Analyse mathématique et approximation numérique des équations de Stokes et de Navier-Stokes avec des conditions aux limites non standard

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ACADÉMIE DE BORDEAUX

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Introduction

La plupart des travaux mathématiques sur les équations de Stokes et de Navier-Stokes dans des domaines bornés ont considéré ces systèmes avec une condition aux limites de type Dirichlet pour le champ de vitesses. Néanmoins, dans les applications, il est possible de se trouver face à des problèmes où il est nécessaire de faire intervenir d'autres types de conditions aux limites. Pour le cas du système de Stokes :

$$\begin{aligned} -\nu\Delta\,\boldsymbol{u} + \nabla\,\pi &= \boldsymbol{f} \qquad \text{dans } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \qquad \text{dans } \Omega, \end{aligned}$$

si on suppose que le bord Γ a trois parties Γ_1, Γ_2 et Γ_3, ces conditions aux limites peuvent être de la forme :

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}_0 \qquad \text{sur } \Gamma_1, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \ \text{ et } \ \pi &= \pi_0 \qquad \text{sur } \Gamma_2, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \ \text{ et } \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} &= \boldsymbol{h}\times\boldsymbol{n} \qquad \text{sur } \Gamma_3, \end{aligned} \tag{0.1}$$

où $\boldsymbol{f}$, $\boldsymbol{u}_0$, g, π_0, $\boldsymbol{g}$ et $\boldsymbol{h}$ sont données et $\boldsymbol{n}$ est la normale extérieure au bord de Ω. On notera que ces conditions aux limites sont de trois types différents: vitesse donnée sur une partie de la frontière de Ω (notée Γ_1), pression et composante tangentielle de la vitesse données sur une deuxième partie de la frontière (Γ_2), vitesse normale et composante tangentielle du tourbillon données sur une troisième partie du bord (notée Γ_3). On peut poser un probème analogue mais relativement aux équations stationnaires de Navier-Stokes :

$$\begin{aligned} -\nu\Delta\,\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\,\boldsymbol{u} + \nabla\,\pi &= \boldsymbol{f} \qquad \text{dans } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \qquad \text{dans } \Omega, \end{aligned}$$

avec les mêmes conditions dans (0.1), sauf que la pression statique π dans (0.1) va être remplacée par une pression dynamique : $\pi + \frac{1}{2}|\boldsymbol{u}|^2$ qui joue le rôle joué par la pression statique.

La première condition donnée dans (0.1) est une condition classique, c'est l'adhérence à la paroi Γ_1, que celle ci soit fixe ($\boldsymbol{u}_0 = \boldsymbol{0}$) ou mobile (cas d'un obstacle ou d'une paroi qui se déplace dans l'écoulement). Les conditions aux limites données sur Γ_2 et Γ_3 sont en revanche moins considérées dans la littérature. Dans les applications, on trouve fréquemment des problèmes dans lequels les conditions aux limites données sur Γ_1 et sur Γ_2 interviennent d'une façon naturelle. Voici quelques exemples :

Exemple 1: **L'écoulement de Poiseuille**. Cet exemple consiste à étudier l'écoulement dans un cylindre (disons de génératrices parallèles à l'axe x_3). Ici Ω désigne le volume occupé par le fluide, Γ_1 est formé des surfaces latérales du cylindre, Γ_2 a deux parties: l'entrée Γ_{21} et la sortie Γ_{22}, Γ_3 est vide (voir la figure çi-après). Les parois latérales étant supposées rigides et fixes, on suppose ici que $\boldsymbol{g} = \boldsymbol{0}$, l'écoulement est normal à l'entrée et à la sortie, c'est-à-dire ($\boldsymbol{g} = \boldsymbol{0}$) et la pression est constante à l'entrée et à la sortie.

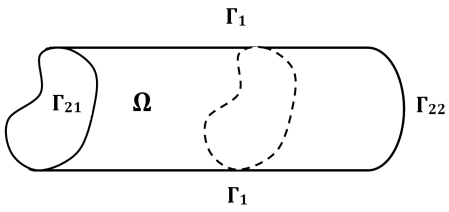


Figure 1:

Exemple 2: **Écoulement dans une tuyauterie**. Le domaine Ω représente l'ensemble de la tuyauterie (voir la figure çi dessus). Dans ce cas, Γ_1 est formé des parois des tuyaux et Γ_2 est l'union de toutes les entrées et sorties du réseau de tuyaux. Ainsi, $\Gamma_2 = \cup_i \Gamma_{2i}$ où chaque Γ_{2i} représente une entrée ou une sortie sur laquelle la pression est connue, Γ_3 est vide.

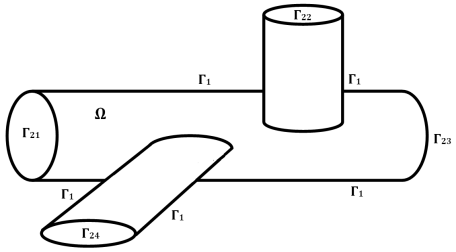


Figure 2:

On peut supposer que $\boldsymbol{u}_0$ et $\boldsymbol{g}$ sont nulles, ce qui veut dire que les parois de la tuyauterie sont rigides et que le fluide adhère aux parois. De plus, le fluide entre et sort du réseau avec une vitesse tangentielle nulle.

Bien que moins vaste que celles portant sur des conditions de type Dirichlet, il existe néanmoins une littérature importante concernant l'analyse mathématique et l'approximation numérique des problèmes de Stokes et de Navier-Stokes avec les conditions aux limites de type (0.1). Concernant l'analyse mathématique, une des premières références que l'on peut citer concerne le travail de Conca *et. al.* [26] où les auteurs étudient les équations de Stokes et de Navier-Stokes avec les conditions aux limites (0.1). Dans un cadre hilbertien, ils obtiennent ainsi des résultats d'existence et d'unicité pour le problème variationnel associé. Des résultats analogues sont également établis dans le cas des équations de Navier-Stokes stationnaires. Leur étude fut ensuite complété par Bernard [15], principalement au sujet de la régularié. Les résultats d'existence et d'unicité démontrés dans [26] sont basés sur le lemme de Lax-Milgram

et dépendent de la géométrie des problèmes, c'est-à-dire de Ω et de la partition $\{\Gamma_1,\ \Gamma_2,\ \Gamma_3\}$. Nous pouvons citer également C. Ebmeyer and J. Frehse [34] qui ont étudié le problème stationnaire de Navier-Stokes dans un domaine polyhédral et lipschitzien, avec des conditions aux limites portant sur la composante normale de la vitesse et la composante tangentielle du tourbillon en dimension 3. Les auteurs ont établi des résultats de régularité $W^{s,2}$ pour $s < 3/2$. De nombreux auteurs se sont intéréssés à des problèmes de type Stokes mais avec d'autres conditions aux limites portant sur le tenseur des contraintes. Nous pouvons citer par exemple [16], [20] et [25].

Concernant l'analyse numérique, nous renvoyons par exemple à [1, 2, 33] où des méthodes d'élements finis pour la formulation tourbillon-vitesse-pression ont été obtenues et analysées en dimension deux. Dans [37, 38], une discrétisation par éléments finis dans l'espace $\boldsymbol{H}(\mathbf{curl})$ pour le probleme de Navier-Stokes est considérée. Dans [37], les inconnues sont la fonction courant, le tourbillon et la pression, tandis que dans [38], les inconnues sont la vitesse et la pression. Des méthodes spectrales ont été considérées dans [14] pour les équations de Stokes dans un domaine bidimensionnel ou tridimensionnel, munies de conditions aux limites portant sur la composante normale de la vitesse et la composante tangentielle du tourbillon. Ces conditions aux limites sont traitées dans [13] pour les équations de Navier-Stokes avec une approche similaire.

Cette thèse est consacrée à l'étude des équations de Stokes et de Navier-Stokes avec différentes conditions aux limites de type (0.1). La thèse se compose de quatre parties. Dans la première, nous nous intéressons en premier lieu à l'obtention d'inégalités de Sobolev pour des champs de vecteurs $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Dans un second temps, nous établissons des résultats d'existence pour les potentiels vecteurs avec diverses conditions aux limites. La seconde partie est consacrée à la résolution de problèmes de Stokes avec des conditions aux limites de type (0.1) et de systèmes elliptiques qui se traduisent par des équations de Stokes sans la pression. Dans la troisième, on étudie les équations d'Oseen et de Navier-Stokes correspondantes et on termine dans une dernière partie par l'analyse numérique de ces équations.

L'objet principal de ce travail consiste à étendre au cadre non hilbertien les résultats concernant les potentiels vecteurs établis dans Amrouche *et al* [3] et ceux concernant les équations de Stokes et de Navier-Stokes établis dans Conca *et al* [26].

Dans les trois premières parties, nous considérons le cas très général où le domaine Ω occupé par un fluide est un ouvert borné tridimensionnel éventuellement multiplement connexe et suffisamment régulier (en général au moins lipschitzien, parfois de classe $\mathcal{C}^{1,1}$, voire de classe $\mathcal{C}^{2,1}$). Dans la dernière, Ω est un ouvert borné polygonal.

La partie I est découpée en trois chapitres.

Le chapitre 1 est naturellement dédié aux notations, définitions et propriétés des espaces fonctionnels et aux résultats fondamentaux sur lesquels nous nous appuyons dans la suite. On commence par donner des résultats de densité de l'espace $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ dans l'espace des champs de vecteurs $\boldsymbol{L}^p$ à divergence L^p ou au rotationnel $\boldsymbol{L}^p$. Ceci permet de définir proprement les traces

de certains champs de vecteurs et des formules de Green appropriées (voir le travail de [22] dans le cas d'un polyhédral lipschitz). Nous donnons ensuite des caractérisations de duaux d'espaces fonctionnels qui correspondent ici aux espaces naturels dans lesquels nous prendrons les fonctions extérieurs. On y établit aussi un autre résultat concernant les traces de fonctions peu régulières à laplacien dans $W^{-1,p}(\Omega)$. Nous donnons également un résultat concernant l'identité algébrique et topologique de l'espace des champs de vecteurs $\boldsymbol{L}^p$ à divergence L^p et à rotationnel $\boldsymbol{L}^p$ avec trace tangentielle et trace normale nulles avec l'espace $\boldsymbol{W}_0^{1,p}(\Omega)$. Ce résultat est connu (voir [3] et [39]) dans le cas $p = 2$ et nous donnons une extension au cas $p \neq 2$ où la situation est différente comme nous allons le voir plus loin.

Dans le chapitre 2, nous nous consacrons à la preuve de deux types d'inégalités, où la norme $\boldsymbol{L}^p$ du gradient peut être contrôlée par celle de la divergence, du rotationnel et d'une quantité qui reflète la structure topologique du domaine suivant la condition aux limites considérée. La première concerne les champs de vecteurs tangents à la frontière :

$$\|\nabla\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}|). \tag{0.2}$$

La deuxième concerne les champs de vecteurs normaux à la frontière :

$$\|\nabla\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|). \tag{0.3}$$

Les preuves sont basées sur l'utilisation des formules classiques de représentations intégrales pour les champs de vecteurs faisant apparaitre les opérateurs divergence et rotationnel. A ces formules de représentations, on applique l'inégalité de Calderon-Zygmund et les résultats classiques sur les traces pour obtenir les estimations (0.2) et (0.3). Nous pouvons citer ici Von Wahl [55] pour une approche similaire dans un domaine simplement connexe et de bord connexe (ce qui implique en fait que le premier et le deuxième nombre de Betti sont nuls), où il estime la norme du gradient par la norme de la divergence et celle du rotationnel. Comme conséquence de ces inégalités, on montre l'identité algébrique et topologique de l'espace des champs de vecteurs $\boldsymbol{L}^p$ à divergence L^p et à rotationnel $\boldsymbol{L}^p$ avec trace tangentielle ou trace normale nulle avec un sous-espace de $\boldsymbol{W}^{1,p}(\Omega)$. On étend ce dernier résultat au cas où la trace tangentielle ou la trace normale ne sont pas nulles mais appartiennent à l'espace de traces $W^{1-1/p,p}(\Gamma)$ ou $\boldsymbol{W}^{1-1/p,p}(\Gamma)$. Nous considérons aussi le cas des espaces de Sobolev fractionnaires. On termine ce chapitre par des propriétés de compacité.

Dans le chapitre 3 de cette première partie, nous donnons une généralisation des résultats concernant les potentiels vecteurs, scalaires et les potentiels vecteurs faibles en théorie $\boldsymbol{L}^p$ avec $1 < p < \infty$, étendant ainsi les résultats établis par Amrouche, Bernardi, Dauge et Girault [3] et par [4] dans le cadre hilbertien (voir aussi les résultats établis par D. Mitrea, M. Mitrea et J. Pipher [47] dans le cas de domain lipschitzien de $\mathbb{R}^3$). Dans un premier temps, nous montrons l'existence d'un potentiel vecteur associé à un champ de vecteurs à divergence nulle et vérifiant

une condition de flux nul. Ce résultat de base nous permettra ensuite de construire d'autres types de potentiels vecteurs. Contrairement au cas hilbertien où la preuve est basée sur la transformée de Fourier, pour $p \neq 2$ on fait intervenir la solution fondamentale du laplacien. Dans un second temps on construit, à l'aide du potentiel vecteur de base, trois types de potentiels vecteurs. En particulier, trois types de conditions aux limites sont proposées :

$$\boldsymbol{\psi} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}, \quad \boldsymbol{\psi} = \mathbf{0} \quad \text{sur } \Gamma.$$

Pour prouver l'existence et l'unicité de ces potentiels vecteurs on passe comme pour le cadre hilbertien par la résolution de certains problèmes elliptiques. Le lemme de Lax-Milgram ne s'appliquant plus, on est amené à établir une condition Inf-Sup permettant d'utiliser le théorème de Babuška-Brezzi :

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}_T^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{\xi} \in \boldsymbol{V}_T^{p}(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_T^p(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}} \geq \beta. \tag{0.4}$$

Cette dernière joue ici un rôle clef pour établir l'existence et l'unicité de solutions pour le problème de Stokes considéré et plus généralement, comme on le sait, pour les problèmes elliptiques linéaires. Nous adaptons aussi des résultats concernant les potentiels scalaires et les potentiels vecteurs faibles établis dans le cadre hilbertien ([3]) au cas $1 < p < \infty$. Enfin, en utilisant les résultats précédents, nous donnons une nouvelle formulation pour le problème de Stokes avec une condition aux limites de type Dirichlet, où l'inconnue est le potentiel vecteur. Nous commençons par le cas du potentiel vecteur normal ($\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad$ sur Γ) et puis le cas du potentiel vecteur tangentiel ($\boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \quad$ sur Γ). L'avantage de cette méthode est que la contrainte de la divergence n'est pas exigée pour l'espace des fonctions tests. Ces résultats de la première partie ont fait l'objet d'une Note soumise aux " Comptes Rendus de l'Académie des Sciences de Paris ".

Dans **la partie II**, nous appliquons les résultats de la partie **I** pour étudier le problème de Stokes toujours en théorie $\boldsymbol{L}^p$ avec différentes conditions aux limites. Contrairement au problème de Stokes avec une condition de bord de Dirichlet, les conditions aux limites données sur Γ_2 et Γ_3 dans (0.1) permettent de découpler la pression π du champ de vitesses $\boldsymbol{u}$ à partir des équations. Des formulations variationnelles peuvent être introduites pour le problème réduit de Stokes (sans le terme en π).

Dans le chapitre 4, nous étudions le problème de Stokes avec des conditions aux limites portant sur la composante normale du champ de vitesses et la composante tangentielle du tourbillon :

$$(\mathcal{S}_T) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{dans } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, \quad \text{et} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{sur } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \qquad 1 \leq j \leq J. \end{cases}$$

Les conditions aux limites données dans $(\mathcal{S}_T)$ permettent en fait d'obtenir la pression π directement comme solution d'un problème de Neumann. C'est la raison pour laquelle nous sommes naturellement conduits à étudier le problème elliptique suivant :

$$(E_T) \quad \begin{cases} -\Delta\, \boldsymbol{\xi} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{\xi} = 0 & \text{dans } \Omega, \\ \boldsymbol{\xi} \cdot \boldsymbol{n} = g, \quad \text{et} \quad \mathbf{curl}\, \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{sur } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \qquad 1 \leq j \leq J. \end{cases}$$

On commence donc par établir l'existence de solutions faibles pour ce dernier problème, ce que l'on fait grâce à la condition Inf-Sup (0.4). Par un argument de régularité, on déduit ensuite l'existence de solutions fortes. Nous nous intéressons aussi à la recherche des solutions très faibles correspondantes que l'on obtient par des arguments de dualité. Une des difficultés consiste à donner un sens aux traces de fonctions très peu régulières et à obtenir par le biais de lemmes de densité les formules de Green adéquates. On est alors en position de faire le même travail pour le problème $(\mathcal{S}_T)$. Les dernières conditions dans $(\mathcal{S}_T)$ sont des conditions de flux nuls à travers les coupures Σ_j permettant d'obtenir l'unicité de $\boldsymbol{u}$.

En suivant le même schéma qu'au chapitre 4, nous abordons ensuite dans le chapitre 5 l'étude du problème elliptique suivant :

$$(E_N) \quad \begin{cases} -\Delta\, \boldsymbol{\xi} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{\xi} = 0 & \text{dans } \Omega, \\ \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, & \text{sur } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \qquad 1 \leq i \leq I, \end{cases}$$

qui va s'avérer très utile pour la résolution du problème de Stokes avec des conditions aux limites portant sur la pression et sur la composante tangentielle du champ de vitesses :

$$(\mathcal{S}_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{dans } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, \quad \text{et} \quad \pi = \pi_0 & \text{sur } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \qquad 1 \leq i \leq I. \end{cases}$$

Ici encore, la pression peut être obtenue directement comme solution d'un problème de Dirichlet et la résolution du problème (E_N) se fait au moyen de l'obtention de la condition Inf-Sup :

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}_N^{p}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_N^{p}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}} \geq \beta. \tag{0.5}$$

Ensuite, nous donnons une variante du système de Stokes $(\mathcal{S}_N)$ dans le cas où la condition de compatibilité n'est pas vérifiée. Ce qui fait apparaitre des constantes comme inconnues supplémentaires. Enfin la dernière section de ce chapitre est consacrée à deux types de décompositions de Helmholtz :

i) si $\boldsymbol{u}\in \boldsymbol{L}^p(\Omega)$, il existe $\chi\in W^{1,p}(\Omega)$, $\boldsymbol{w}\in \boldsymbol{W}_\sigma^{1,p}(\Omega)\cap \boldsymbol{X}_N^{\,p}(\Omega)$, $\boldsymbol{z}\in \boldsymbol{K}_T^{p}(\Omega)$ tel que :

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\,\chi + \mathbf{curl}\,\boldsymbol{w}, \tag{0.6}$$

ii) si $\boldsymbol{u}\in \boldsymbol{L}^p(\Omega)$, il existe $\chi\in W_0^{1,p}(\Omega)$, $\boldsymbol{w}\in \boldsymbol{W}_\sigma^{1,p}(\Omega)\cap \boldsymbol{X}_T^{\,p}(\Omega)$, $\boldsymbol{z}\in \boldsymbol{K}_N^{p}(\Omega)$ tel que :

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\,\chi + \mathbf{curl}\,\boldsymbol{w}. \tag{0.7}$$

Ces conditions Inf-Sup et ces décompositions de Helmholtz ont été récemment établies par Kozono et Yanagisawa [46] dans le cas d'un ouvert de classe C^∞ utilisant pour cela la théorie d'Agmon-Douglis-Nirenberg. Citons également le travail de Buffa et Ciarlet, Jr [21] dans le cas d'un polyhédre lipschitzien en théorie hilbertienne.

Le Chapitre 6 est consacré à la résolution des problèmes elliptiques (E'_T) et (E'_N) qui sont des variantes des problèmes (E_T) et (E_N) :

$$(E'_T)\quad \begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{f} & \text{dans } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{et} \quad \dfrac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{sur } \Gamma, \end{cases}$$

et

$$(E'_N)\quad \begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{f} & \text{dans } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{et} \quad \dfrac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} = h & \text{sur } \Gamma. \end{cases}$$

Les résultats de la seconde partie ont fait l'objet d'une Note soumise aux " Comptes Rendus de l'Academie des Sciences de Paris ".

La partie III concerne les équations stationnaires d'Oseen et de Navier-Stokes et se compose de deux chapitres. Dans le premier chapitre, on commence par étudier le problème linéaire : on cherche $\boldsymbol{u}$, π et des constantes C_i, $1\leq i\leq I$ solution de :

$$(OS_N)\qquad \begin{cases} -\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} & \text{dans } \Omega\\ \operatorname{div}\boldsymbol{u} = 0 & \text{dans } \Omega\\ \boldsymbol{u}\times\boldsymbol{n} = 0 & \text{sur } \Gamma,\\ \pi = \pi_0 + C_i & \text{sur } \Gamma_i,\ i=1,\ldots,I,\\ \int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i=1,\ldots,I, \end{cases}$$

où $\boldsymbol{f}$, $\boldsymbol{a}$ et π_0 sont données. Comme pour le problème de Stokes, on établit des résultats d'existence et d'unicité de solutions faibles et fortes pour (OS_N). Par un argument de point fixe, on résout ensuite le problème non linéaire correspondant. Le second chapitre est consacré à l'étude du même problème avec les conditions aux limites portant sur $\boldsymbol{u} \cdot \boldsymbol{n}$ et $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}$.

La partie IV est divisée en deux chapitres. Le chapitre 9 est consacré à la discretisation par la méthode de Nitsche des équations de Stokes avec diverses conditions aux limites. Il s'organise comme suit: dans la section 9.1, nous rappelons la méthode de Nitsche pour le problème de Poisson tandis que les sections 9.3 et 9.4 sont consacrées à l'analyse de la méthode de Nitsche pour le problème de Stokes avec les diverses conditions aux limites sans introduire la vorticité comme une nouvelle inconnue. Enfin, ces résultats théoriques sont illustrés dans la section 9.5 par quelques résultats numériques. Dans le chapitre 10, nous analysons une méthode de Galerkine discontinue (DG) pour le même problème. Dans la section 10.2, nous écrivons le probème discret. Puis nous démontrons dans la section 10.3, que les estimations de l'erreur d'approximation sont optimales, en vitesse et en pression. Quelques résultats numériques de convergence sont décrits en section 10.4 qui sont parfaitement cohérents avec l'analyse.

Les notations de la dernière partie sont indépendantes des parties précédentes.

Part I

L^{p}-Inequalities for Vector Fields and Vector Potentials

Chapter 1

Basic properties of the functional framework

Let Ω be a bounded connected open set of $\mathbb{R}^3$ and Γ its boundary. **In this chapter, Ω is supposed of class $\mathcal{C}^{0,1}$** except in some cases where we will precise that the boundary can be more regular. Then a unit exterior normal vector to the boundary can be defined almost everywhere on $\partial\Omega$; it is denoted by $\boldsymbol{n}$. The generic point in Ω (or $\mathbb{R}^3$) is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$. We denote by Γ_i, $0 \leq i \leq I$, the connected components of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$. We fix a smooth open set $\mathcal{O}$ with a connected boundary, such that $\overline{\Omega}$ is contained in $\mathcal{O}$, and we denote by Ω_i, $1 \leq i \leq I$, the connected components of $\mathcal{O} \backslash \overline{\Omega}$ with boundary Γ_i ($\Gamma_0 \cup \partial\mathcal{O}$ for $i = 0$). We do not assume that Ω is simply-connected, but we suppose that there exists J connected, oriented and open surfaces Σ_j, $1 \leq j \leq J$, called 'cuts', contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$, the boundary of Σ_j is contained in Γ for $1 \leq j \leq J$, the intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$, and the open set $\Omega^\circ = \Omega \backslash \bigcup_{j=1}^{J} \Sigma_j$ is simply-connected and pseudo-Lipschitz (see [3]). For $J = 1$ with $I = 3$, see for example Figure 1.1. We need Sobolev spaces $\boldsymbol{W}^{s,p}(\Gamma_i)$ on the connected component Γ_i, for $0 \leq i \leq I$, $1 < p < \infty$ and for some real numbers s. We can also define Sobolev spaces on the cuts $\boldsymbol{W}^{s,p}(\Sigma_j)$ as restrictions to Σ_j of the distributions belonging to $\boldsymbol{W}^{s,p}(\mathcal{M}_j)$. We will denote by $\boldsymbol{W}^{s,p}(\Sigma_j)'$ the dual space of $\boldsymbol{W}^{s,p}(\Sigma_j)$.

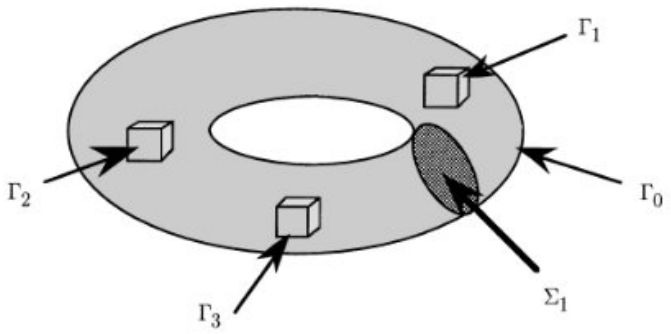


Figure 1.1:

Finally, $[\cdot]_j$ denote the jump of a function over Σ_j, for $1 \leq j \leq J$ and $\langle \cdot, \cdot \rangle_{X,X'}$ denotes the duality pairing between a space X and X'. Using the derivation in the distribution sense, we can define the operators **curl** and div on $\boldsymbol{L}^p(\Omega)$ for $1 \leq p \leq \infty$. Indeed, let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathcal{D}(\Omega)$ and its dual space $\mathcal{D}'(\Omega)$. For any function $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\boldsymbol{L}^p(\Omega)$, we have

$\forall \boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{D}(\Omega)$,

$$\begin{aligned} \langle \mathbf{curl}\, \boldsymbol{v}, \boldsymbol{\varphi} \rangle &= \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\boldsymbol{x} \\ &= \int_\Omega \left(v_1 (\frac{\partial \varphi_3}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_3}) + v_2 (\frac{\partial \varphi_1}{\partial x_3} - \frac{\partial \varphi_3}{\partial x_1}) + v_3 (\frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2}) \right) d\boldsymbol{x}, \end{aligned}$$

$\forall \varphi \in \mathcal{D}(\Omega)$,

$$\langle \mathrm{div}\, \boldsymbol{v},\, \varphi \rangle = -\int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\, \varphi\, d\boldsymbol{x} = -\int_\Omega \left(v_1 \frac{\partial \varphi}{\partial x_1} + v_2 \frac{\partial \varphi}{\partial x_2} + v_3 \frac{\partial \varphi}{\partial x_3} \right) d\boldsymbol{x}.$$

We note that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1,\, v_2,\, v_3)$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = \mathbf{grad}\, (\mathrm{div}\, \boldsymbol{v}) - \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{v} \tag{1.1}$$

or by

$$\Delta \boldsymbol{v} = (\Delta v_1,\, \Delta v_2,\, \Delta v_3).$$

For any function q in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ can be extended to $\boldsymbol{L}^p(\Omega)$. We denote this extension by $\widetilde{\mathbf{grad}}\, q$. In the sequel, the letter C denotes a constant that is not necessarily the same at its various occurrences and p denotes unless it is explicitely mentioned, a real number such that $1 < p < \infty$. This leads to the following definitions.

Definition 1.0.1. For $1 \leq p < \infty$, the space $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ is defined by

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\}\ , \tag{1.2}$$

and is provided with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}\right)^{\frac{1}{p}}.$$

The space $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ is defined by

$$\boldsymbol{H}^p(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)\}\ , \tag{1.3}$$

and is provided with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathrm{div},\Omega)} = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

Finally, we set

$$\boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\mathrm{div}, \Omega). \tag{1.4}$$

It is provided with the norm

$$\boldsymbol{X}^p(\Omega) = \left(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|^p_{L^p(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}\right)^{\frac{1}{p}}.$$

These definitions will also be used with Ω replaced by $\mathbb{R}^3$.

Let us firstly give an adaptation of a basic result which can be found in [39] and [52].

Proposition 1.0.2. *The space $\mathcal{D}(\overline{\Omega})$ of the restrictions to Ω of functions of $\mathcal{D}(\mathbb{R}^3)$ is dense both in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ and in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$, for $1 \leq p < \infty$.*

Proof. We give the proof of the density in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ and the proof for the space $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ is very similar. Let $\boldsymbol{u}$ be some element of $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$. We have to prove that $\boldsymbol{u}$ is a limit in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ of vector functions of $\mathcal{D}(\overline{\Omega})$.

1. Assume for the moment that Ω is strictly star-shaped with respect to one of its points, after translation in $\mathbb{R}^3$, we can suppose this point is 0. This amounts to say that

$$\lambda\overline{\Omega} \subset \Omega,\ \forall 0 \leq \lambda < 1 \qquad \text{and } \overline{\Omega} \subset \lambda\Omega,\ \forall \lambda > 1.$$

Here, we take $\lambda > 1$ and we set $\Omega_\lambda = \lambda\Omega$.
For a function φ defined on Ω, we set:

$$\forall x \in \Omega_\lambda, \qquad \varphi_\lambda(x) = \varphi(\frac{x}{\lambda}). \tag{1.5}$$

Which we extend to distribution, $T \in \mathcal{D}'(\Omega) \longrightarrow T_\lambda \in \mathcal{D}'(\Omega_\lambda)$ by:

$$\langle T_\lambda, \varphi\rangle = \lambda^3 \langle T, \varphi_{\frac{1}{\lambda}}\rangle, \quad \varphi \in \mathcal{D}(\Omega_\lambda)$$

The distribution T_λ belongs then to $\mathcal{D}'(\Omega_\lambda)$. It is easy to check that:

$$\forall T \in \mathcal{D}'(\Omega), \quad \mathbf{curl}\,(T_\lambda) = \frac{1}{\lambda}(\mathbf{curl}\, T)_\lambda, .$$

Due to ([52], Chapter 1, Lemma 1.1), the restriction to Ω of the function $\boldsymbol{u}_\lambda$, $\lambda > 1$, converge to $\boldsymbol{u}$ in $\boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega)$ as $\lambda \mapsto 1$. Let $\varphi \in \mathcal{D}(\Omega_\lambda)$ and $\varphi = 1$ on Ω. the function $\varphi \boldsymbol{u}_\lambda$ extended by 0 outside Ω_λ clearly belongs to $\boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3)$ and has compact support. The result is then proved by regularization. Let $\rho \in \mathcal{D}(\mathbb{R}^3)$, be a smooth $\mathcal{C}^\infty$ function with compact support, such that $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = 1$. For $\varepsilon \in (0,1)$, let ρ_ε denote the function $x \longmapsto (\frac{1}{\varepsilon^3})\rho(\frac{x}{\varepsilon})$. As $\varepsilon \to 0$, ρ_ε converges in the distribution sence to the Dirac distribution and it is a classical result that for any $\boldsymbol{\nu}$ in $\boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3)$

$$\rho_\varepsilon * \boldsymbol{\nu} \longrightarrow \boldsymbol{\nu} \ \text{ in } \boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3), . \tag{1.6}$$

As a consequence, $\rho_\varepsilon * \widetilde{\varphi \boldsymbol{u}}_\lambda$ belongs to $\mathcal{D}(\mathbb{R}^3)$ since this function has a compact support $(\mathrm{supp}(\rho_\varepsilon * \widetilde{\varphi \boldsymbol{u}}_\lambda) \subset (\mathrm{supp}\rho_\varepsilon) + (\mathrm{supp}\widetilde{\varphi \boldsymbol{u}}_\lambda))$ and components which are $\mathcal{C}^\infty$. Moreover,

$$\lim_{\varepsilon \to 0} \rho_\varepsilon * \widetilde{\varphi \boldsymbol{u}}_\lambda = \widetilde{\varphi \boldsymbol{u}}_\lambda \quad \text{in} \quad \boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3).$$

We note:

$$\omega_\lambda = \text{the restriction of the functions } \widetilde{\varphi \boldsymbol{u}}_\lambda \text{ to } \Omega.$$

Hence $\boldsymbol{u}$ is the limit in $\boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega)$ of the functions $\rho_\varepsilon * \omega_\lambda$ as $\lambda \mapsto 1$. the result follows since $\rho_\varepsilon * \omega_\lambda$ belongs to $\mathcal{D}(\overline{\Omega})$.

2. In the general case, we use the following property (cf. for exemple Bernardi [14]).

   A bounded, Lipschitz-continuous open set is the union of a finite number of star-shaped, Lipschitz-continuous open sets.

   Clearly, it suffices to apply the above argument to each of these sets to derive the desired result on the entire domain.

   Indeed the sets Ω, $(\theta_j)_{j\in J}$ form an open covering of $\overline{\Omega}$. Let us consider a partition of unity subordinated to the covering $\theta_j \cap \overline{\Omega}$:

$$1 = \varphi + \sum_{j\in J} \varphi_j, \ \text{ where } \ \varphi \in \mathcal{D}(\Omega), \varphi_j \in \mathcal{D}(\theta_j \cap \Omega).$$

   We may write

$$\boldsymbol{u} = \varphi \boldsymbol{u} + \sum_{j\in J} \varphi_j \boldsymbol{u}.$$

Since the function $\varphi \boldsymbol{u}$ has compact support in Ω it can be shown as in (a) that $\varphi \boldsymbol{u}$ is the limit in $\boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega)$ of functions belonging to $\mathcal{D}(\overline{\Omega})$ (function $\varphi \boldsymbol{u}$ extended by 0 outside Ω belongs to $\boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3)$ and for ε sufficiently small, $\rho_\varepsilon * \varphi \boldsymbol{u}$ has compact support in Ω).

Let us consider now one of the function $\boldsymbol{u}_j = \varphi_j \boldsymbol{u}$. Let σ_λ, $\lambda \neq 0$, be the linear transformation $x \to \lambda x$. The set $\theta'_j = \theta_j \cap \Omega$ is star-shaped with respect to one of its points y, by taking y as origin, it is clear that :

$$\theta'_j \subset \overline{\theta'_j} \subset \sigma_\lambda \theta'_j \qquad \text{for } \lambda > 1,$$

$$\sigma_\lambda \theta'_j \subset \sigma_\lambda \overline{\theta'_j} \subset \theta'_j \qquad \text{for } 0 < \lambda < 1$$

Let $\boldsymbol{\nu} \circ \sigma_\lambda$ denote the function $x \mapsto \boldsymbol{\nu}(\sigma_\lambda(x))$ then, the restriction to θ'_j of the function $\boldsymbol{u}_j \circ \sigma_\lambda$, $\lambda > 1$, converges to $\boldsymbol{u}_j$ in $\boldsymbol{H}^{\,p}(\mathbf{curl}, \theta'_j)$ as $\lambda \to 1$ (cf. [52], Lemma1.1 p 7). But if $\psi_j \in \mathcal{D}(\sigma_\lambda(\theta'_j))$ and $\psi_j = 1$ on θ'_j, the function $\psi_j(\sigma_\lambda \circ \boldsymbol{u}_j)$ clearly belongs to $\boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3)$ (the function $\psi_j(\sigma_\lambda \circ \boldsymbol{u}_j)$ extended by 0 outside $\sigma_\lambda \theta'_j$). Take $w_j = \psi_j(\sigma_\lambda \circ \boldsymbol{u}_j)$. Since this function belongs to $\boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3)$ and has a compact support, by regularization, the function $\rho_\epsilon * w_j$ belongs to $\mathcal{D}(\mathbb{R}^3)$. Moreover, .

$$\lim_{\varepsilon \to 0} \rho_\varepsilon * w_j = w_j \quad \text{in} \quad \boldsymbol{H}^{\,p}(\mathbf{curl}, \mathbb{R}^3).$$

Since the restriction to θ'_j of the function $\rho_\varepsilon * w_j$ converges to the restriction of the function $\sigma_\lambda \circ \boldsymbol{u}_j$ to θ'_j, it can be shown that $\boldsymbol{u}_j$ is is the limit in $\boldsymbol{H}^{\,p}(\mathbf{curl}, \theta'_j)$ of functions belongings to $\mathcal{D}(\overline{\theta'}_j)$, as $\lambda \to 1$.

. □

Remark 1.0.3. Note that the previou proof for the space $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ is general and does not use the particular structure of the differential system **curl**, and we can use the same proof for the space $\boldsymbol{H}^p(\mathrm{div}, \Omega)$. We can easily derive by the same arguments that The space $\mathcal{D}(\overline{\Omega})$ is also dense in $\boldsymbol{X}^p(\Omega)$.

As proven in Reference [39], chapter I, section 2 for the Hilbertian case, these properties of density allow for defining tangential or normal traces for the functions of these spaces. More precisely, any function $\boldsymbol{v}$ in $\boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega)$ has a tangential trace $\boldsymbol{v} \times \boldsymbol{n}$ in $\boldsymbol{W}^{\,-\frac{1}{p},p}(\Gamma)$, defined by

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}^{\,1,p}(\Omega), \qquad \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}, \tag{1.7}$$

where the symbol $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing between $\boldsymbol{W}^{\,-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\,1-\frac{1}{p'},p'}(\Gamma)$. Any function $\boldsymbol{v}$ in $\boldsymbol{H}^{\,p}(\mathrm{div}, \Omega)$ has a normal trace $\boldsymbol{v} \cdot \boldsymbol{n}$ in $\boldsymbol{W}^{\,-\frac{1}{p},p}(\Gamma)$, defined by

$$\forall \varphi \in W^{\,1,p}(\Omega), \qquad \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma = \int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\, \varphi\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathrm{div}\, \boldsymbol{v}) \varphi\, \mathrm{d}\boldsymbol{x}. \tag{1.8}$$

We can define the 'homogeneous' spaces:

$$\boldsymbol{H}_0^p(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},$$

$$\boldsymbol{H}_0^p(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Following the approach in [52] and [39] for the case $p = 2$, we can prove that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^p(\mathbf{curl},\Omega)$ and in $\boldsymbol{H}_0^p(\mathrm{div},\Omega)$ for any $1 \le p < \infty$.

For $1 < p < \infty$, we denote by $[\boldsymbol{H}_0^p(\mathrm{div},\Omega)]'$ and by $[\boldsymbol{H}_0^p(\mathbf{curl},\Omega)]'$ the dual spaces of $\boldsymbol{H}_0^p(\mathrm{div},\Omega)$ and $\boldsymbol{H}_0^p(\mathbf{curl},\Omega)$ respectively. We can characterize theses spaces as it is stated in the following propositions.

Proposition 1.0.4. *A distribution $\boldsymbol{f}$ belongs to $\boldsymbol{H}_0^p(\mathrm{div},\Omega)'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\,\chi$. Moreover, we have the estimate*

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\chi\|_{L^{p'}(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)'}. \tag{1.9}$$

Proof. For any function $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\chi \in L^{p'}(\Omega)$, we have

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega), \quad \langle \boldsymbol{\psi} + \mathbf{grad}\,\chi,\ \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = \int_\Omega (\boldsymbol{\psi}\cdot\boldsymbol{v} - \chi\,\mathrm{div}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}.$$

The linear form in the right-hand side of the above relation is continuous for the norm $\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)$, we deduce by density that $\boldsymbol{\psi} + \mathbf{grad}\,\chi$ is an element of $[\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)]'$.

Conversely, we set $E = \boldsymbol{L}^p(\Omega) \times L^p(\Omega)$ endowed whith the norm

$$\|\boldsymbol{v}\|_E = (\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^p_{L^p(\Omega)})^{\frac{1}{p}}.$$

The operator $T : \boldsymbol{v} \to (\boldsymbol{v},\,\mathrm{div}\,\boldsymbol{v})$ is continuous from $\boldsymbol{H}_0^p(\mathrm{div},\Omega)$ onto E. Its range $\mathcal{R}(T)$ is a closed subspace of E and T is an isomorphism from $\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)$ onto $\mathcal{R}(T)$ and is an isometry. Hence, with each $\boldsymbol{g} \in \boldsymbol{H}_0^p(\mathrm{div},\,\Omega)'$, we associate the element $\boldsymbol{g}^* \in (\mathcal{R}(T))'$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^p(\mathrm{div},\,\Omega), \qquad \langle\, \boldsymbol{g},\ \boldsymbol{v}\,\rangle_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)'\times\boldsymbol{H}_0^p(\mathrm{div},\Omega)} = \langle\, \boldsymbol{g}^*,\ T\boldsymbol{v}\,\rangle_{(\mathcal{R}(T))'\times\mathcal{R}(T)}.$$

Note that $\|\,\boldsymbol{g}\,\|_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)'} = \|\boldsymbol{g}^*\|_{\mathcal{R}(T)'}$ because T is an isometry. By the extension theorem of Hahn-Banach, $\boldsymbol{g}^*$ can be extended in $\boldsymbol{L}^{p'}(\Omega) \times L^{p'}(\Omega)$ to an element called $(\boldsymbol{\psi},\,\chi)$ with $(\|\boldsymbol{\psi}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + \|\chi\|^{p'}_{L^{p'}(\Omega)})^{\frac{1}{p'}} = \|\boldsymbol{g}^*\|_{\mathcal{R}(T)'}$. We deduce that:

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^p(\mathrm{div},\,\Omega), \quad \langle\, \boldsymbol{g},\ \boldsymbol{v}\,\rangle_{\boldsymbol{H}_0^p(\mathrm{div},\Omega)'\times\boldsymbol{H}_0^p(\mathrm{div},\Omega)} = \int_\Omega (\boldsymbol{\psi}\cdot\boldsymbol{v} + \chi\,\mathrm{div}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}.$$

So, $\boldsymbol{g}$ is equal to $\boldsymbol{\psi} - \mathbf{grad}\,\chi$ in Ω with the estimate (1.9). □

We skip the proof of the following result as it is entirely similar to that of Proposition 1.0.4.

Proposition 1.0.5. *A distribution $\boldsymbol{f}$ belongs to $\boldsymbol{H}_0^p(\mathbf{curl},\Omega)'$ if and only if there exist functions $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^{p'}(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi}$, and we have the following estimate:*

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\,\boldsymbol{\xi}\,\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C\|\,\boldsymbol{f}\|_{\boldsymbol{H}_0^p(\mathbf{curl},\Omega)'}.$$

However, we can impose on $\boldsymbol{\xi}$ a boundary condition as in the following Proposition.

Proposition 1.0.6. *For any $\boldsymbol{f} \in \boldsymbol{H}_0^p(\mathbf{curl},\Omega)'$, there exist functions $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{\xi} \in \boldsymbol{L}^{p'}(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi}$ where $\boldsymbol{\xi}$ satisfies*

$$\operatorname{div}\boldsymbol{\xi} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \tag{1.10}$$

and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\boldsymbol{\xi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C\|\,\boldsymbol{f}\|_{\boldsymbol{H}_0^p(\mathbf{curl},\Omega)'}. \tag{1.11}$$

Proof. Let $\boldsymbol{f}$ in $\boldsymbol{H}_0^{\,p}(\mathbf{curl},\Omega)'$. Applying Proposition 1.0.5, we can write $\boldsymbol{f}$ as $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi}$, where $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^{p'}(\Omega)$. The problem: find χ in $W^{\,1,p'}(\Omega)/\mathbb{R}$ such that,

$$\forall q \in W^{\,1,p}(\Omega), \quad \int_\Omega \mathbf{grad}\,\chi\cdot\mathbf{grad}\,q\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\xi}\cdot\mathbf{grad}\,q\,\mathrm{d}\boldsymbol{x}.$$

has a unique solution χ satisfying the estimate:

$$\|\nabla\,\chi\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C\|\,\xi\,\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C\|\,\boldsymbol{f}\,\|_{\boldsymbol{H}_0^{\,p}(\mathbf{curl},\Omega)'}. \tag{1.12}$$

The function $\widetilde{\boldsymbol{\xi}} = \boldsymbol{\xi} - \mathbf{grad}\,\chi$ belongs to $\boldsymbol{L}^{p'}(\Omega)$ and satisfies (1.10). The inequality (1.11) follows easily from (1.12). □

Definition 1.0.7. Let $\boldsymbol{X}_N^p(\Omega)$, $\boldsymbol{X}_T^p(\Omega)$ and $\boldsymbol{X}_0^p(\Omega)$ be the following subspaces of $\boldsymbol{X}^p(\Omega)$:

$$\boldsymbol{X}_N^p(\Omega) = \{\boldsymbol{v}\in\boldsymbol{X}^p(\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}, \tag{1.13}$$

$$\boldsymbol{X}_T^p(\Omega) = \{\boldsymbol{v}\in\boldsymbol{X}^p(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\},$$

and

$$\boldsymbol{X}_0^p(\Omega) = \boldsymbol{X}_N^p(\Omega)\cap\boldsymbol{X}_T^p(\Omega). \tag{1.14}$$

The following theorem gives a characterization of the space $\boldsymbol{X}_0^p(\Omega)$. Using Fourier transform, the same result can be found in [3] or [39] for the case $p = 2$.

Theorem 1.0.8. *The space $\boldsymbol{X}_0^p(\Omega)$ coincides with $\boldsymbol{W}_0^{1,p}(\Omega)$. Moreover, we have the following estimate: for any function $\boldsymbol{v}$ in $\boldsymbol{X}_0^p(\Omega)$*

$$\| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \left(\| \mathbf{curl}\, \boldsymbol{v} \|_{\boldsymbol{L}^p(\Omega)} + \| \operatorname{div} \boldsymbol{v} \|_{L^p(\Omega)} \right), \tag{1.15}$$

where $C \geq 0$ depends only on p and Ω.

Proof. Since the imbedding of $\boldsymbol{W}_0^{1,p}(\Omega)$ in $\boldsymbol{X}_0^p(\Omega)$ is obvious, we study the inverse imbedding. Let $\boldsymbol{v}$ be any function in $\boldsymbol{X}_0^p(\Omega)$. We would prove that $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$.
We define the extension $\widetilde{\boldsymbol{v}}$ of $\boldsymbol{v}$ by $\widetilde{\boldsymbol{v}} = \boldsymbol{v}$ in Ω and $\widetilde{\boldsymbol{v}} = 0$ in $\mathbb{R}^3 \backslash \Omega$. Since $\boldsymbol{v}$ is in $\boldsymbol{X}_N^p(\Omega)$, it is easy to check from (1.7) that $\mathbf{curl}\, \widetilde{\boldsymbol{v}}$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$. Similarly, the fact that $\boldsymbol{v}$ is in $\boldsymbol{X}_T^p(\Omega)$ implies that $\operatorname{div} \widetilde{\boldsymbol{v}}$ belongs to $L^p(\mathbb{R}^3)$. Moreover, we have:

$$\| \operatorname{div} \boldsymbol{v} \|_{L^p(\Omega)} = \| \operatorname{div} \widetilde{\boldsymbol{v}} \|_{L^p(\mathbb{R}^3)}, \quad \| \mathbf{curl}\, \boldsymbol{v} \|_{\boldsymbol{L}^p(\Omega)} = \| \mathbf{curl}\, \widetilde{\boldsymbol{v}} \|_{\boldsymbol{L}^p(\mathbb{R}^3)}.$$

But, $-\Delta \widetilde{\boldsymbol{v}} = \mathbf{curl}\, \mathbf{curl}\, \widetilde{\boldsymbol{v}} - \nabla \operatorname{div} \widetilde{\boldsymbol{v}}$. Then, $\Delta \widetilde{\boldsymbol{v}}$ belongs to $W_0^{-1,p}(\mathbb{R}^3)$ which is the dual space of the weighted Sobolev space (see [7]):

$$W_0^{1,p'}(\mathbb{R}^3) = \{ v \in \mathcal{D}'(\mathbb{R}^3),\ \frac{v}{\omega_0} \in L^{p'}(\mathbb{R}^3),\ \nabla v \in \boldsymbol{L}^{p'}(\mathbb{R}^3) \},$$

where $\omega_0 = 1 + |\boldsymbol{x}|$ if $p' \neq 3$ and $\omega_0 = 1 + |\boldsymbol{x}| \ln(2 + |\boldsymbol{x}|)$. Moreover, for any function φ in $W_0^{1,p'}(\mathbb{R}^3)$ and any $i = 1, 2, 3$ we have:

$$\langle \frac{\partial}{\partial x_i} \operatorname{div} \widetilde{\boldsymbol{v}},\, \varphi \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = - \int_{\mathbb{R}^3} \operatorname{div} \widetilde{\boldsymbol{v}}\, \frac{\partial \varphi}{\partial x_i}\, dx \quad \forall i = 1, 2, 3.$$

Especially if $\varphi = 1$ and $p \leq \frac{3}{2}$, we have $\langle \frac{\partial}{\partial x_i} \operatorname{div} \widetilde{\boldsymbol{v}},\, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0.$

Similarly, we can check that $\langle \mathbf{curl}\, \mathbf{curl}\, \widetilde{\boldsymbol{v}},\, \boldsymbol{e}_i \rangle = 0$, for $i = 1, 2, 3$, where $(\boldsymbol{e}_i)_i$ is the canonical basis of $\mathbb{R}^3$. Hence, $\langle -\Delta \widetilde{\boldsymbol{v}}_i,\, 1 \rangle = 0$, for $p \leq \frac{3}{2}$. Due to the isomorphism:

$$\Delta : W_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-\frac{3}{p}]} \longmapsto W_0^{-1,p}(\mathbb{R}^3) \perp \mathcal{P}_{[1-\frac{3}{p'}]}, \tag{1.16}$$

there exists $\boldsymbol{z} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ such that $\Delta\, \boldsymbol{z} = \Delta\, \widetilde{\boldsymbol{v}}$, with the estimate

$$\begin{aligned} \| \boldsymbol{z} \|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} &\leq C \| \Delta\, \widetilde{\boldsymbol{v}} \|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} \\ &\leq C(\| \operatorname{div} \widetilde{\boldsymbol{v}} \|_{L^p(\mathbb{R}^3)} + \| \mathbf{curl}\, \widetilde{\boldsymbol{v}} \|_{\boldsymbol{L}^p(\mathbb{R}^3)}) \\ &\leq C(\| \operatorname{div} \boldsymbol{v} \|_{L^p(\Omega)} + \| \mathbf{curl}\, \boldsymbol{v} \|_{\boldsymbol{L}^p(\Omega)}), \end{aligned} \tag{1.17}$$

and where we have used relation (1.1). Hence, $\boldsymbol{z} - \widetilde{\boldsymbol{v}}$ is a harmonic polynomial of $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) + \boldsymbol{L}^p(\mathbb{R}^3)$. Consequently, there exists a unique constant vector $\boldsymbol{k}$ such that $\widetilde{\boldsymbol{v}} = \boldsymbol{z} + \boldsymbol{k}$, with $\boldsymbol{k} = \boldsymbol{0}$ if $p < 3$. We then obtain $\widetilde{\boldsymbol{v}} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$. Moreover,

$$\| \boldsymbol{v} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \| \nabla\, \boldsymbol{v} \|_{\boldsymbol{L}^p(\Omega)} \leq C \| \nabla\, \boldsymbol{z} \|_{\boldsymbol{L}^p(\Omega)}.$$

It follows from (1.17) that the inequality (1.15) holds. □

Remark 1.0.9.

By using the isomorphism (1.16), the following formula is checked

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^3)/\mathcal{P}[1-\frac{3}{p}]} \leq C\,\|\Delta\,\widetilde{\boldsymbol{v}}\|_{\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^3)}.$$

But $\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^3)/\mathcal{P}_{[1-\frac{3}{p}]}} \simeq \|\nabla\,\boldsymbol{z}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}$, then

$$\begin{aligned}\|\nabla\,\boldsymbol{z}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} &\leq C\,\|\Delta\,\widetilde{\boldsymbol{v}}\|_{\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^3)}\\ &\leq C\left(\|\mathbf{curl}\,\mathbf{curl}\,\widetilde{\boldsymbol{v}}\|_{\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^3)} + \|\nabla\,\mathrm{div}\,\widetilde{\boldsymbol{v}}\|_{\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^3)}\right).\end{aligned}$$

As it was more $\|\nabla\,\boldsymbol{z}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} = \|\nabla\,\widetilde{\boldsymbol{v}}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}$, we have

$$\|\nabla\,\widetilde{\boldsymbol{v}}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\left(\|\mathbf{curl}\,\widetilde{\boldsymbol{v}}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|\mathrm{div}\,\widetilde{\boldsymbol{v}}\|_{L^p(\mathbb{R}^3)}\right).$$

Finally by restriction on Ω, the following estimate is readily checked for any function of the space $\boldsymbol{X}_0^{\,p}(\Omega)$:

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq \boldsymbol{C}\left(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}\right).$$

Finally, we introduce the space

$$E^p(\Delta,\Omega) = \{h \in L^p(\Omega),\ \Delta\,h \in W^{-1,p}(\Omega)\}, \tag{1.18}$$

which is a Banach space for the norm

$$\|h\|_{E^p(\Delta,\Omega)} = \|h\|_{L^p(\Omega)} + \|\Delta\,h\|_{W^{-1,p}(\Omega)}.$$

Then, let us observe in the following lemma that for $h \in E^p(\Delta,\Omega)$ it is possible to define a trace $h|_\Gamma$ in $W^{-\frac{1}{p},p}(\Gamma)$.

Lemma 1.0.10. *Assume that Ω is of class $\mathcal{C}^{1,1}$. The space $\mathcal{D}(\overline{\Omega})$ is dense in $E^p(\Delta,\Omega)$. Moreover, the mapping $\gamma : h \to h|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted γ from $E^p(\Delta,\Omega)$ into $W^{-\frac{1}{p},p}(\Gamma)$ and the following Green's formula holds:*
for any $\chi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ and any $h \in E^p(\Delta,\Omega)$, we have

$$\int_\Omega h\,\Delta\,\chi\,\mathrm{d}\boldsymbol{x} - \langle\Delta h,\,\chi\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} = \langle h,\,\frac{\partial\,\chi}{\partial\,\boldsymbol{n}}\rangle_{W^{-\frac{1}{p},p}(\Gamma)\times W^{1-\frac{1}{p'},p'}(\Gamma)}. \tag{1.19}$$

Proof. Let ℓ in $(E^p(\Delta,\Omega))'$ such that $\langle \ell,\, v\rangle = 0$ for any v in $\mathcal{D}(\overline{\Omega})$. We known that there exist $f \in L^{p'}(\Omega)$ and $g \in W_0^{1,p'}(\Omega)$ such that for any u in $E^p(\Delta,\Omega)$,

$$\langle \ell, u\rangle = \int_\Omega f\, u \,\mathrm{d}\boldsymbol{x} + \langle \Delta\, u, g\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}.$$

Since ℓ vanishes on $\mathcal{D}(\overline{\Omega})$, we have $f + \Delta\, g = 0 \quad \text{in}\, \Omega$. Then, g belongs to $W^{\,2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$, since $\Delta\, g$ belongs to $L^{p'}(\Omega)$ and Ω is of class $\mathcal{C}^{\,1,1}$. Let now $\widetilde{f}$ and $\widetilde{g}$ the extensions of f and g respectively by zero outside of Ω. Then $\widetilde{f} \in L^{p'}(\mathbb{R}^3)$, $\widetilde{g} \in W_0^{1,p'}(\mathbb{R}^3)$ and for any $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} \widetilde{f}\,\varphi \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3} \widetilde{g}\,\Delta\,\varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega f\,\varphi \,\mathrm{d}\boldsymbol{x} + \int_\Omega g\,\Delta\,\varphi \,\mathrm{d}\boldsymbol{x} = 0.$$

This implies that $\widetilde{f} + \Delta\,\widetilde{g} = 0$ in $\mathbb{R}^3$. So $\widetilde{g} \in W^{\,2,p'}(\mathbb{R}^3)$, since $\Delta\,\widetilde{g} \in L^{p'}(\mathbb{R}^3)$ and $\widetilde{g} \in L^{p'}(\mathbb{R}^3)$. We deduce that g belongs to $W_0^{2,p'}(\Omega)$. As $\mathcal{D}(\Omega)$ is dense in $W_0^{2,p'}(\Omega)$, let $(\psi_k)_k$ be a sequence of $\mathcal{D}(\Omega)$ that tends to g in $W^{\,2,p'}(\Omega)$. Then, for $u \in E^p(\Delta,\Omega)$,

$$\begin{aligned}
\langle \ell,\, u\rangle &= -\int_\Omega u\,\Delta g \,\mathrm{d}\boldsymbol{x} + \langle \Delta\, u, g\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} \\
&= \lim_{k\to\infty}\left(-\int_\Omega u\,\Delta\,\psi_k \mathrm{d}\boldsymbol{x} + \langle \Delta\, u, \psi_k\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)}\right) \\
&= \lim_{k\to\infty}\left(-\int_\Omega u\,\Delta\,\psi_k \mathrm{d}\boldsymbol{x} + \int_\Omega u\,\Delta\,\psi_k \mathrm{d}\boldsymbol{x}\right) = 0.
\end{aligned}$$

Then ℓ also vanishes on $E^p(\Delta,\Omega)$ and $\mathcal{D}(\overline{\Omega})$ is dense in $E^p(\Delta,\,\Omega)$.
Let now $u \in \mathcal{D}(\overline{\Omega})$ and $v \in \mathcal{D}(\overline{\Omega})$. The following Green's formula holds:

$$\int_\Omega u\Delta\, v \,\mathrm{d}\boldsymbol{x} + \int_\Omega v\,\Delta\, u \,\mathrm{d}\boldsymbol{x} = \int_\Gamma u\,\frac{\partial v}{\partial \boldsymbol{n}}\,\mathrm{d}\boldsymbol{x}.$$

As $\mathcal{D}(\overline{\Omega})$ is dense in $W^{\,2,p'}(\Omega)$, this equality is still valid for $v \in W^{\,2,p'}(\Omega)$ and $u \in \mathcal{D}(\overline{\Omega})$. Therefore,

$$\Big|\int_\Gamma u\,\frac{\partial v}{\partial \boldsymbol{n}}\Big| \le \|\, u\,\|_{E^p(\Delta,\Omega)}\|\, v\,\|_{W^{\,2,p'}(\Omega)}.$$

Now, we known that for any μ element of $W^{1-\frac{1}{p'},p'}(\Gamma)$, there exists an element v of $W^{\,2,p'}(\Omega)\cap W_0^{1,p'}(\Omega)$ such that

$$v = 0, \quad \frac{\partial v}{\partial \boldsymbol{n}} = \mu \ \text{on}\,\Gamma \quad \text{and} \quad \|v\|_{W^{\,2,p'}(\Omega)} \le C\|\mu\|_{W^{\,1-\frac{1}{p'},p'}(\Gamma)}.$$

Hence the above inequality implies that for $\mu \in W^{1-\frac{1}{p'},p'}(\Gamma)$

$$\Big|\int_\Gamma u\,\mu\Big| \le C\|u\|_{E^p(\Delta,\Omega)}\|\mu\|_{W^{\,1-\frac{1}{p'},p'}(\Gamma)}.$$

Thus,

$$\|u\|_{W^{-\frac{1}{p},p}(\Gamma)} \le C\|u\|_{E^p(\Delta,\Omega)}.$$

Therefore, the linear mapping $\gamma : u \to u|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ is continuous for the norm of $E^p(\Delta, \Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $E^p(\Delta, \Omega)$, γ can be extended by continuity to a mapping still called $\gamma \in \mathcal{L}(E^p(\Delta, \Omega), W^{-\frac{1}{p},p}(\Gamma))$ and we have the Green's formula (1.19). □

Chapter 2

Regularity and compactness results

2.1 Introduction and preliminaries

This chapter is devoted to the proof of some regularity and compactness properties concerning the spaces $\boldsymbol{X}_T^p(\Omega)$ and $\boldsymbol{X}_N^p(\Omega)$. It is organised as follows. In Section 2.2, we will prove an estimate of $\nabla \boldsymbol{u}$ in terms of $\operatorname{div} \boldsymbol{u}$, $\mathbf{curl}\, \boldsymbol{u}$ and additional terms which reflect the topological structure of Ω. Firstly, we consider homogeneous boundary conditions: either the tangential component of $\boldsymbol{u}$ vanishes on Γ or the normal one does, which leads to Theorem 2.2.2 and Theorem 2.2.7. Secondly, we give in Corollary 2.2.12 an extention to the case of inhomogeneous normal boundary conditions. We refer to Corollary 3.3.6 in Chapter 3 for the case of inhomogeneous tangential boundary conditions. We deduce from the above inequalities two regularity results concerning the continuous imbeddings of both spaces $\boldsymbol{X}_N^p(\Omega)$ and $\boldsymbol{X}_T^p(\Omega)$ into $\boldsymbol{W}^{1,p}(\Omega)$, see Theorem 2.2.4 and Theorem 2.2.8. Finally, in Section 2.3, we prove in Theorem 2.3.2 that the imbeddings of the spaces $\boldsymbol{X}_T^p(\Omega)$ and $\boldsymbol{X}_N^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$ are compact. **In all this chapter, we suppose that Ω is of class $\mathcal{C}^{1,1}$.**

We start off with giving two density results. These results are proven in [3] for $p = 2$ and we give here a generalization for any $1 < p < \infty$. Note that in our work the first lemma is proven in a different way.

Lemma 2.1.1. *The space $\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_N^p(\Omega)$ is dense in the space $\boldsymbol{X}_N^p(\Omega)$.*

Proof. Let $\boldsymbol{\ell}$ belongs to $(\boldsymbol{X}_N^p(\Omega))'$, the dual space of $\boldsymbol{X}_N^p(\Omega)$. We know that there exist $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{L}^{p'}(\Omega)$ and $h \in L^{p'}(\Omega)$ such that

$$\langle \boldsymbol{\ell}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega h \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{g} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega). \tag{2.1}$$

We suppose that

$$\langle \boldsymbol{\ell},\, \boldsymbol{v} \rangle = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_N^p(\Omega). \tag{2.2}$$

So, we have in the sense of distributions in Ω

$$\boldsymbol{f} - \nabla h + \mathbf{curl}\, \boldsymbol{g} = \mathbf{0}. \tag{2.3}$$

Therefore, due to (2.2) we have for any $\chi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\int_\Omega \boldsymbol{f} \cdot \nabla \chi \, \mathrm{d}\boldsymbol{x} + \int_\Omega h \, \Delta \chi \, \mathrm{d}\boldsymbol{x} = 0. \tag{2.4}$$

Then it follows that $\mathrm{div}\, \boldsymbol{f} = \Delta h$ in Ω. Thus, $h \in E^{p'}(\Delta, \Omega)$, since $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$. Applying (1.19), we obtain for any $\chi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\int_\Omega h \, \Delta \, \chi \, \mathrm{d}\boldsymbol{x} - \langle \mathrm{div}\, \boldsymbol{f}, \, \chi \rangle_{W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)} = \langle \, h, \, \frac{\partial \chi}{\partial \boldsymbol{n}} \, \rangle_\Gamma. \tag{2.5}$$

As

$$\int_\Omega \boldsymbol{f} \cdot \nabla \chi \, \mathrm{d}\boldsymbol{x} = -\langle \mathrm{div}\, \boldsymbol{f}, \, \chi \rangle_{W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)},$$

it follows from (2.4) that

$$\langle \, h, \, \frac{\partial \chi}{\partial \boldsymbol{n}} \, \rangle_\Gamma = 0, \qquad \forall \chi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{2.6}$$

Now, let μ be any element of $W^{1-\frac{1}{p},p}(\Gamma)$. Then, there exists an element χ of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\frac{\partial \chi}{\partial \boldsymbol{n}} = \mu$ on Γ. Hence, (2.6) implies that

$$\langle \, h, \, \mu \, \rangle_{W^{-\frac{1}{p'},p'}(\Gamma) \times W^{1-\frac{1}{p},p}(\Gamma)} = 0, \tag{2.7}$$

and $h = 0$ in $W^{-\frac{1}{p'},p'}(\Gamma)$. Because Δh belongs to $W^{-1,p'}(\Omega)$ and $h \in L^{p'}(\Omega)$, then $h \in W_0^{1,p'}(\Omega)$. As a consequence, due to (2.3), $\mathbf{curl}\, \boldsymbol{g}$ belongs to $\boldsymbol{L}^{p'}(\Omega)$. Finally, let $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$. From (2.3) and since $h \in W_0^{1,p}(\Omega)$, we can write

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega h \, \mathrm{div}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{curl}\, \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega). \tag{2.8}$$

As $\boldsymbol{g} \in \boldsymbol{H}^{p'}(\mathbf{curl}, \Omega)$, we have also

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^p(\mathbf{curl}, \Omega), \qquad \int_\Omega \mathbf{curl}\, \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{g} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

Then it follows from the last equality and (2.8) that $\boldsymbol{\ell}$ vanishes on $\boldsymbol{X}_N^p(\Omega)$, thus proving the required density. □

We give now the corresponding result for the space $\boldsymbol{X}_T(\Omega)$

Lemma 2.1.2. *The space* $\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^p(\Omega)$ *is dense in the space* $\boldsymbol{X}_T^p(\Omega)$.

Proof. Let $\boldsymbol{v}$ be any function in $\boldsymbol{X}^p_T(\Omega)$. Applying Remark **??**0$Choixde\pi_0$ yields that there is a sequence $(\boldsymbol{v}_k)_k$ of $\mathcal{D}(\overline{\Omega})$ which converges to $\boldsymbol{v}$ in $\boldsymbol{X}^{\,p}(\Omega)$. Next, for each k, we consider the unique solution χ_k in $W^{\,1,p}(\Omega)$, with zero mean value, of the Neumann problem:

$$\Delta\chi_k = \operatorname{div} \boldsymbol{v}_k \quad \text{in}\,\Omega \quad \text{and} \qquad \partial_n\chi_k = \boldsymbol{v}_k \cdot \boldsymbol{n} \quad \text{on}\,\Gamma.$$

Due to the regularity assumption on the domain Ω, for each k, the function χ_k belongs to $W^{\,2,p}(\Omega)$, so that the function $\boldsymbol{v}_k - \mathbf{grad}\,\chi_k$ is in $\boldsymbol{W}^{\,1,\,p}(\Omega)$. Finally, due to the convergence of $(\boldsymbol{v}_k)_k$ in $\boldsymbol{X}^{\,p}(\Omega)$, it is easy to check that the sequence $(\chi_k)_k$ converges in $W^{\,1,p}(\Omega)$ towards the solution χ of the problem:

$$\Delta\chi = \operatorname{div} \boldsymbol{v} \quad \text{in}\,\Omega \quad \text{and} \qquad \partial_n\chi = \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on}\,\Gamma.$$

This solution also belongs to $W^{\,2,p}(\Omega)$. Hence, the sequence $(\boldsymbol{v}_k - \mathbf{grad}\,\chi_k + \mathbf{grad}\,\chi)_k$ is in $\boldsymbol{W}^{\,1,\,p}(\Omega) \cap \boldsymbol{X}^{\,p}_T(\Omega)$ and converges to $\boldsymbol{v}$ in $\boldsymbol{X}^{\,p}_T(\Omega)$, which proves the lemma. □

In the sequel, we require a basic result which is known as the Peetre-Tartar Lemma, (*cf.* references [39, Chapter I, Theorem 2.1])

Theorem 2.1.3. *(Peetre-Tartar) Let E_1, E_2, E_3 be three Banach spaces, A operator in $\mathcal{L}(E_1,\, E_2)$ and B a compact operator in $\mathcal{L}(E_1,\, E_3)$ satisfying:*

$$\forall \boldsymbol{u} \in E_1, \quad \|\boldsymbol{u}\|_{E_1} \simeq \|A\boldsymbol{u}\|_{E_2} + \|B\boldsymbol{u}\|_{E_3} \tag{2.9}$$

then the following properties hold:

- *the dimension of $Ker\,A$ is finite. The range space $R(A)$ of the operator A is a closed subspace of E_2 and the mapping A: $E_2/Ker\,A \mapsto R(A)$ is an isomorphism.*
- *if G is a Banach space and $M \in \mathcal{L}(E_1, G)$ satisfies*

$$\forall \boldsymbol{u} \in Ker\,A\backslash\{0\}, \quad M\boldsymbol{u} \neq 0, \tag{2.10}$$

then

$$\forall \boldsymbol{u} \in E_1, \quad \|\boldsymbol{u}\|_{E_1} \simeq \|A\boldsymbol{u}\|_{E_2} + \|M\boldsymbol{u}\|_{G} \tag{2.11}$$

2.2 Sobolev's inequalities and regularity results

The aim of this subsection is to prove continuous imbeddings of both spaces $\boldsymbol{X}^p_T(\Omega)$ and $\boldsymbol{X}^p_N(\Omega)$ in $\boldsymbol{W}^{\,1,p}(\Omega)$. In a first step, we give density results concerning the two spaces. In a second step, we introduce two integral operators that allow to estimate $\nabla\,\boldsymbol{v}$ by $\mathbf{curl}\,\boldsymbol{v}$, $\operatorname{div}\boldsymbol{v}$ and the flux of $\boldsymbol{v}$ past the boundary Γ_i for $1 < i < I$ provided that $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, and by

$\mathbf{curl}\,\boldsymbol{v}$, $\operatorname{div}\boldsymbol{v}$ and the flux of $\boldsymbol{v}$ past the cuts Σ_j, $1<j<J$ provided that $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ which are useful to deduce the continuity of the imbedding.

We introduce now the linear integral operator

$$T\,\lambda(\boldsymbol{x})=-\frac{1}{2\pi}\int_\Gamma \lambda(\boldsymbol{\xi})\,\frac{\partial}{\partial\boldsymbol{n}}|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\,\mathrm{d}\sigma_{\boldsymbol{\xi}}. \tag{2.12}$$

The next lemma gives some properties of this operator.

Lemma 2.2.1. *We have the following properties:*

1. *The operator T is compact from $L^p(\Gamma)$ into $L^p(\Gamma)$.*
2. *The space $\operatorname{Im}(Id+T)$ is a closed subspace of $L^p(\Gamma)$ and $\operatorname{Ker}(Id+T)$ is of finite dimension. It is spanned by the traces of the functions $\mathbf{grad}\,q_i^N\cdot\boldsymbol{n}|_\Gamma$, $1\le i\le I$, where each q_i^N is the unique solution in $W^{2,p}(\Omega)$ of the problem*

$$\begin{cases} -\Delta q_i^N=0 \qquad \text{in } \Omega,\\ q_i^N|_{\Gamma_0}=0 \quad\text{and}\quad q_i^N|_{\Gamma_k}=\text{constant},\ 1\le k\le I,\\ \big\langle \partial_n\, q_i^N,\,1\big\rangle_{\Gamma_k}=\delta_{i\,k},\ 1\le k\le I, \quad\text{and}\quad \big\langle \partial_n\, q_i^N,\,1\big\rangle_{\Gamma_0}=-1, \end{cases} \tag{2.13}$$

   (see section 3.3).
3. *For any $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$:*

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)}\le C\big(\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)}+\sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\big). \tag{2.14}$$

Proof. **1.** According to [55], we have $T\in\mathcal{L}(L^p(\Gamma),W^{1,p}(\Gamma))$. Since the embedding of the space $W^{1,p}(\Gamma)$ in $L^p(\Gamma)$ is compact, we obtain that T is compact from $L^p(\Gamma)$ into $L^p(\Gamma)$.

2. By virtue of the first point and the Fredholm alternative, we have that the space $\operatorname{Ker}(Id+T)$ is of finite dimension and $\operatorname{Im}(Id+T)$ is a closed subspace of $L^p(\Gamma)$. Since the functions $\mathbf{grad}\,q_i^N$, $1\le i\le I$ are linearly independent, it is readily checked that $\mathbf{grad}\,q_i^N\cdot\boldsymbol{n}|_\Gamma$ are also linearly independent for $1\le i\le I$ (for the properties of q_i^N, see section 3.3).
Now, let $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$. Then the quantity $\boldsymbol{v}\cdot\boldsymbol{n}$ satisfies on Γ the following representation (see [55]):

$$\begin{aligned}(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n}) &= -\frac{1}{2\pi}\left(\mathbf{grad}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\operatorname{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n}\\ &- \frac{1}{2\pi}\left(\mathbf{curl}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n}\\ &+ \frac{1}{2\pi}\left(\mathbf{curl}\int_\Gamma\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}(\boldsymbol{v}\times\boldsymbol{n})(\boldsymbol{\xi})\,\mathrm{d}\sigma_{\boldsymbol{\xi}}\right)\cdot\boldsymbol{n}.\end{aligned}$$

As $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{W}^{1,p}(\Omega)$, this relation is still valid for $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. In particular, if $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ, then we have

$$\begin{aligned}(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n}) &= -\frac{1}{2\pi}\left(\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n} \\ &- \frac{1}{2\pi}\left(\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n}. \qquad (2.15)\end{aligned}$$

Since $\mathbf{grad}\, q_i^N$ belong to $\boldsymbol{W}^{1,p}(\Omega)$ and $\mathbf{grad}\, q_i^N \times \boldsymbol{n} = 0$ on Γ, due to (2.15), for each $1 \le i \le I$ the function $\mathbf{grad}\, q_i^N \cdot \boldsymbol{n}$ belongs to $\mathrm{Ker}(Id+T)$. Since the dimension of $\mathrm{Ker}(Id+T)$ is equal to I (see [55], the set $\{\mathbf{grad}\, q_i^N \cdot \boldsymbol{n}|_\Gamma,\ 1 \le i \le I\}$ is a basis of $\mathrm{Ker}(Id+T)$.

3. The operator $Id+T$ is linear, continuous and surjective from $L^p(\Gamma)$ onto $\mathrm{Im}(Id+T)$. Since $\mathrm{Ker}(Id+T)$ is of finite dimension, through the theorem of open application we deduce the existence of a constant $C > 0$ such that (2.14) holds. □

The result of the next theorem is a generalization of the one in [55] to the case $I \ge 1$. So, we expect that for an estimate of $\nabla\,\boldsymbol{u}$ in addition to $\mathrm{div}\,\boldsymbol{v}$ and $\mathbf{curl}\,\boldsymbol{u}$ the quantity $\sum_{i=1}^{I}\langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}$ if $\boldsymbol{v}\times\boldsymbol{n}$ vanish on Γ.

Theorem 2.2.2. *Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{v}\times\boldsymbol{n} = 0$ on Γ. Then the following estimate holds*

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}|\big), \qquad (2.16)$$

where the constant C depends only on p and Ω.

Proof. We use the same arguments as in [55] and we proceed in two steps. First, we prove that for any function $\boldsymbol{v}$ of $\boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n} = 0$ on Γ we have:

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \le C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle \boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\big). \qquad (2.17)$$

Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n} = 0$ on Γ. By using the integral representation (2.15) we obtain

$$\begin{aligned}\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)} &\le C\Big(\Big\|\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{L}^p(\Gamma)} \\ &+ \Big\|\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{L}^p(\Gamma)}\Big).\end{aligned}$$

By the trace inequality, we have

$$\begin{aligned}\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{\boldsymbol{L}^p(\Gamma)} &\leq C\Big(\Big\|\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ &+ \Big\|\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).\end{aligned}$$

By using the Calderón-Zygmund inequalities, we obtain

$$\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)} \leq C\left(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\right).$$

Thus, it follows directly by using (2.14) that

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} \leq C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\big). \tag{2.18}$$

Moreover, from the equality (2.15) we obtain

$$\begin{aligned}\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} &\leq \Big(\|T(\boldsymbol{v}\cdot\boldsymbol{n})\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \Big\|\mathbf{grad}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\\ &+ \Big\|\mathbf{curl}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big).\end{aligned}$$

It is well known (see [55]), that

$$\Big\|T(\boldsymbol{v}\cdot\boldsymbol{n})\Big\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)}.$$

Then, by the trace theorem

$$\begin{aligned}\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} &\leq C\Big(\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} + \Big\|\mathbf{grad}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ &+ \Big\|\mathbf{curl}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).\end{aligned}$$

We use again the Calderón-Zygmund inequalities and (2.18) to obtain (2.17), which completes the proof of the first step.

Secondly, as $\boldsymbol{v}$ belongs to $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, due to the trace theorem, there exists a $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\boldsymbol{v}=\boldsymbol{u}\quad\text{on}\,\Gamma\qquad\text{and}\quad\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Since $\boldsymbol{v}\times\boldsymbol{n}=0$ on Γ, $\boldsymbol{v}|_\Gamma=(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$. Then, by using (2.17) we have

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\\ &\leq C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\Big).\end{aligned}\tag{2.19}$$

We know that, for any function $\boldsymbol{w}$ of $\boldsymbol{W}_0^{1,p}(\Omega)$, we have the following integral representaion (see [55])

$$\boldsymbol{w} = -\mathbf{grad}\,\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{w}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \mathbf{curl}\,\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{w}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Using the Calderón-Zygmund inequalities or inequality (1.15), we have

$$\|\nabla\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \le C\big(\|\mathrm{div}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\big). \tag{2.20}$$

Applying (2.20) to $\boldsymbol{v}-\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega)$, we obtain

$$\|\nabla\,(\boldsymbol{v}-\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \le C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\Big).$$

Finally, the assertion (2.16) follows directly by using (2.19). □

Remark 2.2.3. We recall that if $p=2$, any function $\boldsymbol{v}$ of $\boldsymbol{H}^1(\Omega)\cap\boldsymbol{X}_N^2(\Omega)$ satisfies (see [3], Lemma 2.11)

$$\|\nabla\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)} - \int_\Gamma (\mathrm{Tr}\,\mathcal{B})(\boldsymbol{v}\cdot\boldsymbol{n})^2\,\mathrm{d}\boldsymbol{\tau},$$

where $\mathcal{B}$ is the curvature tensor of the boundary and $\mathrm{Tr}\,\mathcal{B}$ denote the trace of the operator $\mathcal{B}$. In the case $p\neq 2$, we have the inequality (2.16) which allows us to estimate $\nabla\,\boldsymbol{v}$ by $\mathbf{curl}\,\boldsymbol{v}$, $\mathrm{div}\,\boldsymbol{v}$ and $\sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}|$.

We can now prove the following theorem.

Theorem 2.2.4. *The space $\boldsymbol{X}_N^p(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant C, such that for any $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}|\big), \tag{2.21}$$

Proof. Let $\boldsymbol{v}$ be any function in $\boldsymbol{X}_N^p(\Omega)$. Due to Lemma 2.1.1, there exists a sequence $(\boldsymbol{v}_k)_k$ of $\boldsymbol{W}^{1,p}(\Omega)\cap\boldsymbol{X}_N^p(\Omega)$ which converges to $\boldsymbol{v}$ in $\boldsymbol{X}^p(\Omega)$. Applying the estimate (2.16) to $\boldsymbol{v}_k$ for each k, we see that the sequence $(\boldsymbol{v}_k)_k$ is bounded in $\boldsymbol{W}^{1,p}(\Omega)$. Hence it admits a subsequence which converges weakly in $\boldsymbol{W}^{1,p}(\Omega)$ and the limit is nothing else but $\boldsymbol{v}$. The inequality (2.21) follows directly from (2.16) and it gives the continuity of the imbedding. □

Remark 2.2.5. It is proved in [30], when the set Ω is a convex polyhedra that there is a real number $p_\Omega > 2$ such that for all p, $2<p<p_\Omega$, any function $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate (2.21) with p_Ω depends on the geometry of the domain Ω. Theorem 2.2.4 is an extension of this result to any p, $1<p<\infty$ when Ω is of class $\mathcal{C}^{1,1}$.

In order to prove the corresponding theorem for the space $\boldsymbol{X}_T^p(\Omega)$, we introduce the following linear integral operator

$$R\,\boldsymbol{\lambda}(\boldsymbol{x}) = \frac{1}{2\pi}\int_\Gamma \mathbf{curl}\,\Big(\frac{\boldsymbol{\lambda}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}\Big)\times \boldsymbol{n}\,\mathrm{d}\sigma_{\boldsymbol{\xi}}. \tag{2.22}$$

We give some properties of this operator.

Lemma 2.2.6. *We have the following properties:*

1. *The operator R is compact from $\boldsymbol{L}^p(\Gamma)$ into $\boldsymbol{L}^p(\Gamma)$.*
2. *The space $\mathrm{Im}(Id+R)$ is a closed subspace of $\boldsymbol{L}^p(\Gamma)$ and $\mathrm{Ker}(Id+R)$ is of finite dimension. It is spaned by the the traces of the functions $\widetilde{\mathbf{grad}}\,q_j^T\times\boldsymbol{n}|_\Gamma$, $1\le j\le J$, where each q_j^T is the unique solution in $W^{2,p}(\Omega^\circ)$ of the problem*

$$\begin{cases} -\Delta q_j^T = 0 \qquad \text{in } \Omega^\circ,\\ \partial_n\, q_j^T = 0 \qquad \text{on } \Gamma,\\ \Big[\,q_j^T\,\Big]_k = \text{constant} \quad \text{and} \quad [\,\partial_n\, q_j^T\,]_k = 0,\ 1\le k\le J,\\ \Big\langle \partial_n\, q_j^T,\,1\Big\rangle_{\Sigma_k} = \delta_{j\,k},\ 1\le k\le J, \end{cases} \tag{2.23}$$

   (see section 3.2).
3. *For any $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ we have:*

$$\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{L}^p(\Gamma)} \le C\big(\|(Id+R)(\boldsymbol{v}\times\boldsymbol{n})\|_{L^p(\Gamma)} + \sum_{j=1}^J |\langle \boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}|\big). \tag{2.24}$$

Proof.

1. According to [55], we have $R\in\mathcal{L}(\boldsymbol{L}^p(\Gamma),\boldsymbol{W}^{\,1,p}(\Gamma))$. The compact imbedding of the space $\boldsymbol{W}^{\,1,p}(\Gamma)$ in $\boldsymbol{L}^p(\Gamma)$ implies that T is compact from $\boldsymbol{L}^p(\Gamma)$ into $\boldsymbol{L}^p(\Gamma)$.

2. By virtue of the first point and the Fredholm alternative, we have that the space $\mathrm{Ker}(Id+R)$ is of finite dimension and $\mathrm{Im}(Id+R)$ is a closed subspace of $L^p(\Gamma)$. We will see later in section 3.2 that the functions $\widetilde{\mathbf{grad}}\,q_j^T$ belong to $\boldsymbol{W}^{\,1,p}(\Omega)$. Since the functions $\widetilde{\mathbf{grad}}\,q_j^T$, $1\le j\le J$ are linearly independent, it is readily checked that $\widetilde{\mathbf{grad}}\,q_j^T\times\boldsymbol{n}|_\Gamma$ are also linearly independent for $1\le j\le J$.
For $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$, the quantity $\boldsymbol{v}\times\boldsymbol{n}$ satisfies on Γ the following representation (see [55]):

$$\begin{aligned}(Id+R)(\boldsymbol{v}\times\boldsymbol{n}) &= \frac{1}{2\pi}\left(\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\times\boldsymbol{n}\\ &+ \frac{1}{2\pi}\left(\mathbf{grad}\int_\Gamma \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{\xi})\,\mathrm{d}\sigma_{\boldsymbol{\xi}}\right)\times\boldsymbol{n}\\ &- \frac{1}{2\pi}\left(\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\times\boldsymbol{n}.\end{aligned}$$

As $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{W}^{1,p}(\Omega)$, this relation is still valid for $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. In particular, if $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ, we have

$$\begin{aligned}(Id+R)(\boldsymbol{v}\times\boldsymbol{n}) &= \frac{1}{2\pi}\left(\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\times\boldsymbol{n}\\ &- \frac{1}{2\pi}\left(\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\times\boldsymbol{n}. \qquad (2.25)\end{aligned}$$

Since $\widetilde{\mathbf{grad}}\, q_j^T$ belong to $\boldsymbol{W}^{1,p}(\Omega)$, due to (2.25), for each $1\le j\le J$ the function $\widetilde{\mathbf{grad}}\, q_j^T\times\boldsymbol{n}$ belongs to $\mathrm{Ker}(Id+R)$. Since the dimension of $\mathrm{Ker}(Id+R)$ is J (see [55]), the set $\{\widetilde{\mathbf{grad}}\, q_j^T\times\boldsymbol{n}|_\Gamma,\ 1\le j\le J\}$ is a basis of $\mathrm{Ker}(Id+R)$.

3. The operator $Id+R$ is linear, continuous and surjective from $\boldsymbol{L}^p(\Gamma)$ onto $\mathrm{Im}(Id+R)$. Since $\mathrm{Ker}(Id+R)$ is of finite dimension, through the theorem of open application we deduce the existence of a constant $C>0$ such that (2.24) holds. □

The result of the next theorem is a generalization of the one in [55] to the case $J\ge 1$. So, we expect that for an estimate of $\nabla\,\boldsymbol{v}$ in addition to div $\boldsymbol{v}$ and $\mathbf{curl}\,\boldsymbol{v}$ the quantity $\sum_{j=1}^{J}\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}$ if $\boldsymbol{v}\cdot\boldsymbol{n}$ vanish on Γ.

Theorem 2.2.7. *Let $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ. Then the following estimate holds*

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|), \qquad (2.26)$$

where the constant C depends only on p and Ω.

Proof. We use the same arguments as for Theorem 2.2.2. First, we prove that for any function $\boldsymbol{v}$ of $\boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ we have:

$$\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} \le C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}|\big). \qquad (2.27)$$

Let $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ. By using the integral representation (2.25), the trace inequality and the Calderón-Zygmund inequalities we obtain

$$\|(Id+R)(\boldsymbol{v}\times\boldsymbol{n})\|_{\boldsymbol{L}^p(\Gamma)} \le C\left(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\right).$$

Thus, it follows directly by using (2.24) that

$$\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{L}^p(\Gamma)} \le C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}|\big). \qquad (2.28)$$

Moreover, from the equality (2.25) we obtain

$$\begin{aligned}\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} &\leq \Big(\|R(\boldsymbol{v}\times\boldsymbol{n})\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}+\Big\|\mathbf{grad}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\operatorname{div}_{\boldsymbol{y}}\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\\ &+ \Big\|\mathbf{curl}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big).\end{aligned}$$

It is well known (see [55]), that

$$\Big\|R(\boldsymbol{v}\times\boldsymbol{n})\Big\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} \leq C\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{L}^p(\Gamma)},$$

Then, by the trace theorem

$$\begin{aligned}\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} &\leq C\Big(\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{L}^p(\Gamma)}+\Big\|\mathbf{grad}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\operatorname{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\\ &+ \Big\|\,\mathbf{curl}\int_\Omega\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).\end{aligned}$$

We use again the Calderón-Zygmund inequalities and (2.28) we obtain (2.27), which completes the proof of the first step.

Secondly, as $\boldsymbol{v}$ belongs to $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, due to the trace theorem, there exists a $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\boldsymbol{v}=\boldsymbol{u}\quad\text{on}\,\Gamma\qquad\text{and}\quad\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Since $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ, $\boldsymbol{v}|_\Gamma=\boldsymbol{v}\times\boldsymbol{n}$. Then, by using (2.27) we have

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\\ &\leq C\Big(\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{j=1}^J|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}|\Big).\end{aligned}\tag{2.29}$$

Then we can proceed exactly as in the end of the proof of Lemma 2.2.2 in order to prove (2.26). □

As a consequence, the following theorem can be proved as in Theorem 2.2.4 by using Lemma 2.1.2 and Lemma 2.2.7.

Theorem 2.2.8. *The space $\boldsymbol{X}_T^p(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and for any function $\boldsymbol{v}$ in $\boldsymbol{X}_T^p(\Omega)$, we have the following estimate:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{j=1}^J|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|\big).\tag{2.30}$$

Remark 2.2.9. We recall that if $p = 2$, any function $\boldsymbol{v}$ of $\boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}^2_T(\Omega)$ satisfies (see [3], Lemma 2.11)

$$\|\nabla\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\mathbf{curl}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{L^2(\Omega)} - \int_\Gamma \mathcal{B}(\boldsymbol{v}\times\boldsymbol{n}, \boldsymbol{v}\times\boldsymbol{n})\, \mathrm{d}\boldsymbol{\tau},$$

but for $p \neq 2$, we have the inequality (2.26) which allows us to estimate $\nabla\, \boldsymbol{v}$ by $\mathbf{curl}\, \boldsymbol{v}$, $\mathrm{div}\, \boldsymbol{v}$ and $\sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}|$. As for the case of the space $\boldsymbol{X}^p_N(\Omega)$, we have the imbedding of the space $\boldsymbol{X}^p_T(\Omega)$ in $\boldsymbol{W}^{1,p}(\Omega)$ for any $1 < p < \infty$ when the domain Ω is of class $\mathcal{C}^{1,1}$, contrary to what is shown in [30], when the domain Ω is a convex polyhedra. The author has proved that there is a real number $p_\Omega > 2$ such that for all p such that $2 < p < p_\Omega$, any function $\boldsymbol{v}$ in $\boldsymbol{X}^p_T(\Omega)$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate (2.30) with p_Ω depending on the geometry of Ω.

Now, in the following we show that the results of Theorem 2.2.8 can be extended to the case where the boundary conditions $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ is replaced by inhomogeneous one. More precisely, we introduce the following space for $s \in \mathbb{R}$, $s \geq 1$:

$$\boldsymbol{X}^{\,s,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\, \boldsymbol{v} \in W^{\,s-1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{\,s-1,p}(\Omega) \ \text{ and } \ \boldsymbol{v}\cdot\boldsymbol{n} \in \boldsymbol{W}^{\,s-\frac{1}{p},p}(\Gamma)\},$$

Theorem 2.2.10. *The space $\boldsymbol{X}^{1,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and we have the following estimate for any $\boldsymbol{v}$ in $\boldsymbol{X}^{1,p}(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big). \tag{2.31}$$

Proof. Let $\boldsymbol{v}$ any function of $\boldsymbol{X}^{\,1,p}(\Omega)$. Due to the regularity of Ω, the following Neumann problem

$$\Delta\chi = \mathrm{div}\, \boldsymbol{v} \ \text{ in}\,\Omega \quad \text{ and } \quad \partial_n\chi = \boldsymbol{v}\cdot\boldsymbol{n} \ \text{ on}\,\Gamma,$$

has a unique solution χ in $W^{\,2,p}(\Omega)$ with the estimate

$$\|\chi\|_{W^{\,2,p}(\Omega)} \leq C\big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W}^{\,1-\frac{1}{p},p}(\Gamma)}\big). \tag{2.32}$$

Hence, the function $\boldsymbol{w} = \boldsymbol{v} - \mathbf{grad}\, \chi$ is a divergence-free function of $\boldsymbol{X}^{\,p}_T(\Omega)$. Applying Theorem 2.2.8, we have $\boldsymbol{w}$ belongs to $\boldsymbol{W}^{\,1,p}(\Omega)$ and then $\boldsymbol{v}$ is in $\boldsymbol{W}^{\,1,p}(\Omega)$. So applying Inequality (2.30) to $\boldsymbol{w}$ we obtain

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{\,1,p}(\Omega)} \leq C\big(\|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\big).$$

Then, the inequality (2.31) follows directly from (2.32). □

Remark 2.2.11. This result can be obtained in another way. Indeed, observe that the identity mapping belongs to $\mathcal{L}(\boldsymbol{W}^{1,p}(\Omega), \boldsymbol{X}^{1,p}(\Omega))$, which implies that its inverse is also continuous. As a consequence, Inequality (2.31) can be readily checked for any function $\boldsymbol{v}$ in $\boldsymbol{X}^{1,p}(\Omega)$ which gives the continuity of the imbedding of $\boldsymbol{X}^{1,p}(\Omega)$ into $\boldsymbol{W}^{1,p}(\Omega)$.

More generally, we can derive the following corollary exactly in the same way.

Corollary 2.2.12.

i) Let $m \in \mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$. Then the space $\boldsymbol{X}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and for any function $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$, we have the following estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{W^{m-1,p}(\Omega)} + \|\boldsymbol{v}\cdot \boldsymbol{n}\|_{W^{m-\frac{1}{p},p}(\Gamma)}\big). \quad (2.33)$$

ii) Let $s = m + \sigma$, $m \in \mathbb{N}^*$ and $0 < \sigma \leq 1$, Assume that Ω is of class $\mathcal{C}^{m+1,1}$. Then, the space $\boldsymbol{X}^{s,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{s,p}(\Omega)$ and and for any function $\boldsymbol{v}$ in $\boldsymbol{X}^{s,p}(\Omega)$, we have the estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{W}^{s-1,p}(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{W^{s-1,p}(\Omega)} + \|\boldsymbol{v}\cdot \boldsymbol{n}\|_{W^{s-\frac{1}{p},p}(\Gamma)}\big). \quad (2.34)$$

2.3 Compactness properties

In the following, we give a non-compactness result of the space $\boldsymbol{X}^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$, where a similar proof for $p = 2$ can be found in [3]. Next, we will prove that the vanishing of the tangential component or the normal component on the boundary implies the compactness.

Proposition 2.3.1. *The imbedding of $\boldsymbol{X}^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$ is not compact.*

Proof. Let $(g_k)_k$ be a sequence which tends to 0 weakly but not strongly in $W^{1-\frac{1}{p},p}(\Gamma)$. For any k, we consider the unique solution χ_k of the problem

$$\Delta\chi_k = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \chi_k = g_k \quad \text{on } \Gamma.$$

Since Ω is of class $C^{1,1}$, χ_k belongs to $W^{1,p}(\Omega)$. Moreover, we have

$$\|\chi_k\|_{W^{1,p}(\Omega)} \leq C\|g_k\|_{W^{1-\frac{1}{p},p}(\Gamma)}.$$

Then, the sequence $(\chi_k)_k$ is bounded in $W^{1,p}(\Omega)$, and it tends to 0 weakly but not strongly in $W^{1,p}(\Omega)$. Finally, for any k, the function $\boldsymbol{v}_k = \mathbf{grad}\, \chi_k$ satisfies

$$\mathbf{curl}\, \boldsymbol{v}_k = \mathbf{0} \quad \text{and} \qquad \mathrm{div}\, \boldsymbol{v}_k = 0 \quad \text{in } \Omega.$$

The sequence $(\boldsymbol{v}_k)_k$ tends to $\mathbf{0}$ in $\boldsymbol{L}^p(\Omega)$ weakly but not strongly, because if it converged strongly, due to the compact imbedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, the sequence $(\chi_k)_k$ would converge strongly to 0 in $W^{1,p}(\Omega)$, which is forbidden by the choice of $(g_k)_k$. □

We state the following result.

Lemma 2.3.2. *The imbeddings of the spaces* $\boldsymbol{X}_T^p(\Omega)$ *and* $\boldsymbol{X}_N^p(\Omega)$ *into* $\boldsymbol{L}^p(\Omega)$ *are compact.*

Proof. Let us give the proof for the case of $\boldsymbol{X}_T^{\,p}(\Omega)$ and with the same arguments, we can derive the result for the case of $\boldsymbol{X}_N^{\,p}(\Omega)$. Let $(\boldsymbol{u}_k)_k$ be a bounded sequence of $\boldsymbol{X}_T^{\,p}(\Omega)$. Due to Theorem 2.2.8, the sequence belongs to $\boldsymbol{W}^{\,1,p}(\Omega)$ and satisfies the estimate (2.26). It follows from this estimate that the sequence $(\boldsymbol{u}_k)_k$ is bounded in $\boldsymbol{W}^{\,1,p}(\Omega)$. Then, by the Sobolev-Rellich compactness theorem, there exists a subsequence $(\boldsymbol{u}_{k'})_{k'}$ which converges strongly in $\boldsymbol{L}^p(\Omega)$. Hence we have derived the result for the space $\boldsymbol{X}_T^p(\Omega)$. The proof for $\boldsymbol{X}_N^p(\Omega)$ is very similar. Indeed, for each bounded sequence $(\boldsymbol{u}_k)_k$ of $\boldsymbol{X}_N^{\,p}(\Omega)$, exactly as previously, due to Lemme 2.2.4, we check that there exist a subsequence $(\boldsymbol{u}_{k'})_{k'}$ which converges strongly in $\boldsymbol{L}^p(\Omega)$.

□

Chapter 3

Vector potentials

In this chapter, we consider the same domain as the previous one and here we want to prove some results concerning the vector potentials. The chapter is organised as follows. In section 3.1, we first give a basic result about the vector potential without boundary conditions useful for the remainder of this chapter (see Lemma 3.1.1). Next, sections 3.2, 3.3 and 3.4 are respectively devoted to the proof of existence and uniqueness of tangential vector potentials, normal vector potentials and an other type with vanishing trace on the boundary. For the construction of these vector potentials, an important tool is the characterization of the kernel. The main results are Theorem 3.2.5 for the tangential vector potentials, Theorem 3.3.7 for the normal vector potentials and Theorem 3.4.2 for the vector potentials with vanishing trace on the boundary. In sections 3.5 and 3.6 we will get interested respectively in the case of scalar potentials and weak vector potentials. Finally, using the previous results about vector potentials, we will consider a new formulation to the Stokes equations with the Dirichlet boundary condition, where the unknown is the vector potential.

3.1 Vector potential without boundary conditions

This section is devoted to proof of the following basic lemma. A detailed proof for the case $p = 2$ can be found in [3], Lemma 3.5 and in [39], Chapter I, Theorem 3.4. For $1 < p < \infty$, we give a different proof using the fundamental solution of the laplacian.

Lemma 3.1.1. *A vector field* $\boldsymbol{u}$ *in* $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ *satisfies*

$$\mathrm{div}\, \boldsymbol{u} = 0 \quad \mathrm{in}\, \Omega \quad \text{and} \qquad \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_\mathrm{i}} = 0, \quad 0 \leq \mathrm{i} \leq \mathrm{I}, \tag{3.1}$$

if and only if there exists a vector potential $\boldsymbol{\psi}_0$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *such that*

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}_0. \tag{3.2}$$

Moreover, we can choose $\boldsymbol{\psi_0}$ *such that* $\operatorname{div}\boldsymbol{\psi}_0 = 0$ *and we have the estimate*

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.3}$$

where $C > 0$ *depends only on* p *and* Ω.

Proof. **1.** Let us show that (3.2) implies (3.1). Clearly, $\operatorname{div}(\mathbf{curl}\,\boldsymbol{\psi}_0)$ is equal to 0, for any function $\boldsymbol{\psi}_0$ in $\boldsymbol{W}^{1,p}(\Omega)$. Next, we must check that $\langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0$. For $0 \leq i \leq I$, let μ_i be a function of $\mathcal{D}(\mathbb{R}^3)$ satisfying:

$$0 \leq \mu_i(x) \leq 1 \text{ in } \mathbb{R}^3 \quad \text{and} \quad \mu_i = \delta_{ij} \quad \text{in a neighbourhood of } \Gamma_j.$$

We set $\boldsymbol{w}_i = \mathbf{curl}(\mu_i\boldsymbol{\psi}_0)$. Obviously $\boldsymbol{w}_i$ belongs to $\boldsymbol{L}^p(\Omega)$ and $\operatorname{div}\boldsymbol{w}_i = 0$. Moreover,

$$\boldsymbol{w}_i\cdot\boldsymbol{n}|_{\Gamma_j} = 0 \quad \text{if } j \neq i, \quad \boldsymbol{w}_i\cdot\boldsymbol{n}|_{\Gamma_i} = \boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_i}.$$

Hence, for $0 \leq i \leq I$ we have

$$\langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = \langle \boldsymbol{w}_i\cdot\boldsymbol{n},\,1\rangle_{\Gamma} = \int_\Omega \operatorname{div}\boldsymbol{w}_i\,\mathrm{d}\boldsymbol{x} = 0$$

which is the desired condition.

2. Conversely, let $\boldsymbol{u}$ be any function satisfying (3.1). The idea is to extend $\boldsymbol{u}$ to the whole space so that the extended function $\widetilde{\boldsymbol{u}}$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$, is divergence-free and has a compact support. Then, it will be easy to construct its stream function by means of the fundamental solution of the Laplacian.
Using the above notation, let χ_0 in $W^{1,p}(\Omega)$ be the unique solution up to an additive constant of the following Neumann problem $-\Delta\chi_0 = 0$ in Ω_0 and $\partial_n\chi_0 = \boldsymbol{u}\cdot\boldsymbol{n}$ on Γ_0, $\partial_n\chi_0 = 0$ on $\partial\mathcal{O}$, and let $\chi_i \in W^{1,p}(\Omega)$, with $1 \leq i \leq I$, be the unique solution up to an additive constant of the problem: $-\Delta\chi_i = 0$ in Ω_i and $\partial_n\chi_i = \boldsymbol{u}\cdot\boldsymbol{n}$ on Γ_i, with the estimate

$$\|\chi_i\|_{\boldsymbol{W}^{1,p}(\Omega_i)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

(recall that $\boldsymbol{n}$ denotes the unit outward normal to Ω and $\mathcal{O}$). Now we can extend $\boldsymbol{u}$ as follows

$$\widetilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega, \\ \mathbf{grad}\,\chi_i & \text{in } \Omega_i, \quad 0 \leq i \leq I, \\ \boldsymbol{0} & \text{in } \mathbb{R}^3\backslash\bar{\mathcal{O}}. \end{cases} \tag{3.4}$$

Clearly, $\widetilde{\boldsymbol{u}}$ belongs to $\boldsymbol{H}^p(\mathrm{div},\,\mathbb{R}^3)$ and is divergence-free in $\mathbb{R}^3$. The function $\boldsymbol{\psi}_0 = \mathbf{curl}(\,E * \widetilde{\boldsymbol{u}}\,)$, with E the fundamental solution of the laplacian, satisfies

$$\mathbf{curl}\,\boldsymbol{\psi}_0 = \widetilde{\boldsymbol{u}} \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi}_0 = 0 \quad \text{in}\,\mathbb{R}^3.$$

Therefore

$$\|\nabla \boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\mathbb{R}^3)} = \|\nabla\,(\mathbf{curl}(\,E * \widetilde{\boldsymbol{u}}))\|_{\boldsymbol{L}^p(\mathbb{R}^3)}.$$

Applying the Calderon Zygmund inequality, we obtain

$$\|\nabla \boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\|\Delta\,(E * \widetilde{\boldsymbol{u}})\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

It follows from this estimate that $\nabla \boldsymbol{\psi}_0$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$, hence its restriction to Ω belongs to $\boldsymbol{L}^p(\Omega)$. Due to [8] (Proposition 2.10), $\boldsymbol{\psi}_0$ belongs to $\boldsymbol{W}^{\,1,p}(\Omega)$. As a consequence, $\boldsymbol{\psi}_0$ satisfies the condition (3.2) and the estimate (3.3). □

3.2 Vector potential tangential

In this subsection we focus our attention on the construction of a vector potential in $\boldsymbol{X}_T^{\,p}(\Omega)$. We require the following preliminaries which are the equivalent to those in [3] for an arbitrary p with $1 < p < \infty$.

Lemma 3.2.1. *If $\boldsymbol{\psi}$ belongs to $\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)$, the restriction of $\boldsymbol{\psi}\cdot\boldsymbol{n}$ to any Σ_j belongs to the dual space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Sigma_j)'$, and the following Green's formula holds:*

$$\forall \chi \in W^{\,1,p'}(\Omega^\circ),\ \sum_{j=1}^{J} \langle \boldsymbol{\psi}\cdot\boldsymbol{n},\,[\,\chi\,]_j\rangle_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{\psi}\cdot\mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\circ} \chi\,\mathrm{div}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}. \tag{3.5}$$

Proof. Let μ be any element of $W^{\,1-\frac{1}{p'},p'}(\Sigma_j)$, for a fixed integer j with $1 \leq j \leq J$. we can find φ in $W^{\,1,p'}(\Omega^\circ)$ such that $[\,\varphi\,]_k$ is equal to 0 for all $k \neq j$, $[\,\varphi\,]_j$ is equal to μ and φ satisfies the estimate

$$\|\varphi\|_{W^{\,1,p'}(\Omega^\circ)} \leq C\|\mu\|_{W^{\,1-\frac{1}{p'},p'}(\Sigma_j)}. \tag{3.6}$$

Next, for any function $\boldsymbol{\psi}$ in $\boldsymbol{\mathcal{D}}(\Omega)$, the following Grenn's formula holds:

$$\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\,\mu\rangle_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{\psi}\cdot\mathbf{grad}\,\varphi\,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\circ} \varphi\,\mathrm{div}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}. \tag{3.7}$$

Hence, the above inequality implies that

$$|\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\,\mu\rangle_{\Sigma_j}| \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{H}^{\,p}(\mathrm{div},\Omega)}\|\mu\|_{W^{\,1-\frac{1}{p'},p'}(\Sigma_j)},$$

where C is the constant of (3.6) . Thus

$$\|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{W^{\,1-\frac{1}{p'},p'}(\Sigma_j)'} \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{H}^{\,p}(\mathrm{div},\Omega)}.$$

Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{\,p}(\mathrm{div},\,\Omega)$, the normal trace of functions in $\mathcal{D}(\Omega)$ on Σ_j can be extended to functions of $\boldsymbol{H}_0^{\,p}(\mathrm{div},\,\Omega)$ and the above inequality holds for any function $\boldsymbol{\psi}$ in $\boldsymbol{H}_0^{\,p}(\mathrm{div},\,\Omega)$. We introduce the partition of unity $\theta_j \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\sum_{j=1}^{J} \theta_j = 1, \quad 0 \leq \theta_j(x) \leq 1 \text{ in } \mathbb{R}^3 \quad \text{and} \qquad \theta_j = \delta_{jk} \quad \text{in a neighbourhood of } \Sigma_k.$$

Then, from (3.7), we can write

$$\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\, \theta_j[\varphi]_j\rangle_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{\psi}\cdot\mathbf{grad}\,(\theta_j\varphi)\,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\circ} \theta_j\varphi\,\mathrm{div}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}. \tag{3.8}$$

The Green's formula (3.5) follows easily from (3.8). □

We introduce the space

$$\Theta^p = \left\{ r \in W^{1,p}(\Omega^\circ);\ [\,r\,]_j = \text{constant},\ 1 \le j \le J \right\}.$$

The next lemma is an extension of Lemma 3.11, [3] for the case $1 < p < \infty$, where the proof is similar and gives a characterization of the space Θ^p.

Lemma 3.2.2. *Let r belong to $W^{1,p}(\Omega^\circ)$. Then r belongs to Θ^p if and only if*

$$\mathbf{curl}(\widetilde{\mathbf{grad}}\, r) = \mathbf{0} \quad \text{in}\,\Omega.$$

Proof. Let r in $W^{1,p}(\Omega^\circ)$ and $\boldsymbol{\varphi}$ in $\boldsymbol{\mathcal{D}}(\Omega)$. We have

$$< \mathbf{curl}(\widetilde{\mathbf{grad}}\, r),\, \boldsymbol{\varphi} >_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_{\Omega^\circ} \mathbf{grad}\, r\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.$$

Then, by applying (3.5), we deduce that for any function $r \in W^{1,p}(\Omega^\circ)$ and $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$:

$$\left\langle \mathbf{curl}(\widetilde{\mathbf{grad}}\, r),\, \boldsymbol{\varphi} \right\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \sum_{j=1}^{J} \int_{\Sigma_j} [\,r\,]_j\,(\mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau}. \tag{3.9}$$

Besides that, we have

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega),\quad \forall \mu \in L^p(\Sigma_j),\quad \int_{\Sigma_j} \mu(\mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau} = \langle\, \mathbf{grad}\,\mu\times\boldsymbol{n},\, \boldsymbol{\varphi}\rangle_{\Sigma_j}. \tag{3.10}$$

Now, let r belong to Θ^p; by applying (3.10) with $\mu = [\,r\,]_j$, we infer from (3.9) that $\mathbf{curl}(\widetilde{\mathbf{grad}}\, r)$ is equal to $\mathbf{0}$.

Conversely, let r belong to $W^{1,p}(\Omega^\circ)$ and satisfy $\mathbf{curl}(\widetilde{\mathbf{grad}}\, r) = \mathbf{0}$. It follows from (3.9) and (3.10) that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega),\quad \sum_{j=1}^{J} \langle\, (\mathbf{grad}\,[\,r\,]_j\times\boldsymbol{n}),\, \boldsymbol{\varphi}\rangle_{\Sigma_j} = 0. \tag{3.11}$$

Therefore, let μ_j is a function of $\mathcal{D}(\mathbb{R}^3)$ for $0 \le j \le J$ satisfying

$$0 \le \mu_j(x) \le 1 \text{ in } \mathbb{R}^3 \quad \text{and} \quad \mu_j = \delta_{jk} \quad \text{in a neighbourhood of } \Gamma_k.$$

Then, by rewriting (3.11) with $\mu_j\boldsymbol{\varphi}$ in place of $\boldsymbol{\varphi}$, we find that

$(\mathbf{grad}\,([\,r\,]_k)\times\boldsymbol{n})\,([\,r\,]_k) = \mathbf{0}$, for $1 \le k \le J$, and $[\,r\,]_k$ is constant. □

We denote $\boldsymbol{K}^p_T(\Omega)$ the space

$$\boldsymbol{K}^p_T(\Omega) = \left\{ \boldsymbol{w} \in \boldsymbol{X}^p_T(\Omega);\ \mathbf{curl}\, \boldsymbol{w} = \mathbf{0} \text{ in}\, \Omega \text{ and } \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega \right\}. \tag{3.12}$$

As shown in [3] Proposition 3.14, the space $\boldsymbol{K}^2_T(\Omega)$ is spanned by the functions $\widetilde{\mathbf{grad}}\, q^T_j$, $1 \le j \le J$, where each q^T_j belongs to $H^1(\Omega^\circ)$ is unique up to an additive constant and satisfies

$$\begin{cases} -\Delta q^T_j = 0 \qquad \text{in } \Omega^\circ, \\ \partial_n\, q^T_j = 0 \qquad \text{on } \Gamma, \\ \left[\, q^T_j \,\right]_k = \text{constant} \quad \text{and} \quad [\, \partial_n\, q^T_j \,]_k = 0;\ 1 \le k \le J, \\ \left\langle \partial_n\, q^T_j,\, 1 \right\rangle_{\Sigma_k} = \delta_{j\,k},\ 1 \le k \le J. \end{cases} \tag{3.13}$$

Corollary 3.2.3. *The functions $\widetilde{\mathbf{grad}}\, q^T_j$, $1 \le j \le J$ belong to $\boldsymbol{W}^{1,q}(\Omega)$ for any $q \ge 2$ and the space $\boldsymbol{K}^p_T(\Omega)$ is spanned by these functions.*

Proof. First, let us check that $\widetilde{\mathbf{grad}}\, q^T_j$ belongs to $\boldsymbol{K}^p_T(\Omega)$ for each $1 \le j \le J$. According to [3], the functions $\widetilde{\mathbf{grad}}\, q^T_j$ belong to $\boldsymbol{K}^2_T(\Omega)$. Then, it suffices to show that $\widetilde{\mathbf{grad}}\, q^T_j$ belong to $\boldsymbol{L}^p(\Omega)$ when $p > 2$. Observe that, thanks to Theorem 2.2.8, we have $\widetilde{\mathbf{grad}}\, q^T_j$ belong to $\boldsymbol{H}^1(\Omega)$.Therefore, using the Sobolev's imbedding, the functions $\widetilde{\mathbf{grad}}\, q^T_j$ belong to $\boldsymbol{L}^6(\Omega)$ and then to $\boldsymbol{X}^6_T(\Omega)$. It follows from Theorem 2.2.8 and the Sobolev's imbedding, that $\widetilde{\mathbf{grad}}\, q^T_j$ belong to $\boldsymbol{L}^\infty(\Omega)$. As a consequence, for any $1 < q < \infty$, we have $\widetilde{\mathbf{grad}}\, q^T_j \in \boldsymbol{L}^q(\Omega)$. We deduce the first part of our statement by using again Theorem 2.2.8. We already know that the functions $\widetilde{\mathbf{grad}}\, q^T_j$ are linearly independent. Let show now that those functions span $\boldsymbol{K}^p_T(\Omega)$ for any $1 < p < \infty$. Let $\boldsymbol{w} \in \boldsymbol{K}^p_T(\Omega)$. The function

$$\boldsymbol{v} = \boldsymbol{w} - \sum_{j=1}^{J} \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q^T_j.$$

belongs to $\boldsymbol{K}^p_T(\Omega)$ and satisfies $\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_k} = 0$, for $1 \le k \le J$. Using (2.26), we deduce that $\nabla\, \boldsymbol{v}$ is equal to zero. Then $\boldsymbol{v} = \boldsymbol{a} \in \mathbb{R}^3$. Now, $\boldsymbol{a} = \mathbf{0}$ because $\boldsymbol{a} \cdot \boldsymbol{n} = 0$ on Γ. Hence $\boldsymbol{v}$ is zero and this finish the proof. □

Let us state the following consequence

Corollary 3.2.4. *On the space $\boldsymbol{X}^p_T(\Omega)$, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{w}\|_{\mathrm{L}^p(\Omega)} + \sum_{\mathrm{j}=1}^{\mathrm{J}} |\langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_\mathrm{j}}|, \tag{3.14}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$. *In particular, we have the following Poincaré inequality for every function* $\boldsymbol{u}$ *in the Sobolev space* $\boldsymbol{W}^{1,p}(\Omega)$ *with* $\boldsymbol{u}\cdot\boldsymbol{n}=0$ *on* Γ*:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \sum_{j=1}^{J}\langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\Big) \tag{3.15}$$

Proof. The proof consistes in applying "Peetre-Tartar Lemma", (cf. references [39], Chapter I, Theorem 2.1), with the following correspondance: $E_1 = \boldsymbol{X}_T^p(\Omega)$, $E_2 = L^p(\Omega)\times\boldsymbol{L}^p(\Omega)$, $E_3 = \boldsymbol{L}^p(\Omega)$, $A\boldsymbol{u} = (\mathrm{div}\,\boldsymbol{u},\,\mathbf{curl}\,\boldsymbol{u})$, $B = Id$, the identity operator. Due to the compactness result of Theorem 2.3.2, the canonical imbedding Id of E_1 into E_3 is compact. Besides, let $G = \boldsymbol{K}_T^p(\Omega)$ and let $M:\ \boldsymbol{X}_T^p(\Omega)\mapsto\boldsymbol{K}_T^p(\Omega)$ be the following mapping: $\boldsymbol{u}\mapsto M\boldsymbol{u} = \sum_{j=1}^{J}\langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\,\widetilde{\mathbf{grad}}\,q_j^T$. We set $\|M\boldsymbol{u}\|_G = \sum_{j=1}^{J}|\langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|$. It is clear that $M\in\mathcal{L}(\boldsymbol{X}_T^p(\Omega),\,\boldsymbol{K}_T^p(\Omega))$. Next, it is clear that if $\boldsymbol{u}\in\mathrm{Ker}\,A\backslash\{0\} = \boldsymbol{K}_T^p(\Omega)\backslash\{0\}$, then $M\boldsymbol{u}=\mathbf{0}\Leftrightarrow\boldsymbol{u}=\mathbf{0}$ and this finish the proof. □

Theorem 3.2.5. *A function* $\boldsymbol{u}$ *in* $\boldsymbol{H}^p(\mathrm{div},\,\Omega)$ *satisfies* (3.1) *if and only if there exists a vector potential* $\boldsymbol{\psi}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *such that*

$$\begin{aligned}
&\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad\text{and}\quad \mathrm{div}\,\boldsymbol{\psi} = 0 \text{ in } \Omega,\\
&\boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma,\\
&\langle\boldsymbol{\psi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ 1\leq j\leq J.
\end{aligned} \tag{3.16}$$

This function $\boldsymbol{\psi}$ *is unique and we have the estimate:*

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.17}$$

Proof. **Step 1: Necessary conditions.** The necessity of conditions (3.1) was estabished in the proof of Lemma 3.1.1.

Step 2: Uniqueness. The uniqueness of this function $\boldsymbol{\psi}$ follows from the characterization of the kernel $\boldsymbol{K}_T^p(\Omega)$. Indeed, Suppose $\boldsymbol{\psi}_1,\boldsymbol{\psi}_2$ are two vector potentials which satisfy (3.16) and set $\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$. The function $\boldsymbol{\psi}$ belongs to $\boldsymbol{K}_T^p(\Omega)$ and since it verifies the last condition in (3.16), we deduce that $\boldsymbol{\psi}=\mathbf{0}$.

Step 3: Existence. Let $\boldsymbol{u}$ be in $\boldsymbol{H}^p(\mathrm{div},\,\Omega)$ and satisfies (3.1). According to Lemma 3.1.1, there exists $\boldsymbol{\psi}_0\in\boldsymbol{W}^{1,p}(\Omega)$ with $\mathrm{div}\,\boldsymbol{\psi}_0 = 0$ in Ω, such that $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}_0$ and with

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

We introduce the solution χ in $W^{2,p}(\Omega)$, unique up to an additive constant, of the problem:

$$-\Delta\,\chi = 0 \text{ in } \Omega, \quad \text{and} \quad \partial_n\,\chi = \boldsymbol{\psi}_0 \cdot \boldsymbol{n} \text{ on } \Gamma,$$

which satisfies the estimate

$$\|\chi\|_{W^{2,p}(\Omega)} \leq C\|\boldsymbol{\psi}_0 \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)}.$$

We set

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad}\,\chi - \sum_{j=1}^{J} \langle\,(\boldsymbol{\psi}_0 - \mathbf{grad}\,\chi)\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\,\widetilde{\mathbf{grad}}\,q_j^T,$$

where q_j^T are the solutions of the problem (3.13). Then, it is easy to check that the function $\boldsymbol{\psi}$ belongs to $\boldsymbol{X}_T^p(\Omega)$ and satisfies (3.16). Hence, $\boldsymbol{\psi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ by applying Theorem 2.2.8 and we have the estimate (3.17). □

3.3 Vector potential normal and first elliptic problem

This subsection is devoted to the normal vector potentials. As previously, we define the kernel

$$\boldsymbol{K}_N^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}_N^p(\Omega),\ \mathbf{curl}\,\boldsymbol{w} = \boldsymbol{0} \text{ and } \operatorname{div}\boldsymbol{w} = 0 \text{ in } \Omega\}.$$

It is shown in [3], Proposition 3.18, that the space $\boldsymbol{K}_N^2(\Omega)$ is spanned by the functions $\mathbf{grad}\,q_i^N$, $1 \leq i \leq I$, where each $q_i^N \in \boldsymbol{H}^1(\Omega)$ is the unique solution of the problem

$$\begin{cases} -\Delta q_i^N = 0 \qquad \text{in } \Omega, \\ q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|_{\Gamma_k} = \text{constant},\ \ 1 \leq k \leq I, \\ \big\langle \partial_n\,q_i^N,\,1\big\rangle_{\Gamma_k} = \delta_{i\,k},\ \ 1 \leq k \leq I, \quad \text{and} \quad \big\langle \partial_n\,q_i^N,\,1\big\rangle_{\Gamma_0} = -1. \end{cases} \tag{3.18}$$

With an argument entirely similar to that used in the proof of Corollary 3.2.3, we have the following result

Corollary 3.3.1. *The functions* $\mathbf{grad}\,q_i^N$ *belongs to* $\boldsymbol{W}^{1,q}(\Omega)$ *for any* $q \geq 2$ *and the space* $\boldsymbol{K}_N^p(\Omega)$ *is spanned by those functions for* $1 \leq i \leq I$.

Proof. Since Ω is of class $\mathcal{C}^{1,1}$, it is clear that each $q_i^N \in W^{2,r}(\Omega)$ for any $1 \leq r < \infty$. We know that the functions $\mathbf{grad}\,q_i^N$ are linearly independent. It remains to prove that they span $\boldsymbol{K}_N^p(\Omega)$. Let $\boldsymbol{w}$ any function in $\boldsymbol{K}_N^p(\Omega)$ and consider the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{i=1}^{I} \langle \boldsymbol{w}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}\,\mathbf{grad}\,q_i^N.$$

Observe that $\boldsymbol{u} \in \boldsymbol{K}_N^p(\Omega)$ and satisfies $\langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0$ for any $1 \leq i \leq I$. We deduce from Theorem 2.2.2 that $\nabla\,\boldsymbol{u} = \boldsymbol{0}$. Hence, $\boldsymbol{u} = \boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{a} = \boldsymbol{0}$ because $\boldsymbol{a} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. That ends the proof. □

We define the space

$$\boldsymbol{V}_T^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}_T^p(\Omega);\ \mathrm{div}\,\boldsymbol{w} = 0 \text{ in}\,\Omega \text{ and } \langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J\},$$

which is a Banach space for the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$. As a consequence, we have the following result

Lemma 3.3.2. *The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_T^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}_T^{p}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_T^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}} \ge \beta. \tag{3.19}$$

Proof. We need the following Helmholtz decomposition: every vector function $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ can be decomposed into a sum $\boldsymbol{g} = \nabla\chi + \boldsymbol{z}$, where $\boldsymbol{z}$ belongs to $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ with $\mathrm{div}\,\boldsymbol{z} = 0$, χ belongs to $\boldsymbol{W}_0^{\,1,p}(\Omega)$ and satisfies the estimate

$$\|\nabla\chi\|_{\boldsymbol{L}^p(\Omega)} \le C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.20}$$

Let $\boldsymbol{\varphi}$ any function of $\boldsymbol{V}_T^{p'}(\Omega)$. Due to Corollary 3.2.4 we can write

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)} \le C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} = C \sup_{\substack{\boldsymbol{g}\in\boldsymbol{L}^p(\Omega)\\ \boldsymbol{g}\neq 0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}}. \tag{3.21}$$

It is clear that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla\chi\,\mathrm{d}\boldsymbol{x} = 0. \tag{3.22}$$

We set

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{i=1}^{I}\langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\nabla\, q_i^N,$$

so, $\widetilde{\boldsymbol{z}} \in \boldsymbol{L}^p(\Omega)$, $\mathrm{div}\,\widetilde{\boldsymbol{z}} = 0$ and $\langle \widetilde{\boldsymbol{z}}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$ for each $0 \le i \le I$. Due to Theorem 3.2.5, there exists a potential vector $\boldsymbol{\psi} \in \boldsymbol{V}_T^p(\Omega)$ such that $\widetilde{\boldsymbol{z}} = \mathbf{curl}\,\boldsymbol{\psi}$ in Ω. This implies that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}.$$

Moreover, we have

$$\begin{aligned}\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} &\le \|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I}|\langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|\|\nabla\, q_i^N\|_{\boldsymbol{L}^p(\Omega)}\\ &\le \|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + C\,\|\boldsymbol{z}\cdot\boldsymbol{n}\|_{W^{-\frac{1}{p},p}(\Gamma)}.\end{aligned}$$

Since $\boldsymbol{z}$ belongs to $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ and $\mathrm{div}\,\boldsymbol{z} = 0$, by using the continuity of the normal trace operator on $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$, we obtain

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \le C\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.23}$$

So, the following inequality can be checked by a combination of (3.20) and (3.23)

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.24}$$

Finally, we can write

$$\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \leq C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}\right|}{\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)}} = C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right|}{\|\mathbf{curl}\,\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)}}.$$

But by Corollary 3.2.4 we have

$$\|\boldsymbol{\psi}\|_{\boldsymbol{X}^p_T(\Omega)} \simeq \|\mathbf{curl}\,\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)}.$$

Then,

$$\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \leq C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^p_T(\Omega)}},$$

and the Inf-Sup Condition (3.19) follows immediately from (3.21). □

In the next, we illustrate the importance goal of the Inf-Sup Condition by using it to resolve the following first elliptic system.

Proposition 3.3.3. *Assume that $\boldsymbol{v}$ belongs to $\boldsymbol{L}^p(\Omega)$. Then, the following problem*

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v} & \text{in}\,\Omega, \\ \operatorname{div}\boldsymbol{\xi} = 0 & \text{in}\,\Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0,\ (\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = 0 & \text{on}\,\Gamma, \\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ 1\leq j\leq J, \end{cases} \tag{3.25}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ and we have:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.26}$$

Moreover, if $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{3.27}$$

Proof. **First step: Existence and uniqueness**. Due to Lemma 3.3.2, the following problem: find $\boldsymbol{\xi}\in\boldsymbol{V}^p_T(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{V}^{p'}_T(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \tag{3.28}$$

satisfies the Inf-Sup condition (3.19). So, it has a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}_T^p(\Omega)$ since the right-hand side belongs to $(\boldsymbol{V}_T^{p'}(\Omega))'$. We known that due to Theorem 2.2.8, this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Next, we want to extend (3.28) to any test function $\widetilde{\boldsymbol{\varphi}}$ in $\boldsymbol{X}_T^{p'}(\Omega)$. We consider the solution χ in $W^{1,p'}(\Omega)$ up to an additive constant of the Neumann problem:

$$\Delta\,\chi = \operatorname{div}\widetilde{\boldsymbol{\varphi}} \ \text{in}\,\Omega \quad \text{and} \quad \frac{\partial\,\chi}{\partial\,\boldsymbol{n}} = 0 \ \text{on}\,\Gamma. \tag{3.29}$$

Then, we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi - \sum_{j=1}^{J} \langle (\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi)\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T. \tag{3.30}$$

Observe that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{V}_T^{p'}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}$. Hence, (3.28) becomes: find $\boldsymbol{\xi} \in \boldsymbol{V}_T^p(\Omega)$ such that

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}. \tag{3.31}$$

It is easy to proof that every solution of (3.25) also solves (3.31). Conversely, let $\boldsymbol{\xi}$ the solution of the problem (3.31). Then,

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega), \quad \langle \mathbf{curl}(\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}),\, \widetilde{\boldsymbol{\varphi}}\rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} = 0.$$

So $-\Delta\,\boldsymbol{\xi}$ is equal to $\mathbf{curl}\,\boldsymbol{v}$ in Ω. Moreover, by the fact that $\boldsymbol{\xi}$ belongs to the space $\boldsymbol{V}_T^p(\Omega)$ we have $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω, $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$ for any $1 \le j \le J$. Then, it remains to verify the boundary condition $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{v}\times\boldsymbol{n}$ on Γ of (3.25). The function $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$ belongs to $\boldsymbol{X}^p(\Omega)$ and $\mathbf{curl}\,\boldsymbol{z} = \boldsymbol{0}$. Consequently, for any $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega)$ we have:

$$\begin{aligned} &\int_\Omega \ \boldsymbol{z}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} \ + \ \langle \boldsymbol{z}\times\boldsymbol{n}, \widetilde{\boldsymbol{\varphi}}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \\ &= \ \langle \mathbf{curl}\,\boldsymbol{z}, \widetilde{\boldsymbol{\varphi}}\rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} = 0. \end{aligned} \tag{3.32}$$

Using (3.31), we deduce that

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{V}_T^{p'}(\Omega), \quad \langle \boldsymbol{z}\times\boldsymbol{n}, \widetilde{\boldsymbol{\varphi}}\rangle_\Gamma = 0.$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$. So, there exists an element $\widetilde{\boldsymbol{\varphi}}$ of $\boldsymbol{W}^{1,p'}(\Omega)$ such that $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\mu}_t$ on Γ, where $\boldsymbol{\mu}_t$ is the tangentiel component of $\boldsymbol{\mu}$ on Γ. It is clear that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{X}_T^{p'}(\Omega)$ and

$$\langle \boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma = \langle \boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\mu}_t\rangle_\Gamma = \langle \boldsymbol{z}\times\boldsymbol{n}, \widetilde{\boldsymbol{\varphi}}\rangle_\Gamma = 0. \tag{3.33}$$

This implies that $\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ which is the last boundary condition in (3.25). Let $B \in \mathcal{L}(\boldsymbol{V}_T^p(\Omega), (\boldsymbol{V}_T^{p'}(\Omega))')$ be the operator defined by

$$\forall \boldsymbol{\mu} \in \boldsymbol{V}_T^p(\Omega), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}_T^{p'}(\Omega), \quad \langle B\,\boldsymbol{\mu},\, \boldsymbol{\varphi}\rangle = \int_\Omega \mathbf{curl}\,\boldsymbol{\mu}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \tag{3.34}$$

It is clear that the operator B is an isomorphism from $\boldsymbol{V}_T^p(\Omega)$ onto $(\boldsymbol{V}_T^{p'}(\Omega))'$, and

$$\|\boldsymbol{\mu}\|_{\boldsymbol{X}_T^p(\Omega)} \backsimeq \|B\,\boldsymbol{\mu}\|_{(\boldsymbol{V}_T^{p'}(\Omega))'}. \tag{3.35}$$

Hence, since $\boldsymbol{\xi}$ is the solution of the problem (3.31) we have:

$$\|B\,\boldsymbol{\xi}\|_{(\boldsymbol{V}_T^{p'}(\Omega))'} = \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_T^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\langle B\,\boldsymbol{\xi},\,\boldsymbol{\varphi}\rangle}{\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}} = \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_T^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}}.$$

Then,

$$\|B\,\boldsymbol{\xi}\|_{(\boldsymbol{V}_T^{p'}(\Omega))'} \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

The estimate (3.26) follows by using (3.35) and Lemma 2.2.7.

ii) **Second step: Regularity**. Now, we suppose that $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ given by the first step and we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$. It is clear that $\boldsymbol{z}$ belongs to $\boldsymbol{X}_N^p(\Omega)$. Applying Theorem 2.2.4, we obtain that $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. This implies that $\mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$. Applying Corollary 2.2.12, we deduce that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. This finish the proof of the proposition.

□

Remark 3.3.4 *)* Note that we can prove directly that the solution $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ of the problem (3.25) is unique. Indeed, we suppose that $\boldsymbol{v} = \mathbf{0}$ and we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$. Then, $\boldsymbol{z}$ belongs to $\boldsymbol{K}_N^p(\Omega)$ and due to Corollary *3.3.1* we can write:

$$\boldsymbol{z} = \sum_{i=1}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} \mathbf{grad}\, q_i^N.$$

Since $\mathbf{grad}\, q_i^N \in \boldsymbol{W}^{1,q}(\Omega)$ for any $q \geq 1$, the function $\boldsymbol{z}$ belongs to $\boldsymbol{L}^2(\Omega)$. So,

$$\int_\Omega |\boldsymbol{z}|^2\,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{grad}\, q_i^N\,\mathrm{d}\boldsymbol{x} = 0.$$

Hence, $\boldsymbol{z}$ is zero and $\boldsymbol{\xi}$ belongs to $\boldsymbol{K}_T^p(\Omega)$. Due to Corollary *3.2.3*, we can write $\boldsymbol{\xi}$ as*:*

$$\boldsymbol{\xi} = \sum_{j=1}^{J} \langle \boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^T$$

and $\boldsymbol{\xi} = \boldsymbol{0}$ since $\boldsymbol{\xi}$ satisfies the last condition in (3.25).

ii) If $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^p(\Omega)$, then $(\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ but neither $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}$ nor $\boldsymbol{v}\times\boldsymbol{n}$ is defined. However, if $\boldsymbol{v}$ belongs to $\boldsymbol{H}^p(\mathbf{curl},\,\Omega)$, then $\boldsymbol{v}\times\boldsymbol{n}$ and $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}$ have a sense in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$.

We give the following corollary which extends Theorem 2.2.4 to the case where the boundary condition $\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ is replaced by inhomogeneous one. we introduce the following space for $s\in\mathbb{R}$, $s\geq 1$:

$$\boldsymbol{Y}^{s,p}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v}\in W^{s-1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{W}^{s-1,p}(\Omega)\ \text{ and }\ \boldsymbol{v}\times\boldsymbol{n}\in\boldsymbol{W}^{s-\frac{1}{p},p}(\Omega)\}.$$

Corollary 3.3.5. *The space $\boldsymbol{Y}^{1,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and we have the following estimate: for any $\boldsymbol{v}$ in $\boldsymbol{Y}^{1,p}(\Omega)$,*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}+\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big). \tag{3.36}$$

Proof. Let $\boldsymbol{v}$ any function of $\boldsymbol{Y}^{1,p}(\Omega)$. We set $\boldsymbol{z}=\boldsymbol{v}-\mathbf{curl}\,\boldsymbol{\xi}$ where $\boldsymbol{\xi}\in\boldsymbol{W}^{2,p}(\Omega)$ is the solution of the problem (3.25). Hence, $\boldsymbol{z}$ belongs to the space $\boldsymbol{X}^p_N(\Omega)$. Applying Theorem 2.2.4, we have $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Then, $\boldsymbol{v}$ is necessarily in $\boldsymbol{W}^{1,p}(\Omega)$ and the inequality (3.36) can be easily deduced by using inequalities (3.27) and (2.16). □

More generally, we can prove:

Corollary 3.3.6.

i) Let $m\in\mathbb{N}^*$ and Ω is of class $\mathcal{C}^{m,1}$. Then the space $\boldsymbol{Y}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and we have the following estimate: for any function $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)}\leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{W^{m-1,p}(\Omega)}+\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)}\Big). \tag{3.37}$$

ii) Let $s=m+\sigma$, $m\in\mathbb{N}^*$ and $0<\sigma\leq 1$, Assume that Ω is of class $\mathcal{C}^{m+1,1}$. Then, the space $\boldsymbol{Y}^{s,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{s,p}(\Omega)$ and for any function $\boldsymbol{v}$ in $\boldsymbol{Y}^{m,p}(\Omega)$ we have the following estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(\Omega)}\leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{s-1,p}(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{W^{s-1,p}(\Omega)}+\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{s-\frac{1}{p},p}(\Gamma)}\Big). \tag{3.38}$$

The following theorem is the main result of this subsection

Theorem 3.3.7. *A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\mathrm{div},\,\Omega)$ satisfies:*

$$\mathrm{div}\,\boldsymbol{u}=0\quad\text{in}\,\Omega,\ \ \boldsymbol{u}\cdot\boldsymbol{n}=0\quad\text{on}\ \Gamma\quad\text{and}\quad\langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}=0,\quad 1\leq j\leq J, \tag{3.39}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{aligned}&\boldsymbol{u}=\mathbf{curl}\,\boldsymbol{\psi}\quad\text{and}\quad\mathrm{div}\,\boldsymbol{\psi}=0\ \text{in}\ \Omega,\\&\boldsymbol{\psi}\times\boldsymbol{n}=\boldsymbol{0}\qquad\text{on}\ \Gamma,\\&\langle\boldsymbol{\psi}\cdot\boldsymbol{n},\ 1\rangle_{\Gamma_i}=0,\ \text{for any}\ \ 1\leq i\leq I.\end{aligned} \tag{3.40}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.41}$$

Proof. **Step 1: Necessary conditions.** Let us show that (3.40) implies (3.39). Clearly, $\operatorname{div}(\mathbf{curl}\,\boldsymbol{\psi}) = 0$. Next, we must check that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$, for any $1 \le j \le J$. From formulas (1.7) and (1.8), because $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p}(\Omega)$ we have for any χ in $W^{2,p'}(\Omega)$ respectively:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{u}\cdot\boldsymbol{n},\, \chi\rangle_\Gamma, \tag{3.42}$$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} = -\langle \boldsymbol{\psi}\times\boldsymbol{n},\, \mathbf{grad}\,\chi\rangle_\Gamma. \tag{3.43}$$

If $\boldsymbol{\psi}\times\boldsymbol{n}$ vanishes on Γ, we have $\langle \boldsymbol{u}\cdot\boldsymbol{n},\, \chi\rangle_\Gamma = 0$ for any function χ in $W^{2,p'}(\Omega)$ and in particular for $\chi \in \mathcal{D}(\overline{\Omega})$. Because $\boldsymbol{u}\cdot\boldsymbol{n} \in W^{-\frac{1}{p},p}(\Gamma)$ and $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p'}(\Omega)$, $\langle \boldsymbol{u}\cdot\boldsymbol{n},\, \chi\rangle_\Gamma = 0$ for any function χ in $W^{1,p'}(\Omega)$ and then $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Since $\mathbf{curl}\,\boldsymbol{\psi}$ belongs to $\boldsymbol{H}^p_0(\mathrm{div},\,\Omega)$, it follows from Lemma 3.2.1 that $\mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n}$ belongs to $W^{1-\frac{1}{p},p}(\Sigma_j)'$, for $1 \le j \le J$. Then, the density of $\mathcal{D}(\Sigma_j)$ in $W^{1-\frac{1}{p},p}(\Sigma_j)'$ and the choice $\mu = 1$ in (3.10) give

$$\langle \mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J, \tag{3.44}$$

which is the last condition in (3.39).

Step 2: Uniqueness. The proof is similar to that of Theorem 3.2.5 and it follows from the caracterization of the kernel $\boldsymbol{K}^p_T(\Omega)$. Indeed, Suppose $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2$ are two vector potentials which satisfy (3.16) and set $\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$. The function $\boldsymbol{\psi}$ belongs to $\boldsymbol{K}^p_T(\Omega)$ and since it verifies the last condition in (3.16), we deduce that $\boldsymbol{\psi} = \boldsymbol{0}$.

Step 3: Existence. Let $\boldsymbol{u} \in \boldsymbol{H}^p(\mathrm{div},\,\Omega)$ be any function satisfying (3.39). According to Lemma 3.1.1, there exist $\boldsymbol{\psi}_0 \in \boldsymbol{W}^{1,p}(\Omega)$ with $\operatorname{div}\boldsymbol{\psi}_0 = 0$ and such that $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}_0$. Due to Lemma 3.3.2, the following problem: find $\boldsymbol{\xi} \in \boldsymbol{V}^p_T(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_T(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}, \tag{3.45}$$

satisfies the Inf-Sup condition (3.19). So, it has a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}^p_T(\Omega)$ since here also the right-hand side belongs to $(\boldsymbol{V}^{p'}_T(\Omega))'$. We known that due to Theorem 2.2.8, this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Next, as in the proof of Proposition 3.3.3 we want to extend (3.45) to any test function $\widetilde{\boldsymbol{\varphi}}$ in $\boldsymbol{X}^{p'}_T(\Omega)$. Let χ in $W^{1,p'}(\Omega)$ satisfying (3.29) and $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{V}^{p'}_T(\Omega)$ satisfying (3.30). Due to (3.39), we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} = 0,$$

and from (3.5) we obtain

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\widetilde{\mathbf{grad}}\,q^T_j\,\mathrm{d}\boldsymbol{x} = \int_{\Omega^\circ} \boldsymbol{u}\cdot\mathbf{grad}\,q^T_j\,\mathrm{d}\boldsymbol{x} = \sum_{k=1}^{J}[q^T_j]_k\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_k} = 0.$$

Hence, (3.45) becomes: find $\boldsymbol{\xi} \in \boldsymbol{V}_T^p(\Omega)$ such that

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0 \cdot \widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}. \tag{3.46}$$

By using the same arguments as in Proposition 3.3.3, we can prove that every solution of (3.46) also solves the problem

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = 0 & \text{in}\,\Omega, \\ \mathrm{div}\,\boldsymbol{\xi} = 0 & \text{in}\,\Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{\psi}_0\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ 1\le j\le J. \end{cases} \tag{3.47}$$

According to Corollary 3.3.5, this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{\,2,p}(\Omega)$. Then, we set

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi} - \sum_{i=1}^{I} \langle (\boldsymbol{\psi}_0 - \mathbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}\,\mathbf{grad}\,q_i^N,$$

and it follows that the function $\boldsymbol{\psi}$ belongs to $\boldsymbol{W}^{\,1,p}(\Omega)$ satisfies (3.40) and the estimate (3.41). □

We skip the proof of the next corollary about equivalent norms, because it uses exactly the same tools as in the proof of Corollary 3.2.4.

Corollary 3.3.8. *On the space $\boldsymbol{X}_N^p(\Omega)$, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{w}\|_{\mathrm{L}^p(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{I}} |\langle \boldsymbol{w}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_\mathrm{i}}| \tag{3.48}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$. In particular, we have the following Poincaré inequality for every function $\boldsymbol{u}$ in the Sobolev space $\boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{u}\times\boldsymbol{n} = 0$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \sum_{i=1}^{I} \langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}\big) \tag{3.49}$$

3.4 Other vector potentials

We define the space

$$\boldsymbol{K}_0^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{W}_0^{\,1,p}(\Omega);\ \mathbf{curl}\,\boldsymbol{w} = \boldsymbol{0} \text{ and } \mathrm{div}\,(\Delta\boldsymbol{w}) = 0 \text{ in}\,\Omega\}.$$

It has proven in [3] that the space $\boldsymbol{K}_0^2(\Omega)$ is spanned by the functions $\mathbf{grad}\, q_i^0$, for $1 \le i \le I$, where q_i^0 is the unique solution in $\boldsymbol{H}^2(\Omega)$ of the problem

$$\begin{cases} -\Delta^2 q_i^0 = 0 & \text{in } \Omega, \\ q_i^0|_{\Gamma_0} = 0 & \text{and} \quad q_i^0|_{\Gamma_k} = \delta_{ik}, \ 1 \le k \le I, \\ \partial_n\, q_i^0 = 0 & \text{on } \Gamma, \\ \langle \partial_n\, (\Delta q_i^0),\, 1 \rangle_{\Gamma_k} = \delta_{i\,k}, \ 1 \le k \le I & \text{and} \quad \langle \partial_n\, (\Delta q_i^0),\, 1 \rangle_{\Gamma_0} = -1. \end{cases} \tag{3.50}$$

We have the following regularity result

Lemma 3.4.1. *The solution q_i^0 of the problem (3.50) belongs to $\boldsymbol{W}^{2,q}(\Omega)$ for any $q \ge 1$.*

Theorem 3.4.2. *A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\mathrm{div},\, \Omega)$ satisfies:*

$$\mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in}\, \Omega, \ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on}\ \Gamma \quad \text{and} \quad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_\mathrm{j}} = 0, \quad 1 \le \mathrm{j} \le \mathrm{J}, \tag{3.51}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\, (\Delta \boldsymbol{\psi}) = 0 \text{ in } \Omega, \\ &\boldsymbol{\psi} = \boldsymbol{0} \qquad \text{on } \Gamma, \qquad \langle \partial_\mathrm{n} (\mathrm{div}\, \boldsymbol{\psi}),\, 1 \rangle_{\Gamma_\mathrm{i}} = 0, \ \text{for any } \ 0 \le \mathrm{i} \le \mathrm{I}. \end{aligned} \tag{3.52}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. We only need to prove that with any function $\boldsymbol{u}$ satisfying (3.51) we can associate the vector potential in (3.52). The inverse can be proved exactly with the same argument as in Theorem 3.3.7. Here also the uniqneness result is a consequence of the characterization of the space $\boldsymbol{K}_0^p(\Omega)$.
Assume that (3.51) holds and let $\widetilde{\boldsymbol{\psi}}$ denote the potential vector associated with $\boldsymbol{u}$ by Theorem 3.3.7. We introduce the fourth-order problem

$$\Delta^2 \chi = 0 \quad \text{in}\, \Omega, \quad \chi = 0 \quad \text{and} \quad \partial_n \chi = \widetilde{\boldsymbol{\psi}} \cdot \boldsymbol{n} \quad \text{on}\, \Gamma.$$

Since Ω is of class $\mathcal{C}^{1,1}$ and $\widetilde{\boldsymbol{\psi}} \cdot \boldsymbol{n}$ belongs to $W^{1-\frac{1}{p},p}(\Gamma)$, this problem has a unique solution χ in $W^{2,p}(\Omega)$. Finally, the function

$$\boldsymbol{\psi} = \widetilde{\boldsymbol{\psi}} - \mathbf{grad}\, \chi + \sum_{i=1}^{I} \langle \partial_n (\Delta \chi),\, 1 \rangle_{\Gamma_i} \mathbf{grad}\, q_i^0,$$

satisfies (3.52). □

By using the same arguments and in this ones the kernel:

$$\boldsymbol{K}_0^{m+1,p}(\Omega) = \{\boldsymbol{w} \in \boldsymbol{W}_0^{m+1,p}(\Omega);\ \mathbf{curl}\,\boldsymbol{w} = \mathbf{0} \text{ and } \operatorname{div}(\Delta^{\mathbf{m+1}}\boldsymbol{w}) = 0 \text{ in } \Omega\},$$

we can prove the following result (see [4] for the case $p = 2$).

Theorem 3.4.3. *Assume that the boundary of Ω is of class $\mathcal{C}^{m+1,1}$ for some integer $m \geq 1$. For any vector field $\boldsymbol{u}$ in $\boldsymbol{W}_0^{m,p}(\Omega)$ that satisfies:*

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega \quad \text{and} \quad \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ \ 1 \leq j \leq J,$$

there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}_0^{m+1,p}(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \operatorname{div}\Delta^{m+1}\boldsymbol{\psi} = 0 \text{ in } \Omega \quad \text{and} \quad \|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{W}^{m,p}(\Omega)}. \tag{3.53}$$

Moreover, there exists a unique vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}_0^{m+1,p}(\Omega)$, satisfying (3.53) and

$$\langle \partial_n \operatorname{div}\Delta\,\boldsymbol{\psi}^{m+1},\, 1\rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I. \tag{3.54}$$

3.5 Scalar potential

In this section we present several results concerning scalar potentials. We begin with the following fundamental result.

Theorem 3.5.1. *Let $\boldsymbol{f} \in \boldsymbol{W}^{-m,p}(\Omega)$ for some integer $m \geqslant 0$. then the following properties are equivalent:*

i) ${}_{\boldsymbol{W}^{-m,p}(\Omega)}\langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{m,p'}(\Omega)} = 0$ for all $\boldsymbol{\varphi} \in \{\boldsymbol{\varphi} \in \boldsymbol{W}_0^{m,p'}(\Omega),\ \operatorname{div}\boldsymbol{\varphi} = 0 \text{ in } \Omega\}$,

ii) ${}_{\boldsymbol{W}^{-m,p}(\Omega)}\langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{m,p'}(\Omega)} = 0$ for all $\boldsymbol{\varphi} \in \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega),\ \operatorname{div}\boldsymbol{\varphi} = 0 \text{ in } \Omega\}$,

iii) There exists a distribution $\chi \in W^{-m+1,p}(\Omega)$, unique up to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\,\chi$ in Ω.

If in addition Ω is simply-connected, the above properties are equivalent to:

iv) $\mathbf{curl}\,\boldsymbol{f} = \mathbf{0}$ *in* Ω.

Proof. We refer to [5] for the equivalence between (i), (ii) and (iii). It is clear that (iii) implies (iv), then it remains to prove that (iv) implies (iii). Let then $\boldsymbol{f} \in \boldsymbol{W}^{-m,p}(\Omega)$ be such

that $\mathbf{curl}\, \boldsymbol{f} = \mathbf{0}$ in Ω. With the same arguments as in [5] we can prove the existence of a unique $\boldsymbol{u} \in \boldsymbol{W}_0^{m,p}(\Omega)$ and a unique $p \in W^{-m+1,p}(\Omega)/\mathbb{R}$ such that

$$\Delta^m \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega.$$

Hence, $\Delta^m \mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ in Ω. Because the hypoellipticity of the pluriharmonic operator Δ^m, this implies that (see [31]) $\mathbf{curl}\, \boldsymbol{u}$ belongs to $\mathcal{C}^\infty(\Omega)$. We deduce that $\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u}$ belongs to $\mathcal{C}^\infty(\Omega)$ which implies that $\Delta^m \boldsymbol{u} \in \mathcal{C}^\infty(\Omega)$. By the classical Poincaré theorem, there exists $q \in \mathcal{C}^\infty(\Omega)$ such that $\Delta^m \boldsymbol{u} = \mathbf{grad}\, q$. Thus, $\boldsymbol{f} = \mathbf{grad}\,(p+q)$ and by using [8], Proposition 2.10, we deduce that the function $\chi = p + q$ belongs to the space $W^{-m+1,p}(\Omega)$. □

Theorem 3.5.2. *For any function $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ that satisfies*

$$\mathbf{curl}\, \boldsymbol{f} = \mathbf{0} \quad \text{and} \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{K}_T^{p'}(\Omega), \tag{3.55}$$

there exists a scalar potential $\chi \in W^{1,p}(\Omega)$, unique up to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\, \chi$ and the following estimate holds:

$$\|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}} \le C \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.56}$$

Proof. First, we prove that for any $\boldsymbol{v} \in \boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω, there holds $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = 0$. For such $\boldsymbol{v} \in \boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)$, let

$$\boldsymbol{z} = \sum_{j=1}^{J} \langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^T$$

and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$. According to Theorem 3.3.7, there exists a vector potential $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p'}(\Omega)$ satisfying $\boldsymbol{w} = \mathbf{curl}\, \boldsymbol{\psi}$, $\operatorname{div} \boldsymbol{\psi} = 0$ in Ω and $\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}$ on Γ. Hence

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \mathbf{curl}\, \boldsymbol{\psi}\, \mathrm{d}\boldsymbol{x} = 0.$$

The result is then a consequence of Lemma 2.7 of [6]: there exists a function $\chi \in W^{1,p}(\Omega)$ satisfying $\boldsymbol{f} = \mathbf{grad}\, \chi$ and the estimate (3.56) holds. □

Theorem 3.5.3. *For any $\boldsymbol{f}$ in the dual space of $\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)$ with $\mathbf{curl}\, \boldsymbol{f} = 0$ in Ω and satisfies*

$${}_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)'}\langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{K}_T^{p'}(\Omega), \tag{3.57}$$

there exists a scalar potential χ in $L^p(\Omega)$, unique up to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\, \chi$ and the following estimate holds:

$$\|\chi\|_{L^p(\Omega)/\mathbb{R}} \le C \|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)'}. \tag{3.58}$$

Proof. Let $\boldsymbol{f}$ in $\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)'$. Due to Proposition 1.0.4, there exist $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\chi_0 \in L^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\,\chi_0$, with the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi_0\|_{L^p(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)'}.$$

By the density of $\mathcal{D}(\Omega)$ in $\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)$, we have ${}_{\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)'}\langle \mathbf{grad}\,\chi_0\,,\ \boldsymbol{v}\rangle_{\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)} = 0$ for all $\boldsymbol{v} \in \boldsymbol{K}_T^{p'}(\Omega)$. Therefore, the function $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ satisfies the conditions (3.55) of Theorem 3.5.2. Hence, there exists a function $q \in W^{1,p}(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{grad}\,q$, with the estimate

$$\|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)'}.$$

Hence, the function $\chi = q + \chi_0$ satisfies the announced properties. □

Remark 3.5.4. If Ω is simply-connected, for a distribution $\boldsymbol{f}$ in the dual space $\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)'$ with $\mathbf{curl}\,\boldsymbol{f} = \mathbf{0}$, there exist a unique function $\chi \in L^p(\Omega)$ to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\,\chi$, without f satisfies the condition (3.57).

3.6 Weak vector potential

We know that for a given function $\boldsymbol{f}$ in $\boldsymbol{W}^{-1,p}(\Omega)$, there exist a unique $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\chi \in L^p(\Omega)$ such that

$$\boldsymbol{f} = -\Delta\boldsymbol{u} + \mathbf{grad}\,\chi \qquad \text{and} \qquad \mathrm{div}\,\boldsymbol{u} = 0 \ \text{ in}\,\Omega, \tag{3.59}$$

and satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\chi\|_{L^p(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p}(\Omega)}.$$

We give the following result where the proof for the case $p = 2$ can be found in [4].

Proposition 3.6.1. *For any $\boldsymbol{f}$ in the dual space $\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)'$, there exist a unique $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)$ and $\chi \in L^p(\Omega)$ solution to (3.59) and satisfying the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\chi\|_{L^p(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)'}. \tag{3.60}$$

Proof. Let $\boldsymbol{f}$ be in the dual space of $\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)$. According to Proposition 1.0.4, there exist $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\chi_0 \in L^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\,\chi_0$ and

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi_0\|_{L^p(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)'}.$$

Thanks to the regularuty of Γ, there exist $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)$ and $q \in W^{1,p}(\Omega)$ satisfying

$$\boldsymbol{\psi} = -\Delta\boldsymbol{u} + \mathbf{grad}\,q \qquad \text{and} \qquad \mathrm{div}\,\boldsymbol{u} = 0 \ \text{ in}\,\Omega, \tag{3.61}$$

with

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|q\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)}.$$

Then $\boldsymbol{u}$ and $\chi = q + \chi_0$ satisfy the announced properties. □

We next consider the following result.

Theorem 3.6.2. *For any $\boldsymbol{f}$ in the dual space of $\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)$ with $\operatorname{div}\boldsymbol{f} = 0$ in Ω and satisfies*

$$_{\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)'}\langle \boldsymbol{f},\ \boldsymbol{v}\rangle_{\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \tag{3.62}$$

there exists a vector potential $\boldsymbol{\xi}$ in $\boldsymbol{L}^p(\Omega)$ such that

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}, \qquad \text{with} \qquad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in}\,\Omega \quad \text{and} \quad \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\Gamma, \tag{3.63}$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)'}. \tag{3.64}$$

Proof. Let $\boldsymbol{f}$ be in the dual space of $\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)$ and satisfies (3.62). According to Theorem 1.0.6, there exist $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{\xi}_0 \in \boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi}_0$, where $\operatorname{div}\boldsymbol{\xi}_0 = 0$ in Ω and $\boldsymbol{\xi}_0\cdot\boldsymbol{n} = 0$ on Γ and we have the following estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{\xi}_0\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)'}$$

holds. By the density of $\mathcal{D}(\Omega)$ in $\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$, we deduce that for all $\boldsymbol{v}$ in $\boldsymbol{K}_N^{p'}(\Omega)$, we have

$$_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)'}\langle \mathbf{curl}\,\boldsymbol{\xi}_0\,,\ \boldsymbol{v}\rangle_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)} = 0.$$

Since $\mathbf{grad}\,q_i^N$ belongs to $\boldsymbol{K}_N^{p'}(\Omega)$, it follows that

$$_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)'}\langle \boldsymbol{f}\ ,\ \mathbf{grad}\,q_i^N\rangle_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)} = 0 \qquad \text{for all } i = 1,...,I.$$

Then, the function $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ satisfies the relation $\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\ 1\rangle_{\Gamma_i} = 0$ for all $1 \leq i \leq I$. According to Theorem 3.2.5, there exists a vector potential $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{\varphi}$, with $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ, and such that

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)}.$$

Hence, the vector field $\boldsymbol{\xi} = \boldsymbol{\xi_0} + \boldsymbol{\varphi}$ satisfy the announced properties. □

Remark 3.6.3. If we assume that Ω has a connected boundary Γ, a distribution $\boldsymbol{f}$ belongs to $\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)'$ such that $\operatorname{div}\boldsymbol{f} = 0$ if and only if there exists a function $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$, such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}$, where $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ, without f satisfies the condition (3.62). Moreover, $\boldsymbol{\xi}$ is unique up to an additive element of $\boldsymbol{K}_T^p(\Omega)$.

For any integer $m \geq 0$, the space

$$\boldsymbol{H}_0^{m,p}(\mathbf{curl},\,\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}_0^{p}(\mathbf{curl},\,\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}_0^{m,p}(\Omega)\}.$$

It can be shown that its dual space $\boldsymbol{H}^{-m,p'}(\mathbf{curl},\,\Omega)$ is characterized by

$$\boldsymbol{H}^{-m,p'}(\mathbf{curl},\,\Omega) = \{\boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi},\ \boldsymbol{\psi} \in \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)',\ \boldsymbol{\xi} \in \boldsymbol{W}^{-m,p'}(\Omega)\}.$$

We can prove that $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{H}_0^{m,p}(\mathbf{curl},\,\Omega)$ and that the following Green formula holds:
for any $\boldsymbol{\xi} \in \boldsymbol{H}^{-m,p'}(\mathbf{curl},\,\Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}_0^{m,p}(\mathbf{curl},\,\Omega)$

$$_{\boldsymbol{H}^{-m,p'}(\mathbf{curl},\,\Omega)}\langle \mathbf{curl}\,\boldsymbol{\xi},\ \boldsymbol{v}\rangle_{\boldsymbol{H}_0^{m,p}(\mathbf{curl},\,\Omega)} + {}_{\boldsymbol{W}^{-m,p'}(\Omega)}\langle\, \boldsymbol{\xi},\ \mathbf{curl}\,\boldsymbol{v}\rangle_{\boldsymbol{W}_0^{m,p}(\Omega)} = 0.$$

Theorem 3.6.4. *For any $\boldsymbol{f}$ in the dual space $\boldsymbol{H}^{-1,p'}(\mathbf{curl}\,,\Omega)$ that satisfies*

$$\operatorname{div}\boldsymbol{f} = 0 \ \text{ in}\,\Omega \quad \text{and} \quad \langle \boldsymbol{f},\ \boldsymbol{v}\rangle = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{K}_N^p(\Omega), \tag{3.65}$$

there exists a vector potential $\boldsymbol{\xi}$ in $\boldsymbol{W}^{-1,p'}(\Omega)$ such that

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}, \qquad \text{with} \qquad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in}\,\Omega, \tag{3.66}$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{-1,p'}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1,p'}(\mathbf{curl}\,,\Omega)}. \tag{3.67}$$

Proof. Let $\boldsymbol{f}$ be in the dual space $\boldsymbol{H}^{-1,p'}(\mathbf{curl}\,,\Omega)$. Then, there exist $\boldsymbol{f}_0 \in \boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)'$ and $\boldsymbol{\xi}_0 \in \boldsymbol{W}^{-1,p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{f}_0 + \mathbf{curl}\,\boldsymbol{\xi}_0$, and satisfying the estimate

$$\|\boldsymbol{f}_0\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)'} + \|\boldsymbol{\xi}_0\|_{\boldsymbol{W}^{-1,p'}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1,p'}(\mathbf{curl}\,,\Omega)}.$$

Since $\boldsymbol{\xi}_0 \in \boldsymbol{W}^{-1,p'}(\Omega)$, there exist $\boldsymbol{\theta}_0 \in \boldsymbol{L}^{p'}(\Omega)$ satisfying $\operatorname{div}\boldsymbol{\theta}_0 = 0$ in Ω, $\boldsymbol{\theta}_0 \cdot \boldsymbol{n} = 0$ on Γ and $\chi \in L^{p'}(\Omega)$ such that $\boldsymbol{\xi}_0 = \mathbf{curl}\,\boldsymbol{\theta}_0 + \mathbf{grad}\,\chi$ and

$$\|\boldsymbol{\theta}_0\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\chi\|_{L^{p'}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\xi}_0\|_{\boldsymbol{W}^{-1,p'}(\Omega)}.$$

Since $\boldsymbol{f}_0 \in \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)'$, then $\boldsymbol{f}_0 = \boldsymbol{\psi}_0 + \mathbf{curl}\,\boldsymbol{\varphi}_0$, with $\boldsymbol{\psi}_0 \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{\varphi}_0 \in \boldsymbol{L}^p(\Omega)$, $\operatorname{div}\boldsymbol{\varphi}_0 = 0$ in Ω, $\boldsymbol{\varphi}_0 \cdot \boldsymbol{n} = 0$ on Γ and:

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{\varphi}_0\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{f}_0\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)'}.$$

Then $\boldsymbol{f} = \boldsymbol{\psi}_0 + \mathbf{curl}\,\boldsymbol{\varphi}_0 + \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\theta}_0 = \boldsymbol{\psi}_0 + \mathbf{curl}\,\boldsymbol{\mu}$, with $\boldsymbol{\mu} = \boldsymbol{\varphi}_0 + \mathbf{curl}\,\boldsymbol{\theta}_0$, $\operatorname{div}\boldsymbol{\mu} = 0$ in Ω. Thanks to the density of $\mathcal{D}(\Omega)$ in $\boldsymbol{H}_0^{1,p}(\mathbf{curl},\,\Omega)$, we infer that

$$\boldsymbol{H}^{-1,p'}(\mathbf{curl},\Omega)\langle \mathbf{curl}\,\boldsymbol{\mu},\,\boldsymbol{v}\rangle_{\boldsymbol{H}_0^{1,p}(\mathbf{curl},\Omega)} = 0, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{K}_N^p(\Omega).$$

Since $\operatorname{div}\boldsymbol{f} = 0$, then $\operatorname{div}\boldsymbol{\psi}_0 = 0$ and therefore the condition $\langle \boldsymbol{\psi}_0 \cdot \boldsymbol{n},\,1\rangle_{\Gamma_i} = 0$ is automatically satisfied for any $i = 1, ..., I$. Then according to Theorem 3.2.5, there exists a vector potential $\boldsymbol{\varphi} \in \boldsymbol{L}^p(\Omega)$ such that

$$\boldsymbol{\psi}_0 = \mathbf{curl}\,\boldsymbol{\varphi}, \quad \operatorname{div}\boldsymbol{\varphi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

and

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)} \le C\|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

Hence, the vector field $\boldsymbol{\xi} = \boldsymbol{\mu} + \boldsymbol{\varphi}$ satisfies the announced properties. □

More generally, we can prove:

Theorem 3.6.5. *Given any integer $m \ge 0$ and any distribution $\boldsymbol{f}$ in the dual sapce $\boldsymbol{H}^{-m,p'}(\mathbf{curl},\,\Omega)$ that satisfies (3.65), there exists a vector potential $\boldsymbol{\xi}$ in $\boldsymbol{W}^{-m,p'}(\Omega)$ such that*

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}, \qquad \text{with} \qquad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in}\,\Omega \quad \text{and} \quad \|\boldsymbol{\xi}\|_{\boldsymbol{W}^{-m,p'}(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{H}^{-m,p'}(\mathbf{curl},\Omega)}.$$

3.7 The Stokes problem in vector potential formulation

We consider the following Stokes problem:

$$\begin{cases} -\nu\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{3.68}$$

where $\boldsymbol{f}$ is a given data in $\boldsymbol{L}^p(\Omega)$, ν is a positive viscosity, $\boldsymbol{u}$ is the velocity and π is the pressure. We know that the problem (3.68) is equivalent to the variational formulation: find $\boldsymbol{u} \in \boldsymbol{V}^p$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{V}^{p'}, \quad \nu\int_\Omega \mathbf{grad}\,\boldsymbol{u}\cdot\mathbf{grad}\,\boldsymbol{v}\,\mathrm{d}\mathbf{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}, \tag{3.69}$$

where $\boldsymbol{V}^p = \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega),\, \operatorname{div}\boldsymbol{v} = 0 \text{ in}\,\Omega\}$.

We will use Theorem 3.2.5 or Theorem 3.3.7 in order to obtain new variational formulations of this problem where the unknown is the vector potential $\boldsymbol{\psi}$. Let begin with the tangential vector potential. We define the space

$$\begin{aligned} \boldsymbol{W}^{\,p}_{T}(\Omega) \;&=\; \{\boldsymbol{\varphi} \in \boldsymbol{X}^{\,p}_{T}(\Omega),\ \Delta\,\boldsymbol{\varphi} \in \boldsymbol{L}^{p}(\Omega),\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi} \in \boldsymbol{L}^{p}(\Omega),\\ &\qquad \langle \boldsymbol{\varphi}\cdot\boldsymbol{n},\ 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J \ \text{ and } \ \mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0}\ \text{on}\,\Gamma\}, \end{aligned}$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,p}(\Omega)_T} = \|\boldsymbol{\varphi}\|_{\boldsymbol{X}^{\,p}(\Omega)} + \|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)}.$$

We consider the problem: find $\boldsymbol{\psi}$ in $\boldsymbol{W}^{\,p}_{T}(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}^{\,p'}_{T}(\Omega), \quad \nu \int_{\Omega} \Delta\,\boldsymbol{\psi}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\mathbf{x} = \int_{\Omega} \boldsymbol{f}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \tag{3.70}$$

By virtue of the continuity of the mapping $\boldsymbol{u} \mapsto \boldsymbol{\psi}$ (see Theorem 3.2.5) from $\boldsymbol{V}^{\,p}$ into $\boldsymbol{W}^{\,p}_{T}(\Omega)$, where $\boldsymbol{\psi}$ is the vector potential defined in (3.16), we can prove easily the following result

Theorem 3.7.1. *If $(\boldsymbol{u},\,\pi)$ is the solution in $\boldsymbol{W}^{1,p}(\Omega)\times L^{p}(\Omega)/\mathbb{R}$ of the Stokes problem (3.68), then vector potential $\boldsymbol{\psi}$ defined in (3.16) is a solution of the problem (3.70).*

In order to study the problem (3.70), we require the following lemma which is proven in [3] for $p = 2$ by the use of the formula:

$$\|\mathbf{grad}\,(\mathrm{div}\,\boldsymbol{\varphi})\|^{2}_{\boldsymbol{L}^{2}(\Omega)} + \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi}\|^{2}_{\boldsymbol{L}^{2}(\Omega)} = \|\Delta\,\boldsymbol{\varphi}\|^{2}_{\boldsymbol{L}^{2}(\Omega)}.$$

For the case $p \neq 2$, we use an other argument.

Lemma 3.7.2. *The mapping: $\boldsymbol{\varphi} \mapsto \|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)}$ is a norm on $\boldsymbol{W}^{p}_{T}(\Omega)$, equivalent to the norm $\|\cdot\|_{\boldsymbol{W}^{p}(\Omega)}$.*

Proof. It is clear that we have only to prove that for any $\boldsymbol{\varphi} \in \boldsymbol{W}^{\,p}_{T}(\Omega)$, we have:

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,p}(\Omega)} \le C\|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)}. \tag{3.71}$$

We can apply Corollary 3.2.4 to $\boldsymbol{\varphi}$, next Corollary 3.3.8 to $\mathbf{curl}\,\boldsymbol{\varphi}$ then we derive

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,p}(\Omega)} \le C(\|\mathrm{div}\,\boldsymbol{\varphi}\|_{L^{p}(\Omega)} + \|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)}).$$

Since $\mathrm{div}\,\boldsymbol{\varphi} \in W^{\,1,p}(\Omega)$ and satisfies $\int_{\Omega}\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = 0$, so the Poincaré inequality implies that

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,p}(\Omega)} \le C(\|\mathbf{grad}\,(\mathrm{div}\,\boldsymbol{\varphi})\|_{\boldsymbol{L}^{p}(\Omega)} + \|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p}(\Omega)}).$$

We set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\varphi}$. Then, $\boldsymbol{z}$ satisfies: $\Delta\,\boldsymbol{z} = \mathbf{curl}\,(\Delta\,\boldsymbol{\varphi}) \in \boldsymbol{W}^{\,-1,p}(\Omega)$ in Ω and $\boldsymbol{z} = \mathbf{0}$ on Γ.

According to [10], the function $\boldsymbol{z}$ belongs to $\boldsymbol{W}_0^{1,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\mathbf{curl}\,(\Delta\,\boldsymbol{\varphi})\|_{\boldsymbol{W}^{-1,p}(\Omega)}.$$

Especially, we have

$$\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.$$

By using the identity (1.1), we can easily derive the same inequality for $\mathbf{gard}\,(\operatorname{div}\boldsymbol{\varphi})$ which gives (3.71) and finishes the proof. □

The problem (3.70) is solved when $p = 2$ (see [3]), where the well-posedness follows immediately from Lax-Milgram's Lemma. For $p > 2$, we will apply Theorem 3.2.5 and a uniqueness result in order to prove the existence of solution of the problem (3.70). And once the problem is solved for all $p > 2$, a duality argument will also solve it for $p < 2$. We have this first result

Theorem 3.7.3. *For $p > 2$, the problem (3.70) has at most one solution.*

Proof. We suppose that $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$ are two solutions of the problem (3.70) and we set $\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$. Then, $\boldsymbol{\psi}$ belongs to $\boldsymbol{W}_T^2(\Omega)$ and satisfies

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}_T^{p'}(\Omega), \qquad \int_\Omega \Delta\,\boldsymbol{\psi} \cdot \Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = 0.$$

So, as $p' < 2$ then

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}_T^2(\Omega), \qquad \int_\Omega \Delta\,\boldsymbol{\psi} \cdot \Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = 0.$$

By choosing $\boldsymbol{\varphi} = \boldsymbol{\psi}$ and applying Lemma 3.7.2 for $p = 2$, we deduce that $\boldsymbol{\psi} = \mathbf{0}$. □

Theorem 3.7.4. *Let $p > 2$. For any $\boldsymbol{f}$ in $\boldsymbol{L}^p(\Omega)$, problem (3.70) has a unique solution in $\boldsymbol{W}_T^p(\Omega)$.*

Proof. We known that the Stokes problem (3.68) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. By applying Theorem 3.7.1, the unique vector potential associated with $\boldsymbol{u}$ and defined in (3.16) is a solution of (3.70). Since due to Theorem 3.7.3 this problem has at most one solution, it is the only solution. □

As a consequence, we have the following result

Corollary 3.7.5. *The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}_T^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{\psi} \in \boldsymbol{W}_T^{p}(\Omega) \\ \boldsymbol{\psi} \neq 0}} \frac{\int_\Omega \Delta\,\boldsymbol{\psi} \cdot \Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\psi}\|_{\boldsymbol{W}^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{p'}(\Omega)}} \geq \beta. \tag{3.72}$$

Proof. Let a be the bilinear form defined on $\boldsymbol{W}_T^{\,p}(\Omega) \times \boldsymbol{W}_T^{\,p'}(\Omega)$ by:

$$a(\boldsymbol{\psi},\, \boldsymbol{\varphi}) = \int_\Omega \Delta\, \boldsymbol{\psi} \cdot \Delta\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}.$$

Let $A \in \mathcal{L}(\boldsymbol{W}_T^{\,p}(\Omega),\, (\boldsymbol{W}_T^{\,p'}(\Omega))')$ and $A' \in \mathcal{L}(\boldsymbol{W}_T^{\,p'}(\Omega),\, (\boldsymbol{W}_T^{\,p}(\Omega))')$ be the operators defined by

$$\forall \boldsymbol{\psi} \in \boldsymbol{W}_T^{\,p}(\Omega),\ \forall \boldsymbol{\varphi} \in \boldsymbol{W}_T^{\,p'}(\Omega), \quad a(\boldsymbol{\psi},\, \boldsymbol{\varphi}) = \langle A\, \boldsymbol{\psi},\, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\psi},\, A'\, \boldsymbol{\varphi}\rangle$$

The Babuška-Brezzi's Theorem implies that, for $p > 2$, the operator A associated with a is an isomorphism from $\boldsymbol{W}_T^{\,p}(\Omega)$ onto $(\boldsymbol{W}_T^{\,p'}(\Omega))'$ and its dual operator A' is an isomorphism from $\boldsymbol{W}_T^{\,p'}(\Omega)$ onto $(\boldsymbol{W}_T^{\,p}(\Omega))'$. But since the bilinear form a is symmetric, the operator A' coincides with A if we interchange p by p'. As $p' < 2$, this means that the above isomorphisms are valid for all real numbers $p > 1$. So, the Babuška-Brezzi's Theorem implies that the Inf-Sup Condition (6.13) holds for all $p > 1$. The next theorem collects these results. □

Theorem 3.7.6. *Let $1 < p < \infty$. For any data $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, the problem* (3.70) *has a unique solution in $\boldsymbol{W}_T^p(\Omega)$.*

To illustrate the relation that exists between the solution of the Stokes problem and the solution of the problem (3.70), let us remark that $\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}$ is the solution of problem (3.69) and we can prove the following corollary.

Corollary 3.7.7. *The solution $\boldsymbol{\psi}$ of problem* (3.70) *is divergence-free. Moreover, there exists π in $L^p(\Omega)/\mathbb{R}$ such that the pair $(\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi},\, \pi)$ is a solution of the Stokes problem* (3.68).

In a manner analogous to one previously used, but more complicated, we will construct a new equivalent problem where the unknown is the normal vector potential. We cannot apply directly Theorem 3.3.7 to $\boldsymbol{u}$ because the necessary condition $\langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$ for $1 \le j \le J$ is not always satisfied. So we introduce the function

$$\boldsymbol{\delta} = \sum_{j=1}^{J} \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T,$$

and since $\langle (\boldsymbol{u} - \boldsymbol{\delta}) \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$ for $1 \le j \le J$, we apply Theorem 3.3.7 to the function $\boldsymbol{u} - \boldsymbol{\delta}$. Then, there exists a vector potential $\boldsymbol{\psi}$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} + \boldsymbol{\delta} \quad \text{in}\, \Omega \qquad \text{and} \qquad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in}\, \Omega, \\ &\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on}\, \Gamma, \qquad\qquad \langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0 \quad \text{for any } 0 \le \mathrm{i} \le \mathrm{I}. \end{aligned} \tag{3.73}$$

Now, we introduce the space

$$\begin{aligned} \boldsymbol{Z}_N^p(\Omega)) &= \{\boldsymbol{\varphi} \in \boldsymbol{X}_N^p(\Omega),\, \Delta\,\boldsymbol{\varphi} \in \boldsymbol{L}^p(\Omega),\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi} \in \boldsymbol{L}^p(\Omega), \\ &\quad \langle \boldsymbol{\varphi}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0,\ 0 \le \mathrm{i} \le \mathrm{I}\}, \end{aligned}$$

provided with the same previous norm $\|\cdot\|_{\boldsymbol{W}^p(\Omega)}$. We will need also the space

$$\boldsymbol{W}_N^p(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{Z}_N^p(\Omega),\ \exists \boldsymbol{\delta} \in \boldsymbol{K}_T^p(\Omega) \text{ such that } (\mathbf{curl}\,\boldsymbol{\varphi} + \boldsymbol{\delta}) \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}.$$

We remark that $\mathbf{curl}\,\boldsymbol{\varphi} + \boldsymbol{\delta}$ belongs to $\boldsymbol{K}_T^p(\Omega)$ with null tangential traces on Γ. Then, it is zero and there exists at most one function $\boldsymbol{\delta}(\boldsymbol{\varphi}) \in \boldsymbol{K}_T^p(\Omega)$ such that $(\mathbf{curl}\,\boldsymbol{\varphi} + \boldsymbol{\delta}(\boldsymbol{\varphi})) \times \boldsymbol{n} = \mathbf{0}$ on Γ.

We consider the problem : find $\boldsymbol{\psi} \in \boldsymbol{W}_N^p(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}_N^{p'}(\Omega), \quad \nu \int_\Omega \Delta\,\boldsymbol{\psi} \cdot \Delta\,\boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot (\mathbf{curl}\,\boldsymbol{\varphi} + \boldsymbol{\delta}(\boldsymbol{\varphi}))\ \mathrm{d}\boldsymbol{x}. \tag{3.74}$$

We can easily prove the following result:

Theorem 3.7.8. *If $(\boldsymbol{u}, \pi)$ is the solution in $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ of the stokes problem (3.68), the vector potential $\boldsymbol{\psi}$ introduced in (3.73) is a solution of problem (3.74).*

By using the same techniques of proof introduced for the case of tangential vector potential, we can prove easily the two following results.

Theorem 3.7.9. *The mapping: $\boldsymbol{\varphi} \mapsto \|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}$ is a norm on $\boldsymbol{W}_N^p(\Omega)$, equivalent to the norm $\|\cdot\|_{\boldsymbol{W}^p(\Omega)}$ and for any $\boldsymbol{f}$ in $\boldsymbol{L}^p(\Omega)$, the problem (3.74) has a unique solution in $\boldsymbol{W}_N^p(\Omega)$.*

Corollary 3.7.10. *The solution $\boldsymbol{\psi}$ of problem (3.74) is divergence-free. Moreover, there exists π in $L^p(\Omega)/\mathbb{R}$ such that the pair $(\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} + \boldsymbol{\delta}(\boldsymbol{\psi}))$ is a solution of the Stokes problem (3.68).*

Part II

The Stokes equations and elliptic systems with non standard boundary conditions

Chapter 4

The Stokes equations with the tangential boundary conditions

In this chapter, Ω is supposed of class $\mathcal{C}^{1,1}$ except in some cases where we will precise that the boundary can be more regular.

4.1 Introduction and preliminaries

In this section we will study the Stokes problem:

$$(\mathcal{S}_T)\quad \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in}\,\Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0, \quad 1\le j\le J. \end{cases}$$

4.2 Weak solutions

The aim of this subsection is to give a variational formulation of problem $(\mathcal{S}_T)$ and to prove the existence and the uniqueness of weaks solutions. Let us consider the space

$$\boldsymbol{E}^{\,p}(\Omega) = \{\boldsymbol{v}\in\ \boldsymbol{W}^{\,1,p}(\Omega),\, \Delta\,\boldsymbol{v}\in[\boldsymbol{H}_0^{\,p'}(\operatorname{div},\,\Omega)]'\}.$$

This is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{E}^{\,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{\,1,p}(\Omega)} + \|\Delta\,\boldsymbol{v}\|_{[\boldsymbol{H}_0^{\,p'}(\operatorname{div},\,\Omega)]'}.$$

We have the following preliminary result.

Lemma 4.2.1. *Assume that Ω is of class $\mathcal{C}^{0,1}$. The space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}^{p}(\Omega)$.*

Proof. Let P is a continuous linear mapping from $\boldsymbol{W}^{1,p}(\Omega)$ to $\boldsymbol{W}^{1,p}(\mathbb{R}^3)$, such that $P\boldsymbol{v}|_\Omega = \boldsymbol{v}$. Then, for every continuous linear form $\boldsymbol{\ell} \in (\boldsymbol{E}^p(\Omega))'$, there exists a pair $(\boldsymbol{f},\, \boldsymbol{g}) \in \boldsymbol{W}^{-1,p'}(\mathbb{R}^3) \times \boldsymbol{H}_0^{p'}(\mathrm{div},\, \Omega)$ such that

$$\langle \boldsymbol{\ell},\, \boldsymbol{v}\rangle = \langle \boldsymbol{f},\, P\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,p'}(\mathbb{R}^3)\times \boldsymbol{W}^{1,p}(\mathbb{R}^3)} + \langle \Delta\, \boldsymbol{v},\, \boldsymbol{g}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)},$$

for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$. Thanks to the Hahn-Banach theorem, it suffices to show that any $\boldsymbol{\ell} = 0$ on $\mathcal{D}(\overline{\Omega})$ is actually zero on $\boldsymbol{E}^p(\Omega)$. Let's suppose that $\boldsymbol{\ell} = 0$ on $\mathcal{D}(\overline{\Omega})$ and let $\widetilde{\boldsymbol{g}} \in \boldsymbol{H}^{p'}(\mathrm{div},\, \mathbb{R}^3)$ be the extension by $\boldsymbol{0}$ of $\boldsymbol{g}$ to $\mathbb{R}^3$. Then, for any $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3)$ we have

$$\langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,p'}(\mathbb{R}^3)\times \boldsymbol{W}^{1,p}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \Delta\, \boldsymbol{\varphi}\cdot \widetilde{\boldsymbol{g}}\, \mathrm{d}\boldsymbol{x} = 0,$$

since $\langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{f},\, P\boldsymbol{v}\rangle$ where $\boldsymbol{v} = \boldsymbol{\varphi}|_\Omega$. It follows that

$$\boldsymbol{f} + \Delta\, \widetilde{\boldsymbol{g}} = \boldsymbol{0} \quad \text{in } \mathbb{R}^3.$$

So, $\widetilde{\boldsymbol{g}}$ belongs to $\boldsymbol{W}^{1,p'}(\mathbb{R}^3)$, since $\boldsymbol{f} \in \boldsymbol{W}^{-1,p'}(\mathbb{R}^3)$. As a consequance $\boldsymbol{g}$ belongs to $\boldsymbol{W}_0^{1,p'}(\Omega)$. Then, by the density of $\mathcal{D}(\Omega)$ in $\boldsymbol{W}_0^{1,p'}(\Omega)$, there exists a sequence $(\boldsymbol{g}_k) \in \mathcal{D}(\Omega)$ such that $\boldsymbol{g}_k \to \boldsymbol{g}$ in $\boldsymbol{W}^{1,p'}(\Omega)$. Thus, for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$, we have

$$\begin{aligned}
\langle \boldsymbol{\ell},\, \boldsymbol{v}\rangle &= \lim_{k\to\infty} \{-\langle \Delta\, \widetilde{\boldsymbol{g}}_k,\, P\boldsymbol{v}\rangle + \langle \Delta\, \boldsymbol{v},\, \boldsymbol{g}_k\rangle\} \\
&= \lim_{k\to\infty} \{\int_\Omega \boldsymbol{v}\cdot \Delta\, \boldsymbol{g}_k\, \mathrm{d}\boldsymbol{x} + \langle \Delta\, \boldsymbol{v},\, \boldsymbol{g}_k\rangle\} \\
&= 0.
\end{aligned}$$

Thus, $\boldsymbol{\ell}$ is identically zero. □

As a consequence, we have the following result.

Corollary 4.2.2. *The linear mapping $\gamma : \boldsymbol{v} \to \mathbf{curl}\, \boldsymbol{v}|_\Gamma \times \boldsymbol{n}$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and any $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{p'}(\Omega)$ with $\mathrm{div}\, \boldsymbol{\varphi} = 0$ *in* Ω *,*

$$-\langle \Delta\, \boldsymbol{v},\, \boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} = \int_\Omega \mathbf{curl}\, \boldsymbol{v}\cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \langle \mathbf{curl}\, \boldsymbol{v}\times\, \boldsymbol{n},\, \boldsymbol{\varphi}\rangle_\Gamma, \tag{4.1}$$

where the duality on Γ *is defined by* $\langle \cdot,\cdot\rangle_\Gamma = \langle \cdot,\cdot\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$.

Proof. Let $\boldsymbol{v} \in \mathcal{D}(\overline{\Omega})$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{p'}(\Omega)$, then formula (4.1) obviously holds. For every $\boldsymbol{\mu} \in \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$, there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ with

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C\|\boldsymbol{\mu}_t\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \leq C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}.$$

Consequently,

$$\left|\langle \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n},\, \boldsymbol{\mu}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}\right| \leq C\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)}\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}.$$

Thus,

$$\|\mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)}.$$

We can deduce that the linear mapping γ is continuous for the norm of $\boldsymbol{E}^p(\Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}^p(\Omega)$, γ can be extended by continuity to $\gamma \in \mathcal{L}(\boldsymbol{E}^p(\Omega);\, \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma))$ and formula (4.1) holds for all $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{p'}(\Omega)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω. □

Proposition 4.2.3. *Let $\boldsymbol{f}$ belongs to $\boldsymbol{L}^p(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω , $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ verify the following compatibility conditions: for any $\boldsymbol{v} \in \boldsymbol{K}_T^{p'}(\Omega)$,*

$$\int_\Omega \boldsymbol{f}\cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{h}\times \boldsymbol{n},\, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} = 0, \tag{4.2}$$

$$\int_\Gamma g\,\mathrm{d}\boldsymbol{\sigma} = 0, \tag{4.3}$$

$$\boldsymbol{f}\cdot \boldsymbol{n} + \operatorname{div}_\Gamma(\boldsymbol{h}\times \boldsymbol{n}) = 0 \quad \text{on } \Gamma, \tag{4.4}$$

where $\operatorname{div}_\Gamma$ is the surface divergence on Γ. Then, the problem

$$(E_T) \qquad \begin{cases} -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{cases}$$

has a unique solution $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big\{\|\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\,g\,\|_{W^{1-1/p,p}(\Gamma)} + \|\,\boldsymbol{h}\times\boldsymbol{n}\,\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big\}. \tag{4.5}$$

Moreover, if $g \in W^{2-1/p,p}(\Gamma)$, $\boldsymbol{h} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ and Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\Big\{\|\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\,g\,\|_{W^{2-1/p,p}(\Gamma)} + \|\,\boldsymbol{h}\times\boldsymbol{n}\,\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\Big\}. \tag{4.6}$$

Proof. We suppose that $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω , $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and we prove that problem (E_T) has a unique solution $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate (4.5).

Step 1: Uniqueness. We begin to prove uniqueness of weak solution of (E_T). We suppose that $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfies: $-\Delta\,\boldsymbol{\xi} = 0$, $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi} \cdot \boldsymbol{n} = 0$, $\mathbf{curl}\,\boldsymbol{\xi} \times \boldsymbol{n} = \mathbf{0}$ on Γ with the condition $\langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$. The function $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{\xi}$ belongs to $\boldsymbol{L}^p(\Omega)$ and satisfies:

$$\operatorname{div} \boldsymbol{w} = 0, \;\; \mathbf{curl}\,\boldsymbol{w} = \mathbf{0} \text{ in}\,\Omega \;\text{ and }\; \boldsymbol{w} \times \boldsymbol{n} = \mathbf{0} \;\text{ on}\,\Gamma.$$

This implies that $\boldsymbol{w} \in \boldsymbol{K}^p_N(\Omega)$. Due to Corollary 2.7, we can write:

$$\boldsymbol{w} = \sum_{i=1}^{I} \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N.$$

So,

$$\int_\Omega |\,\boldsymbol{w}|^2 \,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^{I} \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} \int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{grad}\, q_i^N \,\mathrm{d}\boldsymbol{x} = 0.$$

Hence, $\boldsymbol{w}$ is zero and $\boldsymbol{\xi}$ belongs to $\boldsymbol{K}^p_T(\Omega)$. Thanks to Corollary 2.4 , we can write $\boldsymbol{\xi}$ as:

$$\boldsymbol{\xi} = \sum_{j=1}^{J} \langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^T = 0,$$

and the proof of the uniqueness is completed.

Step 2: Compatibility conditions. Before solving the problem (E_T), we show that the conditions (4.4) and (4.3) are necessary. First, we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$, with $\boldsymbol{\xi}$ solution of the problem (E_T). It is clear that

$$\forall \varphi \in W^{2,p'}(\Omega), \quad \langle \mathbf{curl}\,\boldsymbol{z} \cdot \boldsymbol{n},\, \varphi\rangle_\Gamma = \langle \boldsymbol{z} \times \boldsymbol{n},\, \nabla\,\varphi\rangle_\Gamma,$$

where the bracket denote the duality $W^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)$. So that $\boldsymbol{f}$ must satisfy:

$$\langle \boldsymbol{f} \cdot \boldsymbol{n},\, \varphi\rangle_\Gamma = \langle \boldsymbol{h} \times \boldsymbol{n},\, \nabla\varphi\rangle_\Gamma = -\langle \operatorname{div}_\Gamma (\boldsymbol{h} \times \boldsymbol{n}),\, \varphi\rangle_\Gamma.$$

This shows that $\boldsymbol{f} \cdot \boldsymbol{n} + \operatorname{div}_\Gamma (\boldsymbol{h} \times \boldsymbol{n}) = 0$ in the sense of $W^{-1-\frac{1}{p},p}(\Gamma)$. We deduce in fact that $\operatorname{div}_\Gamma \boldsymbol{h}_t$ belongs to $W^{-\frac{1}{p},p}(\Gamma)$ and the above equation occurs to the sense of the last space. Next, the fact that $\int_\Gamma g \,\mathrm{d}\boldsymbol{\sigma} = 0$ is due to $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω. Before the justification of the compatibility condition (4.2), we consider the Neumann problem:

$$(\mathcal{N}) \qquad \Delta\,\theta = 0 \;\text{ in}\,\Omega \quad \text{and} \quad \frac{\partial\,\theta}{\partial\,\boldsymbol{n}} = g \;\text{ on}\,\Gamma.$$

Since $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and satisfies (4.3), this problem has a unique solution $\theta \in W^{2,p}(\Omega)$ satisfying the estimate:

$$\|\,\theta\,\|_{W^{2,p}(\Omega)} \leq C \|\,g\,\|_{W^{1-1/p,p}(\Gamma)}. \tag{4.7}$$

We set $\boldsymbol{w} = \boldsymbol{\xi} - \nabla\,\theta$ and $\boldsymbol{z} = \boldsymbol{w} - \sum_{j=1}^{J} \langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T$. Then, problem (E_T) becomes:

$$\begin{cases} -\Delta\,\boldsymbol{z} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = 0 & \text{in}\,\Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0, \quad 1\le j\le J. \end{cases} \tag{4.8}$$

It is easy to prove that every solution of (4.8) also solves: find $\boldsymbol{z}\in \boldsymbol{V}_T^{\,p}(\Omega)$ such that

$$\forall \boldsymbol{\varphi}\in \boldsymbol{V}_T^{\,p'}(\Omega) \qquad \begin{aligned} \int_\Omega \mathbf{curl}\,\boldsymbol{z}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \\ &- \langle\, \boldsymbol{h}\times\boldsymbol{n},\, \boldsymbol{\varphi}\,\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \end{aligned} \tag{4.9}$$

We are now able to justify the compatibility condition (4.2). It comes from the variational formulation (4.9) by taking $\boldsymbol{\varphi}$ in $\boldsymbol{K}_T^{\,p'}(\Omega)$.

Step 3: Existence. We next prove existence of weak solutions of (E_T). We know from Lemma 3.3.2, that the problem (4.9) satisfies the Inf-Sup condition (3.19). So, It has a unique solution $\boldsymbol{z}\in \boldsymbol{V}_T^{\,p}(\Omega)$ since the right-hand side belongs to $(\boldsymbol{V}_T^{\,p'}(\Omega))'$. We known due to Theorem 1.16 that this solution $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{\,1,p}(\Omega)$. Next, we want to extend (4.9) to any test function $\widetilde{\boldsymbol{\varphi}}$ in $\boldsymbol{X}_T^{\,p'}(\Omega)$. We consider the solution χ in $W^{\,1,p'}(\Omega)$ up to an additive constant of the Neumann problem:

$$\Delta\,\chi = \operatorname{div}\widetilde{\boldsymbol{\varphi}} \ \text{ in}\,\Omega \quad \text{and} \quad \frac{\partial\,\chi}{\partial\,\boldsymbol{n}} = 0 \ \text{ on}\,\Gamma. \tag{4.10}$$

Then, we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi - \sum_{j=1}^{J} \langle (\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi)\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T. \tag{4.11}$$

Observe that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{V}_T^{\,p'}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}$. Moreover, using the compatibility conditions (4.2) and (4.4), we obtain:

$$\forall \chi\in W^{\,1,p'}(\Omega), \quad \int_\Omega \boldsymbol{f}\cdot\nabla\,\chi\,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{h}\times\boldsymbol{n},\, \nabla\chi\,\rangle_\Gamma = \langle \boldsymbol{f}\cdot\boldsymbol{n} + \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}),\, \chi\rangle_\Gamma = 0,$$

and for $1\le j\le J$,

$$\int_\Omega \boldsymbol{f}\cdot\widetilde{\mathbf{grad}}\, q_j^T\,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{h}\times\boldsymbol{n},\, \widetilde{\mathbf{grad}}\, q_j^T\,\rangle_\Gamma = 0.$$

So, the problem (4.9) becomes: find $\boldsymbol{z}\in \boldsymbol{V}_T^{\,p}(\Omega)$ such that

$$\forall \widetilde{\boldsymbol{\varphi}}\in \boldsymbol{X}_T^{\,p'}(\Omega) \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{z}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} - \langle\, \boldsymbol{h}\times\boldsymbol{n},\, \widetilde{\boldsymbol{\varphi}}\,\rangle_\Gamma. \tag{4.12}$$

It is easy to prove that every solution of (4.8) also solves (4.12). Conversely, let $\boldsymbol{z}$ a solution of the problem (4.12). Then,

$$\forall \widetilde{\boldsymbol{\varphi}}\in \boldsymbol{\mathcal{D}}(\Omega), \quad \langle \mathbf{curl}(\mathbf{curl}\,\boldsymbol{z} - \boldsymbol{f}),\, \widetilde{\boldsymbol{\varphi}}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

So $-\Delta\, \boldsymbol{z}$ is equal to $\boldsymbol{f}$ in Ω. Moreover, by the fact that $\boldsymbol{z}$ belongs to the space $\boldsymbol{V}^p_T(\Omega)$ we have $\operatorname{div} \boldsymbol{z} = 0$ in Ω, $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J$. Then, it remains to verify the boundary condition $\mathbf{curl}\, \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ. We multiply the equation $-\Delta\, \boldsymbol{z} = \boldsymbol{f}$ in Ω by $\boldsymbol{\varphi} \in \boldsymbol{X}^{p'}_T(\Omega)$, we integrate on Ω and we compare with (4.12). Consequently, for any $\boldsymbol{\varphi} \in \boldsymbol{X}^{p'}_T(\Omega)$ we obtain:

$$\langle\, \mathbf{curl}\, \boldsymbol{z}\times\boldsymbol{n},\, \boldsymbol{\varphi}\,\rangle_\Gamma = \langle\, \boldsymbol{h}\times\boldsymbol{n},\, \boldsymbol{\varphi}\,\rangle_\Gamma.$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{W}^{\,1-\frac{1}{p'},p'}(\Gamma)$. So, there exists an element $\boldsymbol{\varphi}$ of $\boldsymbol{W}^{\,1,p'}(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ, where $\boldsymbol{\mu}_t$ is the tangential component of $\boldsymbol{\mu}$ on Γ. It is clear that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{X}^{p'}_T(\Omega)$ and

$$\begin{aligned}\langle \mathbf{curl}\, \boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma - \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma &= \langle \mathbf{curl}\, \boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\mu}_t\rangle_\Gamma - \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\mu}_t\rangle_\Gamma \\ &= \langle \mathbf{curl}\, \boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma - \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma = 0.\end{aligned}$$

This implies that $\mathbf{curl}\, \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ.
We deduce that $\boldsymbol{\xi} = \boldsymbol{z} + \nabla\,\theta \in \boldsymbol{W}^{\,1,p}(\Omega)$ is a solution of the problem (E_T).

Step 4: Estimate. In order to prove the estimate (4.5), let us consider the operator $B \in \mathcal{L}(\boldsymbol{V}^p_T(\Omega), (\boldsymbol{V}^{p'}_T(\Omega))')$ defined by

$$\forall \boldsymbol{\lambda} \in \boldsymbol{V}^p_T(\Omega), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}^{p'}_T(\Omega), \quad \langle B\,\boldsymbol{\lambda},\, \boldsymbol{\varphi}\rangle = \int_\Omega \mathbf{curl}\, \boldsymbol{\lambda}\cdot\mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}. \tag{4.13}$$

Thanks to the Inf-Sup condition (3.19), the operator B is an isomorphism from $\boldsymbol{V}^p_T(\Omega)$ onto $(\boldsymbol{V}^{p'}_T(\Omega))'$, and for any $\boldsymbol{\lambda} \in \boldsymbol{V}^p_T(\Omega)$,

$$\|\boldsymbol{\lambda}\|_{\boldsymbol{X}^p_T(\Omega)} \backsimeq \|B\,\boldsymbol{\lambda}\|_{(\boldsymbol{V}^{p'}_T(\Omega))'}. \tag{4.14}$$

Hence, for $\boldsymbol{z} \in \boldsymbol{V}^p_T(\Omega)$ solution of problem (4.9) we have:

$$\|B\,\boldsymbol{z}\|_{(\boldsymbol{V}^{p'}_T(\Omega))'} = \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_T(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\langle B\,\boldsymbol{z},\, \boldsymbol{\varphi}\rangle}{\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^{p'}_T(\Omega)}} = \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_T(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{h}\times\boldsymbol{n},\, \boldsymbol{\varphi}\rangle_\Gamma}{\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^{p'}_T(\Omega)}}.$$

Then,

$$\|B\,\boldsymbol{z}\|_{(\boldsymbol{V}^{p'}_T(\Omega))'} \le C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{\,-1/p,p}(\Gamma)}\Big).$$

Using (4.14) and the continuous embedding of $\boldsymbol{X}^{\,p}_T(\Omega)$ in $\boldsymbol{W}^{\,1,p}(\Omega)$ we obtain:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{\,1,p}(\Omega)} \le C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{\,-1/p,p}(\Gamma)}\Big). \tag{4.15}$$

The estimate (4.5) follows directly from (4.7) and (4.15).

Step 5: Regularity. Let $\boldsymbol{\xi} \in \boldsymbol{W}^{\,1,p}(\Omega)$ the solution of (E_T). We suppose now that $g \in W^{\,2-1/p,p}(\Gamma)$, $\boldsymbol{h} \in \boldsymbol{W}^{\,1-1/p,p}(\Gamma)$, Ω is of class $\mathcal{C}^{\,2,1}$ and we prove that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{\,2,p}(\Omega)$. We set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{\xi}$. It is clear that $\boldsymbol{z} \in \boldsymbol{X}^{\,1,p}_N(\Omega)$. It follows from Theorem 3.3.5 that $\boldsymbol{z} \in \boldsymbol{W}^{\,1,p}(\Omega)$. As a consequence, $\boldsymbol{\xi}$ belongs to $\boldsymbol{X}^{\,2,p}(\Omega)$. By Corollary 2.2.12, we have $\boldsymbol{\xi} \in \boldsymbol{W}^{\,2,p}(\Omega)$.

□

We now can solve the Stokes problem $(\mathcal{S}_T)$.

Theorem 4.2.4. (**Weak solutions for** $(\mathcal{S}_T)$) *Let* $\boldsymbol{f}$, g, $\boldsymbol{h}$ *with:*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega))',\quad g \in W^{1-\frac{1}{p},p}(\Gamma),\quad \boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \tag{4.16}$$

and verify the compatibility conditions (4.2)-(4.3). *Then, the Stokes problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega))'} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \tag{4.17}$$

Proof. **Step 1: Uniqueness.** We begin to prove uniqueness of weak solution of $(\mathcal{S}_T)$. We suppose that $\boldsymbol{f} = \boldsymbol{0}$ in Ω, $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ on Γ. Since $\Delta\,\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ, we find that π satisfies: $\Delta\,\pi = 0$ in Ω and $\dfrac{\partial\,\pi}{\partial\,\boldsymbol{n}} = \boldsymbol{0}$ on Γ. We deduce that $\pi =$ constant in Ω and this implies the uniqueness, up to an additive constant, of the pressure. So, we are reduced to a problem of type of (E_T) with the unknown $\boldsymbol{u}$. Using the same uniqueness argument in Proposition 5.2.4, we can prove that $\boldsymbol{u} = \boldsymbol{0}$ in Ω.

Step 2: Compatibility conditions.

We note that condition (4.3) is necessary, because $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω. We will justify the necessity of condition (4.2) below. Let $\theta \in W^{2,p}(\Omega)$ the unique solution of the Neumann problem $(\mathcal{N})$. We set $\boldsymbol{w} = \boldsymbol{u} - \nabla\,\theta$ and $\boldsymbol{z} = \boldsymbol{w} - \sum_{j=1}^{J} \langle \boldsymbol{w}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\,\widetilde{\mathbf{grad}}\,q_j^T$. Then, problem $(\mathcal{S}_T)$ becomes

$$\begin{cases} -\Delta\,\boldsymbol{z} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{z} = 0 & \text{in}\,\Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\quad 1 \leq j \leq J. \end{cases} \tag{4.18}$$

It is easy to prove that every solution of (4.18) also solves (4.9) where we replace the integral $\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}$ by the brackets $\langle\cdot,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)}$, which justifies the necessity of the compatibility condition (4.2) by taking $\boldsymbol{\varphi} \in \boldsymbol{K}_T^{p'}(\Omega)$.

Step 3: Existence. We next prove existence of weak solutions of $(\mathcal{S}_T)$. We know that problem (4.9) has a unique solution $\boldsymbol{z} \in \boldsymbol{V}_T^{\,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\{\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)]'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\}. \tag{4.19}$$

We set

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \sum_{j=1}^{J} \langle \boldsymbol{\varphi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\,\widetilde{\mathbf{grad}}\,q_j^T,$$

where $\boldsymbol{\varphi} \in \boldsymbol{X}_T^{p'}(\Omega)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω. It is clear that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{V}_T^{p'}(\Omega)$. Using the compatibility condition (4.2), there exists a unique $\boldsymbol{z} \in \boldsymbol{V}_T^{p}(\Omega)$ such that for any $\boldsymbol{\varphi} \in \boldsymbol{X}_T^{p'}(\Omega)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω:

$$\begin{aligned} & \int_\Omega \mathbf{curl}\, \boldsymbol{z} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} \\ = \; & \langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_{(\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))' \times \boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} - \langle\, \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{\varphi} \,\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}. \end{aligned} \tag{4.20}$$

Let us take $\boldsymbol{\varphi} \in \mathcal{D}_\sigma(\Omega)$ as the test function in (4.20). Then we get

$$\langle -\Delta\, \boldsymbol{z} - \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_{(\mathcal{D}(\Omega))' \times \mathcal{D}(\Omega)} = 0, \quad \text{for any } \boldsymbol{\varphi} \in \mathcal{D}_\sigma(\Omega).$$

By De Rham theorem, there exist a function $\pi \in L^p(\Omega)$ such that

$$-\Delta\, \boldsymbol{z} + \nabla\, \pi = \boldsymbol{f} \quad \text{in } \Omega.$$

Moreover, by the fact that $\boldsymbol{z}$ belongs to the space $\boldsymbol{V}_T^{p}(\Omega)$, we have $\operatorname{div} \boldsymbol{z} = 0$ in Ω, $\boldsymbol{z} \cdot \boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0,\ 1 \le j \le J$. The remainder boundary condition $\mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}$ on Γ is implicitly contained in (4.20). Observe that since $\boldsymbol{f}$ and $\nabla\, \pi$ are two elements of $[\boldsymbol{H}_0^{p'}(\operatorname{div},\, \Omega)]'$, it is the same for $\Delta\, \boldsymbol{z}$. Otherwise, $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{p'}(\operatorname{div},\, \Omega)$, it is clear then that for any $\boldsymbol{\varphi} \in \boldsymbol{H}_0^{p'}(\operatorname{div},\, \Omega)$:

$$\langle \nabla\, \pi,\, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} = 0.$$

Moreover, if $\boldsymbol{\varphi} \in \boldsymbol{X}_T^{p'}(\Omega)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω, using (4.1) we have

$$\begin{aligned} \langle -\Delta\, \boldsymbol{z},\, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} &= \int_\Omega \mathbf{curl}\, \boldsymbol{z} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \\ &+ \langle \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n},\, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}. \end{aligned}$$

We deduce that for all $\boldsymbol{\varphi} \in \boldsymbol{X}_T^{p'}(\Omega)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$

$$\langle \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n},\, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} = \langle\, \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{\varphi} \,\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}.$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$. So, there exists an element $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ. It is clear that $\boldsymbol{\varphi} \in \boldsymbol{X}_T^{p'}(\Omega)$ and

$$\begin{aligned} \langle \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n},\, \boldsymbol{\mu} \rangle_\Gamma - \langle \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{\mu} \rangle_\Gamma &= \langle \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n},\, \boldsymbol{\mu}_t \rangle_\Gamma - \langle \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{\mu}_t \rangle_\Gamma \\ &= \langle \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n},\, \boldsymbol{\varphi} \rangle_\Gamma - \langle \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{\varphi} \rangle_\Gamma = 0. \end{aligned}$$

This implies that $\mathbf{cur}\, \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}$ on Γ. As a consequence, the pair $(\boldsymbol{z},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)/\mathbb{R}$ is the unique solution to the problem (4.18). Finally $(\boldsymbol{u} = \boldsymbol{z} + \nabla\, \theta,\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ is the unique solution of the problem $(\mathcal{S}_T)$. □

Remark 4.2.5. Observe that if we suppose in Theorem 4.2.4 that $\boldsymbol{f} \in L^p(\Omega)$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω and we add the compatibility conditions (4.4), then the pressure π is constant. Indeed, from the first equation in the Stokes problem $(\mathcal{S}_T)$, we obtain that $\Delta\,\pi = 0$ in Ω and $\dfrac{\partial\,\pi}{\partial \boldsymbol{n}} = \boldsymbol{f}\cdot\boldsymbol{n} + \operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) = 0$ on Γ. This implies that π is a constant and we are reduced to solve the problem

$$\begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{f} & \text{in}\,\Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in}\,\Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ \ 1\le j\le J. \end{cases} \tag{4.21}$$

where the well-posedeness of this problem is given by Proposition 4.2.3.

We can also solve the Stokes problem when the divergence operator does not vanish and it is a given function. So, we focus on the study of the problem:

$$\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div}\boldsymbol{u} = \chi & \text{in}\,\Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, \quad 1\le j\le J. \end{cases} \tag{4.22}$$

Corollary 4.2.6. *For every $\boldsymbol{f}$, χ, g, $\boldsymbol{h}$ with*

$$\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]',\ \ \chi\in L^p(\Omega),\ \ g\in W^{1-1/p,p}(\Gamma),\ \ \boldsymbol{h}\in\boldsymbol{W}^{-1/p,p}(\Gamma),$$

with the compatibility condition (4.2) *and*

$$\int_\Omega \chi\,\mathrm{d}\boldsymbol{x} = \langle g,\,1\rangle_\Gamma, \tag{4.23}$$

the Stokes problem (4.22) *has exactly one solution $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ and $\pi\in L^p(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C>0$ depending only on p and Ω such that:*

$$\begin{aligned} \|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \ &\le\ C\Big(\|\,\boldsymbol{f}\,\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} + \|\chi\|_{L^p(\Omega)} + \|\,g\,\|_{W^{1-1/p,p}(\Gamma)} + \\ &+\ \|\,\boldsymbol{h}\times\boldsymbol{n}\,\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big). \end{aligned} \tag{4.24}$$

Proof. We consider the Neumann problem:

$$\Delta\,\theta = \chi \ \ \text{in}\ \Omega, \qquad \frac{\partial\,\theta}{\partial\,\boldsymbol{n}} = g \ \ \text{on}\ \Gamma,$$

which has a unique solution $\theta\in W^{2,p}(\Omega)/\mathbb{R}$ and verifies the estimate:

$$\|\theta\|_{W^{2,p}(\Omega)/\mathbb{R}} \le C\Big(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)}\Big). \tag{4.25}$$

Set $\boldsymbol{u}_0 = \nabla\theta$. By Theorem 4.2.4, there exists a unique $(\boldsymbol{z},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of problem:

$$-\Delta\, \boldsymbol{z} + \nabla\, \pi = \boldsymbol{f} + \nabla\, \chi \text{ and } \operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega, \quad \boldsymbol{z}\cdot\boldsymbol{n} = 0 \text{ and } \mathbf{curl}\, \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \text{ on } \Gamma,$$

satisfying the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq \big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} + \|\chi\|$$

where $\nabla\,\chi \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\,\Omega)]'$ and satisfies $\langle \nabla\,\chi,\, \widetilde{\mathbf{grad}}\, q_j^T\rangle_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)} = 0$. Finally, the pair of functions $(\boldsymbol{u},\, \pi) = (\boldsymbol{z} + \boldsymbol{u}_0,\, \pi)$ is the required solution. □

4.3 Strong solutions and regularity for the Stokes system $(\mathcal{S}_T)$

Theorem 4.3.1. (**Strong solutions for** $(\mathcal{S}_\mathrm{T})$) *Assume that* Ω *is of class* $\mathcal{C}^{2,1}$. *Let* $\boldsymbol{f}$, g $\boldsymbol{h}$ *such that:*

$$\boldsymbol{f}\in \boldsymbol{L}^p(\Omega), \quad g\in W^{2-1/p,p}(\Gamma) \quad \text{and} \quad \boldsymbol{h}\in \boldsymbol{W}^{1-1/p,p}(\Gamma) \tag{4.27}$$

and satisfying the compatibility conditions (4.2)-(4.3). *Then, the solution* $(\boldsymbol{u},\, \pi)$ *of problem* $(\mathcal{S}_T)$ *given by Theorem 4.2.4 belongs to* $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ *and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big). \tag{4.28}$$

Proof. Before, we note that under the hypothesis of Theorem 4.3.1, the data $\boldsymbol{f}$, g ans $\boldsymbol{h}$ satisfy also the hypothesis of Theorem 4.2.4. So, this implies that problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)/\mathbb{R}$. Moreover, since π satisfies:

$$\operatorname{div}(\nabla\,\pi - \boldsymbol{f}) = 0 \quad \text{in}\,\Omega, \qquad (\nabla\,\pi - \boldsymbol{f})\cdot\boldsymbol{n} = -\operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n}) \quad \text{on}\,\Gamma. \tag{4.29}$$

Then, $\pi \in W^{1,p}(\Omega)$. Now, we set $\boldsymbol{F} = \nabla\,\pi - \boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$. So, problem $(\mathcal{S}_T)$ becomes: $-\Delta\,\boldsymbol{u} = \boldsymbol{F}$ and $\operatorname{div}\boldsymbol{u} = 0$ in Ω, $\boldsymbol{u}\cdot\boldsymbol{n} = g$ and $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ and $\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$. Therefore, $\boldsymbol{F}$, g and $\boldsymbol{h}$ satisfy the assumptions of Proposition 4.2.3. Then, $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. □

Remark 4.3.2. Note that in Theorem 4.3.1, to prove the regularity of the velocity, we have used the result of Proposition 4.2.3. We can establish this regularity result by using other argument. Let us explain. We set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Then, $\boldsymbol{z}$ satisfies:

$$\boldsymbol{z}\in \boldsymbol{L}^p(\Omega),\ \operatorname{div}\boldsymbol{z} = 0\ \text{in}\,\Omega,\ \ \mathbf{curl}\,\boldsymbol{z}\in \boldsymbol{L}^p(\Omega),\ \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}\ \text{on}\,\Gamma.$$

Using Theorem 3.3.5, the function $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. As a consequence, thanks to Corollary 2.2.12, the solution $\boldsymbol{u}$ of the problem $(\mathcal{S}_T)$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$.

Corollary 4.3.3. *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)]'$, $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ on Γ with $\boldsymbol{f}$ satisfying the compatibility condition*

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{K}_N^{p'}(\Omega), \quad \langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)} = 0. \tag{4.30}$$

Then, the solution $(\boldsymbol{u},\, \pi)$ of problem $(\mathcal{S}_T)$ given by Theorem 4.2.4 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times L^p(\Omega)$.

Proof. Let $\boldsymbol{f}$ in the dual space of $\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)$. We know (see Proposition 1.2) that there exist $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\chi_0 \in L^p(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{\psi} + \nabla\, \chi_0 \quad \text{and} \quad \|\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi_0\|_{L^p(\Omega)} \leq C \| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega))'}.$$

Since $\langle \nabla\, \chi_0,\, \widetilde{\mathbf{grad}}\, q_j^T \rangle_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega))' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)} = 0$ and $\boldsymbol{f}$ satisfy the compatibility condition (4.30), it is the same for $\boldsymbol{\psi}$. Thanks to Theorem 4.3.1, there exist $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\theta \in W^{1,p}(\Omega)$ satisfying

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \theta = \boldsymbol{\psi} & \text{in}\, \Omega, \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0,\ \ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \ \text{ for any } 1 \leq j \leq J, \end{cases}$$

with

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\theta\|_{W^{1,p}(\Omega)} \leq C \| \boldsymbol{\psi} \|_{\boldsymbol{L}^p(\Omega)}.$$

Then, $\boldsymbol{u}$ and $\pi = \theta + \chi_0$ satisfy the announced properties. □

Using the same arguments in the proof of Corollary 4.2.6, we have the following result concerning strong solution of the Stokes problem (4.22), where the divergence operator does not vanish.

Corollary 4.3.4. *For every $\boldsymbol{f}$, χ, g, $\boldsymbol{h}$ with*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \ \ \chi \in W^{1,p}(\Omega), \ \ g \in W^{2-1/p,p}(\Gamma), \ \ \boldsymbol{h} \in \boldsymbol{W}^{1-1/p,p}(\Gamma),$$

with the compatibility condition (4.2) and (4.23), the solution $(\boldsymbol{u},\, \pi)$ of the Stokes problem (4.22) given by Corollary 4.2.6 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:

$$\begin{aligned} \| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \ &\leq \ C\Big(\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \| g \|_{W^{2-1/p,p}(\Gamma)} + \\ &+ \ \| \boldsymbol{h} \times \boldsymbol{n} \|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} \Big). \end{aligned} \tag{4.31}$$

In the general case we have the following theorem.

Theorem 4.3.5. *Let $m \geq 0$ be an integer. Suppose that Ω is of class $\mathcal{C}^{m+1,1}$ and $\boldsymbol{f}$, g, $\boldsymbol{h}$ with*

$$\boldsymbol{f} \in \boldsymbol{W}^{m,p}(\Omega), \;\; g \in W^{m+2-1/p,p}(\Gamma), \;\; \boldsymbol{h} \in \boldsymbol{W}^{m+1-1/p,p}(\Gamma)$$

satisfying the compatibility condition (4.2)-(4.3). Then, the solution $(\boldsymbol{u}, \pi)$ of the problem $(\mathcal{S}_T)$ given by Theorem 4.3.1 belongs to $\boldsymbol{W}^{m+2,p}(\Omega) \times W^{m+1,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{m+2,p}(\Omega)} + \|\pi\|_{W^{m+1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|g\|_{W^{m+2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{m+1-\frac{1}{p},p}(\Gamma)}\big). \tag{4.32}$$

4.4 Very weak solutions for the Stokes system $(\mathcal{S}_T)$

We are going to study the existence result of very weak solutions for the Stokes problem $(\mathcal{S}_T)$. In the sequel, we will use the following space:

$$\boldsymbol{T}^p(\Omega) = \Big\{\boldsymbol{\varphi} \in \boldsymbol{H}_0^p(\mathrm{div},\, \Omega);\;\; \mathrm{div}\, \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\Big\}.$$

Lemma 4.4.1. *The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and for all $\chi \in W^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{T}^{p'}(\Omega)$, we have:*

$$\langle \nabla\, \chi,\, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} = -\langle \chi,\, \mathrm{div}\, \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)}. \tag{4.33}$$

Proof. In a first step, we consider that Ω is strictely star-shaped with respect to one of its points which is taken as the origin. Then, for any $\boldsymbol{\varphi} \in \boldsymbol{T}^p(\Omega)$, we take $\widetilde{\boldsymbol{\varphi}}$ its extension by zero to $\mathbb{R}^3$. Thus, $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{H}^p(\mathrm{div},\, \mathbb{R}^3)$ and as $\mathrm{div}\, \widetilde{\boldsymbol{\varphi}} = \widetilde{\mathrm{div}\, \boldsymbol{\varphi}} \in W_0^{1,p}(\mathbb{R}^3)$, then $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{T}^p(\mathbb{R}^3)$. For $\theta < 1$, we define the functions:

$$\widetilde{\boldsymbol{\varphi}}_\theta(\boldsymbol{x}) = \widetilde{\boldsymbol{\varphi}}(\frac{\boldsymbol{x}}{\theta}), \quad \text{for a.e } \boldsymbol{x} \in \mathbb{R}^3.$$

Since $\mathrm{supp}\, \widetilde{\boldsymbol{\varphi}}_\theta \subset \theta\overline{\Omega} \subset \Omega$, the function $\widetilde{\boldsymbol{\varphi}}_\theta$ has a compact support in Ω. Moreover, $\widetilde{\boldsymbol{\varphi}}_\theta \in \boldsymbol{T}^p(\mathbb{R}^3)$ and

$$\lim_{\theta \to 1} \widetilde{\boldsymbol{\varphi}}_\theta = \widetilde{\boldsymbol{\varphi}} \;\; \text{in} \;\; \boldsymbol{T}^p(\mathbb{R}^3).$$

The result is then proved by regularization. Let $\rho \in \mathcal{D}(\mathbb{R}^3)$, be a smooth $\mathcal{C}^\infty$ function with compact support, such that $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho(x)\, \mathrm{d}x = 1$. For $\varepsilon > 0$, let ρ_ε denote the function $x \longmapsto (\frac{1}{\varepsilon^3})\rho(\frac{x}{\varepsilon})$. As $\varepsilon \to 0$, ρ_ε converges in the distribution sence to the Dirac distribution. As a consequence, $\rho_\epsilon * \widetilde{\boldsymbol{\varphi}}_\theta|_\Omega$ belongs to $\boldsymbol{\mathcal{D}}(\Omega)$ and

$$\lim_{\epsilon \to 0} \lim_{\theta \to 1} \rho_\epsilon * \widetilde{\boldsymbol{\varphi}}_\theta = \widetilde{\boldsymbol{\varphi}} \;\; \text{in} \;\; \boldsymbol{T}^p(\mathbb{R}^3).$$

The result follows because $\boldsymbol{\varphi}$ is the limit in $\boldsymbol{T}^{\,p}(\Omega)$ of the restriction of the functions $\rho_\epsilon * \widetilde{\boldsymbol{\varphi}}_\theta$ to Ω. In the case where Ω is not star-shaped, we have to recover Ω which is Lipschitz by a finite number of star open sets. Finally the relation (4.33) is a simple consequence of this density. □

Lemma 4.4.2. *A distribution $\boldsymbol{f}$ belongs to $(\boldsymbol{T}^p(\Omega))'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $f_0 \in W^{-1,p'}(\Omega)$, such that*

$$\boldsymbol{f} = \boldsymbol{\psi} + \nabla f_0.$$

Moreover, we have the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|f_0\|_{W^{-1,p'}(\Omega)} \leq C \|\boldsymbol{f}\|_{(\boldsymbol{T}^{\,p}(\Omega))'}. \tag{4.34}$$

Proof. **i**) For any function $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $f_0 \in W^{-1,p'}(\Omega)$, we have

$$\forall \boldsymbol{v} \in \mathcal{D}(\Omega), \quad \langle \boldsymbol{\psi} + \nabla\, f_0,\, \boldsymbol{v} \rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \langle f_0,\, \mathrm{div}\, \boldsymbol{v} \rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)}.$$

The linear form in the right-hand side of the above relation is continuous for the norm of $\boldsymbol{T}^{\,p}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{T}^{\,p}(\Omega)$, we deduce by Hahn-Banach Theorem that $\boldsymbol{\psi} + \nabla\, f_0$ is an element of $\boldsymbol{T}^{\,p}(\Omega)'$.

ii) Conversely, we set $\boldsymbol{E} = \boldsymbol{H}_0^{\,p}(\mathrm{div},\, \Omega) \times W_0^{1,p}(\Omega)$ equipped whith the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{E}} = \|\boldsymbol{v}\|_{\boldsymbol{H}_0^{\,p}(\mathrm{div},\,\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{W_0^{1,p}(\Omega)}.$$

The operator $T : \boldsymbol{v} \to (\boldsymbol{v}, \mathrm{div}\boldsymbol{v})$ is continuous from $\boldsymbol{T}^p(\Omega)$ onto $\boldsymbol{E}$. Its range $\mathcal{R}(T)$ is a closed subspace of $\boldsymbol{E}$ and T is an isomorphism from $\boldsymbol{T}^{\,p}(\Omega)$ onto $\mathcal{R}(T)$ and is an isometry. Hence, for each $\boldsymbol{f} \in \boldsymbol{T}^{\,p}(\Omega)'$, we associate the element $\boldsymbol{f}^* \in (\mathcal{R}(T))'$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{T}^{\,p}(\Omega), \qquad \langle\, \boldsymbol{f},\, \boldsymbol{v}\, \rangle_{\boldsymbol{T}^{\,p}(\Omega)'\times\boldsymbol{T}^{\,p}(\Omega)} = \langle\, \boldsymbol{f}^*,\, T\boldsymbol{v}\, \rangle_{(\mathcal{R}(T))'\times\mathcal{R}(T)}.$$

Note that $\|\,\boldsymbol{f}\,\|_{\boldsymbol{T}^{\,p}(\Omega)'} = \|\boldsymbol{f}^*\|_{\mathcal{R}(T)'}$ because T is an isometry. By the extension theorem of Hahn-Banach, $\boldsymbol{f}^*$ can be extended in $[\boldsymbol{H}_0^{\,p}(\mathrm{div},\, \Omega)]' \times W^{-1,p'}(\Omega)$ to an element called $(\boldsymbol{h}, g)$ with $\|\boldsymbol{h}\|_{[\boldsymbol{H}_0^{\,p}(\mathrm{div},\Omega)]'} + \|g\|_{W^{-1,p'}(\Omega)} = \|\boldsymbol{f}^*\|_{\mathcal{R}(T)'}$. We deduce that there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\chi \in L^{p\prime}(\Omega)$ such that:

$$\begin{aligned} \forall \boldsymbol{v} \in \boldsymbol{T}^p(\Omega), \quad \langle\, \boldsymbol{f},\, \boldsymbol{v}\, \rangle_{\boldsymbol{T}^{\,p}(\Omega)'\times\boldsymbol{T}^{\,p}(\Omega)} &= \int_\Omega \boldsymbol{\psi} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \langle g,\, \mathrm{div}\, \boldsymbol{v} \rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} + \\ &+ \langle \nabla\, \chi,\, \boldsymbol{v} \rangle_{[\boldsymbol{H}_0^{\,p}(\mathrm{div},\,\Omega)]'\times\boldsymbol{H}_0^{\,p}(\mathrm{div},\Omega)}. \end{aligned}$$

So, $\boldsymbol{f}$ is equal to $\boldsymbol{\psi} + \nabla f_0$ in Ω where $f_0 = g + \chi$ and satisfies the estimate (4.34). □

We shall use the space

$$\boldsymbol{H}_p(\Delta;\,\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\,\Delta\,\boldsymbol{v}\in(\boldsymbol{T}^{p'}(\Omega))'\},$$

equipped with the topology given by the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^{\,p}(\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\Delta\,\boldsymbol{v}\|_{(\boldsymbol{T}^{p'}(\Omega))'}.$$

The following lemma will help us to prove a trace result.

Lemma 4.4.3. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}_p(\Delta;\,\Omega)$.*

Proof. Let $\boldsymbol{\ell}\in(\boldsymbol{H}_p(\Delta;\,\Omega))'$ such that for any $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have $\langle\boldsymbol{\ell},\,\boldsymbol{v}\rangle=0$. We want to prove that $\boldsymbol{\ell}=\boldsymbol{0}$. Using the Riesz's representation lemma, there exists $(\boldsymbol{u},\,\boldsymbol{z})\in\boldsymbol{L}^{p'}(\Omega)\times\boldsymbol{T}^{p'}(\Omega)$ such that: for any $\boldsymbol{v}\in\boldsymbol{H}_p(\Delta;\,\Omega)$,

$$\langle\boldsymbol{\ell},\,\boldsymbol{v}\rangle=\int_\Omega\boldsymbol{u}\cdot\boldsymbol{v}+\langle\Delta\,\boldsymbol{v},\,\boldsymbol{z}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}.\tag{4.35}$$

Observe that we can easily extend by zero the functions $\boldsymbol{u}$ and $\boldsymbol{z}$ in such a way that $\widetilde{\boldsymbol{u}}\in\boldsymbol{L}^{p'}(\mathbb{R}^3)$ and $\widetilde{\boldsymbol{z}}\in\boldsymbol{T}^{p'}(\mathbb{R}^3)$ (i.e. $\widetilde{\boldsymbol{z}}\in\boldsymbol{H}^{p'}(\mathrm{div},\,\mathbb{R}^3)$ and $\mathrm{div}\,\widetilde{\boldsymbol{z}}\in W^{1,p'}(\mathbb{R}^3)$). Now, we take $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$. Then, we have by definition that:

$$\int_{\mathbb{R}^3}\widetilde{\boldsymbol{u}}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}^3}\widetilde{\boldsymbol{z}}\cdot\,\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=\langle\boldsymbol{\ell},\,\boldsymbol{\varphi}|_\Omega\rangle=0,\tag{4.36}$$

and thus $\widetilde{\boldsymbol{u}}+\Delta\,\widetilde{\boldsymbol{z}}=0$ in $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$. Using that $\widetilde{\boldsymbol{z}}\in\boldsymbol{L}^{p'}(\mathbb{R}^3)$ and $\Delta\,\widetilde{\boldsymbol{z}}=-\widetilde{\boldsymbol{u}}\in\boldsymbol{L}^{p'}(\mathbb{R}^3)$, we conclude that $\widetilde{\boldsymbol{z}}\in\boldsymbol{W}^{2,p'}(\mathbb{R}^3)$ and therefore $\boldsymbol{z}\in\boldsymbol{W}_0^{2,p'}(\Omega)$. As $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{W}_0^{2,p'}(\Omega)$, there exists a sequence $(\boldsymbol{z}_k)_k\subset\boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{z}_k\to\boldsymbol{z}$ in $\boldsymbol{W}^{2,p'}(\Omega)$, when $k\to\infty$. In particular, $\boldsymbol{z}_k\to\boldsymbol{z}$ in $\boldsymbol{T}^{p'}(\Omega)$. Finally, we consider $\boldsymbol{v}\in\boldsymbol{H}_p(\Delta;\,\Omega)$ and we have to prove that $\langle\boldsymbol{\ell},\,\boldsymbol{v}\rangle=0$. Observe that:

$$\begin{aligned}\langle\boldsymbol{\ell},\,\boldsymbol{v}\rangle&=-\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}+\langle\Delta\,\boldsymbol{v},\,\boldsymbol{z}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}&(4.37)\\&=\lim_{k\to\infty}\Big(-\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{z}_k\,\mathrm{d}\boldsymbol{x}+\langle\Delta\,\boldsymbol{v},\,\boldsymbol{z}_k\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}\Big)&(4.38)\\&=\lim_{k\to\infty}\Big(-\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{z}_k+\int_\Omega\boldsymbol{v}\cdot\Delta\,\boldsymbol{z}_k\,\mathrm{d}\boldsymbol{x}\Big)=0.&(4.39)\end{aligned}$$

Therefore, $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}_p(\Delta;\,\Omega)$. □

We define the space:

$$\boldsymbol{Y}_T^p(\Omega)=\Big\{\boldsymbol{\varphi}\in\boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\varphi}\cdot\boldsymbol{n}=0,\ \mathrm{div}\,\boldsymbol{\varphi}=0,\ \mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n}=\boldsymbol{0}\ \text{ on}\,\Gamma\Big\}.$$

Let us introduce some notations. For any vector field $\boldsymbol{v}$ on Γ, we shall denote by v_n the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{n}$, while we shall denote by $\boldsymbol{v}_t$ the projection of $\boldsymbol{v}$ on the tangent hyperplane to Γ. In other words

$$v_n = \boldsymbol{v}\cdot\boldsymbol{n} \qquad \boldsymbol{v}_t = \boldsymbol{v} - v_n\boldsymbol{n}.$$

Let us now consider any point P on Γ and choose an open neighbourhood W of P in Γ small enough to allow the existence of two families of C^2 curves on W. The lengths s_1, s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves, respectively. With this notations, we have $\boldsymbol{v}_t = \sum_{k=1}^2 v_k\boldsymbol{\tau}_k$, where $v_j = \boldsymbol{v}\cdot\boldsymbol{\tau}_j$. Consequently for any $\boldsymbol{v}\in \boldsymbol{W}^{2,p'}(\Omega)$, we have the following formulas:

$$\operatorname{div}\boldsymbol{v} = \operatorname{div}_\Gamma \boldsymbol{v}_t - 2K\,\boldsymbol{v}\cdot\boldsymbol{n} + \frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} \qquad \text{on}\,\Gamma, \tag{4.40}$$

$$\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} = \sum_{k=1}^2 v_k\,\frac{\partial\boldsymbol{\tau}_k}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} + \frac{\partial v_n}{\partial\boldsymbol{n}} \qquad \text{on}\,\Gamma, \tag{4.41}$$

$$\mathbf{curl}\,\boldsymbol{v} = \sum_{j=1}^2 \frac{\partial\boldsymbol{v}}{\partial s_j}\times\boldsymbol{\tau}_j + \frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\times\boldsymbol{n} \qquad \text{on}\,\Gamma, \tag{4.42}$$

where K denotes the mean curvature of Γ, $\operatorname{div}_\Gamma$ is the surface divergence.

The following lemma proves that the tangential trace of the **curl** of function $\boldsymbol{v}$ of $\boldsymbol{H}^{\,p}(\Omega)$ belongs to $\boldsymbol{W}^{\,-1-\frac{1}{p},p}(\Gamma)$.

Lemma 4.4.4. *The mapping γ : $\boldsymbol{u}\mapsto \mathbf{curl}\,\boldsymbol{u}|_\Gamma\times\boldsymbol{n}$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping still denoted by γ, from $\boldsymbol{H}_p(\Delta;\Omega)$ into $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and we have the following Green formula: for any $\boldsymbol{u}\in\boldsymbol{H}_p(\Delta;\Omega)$ and $\boldsymbol{\varphi}\in \boldsymbol{Y}_T^{p'}(\Omega)$,*

$$\langle\Delta\,\boldsymbol{u},\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma, \tag{4.43}$$

where the dualiy on Γ is given by $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}$.

Proof. Let $\boldsymbol{u}\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$, then formula (4.43) is valid for any $\boldsymbol{\varphi}\in\boldsymbol{Y}_T^{p'}(\Omega)$. Let $\boldsymbol{\mu}\in\boldsymbol{W}^{1+1/p,p'}(\Gamma)$. Then, there exists a function $\boldsymbol{\varphi}\in\boldsymbol{W}^{2,p'}(\Omega)$ such that: $\boldsymbol{\varphi}=\boldsymbol{\mu}_t$ on Γ. Moreover, using (4.40), we must choose $\boldsymbol{\varphi}$ such that $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} = -\operatorname{div}_\Gamma\boldsymbol{\mu}_t$ on Γ, in order to obtain $\operatorname{div}\boldsymbol{\varphi}=0$ on Γ. We can also fix $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\times\boldsymbol{n}$ so that we have $\mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n}=\mathbf{0}$ on Γ. For this, using (4.42) and the fact that $(\boldsymbol{z}\times\boldsymbol{n})\times\boldsymbol{n} = -\boldsymbol{z}_t$ for any vector field $\boldsymbol{z}$, we must choose $\big(\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\big)_t = \sum_{j=1}^2\big(\dfrac{\partial\boldsymbol{\mu_t}}{\partial s_j}\times\boldsymbol{\tau}_j\big)\times\boldsymbol{n}$. As a consequence, $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}} = -\boldsymbol{n}\operatorname{div}_\Gamma\boldsymbol{\mu}_t + \sum_{j=1}^2\big(\dfrac{\partial\boldsymbol{\mu}_t}{\partial s_j}\times\boldsymbol{\tau}_j\big)\times\boldsymbol{n}$ and $\boldsymbol{\varphi}$ satisfies the two conditions:

$$\operatorname{div}\boldsymbol{\varphi}=0 \ \text{ and } \ \mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n}=\mathbf{0} \ \text{ on}\,\Gamma.$$

Let us summarizes. The function $\boldsymbol{\varphi}$ belongs to $\boldsymbol{Y}_T^{p'}(\Omega)$ and satisfies:

$$\begin{cases} \boldsymbol{\varphi} = \boldsymbol{\mu}_t & \text{on}\,\Gamma, \\ \dfrac{\partial\,\boldsymbol{\varphi}}{\partial\,\boldsymbol{n}} = -\boldsymbol{n}\,\mathrm{div}_\Gamma\,\boldsymbol{\mu}_t + \displaystyle\sum_{j=1}^{2}\big(\frac{\partial\,\boldsymbol{\mu}_t}{\partial\,s_j}\times\boldsymbol{\tau}_j\big)\times\boldsymbol{n} & \text{on}\,\Gamma, \end{cases} \tag{4.44}$$

such that

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)} \le C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\,1+1/p,p'}(\Gamma)}. \tag{4.45}$$

Consequently,

$$\begin{aligned} \Big|\langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\mu}\rangle_\Gamma\Big| &= \Big|\langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\mu}_t\rangle_\Gamma\Big| = \Big|\langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma\Big| \\ &\le \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)} + \|\Delta\,\boldsymbol{u}\|_{(\boldsymbol{T}^{\,p'}(\Omega))'}\|\boldsymbol{\varphi}\|_{\boldsymbol{T}^{\,p'}(\Omega)} \\ &\le \|\boldsymbol{u}\|_{\boldsymbol{H}_{\,p}(\Delta;\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)}. \end{aligned}$$

Thus, using (4.45), we obtain for any $\boldsymbol{u}\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$:

$$\|\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\|_{\boldsymbol{W}^{\,-1-\frac{1}{p},p}(\Gamma)} \le C\|\boldsymbol{u}\|_{\boldsymbol{H}_{\,p}(\Delta;\Omega)}.$$

Therefore, the linear continuous mapping $\gamma:\ \boldsymbol{u}\mapsto\mathbf{curl}\,\boldsymbol{u}|_\Gamma\times\boldsymbol{n}$ defined on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{H}_{\,p}(\Delta;\,\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}_{\,p}(\Delta;\,\Omega)$, then we can extend continuously this mapping from $\boldsymbol{H}_{\,p}(\Delta;\,\Omega)$ into $\boldsymbol{W}^{\,-1-\frac{1}{p},p}(\Gamma)$ and we have the Green formula (4.43). □

Theorem 4.4.5. (**Very weak solutions for** $(\mathcal{S}_\mathrm{T})$) *Assume that* Ω *is of class* $\mathcal{C}^{\,2,1}$. *Let* $\boldsymbol{f}$, χ, g *and* $\boldsymbol{h}$ *with:*

$$\boldsymbol{f}\in(\boldsymbol{T}^{p'}(\Omega))',\ \chi\in L^p(\Omega),\ g\in W^{-1/p,p}(\Gamma),\ \boldsymbol{h}\in\boldsymbol{W}^{-1-1/p,p}(\Gamma),$$

and satisfying the compatibility conditions (4.2) *and* (4.23). *Then, the Stokes problem* (4.22) *has exactly one solution* $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ *and* $\pi\in W^{\,-1,p}(\Omega)/\mathbb{R}$. *Moreover, there exists a constant* $C>0$ *depending only on* p *and* Ω *such that:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W^{\,-1,p}(\Omega)/\mathbb{R}} &\le C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{\,-1/p,p}(\Gamma)} + \\ &+ \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\Big). \end{aligned} \tag{4.46}$$

Proof. We proceed in three steps.

i) **First step**: Thanks to the Green formula (4.43), it is easy to verify that $(\boldsymbol{u},\,\pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is solution of problem (4.22) without the last condition, is equivalent to the variational formulation: find $(\boldsymbol{u},\,\pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ such that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_T^{p'}(\Omega)$, and for any $q \in W^{1,p'}(\Omega)$

$$\begin{aligned} -\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} &- \langle\,\pi,\,\mathrm{div}\,\boldsymbol{\varphi}\,\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} = \langle\,\boldsymbol{f},\,\boldsymbol{\varphi}\,\rangle_\Omega + \langle\,\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\,\rangle_\Gamma \\ \int_\Omega \boldsymbol{u}\cdot\nabla\,q\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega \chi\,q\,\mathrm{d}\boldsymbol{x} + \langle\,g,\,q\,\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}, \end{aligned} \tag{4.47}$$

where the dualities on Ω and Γ are defined by:

$$\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\,\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} \qquad \langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\,\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}.$$

Indeed, let $(\boldsymbol{u},\,\pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ be a solution to (4.47). It is clear that:

$$-\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{ and } \quad \mathrm{div}\,\boldsymbol{u} = \chi \text{ in } \Omega.$$

Consequently $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta;\,\Omega)$, because the characterization given by Lemma 4.4.2 implies that $\nabla\,\pi \in (\boldsymbol{T}^{p'}(\Omega))'$. Using Lemma 4.4.1 and Lemma 4.4.4, we obtain for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_T^{p'}(\Omega)$:

$$-\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma - \langle\pi,\,\mathrm{div}\,\boldsymbol{\varphi}\rangle_{W^{-1,p}(\Omega)\times W_0^{1,p'}(\Omega)} = \langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega.$$

Then, by (4.47) we deduce that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_T^{p'}(\Omega)$,

$$\langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma = \langle\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma.$$

Let $\boldsymbol{\mu} \in \boldsymbol{W}^{1+1/p,p'}(\Gamma)$. As in the proof of Lemma 4.4.4, there exists a function $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}(\Omega)$ satisfying (4.44). So, we can write that for any $\boldsymbol{\mu} \in \boldsymbol{W}^{1+1/p,p'}(\Gamma)$,

$$\begin{aligned} \langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\mu}\rangle_\Gamma &= \langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\mu}_t\rangle_\Gamma = \langle\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma = \\ &= \langle\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\mu}_t\rangle_\Gamma = \langle\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\mu}\rangle_\Gamma. \end{aligned}$$

Which implies that $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ. From the equation $\mathrm{div}\,\boldsymbol{u} = \chi$ in Ω, we deduce that for any $q \in W^{1,p'}(\Omega)$, we have

$$\langle\boldsymbol{u}\cdot\boldsymbol{n},\,q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)} = \langle g,\,q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}.$$

Consequently, $\boldsymbol{u}\cdot\boldsymbol{n} = g$ in $W^{-1/p,p}(\Gamma)$. The converse is a simple consequence of Lemma 4.4.1, Lemma 4.4.3 and Lemma 4.4.4.

ii) **Second step** : Let's now solve problem (4.47). We suppose that $g = 0$ on Γ and $\int_\Omega \chi \, \mathrm{d}\boldsymbol{x} = 0$. We know due to Corollary 4.3.4 that for any pair $(\boldsymbol{F}, \xi) \in (\boldsymbol{L}^{p'}(\Omega) \perp \boldsymbol{K}^p_T(\Omega)) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$, there exists a unique $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}(\Omega)$ and $q \in W^{1,p'}(\Omega)/\mathbb{R}$ satisfying:

$$\begin{cases} -\Delta \boldsymbol{\varphi} + \nabla q = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\varphi} = \xi & \text{in } \Omega, \\ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \ \text{ and } \ \mathbf{curl}\, \boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{\varphi} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \ \text{ for any } 1 \le j \le J, \end{cases}$$

with the estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)} + \|q\|_{W^{1,p'}(\Omega)/\mathbb{R}} \le C\{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)}\}. \tag{4.48}$$

Note that for any $K \in \mathbb{R}$,

$$|\int_\Omega \chi \, q \, \mathrm{d}\boldsymbol{x}| = |\int_\Omega \chi \, (q + K) \, \mathrm{d}\boldsymbol{x}\,| \le \|\chi\|_{L^p(\Omega)} \|q\|_{L^{p'}(\Omega)/\mathbb{R}}$$

and

$$|\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega| \le \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} \|\boldsymbol{\varphi}\|_{\boldsymbol{T}^{p'}(\Omega)} \le \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)}.$$

From these bounds, we have

$$\begin{aligned} & \Big| \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma - \int_\Omega \chi \, q \, \mathrm{d}\boldsymbol{x} \Big| \\ \le \ & C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)} + \|\chi\|_{L^p(\Omega)} \Big) \times \Big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)} \Big). \end{aligned}$$

In other words, we can say that the linear mapping:

$$(\boldsymbol{F}, \xi) \mapsto \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma - \int_\Omega \chi \, q \, \mathrm{d}\boldsymbol{x}$$

defines an element of the dual space of $\big(\boldsymbol{L}^{p'}(\Omega) \perp \boldsymbol{K}^p_T(\Omega)\big) \times W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega)$ and according to the Riesz's representation theorem, there exists a unique $(\boldsymbol{u}, \pi) \in (\boldsymbol{L}^p(\Omega)/\boldsymbol{K}^p_T(\Omega)) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of problem (4.47) satisfying the bound (4.46).

iii) **Third step** : Now, we suppose that $g \neq 0$ and the compatibility condition (4.23) holds. We consider the Neumann problem:

$$\Delta \theta = \chi \ \text{ in } \Omega \qquad \text{and} \qquad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \ \text{ on } \Gamma,$$

which has a unique solution $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \le C\big(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} \big). \tag{4.49}$$

Set $\boldsymbol{w} = \nabla\theta$. By step ii), there exists a unique $(\boldsymbol{z}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ solution of problem:

$$\begin{cases} -\Delta\,\boldsymbol{z} + \nabla\,\pi = \boldsymbol{f} + \Delta\,\boldsymbol{w} & \text{in}\,\Omega, \\ \operatorname{div}\boldsymbol{z} = 0 & \text{in}\,\Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \text{ and } \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0, \text{ for any } 1 \le j \le J. \end{cases}$$

Indeed, we have $\Delta\,\boldsymbol{w} = \nabla\,\chi$ and the characterization given by Lemma 4.4.2 implies that $\Delta\,\boldsymbol{w} \in (\boldsymbol{N}^{p'}(\Omega))'$. Moreover, since $\int_\Omega \Delta\,\boldsymbol{w}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = 0$ for any $\boldsymbol{v} \in \boldsymbol{K}^{p'}_T(\Omega)$, then $\boldsymbol{f} + \Delta\,\boldsymbol{w}$ and $\boldsymbol{h}\times\boldsymbol{n}$ satisfies the compatibility condition (4.2). Finally, the pair of functions $(\boldsymbol{u}, \pi) = (\boldsymbol{z} + \boldsymbol{w}, \pi)$ is the required solution. □

□

Corollary 4.4.6. *Assume that Ω is of class $\mathcal{C}^{2,1}$. For $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$, $\boldsymbol{g} \in W^{-1/p,p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-1-1/p,p}(\Gamma)$ satisfying the compatibility condition (4.2) and (4.3), the problem (E_T) has a unique solution $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le C\big(\|g\|_{W^{-1/p,p}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\big). \tag{4.50}$$

Proof. we proceed in two steps.

i) **First step.** We suppose that $g = 0$ on Γ. Thanks to the Green formula (4.43), it is easy to verify that $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ is solution of problem (E_T) without the last condition, is equivalent ot the variational formulation: Find $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ such that

$$\forall\,\boldsymbol{\varphi} \in \boldsymbol{Y}^{p'}_T(\Omega),\ \int_\Omega \boldsymbol{\xi}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} - \langle \boldsymbol{h}\times\boldsymbol{n},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}. \tag{4.51}$$

Let's then solve problem (4.51). We know from Proposition 4.2.3 that for all $\boldsymbol{F} \in \boldsymbol{L}^{p'}(\Omega) \perp \boldsymbol{K}^{p'}_T(\Omega)$, there exists a unique $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}(\Omega)$ satisfying

$$\begin{cases} -\Delta\,\boldsymbol{\varphi} = \boldsymbol{F} \text{ and } \operatorname{div}\boldsymbol{\varphi} = 0 & \text{in } \Omega, \\ \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0, \text{ and } \mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{\varphi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0 \text{ for any } 1 \le j \le J. \end{cases}$$

with the estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)} \le C\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Then, we have:

$$\begin{aligned}\left|\langle \boldsymbol{f},\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}-\langle \boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_{\Gamma}\right| &\leq \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{\varphi}\|_{\boldsymbol{T}^{p'}(\Omega)}+\\ &+ \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1+1/p,p'}(\Gamma)}\\ &\leq \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)}\\ &+ C\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}(\Omega)}+\\ &\leq C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\Big)\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.\end{aligned}$$

In other words, we can say that the linear mapping

$$T:\ \boldsymbol{F}\longmapsto\langle \boldsymbol{f},\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}-\langle \boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)},$$

is continuous on $\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{K}^{p'}_{T}(\Omega)$. According to the Riesz representation theorem, there exists a unique $\boldsymbol{\xi}\in\boldsymbol{L}^{p}(\Omega)/\boldsymbol{K}^{p}_{T}(\Omega)$ such that:

$$\forall\boldsymbol{F}\in\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{K}^{p'}_{T}(\Omega),\quad T(\boldsymbol{F})=\langle\boldsymbol{\xi},\,\boldsymbol{F}\rangle_{\boldsymbol{L}^{p}(\Omega)\times\boldsymbol{L}^{p'}(\Omega)},$$

i.e $\boldsymbol{\xi}$ is solution of (E_T).

ii) **Second step:** Now, we suppose that $g\in W^{-1/p,p}(\Gamma)$ and consider the Neumann problem: Find $\theta\in W^{1,p}(\Omega)/\mathbb{R}$ such that

$$\Delta\,\theta=\lambda\qquad\text{in }\Omega\qquad\text{and}\qquad\frac{\partial\,\theta}{\partial\,\boldsymbol{n}}=g\qquad\text{on }\Gamma,$$

where $\lambda\in\mathbb{R}$ is chosen such that $\lambda\text{mes}\,\Omega+\int_{\Gamma}g\,\mathrm{d}\boldsymbol{\sigma}=0$. We set $\boldsymbol{u}_0=\nabla\,\theta$. By step i), there exist a unique $\boldsymbol{z}\in\boldsymbol{L}^{p}(\Omega)/K^{p}_{T}(\Omega)$ solution of problem:

$$\begin{cases}-\Delta\,\boldsymbol{z}=\boldsymbol{0}\quad\text{and}\quad\text{div}\,\boldsymbol{z}=0 & \text{in }\Omega,\\ \boldsymbol{z}\cdot\boldsymbol{n}=0\quad\text{and}\quad\mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n} & \text{on }\Gamma.\end{cases}$$

Finally, $\boldsymbol{\xi}=\boldsymbol{z}+\boldsymbol{u}_0$ is the required solution.

□

Chapter 5

The Stokes equations with the normal boundary conditions

In this chapter, Ω is supposed of class $\mathcal{C}^{1,1}$ except in some cases where we will precise that the boundary can be more regular.

5.1 Introduction and preliminaries

In this section we will study the Stokes problem:

$$(\mathcal{S}_N)\quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

5.2 Weak solutions

The aim of this subsection is to give a variational formulation of problem $(\mathcal{S}_N)$ and to prove a theorem of existence and uniqueness of weak solutions.

We define the space

$$\boldsymbol{V}_N^p(\Omega) = \{ \boldsymbol{w} \in \boldsymbol{X}_N^p(\Omega);\, \operatorname{div} \boldsymbol{w} = 0 \,\text{in}\, \Omega \text{ and } \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0,\ 1 \leq i \leq I \},$$

which is a Banach space for the norm $\| \cdot \|_{\boldsymbol{X}^p(\Omega)}$.

We begin with the following useful result.

Lemma 5.2.1. *The following Inf-Sup condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}_N^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{\xi} \in \boldsymbol{V}_N^{p}(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_N^p(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}} \geq \beta. \tag{5.1}$$

Proof. The proof is very similar to that of Lemma 3.3.2. Let $\boldsymbol{\varphi}$ any function of $\boldsymbol{V}_N^{p'}(\Omega)$. Due to Corollary 3.3.8, we can write: for any $\boldsymbol{\varphi} \in \boldsymbol{V}_N^{p'}(\Omega)$

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} = C \sup_{\substack{\boldsymbol{g}\in\boldsymbol{L}^p(\Omega)\\ \boldsymbol{g}\neq 0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}}. \tag{5.2}$$

We use now the Helmholtz decomposition $\boldsymbol{g} = \nabla\,\chi + \boldsymbol{z}$, where $\chi \in W^{\,1,p}(\Omega)$ and $\boldsymbol{z}$ belongs to $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ with $\mathrm{div}\,\boldsymbol{z} = 0$ in Ω and $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ on Γ. Moreover, we have the estimate

$$\|\nabla\,\chi\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}.$$

We set

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{j=1}^{J} \langle \boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\,\widetilde{\mathbf{grad}}\,q_j^T,$$

so, $\widetilde{\boldsymbol{z}} \in \boldsymbol{L}^p(\Omega)$, $\mathrm{div}\,\widetilde{\boldsymbol{z}} = 0$ in Ω, $\widetilde{\boldsymbol{z}}\cdot\boldsymbol{n} = 0$ on Γ and $\langle\widetilde{\boldsymbol{z}}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0$. Due to Theorem 3.3.7, there exists a potential vector $\boldsymbol{\psi} \in \boldsymbol{V}_N^p(\Omega)$ such that $\widetilde{\boldsymbol{z}} = \mathbf{curl}\,\boldsymbol{\psi}$ in Ω with the estimate $\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{\,1,p}(\Omega)} \leq C\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)}$. This implies that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}.$$

Moreover, we have

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} \left|\langle\boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\right| \|\widetilde{\mathbf{grad}}\,q_j^T\|_{\boldsymbol{L}^p(\Omega)}$$

Since $\boldsymbol{z}$ belongs to $\boldsymbol{H}_0^{\,p}(\mathrm{div},\,\Omega)$, then due to Lemma 3.2.1, the restriction of $\boldsymbol{z}\cdot\boldsymbol{n}$ to any Σ_j belongs to the dual space $\boldsymbol{W}^{\,1-\frac{1}{p'},p'}(\Sigma_j)'$. Thus, we obtain

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} \|\boldsymbol{z}\|_{\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)} \|\widetilde{\mathbf{grad}}\,q_j^T\|_{\boldsymbol{L}^p(\Omega)}.$$

So, since $\mathrm{div}\,\boldsymbol{z} = 0$ we deduce that:

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally, we can write

$$\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \leq C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}\right|}{\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)}} = C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right|}{\|\mathbf{curl}\,\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)}}.$$

But by Corollary 3.3.8 we have

$$\|\boldsymbol{\psi}\|_{\boldsymbol{X}_N^{\,p}(\Omega)} \simeq \|\mathbf{curl}\,\boldsymbol{\psi}\|_{\boldsymbol{L}^p(\Omega)}.$$

Then,

$$\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \leq C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}_N^{\,p}(\Omega)}},$$

and the Inf-Sup condition (5.1) follows immediately. □

We are now interested to the resolution of a second type of elliptic system.

Proposition 5.2.2. *Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega))'$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω and satisfying the compatibility condition:*

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} = 0. \tag{5.3}$$

Then, the following problem

$$\begin{cases} -\Delta \boldsymbol{\xi} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0,\ 1 \le i \le I, \end{cases} \tag{5.4}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ and we have:

$$\| \boldsymbol{\xi} \|_{\boldsymbol{W}^{1,p}(\Omega)} \le C \| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'}. \tag{5.5}$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate:

$$\| \boldsymbol{\xi} \|_{\boldsymbol{W}^{2,p}(\Omega)} \le C \| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)}. \tag{5.6}$$

Proof. **i)** Due to Theorem 5.2.1, the following problem: find $\boldsymbol{\xi} \in \boldsymbol{V}_N^p(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}_N^{p'}(\Omega),\ \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_\Omega, \tag{5.7}$$

satisfies the Inf-Sup condition (5.1), where the duality on Ω is

$$\langle \cdot,\, \cdot \rangle_\Omega = \langle \cdot,\, \cdot \rangle_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}.$$

So, it has a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}_N^p(\Omega)$ since the right-hand sides belongs to $(\boldsymbol{V}_N^{p'}(\Omega))'$. We known that due to Theorem 3.3.7, this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Next, we want to extend (5.7) to any test function $\boldsymbol{\varphi}$ in $\boldsymbol{X}_N^{p'}(\Omega)$. Let us $\boldsymbol{\varphi} \in \boldsymbol{X}_N^{p'}(\Omega)$ and we consider the solution χ in $W^{1,p'}(\Omega)$ satisfying: $\Delta \chi = \operatorname{div} \boldsymbol{\varphi}$ in Ω and $\chi = 0$ on Γ. We set

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \mathbf{grad}\, \chi - \sum_{i=1}^{I} \langle (\boldsymbol{\varphi} - \mathbf{grad}\, \chi) \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N.$$

Observe that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{V}_N^{p'}(\Omega)$ and $\mathbf{curl}\, \boldsymbol{\varphi} = \mathbf{curl}\, \widetilde{\boldsymbol{\varphi}}$. Hence, thanks to (5.3) problem (5.7) becomes: find $\boldsymbol{\xi} \in \boldsymbol{V}_N^p(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{X}_N^{p'}(\Omega),\ \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_\Omega. \tag{5.8}$$

That means that problem (5.7) and problem (5.8) are equivalent. It is easy to prove that every solution of (5.4) also solves (5.8). From now on, we are in position to justify the compatibility condition (5.3). It comes from the variational formulation (5.8) by taking the test function $\boldsymbol{\varphi} \in \boldsymbol{K}_N^{p'}(\Omega)$. Conversely, let $\boldsymbol{\xi} \in \boldsymbol{V}_N^{p}(\Omega)$ the solution of the problem (5.8). Then,

$$\forall \boldsymbol{\varphi} \in \boldsymbol{X}_N^{p'}(\Omega), \quad \langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} = 0.$$

So $-\Delta\,\boldsymbol{\xi}$ is equal to $\boldsymbol{f}$ in Ω. Moreover, by the fact that $\boldsymbol{\xi}$ belongs to the space $\boldsymbol{V}_N^p(\Omega)$ we have $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω, $\boldsymbol{\xi} \times \boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0,\ 1 \le i \le I$. This prove that $\boldsymbol{\xi}$ solves (5.4). Let $B \in \mathcal{L}(\boldsymbol{V}_N^p(\Omega), (\boldsymbol{V}_N^p(\Omega))')$ be the operator defined by

$$\forall \boldsymbol{\mu} \in \boldsymbol{V}_N^p(\Omega), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}_N^p(\Omega), \quad \langle B\,\boldsymbol{\mu},\, \boldsymbol{\varphi}\rangle = \int_\Omega \mathbf{curl}\,\boldsymbol{\mu}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \tag{5.9}$$

It is clear that the operator B is an isomorphism from $\boldsymbol{V}_N^p(\Omega)$ onto $(\boldsymbol{V}_N^p(\Omega))'$, and

$$\|\boldsymbol{\mu}\|_{\boldsymbol{X}_N^p(\Omega)} \backsimeq \|B\,\boldsymbol{\mu}\|_{(\boldsymbol{V}_N^p(\Omega))'}. \tag{5.10}$$

Hence, since $\boldsymbol{\xi}$ is the solution of the problem (5.7) we have:

$$\|B\,\boldsymbol{\xi}\|_{(\boldsymbol{V}_N^p(\Omega))'} = \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\langle B\,\boldsymbol{\xi},\, \boldsymbol{\varphi}\rangle}{\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}} = \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}}.$$

Then,

$$\|B\,\boldsymbol{\xi}\|_{(\boldsymbol{V}_N^{p'}(\Omega))'} \le \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}$$

The estimate (5.5) follows by using (5.10) and Theorem 2.2.4.

ii) We suppose that $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, Ω is of class $\mathcal{C}^{2,1}$ and we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$. Then, the function $\boldsymbol{z}$ satisfies:

$$\boldsymbol{z} \in \boldsymbol{L}^p(\Omega),\ \mathbf{curl}\,\boldsymbol{z} = \boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div}\boldsymbol{z} = 0 \ \text{ and } \ \boldsymbol{z}\cdot\boldsymbol{n} = 0.$$

Due to Theorem 2.2.8, $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. As a consequence $\boldsymbol{\xi}$ satisfies:

$$\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega),\ \mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega),\ \operatorname{div}\boldsymbol{\xi} = 0 \ \text{ and } \ \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{0}.$$

We deduce by using Corollary 3.3.6 that the solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (5.6). □

Remark 5.2.3.

i) Thanks to the characterization of the kernels $\boldsymbol{K}_T^p(\Omega)$ and $\boldsymbol{K}_N^p(\Omega)$, we can in fact show directly the uniqueness of the solution $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ of the problem (5.4). Indeed, we suppose that

$\boldsymbol{f} = \mathbf{0}$ and we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$. Then, $\boldsymbol{z}$ belongs to $\boldsymbol{K}_T^p(\Omega)$ and due to Corollary 3.2.3 we can write:

$$\boldsymbol{z} = \sum_{j=1}^{J} \langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^T.$$

Since $\boldsymbol{K}_T^p(\Omega) \subset \boldsymbol{W}^{1,q}(\Omega)$ for any $q \geq 1$, the function $\boldsymbol{z}$ belongs to $\boldsymbol{L}^2(\Omega)$. So,

$$\int_\Omega |\boldsymbol{z}|^2 \,\mathrm{d}\boldsymbol{x} = \sum_{j=1}^{J} \langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\widetilde{\mathbf{grad}}\, q_j^T \,\mathrm{d}\boldsymbol{x} = 0.$$

Hence, $\boldsymbol{z}$ is zero and $\boldsymbol{\xi}$ belongs to $\boldsymbol{K}_N^p(\Omega)$. Due to Corollary 3.3.1, we can write $\boldsymbol{\xi}$ as:

$$\boldsymbol{\xi} = \sum_{i=1}^{I} \langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} \mathbf{grad}\, q_i^N$$

and $\boldsymbol{\xi} = \mathbf{0}$ since $\boldsymbol{\xi}$ satisfies the last condition in (5.4).

ii) We can replace in (5.4) the right hand side by the **curl** of an element $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$. Indeed, due to Theorem 3.6.2, every element $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))'$ with $\operatorname{div}\boldsymbol{f} = 0$ in Ω and satisying the compatibility condition (5.3), can be written as the **curl** of a function $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$.

Corollary 5.2.4. *Let $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))'$ with $\operatorname{div}\boldsymbol{f} = 0$ in Ω and satisfying the compatibility condition* (5.3) *and $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. Then, the following problem*

$$(E_N) \qquad \begin{cases} -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} & \text{in}\,\Omega, \\ \operatorname{div}\boldsymbol{\xi} = 0 & \text{in}\,\Omega, \\ \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0,\ 1 \leq i \leq I, \end{cases}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ and we have:

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\,\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} + \|\,\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\Big). \tag{5.11}$$

Proof. **i)** Let $\boldsymbol{\xi}_0$ be the divergence free lift of $\boldsymbol{g}$:

$$\boldsymbol{\xi}_0 = \boldsymbol{g}_t \text{ on}\,\Gamma,\ \operatorname{div}\boldsymbol{\xi}_0 = 0 \text{ in}\,\Omega \quad \text{and} \quad \boldsymbol{\xi}_0 \in \boldsymbol{W}^{1,p}(\Omega),$$

whit the estimate:

$$\|\boldsymbol{\xi}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{g}_t\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}$$

Then setting $\boldsymbol{z} = \boldsymbol{\xi} - \boldsymbol{\xi}_0$ problem (E_N) is equivalent to: find $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta\, \boldsymbol{z} = \boldsymbol{f} - \Delta\, \boldsymbol{\xi}_0 & \text{in}\,\Omega, \\ \operatorname{div} \boldsymbol{z} = 0 & \text{in}\,\Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma, \\ \langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I. \end{cases} \tag{5.12}$$

Observe that $\boldsymbol{F} = \boldsymbol{f} + \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}_0$ belongs to $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ and satisfies the compatibility condition (5.3). Then, due to Proposition 5.2.2, there exists a unique $\boldsymbol{z} \in \boldsymbol{W}^{\,1,p}(\Omega)$ solution of (5.12). Then, $\boldsymbol{\xi} = \boldsymbol{z} + \boldsymbol{\xi}_0 \in \boldsymbol{W}^{\,1,p}(\Omega)$ is a the unique solution of (E_N) which satisfies the estimate (5.11). □

The next theorem provided the information on the solvability, in weak sense, of the inhomogeneous Stokes problem $(\mathcal{S}_N)$.

Theorem 5.2.5. (**Weak solutions for** $(\mathcal{S}_{\mathrm{N}})$) *Let* $\boldsymbol{f}$, $\boldsymbol{g}$, π_0 *with*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))', \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{\,1-1/p,p}(\Gamma), \tag{5.13}$$

satisfying the compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega - \int_\Gamma \pi_0\, \boldsymbol{v}\cdot\boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \tag{5.14}$$

where $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}$. *Then, the Stokes problem* $(\mathcal{S}_N)$ *has a unique solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{\,1,p}(\Omega)$ *satisfying the estimate*

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\,\pi\,\|_{W^{\,1,p}(\Omega)} \le C\Big(\|\,\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\Big). \tag{5.15}$$

Proof. First, we consider the problem

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in}\,\Omega, \qquad \pi = \pi_0 \quad \text{on}\,\Gamma. \tag{5.16}$$

Because $\operatorname{div}\boldsymbol{f} \in W^{\,-1,p}(\Omega)$, this problem has a unique solution $\pi \in W^{\,1,p}(\Omega)$ satisfying the estimate

$$\|\pi\|_{W^{\,1,p}(\Omega)} \le C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega))'} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\big). \tag{5.17}$$

Next, observe that $\boldsymbol{F} = \boldsymbol{f} - \nabla\,\pi$ is an element of $[\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)]'$. Moreover, since $\boldsymbol{f}$ satisfies the compatibility condition (5.14), we have then

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{\,p}(\Omega), \quad \langle \boldsymbol{F},\, \boldsymbol{v}\rangle_{(\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega))'\times\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)} = 0.$$

So problem $(\mathcal{S}_N)$ becomes: $-\Delta\,\boldsymbol{u} = \boldsymbol{F}$ in Ω, $\operatorname{div}\boldsymbol{u} = 0$ in Ω, $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ on Γ and $\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$ for any $1 \le i \le I$ which is equivalent to: Find $\boldsymbol{u} \in \boldsymbol{X}_N^{\,p}(\Omega)$ such that:

$$\begin{aligned} &\boldsymbol{u} - \boldsymbol{g} \in \boldsymbol{V}_N^{\,p}(\Omega) \\ \forall \boldsymbol{v} \in \boldsymbol{V}_N^{\,p'}(\Omega), \quad &\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \langle\, \boldsymbol{F}\,,\, \boldsymbol{v}\,\rangle_\Omega. \end{aligned}$$

We are now able to justify the compatibility condition (5.14). It comes from the last variational formulation by taking $\boldsymbol{v} \in \boldsymbol{K}_N^{\,p'}(\Omega)$. Moreover, since $\boldsymbol{F}$ satisfies the assumptions of Corollary 5.2.4, this problem has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{\,1,p}(\Omega)$ satisfying the estimate

$$\|\,\boldsymbol{u}\,\|_{W^{\,1,p}(\Omega)} \le C\big(\|\,\boldsymbol{F}\,\|_{[\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{\,1-1/p,p}(\Gamma)}\big). \tag{5.18}$$

Finally, the pair $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{\,1,p}(\Omega)\times W^{\,1,p}(\Omega)$ is the unique solution of the problem $(\mathcal{S}_N)$ and the estimate (5.15) follows easily from (5.17) and (5.18). □

Remark 5.2.6. If we take $\pi_0 \in W^{\,-1/p,p}(\Gamma)$ we obtain that $\pi \in L^p(\Omega)$ a unique solution of the problem:

$$-\Delta\,\pi = \operatorname{div}\boldsymbol{f} \ \text{ in } \Omega \quad \text{and} \quad \pi = \pi_0 \ \text{ on } \Gamma.$$

But we are not able to solve problem $(\mathcal{S}_N)$ because in this case $\boldsymbol{f} = \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}) + \nabla\,\pi \notin [\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)]'$.

We can also solve the Stokes problem $(\mathcal{S}_N)$ when the divergence operator does not vanish and it is a given function. So, we focus on the study of the problem:

$$\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div}\boldsymbol{u} = \chi & \text{in}\,\Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\,\Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I. \end{cases} \tag{5.19}$$

Corollary 5.2.7. *Let $\boldsymbol{f}$, χ, $\boldsymbol{g}$, π_0 with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)]',\ \ \chi \in W^{\,1,p}(\Omega),\ \ \boldsymbol{g} \in W^{\,1-1/p,p}(\Gamma),\ \ \pi_0 \in W^{\,1-1/p,p}(\Gamma)$$

and satisfying the compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{\,p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega - \int_\Gamma (\pi_0 - \chi)\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0. \tag{5.20}$$

Then, the Stokes problem (5.19) *has exactly one solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in W^{\,1,p}(\Omega)$. Moreover, there exists a constant $C > 0$ depending only on p and Ω such that:*

$$\begin{aligned} \|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{\,1,p}(\Omega)} \ &\le\ C\Big(\|\,\boldsymbol{f}\,\|_{[\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)]'} + \|\chi\|_{W^{\,1,p}(\Omega)} + \|\,\boldsymbol{g}\,\|_{W^{\,1-1/p,p}(\Gamma)} + \\ &+\ \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\Big). \end{aligned} \tag{5.21}$$

Proof. We consider the problem:

$$\Delta\,\pi = \operatorname{div} \boldsymbol{f} + \Delta\,\chi \quad \text{in}\,\Omega, \qquad \pi = \pi_0 \quad \text{on}\,\Gamma. \tag{5.22}$$

Because $\operatorname{div} \boldsymbol{f} + \Delta\,\chi \in W^{-1,p}(\Omega)$, this problem has a unique solution $\pi \in W^{1,p}(\Omega)$ satisfying the estimate

$$\|\pi\|_{W^{1,p}(\Omega)} \le C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))'} + \|\chi\|_{W^{1,p}(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big). \tag{5.23}$$

So problem (5.19) becomes:

$$\begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{f} - \nabla\,\pi \;\text{ and }\; \operatorname{div}\boldsymbol{u} = \chi & \text{in}\,\Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \;\; \text{for any } 1 \le i \le I. \end{cases} \tag{5.24}$$

To solve (5.24), we shall reduce it to the homogeneous condition on the divergence in Ω. For this, we consider the problem

$$\Delta\,\theta = \chi \quad \text{in}\,\Omega \quad \text{and} \quad \theta = 0 \quad \text{on}\,\Gamma.$$

Since $\chi \in L^p(\Omega)$, it has a unique solution $\theta \in W^{2,p}(\Omega)$, with

$$\|\theta\|_{W^{2,p}(\Omega)} \le C\|\chi\|_{L^p(\Omega)}. \tag{5.25}$$

Taking $\boldsymbol{w} = \nabla\,\theta$ and defining

$$\widetilde{\boldsymbol{w}} = \boldsymbol{w} - \sum_{i=1}^{I} \langle \boldsymbol{w}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}\,\mathbf{grad}\,q_i^N,$$

we see that $\widetilde{\boldsymbol{w}} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\operatorname{div}\widetilde{\boldsymbol{w}} = \chi$, $\mathbf{curl}\,\widetilde{\boldsymbol{w}} = \boldsymbol{0}$, in Ω, $\widetilde{\boldsymbol{w}}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ and $\langle \widetilde{\boldsymbol{w}}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0$ for any $1 \le i \le I$. Finally, taking $\boldsymbol{z} = \boldsymbol{u} - \widetilde{\boldsymbol{w}}$, we see that the problem (5.24) can be reduced to the following problem for $\boldsymbol{z}$:

$$\begin{cases} -\Delta\,\boldsymbol{z} = \boldsymbol{F} \;\text{ and }\; \operatorname{div}\boldsymbol{z} = 0 & \text{in}\,\Omega, \\ \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} & \text{on}\,\Gamma, \\ \langle \boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \;\; \text{for any } 1 \le i \le I, \end{cases} \tag{5.26}$$

where $\boldsymbol{F} = \boldsymbol{f} - \nabla\,\pi + \Delta\,\widetilde{\boldsymbol{w}}$. Under our assumptions and since $\Delta\,\widetilde{\boldsymbol{w}} = \nabla\,\chi \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{F} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ with $\operatorname{div}\boldsymbol{F} = 0$ in Ω and satisfies the compatibility condition (5.3). Applying Corollary 5.2.4, we deduce the existence of a unique solution $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$ of (5.26). Moreover, using (5.23) we have the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\chi\|_{W^{1,p}(\Omega)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big). \tag{5.27}$$

As a consequence, $\boldsymbol{u} = \boldsymbol{z} + \widetilde{\boldsymbol{w}} \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution of (5.24). It remains to show (5.21). Observe that $\widetilde{\boldsymbol{w}}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and satisfies $\widetilde{\boldsymbol{w}} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. From Corollary 3.3.8, we see that the semi norm

$$\widetilde{\boldsymbol{w}} \mapsto \|\mathbf{curl}\, \widetilde{\boldsymbol{w}}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\, \widetilde{\boldsymbol{w}}\|_{L^p(\Omega)} + \sum_{i=1}^{I} |\langle \widetilde{\boldsymbol{w}} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}|$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{W}^{1,p}(\Omega)}$. Since $\mathbf{curl}\, \widetilde{\boldsymbol{w}} = \boldsymbol{0}$ in Ω and $\langle \widetilde{\boldsymbol{w}} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$, this yields the existence of a constant $C > 0$ such that:

$$\|\widetilde{\boldsymbol{w}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \|\chi\|_{L^p(\Omega)}. \tag{5.28}$$

We obtain the desired estimate (5.21) from (5.23), (5.27) and (5.28). □

Remark 5.2.8. Notice that in the corresponding theorem for the problem $(\mathcal{S}_T)$, we took $\chi \in L^p(\Omega)$. In the case of the problem $(\mathcal{S}_N)$, we can not suppose the same, because we need to solve (5.19), the fact that $\nabla \chi \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$, which is not checked because χ is only $L^p(\Omega)$. To get around such difficulty, we took $\chi \in W^{1,p}(\Omega)$.

5.3 Strong solutions and regularity for the Stokes system $(\mathcal{S}_N)$

We give the following preliminary result.

Lemma 5.3.1. *The mapping $\boldsymbol{v} \longmapsto \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}$ is continuous from $\boldsymbol{W}^{1,p}(\Omega)$ into $W^{-1/p,p}(\Gamma)$ and we have the relation:*

$$\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = \Big(\sum_{j=1}^{2} \frac{\partial\, \boldsymbol{v}}{\partial\, s_j} \times \boldsymbol{\tau}_j\Big) \cdot \boldsymbol{n} \quad \text{on}\, \Gamma, \quad \text{in the sense of } W^{-1/p,p}(\Gamma). \tag{5.29}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. By the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{W}^{1,p}(\Omega)$, there exits a sequence $\boldsymbol{v}_k \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ which converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{1,p}(\Omega)$. We know from (4.42), that for any $\boldsymbol{v}_k \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have

$$\mathbf{curl}\, \boldsymbol{v}_k = \sum_{j=1}^{2} \frac{\partial\, \boldsymbol{v}_k}{\partial\, s_j} \times \boldsymbol{\tau}_j + \frac{\partial\, \boldsymbol{v}_k}{\partial\, \boldsymbol{n}} \times \boldsymbol{n} \qquad \text{on}\, \Gamma. \tag{5.30}$$

So,

$$\mathbf{curl}\, \boldsymbol{v}_k \cdot \boldsymbol{n} = \Big(\sum_{j=1}^{2} \frac{\partial\, \boldsymbol{v}_k}{\partial\, s_j} \times \boldsymbol{\tau}_j\Big) \cdot \boldsymbol{n} \quad \text{on}\, \Gamma.$$

Since $\boldsymbol{v}_k$ converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{1-1/p,p}(\Gamma)$, we deduce that the term $\Big(\sum\limits_{j=1}^{2} \dfrac{\partial\, \boldsymbol{v}_k}{\partial\, s_j} \times \boldsymbol{\tau}_j\Big) \cdot \boldsymbol{n}$ converges to $\Big(\sum\limits_{j=1}^{2} \dfrac{\partial\, \boldsymbol{v}}{\partial\, s_j} \times \boldsymbol{\tau}_j\Big) \cdot \boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$. Moreover, $\mathbf{curl}\, \boldsymbol{v}$ belongs to $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ and

by the continuity of the normal trace operator, we have the convergence of $\mathbf{curl}\, \boldsymbol{v}_k \cdot \boldsymbol{n}$ to $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$, which proves formula (5.29). □

Corollary 5.3.2. *Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. If $\boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, then $\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} \in W^{1-1/p,p}(\Gamma)$ and*

$$\|\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)} \leq C \|\boldsymbol{v} \times \boldsymbol{n}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}$$

Proof. Due to Lemma 5.3.1, we have

$$\mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} = \Big(\sum_{j=1}^{2} \frac{\partial\, \boldsymbol{v}}{\partial\, s_j} \times \boldsymbol{\tau}_j \Big) \cdot \boldsymbol{n} \quad \text{on}\, \Gamma,$$

and consequently we have:

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} &= \Big(\sum_{j=1}^{2} \frac{\partial\, \boldsymbol{v}_t}{\partial\, s_j} \times \boldsymbol{\tau}_j \Big) \cdot \boldsymbol{n} + \Big(\sum_{j=1}^{2} \frac{\partial\, [(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}]}{\partial\, s_j} \times \boldsymbol{\tau}_j \Big) \cdot \boldsymbol{n} \qquad (5.31) \\ &= \Big(\sum_{j=1}^{2} \frac{\partial\, \boldsymbol{v}_t}{\partial\, s_j} \times \boldsymbol{\tau}_j \Big) \cdot \boldsymbol{n} + \Big(\sum_{j=1}^{2} \frac{\partial\, (\boldsymbol{v} \cdot \boldsymbol{n})}{\partial\, s_j}\, \boldsymbol{n} \times \boldsymbol{\tau}_j \Big) \cdot \boldsymbol{n} + \\ &+ \Big(\sum_{j=1}^{2} (\boldsymbol{v} \cdot \boldsymbol{n}) \frac{\partial\, \boldsymbol{n}}{\partial\, s_j} \times \boldsymbol{\tau}_j \Big) \cdot \boldsymbol{n} \end{aligned}$$

Observe that the two last terms vanish. Moreover, since $\boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, the tangential derivation on Γ belongs to $\boldsymbol{W}^{1-1/p,p}(\Gamma)$. By the regularity of Γ, we deduce that the first term in (5.31) belongs to $\boldsymbol{W}^{1-1/p,p}(\Gamma)$. This prove Corollary 5.3.2 □

Theorem 5.3.3. (**Strong solutions for** $(\mathcal{S}_{\mathrm{N}})$) *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 with:*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma) \qquad (5.32)$$

satisfying the compatibility condition (5.14). *Then, the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ of the Stokes problem $(\mathcal{S}_N)$ given by Theorem 5.2.5 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.*

Proof. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ be the solution given by Theorem 5.2.5. It suffices to prove that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. The function $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$ satisfies:

$$\boldsymbol{z} \in \boldsymbol{L}^p(\Omega), \ \operatorname{div} \boldsymbol{z} = 0, \ \ \mathbf{curl}\, \boldsymbol{z} \in \boldsymbol{L}^p(\Omega).$$

Moreover, since $\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, due to Corollary 5.3.2, $\boldsymbol{z} \cdot \boldsymbol{n}$ belongs to $W^{1-1/p,p}(\Gamma)$. We deduce from Theorem 1.18 that $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. As a consequence, it follows from Theorem 2.12 that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. □

We can also consider strong solutions in the case when the divergence operator does not vanish and we have only to consider regular boundary data for the velocity. So, the proof of the following result is quite similar to that of Theorem 5.3.3 above.

Corollary 5.3.4. *Let $\boldsymbol{f}$, $\boldsymbol{g}$, χ, π_0 with:*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma), \quad \chi \in W^{1,p}(\Omega), \quad \pi_0 \in W^{1-1/p,p}(\Gamma) \tag{5.33}$$

satisfying the compatibility condition (5.20). *Then, the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ of the Stokes problem* (5.19) *given by Corollary 5.2.7 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.*

5.4 Very weak solutions for the Stokes system $(\mathcal{S}_N)$

In this subsection, we are going to study the existence of very weak solutions for the Stokes equations $(\mathcal{S}_N)$. Before, we give some preliminary results.

We introduce the space:

$$\boldsymbol{N}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega), \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{H}_0^p(\mathbf{curl}, \Omega)\},$$

equipped with the topology given by the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{N}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)}.$$

We have the following density result:

Lemma 5.4.1. *The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{N}^p(\Omega)$.*

Proof. In a first step, we consider that Ω is strictely star-shaped with respect to one of its points which is taken as the origin. Then, for any $\boldsymbol{v} \in \boldsymbol{N}^p(\Omega)$, we take $\widetilde{\boldsymbol{v}}$ its extension by zero to $\mathbb{R}^3$. Thus, $\widetilde{\boldsymbol{v}} \in \boldsymbol{W}^{1,p}(\mathbb{R}^3)$ and as $\mathbf{curl}\, \widetilde{\boldsymbol{v}} = \widetilde{\mathbf{curl}\, \boldsymbol{v}} \in \boldsymbol{H}^p(\mathbf{curl}, \mathbb{R}^3)$, then $\widetilde{\boldsymbol{v}} \in \boldsymbol{N}^p(\mathbb{R}^3)$. For $\theta < 1$, we define the functions:

$$\boldsymbol{v}_\theta(\boldsymbol{x}) = \boldsymbol{v}(\frac{\boldsymbol{x}}{\theta}), \quad \text{for almost all } \boldsymbol{x} \in \mathbb{R}^3.$$

Since $\operatorname{supp} \boldsymbol{v}_\theta \subset \theta\overline{\Omega} \subset \Omega$, $\widetilde{\boldsymbol{v}}_\theta$ has a compact support in Ω. Moreover, $\widetilde{\boldsymbol{v}}_\theta \in \boldsymbol{N}^p(\mathbb{R}^3)$ and

$$\lim_{\theta \to 1} \widetilde{\boldsymbol{v}}_\theta = \widetilde{\boldsymbol{v}} \ \text{ in } \ \boldsymbol{N}^p(\mathbb{R}^3).$$

The result is then proved by regularization. Let $\rho \in \mathcal{D}(\mathbb{R}^3)$, be a smooth $\mathcal{C}^\infty$ function with compact support, such that $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho(x)\, \mathrm{d}x = 1$. For $\varepsilon > 0$, let ρ_ε denote the function

$x \longmapsto (\frac{1}{\varepsilon^3})\rho(\frac{x}{\varepsilon})$. As $\varepsilon \to 0$, ρ_ε converges in the distribution sence to the Dirac distribution. As a consequence, $\rho_\epsilon * \widetilde{\boldsymbol{v}}_\theta|_\Omega$ belongs to $\boldsymbol{\mathcal{D}}(\Omega)$ and

$$\lim_{\epsilon \to 0} \lim_{\theta \to 1} \rho_\epsilon * \widetilde{\boldsymbol{v}}_\theta = \widetilde{\boldsymbol{v}} \ \text{ in } \ \boldsymbol{N}^{\,p}(\mathbb{R}^3).$$

The result follows where $\boldsymbol{v}$ is the limit in $\boldsymbol{N}^{\,p}(\Omega)$ of the restriction of the functions $\rho_\epsilon * \widetilde{\boldsymbol{v}}_\theta$ to Ω. In the case where Ω is not star-shaped, we have to recover Ω which is Lipschitz by a finite number of star open sets. □

We introduce the following spaces

$$\boldsymbol{L}^p_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \text{div}\, \boldsymbol{v} = 0\},$$

which is Banach space for the norm $\|\cdot\|_{\boldsymbol{L}^p(\Omega)}$, and

$$\boldsymbol{G}^{\,p}(\Omega) = \{\nabla\theta,\ \theta \in W_0^{1,p}(\Omega)\}.$$

The following lemma gives a characterization for the dual space of $\boldsymbol{L}^p_\sigma(\Omega)$.

Lemma 5.4.2. *We have the following properties:*

i) $\boldsymbol{L}^p(\Omega) = \boldsymbol{L}^p_\sigma(\Omega) \oplus \boldsymbol{G}^{\,p}(\Omega)$.

ii) $(\boldsymbol{L}^p_\sigma(\Omega))' = \boldsymbol{L}^{p'}_\sigma(\Omega)$.

Proof i) Let $\boldsymbol{v}$ any element of $\boldsymbol{L}^p(\Omega)$ and $\chi \in W_0^{1,p}(\Omega)$ the solution of the Dirichlet problem: $\Delta\chi = \text{div}\, \boldsymbol{v}$ in Ω. We set $\boldsymbol{u} = \boldsymbol{v} - \nabla\chi$. We remark that $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ and $\nabla\chi \in \boldsymbol{G}^{\,p}(\Omega)$. So, we have:

$$\boldsymbol{L}^p(\Omega) = \boldsymbol{L}^p_\sigma(\Omega) + \boldsymbol{G}^{\,p}(\Omega).$$

We next show that $\boldsymbol{L}^p_\sigma(\Omega) \cap \boldsymbol{G}^{\,p}(\Omega) = 0$. Let $\boldsymbol{v} \in \boldsymbol{L}^p_\sigma(\Omega) \cap \boldsymbol{G}^{\,p}(\Omega)$. Then, we can write $\boldsymbol{v} = \nabla\chi$ with $\chi \in W_0^{1,p}(\Omega)$. Since $\text{div}\, \boldsymbol{v} = 0$ in Ω, we deduce that $\boldsymbol{v} = \boldsymbol{0}$ and consequently we have

$$\boldsymbol{L}^p(\Omega) = \boldsymbol{L}^p_\sigma(\Omega) \oplus \boldsymbol{G}^{\,p}(\Omega).$$

ii) We observe that $\boldsymbol{L}^p_\sigma(\Omega) = \boldsymbol{L}^p(\Omega)/\boldsymbol{G}^{\,p}(\Omega)$ and $(\boldsymbol{L}^p_\sigma(\Omega))' = \boldsymbol{G}^{\,p}(\Omega)^\perp$. Moreover, $\boldsymbol{L}^p_\sigma(\Omega)$ is a closed subspace of $\boldsymbol{L}^p(\Omega)$. Hence, if we prove that $\boldsymbol{L}^{p'}_\sigma(\Omega)^\perp = \boldsymbol{G}^{\,p}(\Omega)$ this will imply

$$\boldsymbol{G}^{\,p}(\Omega)^\perp = (\boldsymbol{L}^{p'}_\sigma(\Omega)^\perp)^\perp = \overline{\boldsymbol{L}^{p'}_\sigma(\Omega)} = \boldsymbol{L}^{p'}_\sigma(\Omega),$$

which is the required result because $\boldsymbol{G}^{\,p}(\Omega)^\perp = (\boldsymbol{L}^p_\sigma(\Omega))'$.
First, let $\boldsymbol{u} \in \boldsymbol{G}^{\,p}(\Omega)$. Then, we have for any $\boldsymbol{v} \in \boldsymbol{L}^{p'}_\sigma(\Omega)$

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\pi \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = 0,$$

because π belongs to $W_0^{1,p}(\Omega)$. Hence $\boldsymbol{u} \in \boldsymbol{L}^{p'}_{\sigma}(\Omega)^{\perp}$ and $\boldsymbol{G}^{\,p}(\Omega) \subset \boldsymbol{L}^{p'}_{\sigma}(\Omega)^{\perp}$. Conversely, let $\boldsymbol{u} \in \boldsymbol{L}^{p}(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{L}^{p'}_{\sigma}(\Omega)$:

$$\int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0. \tag{5.34}$$

By choosing $\boldsymbol{v}$ in the space $\mathcal{V} = \{\boldsymbol{v} \in \mathcal{D}(\Omega),\ \mathrm{div}\, \boldsymbol{v} = 0 \ \mathrm{in}\, \Omega\}$ and using De Rham's Lemma, we deduce that $\boldsymbol{u} = \nabla\, \pi$, where $\pi \in W^{\,1,p}(\Omega)$. As π is unique up to an additive constant, we can choose this constant in such a way that $\int_{\Gamma} \pi \, \mathrm{d}\boldsymbol{\sigma} = 0$. From (5.34), we obtain

$$\forall\, \boldsymbol{v} \in \boldsymbol{L}^{p'}_{\sigma}(\Omega), \quad \langle \pi,\ \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{W^{\,1-1/p,p}(\Gamma) \times W^{\,-1/p',p'}(\Gamma)} = 0. \tag{5.35}$$

Let now $\mu \in W^{\,-1/p',p'}(\Gamma)$ and $\theta \in W^{\,1,p'}(\Omega)$ a solution of the followin Neumann problem

$$\begin{cases} \Delta\, \theta = 0 & \text{in } \Omega, \\ \dfrac{\partial\, \theta}{\partial\, \boldsymbol{n}} = \mu - \frac{1}{|\mu|} \langle \mu,\ 1 \rangle_{W^{\,1-1/p,p}(\Gamma) \times W^{\,-1/p',p'}(\Gamma)} & \text{on } \Gamma. \end{cases} \tag{5.36}$$

Next, we set $\boldsymbol{v} = \nabla\, \theta$. We deduce from (5.36) that:

$$\langle \pi,\ \mu - \frac{1}{|\mu|} \langle \mu,\ 1 \rangle_{\Gamma} \rangle_{W^{\,1-1/p,p}(\Gamma) \times W^{\,-1/p',p'}(\Gamma)} = 0.$$

Then

$$\forall \mu \in W^{\,-1/p',p'}(\Gamma), \quad \langle \pi,\ \mu \rangle_{W^{\,1-1/p,p}(\Gamma) \times W^{\,-1/p',p'}(\Gamma)} = 0,$$

which implies that $\pi = 0$ on Γ. Consequently, $\boldsymbol{u}$ belongs to $\boldsymbol{G}^{\,p}(\Omega)$. Therefore, $\boldsymbol{L}^{p'}_{\sigma}(\Omega)^{\perp} \subset \boldsymbol{G}^{\,p}(\Omega)$ which finishes the proof.

□

Now, we introduce the space:

$$\boldsymbol{M}^{\,p}(\Omega) = \big\{(\boldsymbol{v},\ \pi) \in \boldsymbol{L}^{p}_{\sigma}(\Omega) \times L^{p}(\Omega);\ \ -\Delta\, \boldsymbol{v} + \nabla\, \pi \in [\boldsymbol{H}^{\,p'}_{0}(\mathbf{curl},\ \Omega)]'\big\},$$

which is a Banach space for the norm:

$$\|(\boldsymbol{v},\ \pi)\|_{\boldsymbol{M}^{\,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\pi\|_{L^{p}(\Omega)} + \| - \Delta\, \boldsymbol{v} + \nabla\, \pi\|_{[\boldsymbol{H}^{\,p'}_{0}(\mathbf{curl},\, \Omega)]'}.$$

Lemma 5.4.3. *The space $\boldsymbol{\mathcal{D}}_{\sigma}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{M}^{p}(\Omega)$.*

Proof. Let $\boldsymbol{\ell}$ in $[\boldsymbol{M}^{\,p}(\Omega)]'$ such that:

$$\forall\, (\boldsymbol{v},\ \pi) \in \boldsymbol{\mathcal{D}}_{\sigma}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega}), \quad \langle \boldsymbol{\ell},\ (\boldsymbol{v},\ \pi) \rangle = 0. \tag{5.37}$$

Using the Riesz's Representation Lemma, there exist $\boldsymbol{f} \in \boldsymbol{L}^{p'}_{\sigma}(\Omega)$, $\lambda \in L^{p'}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{H}^{\,p'}_{0}(\mathbf{curl},\ \Omega)$, such that for any $(\boldsymbol{v}, \pi) \in \boldsymbol{M}^{\,p}(\Omega)$,

$$\langle \boldsymbol{\ell}, (\boldsymbol{v}, \pi)\rangle = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \lambda\,\pi\,\mathrm{d}\boldsymbol{x} + \langle -\Delta\,\boldsymbol{v} + \nabla\,\pi,\ \boldsymbol{g}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} \tag{5.38}$$

where we have used that $[\boldsymbol{L}^p_\sigma(\Omega)]' = \boldsymbol{L}^{p'}_\sigma(\Omega)$ (as in Lemma 5.4.2). In particular, if $(\boldsymbol{v},\,\pi) \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)\times\mathcal{D}(\Omega)$, we have

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \lambda\,\pi\,\mathrm{d}\boldsymbol{x} + \langle -\Delta\,\boldsymbol{g},\ \boldsymbol{v}\rangle_\Omega - \langle \operatorname{div}\boldsymbol{g},\ \pi\rangle_\Omega = 0,$$

where $\langle\cdot,\,\cdot\rangle_\Omega$ is the duality bracket between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Particularly, if $\pi = 0$, we obtain for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$:

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \langle -\Delta\,\boldsymbol{g},\ \boldsymbol{v}\rangle_\Omega = 0.$$

Since $\boldsymbol{f} - \Delta\,\boldsymbol{g}$ belongs to $\boldsymbol{W}^{-2,p}(\Omega)$, using De Rham's Lemma (see [6]), there exists $\theta \in W^{-1,p}(\Omega)$, unique up to an additive constant, such that

$$\boldsymbol{f} - \Delta\,\boldsymbol{g} = \nabla\,\theta \ \text{ in}\,\Omega. \tag{5.39}$$

We can choose also in particular $\boldsymbol{v} = \mathbf{0}$ and this time, we obtain for any $\pi \in \mathcal{D}(\Omega)$:

$$\int_\Omega \lambda\,\pi\,\mathrm{d}\boldsymbol{x} - \langle \operatorname{div}\boldsymbol{g},\ \pi\rangle_\Omega = 0,$$

which implies that

$$\lambda = \operatorname{div}\boldsymbol{g} \ \text{ in}\,\Omega. \tag{5.40}$$

Observe that we can extend by zero the functions $\boldsymbol{f}$, λ and $\boldsymbol{g}$ in such a way that

$$\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^{p'}(\mathbb{R}^3),\ \lambda \in L^p(\mathbb{R}^3) \quad\text{and}\quad \widetilde{\boldsymbol{g}} \in \boldsymbol{H}^{p'}(\mathbf{curl},\,\mathbb{R}^3).$$

Moreover, for any $\chi \in \mathcal{D}(\mathbb{R}^3)$ such that $\Delta\,\chi = 0$ in Ω, we have by (5.38) with $\boldsymbol{v} = \nabla\,\chi|_\Omega$:

$$\int_\Omega \boldsymbol{f}\cdot\nabla\,\chi\,\mathrm{d}\boldsymbol{x} = 0.$$

Let $\mu \in W^{\frac{1}{p'},p}(\Gamma)$. By the density of $\mathcal{D}(\Gamma)$ in $W^{\frac{1}{p'},p}(\Gamma)$, there exist a sequence $\mu_k \in \mathcal{D}(\Gamma)$ such that μ_k converges to μ in $W^{\frac{1}{p'},p}(\Gamma)$. Let now φ_k be a solution of the problem

$$\begin{cases} -\Delta\,\varphi_k = 0 & \text{in}\,\Omega,\\ \varphi_k = \mu_k & \text{on}\,\Gamma. \end{cases} \tag{5.41}$$

We know that φ_k belongs to $\mathcal{C}^\infty(\Omega)$. Let $\psi_k \in \mathcal{D}(\mathbb{R}^3)$ the extension of φ_k to $\mathbb{R}^3$. Then φ_k belongs to $\mathcal{D}(\overline{\Omega})$ and we have,

$$0 = \int_\Omega \boldsymbol{f}\cdot\nabla\,\varphi_k \quad = \quad \langle \boldsymbol{f}\cdot\boldsymbol{n},\ \mu_k\rangle_{W^{-\frac{1}{p'},p'}(\Gamma)\times W^{\frac{1}{p'},p}(\Gamma)}.$$

So, $\langle \boldsymbol{f}\cdot\boldsymbol{n},\,\mu\rangle_{W^{-\frac{1}{p'},p'}(\Gamma)\times W^{\frac{1}{p'},p}(\Gamma)}=0$ for any $\mu\in W^{\frac{1}{p'},p}(\Gamma)$. Consequently, $\boldsymbol{f}\cdot\boldsymbol{n}=0$ on Γ and $\operatorname{div}\widetilde{\boldsymbol{f}}=0$ in $\mathbb{R}^3$. Now, we take $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ with $\operatorname{div}\boldsymbol{\varphi}=0$ and $q\in\mathcal{D}(\mathbb{R}^3)$, we obtain by (5.37) and (5.38):

$$\int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}^3}\widetilde{\lambda}\,q\,\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}^3}(-\Delta\,\boldsymbol{\varphi}+\nabla\,q)\cdot\widetilde{\boldsymbol{g}}\,\mathrm{d}\boldsymbol{x}=0. \tag{5.42}$$

In particular, if $q=0$, then by De Rham's Lemma:

$$\widetilde{\boldsymbol{f}}-\Delta\,\widetilde{\boldsymbol{g}}=\nabla\,\theta_0\quad\text{in }\mathbb{R}^3, \tag{5.43}$$

with $\theta_0\in\mathcal{D}'(\mathbb{R}^3)$. Since $\operatorname{div}\widetilde{\boldsymbol{f}}=0$ in $\mathbb{R}^3$, then $-\Delta\operatorname{div}\widetilde{\boldsymbol{g}}=\Delta\,\theta_0$ in $\mathbb{R}^3$. But $\widetilde{\boldsymbol{f}}\in\boldsymbol{L}^{p'}(\mathbb{R}^3)$ and $\operatorname{supp}\widetilde{\boldsymbol{f}}$ is compact, then $\widetilde{\boldsymbol{f}}\in\boldsymbol{W}_0^{-2,p'}(\mathbb{R}^3)$ where $\boldsymbol{W}_0^{-2,p'}(\mathbb{R}^3)$ is the dual space of the weighted sobolev space

$$\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)=\{\boldsymbol{v}\in\boldsymbol{D}'(\mathbb{R}^3),\ \frac{\boldsymbol{v}}{\omega_0}\in\boldsymbol{L}^p(\mathbb{R}^3),\ \frac{\nabla\,\boldsymbol{v}}{\omega_1}\in\boldsymbol{L}^p(\mathbb{R}^3),\ D^2\boldsymbol{v}\in\boldsymbol{L}^p(\mathbb{R}^3)\},$$

with $\omega_0=(1+|\boldsymbol{x}|)^2$ if $p\notin\{3/2,3\}$, $\omega_0=(1+|\boldsymbol{x}|)^2\ln(2+|\boldsymbol{x}|)$ if $p\in\{3/2,3\}$, $\omega_1=(1+|\boldsymbol{x}|)$ if $p\neq 3$, $\omega_1=(1+|\boldsymbol{x}|)\ln(2+|\boldsymbol{x}|)$ if $p=3$. Consequently $\nabla\,\theta_0\in\boldsymbol{W}_0^{-2,p'}(\mathbb{R}^3)$. We deduce that $\theta_0\in\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$ and then $\theta_0=-\operatorname{div}\widetilde{\boldsymbol{g}}$ in $\mathbb{R}^3$. By taking in (5.43) the restriction to Ω, we obtain $\boldsymbol{f}-\Delta\,\boldsymbol{g}=\nabla\,\theta_0|_\Omega$. As Ω is connected, there exists a unique constant a such that $\theta=\theta_0+a$ in Ω up to an additive constant. Thus relation (5.43) becomes $\Delta\,\widetilde{\boldsymbol{g}}-\nabla\operatorname{div}\widetilde{\boldsymbol{g}}=\widetilde{\boldsymbol{f}}$ in $\mathbb{R}^3$. As $\widetilde{\mathbf{curl}\,\boldsymbol{g}}=\mathbf{curl}\,\widetilde{\boldsymbol{g}}$ and $\mathbf{curl}\,\widetilde{\boldsymbol{g}}\in\boldsymbol{H}^{p'}(\mathbf{curl},\mathbb{R}^3)$, then $\mathbf{curl}\,\boldsymbol{g}\in\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)$.

Moreover, if $\boldsymbol{\varphi}=\boldsymbol{0}$, we obtain from (5.42) that:

$$\int_{\mathbb{R}^3}\widetilde{\lambda}\,q\,\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}^3}\nabla\,q\cdot\widetilde{\boldsymbol{g}}\,\mathrm{d}\boldsymbol{x}=0,$$

which implies that: $\widetilde{\lambda}=\operatorname{div}\widetilde{\boldsymbol{g}}$ in $\mathbb{R}^3$. We deduce that $\operatorname{div}\widetilde{\boldsymbol{g}}$ belongs to $L^{p'}(\mathbb{R}^3)$ and to $L^{p'}(\Omega)$ by restriction on Ω. Then, $\boldsymbol{g}\in\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)$. As a consequence, $\boldsymbol{g}$ belongs to $\boldsymbol{W}_0^{1,p'}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{g}\in\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)$. So, $\boldsymbol{g}$ belongs to $\boldsymbol{N}^{p'}(\Omega)$ and due to Lemma 5.4.1, there exists a sequence $(\boldsymbol{g}_k)_k\in\boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{g}_k$ converges to $\boldsymbol{g}$ in $\boldsymbol{N}^{p'}(\Omega)$ when $k\to\infty$. Now, we consider $(\boldsymbol{v},\pi)\in\boldsymbol{M}^p(\Omega)$. Observe that

$$\begin{aligned}
\langle \boldsymbol{\ell}, (\boldsymbol{v},\, \pi)\rangle &= -\int_\Omega \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{g}\,\mathrm{d}\boldsymbol{x} + \\
&+ \langle -\Delta\,\boldsymbol{v} + \nabla\,\pi,\, \boldsymbol{g}\rangle_{[\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\Omega)} \\
&= \lim_{k\to\infty}\Big(-\int_\Omega \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{g}_k\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{g}_k\,\mathrm{d}\boldsymbol{x} + \\
&+ \langle -\Delta\,\boldsymbol{v} + \nabla\,\pi,\, \boldsymbol{g}_k\rangle_{[\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\Omega)}\Big) \\
&= \lim_{k\to\infty}\Big(-\int_\Omega \mathbf{curl}\,\boldsymbol{g}_k\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{g}_k\,\mathrm{d}\boldsymbol{x} + \\
&+ \int_\Omega \mathbf{curl}\,\boldsymbol{g}_k\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{g}_k\,\mathrm{d}\boldsymbol{x}\Big) = 0.
\end{aligned}$$

Therefore, $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{M}^{\,p}(\Omega)$ □

In order to give meaning to the trace of a very weak solution of the Stokes problem $(\mathcal{S}_N)$, we need to introduce the space:

$$\boldsymbol{Y}_N^{\,p}(\Omega) = \big\{\boldsymbol{\varphi}\in\boldsymbol{W}^{\,2,p}(\Omega),\ \boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0}\ \text{ and }\ \mathrm{div}\,\boldsymbol{\varphi} = \mathbf{0}\ \text{ on}\,\Gamma\big\}.$$

Theorem 5.4.4. *The linear mapping $\gamma : (\boldsymbol{u},\,\pi)\mapsto(\boldsymbol{u}\times\boldsymbol{n},\,\pi|_\Gamma)$ defined on $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ, from $\boldsymbol{M}^p(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma)\times W^{-1/p,p}(\Gamma)$, and we have the Green formula: for any $(\boldsymbol{u},\,\pi)\in\boldsymbol{M}^p(\Omega)$ and $\boldsymbol{\varphi}\in\boldsymbol{Y}_N^{p'}(\Omega)$,*

$$\begin{aligned}
\langle -\Delta\,\boldsymbol{u} + \nabla\,\pi,\ \boldsymbol{\varphi}\rangle_\Omega &= -\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{u}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma - \\
&- \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \pi,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{W^{\,-1/p,p}(\Gamma)\times W^{\,1/p,p'}(\Gamma)},
\end{aligned}\tag{5.44}$$

where

$$\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)},\qquad \langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

Proof. Let $(\boldsymbol{u},\pi)\in\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$, then formula (5.44) is valid for any $\boldsymbol{\varphi}\in\boldsymbol{Y}_N^{\,p'}(\Omega)$. Let $\boldsymbol{\mu}\in\boldsymbol{W}^{\,1/p,p'}(\Gamma)$. Then, there exists a function $\boldsymbol{\varphi}\in\boldsymbol{W}^{\,2,p'}(\Omega)$ such that:

$$\boldsymbol{\varphi} = \mathbf{0}\ \text{ and }\ \frac{\partial\,\boldsymbol{\varphi}}{\partial\,\boldsymbol{n}} = \boldsymbol{\mu}\times\boldsymbol{n}\quad\text{on}\,\Gamma$$

and verifying:

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)} \le C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\,1/p,p'}(\Gamma)}\tag{5.45}$$

Moreover, since $\boldsymbol{\varphi} = \mathbf{0}$ on Γ, using (4.42) we obtain: $\mathbf{curl}\,\boldsymbol{\varphi} = -\boldsymbol{\mu}_t$ on Γ. Then, the function $\boldsymbol{\varphi}$ belongs to $\boldsymbol{Y}_N^{p'}(\Omega)$. Consequently,

$$\begin{aligned} \Big|\langle \boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\mu}\rangle_\Gamma\Big| &= \Big|\langle \boldsymbol{u}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma\Big| \qquad (5.46)\\ &\le \|-\Delta\,\boldsymbol{u}+\nabla\,\pi\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}\|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{\,p'}(\mathbf{curl},\,\Omega)} + \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\Delta\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \\ &+ \|\pi\|_{L^{\,p}(\Omega)}\|\mathrm{div}\,\boldsymbol{\varphi}\|_{L^{\,p'}(\Omega)} \\ &\le C\|(\boldsymbol{u},\,\pi)\|_{\boldsymbol{M}^{\,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)}. \qquad (5.47) \end{aligned}$$

Thus, using (5.45), we obtain for any $(\boldsymbol{u},\,\pi)\in\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$:

$$\|\boldsymbol{u}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \le C\|(\boldsymbol{u},\pi)\|_{\boldsymbol{M}^{\,p}(\Omega)}.$$

Next, we have $\Delta\,\pi\in W^{-1,p}(\Omega)$ and $\pi\in L^p(\Omega)$, then due to [10], the trace of π on Γ belongs to $W^{-1/p,p}(\Gamma)$. Moreover, we have:

$$\begin{aligned} \|\pi\|_{W^{-1/p,p}(\Gamma)} &\le C\Big(\|\pi\|_{L^p(\Omega)} + \|\Delta\,\pi\|_{W^{-1,p}(\Omega)}\Big)\\ &\le C\Big(\|\pi\|_{L^p(\Omega)} + \|-\Delta\,\boldsymbol{u}+\nabla\pi\|_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))'}\Big)\\ &\le C\|(\boldsymbol{u},\,\pi)\|_{\boldsymbol{M}^{\,p}(\Omega)} \end{aligned}$$

Therefore, we obtain that the linear continuous mapping $\gamma:\ (\boldsymbol{u},\,\pi)\mapsto(\boldsymbol{u}_\Gamma\times\boldsymbol{n},\,\pi|_\Gamma)$ defined on the space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{M}^{\,p}(\Omega)$. Since $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})\times\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{M}^{\,p}(\Omega)$, then we can extend this mapping from $\boldsymbol{M}^{\,p}(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma)\times W^{-1/p,p}(\Gamma)$ and the Green formula (5.44) holds for any $(\boldsymbol{u},\,\pi)\in\boldsymbol{M}^{\,p}(\Omega)$ and for any $\boldsymbol{\varphi}\in\boldsymbol{Y}_N^{p'}(\Omega)$. □

Theorem 5.4.5. (**Very weak solutions for** $(\mathcal{S}_\mathrm{N})$) *Assume that* Ω *is of class* $\mathcal{C}^{\,2,1}$. *Let* $\boldsymbol{f}$, $\boldsymbol{g}$ *and* π_0 *with:*

$$\boldsymbol{f}\in[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]',\ \boldsymbol{g}\in\boldsymbol{W}^{-1/p,p}(\Gamma),\ \pi_0\in W^{-1/p,p}(\Gamma),$$

and satisfying the compatibility conditions (4.2) *and* (5.14). *Then, the Stokes problem* $(\mathcal{S}_N)$ *has exactly one solution* $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ *and* $\pi\in L^p(\Omega)$. *Moreover, there exists a constant* $C>0$ *depending only on* p *and* Ω *such that:*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\,\boldsymbol{g}\,\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} + \|\pi_0\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big). \qquad (5.48)$$

Proof) **First step:** Thanks to the Green formula (5.44), it is easy to verify that $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times L^p(\Omega)$ is solution of problem $(\mathcal{S}_N)$, without the last flux condition, is equivalent to the variational formulation: Find $(\boldsymbol{u},\, \pi) \in \boldsymbol{L}^p(\Omega) \times L^p(\Omega)$ such that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{p'}(\Omega)$ and $q \in W_0^{1,p'}(\Omega)$,

$$\begin{aligned} \int_\Omega \boldsymbol{u} \cdot \Delta\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} \quad &+ \quad \int_\Omega \pi\, \mathrm{div}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = -\langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_\Omega + \langle\, \boldsymbol{g} \times \boldsymbol{n}, \mathbf{curl}\, \boldsymbol{\varphi} \rangle_\Gamma + \langle \pi_0,\, \boldsymbol{\varphi} \cdot \boldsymbol{n} \rangle_\Gamma \\ \int_\Omega \boldsymbol{u} \cdot \nabla\, q\, \mathrm{d}\boldsymbol{x} \quad &= \quad 0. \end{aligned} \tag{5.49}$$

Indeed, let $(\boldsymbol{u},\, \pi) \in \boldsymbol{L}^p(\Omega) \times L^p(\Omega)$ be a solution to (5.49). It is clear that:

$$-\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{ and } \quad \mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in } \Omega.$$

Using Green formula (5.44), we obtain for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{p'}(\Omega)$:

$$-\int_\Omega \boldsymbol{u} \cdot \Delta\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{u} \times \boldsymbol{n}, \mathbf{curl}\, \boldsymbol{\varphi} \rangle_\Gamma - \int_\Omega \pi \mathrm{div}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \int_\Gamma \pi_0\, \boldsymbol{\varphi} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = \langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_\Omega.$$

Then, we deduce that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{p'}(\Omega)$,

$$\langle\, \boldsymbol{u} \times \boldsymbol{n},\, \mathbf{curl}\, \boldsymbol{\varphi} \rangle_\Gamma + \langle \pi,\, \boldsymbol{\varphi} \cdot \boldsymbol{n} \rangle_\Gamma = \langle \boldsymbol{g} \times \boldsymbol{n},\, \mathbf{curl}\, \boldsymbol{\varphi} \rangle_\Gamma + \langle \pi_0,\, \boldsymbol{\varphi} \cdot \boldsymbol{n} \rangle_\Gamma.$$

Let $\boldsymbol{\mu} \in \boldsymbol{W}^{1/p,p'}(\Gamma)$. Then, there exists a function $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}(\Omega)$ such that:

$$\boldsymbol{\varphi} = \mathbf{0} \ \text{ and } \ \frac{\partial\, \boldsymbol{\varphi}}{\partial\, \boldsymbol{n}} = \boldsymbol{\mu_t} \ \text{ on } \Gamma,$$

and this implies that $\mathbf{curl}\, \boldsymbol{\varphi} \times \boldsymbol{n} = -\boldsymbol{\mu}_t$ on Γ and $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{p'}(\Omega)$. We deduce that for all $\boldsymbol{\mu} \in \boldsymbol{W}^{1/p,p'}(\Gamma)$,

$$\langle\, \boldsymbol{u} \times \boldsymbol{n},\, \boldsymbol{\mu} \rangle_\Gamma = \langle \boldsymbol{g} \times \boldsymbol{n},\, \boldsymbol{\mu} \rangle_\Gamma.$$

Consequently $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}$ on Γ. Let us prove now that $\pi = \pi_0$ on Γ. For any $\lambda \in W^{1+1/p,p'}(\Gamma)$, there exists a function $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}(\Omega)$ such that: $\boldsymbol{\varphi} = \lambda \boldsymbol{n}$ and $\dfrac{\partial\, \boldsymbol{\varphi}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} = -K\lambda$ on Γ. Observe that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{Y}_N^{p'}(\Omega)$ and then for any $\lambda \in W^{1+1/p,p'}(\Gamma)$ we have:

$$\langle \pi,\, \lambda \rangle_{W^{-1-1/p,p}(\Gamma) \times W^{1+1/p,p'}(\Gamma)} = \langle \pi_0,\, \lambda \rangle_{W^{-1-1/p,p}(\Gamma) \times W^{1+1/p,p'}(\Gamma)},$$

and then, $\pi = \pi_0$ on Γ.

The converse is a simple consequence of the Green formula (5.44) and the fact that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{p'}(\Omega)$:

$$\int_\Omega \boldsymbol{u} \cdot \nabla\, \mathrm{div} \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{u} \cdot \boldsymbol{n},\, \mathrm{div}\, \boldsymbol{\varphi} \rangle_\Omega = 0.$$

ii) **Second step** : Let's now solve problem (5.49). We know due to Corollary 5.3.4 that for any $(\boldsymbol{F},\,\chi)\in \boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{K}^{\,p}_{N}(\Omega)\times W_0^{1,p'}(\Omega)$, there exists a unique $\boldsymbol{\varphi}\in \boldsymbol{W}^{\,2,p'}(\Omega)$ and $q\in W_0^{1,p'}(\Omega)$ satisfying :

$$\begin{cases} -\Delta\,\boldsymbol{\varphi}+\nabla\,q=\boldsymbol{F} & \text{in}\,\Omega,\\ \operatorname{div}\boldsymbol{\varphi}=\chi & \text{in}\,\Omega,\\ \boldsymbol{\varphi}\times\boldsymbol{n}=\boldsymbol{0}\ \text{ and }\ q=0 & \text{on}\,\Gamma,\\ \langle\boldsymbol{\varphi}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}=0\ \text{ for any }\ 1\leq i\leq I, \end{cases}$$

with the estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)}+\|q\|_{W^{\,1,p'}(\Omega)}\leq\big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}+\|\chi\|_{W_0^{1,p'}(\Omega)}\big). \tag{5.50}$$

From this bound, we have

$$\begin{aligned} &\Big|\langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega-\langle\boldsymbol{g}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma-\langle\pi_0,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma\Big| \qquad (5.51)\\ \leq\ &\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{\,-1/p,p}(\Gamma)}+\|\pi_0\|_{W^{\,-1/p,p}(\Gamma)}\Big)\Big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}+\|\chi\|_{W_0^{1,p'}(\Omega)}\Big). \end{aligned}$$

In other words, we can say that the linear mapping:

$$(\boldsymbol{F},\,\chi)\mapsto\langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega-\langle\boldsymbol{g}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma-\langle\pi_0,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma$$

defines an element of the dual space of $\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{K}^{\,p}_{N}(\Omega)\times W_0^{1,p'}(\Omega)$ and according to the Riesz's representation theorem, there exists a unique $(\boldsymbol{u},\,\pi)\in\boldsymbol{L}^p(\Omega)/\boldsymbol{K}^{\,p}_{N}(\Omega)\times W^{\,-1,p}(\Omega)$ satisfying

$$\int_\Omega\boldsymbol{u}\cdot\boldsymbol{F}\,\mathrm{d}\boldsymbol{x}-\int_\Omega\pi\chi\,\mathrm{d}\boldsymbol{x}=\langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega-\langle\boldsymbol{g}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma-\langle\pi_0,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma.$$

A such solution $(\boldsymbol{u},\,\pi)$ satisfies the problem $(\mathcal{S}_N)$ without the last condition but we have only to set

$$\widetilde{\boldsymbol{u}}=\boldsymbol{u}-\sum_{i=1}^{I}\langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}\,\nabla\,q_i^N.$$

It is clear that $\widetilde{\boldsymbol{u}}\in\boldsymbol{L}^p(\Omega)$ is also solution of $(\mathcal{S}_N)$ and satisfies its last condition. Moreover, $\pi\in W^{\,-1,p}(\Omega)$ satisfies:

$$\Delta\,\pi=\operatorname{div}\boldsymbol{f}\ \text{ in}\,\Omega\quad\text{and}\quad\pi=\pi_0\ \text{ on}\,\Gamma.$$

Since $\operatorname{div}\boldsymbol{f}\in W^{\,-1,p}(\Omega)$ and $\pi_0\in W^{\,-1/p,p}(\Gamma)$, we deduce from [10] that π belongs to $L^p(\Omega)$. Finally, we can prove easily the estimate (5.48).

□

Now we set the space:

$$\boldsymbol{\mathcal{M}}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p_\sigma(\Omega);\ \ \Delta\, \boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'\},$$

which is a Banach space for the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{\mathcal{M}}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta\, \boldsymbol{v}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'}.$$

Using the same arguments given in the Lemma 5.4.3, we can prove that the space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\boldsymbol{M}^p(\Omega)$. To give a sense to the trace of functions which belong to $\boldsymbol{\mathcal{M}}^p(\Omega)$, we have the following lemma, where the proof is very similar to that of Theorem 5.4.4.

Lemma 5.4.6. *The linear mapping $\gamma\ :\ \boldsymbol{u} \longmapsto \boldsymbol{u} \times \boldsymbol{n}|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma\ :\ \boldsymbol{\mathcal{M}}^p(\Omega) \longmapsto \boldsymbol{W}^{-1/p,p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{u} \in \boldsymbol{\mathcal{M}}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{p'}(\Omega)$,

$$\langle \Delta\, \boldsymbol{u},\, \boldsymbol{\varphi}\rangle_\Omega = \int_\Omega \boldsymbol{u} \cdot \Delta\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{u} \times \boldsymbol{n},\, \mathbf{curl}\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}, \tag{5.52}$$

where the duality on Ω is the following

$$\langle \cdot, \cdot\rangle_\Omega = \langle \cdot,\ \ \cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)}. \tag{5.53}$$

As a consequence, we have the following result concerning very weak solutions for the elliptic problem (E_N).

Corollary 5.4.7. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'$ with $\operatorname{div} \boldsymbol{f} = 0$ in Ω and $\boldsymbol{g} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Then the problem (E_N) has a unique solution $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$, with the estimate*

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\,\boldsymbol{f}\,\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big). \tag{5.54}$$

Proof. Thanks to Green formula (5.52), it is easy to verify that $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ is solution of problem (E_N) is equivalent to the following variational formulation: Find $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ such that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{p'}(\Omega)$ and for any $q \in W_0^{1,p'}(\Omega)$:

$$\begin{aligned} -\int_\Omega \boldsymbol{\xi} \cdot \Delta\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} &= \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g} \times \boldsymbol{n},\, \mathbf{curl}\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}, \\ \int_\Omega \boldsymbol{\xi} \cdot \nabla\, q \mathrm{d}\boldsymbol{x} &= 0. \end{aligned} \tag{5.55}$$

where the duality on Ω is defined in (5.53). We are now able to conclude by using the regularity of its dual problem presented in Proposition 5.2.2. □

5.5 A variant of the system $(\mathcal{S}_N)$

As it is shown in the previous sections, in order to solve problem $(\mathcal{S}_N)$, the data must satisfy the compatibility condition (5.14). Now, what happen if this condition is not satisfied? As will appear, the answer strongly depends on the following variant of the Stokes problem $(\mathcal{S}_N)$: Find functions $\boldsymbol{u}$, π and constants C_i for $i = 1, \ldots, I$, such that:

$$(\mathcal{S}'_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + C_i & \text{on } \Gamma_{\mathrm{i}}, \ \ 1 \le i \le I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \ \ 1 \le i \le I, & \end{cases}$$

situation that we can be found in the paper [26]. Let us compare with our approach.

Theorem 5.5.1. *Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 such that:*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{\,1-1/p,p}(\Gamma).$$

Then, the problem $(\mathcal{S}'_N)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{\,1,p}(\Omega)$ and constants $C_1, \ldots, C_I$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{\,1,p}(\Omega)} \le C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\big), \quad (5.56)$$

and where $C_1, \ldots, C_I$ are given by (5.58)

Proof. Observe that the following problem

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 & \text{on } \Gamma_0, \\ \pi = \pi_0 + \langle \boldsymbol{f},\, \nabla\, q_i^N \rangle_\Omega - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma & \text{on } \Gamma_{\mathrm{i}},, \ \ 1 \le i \le I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, & \end{cases} \quad (5.57)$$

has a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{\,1,p}(\Omega) \times W^{\,1,p}(\Omega)$ since the compatibility condition (5.14) is verified. For $i = 0, \ldots, I$, we set

$$C_i = \langle \boldsymbol{f},\, \nabla\, q_i^N \rangle_\Omega - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma, \quad (5.58)$$

and finally, $\boldsymbol{u}$, π and the constants $C_1, \ldots, C_I$ are solutions of $(\mathcal{S}'_N)$. □

Remark 5.5.2.

i) Observe that if we suppose that the compatibility condition (5.14) is verified, we have that $C_i = 0$ for all $i = 1, \dots, I$ and we have reduced to solve the problem $(\mathcal{S}'_N)$ without the constant C_i.

ii) We can also give a similar variant of system $(\mathcal{S}_T)$.

5.6 Helmholtz decompositions

According to the two types $\boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u} \times \boldsymbol{n}$ of boundary conditions on Γ, we give decompositions of vector fields $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$. Our results may be regarded as an extension of the well-known De Rham-Hodge-Kodaira decomposition of $\mathcal{C}^\infty$-forms on compact Riemannian manifolds into $\boldsymbol{L}^p$-vector fields on Ω. We can find similar decompositions in [46], where the authors consider more regular domain with $\mathcal{C}^\infty$-boundary Γ. We can see also [52] for the case $p = 2$.

Theorem 5.6.1.

i) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then, there exist $\chi \in W^{1,p}(\Omega)$, $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_N(\Omega)$, $\boldsymbol{z} \in \boldsymbol{K}^p_T(\Omega)$ such that $\boldsymbol{u}$ can be represented as:

$$\boldsymbol{u} = \boldsymbol{z} + \nabla \chi + \mathbf{curl}\, \boldsymbol{w}, \tag{5.59}$$

where $\boldsymbol{z}$ is unique, χ is unique up to an additive constant and $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_N(\Omega)$. Moreover, we have the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}} + \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_N(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.60}$$

ii) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then, there exist $\chi \in W^{1,p}_0(\Omega)$, $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_T(\Omega)$, $\boldsymbol{z} \in \boldsymbol{K}^p_N(\Omega)$ such that $\boldsymbol{u}$ can be represented as:

$$\boldsymbol{u} = \boldsymbol{z} + \nabla \chi + \mathbf{curl}\, \boldsymbol{w}, \tag{5.61}$$

where $\boldsymbol{z}$ and χ are unique and $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_T(\Omega)$. Moreover, we have the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_T(\Omega)} \leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.62}$$

Proof.

i) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. The scalar potential $\chi \in W^{1,p}(\Omega)$ is taken as a weak solution of the following problem:

$$\operatorname{div}(\nabla \chi - \boldsymbol{u}) = 0 \text{ in } \Omega, \quad (\nabla \chi - \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, \tag{5.63}$$

or equivalently of

$$\forall \mu \in W^{1,p'}(\Omega), \quad \int_\Omega \nabla \chi \cdot \nabla \mu \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \nabla \mu \, \mathrm{d}\boldsymbol{x}. \tag{5.64}$$

Such a scalar function χ as (5.64) is unique up to an additive constant and satisfies the estimate:

$$\|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.65}$$

Next, the vector potential $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{\sigma}(\Omega) \cap \boldsymbol{X}^p_N(\Omega)$ can be derived from the point *ii)* of Remark 5.2.3. For $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, we take $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$\begin{cases} -\Delta\, \boldsymbol{w} = \mathbf{curl}\, \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{5.66}$$

The vector potential $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_N(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{w}\|_{W^{1,p}(\Omega)/\boldsymbol{K}^p_N(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.67}$$

Finally, let us define $\boldsymbol{z} = \boldsymbol{u} - \nabla\,\chi - \mathbf{curl}\, \boldsymbol{w}$. Then, $\boldsymbol{z} \in \boldsymbol{K}^p_T(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla\,\chi\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}, \tag{5.68}$$

which yields the representation (5.59) of $\boldsymbol{u}$. The estimate (5.60) is a consequence of (5.65), (5.67) and (5.68).

ii) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Compared with (5.63), the scalar potential $\chi \in W^{1,p}_0(\Omega)$ is taken as the weak solution of the Dirichlet problem:

$$\Delta\,\chi = \operatorname{div} \boldsymbol{u} \ \text{ in } \Omega, \quad \chi = 0 \ \text{ on } \Gamma. \tag{5.69}$$

Such a scalar function χ is unique and satisfies the estimate:

$$\|\chi\|_{W^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.70}$$

The vector potential $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{\sigma}(\Omega) \cap \boldsymbol{X}^p_T(\Omega)$ can be derived from Lemma 3.3.3. For $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ we take $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$\begin{cases} \Delta\, \boldsymbol{w} = \mathbf{curl}\, \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad (\mathbf{curl}\, \boldsymbol{w} - \boldsymbol{u}) \times \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases} \tag{5.71}$$

The vector potential $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_T(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{w}\|_{W^{1,p}(\Omega)/\boldsymbol{K}^p_T(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.72}$$

Let us define $\boldsymbol{z} = \boldsymbol{u} - \nabla\,\chi - \mathbf{curl}\, \boldsymbol{w}$. Then, similarly to the proof of the above i) we obtain the representation (5.61) of $\boldsymbol{u}$ and the estimate (5.62).

□

Remark 5.6.2.

i) Note that in the representation (5.59), if Ω is simply connected then $\boldsymbol{z} = \boldsymbol{0}$, situation that can be the same in (5.61) if we suppose that the boundary Γ is connected or in other words without holes.

ii) In the decomposition (5.59), $\boldsymbol{z}$ being a divergence-free vector field with a zero normal trace on the boundary. We know from Theorem 3.2.5 that $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\psi}$ with $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p}(\Omega)$, $\operatorname{div}\boldsymbol{\psi} = 0$ in Ω and $\boldsymbol{\psi} \cdot \boldsymbol{n} = 0$ on Γ.

iii) In the decomposition (5.61), $\boldsymbol{z}$ being an element of $\boldsymbol{K}^p_N(\Omega)$. We know then that $\boldsymbol{z}$ is a gradient of a function of $W^{1,p}(\Omega)$.

Chapter 6

Elliptic systems with non standard boundary conditions

6.1 Study of the problem $(\mathcal{P}_T)$

Lemma 6.1.1. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f}$ belongs to $\boldsymbol{L}^p(\Omega)$ with $p \geq 6/5$. Then, the following problem*

$$(\mathcal{P}_T) \quad \begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} & \text{in}\,\Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \dfrac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}} \times \boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma, \end{cases}$$

has a unique solution $\boldsymbol{u}$ in $\boldsymbol{W}^{2,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.1}$$

Proof. **First case:** $p = 2$. A variational formulation of the problem $(\mathcal{P}_N)$ can be introduced by the bilinear form $a(\cdot,\,\cdot):\ \boldsymbol{H}^{\,1}(\Omega) \cap \boldsymbol{X}_T^{\,2}(\Omega) \times \boldsymbol{H}^{\,1}(\Omega) \cap \boldsymbol{X}_T^{\,2}(\Omega) \to \mathbb{R}$, which is defined by

$$a(\boldsymbol{u},\,\boldsymbol{v}) = \int_\Omega \nabla\, \boldsymbol{u} : \nabla\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}.$$

Obviousely, $a(\cdot,\,\cdot)$ is continuous over $\boldsymbol{H}^{\,1}(\Omega) \cap \boldsymbol{X}_T^{\,2}(\Omega)$ and according the Poincare's Inequality (see [9], Proposition 2.1), it is also coercive over $\boldsymbol{H}^{\,1}(\Omega) \cap \boldsymbol{X}_T^{\,2}(\Omega)$. Using a Green's formula, we have that the problem $(\mathcal{P}_T)$ is equivalent to the following variational formulation: Find $\boldsymbol{u} \in \boldsymbol{H}^{\,1}(\Omega)$ with $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ such that:

$$\forall\, \boldsymbol{v} \in \boldsymbol{H}^{\,1}(\Omega) \cap \boldsymbol{X}_T^{\,2}(\Omega), \qquad a(\boldsymbol{u},\,\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, \mathrm{d}\boldsymbol{x}. \tag{6.2}$$

Since the right hand-side defines a linear continuous form on $[\boldsymbol{H}^{\,1}(\Omega) \cap \boldsymbol{X}_T^{\,2}(\Omega)]'$, we deduce by the Lax-Milgram's Lemma that problem (6.2) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_T^{\,2}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\,\boldsymbol{f}\,\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.3}$$

Moreover, we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Then $\boldsymbol{z}$ satisfies the problem

$$\begin{cases} -\Delta\,\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \ \text{ for any } 0 \le i \le I. \end{cases} \tag{6.4}$$

Since $\mathbf{curl}\,\boldsymbol{f}$ satisfies the compatibility condition (5.3), it follows from Proposition 5.2.2 that $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.5}$$

Next, $\nabla \operatorname{div}\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$. As a consequence $\operatorname{div}\boldsymbol{u}$ belongs to $H^1(\Omega)$ and using (6.3) and (6.5), we obtain

$$\|\operatorname{div}\boldsymbol{u}\|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{6.6}$$

Finally, applying Corollary 2.2.12 leads to deduce that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$. Using (6.5) and (6.6) gives directly (6.1) for $p = 2$.

ii) **Second case:** $p > 2$. We know that problem $(\mathcal{P}_N)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Then $\boldsymbol{u}$ belongs to $\boldsymbol{L}^q(\Omega)$ for all $1 \le q \le \infty$. Moreover as for the case $p = 2$, the function $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ is a solution in $\boldsymbol{W}^{1,p}(\Omega)$ of the problem (6.4). This implies that $\nabla \operatorname{div}\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and by [5, Proposition 2.10], we have $\operatorname{div}\boldsymbol{u} \in W^{1,p}(\Omega)$. Applying again Corollary 2.2.12, we deduce that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (6.1).

iii) **Third case:** $6/5 \le p < 2$. Observe that since $\mathbf{f} \in \boldsymbol{L}^{6/5}(\Omega)$, as in point i) by using the Lax-Milgram lemma, the problem $(\mathcal{P}_T)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C\|\,\boldsymbol{f}\,\|_{\boldsymbol{L}^{6/5}(\Omega)}.$$

Next, we use the same argument as in the second step in order to prove that $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and then $\nabla \operatorname{div}\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Now, since $\boldsymbol{u} \in \boldsymbol{L}^6(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ then due to Theorem 2.2.12, $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (6.1).

□

We can also, consider the case of the non homogenuous boundary conditions.

Theorem 6.1.2. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $g \in W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$. Then, the following problem*

$$(\mathcal{P}'_T) \quad \begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{f} & \text{in}\,\Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = g, \ \dfrac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}} \times \boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \end{cases}$$

has a unique solution $\boldsymbol{u}$ in $\boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\big(\|\,\boldsymbol{f}\,\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\big). \tag{6.7}$$

Moreover, if $g \in W^{2-1/p,p}(\Gamma)$ *and* $\boldsymbol{h} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, *then the solution* $\boldsymbol{u}$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega)$ *and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|g\|_{W^{2-1/p,p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big). \tag{6.8}$$

Proof.

1. We suppose that $g \in W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$. Let $\theta \in W^{2,p}(\Omega)$ a solution of the Neumann problem:
$$\Delta\,\theta = \lambda \ \text{ in } \Omega, \quad \text{and} \quad \frac{\partial\,\theta}{\partial\,\boldsymbol{n}} = g \ \text{ on } \Gamma,$$
where $\lambda \in \mathbb{R}$ is chosen such that $\lambda \operatorname{mes}\Omega + \int_\Gamma g\,\mathrm{d}\boldsymbol{x} = 0$. We set $\boldsymbol{u}_0 = \nabla\,\theta$. It is clear that $\dfrac{\partial\,\boldsymbol{u}_0}{\partial\,\boldsymbol{n}} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$. We are reduced to solve the following problem:
$$(\mathcal{P}'_{T_0}) \quad \begin{cases} -\Delta\,\boldsymbol{z} = \boldsymbol{f} & \text{in}\,\Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0, \quad \dfrac{\partial\,\boldsymbol{z}}{\partial\,\boldsymbol{n}} \times \boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} - \dfrac{\partial\,\boldsymbol{u}_0}{\partial\,\boldsymbol{n}} \times \boldsymbol{n} & \text{on}\,\Gamma, \end{cases} \tag{6.9}$$
and we consider three cases:
First case: $p = 2$. We will repeat quickly the reasoning given in Lemme 6.1.1. The problem $(\mathcal{P}'_{N_0})$ is equivalent to the following variational formulation: Find $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}^2_T(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}^2_T(\Omega)$,
$$\int_\Omega \nabla\,\boldsymbol{z} : \nabla\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{h}\times\boldsymbol{n} - \frac{\partial\,\boldsymbol{u}_0}{\partial\,\boldsymbol{n}},\, \boldsymbol{v}\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma)}. \tag{6.10}$$
This problem is well posed thanks to the Poincaré inequality and the Lax-Milgram's Lemma. Moreover, the solution $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$ satisfies the estimate:
$$\|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\big). \tag{6.11}$$

Finally, the function $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega)$ is the unique solution of $(\mathcal{P}'_T)$ and satisfies the estimate (6.7).

Second case: $p > 2$. We know by the first step that problem $(\mathcal{P}'_{T_0})$ has a unique solution $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$. Let prove now that $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. We set $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{z}$. Then, $\boldsymbol{w}$ is solution of the elliptic problem:
$$\begin{cases} -\Delta\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{f} & \text{in}\Omega, \\ \operatorname{div}\boldsymbol{w} = 0 & \text{in}\Omega, \\ \boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} - \dfrac{\partial\,\boldsymbol{u}_0}{\partial\,\boldsymbol{n}} \times \boldsymbol{n} & \text{on}\Gamma, \\ \langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0 \ \text{ for any } \ 0 \leq i \leq I. \end{cases} \tag{6.12}$$

According to Corollary 5.4.7, this problem has a unique solution $\boldsymbol{w} \in \boldsymbol{L}^p(\Omega)$. This implies that $\nabla \operatorname{div} \boldsymbol{z} \in \boldsymbol{W}^{-1,p}(\Omega)$ and then $\operatorname{div} \boldsymbol{z} \in L^p(\Omega)$. Moreover, using the Sobolev's imbedding, the function $\boldsymbol{z}$ belongs to $\boldsymbol{L}^6(\Omega)$. Then, $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ for $p \leq 6$. If $p > 6$, then $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. So, $\boldsymbol{z} \in \boldsymbol{L}^q(\Omega)$ for any $1 \leq q \leq \infty$. So, due to Theorem 2.2.8, $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Finally, the function $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution of $(\mathcal{P}'_T)$ and satisfies the estimate (6.7).

Third case: $p < 2$. Let a be the bilinear form defined on $\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^{\,p}(\Omega) \times \boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{\,p'}(\Omega)$ by:

$$a(\boldsymbol{z},\, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

Let

$$A \in \mathcal{L}(\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^{\,p}(\Omega),\, (\boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{\,p'}(\Omega))')$$

and

$$A' \in \mathcal{L}(\boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{\,p'}(\Omega),\, (\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^{\,p}(\Omega))')$$

be the operators defined by

$$\forall \boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^{\,p}(\Omega),\ \forall \boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{\,p'}(\Omega),$$

$$a(\boldsymbol{z},\, \boldsymbol{v}) = \langle A\, \boldsymbol{z},\, \boldsymbol{v} \rangle = \langle \boldsymbol{z},\, A'\, \boldsymbol{v} \rangle.$$

The Babuška-Brezzi's Theorem implies that, for $p > 2$, the operator A associated with a is an isomorphism from $\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^{\,p}(\Omega)$ onto $(\boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{\,p'}(\Omega))'$ and its dual operator A' is an isomorphism from $\boldsymbol{W}^{1,p'}(\Omega) \cap \boldsymbol{X}_T^{\,p'}(\Omega)$ onto $(\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_T^{\,p}(\Omega))'$. As a consequence, the following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{\substack{\boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega) \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0}} \sup_{\substack{\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega) \\ \boldsymbol{z} \cdot \boldsymbol{n} = 0}} \frac{\int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}}{\|\nabla \boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}} \geq \beta. \tag{6.13}$$

But since the bilinear form a is symmetric, the operator A' coincides with A if we interchange p by p'. As $p' < 2$, this means that the above isomorphisms are valid for all real numbers $p > 1$. Again, the Babuška-Brezzi's Theorem implies that the Inf-Sup Condition (6.13) holds for all $p > 1$. Finally, the function $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}(\Omega)$ is the required solution of the problem $(\mathcal{P}'_T)$.

2. Now, we suppose that $g \in W^{2-1/p,p}(\Gamma)$ and $\boldsymbol{h} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. Let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ the unique solution of $(\mathcal{P}'_T)$ and we set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. Then, $\boldsymbol{z}$ satisfies the following problem:

$$\begin{cases} -\Delta \boldsymbol{z} = \mathbf{curl}\, \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{z} = 0 & \text{in } \Omega \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma \\ \langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \ \text{ for any } 0 \leq i \leq I. \end{cases} \tag{6.14}$$

Due to Corollary 5.4.7, $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big). \tag{6.15}$$

This implies that $\nabla \operatorname{div} \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and again due to [5, Proposition 2.10], we have $\operatorname{div} \boldsymbol{u} \in W^{1,p}(\Omega)$. Finally, we are able to conclude by applaying Corollary 2.2.12.

□

6.2 Study of the problem $(\mathcal{P}_N)$

We now interested to solve an elliptic system with an other kind of boundary condition.

Lemma 6.2.1. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f}$ belongs to $\boldsymbol{L}^p(\Omega)$ with $p \geq 6/5$. Then, the following problem*

$$(\mathcal{P}_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0, \quad \dfrac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

has a unique solution $\boldsymbol{u}$ in $\boldsymbol{W}^{2,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.16}$$

Proof. i) **First case:** $p = 2$. We can prove exactly as for the problem $(\mathcal{P}_T)$, the existence of a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ of problem $(\mathcal{P}_T)$. Next, let prove that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$. Let $E(\Delta, \Omega)$ the space defined by:

$$E(\Delta, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \quad \Delta\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$$

Observe that $\boldsymbol{u} \in E(\Delta, \Omega)$. We know that if $\boldsymbol{v} \in E(\Delta, \Omega)$, then $\nabla\, \boldsymbol{v} \in \boldsymbol{H}^{-1/2}(\Gamma)$ and by the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $E(\Delta, \Omega)$, formula (4.40) can be extended for any function $E(\Delta, \Omega)$. So, applying formula (4.40), we have

$$\operatorname{div} \boldsymbol{u} = K\, \boldsymbol{u} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma)$$

. Since $\Delta \operatorname{div} \boldsymbol{u} \in H^{-1}(\Omega)$, it follows then that $\operatorname{div} \boldsymbol{u} \in H^1(\Omega)$. Now, we set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. Then $\boldsymbol{z} \in \boldsymbol{X}_T^2(\Omega)$ and by Theorem 2.2.8, the function $\boldsymbol{z}$ belongs to $\boldsymbol{H}^1(\Omega)$. Applying Theorem 3.3.6, we deduce that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$.

ii) **Second case:** $p > 2$. We know that problem $(\mathcal{P}_T)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Then $\boldsymbol{u}$ belongs to $\boldsymbol{L}^q(\Omega)$ for all $1 \leq q \leq \infty$. Moreover, $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ for any $p \leq 6$. Now, due to the regularity of Ω, the mean curvature K of Γ belongs to $\mathcal{C}^{1,1}(\Gamma)$. So, $K\boldsymbol{u} \cdot \boldsymbol{n}$ belongs to $W^{1-1/6,6}(\Gamma)$ and $\operatorname{div} \boldsymbol{u}$ belongs to $W^{1-1/6,6}(\Gamma)$. We have also that $\Delta \operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{f} \in W^{-1,p}(\Omega)$. So, we distinguish two cases:

a) We suppose that $p \leq 6$. So we have $\operatorname{div} \boldsymbol{u} \in W^{1,p}(\Omega)$ and then $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{X}_T^{\,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. Applying Corollary 3.3.6 gives that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

b) We consider now the case $p > 6$. We know that $\boldsymbol{u} \in \boldsymbol{W}^{2,6}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\infty}(\Omega)$. So as previously, we can prove that $\operatorname{div} \boldsymbol{u} \in W^{1,p}(\Omega)$ and $\operatorname{curl} \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. By applying again Corollary 3.3.6, we are able to conclude.

iii) **Third case:** $6/5 \leq p < 2$. We proceed as in the first case i) in order to prove that the problem $(\mathcal{P}_N)$ has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. Here again we have $\operatorname{div} \boldsymbol{u} \in H^{1/2}(\Gamma)$ and $\Delta(\operatorname{div} \boldsymbol{u}) \in W^{-1,p}(\Omega)$. As a consequence, $\operatorname{div} \boldsymbol{u}$ belongs to $W^{1,p}(\Omega)$ and we prove as previously that $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ which implies that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

□

We also can consider the case of the non homogenuous boundary conditions.

Theorem 6.2.2. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{g} \in W^{1-1/p,p}(\Gamma)$ and $h \in W^{-1/p,p}(\Gamma)$. Then, the following problem*

$$(\mathcal{P}'_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g}, \quad \dfrac{\partial\, \boldsymbol{u}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} = h & \text{on } \Gamma, \end{cases}$$

has a unique solution $\boldsymbol{u}$ in $\boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|h\|_{W^{-1/p,p}(\Gamma)}\big). \tag{6.17}$$

Moreover, if $\boldsymbol{g} \in W^{2-1/p,p}(\Gamma)$ and $h \in W^{1-1/p,p}(\Gamma)$, then the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{W^{2-1/p,p}(\Gamma)} + \|h\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big). \tag{6.18}$$

Proof.

1. We suppose that $\boldsymbol{f} = 0$, $\boldsymbol{g} \in W^{1-1/p,p}(\Gamma)$ and $h \in \boldsymbol{W}^{-1/p,p}(\Gamma)$. Let $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ a solution of the problem (see Corollary 5.2.4):

$$\Delta\, \boldsymbol{\xi} = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 \ \text{ in } \Omega, \qquad \boldsymbol{\xi} \times \boldsymbol{n} = \boldsymbol{g} \ \text{ on } \Gamma.$$

We are reduced to solve the following problem:

$$(\mathcal{P}'_{N_0}) \quad \begin{cases} -\Delta\, \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0}, \quad \dfrac{\partial\, \boldsymbol{z}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} = h - \dfrac{\partial\, \boldsymbol{\xi}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} & \text{on } \Gamma, \end{cases} \tag{6.19}$$

and we consider three cases:
First case: $p = 2$. Using the Lax-Milgram's Lemma, we can prove that problem $(\mathcal{P}'_{N_0})$ has a unique solution $\boldsymbol{z} \in \boldsymbol{H}^{\,1}(\Omega)$.
Second case: $p > 2$. We know by the first step that problem $(\mathcal{P}'_{N_0})$ has a unique solution $\boldsymbol{z} \in \boldsymbol{H}^{\,1}(\Omega)$. Let prove now that $\boldsymbol{z} \in \boldsymbol{W}^{\,1,p}(\Omega)$. Observe that $\boldsymbol{z} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma) \hookrightarrow W^{\,-1/6,6}(\Gamma)$. Then we observe two cases:

a) We suppose that $p \leq 6$. Using formula (4.40), we obtain that $\operatorname{div} \boldsymbol{z}$ satisfies:

$$\Delta(\operatorname{div} \boldsymbol{z}) = 0 \ \text{ in } \Omega \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = K\boldsymbol{z} \cdot \boldsymbol{n} + h - \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \ \text{ on } \Gamma. \tag{6.20}$$

We note that $\nabla \boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ and $\Delta(\nabla \boldsymbol{\xi}) = 0$ in Ω. Then, $\nabla \boldsymbol{\xi} \in \boldsymbol{W}^{\,-1/p,p}(\Gamma)$ and $\dfrac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \in W^{\,-1/p,p}(\Gamma)$ so that $\operatorname{div} \boldsymbol{z} \in L^p(\Omega)$. Moreover, we set $\boldsymbol{w} = \mathbf{curl}\, \boldsymbol{z}$. Observe that $\mathbf{curl}\, \boldsymbol{w} = \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{z} = -\Delta \boldsymbol{z} + \nabla \operatorname{div} \boldsymbol{z}$. Then, $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{W}^{\,-1,p}(\Omega)$ and $\Delta\, \mathbf{curl}\, \boldsymbol{w} = \boldsymbol{0}$ in Ω. Due to [10] the trace of $\mathbf{curl}\, \boldsymbol{w}$ on Γ belongs to $\boldsymbol{W}^{\,-1-1/p,p}(\Gamma)$. Since the function $\boldsymbol{w}$ satisfies

$$\begin{cases} \Delta\, \boldsymbol{w} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{n} \in \boldsymbol{W}^{\,-1-1/p,p}(\Gamma) & \text{on } \Gamma, \end{cases} \tag{6.21}$$

it follows from Corollary 4.4.6, that $\boldsymbol{w} \in \boldsymbol{L}^p(\Omega)$ and by Corollary 2.2.4, we deduce that $\boldsymbol{z} \in \boldsymbol{W}^{\,1,p}(\Omega)$.

b) Now we suppose that $p > 6$. We know only that the solution $\boldsymbol{z} \in \boldsymbol{W}^{\,1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Then, $\boldsymbol{z} \cdot \boldsymbol{n} \in W^{\,1-1/6,6}(\Gamma) \hookrightarrow W^{-1/q,q}(\Gamma)$ for any finite q. We deduce again from (6.20) that $\operatorname{div} \boldsymbol{z} \in L^p(\Omega)$. Next, as previously, we can prove that $\mathbf{curl}\, \boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$. Finally, since $\boldsymbol{z} \in \boldsymbol{L}^\infty(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, we have that $\boldsymbol{z} \in \boldsymbol{W}^{\,1,p}(\Omega)$.

Third case: $1 < p < 2$. By using a duality argument, as for problem $(\mathcal{P}'_T)$, the problem $(\mathcal{P}'_{N_0})$ has a unique solution $\boldsymbol{z} \in \boldsymbol{W}^{\,1,p}(\Omega)$.

Finally, the function $\boldsymbol{u}_1 = \boldsymbol{z} + \boldsymbol{\xi} \in \boldsymbol{W}^{\,1,p}(\Omega)$ is the unique solution of the problem

$$\begin{cases} -\Delta\, \boldsymbol{u}_1 = \boldsymbol{0} & \text{in } \Omega, \\ \boldsymbol{u}_1 \times \boldsymbol{n} = \boldsymbol{g}, \quad \dfrac{\partial\, \boldsymbol{u}_1}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} = h & \text{on } \Gamma, \end{cases}$$

2. We suppose that $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{g} = \boldsymbol{0}$ and $h = 0$. Thanks to Lemma 6.2.1, there existes a unique solution $\boldsymbol{u}_2 \in \boldsymbol{W}^{\,2,p}(\Omega)$ of the problem $(\mathcal{P}_N)$.
We deduce from the points 1. and 2. that the function $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 \in \boldsymbol{W}^{\,1,p}(\Omega)$ is the required solution of the problem $(\mathcal{P}'_N)$.

3. We suppose now that $\boldsymbol{g} \in W^{2-1/p,p}(\Gamma)$ and $h \in W^{1-1/p,p}(\Gamma)$. Then, $\boldsymbol{\xi} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\dfrac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \in W^{-1/p,p}(\Gamma)$. By (6.20) $\operatorname{div} \boldsymbol{z}$ belongs to $W^{1,p}(\Omega)$. We set $\boldsymbol{w} = \mathbf{curl}\, \boldsymbol{z}$. Then, $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^p(\Omega)$ and $\Delta\,(\mathbf{curl}\, \boldsymbol{w}) = 0$ in Ω. This implies that $\mathbf{curl}\, \boldsymbol{w}$ belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$. Since $\boldsymbol{w}$ satisfies (6.21), then due to Proposition 4.2.3, $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$. Applying Corollary 3.3.6, we obtain that $\boldsymbol{z} \in \boldsymbol{W}^{2,p}(\Omega)$. Finally, the solution $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{\xi}$ of $(\mathcal{P}'_N)$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$.

□

Part III

Oseen and Navier-Stokes problem with non standard boundary conditions

Chapter 7

Oseen problem with normal boundary conditions

7.1 Introdution

In this chapter, we want to study the existence of a generalized and strong solutions for the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = \pi_0 + c_i & \text{on } \Gamma_i,\ i = 0, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{7.1}$$

where $\boldsymbol{f}$, $\boldsymbol{a}$ and π_0 are given functions, c_i are constants and we will denote by $\boldsymbol{c}$ the vector $\boldsymbol{c} = (c_0, \ldots, c_I)$.

7.2 Study of the problem (7.1)

First, we are going to suppose that

$$\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega) \quad \text{and that} \quad \boldsymbol{f} \in [\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]' \quad \text{or} \quad \boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega).$$

We introduce the space:

$$\boldsymbol{V}_N = \Big\{ \boldsymbol{v} \in H^1(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \text{ and } \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0, \forall 1 \leq i \leq I \Big\}.$$

Let us give the following lemma concerning a Poincaré type inequality which is a consequence of Corollary 3.3.8 (see also [26]).

Lemma 7.2.1. *The space $\boldsymbol{V}_N$ is a Hilbertian space and the semi norm:*

$$\boldsymbol{v} \mapsto \Big(\int_\Omega |\mathbf{curl}\, \boldsymbol{v}|^2 \Big)^{1/2} \tag{7.2}$$

is a norm on $\boldsymbol{V}_N$ equivalent to the full norm of $\boldsymbol{H}^1(\Omega)$.

We have the following theorem.

Theorem 7.2.2. *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]'$, $\pi_0 \in H^{-1/2}(\Gamma)$ and $\boldsymbol{a} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$. Then, there exists a unique $\boldsymbol{u} \in \boldsymbol{V}_N$ such that*

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v} = \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_\Omega - \langle \pi_0,\, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_\Gamma, \tag{7.3}$$

where the dualities on Ω and Γ are defined by:

$$\langle \cdot,\, \cdot \rangle_\Omega = \langle \cdot,\, \cdot \rangle_{[\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]' \times \boldsymbol{H}_0^2(\mathbf{curl}), \Omega}, \quad \langle \cdot,\, \cdot \rangle_\Gamma = \langle \cdot,\, \cdot \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}. \tag{7.4}$$

Moreover, we have the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} \Big). \tag{7.5}$$

Proof. Let $a(\cdot, \cdot) : \boldsymbol{V}_N \times \boldsymbol{V}_N \to \mathbb{R}$ be the following bilinear continuous form:

$$\forall \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad a(\boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}. \tag{7.6}$$

The bilinear form $a(\cdot, \cdot)$ is coercive on $\boldsymbol{V}_N \times \boldsymbol{V}_N$. Indeed, by Lemma 7.2.1, the semi norm $\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$ is equivalent to the norm $\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}$ and for any $\boldsymbol{v} \in \boldsymbol{V}_N$ we have:

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{v}) \cdot \boldsymbol{v} = 0.$$

Since the right-hand side in (7.3) belongs to $(\boldsymbol{V}_N)'$, by the Lax-Milgram Lemma, the problem (7.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}_N$ satisfying the estimate (7.5). □

Remark 7.2.3. In the same way, suppose that $\boldsymbol{f}$ belongs to $\boldsymbol{L}^{6/5}(\Omega)$ with π_0, $\boldsymbol{a}$ as in Theorem 7.2.2. Then, the problem (7.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ where we replace the brackets $\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{[\boldsymbol{H}_0^2(\mathbf{curl}, \Omega)]' \times \boldsymbol{H}_0^2(\mathbf{curl}, \Omega)}$ by the integral $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$.

Remark 7.2.4. Before establishing the link between the problem (7.1) and the variational formulation (7.3), we will see in what functional space it is reasonable to find the pressure π appearing in (7.1), knowing that we are first ineresting to velocity fields in $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ (see Remark 7.2.3). With a such vector $\boldsymbol{u}$, we have $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^{6/5}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$. Since $\Delta\, \boldsymbol{u} \in \boldsymbol{H}^{-1}(\Omega)$, we deduce from the first equation in (7.1) that $\nabla\, \pi \in \boldsymbol{H}^{-1}(\Omega)$. According to [6], the pressure π belongs to $L^2(\Omega)$. Furtheremore,

$$-\Delta\, \pi = \operatorname{div} \boldsymbol{f} - \operatorname{div} (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}),$$

so that $\Delta\, \pi \in \boldsymbol{W}^{-1,6/5}(\Omega)$. We can prove as in Lemma 1.0.10 that the trace of π on Γ belongs to $H^{-1/2}(\Gamma)$ so that we must assume that $\pi_0 \in H^{-1/2}(\Gamma)$.

Theorem 7.2.5. *Let $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\pi_0 \in H^{-1/2}(\Gamma)$ and $\boldsymbol{a} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$. Then, the problems:*

$$\textit{Find } (\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{V}_N \times L^2(\Omega) \times \mathbb{R}^{I+1} \textit{ satisfying } (7.1) \textit{ with } \langle \pi,\, 1 \rangle_\Gamma = 0 \tag{7.7}$$

and

Find $\boldsymbol{u} \in \boldsymbol{V}_N$ and constants $c_0, \ldots, c_I$ satisfying $\displaystyle\sum_{i=0}^{I} c_i \operatorname{mes} \Gamma_i + \langle \pi_0,\, 1 \rangle_\Gamma = 0$ and such that

$$\forall \boldsymbol{v} \in \boldsymbol{V}_N, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \langle \pi_0,\, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_\Gamma, \tag{7.8}$$

$$\forall i = 1, \ldots, I, \quad c_i - c_0 = \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma \tag{7.9}$$

are equivalent

Proof. Let $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$, with $\langle \pi,\, 1 \rangle_\Gamma = 0$, solution of the problem (7.1) and we prove that $\boldsymbol{u}$ and the constants $c_0, \ldots, c_I$ are also solution of the variational problem (7.8) and (7.9). It is clear that $\boldsymbol{u} \in \boldsymbol{V}_N$ and that $\sum_{i=0}^{I} c_i \operatorname{mes} \Gamma_i + \langle \pi_0,\, 1 \rangle_\Gamma = 0$. The relation (7.8) is satisfied. Indeed, multiplying the first equation of (7.1) by $\boldsymbol{v} \in \boldsymbol{V}_N$, integrating by parts in Ω, we obtain

$$\int_\Omega (-\Delta\, \boldsymbol{u} + \nabla\, \pi) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \cdot \boldsymbol{v})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}. \tag{7.10}$$

But $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is dense in the space

$$\boldsymbol{\mathcal{M}} = \left\{ (\boldsymbol{u},\, \pi) \in \boldsymbol{H}^1_\sigma(\Omega) \times L^2(\Omega);\ -\Delta\, \boldsymbol{u} + \nabla\, \pi \in \boldsymbol{L}^{6/5}(\Omega) \right\}$$

and we have the following Green formula: For any $(\boldsymbol{u},\, \pi) \in \boldsymbol{\mathcal{M}}$ and $\boldsymbol{\varphi} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\int_\Omega (-\Delta\, \boldsymbol{u} + \nabla\, \pi) \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \langle \pi,\, \boldsymbol{\varphi} \cdot \boldsymbol{n} \rangle_\Gamma. \tag{7.11}$$

Taking into account the boundary condition that verify the pressure π on Γ, we have:

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_N, \quad \langle \pi,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = \langle \pi_0,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma, \tag{7.12}$$

which implies that $\boldsymbol{u}$ satisfies (7.8). It remains to prove the relation (7.9).

Now, let $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ and we set

$$\boldsymbol{v}_0 = \boldsymbol{v} - \sum_{i=1}^{I}\Big(\int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\Big)\,\nabla\, q_i^N. \tag{7.13}$$

Observe that $\boldsymbol{v}_0$ belongs to $\boldsymbol{V}_N$. Multiplying the first equation of (7.1) by $\boldsymbol{v}$, integrating by parts in Ω and using the relation (7.8) with the test function $\boldsymbol{v}_0$ and (7.25), we obtain

$$\begin{aligned}
-\sum_{i=1}^{I}\Big(\int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\Big)\int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\nabla\, q_i^N\,\mathrm{d}\boldsymbol{x} &+ \sum_{i=1}^{I}\Big(\int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\Big)\int_\Omega \boldsymbol{f}\cdot\nabla\, q_i^N\,\mathrm{d}\boldsymbol{x}\\
&= \sum_{i=1}^{I}\Big(\int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\Big)\Big\{\sum_{j=0}^{I} c_j\int_{\Gamma_j}\nabla\, q_i^N\cdot\boldsymbol{n} + \langle\pi_0,\,\nabla\, q_i^N\cdot\boldsymbol{n}\rangle_\Gamma\Big\}\\
&= \sum_{i=1}^{I}\Big(\int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\Big)\Big\{(c_i - c_0) + \langle\pi_0,\,\nabla\, q_i^N\cdot\boldsymbol{n}\rangle_\Gamma\Big\}.
\end{aligned}$$

Particularly, for $\boldsymbol{v} = \nabla\, q_j^N$, we deduce that

$$c_j - c_0 = \int_\Omega \boldsymbol{f}\cdot\nabla\, q_j^N\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\nabla\, q_j^N\,\mathrm{d}\boldsymbol{x} - \langle\pi_0,\,\nabla\, q_j^N\cdot\boldsymbol{n}\rangle_\Gamma,$$

which is the required relation in (7.9).

ii) Reciprocally, let $\boldsymbol{u} \in \boldsymbol{V}_N$ a solution of (7.8) and $c_0,\ldots,c_I$ constants satisfying the condition $\sum_{i=0}^{I} c_i\,\mathrm{mes}\,\Gamma_i + \langle\pi_0,\,1\rangle_\Gamma = 0$ and (7.9). To prove the first equation of the problem (7.1), let us take a function $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ as the test function in (7.8). It is clear that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ we have:

$$\langle -\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} - \boldsymbol{f},\, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

So, by the De Rham's Theorem, there exists a distribution π in $\mathcal{D}'(\Omega)$ defined uniquely up to an additive constant such that

$$-\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} - \boldsymbol{f} = -\nabla\,\pi \tag{7.14}$$

in the sens of distributions in Ω. Moreover, since $\Delta\,\boldsymbol{u} \in \boldsymbol{H}^{-1}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} \in \boldsymbol{L}^{6/5}(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$, we deduce that $\nabla\,\pi \in \boldsymbol{H}^{-1}(\Omega)$. Due to [6], π belongs to $L^2(\Omega)$. Now, applying divergence operator to equations (7.14), we obatin :

$$\Delta\,\pi = \mathrm{div}\,\boldsymbol{f} - \mathrm{div}\,(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}) \in W^{-1,6/5}(\Omega). \tag{7.15}$$

Since $\pi \in L^2(\Omega)$, we can prove as in Lemma 1.0.10 that the trace of π on Γ belongs to $H^{-1/2}(\Gamma)$. Then, since π is defined up to an additive constant, we can choose the this constant so that $\langle \pi,\, 1\rangle_\Gamma = 0$.

It remains to prove the boundary condition on the pressure. Let $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. Multiplying (7.14) by $\boldsymbol{v}$, integrating by parts in Ω we obtain as before:

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \langle \pi,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma, \tag{7.16}$$

Using the decomposition (7.13), we obtain:

$$\begin{aligned} &\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}_0\, \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{v}_0\, \mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_0\, \mathrm{d}\boldsymbol{x} + \langle \pi,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma \\ = \;& -\sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} + \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \\ - \;& \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \langle \pi,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma. \end{aligned} \tag{7.17}$$

Particularly, if $\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} = 0$ for any $i = 1, \ldots, I$, we deduce from (7.8) that:

$$\langle \pi,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma = \langle \pi_0,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma. \tag{7.18}$$

Using again (7.8), the relation (7.17) becomes:

$$\begin{aligned} \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \langle \pi,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma \;=\; & \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \\ & - \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{7.19}$$

Now, using the decomposition (7.13) and (7.18), we can write for any $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\begin{aligned} \langle \pi,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma \;&=\; \langle \pi,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \langle \pi,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma \\ &=\; \langle \pi_0,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \langle \pi,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma. \end{aligned}$$

Using (7.19) this last relation becomes:

$$\begin{aligned} \langle \pi,\, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_\Gamma \;=\; & \langle \pi_0,\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \\ & - \sum_{i=1}^I \big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\big) \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x}. \end{aligned}$$

As a consequence, from the relation (7.9) we obtain:

$$\begin{aligned}\langle \pi,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma &= \langle \pi_0,\, \boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^{I}\big(\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}\big)\langle \pi_0,\, \nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^{I}\big(\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}\big)(c_i-c_0)\\ &= \langle \pi_0,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^{I}\langle c_i-c_0,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma.\end{aligned}$$

Since $\operatorname{div}\boldsymbol{v}=0$ in Ω, we have $\sum_{i=0}^{I}\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}=0$ and then we can write

$$\langle \pi,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = \langle \pi_0,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma + \sum_{i=0}^{I}\langle c_i,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = \langle \pi_0+c,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma, \tag{7.20}$$

with $c=c_i$ on Γ_i for any $i=0,\ldots,I$.

Finally, let $\mu\in H^{1/2}(\Gamma_j)$ for any fixed j with $0\le j\le I$ and we set

$$\mu' = \begin{cases}\mu & \text{on } \Gamma_j,\\ 0 & \text{on } \Gamma-\Gamma_j.\end{cases}$$

We know that there exists a vector $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$ with $\operatorname{div}\boldsymbol{v}=0$ in Ω such that

$$\boldsymbol{v} = \begin{cases}\Big(\mu - \frac{1}{|\Gamma_j|}\int_{\Gamma_j}\mu\Big)\boldsymbol{n} & \text{on } \Gamma_j,\\ \boldsymbol{0} & \text{on } \Gamma-\Gamma_j.\end{cases} \tag{7.21}$$

As a consequence we have,

$$\langle \pi-\pi_0,\, \mu\rangle_{\Gamma_j} = \langle \pi-\pi_0,\, \mu'\rangle_{\Gamma_j} = \langle \pi-\pi_0,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = \langle c_j,\, \mu\rangle_{\Gamma_j} \tag{7.22}$$

and then $\pi=\pi_0+c_j$ on Γ_j.

□

Theorem 7.2.6. *Let $\boldsymbol{f}\in\boldsymbol{L}^{6/5}(\Omega)$, $\mathbf{curl}\,\boldsymbol{a}\in\boldsymbol{L}^{3/2}(\Omega)$ and $\pi_0\in H^{-1/2}(\Gamma)$, then the problem (7.1) has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{c})\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times\mathbb{R}^{I+1}$ with $\langle\pi,\,1\rangle_\Gamma=0$ and we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{7.23}$$

$$\|\pi\|_{L^2(\Omega)} \le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{7.24}$$

where $\boldsymbol{c}=(c_0,\ldots,c_I)$.

Moreover, if $\pi_0\in W^{1/6,6/5}(\Gamma)$ then $\boldsymbol{u}\in\boldsymbol{W}^{2,6/5}(\Omega)$ and $\pi\in W^{1,6/5}(\Omega)$.

Proof. i) Let $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\pi_0 \in H^{-1/2}(\Gamma)$. We know due to Theorem 7.2.5 that the problem (7.1) is equivalent to (7.8)-(7.9). Due to Remark 7.2.3, we deduce that (7.1) has a unique solution $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$ satisfying the estimate (7.23). The estimate (7.24) can be obtained by using the fact that the pressure π verifies:

$$-\Delta\, \pi = \operatorname{div} \boldsymbol{f} + \operatorname{div} (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \ \text{ in } \Omega \quad \text{and} \quad \pi = \pi_0 + c_i \ \text{ on } \Gamma_i. \tag{7.25}$$

ii) Now, we suppose that $\pi_0 \in W^{1/6,6/5}(\Gamma)$. Let $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) \in \boldsymbol{H}^{\,1}(\Omega) \times L^2(\Omega) \times \mathbb{R}^{I+1}$ the solution of (7.1) given by step i). Since the pressure π verifies (7.25), we deduce directly that $\pi \in W^{1,6/5}(\Omega)$. The velocity $\boldsymbol{u}$ is then a solution of the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, & \end{cases}$$

with $\boldsymbol{F} = \boldsymbol{f} - \nabla\, \pi - \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^{6/5}(\Omega)$. We deduce from Proposition 5.2.2, that $\boldsymbol{u} \in \boldsymbol{W}^{\,2,6/5}(\Omega)$.

□

Remark 7.2.7. Even if the pressure π change in $\pi - c_0$, the system (7.1) is equivalent to the following Oseen problem:

$$(\mathcal{OS}_N) \qquad \begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i,\ i = 1, \ldots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, & \end{cases}$$

where the unknowns constants satisfy for any $i = 1, \ldots, I$:

$$\alpha_i = \int_\Omega \boldsymbol{f} \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma.$$

But, it is clear that the new pressure π does not satisfy the condition $\langle \pi,\, 1 \rangle_\Gamma = 0$.

For $1 < r,\, p < \infty$, we introduce the space

$$\boldsymbol{H}_0^{r,p}(\mathbf{curl},\, \Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{L}^r(\Omega),\ \ \mathbf{curl}\, \boldsymbol{\varphi} \in \boldsymbol{L}^p(\Omega),\ \ \boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0} \ \text{on}\, \Gamma\},$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{r,p}(\mathbf{curl},\, \Omega)} = \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathbf{curl}\, \boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.$$

We can prove that the space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)$ and its dual space can be characterized as:

$$[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]' = \{\boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi},\ \ \boldsymbol{F} \in \boldsymbol{L}^r(\Omega),\ \boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)\}. \tag{7.26}$$

Remark 7.2.8. If we suppose that $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\pi_0 \in H^{-1/2}(\Gamma)$, then the problems (7.7) and (7.8)-(7.9) are equivalent. The proof is similar to that of Theorem 7.2.5 with the difference that we use here the duality brackets between $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$ and $\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)$ in place of the integral on Ω in the right hand side of (7.8) and the density of $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ in the space

$$\boldsymbol{\mathcal{M}} = \big\{(\boldsymbol{u},\,\pi) \in \boldsymbol{H}^1_\sigma(\Omega) \times L^2(\Omega);\ \ -\Delta\,\boldsymbol{u} + \nabla\,\pi \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'\big\}.$$

It is easy now to expand Theorem 7.2.6 to the case where $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$.

Theorem 7.2.9. *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\pi_0 \in H^{-1/2}(\Gamma)$, then the problem $(\mathcal{OS}_N)$ has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ satisfying the estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{7.27}$$

$$\|\pi\|_{L^2(\Omega)} \le C\big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\big), \tag{7.28}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$.

Now, we consider the case of non homogenuous boundary conditions and when the divergence operator does not vanish.

Theorem 7.2.10. *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$, $\chi \in W^{1,6/5}(\Omega)$, $\pi_0 \in H^{-1/2}(\Gamma)$ and $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$. Then the problem*

$$\begin{cases} -\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi = \pi_0 + \alpha_i,\ i = 1, \ldots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{7.29}$$

has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ verifying the estimate:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \ &\le\ C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big) \times \\ &\times\ \big(\|\chi\|_{W^{1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big), \end{aligned} \tag{7.30}$$

$$\begin{aligned}\|\pi\|_{L^2(\Omega)} &\leq C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'}+\|\pi_0\|_{H^{-1/2}(\Gamma)}+\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\times\\ &\times\ \big(\|\chi\|_{W^{1,6/5}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big),\end{aligned}$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_I)$. *Moreover, if* $\boldsymbol{f}\in\boldsymbol{L}^{6/5}(\Omega)$, $\pi_0\in W^{1/6,6/5}(\Gamma)$ *and* $\boldsymbol{g}\in\boldsymbol{W}^{7/6,6/5}(\Gamma)$, *then* $\boldsymbol{u}\in\boldsymbol{W}^{2,6/5}(\Omega)$ *and* $\pi\in W^{1,6/5}(\Omega)$.

Proof) Let $\boldsymbol{g}\in\boldsymbol{H}^{1/2}(\Gamma)$. We know that there exists $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{w}=\boldsymbol{g}$ on Γ and verifying:

$$\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}\leq C\|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{7.31}$$

Let $\theta\in H^2(\Omega)$ the unique solution of the problem:

$$\begin{cases}\Delta\,\theta=\operatorname{div}\boldsymbol{w}-\chi & \text{in}\,\Omega,\\ \theta=0 & \text{on}\,\Gamma,\end{cases} \tag{7.32}$$

satisfying the estimate:

$$\|\theta\|_{\boldsymbol{H}^2(\Omega)}\leq C\big(\|\chi\|_{W^{1,6/5}(\Omega)}+\|\operatorname{div}\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}\big). \tag{7.33}$$

We set $\boldsymbol{u}_0=\nabla\,\theta+\boldsymbol{w}$. Then, $\boldsymbol{u}_0\in\boldsymbol{H}^1(\Omega)$ and satisfies:

$$\operatorname{div}\boldsymbol{u}_0=\chi\ \text{in}\ \Omega\ \text{and}\ \boldsymbol{u}_0\times\boldsymbol{n}=\boldsymbol{g}\times\boldsymbol{n}\ \text{on}\ \Gamma,$$

with:

$$\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}\leq C\big(\|\chi\|_{W^{1,6/5}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big). \tag{7.34}$$

We know by Theorem 7.2.9 that there exists a unique $(\boldsymbol{z},\,\pi,\,\boldsymbol{\alpha})\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times\mathbb{R}^I$ solution of

$$\begin{cases}-\Delta\,\boldsymbol{z}+\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{z}+\nabla\,\pi=\boldsymbol{F}_0 & \text{in}\ \Omega,\\ \operatorname{div}\boldsymbol{z}=0 & \text{in}\ \Omega,\\ \boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{0} & \text{on}\ \Gamma,\\ \pi=\pi_0\ \text{on}\ \Gamma_0,\ \ \pi=\pi_0+\alpha_i & \text{on}\ \Gamma_i,\ i=1,\ldots,I,\\ \int_{\Gamma_i}\boldsymbol{z}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma}=0,\ i=1,\ldots,I,\end{cases} \tag{7.35}$$

where $\boldsymbol{F}_0=\boldsymbol{f}+\Delta\,\boldsymbol{u}_0-\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0$. Observe that $\Delta\,\boldsymbol{u}_0=\nabla\,\chi-\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}_0)\in[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$ which implies that $\boldsymbol{F}_0\in[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'$. Moreover, $\boldsymbol{z}$ satisfies the estimate:

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} &\leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'}+\|\Delta\,\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'}+\\ &+\ \|\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\,\Omega)]'}+\|\pi_0\|_{H^{-1/2}(\Gamma)}\Big)\end{aligned} \tag{7.36}$$

Using the estimate (7.34), we obtain:

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{H}^{\,1}(\Omega)} &\leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)} + \big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\times\\ &\times \big(\|\chi\|_{W^{\,1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big)\Big). \qquad (7.37)\end{aligned}$$

In addition the pressure π satisfies the estimate:

$$\|\pi\|_{L^2(\Omega)} \leq C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\Big(\|\boldsymbol{F}_0\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{H^{-1/2}(\Gamma)}\Big).$$

using again (7.34), we obtain:

$$\begin{aligned}\|\pi\|_{L^2(\Omega)} &\leq C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\big)\times\\ &\times \big(\|\chi\|_{W^{\,1,6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big) + \|\pi_0\|_{H^{-1/2}(\Gamma)}\Big).\end{aligned}$$

Finally, the pair of functions $(\boldsymbol{u},\,\pi) = (\boldsymbol{z}+\boldsymbol{u}_0),\,\pi)$ is the required solution.

ii) Now, we suppose that $\boldsymbol{f}\in\boldsymbol{L}^{6/5}(\Omega)$, $\pi_0\in W^{\,1/6,6/5}(\Gamma)$ and $\boldsymbol{g}\in\boldsymbol{W}^{\,7/6,6/5}(\Gamma)$. Let $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha})\in \boldsymbol{H}^{\,1}(\Omega)\times L^2(\Omega)\times\mathbb{R}^I$ the solution of (7.29) given by step i). Since the pressure π verifies

$$-\Delta\,\pi = \operatorname{div}\boldsymbol{f} + \Delta\,\chi - \operatorname{div}\left(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}\right)\ \text{ in }\Omega,$$

which belongs to $\boldsymbol{W}^{\,-1,6/5}(\Omega)$, then $\pi\in W^{\,1,6/5}(\Omega)$. The velocity $\boldsymbol{u}$ is then a solution of the following problem:

$$\begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{F} & \text{in }\Omega,\\ \operatorname{div}\boldsymbol{u} = \chi & \text{in }\Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} & \text{on }\Gamma,\\ \int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i=1,\ldots,I, & \end{cases}$$

with $\boldsymbol{F} = \boldsymbol{f} - \nabla\,\pi - \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}\in\boldsymbol{L}^{6/5}(\Omega)$. We set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Then the function $\boldsymbol{z}$ satisfies: $\boldsymbol{z}\in\boldsymbol{L}^{6/5}(\Omega)$, $\operatorname{div}\boldsymbol{z} = 0$ in Ω and $\mathbf{curl}\,\boldsymbol{z} = \boldsymbol{F}+\nabla\,\chi\in\boldsymbol{L}^{6/5}(\Omega)$. Moreover, since $\boldsymbol{u}\times\boldsymbol{n}\in\boldsymbol{W}^{\,7/6,6/5}(\Gamma)$, applying Corollary 5.3.2 we obtain that $\boldsymbol{z}\cdot\boldsymbol{n}\in W^{1/6,6/5}(\Gamma)$. So, from Theorem 2.2.10, we deduce that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{\,1,6/5}(\Omega)$. Then, $\boldsymbol{u}$ satisfies:

$$\boldsymbol{u}\in\boldsymbol{L}^{6/5}(\Omega),\ \operatorname{div}\boldsymbol{u} = \chi\in W^{\,1,6/5}(\Omega),\ \mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{W}^{\,1,6/5}(\Omega)\text{ and }\boldsymbol{u}\times\boldsymbol{n}\in\boldsymbol{W}^{\,7/6,6/5}(\Gamma).$$

We deduce from Corollary 3.3.6 that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{\,2,6/5}(\Omega)$.

□

7.3 Generalized and strong solutions

We begin by proving strong solution for the problem $(\mathcal{OS}_N)$.

Theorem 7.3.1. *Let $p \geq 6/5$,*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \ \pi_0 \in W^{\,1-1/p,p}(\Gamma),\ \ \mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$$

with

$$s = \frac{3}{2} \text{ if } p < \frac{3}{2}, \qquad s = p \text{ if } p > \frac{3}{2}, \qquad s = \frac{3}{2} + \epsilon \text{ if } p = \frac{3}{2},$$

for some arbitrary $\epsilon > 0$. Then, the solution $(\boldsymbol{u}, \pi)$ given by Theorem 7.2.6 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{\,1,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{\,1,p}(\Omega)} \leq C\big(1 + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\big). \tag{7.38}$$

Proof. We know by Theorem 7.2.6 that the solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{\,2,6/5}(\Omega) \times W^{\,1,6/5}(\Omega)$. We set $\boldsymbol{b} = \mathbf{curl}\,\boldsymbol{a}$. Then $\boldsymbol{b} \in \boldsymbol{L}^s_\sigma(\Omega)$. Since the space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\boldsymbol{L}^s_\sigma(\Omega)$, there exists a sequence $\boldsymbol{b}_\lambda \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ such that $\boldsymbol{b}_\lambda$ converges to $\boldsymbol{b}$ in $\boldsymbol{L}^s(\Omega)$ as $\lambda \to 0$. Therefore, we search for $(\boldsymbol{u}_\lambda, \pi_\lambda, \boldsymbol{\alpha}_\lambda) \in \boldsymbol{W}^{\,2,p}(\Omega) \times W^{\,1,p}(\Omega) \times \mathbb{R}^I$ solution of the problem:

$$(OS_N)_\lambda \qquad \begin{cases} -\Delta\,\boldsymbol{u}_\lambda + \boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda + \nabla\,\pi_\lambda = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u}_\lambda = 0 & \text{in } \Omega \\ \boldsymbol{u}_\lambda \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi_\lambda = \pi_0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \pi_\lambda = \pi_0 + \alpha_i^\lambda,\ i = 1, \ldots, I, & \text{on } \Gamma_i, \\ \int_{\Gamma_i} \boldsymbol{u}_\lambda \cdot \boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases}$$

with $\boldsymbol{\alpha}_\lambda = (\alpha_1^\lambda, \ldots, \alpha_I^\lambda)$. Remember that from above we can obtain a unique solution $(\boldsymbol{u}_\lambda, \pi_\lambda)$ belongs to $\boldsymbol{W}^{\,2,6/5}(\Omega) \times W^{\,1,6/5}(\Omega)$. Since $\boldsymbol{u}_\lambda \in \boldsymbol{L}^6(\Omega)$, then $\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda \in \boldsymbol{L}^6(\Omega)$. Using the Stokes regularity, we deduce that $(\boldsymbol{u}_\lambda, \pi_\lambda)$ belongs to $\boldsymbol{W}^{\,2,p}(\Omega) \times W^{\,1,p}(\Omega)$ if $p \leq 6$. Now, we suppose that $p > 6$. We know that $\boldsymbol{u}_\lambda \in \boldsymbol{W}^{\,2,6}(\Omega)$ and then $\boldsymbol{u}_\lambda \in \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda \in \boldsymbol{L}^\infty(\Omega)$. Again using the Stokes regularity, we deduce that $(\boldsymbol{u}_\lambda, \pi_\lambda)$ belongs to $\boldsymbol{W}^{\,2,p}(\Omega) \times W^{\,1,p}(\Omega)$. Thus, we focus on the obtention of a strong estimate for $(\boldsymbol{u}_\lambda, \pi_\lambda)$ independently from λ. Let $\epsilon > 0$ with $0 < \lambda < \epsilon/2$. We consider:

$$\boldsymbol{b}_\lambda = \boldsymbol{b}_1^\epsilon + \boldsymbol{b}_{\lambda,\,2}^\epsilon \quad \text{where} \quad \boldsymbol{b}_1^\epsilon = \widetilde{\boldsymbol{b}} \star \rho_{\epsilon/2} \quad \text{and} \quad \boldsymbol{b}_{\lambda,\,2}^\epsilon = \boldsymbol{b}_\lambda - \boldsymbol{b}_1^\epsilon, \tag{7.39}$$

being $\widetilde{\boldsymbol{b}}$ the extension by zero of $\boldsymbol{b}$ to $\mathbb{R}^3$ and $\rho_{\epsilon/2}$ the classical mollifier. By regularity estimates for the Stokes problem, we have

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{\,2,p}(\Omega)} + \|\pi_\lambda\|_{W^{\,1,p}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)} + \sum_{i=1}^{I} |\alpha_i^\lambda| + \|\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}\Big), \tag{7.40}$$

where the constant C is independent of λ. We note that

$$\begin{aligned}\sum_{i=1}^{I}|\alpha_i^\lambda| &\leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)}(1+\|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{3/2}(\Omega)})+\|\pi_0\|_{H^{-1/2}(\Gamma)}\Big),\\ &\leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)}(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)})+\|\pi_0\|_{H^{-1/2}(\Gamma)}\Big). \qquad (7.41)\end{aligned}$$

Now, we use the decomposition (7.39) in order to bound the term $\|\boldsymbol{b}_\lambda\times\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}$. We observe first that

$$\|\boldsymbol{b}^\epsilon_{\lambda,2}\|_{\boldsymbol{L}^s(\Omega)}\leq\|\boldsymbol{b}_\lambda-\boldsymbol{b}\|_{\boldsymbol{L}^s(\Omega)}+\|\boldsymbol{b}-\widetilde{\boldsymbol{b}}\star\rho_{\epsilon/2}\|_{\boldsymbol{L}^s(\Omega)}\leq\lambda+\epsilon/2\leq\epsilon. \qquad (7.42)$$

Recall that

$$\boldsymbol{W}^{2,p}(\Omega)\hookrightarrow\boldsymbol{L}^m(\Omega)$$

with $\frac{1}{m}=\frac{1}{p}-\frac{2}{3}$ if $p<\frac{3}{2}$, for any $m\geq 1$ if $p=\frac{3}{2}$ and for any $m\in[1,\,\infty]$ if $p>\frac{3}{2}$. Moreover the imbedding

$$\boldsymbol{W}^{2,p}(\Omega)\hookrightarrow\boldsymbol{L}^q(\Omega)$$

is compact for any $q<m$ if $p<\frac{3}{2}$, for any $q\in[1,\,\infty[$ if $p=\frac{3}{2}$ and for any $q\in[1,\,\infty]$ if $p>\frac{3}{2}$. Then, using the Hölder inequality and the Sobolev imbedding, we obtain

$$\|\boldsymbol{b}^\epsilon_{\lambda,2}\times\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}\leq\|\boldsymbol{b}^\epsilon_{\lambda,2}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^m(\Omega)}\leq C\epsilon\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}, \qquad (7.43)$$

where $\frac{1}{m}=\frac{1}{p}-\frac{1}{s}$, which is well defined because the definition of the real number s. For the second estimate, we consider two cases.

i) Case $p\leq 3/2$. Let $r\in[\frac{3}{2},\,\infty]$ such that $\frac{1}{p}=\frac{1}{r}+\frac{1}{6}$ and $t\geq 1$ such that $1+\frac{1}{r}=\frac{2}{3}+\frac{1}{t}$ satisfying:

$$\begin{aligned}\|\boldsymbol{b}^\epsilon_1\times\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} &\leq \|\boldsymbol{b}^\epsilon_1\|_{\boldsymbol{L}^r(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)}\\ &\leq \|\boldsymbol{b}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\rho_{\epsilon/2}\|_{\boldsymbol{L}^t(\mathbb{R}^3)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)}.\end{aligned}$$

Using the estimate (5.18), we have

$$\|\boldsymbol{b}^\epsilon_1\times\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}\leq C_\epsilon\|\boldsymbol{b}\|_{\boldsymbol{L}^{3/2}(\Omega)}\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)}+\|\pi_0\|_{H^{-1/2}(\Gamma)}\Big). \qquad (7.44)$$

Thanks to the following imbeddings

$$\boldsymbol{L}^p(\Omega)\hookrightarrow\boldsymbol{L}^{6/5}(\Omega),\qquad W^{1-1/p,p}(\Gamma)\hookrightarrow H^{-1/2}(\Gamma),$$

we obtain that

$$\|\boldsymbol{b}^\epsilon_1\times\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}\leq C_\epsilon\|\boldsymbol{b}\|_{\boldsymbol{L}^{3/2}(\Omega)}\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\Big). \qquad (7.45)$$

Using (7.43) and (7.45), we deduce from (7.40) and (7.41) that

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|\pi_\lambda\|_{W^{1,p}(\Omega)}\leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)})+\|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\Big), \qquad (7.46)$$

where in this case $s=\frac{3}{2}$.

ii) Case $p > 3/2$. We know that for any ϵ' there exists $C'_\epsilon > 0$ such that

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{L}^\infty(\Omega)} \leq \varepsilon' \|\boldsymbol{u}_\lambda\|_{\mathbf{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}_\lambda\|_{\mathbf{L}^6(\Omega)},$$

Moreover, we have

$$\|\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^\infty(\Omega)}. \tag{7.47}$$

Thanks to (7.40), (7.41) and (7.23), we deduce the estimate (7.46) where we replace $\boldsymbol{L}^{3/2}(\Omega)$ by $\boldsymbol{L}^p(\Omega)$ because in this case $s = p$. The estimate (7.46) is uniform on λ, and therefore we can extract subsequences, that we still call $\{\boldsymbol{u}_\lambda\}_\lambda$, $\{\pi_\lambda\}_\lambda$ and $\{\alpha_i^\lambda\}_\lambda$, such that if $\lambda \to 0$,

$$\boldsymbol{u}_\lambda \longrightarrow \boldsymbol{u} \quad \text{weakly in } \boldsymbol{W}^{2,p}(\Omega),$$

and

$$\pi_\lambda \longrightarrow \pi \quad \text{weakly in } W^{1,p}(\Omega), \qquad \alpha_i^\lambda \longrightarrow \alpha_i \text{ for any } i = 1, \ldots, I.$$

It is easy to verify that $(\boldsymbol{u}, \pi, \boldsymbol{\alpha})$ is solution of the problem $(\mathcal{OS}_N)$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$, $\pi = \pi_0$ on Γ_0 and $\pi = \pi_0 + \alpha_i$ on Γ_i for any $i = 1, \ldots I$. Moreover, $(\boldsymbol{u}, \pi)$ satisfies the estimate (7.38).

□

We are now interested in the study of generalized solutions of the problem (7.29) where $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ for $1 < p < \infty$. As for the case $p = 2$, we choose $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. Observe that the pressure π is a solution of:

$$\Delta \pi = \operatorname{div} \boldsymbol{f} - \operatorname{div} (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) + \Delta \chi \quad \text{in } \Omega, \qquad \pi = \pi_0 \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + \alpha_i \text{ on } \Gamma_i.$$

If $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, then $\boldsymbol{u} \in \boldsymbol{L}^{p*}(\Omega)$ with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$ if $p < 3$, $p*$ arbitrary in $[1, \infty[$ if $p = 3$ and $p* = \infty$ if $p > 3$. Consequently, if we keep the same hypothesis on $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$ we have $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$ with $t = r$ if $p < 3$, $t < r = \frac{3}{2}$ if $p = 3$ and $t = \frac{3}{2} < r$ if $p > 3$. Consequently, if $p < 3$, we have $\Delta \pi \in W^{-1,r}(\Omega)$, so that if $\pi_0 \in W^{1-1/r,r}(\Gamma)$, then $\pi \in W^{1,r}(\Omega)$. But, if $p \geq 3$, $\Delta \pi \notin \boldsymbol{W}^{-1,r}(\Omega)$ if we suppose only that $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^{3/2}(\Omega)$. The next theorem give the existence of solutions $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$ provided that $\mathbf{curl}\, \boldsymbol{a}$ is in a space $\boldsymbol{L}^s(\Omega)$ with s large enough.

Theorem 7.3.2. *Let $p > 2$. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\chi \in W^{1,r}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. We suppose that $\pi_0 \in W^{1-1/r,r}(\Gamma)$ and $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ and s satisfies:*

$$s = \frac{3}{2} \text{ if } 2 < p < 3, \quad s = \frac{3}{2} + \epsilon \text{ if } p = 3 \text{ and } s = r \text{ if } p > 3,$$

for some arbitrary $\epsilon > 0$. *Then the problem* (7.29) *has a unique solution* $(\boldsymbol{u},\, \pi,\, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ *satisfying the estimate*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} &\leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2 \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \\ &+ \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)} \Big), \end{aligned} \tag{7.48}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$.

Proof. As for the Stokes problem $(\mathcal{S}'_N)$, where the external forces belong to $\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)$, we can prove that the problem:

$$\begin{cases} -\Delta\, \boldsymbol{u}_0 + \nabla\, q_0 = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u}_0 = \chi & \text{in } \Omega \\ \boldsymbol{u}_0 \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ q_0 = 0 \ \text{ on } \Gamma_0 \quad \text{ and } \ q_0 = \alpha_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u}_0 \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{7.49}$$

with $\alpha_i = \langle \boldsymbol{f},\, \nabla\, q_i^N \rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)} + \int_\Gamma \chi\, \nabla\, q_i^N \cdot \boldsymbol{n}$, has a unique solution $(\boldsymbol{u}_0,\, q_0) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ satisfying the estimate

$$\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|q_0\|_{W^{1,r}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)} \Big). \tag{7.50}$$

Now, observe that with the values chosen for s and r, since $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p*}(\Omega)$ and $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$, we can verify that $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_0 \in \boldsymbol{L}^r(\Omega)$. Indeed, if $p < 3$, then $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$ and $\frac{1}{s} + \frac{1}{p*} = \frac{1}{r}$. If $p = 3$, then there exists $\epsilon > 0$ such that $\dfrac{1}{\frac{3}{2} + \epsilon} + \frac{1}{p*} = \frac{2}{3}$ and if $p > 3$, then $p* = \infty$ and $r = s$. We note also that with this choice of r, we have the imbedding $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ and $r > 6/5$ because $p > 2$. Finally, from Theorem 7.3.1, we deduce the existence of a unique $(\boldsymbol{z},\, \theta) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$ solution of

$$\begin{cases} -\Delta\, \boldsymbol{z} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{z} + \nabla\, \theta = -\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_0, \quad \operatorname{div} \boldsymbol{z} = 0 & \text{in } \Omega \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \theta = \pi_0 \ \text{ on } \Gamma_0 \quad \text{ and } \ \theta = \pi_0 + \beta_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{z} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, \end{cases} \tag{7.51}$$

with $\beta_i = -\int_\Omega \mathbf{curl}\, \boldsymbol{a} \times (\boldsymbol{u}_0 + \boldsymbol{z}) \cdot \nabla\, q_i^N\, \mathrm{d}\boldsymbol{x} - \int_\Gamma \pi_0\, \nabla\, q_i^N \cdot \boldsymbol{n}$ and using (7.38) we have the estimate

$$\begin{aligned} \|\boldsymbol{z}\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\theta\|_{W^{1,r}(\Omega)} &\leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big) \Big(\|\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_0\|_{\boldsymbol{L}^r(\Omega)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \Big) \\ &\leq C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big) \Big(\|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)} \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \\ &+ \|\boldsymbol{g}\|_{W^{1-1/p,p}(\Gamma)} + \|\chi\|_{W^{1,r}(\Omega)} \big) + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \Big). \end{aligned} \tag{7.52}$$

The pair $(\boldsymbol{u},\,\pi)=(\boldsymbol{z}+\boldsymbol{u}_0,\,q_0+\theta)\in \boldsymbol{W}^{\,1,p}(\Omega)\times W^{\,1,r}(\Omega)$ is a solution of (7.29). Estimate (7.48) follows from (7.52) and (7.50). □

Now, we will treat the case $p<2$ and first we consider the case where: $\chi=0$ and $\boldsymbol{g}=\boldsymbol{0}$.

Proposition 7.3.3. *We suppose that $p<2$. Let $\boldsymbol{f}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$, $\mathbf{curl}\,\boldsymbol{a}\in\boldsymbol{L}^s(\Omega)$ and $\pi_0\in W^{\,1-1/r,r}(\Gamma)$ with*

$$r=1+\epsilon' \text{ if } p<\frac{3}{2},\ \ r=\frac{9+6\epsilon}{9+2\epsilon} \text{ if } p=\frac{3}{2} \text{ and } r=\frac{3p}{3+p} \text{ if } \frac{3}{2}<p<2, \tag{7.53}$$

$$s=(1+\epsilon')\frac{3p}{4p-3-\epsilon'(3-p)} \text{ if } p<\frac{3}{2},\ \ s=\frac{3}{2}+\epsilon \text{ if } p=\frac{3}{2} \text{ and } s=\frac{3}{2} \text{ if } \frac{3}{2}<p<2, \tag{7.54}$$

where $\epsilon,\,\epsilon'>0$ are arbitrary. Then the problems:

$$\textit{Find } (\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha})\in \boldsymbol{W}^{1,p}(\Omega)\times W^{\,1,r}(\Omega)\times\mathbb{R}^I \textit{ satisfying } (\mathcal{OS}_N) \tag{7.55}$$

with $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_I)$ and the variational formulation:

$$\begin{cases}
\textit{Find } (\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha})\in \boldsymbol{W}^{1,p}(\Omega)\times W^{\,1,r}(\Omega)\times\mathbb{R}^I \textit{ satisfying } \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0} \text{ on } \Gamma \text{ and } \int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}=0 \textit{ such that}:\\
\forall(\boldsymbol{w},\,\theta,\,\boldsymbol{d})\in \boldsymbol{W}^{\,1,p'}(\Omega)\times W^{\,1,p^{*\prime}}(\Omega) \textit{ with } \operatorname{div}\boldsymbol{w}\in W^{\,1,p^{*\prime}}(\Omega) \textit{ satisfying}:\\
\boldsymbol{w}\times\boldsymbol{n}=\boldsymbol{0} \text{ on } \Gamma,\ -\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w}+\nabla\,\theta\in[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]' \text{ and } \int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}=0,\ 1\le i\le I,\\
\theta=0 \text{ on } \Gamma_0,\ \theta=d_i \text{ on } \Gamma_i,\\
\langle\boldsymbol{u},\,-\Delta\,\boldsymbol{w}-\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w}+\nabla\,\theta\rangle_{\Omega_{p^*,p}}-\int_\Omega\pi\operatorname{div}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}=\\
\langle\boldsymbol{f},\,\boldsymbol{w}\rangle_{\Omega_{r',p'}}-\int_\Gamma\pi_0\,\boldsymbol{w}\cdot\boldsymbol{n},\\
\forall i=1,\ldots,I,\quad \alpha_i=\langle\boldsymbol{f},\,\nabla\,q_i^N\rangle_{\Omega_{r',p'}}-\int_\Omega(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\nabla\,q_i^N\,\mathrm{d}\boldsymbol{x}-\int_\Gamma\pi_0\nabla\,q_i^N\cdot\boldsymbol{n}.
\end{cases} \tag{7.56}$$

are equivalent, where $\boldsymbol{d}=(d_1,\ldots,d_I)$ and where the brackets $\langle\cdot,\,\cdot\rangle_{\Omega_{p^*,p}}$ denotes the duality $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]'\times\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)$ and $\langle\cdot,\,\cdot\rangle_{\Omega_{r',p'}}$ denotes the duality $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'\times\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)$.

Proof. Let $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha})\in \boldsymbol{W}^{\,1,p}(\Omega)\times W^{\,1,r}(\Omega)\times\mathbb{R}^I$ a solution of $(\mathcal{OS}_N)$ and let $(\boldsymbol{w},\,\theta,\,\boldsymbol{d})\in \boldsymbol{W}^{\,1,p'}(\Omega)\times W^{\,1,p^{*\prime}}(\Omega)$ with $\operatorname{div}\boldsymbol{w}\in W_0^{1,p^{*\prime}}(\Omega)$ satisfying :
$\boldsymbol{w}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, $-\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{w}+\nabla\,\theta\in[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]'$ and $\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}=0$, $1\le i\le I$, $\theta=0$ on Γ_0, $\theta=d_i$ on Γ_i. By the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in the space $\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)$ and $\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)$, we obtain:

$$\begin{aligned}
\langle-\Delta\,\boldsymbol{u},\,\boldsymbol{w}\rangle_{\Omega_{r',p'}} &= \int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\\
&= \langle\boldsymbol{u},\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w}\rangle_{\Omega_{p^*,p}}\\
&= \langle\boldsymbol{u},\,-\Delta\,\boldsymbol{w}+\nabla\operatorname{div}\boldsymbol{w}\rangle_{\Omega_{p^*,p}}.
\end{aligned}$$

But, since $\nabla \operatorname{div} \boldsymbol{w} \in \boldsymbol{L}^{p^{*\prime}}(\Omega)$, we have $\nabla \operatorname{div} \boldsymbol{w} \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]'$. Moreover, it is clear that $\Delta \boldsymbol{w} \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]'$. Then, we can write:

$$\langle \boldsymbol{u}, -\Delta \boldsymbol{w} + \nabla \operatorname{div} \boldsymbol{w} \rangle_{\Omega_{p^*,p}} = \langle \boldsymbol{u}, -\Delta \boldsymbol{w} \rangle_{\Omega_{p^*,p}}, \tag{7.57}$$

because $\int_\Omega \boldsymbol{u} \cdot \nabla \operatorname{div} \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = 0$ since $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\operatorname{div} \boldsymbol{w} = 0$ on Γ. Next, if $\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, then

$$\langle \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi, \boldsymbol{w} \rangle_{\Omega_{r',p'}} = \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x},$$

where the integral is well defined because $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ with $\frac{1}{s} + \frac{1}{p*} + \frac{1}{p'^*} = 1$ if $3/2 \le p < 2$ and $\frac{1}{s} + \frac{1}{p*} + \frac{1}{p'^*} < 1$ if $1 < p < 3/2$. Moreover, $\frac{1}{r} + \frac{1}{p'^*} = 1$, if $3/2 \le p < 2$ and $\frac{1}{r} + \frac{1}{p'^*} < 1$ if $1 < p < 3/2$. As a consequence, we can write

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla \pi) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla \pi \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x}.$$

Next, it is clear that

$$\int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = -\int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w}) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}.$$

Also, since $\pi \in W^{1,r}(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}^{p'^*,(p^{*\prime})^*}(\operatorname{div}, \Omega)$ we have

$$\int_\Omega \nabla \pi \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = -\int_\Omega \pi \operatorname{div} \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} + \int_\Gamma \pi \, \boldsymbol{w} \cdot \boldsymbol{n}. \tag{7.58}$$

where $\boldsymbol{H}^{p'^*,(p^{*\prime})^*}(\operatorname{div}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^{p'^*}(\Omega), \ \operatorname{div} \boldsymbol{v} \in L^{(p^{*\prime})^*}(\Omega) \right\}$. In order to establish (7.58), we just check that $\frac{1}{r^*} + \frac{1}{p'^*} \le 1$ and we use the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{H}^{p'^*,(p^{*\prime})^*}(\operatorname{div}, \Omega)$ (see [10]).

Now, since $\operatorname{div} \boldsymbol{u} = 0$ in Ω we have for any $\theta \in W^{1,p^{*\prime}}(\Omega)$:

$$0 = -\int_\Omega \theta \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \nabla \theta \, \mathrm{d}\boldsymbol{x} - \int_\Gamma \theta \, \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{\sigma}.$$

Here too, the last Green formula is verified. Indeed, since $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$ wich implies that $\boldsymbol{u} \in \boldsymbol{H}^{p^*,p}(\operatorname{div}, \Omega)$. Let us summarize. We have obtained that:

$$\begin{aligned} \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\Omega_{r',p'}} &= \langle \boldsymbol{u}, -\Delta \boldsymbol{w} \rangle_{\Omega_{p^*,p}} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w}) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} - \int_\Omega \pi \operatorname{div} \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} \\ &+ \int_\Gamma \pi_0 \, \boldsymbol{w} \cdot \boldsymbol{n} + \int_\Omega \boldsymbol{u} \cdot \nabla \theta \, \mathrm{d}\boldsymbol{x} - \int_\Gamma \theta \, \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{\sigma}. \end{aligned}$$

Using the properties of $\boldsymbol{w}$ and θ, we obtain that $(\boldsymbol{u}, \pi)$ satisfies the first equation in (7.56). With the same arguments used in Theorem 7.2.5, we can prove that the constants α_i, for $1 \le i \le I$ satisfy the last relation in (7.56).

ii) Reciprocally, let $(\boldsymbol{u},\, \pi,\, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ a solution of (7.56). First choose $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\theta = 0$, then we deduce that

$$-\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{in } \Omega.$$

Next, we choose $\boldsymbol{w} = \boldsymbol{0}$ and $\theta \in \mathcal{D}(\Omega)$. Then, we obtain that $\operatorname{div} \boldsymbol{u} = 0$ in Ω. We know that $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and that $\int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0$. Then it remains to prove the boundary condition given on the pressure π. Let then $\boldsymbol{w} \in \boldsymbol{W}^{1,p'}(\Omega)$ with $\boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and such that $\operatorname{div} \boldsymbol{w} \in W_0^{1,p^*}(\Omega)$. We obtain as previously,

$$\langle \boldsymbol{f},\, \boldsymbol{w} \rangle_{\Omega_{r',\,p'}} = \langle \boldsymbol{u},\, -\Delta\, \boldsymbol{w} \rangle_{\Omega_{p^*,\,p}} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w}) \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} - \int_\Omega \pi \operatorname{div} \boldsymbol{w}\, \mathrm{d}\boldsymbol{x} + \int_\Gamma \pi\, \boldsymbol{w} \cdot \boldsymbol{n}.$$

Let $\theta \in W^{1,p^{*\prime}}(\Omega)$. Then,

$$\begin{aligned} \langle \boldsymbol{f},\, \boldsymbol{w} \rangle_{\Omega_{r',\,p'}} &= \langle \boldsymbol{u},\, -\Delta\, \boldsymbol{w} \rangle_{\Omega_{p^*,\,p}} + \int_\Omega \big(-\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w} + \nabla\, \theta \big) \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} - \int_\Omega \nabla\, \theta \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} \\ &- \int_\Omega \pi \operatorname{div} \boldsymbol{w}\, \mathrm{d}\boldsymbol{x} + \int_\Gamma \pi\, \boldsymbol{w} \cdot \boldsymbol{n}. \end{aligned}$$

But,

$$\boldsymbol{w} = \boldsymbol{w} - \sum_{i=1}^{I} \big(\int_{\Gamma_i} \boldsymbol{w} \cdot \boldsymbol{n} \big)\, \nabla\, q_i^N + \boldsymbol{z}, \tag{7.59}$$

with $\boldsymbol{z} = \sum_{i=1}^{I} \big(\int_{\Gamma_i} \boldsymbol{w} \cdot \boldsymbol{n} \big)\, \nabla\, q_i^N$. We set $\boldsymbol{w}_0 = \boldsymbol{w} - \boldsymbol{z}$, then if $-\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w}_0 + \nabla\, \theta \in [\boldsymbol{H}_0^{p^*,\,p}(\mathbf{curl},\, \Omega)]'$ we obtain:

$$\begin{aligned} \langle \boldsymbol{f},\, \boldsymbol{w}_0 \rangle_{\Omega_{r',\,p'}} + \langle \boldsymbol{f},\, \boldsymbol{z} \rangle_{\Omega_{r',\,p'}} &= \langle \boldsymbol{u},\, -\Delta\, \boldsymbol{w}_0 - \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w}_0 + \nabla\, \theta \rangle_{\Omega_{p^*,\,p}} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{z}) \cdot \boldsymbol{u} \\ &- \int_\Omega \nabla\, \theta \cdot \boldsymbol{u} - \int_\Omega \pi \operatorname{div} \boldsymbol{w}_0 + \int_\Gamma \pi\, \boldsymbol{w}_0 \cdot \boldsymbol{n} + \int_\Gamma \pi\, \boldsymbol{z} \cdot \boldsymbol{n}. \end{aligned} \tag{7.60}$$

This last relation is valid for $\boldsymbol{z} = \boldsymbol{0}$ and we deduce from (7.56) that:

$$-\int_\Omega \nabla\, \theta \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} + \int_\Gamma \pi\, \boldsymbol{w}_0 \cdot \boldsymbol{n} - \int_\Gamma \pi_0\, \boldsymbol{w}_0 \cdot \boldsymbol{n} = 0.$$

Choosing $\theta = 0$ on Γ_0 and $\theta = d_i$ on Γ_i for any $1 \le i \le I$, we obtain

$$\int_\Gamma (\pi - \pi_0)\, \boldsymbol{w}_0 \cdot \boldsymbol{n} = 0.$$

Now, we return to the relation (7.60) and we can write:

$$\begin{aligned} \int_\Gamma \pi\, \boldsymbol{z} \cdot \boldsymbol{n} &= \langle \boldsymbol{f},\, \boldsymbol{z} \rangle_{\Omega_{r',\,p'}} + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{z}) \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} \\ &= \langle \boldsymbol{f},\, \boldsymbol{z} \rangle_{\Omega_{r',\,p'}} - \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}) \cdot \boldsymbol{z}\, \mathrm{d}\boldsymbol{x} \end{aligned}$$

So that, using the decomposition (7.59) and the last relation in (7.56) we obtain that:

$$\begin{aligned}\int_\Gamma \pi\, \boldsymbol{w}\cdot\boldsymbol{n} &= \int_\Gamma \pi\, \boldsymbol{w}_0\cdot\boldsymbol{n} + \int_\Gamma \pi\, \boldsymbol{z}\cdot\boldsymbol{n}\\ &= \int_\Gamma \pi_0\, \boldsymbol{w}_0\cdot\boldsymbol{n} + \langle \boldsymbol{f},\, \boldsymbol{z}\rangle_{\Omega_{r',\,p'}} - \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}\\ &= \int_\Gamma \pi_0\, \boldsymbol{w}_0\cdot\boldsymbol{n} + \sum_{i=1}^{I}\Big(\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}\Big)\langle \boldsymbol{f},\, \nabla q_i^N\rangle_{\Omega_{r',\,p'}}\\ &- \sum_{i=1}^{I}\Big(\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}\Big)\int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u})\cdot\nabla q_i^N\,\mathrm{d}\boldsymbol{x}. \end{aligned}\tag{7.61}$$

This implies that

$$\int_\Gamma \pi\, \boldsymbol{w}\cdot\boldsymbol{n} = \int_\Gamma \pi_0\, \boldsymbol{w}_0\cdot\boldsymbol{n} + \sum_{i=1}^{I}\Big(\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}\Big)\Big(\int_\Gamma \pi_0\,\nabla q_i^N\cdot\boldsymbol{n} + \alpha_i\Big),$$

and then

$$\begin{aligned}\int_\Gamma \pi\, \boldsymbol{w}\cdot\boldsymbol{n} &= \int_\Gamma \pi_0\, \boldsymbol{w}_0\cdot\boldsymbol{n} + \sum_{i=1}^{I}\alpha_i\int_{\Gamma_i}\boldsymbol{w}\cdot\boldsymbol{n}\\ &= \int_\Gamma (\pi_0 + C)\, \boldsymbol{w}_0\cdot\boldsymbol{n},\end{aligned}$$

with $C = 0$ on Γ_0 and $C = \alpha_i$ on Γ_i.

We use exactly the same arguments as in Theorem 7.2.5 and we obtain that:

$$\pi = \pi_0 \text{ on } \Gamma_0 \text{ and } \pi = \pi_0 + \alpha_i \text{ on } \Gamma_i,$$

which finishes the proof. □

Now, we are going to solve the problem (7.56).

Theorem 7.3.4. *Under the assumptions of Proposition 7.3.3, the problem* (7.56) *has a unique solution* $(\boldsymbol{u}, \pi, \boldsymbol{\alpha})$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^2(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)}),\tag{7.62}$$

$$\|\pi\|_{W^{1,r}(\Omega)} \le C(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^3(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Omega)})\tag{7.63}$$

Proof. Since $p' > 2$, we know due to Theorem 7.3.2 that for any $\boldsymbol{F} \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)]'$ and $\varphi \in W_0^{1,p^{*\prime}}(\Omega)$, there exists a unique $(\boldsymbol{w},\theta,\boldsymbol{d}) \in \boldsymbol{W}^{1,p'}(\Omega)\times W^{1,p^{*\prime}}(\Omega)\times\mathbb{R}^I$ with

$\operatorname{div} \boldsymbol{w} \in W_0^{1,p^{*\prime}}(\Omega)$ such that:

$$\begin{cases} -\Delta \boldsymbol{w} - \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w} + \nabla \theta = \boldsymbol{F}, \quad \operatorname{div} \boldsymbol{w} = \varphi \ \text{ in } \Omega, \\ \\ \boldsymbol{w} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma, \\ \\ \theta = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \theta = d_i \quad \text{on } \Gamma_i,\ 1 \le i \le I, \\ \\ \int_{\Gamma_i} \boldsymbol{w} \cdot \mathbf{n} = 0,\ \ 1 \le i \le I, \end{cases} \tag{7.64}$$

satisfying the estimate:

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\theta\|_{W^{1,p^{*\prime}}(\Omega)} \le C\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2 \big(\|\boldsymbol{F}\|_{[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)]'} + \|\varphi\|_{W^{1,p^{*\prime}}(\Omega)}\big), \tag{7.65}$$

where $\boldsymbol{d} = (d_1, \ldots, d_I)$ and

$$d_i = \langle \boldsymbol{F}, \nabla q_i^N \rangle_\Omega + \int_\Omega (\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w}) \cdot \nabla q_i^N \, \mathrm{d}\boldsymbol{x}, \ \ 1 \le i \le I.$$

Remark that we can apply Theorem 7.3.2 because the real s defined in (7.54) satisfies the assumptions of Theoreme 7.3.2 where however we must replace p by p'. Indeed, since $p' > 2$, we have $s = \frac{3}{2}$ if $2 < p' < 3$, $s = \frac{3}{2} + \epsilon$ if $p' = 3$. For the last condition, $s = p^{*\prime}$ if $p' > 3$. But, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. Then, $\frac{1}{p^{*\prime}} = \frac{4}{3} - \frac{1}{p} = \frac{4p-3}{3p}$. We deduce that $s = \frac{3p}{4p-3}$ if $p' > 3$. So, $s = \frac{3}{4-\frac{3}{p}} = \frac{3p'}{4p'-3\frac{p'}{p}} = \frac{3p'}{4p'-3(p'-1)} = \frac{3p'}{3+p'}$. Now, using (7.65) we obtain:

$$\begin{aligned} & \Big|\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega_{r',p'}} - \int_\Gamma \pi_0 \boldsymbol{w} \cdot \boldsymbol{n}\Big| \\ \le\ & \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \|\boldsymbol{w} \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{p'},p'}(\Gamma)} \\ \le\ & C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\big)\big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2 \times \\ \times\ & \big(\|\boldsymbol{F}\|_{[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)]'} + \|\varphi\|_{W^{1,p^{*\prime}}(\Omega)}\big). \end{aligned}$$

In other words, the linear mapping

$$(\mathbf{F}, \varphi) \longrightarrow \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega_{r',p'}} - \int_\Gamma \pi_0 \boldsymbol{w} \cdot \boldsymbol{n},$$

defines an element $(\boldsymbol{u}, \pi)$ of the dual space of $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]' \times W_0^{1,p^{*\prime}}(\Omega)$ which means that $(\boldsymbol{u}, \pi) \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega) \times W^{-1,p^*}(\Omega)$ and satisfies:

$$\langle \boldsymbol{u}, \boldsymbol{F}\rangle_{\Omega_{p^*,p}} - \langle \pi, \varphi\rangle_{W^{-1,p^*}(\Omega) \times W_0^{1,p^{*\prime}}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\Omega_{r',p'}} - \int_\Gamma \pi_0 \boldsymbol{w} \cdot \boldsymbol{n}, \tag{7.66}$$

with

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{H}^{p^*,p}(\mathbf{curl},\Omega)} + \|\pi\|_{W^{-1,p^*}(\Omega)} \le\ & C(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)})^2 \times \\ \times\ & (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}). \end{aligned} \tag{7.67}$$

We have then

$$\begin{aligned}\langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{w} + \nabla\, \theta\rangle_{\Omega_{p^*,\,p}} - \langle \pi, \operatorname{div} \boldsymbol{w}\rangle_{W^{-1,r}(\Omega)\times W_0^{1,r'}(\Omega)} &= \langle \boldsymbol{f},\ \boldsymbol{w}\rangle_{\Omega_{r',\,p'}} \\ &- \int_\Gamma \pi_0 \boldsymbol{w}\cdot \boldsymbol{n}, \end{aligned} \tag{7.68}$$

for any $(\boldsymbol{w}, \theta) \in \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,p^{*\prime}}(\Omega)$ with $\operatorname{div} \boldsymbol{w} \in W_0^{1,p^{*\prime}}(\Omega)$ satisfying $\boldsymbol{w}\times \boldsymbol{n} = \mathbf{0}$ on Γ, $\int_{\Gamma_i} \boldsymbol{w}\cdot\boldsymbol{n} = 0$ for any $1 \le i \le I$, $-\mathbf{curl}\, \boldsymbol{a}\times\boldsymbol{w} + \nabla\theta \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)]'$, $\theta = 0$ on Γ_0 and finally $\theta = d_i$ on Γ_i. Then, we have proved that $(\boldsymbol{u},\,\pi) \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)\times W^{-1,p^*}(\Omega)$ satisfies the first equation in (7.56)

ii) We prove now that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{1,r}(\Omega)$ and that the second equation in (7.56) is verified. We choose $\boldsymbol{w} = \mathbf{0}$ and $\theta \in \mathcal{D}(\Omega)$. Then, we obtain $\operatorname{div}\boldsymbol{u} = 0$ in Ω. Since $\boldsymbol{u} \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\,\Omega)$ and $\operatorname{div}\boldsymbol{u} = 0$ in Ω, we deduce by, Theorem 2.2.4, we obtain that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. The estimate (7.62) follows from (7.67). Next, we choose $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\theta \in \mathcal{D}(\Omega)$, we obtain

$$-\Delta\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \qquad \text{in } \Omega. \tag{7.69}$$

But $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ and $\boldsymbol{u} \in L^{p^*}(\Omega)$ with s defined in (7.54), we have then $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} \in \boldsymbol{L}^r(\Omega)$ with r defined in (7.53). So, since $\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u} + \nabla\,\pi \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]' \hookrightarrow \boldsymbol{W}^{-1,r^*}(\Omega)$, we have $\nabla\pi \in W^{-1,r^*}(\Omega)$ and then also $\pi \in L^{r^*}(\Omega)$. This means in particular that

$$\langle \pi, \operatorname{div}\boldsymbol{w}\rangle_{W^{-1,p^*}(\Omega)\times W_0^{1,p^{*\prime}}(\Omega)} = \int_\Omega \pi \operatorname{div}\boldsymbol{w},$$

because we have $\operatorname{div}\boldsymbol{w} \in L^{(p^{*\prime})^*}(\Omega) = L^{p'}(\Omega)$ with $\frac{1}{r^*} + \frac{1}{p'} \le 1$. It remains to show that $\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\,\boldsymbol{\sigma} = 0$ and $\pi \in W^{1,r}(\Omega)$. For the first point, we choose in (7.68) $\boldsymbol{w} = 0$ and $\theta \in W^{1,p^{*\prime}}(\Omega)$ with $\theta = 0$ on Γ_0 and $\theta = \delta_{ij}$ on Γ_j for any $1 \le j \le I$ and fixed i in $[1,\,I]$. We deduce from (7.68) that $\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\,\boldsymbol{\sigma} = 0$ for any i in $[1, I]$. For the second point, we know that π satisfies

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} - \operatorname{div}(\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}) \text{ in } \Omega,$$

and as in the point *ii*) in the proof of Theorem 7.2.5, we can prove that π verifies the boundary conditions:

$$\pi = \pi_0 \text{ on } \Gamma_0 \qquad \text{and} \qquad \pi = \pi_0 + \alpha_i \text{ on } \Gamma_i.$$

Then, using (7.53) and (7.54) we can verify that $\Delta\,\pi \in W^{-1,r}(\Omega)$. So, $\pi \in W^{1,r}(\Omega)$ and satisfies the estimate:

$$\|\pi\|_{W^{1,r}(\Omega)} \le (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^{p*}(\Omega)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}). \tag{7.70}$$

Finally the estimate (7.63) follows from (7.70) and (7.67).

□

The next theorem give an extension of the previous one to the case of the problem (7.29).

Theorem 7.3.5. *Let $p < 2$,*

$\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]',\ \chi \in W^{1,r}(\Omega),\ \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ and $\pi_0 \in W^{1-1/r,r}(\Gamma),\ \mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$

with r and s satisfies (7.53) and (7.54). Then the problem (7.29) has a unique solution $(\boldsymbol{u},\,\pi,\,\boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimates

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2\Big(\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big) \qquad (7.71)\end{aligned}$$

$$\begin{aligned}\|\pi\|_{W^{1,r}(\Omega)} &\le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^3\Big(\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big) \qquad (7.72)\end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_1,\,\ldots,\,\alpha_I)$ with $\alpha_i = \langle \boldsymbol{f},\,\nabla\, q_i^N\rangle_{\Omega_{r',\,p'}} + \int_\Gamma (\chi-\pi_0)\,\nabla\, q_i^N\cdot\boldsymbol{n} - \int_\Omega (\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{n})\cdot\nabla q_i^N$.

Proof. Let $(\boldsymbol{u}_0,\,q_0) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ such that (7.49) and satisfies the estimate (7.50). Next, we consider $(\boldsymbol{z},\,\theta) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ solution of (7.51). We note here that we appply Theroem 7.3.4 because $\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}_0 \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$. Indeed, using (7.53) and (7.54), we have $\mathbf{curl}\,\boldsymbol{a} \times \boldsymbol{u}_0 \in \boldsymbol{L}^r(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$. Moreover using (7.62), we obatin that $\boldsymbol{z}$ satisfies the estimate:

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2\big(\|\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big)\\ &\le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2\big(\|\mathbf{curl}\,\boldsymbol{a}\times\boldsymbol{u}_0\|_{\boldsymbol{L}^r(\Omega)}+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big)\\ &\le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2\big(\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big).\end{aligned}$$

Using the estimate (7.50), we deduce that $\boldsymbol{z}$ satisfies:

$$\begin{aligned}\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^2\Big(\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big). \qquad (7.73)\end{aligned}$$

Similarly, using (7.63) and (7.50), we obatin the estimate:

$$\begin{aligned}\|\theta\|_{W^{1,r}(\Omega)} &\le C\big(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)^3\Big(\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\\ &+ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\chi\|_{W^{1,r}(\Omega)}\big)+\|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\Big). \qquad (7.74)\end{aligned}$$

Finally, the pair $(\boldsymbol{u},\,\pi) = (\boldsymbol{z}+\boldsymbol{u}_0,\,q_0+\theta) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ is a solution of (7.29). Estimates (7.71) and (7.72) follows from (7.50), (7.73) and (7.74). □

Proposition 7.3.6. *Let p such that $6/5 \leq p \leq 6$. We suppose that $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\chi \in W^{1,r}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, $\pi_0 \in W^{1-1/r,r}(\Gamma)$ and $\mathbf{curl}\, \boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ with*

$$r = 1+\mu' \text{ if } \frac{6}{5} \leq p < \frac{3}{2},\ r = \frac{9+6\mu}{9+2\mu} \text{ if } p = \frac{3}{2} \text{ and } r = \frac{3p}{3+p} \text{ if } \frac{3}{2} < p \leq 6, \tag{7.75}$$

$$\begin{aligned} &s = (1+\mu')\frac{3p}{4p-3-\mu'(3-p)} \text{ if } \frac{6}{5} \leq p < \frac{3}{2},\ s = \frac{3}{2}+\mu \text{ if } p = \frac{3}{2} \text{ or } p = 3, \text{ and} \\ &s = \frac{3}{2} \text{ if } \frac{3}{2} < p \leq 6, \end{aligned} \tag{7.76}$$

where μ, $\mu' > 0$ are arbitrary. Then, the solution $(\boldsymbol{u}, \pi)$ satisfies the estimate:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r}(\Omega)} &\leq C\big(1+\|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\Big(\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \\ &+ \big(1+\|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\big)\big(\|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big)\Big). \end{aligned} \tag{7.77}$$

Proof. The case $p = 2$ is treated in Theorem 7.2.10. We will repeat quickly the reasoning given in Theorem 7.3.1.

i) **First case:** $2 < p < 3$. We obtain, thanks to the regularity for the Stokes problem

$$\begin{aligned} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,r}(\Omega)} &\leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} \\ &+ \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\boldsymbol{b}_\lambda \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^r(\Omega)}\Big) \end{aligned} \tag{7.78}$$

where we have used the continuous imbedding $\boldsymbol{L}^r(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Using the inequality (7.42), the Hölder inequality and the Sobolev embedding, we obtain

$$\begin{aligned} \|\boldsymbol{b}_{\lambda,2}^\epsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^r(\Omega)} &\leq \|\boldsymbol{b}_{\lambda,2}^\epsilon\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p*}(\Omega)} \\ &\leq C\epsilon \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)}. \end{aligned}$$

For the second estimate, we have:

$$\begin{aligned} \|\boldsymbol{b}_1^\epsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^r(\Omega)} &\leq \|\boldsymbol{b}_1^\epsilon\|_{\boldsymbol{L}^t(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^q(\Omega)} \\ &\leq \|\boldsymbol{b}\|_{\boldsymbol{L}^{3/2}(\Omega)} \|\rho_{\epsilon/2}\|_{L^k(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^q(\Omega)}, \end{aligned}$$

where $k \in]1,2[$ is such that $1 + \frac{1}{t} = \frac{2}{3} + \frac{1}{k}$ and $\frac{1}{r} = \frac{1}{t} + \frac{1}{q}$. We choose $t \in]\frac{3}{2}, \frac{6r}{6-r}[$ for that $6 < q < p*$. So, for any $\epsilon' > 0$, we known that there exists a constant $C'_\epsilon > 0$ such that:

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^q(\Omega)} \leq \epsilon' \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)} + C_{\epsilon'} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)}.$$

Hence, by Theorem 7.2.10, we obtain

$$\begin{aligned}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi_\lambda\|_{W^{1,r}(\Omega)} &\leq C(1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \\ &+ (1+\|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})(\|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)})\Big).\end{aligned}$$

ii) **Second case:** $3 \leq p \leq 6$. Using again the regularity for the Stokes problem, we have the estimate (7.78) with now

$$\|\boldsymbol{b}^\epsilon_{\lambda,2} \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^r(\Omega)} \leq C\epsilon\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^\infty(\Omega)} \leq C\epsilon\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)} \qquad \text{if } p > 3,$$

$$\|\boldsymbol{b}^\epsilon_{\lambda,2} \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{3/2}(\Omega)} \leq C\epsilon\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^q(\Omega)} \leq C\epsilon\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{1,p}(\Omega)} \qquad \text{if } p = 3,$$

where $\dfrac{2}{3} = \dfrac{1}{q} + \dfrac{1}{\frac{3}{2}+\mu}$. For the second estimate, we have:

$$\begin{aligned}\|\boldsymbol{b}^\epsilon_1 \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^r(\Omega)} &\leq \|\boldsymbol{b}^\epsilon_1\|_{\boldsymbol{L}^k(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)} \\ &\leq \|\mathbf{curl}\,\boldsymbol{a}\|_{\boldsymbol{L}^s(\Omega)}\|\rho_{\epsilon/2}\|_{L^t(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^6(\Omega)},\end{aligned}$$

with $\frac{1}{r} = \frac{1}{k} + \frac{1}{6}$ and $1 + \frac{1}{k} = \frac{1}{s} + \frac{1}{t}$. Since $s = r$ if $p > 3$, we have $t = \frac{6}{5}$ and if $p = 3$, $s = \frac{3}{2} + \mu$, $r = \frac{3}{2}$, $k = 2$, $t = \frac{6+4\mu}{5+6\mu} \in]1, \frac{6}{5}[$.

ii) **Second case:** $\frac{6}{5} \leq p < 2$. As in Theorem 7.3.4 , we use a duality argument. □

Corollary 7.3.7. *Let* $1 < p < \frac{6}{5}$, $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\chi \in W^{1,p}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, $\pi_0 \in W^{1-1/p,p}(\Gamma)$ *with* $\mathbf{curl}\,\boldsymbol{a} \in \boldsymbol{L}^s(\Omega)$ *where*

$$s = \frac{3}{2} \text{ if } p < \frac{3}{2}\ ,\ s = p \text{ if } \ p > \frac{3}{2} \text{ and } s = \frac{3}{2} + \epsilon \text{ if } p = \frac{3}{2}, \qquad (\epsilon > 0 \text{ is arbitrary}).$$

Then the problem (7.29) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Chapter 8

Navier-Stokes problem with normal boundary conditions

As a consequence of the previous study in Chapter 7, we want in this chapter to study the following Navier-Stokes equations:

$$(\mathcal{NS}_N) \qquad \begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_i \text{ and } \pi = \pi_0 + c_i & \text{on } \Gamma_i,\ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I, & \end{cases}$$

where $\boldsymbol{f}$, and π_0 are given functions, c_i are constants and we will denote by $\boldsymbol{c}$ the vector $\boldsymbol{c} = (c_1, \ldots, c_I)$.

In the search of a proof of the existence of generalized solution for the Navier-Stokes equations $(\mathcal{NS}_N)$, we consider the case of small enough data.

Theorem 8.0.8. *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\chi \in W^{1,r}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. $\pi_0 \in W^{1-1/r,r}(\Gamma)$ with $\frac{3}{2} < p < 3$ and $r = \frac{3p}{3+p}$.*

i) There exists a constant $\alpha_1 > 0$ such that, if

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_1,$$

then, there exists a solution $(\boldsymbol{u}, \pi, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ to problem $(\mathcal{NS}_N)$ verifying the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}\big), \quad (8.1)$$

with $c_i = \langle \boldsymbol{f}, \nabla\, q_i \rangle_{\Omega_{r',p'}} + \int_\Gamma (\chi - \pi_0)\, \nabla\, q_i^N \cdot \boldsymbol{n} - \int_\Omega (\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}) \cdot \nabla q_i^N$.

ii) Moreover, there exists a constant $\alpha_2 \in]0, \alpha_1]$ *such that this solution is unique, if*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \leq \alpha_2.$$

Proof. i) **Existence:** We begin to prove existence of generalized solutions. We want to apply Banach's fixed point theorem. The idea is to do this fixed point over the Oseen problem (7.29). We are searching for a fixed point for the application T,

$$\begin{array}{rcl} T \ : \ \boldsymbol{W}^{1,p}(\Omega) & \longrightarrow & \boldsymbol{W}^{1,p}(\Omega) \\ \boldsymbol{a} & \longmapsto & \boldsymbol{u}, \end{array}$$

where given $\boldsymbol{a} \in \boldsymbol{W}^{1,p}(\Omega)$, $T\,\boldsymbol{a} = \boldsymbol{u}$ is the unique solution of (7.29), where we replace α_i by c_i, given by Theorem 7.3.2 and Theorem 7.3.5. In order to apply the fixed point result, we have to define a neighborhood $\boldsymbol{B}_\lambda$, in the form:

$$\boldsymbol{B}_\lambda = \left\{ \boldsymbol{a} \in \boldsymbol{W}^{1,p}(\Omega), \quad \|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \lambda \right\}.$$

Let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2 \in \boldsymbol{B}_\lambda$. If we choose a contraction method, we must prove that: there exists a constant $\theta \in]0,\, 1[$ such that

$$\|T\,\boldsymbol{a}_1 - T\,\boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \theta \|\boldsymbol{a}_1 - \boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{8.2}$$

In order to estimate $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}$, we observe that for each $k = 1, 2$, $(\boldsymbol{u}_k,\, \pi_k) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ verifies:

$$\begin{cases} -\Delta\, \boldsymbol{u}_k + \mathbf{curl}\, \boldsymbol{a}_k \times \boldsymbol{u}_k + \nabla\, \pi_k = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_k = \chi & \text{in } \Omega, \\ \boldsymbol{u}_k \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on } \Gamma, \\ \pi_k = \pi_0 \ \text{ on } \Gamma_0 \ \text{ and } \ \pi_k = \pi_0 + c_i & \text{on } \Gamma_i, \ i = 1, \ldots, I, \\ \int_{\Gamma_i} \boldsymbol{u}_k \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, \end{cases}$$

with the estimate:

$$\begin{array}{rcl} \|\boldsymbol{u}_k\|_{\boldsymbol{W}^{1,p}(\Omega)} & \leq & C(1 + \|\mathbf{curl}\, \boldsymbol{a}_k\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}) \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} \\ & + & (1 + \|\mathbf{curl}\, \boldsymbol{a}_k\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})(\|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}) \Big). \end{array}$$

However, in order to estimate the difference $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have to reason differently. We start with the problem verified by $(\boldsymbol{u},\, \pi) = (\boldsymbol{u}_1 - \boldsymbol{u}_2,\, \pi_1 - \pi_2)$, which is the following one:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{a}_1 \times \boldsymbol{u} + \nabla\, \pi = -\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_2 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0, \ i = 1, \ldots, I, \end{cases}$$

where $\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2$. Using the estimate made for the Oseen problem successively for $\boldsymbol{u}$ and $\boldsymbol{u}_2$, we obtain that:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl}\, \boldsymbol{a}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})(\|\mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{u}_2\|_{\boldsymbol{L}^r(\Omega)}).$$

Then

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C(1 + \|\mathbf{curl}\, \boldsymbol{a}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})\|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\boldsymbol{u}_2\|_{\boldsymbol{L}^{p*}(\Omega)} \qquad (8.3)\\ &\leq C^2(1 + \|\mathbf{curl}\, \boldsymbol{a}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})(1 + \|\mathbf{curl}\, \boldsymbol{a}_2\|_{\boldsymbol{L}^{3/2}(\Omega)})\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}\\ &+ \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \big(1 + \|\mathbf{curl}\, \boldsymbol{a}\|_{\boldsymbol{L}^3/2(\Omega)}\big)\big(\|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big)\Big). \end{aligned}$$

We set $\alpha = \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)}$ and $\beta = \|\chi\|_{W^{1,r}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}$. Then, (8.3) becomes:

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C_1C^2\|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p}(\Omega)}(1 + C_1\|\boldsymbol{a}_1\|_{\boldsymbol{W}^{1,p}(\Omega)})(1 + C_1\|\boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}) \times\\ &\times \Big(\alpha + \beta(1 + C_1\|\boldsymbol{a}_2\|_{\boldsymbol{W}^{1,p}(\Omega)})\Big), \qquad (8.4) \end{aligned}$$

where $C_1 > 0$ is such that

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \qquad \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^{3/2}(\Omega)} \leq C_1\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \qquad (8.5)$$

Thus, we can (for instance) obtain estimate (8.2) if we consider r such that

$$C_1C^2(1 + C_1\lambda)^2\big(\alpha + \beta(1 + C_1\lambda)\big) < 1,$$

that is verified, for example, taking:

$$\lambda = C_1^{-1}\Big(\big(2C_1C^2(\alpha + \beta)\big)^{-1/3} - 1\Big) \quad \text{and} \quad \alpha + \beta < (2C_1C^2)^{-1}. \qquad (8.6)$$

Therefore, if (8.6) is verified, then the fixed point $\boldsymbol{u}^* \in \boldsymbol{W}^{1,p}(\Omega)$ satisfies the estimate:

$$\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\alpha + \beta)\big(1 + C_1\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)}\big)^2. \qquad (8.7)$$

i.e

$$\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq x^*,$$

where x^* is the smallest solution of the equation: $x = ax^2 + 1$ with $a = CC_1(\alpha + \beta)$, i.e $x^* = \dfrac{2}{1 + \sqrt{1 - 4a}}$, where we suppose that $a < 1/4$. Thus,

$$\|\boldsymbol{u}^*\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \frac{4a}{(1 + \sqrt{1 - 4a})^2} \leq 4a, \qquad (8.8)$$

which proves the point i) and the estimate (8.1).

ii) **Uniqueness:** We shall next prove uniqueness. Let us denote by $(\boldsymbol{u}_1, \pi_1)$ the solution obtained in step i) and by $(\boldsymbol{u}_2, \pi_2)$ any other solution for $(\mathcal{NS}_N)$ corresponding to the same data. Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\pi = \pi_1 - \pi_2$. We find that:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{u}_1 \times \boldsymbol{u} + \nabla\, \pi = -\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}_2 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma, \\ \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0,\ i = 1, \ldots, I. \end{cases} \tag{8.9}$$

Then,

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl}\, \boldsymbol{u}_1\|_{\boldsymbol{L}^{3/2}(\Omega)})(\|\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}_2\|_{\boldsymbol{L}^r(\Omega)}), \tag{8.10}$$

i.e as in step i):

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq (1 + C_1\|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)})C_1(\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{8.11}$$

But, for $i = 1, 2$

$$\|\boldsymbol{u}_i\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + C_1\|\boldsymbol{u}_i\|_{\boldsymbol{W}^{1,p}(\Omega)})^2(\alpha + \beta), \tag{8.12}$$

and then,

$$\|\boldsymbol{u}_i\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq 4\gamma C C_1, \tag{8.13}$$

where $\gamma = \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'} + \|\chi\|_{W^{1,r}(\Omega)} + \|\pi_0\|_{W^{1-1/r,r}(\Gamma)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}$, we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq CC_1^2\big(1 + 4C_1^2C\gamma\big)4CC_1\gamma\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{8.14}$$

Then, if $CC_1^2\big(1 + 4C_1^2C\gamma\big)4CC_1\gamma < 1$, we deduce that $\boldsymbol{u} = \boldsymbol{0}$ and the proof of uniqueness is completed. □

Part IV

Numerical approximation of the Stokes problem with non-standard boundary conditions

Chapter 9

A Nitsche type Method

In this chapter, we consider a discretization based on continuous finite elements of the Stokes problem with non-standard boundary conditions. In what follows Ω denotes a Lipschitz domain of $\mathbb{R}^2$ with a polygonal boundary $\Gamma = \partial\Omega$. The velocity $\boldsymbol{u}$ and the pressure p are assumed to satisfy:

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega, \tag{9.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega, \tag{9.2}$$

where $\nu > 0$ is the kinematic viscosity of the fluid and $\boldsymbol{f}$ is the density of external forces.

We suppose that Γ is devided into three open and disjoint subsets $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$.

On the boundary Γ, we let $\boldsymbol{n} = (n_1, n_2)$ denote the unit outward normal and $\boldsymbol{t} = (-n_2, n_1)$ denote the unit tangential. Given vector data $\boldsymbol{u}_0$, $\boldsymbol{a}$, $\boldsymbol{b}$ and scalar data p_0, w_0, we consider the following boundary conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{u}_0\cdot\boldsymbol{n}, \qquad \boldsymbol{u}\cdot\boldsymbol{t} = \boldsymbol{u}_0\cdot\boldsymbol{t} \quad \text{on} \quad \Gamma_1, \tag{9.3}$$

$$\boldsymbol{u}\cdot\boldsymbol{t} = \boldsymbol{a}\cdot\boldsymbol{t} \qquad p = p_0 \quad \text{on} \quad \Gamma_2, \tag{9.4}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{b}\cdot\boldsymbol{n} \qquad \operatorname{curl}\boldsymbol{u} = w_0 \quad \text{on} \quad \Gamma_3, \tag{9.5}$$

The chapter is organized as follows. First, in Section 9.1, we recall Nitsche's method for the Poisson problem. Section 9.2 is devoted to the description of fonctional framework. Next, in Section 9.3, we formulate and analyze the Nitsche method for the Stokes problem. In Section 9.4, we establish the stability of the method. Finally, the last section deals with the numerical experiments.

9.1 Definition of the method

Nitsche's Method [48, 53] was originally introduced for the purpose of solving Dirichlet problems without enforcing the boundary conditions in the definition of the finite element

spaces. For simplicity we consider the Poisson problem:

$$\begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} & \text{in}\,\Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on}\,\Gamma, \end{cases} \tag{9.6}$$

which can be written in weak form as: Find $\boldsymbol{u} \in \boldsymbol{u}_g + \boldsymbol{H}_0^1(\Omega)$ such that

$$\forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \quad \mathcal{B}(\boldsymbol{u},\, \boldsymbol{v}) = \mathcal{F}(\boldsymbol{v}),$$

where $\mathcal{B}(\boldsymbol{u},\, \boldsymbol{u}) = \int_\Omega \nabla\, \boldsymbol{u} \cdot \nabla\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$ and $\mathcal{F}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$.

The finite element space is defined, using a shape-regular triangulation of Ω, as:

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{H}^{\,1}(\Omega),\, (\boldsymbol{v})|_T \in \boldsymbol{P}_k,\ \forall T \in \mathcal{T}_h\}.$$

In Nitsche's method no boundary conditions are imposed on the space, but the variational formulation is augmented by appropriate terms. The method is defined as: Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that:

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_h, \quad \mathcal{B}_h(\boldsymbol{u}_h,\, \boldsymbol{v}) = \mathcal{F}(\boldsymbol{v}), \tag{9.7}$$

where

$$\mathcal{B}_h(\boldsymbol{u}_h,\, \boldsymbol{v}) \;=\; \int_\Omega \nabla\, \boldsymbol{u}_h \cdot \nabla\, \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \langle \frac{\partial\, \boldsymbol{u}_h}{\partial\, \boldsymbol{n}},\, \boldsymbol{v}\rangle_\Gamma - \langle \boldsymbol{u}_h,\, \frac{\partial\, \boldsymbol{v}}{\partial\, \boldsymbol{n}}\rangle_\Gamma + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{|e|}\langle \boldsymbol{u}_h,\, \boldsymbol{v}\rangle_e,$$

and

$$\mathcal{F}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{g},\, \frac{\partial\, \boldsymbol{v}}{\partial\, \boldsymbol{n}}\rangle_\Gamma + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{|e|}\langle \boldsymbol{g},\, \boldsymbol{v}\rangle_e,$$

where $\mathcal{F}_h$ is the collection of edges on Γ and $\gamma > 0$ is a stabilization parameter.

By use of Green's formula, we have the following proposition.

Proposition 9.1.1. *The method* (9.7) *is consistent in the sense that:*

$\mathcal{B}_h(\boldsymbol{u}_h - \boldsymbol{u},\, \boldsymbol{v}) = 0$, *for all* $\boldsymbol{v} \in \boldsymbol{V}_h$ *and for* $\boldsymbol{u}$ *sufficiently regular.*

For the purpose of stability and error estimate, we introduce the following mesh dependent norm:

$$|||\boldsymbol{v}|||^2 = \|\nabla\, \boldsymbol{v}\|_{0,\Omega}^2 + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{|e|}\|\boldsymbol{v}\|_{0,e}^2.$$

In order to show that the method (9.7) is stable, we shall show that $\mathcal{B}_h(\cdot,\, \cdot)$ is coercive with respect to the norm $|||\cdot|||$, given that γ is chosen large enough. In order to do so, we need the following inverse inequality:

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}_h, \quad \sum_{e\in\mathcal{F}_h} |e|\|\nabla\, \boldsymbol{v} \cdot \boldsymbol{n}\|_{0,e}^2 \le C_1\|\nabla\, \boldsymbol{v}\|_{0,\Omega}^2. \tag{9.8}$$

We are now ready to give the coercivity result.

Lemma 9.1.2. *If $\gamma > 0$ is large enough, then the following estimate holds:*

$$\mathcal{B}_h(\boldsymbol{u}_h,\ \boldsymbol{u}_h) \geq C|||\boldsymbol{u}_h|||^2, \tag{9.9}$$

for all $\boldsymbol{u}_h \in \boldsymbol{V}_h$ independent of h and we have

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq Ch^k \|\boldsymbol{u}\|_{k+1,\Omega}. \tag{9.10}$$

Proof. Let us here simply indicate the main steps in the proof. First one has

$$\mathcal{B}_h(\boldsymbol{u}_h,\ \boldsymbol{u}_h) = \|\nabla\ \boldsymbol{u}_h\|^2_{0,\Omega} - 2\langle \frac{\partial\ \boldsymbol{u}_h}{\partial\ \boldsymbol{n}},\ \boldsymbol{u}_h\rangle_\Gamma + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{|e|}\|\boldsymbol{u}_h\|^2_{0,e}. \tag{9.11}$$

Next, using the inverse inequality (9.8), it follows that if $\gamma > C_1^2$, the stability estimate (9.9) holds, where C_1 is the constant in the discrete trace inequality (9.8). The error estimate (9.10) then follows from the interpolation estimate in the norm $|||\cdot|||$. □

9.2 Preliminary notations and results

Let $(\mathcal{T}_h)_h$ be a shape-regular family of tiangulations of $\bar{\Omega}$. For each triangle T, we denote by h_T its diameter, by $|T|$ its area and let $h = \max\limits_{T\in\mathcal{T}_h} h_T$. We agree to denote by $\mathcal{F}_h^i$ the set of edges situated on the part Γ_i of the boundary ($i = 1, 2, 3$). Moreover, the meshes are supposed to be shape-regular, i.e., there is a constant $c > 0$, independent of h, such that:

$$\frac{h_T}{\rho_T} \leq c,\ \forall\, h > 0, \quad \forall\, T \in \mathcal{T}_h.$$

Here ρ_T denotes the diameter of the largest circle inscribed in T. For every $k \in \mathbb{N}$ and $T \in \mathcal{T}_h$ we denote by P_k the space of the polynomial functions defined on T of degree less than or equal to k, and by $\boldsymbol{P}_k$ the space $P_k \times P_k$. We introduce the following finite dimentional spaces:

$$\boldsymbol{V}_h = \big\{\boldsymbol{v}_h \in \boldsymbol{\mathcal{C}}^0(\overline{\Omega}),\ \boldsymbol{v}_h|_T \in \boldsymbol{P}_2,\ \forall T \in \mathcal{T}_h\big\},$$
$$Q_h = \big\{q_h \in \mathcal{C}^0(\overline{\Omega}),\ q_h|_T \in P_1,\ \forall T \in \mathcal{T}_h\big\}.$$

We suppose that the pair $(\mathbf{V}_h,\ Q_h)$ satisfies a uniform inf-sup condition:

$$\inf_{q_h\in Q_h} \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_h\cap H_0^1(\Omega)^2} \frac{b_h(q_h, \boldsymbol{v}_h)}{\|q_h\|_{0,\Omega}[[\boldsymbol{v}_h]]} \geq \frac{\beta}{\sqrt{\nu}}, \tag{9.12}$$

where $\beta > 0$ independent of h, ν and γ, the bilinear form b_h is defined in (9.19) and the norm $[[\cdot]]$ in (9.15).

Many finite element pairs are known to satisfy the inf-sup condition such as the classical P^2-P^1 Taylor-Hood method (see [39]).

One needs to define the following Hilbert spaces:

$$\boldsymbol{V} = \{\boldsymbol{v} \in \mathbf{L}^2(\Omega),\, \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega),\, \mathrm{curl}\, \boldsymbol{v} \in L^2(\Omega)\}, \tag{9.13}$$

$$\boldsymbol{V}_0 = \{\boldsymbol{v} \in \boldsymbol{V}(\Omega),\, \boldsymbol{v}\cdot\boldsymbol{n}|_{\Gamma_1\cup\Gamma_3} = \boldsymbol{v}\cdot\boldsymbol{t}|_{\Gamma_1\cup\Gamma_2} = 0\}. \tag{9.14}$$

The spaces $\boldsymbol{V}$ and $\boldsymbol{V}_0$ are both normed by

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} = \big(\big\|\boldsymbol{v}\|_{0,\Omega} + \|\mathrm{div}\, \boldsymbol{v}\|_{0,\Omega} + \|\mathrm{curl}\, \boldsymbol{v}\|_{0,\Omega}\big)^{1/2}.$$

Now, let us introduce a bilinear form on $\boldsymbol{V}_h \times \boldsymbol{V}_h$ representing our stabilization term:

$$J(\boldsymbol{u},\boldsymbol{v}) = \nu\Big(\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}\frac{1}{|e|}\int_e(\boldsymbol{u}\cdot\boldsymbol{t})(\boldsymbol{v}\cdot\boldsymbol{t})\,ds + \sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^3}\frac{1}{|e|}\int_e(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{n})\,ds\Big),$$

and let us also introduce the following semi-norm on $\boldsymbol{V}_h$:

$$[[\boldsymbol{v}]] = \Big(\nu\big(\|\mathrm{div}\, \boldsymbol{v}\,\|_{0,\Omega}^2 + \|\mathrm{curl}\, \boldsymbol{v}\,\|_{0,\Omega}^2\big) + \gamma J(\boldsymbol{v},\boldsymbol{v})\Big)^{\frac{1}{2}}, \tag{9.15}$$

where $\gamma > 0$ is a stabilization parameter. Then we can prove:

Lemma 9.2.1. *The application $\boldsymbol{v} \to [[\boldsymbol{v}]]$ is a norm on $\boldsymbol{V}_h$.*

Proof. We only have to show that $[[\boldsymbol{v}]] = 0$ implies $\boldsymbol{v} = \boldsymbol{0}$. So let $\boldsymbol{v} \in \boldsymbol{V}_h$ such that $\|\mathrm{div}\, \boldsymbol{v}\,\|_{0,\Omega} = \|\mathrm{curl}\, \boldsymbol{v}\,\|_{0,\Omega} = J(\boldsymbol{v},\boldsymbol{v}) = 0$. On the one hand, it follows that $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on every edges $e \in \mathcal{F}_h^1 \cup \mathcal{F}_h^3$. On the other hand, one gets that $\boldsymbol{v}\cdot\boldsymbol{t} = \boldsymbol{0}$ on every edges $e \in \mathcal{F}_h^1 \cup \mathcal{F}_h^2$. Then, $\boldsymbol{v}$ belongs to $\boldsymbol{V}_0$. It is known that for any function $\boldsymbol{v}$ of $\boldsymbol{V}_0$, we have

$$\|\boldsymbol{v}\|_{0,\Omega} \le C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{0,\Omega} + \|\mathrm{curl}\, \boldsymbol{v}\|_{0,\Omega}\Big). \tag{9.16}$$

Then, the fact of $\|\mathrm{div}\, \boldsymbol{v}\,\|_{0,\Omega} = \|\mathrm{curl}\, \boldsymbol{v}\,\|_{0,\Omega} = 0$ implies that $\boldsymbol{v} = \boldsymbol{0}$ on Ω, which concludes the proof. □

9.3 Formulation of the Nitsche's method

Our method can now be defined as follows: Find $(\boldsymbol{u}_h,\, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h,\, \boldsymbol{v}_h) + b_h(p_h,\, \boldsymbol{v}_h) = \boldsymbol{f}_h(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(q_h,\, \boldsymbol{u}_h) = \boldsymbol{g}_h(q_h) & \forall q_h \in Q_h, \end{cases} \tag{9.17}$$

where the bilinear, respectively linear forms are defined as follows:

$$\begin{aligned} a_h(\boldsymbol{u}_h,\, \boldsymbol{v}_h) &= \nu\Big(\int_\Omega \operatorname{curl} \boldsymbol{u}_h \operatorname{curl} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x}\Big) \\ &- \nu \sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2} \Big(\int_e \operatorname{curl}\boldsymbol{u}_h(\boldsymbol{v}_h\cdot\boldsymbol{t})\,\mathrm{d}\boldsymbol{s} + \int_e \operatorname{curl}\boldsymbol{v}_h(\boldsymbol{u}_h\cdot\boldsymbol{t})\,\mathrm{d}\boldsymbol{s}\Big) \\ &- \nu \sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^3} \Big(\int_e \operatorname{div}\boldsymbol{u}_h(\boldsymbol{v}_h\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{s} + \int_e \operatorname{div}\boldsymbol{v}_h(\boldsymbol{u}_h\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{s}\Big) \\ &+ \gamma\nu\Big(\sum_{e\mathcal{F}_h^1\cup\mathcal{F}_h^2} \frac{1}{|e|}\int_e (\boldsymbol{u}_h\cdot\boldsymbol{t})(\boldsymbol{v}_h\cdot\boldsymbol{t})\,\mathrm{d}\boldsymbol{s} + \sum_{e\mathcal{F}_h^1\cup\mathcal{F}_h^3} \frac{1}{|e|}\int_e (\boldsymbol{u}_h\cdot\boldsymbol{n})(\boldsymbol{v}_h\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{s}\Big), \end{aligned} \tag{9.18}$$

$$b_h(q_h,\, \boldsymbol{v}_h) = -\int_\Omega q_h \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} + \sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^3} \int_e q_h\, \boldsymbol{v}_h\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s}. \tag{9.19}$$

$$\begin{aligned} \boldsymbol{f}_h(\boldsymbol{v}_h) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h\,\mathrm{d}\boldsymbol{x} - \nu\sum_{e\in\mathcal{F}_h^1}\Big(\int_e (\operatorname{curl}\boldsymbol{v}_h)\boldsymbol{u}_0\cdot\boldsymbol{t}\,\mathrm{d}\boldsymbol{s} + \int_e (\operatorname{div}\boldsymbol{v}_h)\boldsymbol{u}_0\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s} \\ &- \frac{\gamma}{|e|}\int_e \boldsymbol{u}_0\cdot\boldsymbol{n}\,\boldsymbol{v}_h\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s} - \frac{\gamma}{|e|}\int_e \boldsymbol{u}_0\cdot\boldsymbol{t}\,\boldsymbol{v}_h\cdot\boldsymbol{t}\,\mathrm{d}\boldsymbol{s}\Big) - \nu\sum_{e\in\mathcal{F}_h^2}\Big(\int_e (\operatorname{curl}\boldsymbol{v}_h)\boldsymbol{a}\cdot\boldsymbol{t}\,\mathrm{d}\boldsymbol{s} \\ &+ \int_e p_0\,\boldsymbol{v}_h\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s} - \frac{\gamma}{|e|}\int_e \boldsymbol{a}\cdot\boldsymbol{t}\,\boldsymbol{v}_h\cdot\boldsymbol{t}\,\mathrm{d}\boldsymbol{s}\Big) - \nu\sum_{e\in\mathcal{F}_h^3}\Big(\int_e (\operatorname{div}\boldsymbol{v}_h)\,\boldsymbol{b}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s} \\ &- \int_e w_0\,\boldsymbol{v}_h\cdot\boldsymbol{t}\,\mathrm{d}\boldsymbol{s} - \frac{\gamma}{|e|}\int_e \boldsymbol{b}\cdot\boldsymbol{n}\,\boldsymbol{v}_h\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s}\Big). \end{aligned} \tag{9.20}$$

$$\boldsymbol{g}_h(\boldsymbol{v}_h) = \sum_{e\in\mathcal{F}_h^1}\int_e q_h\,\boldsymbol{u}_0\cdot\boldsymbol{n}\mathrm{d}\boldsymbol{s} + \sum_{e\in\mathcal{F}_h^3}\int_e q_h\,\boldsymbol{b}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s}. \tag{9.21}$$

The following Lemma states that the proposed method is indeed consistent.

Lemma 9.3.1. *For a sufficiently regular solution* $(\boldsymbol{u},\, p)$ *of the continuous problem* (9.1)-(9.5), *we have:*

$$\begin{cases} a_h(\boldsymbol{u},\, \boldsymbol{v}_h) + b_h(p,\, \boldsymbol{v}_h) = l_h(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b_h(q_h,\, \boldsymbol{u}) = g_h(q_h) & \forall q_h \in Q_h. \end{cases} \tag{9.22}$$

Proof. Let $(\boldsymbol{u},\, p)$ be a sufficiently regular solution of (9.1)-(9.5). The second relation is obvious, since $\operatorname{div}\boldsymbol{u} = 0$ in Ω. The first consistency property is obtained after integrating by parts

on each element T and using the supposed regularity of the continuous solution:

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T(\nu\,\mathrm{curl}\,\boldsymbol{u}\,\mathrm{curl}\,\boldsymbol{v}_h - p\,\mathrm{div}\,\boldsymbol{v}_h)\,\mathrm{d}\boldsymbol{x} &= \int_\Omega(-\nu\Delta\,\boldsymbol{u}+\nabla p)\cdot\boldsymbol{v}_h\,\mathrm{d}\boldsymbol{x}\\ &+\sum_{e\in\bigcup_{i=1}^3\mathcal{F}_h^i}\int_e\nu\,\mathrm{curl}\,\boldsymbol{u}\,(\boldsymbol{v}_h\cdot\boldsymbol{t})\,\mathrm{d}\boldsymbol{s}\\ &-\sum_{e\in\bigcup_{i=1}^3\mathcal{F}_h^i}\int_e p\,\boldsymbol{v}_h\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s}.\end{aligned}$$

Therefore, since $\mathrm{div}\,\boldsymbol{u}=0$ in Ω and taking into account the boundary condition given in (9.3)-(9.5), the definition of $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$, we obtain that $(\boldsymbol{u},p)$ verifies the first property in (9.22).

□

9.4 Stability

Lemma 9.4.1. *If γ is large enough, there exists a constant $C>0$ such that for all $\boldsymbol{v}_h\in\boldsymbol{V}_h$*

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h)\geq C[[\boldsymbol{v}_h]]^2. \tag{9.23}$$

Proof. By definition one has that

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h)=[[\boldsymbol{v}_h]]^2-2\nu\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}\int_e\mathrm{curl}\boldsymbol{v}_h(\boldsymbol{v}_h\cdot\boldsymbol{t})-2\nu\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^3}\int_e\mathrm{div}\,\boldsymbol{v}_h(\boldsymbol{v}_h\cdot\boldsymbol{n}). \tag{9.24}$$

So one only has to control the two last terms. Using the Cauchy-Schwarz, it follows that

$$-2\nu\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}\int_e\mathrm{curl}\boldsymbol{v}_h(\boldsymbol{v}_h\cdot\boldsymbol{t})\geq-2\nu\left(\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}|e|\|\mathrm{curl}\boldsymbol{v}_h\|_{0,e}^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}\frac{1}{|e|}\|\boldsymbol{v}_h\cdot\boldsymbol{t}\|_{0,e}^2\right)^{\frac{1}{2}}.$$

A classical scaling argument together with the equivalence of norms in finite dimensional spaces yields for any polynomial function $\boldsymbol{u}$ that

$$\sqrt{|e|}\|\boldsymbol{u}\|_{0,e}\leq c\|\boldsymbol{u}\|_{0,T}$$

where c is a constant independent of the discretization.
Now let $e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2$ such that $e=\partial T$ with $T\in\mathcal{T}_h$. Obviousely,

$$\sqrt{|e|}\|\mathrm{curl}\boldsymbol{v}_h\|_{0,e}\leq c\|\mathrm{curl}\boldsymbol{v}_h\|_{0,\mathrm{T}}.$$

So by summing upon all edges of $\mathcal{F}_h^1 \cup \mathcal{F}_h^2$ it follows that

$$-2\nu \sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}\int_e \mathrm{curl}\boldsymbol{v}_h(\boldsymbol{v}_h\cdot\boldsymbol{t}) \geq -2c_1\sqrt{\nu}\|\mathrm{curl}\,\boldsymbol{v}_h\|_{0,\Omega}\Big(\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}\frac{1}{|e|}\|\boldsymbol{v}_h\cdot\boldsymbol{t}\|_{0,e}^2\Big)^{\frac{1}{2}}.$$

The bound for the last term in (9.24) is very similar and we can check that

$$-2\nu \sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^3}\int_e \mathrm{div}\,\boldsymbol{v}_h(\boldsymbol{v}_h\cdot\boldsymbol{n}) \geq -2c_2\sqrt{\nu}\|\mathrm{div}\,\boldsymbol{v}_h\|_{0,\Omega}\Big(\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^3}\frac{1}{|e|}\|\boldsymbol{v}_h\cdot\boldsymbol{n}\|_{0,e}^2\Big)^{\frac{1}{2}}.$$

Then, we obtain,

$$\begin{aligned} a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \geq [[\boldsymbol{v}_h]]^2 \;&-\; \frac{2c_1}{\sqrt{\gamma}}\big(\sqrt{\nu}\,\|\mathrm{curl}\,\boldsymbol{v}_h\|_{0,\Omega}\big)\Big(\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^2}\frac{\gamma}{|e|}\|\boldsymbol{v}_h\cdot\boldsymbol{t}\|_{0,e}^2\Big)^{\frac{1}{2}} \\ &-\; \frac{2c_2}{\sqrt{\gamma}}\big(\sqrt{\nu}\,\|\mathrm{div}\,\boldsymbol{v}_h\|_{0,\Omega}\big)\Big(\sum_{e\in\mathcal{F}_h^1\cup\mathcal{F}_h^3}\frac{\gamma}{|e|}\|\boldsymbol{v}_h\cdot\boldsymbol{n}\|_{0,e}^2\Big)^{\frac{1}{2}}, \end{aligned}$$

and finally,

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \geq [[\boldsymbol{v}_h]]^2 - \frac{c}{\sqrt{\gamma}}[[\boldsymbol{v}_h]]^2.$$

In conclusion, $a_h(\cdot,\cdot)$ is positive definite for $\gamma > c^2$. So the statement clearly holds. □

We obtain stability of the pressure via the inf-sup condition (9.12). Th details are skipped here, since they will be given later in the context of the discontinuous Galerkin formulation.

The continuity of the linear forms $\boldsymbol{f}_h$ and $\boldsymbol{g}_h$ follows with the same arguments. So, we are able to state the following result.

Theorem 9.4.2. *For γ sufficiently large, the problem* (9.17) *has a unique solution.*

Proof. Since the hypotheses of the Babuška-Brezi Theorem are satisfied, the mixed variational formulation (9.17) is well-posed. □

9.5 Numerical results

In this section, we present several numerical experiments in order to confirm the theoretical results. We are first interested in the convergence rate and next in the flow arround cylinder confined between two plates. The developed codes are based on the C++-library Concha [51]. We use the $P^2 - P^1$ Taylor-Hood finite element pair.

9.5.1 Mesh convergence

We first study the behavior of the numerical scheme (9.17) with respect to mesh refinement. We consider the exact solution of the Stokes problem:

$$\boldsymbol{u} = \begin{pmatrix} \pi \sin(\pi x)\cos(\pi y) \\ -\pi \cos(\pi x)\sin(\pi y) \end{pmatrix}, \qquad p = \sin(\pi x)\sin(\pi y), \tag{9.25}$$

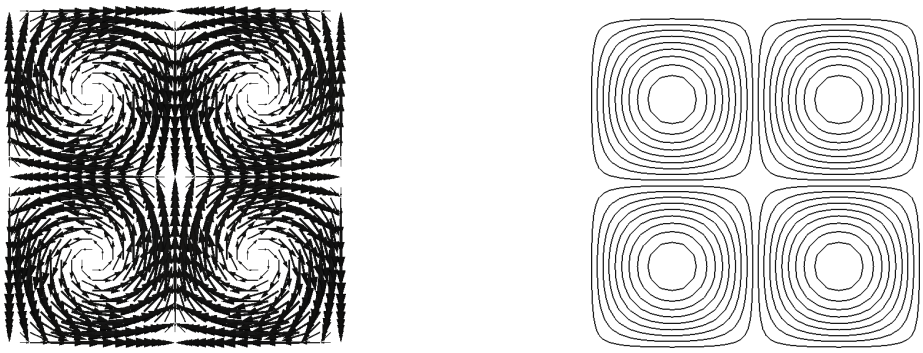
on the sequare $\Omega = [-1,1]\times[-1,1]$. This solution satisfies $\operatorname{div}\boldsymbol{u} = 0$ and $\displaystyle\int_\Omega p\,\mathrm{d}\boldsymbol{x} = 0$.

We take $\gamma = 10$ and $\nu = 1$. We present in Table 9.1 the values of the errors on different meshes obtained for the velocity $\boldsymbol{u}$ together with its curl and div in the L^2-norm, the errors for the velocity in the $\boldsymbol{H}^1-$norm and the boundary errors, while the error for the pressure is given in the L^2-norm.

One may notice that the errors on p and $\boldsymbol{u}$ are both in $O(h^2)$ in the L^2-norm for the pressure and in the energy norm for the velocity. They are in agreement with the theoritical results: the error is divided by 4 when the mesh size h is divided by 2. The exact solution is depicted in Figure 9.1.

N	$\|e\|_{1,\Omega}$	$\|p-p_h\|_{0,\Omega}$	$\|\mathrm{curl}\,(e)\|_{0,\Omega}$	$\|\mathrm{div}\,(e)\|_{0,\Omega}$	$J(e,e)$	$\|e\|_{0,\Omega}$
16	4.4846e-01	3.1296-01	3.4895e-01	2.9378e-01	4.8003e-01	3.1792e-02
64	1.0483e-01	6.3539-02	7.7568e-02	7.2695e-02	8.1571e-02	3.9029e-03
256	2.5710e-02	1.5023-02	1.8546e-02	1.8123e-02	1.4988e-02	4.9024e-04
1024	6.3950e-03	3.7030-03	4.5626e-03	4.5231e-03	2.7033e-03	6.1426e-05
4096	1.5966e-03	9.2248-04	1.1337e-03	1.1296e-03	4.8227e-04	7.6902e-06
16384	3.9902e-04	2.3042-04	2.8271e-04	2.8228e-04	8.7373e-05	9.7799e-07

Table 9.1: Pressure error in L^2 and velocity error $e = u - u_h$ in different norms for $\gamma = 10$.



(a) Velocity (b) Pressure

Figure 9.1: Exact Velocity and Pressure

9.5.2 Test: Flow around cylinder confined between two plates

Now, we consider a flow past a cylinder of radius $R = 0.01m$ symmetrically placed in a rectangle with of length $0.3m$ and height $0.04m$, see Figure 9.2. This geometry is taken from [43]. We compute the drag coefficient C_D along the cylinder. The results are illustrated in Table 9.2. In addition, we give the extrapolated values. The computed values are in good agreement with the reference values given in [51].

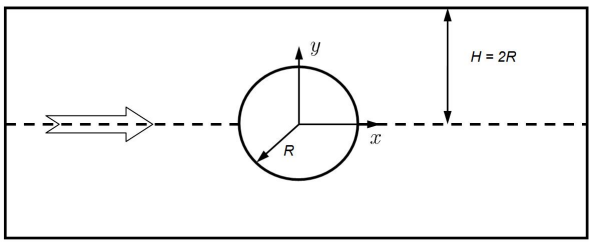


Figure 9.2: Geometry of the cylinder between two flat plates

In this numerical experiments, the normal velocity and the vorticity are imposed on the inlet. On the outlet, the pressure is imposed together with the tagential velocity. We present in Figure 9.3 and Figure 9.4 the velocity field and the pressure.

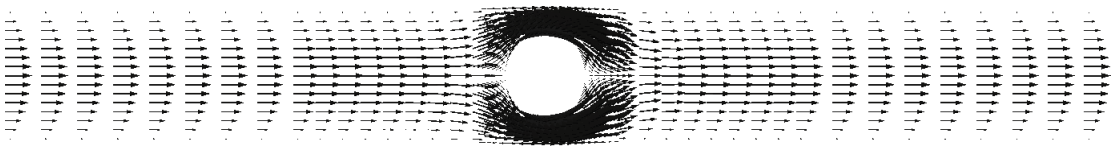


Figure 9.3: Calculated velocity for the flow test around cylinder

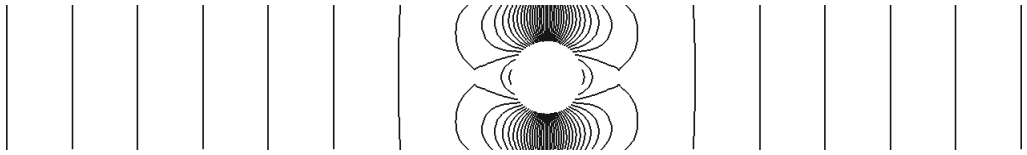


Figure 9.4: Calculated pressure for the flow test around cylinder

ne	C_D	ΔC_D	C_D^*
256	131.757	—	—
1024	132.166	0,409	132,3023
4096	132.274	0,108	132,31
16384	132.301	0,027	132,31
65536	132.308	0,007	132,3103

Table 9.2: Drag results on succesive meshes.

Chapter 10

A Discontinuous Galerkin method

The aim of this chapter is to study a Discontinuous Galerkin method for the Stokes equations (9.1)-(9.5).

10.1 Preliminary notations and results

For defining our DG method for the Stokes problem, we use the notations of Chapter 9. We recall thet $\mathcal{F}_h^i$ denotes the set of edges situated on the boundary Γ_i, $i = 1, 2, 3$. We denote by $\mathcal{E}_h^{int}$ the set of internal edges of $\mathcal{T}_h$ and by $\mathcal{C}_h$ the set of all edges of $\mathcal{T}_h$, $\mathcal{C}_h = \mathcal{E}_h^{int} \cup \mathcal{F}_h^i$. Let e denote a segment of $\mathcal{E}_h^{int}$ shared by two triangles T_1 and T_2 of $\mathcal{T}_h$; we associate with e, once and for all, a unit normal vector $\boldsymbol{n}_e$ directed from T_1 to T_2, and we define the jump and average of function φ on e by

$$[\varphi] = (\varphi\,|_{T_1})\,|_e - (\varphi\,|_{T_2})\,|_e, \qquad \{\varphi\} = \frac{1}{2}(\varphi\,|_{T_1})\,|_e + \frac{1}{2}(\varphi\,|_{T_2})\,|_e\ .$$

If e is adjacent to $\partial\Omega$, then $\boldsymbol{n}_e$ is the unit normal $\boldsymbol{n}$ exterior to Ω and the jump and the average of φ on e coincide with the trace of φ on e.

In what follows, we take $k = 1$, 2 or 3 and we introduce the finite dimensional spaces:

$$\boldsymbol{X}_h = \big\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega),\ \boldsymbol{v}_h|_T \in \boldsymbol{P}_k,\ \forall T \in \mathcal{T}_h\big\},$$

$$Q_h = \big\{q_h \in \mathbf{L}_0^2(\Omega),\ q_h|_T \in P_{k-1},\ \forall T \in \mathcal{T}_h\big\},$$

Let us recall the approximation properties of $\boldsymbol{X}_h$ and Q_h (see also [40]). We denote by $r_h \in \mathcal{L}(L_0^2(\Omega),\, Q_h)$ the L^2-projection such that, for any $T \in \mathcal{T}_h$

$$\forall p \in L_0^2(\Omega),\quad \forall q \in P_{k-1}, \qquad \int_T q(r_h(p) - p) = 0, \tag{10.1}$$

For each integer $k = 1, 2, 3$, it is known that there exists an operator $\boldsymbol{R}_h \in \mathcal{L}(\boldsymbol{H}^1(\Omega),\ \boldsymbol{X}_h)$ such that for any $T \in \mathcal{T}_h$ and any $e \in \mathcal{C}_h$, one has:

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \qquad \forall q \in P_{k-1}, \qquad \int_T q \,\mathrm{div}\,(\boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \tag{10.2}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \qquad \forall q \in P_{k-1}, \qquad \int_T q \,\mathrm{curl}\,(\boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \tag{10.3}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega), \quad \forall \boldsymbol{q} \in \boldsymbol{P}_{k-1}, \qquad \int_e q \cdot [\boldsymbol{R}_h(\boldsymbol{v})] = 0, \tag{10.4}$$

thanks to the fact that the space $\boldsymbol{X}_h$ contains nonconforming finite element spaces. Moreover, for $s \in [0, k]$, the following interpolation estimates holds:

$$\forall p \in H^s(\Omega) \cap L^2_0(\Omega), \qquad \|p - r_h(p)\|_{0,T} \leq C h_T^s \ |p|_{s,T} \tag{10.5}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}^{s+1}(\Omega), \qquad \|\boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}\|_{1,T} \leq C h_T^s |\boldsymbol{v}|_{s+1,\Delta_T}, \tag{10.6}$$

Where Δ_T is a suitable macro-element containing T. When $k = 1$, $\Delta_T = T$, this case follows from [28]. The case $k = 2$ follows from [36] and $k = 3$ from [29].

Before defining the DG method, we consider the variational formulation for the probelm (9.1)-(9.5). We suppose that

$$\begin{cases} \boldsymbol{f} \in \boldsymbol{L}^2(\Omega),\ p_0 \in L^2(\Gamma_2),\ \boldsymbol{h} \in \boldsymbol{L}^2(\Gamma_3), \\ \exists \boldsymbol{u}^* \in \boldsymbol{H}^{\,1}(\Omega) \text{ such that } \mathrm{div}\,\boldsymbol{u}^* = 0,\ \ \boldsymbol{u}^*|_{\Gamma_1} = \boldsymbol{u}_0,\ \ \boldsymbol{u}^* \times \boldsymbol{n}|_{\Gamma_2} = \boldsymbol{a} \times \boldsymbol{n},\ \ \boldsymbol{u}^* \cdot \boldsymbol{n}|_{\Gamma_3} = \boldsymbol{b} \cdot \boldsymbol{n}. \end{cases}$$

Then, the Stokes problem (9.1)-(9.5) is equivalent to the variational formulation: (see [24])

$$\begin{cases} \text{Find} (\boldsymbol{u}, p) \in (\boldsymbol{u}^* + \boldsymbol{V}_0) \times L^2(\Omega) \text{ such that:} \\ a(\boldsymbol{u}, \boldsymbol{v}) + b(p, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0 \\ b(q, \boldsymbol{u}) = 0 \qquad \forall q \in L^2(\Omega) \end{cases}, \tag{10.7}$$

where the bilinear and the linear forms are defined by:

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= \nu \int_\Omega \mathrm{curl}\,\boldsymbol{u} \cdot \mathrm{curl}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x}, \\ b(p, \boldsymbol{v}) &= -\int_\Omega p \mathrm{div}\,\boldsymbol{v}\, \mathrm{d}\boldsymbol{x}, \\ l(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \int_{\Gamma_2} p_0(\boldsymbol{v} \cdot \boldsymbol{n})\, \mathrm{d}\boldsymbol{s} + \int_{\Gamma_3} (\boldsymbol{h} \times \boldsymbol{n}) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{s}. \end{aligned}$$

Now, let us introduce two bilinear forms on $\boldsymbol{X}_h$ representing our stabilization terms:

$$J_1(\boldsymbol{u}, \boldsymbol{v}) = \nu \sum_{e \in \mathcal{C}_h \setminus \mathcal{F}_h^3} \frac{1}{|e|} \int_e [\boldsymbol{u} \cdot \boldsymbol{t}_e] \cdot [\boldsymbol{v} \cdot \boldsymbol{t}_e]\, ds,$$

$$J_2(\boldsymbol{u}_h, \boldsymbol{v}) = \nu \sum_{e \in \mathcal{C}_h \backslash \mathcal{F}_h^2} \frac{1}{|e|} \int_e [\boldsymbol{u} \cdot \boldsymbol{n}_e] \cdot [\boldsymbol{v} \cdot \boldsymbol{n}_e] \, ds$$

Let us also introduce the following mapping :

$$|||\boldsymbol{v}||| = \Big(\nu \sum_{T \in \mathcal{T}_h} \big(\|\text{div } \boldsymbol{v}\|_{0,T}^2 + \|\text{curl } \boldsymbol{v}\|_{0,T}^2 \big) + \sigma \big(J_1(\boldsymbol{v}, \boldsymbol{v}) + J_2(\boldsymbol{v}, \boldsymbol{v}) \big) \Big)^{\frac{1}{2}} \tag{10.8}$$

where $\sigma > 0$ is a stabilization parameter. Then we can prove:

Lemma 10.1.1. *The application $\boldsymbol{v} \to |||\boldsymbol{v}|||$ is a norm on $\boldsymbol{X}_h$.*

Proof. We only have to show that $|||\boldsymbol{v}||| = 0$ implies $\boldsymbol{v} = \boldsymbol{0}$. So let $\boldsymbol{v} \in \boldsymbol{X}_h$ such that $\sum_{T \in \mathcal{T}_h} \big(\|\text{div } \boldsymbol{v}\|_{0,T}^2 + \|\text{curl } \boldsymbol{v}\|_{0,T}^2 \big) = J_1(\boldsymbol{v}, \boldsymbol{v}) = J_2(\boldsymbol{v}, \boldsymbol{v}) = 0$. On the one hand, it follows that $[\boldsymbol{v} \cdot \boldsymbol{n}_e] = 0$ on every edges $e \in \mathcal{C}_h | \mathcal{F}_h^2$. Then, $\boldsymbol{v}$ is continuous across the internel edges and belongs to $\boldsymbol{H}(\text{div}, \Omega)$. On the other hand, one gets that $[\boldsymbol{v} \cdot \boldsymbol{t}_e] = 0$ on every edges $e \in \mathcal{C}_h | \mathcal{F}_h^3$. Then, $\boldsymbol{v}$ belongs to $\boldsymbol{H}(\text{curl}, \Omega)$. As a consequence $\boldsymbol{v}$ belongs to $\boldsymbol{V}_0$. Using the inequality (9.16) and the fact of $\sum_{T \in \mathcal{T}_h} \big(\|\text{div } \boldsymbol{v}\|_{0,T}^2 + \|\text{curl } \boldsymbol{v}\|_{0,T}^2 \big) = 0$ implies that $\boldsymbol{v} = 0$ on Ω, which concludes the proof. □

10.2 The discrete formulation and the well-possedness of the discrete problem

We consider the next discrete DG formulation of (9.1)-(9.5), where Nitsche's method is employed in order to treat the non-homegeneous boundary conditions:

$$\begin{cases} (\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times Q_h, & & \\ a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) & = l_h(\boldsymbol{v}_h) \quad , & \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \\ b_h(q_h, \boldsymbol{u}_h) & = g_h(q_h) \quad , & \forall q_h \in Q_h. \end{cases} \tag{10.9}$$

The bilinear, respectively linear forms are defined as follows:

$$a_h(\cdot, \cdot) = A_0(\cdot, \cdot) + A_1(\cdot, \cdot) + A_2(\cdot, \cdot) + \sigma \big(J_1(\cdot, \cdot) + J_2(\cdot, \cdot) \big).$$

$$A_0(\boldsymbol{u}_h, \boldsymbol{v}_h) = \nu \sum_{\mathrm{T} \in \mathcal{T}_h} \Big(\int_{\mathrm{T}} \text{curl } \boldsymbol{u}_h \, \text{curl } \boldsymbol{v}_h + \int_{\mathrm{T}} \text{div } \boldsymbol{u}_h \, \text{div } \boldsymbol{v}_h \Big).$$

$$A_1(\boldsymbol{u}_h, \boldsymbol{v}_h) = -\nu \sum_{e \in \mathcal{C}_h | \mathcal{F}_h^3} \Big(\int_e \{\text{curl} \boldsymbol{u}_h\} [\boldsymbol{v}_h \cdot \boldsymbol{t}_e] + \int_e \{\text{curl} \boldsymbol{v}_h\} [\boldsymbol{u}_h \cdot \boldsymbol{t}_e] \Big).$$

$$A_2(\boldsymbol{u}_h, \boldsymbol{v}_h) = -\nu \sum_{e \in \mathcal{C}_h | \mathcal{F}_h^2} \Big(\int_e \{\text{div} \boldsymbol{u}_h\} [\boldsymbol{v}_h \cdot \boldsymbol{n}_e] + \int_e \{\text{div} \boldsymbol{v}_h\} [\boldsymbol{u}_h \cdot \boldsymbol{n}_e] \Big).$$

$$b_h(q_h, \boldsymbol{v}_h) = -\nu \sum_{\mathrm{T}\in\mathcal{T}_h} \int_{\mathrm{T}} q_h \mathrm{div}\boldsymbol{v}_h \,\mathrm{d}\,\boldsymbol{x} + \sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2} \int_e \{q_h\}\,[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\,\mathrm{d}\,\gamma,$$

$$\begin{aligned} l_h(\boldsymbol{v}_h) = \sum_{\mathrm{T}\in\mathcal{T}_h} \int_{\mathrm{T}} \boldsymbol{f}\cdot\boldsymbol{v}_h - \nu \sum_{e\in\mathcal{F}_h^1} \Big(\int_e (\mathrm{curl}\,\boldsymbol{v}_h)\boldsymbol{u}_0\cdot\boldsymbol{t}_e + \int_e (\mathrm{div}\,\boldsymbol{v}_h)\boldsymbol{u}_0\cdot\boldsymbol{n}_e + \int_e \frac{\sigma}{|e|}\boldsymbol{u}_0\cdot\boldsymbol{n}_e\,\boldsymbol{v}_h\cdot\boldsymbol{n}_e \\ + \int_e \frac{\sigma}{|e|}\boldsymbol{u}_0\cdot\boldsymbol{t}_e\cdot\boldsymbol{v}_h\cdot\boldsymbol{t}_e\Big) - \nu \sum_{e\in\mathcal{F}_h^2} \Big(\int_e (\mathrm{curl}\,\boldsymbol{v}_h)\boldsymbol{a}\cdot\boldsymbol{t}_e + \int_e p_0\,\boldsymbol{v}_h\cdot\boldsymbol{n}_e - \frac{\sigma}{|e|}\int_e \boldsymbol{a}\cdot\boldsymbol{t}_e\cdot\boldsymbol{v}_h\cdot\boldsymbol{t}_e\Big) \quad , \\ -\nu \sum_{e\in\mathcal{F}_h^3} \Big(\int_e (\mathrm{div}\,\boldsymbol{v}_h)\,\boldsymbol{b}\cdot\boldsymbol{n}_e - \int_e w_0\,\boldsymbol{v}_h\cdot\boldsymbol{t}_e - \frac{\sigma}{|e|}\int_e \boldsymbol{b}\cdot\boldsymbol{n}_e\,\boldsymbol{v}_h\cdot\boldsymbol{n}_e\,d\gamma\Big). \end{aligned}$$

$$g_h(q_h) = \sum_{e\in\mathcal{F}_h^1} \int_e q_h\,\boldsymbol{u}_0\cdot\boldsymbol{n}\mathrm{d}\,\gamma + \sum_{e\in\mathcal{F}_h^3} \int_e q_h\,\boldsymbol{b}\cdot\boldsymbol{n}\,\mathrm{d}\,\gamma.$$

The aim of this section is to prove the well-posedness of the previous discrete formulation (10.9). For this purpose, we shall apply the Babuska-Brezzi theorem. Let us begin by checking the coercivity of the bilinear form $a_h(\cdot,\cdot)$ on the discret Kernel $Ker_h b_h$, as well as the inf-sup condition for $b_h(\cdot,\cdot)$ with respect for the norm $|||\cdot|||$.

Lemma 10.2.1. *If σ is large enough, there exists a constant $C>0$ such that for all $\boldsymbol{v}_h \in \boldsymbol{X}_h$*

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C|||\boldsymbol{v}_h|||^2 \tag{10.10}$$

Proof. One has by definition that

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) = |||\boldsymbol{v}_h|||^2 + A_1(\boldsymbol{v}_h, \boldsymbol{v}_h) + A_2(\boldsymbol{v}_h, \boldsymbol{v}_h). \tag{10.11}$$

So one only has to control the two terms $A_1(\boldsymbol{v}_h, \boldsymbol{v}_h)$ and $A_2(\boldsymbol{v}_h, \boldsymbol{v}_h)$.
By definition

$$A_1(\boldsymbol{v}_h, \boldsymbol{v}_h) = -2\nu \sum_{e\in\mathcal{C}_h|\mathcal{F}_h^3} \int_e \{\mathrm{curl}\boldsymbol{v}_h\}[\boldsymbol{v}_h\cdot\boldsymbol{t}_e].$$

Using the Cauchy-Schwarz, it follows that

$$A_1(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq -2\nu \left(\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^3} |e| \|\{\mathrm{curl}\boldsymbol{v}_h\}\|_{0,e}^2 \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^3} \frac{1}{|e|} \|[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]\|_{0,e}^2 \right)^{\frac{1}{2}}.$$

A classical scaling argument together with the equivalence of norms in finite dimensional spaces yields for any polynomial function $\boldsymbol{u}$ that

$$\sqrt{|e|}\|\boldsymbol{u}\|_{0,e} \leq c\|\boldsymbol{u}\|_{0,T}$$

where c is a constant independent of the discretization.
Now let $e \in \mathcal{E}_h^{int}$ such that $\{e\} = \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in \mathcal{T}_h$. Obviousely,

$$\begin{aligned}\sqrt{|e|}\|\{\mathrm{curl}\boldsymbol{v}_h\}\|_{0,e} &\leq \frac{\sqrt{|e|}}{2}\sum_{i=1}^{2}\|(\mathrm{curl}\,\boldsymbol{v}_h)_{/T_i}\|_{0,e} \\ &\leq \frac{c_1}{2}\Big(\|\mathrm{curl}\boldsymbol{v}_h\|_{0,\mathrm{T}_1} + \|\mathrm{curl}\boldsymbol{v}_h\|_{0,\mathrm{T}_2}\Big)\end{aligned}$$

a completely similar argument holds on a boundary edge $e \in \mathcal{F}_h^i$, $i = 1, 2$. So by summing upon all edges it follows that

$$A_1(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq -2c_1\big(\sqrt{\nu}\sum_{T\in\mathcal{T}_h}\|\mathrm{curl}\,\boldsymbol{v}_h\|_{0,T}\big)\sqrt{J_1(\boldsymbol{v}_h, \boldsymbol{v}_h)}\,.$$

The bound for $A_2(\boldsymbol{v}_h, \boldsymbol{v}_h)$ is very similar and we can check that

$$A_2(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq -2c_2\big(\sqrt{\nu}\sum_{T\in\mathcal{T}_h}\|\mathrm{div}\,\boldsymbol{v}_h\|_{0,T}\big)\sqrt{J_2(\boldsymbol{v}_h, \boldsymbol{v}_h)}\,.$$

As a consequence, we have

$$\begin{aligned}a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) &\geq |||\boldsymbol{v}_h|||^2 - \frac{2c_1}{\sqrt{\sigma}}\big(\sqrt{\nu}\sum_{T\in\mathcal{T}_h}\|\mathrm{curl}\,\boldsymbol{v}_h\|_{0,T}\big)\big(\sqrt{\sigma\,J_1(\boldsymbol{v}_h, \boldsymbol{v}_h)}\big) \\ &- \frac{2c_2}{\sqrt{\sigma}}\big(\sqrt{\nu}\sum_{T\in\mathcal{T}_h}\|\mathrm{div}\,\boldsymbol{v}_h\|_{0,T}\big)\sqrt{\sigma\,J_2(\boldsymbol{v}_h, \boldsymbol{v}_h)},\end{aligned}$$

and finally,

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq |||\boldsymbol{v}_h|||^2 - \frac{c}{\sqrt{\sigma}}|||\boldsymbol{v}_h|||^2.$$

In conclusion, $a_h(\cdot,\cdot)$ is positive definite for $\sigma > c^2$. So the statement clearly holds. □

Let us now focus on the inf-sup condition for $b_h(\cdot,\cdot)$, with respect to the norm $|||\cdot|||$.

Lemma 10.2.2. *There exists a constant $\beta > 0$, independent of h and ν, such that*

$$\inf_{q\in Q_h}\sup_{\boldsymbol{v}\in\boldsymbol{X}_h}\frac{b_h(q, \boldsymbol{v})}{\|q\|_{0,\Omega}|||\boldsymbol{v}|||} \geq \frac{\beta}{\sqrt{\nu}}. \tag{10.12}$$

Proof. The proof is rather classical. With any $q \in Q_h$, we shall associate $\boldsymbol{w} \in \boldsymbol{X}_h$, such that

$$b(q, \boldsymbol{w}) = \|q\|_{0,\Omega}^2 \tag{10.13}$$

$$|||\boldsymbol{w}||| \leq c\|q\|_{0,\Omega}. \tag{10.14}$$

For this purpose, we make use of the continuous inf-sup condition for the Stokes problem (see for instance [39]). Thus, let $q \in Q_h \subset L_0^2(\Omega)$ and let $\boldsymbol{z} \in \boldsymbol{H}_0^1(\Omega)$ such that:

$$\begin{cases}\mathrm{div}\,\boldsymbol{z} = q \\ \|\boldsymbol{z}\|_{1,\Omega} \leq c\|q\|_{0,\Omega}\end{cases}.$$

Then we put $\boldsymbol{w} = \boldsymbol{R}_h(\boldsymbol{z}) \in \boldsymbol{X}_h$. By construction, we have according to (10.2) and (10.4) on every $T \in \mathcal{T}_h$ and every $e \in \mathcal{C}_h$:

$$\int_T q \operatorname{div} \boldsymbol{w} dx = \int_T q \operatorname{div} \boldsymbol{z} = \|q\|^2_{0,\Omega}$$

$$\int_e \{q\}[\boldsymbol{w} \cdot \boldsymbol{n}_e] ds = 0$$

so $b_h(q, \boldsymbol{w}) = \|q\|^2_{0,\Omega}$. Next, since $[\boldsymbol{z} \cdot \boldsymbol{n}_e] = 0$ we have

$$\begin{aligned} J_1(\boldsymbol{w}, \boldsymbol{w}) &= J_1(\boldsymbol{z} - \boldsymbol{R}_h\boldsymbol{z}, \boldsymbol{z} - \boldsymbol{R}_h\boldsymbol{z}) \\ &= \nu \sum_{e \in \mathcal{C}_h | \mathcal{F}_h^3} \frac{1}{|e|} \|[(\boldsymbol{z} - \boldsymbol{R}_h\boldsymbol{z}) \cdot \boldsymbol{t}_e]\|^2_{0,e} \\ &\leq c\nu |\boldsymbol{z}|^2_{1,\Omega}. \end{aligned}$$

The last inequality is obtained in a classical way by using (10.6) after passing to reference element and by making use of the trace inequality. Similarly, we can check that

$$J_2(\boldsymbol{w}, \boldsymbol{w}) \leq c'\nu |\boldsymbol{z}|^2_{1,\Omega}.$$

Therefore, by using the interpolation estimate (10.6) we get

$$\begin{aligned} |||\boldsymbol{w}|||^2 &= \nu \sum_{T \in \mathcal{T}_h} \|\operatorname{div}(\boldsymbol{R}_h\boldsymbol{z})\|^2_{0,\Omega} + \nu \sum_{T \in \mathcal{T}_h} \|\operatorname{curl}(\boldsymbol{R}_h\boldsymbol{z})\|^2_{0,\Omega} + \sigma\big(J_1(\boldsymbol{w}, \boldsymbol{w}) + J_2(\boldsymbol{w}, \boldsymbol{w})\big) \\ &\leq C\nu \|\boldsymbol{z}\|^2_{1,\Omega} \end{aligned}$$

so

$$|||\boldsymbol{w}||| \leq C\sqrt{\nu}\|q\|_{0,\Omega}$$

which finally yields

$$\sup_{\boldsymbol{v} \in \boldsymbol{X}_h} \frac{b_h(q, \boldsymbol{v})}{|||\boldsymbol{v}|||} \geq \frac{b_h(q, \boldsymbol{w})}{|||\boldsymbol{w}|||} \geq \frac{\|q\|^2_{0,\Omega}}{C\sqrt{\nu}\|q\|_{0,\Omega}} \geq \frac{\beta}{\sqrt{\nu}}\|q\|_{0,\Omega}.$$

□

In order to establish the continuity of the bilinear form $a_h(\cdot, \cdot)$, it is sufficient to notice that the following bounds hold true:

$$\begin{aligned} \nu \sum_{e \in \mathcal{C}_h | \mathcal{F}_h^3} \int_e \{\operatorname{curl} \boldsymbol{u}_h\} \cdot [\boldsymbol{v}_h \cdot \boldsymbol{t}_e] ds &\leq \left(\nu \sum_{e \in \mathcal{C}_h | \mathcal{F}_h^3} |e| \, \|\operatorname{curl} \boldsymbol{u}_h\|^2_{0,e} \right)^{\frac{1}{2}} J_1(\boldsymbol{v}_h, \boldsymbol{v}_h)^{\frac{1}{2}} \\ &\leq \left(\nu \sum_{T \in \mathcal{T}_h} \|\operatorname{curl} \boldsymbol{u}_h\|^2_{0,T} \right)^{\frac{1}{2}} J_1(\boldsymbol{v}_h, \boldsymbol{v}_h)^{\frac{1}{2}} \\ &\leq |||\boldsymbol{u}_h||| \, |||\boldsymbol{v}_h|||. \end{aligned}$$

Here above, we have applied to $\psi = \operatorname{curl} \boldsymbol{u}_h$ the next trace inequality on $e \subset \partial T$ combined with an inverse inequality:

$$\frac{1}{\sqrt{|e|}}\|\psi\|_{0,e} \leq c(\frac{1}{h_T}\|\psi\|_{0,T} + |\psi|_{1,T}) \leq \frac{c}{h_T}\|\psi\|_{0,T}.$$

Similar arguments yield

$$\nu \sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2} \int_e \{\operatorname{div} \boldsymbol{u}_h\} \cdot [\boldsymbol{v}_h \cdot \boldsymbol{n}_e]\, ds \leq |||\boldsymbol{u}_h|||\, |||\boldsymbol{v}_h|||,$$

and the continuity of $b_h(\cdot,\cdot)$:

$$\begin{aligned}\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2} \int_e \{q_h\}\, [\boldsymbol{v}_h \cdot \boldsymbol{n}_e]\, ds &\leq \frac{1}{\sqrt{\nu}} \left(\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2} |e|\, \|q_h\|_{0,e}^2 \right)^{\frac{1}{2}} J_2(\boldsymbol{v}_h, \boldsymbol{v}_h)^{\frac{1}{2}} \\ &\leq \frac{c}{\sqrt{\nu}} \|q_h\|_{0,T}\, J_2(\boldsymbol{v}_h, \boldsymbol{v}_h)^{\frac{1}{2}}.\end{aligned}$$

The continuity of the linear forms $l_h(\cdot)$ and $g_h(\cdot)$ follows with the same arguments.

We are now able to state the main results of this section.

Theorem 10.2.3. *For σ sufficiently large, the mixed problem (10.9) has a unique solution.*

Proof. According to Lemmas 10.2.1 and Lemma 10.2.2, the hypotheses of the Babuska-Brezzi theorem are satisfied (cf. [17]). Therefore, the variational formulation (10.9) is well-posed. □

10.3 A priori error estimates

This section is devoted to the derivation of optimal *a priori* error estimates for both the velocity and the pressure for the DG formulation (10.9). We will first consider the energy norm for the error on the velocity and then the L^2-norm for the pressure.

Let us first establish some auxiliary results.

Lemma 10.3.1. *If $(\boldsymbol{u}, p)$ is a sufficiently regular solution of the continuous Stokes problem, then we have the following consistency properties*

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(p, \boldsymbol{v}_h) = l_h(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h \tag{10.15}$$

$$b_h(q_h, \boldsymbol{u}) = g_h(q_h), \qquad \forall q_h \in Q_h. \tag{10.16}$$

Proof. The proof is rather classical. The second relation is obvious, since $\operatorname{div} \boldsymbol{u} = 0$ on any triangle, $[\boldsymbol{u} \cdot \boldsymbol{n}_e] = 0$ across any internel edges and $\boldsymbol{u} \cdot \boldsymbol{n}_e = \boldsymbol{u}_0 \cdot \boldsymbol{n}_e$ for $e \in \mathcal{F}_h^1$, $\boldsymbol{u} \cdot \boldsymbol{n}_e = \boldsymbol{b} \cdot \boldsymbol{n}_e$ for $e \in \mathcal{F}_h^3$. The first consistency property is obtained after integrating by parts on each element

and using the following regularity of the Stokes problem with data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ on a Lipschitz polygon : $(\boldsymbol{u}, p) \in \boldsymbol{H}^{\,2}(\Omega) \times H^1(\Omega)$. This ensures that curl $\boldsymbol{u}$, div $\boldsymbol{u}$ and p have a trace on each line segment e, which moreover belongs to $L^2(e)$. Since curl $\boldsymbol{u} \in \boldsymbol{H}(\text{curl}, \Omega)$ and $\boldsymbol{u} \in \boldsymbol{H}^{\,1}(\Omega)$, it follows that curl $\boldsymbol{u} \cdot \boldsymbol{t}$, $\boldsymbol{u} \cdot \boldsymbol{t}$ and $\boldsymbol{u} \cdot \boldsymbol{n}$ are continuous across any internal edge, so the desired relation holds true. □

Lemma 10.3.2. *Assume that $\boldsymbol{u}$ belong to $\boldsymbol{H}^{k+1}(\Omega)$. Then there exists a constant $c > 0$ independent of h and ν such that*

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \qquad |a_h(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}, \boldsymbol{v}_h)| \leq c\sqrt{\nu} h^k |||\boldsymbol{v}_h||| \, |\boldsymbol{u}|_{k+1,\Omega}. \tag{10.17}$$

Proof. We recall that $a_h(\cdot,\cdot) = A_0(\cdot,\cdot) + A_1(\cdot,\cdot) + A_2(\cdot,\cdot) + \sigma\big(J_1(\cdot,\cdot) + J_2(\cdot,\cdot)\big)$. The Cauchy-Schwarz inequality and the approximation result (10.6) immedialtely give that

$$\begin{aligned} A_0(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u},\ \boldsymbol{v}_h) &\leq \nu \left(\sum_{T \in \mathcal{T}_h} \|\text{curl}(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u})\|_{0,T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\text{curl}\, \boldsymbol{v}_h\|_{0,T}^2 \right)^{\frac{1}{2}} \\ &+ \nu \left(\sum_{T \in \mathcal{T}_h} \|\text{div}(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u})\|_{0,T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\text{div}\, \boldsymbol{v}_h\|_{0,T}^2 \right)^{\frac{1}{2}} \\ &\leq c\sqrt{\nu} h^k |\boldsymbol{u}|_{k+1,\Omega} |||\boldsymbol{v}_h||| \end{aligned}$$

Using Cauchy-Schwarz inequality in the first jump term, we obtain

$$J_1(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}, \boldsymbol{v}_h) \leq c\sqrt{\nu} \left(\sum_{e \in \mathcal{C}_h | \mathcal{F}_h^3} \frac{1}{|e|} \|[\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}]\|_{0,e}^2 \right)^{1/2} |||\boldsymbol{v}_h|||.$$

Let e be an internal edge common to the triangles T_1 and T_2. The proof is completely similar for a boundary edge. using the Trace inequality:

$$\begin{aligned} \frac{1}{\sqrt{|e|}} \|[(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}) \cdot \boldsymbol{n}_e]\|_{0,e} \leq\ & c \quad \Big(\frac{1}{h_{T_1}} \|\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}\|_{0,T_1} + \frac{1}{h_{T_2}} \|\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}\|_{0,T_2} + |\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}|_{1,T_1} \\ &+ \quad |\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}|_{1,T_1} \Big) \end{aligned} \tag{10.18}$$

it follows, thanks to (10.6), that

$$J_1(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}, \boldsymbol{v}_h) \leq c\sqrt{\nu} h^k \left(\sum_{e \in \mathcal{C}_h | \mathcal{F}_h^3} \frac{1}{|e|} \|[\boldsymbol{v}_h]\|_{0,e}^2 \right)^{1/2} |\boldsymbol{u}|_{k+1,\Omega} \leq c\sqrt{\nu} h^k |||\boldsymbol{v}_h||| \, |\boldsymbol{u}|_{k+1,\Omega}.$$

We can easily check that the same bound follows for the second jump term $J_2(\cdot,\cdot)$.
The property (10.3) of the interpolation operator $\boldsymbol{R}_h$ gives on the one hand

$$\sum_{e \in \mathcal{C}_h | \mathcal{F}_h^3} \int_e \{\text{curl}\, \boldsymbol{v}_h\} \cdot [(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}) \cdot \boldsymbol{t}_e] = 0$$

and on the other hand

$$\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2}\int_e\{\mathrm{div}\,\boldsymbol{v}_h\}\cdot[(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\cdot\boldsymbol{n}_e]=0$$

since $\{\mathrm{curl}\boldsymbol{v}_h\}$ and $\{\mathrm{div}\boldsymbol{v}_h\}$ belong to $\boldsymbol{P}_{k-1}$ on every edge.
So, we only have to bound the remainning parts of the terms A_1 and A_2 respectively

$$\nu\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^3}\int_e\{\mathrm{curl}(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{t}_e] \tag{10.19}$$

and

$$\nu\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2}\int_e\{\mathrm{div}(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]. \tag{10.20}$$

For this pupose, let us introduce the classical Lagrange interpolation operator of polynomial degree k, denoted by $\boldsymbol{L}_h$, and let us insert it in the two last terms. Then, we can write that

$$\begin{aligned}\int_e\{\mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]ds &= \int_e\{\mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]ds\\ &+ \int_e\{\mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]ds\\ &\le \|[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]\|_{0,e}\|\{\mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\|_{0,e}\\ &+ \|[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]\|_{0,e}\|\{\mathrm{curl}(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\}\|_{0,e}.\end{aligned}$$

Thanks to the same trace inequality as in (10.18), one obtain

$$\frac{1}{\sqrt{|e|}}\|\{\mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\le ch^{k-1}|\boldsymbol{u}|_{k+1,T_1\cup T_2}.$$

Then,

$$\begin{aligned}\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^3}\int_e\{\mathrm{curl}\,(\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]ds &\le ch^k\left(\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^3}\frac{1}{|e|}\|[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]\|_{0,e}^2\right)^{1/2}|\boldsymbol{u}|_{k+1,\Omega}\\ &\le \frac{c}{\sqrt{\nu}}h^k|||\boldsymbol{v}_h|||\,|\boldsymbol{u}|_{k+1,\Omega}.\end{aligned}$$

Next, using that $\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}$ is a piecewise polynomial, we obtain by means of a scaling argument that :

$$\frac{1}{\sqrt{|e|}}\|\{\mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\}\|_{0,e}\le c\left(\frac{1}{h_{T_1}}\|\mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\|_{0,T_1}+\frac{1}{h_{T_2}}\|\mathrm{curl}\,(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\|_{0,T_2}\right).$$

By means of the triangle inequality, we can deduce that on any triangle T,

$$\begin{aligned}\|\mathrm{curl}(\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})\|_{0,T} &\le |\boldsymbol{L}_h\boldsymbol{u}-\boldsymbol{u}|_{1,T}+|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}|_{1,T}\\ &\le ch_T^k|\boldsymbol{u}|_{k+1,\Delta_T}\end{aligned}$$

so finally,

$$\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^3}\int_e\{\mathrm{curl}\,(\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{t}_e]ds\le\frac{c}{\sqrt{\nu}}h^k\sqrt{J_1(\boldsymbol{v}_h,\boldsymbol{v}_h)}\,|\boldsymbol{u}|_{k+1,\Omega}.$$

The term in (10.20) has the same structure as the one in (10.19), thus it satisfies the bound

$$\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2}\int_e\{\mathrm{div}\,(\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{L}_h\boldsymbol{u})\}\cdot[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]ds\le\frac{c}{\sqrt{\nu}}h^k\sqrt{J_2(\boldsymbol{v}_h,\boldsymbol{v}_h)}\,|\boldsymbol{u}|_{k+1,\Omega}.$$

It is now sufficient to gather together the previous estimates in order to end the proof. □

Lemma 10.3.3. *Suppose $p\in H^k(\Omega)$. Then there exists a constant $c>0$ independent of h and ν such that*

$$\forall\boldsymbol{v}_h\in\boldsymbol{X}_h,\qquad |b_h(p-r_hp,\boldsymbol{v}_h)|\le\frac{c}{\sqrt{\nu}}h^k|||\boldsymbol{v}_h|||\,|p|_{k,\Omega}. \tag{10.21}$$

Proof. We recall that

$$\begin{aligned}b_h(p-r_hp,\boldsymbol{v}_h) &= -\sum_{T\in\mathcal{T}_h}\int_T(p-r_hp)\,\mathrm{div}\,\boldsymbol{v}_h\,dx+\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2}\int_e\{p-r_hp\}\,[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\,ds\\ &= \sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2}\int_e\{p-r_hp\}\,[\boldsymbol{v}_h\cdot\boldsymbol{n}_e]\,ds.\end{aligned}$$

Then,

$$|b_h(p-r_hp,\boldsymbol{v}_h)|\le\frac{c}{\sqrt{\nu}}\left(\sum_{e\in\mathcal{C}_h|\mathcal{F}_h^2}|e|\,\|\{p-r_hp\}\|_{0,e}^2\right)^{1/2}|||\boldsymbol{v}_h|||.$$

On each edge $e\in\mathcal{C}_h|\mathcal{F}_h^2$ such that $e\subset\partial T$, we bound the term $\|\{p-r_hp\}\|_{0,e}$ as in the proof of the Lemma 10.3.2. Denoting by l_h the Lagrange interpolation operator on P_{k-1} for $k=2$ or 3, we obtain:

$$\begin{aligned}\sqrt{|e|}\|p-r_hp\|_{0,\Omega} &\le \sqrt{|e|}(\|p-l_hp\|_{0,\Omega}+\|l_hp-r_hp\|_{0,\Omega})\\ &\le c\,(\|p-l_hp\|_{0,T}+h_T|p-l_hp|_{1,T}+\|l_hp-r_hp\|_{0,T})\\ &\le ch^k|p|_{k,T}.\end{aligned}$$

For $k=1$, we directly have:

$$\sqrt{|e|}\|p-r_hp\|_{0,\Omega}\le c\,(\|p-r_hp\|_{0,T}+h_T|p-r_hp|_{1,T})\le ch|p|_{1,T}.$$

So the announced result holds. □

Theorem 10.3.4. *Let $(\boldsymbol{u},p) \in \boldsymbol{H}^{k+1}(\Omega)\times H^k(\Omega)$ be the solution of the continuous Stokes problem and let σ_1 and σ_2 be sufficiently large (as in Lemma 10.2.1). Then, the solution $(\boldsymbol{u}_h,p_h)$ of the discrete problem (10.9) satisfies the following a priori error bounds*

$$|||\boldsymbol{u}-\boldsymbol{u}_h||| \leq ch^k\big(\sqrt{\nu}|\boldsymbol{u}|_{k+1,\Omega}+\frac{1}{\sqrt{\nu}}|p|_{k,\Omega}\big), \tag{10.22}$$

$$\|p-p_h\|_{0,\Omega} \leq ch^k(\nu|\boldsymbol{u}|_{k+1,\Omega}+|p|_{k,\Omega}), \tag{10.23}$$

with a constant c independent of h and ν.

Proof. According to Lemma 10.2.1, one has for γ large enough that

$$\begin{aligned}\alpha\,|||\,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}\,|||^2 &\leq a_h(\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})\\ &= a_h(\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})-a_h(\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})+a_h(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})\\ &= l_h(\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})-b_h(p_h,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})-a_h(\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})\\ &+ a_h(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}).\end{aligned}$$

Thanks to the consistency property stated in Lemma 10.3.1, one gets that

$$\alpha\,|||\,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}\,|||^2\leq b_h(p-p_h,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})+a_h(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}). \tag{10.24}$$

On the one hand, the continuity property proved in Lemma 10.3.2 implies that

$$a_h(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})\leq c\sqrt{\mu}h^k|\boldsymbol{u}|_{k+1,\Omega}|||\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}|||.$$

On the other hand, we write that

$$b_h(p-p_h,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})=b_h(p-r_hp,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})+b_h(r_hp-p_h,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})$$

and we notice that, thanks to the second equation of (10.9) and to the interpolation properties of $\boldsymbol{R}_h$, we have that

$$\begin{aligned}b_h(r_hp-p_h,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}) &= -b_h(r_hp-p_h,\boldsymbol{R}_h\boldsymbol{u})+g_h(r_hp-p_h)\\ &= -b_h(r_hp-p_h,\boldsymbol{u})+g_h(r_hp-p_h)\\ &= -g_h(r_hp-p_h)+g_h(r_hp-p_h)=0.\end{aligned}$$

Using now Lemma 10.3.3, it follows that

$$b_h(p-p_h,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u})\leq\frac{c}{\sqrt{\mu}}h^k|p|_{k,\Omega}|||\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}|||.$$

From (10.24), we obtain that

$$|||\,\boldsymbol{u}_h-\boldsymbol{R}_h\boldsymbol{u}\,|||\leq ch^k(\sqrt{\mu}|\boldsymbol{u}|_{k+1,\Omega}+\frac{1}{\sqrt{\mu}}|p|_{k,\Omega}). \tag{10.25}$$

Finally, by means of the triangle inequality and thanks to the relations

$$\begin{aligned}J_1(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}) &\leq c\nu h^{2k}|\boldsymbol{u}|^2_{k+1,\Omega}\\ J_2(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u},\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}) &\leq c\nu h^{2k}|\boldsymbol{u}|^2_{k+1,\Omega},\end{aligned}$$

we deduce (10.22).
In order to establish the error estimate for the pressure, we write that

$$\|p - p_h\|_{0,\Omega} \leq \|p - r_h p\|_{0,\Omega} + \|r_h p - p_h\|_{0,\Omega}.$$

According to the discrete inf-sup condition (see Lemma 10.2.2), one has that

$$\|r_h p - p_h\|_{0,\Omega} \leq \frac{\sqrt{\nu}}{\beta} \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{b_h(r_h p - p_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||} = \frac{\sqrt{\nu}}{\beta} \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{b_h(r_h p - p, \boldsymbol{v}_h) + b_h(p - p_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||}.$$

The continuity property of Lemma 10.3.3 gives :

$$b_h(r_h p - p, \boldsymbol{v}_h) \leq \frac{c}{\sqrt{\nu}} h^k |||\boldsymbol{v}_h||| |p|_{k,\Omega}$$

whereas the consistency property of Lemma 10.3.1 together with the first variational equation of (10.9) yield

$$b_h(p - p_h, \boldsymbol{v}_h) = -a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

One can then show, using Lemma 10.3.2 and the continuity of $a_h(\cdot,\cdot)$ on $\boldsymbol{X}_h$, that

$$\begin{aligned} a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) &= a_h(\boldsymbol{u} - \boldsymbol{R}_h \boldsymbol{u}, \boldsymbol{v}_h) + a_h(\boldsymbol{R}_h \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) \\ &\leq c\sqrt{\nu} h^k |\boldsymbol{u}|_{k+1,\Omega} |||\boldsymbol{v}_h||| + 2 \, ||| \, \boldsymbol{R}_h \boldsymbol{u} - \boldsymbol{u}_h \, ||| \, |||\boldsymbol{v}_h|||. \end{aligned}$$

In conclusion, we obtain that

$$\|r_h p - p_h\|_{0,\Omega} \leq c h^k (\nu |\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}) + c\sqrt{\nu} \, ||| \, \boldsymbol{R}_h \boldsymbol{u} - \boldsymbol{u}_h \, |||,$$

which together with (10.25) and (10.5) imply the desired estimate (10.23). □

10.4 Mesh convergence

We study the behavior of the numerical scheme (10.9) with respect to mesh refinement, for $k = 1, 2$. We present in Table (10.1) and (10.2) the values of the errors on different meshes. They are in agreement with the theoretical results.

ne	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	$\|p-p_h\|_{0,\Omega}$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	$J(e,e)$
16	1,36082e-00	6,28132e-01	8,82325e-00	9,08897e-00
64	2,65389e-01	3,18345e-01	4,45542e-00	2,98147e-00
256	6,69665e-02	1,60014e-01	2,23499e-00	1,03609e-00
1024	1,68163e-02	8,01187e-02	1,11850e-00	3,64884e-01
4096	4,21325e-03	4,00744e-02	5,59379e-01	1,28887e-01
16384	1,05444e-03	2,00391e-02	2,79705e-01	4,55587e-02
65536	2,63752e-04	1,00198e-02	1,39854e-01	1,61065e-02

Table 10.1: Velocity and pressure errors on a sequence of uniformly refined meshes for $k = 1$.

ne	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	$\|p-p_h\|_{0,\Omega}$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	$J(e,e)$
16	1,18628e-01	2,03708e-01	1,83275e-00	1,25154e-00
64	1,58137e-02	5,11701e-02	4,55609e-01	2,31933e-01
256	2,01513e-03	9,84020e-03	1,13720e-01	4,08763e-02
1024	2,53415e-04	2,12524e-03	2,84119e-02	7,16563e-03
4096	3,17409e-05	5,12599e-04	7,10052e-03	1,25951e-03
16384	3,97059e-06	1,27238e-04	1,77480e-03	2,21942e-04

Table 10.2: Velocity and pressure errors on a sequence of uniformly refined meshes for $k = 2$.

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Analyse Mathématiques et Approximation Numérique des Équations de Stokes et de Navier-Stokes avec des Conditions aux Limites Non standard

Résumé : Les travaux de la thèse portent sur la résolution des équations de Stokes, d'abord avec des conditions au bord portant sur la composante normale du champ de vitesse et la composante tangentielle du tourbillon, ensuite avec des conditions au bord portant sur la pression et la composante tangentielle du champ de vitesse. Dans chaque cas nous démontrons l'existence, l'unicité et la régularité de la solution. Nous traitons aussi le cas de solutions trés faibles, par dualité. Le cadre fonctionel que nous avons choisi est celui des espaces de Banach du type $H(\mathrm{div})$ et $H(\mathrm{rot})$ ou l'intersection des deux, basés sur l'espace L^p, avec $1 < p < \infty$. En particulier, on se place dans des domaines non simplement connexes, avec des frontières non connexes. Nous nous intéressons en premier lieu à l'obtention d'inégalités de Sobolev pour des champs de vecteurs $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Dans un second temps, nous établissons des résultats d'existence pour les potentiels vecteurs avec diverses conditions aux limites. Ceci nous permet d'abord d'effectuer des décompositions de type Helmholtz et ensuite de démontrer des conditions $Inf - Sup$ l'orsque la forme bilinéaire est un produit de rotationnels. Ces conditions aux limites font que l'équation de la pression est indépendante des autres variables. C'est la raison pour laquelle nous sommes naturellement conduit à étudier les problèmes elliptiques qui se traduisent par les systèmes de Stokes sans la pression. La résolution de ces problèmes se fait au moyen des Conditions $Inf - Sup$ qui jouent un rôle clef pour établir l'éxistence et l'unicité de solutions. Nous donons une applications aux systèmes de Navier-Stokes, où on obtient l'existence d'une solution en effectuant un point fixe autour du problème d'Oseen. Enfin, deux méthodes numériques sont proposées pour approcher le problème de Stokes. Nous analysons d'abord une méthode de Nitsche et puis une méthode de Galerkin discontinu. Quelques résultats numériques de convergence sont décrits qui sont parfaitement cohérents avec l'analyse.

Mots clés : Stokes - Navier-Stokes - Oseen - Conditions aux limites - Potentiels vecteurs - Inégalités de Sobolev - Conditions $Inf - Sup$ - Problèmes elliptiques - Décompositions d'Helmholtz - Méthode de Nitsche - Méthode de Galerkin Discontinu.

Mathematical Analysis and Numerical Approximation of the Stokes and Navier-Stokes Equations with Non Standard Boundary Conditions

Abstract : This work of thesis deals with the solving of the Stokes problem, first with boundary conditions on the normal component of the velocity field and the tangential component of the vorticity, next with boundary conditions on the pressure and the tangential component of the velocity field. In each case, we give existence, uniqueness and regularity of solutions. The case of very weak solutions is also treated by using a duality argument. The functional framework that we have choosed is that of Banach spaces of type $H(\mathrm{div})$ and $H(\mathrm{rot})$ or their intersection based on the space L^p, with $1 < p < \infty$. In particular, we suppose that Ω is multiply connected and that the boundary Γ is not connexe. We are interested in a first time by some Sobolev inequality for vector fields $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. In a second time, we give some results concerning vector potentials with different boundary conditions. This allow to establish Helmholtz decompositions and $Inf - Sup$ condition when the bilinear form is a rotational product. Due to these non standard boundary conditions, the pressure is decoupled from the system. It is the reason whay we are naturally reduced to solving elliptic problems which are the Stokes equations without the pressure term. For this, we use the $Inf - Sup$ conditions, which plays a crutial role in the existence and uniqueness of solutions. We give an application to the Navier-Stokes equations where the proof of solutions is obtained by applying a fixed point theorem over the Oseen equations. Finally, two numerical methods are proposed inorder to approximate the Stokes problem. First, by means of the Nitsche method and next by means of the Discontinuous Galerkin method. Some numerical results of convergence verifying the theoretical predictions are given.

Keywords : Stokes - Navier-Stokes - Oseen - Boundary conditions - Vector potentials - Sobolev inequality - Inf-Sup conditions - Elliptic problems - Helmholtz decomposition - Nitsche method - Discontinuous Galerkin method.

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