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# Periodic and quasi-periodic motions in the many-body problem

Jacques Fejoz

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Mémoire d'Habilitation

de

l'Université Pierre & Marie Curie – Paris VI

spécialité mathématiques

présenté par

Jacques Féjoz

# Mouvements périodiques et quasi-périodiques dans le problème des $n$ corps

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Microscope, chancelant — Ma démission ! Mais, Prince permettez après tout, la lune, ce n'est pas ma partie, ... Moi je m'occupe que de mécanique – vous avez un observatoire ... ça regarde l'observatoire ... C'est lui qui est chargé des relations avec le ciel.
Cosinus — Allons, mon enfant, ne demandez pas l'impossible !
Prince Caprice — L'impossible ! Vous osez dire que c'est impossible ? Mais je m'y oppose. Vous êtes tous des ânes !

*Le Voyage dans la Lune*, J. Offenbach (1875)

I am particularly grateful to

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## A Brief Astronomical Preface

NEWTON's discovery of universal attraction dramatically modified our understanding of the motion of celestial bodies. This law masterly reconciles two seemingly contradictory physical principles: the principle of inertia, put forward by GALILEO and DESCARTES in terrestrial mechanics, and the laws of KEPLER, governing the elliptical motion of planets around the Sun.

In an additional *tour de force*, NEWTON also estimated the first order effect on Mars of the attraction of other planets. Indeed, he soon realized that in the long term the mutual attraction of planets and other celestial bodies, could have a considerable accumulated effect, destroying the Keplerian regularity it had first explained:

> For while comets move in very eccentric orbs in all manner of positions, blind fate could never make all the planets move one and the same way in orbs concentric, some inconsiderable irregularities excepted which may have arisen from the mutual actions of comets and planets on one another, and which will be apt to increase, till this system wants a reformation. [76, Book III, Query 31]

The unforeseen consequence of NEWTON's discovery was to question the belief that the solar system be stable: it was no longer obvious that planets kept moving immutably, without collisions or ejections. And symmetrically, the question remained for a long time, whether universal attraction could explain the irregularities of motion observed in the past. A two-century long competition started between astronomers, who made more and more precise observations, and *geometers*, who had the status and destiny of NEWTON's law in their hands. Two main mysteries kept the mathematical suspense at its highest: the motion of the moon's perigee, and the shift of Jupiter's and Saturn's longitudes, revealed by the comparison between the observations of that time and those which PTOLEMY had recorded almost two thousand years earlier. The first computations of NEWTON, EULER and others were giving wrong results [22, 35]. Infinitesimal calculus was in its infancy and geometers, at first, lacked the necessary mathematical apparatus to understand the long-term influence of mutual attractions.

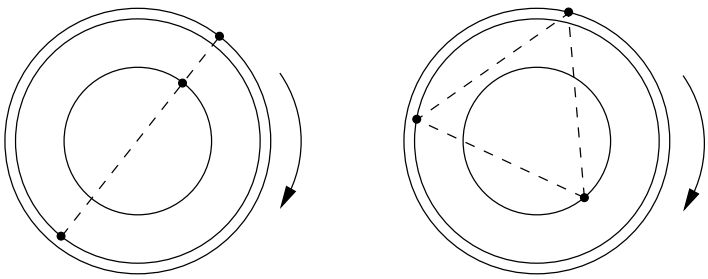
Regarding the moon's perigee, CLAIRAUT and D'ALEMBERT understood that the most glaring discrepancy with observations could be explained by higher order terms [22, 35]. But the theory of perturbations was given its major impulse at the end of the XVIII century, when LAGRANGE transformed mechanics and dynamics into a branch of mathematical analysis, laying the foundations of differential and symplectic geometry (see [100], and footnote (6) in the appendix of the present memoir). In his study of Jupiter's and Saturn's motions, LAPLACE found approximate evolution equations, describing the average variations of elliptical elements of the planets. These variations are called *secular* because they can be detected only over a long time interval, typically one century. LAPLACE computed the secular dynamics at the first order with respect to the masses, eccentricities and inclinations of the planets. His analysis of the spectrum of the linearized vector field, at a time when this chapter of linear algebra did not exist, led him

and LAGRANGE to a resounding theorem on the stability of the solar system, which entails that the observed variations in the motion of Jupiter and Saturn come from resonant terms of large amplitude and long period, but with zero average ([51, p. 164]; in this memoir, see theorem 11). We are back to a regular –namely, quasi-periodic[(1)]– model, however far it is conceptually from PTOLEMY's ancient epicycle theory. Yet it is a mistake, which LAPLACE made, to infer the topological stability of the planetary system, since the theorem deals only with a truncated problem.

At this point, I would like to take the liberty of mentioning that around that time EULER and LAGRANGE found two explicit, simple solutions of the three-body problem, called *relative equilibria* because the bodies rigidly rotate around the center of attraction at constant speed [49]. These solutions, where each body moves as if it were attracted by a unique fictitious body, belong to a larger class of motions, called *homographic*, parametrized by the common eccentricity of bodies. Recently, many new periodic orbits have been found, which share some of the discrete symmetries of EULER's and LAGRANGE's orbits in the equal-mass problem [98], and to which the second half of this memoir is devoted. However, no other explicit solution to the three-body problem has been found ever since!

Relative equilibria of

Euler and Lagrange



The theory of the moon did not reach a satisfactory stage before the work of ADAMS and DELAUNAY in the XIX century. DELAUNAY noticed *un résultat singulier*, already visible in CLAIRAUT's computation: according to the first order secular system, the perigee and the node describe uniform rotations, in opposite directions, with the *same* frequency [37]. This was to play a role later in the proof of ARNOLD's theorem, although higher order terms of large amplitude destroy the resonance. DELAUNAY carried out the Herculean computation of the secular dynamics up to the eighth order of averaging with respect to the semi major axis ratio. At the same time, LE VERRIER pursued LAPLACE's computations, but questioned the astronomical relevance of his stability theorem.

After the failure of formal methods of the XIX century, due to the irreducible presence of small denominators in perturbation series, POINCARÉ has drawn the attention of mathematicians to qualitative questions, concerning the geometry of the phase portrait

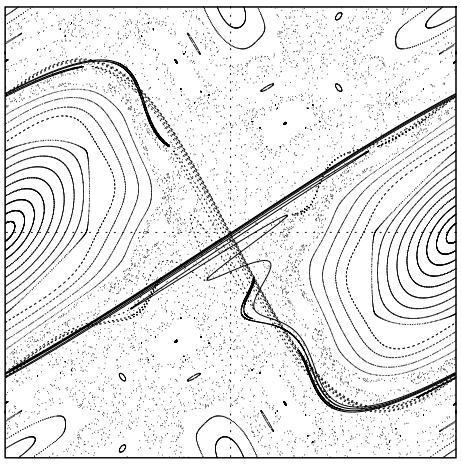
(1) Some authors, e.g. [95], call such motions *conditionally periodic*, restricting the use of the adjective *quasi-periodic* to the non-resonant case.

rather than the analytic expression of particular solutions. In in his epoch-making treatise *Les méthodes nouvelles de la mécanique céleste*, he wrote:

> Les séries [de Gyldén et de Lindstedt] ne sont pas susceptibles de donner une approximation indéfinie. [...] D'ailleurs, certaines conséquences théoriques que l'on pourrait être tenté de tirer de la forme de ces séries ne sont pas légitimes à cause de leur divergence. C'est ainsi qu'elles ne peuvent servir à résoudre la question de la stabilité du système solaire. [81, Introduction]
> (The series [of Gyldén and Lindstedt] are not likely to give arbitrary approximations. [...] Furthermore, certain theoretical consequences which one might be tempted to infer from the properties of these series, are not legitimate because of their divergence. Thus they cannot help to solve the question of the stability of the solar system.)

Poincaré gave arguments against the existence of first integrals other than the energy, in Hamiltonian systems in general. In the case of the three-body problem: "Le problème [...] n'admet pas d'autre intégrale uniforme que celles des forces vives et des aires" (the problem has no uniform integral other than the energy and the angular momentum) [81, Chap. v, § 85] (see [104, p. 241] for a criticism of the shortcomings of this result). Poincaré also uncovered the splitting of separatrices of a hyperbolic equilibrium point and the resulting entanglement (the interesting story of Poincaré's mistake in the first version of his memoir for king Oscar, which later led him to this discovery, is told in [11]):

> On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problème des trois corps et en général de tous les problèmes de Dynamique où il n'y a pas d'intégrale uniforme et où les séries de Bohlin sont divergentes.
> [...] Cette remarque est de nature à nous faire comprendre [...] combien les transcendantes qu'il faudrait imaginer pour résoudre [le problème des trois corps] diffèrent de toutes celles que nous connaissons. [81, § 397–398]



> (One is struck by the complexity of this figure, which I will not even try to draw. Nothing is more appropriate to give an idea of the complexity of the three-body problem, and, in general, of those dynamical systems which do not have uniform integrals and where Bohlin series diverge.
> [...] This remark should make us understand to what extent the transcendants which we would have to imagine, to solve [the three-body problem], depart from all those we know.)

Some facts like the anomalous perihelion advance of the planet Mercury could only be explained in 1915 by EINSTEIN's theory of general relativity [55, 78]. Classical dynamics thus proved to be a limit case of, already inextricably complicated but simpler than, EINSTEIN's infinite dimensional field equations.

On the positive side, POINCARÉ gave a new impulse to the perturbative study of periodic orbits. Adding to the work of HILL and cleverly exploiting the symmetries of the three-body problem, he found several new families, demanding a classification in terms of *genre*, *espèce* and *sorte* (genre, species and kind) [81, Chap. III]. That periodic orbits are dense among bounded motions, as he conjectured, is still an open and highly plausible conjecture (see [85] for $C^1$-generic Hamiltonian systems and [44] for the restricted three-body problem with small mass ratio for the primaries). He famously commented:

> On peut alors avec avantage prendre [les] solutions périodiques comme première approximation, comme *orbite intermédiaire* [...]. Ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable. [81, § 36]
> (One can then advantageously take periodic solutions as first approximation, as *intermediate orbit*. [...] What makes periodic orbits so valuable is that they are the only breach, so to speak, through which we can try to enter a place up to now deemed unapproachable.)

In the XX century, followers like BIRKHOFF, MOSER and MEYER have developed a variety of techniques to establish the existence, and study the stability, of periodic solutions in the many-body problem, and more generally in Hamiltonian systems: analytic continuation (in the presence of symmetries, first integrals and other degeneracies), averaging, normal forms, special fixed point theorems, symplectic topology. BROUCKE, BRUNO, HÉNON, SIMÓ and others have quite systematically explored families of periodic orbits, in particular in the HILL (or lunar) problem. See [69, 103, 91, 70, 87, 60, 98, 42, 46] and references therein.

Among the many kinds of perturbations series, the LINDSTEDT series are the most interesting from the mathematical point of view; up to the correspondence between Cartesian and polar coordinates, nowadays they are called BIRKHOFF series. Astronomers were not fond of them because, while celestial bodies certainly have a prescribed position and velocity, the LINDSTEDT normal forms (of infinite order) are defined on a set of empty interior. This set is foliated in invariant embedded tori, in restriction to which the flow is quasi-periodic. The series are generally divergent ([81, Chap. XII]; see also

[79]), although Poincaré could not preclude that these series, the frequencies being fixed, sometimes converge, non uniformly:

> Nous avons reconnu que les équations canoniques [...] peuvent être satisfaites formellement par des séries de la forme
>
> $$\begin{cases} x_i = x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + ..., \\ y_i = y_i^0 + \mu y_i^1 + \mu^2 y_i^2 + ..., \end{cases}$$
>
> où [...] $w_i = n_i t + \varpi_i$ $(i = 1, 2, ..., n)$, [de quoi] nous avons tiré
>
> $$x_i^k = \sum \frac{B \sin(m_1 w_1 + m_2 w_2 + ... + m_n w_n + h)}{m_1 n_1^0 + m_2 n_2^0 + ... + m_n n_n^0} + A_0.$$
>
> [Cette] série converge-t-elle absolument et uniformément ? [... À] deux degrés de liberté, les séries ne pourraient-elles pas, par exemple, converger quand $x_1^0$ et $x_2^0$ ont été choisis de telle sorte que le rapport $\frac{n_1}{n_2}$ soit incommensurable, et que son carré soit au contraire commensurable (ou quand le rapport $\frac{n_1}{n_2}$ est assujetti à une autre condition analogue à celle que je viens d'énoncer un peu au hasard) ? [81, §§ 146–149]
> (We have realized that canonical equations [...] can be satisfied formally by series of the form
>
> $$\begin{cases} x_i = x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + ..., \\ y_i = y_i^0 + \mu y_i^1 + \mu^2 y_i^2 + ..., \end{cases}$$
>
> where [...] $w_i = n_i t + \varpi_i$ $(i = 1, 2, ..., n)$. From this we have inferred
>
> $$x_i^k = \sum \frac{B \sin(m_1 w_1 + m_2 w_2 + ... + m_n w_n + h)}{m_1 n_1^0 + m_2 n_2^0 + ... + m_n n_n^0} + A_0.$$
>
> Does [this] series converge absolutely and uniformly? [With] two degrees of freedom, couldn't it happen that the series converge when $x_1^0$ and $x_2^0$ have been so chosen that the ratio $\frac{n_1}{n_2}$ be rational and its square on the contrary be irrational (or so that the ratio $\frac{n_1}{n_2}$ satisfy another condition, analogous to the one I have just stated a bit randomly) ?)

Considering the unreasonable consequences of uniform convergence, in terms of existence of periodic orbits at resonances, he speculated:

> Les raisonnements de ce Chapitre ne permettent pas d'affirmer que ce fait ne se présentera pas. Tout ce qu'il m'est permis de dire, c'est qu'il est fort invraisemblable. [*ibid.*]
> (The arguments in this Chapter do not make it possible to assert that this fact will not occur. All I can say is that it is most unlikely.)

A stupendous breakthrough came from Siegel and Kolmogorov, who proved that, respectively for the linearization problem of a one-dimensional complex map and for the perturbation of an invariant torus of fixed frequency in a Hamiltonian system, perturbation series do converge, albeit non uniformly, assuming in particular that the fixed

frequency is Diophantine [48, 97]:

$$|m_1 n_1^0 + \cdots + m_n n_n^0| \geq \frac{\gamma}{(|n_1| + \cdots + |n_n|)^\tau} \quad (\gamma, \tau > 0).$$

SIEGEL's proof overcomes the effect of small denominators by cleverly controlling how they accumulate. KOLMOGOROV uses NEWTON's algorithm in a functional space of infinite dimension and finds quasi-periodic invariant tori by a limiting process. The fast convergence of the algorithm beats the effect of resonances, one of the main ideas which laid the foundations for the so-called KOLMOGOROV-ARNOLD-MOSER theory; see [7, 13, 14, 31, 36, 80, 84, 94, 96] for background and references.

ARNOLD proved a degenerate version of KOLMOGOROV's celebrated theorem, and deduced the existence of a set of positive measure of almost planar and almost circular quasi-periodic solutions when the masses of the planets are small enough [6]. There are several degeneracies in this problem. The most important one comes from the fact that all the bounded orbits of the Kepler problem—a problem with two degrees of freedom—are periodic, which is a very specific feature of the Newtonian potential in $1/r$ (and of the elastic potential in $r^2$, and only them, according to a theorem of BERTRAND). ARNOLD's proof was complete only for the planar two-planet problem. In the spatial case, an unforeseen and mysterious resonance is present: the trace of the linearized secular system is always zero, identically with respect to the semi major axes. As previously mentioned, this was actually known to DELAUNAY in the three-body problem. But it holds in general for the many-planet problem. This was first noticed by HERMAN who, in a series of lectures in the 1990's, sketched a complete and more conceptual proof of ARNOLD's theorem. Two sections of this memoir are devoted to explaining some ideas of such a proof, as detailed in [39, 40]. CHIERCHIA-PINZARI have just proved in general [33] that HERMAN's resonance disappears when one reduces the problem by the rotational symmetry, as ROBUTEL had proved in the three-body problem, using a computer.

However, ARNOLD's theorem hardly applies to our solar system. There is a first difficulty with the upper value of the small parameter $\varepsilon$. A similar issue occurs when semi-classical analysts let the Planck constant tend to zero. HÉNON noticed that, without any additional care, the first proofs of KOLMOGOROV's theorem show the existence of invariant tori only for a derisory $\varepsilon$ of the order of $10^{-300}$ [45]! However, ROBUTEL has shown numerically that some significant parts of the solar system, in particular of the system consisting of the Sun, Jupiter and Saturn [52, 88], display a quasi-periodic behavior. Also, CELLETTI–CHIERCHIA [15, 16] and LOCATELLI-GIORGILLI [54] have proved quantitative versions of the KAM theorem, which they have applied to the systems Sun–Jupiter–asteroid Victoria and Sun–Jupiter–Saturn; these applications are assisted by computer symbolic processors, requiring in the second case the manipulation of series of ten million terms. Whether bounded motions form a set of positive Lebesgue measure for all $\varepsilon$—and not only for $\varepsilon \ll 1$—remains a completely open problem.

Another matter for discontent when applying KAM theory to astronomy, is that the set of KAM invariant tori in phase space fill a transversely Cantor set, parametrized by Diophantine frequencies, which is topologically meager. Given the approximation which is made by substituting the Newtonian planetary system to the real solar system,

whether the planet's mean motions are Diophantine or not, is not a question with any straightforward meaning. Incidentally, MOLCHANOV has speculated on the opposite hypothesis that these mean motions could be totally periodic [65]. Hence the direct conclusion of ARNOLD's theorem over an infinite time interval, is illusory in astronomy. Yet KAM theory provides a fundamental conceptual tool in the study of conservative systems since, as is wildly believed, the conclusion of invariant tori theorems holds under much weaker hypotheses than current theoretical proofs require. To paraphrase POINCARÉ, quasi-periodic orbits too are part of the breach.

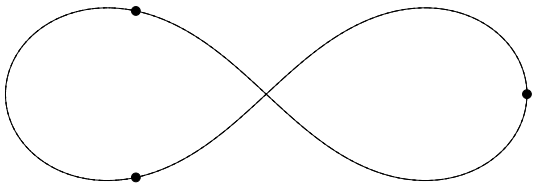
A related and more realistic theorem by NEKHOROSHEV [73], asserts that in the neighborhood of KAM quasi-periodic solutions motions are stable over an exponentially long time interval with respect to the small parameter. By applying a theorem of this type, NIEDERMAN has shown the stability of a solar system with two planets having small masses, not quite equal but much closer to realistic values [77]. In order to describe the slow evolutions more accurately, NEISHTADT has developed the theory of adiabatic invariants [75], and extended related results to non-Hamiltonian perturbations [74].

Over the centuries, geometers have spent an inordinate amount of time and energy proving stronger and stronger stability theorems for dynamical systems more or less closely related to the solar system. It was a huge surprise when the numerical computations of LASKAR showed that over the life span of the Sun, or even over a few hundred million years, collisions and ejections of inner planets are probable (see [53] for a recent account). Our solar system is now wildly believed unstable. Works of SITNIKOV and ALEKSEEV [71], MOECKEL [62], SIMÓ-STUCHI [99] and GALANTE-KALOSHIN [43], among others (see [8] for other references), show the complexity of the simplest non-integrable many-body problem, the restricted three-body problem (*restricted* meaning that the third body has zero mass and thus moves under the influence of, but without influencing, the two primaries). ARNOLD's diffusion and the general mechanisms of instability in large dimension are still to be understood, despite significant progress [7, 12, 30, 38, 58, 59, 64] (see [8] for more references).

Two discoveries have led to another shift of paradigm. First, came the discovery of exoplanets in the early 1990's [92]. This confirmation of an old philosophical speculation has sustained the interest in extraterrestrial life. Many of these exoplanets have larger eccentricities, inclinations or masses (not to mention brown dwarfs), or smaller semi major axes, than planets of our solar system–and there seem to be billions of them in our galaxy alone. Are such orbital elements consistent with a stable dynamics? This wide spectrum of dynamical forms of behavior has considerably broadened the realm of relevant many-body problems in astronomy, and renewed interest in the global understanding of the many-body problems, far from the so-called planetary regime (with small eccentricities, inclinations and masses), and possibly with important tidal or more general dissipating effects (see [18]).

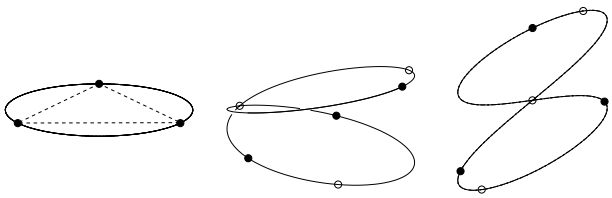
The second discovery is mathematical. Finding periodic geodesics on a Riemannian manifold as length minimizers within a fixed non-trivial homotopy or homology class is commonplace. Yet all attempts to apply the same strategy to the three-body problem had failed because collisions might occur in minimizers, as POINCARÉ had pointed

out [82]. Indeed, the Newtonian potential is weak enough for the Lagrangian action to be finite around collisions. Using variational methods, in 1999 Chenciner-Montgomery managed to prove the existence of a plane periodic solution to the equal-mass three-body problem, whose symmetry group is a 12th-order subgroup of the symmetry group of the Lagrange equilateral triangle. In particular the bodies chase each other along a closed curve—such solutions have been named *choreographies* by Simó. This curve being eight-shaped, the solution has been called the *Eight* [28, 20]. It had been found numerically by Moore [66].

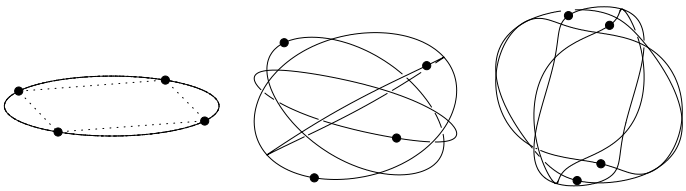


Since then, Simó has searched the phase space for such symmetric orbits quite systematically, and found a whole wealth of them [98]. Theoretical works, in particular from S. Terraccini and her students, have also shown the existence of a large number of symmetric periodic orbits which minimize the Lagrangian action within their symmetry class [10, 42]. And Marchal, helped by Chenciner, remarkably brought the first general answer to the question of collisions: minimizers of the Lagrangian action (among all fixed-end loops) are collision-free [20, 42, 57]. Marchal's theorem thus shows a subtle difference between Cauchy and Dirichlet boundary conditions in the many-body problem.

At the Saari conference in 1999, Marchal realized that the Eight could be related to the equilateral triangle relative equilibria, through a Lyapunov family of spatial orbits, periodic in a rotating frame [27, 24]:



This family has been named $\mathscr{P}_{12}$, after the order of its symmetry group. In fact, such a connection between relative equilibria and symmetric periodic orbits is a very general phenomenon, bringing light to the family tree of all the newly discovered periodic orbits [23, 26]. The second example is that of the Hip-Hop solution of Chenciner-Venturelli [29], which is similarly related to the square relative equilibrium.



Relative equilibria themselves and, in turn, symmetric periodic solutions, have become intermediate orbits in the neighborhood of which local perturbation theory can be applied [27].

## English summary

The first half of the present memoir is devoted to KAM theory and ARNOLD's theorem for several planets in space, as detailed in one preprint and one article:[(2)]

– "Twisted conjugacies and invariant tori theorems [40]. I reprove a normal form theorem due to MOSER [68], for perturbations of a vector field having a Diophantine quasi-periodic invariant torus. This normal form, which I call a *twisted conjugacy*, is a gateway to invariant tori theorems of KOLMOGOROV, ARNOLD, RÜSSMANN and HERMAN, as well as to some other theorems, for example for dissipative vector fields. I introduce a *hypothetical conjugacy*, i.e. a conjugacy depending on arithmetical properties of the perturbed frequency vector, as an intermediate step towards invariant tori theorems with weak non-degeneracy conditions, I improve some estimates on the functional dependance of the normal form, and give some new applications to celestial mechanics.

– "Démonstration du théorème d'Arnold sur la stabilité du système planétaire (d'après Herman)" [39]. Arnold's theorem is proved for $N$ planets in space. (The proof included in [40] is a clarification and an improvement of the abstract part of [39].) ARNOLD remarkably asserted, in the Newtonian model of the planetary problem with $N$ planets, the existence of an invariant set of positive Lebesgue measure, foliated in quasi-periodic invariant tori of dimension $3N-1$ [6]. ARNOLD's suggestion for proving the result in full generality was to fix the direction of the angular momentum vector, in order to get rid of a degeneracy due to the rotational invariance of the problem, and then to apply his degenerate version of Kolmogorov's theorem to find Lagrangian tori in the neighborhood of the elliptic secular singularity (circular horizontal Keplerian ellipses). This strategy of partial reduction fails because of a mysterious resonance, discovered by HERMAN, which generalizes the resonance found by CLAIRAUT in the first order lunar problem. This resonance had not been noticed in the context of KAM theory because in the 2-planet problem, JACOBI's reduction of the node makes it possible to carry out the full symplectic reduction by rotations in DELAUNAY's coordinates (I recall the definition of these coordinates in the appendix of this memoir, and give a new proof of their symplecticity). Here it is proved by induction on the number of planets, following Herman, that the local image of the frequency map of the planetary system (as a function of the semi major axes), is contained in a vector plane of codimension two, and in no vector plane of larger codimension. Using an argument of Lagrangian intersection theory, this allows us to apply an invariant tori theorem with a weak (or Rüssmann-) non-degeneracy condition.

The second half of the memoir deals with periodic and relatively periodic orbits in the global many-body problem. It is based on two publications.

– "The flow of the equal-mass spatial 3-body problem in the neighborhood of the equilateral relative equilibrium" (with A. Chenciner) [24]. It is shown that exactly two families of relatively periodic orbits bifurcate from the Lagrange equilateral triangle, namely the

[(2)] `http://people.math.jussieu.fr/~fejoz/articles.html`

homographic and the $\mathscr{P}_{12}$ families. Moreover, in restriction to the 4-dimensional center manifold, the local dynamics is proved to be a twist map of an annulus of section, bounded by the two families. Another paper shows that the $\mathscr{P}_{12}$ family ends at the Eight [27]. In between, the $\mathscr{P}_{12}$ family is known to exist as a family of minimizers of the Lagrangian action within its symmetry class for all values of the rotation of the frame. Such a family could be non-unique, or not continuous, but numerical experiments indicate that it is not the case (see the pictures above).

– "Unchained polygons and the $N$-body problem" (with A. Chenciner) [26]. The Lagrange relative equilibrium appears above as the organizing center of the Eight. We show that the same phenomenon occurs with the equal-mass relative equilibrium of the square, which appears as the organizing center of the Hip-Hop. More generally, many recently studied classes of periodic solutions bifurcate from symmetric relative equilibria. In a rotating frame where they become periodic, these families acquire remarkable symmetries. We study the possibility of continuing these families globally as action minimizers in a rotating frame, among loops sharing the same symmetries. In a preliminary step we estimate the intervals of the frame rotation frequency over which the relative equilibrium is the sole absolute action minimizer. Then we focus on our main example, the relative equilibrium of the equal-mass regular $N$-gon. The proof of the local existence of the vertical Lyapunov families relies on the fact that the restriction to the corresponding directions of the quadratic part of the energy is positive definite. We compute the symmetry groups $G_{\frac{r}{s}}(N,k,\eta)$ of the vertical Lyapunov families observed in appropriate rotating frames, and use them for continuing the families globally. Paradigmatic families are the Eight families for an odd number of bodies and the Hip-Hop families for an even number. It is precisely for these two kinds of families that global minimization may be used. In the other cases, obstructions to the method come from isomorphisms between the symmetries of different families; this is the case for the so-called chain choreographies, where only a local minimization property is true (except for $N = 3$). Another interesting feature of these chains is the deciding role played by the parity, in particular through the value of the angular momentum. For the Lyapunov families bifurcating from the regular $N$-gon with $N \leq 6$, we check in an appendix that locally the torsion is not zero, which justifies taking the rotation of the frame as a parameter. This article illustrates how fertile symmetry considerations are, not only as a proof technique, but also in the heuristic search for remarkable solutions.

The many-body problem has been at the origin of numerous mathematical theories. And because of the variety of techniques its study demands, it retains all of its fascination.

Paris, Summer 2010

## Résumé en français

La première moitié de ce mémoire est consacrée à la théorie KAM et au théorème d'Arnold sur la stabilité des systèmes planétaires. Ce travail a fait l'objet d'un article en préparation et d'une publication :[(3)]

– "Twisted conjugacies and invariant tori theorems" [40]. Je redémontre une forme normale de champs de vecteurs due à Moser [68], pour les perturbations de champs de vecteurs admettant un tore invariant quasi-périodique diophantien. Cette forme normale, que j'appelle une *conjugaison tordue* est une porte d'entrée pour démontrer des théorèmes de tores invariants dus à Kolmogorov, Arnold, Rüssmann et Herman, ainsi que d'autres théorèmes, par exemple pour des champs de vecteurs dissipatifs. J'introduis une notion de *conjugaison hypothétique*, comme un intermédiaire commun aux théorèmes de tores invariants avec une condition de non-dégénérescence faible, améliore certaines estimations sur la dépendance fonctionnelle de la forme normale, et donne quelques applications nouvelles à la mécanique céleste.

– "Démonstration du théorème d'Arnold sur la stabilité du système planétaire (d'après Herman)" [39]. Cet article donne une démonstration du théorème d'Arnold pour $N$ planètes dans l'espace $\mathbb{R}^3$. La démonstration de [40] est une clarification et une amélioration de la partie abstraite de [39]. Arnold avait publié le résultat remarquable suivant : dans le problème planétaire newtonien à $N$ planètes, si les masses des planètes sont assez petites, il existe dans l'espace des phases un sous-ensemble invariant de mesure de Lebesgue strictement positive, formé de tores invariants quasipériodiques de dimension $3N - 1$ [6]. La suggestion d'Arnold pour le démontrer en toute généralité était de fixer la direction du moment cinétique, pour se débarrasser de la dégénérescence due à l'invariance par rotation, puis d'appliquer sa version dégénérée du théorème de Kolmogorov pour trouver des tores lagrangiens invariants au voisinage de la singularité séculaire elliptique (mouvements képlériens elliptiques circulaires horizontaux). Cette stratégie de réduction partielle ne marche pas à cause d'une résonance mystérieuse, découverte par Herman, qui généralise à $N$ planètes une résonance déjà connue de Clairaut dans le problème de la lune. Cette résonance n'avait pas été remarquée dans le cas de 2 planètes, où la réduction des noeuds de Jacobi permet de réduire complètement le problème par la symétrie de rotation, en coordonnées de Delaunay (je rappelle en appendice la définition de ces coordonnées, et propose une nouvelle démonstration de leur caractère symplectique). Ici, je démontre par récurrence sur le nombre de planètes, en suivant les idées d'Herman, que l'image locale de l'application fréquence (vue comme fonction des demi grands axes des planètes) est contenue dans un plan vectoriel de codimension deux, mais dans aucun plan vectoriel de codimension supérieure. Un argument de la théorie des intersections lagrangiennes permet alors d'appliquer un théorème de tores invariants qui ne requiert qu'une faible condition de non-dégénérescence.

[(3)] `http://people.math.jussieu.fr/~fejoz/articles.html`

La seconde moitié de ce mémoire traite d'orbites périodiques et relativement périodiques (i.e. périodiques en repère tournant), dans le problème global des $N$ corps. Elle aussi est basée sur deux articles.

– "The flow of the equal-mass spatial 3-body problem in the neighborhood of the equilateral relative equilibrium" (avec A. Chenciner) [24]. Nous démontrons qu'exactement deux familles de solutions relativement périodiques bifurquent de la solution d'équilibre relatif de Lagrange : la famille homographique et la famille $\mathscr{P}_{12}$. De plus, en restriction à la variété centrale de dimension 4 de l'équilibre relatif de Lagrange, la dynamique locale est une application twist d'un anneau de section, bordé par les deux familles. Un autre article montre que la famille $\mathscr{P}_{12}$ se termine, de l'autre côté, à la solution en Huit de Chenciner-Montgomery [27]. Entre ces deux extrémités, on sait que la famille $\mathscr{P}_{12}$ existe comme famille des minima de l'action lagrangienne parmi les lacets possédant sa classe de symétrie. Une telle famille pourrait a priori être non unique, ou discontinue, mais les expériences numériques ne laissent guère de doute (voir la figure dans la préface).

– "Unchained polygons and the $N$-body problem" (avec A. Chenciner) [26]. L'équilibre relatif de Lagrange apparaît dans ce qui précède comme le centre organisateur du Huit. Nous montrons que le même phénomène se produit avec l'équilibre relatif du carré à quatre masses égales, qui apparaît comme centre organisateur de la famille du Hip-Hop. Plus généralement, beaucoup de classes de solutions récemment découvertes appartiennent aux familles de Lyapunov issues d'équilibres relatifs symétriques. Dans un repère tournant où elles deviennent périodiques, ces familles acquièrent des symétries remarquables. Nous étudions la possibilité de les prolonger globalement comme minima de l'action lagrangienne en un repère tournant, au sein de leur classe de symétrie. Une étape préliminaire est de déterminer les intervalles de la fréquence de rotation du repère sur lesquels un équilibre relatif est l'unique minimum absolu de l'action. Nous nous focalisons ensuite sur notre exemple principal, l'équilibre relatif du polygone régulier à $N$ sommets. L'existence locale de familles de Lyapunov verticales repose sur le fait que la restriction de la partie quadratique de l'énergie aux directions centrales est définie positive. Nous calculons les groupes de symétrie $G_{\frac{r}{s}}(N,k,\eta)$ des familles de Lyapunov verticales, et les utilisons pour prolonger les familles globalement. Les exemples paradigmatiques sont les familles de Huits pour un nombre impair de corps et les familles de Hip-Hops pour un nombre pair. Ce sont précisément les éléments de ces deux types de familles qui peuvent être des minima globaux. Dans les autres cas, des obstructions apparaissent, qui sont dues à des isomorphismes entre les groupes de symétrie de différentes famille ; c'est le cas des *chaînes chorégraphiques*, dont les éléments sont seulement des minima locaux (sauf pour $N = 3$). Une autre particularité intéressante de ces chaînes est le rôle décisif joué par la parité, en particulier à travers la valeur prise par le moment cinétique. Pour les familles de Lyapunov bifurquant d'un polygone à au plus 6 sommets, nous vérifions en outre que la torsion locale est non dégénérée, ce qui justifie de prendre la rotation du repère comme paramètre. Cet article montre la fécondité des considérations de symétrie, comme technique de démonstration mais aussi comme guide heuristique dans la recherche de solutions remarquables.

Le problème des $n$ corps, depuis longtemps à l'origine de nombreuses théories mathématiques, garde entier, de part la variété des techniques nécessaires à son étude, son pouvoir de fascination.

Paris, été 2010

## 1. A Simple Proof of Invariant Tori Theorems

As Poincaré noticed, a periodic orbit is a fixed point of a return map to a section of the flow. For a quasi-periodic orbit there is no such reduction to finite dimension. In this section we sketch the lines of a simple proof of Kolmogorov's and Rüssman's invariant tori theorem, as detailed in [40] (see also the exposition in [47, Section 15.4]), and as used in the many-planet problem in the next section. The strength of this proof is to reduce Kolmogorov's theorem to a normal form, whose existence and uniqueness follows from a simple, abstract inverse function theorem.

**1.1. Reduction of the invariant torus theorem.** — Let $\mathscr{H}$ be the space of germs along $\mathrm{T}_0^n := \mathbb{T}^n \times \{0\}$ of real holomorphic Hamiltonians in $\mathbb{T}^n \times \mathbb{R}^n = \{(\theta, r)\}$ ($\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$). The vector field associated with $H \in \mathscr{H}$ is

$$X_H : \quad \dot\theta = \partial_r H, \quad \dot r = -\partial_\theta H,$$

and $H$ is said to be close to 0 if the supremum norm of its complex extension over some neighborhood of $\mathrm{T}_0^n$ in $\mathbb{C}^n/(2\pi\mathbb{Z})^n \times \mathbb{C}^n$ is close to 0 (see section 1.2).

For $\alpha \in \mathbb{R}^n$, let $\mathscr{K}^\alpha$ be the affine subspace of Hamiltonians $K \in \mathscr{H}$ such that $K|_{\mathrm{T}_0^n}$ is constant (i.e. $\mathrm{T}_0^n$ is invariant) and $\vec{K}|_{\mathrm{T}_0^n} = \alpha$. Those Hamiltonians are characterized by their first order jet along $\mathrm{T}_0^n$ :

$$j^1_{\mathrm{T}_0^n} K = c + \alpha \cdot r + O(r^2)$$

for some variable $c \in \mathbb{R}$, the notation $O(r^2)$ meaning terms of order 2 or larger with respect of $r$, possibly depending on $\theta$.

In the many-planet problem a resonance relation $k \cdot \alpha = 0$, $k \in \mathbb{Z}^n \setminus \{0\}$, would say that several planets regularly find themselves in the same relative position; over a long time interval, their small mutual attraction, instead of averaging out, would tend to pile up. Here instead, we will consider Diophantine frequencies:

$$\mathrm{D}_{\gamma,\tau} = \{\alpha \in \mathbb{R}^n,\ \forall k \in \mathbb{Z}^n \setminus \{0\}\ |k \cdot \alpha| \geq \gamma |k|^{-\tau}\}, \quad |k| := |k|_1 = |k_1| + \cdots + |k_n|.$$

These arithmetic conditions are by no means necessary [90] (see also [40, Appendix D]).

***Theorem 1* (Kolmogorov [48])**. — *Let $K^o \in \mathscr{K}^\alpha$. Suppose that $\alpha$ belongs to $\mathrm{D}_{\gamma,\tau}$ and that the averaged Hessian $\int_{\mathbb{T}^n} \frac{\partial^2 K^o}{\partial r^2}(\theta, 0)\, d\theta$ non degenerate. Every $H \in \mathscr{H}$ close to $K^o$ possesses an $\alpha^o$-quasi-periodic invariant torus.*

If $K^o$ only had one Diophantine invariant torus without being completely integrable, one would reduce the case to the theorem with an initial normal form along the invariant torus. We refer to [13, 94, 17] for references and background on Kolmogorov's theorem.

Kolmogorov's theorem is a consequence of the following normal form. Let $\mathscr{G}$ be the space of germs along $\mathrm{T}_0^n$ of real holomorphic exact symplectomorphisms $G$ in $\mathbb{T}^n \times \mathbb{R}^n$ of the following form:

$$G(\theta, r) = (\varphi(\theta), {}^t\varphi'(\theta)^{-1}(r + \rho(\theta))),$$

where $\varphi$ is a real holomorphic isomorphism of $\mathbb{T}^n$ fixing the origin (meant to straighten the dynamics along the perturbed torus), and $\rho$ is an exact 1-form on $\mathbb{T}^n$ (meant to straighten the perturbed torus).

***Theorem 2* (Twisted conjugacy).** — *Let $\alpha \in \mathrm{D}_{\gamma,\tau}$ and $K^o \in \mathscr{K}^\alpha$. For every $H \in \mathscr{H}$ close enough to $K^o$, there exists a unique $(K, G, \beta) \in \mathscr{K}^\alpha \times \mathscr{G} \times \mathbb{R}^n$ close to $(K^o, \mathrm{id}, 0)$ such that*

$$H = K \circ G + \beta \cdot r$$

*in some neighborhood of $G^{-1}(\mathbb{T}^n_0)$. Moreover, $\beta$ depends $C^1$-smoothly on $H$.*

Geometrically: the orbits of Hamiltonians $K \in \mathscr{K}^\alpha$ under the action of symplectomorphisms of $\mathscr{G}$ locally form a subspace of finite codimension $n$. In the normal form, the offset $\beta \cdot r$ usually breaks (twists) the dynamical conjugacy between $K$ and $H$.

This theorem is the Hamiltonian particular case of [68]. Here we restrict to this case for the sake of simplicity, although there are some interesting applications —in particular to celestial mechanics— of the more general case.

*Proof of theorem 1 assuming theorem 2.* — We will further assume that $K^o = K^o(r)$ is integrable, which is a slightly simplifying, but non-essential, hypothesis. Write

$$K^o(r) = c + \alpha \cdot r + \frac{1}{2} Q \cdot r^2 + O(r^3), \quad Q := \frac{\partial^2 K^o}{\partial r^2}(0).$$

Applying theorem 2 to $H$ alone is hopeless (because in general $\beta \neq 0$) and naïve (because the cohomology class of the perturbed torus will not be 0 in general). On the other hand, $K^o$ has a whole family of invariant tori, parametrized by their action $r$. So, define $K^o_R(\theta, r) = K^o(\theta, R + r)$ and $H_R(\theta, r) = H(\theta, R + r)$, $R \in \mathbb{R}^n$ small.

Theorem 2 applied to $H_R$ asserts the existence of a triple $(K_R, G_R, \beta_R) \in \mathscr{K}^\alpha \times \mathscr{G} \times \mathbb{R}^n$ such that

$$H_R = K_R \circ G_R + \beta_R \cdot r.$$

In order to prove theorem 1, it suffices to show that there exists $R$ close to 0 such that $\beta_R = 0$.

Claim : The map $R \mapsto \beta_R$ is a local diffeomorphism. So the above conclusion holds.

Since being a local diffeomorphism is an open property and $H_R$ is close to $K^o_R$, it suffices to prove the claim for the trivial perturbation $K^o_R$. But a Taylor expansion of $K^o_R$ immediately gives the normal form of $K^o_R$:

$$\begin{aligned} K^o_R &= [c + O(R)] + \left[\alpha + Q \cdot R + O(R^2)\right] \cdot r + O(r^2) \\ &= \left([c + O(R)] + \alpha \cdot r + O(r^2)\right) \circ \mathrm{id} + (Q \cdot R + O(R^2)) \cdot r. \end{aligned}$$

Hence, $\beta_R = Q \cdot R + O(R^2)$, which is indeed a local diffeomorphism due to the non degeneracy assumption in Kolmogorov's theorem.

Hence if $H$ is close enough to $K^o$ there is a unique small $R$ such that $\beta = 0$. For this $R$, $\|R\| \leq \epsilon$, the equality $H_R = K \circ G$ holds:

$$\begin{tikzcd} \mathrm{T}^n_s + (0,R) \arrow[r, "-(0,R)"] \arrow[drr, "H"'] & \mathrm{T}^n_s \arrow[r, "G"] \arrow[dr, "H_R"] & \mathrm{T}^n_{s-\epsilon} \arrow[d, "K"] \\ & & \mathbb{R} \end{tikzcd}$$

hence the torus obtained by translating $G^{-1}(\mathrm{T}^n_0)$ by $R$ in the direction of actions is invariant and $\alpha$-quasi-periodic for $H$. $\square$

This normal form is the Hamiltonian analogue of the normal form of vector fields on the torus in the neighborhood of Diophantine constant vector fields ([5, 67]). An extensive use of external parameters (like $\beta$ here) was made by Moser [68]. This powerful trick consists in switching the frequency obstruction (obstruction to the conjugacy to the initial dynamics) from one side of the conjugacy to the other. In the 80's, Herman understood the power of this reduction to a finite dimensional problem (see [93]).

**1.2. Proof of the twisted conjugacy.** — It is the aim of this section to show theorem 2, by locally inverting the operator

$$\phi : (K, G, \beta) \mapsto H = K \circ G + \beta \cdot r$$

with adequate source and target spaces. Define the Banach space $\mathscr{H}_s$ of Hamiltonians $H \in \mathscr{H}$ which are continuous on the extension

$$\mathrm{T}^n_s = \{(\theta, r) \in \mathbb{C}^n/\mathbb{Z}^n \times \mathbb{C}^n,\ |\mathrm{Im}\,\theta_j| \leq s,\ |r_j| \leq s\}$$

of $\mathrm{T}^n_0$ of width $s$ and real holomorphic on the interior of $\mathrm{T}^n_s$, and $\mathscr{K}^\alpha_s := \mathscr{K}^\alpha \cap \mathscr{H}_s$. Endow these spaces with the supremum norm $|\cdot|_s$. Note that the strict inductive limit $\mathscr{H} = \varinjlim \mathscr{H}_s$, being the complete countable union of the closed subspaces of empty interior $\mathscr{H}_{1/n}$, is not Baire, thus not metrizable. Also define analogous subsets $\mathscr{G}_s$ of symplectic isomorphisms in $\mathscr{G}$, endowed with their natural structure of a Banach manifold in the neighborhood of the identity (see [40, section 2] for precise definitions). Finally, let $E_s := \mathscr{K}^\alpha_s \times \mathscr{G}_s \times \mathbb{R}^n$ (with norm equal to the maximum of norms on the three factors) and balls $\sigma B^E_s = \{x \in E_s,\ |x|_s \leq \sigma\}$.

The formula above thus defines operators

$$\phi : \sigma B^E_{s+\sigma} \to \mathscr{H}_s$$

commuting with inclusions, in terms of which the local existence and uniqueness of a twisted conjugacy for Hamiltonians takes the following form.[4]

***Theorem 3.*** — *Let $\alpha \in \mathrm{D}_{\gamma,\tau}$. The operator $\phi$ is invertible in the sense that, for all $0 < s < s + \sigma < 1$, if $|H - K^o|_{s+\sigma}$ is small, there is a unique $(K, G, \beta) \in E_s$, $|\cdot|_s$-close*

[4] We could assume that the symplectomorphism $G$ is defined only on $\mathrm{T}^n_s$ (as opposed to $\mathrm{T}^n_{s+\sigma}$), but taking into account the internal structure of the variable $x = (K, G, \beta)$ would result here in a worthless strengthening of the conclusion.

*to* $(K^o, \mathrm{id}, 0)$ *such that* $H = K \circ G + \beta \cdot r$. *Moreover* $\beta \circ \phi^{-1}$ *is a* $C^1$*-function locally in the neighborhood of* $K^o$ *in* $\mathscr{H}_{s+\sigma}$.

This entails theorem 2 and itself follows from the inverse function theorem below. Before stating the latter, we check that the operators $\phi'^{-1}$ and $\phi''$ exist and are bounded.

We will need the following classical lemma in the proof of lemma 5. Let $\mathscr{L}_\alpha$ be the Lie derivative operator in the direction of the constant vector field $\alpha$ :

$$\mathscr{L}_\alpha : \mathscr{A}(\mathbb{T}^n_s) \to \mathscr{A}(\mathbb{T}^n_s), \quad f \mapsto f' \cdot \alpha = \sum_{1 \le j \le n} \alpha_j \frac{\partial f}{\partial \theta_j}.$$

***Lemma 4* (Cohomological equation).** — *If* $g \in \mathscr{A}(\mathbb{T}^n_{s+\sigma})$ *has* 0*-average* ($\int_{\mathbb{T}} g \, d\theta = 0$)*, there exists a unique function* $f \in \mathscr{A}(\mathbb{T}^n_s)$ *of* 0*-average such that* $\mathscr{L}_\alpha f = g$*, and there exists a* $C_0 = C_0(n, \tau)$ *such that, for any* $\sigma$*:*

$$|f|_s \le C_0 \gamma^{-1} \sigma^{-\tau-n} |g|_{s+\sigma}.$$

We will write $x = (K, G, \beta)$ and $\delta x = (\delta K, \delta G, \delta\beta)$. Fix $0 < s < s + \sigma < 1$.

***Lemma 5***. — *There exists* $C' > 0$ *which is locally uniform with respect to* $x \in E_s$ *in the neighborhood of* $G = \mathrm{id}$ *such that the linear map* $\phi'(x)$ *has an inverse* $\phi'(x)^{-1}$ *satisfying*

$$\left|\phi'(x)^{-1} \cdot \delta H\right|_s \le \sigma^{-\tau-n-1} C' \left|\delta H\right|_{G,s+\sigma},$$

*where we have set*

$$\left|\delta H\right|_{G,s+\sigma} := \left|\delta H \circ G^{-1}\right|_{s+\sigma}.$$

It is straightforward to check this lemma, and even more straightforward is the next one.

***Lemma 6***. — *There exists a constant* $C'' > 0$ *which is locally uniform with respect to* $x \in E_{s+\sigma}$ *in the neighborhood of* $G = \mathrm{id}$ *such that the bilinear map* $\phi''(x)$ *satisfies*

$$\left|\phi''(x) \cdot \delta x^{\otimes 2}\right|_{G,s} \le \sigma^{-1} C'' \left|\delta x\right|^2_{s+\sigma}.$$

It is now not difficult to believe that the equation $H = \phi(x)$ can be solved locally, following the remarkable idea of [48], by composing infinitely many times the Newton operator

$$f : x \mapsto x + \phi'(x)^{-1}(H - \phi(x)),$$

on extensions $\mathbb{T}^n_s$ of shrinking width $s$; the two lemmas above indeed allow us to control the convergence of the iterates: $\phi$ is a local diffeomorphism and this depends only on the conclusions of the two latter lemmata.

In order to prove Kolmogorov's theorem, it is crucial that the normal form, and in particular the frequency $\beta$, be unique and depend smoothly on the Hamiltonian. We refer to [40, Appendices A.1 and D] to see how interpolation theory can be used to obtain good uniqueness and smoothness estimates.

**1.3. Degeneracy and hypothetical conjugacy.** — Under the hypotheses of theorem 1, for each unperturbed Diophantine frequency there is a unique value of the parameter $R$ for which the conjugacy is untwisted ($\beta_R = 0$). In more degenerate cases, $R \mapsto \beta_R$ might not be a local diffeomorphism. Then $\alpha$ cannot be fixed arbitrarily among Diophantine vectors and the above argument requires some refinement.

Let $\mathscr{K}_s = \cup_{\alpha\in\mathbb{R}^n}\mathscr{K}_s^\alpha$ be the set of Hamiltonians on $\mathrm{T}_s^n$ for which $\mathrm{T}_0^n$ is invariant and quasi-periodic, yet with unprescribed frequency.

***Theorem 7* (Hypothetical conjugacy).** — *For every $K^o \in \mathscr{K}_{s+\sigma}^{\alpha^o}$ with $\alpha^o \in D_{\gamma,\tau}$, there is a (non unique) germ of $C^\infty$-map*[(5)]

$$\Theta : \mathscr{H}_{s+\sigma} \to \mathscr{K}_s \times \mathscr{G}_s, \quad H \mapsto (K_H, G_H), \quad K_H = cte + \alpha_H \cdot r + O(r^2),$$

*at $K^o \mapsto (K^o, \mathrm{id})$ such that for every $H$ the following implication holds:*

$$\alpha_H \in \mathscr{D}_{\gamma,\tau} \Longrightarrow H = K_H \circ G_H.$$

The pair $(K_H, G_H)$ can rightfully be called a *hypothetical conjugacy* of $H$ because the property $H = K_H \circ G_H$ depends on arithmetical conditions involving the unknown frequency $\alpha_H$.

*Proof.* — Denote $\phi_\alpha$ the operator we have been denoting $\phi$ –because the vector $\alpha$ was fixed while we now want to vary it. Define the map

$$\begin{array}{rlcl} \hat\Theta : & D_{\gamma,\tau} \times \mathscr{H}_{s+\sigma} & \to & \mathscr{K}_s \times \mathscr{G}_s \times \mathbb{R}^n \\ & (\alpha, H) & \mapsto & \hat\Theta_\alpha(H) := (\phi_\alpha)^{-1}(H) = (K, G, \beta) \end{array}$$

locally in the neighborhood of $(\alpha^o, K^o)$, $K^o \in \mathscr{K}_{s+\sigma}^{\alpha^o}$. Using Whitney's extension theorem, one can show the existence of a smooth extension

$$\hat\Theta : \mathbb{R}^n \times \mathscr{H}_{s+\sigma} \to \mathscr{K}_s \times \mathscr{G}_s \times \mathbb{R}^n.$$

Note that the directions where $\hat\Theta$ is extended have finite dimension; on the other hand, that the map $\hat\Theta$ takes values in a Banach space, as opposed to a space of finite dimension as in the initial version of Whitney, causes no difficulty.

Write $K^o = \alpha^o \cdot r + \hat K$, $\hat K = O(r^2)$. In particular, since

$$\phi_\alpha(K^o + (\alpha - \alpha^o) \cdot r, \mathrm{id}, \alpha^o - \alpha) \equiv K^o$$

locally for all $\alpha \in \mathbb{R}^n$ close to $\alpha^o$, we have

$$\hat\Theta(\alpha, K^o) = (K, \mathrm{id}, \beta), \quad \beta = \alpha^o - \alpha,$$

and

$$\frac{\partial\beta}{\partial\alpha} = -\,\mathrm{id}\,.$$

So, by the implicit function theorem, locally for all $H$ there exists a unique $\hat\alpha$ such that $\beta(\hat\alpha, H) = 0$. It then suffices to set $\Theta(H) = \hat\Theta(\hat\alpha, H)$. □

(5) Merci à Jean-Christophe Yoccoz pour sa remarque que $\Theta$ n'a ici aucune raison d'être une difféomorphisme local comme je l'affirmais.

Now assume that the perturbed Hamiltonian $H$ depends on some parameter $t \in \mathbb{B}^T$; if $H$ is close to some completely integrable Hamiltonian, $s$ may be the action coordinate $r$ and, in the case of Arnold's theorem, $s$ represents the semi major axes. By composition with $\Theta$, $H$ determines a non-unique frequency map $t \mapsto \alpha_t := \hat{\alpha}$, which is $C^\infty$-close to the frequency map $t \mapsto \alpha_t^o$ of the unperturbed family of Hamiltonian $K^o$.

In the case of Kolmogorov's theorem, this map is a local diffeomorphism and the set

$$\mathfrak{D}_{\gamma,\tau} := \{t,\ \alpha_t \in \mathscr{D}_{\gamma,\tau}\}$$

has positive measure because $\mathscr{D}_{\gamma,\tau}$ has. In general, the map is not a local diffeomorphism, but the set $\mathfrak{D}_{\gamma,\tau}$ might still have positive measure. This is decided by the following result of Diophantine approximation theory.

***Theorem 8* (Margulis, Pyartli [86])**. — *If some real analytic map $t \in \mathbb{B}^T \mapsto \alpha_s^o \in \mathbb{R}^p$ is non-planar in the sense that its image is nowhere locally contained in some proper vector subspace of $\mathbb{R}^p$, the Lebesgue measure of $\{t \in \mathbb{B}^T,\ \alpha_s^o \in D_{\gamma,\tau}\}$ is positive provided that $\gamma$ is small enough and $\tau$ large enough.*

This kind of non-degeneracy condition is generally attributed to H. Rüssmann, due to the influence of [89]. The two latter statements and the fact that being non-planar is an open property in the $C^k$-topology, with $k$ large enough with respect to the dimension $T$, imply the following invariant tori theorem.

***Theorem 9***. — *If the family $K_t^o \in \mathscr{K}_{t+\sigma}$, $t \in \mathbb{B}^T$, has a non-planar frequency map $t \mapsto \alpha_t^o$ and if $H_t$ is close to $K_t^o$, there exist $(K_t = \alpha_t \cdot r + \delta K_t, G_t) \in \mathscr{K}_t \times \mathscr{G}_t$ such that*

– *For every $t$ such that $\alpha_t \in D_{\gamma,\tau}$,*

$$H_t = K_t \circ G_t.$$

– *The set $\mathfrak{D}_{\gamma,\tau} = \{t \in \mathbb{B}^T,\ \alpha_t \in D_{\gamma,\tau}\}$ has positive Lebesgue measure.*

## 2. Quasi-Periodic Motions in the Planetary Problem

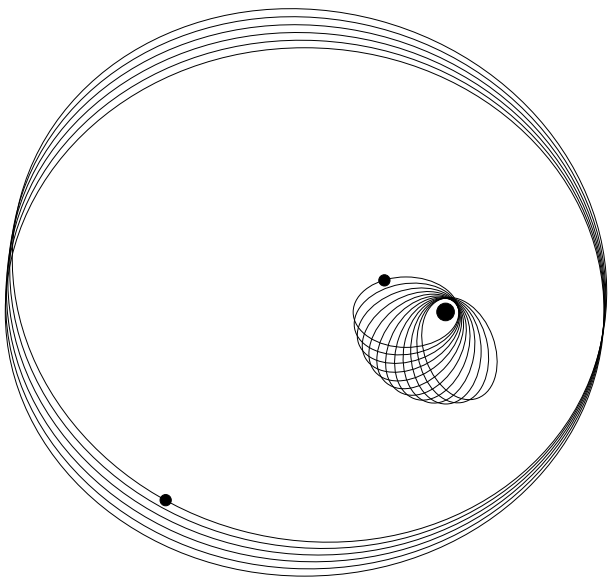
**2.1. Arnold's theorem.** — Consider $1+n$ point bodies with masses $m_0, \varepsilon m_1, ..., \varepsilon m_n > 0$ ($\varepsilon > 0$) and position vectors $x_0, x_1, ..., x_n \in \mathbb{R}^3$. Newton's equations for the planets take the form

$$\ddot{q}_j = m_0 \frac{q_0 - q_j}{||q_0 - q_j||^3} + \varepsilon \sum_{k \neq j} m_k \frac{q_k - q_j}{||q_k - q_j||^3} \quad (j = 1, ..., n).$$

They have a limit when $\varepsilon \to 0$, for which each planet (masses $\varepsilon m_j$) undergoes the only attraction of the sun (mass $m_0$). If their energies are negative, planets describe Keplerian ellipses with some given semi major axes and eccentricities. As a whole, the system is quasi-periodic with $n$ frequencies. In 1963, Arnold published the following remarkable result ([6]).

***Theorem 10***. — *For every $m_0, m_1, ..., m_n > 0$ and for every $a_1 > ... > a_n > 0$ there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, in the phase space in the neighborhood of circular and coplanar Keplerian motions with semi major axes $a_1, ..., a_n$, there is an invariant subset of positive Lebesgue measure consisting of quasi-periodic motions with $3n - 1$ frequencies.*

The proof of this theorem is rendered difficult by the multitudinous degeneracies of the planetary problem; the shift between the number of frequencies, $n$ in the Keplerian approximation and $3n - 1$ in the full problem, reveals one of them. Arnold's initial proof does not fully describe these degeneracies and actually misses one of them. Hence it is wrong in the case of $n \geq 3$ planets in space. In 1998, in a series of lectures M. Herman sketched a complete and more conceptual proof of this theorem, which he never published before his untimely death. We will now review a couple of ideas which make this proof so powerful and, I believe, elegant, referring to [39] for the details, and to [40], described in the previous section, for an improved proof of the abstract invariant torus theorem.



**2.2. The secular dynamics.** — It is not difficult to see that, after reduction by translations, the Hamiltonian is

$$F = \sum_{1 \leq j \leq n} \left( \frac{\|P_j\|^2}{2\mu_j} - \frac{\mu_j M_j}{\|Q_j\|} \right) + \varepsilon \sum_{1 \leq j < k \leq n} \left( -\frac{m_j m_k}{\|Q_j - Q_k\|} + \frac{P_j \cdot P_k}{m_0} \right),$$

where $Q_j$ is the position of the $j$-th planet relatively to the Sun, $P_j$ is its impulsion, and the reduced masses $\mu_j$ and $M_j$ are defined by

$$\frac{1}{\mu_j} = \frac{1}{m_0} + \frac{1}{\varepsilon m_j}, \quad M_j = m_0 + \varepsilon m_j.$$

Let $(\lambda_j, \Lambda_j, \xi_j, \eta_j, x_j, y_j)_{1\le j\le n}$ be the Poincaré coordinates (see appendix A). They are symplectic analytic coordinates in a neighborhood of circular horizontal Keplerian motions. For each planet, the coordinates $(\lambda_j, \Lambda_j)$ are angle-action coordinates of the reduced Kepler problem, so that

$$\frac{\|P_j\|^2}{2\mu_j} - \frac{\mu_j M_j}{\|Q_j\|} = -\frac{\mu_j^3 M_j^2}{2\Lambda_j^2}.$$

These Keplerian Hamiltonians define a *Keplerian action* of the $n$-torus. We will focus on tori which have non resonant, and even Diophantine, Keplerian frequencies. Roughly speaking, these are the initial conditions for which planets will not regularly find themselves in the same relative positions.

Up to an $\varepsilon$-deformation of the Poincaré coordinates, standard normal form theory (and lemma 4) asserts that we may assume that the Hamiltonian is replaced by

$$F = -\frac{\mu_j^3 M_j^2}{2\Lambda_j^2} - \varepsilon \sum_{j<k} m_j m_k \int_{\mathbb{T}^n} \frac{d\lambda_j\, d\lambda_k}{\|Q_j - Q_k\|} + R + O(\varepsilon^2),$$

where the infinite jet of the remainder $R$ vanishes along Keplerian motions whose frequencies are Diophantine (here we lose analyticity with respect to $\varepsilon$); the average of the "complementary part" $\sum P_j \cdot P_k$ indeed equals zero because, up to multiplicative constants, along Keplerian motions we have $P_j = \dot{q}_j = \partial Q_j / \partial \lambda_j$.

The Newtonian potential is degenerate because all its bounded orbits are periodic, as opposed to quasi-periodic on 3-tori, as symplectic geometry would permit in a $6n$-dimensional phase space. This is reflected in the fact that the Keplerian part depends on the $\Lambda_j$'s, and no other action variable. Thus the above averaging procedure allows us to eliminate the fast angles $\lambda_j$'s only. The averaged Hamiltonian

$$F_s = -\varepsilon \sum_{j<k} m_j m_k \int_{\mathbb{T}^n} \frac{d\lambda_j\, d\lambda_k}{\|Q_j - Q_k\|}$$

induces a Hamiltonian system on the space of $n$-uplets of Keplerian ellipses with fixed semi major axes. It is called *secular* (or *Lagrangian*, in [8]) because it describes the slow deformations of the Keplerian ellipses under the influence of the averaged mutual attraction of planets, visible only on the long term, say over one century (= secular in Latin). As Gauss described it, it is the gravitational potential of the system obtained by spreading the masses of the planets along their Keplerian ellipses, according to the third Kepler Law. Roughly speaking, the effect of the secular term is to make the Keplerian ellipses rotate in their planes and to make their planes themselves rotate around the total angular momentum.

The secular Hamiltonian is not integrable, but has an elliptic singularity at circular coplanar ellipses. Moreover Lagrange and Laplace have shown that its quadratic part splits in a remarkable way, which can be partly anticipated thanks to the symmetries [83]. The full computation is long; it is one of those in the subject which made Michael Herman say "BLC" (Bonjour Les Calculs)!

***Theorem 11* (Lagrange-Laplace).** — *Let $m = (m_1, ..., m_n)$, $a = (a_1, ..., a_n)$, $\xi = (\xi_1, ..., \xi_n)$, $\eta = (\eta_1, ..., \eta_n)$, $x = (x_1, ..., , x_n)$ and $y = (y_1, ..., y_n)$. There are two symmetric bilinear forms $Q_h = Q_h(m, a)$ and $Q_v = Q_v(m, a)$ on the tangent space at the origin of the secular space, respectively called* horizontal *and* vertical*, which depend on the masses and semi major axes analytically, and such that*

$$F_s = C_0(m, a) + Q_h \cdot \left(\xi^2 + \eta^2\right) + Q_v \cdot \left(x^2 + y^2\right) + O(4), \tag{1}$$

*with*

$$\left\{\begin{array}{lcl} Q_h \cdot \xi^2 & = & \displaystyle\sum_{1 \le j < k \le n} m_j m_k \left( C_1(a_j, a_k) \left( \frac{\xi_j^2}{\Lambda_j} + \frac{\xi_k^2}{\Lambda_k} \right) + 2C_2(a_j, a_k) \frac{\xi_j \xi_k}{\sqrt{\Lambda_j \Lambda_k}} \right) \\ Q_v \cdot x^2 & = & \displaystyle\sum_{1 \le j < k \le n} -m_j m_k C_1(a_j, a_k) \left( \frac{x_j}{\sqrt{\Lambda_j}} - \frac{y_k}{\sqrt{\Lambda_k}} \right)^2 \end{array}\right. \tag{2}$$

*and the $C_j$'s themselves are explicit linear combinations of the Laplace coefficients.*

The masses and semi major axes being fixed, let $\rho_h, \rho_v \in \mathrm{SO}(n)$ be diagonalizing transformations of $Q_h$ and $Q_v$ :

$$\rho_h^* Q_h = \sum_{1 \le j \le n} \sigma_j \, d\xi_j^2 \quad \text{and} \quad \rho_v^* Q_v = \sum_{1 \le j \le n} \varsigma_j \, dp_j^2, \quad \sigma_1, ..., \sigma_n, \varsigma_1, ..., \varsigma_n \in \mathbb{R}.$$

The map $\rho : (\xi, \eta, p, q) \mapsto (\rho_h \cdot \xi, \rho_h \cdot \eta, \rho_v \cdot p, \rho_v \cdot q)$ is symplectic and we have

$$F_s \circ \rho = C_0 + \sum_{1 \le j \le n} \sigma_j \left(\xi_j^2 + \eta_j^2\right) + \sum_{1 \le j \le n} \varsigma_j \left(p_j^2 + q_j^2\right).$$

In order to investigate the persistence of Lagrangian invariant tori in the neighborhood of the singularity, one only needs to switch to symplectic polar coordinates and apply results of section 1.3.

Let $\mathscr{A} = \{(a_1, ..., a_n) \in \mathbb{R}^n \; ; \quad 0 < a_n < a_{n-1} < ... < a_1\}$. The unperturbed frequency map is the multi-valued map

$$\alpha^o : a \in \mathscr{A} \mapsto \{\nu_1, ..., \nu_n, \sigma_1, ..., \sigma_n, \varsigma_1, ..., \varsigma_n\} \subset \mathbb{R},$$

where $\nu_1, ..., \nu_n$ are the Keplerian frequencies, and $\sigma_1, ..., \sigma_n$ and $\varsigma_1, ..., \varsigma_n$ the eigenvalues of the matrices $Q_h$ and $Q_v$. We are reduced to studying the arithmetic properties of $\alpha^o$, and in particular the measure of the inverse of Diophantine frequencies.

### 2.3. Herman's mysterious resonance. —

***Theorem 12* (M. Herman).** — *The frequency map $\alpha^o$ of the first order secular system, as a function of the semi major axes, has its image lying entirely in a plane $P$ of codimension 2. But its image lies in no plane of higher codimension.*

The theorem can be proved by induction on the number of planets and by complexifying the semi major axes. The first resonance is that one of the frequencies $\varsigma_j$'s is zero. It comes from the Galilean symmetry and disappears when fixing the direction of the angular momentum, e.g. vertically.

The second resonance is that the sum of all the secular frequencies is zero: $\sum(\sigma_j + \varsigma_j) = 0$. This is obvious to check from the above formulae –the quadratic part of $F_s$ is traceless. For two planets revolving around the sun, this means that the plane of each ellipse slowly rotates around the vertical axis in the negative direction, and that the ellipses rotate in their own planes in the positive direction *with the same frequency.* But for $n$ planets the resonance seems to have been unnoticed before. In all cases, it is mysterious, insofar as no symmetry seems to explain it. According to numerical evidence, for small values of $n$ it vanishes in the second order secular system; but one precisely wants to avoid checking this.

It turns out that Herman's resonance too disappears, namely in the system fully reduced by the symmetry of rotations. The difficulty is that the elliptic singularity is a critical level of the angular momentum. A key remark is that the fully reduced system (i.e. the system with fixed angular momentum, quotiented by rotations around the angular momentum) is non planar if and only if there is a rotating frame in which the partially reduced system (i.e. the system with vertical angular momentum) is non planar. But there is one, and actually infinitely many, such rotating frames, because the trace of the quadratic part of the angular momentum is non zero, as can easily be seen, again by an argument of analytic continuation. An alternative proof, using the Deprit coordinates, has recently been provided by [32], which went even further by proving the strong non-degeneracy of the secular system.

The $2n-1$ slow frequencies vanish when $\varepsilon = 0$. Hence, when $\varepsilon$ is small, there is a competition between choosing Diophantine conditions (1) good enough so that (as a quantitative version of the twisted conjugacy theorem shows) the local image of the operator $\phi$ at the secular system of some high enough order contains the full Hamiltonian of the planetary problem; (2) bad enough so that (as a quantitative version of the Arnold-Margulis-Pyartli theorem shows) the frequency map passes through such Diophantine vectors in positive measure in the space of semi major axes. It turns out that fixing $\tau$ large enough and choosing $\gamma$ as some power of $\varepsilon$ fits the bill. The above abstract theory applies to the reduced systems and yields a positive measure of invariant quasi-periodic Diophantine $(3n-2)$-tori (or, as a refinement shows, invariant normally elliptic tori of any dimensions between $n$ and $3n-2$), which lift to a positive measure of invariant quasi-periodic $(3n-1)$-tori of the full system (respectively, to invariant normally elliptic tori of dimensions between $n+1$ and $3n-1$).

## 3. From the Lagrange relative equilibrium to the Eight choreography

Consider three point masses in $\mathbb{R}^3$ of equal mass $m$, undergoing the Newtonian attraction:

$$\ddot{q}_j = m \sum_{k \neq j} \frac{q_k - q_j}{\|q_k - q_j\|^3}, \quad q_j \in \mathbb{R}^3, \quad j = 1, 2, 3.$$

In the absence of equilibria, a natural "breach" to look at is the neighborhood of *relative equilibria*, i.e. equilibria modulo the action of rotations. Focusing on the most symmetric

relative equilibrium, the so-called *Lagrange relative equilibrium*:

$$q_j^L(t) = \left(a \exp i \left(\omega t + j\frac{2\pi}{3}\right), 0\right) \in \mathbb{C} \times \mathbb{R}, \quad \sqrt{3}\omega^2 a^3 = m, \quad \omega = 2\pi,$$

we will describe the neighboring dynamics, determine the families of periodic orbits bifurcating from the relative equilibrium, and show that one of those families continues until the figure-eight solution of Chenciner-Montgomery.

**3.1. Symmetries.** — Newton's equations define a vector field in the 18-dimensional phase space $\{(q, \dot{q})\}$, but the symmetries allow us to decrease the number of dimensions significantly.

– The *linear momentum* $\sum_j \dot{q}_j \in \mathbb{R}^3$ being a first integral, the codimension-6 subspace of equations $\sum_j \dot{q}_j = 0$ and $\sum_j q_j = 0$ is invariant. Moreover we may restrict to this subspace without loss of generality due to the possibility of choosing a frame of reference whose origin is located at the center of mass.

– The *angular momentum* $\sum_j q_j \wedge \dot{q}_j \in \mathbb{R}^3$ is the other known first integral. Its level set containing the Lagrange relative equilibrium is a codimension-3 invariant submanifold, to which we may restrict, again, without loss of generality: the homogeneity of the Newtonian potential allows the rescaling of any motion with non-zero angular momentum to a motion whose angular momentum has a unit norm, and the rotational invariance allows the choice of an inertial frame of reference in which the (non-zero) angular momentum is vertical.

The vector field then descends to the quotient by rotations in the horizontal plane.

All reductions done, the phase space has $18 - 10 = 8$ dimensions, while the invariant horizontal sub-problem has $12 - 6 = 6$ dimensions. That the Galilean symmetry group has 10 dimensions is incidental here, inasmuch as we have not taken advantage of invariance by time shifts (though we will, later), while we have used the homogeneity of Newton's potential, a special property of our problem.

**3.2. Linear analysis.** — The first step towards understanding the local dynamics in the neighborhood of the equilibrium is to study the linearization of the vector field. References are [56, 63, 61]. Recall that for a linear Hamiltonian real vector field the eigenvalues come by (possibly degenerate) quadruples $\pm\lambda, \pm\bar{\lambda}$.

***Lemma 13***. — *The eigenvalues of the Lagrange equilibrium in the plane are:* $\pm i$, $\pm\lambda$ *and* $\pm\bar{\lambda}$, *with* $\lambda = i + \sqrt{2}$. *The two vertical eigenvalues are* $\pm i$.

*In restriction to the* 4*-dimensional central subspace, the Lagrange equilibrium is a non-degenerate minimum of energy.*

That the vertical eigenvalues are $\pm i$, the same as the ones corresponding to the homographic family, will be explained below. But there seems to be no obvious reason why the quadruple is resonant with the others: $\lambda = \bar{\lambda} + 2i$. Due to this resonance the normal

form of the vector field will not be integrable (yet its restriction to the central manifold will).

Thus locally the energy levels are 3-spheres. Each pair of complex conjugate purely imaginary eigenvalues gives rise to periodic first order solutions. Since the eigenvalue $i$ has multiplicity 2, the flow of the linearized vector field consists of two harmonic oscillators in $(1,1)$-resonance, i.e., on each 3-sphere of constant energy, the Hopf flow $S^3 \subset \mathbb{C}^2 \to S^3$, $(u,v) \mapsto (e^{it}u, e^{it}v)$.

It is thrilling to look at the vertical family of first order periodic solutions in the 12-dimensional phase space, since it is here that the 12th-order symmetry group of the (transcendent) Eight solution first appears [56]. The vertical variational equation of the Lagrange relative equilibrium being

$$\delta\ddot{q}_j = \frac{1}{3}\sum_{k\neq j}(\delta q_k - \delta q_j) = -\delta q_j,$$

a basis of solutions is formed by the two particular solutions

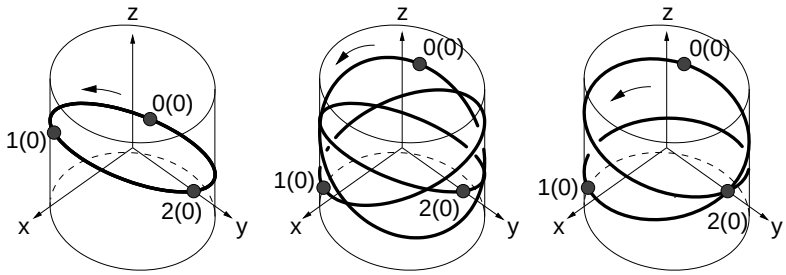
$$z^L(t) = \begin{pmatrix} \operatorname{Re}\zeta e^{it} \\ \operatorname{Re}\zeta^2 e^{it} \\ \operatorname{Re}\zeta^2 e^{it} \end{pmatrix} \quad \text{and} \quad z^P(t) = \begin{pmatrix} \operatorname{Re}\zeta e^{it} \\ \operatorname{Re}\zeta e^{it} \\ \operatorname{Re} e^{it} \end{pmatrix}, \quad \zeta = e^{i\frac{2\pi}{3}}.$$

A short computation shows that $z^L$ corresponds to the Lagrange relative equilibrium itself, after an infinitesimal rotation around the $y$-axis (first part of the figure below). Obviously, all solutions to the vertical variational equation have the same frequency $\omega = 1$. That some of these solutions correspond to infinitesimally rotated Lagrange relative equilibria now explains why the common frequency is the frequency of the Lagrange relative equilibrium.

On the other hand, $z^P$ corresponds to a new family. Each body still describes a circle in a sloping plane, but now the planes don't match: they are obtained from each other by rotations of $2\pi/3$ (second part of the figure below). Now, look at the first order solution $q = q^L \oplus z^P$, rightfully living on a cylinder above the basis $q^L$, in a frame rotating with angular velocity $-\omega$, i.e. making exactly one retrograde rotation per period; each body is then seen to make $+2$ horizontal rotations and one vertical oscillation per period. The corresponding expression is

$$\tilde{q}_j(t) = \left(a \exp i\left(2\omega t + j\frac{2\pi}{3}\right), z_j^P(t)\right).$$

In this frame the first order solution turns out to be *choreographic* (third part of the figure below).




The choreographic symmetry is only the most spectacular symmetry in the 12-th order group

$$\Gamma_1 = \langle s, \sigma | s^6 = 1,\ \sigma^2 = 1,\ s\sigma = \sigma s^{-1} \rangle$$

whose action is defined by

$$sq(t) = \begin{pmatrix} \Sigma q_1(t-\pi/3) \\ \Sigma q_2(t-\pi/3) \\ \Sigma q_0(t-\pi/3) \end{pmatrix} \quad \text{and} \quad \sigma q(t) = \begin{pmatrix} \Delta q_1(-t) \\ \Delta q_0(-t) \\ \Delta q_2(-t) \end{pmatrix},$$

where $\Sigma$ and $\Delta$ are the orthogonal symmetries about the horizontal plane and about $Oy$. (Because we keep the group action of [25], it is the path $\tilde{q}(\cdot + \pi/2)$ which is $\Gamma_1$-symmetric.)

**3.3. Local continuation.** — After quotient by horizontal rotations, the relative equilibrium becomes a true equilibrium. We are concerned with the old problem of finding periodic solutions in its neighborhood. It is well known that the two horizontal eigenvalues $\pm i$ give rise to the homographic family of periodic orbits. We want to show that there is exactly one additional family corresponding to the vertical eigenvalue $i$. The Lyapunov center theorem is the first general result in this direction.

***Theorem 14*** **(Lyapunov center theorem)**. — *Let $H$ be a Hamiltonian on $\mathbb{R}^{2n}$ with an equilibrium point at the origin, and $L$ be the linearization of the Hamiltonian vector field at the origin. If $L$ has some purely imaginary eigenvalues $i\omega_1, \cdots, i\omega_k$ (possibly in addition to non purely imaginary eigenvalues) such that*

$$\frac{\omega_j}{\omega_1} \notin \mathbb{N} \quad \forall j = 1, \cdots, k,$$

*$H$ has a one-parameter family of periodic orbits with period close to $\frac{2\pi}{\omega_1}$.*

The theorem does not apply, due to the fact that the eigenvalue $i$ has multiplicity 2. Yet Weinstein remarkably proved that the non-resonance condition is not necessary (and Moser further generalized the result, showing that the Hamiltonian character of the equations itself is not necessary):

***Theorem 15*** **(Weinstein-Moser center theorem, [70])**. — *Let $H$ be a Hamiltonian on $\mathbb{R}^{2n}$ with an equilibrium point at the origin, and $L$ be the linearization of the Hamiltonian vector field at the origin. Assume that $\mathbb{R}^{2n} = E \oplus F$, where $E$ and $F$ are invariant subspaces of $L$ such that all solutions of $L$ lying in $E$ have some common period $T > 0$, while no nontrivial solution in $F$ has this period. Moreover, assume that the Hessian $D^2H(0)$ restricted to $E$ is positive definite. Then, for sufficiently small $\varepsilon$,*

*on each energy surface $H(z) = H(0) + \varepsilon^2$ the number of periodic orbits of $H$ is at least $\frac{1}{2} dim\, E$.*

This theorem applies, although the convexity condition requires some proof (interestingly, the convexity property generalizes to the regular $N$-gon for all $N$; see next section). However, this does not say if the second solution resembles the first order solution $q^L \oplus z^P$, nor if it is unique for that matter. Yet uniqueness is essential for proving that the periodic orbits belonging to the new family share the symmetries of $q^L \oplus z^P$. As some simple counter-examples show ([70]), this cannot be decided from the linearized vector field, but from the third order normal form:

$$\begin{aligned}
\dot{u} &= iu[1 + \alpha|u|^2 + \beta|v|^2 + \gamma hk + \bar{\gamma}\bar{h}\bar{k}] + O_5 \\
\dot{v} &= iv[1 + a|u|^2 + b|v|^2 + chk + \bar{c}\bar{h}\bar{k}] + A\bar{v}h\bar{k} + O_5 \\
\dot{h} &= \lambda h[1 + r|u|^2 + s|v|^2 + thk + t'\bar{h}\bar{k}] + Rv^2\bar{h} + O_5 \\
\dot{k} &= -\lambda k[1 + r|u|^2 + s|v|^2 + thk + t'\bar{h}\bar{k}] - R\bar{v}^2\bar{k} + O_5,
\end{aligned}$$

where the new coordinates $u$, $v$, $h$ and $k$ are complex (see below), where

$$\alpha = -1,\ \beta = -1, \gamma = \frac{9}{2} + 6i\sqrt{2}, a = -1,\ b = -\frac{21}{19},\ \cdots$$

and where $O_5$ stands for real analytic functions of order 5 in $u, \bar{u}, v, \bar{v}, h, \bar{h}, k, \bar{k}$. The coordinates can be chosen so that the involution $v \mapsto -v$ is a symmetry about the horizontal subspace and the normalization transformation, and hence the normal form, are invariant by this involution. The normal form calls for a few comments:

- For instance the terms $A\bar{v}h\bar{k}, Rv^2\bar{h}$ and $-R\bar{v}^2\bar{k}$ correspond respectively to the resonances $i = -i + \lambda - \bar{\lambda}, \lambda = 2i + \bar{\lambda}$ and $-\lambda = -2i - \bar{\lambda}$.
- The symmetry under $\mathscr{T}$ accounts for the absence of some resonant monomials, e.g. $|u|^2 v$ in $\dot{u}$, or $u|v|^2$ and $\bar{u}v^2$ in $\dot{v}$.
- It remains unclear why the normal form is also invariant with respect to $u \mapsto -u$ (for instance $\dot{u}$ has no term in $u^2\bar{v}$ or $\bar{u}h\bar{k}$); this symmetry holds at order five.
- The equality $a = \alpha$ has an intrinsic meaning and [25, Appendix] explains why this resonance actually persists in normal forms of every order.

Since there are no obstructing cross-resonances from the central directions (coordinates $u, v$) to the hyperbolic directions (coordinates $h, k$), the central space $h = k = 0$ is invariant for the formal infinite-order normal form of the vector field. This holds holomorphically, as proof analogous to [34] shows, and further local analysis proves:

***Proposition 16***. — *There exists a unique symplectic analytic central manifold $\mathscr{C}$, in restriction to which the vector field and the Hamiltonian are*

$$\begin{cases} \dot{u} = iu(1 + \alpha|u|^2 + \beta|v|^2) + O_5 \\ \dot{v} = iv(1 + a|u|^2 + b|v|^2) + O_5 \end{cases} \quad \textit{and} \quad H = -\frac{1}{6} + \frac{|u|^2}{9} + \frac{|v|^2}{9} + O_4.$$

That the Hamiltonian has a local minimum in restriction to $\mathscr{C}$, will generalize to regular $N$-gons, interestingly (see next section). The central manifold contains all the recurrence, and in particular families of periodic orbits.

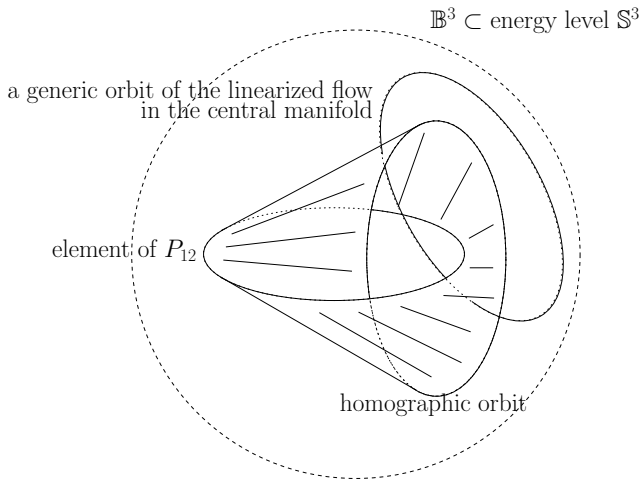
***Theorem 17*** **([25]; see also [56])**. — *Up to the action of similarities and time shifts, exactly two families of relatively periodic solutions bifurcate from the Lagrange relative equilibrium:*

*– the well known horizontal, (absolutely) periodic, homographic family $\mathscr{H}$,*

*– and a vertical family called $\mathscr{P}_{12}$, whose solutions show a 12th-order symmetry which makes them choreographic (i.e. the three bodies chase each other along the same curve) in the frame rotating with some frequency close to $-\omega$.*

The part on the $\mathscr{P}_{12}$ family follows from the analysis of the normal form of the vector field restricted to $\mathscr{C}$ and its complex blow-up, while the local uniqueness of the homographic family, due to the resonance $a = \alpha$ explained in the appendix of [25], follows from the careful analysis of the local dynamics, which boils down to a well known situation:

***Theorem 18*** **([25])**. — *In restriction to the 4-dimensional analytic central manifold $\mathscr{C}$, each energy level is a 3-sphere which possesses a global Poincaré section $\mathscr{A}$ having the topology of a closed annulus and whose two boundary circles are elements of $\mathscr{H}$ and $\mathscr{P}_{12}$ respectively. Moreover, the second return map of $\mathscr{A}$ is twist and monotone.*




**3.4. Global continuation.** — Just as in the two-body problem, the homographic family extends from the Lagrange relative equilibrium to (and past) the homothetic motion of total collision (each body indeed undergoing a two-body-problem motion). But where does the $\mathscr{P}_{12}$ family end?

Christian Marchal had extensively studied the 12th-order symmetry of the $\mathscr{P}_{12}$ family [56]. In 1999, when he heard about the choreographic figure-eight solution of Chenciner-Montgomery, at once he imagined that the Eight could be the unknown end of $\mathscr{P}_{12}$.

***Partially numerical theorem 19*** **([27, 23])**. — *The $\mathscr{P}_{12}$ family interpolates between the Lagrange equilibrium and the Eight. When looked at in a frame rotating with frequency varying in the interval $[-\omega, 0]$, the elements of $\mathscr{P}_{12}$ minimize the Lagrangian action within the class of $\Gamma_1$-symmetric loops.*

What 'to interpolate' means is ambiguous, because the minimization argument (as used) does not show the continuity of the $\mathscr{P}_{12}$ family, nor even its uniqueness for that matter. In this respect, the existing proof relies on a numerical experiment. However some theoretical arguments strongly speak in favor of the result.

Consider the following family of loops in the configuration space:

$$q_j^{\varpi}(t) = \left( \frac{1}{\sqrt{3}} \left( \frac{4\pi + \varpi}{2\pi} \right)^{-2/3} \exp i \left[ (4\pi + \varpi)\, t + j \frac{2\pi}{3} \right], 0 \right), \quad j \in \mathbb{Z}/3\mathbb{Z},$$

parametrized by $\varpi$. They are solutions of Newton's equation, obtained from each other by mere rescaling. In a frame rotating uniformly with angular velocity $\varpi$, each member of the family becomes a $\Gamma_1$-symmetric, periodic orbit, where each body makes two full rotations per period, around the origin. A straightforward computation shows that its action

$$\mathscr{A}_{\varpi} := \int_0^1 L(q^{\varpi}(t), \dot{q}^{\varpi}(t))\, dt,$$

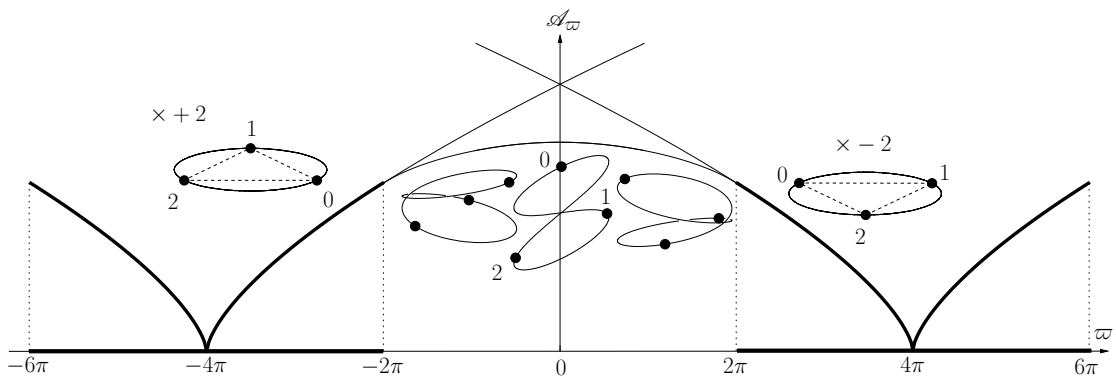
where

$$L(q, v) = \sum_j \frac{1}{2} m_j \|v_j\|^2 + \sum_{j<k} \frac{m_j m_k}{\|q_j - q_k\|}$$

is the Lagrangian, equals

$$\mathscr{A}(q^{\varpi}) = \left( \frac{4\pi + \varpi}{2\pi} \right).$$

In particular, its absolute minimum 0 is attained for $\varpi = -4\pi$, where the three bodies are at rest at infinity. It increases with $\varpi \in [-4\pi, 0]$. It stops being a minimum among loops which are $\Gamma_1$-symmetric in the rotating frame, precisely when the variational equation comes to have a $\Gamma_1$-symmetric, 1-periodic solution in the rotating frame; this solution can indeed be thought of as a Jacobi field. This happens exactly once, when $\varpi = -2\pi$. When $\varpi$ is slightly larger than $-2\pi$, $q^{\varpi}$ is not a minimum anymore, for the corresponding member of the $\mathscr{P}_{12}$ family has a lower action. The global continuation consists, for each value of $\varpi \in [-2\pi, 0]$, in looking for the minimizer among paths which are 1-periodic and $\Gamma_1$-symmetric in the frame rotating with frequency $\varpi$. At the other end, i.e. $\varpi = 0$, the minimum is the figure Eight solution, for which the 12th-order symmetry is the symmetry of the space of similarity classes of plane oriented triangles ([28, 62]).



The thick line on the figure indicates where the Lagrange relative equilibrium is the absolute minimizer [10, 26].

The proof of existence of physically relevant $\Gamma_1$-symmetric minimizers requires to show (by a direct comparison with a well chosen non-collision path) that the minimizers have no collisions [21]. Marchal's theorem ([57, 20, 42]) does not show this directly because the time-reversal symmetry of $\Gamma_1$ prevents us from choosing a fundamental domain of time with arbitrary boundary.

If the two Lyapunov families of the equal-mass Lagrange relative equilibrium are thus quite well understood, there remain many open questions. One of them would be to determine the global advent of the central manifold $\mathscr{C}$, which interpolates between the two families [41], and which strongly constrains the recurrent part of the dynamics.

## 4. Lyapunov families bifurcating from relative equilibria

What are the simplest possible motions for $n$ point bodies in $\mathbb{R}^3$ undergoing Newton's attraction?

– The *homographic solutions* and, among them, the *relative equilibria* (= equilibria modulo translations and rotations). Configurations admitting such motions, the so-called *central configurations*, are very particular and difficult to find. But given such a central configuration, the corresponding relative equilibria are simply obtained by rotating the bodies at constant velocity. Examples of relative equilibria for $n = 3$ bodies comprise the Lagrange solution, where the three bodies form an equilateral triangle at each time, which rotates uniformly around the center of mass. When the three bodies have the same mass, they describe the same circle; such a periodic solution where the bodies chase each other along some common curve at fixed time distances, has been called a *choreography* by C. Simó. This example generalizes obviously to $n$ equal masses located at the vertices of a regular $n$-gon.

– *Action minimizing periodic orbits.* Among critical points of the Lagrangian action

$$\int_0^T L(q, \dot{q})\, dt,$$

minima are the simplest. Many recently discovered symmetric periodic orbits minimize the Lagrangian action within their class of symmetry, e.g. the celebrated $\mathbb{D}_6$-symmetric figure-Eight of Chenciner-Montgomery which we have encountered in the previous section, the $\mathbb{D}_4 \times \mathbb{Z}_2$-symmetric Hip-Hop [29], the $\mathbb{D}_{10}$-symmetric 5-body Eight, or the $\mathbb{D}_{10}$-symmetric 5-body 4-loop chain.

We will see that these two types of solutions are closely related. A connection similar to the one between the Lagrange relative equilibrium and the Eight exists in many cases. Indeed when a relative equilibrium has an elliptic normal direction, infinitesimally there is a 1-parameter family of 2-frequency quasi-periodic solutions bifurcating from the relative equilibrium. In good cases, these infinitesimal solutions can be continued locally.

It turns out that all of the minimizing solutions mentioned above are resonant (hence absolutely periodic) members of such families.

This simple continuation principle happens to be embodied in a variety of ways, and simultaneously to explain the structure of a whole zoology of periodic orbits in the many-body problem. Numerical evidence is spectacular, but the proofs are difficult and reach various degrees of achievement.

**4.1. Infinitesimal vertical variations and their symmetries.** — The first step is to find a relatively periodic solution to the variational equation of the relative equilibrium and determine its symmetry group. There are several reasons for focusing on the *vertical* variational equation: simplicity (the vertical part is time-independent), spectrum (one needs at least 6 bodies for some normal horizontal directions to be spectrally stable), symmetry (usually the symmetry group of horizontal variations is trivial).

Consider a solution of Newton's equations:

$$\ddot{q}_j = \sum_{k \neq j} m_k \frac{q_k - q_j}{\|q_k - q_j\|^3}, \quad j = 1, ..., n.$$

Infinitesimally close, solutions satisfy the so-called variational equations,

$$\delta\ddot{q}_j = \sum_{k \neq j} m_k \left( \frac{\delta q_k - \delta q_j}{\|q_k - q_j\|^2} - 3\frac{(q_k - q_j) \cdot (\delta q_k - \delta q_j)}{\|q_k - q_j\|^5}(q_k - q_j) \right)$$

By identifying $\delta q_j \in \mathbb{R}^3$ to $(h_j, z_j) \in \mathbb{C} \times \mathbb{R}$, and reordering components, these equations split into horizontal and vertical variational equations:

$$\begin{pmatrix} \ddot{h} \\ \ddot{z} \end{pmatrix} = \left( \begin{array}{c|c} HVE & 0 \\ \hline 0 & VVE \end{array} \right) \begin{pmatrix} h \\ z \end{pmatrix};$$

in the endomorphism matrix, the zero of the first line comes from Pythagoras' theorem (infinitesimal vertical variations keep mutual distances constant) and the zero on the second line comes from the invariance of the horizontal problem. The vertical part

$$\ddot{z} = Wz$$

defines the *Wintner-Conley* operator $W$, which is self-adjoint for the mass scalar product, and negative definite after quotient by translations [104, 4].

In order to be more specific, we now restrict to the case of the regular $N$-gon. After reduction by translations, the phase space of the vertical variational equation has dimension $2(N-1)$. Denote by $\lambda_k = -\omega_k^2$, $k = 1, ..., [N/2]$ its eigenvalues. The corresponding eigenspaces have dimension 4 (or 2 if $N$ is even and $k = N/2$). Solutions are of the following form:

$$q_j(t) = (h_j(t), z_j(t)) = \left( \exp i \left( \frac{2\pi}{N} + \omega_1 t \right), \cos \left( \pm \frac{2\pi jk}{N} + \omega_k t \right) \right);$$

the horizontal part is the relative equilibrium (horizontal rotation with frequency $\omega_1$), while the vertical part consists in vertical oscillations of frequency $\omega_k$. Note that the frequency $\omega_1$ of the relative equilibrium is one of the vertical frequencies, since obviously

an infinitesimal rotation of the relative equilibrium around a horizontal axis induces a vertical oscillation of frequency $\omega_1$.

Among quasi-periodic motions with 2 frequencies, these infinitesimal motions are of a very particular kind, since they become periodic in a rotating frame. In a frame rotating horizontally with angular velocity $\varpi$, the above infinitesimal solution becomes

$$q_j^{\varpi}(t) = (h_j(t), z_j(t)) = \left(\exp i\left(\frac{2\pi}{N} + (\omega_1 - \varpi)t\right), \cos\left(\pm\frac{2\pi jk}{N} + \omega_k t\right)\right);$$

In particular, it is $2\pi s/\omega_k$-periodic ($s \in \mathbb{N}_*$) if and only if

$$\frac{\omega_1 - \varpi}{\omega_k} = \frac{r}{s} \in \mathbb{Q}$$

for some integer $r$. The rational number $r/s$ measures the number of horizontal rotations versus the number of vertical oscillations. We will call $S_{r/s}(N, k, \pm)$ the obtained path.

These infinitesimal variations look simple, and as a consequence they have rich symmetry groups, which can be computed as follows. Assuming the path $q^{\varpi}$ is $s$-periodic in the rotating frame, $q^{\varpi}$ can be thought of as a map

$$q^{\varpi} : \mathbb{R}/s\mathbb{Z} \times \{1, ..., n\} \to \mathbb{R}^3.$$

Such paths are being acted on by the group $O_2 \times \mathfrak{S}_n \times O_3$, where $O_d$ is the orthogonal group of $\mathbb{R}^d$ and $\mathfrak{S}_n$ is the permutation group, in the obvious way:

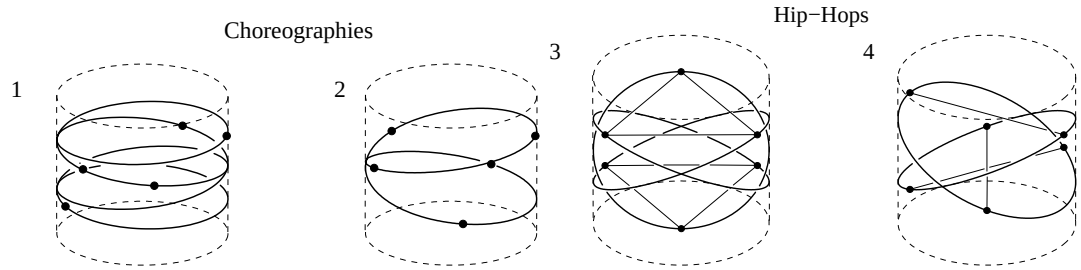
$$\begin{array}{ccccc}
\mathbb{R}/s\mathbb{Z} & \times & \{1, ..., n\} & \xrightarrow{q^{\varpi}} & \mathbb{R}^3 \\
\downarrow{\scriptstyle O_2} & & \downarrow{\scriptstyle \mathfrak{S}_n} & & \downarrow{\scriptstyle O_3} \\
\mathbb{R}/s\mathbb{Z} & \times & \{1, ..., n\} & \longrightarrow & \mathbb{R}^3.
\end{array}$$

Call $G_{r/s}(n, k, \pm) \subset O_2 \times \mathfrak{S}_n \times O_3$ the stabilizer of the above infinitesimal vertical variation. Note that for different integers $r$ and $s$, these groups are isomorphic; only their representation varies. The group structure of $G_{r/s}(n, k, \pm)$ can be described in terms of group extensions which depend on the arithmetic of $n$, $k$ and $s$. Yet one case is simple: for $s = 1$, $G_r(n, k, \pm)$ is isomorphic to $\mathbb{D}_n \times \mathbb{Z}/n\mathbb{Z}$.

More important than the group structure is the presence of some special elements:

– the choreography symmetry: after some fraction of the period, bodies have exchanged relative positions among one another, with one or more cycles,

– or the Hip-Hop symmetry: at all times, some of the bodies share the same position up to some orthogonal transformation of $\mathbb{R}^3$.




1. $S_4(5,1,-1)$ Five-body maximal unchained polygon
2. $S_2(5,2,-1)$ Five-body Eight
3. $S_1(6,1,-1)$ Six-body Hip-Hop
4. $S_2(6,2,1)$ Yet another six-body Hip-Hop.

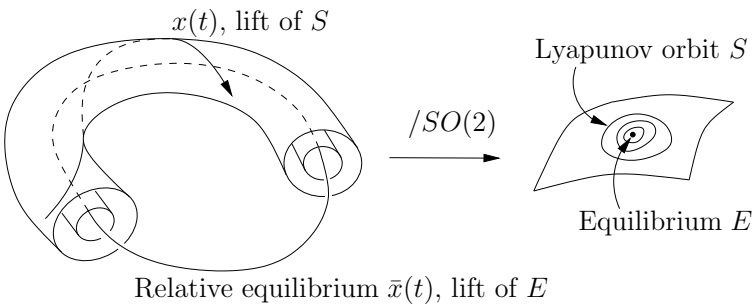
The following examples play a major role in the sequel:

– the *maximal chain* $S_{n-1}(2n+1,1,-1)$, a choreography, where the bodies make $n-1$ horizontal rotations per vertical oscillation,

– the *Eight* $S_2(2n+1,n,-1)$, also a choreography with an odd number of bodies, where the bodies make 2 horizontal rotations per vertical oscillation,

– and the Hip-Hop $S_1(2n,n,1)$.

**4.2. Local continuation.** — One may hope that the infinitesimal variations of the relative equilibrium, described in the previous section, extend to local families of periodic solutions.




As a result of resonances, whether or not they are expected, this step generally does not follow from the Lyapunov center theorem 14. The other related general theorem which might apply is the Weinstein-Moser theorem 15. It requires the existence of a positive definite first integral in restriction to the correspongin generalized eigenspace.

Interestingly, this hypothesis holds in general for the regular $N$-gon, provided there is no unexpected resonance between the horizontal and vertical eigenvalues:

Let $\mathscr{V}_\ell$ be the vertical eigenspace of the frequency $\omega_\ell$ ($1 \le \ell \le N/2$), and $\mathscr{H}_1$ be the plane tangent to the homographic motions. (Recall that the total eigenspace of $\omega_1$ contains $\mathscr{V}_1 \oplus \mathscr{H}_1$.) If no other frequency, horizontal or vertical, is an integer multiple of (possibly equal to) $\omega_\ell$, in order to apply the Weinstein-Moser theorem it is sufficient to prove that

the energy is convex on $\mathscr{V}_1 \oplus \mathscr{H}_1$ for $\ell = 1$ and on $\mathscr{V}_\ell$ for $2 \le \ell \le N/2$. Below we will prove the fact that the quadratic part $H$ is positive definite on the whole vector space

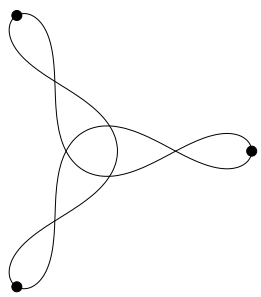
$$\mathscr{F} = \mathscr{H}_1 \bigoplus \oplus_{1 \le \ell \le N/2} \mathscr{V}_\ell.$$

All the first cases, studied in the following section, satisfy the required non-resonance condition. Furthermore, numerical experiment suggests that the purely imaginary horizontal eigenvalues, in general, cannot resonate with the vertical eigenvalue $i\omega_n$, at least, for they are smaller in module. When this is true, the proposition below and the Weinstein-Moser theorem show in a weak sense the local existence of Lyapunov families associated with $\omega_n$, in particular.

***Proposition 20***. — *After reduction by rotations, the restriction of the quadratic part of the energy to $\mathscr{F}$ is definite positive.*

However this theorem does not tell us uniqueness, nor does it give any hint about the shape of the families in the case of higher multiplicity. The most instructive is to look to the first few cases, which are all different from each other. Here we restrict to $N = 3$ or 4.

– The case of $N = 3$ bodies was fully analyzed in section 3 ([25]). Yet we illustrate the content of this section by showing a homographic solution in a choreographic frame:



– With four bodies, the horizontal eigenvalues are

$$\pm i\omega_1, \quad \simeq (\pm 0.8595325038 \pm i)\, \omega_1, \quad \simeq \big( \pm 0.6394812009 \pm 0.9533814590\, i \big)\, \omega_1,$$

and the vertical ones

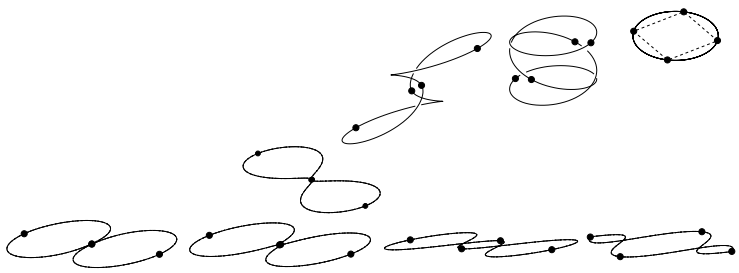
$$\pm i\omega_1, \quad \pm i\omega_2 = \pm i \sqrt[4]{2}\omega_1.$$

There are now two non-trivial vertical Lyapunov families.

The Weinstein-Moser theorem applied to the reduced vector field and to the eigenvalue $\pm i\omega_1$ (using proposition 20 and the fact that $\omega_1/\omega_2 = 2^{-1/4} \notin \mathbb{N}$) implies the existence of at least one more Lyapunov family corresponding to this frequency in addition to the horizontal homographic family.

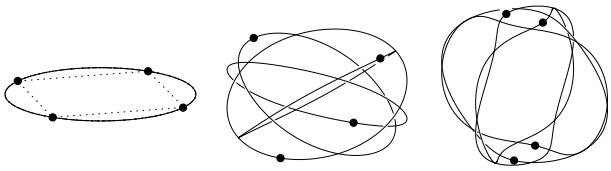
Numerical computations show that the Lyapunov cylinder $S(4, 1, -1)$ is tangent to a family which is choreographic in a rotating frame which starts making two full turns per period in the negative direction; this is a *4-body, 3-lobe chain family ($S(4, 1, -1)$)*. This

does not follow from the Lyapunov theorem, for the frequency $\omega_1$ is also the frequency of the horizontal homographic family.



Gerver's choreography

Since $\omega_2$ has multiplicity one and since the only other frequency, $\omega_1$, is not an integer multiple of $\omega_2$, it follows from Lyapunov's theorem that there exists a unique family of relatively periodic orbits bifurcating from the square with vertical frequency close to $\omega_2$. The eigenmode $S_1(4, 2, \pm 1)$ is tangent to the family and, by uniqueness, the family shares its symmetry. Hence the family is the 4-body Hip-Hop family, with symmetry group $G_1(4, 2, 1) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. It is studied in [9]. In the figure below, the middle orbit is the original Hip-Hop [29].



**4.3. Global continuation.** — We are interested in the following two questions:

– Existence: Does the range of frequency rotation $\varpi$ of the frame over which the family exists contain 0, so that the family contains an absolutely periodic orbit sharing its symmetries?

– Uniqueness: Can one take the frequency $\varpi$ as a monotonous continuous parameter over the whole family, i.e. has the torsion constant sign?

Minimization of the Lagrangian action under the $G_{r/s}(N, k, \eta)$-symmetry is a natural tool for the existence question. Now the richness of the symmetry group is not only a matter of curiosity, but a crucial property to rule out other to-be minimizers completely elsewhere in the phase space.

There are few known results on the absolute minimizing properties of relative equilibria. However, the one below is quite an incentive. Generalizing a remarkable argument first used for choreographies [10], it shows that the first members of some Lyapunov families bifurcating from regular polygons are absolute minimizers within their symmetry class. Let

$$V = \inf_{p>0} \frac{\omega_1}{\omega_{(1+2p)k\eta}} |(1 + 2p)2\pi|.$$

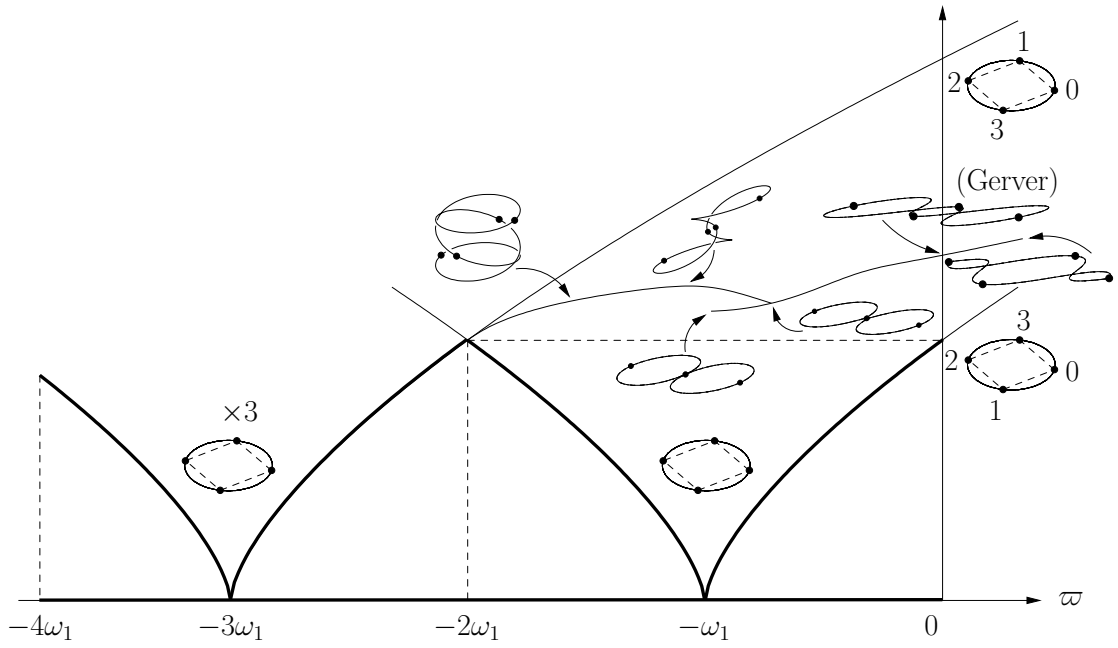
***Theorem 21*** **([26, p. 47])**. — *The following condition implies that the relative equilibrium solution of the equal mass regular $N$-gon with frequency $2\pi\frac{r}{s}$ in a frame rotating with frequency $\varpi$ is the sole absolute minimizer of the action among paths which in the rotating frame are $s$-periodic loops with the $G_{r/s}(N,n,\eta)$-symmetry:*

$$-V \leq \varpi + 2\pi\frac{r}{s} \leq V.$$

In the program, there are many additional difficulties, logically greater than for the local continuation:

1. Minimizers might contain collisions, due to the fact that some symmetries (in particular time-reversal symmetries) may prohibit the use of Marchal-type arguments.
2. In [26, Appendix], the local torsion was checked for all families bifurcating from regular polygons with 6 or fewer bodies. But there is no reason in general for the velocity of the frame of reference to vary monotonically along the family.
3. It turns out that, because of isomorphisms between the actions of different groups $G_{r/s}(N,k,\eta)$, its use is essentially restricted to the case $k = n$, i.e. to the largest vertical frequency.
4. As in general minimization problems, the minimum is not always unique, and hence does not vary continuously with respect to parameters...

Despite all this, numerical evidence does bring back optimism to the general picture. We will limit ourselves, on the next page, to the minimizations diagrams for 4-bodies, analogous to that of section 3 for 3-bodies.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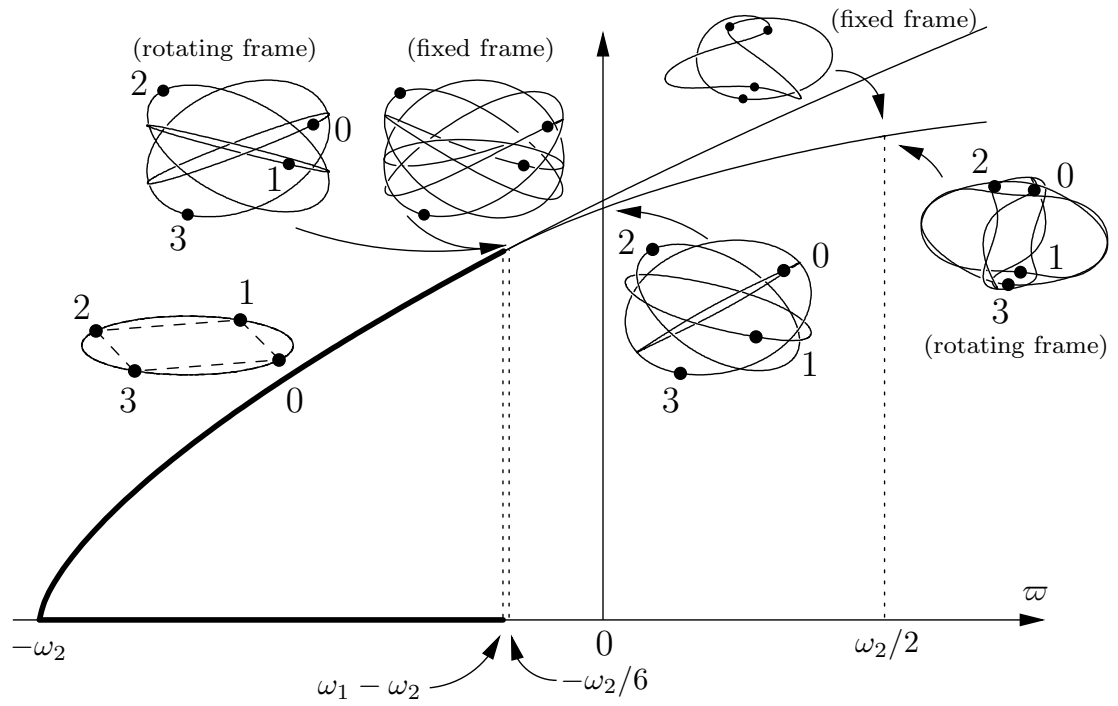
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# A

## The Delaunay and Poincaré coordinates

In this appendix we introduce the (transcendent, symplectic, analytic) Delaunay and Poincaré coordinates, both gateways to celestial mechanics. The Delaunay coordinates are angle-action coordinates of the Kepler problem for negative energies, which blow up circular and horizontal motions; the Poincaré coordinates are analytic in a large neighborhood of prograde circular Keplerian motions (the neighborhood actually includes the whole phase space of negative energy, with the exception of retrograde circular Keplerian motions).

A direct construction of the Delaunay coordinates seems to be hard to find in the literature. In many places, the construction is skipped (see [8], yet a remarkable reference in celestial mechanics) or involves tedious, unnecessary computations (see [2]; the first edition even wrongly proved that the coordinates are not symplectic). One possible strategy is to solve the Hamilton-Jacobi equation by separation of variables ([102, Chap. VII]; the proof is also explained in [83, 101, 19]). Our method is more direct. It requires almost no computation, provided one is familiar with the non symplectic aspects of the Kepler problem (exercise below). It is close to that of Moser [72] and has greatly benefited from discussions with A. Albouy.

**1. Prerequisite.** — Consider the *Keplerian Hamiltonian*, for a point body of position $q \in \mathbb{R}^d_*$, impulsion $p \in \mathbb{R}^d$ and mass $\mu$, attracted by a fixed mass $M$ at the origin:

$$K = \frac{\|p\|^2}{2\mu} - \frac{\mu M}{\|q\|},$$

**Exercise 22** (The Kepler problem)

1. Show that each Keplerian motion lies in a vector plane (use the invariance of $K$ by the symmetry about the line or plane spanned by the initial position and velocity).
2. From now on, restricting to motions taking place in some given $\mathbb{R}^2 = \{q = (x, y)\}$, show that $C = \det(q, \dot{q})$ is a first integral; $\mu C$ is called the *angular momentum*.
3. Show that
$$\ddot{r} = \frac{C^2}{r^3} - \frac{M}{r^2}, \quad r := \sqrt{x^2 + y^2}.$$
4. (Argumentum egregium of Lagrange, [50, Section deuxième]) Fix a solution $(x^o(t), y^o(t))$ and the corresponding $r^o(t)$. Note that $r^o(t)$ and the circular motion $r(t) \equiv C^2$ are particular solutions of the linear, time-dependent equation
$$\ddot{r} = \frac{C^2}{r^o(t)^3} - \frac{Mr}{r^o(t)^3},$$
and that $x^o$ and $y^o$ are solutions of the homogeneous part of the same equation:
$$\ddot{z} = -\frac{Mz}{r^o(t)^3}.$$

Also note that the space of solutions of the non-homogeneous equation is an affine plane, directed by the vector plane of solutions of the homogeneous equation. Deduce that the curve $(x^o, y^o, r^o)$ lies on an affine plane (whether $C = 0$ or not):

$$r = C^2 + \alpha x + \beta y \quad \text{for some } \alpha, \beta \in \mathbb{R},$$

and, by eliminating $r$, that the curve $(x^o, y^o)$ is conic of eccentricity $\epsilon = \sqrt{\alpha^2 + \beta^2}$ with a focus at the origin (*first Kepler law*) and directrix $D : C^2 + \alpha x + \beta y = 0$:

$$\operatorname{dist}(O, q) = \epsilon \operatorname{dist}(D, q).$$

5. Let $\mathscr{E}^4$ be the open subset of initial conditions leading to non-circular (eccentricity $> 0$), non-degenerate (eccentricity $< 1$), prograde (angular momentum $> 0$), elliptic (energy $< 0$), Keplerian motions. Let $v$ the polar angle of the planet from the perihelion (closest point of the ellipse to the Sun), $a$ the semi major axis, $g$ the polar angle of the pericenter from the $x$-axis and $\epsilon$ the eccentricity. Show that the map

$$\mathscr{E}^4 \to \mathbb{S}^1 \times ]0, +\infty[ \times \mathbb{S}^1 \times ]0, 1[, \quad (x, p_x, y, p_y) \mapsto (v, a, g, \epsilon)$$

is a diffeomorphism.[6]

6. Show that

$$K = \mu \left( \frac{\dot{r}^2}{2} + \frac{C^2}{2r^2} \right) - \frac{\mu M}{r} \quad \text{and} \quad C = r^2 \dot{\theta}$$

and, in the open set $\mathscr{E}$,

$$K = -\frac{\mu M}{2a} \quad \text{and} \quad C^2 = Ma(1 - \epsilon^2)$$

(use that $\dot{r} = 0$ when $r$ reaches its extremum values $a(1 \pm \epsilon)$).

7. Using the conservation of the angular momentum, prove that the area swept by the vector $q$ grows linearly in time (*second Kepler law*) and deduce that, if $T$ is the period, $MT^2 = 4\pi^2 a^3$ (*third Kepler law*).

For further description, we refer to the excellent account [3].

Also, the following exercise is the elementary prerequisite to understand the ubiquity of the angular momentum, as the moment map of of $SO_2$-actions.

**Exercise 23** (Symplectic lift of polar coordinates in the plane)

Let $\varphi : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, $(\theta, r) \mapsto r\, e^{i\theta}$ be the polar coordinates map. Show that its cotangent map is

$$\left(\varphi(\theta, r), {}^t\varphi'(\theta, r)^{-1} \cdot (\Theta, R)\right) = \left( r\, e^{i\theta}, \left( R + i\frac{\Theta}{r} \right) e^{i\theta} \right) = (z, Z).$$

Deduce that the flow of the Hamiltonian

$$\Theta = r \operatorname{Im} \left( Z\, e^{-i\theta} \right) = z_1 Z_2 - z_2 Z_1$$

[6] Ancestors of perturbation theory, including Newton, used to compute the variations of the "constants" (elliptic elements $a, g, \epsilon$) under the small influence of other celestial bodies than the Sun. It is remarkable that Lagrange not only expressed variables $(v, a, g, \epsilon)$ as functions of time, but used these variables as coordinates for the vector field (in replacement of the initial positions and impulsions), thus giving birth to modern differential geometry with arguably the first abstract (far-from-the-identity, transcendent) change of coordinates.

in the space $\{(z, Z)\}$ with respect to the standard symplectic structure $\mathrm{Im}\,(\bar{z}Z)$ is $\phi_\theta(z, Z) = (ze^{i\theta}, Ze^{i\theta})$.

**2. Fast Keplerian variables** $(\ell, L)$**.** — Restrict to the set $\{(v, a, g, \epsilon),\ 0 < \epsilon < 1\} \equiv \mathbb{T}^2 \times \mathbb{R}^2$ of non-degenerate, non-circular, elliptical, Keplerian motions. Define coordinate $t$ as the time from the pericenter; it is defined modulo the period $T$ of the orbit. Define $\ell$ as the angle obtained by rescaling time: $\ell := 2\pi t/T \pmod{2\pi}$. Now, if we want an action coordinate $L(K)$ conjugate to $\ell$: $dt \wedge dK = d\ell \wedge dL$, we see that

$$L'(K) = \frac{1}{\dot{\ell}} = \frac{T}{2\pi} = \frac{a^{3/2}}{\sqrt{M}} = \frac{\mu^{3/2}M}{(-2K)^{3/2}}.$$

Conventionally choosing $L = 0$ at infinity where $a = +\infty$, we get

$$L = \frac{\mu^{3/2}M}{\sqrt{-2K}} = \mu\sqrt{Ma};$$

in particular,

$$K = -\frac{\mu^3 M^2}{2L^2}.$$

**3. Slow planar variables** $(g, G)$**.** — We wish to define coordinates on the space of non-circular, non-degenerate, prograde Keplerian ellipses in the plane with fixed $L$. The *angular momentum*

$$G := \mu C = \mu\sqrt{Ma(1-\epsilon^2)} = L\sqrt{1-\epsilon^2},$$

which is a first integral of $K$ (and thus descends to the space of Keplerian orbits) and whose Hamiltonian flow acts by $2\pi$-periodic rotations around the origin, is a natural action coordinate. Define $g$ as the angle, modulo $2\pi$, measuring time along $X_G$-orbits, and vanishing when the pericenter meets the $Ox$-semi-axis.

The coordinates which $(\ell, L, g, G)$ define over $\mathscr{E}^4$ are symplectic:

– $\{\ell, L\} = \{g, G\} = 1$ by definition.

– $\{L, g\} = \{L, G\} = 0$ because $g$ and $G$ are first integrals of $K(L)$.

– $\{\ell, G\} = 0$ because the flow of $G$ rotates the Keplerian ellipse without revolving the planet along the ellipse.

– $\{\ell, g\} = 0$. Due to the Jacobi identity, $\{L, \{\ell, g\}\} = \{G, \{\ell, g\}\} = 0$. Hence it suffices to show that $\{\ell, g\} = 0$ in restriction to the section $\{\ell = g = 0 \pmod{\pi}\}$ of the $L$- and $G$-flows. We may thus assume that the planet is on the major axis and that the major axis itself is the $x$-axis. But then the partial derivatives of $\ell$ and $g$ with respect of $x$ or $p_y$ are zero, and

$$\{\ell, g\} = \underbrace{\frac{\partial \ell}{\partial x}}_{=0}\frac{\partial g}{\partial p_x} - \frac{\partial \ell}{\partial p_x}\underbrace{\frac{\partial g}{\partial x}}_{=0} + \frac{\partial \ell}{\partial y}\underbrace{\frac{\partial g}{\partial p_y}}_{=0} - \underbrace{\frac{\partial \ell}{\partial p_y}}_{=0}\frac{\partial g}{\partial y} = 0.$$

**4. Slow spatial variables $(\theta, \Theta)$ in $\mathbb{R}^3$.** — In the 3-dimensional Kepler problem, choose $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ as a reference plane (called *horizontal*). We will restrict to the set $\mathscr{E}^6$ of initial conditions leading to non-horizontal, non-circular, non-degenerate, prograde elliptic Keplerian motions.

Let $\vec{C} = q \times p$ be the *angular momentum vector* and $\Theta$ be its projection on the vertical axis. By exercise 23, the flow of $X_\Theta$ consists of $2\pi$-periodic rotations in the horizontal plane (diagonally for positions and impulsions, leaving the horizontal 4-planes invariant).

Each Keplerian oriented plane meets the horizontal plane along a half axis, the *ascending line of the node*. Let $\theta$ be the angle measuring time along $X_\Theta$-orbits, vanishing when the line of the node is the $Ox$-semi-axis.

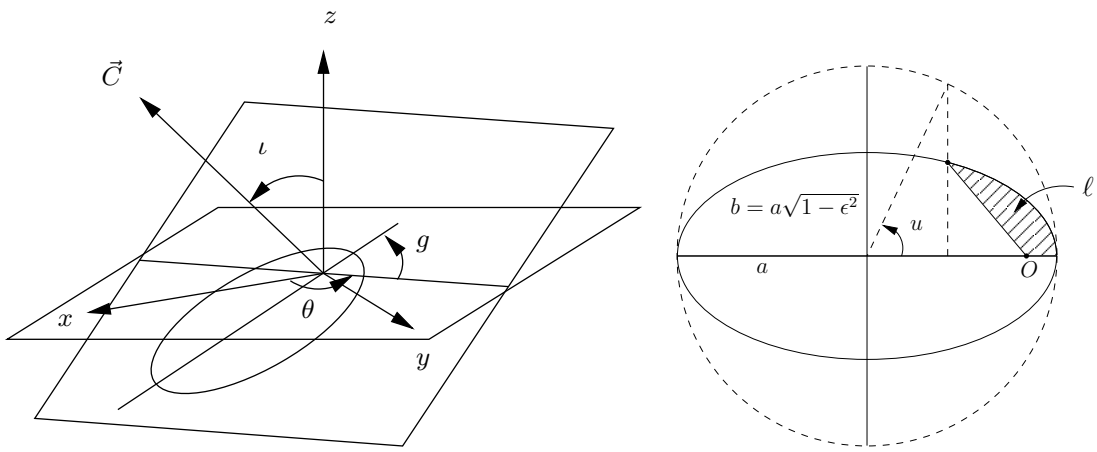
The coordinates which $(\ell, L, g, G, \theta, \Theta)$ define in $\mathscr{E}^6$ are symplectic:

– Poisson brackets with $L$, $G$ and $\Theta$ are what they should : 0, except $\{\ell, L\} = \{g, G\} = \{\theta, \Theta\} = 1$ (we know the flows of $L$, $G$ and $\Theta$).

– The three Poisson brackets between angles can be checked to vanish as above in the plane. Indeed, on the submanifold $\{\ell = g = \theta = 0 \pmod{\pi}\}$, the partial derivatives of any of these angles with respect to $x$, $p_y$ or $p_z$ vanish.

The names of the Delaunay elements are:

| | |
|---|---|
| – $\ell$ | *mean anomaly* |
| – $L = \mu\sqrt{Ma}$ | *circular angular momentum* |
| – $g$ | *argument of pericenter* |
| – $G = L\sqrt{1-\epsilon^2}$ | *angular momentum* |
| – $\theta$ | *longitude of the (ascending) node* |
| – $\Theta = G\cos\iota$ | *vertical component of the angular momentum.* |




**5. The Poincaré coordinates.** — Define the Poincaré coordinates $(\lambda, \Lambda, \zeta, z)$ by the following formulas:

$$\lambda = \ell + g + \theta, \quad \Lambda = L, \quad \zeta = \sqrt{2(L-G)}\, e^{-ig} \quad \text{and} \quad z = \sqrt{2(G-\Theta)}\, e^{-i\theta}$$

(several sign conventions exist). Obviously they are symplectic. From the above formulas, one checks that the Poincaré coordinates extend to differentiable coordinates at direct circular coplanar motions ($\zeta = z = 0$), the elliptic singularity of the secular system. In fact, their extension is analytic, as one can see by expressing the coordinates as explicit analytic functions of analytic first integrals; we refer to [1] for an elegant choice of first integrals.

## References


[1] K. Abdullah and A. Albouy. On a strange resonance noticed by M. Herman. *Regul. Chaotic Dyn.*, 6(4):421–432, 2001.

[2] R. Abraham and J. E. Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of T. Raţiu and R. Cushman.

[3] A. Albouy. Lectures on the two-body problem. In *Classical and celestial mechanics (Recife, 1993/1999)*, pages 63–116. Princeton Univ. Press, Princeton, NJ, 2002.

[4] A. Albouy and A. Chenciner. Le problème des $n$ corps et les distances mutuelles. *Invent. Math.*, 131(1):151–184, 1998.

[5] V. I. Arnold. Small denominators. I. Mapping the circle onto itself. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:21–86, 1961.

[6] V. I. Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. *Uspehi Mat. Nauk*, 18(6 (114)):91–192, 1963.

[7] V. I. Arnold. Instability of dynamical systems with many degrees of freedom. *Dokl. Akad. Nauk SSSR*, 156:9–12, 1964.

[8] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical aspects of classical and celestial mechanics*, volume 3 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, third edition, 2006. [Dynamical systems. III], Translated from the Russian original by E. Khukhro.

[9] E. Barrabés, J. M. Cors, C. Pinyol, and J. Soler. Hip-hop solutions of the $2N$-body problem. *Celestial Mech. Dynam. Astronom.*, 95(1-4):55–66, 2006.

[10] V. Barutello and S. Terracini. Action minimizing orbits in the $n$-body problem with simple choreography constraint. *Nonlinearity*, 17(6):2015–2039, 2004.

[11] F. Béguin. *L'héritage scientifique de Poincaré*, chapter Le mémoire de Poincaré pour le prix du roi Oscar. Belin, 2006.

[12] P. Bernard. The dynamics of pseudographs in convex Hamiltonian systems. *J. Amer. Math. Soc.*, 21(3):615–669, 2008.

[13] J.-B. Bost. Tores invariants des systèmes dynamiques hamiltoniens (d'après Kolmogorov, Arnold, Moser, Rüssmann, Zehnder, Herman, Pöschel, ...). *Astérisque*, 1984/85(133-134):113–157, 1986. Séminaire Bourbaki.

[14] H. W. Broer, G. B. Huitema, and M. B. Sevryuk. *Quasi-periodic motions in families of dynamical systems*, volume 1645 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996. Order amidst chaos.

[15] A. Celletti and L. Chierchia. KAM stability for a three-body problem of the solar system. *Z. Angew. Math. Phys.*, 57(1):33–41, 2006.

[16] A. Celletti and L. Chierchia. KAM stability and celestial mechanics. *Mem. Amer. Math. Soc.*, 187(878):viii+134, 2007.

[17] A. Celletti and L. Chierchia. KAM stability and celestial mechanics. *Mem. Amer. Math. Soc.*, 187(878):viii+134, 2007.

[18] A. Celletti and L. Chierchia. Quasi-periodic attractors in celestial mechanics. *Arch. Ration. Mech. Anal.*, 191(2):311–345, 2009.

[19] A. Chenciner. Intégration du problème de Kepler par la méthode de Hamilton-Jacobi. Technical report, Bureau des Longitudes, 1989.

[20] A. Chenciner. Action minimizing solutions of the Newtonian $n$-body problem: from homology to symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 279–294, Beijing, 2002. Higher Ed. Press.

[21] A. Chenciner. Some facts and more questions about the Eight. In *Topological methods, variational methods and their applications (Taiyuan, 2002)*, pages 77–88. World Sci. Publ., River Edge, NJ, 2003.

[22] A. Chenciner. Rapport sur les premiers textes de mécanique céleste (1747–1749), œuvres complètes de d'alembert. *Gazette des mathématiciens*, 99:107–114, 2004.

[23] A. Chenciner and J. Féjoz. L'équation aux variations verticales d'un équilibre relatif comme source de nouvelles solutions périodiques du problème des $N$ corps. *C. R. Math. Acad. Sci. Paris*, 340(8):593–598, 2005.

[24] A. Chenciner and J. Féjoz. The flow of the equal-mass spatial 3-body problem in the neighborhood of the equilateral relative equilibrium. *Discrete Contin. Dyn. Syst. Ser. B*, 10(2-3):421–438, 2008.

[25] A. Chenciner and J. Féjoz. The flow of the equal-mass spatial 3-body problem in the neighborhood of the equilateral relative equilibrium. *Discrete Contin. Dyn. Syst. Ser. B*, 10(2-3):421–438, 2008.

[26] A. Chenciner and J. Féjoz. Unchained polygons and the $N$-body problem. *Regul. Chaotic Dyn.*, 14(1):64–115, 2009.

[27] A. Chenciner, J. Féjoz, and R. Montgomery. Rotating eights. I. The three $\Gamma_i$ families. *Nonlinearity*, 18(3):1407–1424, 2005.

[28] A. Chenciner and R. Montgomery. A remarkable periodic solution of the three-body problem in the case of equal masses. *Ann. of Math. (2)*, 152(3):881–901, 2000.

[29] A. Chenciner and A. Venturelli. Minima de l'intégrale d'action du problème newtonien de 4 corps de masses égales dans $\mathbf{R}^3$: orbites "hip-hop". *Celestial Mech. Dynam. Astronom.*, 77(2):139–152 (2001), 2000.

[30] Chong-Qing Cheng and Jun Yan. Existence of diffusion orbits in a priori unstable Hamiltonian systems. *J. Differential Geom.*, 67(3):457–517, 2004.

[31] L. Chierchia. KAM lectures. In *Dynamical Systems. Part I*, Pubbl. Cent. Ric. Mat. Ennio Giorgi, pages 1–55. Scuola Norm. Sup., Pisa, 2003.

[32] L. Chierchia and G. Pinzari. Properly degenerate KAM theory (following V. I. Arnold). Submitted to Discrete and Continuous Dynamical Systems, 2009.

[33] L. Chierchia and G. Pinzari. Symplectic structures and quasiperiodic motions in the planetary $n$-body problem. *Preprint*, 2010.

[34] C. Conley. On some new long periodic solutions of the plane restricted three-body problem. *Comm. Pure Appl. Math.*, 16:449–467, 1963.

[35] J. Le Rond d'Alembert. *Premiers textes de mécanique céleste (1747–1749)*, volume 6 of *Œuvres Complètes de d'Alembert. [Complete Works of d'Alembert]*. CNRS Éditions, Paris, 2002. Edited by Michelle Chapront-Touzé.

[36] R. de la Llave. A tutorial on KAM theory. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 175–292. Amer. Math. Soc., Providence, RI, 2001.

[37] C. Delaunay. Note sur les mouvements du périgée et du noeud de la lune. *Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences*, 74:17, 1872.
[38] D. Dolgopyat. Repulsion from resonances. Preprint from Univ. of Maryland, 2004.
[39] J. Féjoz. Démonstration du 'théorème d'Arnold' sur la stabilité du système planétaire (d'après Herman). *Ergodic Theory Dynam. Systems*, 24(5):1521–1582, 2004.
[40] J. Féjoz. A simple proof of invariant tori theorems. Preprint, 2011.
[41] J. Féjoz and C. Simó. The global central manifold of the equal-mass Lagrange relative equilibrium. In preparation, 2010.
[42] D. L. Ferrario and S. Terracini. On the existence of collisionless equivariant minimizers for the classical $n$-body problem. *Invent. Math.*, 155(2):305–362, 2004.
[43] J. Galante and V. Kaloshin. Destruction of invariant curves in the restricted circular planar three-body problem using comparison of action. Preprint, 2010.
[44] G. Gómez and J. Llibre. A note on a conjecture of Poincaré. *Celestial Mech.*, 24(4):335–343, 1981.
[45] M. Hénon. Exploration numérique du problème restreint IV. masses égales, orbites non périodiques. *Bulletin Astronomique*, 3(1:2):49–66, 1966.
[46] M. Hénon. Families of asymmetric periodic orbits in Hill's problem of three bodies. *Celestial Mech. Dynam. Astronom.*, 93(1-4):87–100, 2005.
[47] A. Knauf. Klassische mechanik. To appear, 500 pages, 2010.
[48] A. N. Kolmogorov. On the conservation of conditionally periodic motions for a small change in Hamilton's function. *Dokl. Akad. Nauk SSSR (N.S.)*, 98:527–530, 1954.
[49] J.-L. Lagrange. *Essai sur le problème des trois corps.* Œuvres, Tome VI. Gauthier-Villars (Paris), 1867-1892 edition, 1772.
[50] J.-L. Lagrange. *Œuvres complètes*, volume VI, chapter Recherches sur la théorie des perturbations que les comètes peuvent éprouver par l'action des planètes, pages 403–503. Gauthier-Villars, Paris, 1778. Publiées par les soins de J.-A. Serret, avec une notice sur la vie et les ouvrages de J.-L. Lagrange par J.-B. J. Delambre,.
[51] P.-S. Laplace. Théorie de Jupiter et de Saturne. *Mém. Acad. royale des sciences de Paris*, 1785–1788.
[52] J. Laskar. The chaotic motion of the solar system. a numerical estimate of the size of the chaotic zones. *Icarus*, 88:266–291, 1990.
[53] J. Laskar. Le système solaire est-il stable ? In *Le Chaos*, number XIV in Séminaire Poincaré, pages 221–246. Birkhäuser, 2010.
[54] U. Locatelli and A. Giorgilli. Invariant tori in the Sun-Jupiter-Saturn system. *Discrete Contin. Dyn. Syst. Ser. B*, 7(2):377–398 (electronic), 2007.
[55] H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl. *The principle of relativity.* A collection of original memoirs on the special and general theory of relativity. Dover Publications Inc., New York, N. Y., undated. With notes by A. Sommerfeld, Translated by W. Perrett and G. B. Jeffery.
[56] C. Marchal. *The three-body problem*, volume 4 of *Studies in Astronautics.* Elsevier Science Publishers B.V., Amsterdam, 1990. With a foreword by V. Szebehely, With French, Russian, German, Spanish, Japanese, Chinese and Arabic summaries.

[57] C. Marchal. How the method of minimization of action avoids singularities. *Celestial Mech. Dynam. Astronom.*, 83(1-4):325–353, 2002. Modern celestial mechanics: from theory to applications (Rome, 2001).

[58] J.-P. Marco and D. Sauzin. Wandering domains and random walks in Gevrey near-integrable systems. *Ergodic Theory Dynam. Systems*, 24(5):1619–1666, 2004.

[59] J. N. Mather and G. Forni. Action minimizing orbits in Hamiltonian systems. In *Transition to chaos in classical and quantum mechanics (Montecatini Terme, 1991)*, volume 1589 of *Lecture Notes in Math.*, pages 92–186. Springer, Berlin, 1994.

[60] K. R. Meyer. *Periodic solutions of the $N$-body problem*, volume 1719 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.

[61] K. R. Meyer and D. S. Schmidt. Elliptic relative equilibria in the $N$-body problem. *J. Differential Equations*, 214(2):256–298, 2005.

[62] R. Moeckel. Some qualitative features of the three-body problem. In *Hamiltonian dynamical systems (Boulder, CO, 1987)*, volume 81 of *Contemp. Math.*, pages 1–22. Amer. Math. Soc., Providence, RI, 1988.

[63] R. Moeckel. Celestial Mechanics (Especially Central Configurations). Course at the CIME, 1994.

[64] R. Moeckel. Generic drift on Cantor sets of annuli. In *Celestial mechanics (Evanston, IL, 1999)*, volume 292 of *Contemp. Math.*, pages 163–171. Amer. Math. Soc., Providence, RI, 2002.

[65] A. Moltchanov. The resonant structure of the solar system. *Icarus*, 8(2):203–215, 1968.

[66] C. Moore. Braids in classical dynamics. *Phys. Rev. Lett.*, 70(24):3675–3679, 1993.

[67] J. Moser. A rapidly convergent iteration method and non-linear partial differential equations. I. *Ann. Scuola Norm. Sup. Pisa (3)*, 20:265–315, 1966.

[68] J. Moser. Convergent series expansions for quasi-periodic motions. *Math. Ann.*, 169:136–176, 1967.

[69] J. Moser. Regularization of Kepler's problem and the averaging method on a manifold. *Comm. Pure Appl. Math.*, 23:609–636, 1970.

[70] J. Moser. Periodic orbits near an equilibrium and a theorem by Alan Weinstein. *Comm. Pure Appl. Math.*, 29(6):724–747, 1976.

[71] J. Moser. *Stable and random motions in dynamical systems*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. With special emphasis on celestial mechanics, Reprint of the 1973 original, With a foreword by Philip J. Holmes.

[72] J. Moser and E. J. Zehnder. *Notes on dynamical systems*, volume 12 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2005.

[73] N. N. Nehorošev. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems. II. *Trudy Sem. Petrovsk.*, 5:5–50, 1979.

[74] A. Neishtadt. On averaging in two-frequency systems with small Hamiltonian and much smaller non-Hamiltonian perturbations. *Mosc. Math. J.*, 3(3):1039–1052, 1200, 2003. {Dedicated to V. I. Arnold on the occasion of his 65th birthday}.

[75] A. Neishtadt. Averaging method and adiabatic invariants. In *Hamiltonian dynamical systems and applications*, NATO Sci. Peace Secur. Ser. B Phys. Biophys., pages 53–66. Springer, Dordrecht, 2008.

[76] I. Newton. *Opticks:: Or, A Treatise of the Reflections, Refractions, Inflections and Colours of Light.* Internet Archive, 1730.

[77] L. Niederman. Stability over exponentially long times in the planetary problem. *Nonlinearity*, 9(6):1703–1751, 1996.

[78] W. Pauli. *Theory of relativity.* Translated from the German by G. Field, with supplementary notes by the author. Pergamon Press, New York, 1958.

[79] R. Pérez-Marco. Convergence or generic divergence of the Birkhoff normal form. *Ann. of Math. (2)*, 157(2):557–574, 2003.

[80] R. Pérez-Marco. KAM techniques in PDE. *Astérisque*, 290:Exp. No. 908, ix, 307–317, 2003. Séminaire Bourbaki. Vol. 2001/2002.

[81] H. Poincaré. *Les méthodes nouvelles de la mécanique céleste [microform].* Gauthier-Villars et fils, Paris, 1892.

[82] H. Poincaré. Sur les solutions périodiques et le principe de moindre action. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 123:915 – 918, 1896.

[83] H. Poincaré. *Leçons de mécanique céleste.* Gauthier-Villars, 1905.

[84] J. Pöschel. A lecture on the classical KAM theorem. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 707–732. Amer. Math. Soc., Providence, RI, 2001.

[85] C. C. Pugh and C. Robinson. The $C^1$ closing lemma, including Hamiltonians. *Ergodic Theory Dynam. Systems*, 3(2):261–313, 1983.

[86] A. S. Pyartli. Diophantine approximations of submanifolds of a Euclidean space. *Funkcional. Anal. i Priložen.*, 3(4):59–62, 1969.

[87] P. H. Rabinowitz. Periodic solutions of Hamiltonian systems: a survey. *SIAM J. Math. Anal.*, 13(3):343–352, 1982.

[88] P. Robutel. Stability of the planetary three-body problem. II. KAM theory and existence of quasiperiodic motions. *Celestial Mech. Dynam. Astronom.*, 62(3):219–261, 1995.

[89] H. Rüssmann. Nondegeneracy in the perturbation theory of integrable dynamical systems. In *Stochastics, algebra and analysis in classical and quantum dynamics (Marseille, 1988)*, volume 59 of *Math. Appl.*, pages 211–223. Kluwer Acad. Publ., Dordrecht, 1990.

[90] H. Rüssmann. Invariant tori in non-degenerate nearly integrable Hamiltonian systems. *Regul. Chaotic Dyn.*, 6(2):119–204, 2001.

[91] D. S. Schmidt. Periodic solutions near a resonant equilibrium of a Hamiltonian system. *Celestial Mech.*, 9:81–103, 1974.

[92] J. Schneider. The Extrasolar Planets Encyclopaedia. Paris Observatory, `http://exoplanet.eu/`, 1995–2010.

[93] M. B. Sevryuk. The lack-of-parameters problem in the KAM theory revisited. In *Hamiltonian systems with three or more degrees of freedom (S'Agaró, 1995)*, volume 533 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 568–572. Kluwer Acad. Publ., Dordrecht, 1999.

[94] M. B. Sevryuk. The classical KAM theory at the dawn of the twenty-first century. *Mosc. Math. J.*, 3(3):1113–1144, 1201–1202, 2003. Dedicated to V. I. Arnold on the occasion of his 65th birthday.

[95] M. B. Sevryuk. Invariant tori in quasi-periodic non-autonomous dynamical systems via Herman's method. *Discrete Contin. Dyn. Syst.*, 18(2-3):569–595, 2007.

[96] M. B. Sevryuk. KAM tori: persistence and smoothness. *Nonlinearity*, 21(10):T177–T185, 2008.

[97] C. L. Siegel. Iteration of analytic functions. *Ann. of Math. (2)*, 43:607–612, 1942.

[98] C. Simó. New families of solutions in $N$-body problems. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 101–115. Birkhäuser, Basel, 2001.

[99] C. Simó and T. J. Stuchi. Central stable/unstable manifolds and the destruction of KAM tori in the planar Hill problem. *Phys. D*, 140(1-2):1–32, 2000.

[100] J.-M. Souriau. La structure symplectique de la mécanique décrite par Lagrange en 1811. *Math. Sci. Humaines*, 94:45–54, 1986.

[101] E. L. Stiefel and G. Scheifele. *Linear and regular celestial mechanics. Perturbed two-body motion, numerical methods, canonical theory.* Springer-Verlag, New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 174.

[102] F. Tisserand. *Traité de mécanique céleste.* Gauthier-Villars (Paris), 1896.

[103] A. Weinstein. Normal modes for nonlinear Hamiltonian systems. *Invent. Math.*, 20:47–57, 1973.

[104] A. Wintner. *The analytical foundations of celestial mechanics.* Princeton University Press, 1941.

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