Finite subgroups of extended Morava stabilizer groups

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by

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Introduction

The Morava stabilizer groups

Let n be a positive integer and K a separably closed field of characteristic $p > 0$. If F is a formal group law of height n defined over K, then the Dieudonné-Lubin theorem D.3 says that the K-automorphism group $Aut_K(F)$ of F can be identified with the units in the maximal order $\mathcal{O}_n$ of the central division algebra $\mathbb{D}_n = D(\mathbb{Q}_p, 1/n)$ of invariant $1/n$ over $\mathbb{Q}_p$. In the case where $F = F_n$ is the Honda formal group law of height n, as given by theorem D.1, we have

$$Aut_K(F_n) \cong Aut_{\mathbb{F}_{p^n}}(F_n).$$

We define

$$\mathbb{S}_n := Aut_{\mathbb{F}_{p^n}}(F_n) \cong \mathcal{O}_n^\times$$

to be the *n-th (classical) Morava stabilizer group.*

More generally, we are interested in the category $\mathcal{FGL}_n$ whose objects are pairs (F, k) for k a perfect field of characteristic p and F a formal group law of height n defined over k, and whose morphisms are given by pairs

$$(f, \varphi) : (F_1, k_1) \longrightarrow (F_2, k_2),$$

where $\varphi : k_1 \to k_2$ is a field homomorphism and $f : \varphi_* F_1 \to F_2$ is an isomorphism from the formal group law given by applying φ on the coefficients of F_1. If (f, φ) is an endomorphism of (F, k), then φ is an automorphism of k and $\varphi \in Gal(k/\mathbb{F}_p)$. We let

$$Aut_{\mathcal{FGL}_n}(F, k) = \{(f, \varphi) : (F, k) \to (F, k) \mid \varphi \in Gal(k/\mathbb{F}_p) \text{ and } f : \varphi_* F \cong F\}$$

denote the group of automorphisms of (F, k) in $\mathcal{FGL}_n$. If F is already defined over $\mathbb{F}_p$, the Frobenius automorphism $X^p \in End_K(F)$ defines an element $\xi_F \in \mathcal{O}_n$. Then proposition D.7 says that $End_K(F) = End_{\mathbb{F}_{p^n}}(F)$ if and only if the minimal polynomial of ξ_F over $\mathbb{Z}_p$ is $\xi_F^n - up$ with $u \in \mathbb{Z}_p^\times$. For such an F, we define

$$\mathbb{G}_n(u) := Aut_{\mathcal{FGL}_n}(F, \mathbb{F}_{p^n})$$

to be the *n-th extended Morava stabilizer group* associated to u. We often note $\mathbb{G}_n = \mathbb{G}_n(1)$.

Here $\varphi_* F = F$ for any $\varphi \in Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$. The group $\mathbb{G}_n(u)$ contains $\mathbb{S}_n$ as the subgroup of elements of the form $(f, id_{\mathbb{F}_{p^n}})$, and there is an extension

$$1 \longrightarrow \mathbb{S}_n \longrightarrow \mathbb{G}_n(u) \longrightarrow Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \longrightarrow 1$$

where an element $f \in \mathbb{S}_n$ is mapped to the pair $(f, id_{\mathbb{F}_{p^n}})$ and where the image of a pair $(f, \varphi) \in \mathbb{G}_n(u)$ in the Galois group is the automorphism φ of $\mathbb{F}_{p^n}$. Moreover, the Frobenius automorphism $\sigma \in Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n$ splits as the pair (id_F, σ) in $\mathbb{G}_n(u)$, and we get

$$\mathbb{G}_n(u) \cong \mathbb{S}_n \rtimes_F Gal(\mathbb{F}_{p^n}/\mathbb{F}_p),$$

where the action on $\mathbb{S}_n$ is induced by conjugation by ξ_F. In terms of division algebras (see appendix D), this extension translates into a split exact sequence

$$1 \longrightarrow \mathcal{O}_n^\times \longrightarrow \mathbb{D}_n^\times / \langle \xi_F^n \rangle \longrightarrow \mathbb{Z}/n \longrightarrow 1,$$

so that

$$\mathbb{G}_n(u) \cong \mathbb{D}_n^\times / \langle pu \rangle.$$

In the text we address the problem of classifying the finite subgroups of $\mathbb{G}_n(u)$ up to conjugation. In particular, we give necessary and sufficient conditions on n, p and u for the existence in $\mathbb{G}_n(u)$ of extensions of the form

$$1 \longrightarrow G \longrightarrow F \longrightarrow \mathbb{Z}/n \longrightarrow 1$$

with G maximal finite in $\mathbb{S}_n$, and if such extensions exist, we establish their classification as finite subgroups of $\mathbb{G}_n(u)$ up to conjugation.

Motivation

Given a prime p and for $K(n)$ the n-th Morava K-theory at p, the stable homotopy category of p-local spectra can be analysed from the category of $K(n)$-local spectra in the sense of [9] section 1.1. In particular, letting $L_n = L_{K(0)\vee\ldots\vee K(n)}$ be the localization functor with respect to $K(0) \vee \ldots \vee K(n)$, there is a tower of localization functors

$$\ldots \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \ldots \longrightarrow L_0$$

together with natural maps $X \to L_n X$, such that for every p-local finite spectrum X the natural map $X \to holim L_n X$ is a weak equivalence. Furthermore, the maps $L_n X \to L_{n-1} X$ fit into a natural commutative homotopy pullback square

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X. \end{array}$$

In this way, the Morava K-theory localizations $L_{K(n)}X$ form the basic building blocks for the homotopy type of a p-local finite spectrum X, and of course, the localization of the sphere $L_{K(n)}S^0$ plays a central role in this approach.

The spectrum $L_{K(n)}S^0$ can be identified with the homotopy fixed point spectrum $E_n^{h\mathbb{G}_n}$ of the n-th Lubin-Tate spectrum E_n, and the Adams-Novikov spectral sequence for $L_{K(n)}S^0$ can be identified with the spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_n, (E_n)_t) \implies \pi_{t-s} L_{K(n)} S^0.$$

Here the ring $(E_n)_0$ is isomorphic to the universal deformation ring $E(F, \mathbb{F}_{p^n})$ (in the sense of Lubin and Tate) associated to a formal group law F of height n over $\mathbb{F}_{p^n}$, and $(E_n)_*$ is a graded version of $E(F, \mathbb{F}_{p^n})$. The functor

$$E(\_,\_) : \mathcal{FGL}_n \longrightarrow Rings_{cl}$$

to the category of complete local rings defines the action of $\mathbb{G}_n(u)$ on the universal ring $E(F, \mathbb{F}_{p^n})$, which in turn induces an action on $(E_n)_*$.

There is good hope that $L_{K(n)}S^0$ can be written as the inverse limit of a tower of fibrations whose successive fibers are of the form E_n^{hF} for F a finite subgroup of $\mathbb{G}_n(u)$. This is at least true in the case $n = 2$, $p = 3$ and $u = 1$, which is the object of [6]. In [9] the case $n = p - 1$, $p > 2$ and $u = 1$ is investigated. Moreover, the importance of the

subgroups of $\mathbb{G}_2(-1)$ for $p = 3$ is exemplified in [2]. As shown in the present text, the choice of u plays an important role in the determination of the finite subgroups of $\mathbb{G}_n(u)$.

For example, when $n = 2$ and $p = 3$ theorem 4.29 shows that the maximal finite subgroups of $\mathbb{G}_n(u)$ are represented up to conjugation by SD_{16}, the semidihedral group of order 16, and by a semi-direct product of the cyclic group of order 3 with either the quaternion group Q_8 if $u \equiv 1 \bmod 3$ or the dihedral group D_8 of order 8 if $u \equiv -1 \bmod 3$.

Another example is given by theorem 4.30 in the case $n = 2$ and $p = 2$: the maximal finite conjugacy classes are given by two or four classes depending on u. When $u \equiv 1 \bmod 8$, there are two of them given by a metacyclic group of order 12 and by

$$\begin{cases} O_{48} & \text{if } u \equiv 1 \bmod 8, \\ T_{24} \rtimes C_2 & \text{if } u \equiv -1 \bmod 8, \end{cases}$$

for O_{48} the binary octahedral group of order 48, C_2 the cyclic group of order 2 and T_{24} the binary tetrahedral group of order 24. On the other hand when $u \not\equiv 1 \bmod 8$, there are four of them given by T_{24}, by two distinct metacyclic groups of order 12, and by

$$\begin{cases} D_8 & \text{if } u \equiv 3 \bmod 8, \\ Q_8 & \text{if } u \equiv -3 \bmod 8. \end{cases}$$

The group $\mathbb{G}_2(-1)$ is the Morava stabilizer group associated to the formal group law of a supersingular elliptic curve, while in general $\mathbb{G}_n = \mathbb{G}_n(1)$ is the one associated to the Honda formal group law of height n.

Overview

In the first chapter of the text, we establish a classification up to conjugation of the maximal finite subgroups of $\mathbb{S}_n$ for a prime p and a positive integer n. When n is not a multiple of $p-1$ the situation remains simple as no non-trivial finite p-subgroup exist. In this case, all finite subgroups are subgroups in the unique conjugacy class isomorphic to

$$\begin{cases} C_{p^n-1} & \text{if } p > 2, \\ C_{2(p^n-1)} & \text{if } p = 2, \end{cases}$$

where C_l denotes the cyclic group of order l. Otherwise, $n = (p-1)p^{k-1}m$ with m prime to p. For $\alpha \leq k$ and Euler's totient function φ, we let $n_\alpha = \frac{n}{\varphi(p^\alpha)}$ and we obtain:

Theorem. *If $p > 2$ and $n = (p-1)p^{k-1}m$ with m prime to p, the group $\mathbb{S}_n$ has exactly $k+1$ conjugacy classes of maximal finite subgroups represented by*

$$G_0 = C_{p^n-1} \qquad \text{and} \qquad G_\alpha = C_{p^\alpha} \rtimes C_{(p^{n_\alpha}-1)(p-1)} \qquad \text{for} \quad 1 \leq \alpha \leq k.$$

Theorem. *Let $p = 2$ and $n = 2^{k-1}m$ with m odd. The group $\mathbb{S}_n$, respectively $\mathbb{D}_n^\times$, has exactly k maximal conjugacy classes of finite subgroups. If $k \neq 2$, they are represented by*

$$G_\alpha = C_{2^\alpha(2^{n_\alpha}-1)} \qquad \text{for} \quad 1 \leq \alpha \leq k.$$

If $k = 2$, they are represented by G_α for $\alpha \neq 2$ and by the unique maximal nonabelian conjugacy class

$$Q_8 \rtimes C_{3(2^m-1)} \cong T_{24} \times C_{2^m-1},$$

the latter containing G_2 as a subclass.

The classification of the isomorphism classes of the finite subgroups of $\mathbb{S}_n$ has already been found by Hewett in [10]; it is based on a previous classification made by Amitsur in [1]. Our approach is different: it has the advantage of being more direct, exploiting the structure of $\mathbb{D}_n^\times$ in terms of Witt vectors, and lays the foundations for our study of the extended groups $\mathbb{G}_n(u)$. A further attempt by Hewett to extend his classification from isomorphism classes to conjugacy classes can be found in [11], but the results turn out to be false (see remarks 1.34 and 1.36). In example 1.33, we provide an explicit family of counter examples in the case $p > 2$.

In chapter 2, we present a theoretical framework for the classification of the finite subgroups of $\mathbb{G}_n(u) \cong \mathbb{D}_n^\times/\langle pu\rangle$. Most of the work is done in $\mathbb{D}_n^\times$ via a bijection (see proposition 2.1) between the set of (conjugacy classes of) finite subgroups of $\mathbb{G}_n(u)$ and the set of (conjugacy classes of) subgroups of $\mathbb{D}_n^\times$ containing $\langle pu\rangle$ as a subgroup of finite index. For a finite subgroup F of $\mathbb{G}_n(u)$ for which $F \cap \mathbb{S}_n$ has an abelian p-Sylow subgroup (the remaining case of a quaternionic p-Sylow is quite specific and is treated in chapter 4), we consider its correspondent $\widetilde{F}$ in $\mathbb{D}_n^\times$ via the above bijection. This group fits into a chain of successive extensions

$$\widetilde{F_0} \subseteq \widetilde{F_1} \subseteq \widetilde{F_2} \subseteq \widetilde{F_3} = \widetilde{F},$$

where $F_0 = \langle F \cap S_n, Z_{p'}(F \cap \mathbb{S}_n)\rangle$ is cyclic for S_n the p-Sylow subgroup of $\mathbb{S}_n$ and $Z_{p'}(F \cap \mathbb{S}_n)$ the p'-part of the center of $F \cap \mathbb{S}_n$, and where

$$\widetilde{F_0} = F_0 \times \langle pu\rangle, \qquad \widetilde{F_2} = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(F_0) = C_{\widetilde{F}}(F_0),$$
$$\widetilde{F_1} = \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times, \qquad \widetilde{F_3} = \widetilde{F} \cap N_{\mathbb{D}_n^\times}(F_0) = N_{\widetilde{F}}(F_0).$$

Referring to the above classification of the finite subgroups of $\mathbb{S}_n$, we note that F_0 is a subgroup of a cyclic group of order $p^\alpha(p^{n_\alpha} - 1)$ for an $\alpha \leq k$, and that the whole (nonabelian) groups of type G_α when $p > 2$ can only be recovered in the last stage of the chain of extensions. We then provide cohomological criteria (see theorem 2.16, 2.21, 2.27 and 2.28) for the existence and uniqueness up to conjugation of each of these successive group extensions. We are mostly interested in the cases where each successive $\widetilde{F_i}$ is maximal, that is, F_0 is a maximal abelian finite subgroup of $\mathbb{S}_n$, and for $1 \leq i \leq 3$, each $\widetilde{F_i}$ is a maximal subgroup of the respective group $\mathbb{Q}_p(F_0)^\times$, $C_{\mathbb{D}_n^\times}(F_0)$, $N_{\mathbb{D}_n^\times}(F_0)$ containing $\widetilde{F_0}$ as a subgroup of finite index.

In chapter 3, we treat the abelian cases which are covered up to the second extension type $\widetilde{F_2}$. Given F_0, we let $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ denote the set of all $\widetilde{F_1}$'s which give rise to a finite subgroup F_1 of $\mathbb{G}_n(u)$ extending F_0 by a cyclic group of order r_1. Then:

Theorem. *If F_0 is a maximal abelian finite subgroup of $\mathbb{S}_n$, then $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if*

$$r_1 \quad \textit{divides} \quad \begin{cases} 1 & \textit{if } p > 2 \textit{ with } \zeta_p \notin F_0, \\ p-1 & \textit{if } p > 2 \textit{ with } \zeta_p \in F_0, \\ 1 & \textit{if } p = 2 \textit{ with } \zeta_3 \notin F_0 \textit{ and } u \not\equiv \pm 1 \bmod 8, \textit{ or with } \zeta_4 \notin F_0, \\ 2 & \textit{if } p = 2 \textit{ with } \zeta_4 \in F_0 \textit{ and either } u \equiv \pm 1 \bmod 8 \textit{ or } \zeta_3 \in F_0. \end{cases}$$

Furthermore, given $F_0 \subseteq F_1$, we let $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$ denote the set of all $\widetilde{F_2}$'s which give rise to a finite subgroup F_2 of $\mathbb{G}_n(u)$ extending F_1 by a group of order r_2. Then:

Theorem. *If r_1 is maximal such that $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1) \neq \emptyset$ and if $\widetilde{F_1}$ belongs to this set, then $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$ is non-empty if and only if r_2 divides $\frac{n}{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}$.*

In the particular case where F_0 is a maximal abelian finite subgroup of $\mathbb{S}_n$, we have $\widetilde{F_1} = \widetilde{F_2}$.

In chapter 4, we treat (nonabelian) finite extensions of $\widetilde{F_2}$ in the case where $\mathbb{Q}_p(\widetilde{F_2})$ is a maximal subfield in $\mathbb{D}_n$; any such field is of degree n over $\mathbb{Q}_p$. We provide necessary and sufficient conditions on n, p and u for the existence of $\widetilde{F}$'s such that $|\widetilde{F}/\widetilde{F_0}| = n$ and $\widetilde{F_1} = \widetilde{F_2}$:

Theorem. *Let $p > 2$, $n = (p-1)p^{k-1}m$ with m prime to p, $u \in \mathbb{Z}_p^\times$, $F_0 = C_{p^\alpha} \times C_{p^{n_\alpha}-1}$ be a maximal abelian finite subgroup in $\mathbb{S}_n$, $G = Gal(\mathbb{Q}_p(F_0)/\mathbb{Q}_p)$, $G_{p'}$ be the p'-part of G, and let $\widetilde{F_1} = \langle x_1 \rangle \times F_0 \subseteq \mathbb{Q}_p(F_0)^\times$ be maximal as a subgroup of $\mathbb{Q}_p(F_0)^\times$ having $\widetilde{F_0}$ as subgroup of finite index.*

1) *For any $0 \leq \alpha \leq k$, there is an extension of $\widetilde{F_1}$ by $G_{p'}$; this extension is unique up to conjugation.*

2) *If $\alpha \leq 1$, there is an extension of $\widetilde{F_1}$ by G; this maximal extension is unique up to conjugation.*

3) *If $\alpha \geq 2$, there is an extension of $\widetilde{F_1}$ by G if and only if*

$$\alpha = k \qquad \text{and} \qquad u \notin \mu(\mathbb{Z}_p^\times) \times \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \bmod (p^2)\},$$

in which case this maximal extension is unique up to conjugation.

Theorem. *Let $p = 2$, $n = 2^{k-1}m$ with m odd, $u \in \mathbb{Z}_2^\times$, $F_0 = C_{2^\alpha} \times C_{2^{n_\alpha}-1}$ be a maximal abelian finite subgroup of $\mathbb{S}_n$, $G = Gal(\mathbb{Q}_2(F_0)/\mathbb{Q}_2)$, $G_{2'}$ be the odd part of G, and let $\widetilde{F_1} = \langle x_1 \rangle \times F_0 \subseteq \mathbb{Q}_2(F_0)^\times$ be maximal as a subgroup of $\mathbb{Q}_2(F_0)^\times$ having $\widetilde{F_0}$ as subgroup of finite index.*

1) *For any $1 \leq \alpha \leq k$, there is an extension of $\widetilde{F_1}$ by $G_{2'}$; this extension is unique up to conjugation.*

2) *If $\alpha = 1$, there is an extension of $\widetilde{F_1}$ by G; the number of such extensions up to conjugation is*

$$\begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

3) *If $\alpha = 2$, there is an extension of $\widetilde{F_1}$ by G if and only if $k = 2$; the number of such extensions up to conjugation is*

$$\begin{cases} 1 & \text{if } u \equiv \pm 1 \bmod 8, \\ 2 & \text{if } u \not\equiv \pm 1 \bmod 8. \end{cases}$$

4) *If $\alpha \geq 3$, there is no extension of $\widetilde{F_1}$ by G.*

We then treat the specific remaining case where $F \cap \mathbb{S}_n$ has a quaternionic p-Sylow subgroup; this only occurs when $p = 2$ and $n \equiv 2 \bmod 4$.

Theorem. *Let $p = 2$, $n = 2m$ with m odd, and $u \in \mathbb{Z}_2^\times$. A subgroup G isomorphic to $T_{24} \times C_{2^m-1}$ in $\mathbb{S}_n$ extends to a maximal finite subgroup F of order $n|G| = 48m(2^m - 1)$ in $\mathbb{G}_n(u)$ if and only if $u \equiv \pm 1 \bmod 8$; this extension is unique up to conjugation.*

We end the chapter by explicitly analysing the case $n = 2$, where we obtain:

Theorem. *Let $n = 2$, $p = 3$ and $u \in \mathbb{Z}_p^\times$. The conjugacy classes of maximal finite subgroups of $\mathbb{G}_2(u)$ are represented by*

$$SD_{16} \qquad \text{and} \qquad \begin{cases} C_3 \rtimes Q_8 & \text{if } u \equiv 1 \bmod 3, \\ C_3 \rtimes D_8 & \text{if } u \equiv -1 \bmod 3. \end{cases}$$

Theorem. *Let $n = 2$, $p = 2$ and $u \in \mathbb{Z}_2^\times$. The conjugacy classes of maximal finite subgroups of $\mathbb{G}_2(u)$ are represented by*

$$\begin{cases} C_6 \rtimes C_2,\ O_{48} & \text{if } u \equiv 1 \bmod 8, \\ C_3 \rtimes C_4,\ T_{24} \rtimes C_2 & \text{if } u \equiv -1 \bmod 8, \\ C_3 \rtimes C_4,\ C_6 \rtimes C_2,\ D_8 \text{ and } T_{24} & \text{if } u \equiv 3 \bmod 8, \\ C_3 \rtimes C_4,\ C_6 \rtimes C_2,\ Q_8 \text{ and } T_{24} & \text{if } u \equiv -3 \bmod 8. \end{cases}$$

Chapter 1:

Finite subgroups of $\mathbb{S}_n$

From now on, we will always consider p a prime, n a strictly positive integer, and

$$\mathbb{D}_n := D(\mathbb{Q}_p, 1/n)$$

the central division algebra of invariant $1/n$ over $\mathbb{Q}_p$. The reader may refer to appendix A and C for the essential background on division algebras. We identify $\mathbb{S}_n$ as the group of units $\mathcal{O}_n^\times$ of the maximal order $\mathcal{O}_n$ of $\mathbb{D}_n$.

1.1. The structure of $\mathbb{D}_n$ and its finite subgroups

The structure of $\mathbb{D}_n$ can be explicitly given by the following construction; see appendix C or appendix 2 of [17] for more details. Let $\mathbb{W}_n = \mathbb{W}(\mathbb{F}_{p^n})$ be the ring of Witt vectors on the finite field $\mathbb{F}_{p^n}$ with p^n elements. Here $\mathbb{W}_n$ can be identified with the ring $\mathbb{Z}_p[\zeta_{p^n-1}]$ of integers of the unramified extension of degree n over $\mathbb{Q}_p$. It is a complete local ring with maximal ideal (p) and residue field $\mathbb{F}_{p^n}$ whose elements are written uniquely as

$$w = \sum_{i\geq 0} w_i p^i \qquad \text{with} \quad w_i^{p^n} = w_i.$$

The Frobenius automorphism $x \mapsto x^p \in Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ can be extended to an automorphism $\sigma : w \mapsto w^\sigma$ of $\mathbb{W}_n$ generating $Gal(\mathbb{W}_n/\mathbb{Z}_p)$ by setting

$$w^\sigma = \sum_{i\geq 0} w_i^p p^i \qquad \text{for each} \quad w = \sum_{i\geq 0} w_i p^i \ \in \ \mathbb{W}_n.$$

We then add to $\mathbb{W}_n$ a non-commutative element S satisfying $S^n = p$ and $Sw = w^\sigma S$ for all $w \in \mathbb{W}_n$; the non-commutative ring we obtain in this way can be identified with

$$\mathcal{O}_n \cong \mathbb{W}_n\langle S\rangle/(S^n = p, Sw = w^\sigma S),$$

and

$$\mathbb{D}_n \cong \mathcal{O}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The valuation map $v_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \to \mathbb{Z}$ satisfying $v(p) = 1$ extends uniquely to a valuation $v = v_{\mathbb{D}_n}$ on $\mathbb{D}_n$, with value group

$$v(\mathbb{D}_n^\times) = \frac{1}{n}\mathbb{Z},$$

in such a way that

$$v(S) = \frac{1}{n} \qquad \text{and} \qquad \mathcal{O}_n = \{x \in \mathbb{D}_n \mid v(x) \geq 0\}.$$

Because $v(x^{-1}) = -v(x)$, we have

$$\mathcal{O}_n^\times = \{x \in \mathbb{D}_n \mid v(x) = 0\}.$$

Proposition 1.1. *A finite subgroup of $\mathbb{D}_n^\times$ is a subgroup of $\mathcal{O}_n^\times$.*

Proof. An element $\zeta \in \mathbb{D}_n^\times$ of finite order $i \geq 1$ satisfies

$$0 = v(1) = iv(\zeta),$$

and it follows that $v(\zeta) = 0$. □

As we will now see, the structure of $\mathbb{D}_n$ given above greatly reduces the possibilities of what form a finite subgroup of $\mathcal{O}_n^\times$ can have.

The element $S \in \mathbb{D}_n^\times$ generates a two-sided maximal ideal $\mathfrak{m}$ of $\mathcal{O}_n$ with residue field $\mathcal{O}_n/\mathfrak{m} \cong \mathbb{F}_{p^n}$. This maximal ideal satisfies

$$\mathfrak{m} = \{x \in \mathbb{D}_n \mid v(x) > 0\}.$$

The kernel of the group epimorphism $\mathcal{O}_n^\times \to \mathbb{F}_{p^n}^\times$ which results from this quotient is denoted S_n . We thus have a group extension

$$1 \longrightarrow S_n \longrightarrow \mathcal{O}_n^\times \longrightarrow \mathbb{F}_{p^n}^\times \longrightarrow 1.$$

The groups $\mathcal{O}_n^\times$ and S_n have natural profinite structures induced by the filtration of subgroups

$$S_n = U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$$

given by

$$\begin{aligned} U_i := U_i(\mathbb{W}_n^\times) &= \{x \in S_n \mid v(x-1) \geq \frac{i}{n}\} \\ &= \{x \in S_n \mid x \equiv 1 \bmod S^i\}, \qquad \text{for } i \geq 1. \end{aligned}$$

The intersection of these groups is trivial and $S_n = \lim_i S_n/U_i$. We also have canonical isomorphisms

$$U_i/U_{i+1} \cong \mathbb{F}_{p^n} \qquad \text{given by} \qquad 1 + aS^i \mapsto \overline{a}$$

for $a \in \mathcal{O}_n$ and $\overline{a}$ the residue class of a in $\mathcal{O}_n/\mathfrak{m} = \mathbb{F}_{p^n}$. In particular, all quotients S_n/U_i are finite p-groups and S_n is a profinite p-subgroup of the profinite group $\mathcal{O}_n^\times$. By uniqueness of the maximal ideal $\mathfrak{m}$, we know that S_n is the unique p-Sylow subgroup of $\mathcal{O}_n^\times$. Consequently:

Proposition 1.2. *All p-subgroups of $\mathcal{O}_n^\times$, and only those, are subgroups of S_n.* □

Throughout the text we let φ denote Euler's totient function, which for each positive integer i associates the number $\varphi(i)$ of integers $1 \leq j \leq i$ for which $(i; j) = 1$.

Proposition 1.3. *The group S_n, respectively $\mathcal{O}_n^\times$, has elements of order p^k for $k \geq 1$ if and only if $\varphi(p^k) = (p-1)p^{k-1}$ divides n.*

Proof. This is a straightforward consequence of the embedding theorem C.6, together with proposition C.8 which states that

$$[\mathbb{Q}_p(\zeta_{p^k}) : \mathbb{Q}_p] = \varphi(p^k) = (p-1)p^{k-1},$$

for ζ_{p^k} a primitive p^k-th root of unity. □

Proposition 1.4. *Every abelian finite subgroup of $\mathbb{D}_n^\times$ is cyclic.*

Proof. If G is a finite multiplicative abelian subgroup of a division algebra of type $\mathbb{D}_n$, then it lies within the local commutative field $F = \mathbb{Q}_p(G)$ in $\mathbb{D}_n$ and is a subgroup of $F^\times$. Because G is finite, proposition C.7 implies that G is a subgroup of the cyclic group $\mu(F)$. □

In the text, we are lead to use group cohomology $H^*(G, M)$ extensively for some group G and G-module M. Most often, we will exploit the tools of low dimensional cohomology to study group extensions. A good introduction to the subject is provided in [4] chapter IV. In particular, we will invoke the following classic results; see [4] section IV.4 for proposition 1.5 (with exercise 4), and see [4] chapter IV corollary 3.13 and the following remark for proposition 1.6.

Proposition 1.5. *If G is a finite p-group whose abelian finite subgroups are cyclic, then G is either cyclic or a generalized quaternion group*

$$G \cong Q_{2^k} = \langle x, y \mid x^{2^{k-1}} = 1,\ yxy^{-1} = x^{-1},\ x^{2^{k-2}} = y^2 \rangle,$$

this last possibility being valid only when $p = 2$.

Proposition 1.6 (Schur-Zassenhaus). *If G is a finite group of order mn with m prime to n containing a normal subgroup N of order m, then G has a subgroup of order n and any two such subgroups are conjugate by an element in G.*

It follows that every finite subgroup G of $\mathbb{D}_n^\times$ is contained in $\mathbb{S}_n$ and determines a split extension

$$1 \longrightarrow P \longrightarrow G \longrightarrow C \longrightarrow 1,$$

where $P := G \cap S_n$ is a finite normal p-subgroup which is the p-Sylow subgroup of G, and $C := G/P$ is a cyclic group of order prime to p which embeds into $\mathbb{F}_{p^n}^\times$ via the reduction homomorphism. Moreover, P is either cyclic or a generalized quaternion group if $p = 2$. If P is cyclic of order p^α with $\alpha \geq 1$, we know that n is a multiple of $\varphi(p^\alpha) = (p-1)p^{\alpha-1}$.

Proposition 1.7. *If n is odd or is not divisible by $(p-1)$, then*

$$\begin{cases} C_{p^n-1} \cong \mathbb{F}_{p^n}^\times & \text{if } p > 2, \\ C_{2(p^n-1)} \cong \mathbb{F}_{p^n}^\times \times \{\pm 1\} & \text{if } p = 2, \end{cases}$$

represents the only isomorphic class of maximal finite subgroups of $\mathcal{O}_n^\times$.

Proof. Under the given assumptions, proposition 1.3 implies that the p-Sylow subgroup P of a maximal finite subgroup G of $\mathbb{D}_n^\times$ is trivial if p is odd, and is $\{\pm 1\}$ if $p = 2$. The result then follows from the Skolem-Noether theorem A.9. □

By proposition 1.7, only those cases where n is even and divisible by $p-1$ remain to be studied. From now on, we will adopt the following notations.

Notation 1.8. Fix a prime p and n a multiple of $p-1$. Then we define integers k and m to satisfy

$$n = (p-1)p^{k-1}m \qquad \text{with} \quad (m; p) = 1,$$

and for $0 \leq \alpha \leq k$ we set

$$n_\alpha := \frac{n}{\varphi(p^\alpha)} = \begin{cases} n & \text{if } \alpha = 0, \\ p^{k-\alpha}m & \text{if } \alpha > 0. \end{cases}$$

Notation 1.9. For a finite subgroup $G \subseteq \mathbb{D}_n^\times$ and a commutative ring R extending $\mathbb{Z}_p$ in $\mathbb{D}_n$, respectively a commutative field extending $\mathbb{Q}_p$ in $\mathbb{D}_n$, we denote by

$$R(G) = \{\sum_{g\in G} x_g g \mid x_g \in R\}$$

the R-subalgebra of $\mathbb{D}_n$ generated by G.

For example if $R = \mathbb{Q}_p$, G is a finite cyclic group and ζ is a generator of G, then $\mathbb{Q}_p(G) = \mathbb{Q}_p(\zeta)$ is the cyclotomic field generated by ζ over $\mathbb{Q}_p$.

We note that $R(G)$ is not in general isomorphic to the group ring $R[G]$, although there is a unique surjective homomorphism of R-algebras from $R[G]$ to $R(G)$ extending the embedding of G (seen as abstract group) into $\mathbb{D}_n^\times$.

1.2. Finite subgroups of $\mathbb{D}_n^\times$ with cyclic p-Sylow

Let $n = (p-1)p^{k-1}m$ with m prime to p as in notation 1.8. If G is a finite subgroup of $\mathbb{D}_n^\times$, it is then a subgroup of $\mathcal{O}_n^\times$ which determines an extension

$$1 \longrightarrow P \longrightarrow G \longrightarrow C \longrightarrow 1,$$

and whose p-Sylow subgroup $P = G \cap S_n$ is either cyclic of order p^α for $0 \leq \alpha \leq k$, or a generalized quaternion group. The latter case only occurs when $p = 2$; it is studied in section 1.3.

For now, we fix an integer $1 \leq \alpha \leq k$ and assume that P is cyclic of order p^α. We know from proposition C.8 that $\mathbb{Q}_p(P)$ is a totally ramified extension of degree $\varphi(p^\alpha)$ over $\mathbb{Q}_p$. As P is abelian and normal in G, there are inclusions of subgroups

$$P \subseteq C_G(P) \subseteq N_G(P) = G,$$

and the group $C = G/P$ injects into $\mathbb{F}_{p^n}^\times$. The following result establishes a stronger condition on C.

Proposition 1.10. *The group $C_G(P)/P$ injects into $\mathbb{F}_{p^{n_\alpha}}^\times$ via the reduction homomorphism, and $N_G(P)/C_G(P)$ identifies canonically with a subgroup of the p'-part of $Aut(P)$.*

Proof. First note that P generates a cyclotomic extension $K = \mathbb{Q}_p(P)$, and $C_G(P)$ is contained in $C_{\mathbb{D}_n^\times}(K)$. By the centralizer theorem A.6, $C_{\mathbb{D}_n}(K)$ is itself a central division algebra over K. Since

$$n = \varphi(p^\alpha)n_\alpha = [\mathbb{Q}_p(P) : \mathbb{Q}_p]n_\alpha,$$

it is of dimension n_α^2 over its center K and has residue field $\mathbb{F}_{p^{n_\alpha}}$. The reduction homomorphism in this division algebra induces a map $C_G(P) \to \mathbb{F}_{p^{n_\alpha}}^\times$ whose kernel is P; this shows the first assertion.

The second assertion follows from the facts that

$$P \subseteq C_G(P) \qquad \text{and} \qquad G = N_G(P),$$

and hence that $N_G(P)/C_G(P) \subseteq C$ must be prime to p. □

Corollary 1.11. *The group C is contained in the cyclic subgroup of order $(p^{n_\alpha}-1)(p-1)$ in $\mathbb{F}_{p^n}^\times$.*

Proof. This follows from proposition 1.10 and the fact that the p'-part of

$$Aut(P) \cong \begin{cases} C_{p-1} \times C_{p^{\alpha-1}} & \text{if } p > 2, \\ C_2 \times C_{2^{\alpha-2}} & \text{if } p = 2, \end{cases}$$

is of order $p-1$. □

We now proceed to the existence of such finite groups. Recall from proposition 1.3 that $\mathbb{D}_n^\times$ has cyclic subgroups of order p^α for any $1 \leq \alpha \leq k$.

Proposition 1.12. *If P_α is a cyclic subgroup of order $p^\alpha > 1$ in $\mathbb{D}_n^\times$ and $v = v_{\mathbb{D}_n}$, then*

$$v(C_{\mathbb{D}_n^\times}(P_\alpha)) = \frac{1}{n}\mathbb{Z} \qquad \textit{and} \qquad N_{\mathbb{D}_n^\times}(P_\alpha)/C_{\mathbb{D}_n^\times}(P_\alpha) \cong N_{\mathcal{O}_n^\times}(P_\alpha)/C_{\mathcal{O}_n^\times}(P_\alpha).$$

Proof. From the Skolem-Noether theorem A.9, we know that

$$N_{\mathbb{D}_n^\times}(P_\alpha)/C_{\mathbb{D}_n^\times}(P_\alpha) \cong Aut(P_\alpha).$$

This means that for any f in $Aut(P_\alpha)$, there is an element a in $\mathbb{D}_n^\times$ such that

$$f(x) = axa^{-1} \quad \text{for all } x \in K_\alpha = \mathbb{Q}_p(P_\alpha).$$

As explained in appendix C, the fact that K_α is a totally ramified extension of $\mathbb{Q}_p$ implies that the value group of $C_{\mathbb{D}_n^\times}(K_\alpha)$ is that of $\mathbb{D}_n^\times$; in other words

$$v(C_{\mathbb{D}_n^\times}(P_\alpha)) = \frac{1}{n}\mathbb{Z}.$$

Hence there is an element b in $C_{\mathbb{D}_n^\times}(K_\alpha)$ such that

$$v(ab) = 0 \qquad \text{and} \qquad (ab)x(ab)^{-1} = axa^{-1} = f(x)$$

for all $x \in K_\alpha$. In particular $ab \in \mathcal{O}_n^\times$ and

$$N_{\mathcal{O}_n^\times}(P_\alpha)/C_{\mathcal{O}_n^\times}(P_\alpha) \cong Aut(P_\alpha),$$

as was to be shown. □

Lemma 1.13. *If P_α is a cyclic subgroup of order $p^\alpha > 1$ in $\mathbb{D}_n^\times$, the image of $N_{\mathcal{O}_n^\times}(P_\alpha)$ in $\mathbb{F}_{p^n}^\times$ via the reduction homomorphism is cyclic of order $(p^{n_\alpha} - 1)(p-1)$.*

Proof. Since the residue field of the division algebra

$$C_{\mathbb{D}_n}(\mathbb{Q}_p(P_\alpha)) = C_{\mathbb{D}_n}(P_\alpha)$$

is $\mathbb{F}_{p^{n_\alpha}}$, the image of $C_{\mathcal{O}_n^\times}(P_\alpha) = C_{\mathbb{D}_n^\times}(P_\alpha) \cap \mathcal{O}_n^\times$ via the reduction homomorphism is cyclic of order $p^{n_\alpha} - 1$ in $\mathbb{F}_{p^n}^\times$. Furthermore, there is a canonical surjection

$$N_{\mathcal{O}_n^\times}(P_\alpha) \longrightarrow Aut(P_\alpha) \longrightarrow C_{p-1}.$$

Clearly, $C_{\mathcal{O}_n^\times}(P_\alpha)$ is in the kernel of this projection, and since $p-1$ is prime to p, the p-Sylow subgroup of $N_{\mathcal{O}_n^\times}(P_\alpha)$ must be contained in the kernel as well. It follows that $N_{\mathcal{O}_n^\times}(P_\alpha)$ contains a group which is sent surjectively onto C_{p-1} and whose image in $\mathbb{F}_{p^n}^\times$ is the cyclic subgroup of order $(p^{n_\alpha} - 1)(p-1)$. □

Theorem 1.14. *For each $1 \leq \alpha \leq k$ and each cyclic subgroup P_α of order p^α in $\mathbb{D}_n^\times$, there exists a subgroup G_α of $\mathcal{O}_n^\times$ such that*

$$G_\alpha \cap S_n = P_\alpha \qquad \text{and} \qquad G_\alpha/P_\alpha \cong C_{(p^{n_\alpha}-1)(p-1)} \subseteq \mathbb{F}_{p^n}^\times.$$

Proof. We want to show that the cyclic subgroup of order $(p^{n_\alpha}-1)(p-1)$ in $\mathbb{F}_{p^n}^\times$ obtained from lemma 1.13 can be lifted to an element of finite order in $N_{\mathcal{O}_n^\times}(P_\alpha)$.

Let $\overline{x}$ be an element of order $(p^{n_\alpha}-1)(p-1)$ in $\mathbb{F}_{p^n}^\times$. By lemma 1.13, $\overline{x}$ has a preimage x in $N_{\mathcal{O}_n^\times}(P_\alpha)$ generating by conjugation an element of order $p-1$ in $Aut(P_\alpha)$. The closure $\langle x\rangle$ in $\mathcal{O}_n^\times$ of the group generated by x fits into the exact sequence

$$1 \longrightarrow H \longrightarrow \langle x\rangle \longrightarrow C \longrightarrow 1,$$

where

$$H = \langle x\rangle \cap S_n \qquad \text{and} \qquad C = \langle x\rangle/H.$$

The group H being a cyclic profinite p-group, it must be isomorphic to $\mathbb{Z}_p$ or to a finite cyclic p-group. As $l := |C|$ is prime to p, any element in H is l-divisible, and because $x^l \in H$, there is a $y \in H$ such that $x^l = y^l$. Since $x, y \in \langle x\rangle$ commute with each other, $(xy^{-1})^l = 1$ and xy^{-1} is the desired element of finite order in $N_{\mathbb{D}_n^\times}(P_\alpha)$. □

Remark 1.15. One can show that the isomorphism class of such a G_α is uniquely determined by α. This however is a consequence of the uniqueness of G_α up to conjugation, a fact established in theorem 1.31 and 1.35.

1.3. Finite subgroups of $\mathbb{D}_n^\times$ with quaternionic 2-Sylow

Continuing our investigation of the finite subgroups G of $\mathbb{D}_n^\times$, we now consider the case where the p-Sylow subgroup P of G is non-cyclic. We know from proposition 1.5 that in this case $p = 2$ and P is a generalized quaternion group Q_{2^α} with $\alpha \geq 3$. Throughout this section we assume $p = 2$.

We first look at the case $n = 2$. Consider the filtration of $\mathbb{Z}_2^\times \cong \mathbb{Z}/2 \times \mathbb{Z}_2$ given by

$$U_i = U_i(\mathbb{Z}_2^\times) = 1 + 2^i\mathbb{Z}_2 = \{x \in \mathbb{Z}_2^\times \mid x \equiv 1 \bmod 2^i\}, \qquad \text{for } i \geq 1.$$

As $-7 \equiv 1 \bmod 2^3$, we have

$$-7 \in U_3 = (\mathbb{Z}_2^\times)^2.$$

So let ρ be an element of $\mathbb{Z}_2^\times$ such that $\rho^2 = -7$.

Remark 1.16. By remark C.5, we know that

$$\begin{aligned} \mathbb{D}_2 &\cong \mathbb{Q}_2(\omega)\langle S\rangle/(S^2 = 2, Sx = x^\sigma S) \\ &\cong \mathbb{Q}_2(\omega)\langle T\rangle/(T^2 = -2, Tx = x^\sigma T), \end{aligned}$$

for ω a primitive third root of unity which satisfies

$$1 + \omega + \omega^2 = 0,$$

and for S, T two elements generating the Frobenius σ. Letting $T = x + yS \in \mathbb{D}_2^\times$ for $x, y \in \mathbb{Q}_2(\omega)$, we have

$$-2 = T^2 = (x^2 + 2yy^\sigma) + (xy + yx^\sigma)S \qquad \Leftrightarrow \qquad \begin{cases} x^2 + 2yy^\sigma = -2 \\ xy + yx^\sigma = 0. \end{cases}$$

Taking for solution

$$x = 0 \quad \text{and} \quad y = \frac{3+2\omega}{\rho}, \qquad \text{so that} \quad T = \frac{3+2\omega}{\rho} S,$$

we obtain an isomorphism between these two representations of $\mathbb{D}_2$.

Via these representations, we may further exhibit an explicit embedding of $Q_8 = \langle i, j\rangle$ into $\mathbb{D}_2^\times$ in the following way. We first look for an element $i = a + bT$ with $a, b \in \mathbb{W}(\mathbb{F}_4)$ satisfying

$$-1 = i^2 = (a^2 - 2bb^\sigma) + (a + a^\sigma)bT.$$

Hence either $b = 0$ or $a + a^\sigma = 0$. The first case being impossible as $a^2 = -1$ has no solution in $\mathbb{W}(\mathbb{F}_4)$, we must have $a + a^\sigma = 0$. A possible solution is

$$a = \frac{-1}{1+2\omega} = \frac{1}{3}(1+2\omega) \qquad \text{and} \qquad b = \frac{1}{1+2\omega} = -\frac{1}{3}(1+2\omega),$$

meaning that

$$\begin{aligned} i &= \frac{1}{3}(1+2\omega) - \frac{1}{3}(1+2\omega)T \\ &= \frac{1}{3}(1+2\omega) + \frac{1}{3\rho}(1-4\omega)S. \end{aligned}$$

We then look for an element $j = a' + b'T$ with $a', b' \in \mathbb{W}(\mathbb{F}_4)$ satisfying $j^2 = -1$ and $ij = -ji$, in other words

$$\begin{aligned} & (a'^2 - 2b'b'^\sigma) + (a' + a'^\sigma)b'T = -1 \\ \text{and} \qquad & (aa' - 2bb'^\sigma) + (ab' + ba'^\sigma)T = -(a'a - 2b'b^\sigma) - (a'b + b'a^\sigma)T. \end{aligned}$$

As $a + a^\sigma = 0$ and $a = -b = b^\sigma$, these relations are equivalent to

$$\begin{cases} a' + a'^\sigma = 0 \\ 2aa' = 2(bb'^\sigma + b'b^\sigma) = 2b(b'^\sigma - b') \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} a' + a'^\sigma = 0 \\ a' = b' - b'^\sigma. \end{cases}$$

A possible solution is

$$a' = a = \frac{1}{3}(1+2\omega) \qquad \text{and} \qquad b' = (1+\omega)a' = \frac{1}{3}(-1+\omega),$$

meaning that

$$\begin{aligned} j &= \frac{1}{3}(1+2\omega) + \frac{1}{3}(-1+\omega)T \\ &= \frac{1}{3}(1+2\omega) - \frac{1}{3\rho}(5+\omega)S. \end{aligned}$$

Proposition 1.17. *The quaternion group Q_8 embeds in $\mathbb{D}_n^\times$ if and only if $n \equiv 2 \bmod 4$.*

Proof. The $\mathbb{Q}_2$-algebra $\mathbb{Q}_2(i,j)$ generated by $\langle i, j\rangle \cong Q_8$ is non-commutative and is at least of dimension 4 over $\mathbb{Q}_2$. By remark 1.16 we know that $\mathbb{Q}_2(i,j) \subseteq \mathbb{D}_2$, and it follows that $\mathbb{Q}_2(i,j) = \mathbb{D}_2$. Thus in particular, Q_8 embeds in $\mathbb{D}_n^\times$ if and only if $\mathbb{D}_2^\times$ does, and by corollary C.12 this happens if and only if $n \equiv 2 \bmod 4$. □

Remark 1.18. Using the elements i and j obtained in remark 1.16, and defining

$$k := ij = -\frac{1}{3}(1+2\omega) - \frac{1}{3\rho}(4+5\omega)S,$$

we note that

$$\omega^2 i \omega^{-2} = \omega j \omega^{-1} = -k.$$

This implies that the group

$$T_{24} := Q_8 \rtimes C_3 \cong \langle i, j, \omega \rangle$$

embeds as a maximal finite subgroup of $\mathbb{D}_2^\times$. This group of order 24 is the binary tetrahedral group; it is explicitly given by

$$T_{24} = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}.$$

From proposition 1.17, we have obtained

$$\mathbb{D}_2^\times \cong \mathbb{Q}_2(Q_8) \cong \mathbb{Q}_2(T_{24}).$$

Proposition 1.19. *A generalized quaternion subgroup of $\mathbb{D}_n^\times$ is isomorphic to Q_8.*

Proof. Assume that $Q_{2^{\alpha+1}}$ embeds as a subgroup of $\mathbb{D}_n^\times$ for $\alpha \geq 2$. Then Q_8 embeds and

$$n \equiv 2 \bmod 4$$

by proposition 1.17. On the other hand, the cyclic group C_{2^α} embeds as well and generates a cyclotomic extension of degree $\varphi(2^\alpha) = 2^{\alpha-1}$ over $\mathbb{Q}_p$. Hence

$$n \equiv 0 \bmod 2^{\alpha-1}$$

by the embedding theorem. Therefore $\alpha = 2$. □

Proposition 1.20. *If Q_8 is a quaternion subgroup of $\mathbb{D}_n^\times$ and $v = v_{\mathbb{D}_n}$, then*

$$v(C_{\mathbb{D}_n^\times}(Q_8)) = \frac{2}{n}\mathbb{Z}, \qquad v(N_{\mathbb{D}_n^\times}(Q_8)) = \frac{1}{n}\mathbb{Z},$$

and $N_{\mathcal{O}_n^\times}(Q_8)/C_{\mathcal{O}_n^\times}(Q_8)$ injects into $N_{\mathbb{D}_n^\times}(Q_8)/C_{\mathbb{D}_n^\times}(Q_8)$ as a subgroup of index 2.

Proof. Using the centralizer theorem A.6, together with remark 1.18, we know that

$$\mathbb{D}_n \cong \mathbb{Q}_2(Q_8) \otimes_{\mathbb{Q}_2} C_{\mathbb{D}_n}(Q_8),$$

where $C_{\mathbb{D}_n}(Q_8)$ is a central division algebra of dimension $n^2/4$ over $\mathbb{Q}_2$ whose ramification index is $e(C_{\mathbb{D}_n}(Q_8)/\mathbb{Q}_2) = n/2$ by proposition C.1. In particular,

$$v(C_{\mathbb{D}_n^\times}(Q_8)) = \frac{2}{n}\mathbb{Z}. \qquad (*)$$

Now the existence of Q_8 in $\mathbb{D}_n^\times$ implies by proposition 1.17 that $n \equiv 2 \bmod 4$, so that $n = 2(2r+1)$ for an integer $r \geq 0$. As 2 and $2r+1$ are prime to each other, there are integers $a, b \geq 1$ satisfying

$$(2r+1)a + 2b = 1 \qquad \Leftrightarrow \qquad \frac{a}{2} + \frac{b}{2r+1} = \frac{1}{n}.$$

By $(*)$ we can choose an element $x \in C_{\mathbb{D}_n^\times}(Q_8)$ having valuation $2/n = 1/(2r+1)$. On the other hand since

$$(1+i)j(1+i)^{-1} = k \in Q_8 \qquad \text{and} \qquad (1+i)^2 = 2i,$$

we know that $1+i$ is an element of $N_{\mathbb{D}_n^\times}(Q_8)$ having valuation $1/2$. We thus have found an element $(1+i)^a x^b$ in $N_{\mathbb{D}_n^\times}(Q_8)$ of valuation

$$v((1+i)^a x^b) = av(1+i) + bv(x) = \frac{a}{2} + \frac{b}{2r+1} = \frac{1}{n},$$

so that

$$v(N_{\mathbb{D}_n^\times}(Q_8)) = \frac{1}{n}\mathbb{Z}.$$

This result, together with $(*)$, implies the last assertion of the proposition. □

Proposition 1.21. $|Aut(Q_8)| = |Aut(T_{24})| = 24$.

Proof. Let $Q_8 = \langle i, j\rangle$ and $T_{24} = \langle Q_8, \omega\rangle$ with i, j, k, ω as defined in remark 1.16 and 1.18. Counting on which of the 6 elements $\{\pm i, \pm j, \pm k\}$ of order 4 the generators i and j may be sent via an automorphism, we know that $|Aut(Q_8)|$ divides 24. The inner automorphism group of Q_8 has order $|Q_8/\{\pm 1\}| = 4$; it is generated by conjugation by i and j. Let

$$c_{Q_8} : T_{24} \longrightarrow Aut(Q_8)$$

be the conjugation action of Q_8 by elements of T_{24}. As noted in remark 1.18, the conjugation by ω has order 3, and hence the cardinality of the image of c_{Q_8} is 12. Since the element $(1+i) \in \mathbb{D}_n^\times$ acts by conjugation on Q_8 by $i \mapsto i$ and $j \mapsto k$, it follows that the automorphism of Q_8 induced by $(1+i)$ is not in the image of c_{Q_8}. Because $|Aut(Q_8)| \leq 24$, we obtain $|Aut(Q_8)| = 24$.

Now using that Q_8 is the (normal) 2-Sylow subgroup of T_{24}, consider the canonical map $\varphi : Aut(T_{24}) \to Aut(Q_8)$; it is surjective since $(1+i)$ also induces an automorphism of T_{24}. Let $\sigma \in Aut(T_{24})$ be such that $\sigma|_{Q_8} = id_{Q_8}$. Then for any $t \in T_{24}$ and $q \in Q_8$ we have

$$c_{Q_8}(\sigma(t))(q) = \sigma(t)q\sigma(t)^{-1} = \sigma(tqt^{-1}) = tqt^{-1} = c_{Q_8}(t)(q).$$

Hence $\sigma(t)t^{-1} \in Ker(c_{Q_8}) = \{\pm 1\}$ and $\sigma(t) = \pm t$ for any $t \in T_{24}$. In fact, $t = sq$ with $q \in Q_8$ and s an element of order 3 in T_{24}, and we have

$$\sigma(t)t^{-1} = \sigma(s)\sigma(q)q^{-1}s^{-1} = \sigma(s)s^{-1}.$$

Because s is of order 3 and $-s$ is of order 6, the case $\sigma(t) = -t$ is impossible and we must have $\sigma(t) = t$ for all $t \in T_{24}$. Therefore the map φ is bijective, and as $|Aut(Q_8)| = 24$, it follows that $|Aut(T_{24})| = 24$. □

Now assume $n = 2m$ with m odd and consider a finite subgroup G of $\mathbb{D}_n^\times$ whose 2-Sylow subgroup P is isomorphic to Q_8. Such a group determines a subgroup $C = G/P$ of $\mathbb{F}_{2^n}^\times$.

Proposition 1.22. *If G is a finite subgroup of $\mathbb{D}_n^\times$ with a quaternionic 2-Sylow subgroup $P \cong Q_8$, then G/P embeds into the cyclic subgroup of order $3(2^m - 1)$ in $\mathbb{F}_{2^n}^\times$.*

Proof. Recall that $\mathbb{Q}_2(P) \cong \mathbb{D}_2^\times$ and note that $C_G(P)$ is contained in

$$C_{\mathbb{D}_n^\times}(P) = C_{\mathbb{D}_n^\times}(\mathbb{Q}_2(P)) \cong C_{\mathbb{D}_n^\times}(\mathbb{D}_2)$$

which consists of the non-zero elements of a central division algebra of dimension m^2 over $\mathbb{Q}_2$. Its residue field is $\mathbb{F}_{2^m}$, and $C_G(P)/P \cap C_G(P) \cong P \cdot C_G(P)/P$ injects via the reduction homomorphism into $\mathbb{F}_{2^m}^\times$.

Furthermore, we have an injection

$$N_G(P)/C_G(P) \longrightarrow N_{\mathcal{O}_n^\times}(P)/C_{\mathcal{O}_n^\times}(P) \subseteq N_{\mathbb{D}_n^\times}(P)/C_{\mathbb{D}_n^\times}(P) \cong Aut(Q_8),$$

where the last isomorphism is due to the Skolem-Noether theorem. Since $|Aut(Q_8)| = 24$, proposition 1.20 implies that $|N_G(P)/C_G(P)|$ divides 12. As $P \cap C_G(P) = \{\pm 1\}$ is of index 4 in P, we know that $C_G(P)$ is of index 4 in $P \cdot C_G(P)$, and consequently that $P \cdot C_G(P)$ is of index a divisor of 3 in $N_G(P)$.

We have thus obtained a chain of subgroups

$$P \subseteq P \cdot C_G(P) \subseteq N_G(P) = G,$$

where the first group is of index a divisor of $2^m - 1$ in the second group, and the latter is of index a divisor of 3 in the third group. □

Theorem 1.23. *If $p = 2$ and $n = 2m$ with m odd, the group*

$$T_{24} \times C_{2^m-1} = Q_8 \rtimes C_{3(2^m-1)}$$

embeds as a maximal finite subgroup of $\mathbb{D}_n^\times$.

Proof. By the centralizer theorem

$$\mathbb{D}_n \cong \mathbb{D}_2 \otimes_{\mathbb{Q}_2} C_{\mathbb{D}_n}(\mathbb{D}_2) \cong \mathbb{Q}_2(Q_8) \otimes_{\mathbb{Q}_2} C_{\mathbb{D}_n}(Q_8).$$

By remark 1.18, $T_{24} = Q_8 \rtimes C_3$ embeds as a subgroup of $\mathbb{D}_2^\times$; more precisely $\mathbb{Q}_2(T_{24}) = \mathbb{D}_2$. Moreover, since $C_{\mathbb{D}_n}(\mathbb{D}_2)$ is a central division algebra of dimension m^2 over $\mathbb{Q}_2$, its maximal unramified extension of degree m over $\mathbb{Q}_2$ contains a cyclic subgroup C_{2^m-1} of order $2^m - 1$ which centralizes T_{24}. Since m is odd, $2^m - 1$ is not a multiple of 3 and $\mathbb{D}_n^\times$ contains a subgroup isomorphic to

$$T_{24} \times C_{2^m-1} \cong Q_8 \rtimes C_{3(2^m-1)};$$

its maximality as a finite subgroup then follows from proposition 1.22. □

Corollary 1.24. *The center of $T_{24} \times C_{2^m-1}$ is*

$$Z(T_{24} \times C_{2^m-1}) = \{\pm 1\} \times C_{2^m-1} \cong C_{2(2^m-1)}.$$

Proof. This follows from the proof of theorem 1.23 and the obvious fact that the center of Q_8 is $\{\pm 1\}$. □

1.4. Conjugacy classes in $\mathbb{S}_n$

In this section, we establish a classification of the finite subgroups of $\mathbb{S}_n$ up to conjugation. We say that two subgroups $G_1, G_2 \subseteq \mathbb{D}_n^\times$ are *conjugate* in $\mathbb{D}_n^\times$, respectively in $\mathcal{O}_n^\times$, if there is an element a in $\mathbb{D}_n^\times$, respectively in $\mathcal{O}_n^\times$, satisfying

$$aG_1a^{-1} = G_2.$$

We will see that two finite subgroups G_1 and G_2 whose respective p-Sylow subgroups P_1 and P_2 are isomorphic, and for which the quotient groups G_1/P_1 and G_2/P_2 are also isomorphic, are not only isomorphic but even conjugate in $\mathcal{O}_n^\times$. This will imply that the maximal subgroups of $\mathcal{O}_n^\times$ are classified up to conjugation by the type of their p-Sylow subgroups. To do this, we will exploit the tools of nonabelian cohomology of profinite groups as introduced in [23] chapter I paragraph 5.

For any subgroup G of a group H, we set

$$S_H(G) := \{G' \leq H \mid G' \cong G\} \qquad \text{and} \qquad \mathcal{C}_H(G) := S_H(G)/\sim_H$$

where $\sim_H$ designates the relation of conjugation by an element in H.

Lemma 1.25. *If P is a finite p-subgroup of $\mathcal{O}_n^\times$, then $|\mathcal{C}_{\mathcal{O}_n^\times}(P)| = 1$.*

Proof. Let Q be a finite p-subgroup of $\mathcal{O}_n^\times$ isomorphic to P. We have seen that these two groups are either cyclic or quaternionic. In either case, the Skolem-Noether theorem implies the existence of an element a in $\mathbb{D}_n^\times$ such that

$$\mathbb{Q}_p(Q) = a\mathbb{Q}_p(P)a^{-1}.$$

In the cyclic case, this clearly implies $Q = aPa^{-1}$. In the quaternionic case, this yields two quaternion groups Q and aPa^{-1} within $\mathbb{Q}_2(Q) \cong \mathbb{D}_2^\times$ in which we can use Skolem-Noether once more to obtain an element $a' \in \mathbb{Q}_2(Q)$ such that $Q = a'aP(a'a)^{-1}$. Now by proposition 1.12 and 1.20, we know that

$$v(N_{\mathbb{D}_n^\times}(P)) = \frac{1}{n}\mathbb{Z} = v(\mathbb{D}_n^\times).$$

Thus there is an element b in $\mathbb{D}_n^\times$ such that

$$v(ab) = 0 \qquad \text{and} \qquad P = bPb^{-1},$$

and ab is an element of $\mathcal{O}_n^\times$ conjugating P into Q. □

Lemma 1.26. *Let P be a profinite p-group of the form $P = \lim_n P_n$ where each P_n is a finite p-group and the homomorphisms in the inverse system are surjective, and let R be a finite group of order prime to p which acts by group homomorphisms on all P_n in such a way that the homomorphisms in the inverse system are R-equivariant. Then the (nonabelian) cohomology group $H^1(R, P)$ is trivial.*

Proof. Denote by $j_n : P_n \to P_{n-1}$ the homomorphisms of the inverse system, and consider the map $\delta : \prod_n P_n \to \prod_n P_n$ defined by

$$\delta(f_n) = (-1)^n f_n + (-1)^{n+1} j_{n+1}(f_{n+1}),$$

for $f = (f_n) \in \prod_n P_n$. Then note that δ is surjective and that $Ker(\delta)$ is the set of all $f = (f_n) \in \prod_n P_n$ such that $j_n(f_n) = f_{n-1}$ for all n. Hence there is a short exact sequence

$$1 \longrightarrow P \longrightarrow \prod_n P_n \xrightarrow{\delta} \prod_n P_n \longrightarrow 1$$

which induces a long exact sequence

$$1 \to P^R \to \prod_n P_n^R \to \prod_n P_n^R \to H^1(R, P) \to H^1(R, \prod_n P_n) \to H^1(R, \prod_n P_n),$$

where P^R, respectively P_n^R, denotes the R-invariants. Using the canonical isomorphism

$$H^1(R, \prod_n P_n) \cong \prod_n H^1(R, P_n),$$

and noting that each group $H^1(R, P_n)$ is trivial by the Schur-Zassenhaus theorem 1.6, it is enough to show that the homomorphism

$$\prod_n P_n^R \longrightarrow \prod_n P_n^R$$

in the above exact sequence is surjective, and hence that each homomorphism $j_{n+1}^R : P_{n+1}^R \to P_n^R$ is surjective by the definition of δ.

For each n, let K_{n+1} be the kernel of the map $j_{n+1} : P_{n+1} \to P_n$. For each short exact sequence of finite p-groups with action of R

$$1 \longrightarrow K_{n+1} \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow 1,$$

there is an associated exact cohomology sequence

$$1 \longrightarrow K_{n+1}^R \longrightarrow P_{n+1}^R \xrightarrow{j_{n+1}^R} P_n^R \longrightarrow H^1(R, K_{n+1}).$$

Applying the Schur-Zassenhaus theorem once more, we obtain that $H^1(R, K_{n+1})$ is trivial and that the homomorphism j_{n+1}^R is surjective. □

We recall the following fact from [23] chapter I §5.1:

Lemma 1.27. *If P is an R-group with trivial (nonabelian) $H^1(R, P)$, and if*

$$1 \longrightarrow P \longrightarrow N \longrightarrow R \longrightarrow 1$$

is a split extension, then two splittings of R in N are conjugate by an element in P.

Theorem 1.28. *Two finite subgroups G_1 and G_2 of $\mathcal{O}_n^\times$ with respective isomorphic p-Sylow subgroups $P_1 \cong P_2$ and isomorphic quotient groups $G_1/P_1 \cong G_2/P_2$ are conjugate in $\mathcal{O}_n^\times$.*

Proof. The groups G_1 and G_2 fit into exact sequences

$$1 \longrightarrow P_1 \longrightarrow G_1 \longrightarrow C \longrightarrow 1$$

$$1 \longrightarrow P_2 \longrightarrow G_2 \longrightarrow C \longrightarrow 1,$$

where C is the subgroup of $\mathbb{F}_{p^n}^\times$ isomorphic to $G_1/P_1 \cong G_2/P_2$. We know from lemma 1.25 that P_1 and P_2 are conjugate in $\mathcal{O}_n^\times$. By conjugating G_2, we can therefore assume that

$$P_1 = P_2 =: P \qquad \text{and} \qquad G_1, G_2 \subseteq N_{\mathcal{O}_n^\times}(P).$$

Moreover, the latter groups fit into a split exact sequence

$$1 \longrightarrow N_{\mathcal{O}_n^\times}(P) \cap S_n \longrightarrow N_{\mathcal{O}_n^\times}(P) \longrightarrow R \longrightarrow 1,$$

where $R \subseteq \mathbb{F}_{p^n}^\times$ is a finite cyclic group of order prime to p containing C. It follows from lemma 1.26 that $H^1(R, N_{\mathcal{O}_n^\times}(P) \cap S_n)$ is trivial, and hence by lemma 1.27 that G_1 and G_2 are conjugate in $N_{\mathcal{O}_n^\times}(P) \subseteq \mathcal{O}_n^\times$. □

Remark 1.29. Alternatively, we may directly apply [19] theorem 2.3.15, which shows that if K is the p-Sylow subgroup of a profinite group G, then there is up to conjugation in G a unique closed subgroup H of G such that $G = KH$ and $K \cap H = 1$. Indeed, since in our case both extensions

$$1 \longrightarrow P \longrightarrow G_1 \longrightarrow C \longrightarrow 1$$

$$1 \longrightarrow P \longrightarrow G_2 \longrightarrow C \longrightarrow 1$$

are split by the Schur-Zassenhaus theorem, we obtain that both of the corresponding sections are conjugate in $N_{\mathcal{O}_n^\times}(P)$, and hence that G_1 and G_2 are conjugate in $N_{\mathcal{O}_n^\times}(P)$.

Corollary 1.30. *Two finite subgroups of $\mathcal{O}_n^\times$ are conjugate if and only if they are isomorphic.* □

Theorem 1.31. *If p is an odd prime and $n = (p-1)p^{k-1}m$ with m prime to p, the group $\mathbb{S}_n$, respectively $\mathbb{D}_n^\times$, has exactly $k+1$ conjugacy classes of maximal finite subgroups; they are represented by*

$$G_0 = C_{p^n-1} \qquad \textit{and} \qquad G_\alpha = C_{p^\alpha} \rtimes C_{(p^{n\alpha}-1)(p-1)} \qquad \textit{for} \quad 1 \leq \alpha \leq k.$$

Moreover, when $p-1$ does not divide n, the only class of maximal finite subgroups is that of G_0.

Proof. First note that proposition 1.7 and theorem 1.28 imply that there is a unique maximal conjugacy class G_0 of finite subgroups of order prime to p in $\mathcal{O}_n^\times \cong \mathbb{S}_n$, respectively in $\mathbb{D}_n^\times$ by proposition 1.1, and that this class is the only one among finite subgroups if n is not a multiple of $p-1$.

Now assume that $1 \leq \alpha \leq k$. By theorem 1.14, there is a finite subgroup G_α in $\mathbb{D}_n^\times$ realized as an extension

$$1 \longrightarrow C_{p^\alpha} \longrightarrow G_\alpha \longrightarrow C_{(p^{n\alpha}-1)(p-1)} \longrightarrow 1,$$

where

$$C_{p^\alpha} = G_\alpha \cap S_n \qquad \text{and} \qquad G_\alpha/C_{p^\alpha} \cong C_{(p^{n\alpha}-1)(p-1)} \subseteq \mathbb{F}_{p^n}^\times.$$

The Schur-Zassenhaus theorem implies that this extension splits, in other words that

$$G_\alpha = C_{p^\alpha} \rtimes C_{(p^{n\alpha}-1)(p-1)}.$$

Corollary 1.11 and theorem 1.28 ensure that G_α represents the unique maximal conjugacy class of finite subgroups of $\mathcal{O}_n^\times \cong \mathbb{S}_n$ which have a p-Sylow subgroup of order p^α. □

Corollary 1.32. *If $p > 2$ and $1 \leq \alpha \leq k$, then*

$$Z(G_\alpha) \cong C_{(p^{n_\alpha}-1)}.$$

Proof. This follows from theorem 1.31 and proposition 1.10, where the latter shows that $C_{(p^{n_\alpha}-1)}$ embeds into $Z(G_\alpha)$ and that $C_{(p^{n_\alpha}-1)(p-1)}/C_{(p^{n_\alpha}-1)} \cong C_{p-1}$ acts faithfully on C_{p^α}. □

The following case can be explicitly analyzed. As noted in remark 1.34, this provides a counter example to the main results of [11].

Example 1.33. Assume that p is odd and $n = (p-1)p^{k-1}m$ with $(p;m) = 1$. Let $\omega \in \mathbb{D}_n^\times$ be a primitive (p^n-1)-th root of unity in $\mathcal{O}_n^\times$. Define

$$X := \omega^{\frac{p-1}{2}} S \ \in \mathcal{O}_n^\times \qquad \text{and} \qquad Z := X^l \quad \text{with } l = \frac{n}{p-1}.$$

A simple calculation shows

$$Z^{p-1} = X^n = -p.$$

We can show (see [9] lemma 19) that $\mathbb{Q}_p(Z)$ contains a primitive p-th root of unity ζ_p. Because the fields $\mathbb{Q}_p(Z)$ and $\mathbb{Q}_p(\zeta_p)$ are of the same degree $p-1$ over $\mathbb{Q}_p$, they must be identical. We set

$$K := \mathbb{Q}_p(Z) = \mathbb{Q}_p(\zeta_p).$$

We note that $p^n - 1$ is divisible by $(p^l - 1)(p - 1)$ and let

$$\tau := \omega^{\frac{p^n-1}{(p^l-1)(p-1)}} \ \in \mathbb{F}_q^\times.$$

We have

$$\tau Z \tau^{-1} = \omega^{\frac{p^n-1}{p-1}} Z = \zeta_{p-1} Z,$$

for ζ_{p-1} a primitive $(p-1)$-th root of unity in $\mathcal{O}_n^\times$. Hence τ induces an automorphism of K of order $p-1$ which sends ζ_p to another root of unity of the same order, and τ normalizes the group generated by ζ_p. The group G generated by ζ_p and τ is clearly of order $p(p^l-1)(p-1)$; it is therefore maximal. Since X commutes with all elements of K, it necessarily commutes with ζ_p. Moreover the fact that

$$X \tau X^{-1} = \tau^p$$

shows that X belongs to the normalizer $N_{\mathbb{D}_n^\times}(G)$. The valuation of X is $\frac{1}{n}$ by definition, and we have

$$v(N_{\mathbb{D}_n^\times}(G)) = \frac{1}{n}\mathbb{Z}.$$

As in lemma 1.25, we can then apply the Skolem-Noether theorem to obtain that there is only one conjugacy class of subgroups of $\mathcal{O}_n^\times$ that are isomorphic to G.

In particular, if $p = 3$ and $n = 4$, then $k = 1$, $m = 2$, the order of ω is 80, and a maximal finite 3-Sylow subgroup in $\mathcal{O}_n^\times$ is isomorphic to C_3. Here

$$X = \omega S, \qquad Z = \omega^4 S^2 \qquad \text{and} \qquad Z^2 = -3.$$

In order to find an element ζ_3 in $\mathbb{Q}_3(X^2)$, we may solve the equation

$$(x + yZ)^3 = 1 \qquad \text{with} \quad x, y \in \mathbb{Q}_3.$$

We find $x = \pm y$ with $x = -\frac{1}{2}$, from which we obtain the primitive third roots of unity

$$\zeta_3 = -\frac{1}{2}(1 + \omega^4 S^2) \qquad \text{and} \qquad \zeta_3^2 = -\frac{1}{2}(1 - \omega^4 S^2)$$

in the field $\mathbb{Q}_3(Z)$. Here $\tau = \omega^5$ is of order 16 and we easily verify the relations

$$\tau\zeta_3\tau^{-1} = \zeta_3^2, \qquad X\zeta_3 X^{-1} = \zeta_3, \qquad X\tau X^{-1} = \tau^3,$$

showing as expected that

$$v(N_{\mathbb{D}_n^\times}(C_3 \rtimes C_{2(3^2-1)})) = \frac{1}{4}\mathbb{Z} \qquad \text{and} \qquad |\mathcal{C}_{\mathcal{O}_n^\times}(C_3 \rtimes C_{2(3^2-1)})| = 1.$$

Remark 1.34. Theorem 1.31 and example 1.33 (in particular the case where $n = 4$ and $p = 3$) bring a contradiction to the main results of [11]. In the latter, a central result concerning the nonabelian finite groups when $p > 2$ is proposition 3.9: it states that for $\alpha \geq 1$ the normalizer of G_α in $\mathbb{D}_n^\times$ has valuation group

$$v(N_{\mathbb{D}_n^\times}(G_\alpha)) = \frac{f(\mathbb{Q}_p(\zeta_{p^\alpha(p^{n\alpha}-1)}/\mathbb{Q}_p))}{n}\mathbb{Z} = \frac{n_\alpha}{n}\mathbb{Z},$$

where f denotes the residue degree of the given cyclotomic extension. As a consequence of this incorrect result propositions 3.10 to 3.12 in [11] are incorrect as well.

Theorem 1.35. *Let $p = 2$ and $n = 2^{k-1}m$ with m odd. The group $\mathbb{S}_n$, respectively $\mathbb{D}_n^\times$, has exactly k maximal conjugacy classes of finite subgroups. If $k \neq 2$, they are represented by*

$$G_\alpha = C_{2^\alpha(2^{n\alpha}-1)} \qquad \textit{for} \quad 1 \leq \alpha \leq k.$$

If $k = 2$, they are represented by G_α for $\alpha \neq 2$ and by the unique maximal nonabelian conjugacy class

$$Q_8 \rtimes C_{3(2^m-1)} \cong T_{24} \times C_{2^m-1},$$

the latter containing G_2 as a subclass.

Proof. The argument for the cyclic classes G_α is identical to that of theorem 1.31 except that in this case $G_0 = C_{2^n-1}$ is contained in G_1.

Furthermore, proposition 1.19 ensures that a nonabelian finite subgroup may only exist in $\mathcal{O}_n^\times \cong \mathbb{S}_n$, respectively in $\mathbb{D}_n^\times$, when its 2-Sylow subgroup is isomorphic to Q_8, and proposition 1.17 shows that such a group occurs if and only if $k = 2$. In fact, assuming $k = 2$, the group $Q_8 \rtimes C_{3(2^m-1)}$ embeds in $\mathcal{O}_n^\times$ as a maximal finite subgroup by theorem 1.23, and its conjugacy class is unique among maximal nonabelian finite subgroups by theorem 1.28. □

Remark 1.36. Theorem 1.35 contradicts theorem 5.3 in [11]. According to the latter, we should have two distinct conjugacy classes in $\mathcal{O}_n^\times$ for the finite groups containing $T_{24} = Q_8 \rtimes C_3$. Letting $Inn(T_{24})$ and $Out(T_{24})$ denote the inner and outer automorphisms of T_{24}, the error occurs before theorem 5.1 where it is said that $Out(T_{24})$ is trivial. This is absurd given that

$$|Aut(T_{24})| = 24 \qquad \text{and} \qquad Inn(T_{24}) \cong T_{24}/\{\pm 1\}.$$

All results given in section 5 of [11] are then wrong in this case.

Corollary 1.37. *The abelian finite subgroups of $\mathbb{D}_n^\times$ are classified up to conjugation in $\mathcal{O}_n^\times$, respectively in $\mathbb{D}_n^\times$, by the pairs of integers (α, d) satisfying*

$$0 \leq \alpha \leq k \qquad \textit{and} \qquad 1 \leq d \mid p^{n_\alpha} - 1;$$

each such pair represents the cyclic class $C_{p^\alpha d}$.

Proof. By corollary 1.30, the finite cyclic subgroups are classified up to conjugation by their isomorphism classes. The result then follows from the maximal finite classes provided by theorem 1.31 and 1.35. □

Remark 1.38. We restricted ourselves in considering the finite subgroups of $\mathbb{D}_n^\times$ as split extensions of subgroups of $\mathbb{F}_{p^n}^\times$ by finite p-subgroups in S_n. It is also possible to express these finite groups as subextensions of short exact sequences of the form

$$1 \longrightarrow C_{\mathbb{D}_n^\times}(C_{p^\alpha}) \longrightarrow N_{\mathbb{D}_n^\times}(C_{p^\alpha}) \longrightarrow Aut(C_{p^\alpha}) \longrightarrow 1,$$

as induced by the Skolem-Noether theorem. A finite group of type $G_\alpha \subseteq \mathbb{D}_n^\times$ can be seen as a metacyclic extension

$$1 \longrightarrow \langle A \rangle \longrightarrow G_\alpha \longrightarrow \langle B \rangle \longrightarrow 1,$$

with

$$\langle A \rangle = G'_\alpha \times Z(G_\alpha) \qquad \text{and} \qquad \langle B \rangle \cong C_{p-1},$$

where G'_α denotes the commutator subgroup of G_α. The classification given in [10] follows this approach, but has the disadvantages of being less direct and relying on a classification previously established in [1].

Chapter 2:

A classification scheme for finite subgroups

We fix a prime p, a positive integer n which is a multiple of $(p-1)$, and a unit $u \in \mathbb{Z}_p^\times$. Given these, we adopt notation 1.8. In this chapter, we provide necessary and sufficient conditions for the existence of finite subgroups of

$$\mathbb{G}_n(u) = \mathbb{D}_n^\times / \langle pu \rangle$$

whose intersection with $\mathbb{S}_n$ have a cyclic p-Sylow subgroup. The remaining case of a quaternionic 2-Sylow will be treated in chapter 4.

2.1. A canonical bijection

Let

$$\pi : \mathbb{D}_n^\times \longrightarrow \mathbb{G}_n(u)$$

denote the canonical homomorphism. In order to study a finite subgroup F of $\mathbb{G}_n(u)$, it is often more convenient to analyse its preimage

$$\widetilde{F} := \pi^{-1}(F) \ \in \mathbb{D}_n^\times.$$

For any group G we define $\mathcal{F}(G)$ to be the set of all finite subgroups of G; and if G is a subgroup of $\mathbb{D}_n^\times$ we define $\widetilde{\mathcal{F}}_u(G)$ to be the set, eventually empty, of all subgroups of G which contain $\langle pu \rangle$ as a subgroup of finite index.

Proposition 2.1. *The map π induces a canonical bijection*

$$\widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times) \longrightarrow \mathcal{F}(\mathbb{G}_n(u)).$$

This bijection passes to conjugacy classes.

Proof. For any $F \in \mathcal{F}(\mathbb{G}_n(u))$, it is clear that $\langle pu \rangle$ is a subgroup of finite index in $\pi^{-1}(F)$. Moreover, the fact that π is surjective implies that $\pi\pi^{-1}(F) = F$. On the other hand, for $G \in \widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times)$, as $Ker(\pi) = \langle pu \rangle$ is always a subgroup of G, we have $\pi^{-1}\pi(G) = G$.

In order to show the second assertion, let F_1, F_2 be two subgroups of $\mathbb{G}_n(u)$ with $\widetilde{F_i} = \pi^{-1}(F_i)$ for $i \in \{1, 2\}$. If there is an element $a \in \mathbb{D}_n^\times$ such that $\widetilde{F_2} = a\widetilde{F_1}a^{-1}$, then since π is a group homomorphism we have

$$\begin{aligned} F_2 &= \pi(a\widetilde{F_1}a^{-1}) \\ &= \pi(a)\pi(\widetilde{F_1})\pi(a)^{-1} \\ &= \pi(a)F_1\pi(a)^{-1}. \end{aligned}$$

Conversely, if $F_2 = bF_1b^{-1}$ for some $b \in \mathbb{G}_n(u)$, and if $\widetilde{b} \in \mathbb{D}_n^\times$ satisfies $\pi(\widetilde{b}) = b$, then from the above identity we have

$$\pi(\widetilde{b}\widetilde{F_1}\widetilde{b}^{-1}) = F_2,$$

as was to be shown. □

Remark 2.2. In a similar way, the map π induces a bijection between the set of all subgroups of $\mathbb{G}_n(u)$ and the set of all subgroups of $\mathbb{D}_n^\times$ containing $\langle pu \rangle$.

Notation 2.3. For a subgroup G of $\mathbb{G}_n(u)$, we denote by

$$\widetilde{G} = \pi^{-1}(G)$$

its preimage under the canonical map $\pi : \mathbb{D}_n^\times \to \mathbb{G}_n(u)$. From now on, when introducing a tilded group, its non-tilded correspondent will be implicitly defined.

Remark 2.4. The valuation $v = v_{\mathbb{D}_n} : p \mapsto 1$ on $\mathbb{D}_n^\times$ induces a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & \langle pu \rangle & \xrightarrow{v} & \mathbb{Z} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{S}_n & \longrightarrow & \mathbb{D}_n^\times & \xrightarrow{v} & \frac{1}{n}\mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
1 & \longrightarrow & \mathbb{S}_n & \longrightarrow & \mathbb{G}_n(u) & \xrightarrow{v} & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 1.
\end{array}$$

Subgroups of $\mathbb{S}_n$ can therefore be considered as subgroups of both $\mathbb{G}_n(u)$ and $\mathbb{D}_n^\times$.

Proposition 2.5. *If $F \subseteq \mathbb{S}_n$, then $\widetilde{F} = F \times \langle pu \rangle$.*

Proof. This follows from the exact commutative diagram of remark 2.4 and the fact that $\langle pu \rangle$ is central in $\mathbb{D}_n^\times$. □

2.2. Chains of extensions

For F a finite subgroup of $\mathbb{G}_n(u)$ such that $F \cap S_n$ is cyclic, we set

$$G := F \cap \mathbb{S}_n \qquad \text{and} \qquad F_0 := \langle F \cap S_n, Z_{p'}(G) \rangle,$$

for S_n the p-Sylow subgroup of $\mathbb{S}_n$ and $Z_{p'}(G)$ the p'-part of the center $Z(G)$ of G. As previously seen, F_0 is the maximal abelian subgroup of G equal to $P \times Z_{p'}(G)$ for P the cyclic p-Sylow subgroup of G.

Remark 2.6. From proposition 2.5, we know that

$$\widetilde{F_0} = F_0 \times \langle pu \rangle.$$

Remark 2.7. By definition, G consists of the elements of F which are of valuation zero in $\mathbb{D}_n^\times$. Hence G is normal in F and there is a short exact sequence

$$1 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 1,$$

where the quotient embeds via the valuation into $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Proposition 2.8. *We have*

$$\widetilde{C_F(F_0)} = C_{\widetilde{F}}(\widetilde{F_0}) \supseteq \widetilde{F_0} \qquad \textit{and} \qquad C_F(F_0)/F_0 \cong C_{\widetilde{F}}(\widetilde{F_0})/\widetilde{F_0}.$$

Proof. It is obvious that

$$\widetilde{C_F(F_0)} \;\supseteq\; C_{\widetilde{F}}(\widetilde{F_0}) \;\supseteq\; \widetilde{F_0},$$

and the second assertion is a direct consequence of the first one.

It remains to check that $\widetilde{C_F(F_0)} \subseteq C_{\widetilde{F}}(\widetilde{F_0})$. If $\widetilde{x} \in \widetilde{C_F(F_0)}$ and $\widetilde{f} \in \widetilde{F_0}$, then there is a unique element $z = z(\widetilde{x}, \widetilde{f}) \in \langle pu \rangle$ such that

$$z\widetilde{f} = \widetilde{x}\widetilde{f}\widetilde{x}^{-1}.$$

Because $\langle pu \rangle$ is central, we have

$$z(\widetilde{x}, \widetilde{f}\widetilde{g}) = z(\widetilde{x}, \widetilde{f})z(\widetilde{x}, \widetilde{g}),$$

for every $\widetilde{f}, \widetilde{g} \in \widetilde{F_0}$. This yields an exact sequence

$$1 \longrightarrow C_{\widetilde{F}}(\widetilde{F_0}) \longrightarrow \widetilde{C_F(F_0)} \longrightarrow Hom(\widetilde{F_0}, \langle pu \rangle),$$

where the image of $\widetilde{x}$ is the homomorphism $\widetilde{f} \mapsto z(\widetilde{x}, \widetilde{f})$. As stated in remark 2.6 we know that $\widetilde{F_0} = F_0 \times \langle pu \rangle$. Because $\langle pu \rangle$ is central, the image of $\widetilde{C_F(F_0)}$ in $Hom(\widetilde{F_0}, \langle pu \rangle)$ is contained in the subgroup of those homomorphisms which are trivial on $\langle pu \rangle \subseteq \widetilde{F_0}$ and hence factors through F_0. Because F_0 is finite and $\langle pu \rangle$ is torsion free, it follows that this image is trivial and $C_{\widetilde{F}}(\widetilde{F_0}) = \widetilde{C_F(F_0)}$. □

Proposition 2.9. *We have*

$$F = N_F(F_0) \qquad \textit{and} \qquad \widetilde{F} = N_{\widetilde{F}}(\widetilde{F_0}).$$

Proof. Because P is the unique p-Sylow subgroup of $G = F \cap \mathbb{S}_n$, it is a characteristic subgroup of G. Moreover as $F_0 = P \times Z_{p'}(G)$ and $Z_{p'}(G)$ is also a characteristic subgroup of G, it follows that F_0 is a characteristic subgroup of G; in other words

$$N_{\mathbb{G}_n(u)}(G) \subseteq N_{\mathbb{G}_n(u)}(F_0) \qquad \text{and} \qquad N_{\mathbb{D}_n^\times}(G) \subseteq N_{\mathbb{D}_n^\times}(F_0).$$

Since G is by definition normal in F, its subgroup F_0 is normal in F. Proposition 2.1 finally implies that $\widetilde{F_0}$ is normal in $\widetilde{F}$. □

Corollary 2.10. *There are short exact sequences*

$$1 \longrightarrow F_0 \longrightarrow F \longrightarrow F/F_0 \longrightarrow 1,$$

$$1 \longrightarrow \widetilde{F_0} \longrightarrow \widetilde{F} \longrightarrow F/F_0 \longrightarrow 1.$$

Proof. This follows from that facts that F_0 is normal in F, $\widetilde{F_0}$ is normal in $\widetilde{F}$, and that $\widetilde{F}/\widetilde{F_0} \cong F/F_0$. □

Note that in $\mathbb{D}_n^\times$ we have $\mathbb{Q}_p(F_0) = \mathbb{Q}_p(\widetilde{F_0})$, and there are inclusions

$$\widetilde{F_0} \subseteq \mathbb{Q}_p(F_0)^\times \subseteq C_{\mathbb{D}_n^\times}(F_0) \subseteq N_{\mathbb{D}_n^\times}(F_0).$$

Given F and F_0, the second extension of corollary 2.10 can then be broken into three pieces via the chain of subgroups

$$\widetilde{F_0} \subseteq \widetilde{F_1} \subseteq \widetilde{F_2} \subseteq \widetilde{F_3} = \widetilde{F}$$

defined by:

- $\widetilde{F_1} := \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times$;
- $\widetilde{F_2} := \widetilde{F} \cap C_{\mathbb{D}_n^\times}(F_0) = C_{\widetilde{F}}(F_0)$;
- $\widetilde{F_3} := \widetilde{F} \cap N_{\mathbb{D}_n^\times}(F_0) = N_{\widetilde{F}}(F_0) = \widetilde{F}$.

Clearly the groups $\widetilde{F_0}$, $\widetilde{F_1}$ are abelian, and since $\widetilde{F_2}/\widetilde{F_0} \cong F_2/F_0 \subseteq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ is cyclic by proposition 2.8 and remark 2.7, the group $\widetilde{F_2}$ is also abelian. Moreover we note that for $0 \leq i \leq 2$, each $\widetilde{F_i}$ is normal in $\widetilde{F_{i+1}}$. In particular, any finite subgroup $F \subseteq \mathbb{G}_n(u)$ determines successive group extensions with abelian kernel

$$1 \longrightarrow \widetilde{F_i} \longrightarrow \widetilde{F_{i+1}} \longrightarrow \widetilde{F_{i+1}}/\widetilde{F_i} \longrightarrow 1 \qquad \text{for} \quad 0 \leq i \leq 2.$$

In the following sections, we analyse these extensions recursively.

Remark 2.11. For F_0 the situation is completely understood from chapter 1 (see corollary 1.37), where we have shown that the conjugacy classes of

$$F_0 \cong C_{p^\alpha} \times C_d$$

are classified by the pairs of integers (α, d) satisfying

$$0 \leq \alpha \leq k \qquad \text{and} \qquad 1 \leq d \mid p^{n\alpha} - 1.$$

2.3. Existence and uniqueness in cohomological terms

The following general approach will be applied to the F_i's that can be understood through extensions with abelian kernel.

Let $\rho : G \to Q$ be a group homomorphism whose kernel $Ker(\rho)$ is not necessarily supposed to be abelian. Let A be an abelian normal subgroup of G which is contained in the center of $ker(\rho)$, and let B be a subgroup of $Im(\rho)$.

$$\mathcal{G}_\rho(G, A, B) := \{H \leq G \mid H \cap Ker(\rho) = A \text{ and } H/A \cong B \text{ via } \rho\}.$$

When $Ker(\rho)$ is abelian, we let $e_\rho \in H^2(Im(\rho), Ker(\rho))$ denote the cohomology class of the extension

$$1 \longrightarrow Ker(\rho) \longrightarrow G \longrightarrow Im(\rho) \longrightarrow 1,$$

and we define $e_\rho(B) \in H^2(B, Ker(\rho))$ to be the image of e_ρ under the map

$$j^* = H^2(j, Ker(\rho))$$

induced by the inclusion j of B into $Im(\rho)$.

Theorem 2.12. *If $Ker(\rho)$ is abelian, then the set $\mathcal{G}_\rho(G, A, B)$ is non-empty if and only if $e_\rho(B)$ becomes trivial in $H^2(B, Ker(\rho)/A)$.*

Proof. Let H be an element of $\mathcal{G}_\rho(G, A, B)$, and let $e_H \in H^2(B, A)$ be the extension class of

$$1 \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow 1.$$

Define H' to be the pushout of the diagram

$$Ker(\rho) \xleftarrow{\ i\ } A \longrightarrow H$$

given by the canonical inclusions of A into $Ker(\rho)$ and H respectively. Then H' fits into the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow & & \| & & \\ 1 & \longrightarrow & Ker(\rho) & \longrightarrow & H' & \longrightarrow & B & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact; the top extension class being e_H, and the bottom extension class being $i_B^*(e_H)$, the image of e_H via the map $i_B^* = H^2(B, i)$. Furthermore, define G' to be the pullback of the diagram

$$G \longrightarrow Im(\rho) \xleftarrow{j} B$$

given by the canonical inclusions of G and B into $Im(\rho)$. Then G' fits into the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Ker(\rho) & \longrightarrow & G' & \longrightarrow & B & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 1 & \longrightarrow & Ker(\rho) & \longrightarrow & G & \longrightarrow & Im(\rho) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact; the top extension class being $e_\rho(B)$, and the bottom extension class being e_ρ. From the universal properties of the pushout and the pullback, the above maps

$$H \longrightarrow B \qquad \text{and} \qquad Ker(\rho) \longrightarrow G$$

determine a homomorphism from H' to G' merging the above diagrams into

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow & & \| & & \\ 1 & \longrightarrow & Ker(\rho) & \longrightarrow & H' & \longrightarrow & B & \longrightarrow & 1 \\ & & \| & & \cong\downarrow & & \| & & \\ 1 & \longrightarrow & Ker(\rho) & \longrightarrow & G' & \longrightarrow & B & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 1 & \longrightarrow & Ker(\rho) & \longrightarrow & G & \longrightarrow & Im(\rho) & \longrightarrow & 1, \end{array}$$

so that $i_B^*(e_H) = e_\rho(B)$. Now we have a short exact sequence

$$1 \longrightarrow A \xrightarrow{i} Ker(\rho) \longrightarrow Ker(\rho)/A \longrightarrow 1,$$

which induces an exact sequence in cohomology

$$H^2(B, A) \xrightarrow{i_B^*} H^2(B, Ker(\rho)) \longrightarrow H^2(B, Ker(\rho)/A).$$

Since

$$e_\rho(B) \in H^2(B, Ker(\rho))$$

is in the image of i_B^*, it must become trivial in $H^2(B, Ker(\rho)/A)$.

Conversely, if $e_\rho(B)$ becomes trivial in $H^2(B, Ker(\rho)/A)$, then there is an element e_H in $H^2(B, A)$ satisfying $i_B^*(e_H) = e_\rho(B)$. This means that there is an extension

$$1 \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow 1,$$

and a connecting map from the pushout H' to the pullback G' which induces the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 1 & \longrightarrow & Ker(\rho) & \longrightarrow & G & \longrightarrow & Im(\rho) & \longrightarrow & 1, \end{array}$$

with $H \in \mathcal{G}_\rho(G, A, B)$. □

Remark 2.13. We may interpret theorem 2.12 by saying that $\mathcal{G}_\rho(G, A, B)$ is non-empty if and only if the associated extension

$$1 \longrightarrow Ker(\rho)/A \longrightarrow G/A \longrightarrow Im(\rho) \longrightarrow 1$$

splits when pulled back to $B \subseteq Im(\rho)$.

Denote by $\mathcal{G}_\rho(G, A, B)/\sim_{Ker(\rho)}$ the set of orbits with respect to the conjugation action of $Ker(\rho)$ on $\mathcal{G}_\rho(G, A, B)$. Given a distinguished element H_0 in $\mathcal{G}_\rho(G, A, B)$, we have an action of B on $Ker(\rho)/A$ induced by the conjugation action of G on $Ker(\rho)$. Indeed, since A is normal in G, this conjugation action determines a homomorphism $G \to Aut(Ker(\rho)/A)$, which in turn descends to a homomorphism $G/A \to Aut(Ker(\rho)/A)$ as A is in the center of $Ker(\rho)$. We thus obtain a canonical homomorphism

$$B \cong H_0/A \subseteq G/A \longrightarrow Aut(Ker(\rho)/A),$$

which allows us to consider $H^1(B, Ker(\rho)/A)$. The latter can be identified with the set of $Ker(\rho)/A$-conjugacy classes of sections of the split extension of remark 2.13, as explained in [4] chapter IV proposition 2.3 for the abelian case and [23] chapter I section 5.1 (see exercise 1) for the nonabelian case.

Theorem 2.14. *If $\mathcal{G}_\rho(G, A, B)$ is non-empty and H_0 is an element of $\mathcal{G}_\rho(G, A, B)$, then there exists a bijection*

$$\psi_{H_0} : H^1(B, Ker(\rho)/A) \longrightarrow \mathcal{G}_\rho(G, A, B)/\sim_{Ker(\rho)},$$

which depends on the choice of H_0.

Proof. Let $\mathcal{S}(B, \pi)$ denote the set of all sections $s : B \to G/A$ of the canonical projection $\pi : G/A \to G/Ker(\rho)$, that is

$$\mathcal{S}(B, \pi) := \{\text{group homomorphism } s : B \to G/A \mid (\pi \circ s)(b) = b \text{ for all } b \in B\}.$$

For any $s \in \mathcal{S}(B, \pi)$, we denote by $\widetilde{s} : B \to G$ the choice of a set theoretical lift of s. This defines maps

$$\begin{aligned} \mathcal{G}_\rho(G, A, B) &\longrightarrow \mathcal{S}(B, \pi) \\ H &\longmapsto s : B \cong H/A \to G/A, \end{aligned}$$

$$\begin{aligned} \mathcal{S}(B, \pi) &\longrightarrow \mathcal{G}_\rho(G, A, B) \\ s &\longmapsto \langle A, \widetilde{s}(B) \rangle, \end{aligned}$$

which can easily be checked to be mutually inverse to each other and compatible with the obvious actions of $Ker(\rho)$ by conjugation. The desired result then follows from the usual interpretation of $H^1(B, Ker(\rho)/A)$ as conjugacy classes of sections. □

2.4. The first extension type

In this section we consider the first extension in the chain of section 2.2. Recall that for a given subgroup $\widetilde{F} \in \widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times)$, we let $\widetilde{F_0}$ be such that $F_0 = \langle \widetilde{F} \cap S_n, Z_{p'}(\widetilde{F} \cap \mathbb{S}_n)\rangle$.

Lemma 2.15. *Let H_0 be an abelian finite subgroup of $\mathbb{S}_n$, $\widetilde{H_0} = H_0 \times \langle pu\rangle$, and let $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(H_0)^\times, \widetilde{H_0})$ be the set of all $\widetilde{F} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(H_0)^\times)$ such that*

- *$\widetilde{H_0} \subseteq \widetilde{F}$, and*
- *the valuation $v : \widetilde{F} \to \frac{1}{n}\mathbb{Z}$ induces a monomorphism $\widetilde{F}/\widetilde{H_0} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.*

If $\widetilde{F} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(H_0)^\times, \widetilde{H_0})$, then $\widetilde{F_0} = \widetilde{H_0}$.

Proof. By assumption on $\widetilde{F}$, we have in $\mathbb{G}_n(u)$

$$H_0 = Ker(v : F \to \mathbb{Z}/n) = F \cap \mathbb{S}_n = F_0,$$

and therefore $\widetilde{H_0} = \widetilde{F_0}$. □

We now fix F_0 and analyse the set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0})$ of all $\widetilde{F_1} \subseteq \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times)$ such that

- $\widetilde{F_1}$ contains $\widetilde{F_0}$ as a subgroups of finite index, and
- $\widetilde{F_1}/\widetilde{F_0}$ injects via v into $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Lemma 2.15 ensures that the elements of $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0})$ are extensions of the form

$$1 \longrightarrow \widetilde{F_0} \longrightarrow \widetilde{F_1} \longrightarrow \widetilde{F_1}/\widetilde{F_0} \longrightarrow 1$$

with $\langle \widetilde{F_1} \cap S_n, Z_{p'}(\widetilde{F_1} \cap \mathbb{S}_n)\rangle = F_0$. By definition, such an extension fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{F_0} & \longrightarrow & \widetilde{F_1} & \longrightarrow & \widetilde{F_1}/\widetilde{F_0} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 1 & \longrightarrow & \mathbb{Z}_p(F_0)^\times \times \langle pu\rangle & \longrightarrow & \mathbb{Q}_p(F_0)^\times & \longrightarrow & \frac{1}{e(\mathbb{Q}_p(F_0))}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 1, \end{array}$$

where $e(\mathbb{Q}_p(F_0))$ denotes the ramification index of $\mathbb{Q}_p(F_0)$ over $\mathbb{Q}_p$, where the horizontal maps form exact sequences and where the vertical maps are the canonical inclusions. As

$$\frac{1}{e(\mathbb{Q}_p(F_0))}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/e(\mathbb{Q}_p(F_0)),$$

the quotient group $\widetilde{F_1}/\widetilde{F_0}$ must be cyclic of order a divisor of $e(\mathbb{Q}_p(F_0))$. Let

$$e_u(F_0) \in H^2(\mathbb{Z}/e(\mathbb{Q}_p(F_0)),\ \mathbb{Z}_p(F_0)^\times \times \langle pu\rangle)$$

denote the class of this last extension. Furthermore, for r_1 a divisor of $e(\mathbb{Q}_p(F_0))$, we let

$$\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}, r_1) := \{\widetilde{F_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}) \mid |\widetilde{F_1}/\widetilde{F_0}| = r_1\},$$

and we define the cohomology class

$$e_u(F_0, r_1) \in H^2(\mathbb{Z}/r_1,\ \mathbb{Z}_p(F_0)^\times \times \langle pu\rangle)$$

to be the image of $e_u(F_0)$ under the induced homomorphism

$$j^* = H^2(j,\ \mathbb{Z}_p(F_0)^\times \times \langle pu\rangle),$$

for j the canonical inclusion of $\widetilde{F_1}/\widetilde{F_0}$ into $\frac{1}{e(\mathbb{Q}_p(F_0))}\mathbb{Z}/\mathbb{Z}$.

Theorem 2.16. *Let r_1 be a divisor of $e(\mathbb{Q}_p(F_0))$.*

1) *The set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}, r_1)$ is non-empty if and only if $e_u(F_0, r_1)$ becomes trivial in $H^2(\mathbb{Z}/r_1, \mathbb{Z}_p(F_0)^\times/F_0)$.*

2) *If $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}, r_1)$ is non-empty and if $\widetilde{F_1} = \langle \widetilde{F_0}, x_1\rangle$ belongs to this set with $v(x_1) = \frac{1}{r_1}$, then there is a bijection*

$$\begin{aligned} \psi_1 : H^1(\mathbb{Z}/r_1, \mathbb{Z}_p(F_0)^\times/F_0) &\longrightarrow \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}, r_1) \\ y &\longmapsto \langle \widetilde{F_0}, yx_1\rangle. \end{aligned}$$

Proof. Statements 1) and 2) are the respective specializations of theorem 2.12 and 2.14 in the case where

$$\rho : G = \mathbb{Q}_p(F_0)^\times \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z} = Q$$

is induced by the valuation, G (and hence $Ker(\rho)$) acts trivially on $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}, r_1)$, and

$$A = \widetilde{F_0} = F_0 \times \langle pu\rangle, \qquad B = \frac{1}{r_1}\mathbb{Z}/\mathbb{Z};$$

in particular,

$$Ker(\rho) = \mathbb{Z}_p(F_0)^\times \times \langle pu\rangle, \qquad Im(\rho) = \frac{1}{e(\mathbb{Q}_p(F_0))}\mathbb{Z}/\mathbb{Z},$$

and

$$e_\rho = e_u(F_0), \qquad e_\rho(B) = e_u(F_0, r_1).$$

□

Remark 2.17. Note that $\widetilde{F_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times)$ belongs to $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}, r_1)$ if and only if there exists an element $x_1 \in \mathbb{Q}_p(F_0)^\times$ with $\widetilde{F_1} = \langle \widetilde{F_0}, x_1\rangle$ satisfying

$$v(x_1) = \frac{1}{r_1} \qquad \text{and} \qquad x_1^{r_1} \in \widetilde{F_0}.$$

Clearly, $\widetilde{F_1}$ uniquely determines x_1 modulo F_0.

Corollary 2.18. *If $F_0 = \mu(\mathbb{Q}_p(F_0))$ is the group of roots of unity in $\mathbb{Q}_p(F_0)$, then*

$$|\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}, r_1)| \leq 1.$$

Proof. By proposition C.7 we have $\mathbb{Z}_p(F_0)^\times \cong \mu(\mathbb{Q}_p(F_0)) \times \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}$. Since the action of $\mathbb{Z}/r_1$ is trivial on $\mathbb{Z}_p(F_0)^\times$, we obtain

$$\begin{aligned} H^1(\mathbb{Z}/r_1,\ \mathbb{Z}_p(F_0)^\times/F_0) &\cong H^1(\mathbb{Z}/r_1,\ \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]} \times \mu(\mathbb{Q}_p(F_0))/F_0) \\ &\cong H^1(\mathbb{Z}/r_1,\ \mu(\mathbb{Q}_p(F_0))/F_0) \\ &\cong \{1\}. \end{aligned}$$

The result then follows from theorem 2.16.2. □

Remark 2.19. The condition $F_0 = \mu(\mathbb{Q}_p(F_0))$ is equivalent to the maximality of F_0 as a finite subgroup of $\mathbb{Q}_p(F_0)^\times$. In section 3.3 we will see that if p is odd and $F_0 = \mu(\mathbb{Q}_p(F_0))$, then the set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if p does not divide r_1. As for the case $p = 2$, we will see in section 3.4 that this depends on u and F_0.

2.5. The second extension type

In this section we consider the second extension in the chain of section 2.2. Recall that for a given subgroup $\widetilde{F} \in \widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times)$, we let $\widetilde{F_1} = \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times$.

Lemma 2.20. *Let H_0 be an abelian finite subgroup of $\mathbb{S}_n$, $\widetilde{H_0} = H_0 \times \langle pu \rangle$, $\widetilde{H_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(H_0)^\times, \widetilde{H_0})$, and let $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(H_0), \widetilde{H_1})$ be the set of all $\widetilde{F} \in \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(H_0))$ such that*

- *$\widetilde{H_1} = \widetilde{F} \cap \mathbb{Q}_p(H_0)^\times$, and*
- *the valuation $v : \widetilde{F} \to \frac{1}{n}\mathbb{Z}$ induces a monomorphism $\widetilde{F}/\widetilde{H_1} \to \frac{1}{n}\mathbb{Z}/v(\widetilde{H_1})$.*

If $\widetilde{F} \in \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(H_0), \widetilde{H_1})$, then $\widetilde{F_1} = \widetilde{H_1}$.

Proof. Clearly $\widetilde{F}/\widetilde{H_1}$ injects via v into $\frac{1}{n}\mathbb{Z}/v(\widetilde{H_1})$ if and only if we have in $\mathbb{G}_n(u)$ a monomorphism $F/H_0 \to \mathbb{Z}/n$, and this is true if and only if $H_0 = F \cap \mathbb{S}_n$. Therefore $F_0 = H_0$ and consequently

$$\widetilde{F_1} = \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times = \widetilde{F} \cap \mathbb{Q}_p(H_0)^\times = \widetilde{H_1}.$$

□

We now fix F_0 and r_1 such that $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty, and fix a group $\widetilde{F_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$. We consider the set $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1})$ of all $\widetilde{F_2} \subseteq \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0))$ such that

- $\widetilde{F_2} \cap \mathbb{Q}_p(F_0)^\times = \widetilde{F_1}$, and
- $\widetilde{F_2}/\widetilde{F_1}$ injects via v into $\frac{1}{n}\mathbb{Z}/v(\widetilde{F_1})$.

Lemma 2.20 ensures that the elements of $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1})$ are extensions of the form

$$1 \longrightarrow \widetilde{F_1} \longrightarrow \widetilde{F_2} \longrightarrow \widetilde{F_2}/\widetilde{F_1} \longrightarrow 1$$

with $\widetilde{F_2} \cap \mathbb{Q}_p(F_0)^\times = \widetilde{F_1}$ and $\langle \widetilde{F_2} \cap S_n, Z_{p'}(\widetilde{F_2} \cap \mathbb{S}_n) \rangle = F_0$. For r_2 a divisor of n/r_1, we define

$$\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2) := \{\widetilde{F_2} \in \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}) \mid |\widetilde{F_2}/\widetilde{F_1}| = r_2\}.$$

Note that any $\widetilde{F_2} \in \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$ determines a commutative field extension $L = \mathbb{Q}_p(\widetilde{F_2})$ of degree r_2 over $\mathbb{Q}_p(F_0)$ which is obtained by adjoining to $\mathbb{Q}_p(F_0)$ an element $x_2 \in C_{\mathbb{D}_n^\times}(F_0)$ which satisfies

$$v(x_2) = \frac{1}{r_1 r_2} \qquad \text{and} \qquad x_2^{r_2} \in \widetilde{F_1}.$$

We can thus partition our sets

$$\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2) = \coprod_{\substack{L \supseteq \mathbb{Q}_p(F_0) \\ [L:\mathbb{Q}_p(F_0)]=r_2}} \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L),$$

according to all $L \supseteq \mathbb{Q}_p(F_0)$ obtained from $\mathbb{Q}_p(F_0)$ via irreducible equations of the form $X^{r_2} - x_1$ for x_1 an element of valuation $\frac{1}{r_1}$ in $\widetilde{F_1}$, where

$$\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L) := \{\widetilde{F_2} \in \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}) \mid \mathbb{Q}_p(\widetilde{F_2}) = L\}.$$

Clearly, L determines r_2 and we have

$$\begin{aligned} \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}) &= \coprod_{r_2 | \frac{n}{r_1}} \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2) \\ &= \coprod_{[L:\mathbb{Q}_p(F_0)] | \frac{n}{r_1}} \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L). \end{aligned}$$

Theorem 2.21. *Let x_1 be an element of $\widetilde{F_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ with $v(x_1) = \frac{1}{r_1}$, let L be an extension of $\mathbb{Q}_p(F_0)$ of degree r_2, and let $L_{r_1}^\times$ denote the group of all $x \in L^\times$ such that $v(x) \in \frac{1}{r_1}\mathbb{Z}$.*

1) *The set $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L)$ is non-empty if and only if $r_2[\mathbb{Q}_p(F_0) : \mathbb{Q}_p]$ divides n, there exists a $\delta \in F_0$ such that the equation $X^{r_2} - \delta x_1$ is irreducible over $\mathbb{Q}_p(F_0)$ and $L = \mathbb{Q}_p(x_2)$ for x_2 a root of this equation.*

2) *If $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L)$ is non-empty and if $\widetilde{F_2} = \langle \widetilde{F_1}, x_2 \rangle$ belongs to this set with $v(x_2) = \frac{1}{r_1 r_2}$, then there is a bijection*

$$\begin{aligned} \psi_2 : H^1(\mathbb{Z}/r_2, L_{r_1}^\times/\widetilde{F_1}) &\longrightarrow \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L) \\ y &\longmapsto \langle \widetilde{F_1}, yx_2 \rangle. \end{aligned}$$

Proof. 1) This is a direct consequence of the embedding theorem.

2) This is a specialization of theorem 2.14 in the case where

$$\rho : G = L^\times \longrightarrow \frac{1}{n}\mathbb{Z}/\frac{1}{r_1}\mathbb{Z} = Q$$

is induced by the valuation, G (and hence $Ker(\rho)$) acts trivially on $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L)$, and

$$A = \widetilde{F_1}, \qquad B = \frac{1}{r_1 r_2}\mathbb{Z}/\frac{1}{r_1}\mathbb{Z};$$

in particular, $Ker(\rho) = L_{r_1}^\times$. □

Remark 2.22. Note that

$$\widetilde{F_2} \in \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1})$$

satisfies $\mathbb{Q}_p(\widetilde{F_2}) = L$ if and only if there exists an element $x_2 \in L$ with $\widetilde{F_2} = \langle \widetilde{F_1}, x_2 \rangle$ satisfying

$$v(x_2) = \frac{1}{r_1 r_2} \qquad \text{and} \qquad x_2^{r_2} \in \widetilde{F_1}.$$

Moreover, $\widetilde{F_2}$ uniquely determines such an x_2 modulo F_0.

Corollary 2.23. *If $F_0 = \mu(L)$ is the group of roots of unity in L, then*

$$|\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L)| \leq 1.$$

Proof. We know from proposition C.7 that $L_{r_1}^\times \cong \mathbb{Z}\langle x_1 \rangle \times \mu(L) \times \mathbb{Z}_p^{[L:\mathbb{Q}_p]}$. Since the action of $\mathbb{Z}/r_2$ is trivial on $L_{r_1}^\times$, we obtain

$$\begin{aligned} H^1(\mathbb{Z}/r_2,\ L_{r_1}^\times/\widetilde{F_1}) &\cong H^1(\mathbb{Z}/r_2,\ \mathbb{Z}_p^{[L:\mathbb{Q}_p]} \times \mu(L)/F_0) \\ &\cong H^1(\mathbb{Z}/r_2,\ \mu(L)/F_0) \\ &\cong \{1\}. \end{aligned}$$

The result then follows from theorem 2.21.2. □

2.6. The third extension type

In this section we consider the third extension in the chain of section 2.2. Recall that for a given subgroup $\widetilde{F} \in \widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times)$, we let $\widetilde{F_2} = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(F_0)$.

Lemma 2.24. *Let H_0 be an abelian finite subgroup of $\mathbb{S}_n$, $\widetilde{H_0} = H_0 \times \langle pu \rangle$, $\widetilde{H_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(H_0)^\times, \widetilde{H_0})$, $\widetilde{H_2} \in \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(H_0), \widetilde{H_1})$, and let $\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(H_0), \widetilde{H_2})$ be the set of all $\widetilde{F} \in \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(H_0))$ such that*

- *$\widetilde{H_2} = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(H_0)$, and*
- *$\widetilde{H_2}$ is normal in $\widetilde{F}$.*

If $\widetilde{F} \in \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(H_0), \widetilde{H_2})$, then

a) *$\widetilde{F_i} = \widetilde{H_i}$ for $0 \leq i \leq 2$, or*

b) *$p = 2$, $n \equiv 2 \bmod 4$, $H_0 \cap S_n \cong C_4$, $\widetilde{F} \cap S_n \cong Q_8$ and $Z_{p'}(\widetilde{F} \cap \mathbb{S}_n) \cong Z_{p'}(\widetilde{H_0} \cap \mathbb{S}_n)$.*

Proof. First note that the condition $\widetilde{H_2} = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(H_0)$ implies that the canonical homomorphism $\widetilde{F} \to Aut(H_0)$ induces an injective homomorphism $\widetilde{F}/\widetilde{H_2} \to Aut(H_0)$. In particular, we have a monomorphism $F \cap \mathbb{S}_n/H_0 \to Aut(H_0)$ where

$$(H_0 \cap S_n) \times Z_{p'}(H_0) = H_0 \subseteq F \cap \mathbb{S}_n,$$

for $Z_{p'}$ the p'-part of (the center of) H_0. By theorem 1.31 and 1.35 we know that $F \cap \mathbb{S}_n$ acts trivially on $Z_{p'}(H_0)$, so that we have a monomorphism

$$F \cap \mathbb{S}_n/H_0 \longrightarrow Aut(H_0 \cap S_n). \qquad (*)$$

Assume for the moment that $F \cap S_n$ is abelian; this must be the case if $p > 2$, or if $p = 2$ with either $n \not\equiv 2 \bmod 4$ or $H_0 \cap S_n \not\cong C_4$. Then $F \cap \mathbb{S}_n = (F \cap S_n) \times Z_{p'}(F \cap \mathbb{S}_n)$ is cyclic. Since $(F \cap S_n)/(H_0 \cap S_n)$ injects into the kernel of the injective map $(*)$, we have $F \cap S_n = H_0 \cap S_n$. Furthermore, the p'-part of $Aut(H_0 \cap S_n)$ is a cyclic group of order $p-1$, and the quotient group $Z_{p'}(F \cap \mathbb{S}_n)/Z_{p'}(H_0)$ injects into $C_{p-1} \subseteq Aut(H_0 \cap S_n)$. Hence $Z_{p'}(F \cap \mathbb{S}_n) = Z_{p'}(H_0)$ by theorem 1.31 and 1.35, so that $F_0 = H_0$ and $\widetilde{F_0} = \widetilde{H_0}$. Therefore

$$\widetilde{F_2} = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(F_0) = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(H_0) = \widetilde{H_2},$$

and

$$\widetilde{F_1} = \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times = \widetilde{F} \cap \mathbb{Q}_p(H_0)^\times = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(H_0) \cap \mathbb{Q}_p(H_0)^\times = \widetilde{H_2} \cap \mathbb{Q}_p(H_0)^\times = \widetilde{H_1}.$$

Finally, if $F \cap S_n$ is not abelian, then $p = 2$, $n \equiv 2 \bmod 4$, $H_0 \cap S_n \cong C_4$ and $F \cap S_n \cong Q_8$ by theorem 1.35. As seen above, the quotient group of the $2'$-part of $F \cap \mathbb{S}_n$ by the $2'$-part of H_0 injects into the trivial group. □

We now fix a chain $F_0 \subseteq F_1 \subseteq F_2$ such that condition b) of lemma 2.24 is not satisfied, and we let L be a subfield of $\mathbb{D}_n$ such that $\widetilde{F_2}$ belongs to $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L)$; in particular $L = \mathbb{Q}_p(\widetilde{F_2})$. We consider the set $\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2})$ of all $\widetilde{F_3} \subseteq \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0))$ such that

- $\widetilde{F_3} \cap C_{\mathbb{D}_n^\times}(F_0) = \widetilde{F_2}$, and
- $\widetilde{F_2}$ is normal in $\widetilde{F_3}$.

Proposition 2.25. *If $\widetilde{F_3} \in \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2})$, there is a commutative diagram of obvious group homomorphisms*

$$\begin{array}{ccccc} \widetilde{F_3}/\widetilde{F_2} & \longrightarrow & Aut(\mathbb{Q}_p(\widetilde{F_2}), \mathbb{Q}_p(F_0)) & \longrightarrow & Aut(\mathbb{Q}_p(F_0)) \\ \| & & & & \downarrow \\ \widetilde{F_3}/\widetilde{F_2} & \longrightarrow & Aut(\widetilde{F_2}, F_0) & \longrightarrow & Aut(F_0) \end{array}$$

in which all compositions starting at $\widetilde{F_3}/\widetilde{F_2}$ are injective.

Proof. Clearly, the condition $\widetilde{F_2} = \widetilde{F_3} \cap C_{\mathbb{D}_n^\times}(F_0)$ is equivalent to the fact that the canonical homomorphism $\widetilde{F_3} \to Aut(F_0)$ induces an injective homomorphism $\widetilde{F_3}/\widetilde{F_2} \to Aut(F_0)$. Furthermore, an automorphism of the field $\mathbb{Q}_p(F_0)$ induces an automorphism of the group $\mu(\mathbb{Q}_p(F_0))$ of roots of unity in $\mathbb{Q}_p(F_0)$, and since this group is cyclic and contains F_0, it also induces an automorphism of F_0. This determines an injective homomorphism $Aut(\mathbb{Q}_p(F_0)) \to Aut(F_0)$. The homomorphism $\widetilde{F_3}/\widetilde{F_2} \to Aut(F_0)$ clearly takes its values into the subgroup $Aut(\mathbb{Q}_p(F_0))$.

The condition that $\widetilde{F_2}$ is normal in $\widetilde{F_3}$ yields canonical homomorphisms $\widetilde{F_3} \to Aut(\widetilde{F_2})$ and $\widetilde{F_3} \to Aut(\mathbb{Q}_p(\widetilde{F_2}))$. Since $\widetilde{F_2}$ is abelian, these induce canonical homomorphisms $\widetilde{F_3}/\widetilde{F_2} \to Aut(\widetilde{F_2})$ and $\widetilde{F_3}/\widetilde{F_2} \to Aut(\mathbb{Q}_p(\widetilde{F_2}))$. Moreover, as $\widetilde{F_3}/\widetilde{F_2} \to Aut(\widetilde{F_2})$ takes its values into the subgroup $Aut(\widetilde{F_2}, F_0)$ of those automorphisms of $\widetilde{F_2}$ which leave F_0 invariant, and as $\widetilde{F_3}/\widetilde{F_2} \to Aut(\mathbb{Q}_p(\widetilde{F_2}))$ takes its values into the subgroup $Aut(\mathbb{Q}_p(\widetilde{F_2}), \mathbb{Q}_p(F_0))$ of those automorphisms which leave $\mathbb{Q}_p(F_0)$ invariant, we end up with the given commutative diagram.

From the injectivity of the map $\widetilde{F_3}/\widetilde{F_2} \to Aut(F_0)$, we then obtain that all compositions of homomorphism in the diagram starting at $\widetilde{F_3}/\widetilde{F_2}$ are injective. □

Let $Aut(L, \widetilde{F_2}, F_0)$ denote the subgroup of all elements of $Aut(L)$ which leave both $\widetilde{F_2}$ and F_0 invariant. By proposition 2.25, we may partition the set

$$\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}) = \coprod_W \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$$

according to all subgroups W of $Aut(F_0)$ which lift to $Aut(L, \widetilde{F_2}, F_0)$, where

$$\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W) := \{\widetilde{F_3} \in \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}) \mid \widetilde{F_3}/\widetilde{F_2} = W\}.$$

Let us fix such a W. Under our assumptions, lemma 2.24 and proposition 2.25 ensure that the elements of $\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$ are extensions of the form

$$1 \longrightarrow \widetilde{F_2} \longrightarrow \widetilde{F_3} \longrightarrow W \longrightarrow 1$$

with $\widetilde{F_3} \cap C_{\mathbb{D}_n^\times}(F_0) = \widetilde{F_2}$, $\widetilde{F_3} \cap \mathbb{Q}_p(F_0)^\times = \widetilde{F_1}$ and $\langle \widetilde{F_3} \cap S_n, Z_{p'}(\widetilde{F_3} \cap \mathbb{S}_n)\rangle = F_0$. Define

$$K := L^W \subseteq L = \mathbb{Q}_p(\widetilde{F_2})$$

to be the subfield of all elements of L that are fixed by the action of W. Clearly, K is an extension of $\mathbb{Q}_p$ and the respective dimensions of K and L over $\mathbb{Q}_p$ divide n. Recall from section B.2 that an element $e \in H^2(W, L^\times)$ defines a central simple crossed K-algebra $(L/K, e)$ up to isomorphism.

Lemma 2.26. *There is a generator of $H^2(W, L^\times)$ whose associated crossed algebra embeds into $\mathbb{D}_n$ if and only if $|W|$ is prime to $n[L : \mathbb{Q}_p]^{-1}$.*

Proof. Consider the tower of extensions $\mathbb{Q}_p \subseteq K := L^W \subseteq L$. Let $k := [K : \mathbb{Q}_p]$, $l := [L : \mathbb{Q}_p]$, $w := |W|$, and let e be a generator of $H^2(W, L) \cong \mathbb{Z}/w \subseteq \mathbb{Q}/\mathbb{Z}$. By proposition B.3, we know that the crossed algebra $(L/K, e) = \sum_{\sigma \in W} Lu_\sigma$ is a central division algebra over K of invariant $r/w \in Br(K)$ for some integer r prime to w. If $q = n/l$, then the invariant of $D := C_{\mathbb{D}_n}(K)$ is $1/qw \in Br(K)$ by proposition C.10.

Suppose $(L/K, e)$ can be embedded into $\mathbb{D}_n$. Then it embeds into D, and by the centralizer theorem

$$D \cong (L/K, e) \otimes_K C_D(L/K, e),$$

where $C_D(L/K, e)$ is central of dimension q^2 over K. On the level of Hasse invariants we get a relation of the form

$$\frac{1}{qw} \equiv \frac{r}{w} + \frac{s}{q} \mod \mathbb{Z} \qquad (*)$$

for a suitable integer s which is prime to q. Hence

$$1 \equiv rq + sw \mod q\mathbb{Z},$$

and it follows that q is prime to w. Conversely if q is prime to w, then there is an r prime to w and an s prime to q such that $(*)$ holds. Therefore the algebra $(L/K, e)$ embeds into D, and consequently into $\mathbb{D}_n$. □

Theorem 2.27. *Let W be a subgroup of $Aut(F_0)$ which lifts to $Aut(L, \widetilde{F_2}, F_0)$ and let*

$$i_W^* : H^2(W, \widetilde{F_2}) \longrightarrow H^2(W, L^\times)$$

be the map induced by the inclusion of $\widetilde{F_2}$ into $L^\times$. Then $\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$ is non-empty if and only if $|W|$ is prime to $n[L : \mathbb{Q}_p]^{-1}$ and i_W^* is surjective.

Proof. Suppose that $|W|$ is prime to $n[L:\mathbb{Q}_p]^{-1}$ and i_W^* is surjective. By lemma 2.26, there is a generator $e \in H^2(W, L^\times)$ whose associated algebra $(L/L^W, e)$ embeds into $\mathbb{D}_n$. The group of units $(L/L^W, e)^\times$ contains

$$L^\times W = \coprod_{\sigma \in W} L^\times u_\sigma$$

as a subgroup, and we get an embedding of $L^\times W$ into $\mathbb{D}_n^\times$. By the Skolem-Noether theorem we can assume that this embedding restricts to the given embedding of L into $\mathbb{D}_n$. Since $L = \mathbb{Q}_p(\widetilde{F_2})$, we have $L^\times W \subseteq N_{\mathbb{D}_n^\times}(\widetilde{F_2})$ and there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & L^\times & \longrightarrow & L^\times W & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & N_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & Aut(L) & \longrightarrow & 1,
\end{array}$$

whose vertical maps are inclusions and whose horizontal sequences are exact. Now, the surjectivity of i_W^* implies the existence of an element $e' \in H^2(W, \widetilde{F_2})$ such that $i_W^*(e') = e$, in which case the above diagram extends to a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{F_2} & \longrightarrow & \widetilde{F} & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & L^\times & \longrightarrow & L^\times W & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & N_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & Aut(L) & \longrightarrow & 1,
\end{array}$$

where the top exact sequence has extension class e'. Because of our assumption that W injects into $Aut(F_0)$, we have $\widetilde{F} \cap C_{\mathbb{D}_n^\times}(F_0) = \widetilde{F_2}$, and therefore $\widetilde{F} \in \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$.

Conversely, if $\widetilde{F} \in \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$, then $\widetilde{F}$ extends $\widetilde{F_2}$ by W and there are commutative diagrams

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{F_2} & \longrightarrow & \widetilde{F} & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & N_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & Aut(L) & \longrightarrow & 1,
\end{array}$$

and

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{F_2} & \longrightarrow & \widetilde{F} & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & L^\times & \longrightarrow & L^\times W & \longrightarrow & W & \longrightarrow & 1,
\end{array}$$

whose vertical maps are inclusions and horizontal sequences are exact. Using the universal property of the lower left pushout square, we may extend the latter diagram in an obvious

way to obtain an embedding of extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{F_2} & \longrightarrow & \widetilde{F} & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & L^\times & \longrightarrow & L^\times W & \longrightarrow & W & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow i & & \downarrow & & \\
1 & \longrightarrow & C_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & N_{\mathbb{D}_n^\times}(\widetilde{F_2}) & \longrightarrow & Aut(L) & \longrightarrow & 1,
\end{array}$$

where $L^\times W \subseteq (L/L^W, e) = \sum_{\sigma\in W} Lu_\sigma$ for e the image of the extension class of $\widetilde{F}$ in $H^2(W, L^\times)$. By definition of $L^\times W$, the map i extends uniquely to an algebra homomorphism

$$\widetilde{i} : (L/L^W, e) \longrightarrow \mathbb{D}_n : \sum_\sigma x_\sigma u_\sigma \longmapsto \sum_\sigma i(x_\sigma u_\sigma), \qquad x_\sigma \in L^\times.$$

Moreover since $(L/L^W, e)$ is simple and i is non-trivial, the kernel of $\widetilde{i}$ is trivial. Hence $\widetilde{i}$ is injective and $(L/L^W, e)$, which embeds into $\mathbb{D}_n$, is a division algebra by proposition A.3. It follows that e is a generator of $H^2(W, L^\times)$ and i_W^* is surjective. Applying lemma 2.26 we finally obtain that $|W|$ is prime to $n[L : \mathbb{Q}_p]^{-1}$. □

Theorem 2.28. *Let W be a subgroup of $Aut(F_0)$ which lifts to $Aut(L, \widetilde{F_2}, F_0)$. If the set $\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$ is non-empty and contains $\widetilde{F_3}$, then there is a bijection*

$$\psi_3 : H^1(W, C_{\mathbb{D}_n^\times}(\widetilde{F_2})/\widetilde{F_2}) \longrightarrow \widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_2})}$$
$$c \longmapsto \langle \widetilde{F_2}, cs_{\widetilde{F_3}} \rangle,$$

for c a cocycle and $s_{\widetilde{F_3}} : W \to \widetilde{F_3}$ a set theoretic section of the epimorphism $\widetilde{F_3} \to W$.

Proof. By proposition 2.25, we know that W lifts to an automorphism of $\widetilde{F_2}$. The result is then a specialization of theorem 2.14 in the case where

$$\rho : G = N_{\mathbb{D}_n^\times}(\widetilde{F_2}) \longrightarrow Aut(\widetilde{F_2}) = Q$$

is given by the canonical homomorphism induced by conjugation and

$$A = \widetilde{F_2}, \qquad B = W;$$

in particular, $Ker(\rho) = C_{\mathbb{D}_n^\times}(\widetilde{F_2})$. □

Corollary 2.29. *If $\mathbb{Q}_p(F_0)$ is a maximal subfield of $\mathbb{D}_n$ such that $\mu(\mathbb{Q}_p(F_0)) = F_0$, and if $i_W^* : H^2(W, \widetilde{F_2}) \to H^2(W, L^\times)$ is an epimorphism for W a subgroup of $Aut(F_0)$ which lifts to $Aut(L, \widetilde{F_2}, F_0)$, then there is a bijection between the conjugacy classes of elements of $\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$ and the kernel of i_W^*.*

Proof. Under the stated assumptions, we have $\widetilde{F_2} = \widetilde{F_1}$, $L = \mathbb{Q}_p(F_0)$, $[L : \mathbb{Q}_p] = n$ and $C_{\mathbb{D}_n^\times}(\widetilde{F_2}) = L^\times$. Hence there is a short exact sequence

$$1 \longrightarrow \widetilde{F_2} \longrightarrow L^\times \longrightarrow L^\times/\widetilde{F_2} \longrightarrow 1,$$

which for $W \subseteq Aut(L, \widetilde{F_2}) \subseteq Gal(L/\mathbb{Q}_p)$ induces the long exact sequence

$$\begin{array}{ccccc} \cdots \longrightarrow H^1(W, L^\times) & \longrightarrow & H^1(W, L^\times/\widetilde{F_2}) & & Br(L/L^W) \\ \| & & \downarrow & & \| \\ 0 & & H^2(W, \widetilde{F_2}) & \xrightarrow{i_W^*} & H^2(W, L^\times) \longrightarrow \cdots, \end{array}$$

where the left hand term is trivial by Hilbert's theorem 90. The group $H^1(W, L^\times/\widetilde{F_2})$ is therefore the kernel of i_W^* and the result follows from theorem 2.28. □

Theorem 2.30. *Let $\widetilde{H} \in \widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times)$ be such that $H \cap S_n$ is abelian and $\mu(\mathbb{Q}_p(H_0)) = \mu(\mathbb{Q}_p(\widetilde{H_2}))$. Then there is a subgroup $\widetilde{F} \in \widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times)$ such that*

$$F_0 = \mu(\mathbb{Q}_p(H_0)), \qquad \mathbb{Q}_p(\widetilde{F_2}) = \mathbb{Q}_p(\widetilde{H_2}) \qquad \text{and} \qquad \widetilde{H_i} \subseteq \widetilde{F_i} \ \text{ for } 0 \le i \le 3.$$

Proof. We know that $\widetilde{H_1} = \langle \widetilde{H_0}, x_1 \rangle$ where x_1 commutes with $\widetilde{H_0}$, $v(x_1) = \frac{1}{r_1}$ and $x_1^{r_1} \in \widetilde{H_0}$, and furthermore that $\widetilde{H_2} = \langle \widetilde{H_1}, x_2 \rangle$ where x_2 commutes with $\widetilde{H_1}$, $v(x_2) = \frac{1}{r_1 r_2}$ and $x^{r_2} \in \widetilde{H_1}$. Defining $F_0 = \mu(\mathbb{Q}_p(H_0))$, $\widetilde{F_0} = \langle F_0, pu \rangle$, $\widetilde{F_1} = \langle \widetilde{F_0}, x_1 \rangle$ and $\widetilde{F_2} = \langle \widetilde{F_1}, x_2 \rangle$, we have

$$F_0 = \widetilde{F_2} \cap \mathbb{S}_n, \qquad \widetilde{F_1} = \widetilde{F_2} \cap \mathbb{Q}_p(F_0) \qquad \text{and} \qquad \widetilde{F_2} \subseteq C_{\mathbb{D}_n^\times}(F_0).$$

It remains to show that the extension

$$1 \longrightarrow \widetilde{H_2} \longrightarrow \widetilde{H_3} = \widetilde{H} \longrightarrow W \longrightarrow 1 \qquad (*)$$

can be extended to an extension in $\mathbb{D}_n^\times$

$$1 \longrightarrow \widetilde{F_2} \longrightarrow \widetilde{F_3} = \widetilde{F} \longrightarrow W \longrightarrow 1. \qquad (**)$$

Let $L := \mathbb{Q}_p(\widetilde{H_2}) = \mathbb{Q}_p(\widetilde{F_2})$. The existence of $(*)$ implies that $W \subseteq Aut(H_0) \subseteq Aut(F_0)$ lifts to $Aut(L, \widetilde{H_2}, H_0)$. An automorphism σ of L which leaves H_0 invariant, also leaves $\mathbb{Q}_p(H_0)$ invariant, and therefore the subgroups $F_0 = \mu(\mathbb{Q}_p(H_0))$ and $\widetilde{F_0} = \langle F_0, pu \rangle$ are also left invariant. Hence

$$\sigma(x_1)^{r_1} \in \widetilde{F_0} = \langle F_0, pu \rangle \qquad \text{and} \qquad \left(\frac{\sigma(x_1)}{x_1}\right)^{r_1} \in F_0.$$

Since F_0 is the (unique) maximal finite subgroup of $\mathbb{Q}_p(F_0)^\times$, we have

$$\frac{\sigma(x_1)}{x_1} \in F_0 \qquad \text{and} \qquad \sigma(x_1) \in \langle F_0, x_1 \rangle = \widetilde{F_1},$$

and therefore σ leaves $\widetilde{F_1}$ invariant. This implies

$$\sigma(x_2)^{r_2} \in \widetilde{F_1} \qquad \text{and} \qquad \left(\frac{\sigma(x_2)}{x_2}\right)^{r_2} \in \widetilde{F_1} \cap \mathbb{S}_n = F_0.$$

Using that $\mu(\mathbb{Q}_p(F_0)) = \mu(\mathbb{Q}_p(\widetilde{F_2}))$, we obtain as before that $\sigma(x_2) \in \langle F_0, x_2 \rangle = \widetilde{F_2}$ and consequently that σ leaves $\widetilde{F_2}$ invariant. It follows that W lifts to $Aut(L, \widetilde{F_2}, F_0)$. The chain of inclusions $\widetilde{H_2} \subseteq \widetilde{F_2} \subseteq L^\times$ induces a commutative diagram

$$\begin{array}{ccc} H^2(W, \widetilde{H_2}) & & \\ \downarrow & \searrow & \\ H^2(W, \widetilde{F_2}) & \longrightarrow & H^2(W, L^\times) \end{array}$$

whose oblique arrow is an epimorphism by theorem 2.27. The horizontal homomorphism is therefore surjective and theorem 2.27 implies the existence of $(**)$ in $\mathbb{D}_n^\times$. □

2.7. Classification of embeddings up to conjugation

In this section, we use the results obtained in this chapter to classify the chains of subgroups

$$\widetilde{F_0} \subseteq \widetilde{F_1} \subseteq \widetilde{F_2} \subseteq \widetilde{F_3}$$

that occur in $\mathbb{D}_n^\times$. In order to do this we proceed in four steps.

For a group G and a G-set $\mathcal{S}$, we denote by $\mathcal{S}/\sim_G$ the set of orbits with respect to the G-action on $\mathcal{S}$.

Classifying $\widetilde{F_0}$'s

As explained in remark 2.11, the map

$$\widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times) \longrightarrow \widetilde{\mathcal{F}}_u(\mathbb{S}_n) \ : \ \widetilde{F} \longmapsto \widetilde{F_0} := \langle \widetilde{F} \cap S_n, Z_{p'}(G) \rangle$$

induces a well defined map

$$\varphi_0 : \ \widetilde{\mathcal{F}}_u(\mathbb{D}_n^\times)/\sim_{\mathbb{D}_n^\times} \longrightarrow \widetilde{\mathcal{F}}_u(\mathbb{S}_n)/\sim_{\mathbb{D}_n^\times},$$

whose image can be identified with the set

$$\{(\alpha, d) \in \mathbb{N} \times \mathbb{N}^* \mid 0 \leq \alpha \leq k, \ d \mid p^{n_\alpha} - 1\}.$$

Classifying $\widetilde{F_1}$'s

Pick $\widetilde{F_0} \in \widetilde{\mathcal{F}}_u(\mathbb{S}_n)$ and define the sets

- $\widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_0})$ of all subgroups $\widetilde{F}$ of $\mathbb{D}_n^\times$ such that $\widetilde{F_0} = \widetilde{F} \cap \mathbb{S}_n$ is of finite index in $\widetilde{F}$;
- $\widetilde{\mathcal{F}}(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0})$ of all subgroups $\widetilde{F}$ of $\mathbb{Q}_p(F_0)^\times$ such that $\widetilde{F_0} = \widetilde{F} \cap \mathbb{S}_n$ is of finite index in $\widetilde{F}$.

Clearly the map

$$\widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_0}) \longrightarrow \widetilde{\mathcal{F}}(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}) \ : \ \widetilde{F} \longmapsto \widetilde{F_1} := \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times$$

induces a well defined map

$$\varphi_1 : \ \widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_0})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_0})} \longrightarrow \widetilde{\mathcal{F}}(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}).$$

As seen in section 2.4, every $\widetilde{F_1} \in \widetilde{\mathcal{F}}(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0})$ determines an integer $r_1 = |\widetilde{F_1}/\widetilde{F_0}|$ which is a divisor of n. Furthermore, according to theorem 2.16, if such a divisor r_1 is realized by a subgroup $\widetilde{F_1} \in \widetilde{\mathcal{F}}(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0})$, then the set

$$\{\widetilde{F_1} \in \widetilde{\mathcal{F}}(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0}) \mid |\widetilde{F_1}/\widetilde{F_0}| = r_1\}$$

is in bijection with the set $H^1(\mathbb{Z}/r_1, \mathbb{Z}_p[F_0]^\times/F_0)$.

Classifying $\widetilde{F_2}$'s

Pick $\widetilde{F_1} \in \widetilde{\mathcal{F}}(\mathbb{Q}_p(F_0)^\times, \widetilde{F_0})$ and define the sets

- $\widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_1})$ of all subgroups $\widetilde{F}$ of $\mathbb{D}_n^\times$ such that $\widetilde{F_1} = \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times$ is of finite index in $\widetilde{F}$;
- $\widetilde{\mathcal{F}}(C_{\mathbb{D}_n^\times}(\widetilde{F_1}), \widetilde{F_1})$ of all subgroups $\widetilde{F}$ of $C_{\mathbb{D}_n^\times}(\widetilde{F_1})$ such that $\widetilde{F_1} = \widetilde{F} \cap \mathbb{Q}_p(F_0)^\times$ is of finite index in $\widetilde{F}$.

Then the map

$$\widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_1}) \longrightarrow \widetilde{\mathcal{F}}(C_{\mathbb{D}_n^\times}(\widetilde{F_1}), \widetilde{F_1}) \ : \ \widetilde{F} \longmapsto \widetilde{F_2} := \widetilde{F} \cap C_{\mathbb{D}_n^\times}(\widetilde{F_1})$$

induces a well defined map

$$\varphi_2 : \ \widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_1})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_1})} \longrightarrow \widetilde{\mathcal{F}}(C_{\mathbb{D}_n^\times}(\widetilde{F_1}), \widetilde{F_1})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_1})} .$$

In order to describe the image of φ_2, we recall that every $\widetilde{F_2} \in \widetilde{\mathcal{F}}(C_{\mathbb{D}_n^\times}(\widetilde{F_1}), \widetilde{F_1})$ determines an extension $L := \mathbb{Q}_p(\widetilde{F_2})$ of $\mathbb{Q}_p(\widetilde{F_1})$. Clearly, the isomorphism class of L is constant on each conjugacy class of $\widetilde{F_2}$'s by elements in $C_{\mathbb{D}_n^\times}(\widetilde{F_1})$, and hence determine the integer $r_2 = [L : \mathbb{Q}_p(\widetilde{F_1})]$ dividing $\frac{n}{r_1}$. By the Skolem-Noether theorem, the set of isomorphism classes of extensions $\mathbb{Q}_p(\widetilde{F_1}) \subseteq L$ is in bijection with the set of $C_{\mathbb{D}_n^\times}(\widetilde{F_1})$-conjugacy classes of L's. Thus denoting

$$\widetilde{\mathcal{F}}(L^\times, \widetilde{F_1}) := \{\widetilde{F_2} \in \widetilde{\mathcal{F}}(C_{\mathbb{D}_n^\times}(\widetilde{F_1}), \widetilde{F_1}) \mid \mathbb{Q}_p(\widetilde{F_2}) = L\},$$

we have a bijection

$$\widetilde{\mathcal{F}}(C_{\mathbb{D}_n^\times}(\widetilde{F_1}), \widetilde{F_1})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_1})} \ \cong \ \coprod_{[L]} \widetilde{\mathcal{F}}(L^\times, \widetilde{F_1}),$$

where the union is taken over all isomorphism classes of extensions $\mathbb{Q}_p(\widetilde{F_1}) \subseteq L$. Finally, if for a given L the set $\widetilde{\mathcal{F}}(L^\times, \widetilde{F_1})$ is non-empty, then by theorem 2.21 it is in bijection with the set $H^1(\mathbb{Z}/r_2, L_{r_1}^\times/\widetilde{F_1})$, and we have

$$\widetilde{\mathcal{F}}(C_{\mathbb{D}_n^\times}(\widetilde{F_1}), \widetilde{F_1})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_1})} \ \cong \ \coprod_{[L]} H^1(\mathbb{Z}/r_2, L_{r_1}^\times/\widetilde{F_1}).$$

Classifying $\widetilde{F_3}$'s

Pick $\widetilde{F_2} \in \widetilde{\mathcal{F}}(L^\times, \widetilde{F_1})$ and define the sets

- $\widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_2})$ of all subgroups $\widetilde{F}$ of $\mathbb{D}_n^\times$ such that $\widetilde{F} \cap S_n$ is abelian and $\widetilde{F_2} = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(F_0)$ is of finite index in $\widetilde{F}$;
- $\widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2})$ of all subgroups $\widetilde{F}$ of $N_{\mathbb{D}_n^\times}(\widetilde{F_2})$ such that $\widetilde{F} \cap S_n$ is abelian and $\widetilde{F_2} = \widetilde{F} \cap C_{\mathbb{D}_n^\times}(F_0)$ is of finite index in $\widetilde{F}$.

By proposition 2.9, each $\widetilde{F}$ in $\widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_2})$ satisfies $\widetilde{F} \subseteq N_{\mathbb{D}_n^\times}(F_0)$, in which case $\widetilde{F_2}$ is normal in $\widetilde{F}$. Thus the map

$$\widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_2}) \xrightarrow{\cong} \widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2}) \ : \ \widetilde{F} \longmapsto \widetilde{F_3} := \widetilde{F} \cap N_{\mathbb{D}_n^\times}(\widetilde{F_2})$$

is a bijection and induces a well defined bijection

$$\varphi_3 : \ \widetilde{\mathcal{F}}(\mathbb{D}_n^\times, \widetilde{F_2})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_2})} \xrightarrow{\cong} \widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_2})} .$$

In order to describe the image of φ_3, we recall that every $\widetilde{F_3} \in \widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2})$ determines an extension

$$1 \longrightarrow \widetilde{F_2} \longrightarrow \widetilde{F_3} \longrightarrow W \longrightarrow 1,$$

where W canonically injects into $Aut(\widetilde{F_2}, F_0)$. Via this injection, W is independent of the given representative in the $C_{\mathbb{D}_n^\times}(\widetilde{F_2})$-conjugacy class of $\widetilde{F_3}$. Thus denoting

$$\widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2}, W) := \{\widetilde{F_3} \in \widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2}) \mid \widetilde{F_3}/\widetilde{F_2} = W\},$$

we have a bijection

$$\widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_2})} \ \cong \ \coprod_W \widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2}, W)/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_2})} .$$

Finally, if for a given W the set $\widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2}, W)/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_2})}$ is non-empty, then by theorem 2.28 it is in bijection with the set $H^1(W, C_{\mathbb{D}_n^\times}(\widetilde{F_2})/\widetilde{F_2})$, and we have

$$\widetilde{\mathcal{F}}(N_{\mathbb{D}_n^\times}(\widetilde{F_2}), \widetilde{F_2})/\sim_{C_{\mathbb{D}_n^\times}(\widetilde{F_2})} \ \cong \ \coprod_W H^1(W, C_{\mathbb{D}_n^\times}(\widetilde{F_2})/\widetilde{F_2}).$$

Chapter 3:

On abelian finite subgroups of $\mathbb{G}_n(u)$

Throughout this chapter we assume that $n = (p-1)p^{k-1}m$ with m prime to p. Given an abelian finite subgroup F_0 of $\mathbb{S}_n$ whose p-Sylow subgroup is cyclic of order p^α for $1 \leq \alpha \leq k$, we want to determine what sequences of groups

$$F_0 \subseteq F_1 \subseteq F_2$$

are realized in $\mathbb{G}_n(u) = \mathbb{D}_n^\times/\langle pu\rangle$; here F_2 is an abelian finite subgroup of $\mathbb{G}_n(u)$ containing F_0 and F_1 is such that $\widetilde{F_1} = \widetilde{F_2} \cap \mathbb{Q}_p(F_0)$. We know from chapter 2 that the tilded correspondents of these groups in $\mathbb{D}_n^\times$ are given by

$$\widetilde{F_1} = \langle F_0, x_1\rangle \qquad \text{and} \qquad \widetilde{F_2} = \langle F_0, x_2\rangle$$

with $x_1, x_2 \in \mathbb{D}_n^\times$ such that

$$v(x_1) = \frac{1}{r_1}, \qquad x_1^{r_1} \in \widetilde{F_0}, \qquad v(x_2) = \frac{1}{r_1 r_2} \qquad \text{and} \qquad x_2^{r_2} \in \widetilde{F_1}.$$

We want to determine for what pairs of positive integers (r_1, r_2) the sets

$$\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1) \qquad \text{and} \qquad \widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$$

are non-empty.

3.1. Elementary conditions on r_1

The question of determining for what r_1 the set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty naturally leads to studying the r_1-th roots of pu in $\mathbb{Q}_p(F_0)$. Clearly r_1 must be a divisor of $\varphi(p^\alpha)$, the ramification index of $\mathbb{Q}_p(F_0)$ over $\mathbb{Q}_p$.

Proposition 3.1. *Let ζ_{p^α} be a primitive p^α-th root of unity in $\mathbb{Q}_p(F_0)^\times$. The principal ideal generated by $\zeta_{p^\alpha} - 1$ is maximal in $\mathbb{Z}_p(F_0)$ and satisfies*

$$(p) = (\zeta_{p^\alpha} - 1)^{\varphi(p^\alpha)}.$$

Proof. If a and b are integers prime to p, one can solve the equation $a \equiv bs \bmod p^\alpha$, so that

$$\frac{\zeta_{p^\alpha}^a - 1}{\zeta_{p^\alpha}^b - 1} = \frac{1 - \zeta_{p^\alpha}^{bs}}{1 - \zeta_{p^\alpha}^b} = 1 + \zeta_{p^\alpha}^b + \ldots + \zeta_{p^\alpha}^{(s-1)b} \in \mathbb{Z}_p[\zeta_{p^\alpha}].$$

The same is true for $\frac{\zeta_{p^\alpha}^b - 1}{\zeta_{p^\alpha}^a - 1}$, and

$$\frac{\zeta_{p^\alpha}^a - 1}{\zeta_{p^\alpha}^b - 1} \in \mathbb{Z}_p[\zeta_{p^\alpha}]^\times \qquad \text{whenever} \quad (a;p) = (b;p) = 1.$$

Moreover since

$$\sum_{i=0}^{p-1} x^{ip^{\alpha-1}} = \frac{1-x^{p^\alpha}}{1-x^{p^{\alpha-1}}} = \prod_{\substack{(a;p)=1\\ 1\leq a<p^\alpha}} (\zeta_{p^\alpha}^a - x),$$

for $x = 1$ we get

$$p = \prod_{\substack{(a;p)=1\\ 1\leq a<p^\alpha}} (\zeta_{p^\alpha}^a - 1) = (\zeta_{p^\alpha} - 1)^{\varphi(p^\alpha)} \prod_{\substack{(a;p)=1\\ 1\leq a<p^\alpha}} \frac{\zeta_{p^\alpha}^a - 1}{\zeta_{p^\alpha} - 1},$$

showing that $(p) = (\zeta_{p^\alpha} - 1)^{\varphi(p^\alpha)}$. The ideal generated by $\zeta_{p^\alpha} - 1$ is hence maximal in $\mathbb{Z}_p(F_0)$. □

Corollary 3.2. *We have*

$$p = \prod_{\substack{(a;p)=1\\ 1\leq a<p^\alpha}} (\zeta_{p^\alpha}^a - 1) \qquad \textit{and} \qquad v(\zeta_{p^\alpha} - 1) = \frac{1}{\varphi(p^\alpha)}.$$

□

Let $\mu(\mathbb{Q}_p(F_0))$ denote the roots of unity in $\mathbb{Q}_p(F_0)$ and fix ζ_{p^α} a primitive p^α-th root of unity in $\mu(\mathbb{Q}_p(F_0))$. Define the unit

$$\varepsilon_\alpha \in \mathbb{Z}_p(\zeta_{p^\alpha})^\times \subseteq \mathbb{Q}_p(F_0)^\times \qquad \text{by} \qquad (\zeta_{p^\alpha} - 1)^{\varphi(p^\alpha)} = p\varepsilon_\alpha.$$

Obviously as $u \in \mathbb{Z}_p^\times$, we know that $\frac{\varepsilon_\alpha}{u}$ belongs to $\mathbb{Z}_p(\zeta_{p^\alpha})^\times$. Let $\pi(e_u(F_0))$ denote the class of

$$e_u(F_0) \in H^2(\mathbb{Z}/\varphi(p^\alpha),\ \mathbb{Z}_p(F_0)^\times \times \langle pu\rangle)$$

in $H^2(\mathbb{Z}/\varphi(p^\alpha),\ \mathbb{Z}_p(F_0)^\times)$ as defined in section 2.4.

Proposition 3.3. *We have*

$$\pi(e_u(F_0)) = \frac{\varepsilon_\alpha}{u} \quad \textit{in} \quad H^2(\mathbb{Z}/\varphi(p^\alpha),\ \mathbb{Z}_p(F_0)^\times) \cong \mathbb{Z}_p(F_0)^\times/(\mathbb{Z}_p(F_0)^\times)^{\varphi(p^\alpha)}.$$

Proof. This is a straightforward consequence of the fact that

$$p\varepsilon_\alpha = pu\frac{\varepsilon_\alpha}{u}$$

belongs to the class of $e_u(F_0) \in H^2(\mathbb{Z}/\varphi(p^\alpha),\ \mathbb{Z}_p(F_0)^\times \times \langle pu\rangle)$. □

Recall from proposition C.7 that

$$\mathbb{Z}_p(F_0)^\times \cong \mu(\mathbb{Q}_p(F_0)) \times \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]},$$

so that

$$\begin{aligned} H^2(\mathbb{Z}/r_1, \mathbb{Z}_p(F_0)^\times) &\cong \mathbb{Z}_p(F_0)^\times/(\mathbb{Z}_p(F_0)^\times)^{r_1} \\ &\cong \mu(\mathbb{Q}_p(F_0))/\mu(\mathbb{Q}_p(F_0))^{r_1} \times (\mathbb{Z}_p/r_1\mathbb{Z}_p)^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}. \end{aligned} \tag{3.1}$$

Theorem 3.4. *The set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if*

$$\frac{\varepsilon_\alpha}{u} \equiv 1 \qquad \text{in} \quad \mathbb{Z}_p(F_0)^\times / \langle F_0, (\mathbb{Z}_p(F_0)^\times)^{r_1} \rangle.$$

Proof. The unit

$$\frac{\varepsilon_\alpha}{u} \in \mathbb{Z}_p(\zeta_{p^\alpha})^\times \subseteq \mathbb{Z}_p(F_0)^\times,$$

is equivalent to the trivial element if and only if $q_*(e_u(F_0, r_1))$ is trivial in

$$H^2(\mathbb{Z}/r_1, \mathbb{Z}_p(F_0)^\times / F_0),$$

for $q_* = H^2(\mathbb{Z}/r_1, q)$ the map induced by the canonical homomorphism

$$q : \ \mathbb{Z}_p(F_0)^\times \times \langle pu \rangle \ \longrightarrow \ \mathbb{Z}_p(F_0)^\times \times \langle pu \rangle / \widetilde{F_0} = \mathbb{Z}_p(F_0)^\times / F_0.$$

By theorem 2.16, this is true if and only if $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty. □

Corollary 3.5. *If $\langle F_0, x_1 \rangle \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ with $v(x_1) = \frac{1}{r_1}$, then $x_1^{r_1} = pu\delta$ for a δ in F_0 such that $\delta \equiv \frac{\varepsilon_\alpha}{u}$ modulo $(\mathbb{Z}_p(F_0)^\times)^{r_1}$.*

Proof. This follows from remark 2.17 and theorem 3.4. □

Corollary 3.6. *Let $F_0 = \mu(\mathbb{Q}_p(F_0))$ and r_1 be prime to p.*

1) *The set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if r_1 divides $p-1$.*

2) *If $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty with $r_1 > 1$, then p is odd, and there are elements $\zeta_p \in F_0$ and $t \in \mathbb{Z}_p(\zeta_p)^\times$ such that*

$$x_1 = (\zeta_p - 1)t \qquad \text{and} \qquad x_1^{p-1} \equiv pu \mod \mu_{p-1}.$$

Proof. 1) As r_1 divides the ramification index of $\mathbb{Q}_p(F_0)$, it must be a divisor of $p-1$. The result then follows from corollary 3.5, the isomorphism (3.1) and the fact that $\mathbb{Z}_p = (p-1)\mathbb{Z}_p$.

2) The condition $r_1 > 1$ ensures that $p > 2$ and $\zeta_p \in F_0$. By 1) and theorem 3.4, we know that

$$\frac{u}{\varepsilon_1} \in \langle \mu(\mathbb{Q}_p(\zeta_p)), (\mathbb{Z}_p(\zeta_p)^\times)^{p-1} \rangle = \langle \mu_{p-1}, (\mathbb{Z}_p(\zeta_p)^\times)^{p-1} \rangle.$$

Hence there exists a $t \in \mathbb{Z}_p(\zeta_p)^\times$ such that $u\varepsilon_1^{-1} = t^{p-1}\delta$ for some $(p-1)$-th root of unity $\delta \in \mu_{p-1}$. For $x_1 = (\zeta_p - 1)t$, we then have

$$x_1^{p-1} = (\zeta_p - 1)^{p-1} t^{p-1} = p\varepsilon_1 \cdot \frac{u}{\varepsilon_1}\delta^{-1} \equiv pu \mod \mu_{p-1}.$$

□

Remark 3.7. When $F_0 = \mu(\mathbb{Q}_p(F_0))$, we know by corollary 2.18 that $\widetilde{F_1}$ is unique in the set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$. If r_1 divides $p-1$, we may therefore always assume $\widetilde{F_1} = \langle F_0, x_1 \rangle$ with x_1 as given in corollary 3.6.

Example 3.8. If p is odd, then

$$\varepsilon_\alpha \equiv -1 \mod (\mathbb{Z}_p(F_0)^\times)^{p-1}.$$

Indeed, by example 1.33 we know that

$$\mathbb{Q}_p(\zeta_p) \cong \mathbb{Q}_p(X^{\frac{n}{p-1}}),$$

where $X = \omega^{\frac{p-1}{2}} S$ satisfies $X^n = -p$ for ω a primitive (p^n-1)-th root of unity in $\mathbb{D}_n^\times$. Furthermore, both elements X and $(\zeta_{p^\alpha} - 1)$ belong to the field $\mathbb{Q}_p(\zeta_{p^\alpha})$, and there is a $z \in \mathbb{Z}_p(\zeta_{p^\alpha})^\times$ with

$$(\zeta_{p^\alpha} - 1)^{p^{\alpha-1}} = X^{\frac{n}{p-1}} z.$$

Since $\mathbb{Q}_p(\zeta_{p^\alpha})^\times \subseteq \mathbb{Q}_p(F_0)^\times$, we obtain

$$\varepsilon_\alpha p = (\zeta_{p^\alpha} - 1)^{\varphi(p^\alpha)} = X^n z^{p-1} \equiv -p \mod (\mathbb{Z}_p(F_0)^\times)^{p-1}.$$

Thus if u is a root of unity, the set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, p-1)$ is non-empty.

Example 3.9. If $p = 2$, it is obvious that $\varepsilon_1 = -1$. The case $\alpha = 1$ however is not interesting since then r_1 must divide the trivial ramification index of $\mathbb{Q}_2(F_0)$ over $\mathbb{Q}_2$.

If $p = 2$ and $\alpha \geq 2$, we have

$$\varepsilon_\alpha \equiv -\zeta_4 \mod (\mathbb{Z}_2(F_0)^\times)^2$$

for a primitive 4-th root of unity $\zeta_4 \in \mathbb{Z}_2(F_0)^\times$. Indeed, the element $(\zeta_4 - 1)$ has valuation $\frac{1}{2}$ and

$$(\zeta_4 - 1)^2 = \zeta_4^2 - 2\zeta_4 + 1 = -2\zeta_4.$$

Hence for $z \in \mathbb{Z}_2(F_0)^\times$ satisfying

$$(\zeta_{2^\alpha} - 1)^{2^{\alpha-2}} = (\zeta_4 - 1)z,$$

we obtain

$$2\varepsilon_\alpha = (\zeta_{2^\alpha} - 1)^{2^{\alpha-1}} = (\zeta_4 - 1)^2 z^2 \equiv -2\zeta_4 \mod (\mathbb{Z}_2(F_0)^\times)^2.$$

This shows that if $u = \pm 1$, the set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_2(F_0), \widetilde{F_0}, 2)$ is non-empty.

3.2. Change of rings

Assume p to be any prime. For each $1 \leq \alpha \leq k$, we fix a root of unity ζ_{p^α} in $\mathbb{Q}_p(F_0)$, and we define

$$\pi_\alpha := \zeta_{p^\alpha} - 1 \qquad \text{and} \qquad R_\alpha := \mathbb{Z}_p[\zeta_{p^\alpha}].$$

Recall from proposition 3.1 that π_α is a uniformizing element of R_α where $(\pi_\alpha^{\varphi(p^\alpha)}) = (p)$. Let

$$i_\alpha : R_\alpha \longrightarrow R_{\alpha+1}$$

be the ring homomorphism defined by $i_\alpha(\zeta_{p^\alpha}) = \zeta_{p^{\alpha+1}}^p$. By definition, $\varepsilon_\alpha \in R_\alpha$ for each α. In this section, we compare $\varepsilon_{\alpha+1}$ with the image of ε_α in $R_{\alpha+1}$.

Lemma 3.10. *For any prime p and any $\alpha \geq 1$, we have*

$$i_\alpha(\pi_\alpha) = \sum_{j=1}^{p} \binom{p}{j} \pi_{\alpha+1}^j.$$

Proof. Clearly $i_\alpha(\pi_\alpha) = \zeta_{p^{\alpha+1}}^p - 1$. This, together with the identity

$$X^p - 1 = (X - 1 + 1)^p - 1 = \sum_{j=1}^{p} \binom{p}{j} (X-1)^j$$

applied to the case $X = \zeta_{p^{\alpha+1}}$, yields the result. □

Corollary 3.11. *For any prime p and any $\alpha \geq 1$, we have*

$$i_\alpha(\pi_\alpha) \equiv \pi_{\alpha+1}^p \mod (p\pi_{\alpha+1}).$$

Proof. This follows from lemma 3.10 and the p-divisibility of the binomial coefficients for $1 \leq j < p$. □

For any prime p and any $\alpha \geq 2$, define the positive integer

$$k_\alpha := \begin{cases} p^\alpha - 2p + 1 & \text{if } p > 2, \\ 2^\alpha - 2 & \text{if } p = 2. \end{cases}$$

Lemma 3.12. *If $p > 2$ and $\alpha \geq 2$, or if $p = 2$ and $\alpha \geq 3$, then*

$$i_\alpha(\pi_\alpha^j) \equiv \pi_{\alpha+1}^{jp} \mod (\pi_{\alpha+1}^{p^{\alpha+1}+1}) \qquad \textit{for any } j \geq k_\alpha.$$

Proof. Let $j \geq k_\alpha$. Combining corollary 3.11 with the binomial formula yields

$$i_\alpha(\pi_\alpha^j) = (\pi_{\alpha+1}^p + p\pi_{\alpha+1}z)^j = \pi_{\alpha+1}^{jp} + w$$

with

$$z \in R_{\alpha+1} \qquad \text{and} \qquad w = \sum_{k=0}^{j-1} \binom{j}{k} \pi_{\alpha+1}^{kp} (p\pi_{\alpha+1}z)^{j-k}.$$

Note that the valuation of the k-th term is at least $kp + (j-k)(\varphi(p^{\alpha+1}) + 1)$. Hence for $0 \leq k \leq j-1$ its valuation is at least

$$(j-1)p + \varphi(p^{\alpha+1}) + 1 \geq (k_\alpha - 1)p + \varphi(p^{\alpha+1}) + 1.$$

If $p > 2$ and $\alpha \geq 2$, we have

$$\begin{aligned} (j-1)p + \varphi(p^{\alpha+1}) + 1 &\geq (p^\alpha - 2p)p + \varphi(p^{\alpha+1}) + 1 \\ &= p^{\alpha+1} - 2p^2 + p^{\alpha+1} - p^\alpha + 1 \\ &= p^{\alpha+1} + p^2(p^{\alpha-1} - p^{\alpha-2} - 2) + 1 \\ &\geq p^{\alpha+1} + 1. \end{aligned}$$

Otherwise if $p = 2$ and $\alpha \geq 3$, we obtain

$$\begin{aligned} (j-1)p + \varphi(p^{\alpha+1}) + 1 &\geq (2^\alpha - 3)2 + 2^\alpha + 1 \\ &= 2^{\alpha+1} - 6 + 2^\alpha + 1 \\ &\geq 2^{\alpha+1} + 1. \end{aligned}$$

□

Lemma 3.13. *If $p > 2$ and $\alpha \geq 2$, or if $p = 2$ and $\alpha \geq 3$, then*

$$i_\alpha(\pi_\alpha^{\varphi(p^\alpha)}) \equiv \pi_{\alpha+1}^{\varphi(p^{\alpha+1})} \mod (\pi_{\alpha+1}^{\varphi(p^{\alpha+1})+p^{\alpha+1}+1}).$$

Proof. Combining corollary 3.11 with the binomial formula yields

$$\begin{aligned} i_\alpha(\pi_\alpha^p) &= (\pi_{\alpha+1}^p + \pi_{\alpha+1}^{\varphi(p^{\alpha+1})+1} z_0)^p \\ &= \pi_{\alpha+1}^{p^2} + \sum_{j=1}^{p-1} \binom{p}{j} \pi_{\alpha+1}^{jp+(\varphi(p^{\alpha+1})+1)(p-j)} z_0^{p-j} + \pi_{\alpha+1}^{(\varphi(p^{\alpha+1})+1)p} z_0^p \\ &= \pi_{\alpha+1}^{p^2} + \pi_{\alpha+1}^{2\varphi(p^{\alpha+1})+1} z_1 \end{aligned}$$

for some suitable $z_0, z_1 \in R_{\alpha+1}$, where we have used that the valuation of each term in the middle sum is greater or equal to $(p-1)p + 2\varphi(p^{\alpha+1}) + 1$, while that of the last term is $(\varphi(p^{\alpha+1}) + 1)p > 2\varphi(p^{\alpha+1}) + 1$. By iterating this procedure we obtain some z_k with

$$i_\alpha(\pi_\alpha^{p^k}) = \pi_{\alpha+1}^{p^{k+1}} + \pi_{\alpha+1}^{(k+1)\varphi(p^{\alpha+1})+1} z_k \qquad \text{for every } k \geq 0.$$

The required formula for $p = 2$ and $\alpha \geq 3$ directly follows from the case $k = \alpha - 1$. Again, by taking $k = \alpha - 1$ if $p > 2$, we get

$$\begin{aligned} i_\alpha(\pi_\alpha^{\varphi(p^\alpha)}) &= (\pi_{\alpha+1}^{p^\alpha} + \pi_{\alpha+1}^{\alpha\varphi(p^{\alpha+1})+1} z_{\alpha-1})^{p-1} \\ &= \pi_{\alpha+1}^{\varphi(p^{\alpha+1})} + \pi_{\alpha+1}^{\alpha\varphi(p^{\alpha+1})+1+(p-2)p^\alpha} z \end{aligned}$$

for some $z \in R_{\alpha+1}$. The desired result for $p \geq 3$ and $\alpha \geq 2$ then follows from the fact that

$$\begin{aligned} \alpha\varphi(p^{\alpha+1}) + 1 + (p-2)p^\alpha &= \varphi(p^{\alpha+1}) + (\alpha-1)\varphi(p^{\alpha+1}) + (p-2)p^\alpha + 1 \\ &= \varphi(p^{\alpha+1}) + p^\alpha[(\alpha-1)(p-1) + p - 2] + 1 \\ &\geq \varphi(p^{\alpha+1}) + p^\alpha(2p-3) + 1 \\ &\geq \varphi(p^{\alpha+1}) + p^{\alpha+1} + 1. \end{aligned}$$

□

Corollary 3.14. *If $p > 2$ and $\alpha \geq 2$, or if $p = 2$ and $\alpha \geq 3$, then*

$$i_\alpha(\varepsilon_\alpha) \equiv \varepsilon_{\alpha+1} \mod (\pi_{\alpha+1}^{p^{\alpha+1}+1}).$$

Proof. This follows from lemma 3.13 together with the fact that $p\varepsilon_\alpha = \pi_\alpha^{\varphi(p^\alpha)}$. □

3.3. The p-part of r_1 for p odd

Using notations introduced in sections 3.1 and 3.2, we assume p to be an odd prime. Recall that $\alpha \geq 0$ is defined to satisfy $|F_0 \cap S_n| = p^\alpha$. The goal of the section is to establish that for $\alpha \geq 1$ and $F_0 = \mu(\mathbb{Q}_p(F_0))$, the set $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if r_1 divides $p-1$. This is done by showing that $\frac{\varepsilon_\alpha}{u}$ is non-trivial in the group $\mathbb{Z}_p(F_0)^\times / \langle \mu(\mathbb{Q}_p(F_0)), (\mathbb{Z}_p(F_0)^\times)^p \rangle$ when $\alpha \geq 2$, and hence that p does not divide r_1.

We know from proposition 3.1 that (π_α) is the maximal ideal of $\mathbb{Z}_p(F_0)$. The situation is clear when $\alpha = 1$, because the ramification index of $\mathbb{Q}_p(F_0)$ over $\mathbb{Q}_p$ is prime to p and hence the p-part of r_1 is trivial.

We need to establish a formula for the π_α-adic expansion of $\varepsilon_\alpha = p^{-1}\pi_\alpha^{\varphi(p^\alpha)}$. For this we begin by analysing the cyclotomic polynomials

$$Q_\alpha(X) := \frac{(X+1)^{p^\alpha} - 1}{(X+1)^{p^{\alpha-1}} - 1} \quad \in \mathbb{Z}[X].$$

Note that $Q_\alpha(X)$ is the minimal polynomial of π_α over $\mathbb{Q}_p$. We have

$$Q_\alpha(X) = \sum_{k=0}^{p-1} (X+1)^{p^{\alpha-1}k} = \sum_{i=0}^{\varphi(p^\alpha)} \left(\sum_{k=0}^{p-1} \binom{p^{\alpha-1}k}{i} \right) X^i.$$

Define $a_i^{(\alpha)}$ to be the coefficient of X^i in $Q_\alpha(X)$, and let

$$b_i^{(\alpha)} := \begin{cases} \binom{p^{\alpha-1}}{i} & \text{if } 0 \le i \le p^{\alpha-1}, \\ 0 & \text{if } i > p^{\alpha-1}, \end{cases}$$

be the coefficient of X^i in $(X+1)^{p^{\alpha-1}}$.

Lemma 3.15. *For $\alpha, i \ge 1$, we have a strict identity*

$$a_i^{(\alpha)} = b_i^{(\alpha)} + \sum_{k=2}^{p-1} \sum_{i_1+\ldots+i_k=i} b_{i_1}^{(\alpha)} \ldots b_{i_k}^{(\alpha)}.$$

Proof. This follows form the fact that

$$Q_\alpha(X) = \sum_{k=0}^{p-1} (X+1)^{p^{\alpha-1}k}.$$

□

Lemma 3.16. *For $\alpha \ge 3$ and $i \ge 1$, we have*

$$b_i^{(\alpha)} \equiv \begin{cases} b_j^{(\alpha-1)} \bmod p^2 & \text{if } i = pj, \\ 0 \bmod p^2 & \text{if } i \not\equiv 0 \bmod p. \end{cases}$$

Proof. This is a consequence of the identity

$$\begin{aligned} (1+X)^{p^{\alpha-1}} &= (1 + X^p + pX(1+\ldots+X^{p-2}))^{p^{\alpha-2}} \\ &\equiv (1+X^p)^{p^{\alpha-2}} + p^{\alpha-1}X(1+\ldots+X^{p-2})(1+X^p)^{p^{\alpha-2}-1} \mod (p^\alpha). \end{aligned}$$

□

Lemma 3.17. *For $\alpha \ge 3$ and $i \ge 1$, we have*

$$a_i^{(\alpha)} \equiv \begin{cases} a_j^{(\alpha-1)} \bmod p^2 & \text{if } i = pj, \\ 0 \bmod p^2 & \text{if } i \not\equiv 0 \bmod p. \end{cases}$$

Proof. If $i \not\equiv 0 \bmod p$, the result is a direct consequence of lemma 3.16. It remains to consider the case where $i = pj$ for some integer $j \geq 1$. By lemma 3.15 and 3.16, it suffices to show that

$$\sum_{k=2}^{p-1} \sum_{i_1+\ldots+i_k=pj} b_{i_1}^{(\alpha)} \ldots b_{i_k}^{(\alpha)} \equiv \sum_{k=2}^{p-1} \sum_{j_1+\ldots+j_k=j} b_{j_1}^{(\alpha-1)} \ldots b_{j_k}^{(\alpha-1)} \mod p^2.$$

Using lemma 3.16 once again it suffices to show that

$$\sum_{k=2}^{p-1} \sum_{i_1+\ldots+i_k=pj}^{*} b_{i_1}^{(\alpha)} \ldots b_{i_k}^{(\alpha)} \equiv 0 \mod p^2,$$

where the symbol $\sum^*_{i_1+\ldots+i_k=pj}$ denotes the sum over all k-tuples $(i_1, \ldots, i_k)$ for which at least one (and hence at least two) of the i_k are not divisible by p. Then lemma 3.16 implies that this sum is congruent to 0 modulo p^2. □

The case $\alpha = 2$ is of particular interest.

Remark 3.18. We note that

$$b_i^{(2)} \begin{cases} \equiv 0 \mod p & \text{if } 0 < i < p, \\ = 1 & \text{if } i \in \{0, p\}, \\ = 0 & \text{if } i > p. \end{cases}$$

Lemma 3.19. *We have*

$$a_{(p-2)p+1}^{(2)} \equiv -p \mod p^2.$$

Proof. By lemma 3.15

$$a_{(p-2)p+1}^{(2)} = b_{(p-2)p+1}^{(2)} + \sum_{k=2}^{p-1} \sum_{i_1+\ldots+i_k=(p-2)p+1} b_{i_1}^{(2)} \ldots b_{i_k}^{(2)}.$$

According to remark 3.18, the only nontrivial contributions in this sum modulo p^2 happen when $k = p-1$ and come from tuples where all but one i_k are equal to p (and hence the remaining one equal to 1). As there are $p-1$ of such contributions we obtain

$$a_{(p-2)p+1}^{(2)} \equiv (p-1)b_1^{(2)} \equiv (p-1)p \equiv -p \mod p^2.$$

□

Lemma 3.20. *If* $0 < j < p-1$, *then*

$$\sum_{k=1}^{p-1} \binom{k}{j} \equiv 0 \mod p.$$

Proof. For a fixed $0 < j < p-1$, we have

$$\sum_{k=1}^{p-1} \binom{k}{j} = \frac{1}{j!} \sum_{k=1}^{p-1} k(k-1)\ldots(k-j+1),$$

where the expression $k(k-1)\dots(k-j+1)$ is a polynomial of degree j in $\mathbb{Z}[k]$ with zero constant term. It is consequently enough to check that

$$\sum_{k=1}^{p-1} k^r \equiv 0 \mod p \quad \text{for every } 0 < r < p-1.$$

Given $a \in \mathbb{F}_p^\times$ such that $a^r \neq 1$, we have

$$\sum_{x\in\mathbb{F}_p} x^r = \sum_{x\in\mathbb{F}_p} (ax)^r = a^r \sum_{x\in\mathbb{F}_p} x^r,$$

so that

$$\sum_{x\in\mathbb{F}_p} x^r = 0 \qquad \text{and} \qquad \sum_{k=1}^{p-1} k^r \equiv 0 \mod p.$$

□

Lemma 3.21. *If $0 \leq r < p-2$ and $0 < j < p$, then*

$$a^{(2)}_{pr+j} \equiv 0 \mod p^2.$$

Proof. By lemma 3.15, we have

$$\begin{aligned} a^{(2)}_{pr+j} &= b^{(2)}_{pr+j} + \sum_{k=2}^{p-1} \sum_{i_1+\dots+i_k=pr+j} b^{(2)}_{i_1} \dots b^{(2)}_{i_k} \\ &\equiv b^{(2)}_{pr+j} + \sum_{k=2}^{p-1} \sum_{i_1+\dots+i_k=pr+j}^{*} b^{(2)}_{i_1} \dots b^{(2)}_{i_k} \qquad \mod p^2, \end{aligned}$$

where the last sum is taken over all k-tuples $(i_1, \dots, i_k)$ in which there is exactly one element $b^{(2)}_i$ with $i \notin \{0, p\}$. Furthermore, this $b^{(2)}_i$ is in fact $b^{(2)}_j$ and $b^{(2)}_{i_1} \dots b^{(2)}_{i_k} = b^{(2)}_j$. We hence get

$$a^{(2)}_{pr+j} \equiv b^{(2)}_{pr+j} + b^{(2)}_j \sum_{k=2}^{p-1} k \binom{k-1}{r} \mod p^2.$$

If $r = 0$, then

$$a^{(2)}_j \equiv b^{(2)}_j + b^{(2)}_j \sum_{k=2}^{p-1} k \equiv b^{(2)}_j \frac{p(p-1)}{2} \equiv 0 \mod p^2.$$

If $r > 0$, then $b^{(2)}_{pr+j} = 0$ and we have

$$a^{(2)}_{pr+j} \equiv b^{(2)}_j \sum_{k=2}^{p-1} k \binom{k-1}{r} \mod p^2.$$

Since

$$\sum_{k=2}^{p-1} k \binom{k-1}{r} = \sum_{k=1}^{p-1} k \binom{k-1}{r} = (r+1) \sum_{k=1}^{p-1} \binom{k}{r+1} \equiv 0 \mod p$$

by lemma 3.20, we get $a^{(2)}_{pr+j} \equiv 0 \mod p^2$. □

Let F_0' denote the p'-part of F_0. Since $Q_\alpha(X)$ is the minimal polynomial of π_α in $\mathbb{Z}_p(F_0')$, we have an isomorphism of algebras

$$\varphi_{F_0} : (\mathbb{Z}_p(F_0')[X]/(Q_\alpha(X))) \xrightarrow{\cong} \mathbb{Z}_p(F_0) \quad \text{given by} \quad X \longmapsto \pi_\alpha,$$

which restricts to an isomorphism on the groups of units

$$\varphi_{F_0} : (\mathbb{Z}_p(F_0')[X]/(Q_\alpha(X)))^\times \xrightarrow{\cong} \mathbb{Z}_p(F_0)^\times.$$

Furthermore, there is a polynomial $\widetilde{Q}_\alpha(X) \in \mathbb{Z}[X]$ of degree $\varphi(p^\alpha) - 1$ such that

$$Q_\alpha(X) = X^{\varphi(p^\alpha)} + p\widetilde{Q}_\alpha(X) \qquad \text{and} \qquad \widetilde{Q}_\alpha(0) = 1,$$

and therefore we have

$$\varphi_{F_0}(\widetilde{Q}_\alpha(X)) = -p^{-1}\pi_\alpha^{\varphi(p^\alpha)} = -\varepsilon_\alpha.$$

Recall from proposition 3.1 that π_α is a uniformizing element in $\mathbb{Z}_p(F_0)$, so that (π_α) is the maximal ideal of this ring, and that $(\pi_\alpha^{\varphi(p^\alpha)}) = (p)$. More precisely, the π_α-adic expansion of p in $R_\alpha = \mathbb{Z}_p[\pi_\alpha] \subseteq \mathbb{Z}_p(F_0)$ is given below.

Proposition 3.22. *If $p > 2$ and $\alpha \geq 2$, then*

$$p \equiv -\pi_\alpha^{\varphi(p^\alpha)} + \frac{p-1}{2}\pi_\alpha^{p^\alpha} \mod (\pi_\alpha^{p^\alpha+1}).$$

Proof. Recall that

$$Q_\alpha(X) = p + \sum_{i=1}^{\varphi(p^\alpha)-1} a_i^{(\alpha)} X^i + X^{\varphi(p^\alpha)}.$$

By lemma 3.17,

$$Q_\alpha(X) \equiv Q_{\alpha-1}(X^p) \equiv \ldots \equiv Q_2(X^{p^{\alpha-2}}) \mod (p^2X).$$

By lemma 3.21 we know that $a_i^{(2)} \equiv 0 \bmod p^2$ if $0 < i < p$. Furthermore, by lemma 3.15 and remark 3.18 we have

$$a_i^{(2)} = \sum_{k=1}^{p-1} \sum_{i_1+\ldots+i_k=i} b_{i_1}^{(2)} \ldots b_{i_k}^{(2)} \qquad \text{with} \quad b_i^{(2)} \begin{cases} \equiv 0 \mod p & \text{if } 0 < i < p, \\ = 1 & \text{if } i \in \{0, p\}, \\ = 0 & \text{if } i > p. \end{cases}$$

Then obviously

$$a_p^{(2)} \equiv \sum_{k=1}^{p-1} k = \frac{p(p-1)}{2} \mod p^2, \qquad a_i^{(2)} \equiv 0 \mod p \quad \text{if } i \not\equiv 0 \bmod p,$$

and if $i = pj$ we have

$$\sum_{i_1+\ldots+i_k=i} b_{i_1}^{(2)} \ldots b_{i_k}^{(2)} \equiv \binom{k}{j} \mod p^2.$$

For a fixed $0 < j < p-1$ we have by lemma 3.20

$$\sum_{k=1}^{p-1} \binom{k}{j} \equiv 0 \mod p,$$

so that

$$a_i^{(2)} \equiv \begin{cases} 0 \mod p & \text{if } i \not\equiv 0 \bmod p, \\ 0 \mod p & \text{if } i = jp \text{ for } 0 < j < p-1, \\ 1 \mod p & \text{if } i = p(p-1). \end{cases}$$

Therefore

$$Q_2(X) \equiv p + p\frac{p-1}{2}X^p + X^{\varphi(p^2)} \mod (p^2X, pX^{p+1}).$$

Finally

$$Q_\alpha(\pi_\alpha) \equiv Q_2(\pi_\alpha^{p^{\alpha-2}}) \equiv p + p\frac{p-1}{2}\pi_\alpha^{p^{\alpha-1}} + \pi^{\varphi(p^\alpha)} \mod (\pi_\alpha^{p^\alpha+1}),$$

and consequently

$$\begin{aligned} p &\equiv -\pi_\alpha^{\varphi(p^\alpha)} - p\frac{p-1}{2}\pi_\alpha^{p^{\alpha-1}} \\ &\equiv -\pi_\alpha^{\varphi(p^\alpha)} - \left(-\pi_\alpha^{\varphi(p^\alpha)} - p\frac{p-1}{2}\pi_\alpha^{p^{\alpha-1}}\right)\frac{p-1}{2}\pi^{p^{\alpha-1}} \\ &\equiv -\pi_\alpha^{\varphi(p^\alpha)} + \frac{p-1}{2}\pi_\alpha^{\varphi(p^\alpha)}\pi_\alpha^{p^{\alpha-1}} \\ &\equiv -\pi_\alpha^{\varphi(p^\alpha)} + \frac{p-1}{2}\pi_\alpha^{p^\alpha} \qquad \mod(\pi_\alpha^{p^\alpha+1}). \end{aligned}$$

□

Our interest in approximating modulo the ideal generated by $\pi_\alpha^{p^\alpha+1}$ is explained in the following remark. Consider the decreasing filtration

$$\mathbb{Z}_p(F_0)^\times = U_0 \supseteq U_1 \supseteq U_2 \supseteq \dots$$

given by $U_0 = \mathbb{Z}_p(F_0)^\times$ and

$$U_i = U_i(\mathbb{Z}_p(F_0)^\times) = \{x \in U_0 \mid x \equiv 1 \bmod (\pi_\alpha^i)\} \qquad \text{for } i \geq 1,$$

where $U_0/U_1 = \mu_{p'}(\mathbb{Q}_p(F_0))$, and where U_i/U_{i+1} is isomorphic to the residue field of $\mathbb{Q}_p(F_0)$ for each $i \geq 1$. Because $\mathbb{Z}_p(F_0)$ is complete with respect to the filtration, any $x \in \mathbb{Z}_p(F_0)^\times$ admits a π_α-adic expansion

$$x = \sum_{i \geq 0} \lambda_i \pi_\alpha^i,$$

where the λ_i's run in a given set of representative of the residue field chosen in such a way that the representative in $\mathbb{F}_p$ are integers between 0 and $p-1$.

Remark 3.23. For any a in U_1, we have

$$\begin{aligned} (1 + a\pi_\alpha^i)^p &\equiv 1 + a^p\pi_\alpha^{ip} + ap\pi_\alpha^i \\ &\equiv 1 + a^p\pi_\alpha^{ip} - a\pi_\alpha^{\varphi(p^\alpha)+i} \qquad \mod (\pi_\alpha^{\varphi(p^\alpha)+i+1}). \end{aligned}$$

As

$$ip < \varphi(p^\alpha) + i \qquad \Leftrightarrow \qquad i < p^{\alpha-1},$$

we obtain

$$(1 + a\pi_\alpha^i)^p \equiv \begin{cases} 1 + a^p\pi_\alpha^{ip} & \mod (\pi_\alpha^{ip+1}) & \text{if } i < p^{\alpha-1}, \\ 1 + (a^p - a)\pi_\alpha^{p^\alpha} & \mod (\pi_\alpha^{p^\alpha+1}) & \text{if } i = p^{\alpha-1}, \\ 1 + a\pi_\alpha^{\varphi(p^\alpha)+i} & \mod (\pi_\alpha^{\varphi(p^\alpha)+i+1}) & \text{if } i > p^{\alpha-1}. \end{cases}$$

The last congruence and the completeness of the filtration imply that every element of U_i for $i > \varphi(p^\alpha) + p^{\alpha-1} = p^\alpha$ is a p-th power, and it follows that $U_{p^\alpha+1} \subseteq U_1^p$.

Setting $\mu := \mu(\mathbb{Q}_p(F_0))$, we have

$$U_0/\langle \mu, U_0^p\rangle \cong U_1/\langle \mu \cap U_1, U_1^p\rangle,$$

where $\mu \cap U_1$ is the subgroup generated by $\zeta_{p^\alpha} = \pi_\alpha + 1$. By remark 3.23, there is a commutative diagram

$$\begin{array}{ccc}
(\mathbb{Z}_p(F_0')[X]/(Q_\alpha(X)))^\times & \xrightarrow[\cong]{\varphi_{F_0}} & \mathbb{Z}_p(F_0)^\times \\
\downarrow & & \downarrow \\
(\mathbb{Z}_p(F_0')[X]/(Q_\alpha(X), X^{p^\alpha+1}))^\times & \xrightarrow[\cong]{} & \mathbb{Z}_p(F_0)^\times/(\pi_\alpha^{p^\alpha+1}) \\
\downarrow & & \downarrow \\
(\mathbb{Z}_p(F_0')[X]/(Q_\alpha(X), X^{p^\alpha+1}))^\times/\langle \mu, \text{p-th powers}\rangle & \xrightarrow[\cong]{} & U_1/\langle \mu \cap U_1, U_1^p\rangle,
\end{array}$$

in which the vertical maps are the canonical projections and the horizontal maps are isomorphisms. Since -1 becomes trivial in U_1, the images of $\widetilde{Q}_\alpha(X)$ and ε_α in the quotient group $U_1/\langle \mu \cap U_1, U_1^p\rangle$ are equal. We want to prove that this image is non-trivial. In order to do so, we consider the π_α-adic expansion of $-\varepsilon_\alpha$ in $\mathbb{Z}_p(\pi_\alpha)^\times \subseteq \mathbb{Z}_p(F_0)^\times$:

$$-\varepsilon_\alpha = \varphi_{F_0}(\widetilde{Q}_\alpha(X)) = \sum_{i\geq 0} c_i^{(\alpha)} \pi_\alpha^i,$$

where $c_0^{(\alpha)} = 1$ and $0 \leq c_i^{(\alpha)} < p$ for each $i \geq 0$. For $0 \leq i \leq p^2$, we let $c_i := c_i^{(2)}$.

Remark 3.24. By definition

$$c_i^{(\alpha)} \equiv \frac{a_i^{(\alpha)}}{p} \mod p \qquad \text{for } \alpha \geq 2 \text{ and } 0 \leq i < \varphi(p^\alpha).$$

Lemma 3.25. *Let $k_2 = (p-2)p + 1$. Then $c_{k_2} = -1$ and*

$$-\varepsilon_2 \equiv \sum_{i=0}^{p-2} c_{ip} \pi_2^{ip} - \pi_2^{k_2} + \sum_{i=k_2+1}^{p^2} c_i \pi_2^i \mod (\pi_2^{p^2+1}).$$

Proof. This follows from lemma 3.19 and 3.21. □

Lemma 3.26. *If*

$$x = 1 + \sum_{i=1}^{\infty} a_i \pi_\alpha^i \qquad \text{and} \qquad y = 1 + \sum_{i=1}^{\infty} b_i \pi_\alpha^i$$

are two elements of $U_1 \subseteq \mathbb{Z}_p(F_0)^\times$ such that $0 \leq a_i, b_i < p$ and $x \equiv y$ modulo (π_α^k) for an integer $k \geq 1$, then

$$\frac{x}{y} \equiv 1 + (a_k - b_k)\pi_\alpha^k \mod (\pi_\alpha^{k+1}).$$

Proof. Let $z = x - y$. Then

$$z \equiv 0 \mod (\pi_\alpha^k) \qquad \text{and} \qquad \frac{z}{x} \equiv 0 \mod (\pi_\alpha^k).$$

Therefore

$$\begin{aligned}\frac{x}{y} &= \frac{1}{1-\frac{z}{x}} = 1+\frac{z}{x}+\frac{z^2}{x^2}+\dots \\ &\equiv 1+\frac{z}{x} \\ &\equiv 1+\frac{(a_k-b_k)\pi_\alpha^k}{x} \\ &\equiv 1+(a_k-b_k)\pi_\alpha^k \qquad \mod(\pi_\alpha^{k+1}).\end{aligned}$$

□

Lemma 3.27. *If $x \in U_1 \subseteq \mathbb{Z}_p(F_0)^\times$ is such that*

$$x \equiv 1 + a_k\pi_\alpha^k \mod(\pi_\alpha^{k+1})$$

with $2 \le k < p^\alpha$ prime to p and $a_k \not\equiv 0$ modulo (π_α), then $x \notin \langle \mu \cap U_1, U_1^p\rangle$.

Proof. Recall that $\mu \cap U_1$ is generated by $1+\pi_\alpha$. One must check that x cannot be written in the form

$$x = (1+\pi_\alpha)^i y^p$$

with $0 \le i \le p-1$ and $y \in U_1$. Since $k \ge 2$ by assumption, it follows that $i = 0$ and it remains to verify that x is not a p-th power. As a direct consequence of remark 3.23, we have

$$(1+a\pi_\alpha^i)^p \equiv 1 + a^p\pi_\alpha^{ip} \mod(\pi_\alpha^{ip+1})$$

for any $a \in U_1$ and any $i \le \frac{k}{p}$. Hence $x \notin U_1^p$. □

We are now in position to prove that $-\varepsilon_\alpha$ (and hence ε_α) is non-trivial in the group $U_1/\langle \mu \cap U_1, U_1^p\rangle$.

Theorem 3.28. *If p is odd and $\alpha \ge 2$, then*

$$\varepsilon_\alpha \notin \langle \mu(\mathbb{Q}_p(F_0)), (\mathbb{Z}_p(F_0)^\times)^p\rangle.$$

Proof. In order to obtain the result, it suffices to show that $-\varepsilon_\alpha \in U_1$ is non-trivial in the quotient $U_1/\langle \mu \cap U_1, U_1^p\rangle$. This is done by induction on $\alpha \ge 2$.

First consider the case $\alpha = 2$, and let $k_2 := (p-2)p+1$. According to lemma 3.25

$$-\varepsilon_2 \equiv \sum_{i=0}^{p-2} c_{pi}\pi_2^{pi} \equiv \prod_{i=0}^{p-2}(1+\widetilde{c}_{pi}\pi_2^i)^p \qquad \mod(\pi_2^{k_2}),$$

where each $0 \le \widetilde{c}_j < p$ is such that $(\widetilde{c}_j)^p \equiv c_j \bmod p$, and where the second equivalence is due to the facts that $k_2 - 1 < \varphi(p^2)$ and $(p) = (\pi_2^{\varphi(p^2)})$. Letting

$$z_2 = \prod_{i=1}^{p-2}(1+\widetilde{c}_{pi}\pi_2^i) \quad \in \mathbb{Z}_p(\pi_2)^\times,$$

we get by lemma 3.26 and 3.25

$$\frac{-\varepsilon_2}{z_2^p} \equiv 1 + \widetilde{c}_{k_2}\pi_2^{k_2} \equiv 1 - \pi_2^{k_2} \qquad \mod(\pi_2^{k_2+1}).$$

As $k_2 < p^2$, it follows from lemma 3.27 that $-\varepsilon_2$ does not belong to $\langle \mu \cap U_1, U_1^p \rangle$.

Now let $\alpha \geq 2$ and $k_\alpha = p^\alpha - 2p + 1$. Suppose there is an element z_α in $\mathbb{Z}_p(\pi_\alpha)^\times$ such that

$$\frac{-\varepsilon_\alpha}{z_\alpha^p} \equiv 1 + \sum_{i=k_\alpha}^{p^\alpha} d_i \pi_\alpha^i \qquad \bmod (\pi_\alpha^{p^\alpha+1}),$$

with $d_{k_\alpha} \not\equiv 0 \bmod p$. By corollary 3.14 and lemma 3.12 we have

$$\varepsilon_{\alpha+1} \equiv i_\alpha(\varepsilon_\alpha) \equiv -(1 + \sum_{i=k_\alpha}^{p^\alpha} d_i \pi_{\alpha+1}^{pi}) i_\alpha(z_\alpha)^p \qquad \bmod (\pi_{\alpha+1}^{p^{\alpha+1}+1}),$$

so that

$$\frac{-\varepsilon_{\alpha+1}}{(z'_{\alpha+1})^p} \equiv 1 + \sum_{i=k_\alpha}^{p^\alpha} d_i \pi_{\alpha+1}^{pi} \qquad \bmod (\pi_{\alpha+1}^{p^{\alpha+1}+1})$$

for some $z'_{\alpha+1}$ in $\mathbb{Z}_p(\pi_{\alpha+1})^\times$. Let

$$k_{\alpha+1} := k_\alpha + \varphi(p^{\alpha+1}) \qquad \text{and} \qquad \widetilde{z}_{\alpha+1} := \prod_{i=k_\alpha}^{k_{\alpha+1}-1} (1 + \widetilde{d}_i \pi_{\alpha+1}^i) \quad \in \mathbb{Z}_p(\pi_{\alpha+1})^\times$$

with each $0 \leq \widetilde{d}_i < p$ such that $\widetilde{d}_i^p \equiv d_i \bmod p$. Then by proposition 3.22, there is an element

$$z_{\alpha+1} = z'_{\alpha+1} \widetilde{z}_{\alpha+1} \quad \in \mathbb{Z}_p(\pi_{\alpha+1})$$

such that

$$\frac{-\varepsilon_{\alpha+1}}{z_{\alpha+1}^p} \equiv 1 + \sum_{i=k_{\alpha+1}}^{p^{\alpha+1}} \widetilde{d}_i \pi_{\alpha+1}^i \qquad \bmod (\pi_{\alpha+1}^{p^{\alpha+1}+1})$$

with $\widetilde{d}_{k_{\alpha+1}} \not\equiv 0 \bmod p$. Since

$$k_{\alpha+1} = k_\alpha + \varphi(p^{\alpha+1}) = p^{\alpha+1} - 2p + 1 < p^{\alpha+1},$$

we can apply lemma 3.27 to obtain that $-\varepsilon_{\alpha+1}$ is non-trivial in $U_1/\langle \mu, U_1^p \rangle$. □

Example 3.29. Let us have a look at the case $p = 3$. A straightforward calculation yields

$$-\varepsilon_2 \equiv 1 + \pi_2^3 - \pi_2^4 - \pi_2^5 - \pi_2^6 - \pi_2^7 + \pi_2^8 + \pi_2^9 \qquad \bmod (\pi_2^{10}).$$

Here $(p-2)p + 1 = 4$, so letting $z_2 = (1 + \pi_2)$ we get

$$\frac{-\varepsilon_2}{z_2^3} \equiv 1 - \pi_2^4 \quad \bmod (\pi_2^5).$$

As $4 < 3^2$, we may apply lemma 3.27 to obtain the result. Then if $\alpha = 3$ we have

$$-\varepsilon_3 \equiv 1 + \pi_3^9 - \pi_3^{12} - \pi_3^{15} - \pi_3^{18} - \pi_3^{21} + \pi_3^{24} + \pi_3^{27} \qquad \bmod (\pi_3^{28}).$$

Letting

$$z_3 = (1 + \pi_3^3)(1 - \pi_3^4)(1 - \pi_3^5)(1 - \pi_3^6),$$

so that

$$\begin{aligned} z_3^3 &\equiv (1 + \pi_3^3)^3 (1 - \pi_3^4)^3 (1 - \pi_3^5)^3 (1 - \pi_3^6)^3 \\ &\equiv 1 + \pi_3^9 - \pi_3^{12} - \pi_3^{15} - \pi_3^{18} - \pi_3^{21} + \pi_3^{22} \quad \bmod (\pi_3^{23}), \end{aligned}$$

we find

$$\frac{\varepsilon_3}{z_3^3} \equiv 1 - \pi_3^{22} \mod (\pi_3^{23}).$$

As $22 < 3^3$, we may again apply lemma 3.27 to obtain the result.

Corollary 3.30. *If p is odd, $\alpha \geq 2$ and $u \in \mathbb{Z}_p^\times$, then*

$$\frac{\varepsilon_\alpha}{u} \quad \textit{is non-trivial in} \quad \mathbb{Z}_p(F_0)^\times / \langle \mu(\mathbb{Q}_p(F_0)), (\mathbb{Z}_p(F_0)^\times)^p \rangle.$$

Proof. Let u_1 denote the projection of u onto $U_1(\mathbb{Z}_p^\times) \subseteq \mathbb{Z}_p^\times$. Then

$$u_1^{-1} = 1 + \sum_{i \geq 1} v_i p^i \equiv 1 + v_1 p \mod (\pi_\alpha^{p^\alpha + 1}),$$

for some $0 \leq v_i < p$. Clearly, the projection of $\varepsilon_\alpha u^{-1}$ in the group U_1 via the canonical decomposition $\mathbb{Z}_p^\times = \mu_{p'} \times U_1$ is equal to $-\varepsilon_\alpha u_1^{-1}$, and it is enough to check that $-\varepsilon_\alpha u_1^{-1}$ does not belong to $\langle \mu \cap U_1, U_1^p \rangle$.

If $\alpha = 2$, then

$$\begin{aligned} -\varepsilon_2 u_1^{-1} &\equiv -\varepsilon_2 && \mod (p) = (\pi_\alpha^{\varphi(p^2)}) \\ &\equiv 1 - \pi_\alpha^{(p-2)p+1} && \mod \langle \mu \cap U_1, U_1^p, (\pi_\alpha^{\varphi(p^2)}) \rangle, \end{aligned}$$

and the result follows from theorem 3.28.

If $\alpha \geq 3$, we know from the proof of theorem 3.28 that for a suitable $z_\alpha \in \mathbb{Z}_p(\pi_\alpha)^\times$ we have

$$\frac{-\varepsilon_\alpha}{z_\alpha^p} \equiv 1 + (-1)^\alpha \pi_\alpha^{k_\alpha} \mod (\pi_\alpha^{k_\alpha + 1}),$$

for $k_\alpha = p^\alpha - 2p + 1$, so that by proposition 3.22

$$\begin{aligned} \frac{-\varepsilon_\alpha}{u_1 z_\alpha^p} &\equiv (1 + (-1)^\alpha \pi_\alpha^{k_\alpha})(1 + v_1 p) \\ &\equiv 1 - v_1 \pi_\alpha^{\varphi(p^\alpha)} + (-1)^\alpha \pi_\alpha^{k_\alpha} \qquad \mod (\pi_\alpha^{k_\alpha + 1}). \end{aligned}$$

As

$$k_\alpha = (p-2)p + 1 + \sum_{i=3}^{\alpha} \varphi(p^i) < \sum_{i=2}^{\alpha} \varphi(p^i),$$

we obtain

$$\frac{-\varepsilon_\alpha}{u_1 z_\alpha^p y_\alpha^p} \equiv 1 + (-1)^\alpha \pi_\alpha^{k_\alpha} \mod (\pi_\alpha^{k_\alpha + 1}),$$

where after successive multiplications eliminating all terms in π_α^k for $1 < k < k_\alpha$, we have have used the element

$$y_\alpha = (1 - \widetilde{v}_1 \pi_\alpha^{\varphi(p^{\alpha-1})}) \prod_{k=2}^{\alpha-1} (1 + (-1)^{k-1} \pi_\alpha^{[\sum_{i=1}^{k} \varphi(p^{\alpha-i})]})$$

with $\widetilde{v}_1 \in \mathbb{Z}$ such that $(\widetilde{v}_1)^p \equiv v_1 \bmod p$. It follows that $-\varepsilon_\alpha u_1^{-1} \notin \langle \mu \cap U_1, U_1^p \rangle$. □

Corollary 3.31. *If p is odd and $F_0 = \mu(\mathbb{Q}_p(F_0))$, then $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if*

$$r_1 \quad \textit{divides} \quad \begin{cases} 1 & \textit{if } \alpha = 0, \\ p-1 & \textit{if } \alpha \geq 1. \end{cases}$$

Proof. The result for $\alpha = 0$ is obvious; so let $\alpha \geq 1$. Since $F_0 = \mu(\mathbb{Q}_p(F_0))$, corollary 3.6 applies if r_1 divides $p-1$. Because r_1 must divide the ramification index $\varphi(p^\alpha) = (p-1)p^{\alpha-1}$ of $\mathbb{Q}_p(F_0)$ over $\mathbb{Q}_p$, it remains to show that $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is empty whenever p divides r_1 with $\alpha \geq 2$. This however is a direct consequence of theorem 3.4 and corollary 3.30. □

3.4. The p-part of r_1 for $p = 2$

We now investigate the case where $p = 2$. Recall that $\alpha \geq 1$ is defined to satisfy $|F_0 \cap S_n| = 2^\alpha$. We know from example 3.9 that $\varepsilon_\alpha = 2^{-1}\pi_\alpha^{\varphi(2^\alpha)}$ always belongs to $(\mathbb{Z}_2(F_0)^\times)^2$ modulo the subgroup generated by $-\zeta_4$ if $\alpha \geq 2$. We will hence look for 4-th powers when $\alpha \geq 3$.

Let $\mu := \mu(\mathbb{Q}_2(F_0))$ denote the group of roots of unity in $\mathbb{Q}_2(F_0)$ and fix ζ_{2^α} a primitive 2^α-th root of unity in μ. As in the previous section we consider the decreasing filtration

$$\mathbb{Z}_2(F_0)^\times = U_0 \supseteq U_1 \supseteq U_2 \supseteq \dots$$

given by $U_0 = \mathbb{Z}_2(F_0)^\times$ and

$$U_i = \{x \in U_0 \mid x \equiv 1 \mod (\pi_\alpha^i)\} \qquad \text{for } i \geq 1,$$

where $U_0/U_1 = \mu_{2'}(\mathbb{Q}_2(F_0))$, and where U_i/U_{i+1} is isomorphic to the residue field of $\mathbb{Q}_2(F_0)$ for each $i \geq 1$. Recall that $(\pi_\alpha^{2^{\alpha-1}}) = (2)$. Define

$$Q_\alpha(X) := \frac{(X+1)^{2^\alpha} - 1}{(X+1)^{2^{\alpha-1}} - 1} = (X+1)^{2^{\alpha-1}} + 1 \quad \in \mathbb{Z}[X]$$

to be the minimal polynomial of π_α over $\mathbb{Q}_2$.

Lemma 3.32. *If $p = 2$ and $\alpha \geq 3$, then*

$$Q_\alpha(\pi_\alpha) \begin{cases} = 2 + 4\pi_\alpha + 6\pi_\alpha^2 + 4\pi_\alpha^3 + \pi_\alpha^4 & \text{if } \alpha = 3, \\ \equiv 2 + 4\pi_\alpha^{2^{\alpha-3}} + 6\pi_\alpha^{2^{\alpha-2}} + 4\pi_\alpha^{3\cdot 2^{\alpha-3}} + \pi_\alpha^{2^{\alpha-1}} \mod (8) & \text{if } \alpha \geq 4. \end{cases}$$

Proof. The result for $\alpha = 3$ is clear. When $\alpha \geq 4$, the result follows by induction on $4 \leq h \leq \alpha$ using the identity

$$\begin{aligned} (1+X)^{2^{h-1}} &= (1 + X^2 + 2X)^{2^{h-2}} \\ &\equiv (1+X^2)^{2^{h-2}} + 2^{h-1}X(1+X^2)^{2^{h-2}-1} \mod (2^h). \end{aligned}$$

□

Proposition 3.33. *If $p = 2$ and $\alpha \geq 2$, then*

$$2 \equiv \pi_\alpha^{\varphi(2^\alpha)} + \pi_\alpha^{\varphi(2^\alpha)+2^{\alpha-2}} \mod (\pi_\alpha^{2^\alpha}).$$

Proof. By lemma 3.32 we have

$$Q_\alpha(\pi_\alpha) \equiv 2 + 2\pi_\alpha^{2^{\alpha-2}} + \pi_\alpha^{2^{\alpha-1}} \mod (4).$$

Hence

$$\begin{aligned}
2 &\equiv -\pi_\alpha^{2^{\alpha-1}} - 2\pi_\alpha^{2^{\alpha-2}} \\
&\equiv \pi_\alpha^{2^{\alpha-1}} - (-\pi_\alpha^{2^{\alpha-1}} - 2\pi_\alpha^{2^{\alpha-2}})\pi_\alpha^{2^{\alpha-2}} - (-\pi_\alpha^{2^{\alpha-1}} - 2\pi_\alpha^{2^{\alpha-2}})\pi_\alpha^{2^{\alpha-1}} \\
&\equiv \pi_\alpha^{2^{\alpha-1}} + \pi_\alpha^{2^{\alpha-1}+2^{\alpha-2}} + 2\pi_\alpha^{2^{\alpha-1}} + \pi_\alpha^{2^\alpha} + 2\pi_\alpha^{2^{\alpha-1}+2^{\alpha-2}} \\
&\equiv \pi_\alpha^{2^{\alpha-1}} + \pi_\alpha^{2^{\alpha-1}+2^{\alpha-2}} - \pi_\alpha^{2^\alpha} + \pi_\alpha^{2^\alpha} \\
&\equiv \pi_\alpha^{2^{\alpha-1}} + \pi_\alpha^{2^{\alpha-1}+2^{\alpha-2}} \qquad \mathrm{mod}(\pi_\alpha^{2^\alpha}).
\end{aligned}$$

□

In the cases where $\mathbb{Q}_2(F_0)$ is completely ramified over $\mathbb{Q}_2$, for example if $\mu_{2'}(\mathbb{Q}_2(F_0))$ is trivial, we have $U_0 = U_1$. From the fact that $\varepsilon_\alpha \in \mathbb{Z}_2(\pi_\alpha) \subseteq \mathbb{Z}_2(F_0)^\times$ in general, it follows that ε_α always belongs to U_1.

Proposition 3.34. *If $p = 2$ and $\alpha \geq 3$, then*

$$\varepsilon_\alpha \equiv 1 + \pi_\alpha^{2\cdot 2^{\alpha-3}} + \pi_\alpha^{4\cdot 2^{\alpha-3}} + \pi_\alpha^{5\cdot 2^{\alpha-3}} + \pi_\alpha^{6\cdot 2^{\alpha-3}} \mod (\pi_\alpha^{2^\alpha}).$$

Proof. For simplicity we let $Z = \pi_\alpha^{2^{\alpha-3}}$. Since

$$-2 \equiv 4Z + 6Z^2 + 4Z^3 + Z^4 \mod (8)$$

by lemma 3.32, we obtain

$$\begin{aligned}
\frac{\pi_\alpha^{2^{\alpha-1}}}{2} &\equiv -1 - 2Z - 3Z^2 - 2Z^3 \\
&\equiv 1 + (Z^4 + 6Z^2 + 4Z + 4Z^3) + Z(Z^4 + 6Z^2 + 4Z + 4Z^3) \\
&\quad + Z^2 + Z^3(Z^4 + 6Z^2 + 4Z + 4Z^3) \\
&\equiv 1 + Z^2 + Z^4 + Z^5 + Z^7 + 2Z^2 + 2Z^3 + 2Z^5 \\
&\equiv 1 + Z^2 + Z^4 + Z^5 + Z^7 - Z^2(Z^4 + 6Z^2 + 4Z + 4Z^3) \\
&\quad - Z^3(Z^4 + 6Z^2 + 4Z + 4Z^3) - Z^5(Z^4 + 6Z^2 + 4Z + 4Z^3) \\
&\equiv 1 + Z^2 + Z^4 + Z^5 + Z^6 \qquad \mathrm{mod}(4).
\end{aligned}$$

□

We will now prove that ε_α is non-trivial in the quotient group $U_1/\langle \mu \cap U_1, U_1^4 \rangle$.

Lemma 3.35. *Let*

$$x = 1 + \sum_{i \geq k} \lambda_i \pi_\alpha^i \ \in \langle \mu \cap U_1, U_1^4 \rangle$$

with $\lambda_k \not\equiv 0 \bmod \pi_\alpha$ and each λ_i satisfying $\lambda_i^q = \lambda_i$ for q the cardinality of the residue field of $\mathbb{Q}_2(F_0)$. If $3 \leq k < 2^\alpha$, then $k \equiv 0 \bmod 4$. If $\alpha = 3$ and $k = 2$, then $\lambda_6 = 0$.

Proof. Recall that $\mu \cap U_1$ is generated by $1 + \pi_\alpha$, so that x is of the form

$$x = (1 + \pi_\alpha)^h y^4 \qquad \text{with } 0 \leq h \leq 3 \text{ and } y \in U_1.$$

If $3 \leq k < 2^\alpha$, it easily follows that $h = 0$ and x is a 4-th power. Furthermore, for any $a \in \mathbb{Z}_2(F_0)$ we have

$$\begin{aligned}(1 + a\pi_\alpha^i)^4 &= 1 + 4a\pi_\alpha^i + 6a^2\pi_\alpha^{2i} + 4a^3\pi_\alpha^{3i} + a^4\pi_\alpha^{4i} \\ &\equiv 1 + 6a^2\pi_\alpha^{2i} + a^4\pi_\alpha^{4i} \qquad \mod(\pi_\alpha^{i+2\varphi(2^\alpha)}).\end{aligned}$$

As

$$4i \leq \varphi(2^\alpha) + 2i \qquad \Leftrightarrow \qquad i \leq 2^{\alpha-2},$$

we obtain

$$(1 + a\pi_\alpha^i)^4 \equiv \begin{cases} 1 + a^4\pi_\alpha^{4i} & \mod(\pi_\alpha^{4i+1}) \quad \text{if } i < 2^{\alpha-2}, \\ 1 + (a^4 + a^2)\pi_\alpha^{2^\alpha} & \mod(\pi_\alpha^{2^\alpha+1}) \quad \text{if } i = 2^{\alpha-2}, \end{cases}$$

and the result for $3 \leq k < 2^\alpha$ follows.

Now suppose that $\alpha = 3$ and $k = 2$. In this case $h = 2$ and $y = 1 + b\pi_3$ for some $b \in \mathbb{Z}_2(F_0)$. Using proposition 3.33 we get

$$\begin{aligned}x &\equiv (1 + \pi_3)^2(1 + b\pi_3)^4 \\ &\equiv (1 + \pi_3^2 + \pi_3^5 + \pi_3^7)(1 + b^4\pi_3^4 + b^2\pi_3^6) \\ &\equiv 1 + \pi_3^2 + b^4\pi_3^4 + \pi_3^5 + (b^2 + b^4)\pi_3^6 + \pi_3^7 \qquad \mod(\pi_3^8).\end{aligned}$$

If $\lambda_4 = 0$, then $b \equiv 0 \mod(\pi_3)$ and consequently $\lambda_6 = 0$. On the other hand if $\lambda_4 = 1$, then $b \equiv 1 \mod(\pi_3)$ and once again $\lambda_6 = 0$. □

The idea of theorem 3.36 is the same as in the case $p > 2$: we divide the π_α-adic expression of ε_α by a 4-th power in U_1 in such a way that the resulting expression is in a form that allows lemma 3.35 to be used.

Theorem 3.36. *If $p = 2$ and $\alpha \geq 3$, then*

$$\varepsilon_\alpha \notin \langle \mu(\mathbb{Q}_2(F_0)), (\mathbb{Z}_2(F_0)^\times)^4 \rangle.$$

Proof. Since $\varepsilon_\alpha \in U_1$ it is enough to show that $\varepsilon_\alpha \notin \langle \mu \cap U_1, U_1^4 \rangle$.

In case $\alpha = 3$, we know from proposition 3.34 that

$$\varepsilon_3 \equiv 1 + \pi_3^2 + \pi_3^4 + \pi_3^5 + \pi_3^6 \mod(\pi_3^8),$$

and a direct application of lemma 3.35 yields the result.

Now assume $\alpha = 4$, and let $k_4 := 2^4 - 2 = 14$. By proposition 3.34

$$\varepsilon_4 \equiv 1 + \pi_4^4 + \pi_4^8 + \pi_4^{10} + \pi_4^{12} \mod(\pi_4^{16}).$$

As

$$\begin{aligned}(1 + \pi_4)^4(1 + \pi_4^2)^4 &\equiv (1 + \pi_4^4 + \pi_4^{10} + \pi_4^{14})(1 + \pi_4^8 + \pi_4^{12}) \\ &\equiv 1 + \pi_4^4 + \pi_4^8 + \pi_4^{10} + \pi_4^{12} + \pi_4^{14} \qquad \mod(\pi_4^{16}),\end{aligned}$$

letting

$$z_4 = (1 + \pi_4)(1 + \pi_4^2)(1 + \pi_4^3) \quad \text{in} \quad \mathbb{Z}_2(\pi_4)^\times,$$

we get

$$\frac{\varepsilon_4}{z_4^4} \equiv 1 + \pi_4^{k_4} \mod(\pi_4^{16}).$$

Hence by lemma 3.35, it follows that ε_4 does not belong to $\langle \mu \cap U_1, U_1^4 \rangle$.

Furthermore assume $\alpha \geq 4$, and let $k_\alpha := 2^\alpha - 2$. Suppose there is an element z_α in $\mathbb{Z}_2(\pi_\alpha)^\times$ such that

$$\frac{\varepsilon_\alpha}{z_\alpha^4} \equiv 1 + d_{k_\alpha} \pi_\alpha^{k_\alpha} \qquad \bmod (\pi_\alpha^{2^\alpha}),$$

with $d_{k_\alpha} \not\equiv 0 \bmod 2$ in the residue field. By corollary 3.14 and lemma 3.12 we have

$$\varepsilon_{\alpha+1} \equiv i_\alpha(\varepsilon_\alpha) \equiv (1 + d_{k_\alpha} \pi_{\alpha+1}^{2k_\alpha}) i_\alpha(z_\alpha)^4 \qquad \bmod (\pi_{\alpha+1}^{2^{\alpha+1}}),$$

so that

$$\frac{\varepsilon_{\alpha+1}}{(z'_{\alpha+1})^4} \equiv 1 + d_{k_\alpha} \pi_{\alpha+1}^{2k_\alpha} \qquad \bmod (\pi_{\alpha+1}^{2^{\alpha+1}})$$

for a suitable $z'_{\alpha+1}$ in $\mathbb{Z}_2(\pi_{\alpha+1})^\times$. Let $k_{\alpha+1} := k_\alpha + \varphi(2^{\alpha+1}) = 2^{\alpha+1} - 2$, and let

$$\widetilde{z}_{\alpha+1} := 1 + \widetilde{d}_{k_\alpha} \pi_{\alpha+1}^{\frac{k_\alpha}{2}} \quad \in \mathbb{Z}_2(\pi_{\alpha+1})^\times$$

with $\widetilde{d}_{k_\alpha}$ in the residue field such that $\widetilde{d}_{k_\alpha}^4 \equiv d_{k_\alpha} \bmod 2$. Note that

$$(1 + \widetilde{d}_{k_\alpha} \pi_{\alpha+1}^{\frac{k_\alpha}{2}})^4 \equiv 1 + d_{k_\alpha} \pi_{\alpha+1}^{2k_\alpha} + \widetilde{d}_{k_\alpha}^2 \pi_{\alpha+1}^{2^\alpha + k_\alpha} \qquad \bmod (\pi_{\alpha+1}^{2^{\alpha+1}}),$$

where $2^\alpha + k_\alpha = 2^{\alpha+1} - 2$. Then by proposition 3.33, letting

$$z_{\alpha+1} = z'_{\alpha+1} \widetilde{z}_{\alpha+1} \quad \in \mathbb{Z}_2(\pi_{\alpha+1})^\times,$$

we have

$$\frac{\varepsilon_{\alpha+1}}{z_{\alpha+1}^4} \equiv 1 + d'_{k_{\alpha+1}} \pi_{\alpha+1}^{k_{\alpha+1}} \qquad \bmod (\pi_{\alpha+1}^{2^{\alpha+1}}),$$

with $d'_{k_{\alpha+1}} \not\equiv 0 \bmod 2$ in the residue field. Since $k_{\alpha+1} < 2^{\alpha+1}$, we can apply lemma 3.35 to obtain that $\varepsilon_{\alpha+1}$ is non-trivial in $U_1 / \langle \mu, U_1^p \rangle$. The result then follows by induction on $\alpha \geq 4$. □

Corollary 3.37. *If $p = 2$, $\alpha \geq 3$ and $u \in \mathbb{Z}_2^\times$, then*

$$\frac{\varepsilon_\alpha}{u} \quad \textit{is non-trivial in} \quad \mathbb{Z}_2(F_0)^\times / \langle \mu(\mathbb{Q}_2(F_0)), (\mathbb{Z}_2(F_0)^\times)^4 \rangle.$$

Proof. Since $u \in \mathbb{Z}_2^\times$, its inverse is of the form

$$\begin{aligned} u^{-1} &= 1 + \sum_{i \geq 1} v_i 2^i \\ &\equiv 1 + v_1 2 \\ &\equiv 1 + v_1 (\pi_\alpha^{2^{\alpha-1}} + \pi_\alpha^{2^{\alpha-1} + 2^{\alpha-2}}) \qquad \bmod (4) = (\pi_\alpha^{2^\alpha}), \end{aligned}$$

where $v_i \in \{0, 1\}$ and where the last equivalence follows from proposition 3.33. As in theorem 3.36 it is enough to show that $\varepsilon_\alpha u^{-1}$ does not belong to $\langle \mu \cap U_1, U_1^4 \rangle$.

If $\alpha = 3$, we know from proposition 3.34 that

$$\varepsilon_3 \equiv 1 + \pi_3^2 + \pi_3^4 + \pi_3^5 + \pi_3^6 \quad \bmod (\pi_3^8).$$

Hence

$$\begin{aligned}\varepsilon_3 u^{-1} &\equiv (1+\pi_3^2+\pi_3^4+\pi_3^5+\pi_3^6)(1+v_1\pi_3^4+v_1\pi_3^6)\\ &\equiv 1+\pi_3^2+(1+v_1)\pi_3^4+\pi_3^5+(1+2v_1)\pi_3^6\\ &\equiv 1+\pi_3^2+(1+v_1)\pi_3^4+\pi_3^5+\pi_3^6 \qquad \bmod(\pi_3^8),\end{aligned}$$

and lemma 3.35 implies $\varepsilon_3 u^{-1} \notin \langle \mu \cap U_1, U_1^4\rangle$.

Now if $\alpha \geq 4$, we know from theorem 3.36 that for a suitable $z_\alpha \in \mathbb{Z}_2(\pi_\alpha)^\times$ we have

$$\frac{\varepsilon_\alpha}{z^4} \equiv 1+\pi_\alpha^{k_\alpha} \mod(\pi_\alpha^{k_\alpha+1})$$

for $k_\alpha = 2^\alpha - 2$, so that by proposition 3.33

$$\frac{\varepsilon_\alpha}{u z_\alpha^4} \equiv 1+v_1\pi_\alpha^{\varphi(2^\alpha)}+v_1\pi_\alpha^{\varphi(2^\alpha)+2^{\alpha-2}}+\pi_\alpha^{k_\alpha} \qquad \bmod(\pi_\alpha^{k_\alpha+1}).$$

Letting

$$y_\alpha = (1+v_1\pi_\alpha^{\varphi(2^{\alpha-2})}),$$

we have

$$\begin{aligned}y_\alpha^4 &\equiv 1+v_1\pi_\alpha^{\varphi(2^\alpha)}+2v_1\pi_\alpha^{2^{\alpha-2}}\\ &\equiv 1+v_1\pi_\alpha^{\varphi(2^\alpha)}+v_1\pi_\alpha^{\varphi(2^\alpha)+2^{\alpha-2}} \qquad \bmod(\pi_\alpha^{2^\alpha}).\end{aligned}$$

Hence by lemma 3.35

$$\frac{\varepsilon_\alpha}{u z^4 y_\alpha^4} \equiv 1+\pi_\alpha^{k_\alpha} \mod(\pi_\alpha^{k_\alpha+1}),$$

and $\varepsilon_\alpha u^{-1} \notin \langle \mu \cap U_1, U_1^4\rangle$. □

Corollary 3.38. *If $p=2$, then $\widetilde{\mathcal{F}}_u(\mathbb{Q}_2(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if*

$$r_1 \quad \textit{divides} \quad \begin{cases} 2 & \textit{if } \alpha \geq 2 \textit{ with either } u \equiv \pm 1 \bmod 8 \textit{ or } \zeta_3 \in F_0,\\ 1 & \textit{if } \alpha \leq 1, \textit{ or } u \equiv \pm 3 \bmod 8 \textit{ and } \zeta_3 \notin F_0.\end{cases}$$

Proof. The case $\alpha \leq 1$ is clear; so let $\alpha \geq 2$. A necessary condition for $\widetilde{\mathcal{F}}_u(\mathbb{Q}_2(F_0), \widetilde{F_0}, r_1)$ to be non-empty is that r_1 divides $2^{\alpha-1}$, the ramification index of $\mathbb{Q}_2(F_0)$ over $\mathbb{Q}_2$. If $\alpha = 2$, then r_1 divides 2. Otherwise if $\alpha \geq 3$, corollary 3.37 and theorem 3.4 imply that r_1 must also be a divisor of 2.

By theorem 3.4, the integer r_1 may be any divisor of 2 if and only if $\frac{\varepsilon_\alpha}{u}$ is a square of $\mathbb{Z}_2(F_0)^\times$ modulo F_0. In fact this is true if and only if u is a square of $\mathbb{Z}_2(F_0)^\times$ modulo F_0, since by example 3.9 we have $\varepsilon_\alpha \in \langle (\mathbb{Z}_2(F_0)^\times)^2, F_0\rangle$. The result is then obvious if $u \equiv \pm 1 \bmod 8$. Otherwise if $u \equiv \pm 3 \bmod 8$ we have $u = \pm 3z^2$ for some $z \in \mathbb{Z}_2^\times$, and it remains to verify that -3 belongs to $\langle (\mathbb{Z}_2(F_0)^\times)^2, F_0\rangle$ if and only if F_0 contains a 3rd root of unity ζ_3. If such a ζ_3 exists, we let $\rho = 2\zeta_3 + 1$, so that

$$\rho^2 = -3 \qquad \text{and} \qquad \mathbb{Q}_2(\rho) = \mathbb{Q}_2(\zeta_3).$$

Conversely if there is a ρ such that $\rho^2 = -3$, we take $\zeta_3 = \frac{1}{2}(\rho - 1)$. □

Corollary 3.39. *If $p = 2$, $F_0 = \mu(\mathbb{Q}_2(F_0))$, $[\mathbb{Q}_2(F_0) : \mathbb{Q}_2] = n$, then $\widetilde{\mathcal{F}}_u(\mathbb{Q}_2(F_0), \widetilde{F_0}, r_1)$ is non-empty if and only if*

$$r_1 \quad \textit{divides} \quad \begin{cases} 2 & \textit{if } \alpha \geq 2 \textit{ with either } u \equiv \pm 1 \bmod 8 \textit{ or } n_\alpha \textit{ even,} \\ 1 & \textit{if } \alpha \leq 1, \textit{ or } u \equiv \pm 3 \bmod 8 \textit{ and } n_\alpha \textit{ odd.} \end{cases}$$

Proof. Under these assumptions, $F_0 \cong C_{2^{\alpha}(2^{n_\alpha}-1)}$ by proposition C.8. The result follows from corollary 3.38 and the fact $\zeta_3 \in F_0$ if and only if n_α is even. □

Remark 3.40. When $F_0 = \mu(\mathbb{Q}_2(F_0))$, we know by corollary 2.18 that $\widetilde{F_1}$ is the unique element of $\widetilde{\mathcal{F}}_u(\mathbb{Q}_2(F_0), \widetilde{F_0}, r_1)$. We may therefore assume $\widetilde{F_1}$ to be of the form

$$\widetilde{F_1} = \langle x_1 \rangle \times F_0 \qquad \text{with} \qquad x_1 = \begin{cases} 2u & \text{if } r_1 = 1, \\ (1+i)t & \text{if } r_1 = 2, \end{cases}$$

for i a primitive 4-th root of unity in $\mathbb{Q}_2(F_0)^\times$ and

$$t \in \begin{cases} \mathbb{Z}_2^\times & \text{if } u \equiv \pm 1 \bmod 8, \\ \mathbb{Z}_2(\zeta_3)^\times & \text{if } u \equiv \pm 3 \bmod 8, \end{cases} \qquad \text{with} \qquad t^2 = \begin{cases} u & \text{if } u \equiv 1 \text{ or } -3 \bmod 8, \\ -u & \text{if } u \equiv -1 \text{ or } 3 \bmod 8. \end{cases}$$

3.5. The determination of r_2

We fix F_0 and r_1 such that $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty, and fix an element $\widetilde{F_1}$ in $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$. Corollary 3.31 and 3.38 provide conditions on r_1 for this to happen, in which case, according to remark 2.17, there is an element $x_1 \in \mathbb{Q}_p(F_0)$ satisfying

$$v(x_1) = \frac{1}{r_1} \qquad \text{and} \qquad \widetilde{F_1} = \langle F_0, x_1 \rangle.$$

We want to determine for which integer r_2 the set $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$ is non-empty, that is, for which r_2 dividing $\frac{n}{r_1}$ there exists an element $x_2 \in \mathbb{D}_n^\times$ such that $x_2^{r_2} = a$ with $a \in \widetilde{F_1}$ and $v(a) = v(x_1)$, and such that $\mathbb{Q}_p(F_0, x_2)$ is a commutative field extension of $\mathbb{Q}_p(F_0) = \mathbb{Q}_p(\widetilde{F_1})$.

As seen in theorem 2.21, the existence of such an x_2 is equivalent to the irreducibility of the polynomial $X^{r_2} - a$ over $\mathbb{Q}_p(F_0)$ with r_2 dividing $n[\mathbb{Q}_p(F_0) : \mathbb{Q}_p]^{-1}$.

Theorem 3.41. *Let K be a field, $a \in K^\times$ and $r \geq 2$ an integer. Then $X^r - a$ is irreducible over K if and only if for all primes q dividing r the class $a \in H^2(\mathbb{Z}/r, K^\times)$ is non-trivial in $H^2(\mathbb{Z}/q, K^\times)$, and if $\frac{-a}{4}$ is non-trivial in $H^2(\mathbb{Z}/4, K^\times)$ when 4 divides r, where all cohomology groups are with trivial modules.*

Proof. This is just a cohomological interpretation of [14] chapter VI theorem 9.1. □

In general, there is a well defined map

$$\Xi : \ a \longmapsto K[X]/(X^r - a)$$

from $H^2(\mathbb{Z}/r, K^\times)$ to the set of isomorphism classes of algebra extensions of K by equations of the form $X^r - a = 0$. This map is injective: if there is an extension in which $X^r - a = 0$

and $X^r - b = 0$ both have solutions, then $\frac{a}{b}$ is a r-th power and becomes trivial in $H^2(\mathbb{Z}/r, K^\times)$. Denote by

$$H^2_F(\mathbb{Z}/r, K^\times) \subseteq H^2(\mathbb{Z}/r, K^\times)$$

the subset of all elements of $H^2(\mathbb{Z}/r, K^\times)$ that are sent to a commutative field extension of K via Ξ. Furthermore assuming that $K = \mathbb{Q}_p(F_0)$ has ramification index $e(\mathbb{Q}_p(F_0))$ over $\mathbb{Q}_p$, we consider the homomorphism

$$i^* : H^2(\mathbb{Z}/r, \widetilde{F_1}) \longrightarrow H^2(\mathbb{Z}/r, \mathbb{Q}_p(F_0)^\times)$$

induced by the inclusion $\widetilde{F_1} \subseteq \mathbb{Q}_p(F_0)^\times$. We are interested in understanding the set

$$H^2_F(\mathbb{Z}/r, \mathbb{Q}_p(F_0)^\times) \cap i^*(H^2(\mathbb{Z}/r, \widetilde{F_1})).$$

Note that we have a non-canonically split exact sequence

$$1 \longrightarrow \mu(\mathbb{Q}_p(F_0)) \times \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]} \longrightarrow \mathbb{Q}_p(F_0)^\times \longrightarrow \langle \pi_{F_0} \rangle \longrightarrow 1,$$

for π_{F_0} a uniformizing element in $\mathbb{Q}_p(F_0)$. Moreover there is a commutative diagram

$$\begin{array}{ccc} H^2(\mathbb{Z}/r, \widetilde{F_1}) & \xrightarrow{i^*} & H^2(\mathbb{Z}/r, \mathbb{Q}_p(F_0)^\times) \\ \downarrow & & \downarrow \\ H^2(\mathbb{Z}/r, \langle x_1 \rangle) & \longrightarrow & H^2(\mathbb{Z}/r, \langle \pi_{F_0} \rangle) \\ \| & & \| \\ \mathbb{Z}/r & \longrightarrow & \mathbb{Z}/r, \end{array} \tag{3.2}$$

where the top vertical arrows are non-canonically split surjective homomorphisms respectively induced by the canonical surjections

$$\widetilde{F_1} \cong F_0 \times \langle x_1 \rangle \longrightarrow \langle x_1 \rangle \qquad \text{and} \qquad \mathbb{Q}_p(F_0)^\times \longrightarrow \langle \pi_{F_0} \rangle,$$

and where the bottom horizontal map is the identity if $\mathbb{Q}_p(F_0)$ is unramified over $\mathbb{Q}_p$, or otherwise is by multiplication with

$$\frac{e(\mathbb{Q}_p(F_0))}{r_1} = \begin{cases} p^{\alpha-1}\frac{p-1}{r_1} & \text{if } p > 2, \\ \frac{2^{\alpha-1}}{r_1} & \text{if } p = 2. \end{cases}$$

Define r_{F_0,r_1} to be the greatest divisor of $\frac{n}{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}$ which is prime to

$$\begin{cases} 1 & \text{if } e(\mathbb{Q}_p(F_0)) = 1, \\ \frac{p-1}{r_1} & \text{if } p > 2 \text{ and } \alpha \geq 1, \\ \frac{2}{r_1} & \text{if } p = 2,\ \alpha \geq 2 \text{ and either } u \equiv \pm 1 \bmod 8 \text{ or } \zeta_3 \in F_0, \\ 1 & \text{if } p = 2,\ u \not\equiv \pm 1 \bmod 8 \text{ and } \zeta_3 \notin F_0. \end{cases}$$

Theorem 3.42. *Suppose $p > 2$ and $\widetilde{F_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1) \neq \emptyset$. If r_2 divides r_{F_0,r_1}, then $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$ is non-empty.*

Proof. First note that if $\alpha = 0$, we must have $r_1 = 1$ and x_1 is a uniformizing element of the unramified extension $\mathbb{Q}_p(F_0)/\mathbb{Q}_p$. In this case $r_{F_0,r_1} = \frac{n}{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}$ and the result follows from the embedding theorem.

Now assume that $\alpha \geq 1$, and let $r' = r'_{F_0,r_1}$ denote the p'-part of $r = r_{F_0,r_1}$. Then for any prime q dividing r', diagram (3.2) can be extended to the commutative diagram

$$\begin{array}{ccccc} H^2(\mathbb{Z}/r, \widetilde{F_1}) & \xrightarrow{i^*} & H^2(\mathbb{Z}/r, \mathbb{Q}_p(F_0)^\times) & \xrightarrow{j^*} & H^2(\mathbb{Z}/q, \mathbb{Q}_p(F_0)^\times) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}/r & \longrightarrow & \mathbb{Z}/r & \twoheadrightarrow & \mathbb{Z}/q, \end{array}$$

where $j : \mathbb{Z}/q \to \mathbb{Z}/r$ is the inclusion and the bottom right horizontal map is the canonical projection. Since r is prime to $\frac{p-1}{r_1}$, it follows that r', and hence q, are prime to $\frac{e(\mathbb{Q}_p(F_0))}{r_1}$. Thus for any $\delta \in F_0$, the image of $x_1\delta \in \widetilde{F_1}$ is non-trivial in $\mathbb{Z}/q = H^2(\mathbb{Z}/q, \langle \pi_{F_0} \rangle)$, and consequently non-trivial in $H^2(\mathbb{Z}/q, \mathbb{Q}_p(F_0)^\times)$. In a similar way, it is equally non-trivial in $H^2(\mathbb{Z}/4, \mathbb{Q}_p(F_0)^\times)$ if 4 divides r. The result then follows from theorem 3.41 and corollary 3.30, where we have shown that x_1^{p-1}, and hence x_1, is non-trivial in $H^2(\mathbb{Z}/p, \mathbb{Q}_p(F_0)^\times)$. □

Theorem 3.43. *Suppose $p = 2$ and $\widetilde{F_1} \in \widetilde{\mathcal{F}}_u(\mathbb{Q}_2(F_0), \widetilde{F_0}, r_1) \neq \emptyset$. If r_2 divides r_{F_0,r_1}, then $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$ is non-empty.*

Proof. First note that if $\alpha \leq 1$, we must have $r_1 = 1$ and x_1 is a uniformizing element of the unramified extension $\mathbb{Q}_2(F_0)/\mathbb{Q}_2$. In this case $r_{F_0,r_1} = \frac{n}{[\mathbb{Q}_2(F_0):\mathbb{Q}_2]}$ and the result follows from the embedding theorem.

Now assume that $\alpha \geq 2$. If r_2 is divisible by 2, then so is r_{F_0,r_1} and we know from corollary 3.38 that x_1 is non-trivial in $H^2(\mathbb{Z}/2, \mathbb{Q}_2(F_0)^\times)$. Moreover if r_2 is divisible by 4, the fact that

$$(1 + \zeta_4)^4 = -4$$

imply that $\frac{-x_1}{4}$ is non-trivial in $H^2(\mathbb{Z}/4, \mathbb{Q}_2(F_0)^\times)$. Furthermore for any odd prime q dividing $r = r_{F_0,r_1}$, diagram (3.2) can be extended to the commutative diagram

$$\begin{array}{ccccc} H^2(\mathbb{Z}/r, \widetilde{F_1}) & \xrightarrow{i^*} & H^2(\mathbb{Z}/r, \mathbb{Q}_2(F_0)^\times) & \xrightarrow{j^*} & H^2(\mathbb{Z}/q, \mathbb{Q}_2(F_0)^\times) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}/r & \longrightarrow & \mathbb{Z}/r & \twoheadrightarrow & \mathbb{Z}/q, \end{array}$$

where $j : \mathbb{Z}/q \to \mathbb{Z}/r$ is the inclusion and the bottom right horizontal map is the canonical projection. As q is prime to $\frac{2}{r_1}$, the image of x_1 is non-trivial in $\mathbb{Z}/q = H^2(\mathbb{Z}/q, \langle \pi_{F_0} \rangle)$, and consequently non-trivial in $H^2(\mathbb{Z}/q, \mathbb{Q}_2(F_0)^\times)$. We may thus apply theorem 3.41 to obtain the desired result. □

We say that r_1 is *maximal* if $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$ is non-empty and $\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r)$ is empty whenever $r > r_1$.

Corollary 3.44. *Let p be any prime. If r_1 is maximal, then*

$$\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2) \neq \emptyset \qquad \text{if and only if} \qquad r_2 \mid \frac{n}{[\mathbb{Q}_p(F_0) : \mathbb{Q}_p]},$$

and any element $\widetilde{F_2}$ in $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, n[\mathbb{Q}_p(F_0) : \mathbb{Q}_p]^{-1})$ generates a maximal commutative field $\mathbb{Q}_p(\widetilde{F_2})$ in $\mathbb{D}_n$. Moreover if $F_0 = \mu(\mathbb{Q}_p(F_0))$, the number of such field extensions is equal to

$$|H^2(\mathbb{Z}/r_2, F_0)| = |F_0 \otimes \mathbb{Z}/r_2|.$$

Proof. By the maximality of r_1 we have

$$r_{F_0,r_1} = \frac{n}{[\mathbb{Q}_p(F_0) : \mathbb{Q}_p]}$$

and $\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2) \neq \emptyset$ implies $r_2 \mid \frac{n}{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}$. The first assertion then follows from theorem 3.42 and 3.43.

As for the last assertion, if $F_0 = \mu(\mathbb{Q}_p(F_0))$, diagram (3.2) can be extended, via the short exact sequences

$$1 \longrightarrow F_0 \longrightarrow \widetilde{F_1} \longrightarrow \langle x_1 \rangle \longrightarrow 1,$$
$$1 \longrightarrow F_0 \times \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]} \longrightarrow \mathbb{Q}_p(F_0)^\times \longrightarrow \langle \pi_{F_0} \rangle \longrightarrow 1,$$
$$1 \longrightarrow F_0 \longrightarrow F_0 \times \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]} \longrightarrow \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]} \longrightarrow 1,$$

to the exact diagram

$$\begin{array}{ccccc}
 & & H^1(\mathbb{Z}/r_2, \langle x_1 \rangle) & & H^1(\mathbb{Z}/r_2, \langle \pi_{F_0} \rangle) \\
 & & \downarrow & & \downarrow \\
H^1(\mathbb{Z}/r_2, \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}) & \longrightarrow & H^2(\mathbb{Z}/r_2, F_0) & \longrightarrow & H^2(\mathbb{Z}/r_2, F_0 \times \mathbb{Z}_p^{[\mathbb{Q}_p(F_0):\mathbb{Q}_p]}) \\
 & & \downarrow & & \downarrow \\
 & & H^2(\mathbb{Z}/r_2, \widetilde{F_1}) & \xrightarrow{i^*} & H^2(\mathbb{Z}/r_2, \mathbb{Q}_p(F_0)^\times) \\
 & & \downarrow & & \downarrow \\
 & & H^2(\mathbb{Z}/r_2, \langle x_1 \rangle) & \longrightarrow & H^2(\mathbb{Z}/r_2, \langle \pi_{F_0} \rangle),
\end{array}$$

where all three first cohomology groups are trivial, and where we know from diagram (3.2) that the bottom vertical maps are surjective. In particular i^* is injective on the kernels of the bottom vertical maps, and therefore the number of maximal fields of the form $\mathbb{Q}_p(\widetilde{F_2})$ in $\mathbb{D}_n$ is given by the cardinality of $H^2(\mathbb{Z}/r_2, F_0)$. □

Chapter 4:

On maximal finite subgroups of $\mathbb{G}_n(u)$

We consider a prime p, a positive integer $n = (p-1)p^{k-1}m$ with m prime to p, and a unit $u \in \mathbb{Z}_p^\times$. In this chapter, we work in the context of section 2.6 in order to study the classes of maximal nonabelian finite subgroups of $\mathbb{G}_n(u)$.

More particularly, we consider (nonabelian) finite extensions of $\widetilde{F_2}$ when F_0 is maximal as an abelian finite subgroup of $\mathbb{S}_n$ and the field $L = \mathbb{Q}_p(\widetilde{F_2})$ is maximal in $\mathbb{D}_n$. In this case $C_{\mathbb{D}_n^\times}(\widetilde{F_2}) = L^\times$ and we have a short exact sequence

$$1 \longrightarrow \widetilde{F_2} \longrightarrow L^\times \longrightarrow L^\times/\widetilde{F_2} \longrightarrow 1,$$

which for $W \subseteq Aut(L, \widetilde{F_2}, F_0) \subseteq Gal(L/\mathbb{Q}_p)$ induces the long exact sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow H^1(W, L^\times) & \longrightarrow & H^1(W, L^\times/\widetilde{F_2}) & & Br(L/L^W) & & \\ \| & & \downarrow & & \| & & \\ 0 & & H^2(W, \widetilde{F_2}) & \xrightarrow{i_W^*} & H^2(W, L^\times) & \longrightarrow & \cdots, \end{array}$$

where the left hand term is trivial by Hilbert's theorem 90 (see for example [15] chapter IV theorem 3.5). Then theorem 2.27 and 2.28 become explicit if we can determine the homomorphism i_W^*. We will use the following fact extensively.

Proposition 4.1. *If i_W^* is an epimorphism, then $i_{W'}^*$ is an epimorphism for every subgroup W' of W.*

Proof. The bifunctoriality of the cohomology induces a commutative square

$$\begin{array}{ccc} H^2(W, \widetilde{F_2}) & \xrightarrow{i_W^*} & H^2(W, L^\times) \\ \downarrow & & \downarrow \\ H^2(W', \widetilde{F_2}) & \xrightarrow{i_{W'}^*} & H^2(W', L^\times). \end{array}$$

By corollary B.12, the right hand map of this square is surjective. Hence if i_W^* is surjective, the bottom horizontal homomorphism is surjective as well. □

The cases $p > 2$ and $p = 2$ are treated separately.

4.1. Extensions of maximal abelian finite subgroups of $\mathbb{S}_n$ for $p > 2$

In this section, we assume p to be odd, F_0 to be maximal abelian, and $\widetilde{F_1}$ to be maximal as a subgroup of $\mathbb{Q}_p(F_0)^\times$ having $\widetilde{F_0}$ as a subgroup of finite index; in other words

$$F_0 \cong C_{p^\alpha} \times C_{p^{n_\alpha}-1} \qquad \text{with} \quad 0 \leq \alpha \leq k, \quad n_\alpha = \frac{n}{\varphi(p^\alpha)},$$

and

$$\widetilde{F_1} = \begin{cases} \widetilde{F_0} = F_0 \times \langle pu \rangle & \text{if } \alpha = 0, \\ F_0 \times \langle x_1 \rangle & \text{if } \alpha \geq 1, \end{cases}$$

where in the last case $x_1 \in \mathbb{Q}_p(\zeta_p) \subseteq \mathbb{Q}_p(F_0)$ satisfies

$$v(x_1) = \frac{1}{p-1} \qquad \text{and} \qquad x_1^{p-1} \in \widetilde{F_0} = F_0 \times \langle pu \rangle.$$

In fact we may assume x_1 to satisfy $x_1^{p-1} = pu\delta$ for $\delta \in \mu_{p-1}(\mathbb{Q}_p(\zeta_p))$ as given in corollary 3.6 and remark 3.7. By definition $\mathbb{Q}_p(F_0) = \mathbb{Q}_p(\widetilde{F_1})$, and because the latter is a maximal subfield of $\mathbb{D}_n$ we have $\widetilde{F_1} = \widetilde{F_2}$. We let

$$G := Gal(\mathbb{Q}_p(F_0)/\mathbb{Q}_p) \cong \begin{cases} C_n & \text{if } \alpha = 0, \\ C_{p-1} \times C_{p^{\alpha-1}} \times C_{n_\alpha} & \text{if } \alpha \geq 1, \end{cases}$$

as given by proposition C.8. From our choice of x_1, we know that $\widetilde{F_1}$ is stable under the action of a subgroup $W \subseteq G$; this is because if $\sigma \in W$, then $\frac{\sigma(x_1)}{x_1}$ is a $(p-1)$-th root of unity in $\mathbb{Q}_p^\times$, and hence $\sigma(x_1) \in x_1\langle\zeta_{p-1}\rangle \subseteq \widetilde{F_1}$ for $\zeta_{p-1} \in \mathbb{Q}_p^\times$. The goal of the section is to determine necessary and sufficient conditions on n, p, u and α for the homomorphism

$$i_G^* : H^2(G, \widetilde{F_1}) \longrightarrow H^2(G, \mathbb{Q}_p(F_0)^\times)$$

to be surjective, and whenever this happens, we want to determine its kernel. This is done via the analysis of

$$i_W^* : H^2(W, \widetilde{F_1}) \longrightarrow H^2(W, \mathbb{Q}_p(F_0)^\times)$$

for suitable subgroups $W \subseteq G$.

The case $\alpha = 0$

The situation is much simpler when the p-Sylow subgroup of F_0 is trivial.

Lemma 4.2. *If $\alpha = 0$ and $W = C_n$, then*

$$H^*(W, \widetilde{F_1}) \cong \begin{cases} \langle pu \rangle \times C_{p-1} & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle pu \rangle / \langle (pu)^n \rangle & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Q}_p(F_0)^\times) \cong \begin{cases} \mathbb{Q}_p^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle p \rangle / \langle p^n \rangle & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. The action of $C_n = W$ on $\widetilde{F_1} \cong \langle pu \rangle \times C_{p^n-1}$ is trivial on $\langle pu \rangle$ and acts on C_{p^n-1} by $\zeta \mapsto \zeta^p$.

For t a generator of C_n, written additively, and $N = \sum_{i=0}^{n-1} t^i$, $H^*(C_n, \widetilde{F_1})$ is the cohomology of the complex

$$\widetilde{F_1} \xrightarrow{1-t} \widetilde{F_1} \xrightarrow{N} \widetilde{F_1} \xrightarrow{1-t} \cdots .$$

Using additive notation for $\widetilde{F_1} \cong \mathbb{Z} \times \mathbb{Z}/p^n - 1$, we obtain

$$\begin{aligned} (1-t)(1,0) &= (0,0), & (1-t)(0,1) &= (0, 1-p), \\ N(1,0) &= (n,0), & N(0,1) &= (0, \frac{p^n-1}{p-1}), \end{aligned}$$

and the desired result for $H^*(W, \widetilde{F_1})$ follows.

Now let $L = \mathbb{Q}_p(F_0) = \mathbb{Q}_p(\widetilde{F_1})$ and $K = L^W$. Then

$$H^0(W, \mathbb{Q}_p(F_0)^\times) = K^\times = \mathbb{Q}_p^\times$$

and $H^1(W, \mathbb{Q}_p(\widetilde{F_1})) = 0$ by Hilbert's theorem 90. Furthermore, since L/K is unramified, we know from proposition B.13 that the valuation map induces an isomorphism

$$H^2(W, L^\times) \cong H^2(W, \frac{1}{e(L)}\mathbb{Z}) \cong \mathbb{Z}/|W|\mathbb{Z}.$$

Here $e(L) = 1$, and as $v(p) = 1$, the element p represents a generator of the cyclic group $H^2(W, L^\times)$. The result then follows from the periodicity of the cohomology. □

Corollary 4.3. *If $\alpha = 0$ and $W \subseteq C_n$, then i_W^* is an isomorphism.*

Proof. Let $L = \mathbb{Q}_p(F_0)$ and $K = L^{C_n} = \mathbb{Q}_p$. Since L/K is unramified, $\mathcal{O}_K^\times$ is in the image of the norm by proposition B.13 and $u \in N_{L/K}(L^\times)$. Hence $i_{C_n}^*(pu) = p$ and $i_{C_n}^*$ is an isomorphism. For any subgroup $W \subseteq C_n$, it follows from proposition 4.1 that i_W^* is an epimorphism, and hence from lemma 4.2 that it is an isomorphism. □

Example 4.4. Let $\alpha = 0$ and $F_0 \cong C_{p^n-1}$ generated by a primitive (p^n-1)-th root of unity ω. Since $\mathbb{Q}_p(F_0)/\mathbb{Q}_p$ is a maximal unramified commutative extension in $\mathbb{D}_n$, we have $\widetilde{F_0} = \widetilde{F_1} = \widetilde{F_2}$. Furthermore, as noted in remark C.5, there is an element ξ_u in $\mathbb{D}_n^\times$ that generates the Frobenius σ of $\mathbb{Q}_p(\omega)$ in such a way that

$$\mathbb{D}_n \cong \mathbb{Q}_p(\omega)\langle \xi_u \rangle / (\xi_u^n = pu,\ \xi_u x = x^\sigma \xi_u) \qquad \text{and} \qquad \omega^\sigma = \omega^p.$$

Hence for any $u \in \mathbb{Z}_p^\times$, $\widetilde{F_3} \cong F_0 \rtimes \langle \xi_u \rangle$. In $\mathbb{G}_n(u)$, we therefore have an extension

$$1 \longrightarrow F_0 \longrightarrow F_3 \longrightarrow C_n \longrightarrow 1$$

with $C_n \cong Gal(\mathbb{Q}_p(F_0)/\mathbb{Q}_p)$ acting faithfully on the kernel and

$$F_3 = \langle \omega, \overline{\xi}_u \mid \omega^{p^n-1} = \overline{\xi}_u^n = 1,\ \overline{\xi}_u \omega \overline{\xi}_u^{-1} = \omega^p \rangle \cong C_{p^n-1} \rtimes C_n$$

for $\overline{\xi}_u$ the class of ξ_u in $\mathbb{G}_n(u)$.

The case $\alpha \geq 1$

For the rest of the section we let $\alpha \geq 1$. The Galois group G of $\mathbb{Q}_p(F_0)/\mathbb{Q}_p$ decomposes canonically as

$$G = G_p \times G_{p'},$$

where $G_p = C_{p^{\alpha-1}} \times C_{\frac{n_\alpha}{m}}$ is the p-torsion subgroup in the abelian group G and $G_{p'} = C_{p-1} \times C_m$ is the subgroup of elements of torsion prime to p. If W is any subgroup of G, then W decomposes canonically in the same way as

$$W = W_p \times W_{p'} \qquad \text{with} \quad W_p \subseteq G_p \quad \text{and} \quad W_{p'} \subseteq G_{p'}.$$

For any such $W \subseteq G$ we define the groups

$$W_0 := W_{p'}, \qquad W_1 := W_0 \times (W_p \cap Aut(C_{p^\alpha})) \qquad \text{and} \qquad W_2 := W.$$

Proposition 4.5. *If $\alpha \geq 1$, then $i^*_{W_0}$ is always an isomorphism.*

Proof. The inclusion $\widetilde{F_1} \subseteq \mathbb{Q}_p(F_0)^\times$ induces a short exact sequence

$$1 \longrightarrow \widetilde{F_1} \longrightarrow \mathbb{Q}_p(F_0)^\times \longrightarrow \mathbb{Q}_p(F_0)^\times/\widetilde{F_1} \longrightarrow 1,$$

which induces a long exact sequence

$$\begin{array}{l} \longrightarrow H^1(W_0, \mathbb{Q}_p(F_0)^\times/\widetilde{F_1}) \longrightarrow H^2(W_0, \widetilde{F_1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow i^*_{W_0} \\ \qquad\qquad\qquad\qquad\qquad\qquad H^2(W_0, \mathbb{Q}_p(F_0)^\times) \longrightarrow H^2(W_0, \mathbb{Q}_p(F_0)^\times/\widetilde{F_1}) \longrightarrow \end{array}$$

The group $\mathbb{Q}_p(F_0)^\times/\widetilde{F_1}$ fits into an exact sequence

$$1 \longrightarrow \mathbb{Z}_p(F_0)^\times/F_0 \longrightarrow \mathbb{Q}_p(F_0)^\times/\widetilde{F_1} \longrightarrow \mathbb{Z}/\mathbb{Z}\langle x_1\rangle \longrightarrow 1,$$

induced by the exact sequences

$$1 \longrightarrow F_0 \longrightarrow \widetilde{F_1} \xrightarrow{v} \mathbb{Z}\langle x_1\rangle \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_p(F_0)^\times \longrightarrow \mathbb{Q}_p(F_0)^\times \xrightarrow{v} \mathbb{Z} \longrightarrow 1.$$

Note that the group $\mathbb{Z}_p(F_0)^\times/F_0$ is free over $\mathbb{Z}_p$, while as $\widetilde{F_1}$ is maximal the quotient $\mathbb{Z}/\mathbb{Z}\langle x_1\rangle = \mathbb{Z}/nv(x_1)\mathbb{Z}$ is a p-torsion group. Since $|W_0|$ is prime to p we get

$$H^*(W_0, \mathbb{Z}_p(F_0)^\times/F_0) = H^*(W_0, \mathbb{Z}/\mathbb{Z}\langle x_1\rangle) = 0 \qquad \text{for } * > 0.$$

Thus

$$H^*(W_0, \mathbb{Q}_p(F_0)^\times/\widetilde{F_1}) = 0 \qquad \text{for } * > 0,$$

and the result follows. □

Lemma 4.6. *If $\alpha \geq 1$ and $W = C_{p-1} \subseteq Aut(C_{p^\alpha})$, then*

$$H^*(W, \widetilde{F_1}) \cong \begin{cases} \langle pu \rangle \times C_{p^{n_\alpha}-1} & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ C_{p^{n_\alpha}-1} \otimes C_{p-1} \cong C_{p-1} & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Q}_p(F_0)^\times) \cong \begin{cases} (\mathbb{Q}_p(F_0)^{C_{p-1}})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ (\mathbb{Q}_p(F_0)^{C_{p-1}})^\times / N_W(\mathbb{Q}_p(F_0)^\times) \cong C_{p-1} & \text{if } 0 < * \text{ is even.} \end{cases}$$

Proof. Consider the short exact sequence

$$1 \longrightarrow F_0 \longrightarrow \widetilde{F_1} \longrightarrow \mathbb{Z}\langle x_1 \rangle \longrightarrow 1;$$

it induces a long exact sequence in cohomology

$$H^0(W, F_0) \longrightarrow H^0(W, \widetilde{F_1}) \longrightarrow H^0(W, \mathbb{Z}\langle x_1 \rangle) \longrightarrow H^1(W, F_0) \longrightarrow \dots,$$

where the action of W is trivial on $\mathbb{Z}\langle x_1 \rangle$, while faithful on the first factor of $F_0 \cong C_{p^\alpha} \times C_{p^{n_\alpha}-1}$ and trivial on the second factor. Note that the first factor of F_0 splits off and has trivial cohomology. Hence for t a generator of W, written additively, and $N = \sum_{i=0}^{p-2} t^i$, the cohomology $H^*(W, F_0)$ can be calculated from the additive complex

$$\mathbb{Z}/(p^{n_\alpha} - 1) \xrightarrow{1-t} \mathbb{Z}/(p^{n_\alpha} - 1) \xrightarrow{N} \mathbb{Z}/(p^{n_\alpha} - 1) \xrightarrow{1-t} \cdots$$

with

$$(1 - t)(1) = 0 \qquad \text{and} \qquad N(1) = p - 1;$$

while $H^*(W, \mathbb{Z}\langle x_1 \rangle)$ can be calculated from the additive complex

$$\mathbb{Z} \xrightarrow{1-t} \mathbb{Z} \xrightarrow{N} \mathbb{Z} \xrightarrow{1-t} \cdots$$

with

$$(1 - t)(1) = 0 \qquad \text{and} \qquad N(1) = p - 1.$$

Consequently

$$H^*(W, F_0) \cong \begin{cases} C_{p^{n_\alpha}-1} & \text{if } * = 0, \\ C_{p^{n_\alpha}-1} * C_{p-1} \cong C_{p-1} & \text{if } 0 < * \text{ is odd}, \\ C_{p^{n_\alpha}-1} \otimes C_{p-1} \cong C_{p-1} & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Z}\langle x_1 \rangle) \cong \begin{cases} \mathbb{Z}\langle x_1 \rangle & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ C_{p-1} & \text{if } 0 < * \text{ is even}, \end{cases}$$

where $C_{p^{n_\alpha}-1} * C_{p-1}$ denotes the kernel of the $(p-1)$-th power map on $C_{p^{n_\alpha}-1}$.

Clearly, $\langle pu \rangle \times C_{p^{n_\alpha}-1} \subseteq \widetilde{F_1}^W$. Since $x_1^{p-1} = pu\delta$ with δ a $(p-1)$-th root of unity in $\mathbb{Q}_p^\times$, we know that $\mathbb{Q}_p(x_1)^W = \mathbb{Q}_p$. Consider an element $z = x_1^l y_1 y_2$ in $\widetilde{F_1}^W$ with $y_1 \in \langle \zeta_{p^\alpha} \rangle$ and $y_2 \in \langle \zeta_{p^{n_\alpha}-1} \rangle$. Then for $\sigma \in W$, y_2 is invariant under σ, and we have

$$\left(\frac{\sigma(x_1)}{x_1} \right)^l = \frac{y_1}{\sigma(y_1)}.$$

Since the order of $\frac{y_1}{\sigma(y_1)}$ divides a power of p and the order of $\frac{\sigma(x_1)}{x_1}$ divides $p-1$, we must have $\frac{y_1}{\sigma(y_1)} = 1$, and hence $y_1 = 1$. Therefore x_1^l is invariant under σ, and as $\mathbb{Q}_p(x_1)^W = \mathbb{Q}_p(\zeta_p)^W = \mathbb{Q}_p$, we know that $l \equiv 0 \bmod p-1$. It follows that the valuation of $\widetilde{F_1}^W$ is integral, and therefore

$$H^0(W, \widetilde{F_1}) = \widetilde{F_1}^W = \langle pu \rangle \times C_{p^{n_\alpha}-1}.$$

Since the image of $H^0(W, \widetilde{F_1})$ in $H^0(W, \mathbb{Z}\langle x_1 \rangle) \cong \mathbb{Z}\langle x_1 \rangle$ is $\mathbb{Z}\langle pu \rangle$, the group $H^0(W, \mathbb{Z}\langle x_1 \rangle)$ surjects onto $H^1(W, F_0) \cong C_{p-1}$, and therefore $H^1(W, \widetilde{F_1}) = 0$. By the periodicity of the cohomology, the map

$$H^2(W, \mathbb{Z}\langle x_1 \rangle) \longrightarrow H^3(W, F_0)$$

is an isomorphism, and as $H^1(W, \mathbb{Z}\langle x_1 \rangle) = 0$ we obtain

$$H^2(W, \widetilde{F_1}) \cong H^2(W, F_0) \cong C_{p^{n_\alpha}-1} \otimes C_{p-1}.$$

Finally, the triviality of $H^1(W, \mathbb{Q}_p(F_0)^\times)$ is a direct consequence of Hilbert's theorem 90, while the remaining cases for $H^*(W, \mathbb{Q}_p(F_0)^\times)$ follow from the characterisation of the Brauer group in terms of the invariants and the norm relative to the Galois group W as given by theorem B.8 and corollary B.11. □

Lemma 4.7. *If $\alpha = 1$, $C_{n_1} = Gal(\mathbb{Q}_p(C_{p^{n_1}-1})/\mathbb{Q}_p)$, and $C_{p-1} = Aut(C_p)$, then*

$$H^*(C_{n_1}, \widetilde{F_1}^{C_{p-1}}) \cong \begin{cases} \langle pu \rangle \times C_{p-1} & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle pu \rangle / \langle (pu)^{n_1} \rangle & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(C_{n_1}, (\mathbb{Q}_p(F_0)^{C_{p-1}})^\times) \cong \begin{cases} \mathbb{Q}_p^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle p \rangle / \langle p^{n_1} \rangle & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. The action of C_{n_1} on $\widetilde{F_1}^{C_{p-1}} \cong \langle pu \rangle \times C_{p^{n_1}-1}$ is trivial on the first factor and acts on $C_{p^{n_1}-1}$ by $\zeta \mapsto \zeta^p$.

Let t be generator of C_{n_1}, written additively, and $N = \sum_{i=0}^{n_1-1} t^i$. Using additive notation for $\widetilde{F_0}^{C_{p-1}} \cong \mathbb{Z} \times \mathbb{Z}/p^{n_1}-1$, we obtain

$$(1-t)(1,0) = (0,0) \qquad (1-t)(0,1) = (0, 1-p),$$
$$N(1,0) = (n_1, 0), \qquad N(0,1) = (0, \frac{p^{n_1}-1}{p-1}),$$

and the desired result for $H^*(C_{n_1}, \widetilde{F_1}^{C_{p-1}})$ follows.

Now for $L = \mathbb{Q}_p(\widetilde{F_0})^{C_{p-1}}$ and $K = L^{C_{n_1}}$, we have

$$H^0(C_{n_1}, L^\times) = K^\times = \mathbb{Q}_p^\times$$

and $H^1(C_{n_1}, L^\times) = 0$ by Hilbert's theorem 90. Furthermore as L/K is unramified, we know from proposition B.13 that the valuation map induces an isomorphism

$$H^2(C_{n_1}, L^\times) \cong H^2(C_{n_1}, \frac{1}{e(L)}\mathbb{Z}) \cong \mathbb{Z}/n_1\mathbb{Z}.$$

Here $e(L) = 1$, and as $v(p) = 1$, the element p represents a generator of the cyclic group $H^2(C_{n_1}, L^\times)$. The result then follows from the periodicity of the cohomology. □

Corollary 4.8. *If $\alpha = 1$, then $H^*(C_{n_1}, \widetilde{F_1}^{C_{p-1}}) \to H^*(C_{n_1}, (\mathbb{Q}_p(F_0)^{C_{p-1}})^\times)$ is an isomorphism for $0 < * $ even.*

Proof. Let $L = \mathbb{Q}_p(\widetilde{F_0})^{C_{p-1}}$ and $K = L^{C_{n_1}} = \mathbb{Q}_p$. Because the extension L/K is unramified, proposition B.13 implies that the group of units of the ring of integers $\mathcal{O}_K$ of K is contained in the norm $N_{L/K}(L^\times)$. As p is a uniformizing element of $\mathbb{Q}_p^\times = K^\times$, it is a generator of the cyclic group $K^\times/N_{L/K}(L^\times) \cong H^2(C_{n_1}, L)$, and the result follows. □

Lemma 4.9. *If $\alpha \geq 2$, $W_0 = G_{p'}$ and $C_{p^{\alpha-1}} \subseteq Aut(C_{p^\alpha})$, then*

$$H^*(C_{p^{\alpha-1}}, \widetilde{F_1}^{W_0}) \cong \begin{cases} \langle pu \rangle \times C_{p^{\frac{n_\alpha}{m}}-1} & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle pu \rangle / \langle (pu)^{p^{\alpha-1}} \rangle \cong C_{p^{\alpha-1}} & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(C_{p^{\alpha-1}}, (\mathbb{Q}_p(F_0)^{W_0})^\times) \cong \begin{cases} (\mathbb{Q}_p(F_0)^{W_0 \times C_{p^{\alpha-1}}})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ C_{p^{\alpha-1}} & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. The action of $C_{p-1} \subseteq W_0 \cap Aut(C_{p^\alpha})$ on $\widetilde{F_1} = \langle x_1 \rangle \times C_{p^{n_\alpha}-1} \times C_{p^\alpha}$ being faithful on the first and last factors, we have $\widetilde{F_1}^{W_0} \cong \langle pu \rangle \times C_{p^{\frac{n_\alpha}{m}}-1}$, and consequently the action of $C_{p^{\alpha-1}}$ on $\widetilde{F_1}^{W_0}$ is trivial.

Let t be generator of $C_{p^{\alpha-1}}$, written additively, and $N = \sum_{i=0}^{p^{\alpha-1}-1} t^i$. Using additive notation for $\widetilde{F_1}^{W_0} \cong \mathbb{Z} \times \mathbb{Z}/(p^{\frac{n_\alpha}{m}} - 1)$ we obtain

$$(1-t)(1,0) = (0,0), \qquad (1-t)(0,1) = (0,0),$$
$$N(1,0) = (p^{\alpha-1}, 0), \qquad N(0,1) = (0, p^{\alpha-1}),$$

and the desired result for $H^*(C_{p^{\alpha-1}}, \widetilde{F_1}^{W_0})$ follows.

The case of $H^*(C_{p^{\alpha-1}}, (\mathbb{Q}_p(F_0)^{W_0})^\times)$ follows from Hilbert's theorem 90 when $*$ is odd, while the case where $0 < *$ is even is given by the isomorphism

$$(\mathbb{Q}_p(F_0)^{W_0 \times C_{p^{\alpha-1}}})^\times / N_{C_{p^{\alpha-1}}}((\mathbb{Q}_p(F_0)^{W_0})^\times) \cong C_{p^{\alpha-1}}.$$

□

In view of theorem 4.13, we are only interested in the case where W_1 is maximal, that is $W_1 = G_{p'} \times (G_p \cap Aut(C_{p^\alpha}))$.

Corollary 4.10. *If $\alpha \geq 2$, $W_0 = G_{p'}$ and $|W_1/W_0| = p^{\alpha-1}$, then*

$$H^*(W_1/W_0, \widetilde{F_1}^{W_0}) \longrightarrow H^*(W_1/W_0, (\mathbb{Q}_p(F_0)^{W_0})^\times) \quad \text{for } 0 < * \text{ even}$$

is surjective if and only if it is an isomorphism, and this is true if and only if

$$u \notin \mu(\mathbb{Z}_p^\times) \times \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \bmod (p^2)\} \qquad \text{and} \qquad \alpha = k.$$

Proof. The first assertion is an obvious consequence of lemma 4.9. Let

$$M := \mathbb{Q}_p(F_0), \qquad L := M^{W_0} \qquad \text{and} \qquad K := L^{C_{p^{\alpha-1}}} = M^{W_1}.$$

Since L/K is totally ramified, we know from proposition B.13 that

$$H^2(C_{p^{\alpha-1}}, L^\times) \cong H^2(C_{p^{\alpha-1}}, \mathcal{O}_L^\times),$$

and as $N_{G/W_1} \circ N_{C_{p^{\alpha-1}}}(\mathcal{O}_L^\times) = N_{G/W_0}(\mathcal{O}_L^\times)$, we may consider the homomorphism

$$\tau : H^2(C_{p^{\alpha-1}}, L^\times) \longrightarrow \mathbb{Z}_p^\times / N_{G/W_0}(\mathcal{O}_L^\times)$$

given as the composite

$$H^2(C_{p^{\alpha-1}}, L^\times) \cong H^2(C_{p^{\alpha-1}}, \mathcal{O}_L^\times) \cong (\mathcal{O}_L^\times)^{C_{p^{\alpha-1}}} / N_{C_{p^{\alpha-1}}}(\mathcal{O}_L^\times) \overset{N_{G/W_1}}{\longrightarrow} \mathbb{Z}_p^\times / N_{G/W_0}(\mathcal{O}_L^\times).$$

We claim that τ is an isomorphism. Because $Gal(L/\mathbb{Q}_p)$ preserves $\mathcal{O}_L^\times$, and hence $l^\times$ for l the residue field of L, we have an epimorphism $Gal(L/\mathbb{Q}_p) \to Gal(l/\mathbb{F}_p)$ whose kernel will be denoted A. Since K is the maximal unramified subextension of $L/\mathbb{Q}_p$, we may consider the short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_p^\times / N_{G/W_0}(\mathcal{O}_L^\times) & \longrightarrow & \mathbb{Q}_p^\times / N_{G/W_0}(L^\times) & \xrightarrow{v} & \mathbb{Z}/v(N_{G/W_0}(L^\times)) & \longrightarrow & 1 \\
 & & \downarrow & & \cong \downarrow (_, L/\mathbb{Q}_p) & & \cong \downarrow \sigma^{(_)} & & \\
1 & \longrightarrow & A & \longrightarrow & Gal(L/\mathbb{Q}_p) & \xrightarrow{red} & Gal(l/\mathbb{F}_p) & \longrightarrow & 1 \\
 & & \cong \downarrow & & \| & & \cong \downarrow & & \\
1 & \longrightarrow & Gal(L/K) & \longrightarrow & Gal(L/\mathbb{Q}_p) & \xrightarrow{pr} & Gal(K/\mathbb{Q}_p) & \longrightarrow & 1,
\end{array}$$

where the bottom two squares commute, the middle vertical isomorphism is the norm residue symbol of $L/\mathbb{Q}_p$ as defined in [20] section 2.2, the top left hand vertical map is its restriction, and where the top right hand vertical isomorphism is given by the power map of the Frobenius automorphism $\sigma \in Gal(l/\mathbb{F}_p)$. By local class field theory (see for example [13] chapter 2 §1.3) we have

$$pr(x, L/\mathbb{Q}_p) = (x, K/\mathbb{Q}_p) \qquad \text{for all } x \in \mathbb{Q}_p^\times / N_{G/W_0}(L^\times).$$

On the other hand [20] proposition 2 shows that

$$(x, K/\mathbb{Q}_p) = \sigma^{v(x)} \qquad \text{for all } x \in \mathbb{Q}_p^\times / N_{G/W_0}(L^\times).$$

It follows that the top right hand square in the above diagram, and hence the diagram itself, is commutative. From the five lemma, the top left hand vertical map of the diagram is an isomorphism, and as $Gal(L/K) \cong C_{p^{\alpha-1}}$, we get

$$N_{G/W_0}(\mathcal{O}_L^\times) = \mu(\mathbb{Z}_p^\times) \times U_\alpha(\mathbb{Z}_p^\times)$$

as a subgroup of index $p^{\alpha-1}$ in $\mathbb{Z}_p^\times$. By corollary B.11, we know that $H^2(C_{p^{\alpha-1}}, L^\times)$ has order $p^{\alpha-1}$. The norm $N_{G/W_1} : \mathcal{O}_K^\times \to \mathbb{Z}_p^\times$ being surjective by proposition B.15, it follows that τ is an isomorphism.

As a consequence of this latter result, our map of interest

$$i^* : H^2(C_{p^{\alpha-1}}, \widetilde{F_1}^{W_0}) \cong \langle pu \rangle / \langle (pu)^{p^{\alpha-1}} \rangle \longrightarrow H^2(C_{p^{\alpha-1}}, L^\times)$$

is surjective (and hence an isomorphism) if and only if τi^* is surjective, that is, if and only if $\tau(i^*(pu))$ is a generator of the cyclic group $\mathbb{Z}_p^\times / N_{G/W_0}(\mathcal{O}_L^\times) \cong U_1(\mathbb{Z}_p^\times)/U_\alpha(\mathbb{Z}_p^\times)$. In fact $i^*(pu) = i^*(u)$. Indeed, as seen in example B.14, there is an element $\widetilde{x} \in \mathbb{Q}_p(\zeta_{p^\alpha})$ such that

$$p = N_{\mathbb{Q}_p(\zeta_{p^\alpha})/\mathbb{Q}_p}(\widetilde{x}) = N_{C_{p^{\alpha-1}}}(x)$$

for $x = N_{C_{p-1}}(\widetilde{x})$ in $\mathbb{Q}_p(\zeta_{p^\alpha})^{C_{p-1}} \subseteq L$; by remark B.16 we then have

$$p \in N_{C_{p-1}}(L^\times) \qquad \text{and} \qquad i^*(pu) = i^*(u).$$

Furthermore as $\mu(\mathbb{Z}_p^\times) \subseteq N_{G/W_0}(\mathcal{O}_L^\times)$, we have $\tau(u) = \tau(u_1)$ for u_1 the component of $u \in \mathbb{Z}_p^\times$ in $U_1(\mathbb{Z}_p^\times)$ via the isomorphism $\mathbb{Z}_p^\times \cong \mu(\mathbb{Z}_p) \times U_1(\mathbb{Z}_p^\times)$ of proposition C.7. Letting $z \in \mathbb{Z}_p$ be such that $u_1 = 1 + zp$, we finally obtain that

$$\tau(pu) = N_{G/W_1}(u_1) = u_1^{|G/W_1|} \equiv 1 + z|G/W_1|p \mod p^2$$

is a generator if and only if

$$u \notin \mu(\mathbb{Z}_p^\times) \times \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \bmod (p^2)\} \qquad \text{and} \qquad |G/W_1| \not\equiv 0 \bmod p,$$

the latter condition being equivalent to $\alpha = k$ (or $n_\alpha = m$). □

Lemma 4.11. *If $\alpha \geq 2$, $W_0 = G_{p'}$, $|W/W_1| \neq 1$ and $L = \mathbb{Q}_p(F_0)^{W_1}$ with $v(L) = \frac{1}{e(L)}\mathbb{Z}$, then*

$$H^*(W/W_1, \widetilde{F_0}^{W_1}) \cong \begin{cases} \langle pu \rangle \times C_{p-1} & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle pu \rangle / \langle (pu)^{|W/W_1|} \rangle & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W/W_1, L^\times) \cong \begin{cases} (L^{W/W_1})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle \pi \rangle / \langle \pi^{|W/W_1|} \rangle & \text{if } 0 < * \text{ is even}, \end{cases}$$

for π a unifomizing element of L^{W/W_1}.

Proof. Since $W_0 = G_{p'} \subseteq W_1$, none of the elements of C_{p^α} are left invariant by W_1, and we have $\widetilde{F_0}^{W_1} \cong \langle pu \rangle \times C_{p^{\frac{n_\alpha}{m}}-1}$. The action of W/W_1 on this group is trivial on the first factor and acts on $C_{p^{\frac{n_\alpha}{m}}-1}$ by $\zeta \mapsto \zeta^p$.

Let t be generator of W/W_1, written additively, and $N = \sum_{i=0}^{|W/W_1|-1} t^i$. Using additive notation for $\widetilde{F_0}^{W_1} \cong \mathbb{Z} \times \mathbb{Z}/(p^{\frac{n_\alpha}{m}} - 1)$, we get

$$(1-t)(1,0) = (0,0), \qquad (1-t)(0,1) = (0, 1-p),$$

$$N(1,0) = (|W/W_1|, 0), \qquad N(0,1) = (0, \frac{p^{\frac{n_\alpha}{m}} - 1}{p-1}),$$

and the desired result for $H^*(W/W_1, \widetilde{F_0}^{W_1})$ follows.

Now let $K = L^{W/W_1}$. Then

$$H^0(W/W_1, L^\times) = K^\times \qquad \text{and} \qquad H^1(W/W_1, L^\times) = 0$$

by Hilbert's theorem 90. Finally, as L/K is unramified, proposition B.13 yields

$$H^2(W/W_1, L) \cong H^2(G, \frac{1}{e(L)}\mathbb{Z}) \cong \frac{1}{e(L)}\mathbb{Z}/|W/W_1| \cdot \frac{1}{e(L)}\mathbb{Z} \cong \langle \pi_K \rangle / \langle \pi_K^{|W/W_1|} \rangle$$

for π_K a uniformizing element of K. The result then follows from periodicity of the cohomology. □

Corollary 4.12. *If $\alpha \geq 2$, $W_0 = G_{p'}$, $|W/W_1| \neq 1$ and $L = \mathbb{Q}_p(F_0)^{W_1}$ with $v(L) = \frac{1}{e(L)}\mathbb{Z}$, then*

$$H^*(W/W_1, \widetilde{F_1}^{W_1}) \longrightarrow H^*(W/W_1, L^\times) \quad \textit{for } 0 < * \textit{ even}$$

is surjective if and only if it is an isomorphism, and this is true if and only if $e(L)$ divides $p-1$.

Proof. The short exact sequence

$$1 \longrightarrow \widetilde{F_0} \longrightarrow \widetilde{F_1} \longrightarrow \mathbb{Z}/p-1 \longrightarrow 1$$

induces a long exact sequence

$$1 \longrightarrow \widetilde{F_0}^{W_1} \longrightarrow \widetilde{F_1}^{W_1} \longrightarrow (\mathbb{Z}/p-1)^{W_1} \longrightarrow H^1(W_1, \widetilde{F_0}^{W_1}) \longrightarrow \dots,$$

which in turn induces a short exact sequence

$$1 \longrightarrow \widetilde{F_0}^{W_1} \longrightarrow \widetilde{F_1}^{W_1} \longrightarrow I \longrightarrow 1$$

where $|I|$ divides $p-1$. Since W/W_1 is a p-group, $|W/W_1|$ is prime to $p-1$ and we have $H^*(W/W_1, I) = 0$ for $* \geq 1$. Hence

$$H^*(W/W_1, \widetilde{F_1}^{W_1}) \cong H^*(W/W_1, \widetilde{F_0}^{W_1}) \quad \text{for } * \geq 2,$$

and by the periodicity of the cohomology of the finite cyclic group W/W_1 this is also true for $* = 1$. For $* = 2$, we are interested in the image of this group in $H^2(W/W_1, L^\times)$. Let $K = L^{W/W_1}$ and $M = \mathbb{Q}_p(F_0)$. From lemma 4.11 we have

$$H^2(W/W_1, L^\times) \cong \frac{1}{e(L)}\mathbb{Z}/\frac{|W/W_1|}{e(L)}\mathbb{Z},$$

and we know that $e(L)$ divides $e(M) = (p-1)p^{\alpha-1}$. Because L/K is unramified, the group $\mathcal{O}_K^\times$ is contained in the norm $N_{L/K}(L^\times)$ by proposition B.13. The map

$$H^2(W/W_1, \widetilde{F_1}^{W_1}) \longrightarrow H^2(W/W_1, L^\times)$$

is therefore surjective if and only if $v(pu) = v(p) = 1$ is a generator of $\frac{1}{e(L)}\mathbb{Z}/\frac{|W/W_1|}{e(L)}\mathbb{Z}$, and this is true if and only if p does not divide $e(L)$. □

Theorem 4.13. *Let p be an odd prime, $n = (p-1)p^{k-1}m$ with m prime to p, $u \in \mathbb{Z}_p^\times$, $F_0 = C_{p^\alpha} \times C_{p^{n_\alpha}-1}$ be a maximal abelian finite subgroup in $\mathbb{S}_n$, $G = Gal(\mathbb{Q}_p(F_0)/\mathbb{Q}_p)$, $G_{p'}$ be the p'-part of G, and let $\widetilde{F_1} = \langle x_1 \rangle \times F_0 \subseteq \mathbb{Q}_p(F_0)^\times$ be maximal as a subgroup of $\mathbb{Q}_p(F_0)^\times$ having $\widetilde{F_0}$ as subgroup of finite index.*

1) *For any $0 \leq \alpha \leq k$, there is an extension of $\widetilde{F_1}$ by $G_{p'}$; this extension is unique up to conjugation.*

2) *If $\alpha \leq 1$, there is an extension of $\widetilde{F_1}$ by G; this maximal extension is unique up to conjugation.*

3) *If $\alpha \geq 2$, there is an extension of $\widetilde{F_1}$ by G if and only if*

$$\alpha = k \qquad \text{and} \qquad u \notin \mu(\mathbb{Z}_p^\times) \times \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \bmod (p^2)\},$$

in which case this maximal extension is unique up to conjugation.

Proof. 1) From corollary 4.3 and proposition 4.5 we know that the map

$$i^*_{G_{p'}} : H^2(G_{p'}, \widetilde{F_1}) \longrightarrow H^2(G_{p'}, \mathbb{Q}_p(F_0)^\times)$$

is an isomorphism. Existence and uniqueness up to conjugation of an extension of $\widetilde{F_1}$ by $G_{p'}$ then follow from corollary 2.29.

2) The case $\alpha = 0$ follows from corollary 4.3 and corollary 2.29. Now assume that $\alpha = 1$ and that $W = G = C_{p-1} \times C_{n_1}$. We have a short exact sequence

$$1 \longrightarrow C_{p-1} \longrightarrow W \longrightarrow C_{n_1} \longrightarrow 1,$$

which gives rise to the Hochschild-Serre spectral sequences (see [4] section VII.6)

$$E_2^{s,t} \cong H^s(C_{n_1}, H^t(C_{p-1}, \widetilde{F_1})) \quad \Longrightarrow \quad H^{s+t}(W, \widetilde{F_1}),$$
$$E_2^{s,t} \cong H^s(C_{n_1}, H^t(C_{p-1}, \mathbb{Q}_p(F_0)^\times)) \quad \Longrightarrow \quad H^{s+t}(W, \mathbb{Q}_p(F_0)^\times).$$

By lemma 4.6 and proposition 4.5, each map $E_2^{s,t} \to E_2^{s,t}$ is an isomorphism for $t > 0$. Moreover, by lemma 4.6 we have

$$H^0(C_{p-1}, \widetilde{F_1}) = \widetilde{F_1}^{C_{p-1}} \cong \langle pu \rangle \times C_{p^{n_1}-1}$$

and

$$H^0(C_{p-1}, \mathbb{Q}_p(F_0)^\times) = (\mathbb{Q}_p(F_0)^{C_{p-1}})^\times \cong \mathbb{Q}_p(C_{p^{n_1}-1})^\times.$$

Then lemma 4.7 and corollary 4.8 imply that the map $E_2^{s,t} \to E_2^{s,t}$ is an isomorphism as well for $t = 0$ and $s > 0$. It follows that

$$i^*_W : H^*(W, \widetilde{F_1}) \longrightarrow H^*(W, \mathbb{Q}_p(F_0)^\times)$$

is an isomorphism for $* > 0$. Existence and uniqueness up to conjugation then follows from corollary 2.29.

3) Assume that $\alpha \geq 2$ and that $W \subseteq G$ is such that $W_0 = G_{p'}$ with $|W_1/W_0| \neq 1$ and $|W/W_1| \neq 1$. We have a short exact sequence

$$1 \longrightarrow W_0 \longrightarrow W_1 \longrightarrow W_1/W_0 \longrightarrow 1,$$

which gives rise to the spectral sequences

$$E_2^{s,t} \cong H^s(W_1/W_0, H^t(W_0, \widetilde{F_1})) \quad \Longrightarrow \quad H^{s+t}(W_1, \widetilde{F_1}),$$
$$E_2^{s,t} \cong H^s(W_1/W_0, H^t(W_0, \mathbb{Q}_p(F_0)^\times)) \quad \Longrightarrow \quad H^{s+t}(W_1, \mathbb{Q}_p(F_0)^\times).$$

By proposition 4.5 each map $E_2^{s,t} \to E_2^{s,t}$ is an isomorphism for $t > 0$. Moreover, we know from lemma 4.9 and corollary 4.10 that when $t = 0$ and $s > 0$, a necessary and sufficient condition for

$$H^s(W_1/W_0, \widetilde{F_1}^{W_0}) \longrightarrow H^s(W_1/W_0, (\mathbb{Q}_p(F_0)^{W_0})^\times)$$

to be surjective (and hence an isomorphism) is that u is a topological generator in $\mathbb{Z}_p^\times/\mu(\mathbb{Z}_p)$ and $\alpha = k$. The map

$$H^*(W_1, \widetilde{F_1}) \longrightarrow H^*(W_1, \mathbb{Q}_p(F_0)^\times), \quad \text{for } * > 0$$

is thus surjective if and only if it is an isomorphism, and this is true if and only if

$$u \notin \mu(\mathbb{Z}_p^\times) \times \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \bmod (p^2)\} \qquad \text{and} \qquad \alpha = k.$$

Now assuming these conditions are satisfied, the short exact sequence

$$1 \longrightarrow W_1 \longrightarrow W \longrightarrow W/W_1 \longrightarrow 1$$

induces spectral sequences

$$E_2^{s,t} \cong H^s(W/W_1, H^t(W_1, \widetilde{F_1})) \quad \Longrightarrow \quad H^{s+t}(W, \widetilde{F_1}),$$
$$E_2^{s,t} \cong H^s(W/W_1, H^t(W_1, \mathbb{Q}_p(F_0)^\times)) \quad \Longrightarrow \quad H^{s+t}(W, \mathbb{Q}_p(F_0)^\times),$$

where each map $E_2^{s,t} \to E_2^{s,t}$ is an isomorphism for $t > 0$. Furthermore, lemma 4.11 and corollary 4.12 imply that in case $t = 0$ and $s > 0$, the map

$$H^s(W/W_1, \widetilde{F_1}^{W_1}) \longrightarrow H^s(W/W_1, (\mathbb{Q}_p(F_0)^{W_1})^\times)$$

is surjective (and hence an isomorphism) if and only if $e(\mathbb{Q}_p(F_0)^{W_1})$ divides $p - 1$. In particular for $W = G$, the map

$$i_G^* : H^2(G, \widetilde{F_1}) \longrightarrow H^2(G, \mathbb{Q}_p(F_0)^\times)$$

is surjective (and hence an isomorphism) if and only if W_1 is realized and $e(\mathbb{Q}_p(F_0)^{W_1})$ divides $p - 1$, that is, if and only if W_1 is realized and $W_1/W_0 \cong C_{p^\alpha}$. The result then follows from theorem 2.27 and corollary 2.29. □

4.2. Extensions of maximal abelian finite subgroups of $\mathbb{S}_n$ for $p = 2$

In this section, we assume $p = 2$, F_0 to be a maximal abelian finite subgroup of $\mathbb{S}_n$, and $\widetilde{F_1}$ to be maximal as a subgroup of $\mathbb{Q}_2(F_0)^\times$ having $\widetilde{F_0}$ as a subgroup of finite index; in other words

$$F_0 \cong C_{2^\alpha} \times C_{2^{n_\alpha}-1} \qquad \text{with} \quad 1 \le \alpha \le k,\ n_\alpha = \frac{n}{\varphi(2^\alpha)}.$$

By corollary 3.39, we have $\widetilde{F_1} = \langle x_1 \rangle \times F_0$ with

$$v(x_1) = \begin{cases} 1 & \text{if } \alpha \le 1, \text{ or if } u \equiv \pm 3 \bmod 8 \text{ and } n_\alpha \text{ is odd,} \\ \frac{1}{2} & \text{if } \alpha \ge 2 \text{ and either } u \equiv \pm 1 \bmod 8 \text{ or } n_\alpha \text{ is even.} \end{cases}$$

Remark 4.14. Since $n_\alpha = 2^{k-\alpha}m$ with m odd, we have

$$n_\alpha \equiv 1 \bmod 2 \qquad \Leftrightarrow \qquad \alpha = k.$$

By remark 3.40, we may in fact choose $x_1 \in \widetilde{F_1}$ to be $x_1 = 2u$ in the cases where its valuation is 1, otherwise to be $x_1 = (1+i)t$ for $i \in \mathbb{Q}_2(F_0)^\times$ a primitive 4-th root of unity and

$$t \in \begin{cases} \mathbb{Z}_2^\times & \text{if } u \equiv \pm 1 \bmod 8, \\ \mathbb{Z}_2(\zeta_3)^\times & \text{if } u \equiv \pm 3 \bmod 8, \end{cases} \qquad \text{with} \qquad t^2 = \begin{cases} u & \text{if } u \equiv 1 \text{ or } -3 \bmod 8, \\ -u & \text{if } u \equiv -1 \text{ or } 3 \bmod 8. \end{cases}$$

By definition $\mathbb{Q}_2(F_0) = \mathbb{Q}_2(\widetilde{F_1})$, and because the latter is a maximal subfield of $\mathbb{D}_n$ we have $\widetilde{F_1} = \widetilde{F_2}$. We let

$$G := Gal(\mathbb{Q}_2(F_0)/\mathbb{Q}_2) \cong \begin{cases} C_n & \text{if } \alpha = 1, \\ C_{n_\alpha} \times C_{2^{\alpha-2}} \times C_2 & \text{if } \alpha \geq 2, \end{cases}$$

as given by proposition C.8. From our choice of x_1, we know that $\widetilde{F_1}$ is stable under the action of a subgroup $W \subseteq G$: if $x_1 = 2u$ this is clear, and if $v(x_1) = \frac{1}{2}$ and $\sigma \in W$ we have $\frac{\sigma(x_1)}{x_1} \in F_0$ and hence $\sigma(x_1) \in x_1 F_0 \subseteq \widetilde{F_1}$. The goal of the section is to determine necessary and sufficient conditions on n, u and α for the homomorphism

$$i_G^* : H^2(G, \widetilde{F_1}) \longrightarrow H^2(G, \mathbb{Q}_2(F_0)^\times)$$

to be surjective, and whenever this happens, we want to determine its kernel. This is done via the analysis of

$$i_W^* : H^2(W, \widetilde{F_1}) \longrightarrow H^2(W, \mathbb{Q}_2(F_0)^\times)$$

for suitable subgroups $W \subseteq G$.

The case $\alpha = 1$

The situation is much simpler when the 2-Sylow subgroup of F_0 is contained in $\mathbb{Q}_2^\times$. Recall that $C_{2^\alpha} * C_n$ denotes the kernel of the n-th power map on C_{2^α}.

Lemma 4.15. *If $\alpha \leq 1$ and $W = C_n$, then $\widetilde{F_1} = \langle 2u \rangle \times F_0$ and*

$$H^*(W, \widetilde{F_1}) \cong \begin{cases} \langle 2u \rangle \times C_{2^\alpha} & \text{if } * = 0, \\ C_{2^\alpha} * C_n & \text{if } 0 < * \text{ is odd}, \\ \langle 2u \rangle / \langle (2u)^n \rangle \times C_{2^\alpha} \otimes C_n & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Q}_2(F_0)^\times) \cong \begin{cases} \mathbb{Q}_2^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle 2 \rangle / \langle 2^n \rangle & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. We know from corollary 3.39 that $\widetilde{F_1} = \widetilde{F_0}$. The action of $W = C_n$ on $\widetilde{F_1} \cong \langle 2u \rangle \times C_{2^\alpha} \times C_{2^n - 1}$ is trivial on $\langle 2u \rangle \times C_{2^\alpha}$ and acts on C_{2^n-1} by $\zeta \mapsto \zeta^2$.

For t a generator of C_n, written additively, and $N = \sum_{i=0}^{n-1} t^i$, $H^*(C_n, \widetilde{F_1})$ is the cohomology of the complex

$$\widetilde{F_1} \xrightarrow{1-t} \widetilde{F_1} \xrightarrow{N} \widetilde{F_1} \xrightarrow{1-t} \cdots .$$

Using additive notation for $\widetilde{F_1} \cong \mathbb{Z} \times \mathbb{Z}/2^\alpha \times \mathbb{Z}/2^n - 1$, we obtain

$$\begin{aligned} (1-t)(1,0,0) &= (0,0,0), & N(1,0,0) &= (n,0,0),\\ (1-t)(0,1,0) &= (0,0,0), & N(0,1,0) &= (0,n,0),\\ (1-t)(0,0,1) &= (0,0,-1), & N(0,0,1) &= (0,0,2^n-1), \end{aligned}$$

and the desired result for $H^*(W, \widetilde{F_1})$ follows.

Now let $L = \mathbb{Q}_2(F_0) = \mathbb{Q}_2(\widetilde{F_1})$ and $K = L^W$. Then

$$H^0(W, \mathbb{Q}_2(F_0)^\times) = K^\times = \mathbb{Q}_2^\times$$

and $H^1(W, \mathbb{Q}_2(F_0)^\times) = 0$ by Hilbert's theorem 90. Furthermore as L/K is unramified, proposition B.13 imply

$$H^2(W, \mathbb{Q}_2(F_0)^\times) \cong \langle 2 \rangle / \langle 2^n \rangle$$

as desired. □

Corollary 4.16. *If $\alpha = 1$ and $W \subseteq C_n$, then $i_W^* : H^2(W, \widetilde{F_1}) \to H^2(W, \mathbb{Q}_2(F_0)^\times)$ is an epimorphism. It is an isomorphism if and only if n is odd. If n is even, its kernel is $\{\pm 1\}$.*

Proof. First assume that $W = C_n$ with $L = \mathbb{Q}_2(F_0)$ and $K = L^W$. As L/K is unramified, proposition B.13 yields $u \in N_{C_n}(L^\times)$. Hence $i_{C_n}^*$ is surjective by lemma 4.15. The case $W \subseteq C_n$ follows from proposition 4.1, and the other assertions are clear. □

Example 4.17. When $\alpha = 1$, the group $F_0 \cong C_2 \times C_{2^n-1}$ is generated by $-\omega$ for ω a (2^n-1)-th root of unity in $\mathbb{S}_n$. Here $\mathbb{Q}_2(F_0)/\mathbb{Q}_2$ is a maximal unramified commutative extension in $\mathbb{D}_n$ and $\widetilde{F_0} = \widetilde{F_1} = \widetilde{F_2}$. Now for any $u \in \mathbb{Z}_2^\times$ there are elements ξ_u and ξ_{-u} of valuation $\frac{1}{n}$ in $N_{\mathbb{D}_n^\times}(F_0)$ such that

$$\xi_u^n = 2u, \qquad \xi_{-u}^n = -2u \qquad \text{and} \qquad \xi_{\pm u} \omega \xi_{\pm u}^{-1} = \omega^2,$$

with $\widetilde{F_3^+} = \langle \xi_u \rangle \times F_0$ and $\widetilde{F_3^-} = \langle \xi_{-u} \rangle \times F_0$. In $\mathbb{G}_n(u)$, this gives extensions

$$1 \longrightarrow F_0 \longrightarrow F_3^\pm \longrightarrow C_n \longrightarrow 1,$$

having classes in

$$H^2(C_n, F_0) \cong H^2(C_n, C_2) \oplus H^2(C_n, C_{2^n-1}) \cong H^2(C_n, C_2) \cong \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{Z}/2 & \text{if } n \text{ is even.} \end{cases}$$

One of the extensions is a semi-direct product, represented by

$$\langle -\omega, \overline{\xi}_u \rangle \cong C_{2(2^n-1)} \rtimes C_n,$$

for $\overline{\xi}_u$ the class of ξ_u in $\mathbb{G}_n(u)$. When n is even, we have

$$(-\overline{\xi}_{-u})^n = (-1)^n (\overline{\xi}_{-u})^n = -1$$

for $\overline{\xi}_{-u}$ the class of ξ_{-u} in $\mathbb{G}_n(u)$. The respective 2-Sylow subgroups of $\langle -\omega, \overline{\xi}_u \rangle$ and $\langle -\omega, \overline{\xi}_{-u} \rangle$ are $C_2 \times C_{2^{k-1}}$ and C_{2^k} which are clearly not isomorphic.

The case $\alpha \geq 2$

We let $\alpha \geq 2$. In this case $\mathbb{Q}_2(i) \subseteq \mathbb{Q}_2(F_0)$.

Proposition 4.18. *If $\alpha \geq 2$ and W_0 is a subgroup of odd order in $C_{n_\alpha} \subseteq Aut(C_{2^{n_\alpha}-1})$, then $i_W^* : H^2(W_0, \widetilde{F_1}) \to H^2(W_0, \mathbb{Q}_2(F_0)^\times)$ is an isomorphism.*

Proof. We may use the same argument as proposition 4.5. Using that $\alpha \geq 2$, we know that $\mathbb{Z}/\mathbb{Z}\langle x_1\rangle$ is either trivial or a 2-torsion group, while $\mathbb{Z}_2(F_0)^\times/F_0$ is free over $\mathbb{Z}_2$. Hence

$$H^*(W_0, \mathbb{Z}_2(F_0)^\times/F_0) = H^*(W_0, \mathbb{Z}/\mathbb{Z}\langle x_1\rangle) = H^*(W_0, \mathbb{Q}_2(F_0)^\times/\widetilde{F_1}) = 0 \qquad \text{for } * > 0,$$

and the result follows. □

Lemma 4.19. *If $\alpha \geq 2$, $u \equiv \pm 3 \bmod 8$, n_α is odd and $W = C_{n_\alpha} \subseteq Aut(C_{2^{n_\alpha}-1})$, then $\widetilde{F_1} = \widetilde{F_0}$ and*

$$H^*(W, \widetilde{F_1}) \cong \begin{cases} \langle 2u\rangle \times C_{2^\alpha} & \text{if } * = 0, \\ C_{2^\alpha} * C_{n_\alpha} & \text{if } 0 < * \text{ is odd}, \\ \langle 2u\rangle/\langle (2u)^{n_\alpha}\rangle \times C_{2^\alpha} \otimes C_{n_\alpha} & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Q}_2(F_0)^\times) \cong \begin{cases} \mathbb{Q}_2(\zeta_{2^\alpha})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle \zeta_{2^\alpha} - 1\rangle/\langle (\zeta_{2^\alpha} - 1)^{n_\alpha}\rangle & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. We know from corollary 3.39 that $\widetilde{F_1} = \widetilde{F_0}$. The calculations for $H^*(W, \widetilde{F_1})$ and $H^*(W, \mathbb{Q}_2(F_0))$ are identical to that of lemma 4.15, except that 2 is replaced with $(\zeta_{2^\alpha} - 1)$ in the second case. □

Lemma 4.20. *If $\alpha \geq 2$, $u \equiv \pm 1 \bmod 8$ and $W = C_{n_\alpha} \subseteq Aut(C_{2^{n_\alpha}-1})$, then $\widetilde{F_1} = \langle x_1\rangle \times F_0$ with $v(x_1) = \frac{1}{2}$ and*

$$H^*(W, \widetilde{F_1}) \cong \begin{cases} \langle x_1\rangle \times C_{2^\alpha} & \text{if } * = 0, \\ C_{2^\alpha} * C_{n_\alpha} & \text{if } 0 < * \text{ is odd}, \\ \langle x_1\rangle/\langle x_1^{n_\alpha}\rangle \times C_{2^\alpha} \otimes C_{n_\alpha} & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Q}_2(F_0)^\times) \cong \begin{cases} \mathbb{Q}_2(\zeta_{2^\alpha})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle \zeta_{2^\alpha} - 1\rangle/\langle (\zeta_{2^\alpha} - 1)^{n_\alpha}\rangle & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. We know from corollary 3.39 that $\widetilde{F_1} = \langle x_1\rangle \times F_0$ with $v(x_1) = \frac{1}{2}$. The action of C_{n_α} on $\widetilde{F_1} \cong \langle x_1\rangle \times C_{2^\alpha} \times C_{2^{n_\alpha}-1}$ is trivial on the first two factors and acts on the third by $\zeta \mapsto \zeta^2$.

Let t be generator of C_{n_α}, written additively, and $N = \sum_{i=0}^{n_\alpha - 1} t^i$. Using additive notation for $\widetilde{F_1} \cong \mathbb{Z} \times \mathbb{Z}/2^\alpha \times \mathbb{Z}/2^{n_\alpha} - 1$, we obtain

$$(1-t)(1,0,0) = (0,0,0), \qquad (1-t)(0,1,0) = (0,0,0), \qquad (1-t)(0,0,1) = (0,0,-1),$$
$$N(1,0,0) = (n_\alpha, 0, 0), \qquad N(0,1,0) = (0, n_\alpha, 0), \qquad N(0,0,1) = (0,0,0),$$

and the desired result for $H^*(W, \widetilde{F_1})$ follows.

Now for $L = \mathbb{Q}_2(F_0)$ and $K = L^W = \mathbb{Q}_2(\zeta_{2^\alpha})$, we have

$$H^0(W, \mathbb{Q}_2(\widetilde{F_1})) = \mathbb{Q}_2(Ker(1-t))^\times = \mathbb{Q}_2(\zeta_{2^\alpha})^\times$$

and $H^1(W, \mathbb{Q}_2(F_0)^\times) = 0$ by Hilbert's theorem 90. Furthermore, as L/K is unramified, $\zeta_{2^\alpha} - 1$ is a uniformizing element of L and proposition B.13 implies

$$H^2(W, \mathbb{Q}_2(F_0)^\times) \cong \langle \zeta_{2^\alpha} - 1 \rangle / \langle (\zeta_{2^\alpha} - 1)^{n_\alpha} \rangle$$

as desired. □

Lemma 4.21. *If $\alpha \geq 3$, $u \equiv \pm 3 \bmod 8$, n_α is odd and $W = C_{2^{\alpha-2}} \subseteq Aut(C_{2^\alpha})$ is generated by $\zeta \mapsto \zeta^5$, then $\widetilde{F_1} = \widetilde{F_0}$ and*

$$H^*(W, \widetilde{F_1}) \cong \begin{cases} \langle 2u \rangle \times C_4 \times C_{2^{n_\alpha - 1}} & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle 2u \rangle / \langle (2u)^{2^{\alpha-2}} \rangle \cong C_{2^{\alpha-2}} & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Q}_2(F_0)^\times) \cong \begin{cases} (\mathbb{Q}_2(F_0)^{C_{2^{\alpha-2}}})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ (\mathbb{Q}_2(F_0)^{C_{2^{\alpha-2}}})^\times / N_W(\mathbb{Q}_2(F_0)^\times) \cong C_{2^{\alpha-2}} & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. We know from corollary 3.39 that $\widetilde{F_1} = \widetilde{F_0}$. The action of $C_{2^{\alpha-2}}$ on $\widetilde{F_1} \cong \langle 2u \rangle \times C_{2^\alpha} \times C_{2^{n_\alpha - 1}}$ is trivial on $\langle 2u \rangle \times C_{2^{n_\alpha - 1}}$ and acts on C_{2^α} by $\zeta \mapsto \zeta^5$.

For t a generator of $C_{2^{\alpha-2}}$, written additively, and $N = \sum_{i=0}^{2^{\alpha-2}-1} t^i$, we obtain

$$\begin{aligned} (1-t)(1,0,0) &= (0,0,0), & N(1,0,0) &= (2^{\alpha-2}, 0, 0), \\ (1-t)(0,1,0) &= (0,-4,0), & N(0,1,0) &= (0, 2^{\alpha-2}, 0), \\ (1-t)(0,0,1) &= (0,0,0), & N(0,0,1) &= (0, 0, 2^{\alpha-2}), \end{aligned}$$

and the desired result for $H^*(W, \widetilde{F_1})$ follows.

The case of $H^*(W, \mathbb{Q}_2(F_0)^\times)$ for $0 < *$ odd follows from Hilbert's theorem 90, and the rest is clear. □

Lemma 4.22. *Let $\alpha \geq 3$, and assume either $u \equiv \pm 1 \bmod 8$ or $u \equiv \pm 3 \bmod 8$ with n_α even. If $W = C_{2^{\alpha-2}} \subseteq Aut(C_{2^\alpha})$ is generated by $\zeta \mapsto \zeta^5$, then $\widetilde{F_1} = \langle x_1 \rangle \times F_0$ with $v(x_1) = \frac{1}{2}$ and*

$$H^*(W, \widetilde{F_1}) \cong \begin{cases} \langle x_1 \rangle \times C_4 \times C_{2^{n_\alpha - 1}} & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ \langle x_1 \rangle / \langle x_1^{2^{\alpha-2}} \rangle \cong C_{2^{\alpha-2}} & \text{if } 0 < * \text{ is even}; \end{cases}$$

$$H^*(W, \mathbb{Q}_2(F_0)^\times) \cong \begin{cases} (\mathbb{Q}_2(F_0)^{C_{2^{\alpha-2}}})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd}, \\ (\mathbb{Q}_2(F_0)^{C_{2^{\alpha-2}}})^\times / N_W(\mathbb{Q}_2(F_0)^\times) \cong C_{2^{\alpha-2}} & \text{if } 0 < * \text{ is even}. \end{cases}$$

Proof. We know from corollary 3.39 that $\widetilde{F_1} = \langle x_1 \rangle \times F_0$ with $v(x_1) = \frac{1}{2}$. The calculations are identical to that of lemma 4.21, except that $2u$ is replaced by x_1 for the calculation of $H^*(W, \widetilde{F_1})$. □

Corollary 4.23. *If $\alpha \geq 3$ and $W = C_{2^{\alpha-2}} \subseteq Aut(C_{2^\alpha})$ is generated by $\zeta \mapsto \zeta^5$, then $i_W^* : H^2(W, \widetilde{F_1}) \to H^2(W, \mathbb{Q}_2(F_0)^\times)$ is never surjective.*

Proof. Let $L := \mathbb{Q}_2(F_0)$ and $K := L^W$. Since L/K is totally ramified, we know from proposition B.13 that $H^2(W, L^\times) \cong H^2(W, \mathcal{O}_L^\times)$. As $N_{G/W} \circ N_W(\mathcal{O}_L^\times) = N_G(\mathcal{O}_L^\times)$, we may consider the homomorphism

$$\tau : H^2(W, L^\times) \longrightarrow \mathbb{Z}_2^\times / N_G(\mathcal{O}_L^\times)$$

given by the norm

$$N_{G/W} : H^2(W, \mathcal{O}_L^\times) \cong (\mathcal{O}_K^\times)/N_W(\mathcal{O}_L^\times) \longrightarrow \mathbb{Z}_2^\times / N_G(\mathcal{O}_L^\times).$$

In order to analyse this homomorphism, we consider the short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_2^\times / N_G(\mathcal{O}_L^\times) & \longrightarrow & \mathbb{Q}_2^\times / N_G(L^\times) & \xrightarrow{v} & \mathbb{Z}/v(N_G(L^\times)) & \longrightarrow & 1 \\
 & & \downarrow & & \cong \downarrow (_, L/\mathbb{Q}_2) & & \cong \downarrow \sigma^{(_)} & & \\
1 & \longrightarrow & Gal(L/L^{C_{2^{\alpha-2}} \times C_2}) & \longrightarrow & Gal(L/\mathbb{Q}_2) & \xrightarrow{pr} & Gal(l/\mathbb{F}_2) & \longrightarrow & 1
\end{array}$$

where

$$Gal(L/L^{C_{2^{\alpha-2}} \times C_2}) \cong C_{2^{\alpha-2}} \times C_2, \qquad Gal(L^{C_{2^{\alpha-2}} \times C_2}/\mathbb{Q}_2) \cong Gal(l/\mathbb{F}_2) \cong C_{n_\alpha}$$

for l the residue field of L, where the middle vertical isomorphism is the norm residue symbol of $L/\mathbb{Q}_2$ as defined in [20] section 2.2, the left hand vertical map is its restriction, and where the right hand vertical isomorphism is given by the power map of the Frobenius automorphism $\sigma \in Gal(l/\mathbb{F}_2)$. We know from local class field theory (see for example [13] chapter 2 §1.3) that

$$pr(x, L/\mathbb{Q}_2) = (x, L^{C_{2^{\alpha-2}} \times C_2}/\mathbb{Q}_2) \qquad \text{for all } x \in \mathbb{Q}_2^\times / N_G(L^\times).$$

On the other hand [20] proposition 2 shows that

$$(x, L^{C_{2^{\alpha-2}} \times C_2}/\mathbb{Q}_2) = \sigma^{v(x)} \qquad \text{for all } x \in \mathbb{Q}_2^\times / N_G(L^\times).$$

Thus the right hand square in the above diagram, and hence the diagram itself, is commutative. The five lemma then implies

$$\mathbb{Z}_2^\times / N_G(\mathcal{O}_L^\times) \cong C_{2^{\alpha-2}} \times C_2 \qquad \text{and} \qquad N_G(\mathcal{O}_L^\times) = U_\alpha(\mathbb{Z}_2^\times).$$

The image of τ however is $U_2(\mathbb{Z}_2^\times)/U_\alpha(\mathbb{Z}_2^\times)$. To see this, consider the tower of extensions

$$\mathbb{Q}_2 \overset{C_2}{\text{——}} \mathbb{Q}_2(i) \overset{C_{n_\alpha}}{\text{——}} K \overset{W}{\text{——}} L.$$

Since $K/\mathbb{Q}_2(i)$ is unramified, we know from proposition B.15 that

$$N_{C_{n_\alpha}} : \mathcal{O}_K^\times \longrightarrow \mathbb{Z}_2(i)^\times$$

is surjective. Hence for any $a_1, a_2 \in \mathbb{Z}_2$, there exists an element $x = 1 + a(1+i)$ in $\mathcal{O}_K^\times$ with $a \in \mathbb{Z}_2$ such that

$$N_{C_{n_\alpha}}(x) = 1 + (a_1 + a_2 i)(1+i) \ \in \mathbb{Z}_2(i)^\times.$$

Therefore

$$\begin{aligned} N_{G/W}(x) &= N_{C_2}(1 + (a_1 + a_2 i)(1+i)) \\ &= [1 + (a_1 + a_2 i)(1+i)][1 + (a_1 - a_2 i)(1-i)] \\ &= 1 + 2(a_1^2 + a_2^2 + a_1 - a_2) \\ &= 1 + 2(a_1^2 + a_1) + 2(a_2^2 - a_2) \\ &\equiv 1 \bmod 4, \end{aligned} \tag{4.1}$$

and the map $\tau : H^2(W, \mathcal{O}_L^\times) \to U_2(\mathbb{Z}_2^\times)/U_\alpha(\mathbb{Z}_2^\times)$ is an isomorphism.

By to lemma 4.21 and 4.22, the map i_W^* is therefore surjective if and only if $\tau(x_1)$ is a generator of $U_2(\mathbb{Z}_2^\times)/U_\alpha(\mathbb{Z}_2^\times)$. Recall that

$$x_1 = \begin{cases} 2u & \text{if } u \equiv \pm 3 \bmod 8 \text{ and } n_\alpha \text{ is odd,} \\ (1+i)t & \text{otherwise,} \end{cases}$$

with

$$t^2 = \begin{cases} u & \text{if } u \equiv 1 \text{ or } -3 \bmod 8, \\ -u & \text{if } u \equiv -1 \text{ or } 3 \bmod 8. \end{cases}$$

Since both 2 and $1+i$ belong to $N_{\mathbb{Q}_2(\zeta_{2^\alpha})/\mathbb{Q}_2(i)}(\mathbb{Q}_2(\zeta_{2^\alpha}))$ according to example B.14, it follows by remark B.16 that 2 and $1+i$ both belong to $N_{L/K}(L^\times)$. Thus if $u \equiv \pm 3 \bmod 8$ with n_α odd, we have

$$\tau(2u) = \tau(u) = u^{2n_\alpha} \equiv 1 \mod 8.$$

On the other hand if $u \equiv \pm 1 \bmod 8$, then

$$\begin{aligned} \tau(x_1) = \tau(t) = t^{2n_\alpha} &= \begin{cases} u^{n_\alpha} & \text{if } u \equiv 1 \bmod 8, \\ (-u)^{n_\alpha} & \text{if } (-u) \equiv 1 \bmod 8, \end{cases} \\ &\equiv 1 \mod 8. \end{aligned}$$

Finally if $u \equiv \pm 3 \bmod 8$ with n_α even, there is a subgroup of index 2 in G/W which acts trivially on t, and we have

$$\tau(x_1) = \tau(t) = (t(-t))^{n_\alpha} = (-1)^{n_\alpha} t^{2n_\alpha} \equiv (\pm 3)^{n_\alpha} \equiv 1 \mod 8.$$

In any case, the map i_W^* is never surjective. □

Lemma 4.24. *Let $\alpha \geq 2$, $u \equiv \pm 3 \bmod 8$, n_α be odd, and let $C_2 \subseteq Aut(C_{2^\alpha})$ be generated by $\zeta \mapsto \zeta^{-1}$. If W_0 is a subgroup of odd order in G, then $\widetilde{F_1}^{W_0} = \widetilde{F_0}^{W_0}$ and*

$$H^*(C_2, \widetilde{F_1}^{W_0}) \cong \begin{cases} \langle 2u \rangle \times (C_{2^\alpha} * C_2) \times C_{\frac{n_\alpha}{2^{|W_0|}-1}} & \text{if } * = 0, \\ C_{2^\alpha} \otimes C_2 & \text{if } 0 < * \text{ is odd,} \\ \langle 2u \rangle / \langle (2u)^2 \rangle \times (C_{2^\alpha} * C_2) & \text{if } 0 < * \text{ is even;} \end{cases}$$

$$H^*(C_2, (\mathbb{Q}_2(F_0)^{W_0})^\times) \cong \begin{cases} (\mathbb{Q}_2(F_0)^{C_2})^\times & \text{if } * = 0, \\ 0 & \text{if } 0 < * \text{ is odd,} \\ (\mathbb{Q}_2(F_0)^{C_2})^\times / N_W(\mathbb{Q}_2(F_0)^\times) \cong C_2 & \text{if } 0 < * \text{ is even.} \end{cases}$$

Proof. We know from corollary 3.39 that $\widetilde{F_1} = \widetilde{F_0}$. The action of C_2 on $\widetilde{F_1}^{W_0} \cong \langle 2u \rangle \times C_{2^\alpha} \times C_{2^{\frac{n_\alpha}{|W_0|}}-1}$ is trivial on the first and last factors and acts on the second by $\zeta \mapsto \zeta^{-1}$.

Let t the generator of C_2, written additively, and $N = 1 + t$. Using additive notation for $\widetilde{F_1}^{W_0} \cong \mathbb{Z} \times \mathbb{Z}/2^\alpha \times \mathbb{Z}/(2^{\frac{n_\alpha}{|W_0|}} - 1)$, we obtain

$$\begin{aligned} (1-t)(1,0,0) &= (0,0,0), & N(1,0,0) &= (2,0,0),\\ (1-t)(0,1,0) &= (0,2,0), & N(0,1,0) &= (0,0,0),\\ (1-t)(0,0,1) &= (0,0,0), & N(0,0,1) &= (0,0,2), \end{aligned}$$

and the desired result for $H^*(C_2, \widetilde{F_1}^{W_0})$ follows.

The case of $H^*(C_2, (\mathbb{Q}_2(F_0)^{W_0})^\times)$ for $0 < *$ odd follows from Hilbert's theorem 90, and the rest is clear. □

Lemma 4.25. *Let $\alpha = 2$ and either $u \equiv \pm 1 \bmod 8$ or $u \equiv \pm 3 \bmod 8$ with n_α even. If $C_2 \subseteq Aut(C_{2^\alpha})$ is generated by $\zeta \mapsto \zeta^{-1}$, and if W_0 is a subgroup of odd order in G, then $\widetilde{F_1}^{W_0} = \langle x_1 \rangle \times F_0^{W_0}$ with $v(x_1) = \frac{1}{2}$ and*

$$H^*(C_2, \widetilde{F_1}^{W_0}) \cong \begin{cases} \langle 2u \rangle \times (C_{2^\alpha} * C_2) \times C_{2^{\frac{n_\alpha}{|W_0|}}-1} & \text{if } * = 0,\\ 0 & \text{if } 0 < * \text{ odd},\\ C_{2^\alpha} * C_2 & \text{if } 0 < * \text{ even}; \end{cases}$$

$$H^*(C_2, (\mathbb{Q}_2(F_0)^{W_0})^\times) \cong \begin{cases} (\mathbb{Q}_2(F_0)^{C_2})^\times & \text{if } * = 0,\\ 0 & \text{if } 0 < * \text{ odd},\\ (\mathbb{Q}_2(F_0)^{C_2})^\times / N_W(\mathbb{Q}_2(F_0)^\times) \cong C_2 & \text{if } 0 < * \text{ even}. \end{cases}$$

Proof. We know that $\widetilde{F_1}^{W_0} = \langle x_1 \rangle \times F_0^{W_0}$ with $v(x_1) = \frac{1}{2}$. The action of C_2 on $\widetilde{F_1}^{W_0} \cong \langle x_1 \rangle \times C_{2^\alpha} \times C_{2^{\frac{n_\alpha}{|W_0|}}-1}$ is trivial on the last factor, acts on C_{2^α} by $\zeta_{2^\alpha} \mapsto \zeta_{2^\alpha}^{-1}$ on the second, and sends x_1 to $-ix_1$.

Note that the last factor splits off and has trivial cohomology. Hence for t a generator of C_2, written additively, and $N = 1 + t$, the cohomology $H^*(C_2, \widetilde{F_1}^{W_0})$ can be calculated from the additive complex

$$\mathbb{Z} \times \mathbb{Z}/4 \xrightarrow{1-t} \mathbb{Z} \times \mathbb{Z}/4 \xrightarrow{N} \mathbb{Z} \times \mathbb{Z}/4 \xrightarrow{1-t} \cdots,$$

where

$$t(1,0) = (1,1) \qquad \text{and} \qquad t(0,1) = (0,-1).$$

Therefore

$$\begin{aligned} (1-t)(1,0) &= (0,-1), & (1-t)(0,1) &= (0,2),\\ N(1,0) &= (2,1), & N(0,1) &= (0,0). \end{aligned}$$

Hence

$$\begin{aligned} Ker(1-t) &= \langle (2,1), (0,2) \rangle, & Im(1-t) &= \langle (0,1) \rangle,\\ Ker(N) &= \langle (0,1) \rangle, & Im(N) &= \langle (2,1) \rangle, \end{aligned}$$

and the desired result for $H^*(C_2, \widetilde{F_1}^{W_0})$ follows.

The case of $H^*(C_2, (\mathbb{Q}_p(F_0)^{W_0})^\times)$ for $0 < *$ odd follows from Hilbert's theorem 90, and the rest is clear. □

We have seen in corollary 4.23 that i_G^* is not surjective whenever $\alpha \geq 3$. Thus the case $\alpha = 2$ is all that we want to consider in the following corollary.

Corollary 4.26. *Let $\alpha = 2$, $C_2 = Aut(C_{2^\alpha})$ and let W_0 be a subgroup of odd order in G. Then $H^2(C_2, \widetilde{F_1}^{W_0}) \to H^2(C_2, (\mathbb{Q}_2(F_0)^{W_0})^\times)$ is surjective if and only if $\alpha = k$. In this case, its kernel is isomorphic to C_2 if $u \equiv \pm 3 \bmod 8$ and it is an isomorphism if $u \equiv \pm 1 \bmod 8$.*

Proof. Let $L := \mathbb{Q}_2(F_0)^{W_0}$, $K := L^{C_2}$ and $H := G/W_0 = Gal(L/\mathbb{Q}_p)$. Note that L/K is totally ramified. Similarly to corollary 4.23, we may consider the homomorphism

$$\tau : H^2(C_2, L^\times) \longrightarrow \mathbb{Z}_2^\times / N_H(\mathcal{O}_L^\times)$$

given by the norm

$$N_{H/C_2} : H^2(C_2, L^\times) \cong H^2(C_2, \mathcal{O}_L^\times) \cong (\mathcal{O}_K^\times)/N_{C_2}(\mathcal{O}_L^\times) \longrightarrow \mathbb{Z}_2^\times / N_H(\mathcal{O}_L^\times).$$

Here again, as in corollary 4.23, we have short exact sequences forming a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_2^\times / N_H(\mathcal{O}_L^\times) & \longrightarrow & \mathbb{Q}_2^\times / N_H(L^\times) & \xrightarrow{v} & \mathbb{Z}/v(N_H(L^\times)) & \longrightarrow & 1 \\ & & \cong \downarrow & & \cong \downarrow (\_, L/\mathbb{Q}_2) & & \cong \downarrow \sigma(\_) & & \\ 1 & \longrightarrow & Gal(L/K) & \longrightarrow & Gal(L/\mathbb{Q}_2) & \xrightarrow{pr} & Gal(l/\mathbb{F}_2) & \longrightarrow & 1 \end{array}$$

where

$$Gal(L/K) \cong C_2 \qquad \text{and} \qquad Gal(l/\mathbb{F}_2) \cong C_{\frac{n_\alpha}{|W_0|}},$$

for l the residue field of L. Since $L^{C_{n_\alpha}/W_0} = \mathbb{Q}_2(F_0)^{C_{n_\alpha}} = \mathbb{Q}_2(i)$, and since $L/\mathbb{Q}_2(i)$ is unramified, we know from proposition B.15 that

$$N_{H/C_2} : \mathcal{O}_L^\times \longrightarrow \mathbb{Z}_2(i)^\times$$

is surjective; consequently

$$N_H(\mathcal{O}_L^\times) = N_{C_2} \circ N_{H/C_2}(\mathcal{O}_L^\times) = N_{C_2}(\mathbb{Z}_2(i)^\times).$$

Furthermore, as in (4.1), for any elements $a_1, a_2 \in \mathbb{Z}_2$ we have

$$N_{C_2}(1 + (a_1 + a_2 i)(1 + i)) \equiv 1 \bmod 4.$$

Hence $N_H(\mathcal{O}_L^\times) = U_2(\mathbb{Z}_2^\times)$ and the map

$$\tau : H^2(C_2, L^\times) \longrightarrow \mathbb{Z}_2^\times / U_2(\mathbb{Z}_2^\times) = \{\pm 1\}$$

is surjective by proposition B.15.

Using lemma 4.24 and 4.25, the map $i^* : H^2(C_2, \widetilde{F_1}^{W_0}) \to H^2(C_2, L^\times)$ is therefore surjective if and only if

$$-1 = \begin{cases} \tau(2u) \text{ or } \tau(-1) & \text{if } u \equiv \pm 3 \bmod 8 \text{ and } n_\alpha \text{ odd}, \\ \tau(-1) & \text{otherwise.} \end{cases}$$

Since $N_{C_2}(1+i) = (1+i)(1-i) = 2$, remark B.16 implies

$$\tau(2u) = \tau(u) = u^{|H/C_2|} \qquad \text{and} \qquad \tau(-1) = (-1)^{|H/C_2|}.$$

Hence $\tau(-1) = -1$ if and only if

$$|H/C_2| = |C_{n_\alpha}/W_0| \text{ is odd} \qquad \Leftrightarrow \qquad n_\alpha \text{ is odd} \qquad \Leftrightarrow \qquad \alpha = k,$$

and the result follows. □

Theorem 4.27. *Let $p = 2$, $n = 2^{k-1}m$ with m odd, $u \in \mathbb{Z}_2^\times$, $F_0 = C_{2^\alpha} \times C_{2^{n_\alpha}-1}$ be a maximal abelian finite subgroup of $\mathbb{S}_n$, $G = Gal(\mathbb{Q}_2(F_0)/\mathbb{Q}_2)$, $G_{2'}$ be the odd part of G, and let $\widetilde{F_1} = \langle x_1 \rangle \times F_0 \subseteq \mathbb{Q}_2(F_0)^\times$ be maximal as a subgroup of $\mathbb{Q}_2(F_0)^\times$ having $\widetilde{F_0}$ as subgroup of finite index.*

1) *For any $1 \leq \alpha \leq k$, there is an extension of $\widetilde{F_1}$ by $G_{2'}$; this extension is unique up to conjugation.*

2) *If $\alpha = 1$, there is an extension of $\widetilde{F_1}$ by G; the number of such extensions up to conjugation is*
$$\begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

3) *If $\alpha = 2$, there is an extension of $\widetilde{F_1}$ by G if and only if $k = 2$; the number of such extensions up to conjugation is*
$$\begin{cases} 1 & \text{if } u \equiv \pm 1 \bmod 8, \\ 2 & \text{if } u \not\equiv \pm 1 \bmod 8. \end{cases}$$

4) *If $\alpha \geq 3$, there is no extension of $\widetilde{F_1}$ by G.*

Proof. 1) From corollary 4.16 and proposition 4.18 we know that

$$i^*_{G_{2'}} : H^2(G_{2'}, \widetilde{F_1}) \longrightarrow H^2(G_{2'}, \mathbb{Q}_2(F_0)^\times)$$

is an isomorphism. Existence and uniqueness up to conjugation then follows from corollary 2.29.

2) This follows from corollary 4.16 and 2.29.

3) Let $\alpha = 2$. Applying proposition 4.1 and corollary 2.29 together with corollary 4.26 in the case where W_0 is trivial, we obtain that $\widetilde{F_1}$ can never be extended by G when n_α is even. Assume then that n_α is odd. In this case G decomposes canonically as

$$G = G_{2'} \times C_2 \qquad \text{with} \quad G_{2'} = C_{n_\alpha}.$$

In particular, there is a short exact sequence

$$1 \longrightarrow C_{n_\alpha} \longrightarrow G \longrightarrow C_2 \longrightarrow 1$$

which gives rise to the Hochschild-Serre spectral sequences (see [4] section VII.6)

$$E_2^{s,t} \cong H^s(C_2, H^t(C_{n_\alpha}, \widetilde{F_1})) \quad \Longrightarrow \quad H^{s+t}(G, \widetilde{F_1}),$$

$$E_2^{s,t} \cong H^s(C_2, H^t(C_{n_\alpha}, \mathbb{Q}_2(F_0)^\times)) \quad \Longrightarrow \quad H^{s+t}(G, \mathbb{Q}_2(F_0)^\times).$$

From lemma 4.19, 4.20 and proposition 4.18, each map $E_2^{s,t} \to E_2^{s,t}$ is an isomorphism for $t > 0$. We also have

$$H^0(C_{n_\alpha}, \widetilde{F_1}) = \widetilde{F_1}^{C_{n_\alpha}} = \langle x_1 \rangle \times C_{2^\alpha}$$

and

$$H^0(C_{n_\alpha}, \mathbb{Q}_2(F_0)^\times) = (\mathbb{Q}_2(F_0)^{C_{n_\alpha}})^\times = \mathbb{Q}_2(i)^\times.$$

Then corollary 4.26 applied to the case $W_0 = C_{n_\alpha}$ shows that the map

$$H^s(C_2, \widetilde{F_1}^{C_{n_\alpha}}) \longrightarrow H^s(C_2, (\mathbb{Q}_2(F_0)^{C_{n_\alpha}})^\times)$$

is surjective when $t = 0$ and $s > 0$; its kernel is trivial if $u \equiv \pm 1 \bmod 8$, otherwise it is of cardinality 2. In fact, since n_α is odd, all the terms $E_2^{s,t}$ for which $s > 0$ and $t > 0$ are trivial. By the results of lemma 4.19 and 4.20, the non-trivial terms for which $s = 0$ are of odd order, and the non-trivial terms for which $t = 0$ are powers of 2. Hence all differentials of the spectral sequences are trivial and $E_2^{*,*} = E_\infty^{*,*}$. Consequently, i_G^* is surjective if and only if n_α is odd, that is, if and only if $\alpha = k$. The result then follows from corollary 2.29.

4) By corollary 4.23 and proposition 4.1 the map

$$i_G^* : H^2(G, \widetilde{F_1}) \longrightarrow H^2(G, \mathbb{Q}_2(F_0)^\times)$$

is never surjective if $\alpha \geq 3$. The result is then a consequence of corollary 2.29. □

4.3. Extensions of maximal finite subgroups of $\mathbb{S}_n$ containing Q_8

In this section, we establish under what condition a maximal finite subgroup G of $\mathbb{S}_n$ with a quaternionic 2-Sylow subgroup extends to a subgroup of order $n|G|$ in $\mathbb{G}_n(u)$. Recall from theorem 1.35 that such a G exists if and only if $p = 2$ and $n = 2m$ with m odd, in which case

$$G \cong Q_8 \rtimes C_{3(2^m-1)} \cong T_{24} \times C_{2^m-1}.$$

Theorem 4.28. *Let $p = 2$, $n = 2m$ with m odd, and $u \in \mathbb{Z}_2^\times$. A subgroup G isomorphic to $T_{24} \times C_{2^m-1}$ in $\mathbb{S}_n$ extends to a maximal finite subgroup F of order $n|G| = 48m(2^m - 1)$ in $\mathbb{G}_n(u)$ if and only if $u \equiv \pm 1 \bmod 8$; this extension is unique up to conjugation. Moreover if $u \not\equiv \pm 1 \bmod 8$ and G' is a subgroup isomorphic to $Q_8 \times C_{2^m-1}$ in $\mathbb{S}_n$, there is no extension of G' of order $n|G'|$ in $\mathbb{G}_n(u)$.*

Proof. Let $i, j, \zeta_3, \zeta_{2^m-1}$ be elements of respective order 4, 4, 3 and $2^m - 1$ generating G, and let $T := \langle i, j, \zeta_3 \rangle \cong T_{24}$. We first establish the structure of the centralizer of G. By the centralizer theorem A.6, there is a $\mathbb{Q}_2$-algebra isomorphism

$$\mathbb{D}_n \cong \mathbb{Q}_2(T) \otimes_{\mathbb{Q}_2} C_{\mathbb{D}_n}(T),$$

where $C_{\mathbb{D}_n}(T)$ is a central division algebra of dimension m^2 over $\mathbb{Q}_2$. Note that the commutative extension $\mathbb{Q}_2(\zeta_{2^m-1})/\mathbb{Q}_2$ is maximal unramified in $C_{\mathbb{D}_n}(T)$. Consequently

$$C_{\mathbb{D}_n^\times}(G) \cong \mathbb{Q}_2(\zeta_{2^m-1})^\times, \qquad C_{\mathbb{S}_n}(G) \cong \mathbb{Z}_2(\zeta_{2^m-1})^\times,$$

and as $\mathbb{Q}_2(\zeta_{2^m-1})/\mathbb{Q}_2$ is unramified we have $C_{\mathbb{S}_n}(G) \cong C_{\mathbb{G}_n(u)}(G)$.

We now show the existence of the desired extension of order $n|G|$ assuming $u \equiv \pm 1 \bmod 8$, that is $u \equiv \pm 1 \bmod (\mathbb{Z}_2^\times)^2$. Let $t \in \mathbb{Z}_2^\times$ be such that

$$t^2 = \begin{cases} u & \text{if } u \equiv 1 \bmod 8, \\ -u & \text{if } u \equiv -1 \bmod 8. \end{cases}$$

The valuation map gives rise to a short exact sequence

$$1 \longrightarrow N_{\mathbb{S}_n}(G) \longrightarrow N_{\mathbb{G}_n(u)}(G) \xrightarrow{v} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/n \longrightarrow 1.$$

Let $\xi_u \in C_{\mathbb{D}_n^\times}(T)$ be an element satisfying $\xi_u^m = 2u$ and acting on ζ_{2^m-1} by raising it to its square. Consider the element $(1+i)tj\xi_u \in \mathbb{D}_n^\times$. It becomes a generator in $\mathbb{Z}/n$ as

$$v((1+i)tj\xi_u) = v(1+i) + v(\xi_u) = \frac{1}{2} + \frac{1}{m} = \frac{m+2}{n},$$

where $m+2$ is prime to n. Furthermore as t, ξ_u commute with i, j, we have

$$\begin{aligned} [(1+i)tj\xi_u]^n &= [(1+i)j(1+i)jt^2\xi_u^2]^m \\ &= [(1+i)(1-i)j^2t^2\xi_u^2]^m \\ &= [-2t^2\xi_u^2]^m \\ &= \begin{cases} -(2u)^{m+2} & \text{if } u \equiv 1 \bmod 8, \\ (2u)^{m+2} & \text{if } u \equiv -1 \bmod 8, \end{cases} \end{aligned}$$

and it is easy to check that $(1+i)tj\xi_u \in N_{\mathbb{D}_n^\times}(G)$. This shows the existence of F in the case $u \equiv \pm 1 \bmod 8$.

We proceed to the non-existence part of the result for $u \not\equiv 1 \bmod 8$. First note that there is a short exact sequence

$$1 \longrightarrow C_{\mathbb{D}_n^\times}(G) \longrightarrow N_{\mathbb{D}_n^\times}(G) \xrightarrow{\rho} Aut(T_{24}) \times Gal(\mathbb{Q}_2(\zeta_{2^m-1})/\mathbb{Q}_2) \longrightarrow 1,$$

where $|Aut(T_{24})| = 24$ and $Gal(\mathbb{Q}_2(\zeta_{2^m-1})/\mathbb{Q}_2)$ is cyclic of order m. Indeed, if $x \in N_{\mathbb{D}_n^\times}(G)$, then the conjugation action by x preserves both G and its 2-Sylow subgroup Q. Consequently $\mathbb{Q}_2(Q)^\times = \mathbb{Q}_2(T)^\times$ and $C_{\mathbb{D}_n^\times}(G)$ are preserved as well. As for the surjectivity of ρ, we know from the Skolem-Noether theorem that the restriction of ρ to $\mathbb{Q}_2(T)^\times \subseteq N_{\mathbb{D}_n^\times}(G)$ is surjective on $Aut(T_{24})$, while by definition the element $\xi_u \in N_{\mathbb{D}_n^\times}(G)$ maps to a generator of $Gal(\mathbb{Q}_2(\zeta_{2^m-1})/\mathbb{Q}_2)$. Now since

$$C_{\mathbb{D}_n^\times}(G) = \mathbb{Q}_2(\zeta_{2^m-1})^\times \qquad \text{and} \qquad v(N_{\mathbb{D}_n^\times}(G)) = \frac{1}{n}\mathbb{Z}$$

as shown in proposition 1.20, we know that

$$N_{\mathbb{D}_n^\times}(G) = \langle C_{\mathbb{D}_n^\times}(G), G, (1+i), \xi_u \rangle = \langle \mathbb{Z}_2[\zeta_{2^m-1}]^\times, T, (1+i), \xi_u \rangle.$$

In the case $u \not\equiv \pm 1 \bmod 8$, we claim that there is no $x \in N_{\mathbb{D}_n^\times}(G)$ such that

$$v(x) = \frac{1}{n} \qquad \text{and} \qquad x^n \in \langle G, 2u \rangle.$$

Indeed, if such an x existed, there would be a $y \in T$ and a $z \in \mathbb{Z}_2[\zeta_{2^m-1}]^\times$ such that $x^m = (1+i)yz$, in which case

$$\begin{aligned} x^{2m} &= (1+i)y(1+i)yz^2 \\ &= (1+i)^2(1+i)^{-1}y(1+i)yz^2 \\ &= 2i\sigma(y)yz^2 \end{aligned}$$

would belong to $2z^2T$, for σ the automorphism of T induced by the conjugation by $(1+i)^{-1}$. In this case $2z^2 \in \langle G, 2u\rangle \cap \mathbb{Q}_2(\zeta_{2^m-1})^\times$, and there would be a $g \in G$ with

$$2z^2 = g(2u) \qquad \Leftrightarrow \qquad z^2 = gu.$$

Since both z^2 and u are in $\mathbb{Z}_2(\zeta_{2^m-1})^\times$, so does g and $z^2 = \pm u$. As shown in corollary 3.38, this is impossible since m is odd and $u \not\equiv \pm 1 \bmod (\mathbb{Z}_2^\times)^2$. It follows that G cannot be extended as a subgroup of order $n|G|$ in $\mathbb{G}_n(u)$ when $u \not\equiv \pm 1 \bmod 8$. In fact, the argument also shows the corresponding result for G': since $\mathbb{Q}_2(Q_8) = \mathbb{Q}_2(T_{24})$, we have

$$\mathbb{Q}_2(G') = \mathbb{Q}_2(G) \qquad \text{and} \qquad N_{\mathbb{D}_n^\times}(G') = N_{\mathbb{D}_n^\times}(G),$$

and there is no x of valuation $\frac{1}{n}$ in $N_{\mathbb{D}_n^\times}(G')$ such that $x^n \in \langle G', 2u\rangle \subseteq \langle G, 2u\rangle$.

It remains to verify the statement on uniqueness when $u \equiv \pm 1 \bmod 8$. For a finite group F of order $n|G|$ extending G, we have $F \in N_{\mathbb{D}_n^\times}(G)$. Let

$$A := F \cap Ker(\rho) = \langle 2u, -1, \zeta_{2^m-1}\rangle \qquad \text{and} \qquad B := F/A$$

Applying theorem 2.14 to the case $F \in \mathcal{G}_\rho(N_{\mathbb{D}_n^\times}(G), A, B)$, it is enough to check that the cohomology group $H^1(B, Ker(\rho)/A)$ is trivial. As

$$|B| < \infty, \qquad Ker(\rho)/A = \mathbb{Q}_2(\zeta_{2^m-1})^\times/A \cong \mathbb{Z}_2^m,$$

and because the B-module structure is trivial, we obtain

$$H^1(B, Ker(\rho)/A) \cong Hom(B, \mathbb{Z}_2^m) = 0.$$

□

4.4. Example of the case $n = 2$

In this section, we illustrate the situation for $n = 2$ and we find the finite subgroups of $\mathbb{G}_2(u)$ up to conjugation for $p \in \{2, 3\}$, that is for those primes p for which $p-1$ divides n.

For a given p, we let $\omega \in \mathbb{S}_2$ be a primitive (p^2-1)-th root of unity and σ be the Frobenius automorphism of $\mathbb{Q}_p(\omega)/\mathbb{Q}_p$. For each $u \in \mathbb{Z}_p^\times$, we let $\xi_u \in \mathbb{D}_2^\times$ be an element associated to σ such that $\xi_u^2 = pu$. As in example 4.4 and 4.17 the multiplicative subgroups in the division algebra

$$\mathbb{D}_2 \cong \mathbb{Q}_p(\omega)\langle\xi_u\rangle/(\xi_u^2 = pu, \xi_u x = x^\sigma \xi_u), \qquad x \in \mathbb{Q}_p(\omega),$$

which correspond to finite subgroups of $\mathbb{G}_2(u)$ are easily expressible in terms of ξ_u and ω. This allows to determine the conjugacy classes of those finite subgroups explicitly.

The case $p = 3$

Let $p = 3$. Here $k = 1$, $m = 1$ and $\alpha \in \{0, 1\}$.

1) If $\alpha = 0$, then $F_0 = \langle\omega\rangle \cong C_8$ and $\widetilde{F_0} = F_0 \times \langle 3u\rangle$. As shown in example 4.4

$$\widetilde{F_0} = \widetilde{F_1} = \widetilde{F_2}, \qquad \widetilde{F_3} = \langle \widetilde{F_0}, \xi_u\rangle \quad \text{with} \quad \xi_u^2 = 3u,$$

and for $\overline{\xi}_u$ the class of ξ_u in $\mathbb{G}_n(u)$ the group

$$F_3 = \langle \omega, \overline{\xi}_u\rangle \cong \ SD_{16}$$

is a semidihedral group of order 16.

2) If $\alpha = 1$, then $F_0 = \langle\zeta_3\rangle \times \langle\omega^4\rangle \cong C_6$ where $\omega^4 = -1$. The primitive third root of unity $\zeta_3 \in \mathbb{S}_2$ may be given by

$$\zeta_3 = -\frac{1}{2}(1 + \omega S) \qquad \text{for} \quad S = \xi_1.$$

In this case

$$\zeta_3^{-1} = \zeta_3^2 = -\frac{1}{2}(1 - \omega S) \qquad \text{and} \qquad \zeta_3^2 - \zeta_3 = \omega S.$$

According to theorem 4.13 there is no restriction on u, and x_1 can be chosen as $x_1 = (\zeta_3^2 - \zeta_3)t$ with $t \in \mathbb{Z}_3^\times$ such that

$$t^2 = \begin{cases} u & \text{if } u \equiv 1 \bmod 3, \\ -u & \text{if } u \equiv -1 \bmod 3. \end{cases}$$

Indeed,

$$x_1^2 = (\omega S)^2 t^2 = \omega^4 S^2 t^2 = -3t^2,$$

so that $v(x_1) = \frac{1}{2}$, and we have

$$x_1 \zeta_3 x_1^{-1} = -\frac{1}{2}(1 + (\omega S)^2 (\omega S)^{-1}) = \zeta_3,$$
$$x_1 \omega^2 x_1^{-1} = \omega S \omega^2 S^{-1} \omega^{-1} = (\omega^2)^3.$$

Hence $\widetilde{F_1} = \widetilde{F_2} = \langle x_1\rangle \times \langle\zeta_3\rangle \times \langle\omega^4\rangle$, where

$$x_1^2 = \begin{cases} -3u & \text{if } u \equiv 1 \bmod 3, \\ 3u & \text{if } u \equiv -1 \bmod 3, \end{cases} \qquad \text{and} \qquad \overline{x}_1^2 = \begin{cases} -1 & \text{if } u \equiv 1 \bmod 3, \\ 1 & \text{if } u \equiv -1 \bmod 3, \end{cases}$$

for $\overline{x}_1$ the class of x_1 in $\mathbb{G}_2(u)$. Furthermore $\omega^2 \in N_{\mathbb{D}_2^\times}(\widetilde{F_1})$ given that $\omega^2 \zeta_3 \omega^{-2} = \zeta_3^2$, and

$$\omega^2 x_1 \omega^{-2} = \omega^2 \zeta_3 (1 - \zeta_3) \omega^{-2} = \zeta_3^2 (1 - \zeta_3^2) = (\zeta_3 + \zeta_3^2)(\zeta_3 - \zeta_3^2) = -x_1.$$

Thus $\widetilde{F_3} = \langle \widetilde{F_1}, \omega^2\rangle$ and $F_3 = \langle \overline{x}_1, \zeta_3, \omega^2\rangle$ is a maximal finite subgroup of order 24 in $\mathbb{G}_2(u)$.

We let

$$D_8 \cong \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$$

denote the dihedral group of order 8.

Theorem 4.29. *Let $n = 2$, $p = 3$ and $u \in \mathbb{Z}_p^\times$. The conjugacy classes of maximal finite subgroups F of $\mathbb{G}_2(u)$ are represented by*

$$SD_{16} \qquad \text{and} \qquad \begin{cases} C_3 \rtimes Q_8 & \text{if } u \equiv 1 \bmod 3, \\ C_3 \rtimes D_8 & \text{if } u \equiv -1 \bmod 3. \end{cases}$$

Proof. We first consider the cases where F_0 is such that $[\mathbb{Q}_3(F_0) : \mathbb{Q}_3] = 2$; by theorem 2.30 we may assume that F_0 is maximal. The first class originates from the case $\alpha = 0$; its existence and uniqueness follow from example 4.4 and theorem 4.13.

Suppose then that $\alpha = 1$. If $u \equiv 1 \bmod 3$, the 2-Sylow subgroup $\langle \omega^2, \overline{x}_1 \rangle$ of F_3 is isomorphic to Q_8. As the latter does not contain a subgroup isomorphic to $C_2 \times C_2$, the short exact sequence

$$1 \longrightarrow F_2 = \langle \zeta_3, \overline{x}_1 \rangle \longrightarrow F_3 \longrightarrow C_2 \longrightarrow 1$$

does not split. However, $\langle \zeta_3 \rangle$ being normal in F_3, we obtain $F_3 \cong C_3 \rtimes Q_8$. On the other hand if $u \equiv -1 \bmod 3$, the group F_3 contains a subgroup isomorphic to $C_2 \times C_2$. In this case we have a split extension

$$1 \longrightarrow F_2 = \langle \zeta_3, -1, \overline{x}_1 \rangle \longrightarrow F_3 \longrightarrow C_2 \longrightarrow 1$$

with a 2-Sylow subgroup isomorphic to $D_8 \cong \langle \omega^2, \overline{x}_1 \mid (\omega^2)^4 = \overline{x}_1^2 = 1, \overline{x}_1 \omega^2 \overline{x}_1^{-1} = \omega^{-2} \rangle$, and $F_3 \cong C_3 \rtimes D_8$. Uniqueness of the class of F_3 in $\mathbb{G}_2(u)$ follows from theorem 4.13.

It remains to consider the case where $F_0 = \{\pm 1\} \cong C_2$, that is, F_0 is maximal such that $\mathbb{Q}_3(F_0) = \mathbb{Q}_3$. Then obviously $\widetilde{F_0} = \widetilde{F_1}$. Because $\mathbb{Q}_3^\times / (\mathbb{Q}_3^\times)^2 \cong \mathbb{Z}/2\mathbb{Z} \times \{\pm 1\}$ is represented by the elements of the set $\{\pm 1, \pm 3\}$, we know that there are three possible quadratic extensions of $\mathbb{Q}_3$ given by

$$L_v := \mathbb{Q}_3/(X^2 - v) \qquad \text{for } v \in \{-1, \pm 3\};$$

each of them is unique up to conjugation. Among these $L_{-1} = \mathbb{Q}_3(\zeta_8)$ and $L_{-3} = \mathbb{Q}_3(\zeta_3)$ have already been considered.

Hence suppose $v = 3$ and let $x_2 := Xt$ with $t \in \mathbb{Z}_3^\times$ such that

$$t^2 = \begin{cases} u & \text{if } u \equiv 1 \bmod 3, \\ -u & \text{if } u \equiv -1 \bmod 3. \end{cases}$$

Then

$$x_2^2 = 3t^2 = \begin{cases} 3u \equiv 1 \bmod \langle 3u \rangle & \text{if } u \equiv 1 \bmod 3, \\ -3u \equiv -1 \bmod \langle 3u \rangle & \text{if } u \equiv -1 \bmod 3, \end{cases}$$

and we have an extension

$$1 \longrightarrow \widetilde{F_1} = \langle 2u, \pm 1 \rangle \longrightarrow \widetilde{F_2} = \langle x_2, \pm 1 \rangle \longrightarrow C_2 \longrightarrow 1,$$

where

$$F_2 \cong \begin{cases} C_2 \times C_2 & \text{if } u \equiv 1 \bmod 3, \\ C_4 & \text{if } u \equiv -1 \bmod 3. \end{cases}$$

By corollary 2.23, this group is unique up to conjugation. Because the group $Aut(F_0)$ is trivial, proposition 2.25 implies $F_3 = F_2$. This class however is neither new nor maximal.

Indeed, for the group $\langle \omega, \xi_u \rangle \subseteq \mathbb{D}_2^\times$ whose corresponding group $\langle \omega, \overline{\xi}_u \rangle$ in $\mathbb{G}_n(u)$ represents the class SD_{16} found above, one can take

$$x_2 = \begin{cases} \xi_u & \text{if } u \equiv 1 \bmod 3, \\ \omega\xi_u & \text{if } u \equiv -1 \bmod 3, \end{cases}$$

in order to see that $F_2 \subseteq SD_{16}$. □

The case $p = 2$

Let $p = 2$. Here $k = 2$, $m = 1$ and $\alpha \in \{1, 2\}$.

1) If $\alpha = 1$, then $F_0 = \langle -\omega \rangle \cong C_6$ and $\widetilde{F_0} = F_0 \times \langle 2u \rangle$. As shown in example 4.17

$$\widetilde{F_0} = \widetilde{F_1} = \widetilde{F_2}, \qquad \widetilde{F_3^\pm} = \langle \widetilde{F_0}, \xi_{\pm u} \rangle \quad \text{with } \xi_{\pm u}^2 = \pm 2u,$$

and we have

$$F_3^+ = \langle -\omega, \overline{\xi}_u \rangle \cong C_6 \rtimes C_2, \qquad F_3^- = \langle -\omega, \overline{\xi}_{-u} \rangle \cong C_3 \rtimes C_4,$$

for $\overline{\xi}_{\pm u}$ the class of $\xi_{\pm u}$ in $\mathbb{G}_n(u)$.

2) If $\alpha = 2$, then $F_0 = C_4 \subseteq T_{24}$ with $C_4 = \langle i \rangle$ and $T_{24} = \langle i, j \rangle \rtimes \langle \zeta_3 \rangle$. According to theorem 4.27 and 4.28, a finite maximal extension of F_0 in $\mathbb{G}_2(u)$ is an extension of T_{24} if and only if $u \equiv \pm 1 \bmod 8$. Let

$$x_1 = \begin{cases} (1+i)t \text{ with } t^2 = u & \text{if } u \equiv 1 \bmod 8, \\ (1+i)t \text{ with } t^2 = -u & \text{if } u \equiv -1 \bmod 8, \\ 2u & \text{if } u \equiv \pm 3 \bmod 8. \end{cases}$$

Then we know that $\widetilde{F_1} = \widetilde{F_2} = \langle x_1 \rangle \times F_0$. In case $u \equiv \pm 1 \bmod 8$, we have $\widetilde{F_3} = \widetilde{F_2}$ and we find $x_1^2 = 2it^2$, $x_1^4 = -4u^2$ and $x_1^8 = (2u)^4$, so that the group F_3 is cyclic of order 8; it is unique up to conjugation by corollary 2.18.

We let

$$O_{48} \cong \langle a, b, c \mid a^2 = b^3 = c^4 = abc \rangle$$

denote the binary octahedral group of order 48.

Theorem 4.30. *Let $n = 2$, $p = 2$ and $u \in \mathbb{Z}_2^\times$. The conjugacy classes of maximal finite subgroups F of $\mathbb{G}_2(u)$ are represented by*

$$\begin{cases} C_6 \rtimes C_2,\ O_{48} & \text{if } u \equiv 1 \bmod 8, \\ C_3 \rtimes C_4,\ T_{24} \rtimes C_2 & \text{if } u \equiv -1 \bmod 8, \\ C_3 \rtimes C_4,\ C_6 \rtimes C_2,\ D_8 \text{ and } T_{24} & \text{if } u \equiv 3 \bmod 8, \\ C_3 \rtimes C_4,\ C_6 \rtimes C_2,\ Q_8 \text{ and } T_{24} & \text{if } u \equiv -3 \bmod 8. \end{cases}$$

Proof. We first consider the cases where F_0 is such that $[\mathbb{Q}_2(F_0) : \mathbb{Q}_2] = 2$; by theorem 2.30 we may assume that F_0 is maximal. The classes $C_6 \rtimes C_2$ and $C_3 \rtimes C_4$ originate from the case $\alpha = 1$. They are respectively represented by

$$F_3^+ = \langle -\omega, \overline{\xi}_u \rangle \qquad \text{and} \qquad F_3^- = \langle -\omega, \overline{\xi}_{-u} \rangle.$$

Their existence and uniqueness follow from example 4.17 and theorem 4.27. We will now analyse the case where $F_0 = \langle i \rangle \cong C_4$.

Suppose that $u \equiv \pm 1 \bmod 8$. Then

$$x_1^2 = (1+i)^2 t^2 = 2it^2 \equiv \begin{cases} i \bmod \langle 2u \rangle & \text{if } u \equiv 1 \bmod 8, \\ -i \bmod \langle 2u \rangle & \text{if } u \equiv -1 \bmod 8, \end{cases} \qquad x_1^4 \equiv -1 \bmod \langle 2u \rangle,$$

and

$$x_1 i x_1^{-1} = i, \qquad x_1 j x_1^{-1} = (1+i)j\frac{(1-i)}{2} = \frac{(1+i)^2}{2} j = ij = k.$$

Therefore, we have a chain of subgroups

$$\widetilde{F_0} = \langle i, 2u \rangle \subsetneq \widetilde{F_1} = \widetilde{F_2} = \langle i, x_1 \rangle \subsetneq \widetilde{F_3} = \langle i, j, x_1 \rangle,$$

where $\widetilde{F_i}$ is normal in $\widetilde{F_{i+1}}$ for $1 \leq i \leq 3$, and where $|\widetilde{F_1}/\widetilde{F_0}| = |\widetilde{F_3}/\widetilde{F_2}| = 2$. Because $x_1^2 \equiv \pm i \bmod \langle 2u \rangle$ and $x_1^4 \equiv -1 \bmod \langle 2u \rangle$, we know that for $\overline{x}_1$ the class of x_1 in $\mathbb{G}_n(u)$ we have $F_1 \cong C_8$ and there is an extension

$$1 \longrightarrow F_1 = \langle \overline{x}_1 \rangle \longrightarrow F_3 = \langle \overline{x}_1, j \rangle \longrightarrow C_2 \longrightarrow 1,$$

where $j\overline{x}_1 \in F_3$ maps non-trivially to the quotient group. As

$$(jx_1)^2 = j(x_1 j x_1^{-1}) x_1^2 = j(ij)(2it^2) = -2t^2$$
$$\equiv \begin{cases} -1 \bmod \langle 2u \rangle & \text{if } u \equiv 1 \bmod 8, \\ 1 \bmod \langle 2u \rangle & \text{if } u \equiv -1 \bmod 8, \end{cases}$$

and since

$$(jx_1)x_1(jx_1)^{-1} = jx_1 j^{-1} = -(jx_1)^2 x_1^{-1} = 2t^2 x_1^{-1}$$
$$\equiv \begin{cases} x_1^{-1} \bmod \langle 2u \rangle & \text{if } u \equiv 1 \bmod 8, \\ -x_1^{-1} \bmod \langle 2u \rangle & \text{if } u \equiv -1 \bmod 8, \end{cases}$$

we find

$$F_3 \cong \begin{cases} Q_{16} & \text{if } u \equiv 1 \bmod 8, \\ C_8 \rtimes C_2 = SD_{16} & \text{if } u \equiv -1 \bmod 8. \end{cases}$$

Clearly, F_3 is a 2-Sylow subgroup of $F := \langle F_3, \omega \rangle$ and $T_{24} = \langle i, j, \omega \rangle \subseteq F$. As seen above, x_1 and jx_1 both belong to $N_{\mathbb{D}_n^\times}(\langle i, j \rangle) = N_{\mathbb{D}_n^\times}(\langle i, j, \omega \rangle)$, and there is an extension

$$1 \longrightarrow T_{24} = \langle i, j, \omega \rangle \longrightarrow F \longrightarrow C_2 \longrightarrow 1,$$

where $x_1, jx_1 \in F$ are mapped non-trivially to the quotient group.

Assume for the moment that $u \equiv 1 \bmod 8$. We let $a := \overline{x}_1$, so that $a^2 = i$, and we consider the element of order 6

$$b := \frac{1}{2}(1 + i + j + k) \ \in T_{24} \subseteq F.$$

Then we can take $\omega = -b$ and we easily check that

$$b^{-1} = -\omega^2 = \frac{1}{2}(1 - i - j - k), \qquad b^{-1} a^2 b = j.$$

In particular $F = \langle a, b\rangle$ is generated by the elements a and b of respective order 8 and 6. These two elements interact via $ba = -a^{-1}b^{-1}$ since

$$(bx_1)^2 = \frac{1}{4}(2i + 2j)^2 u = (i + j)^2 u = -2u \equiv -1 \mod \langle 2u\rangle.$$

Letting $c := ba$, it follows that

$$F = \langle a, b \mid (ba)^2 = b^3 = a^4 = -1\rangle = \langle a, b, c \mid c^2 = b^3 = a^4 = cba\rangle$$

is isomorphic to the binary octahedral group O_{48}. Uniqueness of F up to conjugation is given by theorem 4.28; its class is clearly maximal. In fact since

$$\begin{aligned}(jx_1)\omega(jx_1)^{-1} &= -\frac{1}{2}jx_1(1 + i + j + k)x_1^{-1}j^{-1}\\ &= -\frac{1}{2}j(1 + i + k - j)j^{-1}\\ &= -\frac{1}{2}(1 - i - k - j)\\ &= \omega^2,\end{aligned}$$

we may take $\xi_{-u} = jx_1$ in order to find that F contains $F_3^- = \langle b, j\overline{x}_1\rangle$. On the other hand, F does not have a subgroup isomorphic to F_3^+ since its 2-Sylow subgroup $F_3 \cong Q_{16}$ has no subgroup isomorphic to $C_2 \times C_2$. The class of F_3^+ is therefore maximal when $[\mathbb{Q}_2(F_0) : \mathbb{Q}_2] = 2$ and $u \equiv 1 \bmod 8$.

Now assume $u \equiv -1 \bmod 8$. Then $(jx_1)^2 \equiv 1 \bmod \langle 2u\rangle$, in which case

$$F = \langle \overline{x}_1, j, \omega\rangle \cong T_{24} \rtimes C_2.$$

The above calculations show that we may take $\xi_u = jx_1$ in order to find that $F_3^+ = \langle b, j\overline{x}_1\rangle$ is a subgroup of F. On the other hand one easily verifies that the group $\langle x_1, i, j\rangle$ does not have an element of valuation $\frac{1}{2}$ which has order 4 modulo $\langle 2u\rangle$. This means that the class of F does not contain that of F_3^-. The latter is therefore maximal when $[\mathbb{Q}_2(F_0) : \mathbb{Q}_2] = 2$ and $u \equiv -1 \bmod 8$.

We now suppose $u \equiv \pm 3 \bmod 8$. By theorem 1.35, a maximal finite subgroup F of $\mathbb{G}_2(u)$ containing $F_0 = \langle i\rangle \cong C_4$ satisfies $C_4 \subseteq F \cap \mathbb{S}_2 \subseteq T_{24}$. If $F \subseteq \mathbb{S}_2$, then $F \cong T_{24}$ contains the subgroup $F \cap S_n \cong Q_8$ as in lemma 2.24.b. Otherwise if $F \not\subseteq \mathbb{S}_2$, the 2-Sylow subgroup of $F \cap \mathbb{S}_2$ must be C_4 by theorem 4.28, and we have a chain of subgroups

$$\widetilde{F_0} = \widetilde{F_1} = \widetilde{F_2} \subseteq \widetilde{F_3} = \widetilde{F},$$

where $\widetilde{F_3}/\widetilde{F_0}$ is a cyclic group of order at most 2. We are thus looking for an element $x_3 \in \mathbb{D}_2^\times$ such that $x_3^2 \in \widetilde{F_0} = F_0 \times \langle 2u\rangle$. By the Skolem-Noether theorem, there is a short exact sequence

$$1 \longrightarrow C_{\mathbb{D}_2^\times}(F_0) = \mathbb{Q}_2(i)^\times \longrightarrow N_{\mathbb{D}_2^\times}(F_0) = \langle \mathbb{Q}_2(i)^\times, j\rangle \longrightarrow C_2 \longrightarrow 1,$$

where j is mapped non-trivially to the quotient group. Hence x_3 is of the form $x_3 = j^\varepsilon z$ for $\varepsilon \in \{\pm 1\}$ and $z \in \mathbb{Q}_2(i)^\times$. We have

$$x_3^2 = j^\varepsilon z j^\varepsilon z = -(j^\varepsilon z j^{-\varepsilon})z = -N(z)$$

for $N : \mathbb{Q}_2(i)^\times \to \mathbb{Q}_2^\times$ the norm of the extension $\mathbb{Q}_2(i)/\mathbb{Q}_2$. In the proof of corollary 4.26 we have shown that $N(\mathbb{Q}_2(i)^\times) = \langle 2\rangle \times U_2(\mathbb{Z}_2(i)^\times)$. Since

$$N(2+i) = (2+i)(2-i) = 5 \equiv -3 \mod 8,$$

we have $-6 \in N(\mathbb{Q}_2(i)^\times)$. We may therefore choose z such that

$$x_3^2 = \begin{cases} 2u & \text{if } u \equiv 3 \bmod 8, \\ -2u & \text{if } u \equiv -3 \bmod 8. \end{cases}$$

In this case $\widetilde{F_3} = \langle i, x_3\rangle$, and for $\overline{x}_3$ the class of x_3 in $\mathbb{G}_2(u)$ we get

$$F_3 = \begin{cases} \langle i, \overline{x}_3 \mid i^4 = 1,\ \overline{x}_3 i \overline{x}_3^{-1} = i^{-1},\ \overline{x}_3^2 = 1\rangle \cong D_8 & \text{if } u \equiv 3 \bmod 8, \\ \langle i, \overline{x}_3 \mid i^4 = 1,\ \overline{x}_3 i \overline{x}_3^{-1} = i^{-1},\ \overline{x}_3^2 = -1\rangle \cong Q_8 & \text{if } u \equiv -3 \bmod 8, \end{cases}$$

as a maximal finite subgroup of $\mathbb{G}_2(u)$. Since $v(x_3) = \frac{1}{2}$, the conjugacy classes of F_3 and $T_{24} \cap S_2 \cong Q_8$ must be distinct (although they are isomorphic if $u \equiv -3 \bmod 8$). By theorem 4.27, F_3 and T_{24} represent the only two maximal classes containing $\langle i\rangle$ when $u \equiv \pm 3 \bmod 8$. The maximality of F_3^+ and F_3^- in this case is obvious.

It remains to consider the cases where $F_0 = \{\pm 1\} \cong C_2$, that is, F_0 is maximal such that $\mathbb{Q}_2(F_0) = \mathbb{Q}_2$. Then obviously $\widetilde{F_0} = \widetilde{F_1} = \langle 2u, \pm 1\rangle \cong \mathbb{Z} \times C_2$. Because $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2^\times/U_3(\mathbb{Z}_2^\times)$ is represented by the elements of the set $\{\pm 1, \pm 2, \pm 3, \pm 6\}$, we know that there are seven possible quadratic extensions of $\mathbb{Q}_2$ given by

$$L_v := \mathbb{Q}_2/(X^2 - v) \qquad \text{for } v \in \{-1, \pm 2, \pm 3, \pm 6\};$$

each of them is unique up to conjugation. Among these $L_{-1} = \mathbb{Q}_2(\zeta_4)$ and $L_{-3} = \mathbb{Q}_2(\zeta_3)$ have already been considered. Furthermore if $v = 3$, and if $a, b \in \mathbb{Q}_2$, the element

$$(a + bX)^2 = a^2 + 3b^2 + 2abX$$

cannot belong to $\widetilde{F_1} = \langle 2u, \pm 1\rangle$ and the later can never be extended non-trivially to some $\widetilde{F_2}$.

Let us then consider the cases where $v \in \{\pm 2, \pm 6\}$. If $u \equiv \pm\frac{v}{2} \bmod 8$, we let $x_2 := Xt$ with $t \in \mathbb{Z}_2^\times$ such that

$$t^2 = \begin{cases} \frac{2u}{v} & \text{if } u \equiv \frac{v}{2} \bmod 8, \\ -\frac{2u}{v} & \text{if } u \equiv -\frac{v}{2} \bmod 8. \end{cases}$$

Then

$$x_2^2 = vt^2 = \begin{cases} 2u \equiv 1 \bmod \langle 2u\rangle & \text{if } u \equiv \frac{v}{2} \bmod 8, \\ -2u \equiv -1 \bmod \langle 2u\rangle & \text{if } u \equiv -\frac{v}{2} \bmod 8, \end{cases}$$

and for $\overline{x}_2$ the class of x_2 in $\mathbb{G}_2(u)$ we have

$$F_2 = \langle \overline{x}_2, \pm 1\rangle \cong \begin{cases} C_2 \times C_2 & \text{if } u \equiv \frac{v}{2} \bmod 8, \\ C_4 & \text{if } u \equiv -\frac{v}{2} \bmod 8. \end{cases}$$

These classes however are not new: in the case $[\mathbb{Q}_2(F_0) : \mathbb{Q}_2] = 2$ and $\alpha = 1$ treated above, considering the situation where

$$x_2 = \begin{cases} \xi_u & \text{if } u \equiv 1 \text{ or } -3 \bmod 8, \\ \xi_{-u} & \text{if } u \equiv -1 \text{ or } 3 \bmod 8, \end{cases}$$

we see that $C_2 \times C_2 \subseteq F_3^+$ and $C_4 \subseteq F_3^-$. We also know from corollary 2.23 that the group F_2 is unique up to conjugation. On the other hand if $u \not\equiv \pm\frac{v}{2} \bmod 8$, that is if $v \not\equiv \pm 2u \bmod 8$, there is no $x \in L_v$ such that $x^2 \in \langle 2u \rangle \bmod \{\pm 1\}$ and we have $\widetilde{F_2} = \widetilde{F_1} = \widetilde{F_0}$.

Finally, because $Aut(F_0)$ is trivial independently of the value of u, it follows from proposition 2.25 that $F_3 = F_2$. □

Remark 4.31. For $\alpha = 2$, we have shown

$$F_3 \cong \begin{cases} Q_{16} & \text{if } u \equiv 1 \bmod 8, \\ SD_{16} & \text{if } u \equiv -1 \bmod 8, \\ D_8 & \text{if } u \equiv 3 \bmod 8, \\ Q_8 & \text{if } u \equiv -3 \bmod 8. \end{cases}$$

When $u \equiv \pm 3$, the second conjugacy class obtained in theorem 4.27.3 is not maximal as a finite subgroup of $\mathbb{G}_2(u)$. It is contained in T_{24} and is represented by $T_{24} \cap S_2 = Q_8$. It comes from the existence of an element j of valuation zero in $\mathbb{D}_2^\times$ which induces the action of $Gal(\mathbb{Q}_2(i)/\mathbb{Q}_2)$ on $F_0 = \langle i \rangle$ given by $i \mapsto -i$.

Appendix A:

Simple algebras

We provide here the essential background and some classic results on finite dimensional simple algebras. An overview of the subject can be found in [16].

Definition. Let A be an associative ring with unit.

- A is called *simple* if the only two sided ideals of A are A itself and the zero ideal.
- A is a *skew field* if for every non-zero element a of A there is an element $a^{-1} \in A$ satisfying
$$aa^{-1} = 1 = a^{-1}a.$$

Clearly, a commutative skew field is a field, and the set of non-zero elements $A^\times$ of a skew field A forms a group under multiplication. On the other hand, the center $Z(A)$ of a simple ring A is a field, as for any non-zero element a in $Z(A)$ the two sided ideal aA is A by simplicity, and its inverse a^{-1} exists in $Z(A)$. In particular, a simple ring A is an algebra over any subfield K of $Z(A)$.

Definition. A finite dimensional simple algebra A over a field K which is also a skew field is a *division algebra* over K. When $K = Z(A)$, the division algebra A is said to be *central* and is also referred to as an *Azumaya algebra*.

Example A.1. The algebra $M_n(K)$ of all $n \times n$ matrices over a field K is a simple algebra. To see this consider the canonical basis $\{e_{ij}\}$ of $M_n(K)$, where e_{ij} denotes the matrix having zero coefficients everywhere except 1 for the entry on the i-th row and j-th column. We need to show that given a non-zero two-sided ideal I of $M_n(K)$, every e_{ij} belongs to I. Since
$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$
we only have to show that I contains at least on of the e_{ij}. Let
$$a = \sum_{i,j=1}^{n} a_{ij}e_{ij} \ \in I$$
be an element of I with $a_{ij} \in K$ and $a_{kl} \neq 0$ for some $1 \leq k, l \leq n$. Then
$$a_{kl}e_{kl} = e_{kk}ae_{ll} \in I$$
and $e_{kl} \in I$ as desired. It is clear however that when $n \geq 2$, $M_n(K)$ is not a division algebra.

Example A.2. When K is an algebraically closed field, there is no K-division algebra other than K itself, for if A is such an algebra we must have $K(a) = K$ for every element a in A.

Proposition A.3. *If A is a division algebra over a field K, then any K-subalgebra B of A is itself a division algebra.*

Proof. For any non-zero element $x \in B$, we must show that $x^{-1} \in A$ is an element of B. Since B is of finite dimension over K, the elements of the sequence $1, x, x^2, \ldots$ are linearly dependent via a polynomial in B we can assume to be unitary and with a non-zero constant term; in other words

$$x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0 = 0 \quad \text{with } b_i \in B \text{ and } b_0 \neq 0.$$

Hence

$$x(x^{m-1} + b_{n-1}x^{n-2} + \ldots + b_1) = -b_0,$$

and therefore

$$x^{-1} = -b_0^{-1}(x^{m-1} + b_{n-1}x^{n-2} + \ldots + b_1) \ \in B$$

as desired. □

The following classic result reduces the study of finite dimensional simple algebras to the particular case of division algebras. A proof can be found in [12] theorem 2.5 or [18] section 7a.

Theorem A.4 (Wedderburn). *A finite dimensional simple algebra A over a field K is isomorphic as a K-algebra to $M_n(D)$ for D a K-division algebra. The integer n is unique and D is unique up to isomorphism.*

Corollary A.5. *The dimension of a central simple algebra is a square.*

Proof. If A is a central simple algebra of dimension $[A : K]$ over a field K and if $\overline{K}$ denotes the algebraic closure of the latter, we obtain a central simple algebra $A \otimes_K \overline{K}$ of the same dimension

$$[A \otimes_K \overline{K} : \overline{K}] = [A : K].$$

By Wedderburn's theorem $A \otimes_K \overline{K}$ is $\overline{K}$-isomorphic to $M_n(D)$ for D a central division algebra over $\overline{K}$. Because $\overline{K}$ is algebraically closed, we have $D = \overline{K}$ by example A.2. This implies that $A \otimes_K \overline{K}$ has dimension n^2 over $\overline{K}$. □

From the Wedderburn theorem, we know that if A is a central simple algebra of dimension n^2 over K, then $A \cong M_r(D)$ for D an Azumaya algebra over K, and there is an integer m with

$$n^2 = [A : K] = r^2[D : K] = r^2m^2.$$

The skewfield D is called the *skewfield part* of A, the integer $\deg(A) = n$ is the *degree* of A and $\mathrm{ind}(A) = m$ is its *index*.

Another classic result we use in the text is the following. For an algebra A and a subalgebra B of A, we denote by

$$C_A(B) = \{a \in A \mid ab = ba \text{ for any } b \in B\}$$

the centralizer of B in A, and we denote by B^{op} the opposite ring of B. As shown in [12] theorem 8.4, we have:

Theorem A.6 (Centralizer). *Let A be a central simple algebra of finite dimension over a field K, and let B be a simple subalgebra of A. Then*

1) there is a K-algebra homomorphism $C_A(B) \otimes_K M_{[B:K]}(K) \cong A \otimes_K B^{op}$;

2) $C_A(B)$ is a central simple algebra over $Z(B)$;

3) $C_A(C_A(B)) = B$;

4) $C_A(B) \otimes_{Z(B)} B \cong C_A(Z(B))$ via the map

$$C_A(B) \times B \to C_A(Z(B)) \ : \ (x, b) \mapsto xb.$$

In particular if B is central over K, then

$$Z(B) = K, \quad C_A(Z(B)) = A \quad \text{and} \quad [A : K] = [B : K][C_A(B) : K].$$

Corollary A.7. *The degree of a commutative extension L of K contained in a finite dimensional central simple K-algebra A divides $deg(A)$.*

Proof. Because $L \subseteq C_A(L)$, we have

$$[C_A(L) : K] = [C_A(L) : L][L : K],$$

and therefore

$$[A : K] = [L : K][C_A(L) : K] = [L : K]^2[C_A(L) : L].$$

□

Thus the problem of describing subfields of finite dimensional central simple algebras is reduced to the problem of describing their maximal subfields, that is, those subfields of A containing K that are not properly contained in a subfield of A. Because A is assumed to be of finite dimension, maximal subfields always exist in A.

Proposition A.8. *If L is a maximal subfield of a finite dimensional central simple K-algebra A, then $C_A(L) \cong M_n(L)$. In particular, if A is an Azumaya algebra, then*

$$C_A(L) = L \qquad \text{and} \qquad [L : K] = [A : K]^{\frac{1}{2}} = ind(A).$$

Proof. According to the Wedderburn theorem, if the first assertion was not true we would have $C_A(L) \cong M_n(D)$ for a noncommutative division algebra D over L. This division algebra would then contain a subfield properly containing L, and this would contradict the maximality of L in A. Furthermore if A is a skew field, we must have $n = 1$, so that $C_A(L) = L$. By the centralizer theorem,

$$[A : K] = [C_A(L) : K][L : K] = [L : K]^2,$$

as desired. □

We end the section by stating one of the most useful results in the theory of simple algebras. See [18] section 7d or [12] section 8 for proofs.

Theorem A.9 (Skolem-Noether). *Let A be a finite dimensional central simple algebra over a field K and let B be a simple K-subalgebra of A. If $\varphi : B \to A$ is a K-algebra homomorphism, then there exists a unit $a \in A^\times$ satisfying*

$$\varphi(b) = aba^{-1} \qquad \text{for all } b \in B.$$

In particular, every K-isomorphism between subalgebras of A can be extended to an inner automorphism of A.

Appendix B:

Brauer groups of local fields

We collect here the needed results on Brauer groups, cyclic algebras and local class field theory. More details can be found in [18] chapter 7.

B.1. Brauer groups

Let K be a field and let A, B be a central simple K-algebras. We say that A and B are equivalent, denoted $A \sim B$, if their skewfield parts are K-isomorphic, in other words if there is an isomorphism of K-algebras

$$A \otimes_K M_r(K) \cong B \otimes_K M_s(K)$$

for some integers r and s. Let $[A]$ and $[B]$ denote the respective equivalence classes of A and B. Under multiplication defined by

$$[A] \cdot [B] = [A \otimes_K B],$$

the set of classes of central simple K-algebras forms an abelian group denoted $Br(K)$; it is called the *Brauer group* of K. Clearly, its unit is $[K]$.

For an extension L of K, there is a group homomorphism

$$Br(K) \longrightarrow Br(L) \ : \ [A] \longmapsto [L \otimes_K A],$$

whose kernel $Br(L/K) = Br(L, K)$ is the *relative Brauer group* of L over K. Thus $[A] \in Br(L/K)$ if and only if $L \otimes_K A \cong M_r(L)$ for some integer r, in which case we say that L *splits* A, or is a *splitting field* of A. As shown in [18] theorem 28.5 and remark 28.9, we have the following:

Proposition B.1. *For D a central division algebra over K, a field L splits D if and only if it embeds as a maximal subfield of D.*

For $[A] \in Br(K)$, we define the *exponent* $exp[A]$ of $[A]$ to be the order of $[A]$ in $Br(K)$, and we define the *index* $ind[A]$ of $[A]$ to be the index of the skewfield part of A, that is

$$ind[A] = ind(D) = [D : K]^{\frac{1}{2}}$$

for D a division algebra equivalent to A in $Br(K)$. As given in [18] theorem 29.22, we have:

Proposition B.2. *For any $[A]$ in $Br(K)$, $ind[A]$ is a multiple of $exp[A]$.*

B.2. Crossed algebras

Let L be a Galois extension of K with Galois group $G = Gal(L/K)$. We define an algebra

$$A = \sum_{\sigma \in G} L u_\sigma$$

having as L-basis a set of symbols $\{u_\sigma \mid \sigma \in G\}$ satisfying

$$\sigma(x)u_\sigma = u_\sigma x, \qquad u_\sigma u_\tau = f_{\sigma,\tau} u_{\sigma\tau}, \qquad \text{and} \qquad \rho(f_{\sigma,\tau}) f_{\rho,\sigma\tau} = f_{\rho,\sigma} f_{\rho\sigma,\tau}$$

for $x \in L$, $\rho, \sigma, \tau \in G$ and $f_{\sigma,\tau} \in L^\times$. A map $f : G \times G \to L^\times$ satisfying this third condition is a *factor set* from G to $L^\times$. Given such an f, the algebra A thus constructed is a *crossed(-product) algebra* and is denoted $(L/K, f)$.

According to [18] theorem 29.6, for each f, $(L/K, f)$ is a finite dimensional central simple algebra over K having L as maximal subfield.

Proposition B.3. *If $A = (L/K, f)$ and $exp[A] = [L : K]$, then A is a division algebra.*

Proof. Let $n = [L : K]$, so that $[A : K] = n^2$, and let D be the skewfield part of A with $A \cong M_r(D)$ and $m = ind[D]$. Then $n = mr$, and $exp[A]$ divides m by proposition B.2. Because $exp[A] = n$, we have $m = n$ and $r = 1$, in which case A is a division algebra. □

We also know from [18] theorem 29.6 that the set of factor sets from G to $L^\times$ can be partitioned under an equivalence relation to form a multiplicative group of classes $[f]$, isomorphic to the second cohomology group $H^2(G, L^\times)$, in such a way that two crossed algebras $(L/K, e)$, $(L/K, f)$ are K-isomorphic if and only if $[e] = [f]$. Then by [18] theorem 29.12 we have the following:

Theorem B.4. *Let L be a finite Galois extension of a field K with Galois group G. Then*

$$H^2(G, L^\times) \cong Br(L/K)$$

given by mapping $[f] \in H^2(G, L^\times)$ onto the class $[(L/K, f)] \in Br(L/K)$.

Remark B.5. As noted in remark (i) following theorem 29.13 of [18], if $K \subseteq K' \subseteq L$ are finite Galois extensions with Galois groups $G = Gal(K/L)$ and $G' = Gal(K'/L)$, then there is a commutative diagram

$$\begin{array}{ccc} H^2(G, L^\times) & \xrightarrow{\cong} & Br(L/K) \\ {\scriptstyle \text{res}}\downarrow & & \downarrow{\scriptstyle -\otimes_K K'} \\ H^2(G', L^\times) & \xrightarrow{\cong} & Br(L/K') \end{array}$$

where the left hand vertical map is the restriction homomorphism induced by the inclusion $G \subseteq G'$.

B.3. Cyclic algebras

Let L be a finite Galois extension of a field K with cyclic Galois group $G = Gal(L/K)$ of order n generated by σ; such an extension is called *cyclic*. Let a be an element of $K^\times$ and form the associative K-algebra

$$A = (L/K, \sigma, a) = \sum_{i=0}^{n-1} L u^i,$$

for an element u satisfying $ux = \sigma(x)u$ and $u^n = a$ for all $x \in L$, where u^0 is identified with the unit of A. Such a K-algebra is called *cyclic*.

As explained in [18] section 30, A is isomorphic to the crossed algebra $(L/K, f)$ where the factor set f from G to $L^\times$ is given by

$$f_{\sigma^i,\sigma^j} = \begin{cases} 1 & \text{if } i+j < n, \\ a & \text{if } i+j \geq n, \end{cases}$$

for $0 \leq i, j \leq n-1$. In particular, A is a central simple K-algebra split by L. Conversely, [18] theorem 30.3 establishes that if L/K is a cyclic extension with Galois group G of order n generated by σ, and if f is a factor set from G to $L^\times$, then the crossed algebra $(L/K, f)$ is isomorphic to the cyclic algebra $(L/K, \sigma, a)$ for

$$a = \prod_{i=0}^{n-1} f_{\sigma^i,\sigma} \ \in K^\times.$$

According to [18] theorem 30.4, we have:

Proposition B.6. *Let L/K be a cyclic extension with Galois group of order n generated by σ, and let $a, b \in K^\times$. Then*

1) $(L/K, \sigma, a) \cong (L/K, \sigma^s, a^s)$ *for any integer s prime to n;*

2) $(L/K, \sigma, 1) \cong M_n(K)$*;*

3) $(L/K, \sigma, a) \cong (L/K, \sigma, b)$ *if and only if $\frac{a}{b}$ belongs to the norm $N_{L/K}(L^\times)$. In particular, $(L/K, \sigma, a) \cong K$ if and only if $a \in N_{L/K}(L^\times)$;*

4) $(L/K, \sigma, a) \otimes_K (L/K, \sigma, b) \cong (L/K, \sigma, ab)$.

Corollary B.7. *Let $A = (L/K, \sigma, a)$ be a cyclic algebra. Then $exp[A]$ is the smallest positive integer s such that $a^s \in N_{L/K}(L^\times)$.*

Proof. Since $[A]^s = [(L/K, \sigma, a^s)]$, we have $[A]^s = 1$ if and only if $a^s \in N_{L/K}(L^\times)$. □

We know from class field theory and theorem B.4 that the map

$$K^\times \longrightarrow Br(L/K) \ : \ a \longmapsto [(L/K, \sigma, a)]$$

is an epimorphism of group which induces an isomorphism:

Theorem B.8. *If L/K is a cyclic extension with Galois group G, then*

$$H^2(G, L^\times) \cong Br(L/K) \cong K^\times / N_{L/K}(L^\times).$$

□

B.4. The local case

Suppose that K is a local field with residue field of cardinality q and a uniformizing element π_K. Let n be a positive integer, K_n an unramified extension of degree n over K, and let $\sigma \in Gal(K_n/K) \cong \mathbb{Z}/n$ be the Frobenius of this extension. For a positive integer r, we consider the cyclic algebra $A = (K_n/K, \sigma, \pi_K^r)$ and we define the *Hasse invariant* of A to be

$$inv_K(K_n/K, \sigma, \pi_K^r) = \frac{r}{n}.$$

By [18] theorem 31.1 and 31.5, we know that the isomorphism class of A only depends on r modulo n, and that the skewfield part of A has the same invariant as A. Consequently, the invariant of A only depends on the class $[A]$ in $Br(K)$ and there is a well defined map

$$inv_K : \ Br(K) \longrightarrow \mathbb{Q}/\mathbb{Z};$$

it is in fact an isomorphism by [18] theorem 31.8:

Theorem B.9. *If K is a local field, then $Br(K) \cong \mathbb{Q}/\mathbb{Z}$ via inv_K.*

By [18] theorem 31.9, we have:

Theorem B.10. *Let L be a finite extension of degree m over a local field K. There is a commutative diagram*

$$\begin{array}{ccc} Br(K) & \xrightarrow[\cong]{inv_K} & \mathbb{Q}/\mathbb{Z} \\ {\scriptstyle L\otimes_K -}\downarrow & & \downarrow{\scriptstyle \cdot m} \\ Br(L) & \xrightarrow[\cong]{inv_L} & \mathbb{Q}/\mathbb{Z} \end{array}$$

where the right hand vertical map is multiplication by m.

Corollary B.11. *Let L be a finite Galois extension of degree m over a local field K with Galois group G. Then*

$$H^2(G, L^\times) \cong Br(L/K) \cong \mathbb{Z}/m.$$

Proof. By theorem B.10 and the definition of $Br(L/K)$, there is a commutative diagram

$$\begin{array}{ccccc} 1 \longrightarrow & Br(L/K) & \longrightarrow & Br(K) & \longrightarrow Br(L) \\ & & & \downarrow\cong & \downarrow\cong \\ & & & \mathbb{Q}/\mathbb{Z} & \xrightarrow[m]{} \mathbb{Q}/\mathbb{Z} \end{array}$$

where the top row is exact, the bottom map is multiplication by m, and the vertical maps are isomorphisms. Hence $Br(L/K)$ is isomorphic to the kernel of the bottom map. □

Corollary B.12. *If $K \subseteq K' \subseteq L$ are finite Galois extensions of local fields with Galois groups $G = Gal(L/K)$ and $G' = Gal(L/K')$, then the restriction map*

$$H^2(G, L^\times) \longrightarrow H^2(G', L^\times)$$

induced by the inclusion $G' \subseteq G$ is surjective.

Proof. The diagram

$$\begin{array}{ccccccccc} H^2(G,L^\times) & \xrightarrow{\cong} & Br(L/K) & \xrightarrow{\text{inc}} & Br(K) & \xrightarrow{-\otimes_K L} & Br(L) & \xrightarrow{\cong} & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle \text{res}} & & \downarrow & & \downarrow{\scriptstyle -\otimes_K K'} & & \| & & \| \\ H^2(G',L^\times) & \xrightarrow{\cong} & Br(L/K') & \xrightarrow{\text{inc}} & Br(K') & \xrightarrow{-\otimes'_K L} & Br(L) & \xrightarrow{\cong} & \mathbb{Q}/\mathbb{Z} \end{array}$$

given by theorem B.4 and B.10 is commutative by remark B.5. By corollary B.11, the relative Brauer groups $Br(L/K)$ and $Br(L/K')$ are cyclic of order $|G|$ and $|G'|$ respectively, and the second square in the above diagram may be identified with the commutative square

$$\begin{array}{ccc} \mathbb{Z}/|G| & \xrightarrow{\text{inc}} & \mathbb{Q}/\mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \cdot\frac{|G|}{|G'|}} \\ \mathbb{Z}/|G'| & \xrightarrow{\text{inc}} & \mathbb{Q}/\mathbb{Z}, \end{array}$$

where the right hand vertical map is multiplication by $\frac{|G|}{|G'|}$ according to theorem B.10. In particular this latter map is surjective and sends $\frac{1}{|G|}$ to $\frac{1}{|G'|}$. Hence the generator of $\mathbb{Z}/|G|$ associated to $\frac{1}{|G|}$ must be sent to a generator of $\mathbb{Z}/|G'|$. The second vertical map in the first diagram given above is therefore surjective and the result follows. □

Proposition B.13. *Let L/K be a finite Galois extension of local fields of characteristic zero with cyclic Galois group G.*

1) If L/K is unramified, the valuation map induces an isomorphism

$$H^2(G,L^\times) \cong H^2(G,\frac{1}{e(L)}\mathbb{Z}) \cong \langle \pi_K \rangle / \langle \pi_K^{|G|} \rangle,$$

for $e(L)$ the ramification index of $L/\mathbb{Q}_p$ and π_K a uniformizing element of K.

2) If L/K is totally ramified, the valuation map induces an isomorphism

$$H^2(G,L^\times) \cong H^2(G,\mathcal{O}_L^\times),$$

for $\mathcal{O}_L$ the ring of integers of L.

Proof. The valuation map $v = v_{\mathbb{Q}_p} : L^\times \to \frac{1}{e(L)}\mathbb{Z}$ is surjective and induces a short exact sequence

$$1 \longrightarrow \mathcal{O}_L^\times \longrightarrow L^\times \longrightarrow \frac{1}{e(L)}\mathbb{Z} \longrightarrow 1,$$

which in turns induces a long exact sequence

$$H^1(G,\frac{1}{e(L)}\mathbb{Z}) \to H^2(G,\mathcal{O}_L^\times) \to H^2(G,L^\times) \to H^2(G,\frac{1}{e(L)}\mathbb{Z}) \to H^3(G,\mathcal{O}_L^\times).$$

If L/K is unramified, [20] proposition 1 says that $H^i(G,\mathcal{O}_L^\times)$ is trivial for all $i \in \mathbb{Z}$ and hence yields the result.

If L/K is totally ramified, there are unifomizing elements π_K of K and π_L of L such that

$$\pi_K = (\pi_L)^{|G|}.$$

Therefore

$$v(\pi_K) = |G|v(\pi_L) = |G| \cdot \frac{1}{e(L)}\mathbb{Z},$$

and consequently the map $H^2(G, L^\times) \to H^2(G, \frac{1}{e(L)}\mathbb{Z})$ is trivial. Moreover, since G is finite and $\frac{1}{e(L)}\mathbb{Z}$ is infinite, we have $H^1(G, \frac{1}{e(L)}\mathbb{Z}) = 0$ and the result follows. □

Example B.14. For any prime p and $\alpha \geq 1$, we have

$$p \in N_{\mathbb{Q}_p(\zeta_{p^\alpha})/\mathbb{Q}_p}(\mathbb{Q}_p(\zeta_{p^\alpha})^\times).$$

Indeed, for $1 \leq r \leq \alpha - 1$ let σ be a generator of $Gal(\mathbb{Q}_p(\zeta_{p^{r+1}})/\mathbb{Q}_p(\zeta_{p^r}))$ satisfying

$$\sigma(\zeta_{p^{r+1}}) = \zeta_{p^{r+1}}\zeta_p,$$

and define

$$\Sigma_i(X_1, \ldots, X_p)$$

to be the homogeneous symmetric polynomial of degree i in p variables $X_1, \ldots, X_p$, so that

$$\prod_{i=1}^{p}(X - X_i) \;=\; \sum_{i=1}^{p}(-1)^i \Sigma_i(X_1, \ldots, X_n)X^{n-i}.$$

Then for $1 \leq k \leq p-1$ we have

$$\begin{aligned}
N_{\mathbb{Q}_p(\zeta_{p^{r+1}})/\mathbb{Q}_p(\zeta_{p^r})}(1 - \zeta_{p^{r+1}}^k) \;&=\; \prod_{j=0}^{p-1}(1 - \sigma^j(\zeta_{p^{r+1}}^k)) \\
&=\; \sum_{i=0}^{p}(-1)^i \;\Sigma_i(\zeta_{p^{r+1}}^k,\; \sigma(\zeta_{p^{r+1}}^k),\; \ldots,\; \sigma^{p-1}(\zeta_{p^{r+1}}^k)) \\
&=\; \sum_{i=0}^{p}(-1)^i \;\Sigma_i(\zeta_{p^{r+1}}^k,\; \sigma(\zeta_{p^{r+1}}\zeta_p)^k,\; \ldots,\; \sigma(\zeta_{p^{r+1}}\zeta_p^{p-1})^k) \\
&=\; \sum_{i=0}^{p}(-1)^i \;\zeta_{p^{r+1}}^{ik}\;\Sigma_i(1,\; \sigma(\zeta_p^k),\; \ldots,\; \sigma(\zeta_p^{(p-1)k})) \\
&=\; 1 - \zeta_{p^r}^k,
\end{aligned}$$

where the last equality is a consequence of the fact that

$$\Sigma_i(1,\; \sigma(\zeta_p^k),\; \ldots,\; \sigma(\zeta_p^{(p-1)k})) \;=\; \begin{cases} 1 & \text{if } i = 0, p, \\ 0 & \text{if } i \neq 0, p. \end{cases}$$

As shown in corollary 3.2

$$p = \prod_{k=1}^{p-1}(\zeta_p^k - 1) = N_{\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p}(\zeta_p - 1).$$

Consequently

$$p \in N_{\mathbb{Q}_p(\zeta_{p^\alpha})/\mathbb{Q}_p}(\mathbb{Q}_p(\zeta_{p^\alpha})^\times) \qquad \text{and} \qquad p \in N_{\mathbb{Q}_p(\zeta_{p^\alpha})/\mathbb{Q}_p(\zeta_p)}(\mathbb{Q}_p(\zeta_{p^\alpha})^\times).$$

Moreover if $p = 2$, we have

$$2, (1 \pm \zeta_4) \in N_{\mathbb{Q}_2(\zeta_{2^\alpha})/\mathbb{Q}_2(\zeta_4)}(\mathbb{Q}_2(\zeta_{2^\alpha})^\times) \qquad \text{for} \quad \alpha \geq 2.$$

For a local field K of characteristic zero with uniformizing element π_K and ring of integers $\mathcal{O}_K$, we let

$$\begin{aligned} U_i(\mathcal{O}_K^\times) &= \{x \in \mathcal{O}_K^\times \mid v_K(x-1) \geq i\} \\ &= \{x \in \mathcal{O}_K^\times \mid x \equiv 1 \bmod \pi_K^i\}, \quad i \geq 0, \end{aligned}$$

be the i-th group in the filtration

$$\mathcal{O}_K^\times = U_0(\mathcal{O}_K^\times) \supseteq U_1(\mathcal{O}_K^\times) \supseteq U_2(\mathcal{O}_K^\times) \supseteq \dots$$

Proposition B.15. *Let L/K be a finite Galois extension of local fields of characteristic zero with Galois group G. If L/K is unramified, then the trace*

$$Tr_G = Tr_{L/K} : l \longrightarrow k$$

is surjective on the residue fields, and the norm

$$N_G = N_{L/K} : \mathcal{O}_L^\times \longrightarrow \mathcal{O}_K^\times$$

is surjective on the groups of units of the rings of integers.

Proof. Since $G = Gal(l/k)$ is cyclic, Hilbert's theorem 90 yields $H^1(G, l) = 0$. Let t denote a generator of G, and let $Tr := Tr_G$. In the periodic complex

$$l \xrightarrow{1-t} l \xrightarrow{Tr} l \xrightarrow{1-t} l \xrightarrow{Tr} \dots$$

we have $Ker(Tr) = Im(1-t)$. Hence

$$|Ker(1-t)| = \frac{|l|}{|Im(1-t)|} = \frac{|l|}{|Ker(Tr)|} = |Im(Tr)|,$$

and $H^2(G, l) = 0$. Because $Ker(1-t) = k$, it follows that $Im(Tr_G) = k$.

In order to show the second assertion, we first note that for any $i \geq 1$ the norm N_G becomes the trace

$$Tr : U_i(\mathcal{O}_L^\times)/U_{i+1}(\mathcal{O}_L^\times) \longrightarrow U_i(\mathcal{O}_K^\times)/U_{i+1}(\mathcal{O}_K^\times)$$

on the successive quotients of the filtration of the units of the rings of integers; these maps are surjective by the first assertion. For each $i \geq 1$, consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_i(\mathcal{O}_L^\times)/U_{i+1}(\mathcal{O}_L^\times) & \longrightarrow & U_1(\mathcal{O}_L^\times)/U_{i+1}(\mathcal{O}_L^\times) & \longrightarrow & U_1(\mathcal{O}_L^\times)/U_i(\mathcal{O}_L^\times) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle Tr} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U_i(\mathcal{O}_K^\times)/U_{i+1}(\mathcal{O}_K^\times) & \longrightarrow & U_1(\mathcal{O}_K^\times)/U_{i+1}(\mathcal{O}_K^\times) & \longrightarrow & U_1(\mathcal{O}_K^\times)/U_i(\mathcal{O}_K^\times) & \longrightarrow & 1, \end{array}$$

where the horizontal lines are exact and the vertical maps are induced by the norm. If $i = 1$ the vertical maps are obviously surjective. Moreover if $i \geq 2$ and if the vertical map on the right hand side is surjective, then the middle one is also surjective by the five lemma. We conclude by induction on i that N_G is surjective on $U_1(\mathcal{O}_K^\times)$, and consequently on $\mathcal{O}_K^\times$. □

Remark B.16. According to classical Galois theory (see for example [14] chapter VI theorem 1.12), if K_1 and K_2 are extensions of $\mathbb{Q}_p$ such that $K_1K_2 = L$, $K_1 \cap K_2 = K$ and L/K is Galois with abelian Galois group, then any $x \in K$ such that $x \in N_{K_1/K}(K_1^\times)$ satisfies $x \in N_{L/K_2}(K^\times)$.

Appendix C:

Division algebras over local fields

We provide here a short account on division algebras over local fields. The reader may refer to [18] chapter 3 for more details.

Let K be a local field with residue field of cardinality q, let π_K be a uniformizing element of K, and let D be a central division algebra of dimension n^2 over K. As shown in [18] theorem 12.10, the normalized valuation $v_K : \pi_K \mapsto 1$ on K extends in a unique way to a valuation $v = v_D$ on D. By [18] section 13, we know that the skew field D is complete with respect to v and that the maximal order $\mathcal{O}_D$ of D is of degree n^2 over the ring of integers $\mathcal{O}_K$ of K. Let d and k denote the residue fields of D and K respectively. By [18] theorem 13.3 we have

$$n^2 = ef,$$

where

- $e = e(D/K) = |v(D^\times)/v(K^\times)|$ denotes the *ramification index* of D over K;
- $f = f(D/K) = [d : k]$ denotes the *inertial degree* of D over K.

Proposition C.1. *If D is a central division algebra of dimension n^2 over a local field K, then*

$$e(D/K) = f(D/K) = n.$$

Proof. Because there exists an element $x \in D$ such that $v(x) = e(D/K)^{-1}$ and as x belongs to a commutative subfield of degree at most n over K, it follows that $e(D/K) \leq n$. On the other hand k is a finite field and $d = k(\overline{y})$ is a commutative field, for $\overline{y}$ the image in d of some suitable $y \in D$. Hence $f(D/K) \leq n$ and the result follows. □

Since $[d : k] = n$, we can find an $x \in \mathcal{O}_D$ such that $k(\overline{x}) = d$. Let $K_n = K(x)$. Because K_n is commutative, $[K_n : K] \leq n$. On the other hand, $\overline{x}$ is an element of the residue field k_n of K_n, while $k_n = d$, so that $[k_n : k] = n$. It follows that K_n is a maximal unramified extension of degree n over K in D. Such a K_n is referred to as an *inertia field* of D. Of course the above construction of K_n is not unique, but the Skolem-Noether theorem implies that all inertia fields are conjugate.

Let $\omega \in D^\times$ be a root of unity satisfying

$$K(\omega) = K_n;$$

in particular ω is of order $q^n - 1$. According to [18] theorem 14.5, there exists a uniformizing element π of D satisfying

$$\pi^n = \pi_K \qquad \text{and} \qquad \pi\omega\pi^{-1} = \omega^{q^s},$$

where $s < n$ is a positive integer prime to n, uniquely determined by D, which does not depend upon the choice of ω or π. Let $r \in \mathbb{Z}$ be such that $rs \equiv 1 \bmod n$; in particular r is

prime to n. Using [18] theorem 31.1 and proposition B.6, we know that D is isomorphic to the cyclic algebra

$$D \cong (K_n/K, \sigma^s, \pi_K) \cong (K_n/K, \sigma, \pi_K^r),$$

and is classified up to isomorphism by its invariant

$$inv_K(D) = \frac{r}{n} \in \mathbb{Q}/\mathbb{Z}.$$

In other words we have:

Theorem C.2. *All Azumaya algebras over a local field K are classified up to isomorphism, via inv_K, by the elements of the additive group $\mathbb{Q}/\mathbb{Z}$.*

Notation C.3. For a class in $\mathbb{Q}/\mathbb{Z}$ represented by an element $r/n \in \mathbb{Q}$ with $(r;n) = 1$ and $1 \leq r < n$, the corresponding Azumaya algebra is denoted $D(K, r/n)$. When $K = \mathbb{Q}_p$, $r = 1$ and p is understood, we write $\mathbb{D}_n = D(\mathbb{Q}_p, 1/n)$.

Corollary C.4. *If D is a central division algebra over a local field, then $exp[D] = ind[D]$.*

Proof. Suppose $inv_K(D) = \frac{r}{n}$, where K denotes the center of D and $[D : K] = n^2$. By definition $ind[D] = n$. We know from proposition B.2 that $exp[A]$ must divide n. Because r is prime to n, it follows that $exp[A] = n$. □

Remark C.5. Suppose $inv_K(D) = \frac{r}{n}$. By the Skolem-Noether theorem, the Frobenius automorphism σ of $K(\omega) = K_n$ is given by

$$\sigma(x) = \xi x \xi^{-1}$$

for a suitable element $\xi \in D^\times$ determined up to multiplication by an element of $K(\omega)^\times$. Then clearly the image of $v(\xi)$ in

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$$

is none other than the invariant of D. Furthermore, as σ^n is the identity on the inertia field $K(\omega)$, we know that ξ^n commutes with all elements of $K(\omega)$ and hence belongs to $K(\omega)$. Because

$$v(\xi) = \frac{1}{n} v(\xi^n),$$

we have $v(\xi) = r/n$. Hence $\xi^n = \pi_K^r u$ for a unit $u \in K(\omega)^\times$. In this case,

$$D \cong D(K, r/n) \cong K(\omega)\langle\xi\rangle/(\xi^n = \pi_K^r, \xi x = x^\sigma \xi)$$

as mentioned in the paragraph following the proof of [18] theorem 14.5.

So far, we have dealt with unramified extensions of the base field K, but there are in D many more commutative subfields. It can in fact be shown that all extensions of K of degree dividing n exist; see [18] theorem 31.11, [7] 23.1.4 and 23.1.7, or [20] section 1 for proofs.

Theorem C.6 (Embedding). *If D is a central division algebra of dimension n^2 over a local field K, then the degree of a commutative extension L of K in D divides n, and any extension L of K whose degree divides n embeds as a commutative subfield of D.*

In particular, a local field L of characteristic zero embeds in some $\mathbb{D}_n$, in which case its group of units $L^\times$ is a subgroup of $\mathbb{D}_n^\times$. The structure of $L^\times$, both algebraically and topologically, is well known and is recorded below; see for example [15] chapter II proposition 5.3 and 5.7.

Proposition C.7. *Let L be a local field of characteristic zero with residue field $l \cong \mathbb{F}_{p^f}$, roots of unity $\mu(L)$ and uniformizing element π_L. Then*

$$\begin{aligned} L^\times &= \langle \pi_L \rangle \times \mathcal{O}_L^\times \\ &= \langle \pi_L \rangle \times l^\times \times U_1(\mathcal{O}_L^\times) \\ &\cong \mathbb{Z} \times \mu(L) \times \mathbb{Z}_p^{[L:\mathbb{Q}_p]}. \end{aligned}$$

The most frequently encountered fields are the cyclotomic extensions of $\mathbb{Q}_p$. Recall the following result from [15] chapter II proposition 7.12 and 7.13.

Proposition C.8. *Let ζ be a primitive k-th root of unity for $k = \beta p^\alpha \geq 1$ with $(\beta; p) = 1$, and let f be the smallest positive integer such that $p^f \equiv 1 \bmod \beta$. Then $\mathbb{Q}_p(\zeta)/\mathbb{Q}_p$ is a Galois extension with ramification index $\varphi(p^\alpha)$ and residue degree f, where*

$$\mu(\mathbb{Q}_p(\zeta)) \cong \begin{cases} \mathbb{Z}/p^\alpha(p^f-1) & \text{if } p > 2 \text{ or } \alpha \geq 1, \\ \mathbb{Z}/2(2^f-1) & \text{if } p = 2 \text{ and } \alpha = 0, \end{cases}$$

$$Gal(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p) \cong \begin{cases} (\mathbb{Z}/p^\alpha)^\times \times \mathbb{Z}/f & \text{if } \alpha \geq 1, \\ \mathbb{Z}/f & \text{if } \alpha = 0. \end{cases}$$

Corollary C.9. *We have*

$$\mathbb{Q}_p(\zeta)^\times \cong \mathbb{Z} \times \mathbb{Z}_p[\zeta]^\times \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}/p^\alpha(p^f-1) \times \mathbb{Z}_p^{\varphi(p^\alpha)f} & \text{if } p > 2 \text{ or } \alpha \geq 1, \\ \mathbb{Z} \times \mathbb{Z}/2(2^f-1) \times \mathbb{Z}_2^f & \text{if } p = 2 \text{ and } \alpha = 0. \end{cases}$$

Proof. This follows from proposition C.7 and C.8. □

We end the section by analysing the invariant of some embeddings that are useful in the text.

Proposition C.10. *Let D be a central division algebra of invariant $\frac{r}{n}$ over a local field K for r prime to n, let $L \subseteq D$ be a commutative extension of K, and let m be such that $n = m[L:K]$. Then $C_D(L)$ is a central division algebra of invariant $\frac{r}{m}$ over L.*

Proof. Using the centraliser theorem A.6, we know that $C_D(L)$ is a central division algebra of dimension m^2 over L, and we have

$$\begin{aligned} D \otimes_K L &\cong C_D(L) \otimes_K M_{n/m}(K) \\ &\cong C_D(L) \otimes_L L \otimes_K M_{n/m}(K) \\ &\cong C_D(L) \otimes_L M_{n/m}(L). \end{aligned}$$

Hence the invariant of $C_D(L)$ is that of $D \otimes_K L$, which is $\frac{r}{n}[L:K]$ by theorem B.10. □

Proposition C.11. *For any prime p, $\mathbb{D}_m$ embeds as a $\mathbb{Q}_p$-subalgebra of $\mathbb{D}_n$ if and only if $n = km$ with $k \equiv 1 \bmod m$.*

Proof. If $D(\mathbb{Q}_p, 1/m)$ embeds as a $\mathbb{Q}_p$-subalgebra of $D(\mathbb{Q}_p, 1/n)$, then the centralizer theorem provides an isomorphism

$$D(\mathbb{Q}_p, 1/n) \cong D(\mathbb{Q}_p, 1/m) \otimes_{\mathbb{Q}_p} C_{D_n}(D(\mathbb{Q}_p, 1/m)),$$

so that there is an integer k satisfying $n = km$. Because $C_{D_n}(D(\mathbb{Q}_p, 1/m))$ is a central division algebra over $\mathbb{Q}_p$, we also know the existence of an integer l such that

$$C_{D_n}(D(\mathbb{Q}_p, 1/m)) \cong D(\mathbb{Q}_p, l/k).$$

The law on the Brauer group $\mathbb{Q}/\mathbb{Z}$ being defined as such a tensor product over the $\mathbb{Q}_p$-Azumaya algebra classes (see appendix B), it follows that

$$\frac{1}{n} \equiv \frac{1}{m} + \frac{l}{k} \mod \mathbb{Z}. \qquad (*)$$

Consequently $1 \equiv k + lm \bmod n$, and $k \equiv 1 \bmod m$.

Conversely, if $n = km$ with $k \equiv 1 \bmod m$, there is an integer l prime to k such that $1 \equiv k+lm \bmod n$. It follows that $(*)$ is verified and $D(\mathbb{Q}_p, 1/m)$ embeds as a $\mathbb{Q}_p$-subalgebra of $D(\mathbb{Q}, 1/n)$. □

Corollary C.12. *When $p = 2$, $\mathbb{D}_2$ embeds in $\mathbb{D}_n$ if and only if $n \equiv 2 \bmod 4$.* □

Appendix D:
Endomorphisms of formal group laws

We give here a short account on endomorphisms of formal group laws of finite height n defined over a field of characteristic $p > 0$. We summarize how these occurs as elements of the central division algebra $\mathbb{D}_n = D(\mathbb{Q}_p, 1/n)$ of invariant $\frac{1}{n}$ over $\mathbb{Q}_p$. The reader may refer to [7] or [5] for more details.

Definition. Let R be a commutative ring with unit. A *formal group law* over R is a power series $F = F(X,Y) = X +_F Y \in R[[X,Y]]$ satisfying

- $F(X,0) = F(0,X) = X$,
- $F(X,Y) = F(Y,X)$, and
- $F(X,F(Y,Z)) = F(F(X,Y),Z)$ in $R[[X,Y,Z]]$.

We denote by $FGL(R)$ the set of formal group laws defined over R. For $F, G \in FGL(R)$, a *homomorphism* from F to G is a power series $f = f(X) \in R[[X]]$ without constant term such that $f(F(X,Y)) = G(f(X), f(Y))$. It is an *isomorphism* if it is invertible, that is, if the coefficient of X is a unit in R.

The set $Hom_R(F,G)$ of homomorphisms from F to G forms an abelian group under formal addition

$$G(f(X), g(X)) = f(X) +_G g(X).$$

When $F = G$, the group $End_R(F) = Hom_R(F,F)$ becomes a ring via the composition of series. Its group of units is written $End_R(F)^\times = Aut_R(F)$. For an integer $n \in \mathbb{Z}$, we define the *n-series* $[n]_F$ to be the image of n in $End_R(F)$ via the canonical ring homomorphism $\mathbb{Z} \to End_R(F)$, in other words

$$[n]_F(X) = \underbrace{X +_F \ldots +_F X}_{\text{n times}}.$$

As shown in [5] chapter I §3, when $R = k$ is a field of characteristic $p > 0$, any homomorphism $f \in Hom_k(F,G)$ can be written as a series

$$f(X) = \sum_{i \geq 1} a_i X^{ip^n}$$

for some integer $n = ht(f) \in \mathbb{N}^* \cup \{\infty\}$ defined as the *height* of f, where by convention $ht(f) = \infty$ if $f = 0$. For $F \in FGL(k)$ we then define $ht(F)$ to be the height of $[p]_F$. As shown in [5] chapter III §2, this induces a valuation ht on $End_k(F)$ which turns $End_k(F)$ into a complete local ring. In particular, the definition of $[n]_F$ extends to the p-adic integers $\mathbb{Z}_p$, and $ht(f) = 0$ if and only if f is invertible.

Let us fix a separably closed field K of characteristic $p > 0$. As shown in [5] chapter III §2, we have the following three results: the first two provide a classification of the K-isomorphism classes of formal group laws defined over K and the third one describes the endomorphism ring as a subring of the central division algebra of Hasse invariant $1/ht(F)$ over $\mathbb{Q}_p$.

Theorem D.1 (existence). *For a positive integer n, there exists a formal group law $F_n \in FGL(\mathbb{F}_p)$ such that $[p]_{F_n}(X) = X^{p^n}$; it is the Honda formal group law of height n.*

Theorem D.2 (Lazard). *Two formal group laws $F, G \in FGL(K)$ are K-isomorphic if and only if $ht(F) = ht(G)$.*

Theorem D.3 (Dieudonné - Lubin). *For a formal group law $F \in FGL(K)$ of finite height n, the ring $End_K(F)$ is isomorphic to the maximal order $\mathcal{O}_n$ of the central division algebra $\mathbb{D}_n = D(\mathbb{Q}_p, 1/n)$ of invariant $\frac{1}{n}$ over $\mathbb{Q}_p$.*

We now describe the image in $\mathbb{D}_n$ of the ring of endomorphisms defined over a finite subfield of K. For this we identify $\mathcal{O}_n$ with $End_K(F_n)$ and fix two integers $n, r \geq 1$. Let v denote the unique extension to $\mathbb{D}_n^\times$ of the p-adic valuation $p \mapsto 1$ on $\mathbb{Q}_p^\times$. Let $\mathcal{C}_r$ be the set of conjugacy classes of elements of valuation $\frac{r}{n}$ in $\mathcal{O}_n$, and let $\mathcal{I}(\mathbb{F}_{p^r}, n)$ denote the set of $\mathbb{F}_{p^r}$-isomorphism classes of formal group laws of height n. Define the map

$$\Phi : \mathcal{I}(\mathbb{F}_{p^r}, n) \longrightarrow \mathcal{C}_r$$

by assigning to a formal group law $F \in FGL(\mathbb{F}_{p^r})$ of height n and a K-isomorphism $f : F_n \to F$, the conjugacy class of $\xi_F^r \in \mathcal{O}_n^\times$ the element associated to the endomorphism $f^{-1}X^{p^r}f$. Then Φ is a bijection (see [7] 24.4.2, or [5] chapter III §3 theorem 2).

Theorem D.4. *The map*

$$End_{\mathbb{F}_{p^r}}(F) \longrightarrow C_{\mathcal{O}_n}(\xi_F^r) \ : \ x \longmapsto f^{-1}xf$$

is a ring isomorphism from $End_{\mathbb{F}_{p^r}}(F)$ to the subring of all elements of $\mathcal{O}_n$ commuting with ξ_F^r.

Proof. In $End_K(F) \cong \mathcal{O}_n$, the ring $End_{\mathbb{F}_{p^r}}(F)$ is characterized by $\xi_F^r x = x\xi_F^r$, as a series $g(X) \in K[[X]]$ satisfies $g(X)^{p^r} = g(X^{p^r})$ if and only if its coefficients are in $\mathbb{F}_{p^r}$. □

In other words if $m = [\mathbb{Q}_p(\xi_F^r) : \mathbb{Q}_p]$, then m divides n and $End_{\mathbb{F}_{p^r}}(F)$ is isomorphic to the maximal order of the division algebra

$$D(\mathbb{Q}_p(\xi_F^r), m/n) \cong C_{\mathbb{D}_n}(\xi_F^r) \subseteq D_n.$$

In particular $End_{\mathbb{F}_{p^r}}(F)$ is the ring of integers of the $\mathbb{Q}_p$-algebra $End_{\mathbb{F}_{p^r}}(F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Corollary D.5. *There exists a formal group law F defined over $\mathbb{F}_{p^r}$ and of height n such that*

$$End_{\mathbb{F}_{p^r}}(F) \cong End_K(F) \cong \mathcal{O}_n$$

if and only if r is a multiple of n.

Proof. This follows from the fact that the valuation group of the center $\mathbb{Q}_p$ of $\mathbb{D}_n$ is $\mathbb{Z}$, and hence that $End_{\mathbb{F}_{p^r}}(F) \cong \mathcal{O}_n$ if and only if $End_{\mathbb{F}_{p^r}}(F) \subseteq \mathbb{Z}_p$. □

Corollary D.6. *If $r = 1$, then $End_{\mathbb{F}_p}(F)$ is commutative and its field of fractions is totally ramified of degree n over $\mathbb{Q}_p$.*

Proof. In this case, the element $\xi_F \in \mathbb{D}_n$ has valuation $\frac{1}{n}$. Hence $\mathbb{Q}_p(\xi_F)$ has ramification index at least n over $\mathbb{Q}_p$. Since $\mathbb{Q}_p(\xi_F)$ is a commutative subfield of $\mathbb{D}_n$, we have $[\mathbb{Q}_p(\xi_F) : \mathbb{Q}_p] \leq n$ and $\mathbb{Q}_p(\xi_F)/\mathbb{Q}_p$ is totally ramified of degree n. The commutativity of $End_{\mathbb{F}_p}(F)$ follows from the fact that the centralizer of $\mathbb{Q}_p(\xi_F)$ in $\mathbb{D}_n$ is $\mathbb{Q}_p(\xi_F)$ itself. □

Generally $End_K(F) \cong \mathcal{O}_n$ for F a formal group law of height n. If F is already defined over $\mathbb{F}_p$, the element $\xi_F \in \mathcal{O}_n$ corresponds to the Frobenius endomorphism $X^p \in End_K(F)$.

Proposition D.7. *If F is defined over $\mathbb{F}_p$, then $End_K(F) = End_{\mathbb{F}_{p^n}}(F)$ if and only if the minimal polynomial of $\xi_F \in \mathcal{O}_n$ over $\mathbb{Z}_p$ is $\xi_F^n - up$ with $u \in \mathbb{Z}_p^\times$.*

Proof. One has $End_{\mathbb{F}_{p^n}}(F) = C_{End_K(F)}(\xi_F^n)$, and therefore $End_K(F) = End_{\mathbb{F}_{p^n}}(F)$ if and only if ξ_F^n is central. The result then follows form the fact that the center of $End_K(F)$ is $\mathbb{Z}_p$ and the valuation of ξ_F^n is equal to the valuation of p. □

From appendix C, we know that

$$\mathcal{O}_n \cong \mathbb{Z}_p(\omega)\langle\xi_F\rangle/(\xi_F^n = pu, \xi_F x \xi_F^{-1} = \sigma(x)), \qquad x \in \mathbb{Z}_p(\omega),$$

for a primitive (p^n-1)-th root of unity ω and $\sigma \in Gal(\mathbb{Z}_p(\omega)/\mathbb{Z}_p) \cong Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ the Frobenius automorphism. Here σ lifts to an action on $\mathcal{O}_n$ given by

$$\sigma\left(\sum_{i\geq 0} x_i \xi_F^i\right) = \sum_{i\geq 0} \sigma(x_i)\xi_F^i, \qquad x_i \in \mathbb{Z}_p(\omega).$$

Since $\xi_F^n = pu$, we know that $v(\xi_F) = \frac{1}{n}$. Thus the valuation map and the canonical projection $\pi : \mathbb{D}_n^\times \to \mathbb{D}_n^\times/\langle pu\rangle$ induce the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \langle pu\rangle & \longrightarrow & \langle pu\rangle & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}_n^\times & \longrightarrow & \mathbb{D}_n^\times & \xrightarrow{v} & \langle\xi_F\rangle & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}_n^\times & \longrightarrow & \mathbb{D}_n^\times/\langle pu\rangle & \xrightarrow{v} & \langle\xi_F\rangle/\langle pu\rangle & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

in which the bottom horizontal sequence splits and the group $\langle\xi_F\rangle/\langle pu\rangle \cong Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $\mathcal{O}_n^\times \cong \mathbb{S}_n$ by the above given action. It follows that

$$\mathbb{D}_n^\times/\langle pu\rangle \cong \mathbb{S}_n \rtimes_F Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{G}_n(u).$$

Notations

Integers

n	a positive integer	
p	a prime	
k	the maximal integer such that p^k divides n	p. 13
m	the positive integer $\frac{n}{\varphi(p^k)}$ when $p-1$ divides n	p. 13
n_α	the positive integer $\frac{n}{\varphi(p^\alpha)}$ for $0 \leq \alpha \leq k$	p. 13
$(a;b)$	the greatest common divisor of a and b	
r_i	the positive integer $\lvert F_i/F_{i-1}\rvert$ for $1 \leq i \leq 3$	
$[A:K]$	the dimension of A over K	
$deg(A)$	the degree of A	p. 104
$ind(A)$	the index of A	p. 104
$exp(A)$	the exponent of A	p. 107
$e(D/K)$	the ramification index of D over K	p. 115
$f(D/K)$	the inertial degree of D over K	p. 115

Elements

u	a unit in $\mathbb{Z}_p^\times$	
S	an element of $\mathbb{D}_n^\times$ generating the Frobenius such that $S^n = p$	p. 11
ζ_i	a i-th root of unity	
x_i	an element of $\mathbb{D}_n^\times$ such that $v(x_i) = (\prod_{k=1}^{i} r_i)^{-1}$ and $x_i^{r_i} \in \widetilde{F_{i-1}}$	p. 34, 37
ε_{p^α}	an element satisfying $(\zeta_{p^\alpha}-1)^{\varphi(p^\alpha)} = p\varepsilon_{p^\alpha}$	p. 48
π_α	the element $\zeta_{p^\alpha} - 1$	p. 50
π_K	a uniformizing element of K	

Sets

$C_G(H)$	the centralizer of H in G	
$N_G(H)$	the normalizer of H in G	
$S/\sim_G$	the set of orbits with respect to the G-action on S	
$\mathcal{F}(G)$	the set of all finite subgroups of G	p. 27
$\widetilde{\mathcal{F}}_u(G)$	the set of all subgroups of G containing $\langle pu\rangle$ as a subgroup of finite index	p. 27
$\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0})$	as defined in	p. 33
$\widetilde{\mathcal{F}}_u(\mathbb{Q}_p(F_0), \widetilde{F_0}, r_1)$	as defined in	p. 34
$\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1})$	as defined in	p. 35
$\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, r_2)$	as defined in	p. 35
$\widetilde{\mathcal{F}}_u(C_{\mathbb{D}_n^\times}(F_0), \widetilde{F_1}, L)$	as defined in	p. 36
$\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2})$	as defined in	p. 38
$\widetilde{\mathcal{F}}_u(N_{\mathbb{D}_n^\times}(F_0), \widetilde{F_2}, W)$	as defined in	p. 39

Groups

$\mathbb{S}_n$	the n-th (classical) Morava stabilizer group	p. 5
S_n	the p-Sylow subgroup of $\mathbb{S}_n$	p. 12
$\mathbb{G}_n(u)$	the n-th extended Morava stabilizer group associated to u	p. 5
$\mu(R)$	the roots of unity in R	
$\mu_i(R)$	the i-th roots of unity in R	
F_i	the i-th subgroup of $\mathbb{G}_n(u)$ associated to a finite $F \subseteq \mathbb{G}_n(u)$	p. 28
$\widetilde{F_i}$	the i-th subgroup of $\mathbb{D}_n^\times$, associated to a finite $F \subseteq \mathbb{G}_n(u)$	p. 29
$Z(G)$	the center of G	
$\mathbb{Z}\langle x\rangle$	the infinite cyclic group generated by x	
C_n	the cyclic group of order n	
$C_n * C_m$	the kernel of the m-th power map on C_n	
Q_{2^n}	the (generalized) quaternionic group order 2^n	p. 13
T_{24}	the binary tetrahedral group of order 24	p. 18
D_8	the dihedral group of order 8	p. 95
SD_{16}	the semidihedral group of order 16	p. 95
O_{48}	the binary octahedral group of order 48	p. 97
$Br(K)$	the Brauer group of K	p. 107
$Br(L/K)$	the relative Brauer group of L over K	p. 107

Rings, fields

$\mathbb{F}_{p^n}$	the finite field with p^n elements	
$\mathbb{Q}_p$	the field of p-adic numbers	
$\mathbb{Z}_p$	the ring of p-adic integers	
$\mathbb{W}(R)$	the ring of Witt vectors over R	
$D(K, r/n)$	the K-central division algebra of invariant $\frac{r}{n}$	p. 116
$\mathbb{D}_n$	the $\mathbb{Q}_p$-central division algebra of invariant $\frac{1}{n}$	p. 116
$\mathcal{O}_n$	the maximal order of $\mathbb{D}_n$	
$\mathcal{O}_K$	the ring of integers of the field K	
$U_i(\mathcal{O}_K^\times)$	the i-th filtration group $\{x \in \mathcal{O}_K^\times \mid v_K(x-1) \geq i\}$	p. 118
$R(G)$	the R-algebra generated by G	p. 14
$R[G]$	the group ring generated by G	
R_α	the ring $\mathbb{Z}_p[\zeta_{p^\alpha}]$	p. 50

Maps

v	the valuation $p \mapsto 1$ relative to $\mathbb{Q}_p$	
v_K	the valuation $\pi_K \mapsto 1$ relative to the field K	
v_D	the valuation $\pi_{Z(D)} \mapsto 1$ relative to the division algebra D	p. 115
φ	Euler's totient function	p. 12
$N_{L/K}$	the norm of the extension L/K	
$Tr_{L/K}$	the trace of the extension L/K	
N_G	the norm relative to the Galois group G	
Tr_G	the trace relative to the Galois group G	

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If F is a formal group law of finite height n defined over a separably closed field K of characteristic $p > 0$, then the K-automorphism group $Aut_K(F)$ of F is isomorphic to the Morava stabilizer group $\mathbb{S}_n = Aut_{\mathbb{F}_{p^n}}(F_n)$, for F_n the Honda formal group law of height n over $\mathbb{F}_p$. It is well known that $\mathbb{S}_n$ can be identified as the group of units of the maximal order $\mathcal{O}_n$ of the central division algebra $\mathbb{D}_n$ of Hasse invariant $1/n$ over $\mathbb{Q}_p$. Moreover if F is already defined over $\mathbb{F}_p$, the Frobenius endomorphism $X^p \in End_K(F)$ defines en element $\xi_F \in \mathcal{O}_n$ whose minimal polynomial over $\mathbb{Z}_p$ is $\xi_F^n - up$ for some $u \in \mathbb{Z}_p^\times$ if and only if $End_K(F) = End_{\mathbb{F}_{p^n}}(F)$. This element induces an action of the Galois group $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ on $\mathbb{S}_n$ which depends on u, and the extended Morava stabilizer group $\mathbb{G}_n(u) \cong \mathbb{S}_n \rtimes_F Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ associated to F is realized as a split extension of $\mathbb{S}_n$ by $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$ which is isomorphic to the quotient group $\mathbb{D}_n^\times/\langle pu \rangle$.

The object of the thesis is the classification up to conjugation of the finite subgroups of $\mathbb{S}_n$ and more generally $\mathbb{G}_n(u)$ for n a positive integer, p a prime and u a unit in the ring of p-adic integers. More particularly, we provide a complete classification of the finite subgroups of $\mathbb{S}_n$, we give necessary and sufficient conditions on n, p and u for the existence in $\mathbb{G}_n(u)$ of extensions of maximal finite subgroups of $\mathbb{S}_n$ by the Galois group $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$, and whenever such an extension exists we enumerate its conjugacy classes.

In order to do this, we begin by establishing the classification of the finite subgroups of $\mathbb{S}_n$, which are those of $\mathbb{D}_n^\times$, by exploiting its structure in terms of Witt vectors. Based on these results, we provide a theoretical framework for the classification in $\mathbb{G}_n(u)$, which notably breaks a given extension in $\mathbb{G}_n(u)$ into three successive stages. Starting with a maximal abelian finite subgroup of $\mathbb{S}_n$, each stage is then analysed explicitly, first covering the abelian cases before reaching the maximal conjugacy classes. We finally illustrate these methods by providing a complete and explicit classification in the case $n = 2$.

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