \\ v_{k+1} \\ v_{k+2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 & 0 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & \dots & 0 & 0 & 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 & 1 & 1 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

Let us show that it is an NSS. Set

$$\begin{aligned} v &= (\underbrace{0, \dots, 0}_k, \underbrace{1, \dots, 1}_k), \quad v = v_1 + v_{k+1} - v_{k+2}, \\ \frac{1}{k-2}(v_1 + \dots + v_k) &= \frac{1}{k-2}(\underbrace{1, \dots, 1}_k, \underbrace{k-1, \dots, k-1}_k) = \\ &= \frac{1}{k-2}(\underbrace{0, \dots, 0}_k, \underbrace{k-2, \dots, k-2}_k) = (\underbrace{0, \dots, 0}_k, \underbrace{1, \dots, 1}_k) = v. \end{aligned}$$

To explain why v is not a \mathbb{Z}_+ -combination of vectors v_i , consider two cases.

Case 1: $k = 4$. Let $f = -6x_3 - 7x_4 + 5(x_5 + x_6 + x_7) - 2x_8$. Then

$$f(v_1) = f(v_2) = 8, \quad f(v_3) = 2, \quad f(v_4) = 8, \quad f(v_5) = 15, \quad f(v_6) = 10, \quad f(v) = 13.$$

But it is easy to see that 13 cannot be represented as a sum of 2, 8, 10, or 15.

Case 2: $k \geq 5$. Let $f = (k-2)(x_{k+1} + \dots + x_{2k}) - k(x_3 + \dots + x_k)$. Then

$$\begin{aligned} f(v_1) &= f(v_2) = (k-2)(k-1), \\ f(v_3) &= f(v_4) = \dots = f(v_k) = (k-1)(k-2) - k, \\ f(v_{k+1}) &= (k-2)(k-1), \quad f(v_{k+2}) = (k-2)^2, \quad f(v) = k(k-2). \end{aligned}$$

If $k \geq 6$, then two least possible summands give too much: $2((k-1)(k-2) - k) > k(k-2)$, if $k = 5$, then 15 should be represented as a sum of 12, 7, or 9, but this is impossible.

Example 2.4. $n = 2k + 1, k \geq 3$.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_k \\ v_{k+1} \\ \\ v_{k+2} \\ v_{k+3} \\ \vdots \\ v_{2k} \\ v_{2k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \\ 0 & 1 & \dots & 1 & 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 1 & 1 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Let $v = (\underbrace{1, \dots, 1}_{k+1}, \underbrace{0, \dots, 0}_k)$. Then

$$v = \frac{1}{k}(v_1 + \dots + v_{k+1}) = \frac{1}{k}(\underbrace{k, \dots, k}_{k+1}, \underbrace{0, \dots, 0}_k) = (\underbrace{1, \dots, 1}_{k+1}, \underbrace{0, \dots, 0}_k),$$

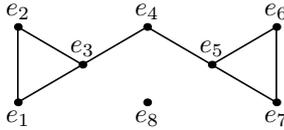
$$\begin{aligned} & (k-1)v_{k+1} - v_{k+2} - \dots - v_{2k+1} = \\ & = (k-1)(\underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{k+1}) - (\underbrace{k-1, \dots, k-1}_k, \underbrace{0, 1, \dots, 1}_k) = \\ & = (\underbrace{0, \dots, 0}_{k+1}, \underbrace{-1, \dots, -1}_k) = (\underbrace{1, \dots, 1}_{k+1}, \underbrace{0, \dots, 0}_k) = v. \end{aligned}$$

It suffices to show that v does not belong to $\mathbb{Z}_+(v_1, v_2, \dots, v_{2k+1})$. Let $f = (k+1)(x_1 + \dots + x_k) - k(x_{k+1} + \dots + x_{2k+1})$. Then

$$\begin{aligned} f(v_1) = \dots = f(v_k) &= k^2 - k - 1, & f(v_{k+1}) &= k(k+1), \\ f(v_{k+2}) = \dots = f(v_{2k+1}) &= k^2 - k - 1, & f(v) &= k^2. \end{aligned}$$

But if $k \geq 3$, then $k^2 < 2(k^2 - k - 1)$, so k^2 cannot be represented as a sum where each summand equals either $(k^2 - k - 1)$ or $k(k+1)$. This means that $v \notin \mathbb{Z}_+(v_1, v_2, \dots, v_{2k+1})$.

Example 2.5. $n = 8, k = 2$.



The NSS consists of vectors which are sums of two quasi-basis vectors connected with an edge in the graph above. Let $v = e_1 + e_2 + e_3 + e_5 + e_6 + e_7$. Then

$$v = \frac{1}{2}((e_1 + e_2) + (e_2 + e_3) + (e_1 + e_3) + (e_5 + e_6) + (e_6 + e_7) + (e_5 + e_7)),$$

$$v = (e_1 + e_2) + (e_3 + e_4) - (e_4 + e_5) + (e_5 + e_6) + (e_5 + e_7).$$

Check that $e_1 + e_2 + e_3 + e_5 + e_6 + e_7$ cannot be represented as a \mathbb{Z}_+ -combination of the vectors of our set. Let

$$f = x_1 + x_2 + x_3 + 2(x_5 + x_6 + x_7) + 9x_4 - 18x_8.$$

Then

$$\begin{aligned} f(e_1 + e_2) &= f(e_2 + e_3) = f(e_1 + e_3) = 2, \\ f(e_5 + e_6) &= f(e_6 + e_7) = f(e_5 + e_7) = 4, \\ f(e_3 + e_4) &= 10, \quad f(e_4 + e_5) = 11, \quad f(v) = 9. \end{aligned}$$

But 9 cannot be represented as the sum of integers 2, 4, 10, or 11.

Example 2.6. $n = 9, k = 3$. Consider the following vectors:

$$\begin{aligned} v_1 &= e_1 + e_2 + e_4, & v_5 &= e_1 + e_3 + e_8, \\ v_2 &= e_1 + e_2 + e_5, & v_6 &= e_1 + e_3 + e_9, \\ v_3 &= e_2 + e_3 + e_6, & v_7 &= e_2 + e_4 + e_6. \\ v_4 &= e_2 + e_3 + e_7, \end{aligned}$$

Then $v = e_1 + e_2 + e_3 = \frac{1}{3}(v_1 + v_2 + v_3 + v_4 + v_5 + v_6) = v_1 + v_3 - v_7$.

Check that v is not a \mathbb{Z}_+ -combination of $v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 . Let $f = 5(x_1 + x_2 + x_3 + x_4) - 4(x_5 + x_6 + x_7 + x_8 + x_9)$. Then $f(v_1) = 15$,

$$f(v_2) = f(v_3) = f(v_4) = f(v_5) = f(v_6) = f(v_7) = 6, \quad f(v) = 15.$$

Note that $v \neq v_1$ and $f(v_1) = f(v)$, so we conclude that if $v \in \mathbb{Z}_+(v_1, \dots, v_7)$, then v_1 does not occur in this decomposition. But $6 \nmid 15$, and this means that v cannot be obtained as a \mathbb{Z}_+ -combination of v_i s.

Example 2.7. $n = 10, k = 4$. Consider the following vectors:

$$\begin{aligned} v_1 &= e_1 + e_2 + e_3 + e_5, & v_4 &= e_5 + e_6 + e_7 + e_8, \\ v_2 &= e_1 + e_2 + e_4 + e_6, & v_5 &= e_5 + e_7 + e_8 + e_9, \\ v_3 &= e_3 + e_4 + e_5 + e_6, & v_6 &= e_6 + e_7 + e_8 + e_{10}, \\ v &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = \frac{1}{2}(v_1 + v_2 + v_3) = v_4 - v_5 - v_6. \end{aligned}$$

Let us show that $v \notin \mathbb{Z}_+(v_1, \dots, v_6)$. Let $f = x_1 + x_3 + x_4 + 6x_7 + 6x_8 - 7x_9 - 8x_{10}$. Then

$$f(v_1) = f(v_2) = f(v_3) = 2, f(v_4) = 12, f(v_5) = 5, f(v_6) = 4, f(v) = 3.$$

But it is clear that 3 cannot be represented as a sum of 2, 4, 5, or 12.

Case when $k \nmid n, (n - k) \nmid n$.

It follows that $n \geq 5$. The exceptional case $\frac{k}{n} \in \{\frac{2}{5}, \frac{3}{5}\}$ will be considered at the end of the section. Below we suppose that $\frac{k}{n} \notin \{\frac{2}{5}, \frac{3}{5}\}$, which gives $n \geq 7$.

Lemma 2.1. *Assume that there exists an NSS for a pair (n, k) , where (n, k) satisfies the conditions above. Then for each $r \in \mathbb{N}$, there exists an NSS for the pair (nr, kr) .*

Proof. Consider an arbitrary vector from $M(\pi_{k,n})$. Write down its quasi-coordinates r times in succession. The result is a vector from $M(\pi_{kr, nr})$: it has kr 1s and $(n - k)r$ 0s. If one takes an NSS for (n, k) and performs this procedure on each vector, the result will be an NSS for (nr, kr) . \square

Thus, if we construct an NSS for all pairs (n, k) where $\text{GCD}(n, k) = 1$, then the NSS for all other pairs will be constructed according to Lemma 2.1.

Lemma 2.2 (the Step procedure). *If there exists an NSS for a pair (n, k) , then there exists an NSS for the pair $(n + k, k)$.*

Proof. The keypoint is that if one takes a weight from $M(\pi_{k,n})$, writes it down in the form where all quasi-coordinates are equal to 0 or to 1, and adds k coordinates equal to 0, then this weight can be considered as a weight from $M(\pi_{k, n+k})$. If one starts with an ENSS $(v; v_1, v_2, \dots, v_m)$ for (n, k) , then one should perform this procedure on all its vectors and after that add one more vector v_{m+1} which has 0s at the first n positions and 1s at the k adjoint positions. Now we show that the obtained set $(v'; v'_1, \dots, v'_m, v_{m+1})$ is indeed an NSS in $M(\pi_{k, n+k})$.

Suppose that a vector v lies in the ENSS for (n, k) , $v = q_1v_1 + q_2v_2 + \dots + q_s v_s$, $q_i \in \mathbb{Q}_+$. If we fix some representations in quasi-basis for v and for all v_i , then this equality can be re-written in the formal basis in the following form:

$$v = q_1v_1 + q_2v_2 + \dots + q_s v_s - \alpha(f_1 + f_2 + \dots + f_n),$$

where f_i s are the preimages of e_i s under the projection

$$\mathbb{Q}^n \rightarrow \Lambda(T) \otimes_{\mathbb{Z}} \mathbb{Q}, (q_1, q_2, \dots, q_n) \mapsto q_1e_1 + q_2e_2 + \dots + q_n e_n.$$

Obviously, f_i s are linearly independent. For each v_i , fix a representation in which it has k coordinates 1 and $n-k$ coordinates 0. The vector v is nonzero, consequently, it has a representation where all coordinates are nonnegative, but some of them are zeroes. Fix this representation. Then $\alpha \geq 0$ (otherwise all coordinates of v are strictly positive), and we get that in \mathbb{Q}^{n+k} the following equality holds:

$$v' = q_1 v'_1 + q_2 v'_2 + \dots + q_s v'_s + \alpha(f_{n+1} + \dots + f_{n+k}) - \alpha(f_1 + f_2 + \dots + f_{n+k}).$$

This shows that v' lies in the \mathbb{Q}_+ -cone generated by $v'_1, \dots, v'_m, v_{m+1}$ (here all vectors taken in quasi-basis $\{e_1, \dots, e_{n+k}\}$).

Similarly one can show that v' lies in the $\mathbb{Z}(v'_1, \dots, v'_m, v_{m+1})$.

To prove that the constructed set is an ENSS, it remains to show that v' does not lie in $\mathbb{Z}_+(v'_1, \dots, v'_m, v_{m+1})$. Suppose the contrary. Let $v' \in \mathbb{Z}_+(v'_1, \dots, v'_m, v_{m+1})$. Omit last k coordinates. We get that $v \in \mathbb{Z}_+(v_1, \dots, v_m)$, so $\{v; v_1, v_2, \dots, v_m\}$ is not an ENSS for (n, k) .

Notice that this proof also shows how to construct an NSS for a pair $(n+k, k)$ if the NSS for the pair (n, k) is given. \square

Now we can explain how, using these Lemmas, the NSSs can be constructed for all pairs (n, k) , for which the following three conditions hold:

- (1) $1 < k < n - 1$,
- (2) $\text{GCD}(n, k) = 1$,
- (3) $n \geq 7$.

Use descent on n . Suppose the NSSs are constructed for all pairs (m, l) with $m < n$, satisfying the conditions above. Take a pair (n, k) . Suppose $k < \frac{n}{2}$ (otherwise change it by $n-k$ and seek for an NSS for the pair $(n, n-k)$, the case $n = 2k$ is impossible because $\text{GCD}(n, k) = 1$). If all the conditions are held for the pair $(n-k, k)$, then we have an NSS for it, and using Lemma 2.2, this NSS can be remade into the NSS for (n, k) . Let us find all the cases when at least one of the conditions fails for $(n-k, k)$.

Condition (1) fails iff $n = 2k + 1$. But we have $n \geq 7$, hence $k \geq 3$. In this case we already have an NSS (Example 2.4).

Condition (2) never fails.

Condition (3) fails iff $n - k \leq 5$. Find these cases. Recall that $k \leq (n-1)/2$. Substitute it: $n \leq (n-1)/2 + 5$. This gives $n \leq 9$. List all these pairs (n, k) (with $k < \frac{n}{2}$).

$n = 7$: pairs $(7, 2)$ and $(7, 3)$; $n = 8$: pair $(8, 3)$; $n = 9$: pairs $(9, 2)$ and $(9, 4)$.

But we already have NSSs for all these pairs. Indeed, the cases $(7, 2)$ and $(8, 3)$ coincide with Examples 2.1 and 2.2, respectively. The cases $(7, 3)$ and $(9, 4)$ are

the particular cases of $n = 2k + 1$ (Example 2.4). The case $(9, 2)$ can be obtained from $(7, 2)$ (Example 2.1) using Lemma 2.2.

Finally, take all the cases where the NSS is already constructed as the base of the descent. In all the other cases the descent is feasible, consequently, we have constructed an NSS for all pairs (n, k) for which conditions (1) – (3) hold.

Now we consider the case $\frac{k}{n} \in \{\frac{2}{5}, \frac{3}{5}\}$. Let $k = 2k_1$, $n = 5k_1$, $k_1 \geq 2$. If $k_1 \geq 4$, we can construct an NSS using Lemma 2.2 and substitution $k \mapsto n - k$: starting with an NSS for $(2k_1, k_1)$, we successively construct NSSs for $(3k_1, k_1)$, $(3k_1, 2k_1)$, and $(5k_1, 2k_1)$. If $k_1 = 2$, Example 2.7 can be applied.

If $k_1 = 3$, the pair $(n, k) = (15, 6)$, and the required NSS can be obtained from Example 2.6 using Lemma 2.2.

Case when $k \mid n$ or $(n - k) \mid n$.

Assume that $k \leq n/2$. Then $k \mid n$, let $d = n/k$. The case $k = 1$ is positive, so here $k \geq 2$.

If $k \geq 4$, Example 2.3 shows that the NSS exists for the pair $(2k, k)$. Using Lemma 2.2, we can easily rebuild this NSS into the NSS for a pair (kd, d) , where $d \geq 2$. It remains to consider cases $k = 2$ and 3.

$k = 2$. It follows from Theorem 1.1 that $d \geq 4$. But we already have an NSS for the pair $(8, 2)$ (Example 2.5). Using Lemma 2.2, we can construct NSSs for all $d > 4$.

$k = 3$. We already have an NSS for $(9, 3)$. Using Lemma 2.2, we can construct an NSS for all n such that $n > 9$ and $3 \mid n$. We are done.

2.2.2 Non-fundamental weights

It follows from Lemma 1.8 that we already have NSSs for the major part of non-fundamental weights (a non-fundamental weight can be reduced to fundamental). However, this method does not cover all the cases, and the size of the resulting NSS grows as $\dim V$ does. Below, NSSs containing only 4 vectors are constructed for all such cases.

There are two situations: all the usual coordinates of λ are integer or all of them are non-integer. Consider these cases independently. The coordinates of vectors in the usual basis are denoted by y_i .

All the coordinates of λ are integer, $\lambda \neq (1, 0, \dots, 0, -1)$

Definition 2.3. By a *shift for A_n* we mean the following procedure: take a point $\lambda = (y_1, \dots, y_n)$, fix two indices $i < j$ such that $|y_i - y_j| \geq 2$ and replace λ with the

point λ' , where $\lambda' = (\dots, y_i - 1, \dots, y_j + 1, \dots)$ if $y_i > y_j$ and $(\dots, y_i + 1, \dots, y_j - 1, \dots)$ otherwise.

The point λ' lies in $M(\lambda)$. Indeed, the set $M(\lambda)$ contains the point

$$(\dots, y_j, \dots, y_i, \dots),$$

the convex hull of λ and $(\dots, y_j, \dots, y_i, \dots)$ contains λ' . Notice that after each use of the shift the value $y_1^2 + \dots + y_n^2$ diminishes by a positive integer. Indeed, let $x = \max\{y_i, y_j\}$, $y = \min\{y_i, y_j\}$. Then $x - y \geq 2$, and

$$(x-1)^2 + (y+1)^2 = x^2 - 2x + 1 + y^2 + 2y + 1 = x^2 + y^2 - 2(x-y-1) \leq x^2 + y^2 - 2.$$

This means that if we apply consequent shifts to λ , then this process cannot be infinite.

Lemma 2.4. *If $n \geq 3$ and λ satisfies the conditions of Subsection 2.2.2, then $M(\lambda)$ contains one of the points*

$$(2, 0, \dots, 0, -1, -1), \quad (1, 1, 0, \dots, 0, -2), \quad \text{or} \quad (1, 1, 0, \dots, 0, -1, -1),$$

and it always contains the point $(1, 0, \dots, 0, -1)$.

Proof. Let $\lambda = (a_1, \dots, a_n)$ (in the usual basis). If $\forall i a_i \in \{-1, 0, 1\}$, then, due to the fact that $\lambda \neq (1, 0, \dots, 0, -1)$, λ has at least 4 nonzero coordinates. Taking into account that $\sum_1^n a_i = 0$, at least two of them are equal to 1 and two are equal to -1 . In this case, $M(\lambda)$ contains the point $(1, 1, 0, \dots, 0, -1, -1)$: split all its other coordinates into pairs $(1, -1)$ and make them zero (using the shift), then permute the remaining 4 coordinates. Applying one more shift, we yield $(1, 0, \dots, 0, -1)$.

Otherwise, if $\exists i, |a_i| > 1$ (one of the coordinates is *big*), then $\max_{i,j} (a_i - a_j) \geq 3$. Keeping at least one coordinate big, perform the shift for the pairs of indices where $|a_i - a_j| \geq 2$. This process is finite. Consider a situation where we can perform no more shifts. If we still have a nonzero coordinate with the same sign as the big coordinate has, we can shift it with the coordinate of the opposite sign (their difference will obviously be ≥ 2). Otherwise we are in the case where we have a big coordinate of one sign (without loss of generality positive) and some coordinates of the opposite sign. If the big coordinate is ≥ 3 , then apply a shift to this coordinate and to some negative coordinate. But we have supposed that shifts are impossible. Then the big coordinate is equal to 2, nonzero ones among the other coordinates are either -2 or two -1 s. But if $M(\lambda)$ contains a point $(2, 0, 0, \dots, 0, -2)$, then it also contains $(2, 0, \dots, 0, -1, -1) = \frac{1}{2}((2, 0, \dots, 0, -2) + (2, 0, \dots, 0, -2, 0))$. We can easily get $(1, 0, \dots, 0, -1)$, performing one more shift. \square

Construct NSSs for the first three points.

Example 2.8. $\lambda = (2, 0, \dots, 0, -1, -1)$, $n \geq 3$. Consider vectors

$$\begin{aligned} v_1 &= (1, -1, 0, 0, \dots, 0), \quad v_2 = (-1, -1, 2, 0, \dots, 0), \quad v_3 = (2, -1, -1, 0, \dots, 0), \\ v &= (0, -1, 1, 0, \dots, 0) = \frac{1}{2}(v_1 + v_2) = v_2 + v_3 - v_1. \end{aligned}$$

Suppose $f = -y_2$, then $f(v_1) = f(v_2) = f(v_3) = f(v) = 1$, but $v \neq v_i$ for any i . We get a contradiction.

The NSS for the point $\lambda = (1, 1, 0, \dots, 0, -2)$, $n \geq 3$ can be constructed similarly (one should multiply all the coordinates by -1).

Example 2.9. $\lambda = (1, 1, 0, \dots, 0, -1, -1) \in M(\lambda)$, $n \geq 4$. Consider vectors

$$\begin{aligned} v_1 &= (1, 1, -1, -1, 0, \dots, 0), \quad v_2 = (1, -1, 1, -1, 0, \dots, 0), \\ v_3 &= (0, 1, 0, -1, 0, \dots, 0), \quad v_4 = (0, 0, 1, -1, 0, \dots, 0), \\ v &= (1, 0, 0, -1, 0, \dots, 0) = \frac{1}{2}(v_1 + v_2) = v_1 + v_4 - v_3. \end{aligned}$$

Suppose $f = -y_4$, then $f(v_1) = f(v_2) = f(v_3) = f(v_4) = f(v) = 1$, but $v \neq v_i$ for any i . We get a contradiction.

Now take an arbitrary dominant weight λ , $n \geq 3$, and the corresponding set $M(\lambda)$. It follows from Lemma 2.4 and Lemma 1.8 that an NSS for λ exists.

It remains to consider the case $n = 2$, $\lambda = (a, -a)$. If $|a| \geq 3$, then the segment $\text{conv}\{(\sigma(a_1, a_2)), \sigma \in S_2\}$ contains the points $(2, -2)$ and $(3, -3)$. But this subset is not saturated:

$$(1, -1) = \frac{1}{2}(2, -2) = (3, -3) - (2, -2),$$

and the vector $(1, -1)$ is not a linear combination of vectors $(2, -2)$ and $(3, -3)$ with integer positive coefficients. If, otherwise, $a \in \{0, \pm 1, \pm 2\}$, then each subset in $M(\lambda)$ is saturated (see Theorem 1.1).

All the coordinates of λ are non-integer

Lemma 2.5. *Take a point $\lambda = (a_1, a_2, \dots, a_n)$ (in the usual basis), $n \geq 4$. If the set $\{a_1, a_2, \dots, a_n\}$ contains simultaneously $\alpha + 1, \alpha$, and $\alpha - 1$ for some $\alpha \in \mathbb{R}$, then the set $M(\lambda)$ contains an NSS.*

Proof. It is easy to see that $M(\lambda)$ contains a point $v_1 = (\alpha + 1, \alpha, \alpha - 1, a_4, \dots, a_n)$, $a_4 \neq 0$ (because $a_4 \notin \mathbb{Z}$). Acting by S_n , we can get the following points from it:

$$\begin{aligned} v_2 &= (\alpha - 1, \alpha, \alpha + 1, a_4, \dots, a_n), \\ v_3 &= (\alpha + 1, \alpha - 1, \alpha, a_4, \dots, a_n), \\ v_4 &= (\alpha, \alpha - 1, \alpha + 1, a_4, \dots, a_n). \end{aligned}$$

Let us show that this set is not saturated. Indeed,

$$\begin{aligned}\frac{1}{2}(v_1 + v_2) &= (\alpha, \alpha, \alpha, a_4, \dots, a_n), \\ v_2 + v_3 - v_4 &= (\alpha, \alpha, \alpha, a_4, \dots, a_n).\end{aligned}$$

If $f = \frac{y_4}{a_4}$, then $f(v_1) = f(v_2) = f(v_3) = f(v_4) = f(v) = 1$. But $v \neq v_i$ for any i , this means that v is not a \mathbb{Z}_+ -combination of v_i . \square

Lemma 2.6 (The Good Triple Lemma). *Let $\lambda = (a_1, \dots, a_n)$, $n \geq 4$, and all a_i be non-integer. If the collection a_1, \dots, a_n contains at least three different values, then $M(\lambda)$ contains a point of form $(\alpha + 1, \alpha, \alpha - 1, a_4, \dots, a_n)$.*

Proof. Perform several shifts preserving the condition that the set $\{a_1, \dots, a_n\}$ contains at least 3 different elements. Suppose further shifts are impossible (we mentioned above that, starting from any position, only a finite number of shifts is possible). Consider $a_{\max} = \max\{a_1, \dots, a_n\}$, $a_{\min} = \min\{a_1, \dots, a_n\}$, $a_{\text{mid}} \in \{a_1, \dots, a_n\}$, $a_{\text{mid}} \neq a_{\max}$, $a_{\text{mid}} \neq a_{\min}$. If $a_{\max} - a_{\text{mid}} \geq 3$, then we can apply the shift to a_{\max} and a_{mid} , thus we obtain three different values of coordinates a_{\min} , $a_{\text{mid}} + 1$, $a_{\max} - 1$. Similarly, if $a_{\text{mid}} - a_{\min} \geq 3$, then at least one more shift is possible. So we yield $a_{\max} - a_{\text{mid}}$, $a_{\text{mid}} - a_{\min} \in \{1, 2\}$. If $a_{\max} - a_{\text{mid}} = a_{\text{mid}} - a_{\min} = 1$, we have already found a point of necessary type in $M(\lambda)$. Up to symmetry, one of the two cases is possible: either $a_{\min} = a_{\text{mid}} - 2$, $a_{\max} = a_{\text{mid}} + 2$, or $a_{\min} = a_{\text{mid}} - 1$, $a_{\max} = a_{\text{mid}} + 2$. Consider these two cases.

In the first case, apply the shift to a_{\max} and a_{\min} . This operation gives us the required triple $(a_{\max} - 1, a_{\text{mid}}, a_{\min} + 1)$.

In the second case, $a_{\min} = a_{\text{mid}} - 1$, $a_{\max} = a_{\text{mid}} + 2$, and we know that λ has at least 4 coordinates. If there are 4 different values among them, the fourth will inevitably form a triple of form $(\alpha + 1, \alpha, \alpha - 1)$ with two of a_{\max} , a_{mid} , a_{\min} . Otherwise $a_i \in \{a_{\max}, a_{\text{mid}}, a_{\min}\}$ for any i . But $n \geq 4$, this means that at least one of the values (a_1, a_2, \dots, a_n) occurs twice. Suppose $n = 4$ (we need only 4 a_i s). The multiplicities of $(a_{\max}, a_{\text{mid}}, a_{\min})$ may be as follows: $(\widehat{1}, \widehat{1}, 2)$, $(\widehat{1}, \widehat{2}, 1)$, $(\widehat{2}, \widehat{1}, 1)$. Apply the shift to the coordinates marked with the hat. We get one of the following collections: $(a_{\text{mid}} + 1, a_{\text{mid}}, a_{\text{mid}}, a_{\text{mid}} - 1)$, $(a_{\text{mid}} + 1, a_{\text{mid}} + 1, a_{\text{mid}}, a_{\text{mid}} - 1)$, $(a_{\text{mid}} + 2, a_{\text{mid}} + 1, a_{\text{mid}}, a_{\text{mid}})$. Each of them contains a triple of form $(\alpha + 1, \alpha, \alpha - 1)$. But this means that here we also find a triple of form $(\alpha + 1, \alpha, \alpha - 1)$. \square

Lemma 2.7. *Let $\lambda = (a_1, \dots, a_n)$ be a dominant weight, $n \geq 4$, $\exists i$ with $|a_i| > 1$, and all $a_i \notin \mathbb{Z}$. If $M(\lambda)$ does not contain a point of the form $(\alpha + 1, \alpha, \alpha - 1, a_4, \dots, a_n)$, then*

$$\lambda = \left(\frac{2n-2}{n}, -\frac{2}{n}, -\frac{2}{n}, \dots, -\frac{2}{n} \right) \text{ or } \lambda = \left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}, -\frac{2n-2}{n} \right).$$

Proof. If the collection a_1, \dots, a_n contains at least 3 different elements, then we can use Lemma 2.6 and show that $M(\lambda)$ contains a point of the desired form. This means that $\forall i \ a_i \in \{a_{\max}, a_{\min}\}$. Without loss of generality, we may suppose that $a_{\max} > 1$, and $a_{\min} < 0$ (otherwise multiply all a_i by -1).

If $a_{\min} < -1$, then apply the shift to a_{\min} and a_{\max} . Thus we get $a_{\min} + 1$ and $a_{\max} - 1$ among the values of the coordinates, and still at least one of a_{\min} and a_{\max} is presented (since $n \geq 4 > 3$). Using Lemma 2.6, we get a contradiction.

We see that $-1 < a_{\min} < 0$. If the collection (a_1, \dots, a_n) contains a_{\max} at least for 2 times, then apply the shift to a_{\min} and a_{\max} . Now we have $a_{\max}, a_{\min} + 1 > 0$ and at least one time a_{\min} among the values of coordinates: all the coordinates cannot be positive. This gives us a contradiction with Lemma 2.6.

Then a_{\max} enters only once in (a_1, \dots, a_n) . If $a_{\max} > 2$, apply the shift to a_{\min} and a_{\max} . We get that $a_{\max} - 1 > 1$, $a_{\min} + 1 < 1$ and a_{\min} are among the values of coordinates, which gives us a contradiction with Lemma 2.6.

We see that the collection has a form $(a_{\max}, a_{\min}, a_{\min}, \dots, a_{\min})$, $1 < a_{\max} < 2$, $-1 < a_{\min} < 0$. Let $a_{\min} = -\frac{k}{n}$. We have $(n-1)a_{\min} + a_{\max} = 0$ from the initial conditions. This yields $a_{\max} = \frac{k(n-1)}{n}$. But $a_{\max} < 2$. Consequently,

$$\frac{k(n-1)}{n} < 2 \Rightarrow k(n-1) < 2n \Rightarrow k < \frac{2n}{n-1} < 3,$$

because $n \geq 4$. Taking into account that $a_{\max} > 1$, we get $k = 2$, $a_{\max} = \frac{2n-2}{n}$, $a_{\min} = -\frac{2}{n}$. But in the beginning of the case we could change the signs at all the coordinates. Thus, we have two cases: $\lambda = (\frac{2n-2}{n}, -\frac{2}{n}, -\frac{2}{n}, \dots, -\frac{2}{n})$ and $\lambda = (-\frac{2n-2}{n}, \frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n})$. \square

Applying the Lemmas, we see that in the case, when $a_i \notin \mathbb{Z}$, $\exists i, |a_i| > 1$, $n \geq 4$, we have not constructed an NSS only in these two cases. In all the other cases the NSS exists due to Lemma 2.5. Let us construct an NSS in these two cases. We may assume that $\lambda = (\frac{2n-2}{n}, -\frac{2}{n}, -\frac{2}{n}, \dots, -\frac{2}{n}) = 2e_1$. Let

$$v_1 = 2e_1, \quad v_2 = 2e_2, \quad w = 2e_3, \quad v_3 = e_1 + e_3 \in M(\lambda), \quad v_4 = e_2 + e_3 \in M(\lambda).$$

Then v_1, v_2, v_3, v_4 form an NSS. Indeed, we have

$$v = e_1 + e_2 = \frac{1}{2}(v_1 + v_2) = v_1 + v_4 - v_3, \quad f = x_1 + x_2 + x_3 - 3x_n.$$

Then $f(v_1) = f(v_2) = f(v_3) = f(v_4) = f(v) = 2$, and $v \neq v_i$ for any i . But 2 cannot be represented as a sum of more than one 2s. We get a contradiction.

Now it remains to consider the cases $n = 3$ and $n = 2$.

In the case $n = 3$ we suppose that the fractional parts of all coordinates are equal to $\frac{2}{3}$ (otherwise change λ for $-\lambda$, as we have done earlier). If $\lambda = (\frac{2}{3}, \frac{2}{3}, -\frac{4}{3})$,

then $M(\lambda)$ is hereditarily normal (see Theorem 1.1). Below we construct an NSS for $\lambda = \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right) = 3e_1 + e_2$, then, using Remark 1.9, show the existence of an NSS for all other points λ . Let

$$\begin{aligned} v_1 = e_1 &= \frac{2}{3}(3e_1 + e_2) + \frac{1}{3}(e_2 + 3e_3), \\ v_2 = 2e_1 + 2e_2 &= \frac{1}{2}(3e_1 + e_2) + \frac{1}{2}(e_1 + 3e_2), \\ v_3 = 3e_1 + e_2, \quad v &= 2e_1 + e_2 = v_1 + \frac{1}{2}v_2 = v_3 - v_1. \end{aligned}$$

If $f = x_1 - x_3$, then $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, and $f(v) = 2$. But v is equal neither to v_2 , nor to $2v_1$. We get a contradiction.

Lemma 2.8. *Suppose that $\lambda = (a_1, a_2, a_3)$ (in the usual basis) is a dominant weight such that the fractional parts of all a_i are equal to $\frac{2}{3}$. Suppose also that there exists an index i with $|a_i| > 1$, and $\lambda \neq \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right)$. Then $M(\lambda)$ contains a point $\left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$.*

Proof. It follows from the conditions of the Lemma that $\exists i, a_i \geq \frac{5}{3}$. Indeed, otherwise we have at least 2 positive coordinates, each of them $\leq \frac{2}{3}$, but due to the condition of the Lemma there exists an a_i such that $|a_i| > 1$. Suppose it is a_1 . We have $a_1 = -a_2 - a_3 \geq -\frac{4}{3}$. This means that $\lambda = \left(-\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)$. We get a contradiction.

If only one coordinate of λ is positive, and it is equal to $\frac{5}{3}$, then $\lambda = \left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$, and the Lemma is proved. Otherwise either λ has two positive coordinates, or one of them is $\geq \frac{8}{3}$. In both cases we can apply the shift to a positive and a negative coordinate, such that after it λ still has a coordinate $\geq \frac{5}{3}$, and so on. \square

Consider the case $n = 2$. Suppose that $\lambda = \left(\frac{a}{2}, -\frac{a}{2}\right)$, $a \in \mathbb{N}$, a is odd. Then for $a = 3$ the set $M(\lambda)$ is hereditarily normal. If $a \geq 5$, let us construct an NSS. Set

$$v_1 = \left(\frac{3}{2}, -\frac{3}{2}\right), \quad v_2 = \left(\frac{5}{2}, -\frac{5}{2}\right).$$

Then $\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{3}v_1 = 2v_1 - v_2 \notin \mathbb{Z}_+(v_1, v_2)$.

So in the case A_n we have constructed non-saturated subsets in the weight systems for all the modules not listed in Theorem 1.1.

Chapter 3

Other classical root systems

3.1 The root system B_n

The root system B_n , where $n \geq 2$, is formed by vectors $\{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid 1 \leq i, j \leq n, i \neq j\}$. With respect to the system of simple roots

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n$$

the fundamental weights have the form

$$\pi_1 = \varepsilon_1, \pi_2 = \varepsilon_1 + \varepsilon_2, \dots, \pi_{n-1} = \varepsilon_1 + \dots + \varepsilon_{n-1}, \pi_n = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n).$$

The root lattice Ξ coincides with the lattice of all integer points in \mathbb{Q}^n . Next, the weight lattice Λ has the form

$$\Lambda = \{(\ell_1, \ell_2, \dots, \ell_n) \mid 2\ell_i \in \mathbb{Z}, \ell_i - \ell_j \in \mathbb{Z}, i, j = 1, \dots, n\}.$$

The Weyl group W acts by permutations on the set of coordinates and by changing signs of an arbitrary set of coordinates. A weight $\lambda = (\ell_1, \ell_2, \dots, \ell_n)$ is dominant if and only if $\ell_1 \geq \dots \geq \ell_n \geq 0$. If all coordinates of λ are integers (or all together half-integers but not integers), then the set $M(\lambda)$ consists of all integer (or half-integer but not integer, respectively) points in the polytope $P(\lambda)$.

3.1.1 Positive results

Case 3.1. $\lambda = \pi_1 = (1, 0, \dots, 0)$. Then $M(\lambda) = \{\pm\varepsilon_i \mid 1 \leq i \leq n\}$. Obviously, this subset is unimodular, and by Lemma 1.12 it is hereditarily normal.

Case 3.2. $\lambda = 2\pi_2 = (1, 1)$, $n = 2$. It is easy to check case-by-case that any subset in the set of vectors $\{\pm\varepsilon_1 \pm \varepsilon_2, \pm\varepsilon_1, \pm\varepsilon_2\}$ is saturated.

Case 3.3. $\lambda = \pi_n = (\frac{1}{2}, \dots, \frac{1}{2})$, $2 \leq n \leq 4$. In this case we have

$$M(\lambda) = \left\{ \underbrace{\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \dots, \pm \frac{1}{2} \right)}_{n \text{ coordinates}} \right\}.$$

One has to check that this set is hereditarily normal.

Multiply all coordinates of all vectors by 2. The problem does not change but now all the coordinates are integers: $M'(\lambda) = \left\{ \underbrace{(\pm 1, \pm 1, \dots, \pm 1)}_{n \text{ coordinates}} \right\}$.

For $n = 2, 3$ the set $M'(\lambda)$ is unimodular, so by Lemma 1.12 it is hereditarily normal.

Consider $n = 4$. The values of all nonzero determinants in $M'(\lambda)$ equal ± 8 and ± 16 . This means that $M'(\lambda)$ is almost unimodular. Find all 4-tuples of vectors such that their determinant equals 16. For any vector v from M' the vector $-v$ also belongs to M' , hence we will look for such 4-tuples up to sign: we will check only one vector from each pair of opposite vectors, namely, the one with the first coordinate equalling 1. Also we may assume that the first vector in this 4-tuple is $(1, 1, 1, 1)$. Using case-by-case consideration, we see that it can be only the following set (given by rows of the following matrix):

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Next, suppose that there exists an ENSS $\{v_0; v_1, \dots, v_r\}$. Using Lemmas 1.16 and 1.15, we obtain that the corresponding $\mathbb{Q}_{\geq 0}$ -combination for v_0 is a sum of some $\pm w_1, \pm w_2, \pm w_3, \pm w_4$ with coefficients 0 and $1/2$, i.e. $q_1 v_1 + q_2 v_2 + q_3 v_3 + q_4 v_4$, where each $q_i \in \{0, \frac{1}{2}\}$ and $v_i = \pm w_i$. Since every two coordinates of v_0 differ by an even integer, $2(\pm q_3 \pm q_4)$ is even, hence $(\pm q_3 \pm q_4)$ is integer, hence q_3 and q_4 are both 0 or $\frac{1}{2}$. For other pairs of coefficients q_i one can proceed analogously. Since $v_0 \neq 0$, all the coefficients equal $\frac{1}{2}$. If we augment $\{v_1, v_2, v_3, v_4\}$ by any new vector v_5 from $M'(\lambda)$ (due to Lemma 1.10(ii) it is not $\pm w_i$, $i = 1, \dots, 4$) we will express v_0 as a $\mathbb{Z}_{\geq 0}$ -combination. Indeed, any vector v_5 from $M(\lambda)$ which is not equal to $\pm w_i$, $i = 1, \dots, 4$ can be represented as $\pm \frac{1}{2} w_1 \pm \frac{1}{2} w_2 \pm \frac{1}{2} w_3 \pm \frac{1}{2} w_4$ (for example, $(1, 1, 1, -1) = 1/2(w_1 + w_2 + w_3 - w_4)$), which gives a similar representation in vectors v_1, v_2, v_3, v_4 . But, if we have any vector v_5 of the form $\pm \frac{1}{2} v_1 \pm \frac{1}{2} v_2 \pm \frac{1}{2} v_3 \pm \frac{1}{2} v_4$, the vector $\frac{1}{2}(v_1 + v_2 + v_3 + v_4)$ can be obtained for any choice of signs: we add to v_5 those vectors v_i at which v_5 has coefficient $-\frac{1}{2}$. This shows that v_0 can be expressed in v_i with $\mathbb{Z}_{\geq 0}$ -coefficients. Hence, it is not an ENSS.

3.1.2 Some negative results

Example 3.1. $\lambda = 2\pi_1 = 2\varepsilon_1$, $n = 2$. Consider the following subset in $M(\lambda)$: $v_1 = 2\varepsilon_1$, $v_2 = \varepsilon_1 + \varepsilon_2$, $v_3 = \varepsilon_2$, $v = \varepsilon_1 = v_1/2 = v_2 - v_3$. Take a discriminating linear function: $f = 3x_1 + 4x_2$ (see Section 1.3), then $f(v_1) = 6$, $f(v_2) = 7$, $f(v_3) = 4$, $f(v) = 3$. It is clear that 3 cannot be represented as a sum of integers 4, 6, and 7.

Example 3.2. $\lambda = \pi_2 = \varepsilon_1 + \varepsilon_2$, $n \geq 3$. Let $v_1 = \varepsilon_1 + \varepsilon_2$, $v_2 = \varepsilon_1 - \varepsilon_2$, $v_3 = \varepsilon_2 - \varepsilon_3$, $v_4 = -\varepsilon_3$. Then $v = \varepsilon_1 = \frac{1}{2}((\varepsilon_1 + \varepsilon_2) + (\varepsilon_1 - \varepsilon_2)) = (\varepsilon_1 - \varepsilon_2) + (\varepsilon_2 - \varepsilon_3) - (-\varepsilon_3)$, but $\varepsilon_1 \notin \mathbb{Z}_{\geq 0}(v_1, v_2, v_3, v_4)$. Indeed, let $f = 3x_1 + x_2 - 5x_3$. Then $f(v_1) = 4$, $f(v_2) = 2$, $f(v_3) = 6$, $f(v_4) = 5$, $f(v) = 3$, but 3 cannot be represented as a sum of integers 2, 4, 5, and 6.

Example 3.3. $\lambda = \pi_1 + \pi_n = (\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, $n \geq 2$. Let

$$v_1 = \left(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right), v_2 = \left(\frac{3}{2}, -\frac{1}{2}, \dots, -\frac{1}{2}\right), v_3 = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right).$$

Then $v = (1, 0, \dots, 0) = 1/3(v_1 + v_2) = v_1 - v_3$, and if one considers the first coordinate, it is clear that

$$v \notin \mathbb{Z}_{\geq 0}(v_1, v_2, v_3).$$

Example 3.4. $\lambda = \pi_n = (\frac{1}{2}, \dots, \frac{1}{2})$, $n = 5$. To simplify the notation, multiply all the coordinates by 2. Let

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \end{pmatrix},$$

$$v = \frac{1}{3}(v_1 + v_2 + v_3 + v_4 + v_5) = (1, 1, 1, 1, 1) = v_1 + v_2 - v_6,$$

$$f = 3x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5,$$

$$f(v_1) = f(v_2) = 9, f(v_3) = f(v_4) = f(v_5) = 7, f(v_6) = 5, f(v) = 13.$$

Note that 13 cannot be decomposed as a sum of integers 9, 7, and 5. Hence, it is an NSS.

3.1.3 Reduction to the already examined cases

By a *shift for B_n* we call the procedure of replacing the vector $\lambda = (\ell_1, \dots, \ell_n)$ with the vector $\lambda' = (\ell_1, \dots, \ell_i - 1, \dots, \ell_n)$, if $\ell_i \geq 1$. Notice that λ' always belongs

to $M(\lambda)$ because $\lambda - \lambda' \in \Xi$ and λ' is a convex linear combination of vectors λ and $(\ell_1, \dots, -\ell_i, \dots, \ell_n)$ with suitable coefficients (these vectors both belong to $M(\lambda)$).

Lemma 3.1. *Let $n \geq 3$. If $\lambda \in \Xi \setminus \Phi$, then the vector $\varepsilon_1 + \varepsilon_2$ belongs to $M(\lambda)$.*

Proof. Let $\lambda = (\ell_1, \dots, \ell_n)$, $\ell_1, \dots, \ell_n \in \mathbb{Z}$. Since λ is a dominant weight, we have $\sum_1^n \ell_i \geq 2$. If $\sum_1^n \ell_i > 2$ and $\ell_i > 0$, then the point $(\ell_1, \dots, \ell_{i-1}, \ell_i - 1, \ell_{i+1}, \dots, \ell_n)$ belongs to $M(\lambda)$ (apply the shift). Repeating this procedure, we show that there is a point $\lambda' \in M(\lambda)$ with $\sum_1^n \ell'_i = 2$. It is either a root $\varepsilon_i + \varepsilon_j$, or $2\varepsilon_i$, in the second case we can obtain $\lambda'' = 2\varepsilon_j$ by acting with the Weyl group, and the midpoint of $\lambda'\lambda''$ is the point $\varepsilon_i + \varepsilon_j \in M(\lambda') \subseteq M(\lambda)$, hence $\varepsilon_1 + \varepsilon_2 \in M(\lambda)$, as well. \square

Now we show how all cases from B_n , which do not appear in Theorem 1.1, can be reduced to Examples 3.1 – 3.4, using Remark 1.9. If all coordinates of λ are integers and $n \geq 3$, then any weight λ which does not belong to Ξ can be reduced to $\varepsilon_1 + \varepsilon_2$ by Lemma 3.1, i.e. Example 3.2 can be applied. If all coordinates of λ are integers and $n = 2$, then $\lambda = (\ell_1, \ell_2) \neq (2, 0)$ but it is not a root, which gives $\ell_1 \geq 2$, hence $(2, 0) \in M(\lambda)$, and we can apply Remark 1.9 to Example 3.1.

If all coordinates of $\lambda = (\ell_1, \dots, \ell_n)$ are not integers, i.e. for all i the value $2\ell_i$ is odd, and if in addition there exists a number i such that $2\ell_i \geq 3$, then the point $(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ belongs to $M(\lambda)$ (apply several shifts), and one can apply Remark 1.9 to Example 3.3. Finally, if $\lambda = (\frac{1}{2}, \dots, \frac{1}{2})$, then we have $M(\lambda) = \{\underbrace{(\pm 1/2, \pm 1/2, \dots, \pm 1/2)}_{n \text{ coordinates}}\}$. Multiply all the coordinates of all the vectors by 2, this

does not change the problem, but now all the coordinates are integers: $M'(\lambda) = \{\underbrace{(\pm 1, \pm 1, \dots, \pm 1)}_{n \text{ coordinates}}\}$. For $n = 5$ see Example 3.4, for $n > 5$ an NSS can be con-

structed in the following way: take Example 3.4 for $n = 5$ and append $n - 5$ coordinates equal to the 5th coordinate to each vector.

3.2 The root system C_n

The root system C_n , $n \geq 3$, is formed by vectors

$$\{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \leq i, j \leq n, i \neq j\}.$$

With respect to the system of simple roots

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = 2\varepsilon_n$$

the fundamental weights have the form

$$\pi_1 = \varepsilon_1, \pi_2 = \varepsilon_1 + \varepsilon_2, \dots, \pi_n = \varepsilon_1 + \dots + \varepsilon_n.$$

The root lattice Ξ coincides with the lattice of all integer points in \mathbb{Q}^n with the even sum of coordinates. The weight lattice Λ consists of all integer points in \mathbb{Q}^n . The Weyl group W acts by permutations on the set of coordinates and by sign changes on an arbitrary subset of coordinates. A weight $\lambda = (\ell_1, \ell_2, \dots, \ell_n)$ is dominant if and only if $\ell_1 \geq \dots \geq \ell_n \geq 0$. The set $M(\lambda)$ coincides with the set of integer points in $P(\lambda)$ such that their sum of coordinates has the same parity as the sum of coordinates of λ .

3.2.1 Positive results

Case 3.4. $\lambda = \pi_1 = \varepsilon_1$. Then M is unimodular, hence hereditarily normal (cf. Case 3.1).

Case 3.5. $\lambda = \pi_2 = \varepsilon_1 + \varepsilon_2$, $n = 3, 4$. For $n = 3$ this set is unimodular, hence it is hereditarily normal. For $n = 4$ it is almost unimodular. All nonzero determinants are equal to ± 2 or ± 4 , and a 4-tuple of vectors with the determinant ± 4 without loss of generality coincides with the set of rows of the matrix

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Now the proof is analogous to Case 3.3. Suppose that there exists an ENSS $\{v_0; v_1, v_2, v_3, v_4\}$. Without loss of generality the vectors v_1, \dots, v_4 are as above, otherwise simultaneously change the sign of some coordinate in all the vectors. Assume that the corresponding $\mathbb{Q}_{\geq 0}$ -combination is $v_0 = q_1 v_1 + q_2 v_2 + q_3 v_3 + q_4 v_4$. Then all $q_i \in \{0, 1/2\}$, $q_1 = q_2$, and $q_3 = q_4$. But the sum of coordinates of v_0 is even, consequently, all q_i equal $\frac{1}{2}$ simultaneously. To obtain v_0 as a \mathbb{Z} -combination, it is necessary to append a vector which is not $\pm v_i$. However, it is easy to see that all the other vectors in $M(\lambda)$ equal $\pm \frac{1}{2} v_1 \pm \frac{1}{2} v_2 \pm \frac{1}{2} v_3 \pm \frac{1}{2} v_4$. If we augment $\{v_1, \dots, v_4\}$ by any vector of this form, then the vector $\frac{1}{2} v_1 + \frac{1}{2} v_2 + \frac{1}{2} v_3 + \frac{1}{2} v_4$ will be easily obtained as a $\mathbb{Z}_{> 0}$ -combination (cf. Section 3.1.1).

3.2.2 Some negative results

Example 3.5. $\lambda = \pi_1 + \pi_2 = (2, 1, 0)$, $n = 3$. Let

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Take $v = (1, 1, 1)$. We have $v = 1/3(v_1 + v_2 + v_3) = v_1 + v_2 - v_4$. To check that v is not a $\mathbb{Z}_{\geq 0}$ -combination of v_i , consider the discriminating function $f = 100x_1 + 10x_2 + x_3$. Then $f(v_1) = 210$, $f(v_2) = 21$, $f(v_3) = 102$, $f(v_4) = 120$, but $f(v) = 111$.

Example 3.6. Let $\lambda = 2\pi_1 = 2\varepsilon_1$, $n = 3$. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $v = \varepsilon_1 + \varepsilon_2 = 1/2(v_1 + v_2) = v_3 - v_4$. To check that v is not a $\mathbb{Z}_{\geq 0}$ -combination of vectors v_i , consider the discriminating function $f = 5x_1 + 3x_2 + 9x_3$. Then $f(v_1) = 10$, $f(v_2) = f(v_4) = 6$, $f(v_3) = 14$, but $f(v) = 8$.

Example 3.7. Take $\lambda = \pi_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $n = 3$. Consider the following vectors:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $v = \varepsilon_1 = 1/2(v_1 + v_2) = v_1 - v_3 + v_4$. To verify that v is not a $\mathbb{Z}_{\geq 0}$ -combination of vectors v_i , consider the discriminating function $f = 11x_1 + 6x_2 - 14x_3$. Then $f(v_1) = 3$, $f(v_2) = 19$, $f(v_3) = 6$, $f(v_4) = 14$, but $f(v) = 11$.

Example 3.8. $\lambda = \pi_4 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, $n = 4$. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Take $v = (1, 1, 0, 0) = \frac{1}{2}(v_1 + v_2) = v_3 - v_4$. Consider the discriminating function $f = 5x_1 + 5x_2 + 8x_3 - x_4$. Then $f(v_1) = 17$, $f(v_2) = f(v_4) = 3$, $f(v_3) = 13$, $f(v) = 10$. It is clear that v_1 and v_3 cannot be used in a $\mathbb{Z}_{\geq 0}$ -combination. But 10 is not divisible by 3, and we cannot obtain v , using only v_2 and v_4 .

Example 3.9. Take $\lambda = \pi_2 = \varepsilon_1 + \varepsilon_2$, $n = 5$. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then $v = \varepsilon_1 + \varepsilon_2 = 1/2(v_1 + v_2 + v_3 + v_4) = v_2 + v_3 + v_5 - v_6$. Take $f = 5x_1 + 6x_2 + x_3 + 2x_4 + 20x_5$. We have $f(v_1) = 6$, $f(v_2) = 4$, $f(v_3) = 8$, $f(v_4) = 4$, $f(v_5) = 21$, $f(v_6) = 22$, but $f(v) = 11$.

Remark 3.2. Examples 3.5 — 3.7 work for all $n \geq 3$, Example 3.8 works for all $n \geq 4$, and Example 3.9 works for all $n \geq 5$. Indeed, we can append $n - 3$ zero coordinates ($n - 4$ and $n - 5$, respectively) to each vector.

3.2.3 Reduction to the already examined cases

Consider two cases: (i) all $\ell_i \in \{0, 1\}$; (ii) there is at least one ℓ_i with $|\ell_i| \geq 2$. First consider case (i): all $\ell_i \in \{0, 1\}$, which means that $\lambda = \pi_k = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k$, $k \leq n$.

Lemma 3.3. *An NSS for the pair (k, n_0) is at the same time an NSS for all the pairs (k, n) , where $n \geq n_0$.*

Proof. Append $n - n_0$ zero coordinates to each vector. \square

Lemma 3.4. *An NSS for the pair (k, n) , where $k + 2 \leq n$, is also an NSS for the pair $(k + 2, n)$.*

Proof. Use Remark 1.9: if $\lambda = \varepsilon_1 + \dots + \varepsilon_{k+2}$, then

$$\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k = \lambda - (\varepsilon_{k+1} + \varepsilon_{k+2}) = \frac{1}{2}(\lambda + (\varepsilon_1 + \dots + \varepsilon_k - \varepsilon_{k+1} - \varepsilon_{k+2})),$$

hence it belongs to $M(\lambda)$. This means that an NSS for (k, n) is also an NSS for $(k + 2, n)$. \square

Now take any pair (k, n) , not equal to $(1, n)$, $(2, 2)$, $(2, 3)$, and $(2, 4)$, where $k \leq n$.

If k is even and $n \leq 4$, then it is the pair $(4, 4)$, i.e. we get Example 3.8. If k is even and $n \geq 5$, then we can modify Example 3.9 to get the required NSS: first apply Lemma 3.3, and then apply Lemma 3.4. If k is odd and $k \geq 3$, then we can modify Example 3.7 to get the required NSS in the same way.

Now consider case (ii).

Definition 3.5. By a *shift* for C_n we denote the procedure of replacing the point

$$\lambda = (\dots, l, \dots, l', \dots)$$

with the point $\lambda' = (\dots, l - 1, \dots, l' + 1, \dots)$ (at the same places) when $l - l' \geq 2$.

The point λ' belongs to $M(\lambda)$. Indeed, the point $(\dots, l', \dots, l, \dots)$ belongs to $M(\lambda)$. Its convex hull with λ with a suitable coefficient equals λ' . Notice that for $l - l' \geq 2$ the sum of squares of coordinates of the point λ decreases after a shift by a positive integer:

$$(l - 1)^2 + (l' + 1)^2 = l^2 - 2l + 1 + l'^2 + 2l' + 1 = l^2 + l'^2 - 2(l - l' - 1) < l^2 + l'^2.$$

Hence, we can consequently apply only finitely many shifts.

Lemma 3.6. *Let $\lambda = (\ell_1, \dots, \ell_n)$, such that $\ell_i \geq 2$ for some i . Then either $(2, 0, \dots, 0)$ or $(2, 1, 0, \dots, 0)$ belongs to $M(\lambda)$.*

Proof. Since λ is dominant, we have $\ell_1 \geq 2$. Now change λ , during this process it can be nondominant. Change sign at any coordinate, e.g. at ℓ_n , in such a way that $\ell_n \leq 0$, and shift it with ℓ_1 several times till the moment when ℓ_1 becomes 2. If meanwhile ℓ_n becomes positive, then change its sign to make it negative, and so on. Then fix $\ell_1 = 2$ and shift other coordinates in any possible way, changing signs at some coordinates, if needed. This process is finite, and if no further shift is possible, then it is either the point $(2, 0, \dots, 0)$, or the point $(2, 1, 0, \dots, 0)$. \square

In case (ii) we can apply Lemma 3.6 and then Remark 1.9: the required NSSs for all highest weights will be produced either from Example 3.5 or from Example 3.6.

3.3 The root system D_n

The root system D_n , $n \geq 4$, consists of vectors

$$\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i, j \leq n, i \neq j\}.$$

With respect to the system of simple roots

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n$$

the fundamental weights have the form

$$\begin{aligned} \pi_1 &= \varepsilon_1, \pi_2 = \varepsilon_1 + \varepsilon_2, \dots, \pi_{n-2} = \varepsilon_1 + \dots + \varepsilon_{n-2}, \\ \pi_{n-1} &= \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{n-1} - \varepsilon_n), \pi_n = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{n-1} + \varepsilon_n). \end{aligned}$$

The root lattice Ξ coincides with the lattice of all integer points in \mathbb{Q}^n with even sum of coordinates. Next, the weight lattice Λ has the form

$$\Lambda = \{(\ell_1, \ell_2, \dots, \ell_n) \mid 2\ell_i \in \mathbb{Z}, \ell_i - \ell_j \in \mathbb{Z}, i, j = 1, \dots, n\}.$$

The Weyl group W acts by permutations of the set of coordinates and by changing signs on any set of coordinates of even cardinality. The weight $\lambda = (\ell_1, \ell_2, \dots, \ell_n)$ is dominant if and only if $\ell_1 \geq \dots \geq \ell_n$, $\ell_{n-1} + \ell_n \geq 0$. If all the coordinates of λ are integers (strictly half-integers), then the set $M(\lambda)$ consists of all integer (strictly half-integer) points in the polytope $P(\lambda)$, such that their sum of coordinates differs with the sum of coordinates of λ by an even number.

The reasoning for D_n has another structure than in the preceding cases. The cases of integer and noninteger coordinates of a fundamental weight are considered separately. Many NSSs are taken from Section 3.2. *Shift for D_n* is the same as Shift for C_n .

3.3.1 Coordinates of all weights are integers

Case 3.6. $\lambda = \pi_1 = \varepsilon_1$. The set $M(\lambda)$ is hereditarily normal, the proof is analogous to Case 3.1 of B_n .

Case 3.7. $\lambda = \pi_2 = \varepsilon_1 + \varepsilon_2$, $n = 4$. The set $M(\lambda)$ coincides with the analogous set from Case 3.5. That set is hereditarily normal.

In all other cases we construct NSSs. We often use NSSs constructed for C_n , it is only necessary to check that for D_n the weights under consideration indeed belong to $M(\lambda)$. If a point v has a zero coordinate, then its orbits under the Weyl groups in cases C_n and D_n coincide, because the coordinate equal to 0 can be, if needed, multiplied by -1 .

Example 3.10. $\lambda = \pi_1 + \pi_2 = (2, 1, 0, 0)$. We can use Example 3.5.

Example 3.11. $\lambda = 2\pi_1 = (2, 0, 0, 0)$, $n = 4$. Example 3.6 with the appended column of zeroes works.

Example 3.12. $\lambda = \pi_3 + \pi_4 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $n = 4$. Example 3.7 with the appended column of zeroes works.

Example 3.13. $\lambda = \pi_2 = \varepsilon_1 + \varepsilon_2$, $n = 5$. Example 3.9 can be applied.

Example 3.14. $\lambda = 2\pi_4 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, $n = 4$. Example 3.8 can be applied.

Now, using Remark 1.9 applied to Examples 3.10–3.14, we show that NSSs exist in all the remaining cases: a) all coordinates of the highest weight equal ± 1 , and their sum is odd; b) all coordinates of the highest weight equal ± 1 , and their sum is even; c) λ has a coordinate such that its absolute value is 2 or more.

In this subsection all coordinates are integer, consequently, any set of weights for any n can be considered as a set of weights for a greater n , filling new coordinates with zeroes. Hence, Examples 3.10, 3.11, 3.12, and 3.14 provide us with NSSs for highest weights of the same form for all $n \geq 4$, and Example 3.13 — for all $n \geq 5$.

In case a), if there are at least 5 nonzero coordinates, make two last of them zero. For this take λ' , which differs from λ by the signs of two last coordinates, and take the midpoint of the interval $\lambda\lambda'$ instead of λ . Then make two more coordinates zero, etc., finally we reduce this case to Example 3.12.

In case b), if we have only two nonzero coordinates, we can obtain an NSS from Example 3.13: just append the required number of zeroes. If there are 4 nonzero coordinates, then an NSS can be obtained from Example 3.14 by appending the required number of zeroes. If there are more than 4 nonzero coordinates (recall that their number is even and each equals ± 1), then make two last of them zero, then two more, and repeat this procedure up to the moment when their number equals 4.

In case c), depending on the parity of $\sum_1^n \ell_i$, one has to show that either the point $(2, 0, \dots, 0)$ or the point $(2, 1, 0, \dots, 0)$ belongs to $M(\lambda)$. The proof is almost the same as in the case of C_n . On the first step change λ to a point having at least one zero coordinate. Actually, take λ' which is obtained from λ by a sign change of ℓ_{n-1} and ℓ_n simultaneously. It is important that λ' has negative coordinates. Now work with λ' . Shift ℓ_2 with any negative coordinate, e.g. with ℓ_{n-1} , till the moment when one of them becomes zero. Permute $n-1$ last coordinates to make $\ell_n = 0$. Secondly, apply the algorithm from the proof of Lemma 3.6 to $n-1$ first coordinates of λ' . If in that algorithm it is required to change λ' with a point $w\lambda'$, w belongs to the Weyl group for C_n , then the same procedure can be applied for D_n . Actually, if w changes the sign of an odd number of coordinates, which is allowed for C_n but not for D_n , then to be applicable for D_n , the element w will also change the sign of the n th coordinate, i.e. the resulting point $w\lambda'$ will be the same since $\ell_n = 0$. This modification of w will belong to the Weyl group of D_n . Now, exactly as in the case of C_n , apply Lemma 3.6 and then Remark 1.9, obtaining the required NSSs for all highest weights either from Example 3.10 or from Example 3.11.

3.3.2 Large noninteger coordinates

Here coordinates of all weights are nonintegers and there exists a coordinate whose absolute value is not less than $\frac{3}{2}$.

Lemma 3.7. *Under these conditions $M(\lambda)$ contains a point of the form*

$$\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \ell'_4, \ell'_5, \dots, \ell'_n \right),$$

where ℓ'_i are half-integers, $i = 4, \dots, n$.

Proof. Since $\lambda = (\ell_1, \dots, \ell_n)$ is dominant, ℓ_1 is one of the coordinates with the maximal absolute value. Now change λ , letting it be nondominant. If $\ell_1 > 3/2$,

we take λ' which is obtained from λ by a sign change of two last coordinates. It is important for us that λ' has negative coordinates. Now work with λ' . Shift ℓ_1 with any negative coordinates till the moment when ℓ_1 becomes equal to $3/2$. If needed, during this process make a sign change of two last coordinates again. Now fix ℓ_1 and perform the same procedure with ℓ_2 till the moment when $\ell_2 = 1/2$. If now ℓ_3 and ℓ_4 have the same sign, then change signs at ℓ_2 and ℓ_4 and shift ℓ_3 and ℓ_4 till the moment when one of them becomes $\pm 1/2$. If at some step they have the same sign, change the sign at ℓ_2 and ℓ_4 . Permuting the coordinates, if needed, we may suppose that we obtained the point $(3/2, \pm 1/2, \pm 1/2, \dots)$. Now, if needed, change the signs at the pairs of coordinates 2, 4 and 3, 4. \square

Now consider the following NSS:

$$\begin{aligned} v_1 &= \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, l'_4, l'_5, \dots, l'_n \right), \quad v_2 = \left(-\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, l'_4, l'_5, \dots, l'_n \right), \\ v_3 &= \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, l'_4, l'_5, \dots, l'_n \right), \quad v_4 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, l'_4, l'_5, \dots, l'_n \right) \\ &= \frac{1}{2} \left(\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, l'_4, l'_5, \dots, l'_n \right) + \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, l'_4, l'_5, \dots, l'_n \right) \right). \end{aligned}$$

Then $v_0 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, l'_4, l'_5, \dots, l'_n \right) = \frac{1}{2}(v_1 + v_2) = v_1 + v_4 - v_3$. If one looks at the third coordinate, it is clear that it is indeed an NSS.

3.3.3 Small noninteger coordinates

Here all coordinates of weights are noninteger and all coordinates of the highest weight are less than 1. Under these conditions

$$\lambda = (1/2, \dots, 1/2, \pm 1/2) \in \{\pi_{n-1}, \pi_n\}.$$

Since π_{n-1} and π_n are dual, it suffices to consider only the case $\lambda = \pi_n$.

Case 3.8. For $n = 4$ the set $M(\lambda)$ is a subset of $M(\pi_4)$ for B_4 (see Case 3.3). Since in the case of C_n all the subsets were saturated, here it is also true.

Now the aim is to show that for $n = 5, 6$ the answer is positive, and for $n \geq 7$ it is negative.

Case 3.9. $\lambda = \pi_5 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$, $n = 5$. Here

$$M(\lambda) = \left\{ \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2} \right) \mid \text{even number of minuses} \right\}.$$

To simplify the notation, multiply all the coordinates by 2. We obtain the set

$$M'(\lambda) = \{(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \mid \text{even number of minuses}\}.$$

Let us show that $M'(\lambda)$ is almost unimodular of volume 16. To compute a determinant of five arbitrary vectors, write them as a matrix and add the first row of this matrix to all the other rows. Now rows 2–5 are even, hence the volume of the determinant is divisible by 16. For the following vectors

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

the determinant equals 16, hence $M'(\lambda)$ is almost unimodular. Notice that each vector has length $\sqrt{5}$. The value of the determinant is at the same time the volume of the parallelepiped generated by these vectors, and the absolute value of the last number does not exceed $(\sqrt{5})^5 < 64$, hence equals 16, 32, or 48.

Letting $m = 16$, we obtain that all possible nonzero values of determinants are $\pm m$, $\pm 2m$, or $\pm 3m$.

Lemma 3.8. *If for some vectors $v_1, \dots, v_5 \in M'(\lambda)$ the scalar product $(v_1, v_2) = -3$, then*

$$|\det(v_1, \dots, v_5)| < 3m.$$

Proof. Each vector from $M'(\lambda)$ has length $\sqrt{5}$. Let S_{12} be the area of the parallelogram generated by vectors v_1 and v_2 . Since $(v_1, v_2) = -3$, we have $S_{12} = 4$. From geometrical reasons

$$|\det(v_1, \dots, v_5)| \leq S_{12} \cdot (\sqrt{5})^3 < 48 = 3m.$$

□

Lemma 3.9. *Take $v_1, \dots, v_6 \in M'(\lambda)$.*

(i) *If $|\det(v_1, \dots, v_5)| = 3m$, then one may suppose that*

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

up to the permutation of lines and the simultaneous sign change in pairs of columns.

(ii) *The following is impossible:*

$$|\det(v_1, v_2, v_3, v_4, v_5)| = |\det(v_1, v_2, v_3, v_4, v_6)| = 3m.$$

Proof. (i) It follows from Lemma 3.8 that no two of these vectors differ in four coordinates. Hence, any two of these vectors differ exactly in 2 coordinates. Without loss of generality $v_1 = (1, 1, 1, 1, 1)$ and $v_2 = (-1, -1, 1, 1, 1)$. Then each of the three other vectors has exactly two -1 s. Say that two first coordinates are *prefix*. To differ with v_2 exactly in two coordinates, each of the remaining vectors must have exactly one prefix coordinate equal to -1 . The pigeonhole principle gives us that two of them (say, v_3 and v_4) have the same prefix coordinate equal to -1 , without loss of generality this prefix coordinate is the first coordinate. Then the first coordinate of v_5 also equals -1 , otherwise v_5 cannot differ simultaneously with v_2 , v_3 , and v_4 in two coordinates. Since all the vectors are pairwise distinct, we obtain the same set as in the formulation of the Lemma.

(ii) It follows from the previous part that v_5 and v_6 cannot differ in two coordinates. Hence, if one supposes that first five vectors are as above, then v_6 has four coordinates equal to -1 . Without loss of generality this is either vector $(1, -1, -1, -1, -1)$ or vector $(-1, 1, -1, -1, -1)$. No one of these vectors works. \square

Lemma 3.10. *Suppose that for some vectors $v_1, \dots, v_6 \in M'(\lambda)$ all the absolute values of their nonzero determinants are greater than m . Then all these determinants equal $\pm 2m$.*

Proof. On the contrary, suppose that there is a determinant equalling $\pm 3m$. Then Lemma 3.9 shows that all the other nonzero determinants equal $\pm 2m$. But the alternating sum of six determinants of 5-tuples of our vectors equals $\det(v_1 - v_2, v_1 - v_3, \dots, v_1 - v_6)$. In the corresponding matrix all the entries are even, hence the determinant is divisible by $32 = 2m$. Contradiction with the fact that $3m \pm 2m \pm \dots \pm 2m$ is not divisible by $2m$. \square

Consider an ENSS $\{v_0; v_1, v_2, \dots, v_s\}$. If the rank d of this set is less than 5, take $5 - d$ vectors from $M'(\lambda)$ to make the rank equal to 5, and assign to them zero coefficients in the corresponding to v_0 \mathbb{Z} - and $\mathbb{Q}_{\geq 0}$ -combinations. Now suppose that this ENSS is $\{v_0; v_1, v_2, v_3, v_4, v_5, \dots, v_s\}$, and only v_1, v_2, v_3, v_4, v_5 appear in the $\mathbb{Q}_{\geq 0}$ -combination (maybe with zero coefficients). We may count all the determinants of the form

$$\det(v_1, \dots, \widehat{v}_i, \dots, v_5, v_s),$$

where one of the first 5 vectors is thrown out and one new vector is taken instead of it. Case a) one of them equals $\pm m$, case b) for all the nonzero determinants their absolute value is greater than m .

In case b) Lemma 3.10 gives us that we have $s - 5$ unimodular six-element subsets

$$\{v_1, \dots, v_5, v_j\}, \quad 6 \leq j \leq s,$$

with $m' = 2m$. In each of them v_j can be expressed in v_1, \dots, v_5 with integer coefficients, hence the determinant of each 5-tuple in the set $\{v_1, \dots, v_s\}$ is divisible by $2m$, hence equals $\pm 2m$. This ENSS is hereditarily normal by Lemma 1.12, a contradiction.

Case a) needs more punctuality. Lemma 1.15 gives us that the determinant $\pm m$ does not coincide with $\det(v_1, v_2, \dots, v_5)$. Without loss of generality $\det(v_1, \dots, v_4, v_6) = 16$ (if it equals -16 , transpose two first vectors, then the determinant will change sign). By our assumption $\det(v_1, \dots, v_5) = \pm 2m$ or $\pm 3m$.

Lemma 3.11. *There are no vectors w_1, \dots, w_6 in $M'(\lambda)$ such that the following is true (simultaneously):*

$$\det(w_1, \dots, w_5) = \pm 2m, \quad \det(w_5, w_2, w_3, w_4, w_6) = \pm 2m,$$

these determinants have different signs, and $\det(w_1, \dots, w_4, w_6) = \pm m$.

Proof. Straightforward check using software Maple 7. □

Lemma 3.12. *There are no vectors w_1, \dots, w_6 in $M'(\lambda)$ such that*

$$\det(w_1, \dots, w_5) = -2m \text{ and } \det(w_1, \dots, w_4, w_6) = -3m.$$

Proof. Using Lemma 3.9, we may assume that $w_6 = (1, 1, 1, 1, 1)$ and

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

The hyperplane $\langle w_1, w_2, w_3, w_4 \rangle$ is defined by the equation $2x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Since $\det(w_1, \dots, w_5) < 0$ and $\det(w_1, \dots, w_6) < 0$, we have that w_5 and w_6 belong to the same half-space with respect to this hyperplane. Hence, exactly two coordinates of w_5 equal -1 . Without loss of generality $w_5 = (1, -1, -1, 1, 1)$, but the corresponding determinant equals $-16 = -m$, a contradiction. □

Recall that we had an ENSS. Let

$$v_0 = q_1 v_1 + \dots + q_5 v_5 \tag{3.1}$$

and

$$v_0 = z_1 v_1 + \dots + z_4 v_4 + z_6 v_6 \tag{3.2}$$

be the initial $\mathbb{Q}_{\geq 0}$ - and \mathbb{Z} -combinations. Consider the decomposition

$$v_5 = y_1 v_1 + \dots + y_4 v_4 + y_6 v_6$$

of the vector v_5 in the basis $\{v_1, v_2, v_3, v_4, v_6\}$. It can be re-written as

$$v_6 = -\frac{y_1}{y_6}v_1 - \dots - \frac{y_4}{y_6}v_4 + \frac{1}{y_6}v_5. \quad (3.3)$$

Substituting (3.3) into (3.2), we obtain

$$v_0 = z_1v_1 + z_2v_2 + z_3v_3 + z_4v_4 + z_6\left(-\frac{y_1}{y_6}v_1 - \dots - \frac{y_4}{y_6}v_4 + \frac{1}{y_6}v_5\right). \quad (3.4)$$

Compare (3.1) and (3.4). From the uniqueness of the decomposition in a basis it follows that

$$q_1 = z_1 - z_6\frac{y_1}{y_6}, \dots, q_4 = z_4 - z_6\frac{y_4}{y_6}, q_5 = z_6\frac{1}{y_6}, \text{ all } q_i \in [0, 1[. \quad (3.5)$$

If $|y_6| = 3$, i.e. $|\det(v_1, v_2, \dots, v_5)| = 3m$, then by Lemma 3.9

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

The linear combination of these vectors with $\mathbb{Q}_{\geq 0}$ -coefficients q_1, \dots, q_5 must belong to the weight lattice multiplied by two, this means that all coordinates of the resulting vector must have the same parity. Subtracting the third coordinate from the second one, we obtain that $2(q_2 - q_3)$ is even, which implies $q_2 = q_3$, and analogously $q_2 = q_3 = q_4 = q_5$. The first coordinate of v_0 equals $q_1 - 4q_2$, while all the others equal $q_1 + 2q_2$. These numbers must also have the same parity, consequently, $q_2 \in \{0, \frac{1}{3}, \frac{2}{3}\}$. Since $q_1 - 4q_2$ and $q_1 + 2q_2$ are both integers and cannot simultaneously equal 0, we obtain that $q_1 = q_2 \in \{\frac{1}{3}, \frac{2}{3}\}$. Hence, v_0 equals either $(-1, 1, 1, 1, 1)$ or $(-2, 2, 2, 2, 2)$.

Lemma 3.13. *Take vectors $(v_1, v_2, v_3, v_4, v_5)$ from Lemma 3.9 and any vector*

$$v_6 \in M'(\lambda) \setminus \{v_1, \dots, v_5\}.$$

Then the vector $(-1, 1, 1, 1, 1)$ can be represented as a $\mathbb{Z}_{\geq 0}$ -combination of vectors

$$(v_1, v_2, v_3, v_4, v_5, v_6).$$

Proof. Up to the permutation of the last four coordinates, we can assume that v_6 is either $(1, 1, 1, -1, -1)$, or $(1, -1, -1, -1, -1)$, or $(-1, -1, -1, -1, 1)$. Consider these cases separately.

(i) $v_6 = (1, 1, 1, -1, -1)$. Then

$$(-1, 1, 1, 1, 1) = (1, 1, 1, -1, -1) + (-1, -1, 1, 1, 1) + (-1, 1, -1, 1, 1).$$

(ii) $v_6 = (1, -1, -1, -1, -1)$. Then

$$\begin{aligned} (-1, 1, 1, 1, 1) &= 2(1, -1, -1, -1, -1) + (1, 1, 1, 1, 1) + (-1, -1, 1, 1, 1) + \\ &\quad + (-1, 1, -1, 1, 1) + (-1, 1, 1, -1, 1) + (-1, 1, 1, 1, -1). \end{aligned}$$

(iii) $v_6 = (-1, -1, -1, -1, 1)$. Then

$$(-1, 1, 1, 1, 1) = (-1, -1, -1, -1, 1) + (1, 1, 1, 1, 1) + (-1, 1, 1, 1, -1).$$

□

It remains to consider the case when $|y_6| = 2$. Here all $q_i \in \{0, \frac{1}{2}\}$.

If z_6 is even, then $q_1 = z_1 - z_6 \frac{y_1}{y_6} = z_1 - y_1 \frac{z_6}{y_6}$ is an integer from the interval $[0, 1[$, hence it equals 0. Analogously all the other q_i s, $i = 2, \dots, 5$, equal 0, consequently, $v_0 = 0$. A contradiction.

If z_6 is odd, then the saturation property is checked in the following way. We need to construct a $\mathbb{Z}_{\geq 0}$ -combination for v_0 , to do this take the vector v_6 and add several vectors from v_1, \dots, v_5 with suitable positive coefficients to obtain v_0 . To show that it is possible, it is enough to verify that if we decompose both v_0 and v_6 in the basis v_1, \dots, v_5 , then any pair of corresponding coordinates differs by an integer and that all coordinates of v_6 are strictly less than 1. This guarantees that they do not exceed the corresponding coordinates of v_0 , since we know that the coordinates of v_0 equal q_i and belong to the interval $[0, 1[$.

By (3.5), cases $i = 1, 2, 3, 4$ and $i = 5$ should be considered separately. Since cases $i = 1, 2, 3, 4$ are symmetrical, consider only cases $i = 1$ and $i = 5$. Since $\frac{z_6-1}{y_6}$ is integer, we have that

$$q_1 - \left(-\frac{y_1}{y_6}\right) = z_1 - z_6 \frac{y_1}{y_6} + \frac{y_1}{y_6} = z_1 + \frac{(1 - z_6)y_1}{y_6}$$

is integer, analogously $q_5 - \frac{1}{y_6} = \frac{z_6-1}{y_6}$ is integer, i.e. all the differences of the corresponding coordinates are integer. We also know that $y_1 = \det(v_5, v_2, v_3, v_4, v_6)$ and

$$y_6 = \det(v_1, v_2, v_3, v_4, v_5),$$

which means that $|y_1| \in \{0, 1, 2, 3\}$ and $|y_6| = 2$. It follows from Lemmas 3.9, 3.11, and 3.12 that the number $-\frac{y_1}{y_6}$ is neither 1 nor $\frac{3}{2}$. In all the other cases the inequality $-\frac{y_i}{y_6} < 1$ is held for all i , $1 \leq i \leq 4$. It is also clear that $\frac{1}{y_6} < 1$. Hence, adding some v_i s ($1 \leq i \leq 5$), we can obtain v_0 from v_6 , and the ENSS under consideration is not an ENSS. Therefore $M'(\lambda)$ is hereditarily normal.

Case 3.10. $\lambda = \pi_6 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $n = 6$. Here

$$M(\lambda) = \{(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2) \mid \text{even number of minuses}\}.$$

For convenience let us work with the set

$$M'(\lambda) = 2M(\lambda) = \{(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \mid \text{even number of minuses}\}.$$

Lemma 3.14. *The set $M'(\lambda)$ is almost unimodular of volume 64. The values of determinants equal ± 64 and ± 128 , or, equivalently, $\pm m$ and $\pm 2m$.*

Proof. Consider a subset $\{v_1, v_2, \dots, v_6\} \subseteq M'$. Without loss of generality $v_1 = (1, 1, 1, 1, 1, 1)$. Add v_1 to each of the other vectors and write down the obtained 6 vectors as rows of a matrix. The rows from the second till the sixth are even, hence the determinant is divisible by 32, and if we divide the rows from the second till the sixth by 2, the number of 1s in each of the rows of the remaining matrix will be even. Now add to the first column of the new matrix the sum of all other columns. The new first column is even, hence the determinant of the original matrix is divisible by 64. Now find an upper bound for it. Split the vectors in three pairs and generate a parallelogram with each pair, then the volume of the parallelepiped does not exceed the product of areas of these three parallelograms. Each vector in $M'(\lambda)$ has length $\sqrt{6}$, the absolute value of the scalar product of two arbitrary vectors equals 2, hence the area of each parallelogram equals $6^{2/2} \sqrt{1 - (1/3)^2} = 2^{5/2}$. Finally, the volume does not exceed $2^{15/2} < 192$, consequently, its absolute value equals 64 or 128. \square

Suppose that we have an ENSS $\{v_0; v_1, v_2, v_3, v_4, v_5, v_6\}$ in $M'(\lambda)$. Consider a $\mathbb{Q}_{\geq 0}$ -combination corresponding to the vector v_0 . By Lemma 1.15 we have that $|\det(v_1, v_2, v_3, v_4, v_5, v_6)|$ equals 128, consequently, by Lemma 1.14 the coefficients of the initial $\mathbb{Q}_{\geq 0}$ -combination equal 0 or $\frac{1}{2}$.

Lemma 3.15. *If the determinant of the set of vectors $\{v_1, v_2, \dots, v_6\} \subseteq M'$ equals 128, then up to multiplying vectors v_i by -1 , multiplying pairs of coordinates simultaneously by -1 and interchanging columns and rows we may assume that it is the set of rows*

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}, \quad w_i \in \{v_i, -v_i\}.$$

Proof. The set $M'(\lambda)$ contains a vector $-v$ for each vector v . Hence, to compute the determinants, we may consider only 16 vectors instead of 32. Indeed, take only vectors with the positive sum of coordinates:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Without loss of generality the minor of size six contains two first rows of this matrix. By the direct check in Maple 7 we obtain that if its determinant is 128, then it is either the set of rows (1 2 3 4 5 6) or the set (1 2 7 8 9 10). Now notice that these two minors interchange if we transpose columns 5 and 6. \square

Now reasoning is analogous to Case 3.3: to obtain v_0 as a \mathbb{Z} -combination, we have to use at least one more vector v_7 . According to Lemma 1.10(ii), $v_7 \neq -v_i$, $1 \leq i \leq 6$. Show that if we augment the given six vectors by any other vector v_7 , we obtain v_0 as a $\mathbb{Z}_{\geq 0}$ -combination. For convenience till the end of this proof we suppose that $M'(\lambda)$ consists of points $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ having odd number of -1 s. Then we can reformulate the result of the preceding lemma in the following way: if the determinant of the given 6 vectors v_1, \dots, v_6 equals 128, then, acting with the Weyl group, we can map one of the (unordered) sets $(\pm v_1, \pm v_2, \dots, \pm v_6)$ to the set of rows

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}. \quad (3.6)$$

The vector $v_7 \in M'(\lambda)$ does not equal $\pm v_i$, hence three of its coordinates equal 1, and three other equal -1 . It is easy to see that any such vector can be expressed as a \mathbb{Q} -linear combination of rows of matrix (3.6) if one takes three rows with the coefficient $1/2$, and three others with the coefficient $-1/2$. Hence, the decomposition of the vector v_7 in vectors v_1, \dots, v_6 has the form $(\pm \frac{1}{2}v_1 \pm \frac{1}{2}v_2 \pm \frac{1}{2}v_3 \pm \frac{1}{2}v_4 \pm \frac{1}{2}v_5 \pm \frac{1}{2}v_6)$.

Now express v_0 as a \mathbb{Q} -linear combination of rows of (3.6). Let r_i be the coefficient of the i th row. By Lemma 1.14, each r_i equals 0 or $\pm \frac{1}{2}$ (because the rows of (3.6) may be not the initial v_i s but $-v_i$ s). Show that all r_i are zero. Compare the first and the second coordinates of v_0 . Their difference is even. On the other hand, it equals $(-r_1 + r_2 + r_3 + r_4 + r_5 + r_6) - (r_1 - r_2 + r_3 + r_4 + r_5 + r_6) = 2(r_2 - r_1)$. Consequently, either r_1 and r_2 are both zero or both nonzero. Repeating this procedure for other pairs of columns, we obtain that if $v_0 \neq 0$, then all r_i are nonzero. Hence, $v_0 = \frac{1}{2}(v_1 + v_2 + v_3 + v_4 + v_5 + v_6)$. Notice that the vector $\frac{1}{2}(v_1 + v_2 + v_3 + v_4 + v_5 + v_6)$ can be obtained from any vector v_7 of the form $(\pm \frac{1}{2}v_1 \pm \frac{1}{2}v_2 \pm \frac{1}{2}v_3 \pm \frac{1}{2}v_4 \pm \frac{1}{2}v_5 \pm \frac{1}{2}v_6)$ by adding several v_i s (cf. Section 3.1.1). Consequently, $M'(\lambda)$ is hereditarily normal.

For $n \geq 7$ multiply all the coordinates by 2. After this all the coordinates of the initial vectors become ± 1 . Now construct an NSS.

Example 3.15. Consider vectors

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

Then $v = (2, 0, 0, 0, 0, 0, 0) = \frac{1}{2}(v_1 + v_2 + v_3 + v_4) = v_5 + v_6 + v_7 - v_1$. If we consider the first coordinate, then if v is a $\mathbb{Z}_{\geq 0}$ -combination of some v_i , it is the sum of two v_i s. But no pairwise sum equals v , and it is indeed an NSS.

Example 3.15 can be easily modified for the greater values of n . Indeed, append $n - 7$ coordinates equalling 1 to each vector. It is easy to see that for $n \geq 7$ these vectors belong to $M(\lambda)$ for $\lambda = \pi_{n-1}$. Since 1 is at the same time the first coordinate of all v_i , each linear combination of v_i s will have the same value on each appended coordinate and on the first coordinate.

This proves Theorem 1.1 for root systems B_n , C_n , and D_n .

Chapter 4

Exceptional root systems

To finish the proof of Theorem 1.1 for exceptional root systems, we consider these root systems individually, in alphabetical order. Since the construction of E_7 and E_6 uses the structure of E_8 , we first treat E_8 , and then E_7 and E_6 . In these cases none of the representations has all T -orbit closures normal. Then we consider the root systems F_4 and G_2 . In both of these cases we show that only for the first fundamental representation all T -orbit closures are normal.

4.1 The root system E_8

Consider \mathbb{Q}^8 with the standard scalar product. The set of vectors

$$\pm\varepsilon_i \pm \varepsilon_j \quad (1 \leq i < j \leq 8) \quad \text{and} \quad \frac{1}{2} \sum_{i=1}^8 (-1)^{\nu_i} \varepsilon_i, \quad \text{where } \sum_{i=1}^8 \nu_i \text{ is even,}$$

form the root system E_8 , see [10, Ch. 6, §4]. The weight lattice Λ coincides with the root lattice Ξ . It has the following structure:

$$\Lambda = \Lambda_0 \cup \Lambda_1, \quad \Lambda_0 = \left\{ v \in \mathbb{Z}^8 : \sum_{i=1}^8 v_i \text{ is even} \right\}, \quad \Lambda_1 = \Lambda_0 + \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1).$$

The set Λ_1 consists of all vectors which have strictly semiinteger coordinates with the even sum. We show below that none of the sets $M(\lambda)$ for an arbitrary weight $\lambda \in \Lambda \setminus \{0\}$ (not only for the dominant weight, see Remark 1.6) is hereditary normal. This implies that for every irreducible representation there exists a non-normal T -orbit closure.

Note that the Weyl group W of the root system E_8 contains, in particular, all the permutations of coordinates, since they are generated by the reflections s_α for the vectors α of the form $\varepsilon_i - \varepsilon_j$. The group W also contains sign changes in the even

number of occurrences. E.g. the sign change of the first and the second coordinates can be obtained as the superposition of reflections $s_{\alpha_1}s_{\alpha_2}$, where $\alpha_{1,2} = \varepsilon_1 \pm \varepsilon_2$.

Consider a "marked" weight $\lambda^\circ = \varepsilon_1 + \varepsilon_2$. We show that this weight belongs to all the $M(\lambda)$. Also we show that $M(\lambda^\circ)$ contains an ENSS (for the definition of an ENSS, see Section 1.3).

Example 4.1. $\lambda = \lambda^\circ = \varepsilon_1 + \varepsilon_2$. Consider the following vectors of $M(\lambda^\circ)$:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and the vector $v_0 = (1, 1, 0, 0, 0, 0, 0, 0) = \frac{1}{2}(v_1 + v_2 + v_3 + v_4 + v_5 + v_6) = v_7 - v_8$. Suppose that v_0 is represented as a $\mathbb{Z}_{\geq 0}$ -combination of other vectors. Consider the seventh coordinate. It is clear that the coefficients at v_7 and v_8 are zero. Since the vectors v_1, \dots, v_6 are linearly independent, v_0 has the unique decomposition as a linear combination of these six vectors. The coefficients of this decomposition are non-integers. Hence the set $\{v_0; v_1, \dots, v_8\}$ is indeed an ENSS.

Next we show that for each nonzero weight λ , the weight λ° is contained in $M(\lambda)$. Consider the cases $\lambda \in \Lambda_0$ and $\lambda \in \Lambda_1$ separately.

Lemma 4.1. *If $\lambda = (a_1, \dots, a_8) \in \Lambda_0 \setminus \{0\}$, then $\lambda^\circ \in M(\lambda)$.*

Proof. Obviously, there are two coordinates of the same parity a_i, a_j such that $|a_i| + |a_j| \geq 2$. Without loss of generality, we may suppose that these coordinates are a_1 and a_2 . Acting with the Weyl group, we can obtain $a_1 \geq 0, a_2 \geq 0$. This gives $a_1 + a_2 \geq 2$. Let $a = \frac{a_1 + a_2}{2}$. Obviously, $a \geq 1$ and a is an integer. Next, the set $M(\lambda)$ contains the point $\lambda' = (a_2, a_1, -a_3, -a_4, -a_5, -a_6, -a_7, -a_8)$. Also it contains the point

$$\lambda'' = (a, a, 0, \dots, 0)$$

because λ'' is the midpoint of the segment $[\lambda, \lambda']$ and $\lambda - \lambda'' \in \Xi$. Obviously, the point λ° belongs to the segment $[\lambda'', -\lambda'']$, moreover, $\lambda^\circ - \lambda'' \in \Xi$. Finally we obtain that $\lambda^\circ \in M(\lambda'') \subseteq M(\lambda)$. \square

Lemma 4.2. *If $\lambda = (a_1, \dots, a_8) \in \Lambda_1$, then $\lambda^\circ \in M(\lambda)$.*

Proof. Take the root $\alpha_0 = (\frac{1}{2}, \dots, \frac{1}{2})$. Consider the reflection $s_{\alpha_0} \in W$. Let $a_i = b_i/2$. Then $b_i \in \mathbb{Z}$, the values b_i are odd and such that $\sum_{i=1}^8 b_i$ is divisible by 4. Thus

$$s_{\alpha_0}(\lambda) = \lambda - \frac{a_1 + \dots + a_8}{4}(1, 1, \dots, 1) = \lambda - \frac{b_1 + \dots + b_8}{8}(1, 1, \dots, 1).$$

If $\sum_{i=1}^8 b_i$ is not divisible by 8, then the vector $s_{\alpha_0}(\lambda)$ is nonzero and has integer coordinates. Hence $M(\lambda) = M(s_{\alpha_0}(\lambda)) \ni \lambda^\circ$ due to Lemma 4.1. If $\sum_{i=1}^8 b_i$ is divisible by 8, choose two coordinates having the same residues modulo 4 among b_i . We may assume that these coordinates are b_1 and b_2 . Changing their signs, we obtain the vector $\lambda' \in M(\lambda)$ such that its doubled sum of coordinates is not divisible by 8. Hence the vector $s_{\alpha_0}(\lambda')$ is nonzero and has integer coordinates. With the same reasoning as above we obtain $M(\lambda) = M(s_{\alpha_0}(\lambda')) \ni \lambda^\circ$. \square

We have shown that for E_8 and for all $\lambda \in \Lambda \setminus \{0\}$ the set $M(\lambda)$ contains a non-saturated subset. Therefore, any rational E_8 -module contains a T -orbit with nonnormal closure.

4.2 The root system E_7

For the root system E_7 , we use a realization which is slightly different from the one in [10, Ch. 6, §4]. Namely, we change the signs of the first and the last coordinates. Now the set of roots belongs to the 7-dimensional subspace $L \subset \mathbb{R}^8$, orthogonal to the vector $\varepsilon_7 - \varepsilon_8$. It contains the following vectors:

$$\begin{aligned} & \pm \varepsilon_i \pm \varepsilon_j \quad (1 \leq i < j \leq 6); \quad \pm(\varepsilon_7 + \varepsilon_8); \\ & \pm \frac{1}{2}(\varepsilon_7 + \varepsilon_8 + \sum_{i=1}^6 (-1)^{\nu_i} \varepsilon_i), \quad \text{where } \sum_{i=1}^6 \nu_i \text{ is even.} \end{aligned}$$

The root lattice Ξ is the intersection of the root lattice of E_8 with L , namely, $\Xi = \Xi_0 \cup \Xi_1$, where

$$\Xi_0 = \left\{ \ell \in \mathbb{Z}^8 : \sum_{i=1}^6 \ell_i \text{ even, } \ell_7 = \ell_8 \right\}, \quad \Xi_1 = \Xi_0 + \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1),$$

and Ξ_1 again consists of all vectors in L having strictly semiinteger coordinates with even sum.

The weight lattice Λ is an overlattice of index 2. Namely,

$$\Lambda = \Xi \cup \left(\Xi + \frac{1}{2}(1, 1, 1, -1, -1, -1, 0, 0) \right).$$

It consists of all vectors $\ell \in L$ such that all their coordinates are either integers or semiintegers, their sum is even, and for all $1 \leq i < j \leq 6$ the value $(\ell_i - \ell_j)$ is integer.

The Weyl group W contains, in particular, all permutations of the first six coordinates. It also contains the sign changes in the even number of occurrences, where the two last coordinates are either both present or both absent. The Weyl group W acts on Φ transitively.

We need analogs of Lemmas 4.1 and 4.2 from Section 4.1. The following lemmas will also be useful in the construction of examples.

Lemma 4.3. *If $\lambda = (a_1, \dots, a_8) \in \Xi \setminus \{0\}$, then $M(\lambda)$ contains all the roots.*

Proof. Consider first the case $\lambda \in \Xi_0$. We claim that $M(\lambda)$ contains either the vector $\varepsilon_1 + \varepsilon_2$ or the vector $\varepsilon_7 + \varepsilon_8$. Then by the transitivity of the Weyl group it contains actually all the roots. To show the claim, we note that if $\sum_{i=1}^6 |a_i| \neq 0$, then this sum is even. Choose two numbers of the same parity such that the sum of their absolute values is not smaller than two among a_1, \dots, a_6 . Now proceed as in the proof of Lemma 4.1. We obtain that $\varepsilon_1 + \varepsilon_2 \in M(\lambda)$. Otherwise, if $\sum_{i=1}^6 |a_i| = 0$, we have $\lambda = a_7(\varepsilon_7 + \varepsilon_8)$, and the point $\varepsilon_7 + \varepsilon_8$ belongs to the interval connecting λ'' and $(-\lambda'')$. Hence it also belongs to $M(\lambda)$.

Now consider the case $\lambda \in \Xi_1$. Write down $\lambda = \frac{1}{2}(b_1, \dots, b_6, b_7, b_7)$, where all the b_i s are odd. If $B = \sum_{i=1}^6 b_i + 2b_7$ is not divisible by 8, then $s_{\alpha_0}(\lambda) \in \Xi_0$, and this case follows from the case considered above. If B is divisible by 8, then we can choose two numbers among b_1, \dots, b_6 having the same residues modulo 4. As in the proof of Lemma 4.2, this allows to reduce the latter case to the preceding one. \square

Lemma 4.4. *Let $\lambda = (a_1, \dots, a_8) \in \Lambda \setminus \Xi$. Then $M(\lambda)$ contains the vector*

$$\lambda_* = (1, 0, 0, 0, 0, 0, 1/2, 1/2).$$

Proof. Since $\lambda \notin \Xi$, then either the six first coordinates of λ are integers and two last are semiintegers, or vice versa. If the last coordinates are integers, we can apply the reflection s_{α_0} , having changed the sign at some two of the first six coordinates, if needed, in an order to obtain the vector such that its first six coordinates are integers, and the two last are semiintegers. Now we may assume that two last coordinates are semiintegers, i.e. $\lambda = (a_1, a_2, \dots, a_6, b_7/2, b_7/2)$ for some integers a_1, \dots, a_6 and an odd integer b_7 .

Consider the vector $\lambda' = (a_1, a_2, \dots, a_6, -b_7/2, -b_7/2) \in M(\lambda)$. Since b_7 is odd, the set $M(\lambda)$ contains the vector $\lambda'' = (a_1, a_2, \dots, a_6, 1/2, 1/2) \in [\lambda, \lambda']$. Notice that $a_1 + \dots + a_6$ is odd. Consider an odd coordinate among a_i ; without loss of generality we may assume that this is a_1 . Consider all the points which can be

obtained from λ'' by the sign change in an even number of occurrences from the second till the sixth. Their barycenter will be the point $(a_1, 0, \dots, 0, 1/2, 1/2) \in M(\lambda)$.

Finally, the point $(1, 0, \dots, 0, 1/2, 1/2)$ belongs to the segment joining the points $(a_1, 0, \dots, 0, 1/2, 1/2)$ and $(-a_1, 0, \dots, 0, 1/2, 1/2)$. It follows that $M(\lambda)$ contains the point $\lambda_* \in M(\lambda)$. \square

Now we are ready to present the NSSs for all the fundamental weights.

Example 4.2. The weight $\lambda \in \Xi$. By Lemma 4.3, the set $M(\lambda)$ contains all the roots. Consider the following subset of $M(\lambda)$:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix},$$

$$v_0 = \frac{1}{2}(1, 1, 1, -1, -1, 1, 1, 1) = \frac{1}{2}(v_1 + v_2 + v_3 + v_4) = v_6 - v_5.$$

Suppose that v_0 is a $\mathbb{Z}_{\geq 0}$ -combination of vectors v_1, \dots, v_6 . The first three coordinates of v_0 being smaller than one, the coefficients at v_1, v_2, v_3 must be zero. However, in this case the fourth coordinate of this linear combination will inevitably be nonnegative, which gives a contradiction. This means that $\{v_0; v_1, \dots, v_6\}$ is indeed an ENSS.

Example 4.3. The weight $\lambda \in \Lambda \setminus \Xi$. By Lemma 4.4, the set $M(\lambda)$ contains the vector

$$\lambda_* = (1, 0, 0, 0, 0, 0, 1/2, 1/2),$$

hence it also contains the vector $-s_{\alpha_0}\lambda_* = \frac{1}{2}(-1, 1, 1, 1, 1, 1, 0, 0)$. Now we can give the NSS.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

We have:

$$\begin{aligned} v_0 = (1, 1, 1, 1, 1, 1, 0, 0) &= \frac{1}{2}(v_1 + v_2 + v_3 + v_4 + v_5 + v_6) \\ &= v_7 + v_8 + v_9 - v_{10} - v_{11} - v_{12}. \end{aligned}$$

If we consider the 7th coordinate, we see that the $\mathbb{Z}_{\geq 0}$ -combination can contain only the first 6 vectors with nonzero coefficients. Since they are linearly independent, we cannot obtain yet another decomposition of v_0 in these vectors.

Thus we have constructed an NSS for each nonzero $\lambda \in \Lambda$.

4.3 The root system E_6

Instead of the standard realization, we will use another more symmetric realization in the 6-dimensional subspace of the 9-dimensional space. The coordinates of each vector split into three triples of subsequent coordinates, and the sum in each triple equals zero. It is easy to check that the set given below is indeed a root system. Moreover, the Gram matrix for its simple roots coincides with the Gram matrix for the simple roots of the root system E_6 in its standard realization. Hence it is indeed E_6 .

The roots are: 18 vectors of the form $\varepsilon_i - \varepsilon_j$, where pairwise distinct indices i, j belong to one of the triples $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$; 27 vectors of the form $(\bar{a}; \bar{b}; \bar{c})$, where

$$\bar{a}, \bar{b}, \bar{c} \in \left\{ \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right), \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3} \right), \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3} \right) \right\},$$

and 27 vectors of the form $(\bar{a}; \bar{b}; \bar{c})$, where

$$\bar{a}, \bar{b}, \bar{c} \in \left\{ \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right), \left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3} \right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right) \right\}.$$

The simple roots are

$$\begin{aligned} \alpha_1 &= (0, 0, 0; 0, 0, 0; 0, 1, -1), & \alpha_2 &= (0, 1, -1; 0, 0, 0; 0, 0, 0), \\ \alpha_3 &= (0, 0, 0; 0, 0, 0; 1, -1, 0), & \alpha_4 &= \left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}; -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}; -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right), \\ \alpha_5 &= (0, 0, 0; 1, -1, 0; 0, 0, 0), & \alpha_6 &= (0, 0, 0; 0, 1, -1; 0, 0, 0). \end{aligned}$$

The corresponding fundamental weights are

$$\begin{aligned}\pi_1 &= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; 0, 0, 0; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right), & \pi_2 &= (1, 0, -1; 0, 0, 0; 0, 0, 0), \\ \pi_3 &= \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; 0, 0, 0; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right), & \pi_4 &= (2, -1, -1; 0, 0, 0; 0, 0, 0), \\ \pi_5 &= \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}; \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; 0, 0, 0 \right), & \pi_6 &= \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}; \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}; 0, 0, 0 \right).\end{aligned}$$

The Weyl group W of the root system E_6 contains all the permutations of coordinates in triples. Moreover, W acts on the set of roots transitively.

Let us show that for all $\lambda \in \Lambda \setminus \{0\}$ the set $M(\lambda)$ is not hereditary normal. By Lemma 1.8 and Remark 1.9 it is enough to show that already in the fundamental representations not all T -orbit closures are normal, i.e. that all the sets $M(\pi_i)$ are not hereditary normal.

Example 4.4. The cases $\lambda = \pi_1$ and $\lambda = \pi_6$ are similar, so we consider just the case $\lambda = \pi_1$. Take the root $\beta_0 = \frac{1}{3}(1, -2, 1; -2, 1, 1; 1, 1, -2)$. Then $M(\lambda)$ contains the vector

$$s_{\beta_0}(\lambda) = \frac{1}{3}(1, 1, -2; 2, -1, -1; 0, 0, 0).$$

Hence, the set $M(\lambda)$ contains the set

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & -2 \\ 2 & -1 & -1 & 0 & 0 & 0 & 1 & -2 & 1 \\ 2 & -1 & -1 & 0 & 0 & 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 2 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -2 & -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & 1 & -2 & -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 1 & 1 & -2 \end{pmatrix}.$$

We have:

$$v_0 = (1, 0, -1, 0, 0, 0, 0, 0, 0) = 1/3(v_1 + v_2 + v_3 + v_4 + v_5 + v_6) = v_1 - v_7.$$

To show that $\{v_0; v_1, \dots, v_7\}$ is an ENSS, it remains to verify that v_0 is not a $\mathbb{Z}_{\geq 0}$ -combination of vectors v_1, \dots, v_7 . Consider the discriminating linear function (see Section 1.3)

$$f = 45n_1 + 15n_2 + 30n_3 - 3n_7 - 3n_8 - 6n_9.$$

Obviously, 15 cannot be decomposed as a sum of integers 17, 14, and 2.

Example 4.5. $\lambda = \pi_2$. The set $M(\pi_2)$ contains all the roots. The following set is an ENSS:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1/3 & 1/3 & -2/3 & -2/3 & 1/3 & 1/3 & -2/3 & 1/3 & 1/3 \end{pmatrix},$$

$$v_0 = \frac{1}{3}(2, -1, -1; 2, -1, -1; 2, -1, -1) = \frac{1}{3} \sum_{i=1}^6 v_i = v_1 - v_7.$$

Using the discriminating linear function $f = 12n_1 + 3n_2$, we easily show that v_0 is not a $\mathbb{Z}_{\geq 0}$ -combination of vectors v_1, \dots, v_7 . Indeed, the values of f on vectors v_1, \dots, v_7 equal 0, 12, 9, and 5, while $f(v_0) = 7$.

Non-saturated subsets for the other fundamental weights can be constructed with the help of Examples 4.4 and 4.5. The weights π_3 and π_5 can be reduced to Example 4.4. For instance, for π_3 the vector $\pi'_3 = \frac{1}{3}(-2, 4, -2; 0, 0, 0; 2, -1, -1)$ belongs to $M(\pi_3)$, so the vector

$$\frac{1}{3}(1, 1, -2; 0, 0, 0; 2, -1, -1),$$

which is the midpoint of the segment $\pi_3\pi'_3$, also belongs to $M(\pi_3)$. Up to a cyclic permutation of the triples, this midpoint is π_6 . The vector π_5 can be reduced to π_6 in the same way. Finally, the weight π_4 can be reduced to π_2 , because

$$\pi_2 = \frac{1}{3}(2 \cdot (2, -1, -1; 0, 0, 0; 0, 0, 0) + (-1, 2, -1; 0, 0, 0; 0, 0, 0)) \in M(\pi_4).$$

We have shown that for the root system E_6 the set of weights of each nonzero representation contains an NSS.

4.4 The root system F_4

The set of roots is the following subset of \mathbb{Q}^4 :

$$\pm \varepsilon_i \quad (1 \leq i \leq 4); \quad \pm \varepsilon_i \pm \varepsilon_j \quad (1 \leq i < j \leq 4); \quad \frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4).$$

The simple roots are $\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$, ε_4 , $\varepsilon_3 - \varepsilon_4$, $\varepsilon_2 - \varepsilon_3$. The corresponding fundamental weights are:

$$\pi_1 = (1, 0, 0, 0), \pi_2 = (3/2, 1/2, 1/2, 1/2), \pi_3 = (2, 1, 1, 0), \pi_4 = (1, 1, 0, 0).$$

The root and weight lattices coincide. The cardinality of the Weyl group is 1152. It contains all the permutations of coordinates and the sign change of any subset of coordinates. Moreover, it acts transitively on the set of roots of a given length.

Case 4.1. $\lambda = \pi_1$. (Cf. Case 3.3.) Here $M(\lambda)$ is the set of all roots of length 1. Suppose on the contrary that we have an ENSS $(v_0; v_1, \dots)$.

Let us verify that all nonzero volumes of 4-tuples of vectors in $M(\lambda)$ equal 1 or $1/2$. Consider any vectors v_1, v_2, v_3, v_4 . If at least two of them are of the form $\pm\varepsilon_i$, then the statement is true. Indeed, if there are only two such vectors, then $D = \det(v_1, v_2, v_3, v_4) = 1 \cdot (\pm 1/4 \pm 1/4)$. If there are exactly three such vectors, then $D = 1 \cdot (\pm 1/2)$; if four, then $D = \pm 1$. Otherwise at least three vectors among the v_i s have the form $1/2(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$. At least two of these vectors have the same parity of the sum of coordinates. Assume that these are vectors v_1, v_2 , and their sums of coordinates are odd. Then apply the symmetry s_{α_0} with respect to the root $\alpha_0 = \frac{1}{2}(1, 1, 1, 1)$ to all the four vectors. Then the vectors v_1 and v_2 will have form $\pm\varepsilon_i$, and we have already checked the statement for the new 4-tuple of vectors.

Now apply the properties of almost unimodular sets (see Section 1.4). If we have an ENSS $\{v_0; v_1, \dots, v_s\}$, then the corresponding $\mathbb{Q}_{\geq 0}$ -combination consists of vectors with volume 1 (otherwise this combination will be integer by the Cramer formulae). From the previous reasoning we may assume that these vectors are $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$. Looking at the form of the weight lattice, we see that $v_0 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$. Note that after adding any vector v_5 , $v_5 \neq -\varepsilon_i$, we will be able to represent v_0 as a $\mathbb{Z}_{\geq 0}$ -combination of vectors v_1, \dots, v_5 . Indeed, adding to the vector v_5 of the form $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ the vectors ε_i for the indices i such that the i th coordinate of the vector v_5 is negative, we get a contradiction.

Example 4.6. $\lambda = \pi_4$. Let $v_1 = \varepsilon_1 + \varepsilon_2$, $v_2 = \varepsilon_1 - \varepsilon_2$, $v_3 = \varepsilon_1 + \varepsilon_3$, $v_4 = \varepsilon_3$, $v_0 = 1/2(v_1 + v_2) = v_3 - v_4$. Obviously, $\{v_0; v_1, \dots, v_4\}$ is an ENSS.

Fix $\alpha_0 = (1/2, 1/2, 1/2, 1/2)$. We show that π_4 is contained in the sets $M(2\pi_1)$, $M(\pi_2)$, $M(\pi_3)$. Indeed,

$$\pi_4 = \frac{1}{2}((1, 1, 2, 0) + (1, 1, -2, 0)) \in M(\pi_3).$$

Next, $s_{\alpha_0}(\pi_2) = (0, -1, -1, -1)$, hence $\pi_4 = \frac{1}{2}((1, 1, 1, 0) + (1, 1, -1, 0)) \in M(\pi_2)$. Finally $s_{\alpha_0}(2\pi_1) = (1, -1, -1, -1)$, which implies

$$\pi_4 = \frac{1}{2}((1, 1, 1, -1) + (1, 1, -1, 1)) \in M(2\pi_1).$$

For all the other dominant weights, their decomposition in the fundamental weights contains either $2\pi_1$, or π_2 , or π_3 , or π_4 , and applying Lemma 1.8 we deduce the desired conclusion.

4.5 The root system G_2

We realize G_2 in the 2-dimensional subspace $\{\ell_1\varepsilon_1 + \ell_2\varepsilon_2 + \ell_3\varepsilon_3 \mid \ell_1 + \ell_2 + \ell_3 = 0\}$ of the space \mathbb{Q}^3 . The following vectors are the roots:

$$\begin{aligned} &\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \\ &\pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2). \end{aligned}$$

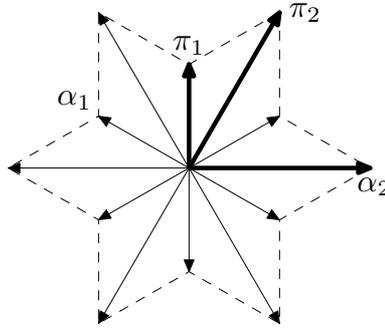


Fig. 5

The simple roots are the vectors $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (-2, 1, 1)$. The fundamental weights are

$$\pi_1 = (0, -1, 1), \quad \pi_2 = (-1, -1, 2).$$

It is clear from Fig. 5 that the Weyl group is the dihedral group of order 12. The root and weight lattices coincide.

Case 4.2. $\lambda = \pi_1$. Then $M(\lambda) = \{\pm(0, 1, -1), \pm(1, 0, -1), \pm(1, -1, 0)\}$, which is hereditary normal.

Example 4.7. $\lambda = \pi_2$. It is the long (dominant) root, from Fig. 5 it is clear that $M(\lambda)$ coincides with the whole root system. Consider the following vectors (the thick edges in Fig. 5):

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $v_0 = 1/3(v_1 + v_2) = (-1, 0, 1) = v_1 - v_3$. Consider the last coordinate. It is clear that v_0 is not a $\mathbb{Z}_{\geq 0}$ -combination of vectors v_1, v_2, v_3 .

All the other dominant weights can be reduced to π_2 . Indeed, if the vector $\lambda = c_1\pi_1 + c_2\pi_2$ has a nonzero coefficient c_2 , then the set $M(\lambda)$ contains $M(\pi_2)$, hence $M(\lambda)$ is not hereditary normal. If $c_2 = 0$ but $\lambda \neq \pi_1$, then $c_2 \geq 2$. Then the vector $2\pi_1 = (0, -2, 2)$ belongs to $M(\lambda)$. Consequently, the vectors $(-2, 0, 2)$ and $(-1, -1, 2) = \pi_2$ belong to $M(\lambda)$, see Fig. 5.

Now the proof of Theorem 1.1 is completed.

Part II

Infinitely transitive actions on affine varieties

Chapter 5

Infinitely transitive actions over \mathbb{C}

Definition 5.1. The action of a group G on a set A is called *m-transitive*, if for any two tuples of pairwise distinct points (a_1, a_2, \dots, a_m) and (b_1, b_2, \dots, b_m) in A there exists an element $g \in G$ such that $g(a_i) = b_i$, $i = 1, 2, \dots, m$. By abuse of language, the actions which are *m-transitive* for all $m \in \mathbb{N}$ will be called *infinitely transitive*.

In this chapter we study infinitely transitive actions of the special automorphism group $\text{SAut}(X)$ on an irreducible affine algebraic variety X over an algebraically closed field of characteristic zero. Infinite transitivity of $\text{SAut}(X)$ is *a priori* stronger than infinite transitivity of the whole automorphism group $\text{Aut}(X)$. However, it is more convenient for us since our tools describe $\text{SAut}(X)$ in a rather comfortable manner.

All varieties are assumed being reduced and irreducible. Let X be such a variety.

Definition 5.2. By a *special automorphism group* $\text{SAut}(X)$ we mean the subgroup of $\text{Aut}(X)$ generated by all its one-parameter unipotent subgroups.

Definition 5.3. Let X be an algebraic variety over \mathbb{k} . We say that a point $x \in X$ is *flexible* if the tangent space $T_x X$ is spanned by the tangent vectors to the orbits $H.x$ of one-parameter unipotent subgroups $H \subseteq \text{Aut}(X)$. The variety X is called *flexible* if every smooth point $x \in X_{\text{reg}}$ is.

Recall the following definition.

Definition 5.4. We call a *suspension* over an affine variety Y the hypersurface $X \subseteq Y \times \mathbb{A}^2$ given by equation $uv - f(y) = 0$, where $\mathbb{A}^2 = \text{Spec } \mathbb{k}[u, v]$ and $f \in \mathbb{k}[Y]$ is non-constant. In particular, $\dim X = 1 + \dim Y$.

5.1 Affine cones over flag varieties

Our goal here is to prove the following statement.

Theorem 5.5. *Consider a flag variety G/P over \mathbb{k} . Then every normal affine cone X over G/P is flexible and its special automorphism group $\text{SAut}(X)$ acts infinitely transitively on the smooth locus $X_{\text{reg}} = X \setminus \{0\}$.*

Given a connected simply connected semisimple linear algebraic group G over \mathbb{k} , we consider an irreducible representation $V(\lambda)$ of G with highest weight λ and a highest weight vector $v_0 \in V(\lambda)$. Let

$$Y = G[v_0] \subseteq \mathbb{P}(V(\lambda))$$

be the closed G -orbit of $[v_0]$ in the associate projective representation, and

$$X = \text{AffCone}(Y) = \overline{Gv_0} = Gv_0 \cup \{0\}$$

be the affine cone over Y (such a cone X is called an *HV-variety* in terminology of [58]).

Remark 5.6. Actually every projective embedding $\varphi : G/P \xrightarrow{\cong} Y \subseteq \mathbb{P}^n$ with a projectively normal image Y , where $P \subseteq G$ is a parabolic subgroup, arises in this way. Indeed, being projectively normal Y is as well linearly normal, i.e. $\varphi = \varphi|_{D|}$, where $D \in \text{Pic}(G/P)$ is very ample. Hence $D \sim \sum_{i=1}^s a_i D_i$, where D_1, \dots, D_s are the Schubert divisorial cycles on G/P , and $a_i > 0 \forall i = 1, \dots, s$. Then $V(\lambda) = H^0(G/P, \mathcal{O}_{G/P}(D))^\vee$ is a simple G -module with a highest weight $\lambda = \sum_{i=1}^s a_i \omega_i$, where $\omega_1, \dots, \omega_s$ are fundamental weights, and $Y = G[v_0]$ for a highest weight vector v_0 ; see e.g. [55, Theorem 5].

Now we prove Theorem 5.5. The flexibility follows from the next general observation.

Proposition 5.7. *If a semisimple linear algebraic group G acts on an affine variety X and this action is transitive in X_{reg} , then X is flexible.*

Proof. The group G acts on X with an open orbit $X_{\text{reg}} = G.x_0$. The dominant morphism onto this orbit $\varphi : G \rightarrow X$, $g \mapsto g.x_0$, yields a surjection $d\varphi : \mathfrak{g} \rightarrow T_{x_0}X$, where $\mathfrak{g} = \text{Lie}(G)$. We claim that \mathfrak{g} is spanned over \mathbb{k} by nilpotent elements, which implies the assertion. Indeed, consider the decomposition $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ of \mathfrak{g} into simple subalgebras. Let \mathfrak{h} be the span of the set of all nilpotent elements in \mathfrak{g} . This is an ad-submodule of \mathfrak{g} and so an ideal of \mathfrak{g} , hence a direct sum of some of the simple ideals \mathfrak{g}_i . However every simple ideal \mathfrak{g}_i , $i = 1, \dots, k$ contains at least one nonzero nilpotent element. Therefore $\mathfrak{h} = \mathfrak{g}$, as claimed. \square

In the setting of Theorem 5.5, $X_{\text{reg}} = X$ if $X \cong \mathbb{A}^n$ and $X_{\text{reg}} = X \setminus \{0\}$ otherwise. Anyhow, the group G acts transitively on $X \setminus \{0\}$, see e.g. [58, Theorem 1]. Hence by Proposition 5.7 X is flexible.

Before passing to the proof of infinite transitivity, we need some preparation.

Let $P \subseteq G$ be the stabilizer of the line $\langle v_0 \rangle \subseteq V(\lambda)$, $B = TB_u \subseteq P$ be a Borel subgroup of G with the maximal torus T and the unipotent radical B_u , and $\Lambda(T)$ be the character lattice of T . Consider the T -invariant (weight) decompositions

$$V(\lambda) = \bigoplus_{\nu \in \Lambda(T)} V(\lambda)_\nu = \langle v_0 \rangle \oplus H(\lambda),$$

where $\langle v_0 \rangle = V(\lambda)_\lambda$ and $H(\lambda) \subseteq V(\lambda)$ is the hyperplane

$$H(\lambda) = \bigoplus_{\nu \in \Lambda(T) \setminus \{\lambda\}} V(\lambda)_\nu.$$

The coordinate function $l_\lambda \in V(\lambda)^*$ of the first projection $p_1 : v \mapsto l_\lambda(v)v_0$ defines a non-trivial character of P .

Let $B^- = TB_u^-$ be the Borel subgroup of G opposite to $B = B^+$. The flag variety G/P contains an open B^- -orbit (the big Schubert cell) isomorphic to the affine space \mathbb{A}^n , where $n = \dim G/P$. Its complement is a union of the divisorial Schubert cycles D_1, D_2, \dots, D_s , see e.g. [45, pp. 22–24].

The orbit map $G \rightarrow \mathbb{P}(V(\lambda))$, $g \mapsto g \cdot [v_0]$, embeds G/P onto a subvariety $Y \subseteq \mathbb{P}(V(\lambda))$. Let $\omega_\lambda \subseteq Y$ be the image of the big Schubert cell under this embedding. By [55, Thm. 2] the hyperplane

$$\mathcal{H}(\lambda) = \mathbb{P}(H(\lambda)) = l_\lambda^{-1}(0) \subseteq \mathbb{P}(V(\lambda))$$

is supported by the union of the Schubert divisors $\bigcup_{i=1}^s D_i$. In particular, $\omega_\lambda = Y \setminus \mathcal{H}(\lambda)$.

Let $\sigma : \widehat{X} \rightarrow X$ be the blow-up of the cone X at the vertex 0. The exceptional divisor $E \subseteq \widehat{X}$ is isomorphic to Y . Moreover, the natural map $\pi : X \setminus \{0\} \rightarrow Y$ yields the projection $p : \widehat{X} \rightarrow Y$ of the line bundle $\mathcal{O}_Y(-1)$ on Y with E being the zero section. Since $\omega_\lambda \cong \mathbb{A}^n$, the restriction $p : p^{-1}(\omega_\lambda) \rightarrow \omega_\lambda$ is a trivial line bundle. Hence the open set

$$\Omega_\lambda := \pi^{-1}(\omega_\lambda) = X \setminus H(\lambda) \subseteq X \setminus \{0\} \cong \widehat{X} \setminus E$$

is isomorphic to $\mathbb{A}^n \times \mathbb{A}_*^1$.

For every $c \in \mathbb{A}_*^1$ the invertible function $l_\lambda(\cdot, c)$ is constant on the affine space \mathbb{A}^n . Thus on $\Omega_\lambda \cong \mathbb{A}^n \times \mathbb{A}_*^1$ we have $l_\lambda = az^k$ for some $a \in \mathbb{k}^\times$, where z is a coordinate in \mathbb{A}_*^1 . Here actually $k = 1$ since l_λ gives a coordinate on $\langle v_0 \rangle$. We may assume that also $a = 1$ and so $l_\lambda|_{\Omega_\lambda} : \Omega_\lambda \rightarrow \mathbb{A}_*^1$ is the second projection.

To prove the infinite transitivity of the group $\text{SAut}(X)$ in $X \setminus \{0\}$ as stated in Theorem 5.5, let us first show the infinite transitivity of $\text{SAut}(X)$ in each hyperplane section $\Omega_\lambda(c_0) := l_\lambda^{-1}(c_0) \subseteq X$, where $c_0 \neq 0$; cf. [31, Lemma 5.6]. More precisely, given a k -tuple of distinct points $c_1, \dots, c_k \in \mathbb{k}$ different from c_0 , we consider the subgroup $\text{Stab}_{c_1, \dots, c_k}^\lambda \subseteq \text{SAut}(X)$ of all automorphisms fixing pointwise the subvarieties $\Omega_\lambda(c_i)$ for all $i = 1, \dots, k$ and leaving invariant the function l_λ .

Proposition 5.8. *In the notation as above, for any $n \geq 2$ and any $m \in \mathbb{N}$ the group $\text{Stab}_{c_1, \dots, c_k}^\lambda$ acts m -transitively on $\Omega_\lambda(c_0) \cong \mathbb{A}^n$.*

Proof. Let Q_1, Q_2, \dots, Q_m and Q'_1, Q'_2, \dots, Q'_m be two tuples of pairwise distinct points in $\Omega_\lambda(c_0)$. For any $n \geq 2$ the group $\text{SAut}(\mathbb{A}^n)$ acts m -transitively on \mathbb{A}^n ; see e.g. [31, Lemma 5.5]. Since $\Omega_\lambda(c_0) \cong \mathbb{A}^n$, we can find $g \in \text{SAut}(\Omega_\lambda(c_0))$ mapping (Q_1, Q_2, \dots, Q_m) to $(Q'_1, Q'_2, \dots, Q'_m)$. By definition, $g = \delta_1(1)\delta_2(1)\dots\delta_s(1)$ for some one-parameter unipotent subgroups $\delta_1, \delta_2, \dots, \delta_s \subseteq \text{SAut}(\Omega_\lambda(c_0))$. Let $\partial_1, \partial_2, \dots, \partial_s$ be the corresponding locally nilpotent derivations¹ (LNDs for short). First we extend them to LNDs $\bar{\partial}_1, \bar{\partial}_2, \dots, \bar{\partial}_s$ of $\mathbb{k}[\Omega_\lambda] \cong \mathbb{k}[\mathbb{A}^n \times \mathbb{A}_*^1]$ by putting $\bar{\partial}_i(l_\lambda) = 0$.

Recall that Ω_λ is a principal Zariski open subset in X defined by the function l_λ . In particular, for every $i = 1, \dots, s$ we have $\bar{\partial}_i: \mathbb{k}[X] \rightarrow \mathbb{k}[X][1/l_\lambda]$. Since $\mathbb{k}[X]$ is finitely generated, there exists $N \in \mathbb{N}$ such that $(l_\lambda)^N \bar{\partial}_i$ is an LND of $\mathbb{k}[X]$ for all $i = 1, \dots, s$; cf. [33, Proposition 3.5].

Let $q[z] \in \mathbb{k}[z]$ be a polynomial with $q(c_0) = 1$ which has simple roots at c_1, \dots, c_k and a root $z = 0$ of multiplicity N (we recall that $c_0 \neq 0$). Then for every $i = 1, \dots, s$, $q(l_\lambda) \bar{\partial}_i$ is an LND of $\mathbb{k}[X]$ such that the corresponding one-parameter subgroup in $\text{Stab}_{c_1, \dots, c_k}^\lambda$ extends the subgroup δ_i . Thus g extends to an element of the group $\text{Stab}_{c_1, \dots, c_k}^\lambda$. Now the assertion follows. \square

Let μ be an extremal weight of the simple G -module $V(\lambda)$ different from λ . Then μ defines a parabolic subgroup P' conjugated to P , the corresponding linear form $l_\mu \in V(\lambda)^*$, and the principal Zariski open subset $\Omega_\mu = \{l_\mu \neq 0\}$ of X , where $X \setminus \Omega_\mu = H(\mu) := l_\mu^{-1}(0)$.

Lemma 5.9. *For every set of m distinct points $Q_1, Q_2, \dots, Q_m \in X \setminus \{0\}$ there exists $g \in \text{SAut}(X)$ such that $g(Q_i) \in \Omega_\mu$ for all $i = 1, \dots, m$.*

Proof. Since the group G is semisimple, it is contained in $\text{SAut}(X)$ (see [56, Lemma 1.1]). Clearly, $G_i := \{g \in G \mid g(Q_i) \in H(\mu)\}$ ($i = 1, \dots, m$) are proper closed subsets of G . Hence the conclusion of the lemma holds for any $g \in G \setminus (G_1 \cup \dots \cup G_m)$. \square

¹A derivation ∂ of a ring A is called locally nilpotent if $\forall a \in A, \partial^n a = 0$ for $n \geq 1$.

Lemma 5.10. *For every $c \neq 0$ the restriction $l_\lambda|_{\Omega_\mu(c)}$ is non-constant.*

Proof. If the restriction $l_\lambda|_{\Omega_\mu(c)}$ were a constant equal, say, a , then the cone X would be contained in the hyperplane $al_\mu - cl_\lambda = 0$ in $V(\lambda)$, which is not the case. \square

Proof of Theorem 5.5. If $n = \dim G/P = 1$ and so $G/P \cong \mathbb{P}^1$, then X is a normal affine toric surface (a Veronese cone). The infinite transitivity in this case follows from Theorem 5.11 below.

From now on we suppose that $n \geq 2$. Given $m \in \mathbb{N}$, we fix an m -tuple of distinct points $Q_1^0, Q_2^0, \dots, Q_m^0 \in \Omega_\lambda(1)$. Let us show that for any m -tuple of distinct points $Q_1, Q_2, \dots, Q_m \in X \setminus \{0\}$ there exists $\psi \in \text{SAut}(X)$ such that $\psi(Q_1) = Q_1^0, \dots, \psi(Q_m) = Q_m^0$.

According to Lemma 5.9 we may suppose that $Q_i \in \Omega_\mu$ for all $i = 1, \dots, m$. Divide the set $\{Q_1, Q_2, \dots, Q_m\}$ into several pieces according to the values of $l_\mu(Q_i)$:

$$\{Q_1, Q_2, \dots, Q_m\} = \bigcup_{j=1}^k M_j, \quad M_j = \{Q_i \mid Q_i \in \Omega_\mu(c_j)\},$$

where $c_1, \dots, c_k \in \mathbb{k}^\times = \mathbb{k} \setminus \{0\}$ are distinct. By Lemma 5.10, every intersection $\Omega_\lambda(1) \cap \Omega_\mu(c_i)$ contains infinitely many points. Acting with the subgroups $\text{Stab}_{c_1, \dots, \widehat{c}_i, \dots, c_k}^\mu \subseteq \text{SAut}(X)$ (see Proposition 5.8), we can successively send the pieces M_i ($i = 1, \dots, k$) to the affine hyperplane section $\Omega_\lambda(1)$. Further, the resulting m -tuple can be sent to the standard one $(Q_1^0, Q_2^0, \dots, Q_m^0)$ using an automorphism from Proposition 5.8 with $c_0 = 1$ and arbitrary c_1, \dots, c_k . Now the proof is completed. \square

5.2 Automorphisms of affine toric varieties

In this section we consider an affine toric variety X over \mathbb{k} with a torus \mathbb{T} acting effectively on X . We assume that X is non-degenerate, i.e. the only invertible regular functions on X are constants or, which is equivalent, that $X \not\cong Y \times \mathbb{A}_*^1$, where $\mathbb{A}_*^1 = \text{Spec } \mathbb{k}[t, t^{-1}] \cong \mathbb{k}^\times$.

Theorem 5.11. *Every non-degenerate affine toric variety X of dimension $n \geq 1$ is flexible. Furthermore, for $n \geq 2$ and for any $m \in \mathbb{N}$ the group $\text{SAut}(X)$ acts m -transitively on the smooth locus X_{reg} of X .*

We note that X is flexible if $\text{Aut}(X)$ acts transitively on X_{reg} and at least one smooth point is flexible on X . Both properties will be established below. Let us recall first some necessary generalities on toric varieties.

1. *Ray generators.*

Let N be the lattice of one-parameter subgroups of the torus \mathbb{T} , $M = \Lambda(\mathbb{T})$ the dual lattice of characters, and $\langle \cdot, \cdot \rangle : N \times M \rightarrow \mathbb{Z}$ the natural pairing. Let χ^m be the character of \mathbb{T} which corresponds to a lattice point $m \in M$ (so that $\chi^m \chi^{m'} = \chi^{m+m'}$). The group algebra

$$\mathbb{k}[M] := \bigoplus_{m \in M} \mathbb{k}\chi^m$$

can be identified with the algebra $\mathbb{k}[\mathbb{T}]$ of regular functions on the torus \mathbb{T} . Let $\mathbb{T}.x_0$ be the open \mathbb{T} -orbit in X . Since the orbit map $\mathbb{T} \rightarrow X$, $t \mapsto t.x_0$, is dominant, we may identify $\mathbb{k}[X]$ with a subalgebra of $\mathbb{k}[M]$. More precisely, there exists a convex polyhedral cone $\sigma^\vee \subseteq M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathbb{k}[X]$ coincides with the semigroup algebra of σ^\vee , i.e.

$$\mathbb{k}[X] = \bigoplus_{m \in \sigma^\vee \cap M} \mathbb{k}\chi^m, \quad (5.1)$$

see [24] for details. We denote by $\sigma \subseteq N_{\mathbb{Q}}$ the cone dual to σ^\vee . The cone σ is pointed and of full dimension in $N_{\mathbb{Q}}$. Let $\Xi = \{\rho_1, \dots, \rho_r\}$ be the set of ray generators, i.e., the primitive vectors on extremal rays of the cone σ . In this section Ξ is not the same as in Part I. Given a ray generator $\rho \in \Xi$, we denote by R_ρ the associate one-parameter subgroup of \mathbb{T} .

2. *The Orbit-Cone correspondence [15, §3.2].*

There exist two natural one to one correspondences $\delta \xleftrightarrow{1-1} \tau \xleftrightarrow{1-1} O_\tau$ between the faces δ of σ , the dual² faces $\tau = \delta^\perp$ of σ^\vee , and the \mathbb{T} -orbits O_τ on X such that $\dim O_\tau = \dim \tau = \dim \sigma - \dim \delta$. In particular, the unique fixed point of \mathbb{T} in X corresponds to the vertex of σ^\vee , and the open \mathbb{T} -orbit to the cone σ^\vee itself. These correspondences respect the inclusions: the \mathbb{T} -orbit O_μ meets the orbit closure $\overline{O_\tau}$ if and only if $O_\mu \subseteq \overline{O_\tau}$, if and only if $\mu \subseteq \tau$, if and only if $\mu^\perp \supseteq \tau^\perp$; cf. [15, §3.2].

Given a face $\tau \subseteq \sigma^\vee$, there is a direct sum decomposition

$$\mathbb{k}[X] = \mathbb{k}[\overline{O_\tau}] \oplus I(\overline{O_\tau}),$$

where

$$\mathbb{k}[\overline{O_\tau}] = \bigoplus_{m \in \tau \cap M} \mathbb{k}\chi^m, \quad \text{and} \quad I(\overline{O_\tau}) = \bigoplus_{m \in (\sigma^\vee \setminus \tau) \cap M} \mathbb{k}\chi^m \quad (5.2)$$

is the graded ideal of $\overline{O_\tau}$ in $\mathbb{k}[X]$.

²By abuse of notation, here $\delta^\perp \cap \sigma^\vee$ is denoted simply by δ^\perp .

A stabilizer $\mathbb{T}_p = \text{Stab}_{\mathbb{T}}(p)$ of any point $p \in X$ is connected, hence $\mathbb{T}_p \subseteq \mathbb{T}$ is a subtorus. Furthermore, $\mathbb{T}_p \subseteq \mathbb{T}_q$ if and only if $\mathbb{T}.q \subseteq \overline{\mathbb{T}.p}$. Moreover, $\overline{\mathbb{T}.p} = X^{\mathbb{T}_p}$ (here X^G stands, as usual, for the set of fixed points of the group G acting on X).

3. *Roots and associate one parameter unipotent subgroups.*

Definition 5.12. (M. Demazure [18]) A *root* of the cone σ is a vector $e \in M$ such that for some index i with $1 \leq i \leq r$, where $r = \text{card } \Xi$, we have

$$\langle \rho_i, e \rangle = -1 \quad \text{and} \quad \langle \rho_j, e \rangle \geq 0 \quad \text{for every } j \neq i. \quad (5.3)$$

Let $\mathcal{R}(\sigma)$ be the set of all roots of the cone σ . There is a one to one correspondence $e \xleftrightarrow{1-1} H_e$ between the roots of σ and the one-parameter unipotent subgroups of $\text{Aut}(X)$ normalized by the torus, see [18] or [46]. Letting $\rho_e := \rho_i$, the root $e \in \mathcal{R}(\sigma)$ defines an LND ∂_e of the M -graded algebra $\mathbb{k}[X]$ given by

$$\partial_e(\chi^m) = \langle \rho_e, m \rangle \chi^{m+e}. \quad (5.4)$$

Its kernel is a (finitely generated) graded subalgebra of $\mathbb{k}[X]$ (see [46]):

$$\ker \partial_e = \bigoplus_{m \in \rho_e^\perp \cap M} \mathbb{k}\chi^m, \quad (5.5)$$

where $\rho_e^\perp = \{m \in \sigma^\vee \cap M, \langle \rho_e, m \rangle = 0\}$ is the facet³ of σ^\vee orthogonal to ρ_e .

Definition 5.13. (see [23], [46]) Two roots e and e' with $\rho_e = \rho_{e'}$ are called *equivalent*; we write $e \sim e'$. Two roots e and e' are equivalent if and only if $\ker \partial_e = \ker \partial_{e'}$.

Remark 5.14. Enumerating the ray generators $\Xi = \{\rho_1, \dots, \rho_r\}$ yields a disjoint partition

$$\mathcal{R}(\sigma) = \bigcup_{i=1}^r \mathcal{R}_i, \quad \text{where } \mathcal{R}_i = \{e \in \mathcal{R}(\sigma) \mid \rho_e = \rho_i\}$$

are nonempty. Indeed, consider the facet τ_i of σ^\vee orthogonal to the ray generator ρ_i . For every v in the relative interior $\text{Int}_{\text{rel}}(\tau_i)$ the inequalities $\langle \rho_j, m \rangle \geq 0$ hold for all $j \neq i$. Let $e_0 \in M$ be such that $\langle \rho_i, e_0 \rangle = -1$, and let $v_0 \in \text{Int}_{\text{rel}}(\tau_i) \cap M$. Letting $e = e_0 + kv_0$ with $k \geq 1$, we obtain $\langle \rho_j, e \rangle > 0$ for all $j \neq i$ and $\langle \rho_i, e \rangle = -1$. Hence $e \in \mathcal{R}_i$, i.e., $\rho_e = \rho_i$.

For instance, let $X = \mathbb{A}^2$ be the affine plane with the standard 2-torus action so that σ and σ^\vee coincide with the first quadrants. Then $\mathcal{R}(\sigma)$ consists of two equivalence classes

$$\mathcal{R}_1 = \{(x, -1) \mid x \in \mathbb{Z}_{\geq 0}\} \quad \text{and} \quad \mathcal{R}_2 = \{(-1, y) \mid y \in \mathbb{Z}_{\geq 0}\}.$$

³Facets of the cone σ^\vee are its codimension one faces.

4. *One-parameter groups of automorphisms.*

The derivation ∂_e generates a one-parameter unipotent subgroup

$$H_e = \lambda_e(G_a(\mathbb{k})) \subseteq \text{Aut}(X),$$

where $\lambda_e : t \mapsto \exp(t\partial_e)$. The algebra of invariants $\mathbb{k}[X]^{H_e}$ coincides with $\ker \partial_e$. The inclusion $\mathbb{k}[X]^{H_e} \subseteq \mathbb{k}[X]$ induces a morphism $\pi : X \rightarrow Z = \text{Spec } \mathbb{k}[X]^{H_e}$ whose general fibers are one-dimensional H_e -orbits isomorphic to \mathbb{A}^1 , cf. [59, Theorems 2.3 and 3.3]. The torus \mathbb{T} normalizes the subgroup H_e , i.e. $\mathbb{T} \subseteq N(H_e)$. Hence \mathbb{T} stabilizes the fixed point set X^{H_e} .

Let $R_e = R_{\rho_e} \subseteq \mathbb{T}$ be the one-parameter subgroup which corresponds to the vector $\rho_e \in N$. The action of R_e on the graded algebra $\mathbb{k}[X]$ can be given, under a suitable parametrization $\rho_e : G_m(\mathbb{k}) \ni t \mapsto \rho_e(t) \in R_e$, by

$$t \cdot \chi^m = t^{\langle \rho_e, m \rangle} \chi^m, \quad t \in G_m(\mathbb{k}). \quad (5.6)$$

In particular, $\mathbb{k}[X]^{R_e} = \mathbb{k}[X]^{H_e}$. Hence the morphism $\pi : X \rightarrow Z$ coincides with the quotient map $X \rightarrow X//R_e$. So the general H_e -orbits are closures of general R_e -orbits. By Proposition 5.17 below the latter holds actually for every one-dimensional H_e -orbit.⁴

There is a direct sum decomposition

$$\mathbb{k}[X] = \mathbb{k}[X]^{R_e} \oplus \bigoplus_{m \in \sigma^\vee \cap M \setminus \rho_e^\perp} \mathbb{k}\chi^m = \mathbb{k}[X]^{R_e} \oplus I(D_e), \quad (5.7)$$

where $D_e := X^{R_e} \cong Z$. The divisor D_e coincides with the attractive set of the action of R_e on X . So every one-dimensional R_e -orbit has a limit point on D_e . The following simple lemma completes the picture.

Lemma 5.15. *Let τ be a face of σ^\vee , \mathcal{O}_τ the corresponding orbit, \mathbb{T}_τ the stabilizer of a point in \mathcal{O}_τ , and Ξ_τ the set of ray generators orthogonal to τ , i.e., the ray generators of the dual face $\tau^\perp \subseteq \sigma$. Then the following hold.*

(a) *The orbit closure $\overline{\mathcal{O}_\tau}$ is stable under H_e if and only if*

$$m + e \in \sigma^\vee \setminus \tau \quad \forall m \in (\sigma^\vee \setminus \tau) \cap M \quad \text{such that} \quad \langle \rho_e, m \rangle > 0. \quad (5.8)$$

(b) *The closure $\overline{\mathcal{O}_\tau}$ is $H_{e'}$ -stable for any root $e' \sim e$ of σ if one of the following equivalent conditions is fulfilled:*

⁴If X is a surface then all fibers of π are R_e -orbit closures isomorphic to \mathbb{A}^1 (see [22], the parabolic case). While for a toric affine 3-fold X some degenerate fibers of π can be two-dimensional.

- (i) $\rho_e \notin \Xi_\tau$,
- (ii) $\overline{\mathcal{O}_\tau} \not\subseteq D_e$,
- (iii) $R_e \not\subseteq \mathbb{T}_\tau$.

Proof. (a) By virtue of (5.4) the ideal $I(\overline{\mathcal{O}_\tau})$ is ∂_e -invariant if and only if

$$\chi^{m+e} \in I(\overline{\mathcal{O}_\tau}) \quad \forall \chi^m \in I(\overline{\mathcal{O}_\tau}) \quad \text{such that} \quad \langle \rho_e, m \rangle > 0, \quad (5.9)$$

which is equivalent to (5.8) (see (5.2)). This proves (a).

(b) For $m \in M$,

$$m \in \sigma^\vee \setminus \tau \Leftrightarrow \langle \rho, m \rangle \geq 0 \quad \forall \rho \in \Xi \quad \text{and} \quad \exists \rho \in \Xi_\tau : \langle \rho, m \rangle > 0.$$

For any $\rho \neq \rho_e$ we have $\langle \rho, m+e \rangle \geq \langle \rho, m \rangle$. Hence (i) \Rightarrow (5.8).

We have $\overline{\mathcal{O}_\tau} = X^{\mathbb{T}_\tau}$ and $D_e = X^{R_e} = \overline{\mathcal{O}_{\rho_e^\perp}}$, where $\rho_e^\perp = (\mathbb{R}_+\rho_e)^\perp$. So the equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) are due to the Orbit-Cone correspondence. \square

Remark 5.16. Consequently, $\rho_e \in \Xi_\tau$ if $\overline{\mathcal{O}_\tau}$ is not H_e -stable. In general, the converse is not true. For instance, let $X = \mathbb{A}^2$ be the plane with the standard torus action so that $\Xi = \{(1,0), (0,1)\}$. Let

$\tau = \{\bar{0}\}$ and $e = (0, -1)$, $e' = (a, -1) \sim e$, where $a > 0$ and $\rho_e = (0, 1)$. Then (5.8) holds for $H_{e'}$ and not for H_e . Hence, $\rho_e \in \Xi_\tau$ and the \mathbb{T} -fixed point $\overline{\mathcal{O}_\tau} = \{(0, 0)\}$ is $H_{e'}$ -stable but not H_e -stable.

One can also construct an example with $\dim X = 4$ such that the closure $\overline{\mathcal{O}_\tau}$ is $H_{e'}$ -stable for every root $e' \sim e$, whereas the equivalent conditions (i)-(iii) are not fulfilled.

For the proof of infinite transitivity we need somewhat more precise information concerning one-parameter group actions on toric varieties, see Proposition 5.17 and Lemmas 5.20-5.22 below.

Proposition 5.17. *Given a root $e \in \mathcal{R}(\sigma)$. Let, as above, $H_e \subseteq \text{SAut}(X)$ be the associate one-parameter unipotent subgroup. Then the following hold.*

(a) *For every point $x \in X \setminus X^{H_e}$ the orbit $H_e \cdot x$ meets exactly two \mathbb{T} -orbits \mathcal{O}_1 and \mathcal{O}_2 on X , where $\dim \mathcal{O}_1 = 1 + \dim \mathcal{O}_2$.*

(b) *The intersection $\mathcal{O}_2 \cap H_e \cdot x$ consists of a single point, while*

$$\mathcal{O}_1 \cap H_e \cdot x = R_e \cdot y \quad \forall y \in \mathcal{O}_1 \cap H_e \cdot x.$$

Proof. (a) The number of \mathbb{T} -orbits in X being finite, there exists a \mathbb{T} -orbit \mathcal{O}_1 such that $\mathcal{O}_1 \cap H_e.x$ is a Zariski open subset of $H_e.x$. So $H_e.x \subseteq \overline{\mathcal{O}_1}$. There is also another \mathbb{T} -orbit \mathcal{O}_2 that meets $H_e.x$. Indeed, otherwise $H_e.x \cong \mathbb{A}^1$ would be contained in a single \mathbb{T} -orbit \mathcal{O}_1 . However, this is impossible because the algebra of regular functions on \mathcal{O}_1 is generated by invertible elements. Since \mathcal{O}_2 meets $\overline{\mathcal{O}_1}$ we have $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ and so $\dim \mathcal{O}_2 < \dim \mathcal{O}_1$.

The torus \mathbb{T} normalizes the unipotent subgroup H_e . Hence the elements of \mathbb{T} send the H_e -orbits into H_e -orbits. In particular, for every point $q \in H_e.x$ the stabilizer \mathbb{T}_q preserves the orbit $H_e.x$. For all $q \in \mathcal{O}_1 \cap H_e.x$ the stabilizer is the same. Since $H_e.x \subseteq \overline{\mathcal{O}_1} = X^{\mathbb{T}_q}$, this stabilizer acts trivially on $H_e.x$. Thus $\mathbb{T}_r \supseteq \mathbb{T}_q$ for any point $r \in H_e.x$, and $\mathbb{T}_r = \mathbb{T}_q$ if and only if $r \in \mathcal{O}_1 \cap H_e.x$.

Fix a point $p \in \mathcal{O}_2 \cap H_e.x$. If $\mathbb{T}_p \subseteq \mathbb{T}_q$ then $\mathbb{T}_p = \mathbb{T}_q$ and so $\dim \mathcal{O}_2 = \dim \mathcal{O}_1$, a contradiction. Consequently, the stabilizer \mathbb{T}_p acts on $H_e.x$ with two orbits, i.e. $H_e.x = \mathbb{T}_p.q \cup \{p\}$, where $q \in H_e.x \setminus \{p\}$. From the exact sequence

$$1 \rightarrow \mathbb{T}_q \rightarrow \mathbb{T}_p \rightarrow G_m(\mathbb{k}) \rightarrow 1$$

we deduce that $\dim \mathbb{T}_p = 1 + \dim \mathbb{T}_q$. Finally $H_e.x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ and $\dim \mathcal{O}_1 = 1 + \dim \mathcal{O}_2$, as stated in (a).

(b) We may assume that $\mathcal{O}_1 = \mathbb{T}.x$. Since $H_e.x \subseteq \overline{\mathcal{O}_1}$ and the torus \mathbb{T} normalizes the subgroup H_e , we have $H_e(\mathcal{O}_1) \subseteq \overline{\mathcal{O}_1}$. Thus $\overline{\mathcal{O}_1}$ is H_e -stable. On the other hand, since $H_e.p = H_e.x \not\subseteq \overline{\mathcal{O}_2}$, the closure $\overline{\mathcal{O}_2}$ is not H_e -stable. In particular, by Lemma 5.15 $\rho_e \in \Xi_{\tau_2}$, where τ_i is the face of σ^\vee which corresponds to $\mathcal{O}_i = \mathcal{O}_{\tau_i}$, $i = 1, 2$. Hence by the same lemma $R_e \subseteq \mathbb{T}_{\tau_2} = \mathbb{T}_p$. Let us show that $R_e \not\subseteq \mathbb{T}_{\tau_1} = \mathbb{T}_q$, where $q \in H_e.x \setminus \{p\}$.

Applying again Lemma 5.15 we obtain that (5.8) holds for $\tau = \tau_1$ but not for $\tau = \tau_2$. Since $\tau_2 \subseteq \tau_1$ this implies that (5.8) does not hold for some $m \in \tau_1 \setminus \tau_2$. The latter is only possible if $\rho_e \in \tau_2^\perp \setminus \tau_1^\perp$. Thus by Lemma 5.15 $R_e \not\subseteq \mathbb{T}_{\tau_1} = \mathbb{T}_q$.

Finally, the one-dimensional orbit $\mathbb{T}_p.q$ coincides with $R_e.q$. This ends the proof. \square

Definition 5.18. We say that a pair of \mathbb{T} -orbits $(\mathcal{O}_1, \mathcal{O}_2)$ in X is H_e -connected if $H_e.x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ for some $x \in X \setminus X^{H_e}$. By Proposition 5.17 $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ and $\dim \mathcal{O}_2 = 1 + \dim \mathcal{O}_1$ for such a pair (up to a permutation). Clearly, we can choose a point x on the orbit \mathcal{O}_2 , as above. Since the torus normalizes the subgroup H_e , any point of \mathcal{O}_2 can actually serve as such a point x .

Example 5.19. Given a root $e \in \mathcal{R}(\sigma)$, the derivation ∂_e as in (5.4) extends to an LND of a bigger graded algebra

$$A(\rho_e) = \bigoplus_{m \in M, \langle \rho_e, m \rangle \geq 0} \mathbb{k}\chi^m.$$

Indeed, letting $k = \langle \rho_e, m \rangle \geq 0$ yields $\langle \rho_e, m + ke \rangle = 0$ and so $\partial_e^k(\chi^m) \in \ker \partial_e$. This provides a \mathbb{T} - and H_e -stable open subset

$$U = \operatorname{Spec} A(\rho_e) \cong (\mathbb{A}_*^1)^{n-1} \times \mathbb{A}^1 \subseteq X,$$

where $n = \dim X$, $\mathbb{A}_*^1 = \operatorname{Spec} \mathbb{k}[t, t^{-1}]$, $\mathbb{A}^1 = \operatorname{Spec} \mathbb{k}[u]$, where $u = \chi^{-e}$, and H_e acts by the shifts along the second factor. The only \mathbb{T} -orbits in U are the open orbit $\mathcal{O}_1 = \{u \neq 0\}$ (which corresponds to the vertex of σ) and the codimension one orbit $\mathcal{O}_2 = \{u = 0\}$ (which corresponds to the ray $\mathbb{k}\rho_e$ of σ). It is easily seen that the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected.

From Proposition 5.17 and its proof we deduce the following criterion of H_e -connectedness.

Lemma 5.20. *Let $(\mathcal{O}_1, \mathcal{O}_2)$ be a pair of \mathbb{T} -orbits on X with $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$, where $\mathcal{O}_i = \mathcal{O}_{\sigma_i^\perp}$ for a face σ_i of σ , $i = 1, 2$. Given a root $e \in \mathcal{R}(\sigma)$, the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected if and only if $e|_{\sigma_2} \leq 0$ and σ_1 is a facet of σ_2 given by equation $\langle v, e \rangle = 0$.*

Proof. In course of the proof of Proposition 5.17(b) we established that the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected if and only if $\overline{\mathcal{O}_1}$ is H_e -invariant, $\overline{\mathcal{O}_2}$ is not, and $\dim \mathcal{O}_1 = 1 + \dim \mathcal{O}_2$. Moreover, if $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected then σ_2^\perp is a facet of σ_1^\perp (and so σ_1 is a facet of σ_2), and there exists $m_0 \in \sigma_1^\perp \setminus \sigma_2^\perp$ such that

$$\langle \rho_e, m_0 \rangle > 0 \quad \text{and} \quad m_0 + e \in \sigma_2^\perp.$$

Since $\langle \rho_i, e \rangle \geq 0 \forall \rho_i \neq \rho_e$, we obtain that $\sigma_2 = \operatorname{cone}(\sigma_1, \rho_e)$. We have also $e|_{\sigma_1} = 0$ because $e = m_0 + e - m_0 \in \operatorname{span} \sigma_1^\perp$. Thus $e|_{\sigma_2} \leq 0$ and σ_1 is given in σ_2 by equation $\langle v, e \rangle = 0$.

Conversely, assume that $e|_{\sigma_2} \leq 0$ and σ_1 is given in σ_2 by equation $\langle v, e \rangle = 0$. Then for any $m \in \sigma^\vee \setminus \sigma_1^\perp$ with $\langle \rho_e, m \rangle > 0$ we have $m + e \notin \sigma_1^\perp$ (indeed, $e|_{\sigma_1} = 0$ and so $e \in \sigma_1^\perp$). Thus (5.8) holds for σ_1^\perp . Furthermore, $\langle \rho_e, m' \rangle > 0$ for any $m' \in \sigma_1^\perp \setminus \sigma_2^\perp$. It follows that

$$m_0 := m' + (\langle \rho_e, m' \rangle - 1) \cdot e \in \sigma_1^\perp \setminus \sigma_2^\perp \quad \text{and} \quad m_0 + e \in \sigma_2^\perp.$$

Indeed, $\langle \rho_e, m_0 \rangle = 1$ and $\langle \rho_e, m_0 + e \rangle = 0$ while $\langle \rho_i, m_0 \rangle \geq 0$ and $\langle \rho_i, m_0 + e \rangle \geq 0$ for every $\rho_i \neq \rho_e$. Therefore (5.8) is fulfilled for σ_1^\perp and not for σ_2^\perp . Consequently, the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H_e -connected. \square

Remark 5.21. Given a one-parameter subgroup $R \subseteq \mathbb{T}$ and a point $x \in X \setminus X^R$, the orbit closure $\overline{R.x}$ coincides with an H_e -orbit if and only if $\overline{R.x}$ is covered by a pair of H_e -connected \mathbb{T} -orbits. For instance, for $X = \mathbb{A}^2$ with the standard torus action and $R \subseteq \mathbb{T}$ being the subgroup of scalar matrices, the latter condition holds only for the points $x \neq 0$ on one of the coordinate axis, which are \mathbb{T} -orbits as well.

Recall that $\mathcal{O}_{\sigma^\vee}$ is the open \mathbb{T} -orbit in X .

Lemma 5.22. *For any $x \in X_{\text{reg}} \setminus \mathcal{O}_{\sigma^\vee}$ there is a root $e \in \mathcal{R}(\sigma)$ such that*

$$\dim \mathbb{T}.y > \dim \mathbb{T}.x$$

for a general point $y \in H_e.x$. In particular, the pair $(\mathbb{T}.y, \mathbb{T}.x)$ is H_e -connected.

Proof. Since $x \notin \mathcal{O}_{\sigma^\vee}$, by the Orbit-Cone Correspondence there exists a proper face, say, $\sigma_2 \subseteq \sigma$ such that $\mathbb{T}.x = \mathcal{O}_{\sigma_2^\perp}$. The point $x \in X$ being regular the ray generators, say, ρ_1, \dots, ρ_s of σ_2 form a base of a primitive sublattice $N' \subseteq N$, see [24, §2.1]. Let σ_1 be the facet of $\overline{\sigma_2}$ spanned by ρ_2, \dots, ρ_s . Again by the Orbit-Cone Correspondence, $\mathcal{O}_{\sigma_2^\perp} \subseteq \overline{\mathcal{O}_{\sigma_1^\perp}}$ and $\dim \mathcal{O}_{\sigma_1^\perp} = 1 + \dim \mathcal{O}_{\sigma_2^\perp}$. Let us show that the pair $(\mathcal{O}_{\sigma_1^\perp}, \mathcal{O}_{\sigma_2^\perp})$ is H_e -connected for some root $e \in \mathcal{R}(\sigma)$ satisfying the assumptions of Lemma 5.20.

Choosing a σ -supporting hyperplane $L \subseteq N_{\mathbb{Q}}$ such that $\sigma_2 = \sigma \cap L$ we obtain a splitting $N = N' \oplus N'' \oplus N'''$, where $N \cap L = N' \oplus N''$ and $N''' \cong \mathbb{Z}$. Consider a linear form e_1 on N' defined by

$$\langle \rho_1, e_1 \rangle = -1 \quad \text{and} \quad \langle \rho_2, e_1 \rangle = \dots = \langle \rho_s, e_1 \rangle = 0.$$

Let e_2 be a non-zero linear form on N''' . Extending e_1 and e_2 to the whole lattice N by zero on the complementary sublattices we obtain a linear form $e = e_1 + e_2$ on N . Multiplying e_2 by a suitable integer we can achieve that $\langle \rho_j, e \rangle > 0$ for every $\rho_j \notin \sigma_2$. Then e is a root of the cone σ such that $\rho_e = \rho_1$ and the condition of Lemma 5.20 holds for e . By Lemma 5.20 the pair $(\mathcal{O}_{\sigma_1^\perp}, \mathcal{O}_{\sigma_2^\perp})$ is H_e -connected, as claimed. Since $\mathbb{T}.x = \mathcal{O}_{\sigma_2^\perp}$ and the torus \mathbb{T} normalizes the group H_e , the desired conclusion follows from Proposition 5.17 and the observation in Definition 5.18. \square

The proof of infinite transitivity in Theorem 5.11 is based on Lemmas 5.23-5.27 below.

Lemma 5.23. *For any collection of m distinct points $Q_1, \dots, Q_m \in X_{\text{reg}}$ there exists an automorphism $\phi \in \text{SAut}(X)$ such that the images*

$$\phi(Q_1), \dots, \phi(Q_m)$$

are contained in the open \mathbb{T} -orbit.

Proof. Let

$$d(Q_1, \dots, Q_m) = \dim \mathbb{T}.Q_1 + \dots + \dim \mathbb{T}.Q_m.$$

Assume that $\dim \mathbb{T}.Q_i < \dim X$ for some i . By Lemma 5.22 there exists a root $e \in \mathcal{R}(\sigma)$ such that $\dim \mathbb{T}.P_i > \dim \mathbb{T}.Q_i$ for a general point $P_i \in H_e.Q_i$. Fix an

isomorphism $\lambda_e : G_a(\mathbb{k}) \xrightarrow{\cong} H_e$. There is a finite set of values $t \in G_a(\mathbb{k})$ such that $\dim \mathbb{T} \cdot (\lambda_e(t) \cdot Q_j) < \dim \mathbb{T} \cdot Q_j$ for some $j \neq i$. Thus for a general $t \in G_a(\mathbb{k})$

$$d(\lambda_e(t) \cdot Q_1, \dots, \lambda_e(t) \cdot Q_m) > d(Q_1, \dots, Q_m).$$

Applying recursion, we get the result. □

From now on we assume that Q_1, \dots, Q_m are contained in the open \mathbb{T} -orbit, say, $\mathbb{T} \cdot x_0$. We fix a maximal subset of linearly independent ray generators $\{\rho_1, \dots, \rho_n\} =: \Xi^{(0)} \subseteq \Xi$, where $n = \dim X$. For every $i = 1, \dots, n$ we choose an isomorphism $\rho_i : G_m(\mathbb{k}) \xrightarrow{\cong} R_{\rho_i}$ (denoted by the same letter as the ray generator). Recall that for a root $e \in \mathcal{R}(\sigma)$ the inclusion $\mathbb{k}[X]^{H_e} \subseteq \mathbb{k}[X]$ induces a morphism $\tau : X \rightarrow Z$, where $Z = \text{Spec } \mathbb{k}[X]^{H_e}$.

Lemma 5.24. *Given $\rho_i \in \Xi^{(0)}$ and a root $e \in \mathcal{R}(\sigma)$ with $\rho_e = \rho_i$, for every finite set $\mathcal{T}_0, \dots, \mathcal{T}_k$ of pairwise distinct R_e -orbits in $\mathbb{T} \cdot x_0$ there exists a regular invariant $q \in \mathbb{k}[X]^{H_e}$ which equals identically 1 on \mathcal{T}_0 and vanishes on $\mathcal{T}_1, \dots, \mathcal{T}_k$.*

Proof. The quotient morphism $\tau : X \rightarrow Z$ separates typical H_e -orbits, see [59, Theorems 2.3 and 3.3]. Since the torus \mathbb{T} normalizes the group H_e , there is a \mathbb{T} -action on Z such that the morphism τ is \mathbb{T} -equivariant. In particular, for every $x \in \mathbb{T} \cdot x_0$ the fiber of τ through x is an H_e -orbit. According to Proposition 5.17 the R_e -orbits $\mathcal{T}_0, \dots, \mathcal{T}_k$ are intersections of the corresponding H_e -orbits with the open orbit $\mathbb{T} \cdot x_0$. Thus for every $j = 1, \dots, k$ there exists an invariant $q_j \in \mathbb{k}[X]^{H_e}$ which vanishes on \mathcal{T}_j and restricts to \mathcal{T}_0 as the constant function 1. It is easily seen that the product $q = q_1 \cdot \dots \cdot q_k \in \mathbb{k}[X]^{H_e}$ has the desired properties. □

In the notation of Lemma 5.24, we denote by $\text{Stab}_{\mathcal{T}_1, \dots, \mathcal{T}_k}(\mathcal{T}_0) \subseteq \text{SAut}(X)$ the subgroup of all transformations that fix pointwise the orbits $\mathcal{T}_1, \dots, \mathcal{T}_k$ and stabilize the closure $\overline{\mathcal{T}_0}$ in X .

Lemma 5.25. *There exists a one-parameter unipotent subgroup*

$$H \subseteq \text{Stab}_{\mathcal{T}_1, \dots, \mathcal{T}_k}(\mathcal{T}_0)$$

which acts transitively on $\overline{\mathcal{T}_0}$.

Proof. We let, as above, $e \in \mathcal{R}(\sigma)$ be a root with $\rho_e = \rho_i$, and q be a regular H_e -invariant, as in Lemma 5.24. The LND $q\partial_e \in \text{Der } \mathbb{k}[X]$ defines a one-parameter unipotent subgroup $H \subseteq \text{Stab}_{\mathcal{T}_1, \dots, \mathcal{T}_k}(\mathcal{T}_0)$. Clearly the restriction $H|_{\mathcal{T}_0} = H_e|_{\mathcal{T}_0}$ acts transitively on $\overline{\mathcal{T}_0} \cong \mathbb{A}^1$ by shifts. □

In the remaining part of the proof of Theorem 5.11 we use the following notation. For a basis $\Xi^{(0)} = \{\rho_1, \dots, \rho_n\}$ in $N_{\mathbb{Q}}$ formed by ray generators, we consider the homomorphism $\theta: (\mathbb{k}^{\times})^n \rightarrow \mathbb{T}$ of the standard n -torus to \mathbb{T} given by

$$\theta: (t_1, \dots, t_n) \mapsto (\rho_1(t_1) \cdot \dots \cdot \rho_n(t_n)). \quad (5.10)$$

It is easily seen that θ is surjective and its kernel $\Theta = \ker(\theta)$ is a finite subgroup in $(\mathbb{k}^{\times})^n$. We consider as well the induced surjective morphism of $(\mathbb{k}^{\times})^n$ to the open orbit $\mathbb{T}.x_0$. In particular, given m distinct points $Q_1, \dots, Q_m \in \mathbb{T}.x_0$, we can write

$$Q_j = \theta(t_{1,j}, \dots, t_{n,j}).x_0, \quad j = 1, \dots, m, \quad (5.11)$$

where the point $(t_{1,j}, \dots, t_{n,j}) \in (\mathbb{k}^{\times})^n$ is determined by Q_j up to the diagonal action of Θ on $(\mathbb{k}^{\times})^n$:

$$\vartheta.(t_1, \dots, t_n) = (\vartheta_1 t_1, \dots, \vartheta_n t_n), \quad \text{where } \vartheta = (\vartheta_1, \dots, \vartheta_n) \in \Theta. \quad (5.12)$$

Letting $\kappa = \text{ord } \Theta$, by the Lagrange Theorem we have $\vartheta_i^{\kappa} = 1 \ \forall i = 1, \dots, n$.

We fix a standard collection of m points in $\mathbb{T}.x_0$:

$$Q_j^0 = \theta(j, \dots, j).x_0, \quad j = 1, \dots, m. \quad (5.13)$$

Since by our assumptions $\text{char}(\mathbb{k}) = 0$ and $\overline{\mathbb{k}} = \mathbb{k}$, these points are distinct. It remains to find a special automorphism $\varphi \in \text{SAut}(X)$ such that $\varphi(Q_j) = Q_j^0$ for every $j = 1, \dots, m$. To this end we use the following Lemmas 5.26 and 5.27.

We will say that $t, t' \in \mathbb{k}^{\times}$ are κ -equivalent if $t' = \varepsilon t$ for some κ th root of unity $\varepsilon \in \mathbb{k}^{\times}$.

Lemma 5.26. (a) For any distinct elements $t_1, \dots, t_n \in \mathbb{k}^{\times}$ the set of values $a \in \mathbb{k}$ such that $t_i + a$ and $t_j + a$ are κ -equivalent for some $i \neq j$ is finite.

(b) Fix $s \in \{1, \dots, n\}$. If the points Q_i and Q_j lie on the same R_s -orbit then their r th components $t_{i,r}$ and $t_{j,r}$ are κ -equivalent for every $r \neq s$.

(c) Suppose that the points Q_{j_1}, \dots, Q_{j_l} lie on the same R_s -orbit $\mathcal{T}^{(s)}$. Then their images under a general shift on the line $\overline{\mathcal{T}^{(s)}} \cong \mathbb{A}^1$ belong to distinct R_r -orbits for every $r \neq s$.

Proof. (a) Given a κ th root of unity ε the linear equation

$$t_i + a = \varepsilon(t_j + a)$$

is satisfied for at most one value of a . Now (a) follows.

The assertion of (b) holds since the R_s -action on X lifted via (5.10) affects only the component $t_{i,s}$ of Q_i in (5.11), while the Θ -action on $(\mathbb{k}^{\times})^n$ replaces the component $t_{i,r}$ ($r \neq s$) by a κ -equivalent one.

Now (c) is immediate from (a) and (b). Indeed, for $i \neq j$ the intersection of any R_i - and R_j -orbits is at most finite. \square

Lemma 5.27. *In the notation as above there exists $\psi \in \text{SAut}(X)$ such that the points $\psi(Q_1), \dots, \psi(Q_m)$ lie in different R_1 -orbits in $\mathbb{T}.x_0$.*

Proof. By our assumption $n \geq 2$, so there is an R_2 -action on X . Let $\mathcal{T}_0^{(2)}, \dots, \mathcal{T}_k^{(2)}$ be the distinct R_2 -orbits passing through the points Q_1, \dots, Q_m so that this collection splits into $k + 1$ disjoint pieces. We may assume that the piece on $\mathcal{T}_0^{(2)}$ is Q_1, \dots, Q_l . Applying Lemma 5.25 with $\rho_e = \rho_2$, we can find a one-parameter unipotent subgroup $H \subseteq \text{Stab}_{\mathcal{T}_1^{(2)}, \dots, \mathcal{T}_k^{(2)}}(\mathcal{T}_0^{(2)})$ acting by shifts on $\overline{\mathcal{T}_0^{(2)}} \cong \mathbb{A}^1$. By Lemma 5.26 the images of Q_1, \dots, Q_l under a general such shift lie in different R_r -orbits for every $r \neq 2$, while all the other points Q_j ($j > l$) remain fixed. Applying the same procedure subsequently to the other pieces we obtain finally a special automorphism $\psi \in \text{SAut}(X)$ such that the points $\psi(Q_1), \dots, \psi(Q_m)$ belong to different R_r -orbits for every $r \neq 2$. \square

Proof of infinite transitivity in Theorem 5.11. By virtue of Lemma 5.27, we may assume that the orbits $\mathcal{T}_j^{(1)} = R_1.Q_j$, $j = 1, \dots, m$, are all distinct. By Lemma 5.25 we can change the component $t_{1,j}$ of a point Q_j arbitrarily while fixing the other components and as well the other points of our collection. Thus we can achieve that $t_{1,j} = j$ for all $j = 1, \dots, m$. This guarantees that the orbits $R_l.Q_1, \dots, R_l.Q_m$ are pairwise distinct for any $l \geq 2$. Applying Lemma 5.25 again to every R_l -orbit for $l = 2, \dots, n$, we can reach the standard collection $Q_1^{(0)}, \dots, Q_m^{(0)}$ as in (5.13) with $t_{l,j} = j$ for all $j = 1, \dots, m$, $l = 1, \dots, n$. This proves the infinite transitivity statement in Theorem 5.11. For the proof of flexibility, see the next lemma. \square

Lemma 5.28. *X is flexible.*

Proof. If $\dim X = 1$, then $X \cong \mathbb{A}^1$ and the assertion is evidently true. Suppose further that $\dim X \geq 2$. We know already that the group $\text{SAut}(X)$ acts (infinitely) transitively on X_{reg} . Hence it is enough to find just one flexible point in X_{reg} . Let us show that the point x_0 in the open \mathbb{T} -orbit is flexible. Consider the action of the standard torus $(\mathbb{k}^\times)^n$ on X induced by the \mathbb{T} -action on X via (5.10). The stabilizer $\text{Stab}(x_0) \subseteq (\mathbb{k}^\times)^n$ being finite, the tangent map $T_g(\mathbb{k}^\times)^n \rightarrow T_{x_0}X$ at each point $g \in \text{Stab}(x_0)$ is surjective. Hence the tangent vectors at x_0 to the orbits $R_i.x_0$, $i = 1, \dots, n$, span the tangent space $T_{x_0}X$. By Remark 5.14 for every $i = 1, \dots, n$ there exists a root $e_i \in \mathcal{R}(\sigma)$ such that $\rho_i = \rho_{e_i}$. Since x_0 cannot be fixed by the one-parameter unipotent subgroup H_{e_i} , $H_{e_i}.x_0 = \overline{R_i}.x_0$ by virtue of Proposition 5.17. Parameterizing properly these two orbits, their velocity vectors at x_0 coincide. Therefore, $T_{x_0}X$ is spanned by the tangent vectors of the orbits $H_{e_i}.x_0$, $i = 1, \dots, n$, as well, which means that the point x_0 is flexible on X (see Definition 5.3). \square

Now the proof of Theorem 5.11 is completed.

Example 5.29. Consider an affine toric surface $X_{d,e} = \mathbb{A}^2/G_d$, where d and e are coprime integers with $0 < e < d$, and G_d is the cyclic group generated by a primitive d th root of unity ζ acting on the plane \mathbb{A}^2 via $\zeta \cdot (x, y) = (\zeta x, \zeta^e y)$. It is well known [22], [26], [54] that for $e \geq 2$ the smooth locus $(X_{d,e})_{\text{reg}} = X_{d,e} \setminus \{0\}$ is not isomorphic to a homogeneous space of any affine algebraic group. However, $X_{d,e} \setminus \{0\}$ is homogeneous under the action of the infinite dimensional group $\text{SAut}(X)$.

5.3 Affine suspensions

Let \mathbb{k} be an algebraically closed field of characteristic zero.

Theorem 5.30. *Suppose that an affine variety $X = X^{(0)}$ over \mathbb{k} is flexible and either $X \cong \mathbb{A}^1$, or $\dim X^{(0)} \geq 2$ and the special automorphism group $\text{SAut}(X^{(0)})$ acts infinitely transitively on the smooth locus $X_{\text{reg}}^{(0)}$. Then all iterated suspensions over $X^{(0)}$ have the same properties, namely, they are flexible and the special automorphism group acts infinitely transitively on the set of its regular points.*

In this section we prove Theorem 5.30. Let us first recall necessary notions.

Let $X^{(0)}$ be an affine variety. By a *cylinder* over $X^{(0)}$ we mean the product $X^{(0)} \times \mathbb{A}^1$. Given a nonconstant regular function $f_1 \in \mathbb{k}[X^{(0)}]$, we define a new affine variety

$$X^{(1)} = \text{Susp}(X^{(0)}, f_1) := \{f_1(x) - uv = 0\} \subseteq X^{(0)} \times \mathbb{A}^2.$$

It is a suspension over $X^{(0)}$, see Definition 5.4. By recursion, for any $l \in \mathbb{N}$ we obtain the iterated suspension $X^{(l)} = \text{Susp}(X^{(l-1)}, f_l)$.

For instance, starting with $X^{(0)} = \mathbb{A}^k$, we arrive at the l th suspension $X^{(l)}$ given in the affine space $\mathbb{A}^{k+2l} = \text{Spec } \mathbb{k}[x_1, \dots, x_k, u_1, v_1, \dots, u_l, v_l]$ by equations

$$\begin{cases} u_1 v_1 - f_1(x_1, x_2, \dots, x_k) = 0 \\ u_2 v_2 - f_2(x_1, x_2, \dots, x_k, u_1, v_1) = 0 \\ \vdots \\ u_l v_l - f_l(x_1, x_2, \dots, x_k, u_1, v_1, u_2, v_2, \dots, u_{l-1}, v_{l-1}) = 0, \end{cases} \quad (5.14)$$

where for every $i = 1, 2, \dots, l$ the polynomial

$$f_i \in \mathbb{k}[x_1, \dots, x_k, u_1, v_1, \dots, u_{i-1}, v_{i-1}]$$

is non-constant modulo the ideal $(u_1 v_1 - f_1, \dots, u_{i-1} v_{i-1} - f_{i-1})$.

We separate the two cases in Theorem 5.30 according to the dimension of the base. In dimension 1, the assertion holds true over an arbitrary field of characteristic zero under an additional restriction on the function $f = f_1$.

Theorem 5.31. *Let \mathbb{k} be a field of characteristic zero and $f \in \mathbb{k}[x]$ a polynomial with $f(\mathbb{k}) = \mathbb{k}$. Consider a surface $X \subseteq \mathbb{A}_{\mathbb{k}}^3$ with equation $f(x) - uv = 0$. Then X is flexible and the special automorphism group $\text{SAut}(X)$ acts m -transitively on X_{reg} for every $m \in \mathbb{N}$.*

Since the assumptions of Theorem 5.30 are fulfilled for the affine space $X^{(0)} = \mathbb{A}^k$, $k \geq 1$, we can conclude that for every $k, l \geq 1$ the affine variety $X^{(l)} \subseteq \mathbb{A}^{k+2l}$ defined by (5.14) is flexible and the group $\text{SAut}(X^{(l)})$ acts infinitely transitively on $X_{\text{reg}}^{(l)}$.

5.3.1 Suspension over a line

Here we prove the infinite transitivity in Theorem 5.31. The proof is elementary and based on some explicit formulae from [47].

Proof. We may assume that $d = \deg f \geq 2$. According to [47]⁵, in our case the special automorphism group $\text{SAut}(X)$ contains the abelian subgroups G_u and G_v generated, respectively, by the one parameter unipotent subgroups

$$H_u(q) : (x, u, v) \mapsto \left(x + tq(u), u, v + \frac{f(x + tq(u)) - f(x)}{u} \right) \quad (5.15)$$

and

$$H_v(q) : (x, u, v) \mapsto \left(x + tq(v), u + \frac{f(x + tq(v)) - f(x)}{v}, v \right), \quad (5.16)$$

where $q(z) \in \mathbb{k}[z]$, $q(0) = 0$, and $t \in \mathbb{k}$. So $u \in \mathbb{k}[X]^{G_u}$ and $v \in \mathbb{k}[X]^{G_v}$. We claim that the subgroup $G = \langle G_u, G_v \rangle \subseteq \text{SAut}(X)$ acts m -transitively on X_{reg} for every $m \in \mathbb{N}$. Given an m -tuple of distinct points

$$Q_1 = (x_1, u_1, v_1), \dots, Q_m = (x_m, u_m, v_m) \in X_{\text{reg}},$$

our aim is to find an automorphism $\phi \in G$ which sends this m -tuple to a standard m -tuple

$$Q_i^{(0)} = (x_i^{(0)}, u_i^{(0)}, v_i^{(0)}), \quad i = 1, \dots, m,$$

chosen in such a way that all $v_i^{(0)}$ are nonzero and distinct.

Step 1. Acting with G_u , we can replace the original m -tuple by another one such that $v_i \neq 0$ for each $i = 1, \dots, m$. Indeed, the polynomial in t

$$\frac{f(x + tu) - f(x)}{u} = \frac{f'(x)}{1!}t + \dots + \frac{f^{(d)}(x)}{d!}u^{d-1}t^d \in \mathbb{k}[x, u][t],$$

⁵Cf. also [16], [21], and [48].

where $f^{(d)}(x)$ is a nonzero constant, is non-constant provided that either $f'(x) \neq 0$ or $u \neq 0$. Since the point $Q_s \in X$ is smooth, the equalities $u_s = 0$, $v_s = 0$, $f'(x_s) = 0$ cannot hold simultaneously. Hence, acting by (5.15) with $q = z$ and a general t does change the coordinate $v_s = 0$, while keeping nonzero those v_i that were already nonzero. Now the claim follows.

Step 2. Suppose further that $v_i \neq 0$ for each $i = 1, \dots, m$. Then, acting with G_v , we can send our m -tuple to another one where all the u_i ($i = 1, \dots, m$) are nonzero and distinct. Indeed, let

$$F(Q_i, q, t) = \frac{f'(x_i) q(v_i)}{1! v_i} t + \dots + \frac{f^{(d)}(x_i) q(v_i)^d}{d! v_i} t^d \in \mathbb{k}[t].$$

We have $(x_i, v_i) \neq (x_j, v_j)$ for all $i \neq j$ because $(x_i, u_i, v_i) \neq (x_j, u_j, v_j)$ while $u_i = f(x_i)/v_i$ and $u_j = f(x_j)/v_j$. If $v_i = v_j$, then $f^{(d-1)}(x_i) \neq f^{(d-1)}(x_j)$ since the linear form $f^{(d-1)}(x)$ is nonzero. Thus for a suitable $q \in \mathbb{k}[z]$ such that $q(v_i) \neq 0$ for all i , the polynomials $F(Q_i, q, t)$ and $F(Q_j, q, t)$ are different for every $i \neq j$. Applying an automorphism $H_v(q)$ in (5.16) with a general t , we obtain the result.

Step 3. We assume now that all the coordinates u_j are nonzero and distinct. Let us show that it is possible to achieve the standard values $v_s^{(0)}$, $s = 1, \dots, m$, acting by G_u . To this end, we construct an automorphism that preserves all the points but Q_i and sends Q_i to a new point Q'_i with $v'_i = v_i^{(0)}$. Namely, fix a polynomial $q(z)$ with $q(0) = 0$, $q(u_i) \neq 0$ and $q(u_j) = 0$ for all $j \neq i$. Our assumption on $f(x)$ guarantees that the equation $f(x) = u_i(v_i^{(0)} - v_i) + f(x_i)$ has a root $x = a_i$, where $a_i \in \mathbb{k}$. Applying $H_u(q)$ in (5.15) with $t = (a_i - x_i)/q(u_i)$, we obtain the required.

Step 4. Suppose finally that $v_i = v_i^{(0)}$ for all i . It suffices to reach the values $x_i = x_i^{(0)}$ for all i acting by an automorphism from G_v (indeed, then also $u_i = f(x_i)/v_i = f(x_i^{(0)})/v_i^{(0)} = u_i^{(0)}$). This can be done by applying $H_u(q)$ as in (5.16) with $t = 1$ and a polynomial q satisfying $q(0) = 0$ and $q(v_i^{(0)}) = x_i^{(0)} - x_i$ for all i . Now the proof is completed. \square

5.3.2 Infinite transitivity in higher dimensions

Clearly, it is enough to prove Theorem 5.30 for a non-iterated suspension. Before passing to the proofs, we establish in Lemmas 5.32-5.35 some necessary elementary facts concerning suspensions. In Lemmas 5.32-5.37 we work over an arbitrary field \mathbb{k} of characteristic zero.

Lemma 5.32. *If $X^{(0)}$ is irreducible, then the suspension $X^{(1)} = \text{Susp}(X^{(0)}, f)$ is also irreducible.*

Proof. Suppose to the contrary that there exist nonzero elements

$$F_1, F_2 \in \mathbb{k}[X^{(1)}] = \mathbb{k}[X^{(0)}][u, v]/(uv - f) \quad \text{such that} \quad F_1 F_2 = 0.$$

We may assume that $\deg_{u,v}(F_1) + \deg_{u,v}(F_2)$ is minimal and no monomial in F_i contains the product uv , since otherwise we can replace this product by f according to Definition 5.4. If u occurs in both F_1 and F_2 , then $\deg_u(F_1F_2) > 0$ since the leading term in u cannot cancel. Hence, up to twisting u and v , we may assume that F_1 does not contain v and F_2 does not contain u . Let us write

$$F_1 = \sum_{i=0}^k a_i u^i \quad \text{and} \quad F_2 = \sum_{j=0}^l b_j v^j,$$

where $a_i, b_j \in \mathbb{k}[X^{(0)}]$, and $k + l$ is minimal.

If neither u nor v occurs, i.e. $k = l = 0$, then $F_1, F_2 \in \mathbb{k}[X^{(0)}]$ are zero divisors, which contradicts the irreducibility of $X^{(0)}$. So $k + l > 0$.

If $a_0 = b_0 = 0$ then we can decrease the degree $k + l$ by dividing out u and v . This contradicts the minimality assumption. So we may suppose that $a_0 \neq 0$. Then the product F_1F_2 contains a nonzero term $a_0b_l v^l$, which gives again a contradiction. \square

Lemma 5.33. *We have $\pi(X_{\text{reg}}^{(1)}) = X_{\text{reg}}^{(0)}$, where $\pi : X^{(1)} \rightarrow X^{(0)}$ is the restriction of the canonical projection $X^{(0)} \times \mathbb{A}^2 \rightarrow X^{(0)}$ to $X^{(1)}$.*

Proof. Let $f_1, f_2, \dots, f_m \in \mathbb{k}[x_1, x_2, \dots, x_s]$ generate the ideal of $X^{(0)} \subseteq \mathbb{A}^s$. A point $P \in X^{(0)}$ is regular if and only if the rank of the Jacobian matrix

$$D_0 = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_s} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_s} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_s} \end{pmatrix}$$

attains its maximal value $s - \dim X^{(0)}$ at P . The corresponding matrix for $X^{(1)}$ is

$$D_1 = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_s} & 0 & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_s} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_s} & 0 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_s} & -v & -u \end{pmatrix}.$$

Obviously, $\text{rk } D_1 \leq 1 + \text{rk } D_0$ at every point. Since $\dim X^{(1)} = 1 + \dim X^{(0)}$, any regular point of $X^{(1)}$ is mapped via π to a regular point of $X^{(0)}$. On the other hand, let a square submatrix M of D_0 and a point $P \in X_{\text{reg}}^{(0)}$ be such that $M(P)$ is

of rank $r = s - \dim X^{(0)}$ equal to its order. We extend M to a square submatrix M' of order $r + 1$ by adding the last line and one of the two extra columns of D_1 in such a way that $\text{rk } M'(P, u, v) = 1 + \text{rk } M(P) = r + 1$ for some $(u, v) \neq (0, 0)$, where $(P, u, v) \in X^{(1)}$. Hence $(P, u, v) \in X_{\text{reg}}^{(1)}$. Now the assertion follows. \square

Remark 5.34. Recall [31] that an *affine modification* of an affine algebra A with center (I, v) , where $I \subseteq A$ is an ideal and $v \in I$ is not a zero divisor, is the quotient algebra $A[It]/(1 - vt)$, where

$$\text{Bl}_I(A) = A[It] = A \oplus \bigoplus_{n=1}^{\infty} (It)^n \cong A \oplus I \oplus I^2 \oplus \dots$$

is the blow-up (or the Rees) algebra of the pair (A, I) and t is a formal symbol.

Geometrically, the variety $\text{Spec}(A[It]/(1 - vt))$ is obtained from $X = \text{Spec } A$ as follows. We perform a blowup of X with center I and then remove the proper transform in $\text{Bl}_I(X)$ of the zero divisor $V(v)$ in X . This results again in an affine variety. (We note that this proper transform meets the exceptional divisor E , since $v \in I$.) See [31, §1] for more details.

According to [31, Example 1.4 and §5], the suspension $X^{(1)} = \text{Susp}(X^{(0)}, f)$ can be viewed as an affine modification of $X^{(0)} \times \mathbb{A}^1$ (where $\mathbb{A}^1 = \text{Spec } \mathbb{k}[v]$) with center $(I_1 = (v, f), v)$ along the divisor $v = 0$. Interchanging v and u , the variety $X^{(1)}$ can be regarded also as an affine modification of the product $X^{(0)} \times \mathbb{A}^1$ (where this time $\mathbb{A}^1 = \text{Spec } \mathbb{k}[u]$) with center $(I_2 = (u, f), u)$ along the divisor $u = 0$. The exceptional divisors $v = 0$ and $u = 0$, respectively, of these two modifications are both isomorphic to $X^{(0)} \times \mathbb{A}^1$ but different as subvarieties of $X^{(1)}$. In the sequel for every $c \in \mathbb{k}$ we consider the level hypersurfaces $U_c = \{u = c\}$ and $V_c = \{v = c\}$ in $X^{(1)}$.

In [31, §2] a method was developed which allows to extend an LND ∂ to the affine modification provided that ∂ stabilizes the center of the modification. In Lemma 5.35 below we concretize this in our particular case of affine suspensions.

Given an LND δ_0 of an affine domain A_0 and a polynomial $q \in \mathbb{k}[z]$ with $q(0) = 0$, we can define a new LND $\delta' = \delta'(\delta_0, q)$ on $A' = A_0 \otimes \mathbb{k}[v]$, where v is a new variable, as follows. First we extend δ_0 to A' by letting $\delta_0(v) = 0$, and then we multiply δ_0 by the element $q(v) \in \ker \delta_0$. Suppose that A_0 is generated by x_1, x_2, \dots, x_s . Then δ' is given in coordinates by

$$\delta'(x_i) = q(v)\delta_0(x_i), \quad i = 1, 2, \dots, s, \quad \delta'(v) = 0. \quad (5.17)$$

Let now u be yet another variable and $f \in A_0$ be nonzero. Consider the structure algebra A_1 of the suspension over A_0 :

$$A_1 = (A_0 \otimes \mathbb{k}[u, v])/(uv - f).$$

Lemma 5.35. *In the notation as above, the LND $\delta' \in \text{Der } A'$ can be transformed into an LND $\delta_1 = \delta_1(\delta_0, q) \in \text{Der } A_1$ by letting*

$$\begin{aligned} \delta_1(x_i) &= \delta'(x_i), \quad i = 1, 2, \dots, s, \\ \delta_1(u) &= \frac{q(v)}{v} \delta_0(f), \\ \delta_1(v) &= \delta'(v) = 0. \end{aligned} \tag{5.18}$$

Proof. First, we check that these formulae extend δ' to $\delta_1 = \delta_1(\delta_0, q) \in \text{Der}(A_0 \otimes \mathbb{k}[u, v])$, where δ_1 preserves the ideal $(uv - f)$. Indeed, since $\frac{q(v)}{v} \in \mathbb{k}[v]$ by our choice of q , the derivation δ_1 is well defined on the generators of $A_0 \otimes \mathbb{k}[u, v]$. It is easily seen that δ_1 is still locally nilpotent. The straightforward calculation shows that $\delta_1(uv - f) = 0$. Hence δ_1 descends to an LND of the quotient algebra A_1 denoted by the same symbol δ_1 . \square

Definition 5.36. Denote by G_v the subgroup of the special automorphism group $\text{SAut}(X^{(1)})$ generated by all one-parameter unipotent subgroups

$$H_v(\delta_0, q) = \exp(t\delta_1), \quad \text{where } t \in \mathbb{k}_+ \text{ and } \delta_1 = \delta_1(\delta_0, q),$$

with δ_0 and $q(t)$ as above⁶. Interchanging the roles of v and u , we obtain the second subgroup $G_u \subseteq \text{SAut}(X^{(1)})$. Thus $u \in \mathbb{k}[X^{(1)}]^{G_u}$ and $v \in \mathbb{k}[X^{(1)}]^{G_v}$. We will show that the subgroup $G \subseteq \text{SAut}(X^{(1)})$ generated by G_u and G_v acts infinitely transitively in $X_{\text{reg}}^{(1)}$.

Given k distinct constants $c_1, \dots, c_k \in \mathbb{k}$. Denote by $\text{Stab}_{c_1 \dots c_k}^v$ the subgroup of the group G_v fixing pointwise the hypersurfaces $V_{c_s} \subseteq X^{(1)}$, $s = 1, \dots, k$.

Lemma 5.37. *Suppose that the group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$. Then for any distinct constants $c_0, c_1, \dots, c_k \in \mathbb{k}^\times$ the group $\text{Stab}_{c_1 \dots c_k}^v$ acts m -transitively on $V_{c_0} \cap X_{\text{reg}}^{(1)}$.*

Proof. Given two collections of m distinct points P'_1, \dots, P'_m and Q'_1, \dots, Q'_m in $V_{c_0} \cap X_{\text{reg}}^{(1)}$. Let P_1, \dots, P_m and Q_1, \dots, Q_m be their π -projections to $X^{(0)}$. Notice that the hypersurface $V_{c_0} \subseteq X^{(1)}$ is mapped isomorphically onto $X^{(0)}$ via π , while by Lemma 5.33 we have $\pi(V_{c_0} \cap X_{\text{reg}}^{(1)}) \subseteq X_{\text{reg}}^{(0)}$. Indeed, a point $P' \in V_{c_0}$ can be written as $P' = (P, u, c_0)$, where $P = \pi(P') \in X^{(0)}$ and $u = u(P') = f(P)/c_0$. Conversely, these formulae give an isomorphism $X^{(0)} \xrightarrow{\cong} V_{c_0}$ which sends P to P' .

⁶Notice that for $X^{(0)} = \mathbb{A}^1 = \text{Spec } \mathbb{k}[z]$ and $\delta_0 = d/dz$ we have $H_v(\delta_0, q) = H_v(q)$ from (5.16).

Since by our assumption the group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$, there exists an automorphism $\psi_0 \in \text{SAut}(X^{(0)})$ which sends the ordered collection (P_1, \dots, P_m) to (Q_1, \dots, Q_m) . It can be written as a product

$$\psi_0 = \prod_{i=1}^k \exp(\delta_0^{(i)})$$

for some LNDs $\delta_0^{(1)}, \dots, \delta_0^{(k)} \in \text{Der } \mathbb{k}[X^{(0)}]$.

Letting $q = \alpha z(z - c_1) \dots (z - c_k)$, where $\alpha \in \mathbb{k}^\times$ is such that $q(c_0) = 1$, we can lift the $\delta_0^{(i)}$ to the LNDs

$$\delta_1^{(i)} = \delta_1^{(i)}(\delta_0^{(i)}, q) \in \text{Der } \mathbb{k}[X^{(1)}], \quad i = 1, \dots, k$$

by Lemma 5.35. Respectively, ψ_0 can be lifted to a special automorphism

$$\psi_1 = \prod_{i=1}^k \exp(\delta_1^{(i)}) \in G_v \subseteq \text{SAut}(X^{(1)}).$$

By virtue of (5.17) it is easily seen that the actions on $X^{(1)}$ of the corresponding one-parameter unipotent subgroups $H_v(\delta_0^{(i)}, q)$ restrict to the original actions on $V_{c_0} \cong X^{(0)}$. So the automorphism $\psi_1|_{V_{c_0}} = \psi_0$ sends (P'_1, \dots, P'_m) to (Q'_1, \dots, Q'_m) . Due to our choice of $q(z)$, this automorphism fixes all the other hypersurfaces V_{c_s} pointwise. \square

Lemma 5.38. *Let \mathbb{k} be an algebraically closed field of characteristic zero. Suppose as above that the group $\text{SAut}(X^{(0)})$ acts m -transitively on $X_{\text{reg}}^{(0)}$. Then for any set of distinct points $Q'_1, \dots, Q'_m \in X_{\text{reg}}^{(1)}$ there exists an automorphism $\varphi \in \text{SAut}(X^{(1)})$ such that $\varphi(Q'_i) \notin U_0 \cup V_0$ for all $i = 1, 2, \dots, m$.*

Proof. We say that the point $Q'_i = (Q_i, u_i, v_i) \in X^{(1)}$ is *hyperbolic* if $u_i v_i \neq 0$, i.e. $Q'_i \notin U_0 \cup V_0$. We have to show that the original collection can be moved by means of a special automorphism in such a way that all the points become hyperbolic. Suppose that Q'_1, \dots, Q'_l are already hyperbolic while Q'_{l+1} is not, where $l \geq 0$. By recursion, it is sufficient to move Q'_{l+1} off $U_0 \cup V_0$ while leaving the points Q'_1, \dots, Q'_l hyperbolic. It is enough to consider the following two cases:

Case 1: $u_{l+1} = 0, v_{l+1} \neq 0$, and

Case 2: $u_{l+1} = v_{l+1} = 0$.

We claim that there exists an automorphism $\varphi \in \text{SAut}(X^{(1)})$ which leaves the points Q'_1, \dots, Q'_l hyperbolic such that in Case 1 the point $\varphi(Q'_{l+1})$ is hyperbolic as well, and in Case 2 this point satisfies the assumptions of Case 1.

In Case 1 we divide Q'_1, \dots, Q'_{l+1} into several disjoint pieces M_0, \dots, M_k according to different values of v so that $Q'_i \in M_j \Leftrightarrow v_i = c_j$, where $c_j \neq 0$. Assuming

that $M_0 = \{Q'_{i_1}, \dots, Q'_{i_r}, Q'_{l+1}\}$, where $i_k \leq l$ for all $k = 1, \dots, r$, we can choose an extra point $Q''_{l+1} \in (V_{c_0} \cap X_{\text{reg}}^{(1)}) \setminus U_0$. Indeed, since $c_0 = v_{l+1} \neq 0$, we have $V_{c_0} \cong X^{(0)}$. Under the assumptions of Theorem 5.30 $\dim X^{(0)} \geq 2$, hence also $\dim(V_{c_0} \cap X_{\text{reg}}^{(1)}) \setminus U_0 = \dim X^{(0)} \geq 2$.

By Lemma 5.37 the subgroup $\text{Stab}_{c_1, \dots, c_k}^v \subseteq G_v$ acts $(r + 1)$ -transitively on $V_{c_0} \cap X_{\text{reg}}^{(1)}$. Therefore we can send the $(r + 1)$ -tuple $(Q'_{i_1}, \dots, Q'_{i_r}, Q'_{l+1})$ to the $(r + 1)$ -tuple $(Q'_{i_1}, \dots, Q'_{i_r}, Q''_{l+1})$, fixing the remaining points of $M_1 \cup \dots \cup M_k$. This confirms our claim in Case 1.

In Case 2 we have $Q'_{l+1} = (Q_{l+1}, 0, 0) \in X_{\text{reg}}^{(1)}$. From Lemma 5.33 and its proof it follows that $Q_{l+1} = \pi(Q'_{l+1}) \in X_{\text{reg}}^{(0)}$ and $df(Q_{l+1}) \neq 0$ in the cotangent space $T_{Q_{l+1}}^* X^{(0)}$. The variety $X^{(0)}$ being flexible, there exists an LND $\partial_0 \in \text{Der } \mathbb{k}[X^{(0)}]$ such that $\partial_0(f)(Q_{l+1}) \neq 0$. Letting $q(v) = v(v - v_1)(v - v_2) \dots (v - v_l) \in \mathbb{k}[v]$ and choosing a set of generators x_1, \dots, x_s of the algebra $\mathbb{k}[X^{(0)}]$, we extend ∂_0 to $\partial_1 \in \text{Der } \mathbb{k}[X^{(1)}]$ via

$$\begin{aligned} \partial_1(x_i) &= q(v)\partial_0(x_i), \quad i = 1, 2, \dots, s, \\ \partial_1(u) &= \frac{q(v)}{v} \partial_0(f), \\ \partial_1(v) &= 0 \end{aligned} \tag{5.19}$$

analogously to (5.18). Due to our choice, $\partial_1(u)(Q'_{l+1}) \neq 0$. Hence the action of the associate one-parameter unipotent subgroup $H_v(\partial_0, q) = \exp(t\partial_1)$ pushes the point Q'_{l+1} out of U_0 . So the orbit $H_v(\partial_0, q) \cdot Q'_{l+1}$ meets the hypersurface $U_0 \subseteq X^{(1)}$ in finitely many points. Similarly, for every $j = 1, 2, \dots, l$ the orbit $H_v(\partial_0, q) \cdot Q_j \not\subseteq U_0$ meets U_0 in finitely many points. Letting now $\varphi = \exp(t_0\partial_1) \in H_v(\partial_0, q) \subseteq G_v$, we conclude that for a general value of $t_0 \in \mathbb{k}$ the image $\varphi(Q'_j)$ lies outside U_0 for all $j = 1, 2, \dots, l + 1$. Since the group $H_v(\partial_0, q)$ preserves the coordinate v , the points $\varphi(Q'_1), \dots, \varphi(Q'_l)$ are still hyperbolic. Interchanging the roles of u and v ⁷, we can achieve that the assumptions of Case 1 are fulfilled for the new collection $\varphi(Q'_1), \dots, \varphi(Q'_l), \varphi(Q'_{l+1})$, as required. \square

Proof of infinite transitivity in Theorem 5.30. If $X^{(0)} = \mathbb{A}^1$, then the assertion follows from Theorem 5.31. Let now $\dim X^{(0)} \geq 2$. To show that the action of the group $\text{SAut}(X^{(1)})$ on $X_{\text{reg}}^{(1)}$ is m -transitive for every $m \in \mathbb{N}$, we fix a standard collection of m distinct points $P'_1, \dots, P'_m \in U_1 \cap X_{\text{reg}}^{(1)}$. It suffices to move any other m -tuple of distinct points $Q'_1, \dots, Q'_m \in X_{\text{reg}}^{(1)}$ to the position of P'_1, \dots, P'_m by means of a special automorphism $\psi \in \text{SAut}(X^{(1)})$. In view of Lemma 5.38 we may suppose that $Q'_i \notin U_0 \cup V_0$ for all $i = 1, \dots, m$. Similarly as in the proof of

⁷We have not done this earlier in order to keep our previous notation.

Lemma 5.38, we divide the collection Q'_1, \dots, Q'_m into disjoint pieces M_1, \dots, M_k according to the values of v so that $M_j \subseteq V_{c_j}$, where $c_j \in \mathbb{k}^\times$ for all $j = 1, \dots, k$.

By our assumption the variety $X^{(0)}$ is flexible. It follows that the only units in $\mathbb{k}[X^{(0)}]$ are constants. Consequently, since f is non-constant, we have $f(X^{(0)}) = \mathbb{k}$. In particular, $U_c \cap V_d \neq \emptyset$ for any $c, d \in \mathbb{k}$. Since $\dim X^{(1)} = 1 + \dim X^{(0)} \geq 3$, the intersection $U_c \cap V_d$ has positive dimension, hence is infinite.

Therefore acting with the subgroups $\text{Stab}_{c_1 \dots c_i \dots c_l}^v \subseteq G_v$, by Lemma 5.37 we can send M_i to $U_1 \cap V_{c_i} \cap X_{\text{reg}}^{(1)}$ fixing the other pieces M_j ($j \neq i$) pointwise. So we may assume that $Q'_1, \dots, Q'_m \in U_1 \cap X_{\text{reg}}$. Applying Lemma 5.37 again with u and v interchanged, $k = 0$, and $c_0 = 1$, i.e. acting with the subgroup G_u , we can send the resulting collection to the standard one P'_1, \dots, P'_m . Now the proof is completed. \square

5.3.3 Flexibility

To complete the proof of Theorem 5.30, it remains to establish the flexibility of $X^{(1)}$.

Lemma 5.39. *Under the assumptions of Theorem 5.30, the variety $X^{(1)}$ is flexible.*

Proof. We know already that the group $\text{SAut}(X^{(1)})$ is transitive in $X_{\text{reg}}^{(1)}$. Hence, similarly as in the proof of Lemma 5.28, it suffices to find just one flexible point $P' = (P, u, v) \in X_{\text{reg}}^{(1)}$.

The function $f \in \mathbb{k}[X^{(0)}]$ being non-constant, $df(P) \neq 0$ at some point $P \in X_{\text{reg}}^{(0)}$ with $f(P) \neq 0$. Due to our assumption $X^{(0)}$ is flexible. Hence there exist n locally nilpotent derivations $\partial_0^{(1)}, \dots, \partial_0^{(n)} \in \text{Der } \mathbb{k}[X^{(0)}]$, where $n = \dim X^{(0)}$, such that the corresponding vector fields ξ_1, \dots, ξ_n span the tangent space $T_P X^{(0)}$, i.e.

$$\text{rk} \begin{pmatrix} \xi_1(P) \\ \vdots \\ \xi_n(P) \end{pmatrix} = n.$$

It follows that $\partial_0^{(i)}(f)(P) \neq 0$ for some index $i \in \{1, \dots, n\}$.

Let now $P' = (P, u_0, v_0) \in X_{\text{reg}}^{(1)}$ be a point such that $\pi(P') = P$. Since $u_0 v_0 = f(P) \neq 0$, the point P' is hyperbolic. Letting $q(v) = v$ in Lemma 5.35, we obtain n LNDs

$$\partial_1^{(1)}, \dots, \partial_1^{(n)} \in \text{Der } \mathbb{k}[X^{(1)}], \quad \text{where} \quad \partial_1^{(j)} = \partial_1^{(j)}(\partial_0^{(j)}, v).$$

Interchanging u and v and letting $j = i$ gives yet another LND

$$\partial_2^{(i)} = \partial_2^{(i)}(\partial_0^{(i)}, u) \in \text{Der } \mathbb{k}[X^{(1)}].$$

Let us show that the corresponding $n + 1$ vector fields span the tangent space $T_{P'}X^{(1)}$ at P' , as required. We can view $\partial_1^{(1)}, \dots, \partial_1^{(n)}, \partial_2^{(i)}$ as LNDs in $\text{Der } \mathbb{k}[X^{(0)}][u, v]$ preserving the ideal $(uv - f)$, so that the corresponding vector fields are tangent to the hypersurface

$$X^{(1)} = \{uv - f(P) = 0\} \subseteq X^{(0)} \times \mathbb{A}^2.$$

The values of these vector fields at the point $P' \in X_{\text{reg}}^{(1)}$ yield an $(n + 1) \times (n + 2)$ -matrix

$$E = \begin{pmatrix} v_0 \xi_1(P) & \partial_0^{(1)}(f)(P) & 0 \\ \vdots & \vdots & \vdots \\ v_0 \xi_n(P) & \partial_0^{(n)}(f)(P) & 0 \\ u_0 \xi_i(P) & 0 & \partial_0^{(i)}(f)(P) \end{pmatrix}.$$

The first n rows of E are linearly independent, and the last one is independent from the preceding ones since $\partial_0^{(i)}(f)(P) \neq 0$. Therefore $\text{rk}(E) = n + 1 = \dim X^{(1)}$. So our locally nilpotent vector fields indeed span the tangent space $T_{P'}X^{(1)}$ at P' , as claimed. \square

Now the proof of Theorem 5.30 is completed.

5.4 The Makar-Limanov invariant

Recall the following definition [23, §9].

Definition 5.40. The *Makar-Limanov invariant* $\text{ML}(X)$ of an affine variety X is the intersection of the kernels of all locally nilpotent derivations of $\mathbb{k}[X]$, or, in other words, the subalgebra in $\mathbb{k}[X]$ of common invariants for all one parameter unipotent subgroups of $\text{Aut}(X)$.

From this definition it is straightforward that $\text{ML}(X) = \mathbb{k}[X]^{\text{SAut}(X)}$.

Proposition 5.41. *The Makar-Limanov invariant of X is trivial (that is, $\text{ML}(X) = \mathbb{k}$) provided that the special automorphism group $\text{SAut}(X)$ acts on X with a dense open orbit (cf. [56]).*

This holds, in particular, for the varieties in all three classes from Theorems 5.5, 5.11 and 5.30 (for the first two of them, see also [33, 3.16], [46], and [56]). On the other hand, $\text{ML}(X)$ is trivial if X is flexible. Indeed, if $f \in \mathbb{k}[X]^{\text{SAut}(X)}$ then the differential df vanishes along the orbits of any unipotent subgroup, hence it vanishes on the tangent space at any flexible point of X_{reg} . Since X is flexible, f is constant.

Chapter 6

Infinitely transitive actions over \mathbb{R}

6.1 Transitivity on each connected component

The proof of Theorem 5.30, with minor changes, works also for real algebraic varieties and leads to the following result.

Theorem 6.1. *Let Y be a flexible real algebraic variety. Suppose that the smooth locus Y_{reg} is connected and the special automorphism group $\text{SAut}(Y)$ acts m -transitively on Y_{reg} for every $m \in \mathbb{N}$. Consider the suspension $X = \text{Susp}(Y, f)$, where the function $f \in \mathbb{R}[Y]$ satisfies the condition $f(Y_{\text{reg}}) = \mathbb{R}$. Then the variety X is flexible and the special automorphism group $\text{SAut}(X)$ acts m -transitively on X_{reg} for every $m \in \mathbb{N}$.*

We recall the specific notations and state the main results.

Definition 6.2. We say that the action of a group G on a set $Y = Y^1 \sqcup Y^2 \sqcup \dots \sqcup Y^s$ is *infinitely transitive on each connected component* if for every s -tuple (m_1, \dots, m_s) , it is transitive on $(m_1 + \dots + m_s)$ -tuples of the form

$$(P_1^1, \dots, P_{m_1}^1, P_1^2, \dots, P_{m_2}^2, \dots, P_1^s, \dots, P_{m_s}^s),$$

where $P_j^i \in Y^i$ are pairwise distinct.

Theorem 6.3 (Infinite transitivity on each connected component). *Let Y be an affine algebraic variety defined over \mathbb{R} and $f \in \mathbb{R}[Y]$. Assume that for each connected component Y^i of Y_{reg} , the dimension $\dim Y^i \geq 2$ and f is non-constant on Y^i .*

If Y is flexible and the action of $\text{SAut}(Y)$ on Y_{reg} is infinitely transitive on each connected component, then the suspension $X = \text{Susp}(Y, f)$ is flexible and $\text{SAut}(X)$ acts on X_{reg} infinitely transitively on each connected component.

Remark 6.4. If X_{reg} is not connected, then $\text{SAut}(X)$ is not even one-transitive on X_{reg} . Indeed, the action of $\text{SAut}(X)$ on X fixes each connected component of X : every special automorphism g admits a decomposition $\prod h_j(1)$, where each h_i is a one-parameter additive group. For any $x \in X$, the arc $t \rightarrow \prod h_i(t).x$ then connects x to $g.x$.

Remark 6.5. The number of connected components can grow on each step even if we started with a variety Y whose non-singular part Y_{reg} is connected. Indeed, if f does not attain zero on one of the connected components, say on Y^1 , then the set $\{(y, u, v) \mid uv = f(y), y \in Y^1\}$ splits into $\{(y, u, v), u > 0, v > 0\}$ and $\{(y, u, v), u < 0, v < 0\}$. We may choose one connected component of the suspension and further perform suspensions over this connected component.

As a preliminary part of Theorem 6.3, we prove the following theorem.

Theorem 6.6 (Infinite transitivity on one connected component). *Let Y be an affine algebraic variety defined over \mathbb{R} and $f \in \mathbb{R}[Y]$. Assume that Y_{reg} contains a flexible connected component Y^1 of dimension at least 2 such that $\text{SAut}(Y)$ acts infinitely transitively on Y^1 and $f|_{Y^1}$ is non-constant. Let X^1 be a connected component of the smooth locus of the suspension $\text{Susp}(Y^1, f) \subseteq X = \text{Susp}(Y, f)$. Then X^1 is flexible and $\text{SAut}(X)$ acts infinitely transitively on X^1 .*

6.2 Connected suspensions over real varieties

Here we prove Theorem 6.1. We need the following elementary lemma.

Lemma 6.7. *Let Y be a smooth, connected real manifold of dimension at least two. Then for any continuous function $f : Y \rightarrow \mathbb{R}$ the level set $f^{-1}(c)$ is infinite for each $c \in \text{Int } f(Y)$.*

Proof. We may assume that $f|_Y$ is non-constant. Choose two points $y_1, y_2 \in Y$ such that $f(y_1) = c_1 < c$ and $f(y_2) = c_2 > c$. They can be joined by a smooth path l in Y . There exists a tubular neighborhood U of l diffeomorphic to a cylinder $\Delta \times I$, where $I = [0, 1]$ and Δ is a ball of dimension $\dim \Delta = \dim Y - 1 \geq 1$. So there is a continuous family of paths joining y_1 and y_2 within U such that any two of them meet only at their ends y_1 and y_2 . By continuity, on each of these paths there is a point y' with $f(y') = c$. In particular, the level set $f^{-1}(c)$ is infinite. \square

The proof of Theorem 6.1 differs just slightly from that of Theorem 5.30. Hence it is enough to indicate the necessary changes.

Sketch of the proof of Theorem 6.1. The assumption that the field \mathbb{k} is algebraically closed was actually used in the proof of Theorem 5.30 only on two occasions.

Namely, in the proofs of Lemma 5.38 and of the infinite transitivity in Theorem 5.30 we exploited the fact that under our assumptions the level sets $(V_{c_k} \cap X_{\text{reg}}) \setminus U_0$ and $U_1 \cap V_{c_i} \cap X_{\text{reg}}$ are of positive dimension, hence are infinite. For $\mathbb{k} = \overline{\mathbb{k}}$ the latter follows from the Krull theorem and the dimension count. In the case $\mathbb{k} = \mathbb{R}$, we can deduce the same conclusion using Lemma 6.7. Indeed, in the notation as before, for every $c_i \neq 0$ the restrictions

$$\pi: V_{c_i} \cap X_{\text{reg}} = (V_{c_i})_{\text{reg}} \rightarrow Y_{\text{reg}} \quad \text{and} \quad \pi: U_1 \cap V_{c_i} \cap X_{\text{reg}} \rightarrow f^{-1}(c_i) \cap Y_{\text{reg}}$$

are isomorphisms. Under the assumptions of Theorem 6.1 the smooth real manifold Y_{reg} of dimension ≥ 2 is connected. Since $f(Y_{\text{reg}}) = \mathbb{R}$, by Lemma 6.7 the level set $f^{-1}(c_i) \cap Y_{\text{reg}}$ is infinite. Since $c_k \neq 0$, the set $(V_{c_k} \cap X_{\text{reg}}) \setminus U_0 \supseteq U_1 \cap V_{c_k} \cap X_{\text{reg}}$ is infinite too.

A posteriori, the manifold X_{reg} is also connected. \square

6.3 Affine modifications and lifts of automorphisms

In this section we prove the basic results of the theory over the real numbers. The main part is close to the treatment in Chapter 5.

For every geometrically irreducible algebraic variety X over the ground field \mathbb{C} , there is a natural one-to-one correspondence between locally nilpotent derivations (LND's) δ on $\mathbb{C}[X]$ and algebraic actions of one-parameter subgroups $(\mathbb{C}, +) \cong H_\delta \subseteq \text{SAut}(X)$. Namely, given a locally nilpotent derivation δ , the corresponding action is the exponential $(t, f) \mapsto \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(f)$. Conversely, for every algebraic action σ of a subgroup $(\mathbb{C}, +) \cong H \subseteq \text{SAut}(X)$, the derivation along the tangent vector field to the orbits of σ , given by $\frac{\sigma(t,f)-f}{t}|_{t=0}$ is an LND, see [23, § 1.5]. The lemma below shows that the same is true for \mathbb{R} .

Let G be an algebraic group. Recall that a G -module V is *rational* if each $v \in V$ belongs to a finite dimensional G -invariant linear subspace $W \subseteq V$ and the G -action on W defines a homomorphism of algebraic groups $G \rightarrow GL(W)$. A G -algebra is an algebra with a structure of a rational G -module.

Lemma 6.8. *There are one-to-one correspondences between locally nilpotent derivations of $\mathbb{R}[X]$, unipotent subgroups $(\mathbb{R}, +) \subseteq \text{Aut}(X)$, and structures of rational $(\mathbb{R}, +)$ -algebras on $\mathbb{R}[X]$.*

Proof. For an LND D , the corresponding $(\mathbb{R}, +)$ -algebra is defined by the following formula:

$$t: f = \exp(tD)(f) = f + tD(f) + \frac{t^2}{2!}D^2(f) + \dots$$

For any fixed f there exists an N with $D^N(f) = 0$, so this formula gives us a polynomial in t . Hence, f belongs to a $(\mathbb{R}, +)$ -invariant linear subspace $\langle f, D(f), \dots, D^{N-1}(f) \rangle$, which shows that $\mathbb{R}[X]$ is rational as a $(\mathbb{R}, +)$ -algebra.

Conversely, let $(A, t \mapsto \varphi_t)$ be a rational $(\mathbb{R}, +)$ -algebra. Let us define

$$D(f) = \left. \frac{d}{dt} \right|_{t=0} \varphi_t(f), \quad f \in A.$$

The main point is to prove that for each $f \in A$ some power $D^N(f)$ vanishes. Consider a finite dimensional invariant subspace $W \subseteq \mathbb{R}[X]$, $f \in W$. Obviously, D preserves W and the action of $\exp(D)$ on $W \otimes_{\mathbb{R}} \mathbb{C}$ is unipotent. By the Lie-Kolchin theorem, the action of D is upper-triangular in some basis of $W \otimes_{\mathbb{R}} \mathbb{C}$. This means in particular that $D^N = 0$ for some N . Note that the actions of D on W and of D on $W \otimes_{\mathbb{R}} \mathbb{C}$ were originally given by the same matrix, hence, this matrix is nilpotent, and the derivation D on $\mathbb{R}[X]$ is in fact locally nilpotent. \square

Here is the geometric counterpart of the affine suspensions introduced above. Let $X = \text{Susp}(Y, f)$ be a suspension of Y given by Definition 5.4. Consider the cylinder $Y \times \mathbb{A}^1$ over Y , where $\mathbb{A}^1 = \mathbb{R}[v]$. Then $\text{Susp}(Y, f)$ is the blow-up of $Y \times \mathbb{A}^1$ with center (f, v) along v , which is a particular instance of an affine modification, see [31, Example 1.4].

Let δ_0 be an LND on $\mathbb{R}[Y]$ and let H_{δ_0} be the associated $(\mathbb{R}, +)$ -action on Y . Recall the construction of an LND δ_1 which lifts δ_0 to X (see Lemma 5.35 or [31]). Let δ' be the lift of δ_0 on $\mathbb{R}[Y \times \mathbb{A}^1]$ defined by $\delta'(v) = 0$ and consider a product $\delta_1 = q\delta'$ by a polynomial $q(v)$ such that $q(0) = 0$. Choosing q such that the value of δ_1 on u preserves the relation $\delta_1(uv - f(y)) = 0$, we get an LND on $\mathbb{R}[X]$ which satisfies

$$\begin{aligned} \delta_1(g) &= q(v)\delta_0(g) \quad \text{for all functions } g \in \mathbb{R}[Y], \\ \delta_1(v) &= 0, \\ \delta_1(u) &= \frac{q(v)}{v} \delta_0(f). \end{aligned} \tag{6.1}$$

There is some freedom in the choice of $q(v)$. All the derivations obtained in this way annihilate the function $v \in \mathbb{R}[X]$, and the corresponding actions preserve the sections $V_c = \{v = c\} \cap X$. Notice that X can also be considered as the blow-up of $Y \times \text{Spec } \mathbb{R}[u]$, and the lifts of LNDs obtained in this way annihilate the function $u \in \mathbb{R}[X]$.

We denote by G_v (resp. G_u) the subgroup of $\text{SAut}(X)$ generated by one-parameter subgroup lifted from $Y \times \text{Spec } \mathbb{R}[v]$ (resp. $Y \times \text{Spec } \mathbb{R}[u]$). Recall the following.

Lemma 6.9 (Cf. Lemma 5.33). *Let Y be an affine variety over a field of characteristic 0 and X be a suspension over Y . Then the restriction $\pi: X \subset Y \times \mathbb{A}^2 \rightarrow Y$ of the canonical projection satisfies $\pi(X_{\text{reg}}) = Y_{\text{reg}}$.*

We denote by $Y_{\text{reg}} = Y^1 \sqcup \dots \sqcup Y^s$ the decomposition of Y_{reg} into connected components. If f is not surjective, then the suspension over a connected component Y^i of Y_{reg} is either connected or consists of two components: if $f|_{Y^i}$ does not attain zero, u and v neither attain zero, but can be either both negative, either both positive.

For every $c \in \mathbb{R}$ the hyperplane section $\{v = c\} \subset X$ will be denoted by V_c . We denote by $v(P)$ the v -coordinate of a point $P \in X$.

Given k distinct constants $c_1, \dots, c_k \in \mathbb{R} \setminus \{0\}$, we let $\text{Stab}_{c_1 \dots c_k}^v$ be the subgroup of G_v fixing pointwise the hypersurfaces $V_{c_s} \subseteq X$, $s = 1, \dots, k$. Observe that, as a subgroup of G_v , the group $\text{Stab}_{c_1 \dots c_k}^v$ stabilizes all the levels V_c of the function $v \in \mathbb{R}[X]^{G_v}$. Likewise, let $\text{Stab}_{c_1 \dots c_k}^u \subset G_u$ be the subgroup of maps inducing the identity on the levels U_{c_s} of the function $u \in \mathbb{R}[X]^{G_u}$.

Lemma 6.10. *If the action of $\text{SAut}(Y)$ on Y_{reg} is infinitely transitive on each connected component, then for every distinct values $c_0, c_1, \dots, c_k \in \mathbb{R} \setminus \{0\}$, the group $\text{Stab}_{c_1 \dots c_k}^v$ acts infinitely transitively on each connected component of $V_{c_0} \cap X_{\text{reg}}$. The same is true for the action of $\text{Stab}_{c_1 \dots c_k}^u$ on $U_{c_0} \cap X_{\text{reg}}$.*

Proof. Let P_1, \dots, P_m and Q_1, \dots, Q_m be two m -tuples of distinct points of $V_{c_0} \cap X_{\text{reg}}$. Since π restricts to an isomorphism $V_{c_0} \cap X_{\text{reg}} \rightarrow \pi(X_{\text{reg}}) = Y_{\text{reg}}$, we get $\pi(P_j) \neq \pi(P_l)$ and $\pi(Q_j) \neq \pi(Q_l)$ for $j \neq l$. Moreover, two points belong to the same connected component of $V_{c_0} \cap X_{\text{reg}}$ if and only if their projections belong to the same connected component of Y_{reg} . As a consequence, there exists a special automorphism ψ such that $\psi \cdot \pi(P_j) = \pi(Q_j), \forall j$. The special automorphism ψ decomposes into exponentials of LND's. We lift each of these derivations using the polynomial $q(z) = \alpha z(z - c_1) \dots (z - c_k)$ where $\alpha \in \mathbb{R} \setminus \{0\}$ is determined by $q(c_0) = 1$. (Compare with Lemma 5.37.) \square

Lemma 6.11. *Let Y be an affine variety over \mathbb{R} and X be a suspension of Y . Let $Y^1 \subset Y$ be a connected component of Y_{reg} of dimension at least two, $f \in \mathbb{R}[Y]$ such that $0 \in \text{Int } f(Y^1)$, and X^1 be the suspension over Y^1 . If Y^1 is flexible and the action of $\text{SAut}(Y)$ is infinitely transitive on Y^1 , then for every set of distinct points P_1, \dots, P_m of X^1 there exists a special automorphism $\varphi \in \text{SAut}(X)$ such that $\varphi \cdot P_j \notin U_0 \cup V_0$ for all j .*

Proof. We follow the proof of Lemma 5.38. We say that the point $P_i = (R_i, u_i, v_i) \in X^1$ is *hyperbolic* if $u_i v_i \neq 0$, i.e. $P_i \notin U_0 \cup V_0$. We have to show that the original collection can be moved by means of a special automorphism so that all the points become hyperbolic. Suppose that P_1, \dots, P_l are already hyperbolic while P_{l+1} is

not, where $l \geq 0$. By recursion, it is sufficient to move P_{l+1} off $U_0 \cup V_0$ while leaving the points P_1, \dots, P_l hyperbolic. It is enough to consider the following two cases:

Case 1: $u_{l+1} = 0, v_{l+1} \neq 0$, and

Case 2: $u_{l+1} = v_{l+1} = 0$.

We claim that there exists an automorphism $\varphi \in \text{SAut}(X)$ leaving P_1, \dots, P_l hyperbolic such that in Case 1 the point $\varphi.P_{l+1}$ is hyperbolic as well, and in Case 2 this point satisfies the assumptions of Case 1.

In Case 1 we divide P_1, \dots, P_{l+1} into several disjoint pieces M_0, \dots, M_k according to different values of v so that $P_i \in M_j \Leftrightarrow v_i = c_j$, where $c_j \neq 0$. Assuming that $M_0 = \{P_{i_1}, \dots, P_{i_r}, P_{l+1}\}$, where $i_k \leq l$ for all $k = 1, \dots, r$, we can choose an extra point $P'_{l+1} \in (V_{c_0} \cap X^1) \setminus U_0$. Indeed, since $c_0 = v_{l+1} \neq 0$, we have $V_{c_0} \cong Y^1$. We have $\dim Y^1 \geq 2$, hence

$$\dim(V_{c_0} \cap X^1) \setminus U_0 = \dim Y^1 \geq 2.$$

By Lemma 6.10 the subgroup $\text{Stab}_{c_1, \dots, c_k}^v \subseteq G_v$ acts $(r+1)$ -transitively on $V_{c_0} \cap X^1$. Therefore we can send the $(r+1)$ -tuple $(P_{i_1}, \dots, P_{i_r}, P_{l+1})$ to $(P_{i_1}, \dots, P_{i_r}, P'_{l+1})$ fixing the remaining points of $M_1 \cup \dots \cup M_k$. This confirms our claim in Case 1.

In Case 2 we have $P_{l+1} = (R_{l+1}, 0, 0) \in X^1$. It follows from Lemma 6.9 that $R_{l+1} = \pi(P_{l+1})$ belongs to Y_{reg} and $df(R_{l+1}) \neq 0$ in the cotangent space $T_{R_{l+1}}^* Y$. The variety Y^1 being flexible, there exists an LND $\partial_0 \in \text{Der } \mathbb{R}[Y]$ such that $\partial_0(f)(R_{l+1}) \neq 0$. Let $q(v) = v(v - v_1)(v - v_2) \dots (v - v_l)$ be a polynomial in $\mathbb{R}[v]$ and choose a set of generators x_1, \dots, x_s of the algebra $\mathbb{R}[Y]$. Then, as in (6.1), the derivation ∂_0 can be extended to $\partial_1 \in \text{Der } \mathbb{R}[X]$ via

$$\begin{aligned} \partial_1(x_i) &= q(v)\partial_0(x_i), \quad i = 1, 2, \dots, s, \\ \partial_1(v) &= 0, \\ \partial_1(u) &= \frac{q(v)}{v} \partial_0(f). \end{aligned}$$

Due to our choice, $\partial_1(u)(P_{l+1}) \neq 0$. Hence the action of the associate one-parameter unipotent subgroup $H(\partial_0, q) = \exp(t\partial_1)$ pushes the point P_{l+1} out of U_0 . So the orbit $H(\partial_0, q).P_{l+1}$ meets the hypersurface $U_0 \subseteq X^1$ in finitely many points. Similarly, for every $j = 1, 2, \dots, l$ the orbit $H(\partial_0, q).P_j \not\subseteq U_0$ meets U_0 in finitely many points. Let $\varphi = \exp(t_0\partial_1) \in H(\partial_0, q) \subseteq G_v$. For a general value of $t_0 \in \mathbb{R}$ the image $\varphi.P_j$ lies outside U_0 for all $j = 1, 2, \dots, l+1$. Since the group $H(\partial_0, q)$ preserves v , the points $\varphi.P_1, \dots, \varphi.P_l$ are still hyperbolic. Interchanging the roles of u and v , we achieve that the assumptions of Case 1 are fulfilled for the new collection $\varphi.P_1, \dots, \varphi.P_l, \varphi.P_{l+1}$, as required. \square

6.4 Infinite transitivity on one connected component

This section is devoted to the proof of Theorem 6.6. Recall that Y is an affine variety defined over \mathbb{R} , $f \in \mathbb{R}[Y]$ is non-constant, and $X = \text{Susp}(Y, f)$. Let Y^1 be a connected component of Y_{reg} , and let X^1 be a connected component of the intersection $\text{Susp}(Y^1, f) \cap X_{\text{reg}}$.

Lemma 6.12. *Let m be a positive integer and let P_1, \dots, P_m be m points in X^1 . There exist an automorphism $g \in \text{SAut}(X)$ and a nonzero real number α such that for each $i = 1, \dots, m$ the number $\alpha v(g.P_i)$ is an interior point of $f|_{Y^1}$.*

Moreover, for any finite sets of real numbers \mathcal{U} disjoint from $\{u(P_i)\}$ and \mathcal{V} disjoint from $\{v(P_i)\}$, the automorphism g can be chosen in $\langle \text{Stab}_{\mathcal{U}}^u, \text{Stab}_{\mathcal{V}}^v \rangle$.

Proof. Acting with G_v , we may assume that the m points P_1, \dots, P_m have pairwise distinct u -coordinates. Acting further with G_u , we may assume that these points have also pairwise distinct values of their v -coordinates.

The proof depends on the behavior of f . If 0 is an interior point of $f(Y^1)$, we let $g = \text{Id}$ and choose α small enough. Then all $\alpha v(P_i)$ are close to 0, and thus are interior points of $f(Y^1)$.

If $f|_{Y^1}$ is non-bounded and $f|_{Y^1} > 0$, we let $g = \text{Id}$ and choose α great enough. All $\alpha v(P_i)$ are then large enough and are thus interior points of $f(Y^1)$. In the case $f|_{Y^1}$ is non-bounded and $f|_{Y^1} < 0$, the same argument works.

It remains to consider a bounded function f , that is $\overline{f(Y^1)} = [a, b]$. This splits into two cases: either $ab > 0$, or $ab = 0$.

Case 1. Without loss of generality we may suppose that $0 < a < b$. Let $v_i = v(P_i)$ and $u_i = u(P_i)$. Consider the connected component X^1 of the suspension over Y^1 such that all $v_i > 0$, and all $u_i > 0$. Let v_{\max} and v_{\min} be the maximal and minimal values of $v(P_1), \dots, v(P_m)$.

If $v_{\max}/v_{\min} < b/a$, we let $g = \text{Id}$. Then for any $\alpha \in]a/v_{\min}, b/v_{\max}[$, it is clear that all real numbers αv_i are interior points of $f(Y^1)$.

Otherwise, if $v_{\max}/v_{\min} \geq b/a$, we need a non-trivial automorphism g . Fix $\varepsilon > 0$. Note that, acting with the group G_v , any point $P \in \{P_1, P_2, \dots, P_m\}$ can be mapped to a point P' such that $f(P')$ is very close to b , while all the other points are fixed. Indeed, for a general $\varepsilon_1 < \varepsilon$, the real number $\frac{b-\varepsilon_1}{v(P)} \notin \{u_1, \dots, u_m\}$. Let R in Y^1 be such that $f(R) = b - \varepsilon_1$. We endow R with two extra coordinates $u = \frac{b-\varepsilon_1}{v(P)}$, $v = v(P)$ and get $P' = (R, \frac{b-\varepsilon_1}{v(P)}, v(P)) \in X^1$. The point P can be mapped to P' by an element g_1 of the group $\text{Stab}_{v(P_1), \dots, v(P), \dots, v(P_m)}^v$. Thus for a general $\varepsilon_1 < \varepsilon$, the automorphism g_1 of X satisfies $g_1.P_j = P_j$ for $P_j \neq P$, and $f(g_1.P) > b - \varepsilon$.

Choose $P = P_i$ such that $v(P_i) = v_{\max}$, $P_i = (R_i, u_i, v_{\max})$. As described above, we map P_i to $P'_i = (R', \frac{b-\varepsilon_1}{v_i}, v_i)$ such that $f(R') = b - \varepsilon_1$. Then, interchanging u and v , and interchanging a and b , there is an element of G_u which maps P'_i to $P''_i = (R'', \frac{b-\varepsilon_1}{v_i}, \frac{(a+\varepsilon_2)v_i}{b-\varepsilon_1})$ such that $f(R'') = a + \varepsilon_2$. Note that $v(P''_i) < \frac{a+\varepsilon}{b-\varepsilon}v(P_i)$.

If $v_{\max}/v_{\min} \geq b/a$ for the new set $P_1, \dots, P'_i, \dots, P_m$, we repeat this procedure. This process is finite since at each step the product $v(P_1) \dots v(P_m)$ reduces by a factor at least $\frac{a+\varepsilon}{b-\varepsilon}$. Finally, we get m points $g.P_1, \dots, g.P_m$ such that $v_{\max}/v_{\min} < b/a$.

Case 2. One of a and b equals zero. Using Lemma 6.11, we map the given m -tuple P_1, \dots, P_m in X^1 to points $P'_1, \dots, P'_m \in X^1 \setminus (U_0 \cup V_0)$. A sufficiently small α then fulfills the required condition.

To prove the second part of the lemma, we run the proof once again but we add an extra condition while performing the lift of an automorphism from Y to X . Namely, for an automorphism in G_u , we multiply the polynomial q by $\prod_{u \in U} q(u)$, and for an automorphism in G_v , we multiply the polynomial q by $\prod_{v \in V} q(v)$. \square

Proof of Theorem 6.6. Fix two m -tuples of distinct points P_1, \dots, P_m and Q_1, \dots, Q_m in X^1 . By Lemma 6.12, up to the action of $\text{SAut}(X)$, there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\alpha v(P_i)$ belongs to $\text{Int } f(Y^1)$ and $\alpha v(Q_i) \in \text{Int } f(Y^1)$ for all $i = 1, \dots, m$. We denote by c_1, \dots, c_k the distinct values of the v -coordinates of the given $2m$ points. We split the set $\{P_1, \dots, P_m, Q_1, \dots, Q_m\}$ into k subsets according to the v -coordinate. For each i , the set $V_{c_i} \cap U_\alpha \cap X^1$ is infinite by Lemma 6.7. In particular, for each i , we get $\#(V_{c_i} \cap U_\alpha \cap X^1) \geq 2m$. By Lemma 6.10, there exists $g_i \in \text{Stab}_{c_1, \dots, c_i, \dots, c_k}^v$ such that

$$g_i(\{P_1, \dots, P_m, Q_1, \dots, Q_m\} \cap V_{c_i}) \subset V_{c_i} \cap U_\alpha$$

and g_i fixes all the points belonging to $\cup_{j \neq i} V_{c_j}$. Let us denote by $P'_1, \dots, P'_m, Q'_1, \dots, Q'_m \in U_\alpha$ the images by $g_k \circ \dots \circ g_1$. Since the action of $\text{Stab}_\emptyset^u = G_u$ on U_α is infinitely transitive by Lemma 6.10, there exists a special automorphism mapping the m -tuple P'_1, \dots, P'_m to Q'_1, \dots, Q'_m . \square

Lemma 6.13. *If $\text{SAut } Y$ acts infinitely transitively on Y^1 where $\dim Y^1 \geq 2$, then the flexibility of Y^1 implies the flexibility of any connected component X^1 of $\text{Susp}(Y^1, f)$,*

Proof. Clearly, X^1 is flexible if one point $P = (R, u, v) \in X^1$ is and if the group $\text{SAut}(X)$ acts transitively on X^1 . Since the function $f \in \mathbb{R}[Y^1]$ is non-constant, $df(R) \neq 0$ at some point $R \in Y^1$ with $f(R) \neq 0$. Due to our assumption Y^1 is flexible. Hence there exist n locally nilpotent derivations $\partial_0^{(1)}, \dots, \partial_0^{(n)} \in \text{Der } \mathbb{R}[Y]$, where $n = \dim Y = \dim Y^1$, such that the corresponding vector fields ξ_1, \dots, ξ_n

span the tangent space $T_R Y$, i.e.

$$\operatorname{rk} \begin{pmatrix} \xi_1(R) \\ \vdots \\ \xi_n(R) \end{pmatrix} = n.$$

It follows that $\partial_0^{(i)}(f)(R) \neq 0$ for some index $i \in \{1, \dots, n\}$.

Let now $P = (R, u_0, v_0) \in X^1$ be a point such that $\pi(P) = R$. Since $u_0 v_0 = f(R) \neq 0$, the point P is hyperbolic. Make a lift as in (6.1) with $q(v) = v$. We obtain n LNDs

$$\partial_1^{(1)}, \dots, \partial_1^{(n)} \in \operatorname{Der} \mathbb{R}[X], \quad \text{where} \quad \partial_1^{(j)} = \partial_1^{(j)}(\partial_0^{(j)}, v).$$

If we interchange u and v and let $j = i$, we get another LND

$$\partial_2^{(i)} = \partial_2^{(i)}(\partial_0^{(i)}, u) \in \operatorname{Der} \mathbb{R}[X].$$

Let us show that the corresponding $n + 1$ vector fields span the tangent space $T_R X$ at R , as required. We can view $\partial_1^{(1)}, \dots, \partial_1^{(n)}, \partial_2^{(i)}$ as LNDs in $\operatorname{Der} \mathbb{R}[Y][u, v]$ preserving the ideal $\hat{E}(uv - f)$, so that the corresponding vector fields are tangent to the hypersurface

$$X = \{uv - f(R) = 0\} \subseteq Y \times \mathbb{A}^2.$$

The values of these vector fields at the point $P' \in X$ yield an $(n+1) \times (n+2)$ -matrix

$$E = \begin{pmatrix} v_0 \xi_1(R) & \partial_0^{(1)}(f)(R) & 0 \\ \vdots & \vdots & \vdots \\ v_0 \xi_n(R) & \partial_0^{(n)}(f)(R) & 0 \\ u_0 \xi_i(R) & 0 & \partial_0^{(i)}(f)(R) \end{pmatrix}.$$

The first n rows of E are linearly independent, and the last one is independent from the preceding since $\partial_0^{(i)}(f)(R) \neq 0$. Therefore $\operatorname{rk}(E) = n + 1 = \dim X$. So these locally nilpotent vector fields indeed span the tangent space $T_P X$ at P , as claimed. \square

6.5 Infinite transitivity on each connected component

The purpose of this section is the proof of Theorem 6.3. As above, Y is an affine variety defined over \mathbb{R} , $f \in \mathbb{R}[Y]$ is non-constant, and $X = \operatorname{Susp}(Y, f)$. Moreover, we assume that $\operatorname{SAut}(Y)$ acts infinitely transitively on each connected component of Y_{reg} . Recall that $v(P)$ denotes the v -coordinate of the point $P \in X \subseteq Y \times \operatorname{Spec} \mathbb{R}[u, v]$.

Lemma 6.14. *Assume that Y is a flexible variety. For every finite set of points P_1, \dots, P_m in X_{reg} , there exists an automorphism $g \in \text{SAut}(X)$ such that all $v(g.P_i)$ are pairwise distinct, and that all $u(g.P_i)$ are pairwise distinct.*

Proof. We cannot use the starting argument of the proof of Lemma 6.12, since several points of X may have the same projection in Y .

We denote the set of projections $\pi(P_1), \dots, \pi(P_m)$ by $\{R_1, \dots, R_{m'}\}$. Note that all R_i belong to Y_{reg} by Lemma 6.9. Up to a special automorphism of X we can assume that all $f(R_i)$ are pairwise distinct. This is possible since f is non-constant on each connected component, and the action of $\text{SAut}(Y)$ on Y is infinite transitive on each connected component. Consider the images of P_1, \dots, P_m under the projection $\rho: X \rightarrow \text{Spec } \mathbb{R}[u, v]$. If $\pi(P_i) \neq \pi(P_j)$, the projections $\rho(P_i)$ and $\hat{E}\rho(P_j)$ cannot coincide. Otherwise $f(R_i) = u_i v_i = u_j v_j = f(R_j)$. If $\pi(P_i) = \pi(P_j)$, we get also $\rho(P_i) \neq \rho(P_j)$ since the points P_i and P_j are distinct.

Thanks to Lemma 6.11, keeping $\rho(P_i) \neq \rho(P_j)$ if $P_i \neq P_j$, we may assume that $u(P_i) \neq 0$ and $v(P_i) \neq 0$ for each i . We split the set $\{P_1, \dots, P_m\}$ into several subsets $M_1 \sqcup \dots \sqcup M_k$ according to the v -coordinate. Let $c_i \in \mathbb{R}$ be such that $M_i \subseteq V_{c_i}$. Using Lemma 6.10, we act by an element of $\prod_{i=1}^k \text{Stab}_{c_1, \dots, c_i, \dots, c_k}^v$ to get m points with pairwise distinct u -coordinates. Arguing likewise with Stab^u -actions, we get m points with pairwise distinct u - and v -coordinates. \square

Proof of Theorem 6.3. We denote by s the number of connected components of Y_{reg} and we suppose that the action of $\text{SAut}(Y)$ on Y_{reg} is infinitely transitive on each connected component. Consider a suspension $X = \text{Susp}(Y, f)$ and decompose $X_{\text{reg}} = X^1 \sqcup \dots \sqcup X^{s'}$ into connected components. Recall that over each connected component of Y_{reg} there is either one or two connected components of X_{reg} . Fix some integers $m'_1, m'_2, \dots, m'_{s'}$ such that $\sum m'_j = m$ and choose two $(m'_1 + \dots + m'_{s'})$ -tuples $\mathcal{P} = \{P_1, \dots, P_m\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_m\}$ in X such that for each j , the component X^j contains exactly m'_j points of \mathcal{P} and m'_j points of \mathcal{Q} . Let $\mathcal{S} = \mathcal{P} \cup \mathcal{Q} = \{S_1, \dots, S_{2m}\}$.

By Lemma 6.14 applied to \mathcal{S} , we may suppose that the values of the v -coordinates are pairwise distinct and that the values of the u -coordinates are also pairwise distinct.

We want to choose an s' -tuple of values $\alpha = (\alpha_1, \dots, \alpha_{s'})$ such that for all $S_i \in X^j$ the number $\alpha_j v(S_i)$ is an interior point of $f|_{X^j}$. To this end, we repeatedly apply Lemma 6.12 proceeding on one connected component at each step. Notice that we need to preserve the condition that the values of the v -coordinates are pairwise distinct and that the values of the u -coordinates are also pairwise distinct. At j th step, we let $\mathcal{U} = \{u(S_i), S_i \notin X^j\}$ and $\mathcal{V} = \{v(S_i), S_i \notin X^j\}$ and we use $g \in \langle \text{Stab}_{\mathcal{U}}^u, \text{Stab}_{\mathcal{V}}^v \rangle$ given by Lemma 6.12.

Such a choice of g provides that $g.S_i = S_i$ for $S_i \notin X^j$. To control the condition that all u -values are pairwise distinct and all v -values are pairwise distinct for

$S_i \in X^j$, we require the following. For each one-parameter subgroup $h(t)$ acting in the course of the proof of Lemma 6.12 (recall that it is non-trivial only for $S_i \in X^j$), the conditions on t are

$$\begin{aligned} v(h(t).S_i) &\notin \mathcal{V}, u(h(t).S_i) \notin \mathcal{U} \text{ for } S_i \in X^j; \\ v(h(t).S_{i_1}) &\neq v(h(t).S_{i_2}), u(h(t).S_{i_1}) \neq u(h(t).S_{i_2}) \\ &\text{for distinct } S_{i_1}, S_{i_2} \in X^j. \end{aligned}$$

This is true for generic t . At each step we get an α_j which fits for all $S_i \in X^j$. We may choose the α_i pairwise distinct. At the end, we get a collection $\alpha = (\alpha_1, \dots, \alpha_{s'})$, as required.

We construct an automorphism g_0 mapping \mathcal{S} to $(X^1 \cap U_{\alpha_1}) \sqcup (X^2 \cap U_{\alpha_2}) \sqcup \dots \sqcup (X^{s'} \cap U_{\alpha_{s'}})$ as the product of $2m$ automorphisms, each of them fixing all the points but one. Since all v -values are pairwise distinct, to map S_i , we let $q(v) = \beta v \prod_{k \neq i} (v - v(S_k))$ where β satisfies $q(v(S_i)) = 1$. If $S_i \in X^j$, using the lift defined by q (see (6.1)), we map S_i to $X^j \cap U_{\alpha_j}$. Notice that for $\{g_0.S_1, \dots, g_0.S_{2m}\}$, the u -values are no longer pairwise distinct.

To map $g_0.P_1, \dots, g_0.P_m$ onto $g_0.Q_1, \dots, g_0.Q_m$, we use, for each i , the infinite transitivity of the group $\text{Stab}_{\alpha_1, \dots, \tilde{\alpha}_i, \dots, \alpha_{s'}}^u$, multiplying the corresponding LNDs on

$\mathbb{R}[Y]$ by the polynomial $\hat{E}q(u) = \gamma u(u - \alpha_1) \dots \overbrace{(u - \alpha_i)} \dots (u - \alpha_{s'})$ where γ is such that $q(\alpha_i) = 1$. In this way, for each i , we fix the points lying off the i th connected component. Finally we get an automorphism g which maps $g_0.P_1, \dots, g_0.P_m$ to $g_0.Q_1, \dots, g_0.Q_m$. Hence, $g_0^{-1}gg_0$ maps the m -tuple P_1, \dots, P_m to Q_1, \dots, Q_m , as required. \square

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