



Combinatorial studies of Euler's table on wreath products

Hilarion Faliharimalala

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Hilarion Faliharimalala. Combinatorial studies of Euler's table on wreath products. General Mathematics [math.GM]. Université Claude Bernard - Lyon I; Université d'Antananarivo, 2010. English. NNT : 2010LYO10039 . tel-00702743

HAL Id: tel-00702743

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N° d'ordre : 39-2010

Année 2010

THÈSE

présentée devant

L'UNIVERSITÉ CLAUDE BERNARD-LYON 1

préparée en cotutelle avec

L'UNIVERSITÉ D'ANTANANARIVO

pour l'obtention

du **DIPLÔME DE DOCTORAT**

(arrêté du 7 août 2006)

Spécialité : Mathématiques

présentée et soutenue publiquement le 31 Mars 2010

par

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ÉTUDES COMBINATOIRES DU TABLEAU D'EULER SUR LES PRODUITS EN COURONNE

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REMERCIEMENTS

Je tiens tout d'abord à remercier chaleureusement mon directeur de thèse, Jiang ZENG, pour son encadrement, ses précieux conseils, tout en me laissant une marge d'initiatives très appréciable. Il m'a guidé et soutenu constamment. Il m'a accordé sa confiance, ce qui m'a permis de réussir à prouver des résultats qui ont largement dépassé mes espérances initiales. Je lui en suis infiniment reconnaissant.

Je tiens également à remercier vivement mon codirecteur, Arthur Randrianarivony, pour son encadrement, pour ses conseils et pour sa participation à l'élaboration de cette thèse en cotutelle.

Je tiens à exprimer toute ma reconnaissance à Messieurs Einar STEINGRIMSSON et Masao ISHIKAWA qui ont accepté d'être rapporteurs de cette thèse.

Merci également à Messieurs Dominique FOATA et Riccardo BIAGIOLI, d'avoir bien voulu participer au jury et d'avoir relu les versions préliminaires de cette thèse, et contribué, par leurs commentaires et leurs remarques, à l'améliorer.

C'est un honneur pour moi que Monsieur Alain LASCOUX, ait accepté d'être membre du jury.

Mes remerciements s'adressent également :

à Monsieur Dominique Dumont pour ces encouragements, et pour les enseignements qu'il m'a donnés.

à l'Institut Camille Jordan pour m'avoir accueilli pendant mes trois années de recherche ;

à mes collègues de l'Institut Supérieur de Technologie d'Antananarivo qui m'ont donné la possibilité de poursuivre mes études, et tout particulièrement, au Directeur Général, Josoa RAMAMONJISOA qui a su m'encourager tout au long de ces nombreuses années ;

- à l'Agence Universitaire de la Francophonie, pour son soutien financier durant les trois années de ma formation doctorale. Je suis très reconnaissant et heureux d'avoir répondu à ses espérances. Sans son aide, mes travaux de recherche n'auraient pu aboutir ;

- à Frédéric JOUHET, Yoann GELINEAU pour les multiples discussions qui ont contribué à l'avancement de ma thèse ;

- à Heesung SHIN de m'avoir aidé à résoudre les problèmes informatiques rencontrés lors de la réalisation de ce travail ;

- à Rondro, mon épouse, Mihanta et Heritiana, mes enfants, et à toute ma famille qui ont souffert de mes longues périodes d'absence tout au long de la préparation de ma thèse, je les remercie de m'avoir soutenu moralement et ;

- à tous ceux qui, de près ou de loin, ont contribué au bon déroulement de ma thèse.

Table des matières

0	Introduction	1
0.1	Définitions et notations	3
0.1.1	Groupe de Coxeter	3
0.1.2	Le groupe symétrique	4
0.1.3	Le groupe hyperoctaédral	6
0.1.4	Produit semi-direct	6
0.1.5	Le produit en couronne $C_r \wr S_n$	7
0.1.6	Le Groupe des permutations colorées $G(r, n)$	8
0.1.7	Isomorphisme entre $C_r \wr S_n$ et $G(r, n)$	8
0.1.8	Le groupe de réflexions complexes $G(r, p, n)$	9
0.1.9	Relations d'ordre sur l'alphabet coloré	10
0.1.10	Quelques statistiques	11
0.1.11	Les classes m -fixées des permutations	14
0.1.12	q -notations	14
0.2	Préliminaires	15
0.2.1	Sur un Théorème de MacMahon	15
0.2.2	Sur une identité de Carlitz	17
0.2.3	Autre raffinement du polynôme eulérien	19
0.2.4	Sur le nombre de dérangements	20
0.2.5	Sur une identité de Gessel-Simion	21
0.3	Résultats principaux	22
1	Euler's table for $G(r, n)$	31
1.1	Introduction	31
1.2	Main results	33
1.3	Proof of Theorem 1.2.1	37
1.4	Proof of Theorem 1.2.2	37
1.5	Proof of Theorem 1.2.3.	39
1.6	Proofs of Theorem 1.2.5	42
1.6.1	First Proof	42
1.6.2	Second proof	43

2	Further results	45
2.1	Generating functions	46
2.2	Some recurrence relations	47
2.2.1	Relations concerning $g_{r,n}^m$	47
2.2.2	Relations concerning $d_{r,n}^m$	48
2.3	Combinatorial proof of (1.2), (2.2),(2.16)	48
2.3.1	Combinatorial proof of (1.2)	48
2.3.2	Combinatorial proof of (2.2)	50
2.3.3	Combinatorial proof of (2.16)	52
3	The q-Euler's table for $G(r, n)$	55
3.1	Introduction	55
3.2	Definitions and main results	56
3.3	Proof of Theorem 3.1	59
3.3.1	Clarke et al.'s bijection Ψ	59
3.3.2	Generalization of Ψ to $G(r, n)$	61
3.3.3	Proof of (3.5)	62
3.4	Proof of Theorem 3.2	64
4	The three Transformations	67
4.1	Introduction	67
4.2	Foata-Han's first transformation Φ	68
4.3	Foata-Han's second transformation $\mathbf{F}$	70
4.4	Proof of Theorem 4.1	70
4.4.1	The i_m -th slot of σ is green	72
4.4.2	The i_m -th slot of σ is red	74
4.5	Proofs of the three lemmas	77
4.5.1	Proof of Lemma 4.1	77
4.5.2	Proof of Lemma 4.3	77
4.5.3	Proof of Lemma 4.4	79
5	Flag-inv, Flag-maj on colored words	81
5.1	Introduction	81
5.2	The construction of $\hat{\Phi}$	86
5.3	Proof of Theorem 5.1.1	87
5.4	Proof of Theorem 5.1.2	89
6	Conclusion et perspectives	93

Chapitre 0

Introduction

Cette thèse est située dans le domaine de la combinatoire énumérative, une branche des mathématiques discrètes. L'étude des statistiques de permutations occupe une place importante dans ce domaine. Au cours des deux dernières décennies, des travaux actifs ont été menés pour étendre des résultats classiques connus pour le groupe symétrique à d'autres groupes de Coxeter, aux produits en couronne et plus généralement aux groupes de réflexions complexes. Cette tendance ouvre de nouveaux horizons dans les recherches contemporaines des mathématiques. Nos recherches se dirigent dans cette optique de généralisation.

Rappelons qu'une statistique est une application d'un ensemble vers l'ensemble des entiers naturels. Par exemple, considérons σ , une permutation du groupe symétrique d'ordre n . Une *descente* de σ est un entier i dans $\{1, 2, \dots, n-1\}$ tel que $\sigma(i) > \sigma(i+1)$. Une *inversion* de σ est un couple (i, j) tel que $i < j$ et $\sigma(i) > \sigma(j)$. Les applications “des” et “inv” qui comptent respectivement les nombres de descentes et d'inversions sont des statistiques sur le groupe symétrique. À noter que la statistique “inv” coïncide avec la traditionnelle *fonction longueur* “ ℓ ” qui est la longueur d'une expression réduite d'une permutation. Il y a deux importantes familles de statistiques de permutations : les statistiques *eulériennes* qui sont les statistiques équidistribuées avec “des” et les statistiques *mahoniennes* qui sont les statistiques équidistribuées avec “ ℓ ”. Par exemple l'*indice majeur* “maj”, qui est la somme des descentes d'une permutation, est une statistique mahonienne.

Un des problèmes importants est de chercher des extensions convenables des statistiques définies sur le groupe symétrique, afin de généraliser des résultats classiques à d'autres familles de groupes. Un exemple remarquable est donné par la statistique *flag-major* notée “fmaj”, définie par Adin et Roichman en 2000, qui est une statistique sur le groupe hyperoctaédral, qui étend l'indice majeur et permet de généraliser le célèbre théorème de Mac-

Mahon sur l'équidistribution du nombre d'inversion et l'indice majeur pour les mots (Théorème 0.4). Pour le groupe de Coxeter de type D, la statistique qui joue le même rôle est “ fmaj_D ”, comme nous le verrons plus tard dans le Théorème 0.6.

On peut également définir ces statistiques sur les produits en couronne du groupe cyclique d'ordre r par le groupe symétrique d'ordre n , noté $C_r \wr S_n$. Ces groupes ont une jolie interprétation combinatoire en termes de permutations colorées. Pour définir ces statistiques, on est amené à considérer des relations d'ordre sur l'ensemble des entiers colorés. Il y a plusieurs possibilités pour définir ces ordres, et les statistiques obtenues donnent des résultats qui ne sont pas nécessairement indépendants de l'ordre choisi. Dans la littérature, il y a plusieurs résultats sur ce sujet ; notamment, nous allons voir les travaux de Fire [40] d'une part, et Adin, Gessel, Roichman [4] d'autre part, à propos des extensions sur B_n d'une identité de Gessel-Simion [55, corollaire 2] (voir aussi [40]), et les travaux récents de Biagioli et Zeng [9] sur une extension aux permutations colorées de l'identité de Chow-Gessel (voir Théorème 0.12).

Dans cette thèse, nous étudions le *tableau d'Euler*, qui est une matrice liée au groupe symétrique. Elle est définie par $(g_n^m)_{0 \leq m \leq n}$, où les (g_n^m) satisfont la relation $g_n^m = g_n^{m+1} - g_{n-1}^m$ et la condition initiale $g_n^n = n!$.

Il y a un certain nombre de travaux concernant l'étude du tableau d'Euler, notamment ceux de Dumont-Randrianarivony [30] et Rakotondrajao [48]. Ils ont donné différentes interprétations combinatoires de ce tableau sur le groupe symétrique. Clarke, Han et Zeng [27] en ont construit un *q-analogue*, et ont donné une interprétation à l'aide d'une nouvelle statistique mahonienne “maf”. Rakotondrajao [47] a étudié le tableau dérivé en divisant les éléments de la m -ième colonne par $m!$. Tout récemment, Randrianarivony a étudié le tableau d'Euler dans [49].

Le but de cette thèse est de définir cet objet pour les produits en couronne $C_r \wr S_n$. Pour faire cela, nous étendons sa dimension, en créant un paramètre supplémentaire. Nos définitions nous permettent de généraliser les résultats cités ci-dessus concernant le tableau d'Euler sur le groupe symétrique aux produits en couronne.

Cette thèse se divise en cinq chapitres. Le premier est consacré aux études des aspects combinatoires du tableau d'Euler sur les produits en couronne. Dans le second, nous cherchons les fonctions génératrices et donnons quelques résultats supplémentaires. Ces deux premiers chapitres constituent notre article *Derangement and Euler's difference table for $C_r \wr S_n$* [31]. Nous étudions dans le chapitre 3 un *q-analogue* du tableau d'Euler sur les produits en couronne. Un des nos résultats est l'extension de la formule du *q*-nombre de dérangements. Récemment, Foata et Han [34] ont construit deux importantes transformations et nous démontrons dans le chapitre 4, que ces deux trans-

formations fournissent une factorisation de la bijection-clé de Clarke et al. [27]. Ces chapitres 3 et 4 font l'objet de notre article *Fix-Euler-Mahonian statistics on wreath products* [32]. Dans le dernier chapitre qui correspond à notre article *Flag-major index and Flag-inversion number on colored words and wreath product* [33], nous proposons une extension du nombre d'inversions afin de généraliser sur les mots colorés et sur les permutations colorées le théorème de MacMahon. Notre preuve combinatoire est basée sur l'extension de la seconde transformation fondamentale de Foata, qui échange l'indice majeur et le nombre d'inversions sur le groupe symétrique.

0.1 Définitions et notations

0.1.1 Groupe de Coxeter

Soit S un ensemble. On appelle matrice de Coxeter toute matrice carrée symétrique $M = (m_{i,j})_{(i,j) \in S^2}$ dont les éléments sont des entiers positifs ou $+\infty$ vérifiant la relation :

$$\forall (i, j) \in S \times S, \quad m_{i,j} = 1 \iff i = j. \quad (1)$$

La matrice M peut être représentée par un graphe (de Coxeter) dont les nœuds sont les éléments de S et les arêtes sont les couples non ordonnés (i, j) tels que $m_{i,j} \geq 3$. Chaque arête est étiquetée par la valeur $m_{i,j}$ si celle-ci est supérieure ou égale à 4.

Exemple 1. Soit $S = \{a, b, c, d, e\}$. La figure 1 représente le graphe associé à la matrice de Coxeter suivante :

$$M = \begin{bmatrix} 1 & 2 & 2 & 2 & 3 \\ 2 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 3 & 4 \\ 2 & 2 & 3 & 1 & 5 \\ 3 & 3 & 4 & 5 & 1 \end{bmatrix}.$$

Une matrice de Coxeter définit un groupe W avec la présentation suivante :

$$\begin{cases} \text{Ensemble des générateurs : } S \\ \text{Relations : } (st)^{m_{s,t}} = e \text{ et } m_{s,t} \neq \infty \quad \forall (s, t) \in S^2, \end{cases} \quad (2)$$

où e désigne l'élément neutre du groupe.

On dit que le couple (W, S) forme un système de Coxeter et que W est un groupe de Coxeter.

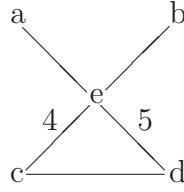


FIGURE 1 – Graphe de Coxeter associé à la matrice M

On classe les groupes de Coxeter selon leurs graphes. Par exemple les graphes des groupes de Coxeter de type A, B et D sont donnés dans la figure 2.

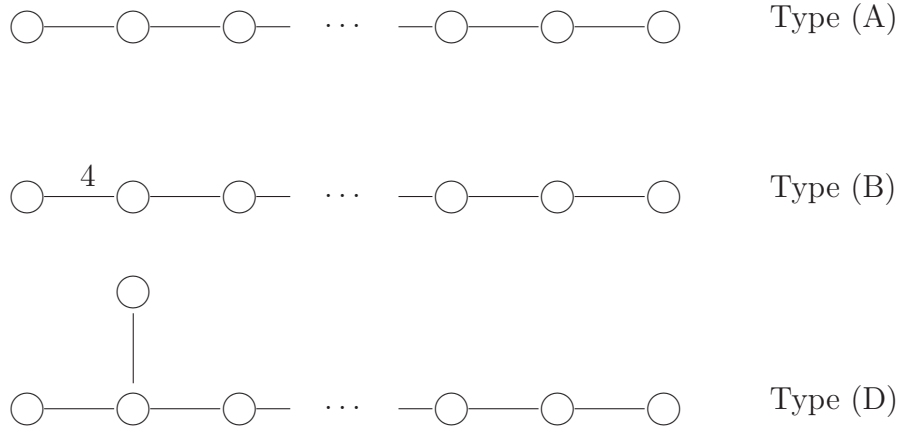


FIGURE 2 – Graphe de Coxeter pour les types A, B, D

Nous travaillons particulièrement sur le groupe symétrique et le groupe hyperoctaédral, qui sont respectivement des groupes de Coxeter de type A et B. Nous donnerons également quelques exemples d'extension des résultats du groupe symétrique au groupe des permutations signées paires D_n , qui est un groupe de Coxeter de type D. Notons que le groupe des permutations colorées à r couleurs n'est pas un groupe de Coxeter si $r > 2$.

Pour ces groupes finis, l'ensemble des générateurs est noté $\{s_i, i \in I\}$ où I est une partie finie de $\mathbb{N}$ et nous notons $m_{i,j}$ le plus petit entier positif tel que $(s_i s_j)^{m_{i,j}} = e$.

0.1.2 Le groupe symétrique

Soit $n \in \mathbb{N}$. On note $[n]$ l'ensemble $\{1, 2, \dots, n\}$ si $n > 0$ et $[0] = \emptyset$. On rappelle qu'une permutation d'ordre n est une bijection de $[n]$ vers $[n]$. Tra-

ditionnellement, on présente une permutation σ sur deux lignes :

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}.$$

Par convention, nous utilisons plutôt l'écriture en une seule ligne :

$$[\sigma(1), \sigma(2), \sigma(3), \dots, \sigma(n)].$$

Par exemple, la permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

peut s'écrire $[3, 1, 2, 5, 4]$ ou tout simplement $\sigma = 3 \ 1 \ 2 \ 5 \ 4$.

Une permutation peut aussi être factorisée en produit de cycles. Le cycle de σ contenant l'entier i est la suite $(i, \sigma(i), \sigma^2(i), \dots, \sigma^{o(i)-1}(i))$ où $o(i)$ est le plus petit entier positif tel que $\sigma^{o(i)}(i) = i$.

Pour l'exemple ci-dessus, la factorisation en produit de cycles est donc

$$\sigma = (1 \ 3 \ 2)(4 \ 5).$$

Le groupe symétrique d'ordre n noté S_n est le groupe constitué des permutations d'ordre n muni de la composition usuelle :

$$\forall \sigma_1, \sigma_2 \in S_n, i \in [n], \quad \sigma_1 \circ \sigma_2(i) = \sigma_1(\sigma_2(i)). \quad (3)$$

Pour chaque $i \in [n-1]$ soit

$$s_i = [1, 2, \dots, i-1, i+1, i, i+2, \dots, n]$$

la transposition qui échange i et $i+1$. Alors $s_1, s_2, \dots, s_{n-1}$ sont des générateurs de S_n . Ils sont liés par les relations :

$$\begin{cases} m_{i,j} = 1 \iff i = j; \\ m_{i,i+1} = 3 \quad \forall i \in [n-1]; \\ m_{i,j} = 2 \quad \text{si } i, j \in [n] \text{ tels que } |i-j| \geq 2. \end{cases} \quad (4)$$

Le groupe symétrique est un groupe de Coxeter de type A comme nous l'avons signalé plus haut.

0.1.3 Le groupe hyperoctaédral

Une permutation signée d'ordre n est une bijection f de $\{\pm 1, \pm 2, \dots, \pm n\}$ vers lui-même vérifiant

$$f(-i) = -f(i) \quad \forall i \in [n]. \quad (5)$$

Le groupe constitué des permutations signées d'ordre n s'appelle le *groupe hyperoctaédral d'ordre n* noté B_n . C'est donc un sous-groupe de S_{2n} . La propriété (5) signifie qu'une permutation signée σ est complètement définie par la donnée de sa restriction $[\sigma(1), \sigma(2), \dots, \sigma(n)]$ sur $[n]$. Le cardinal de B_n est donc $2^n n!$. Le groupe hyperoctaédral est engendré par les générateurs :

$$s_0 = [-1, 2, \dots, n]; \quad s_i = [1, 2, \dots, i-1, i+1, i, i+2, \dots, n] \quad \forall i \in [n-1].$$

Nous avons les relations suivantes :

$$\left\{ \begin{array}{ll} m_{0,1} = 4; \\ m_{i,j} = 1 & \iff i = j; \\ m_{i,i+1} = 3 & \forall i \in [n-1]; \\ m_{i,j} = 2 & \text{si } i, j \in [n] \text{ tels que } |i-j| \geq 2. \end{array} \right. \quad (6)$$

C'est donc un groupe de Coxeter de type B.

Par convention, pour les lettres négatives d'une permutation signée, on écrit $\bar{x}$ au lieu de $-x$. Par exemple $\sigma = [-4, 1, -2, -3, 5]$ peut s'écrire $\bar{4}1\bar{2}\bar{3}5$.

0.1.4 Produit semi-direct

Soient H et K , deux groupes, et f un morphisme de K dans le groupe $\text{Aut}(H)$ des automorphismes de H . A chaque $k \in K$, on associe $f(k) = f_k \in \text{Aut}(H)$. Le **produit semi-direct (externe)** de H et K suivant f que l'on note $H \rtimes_f K$ ou tout simplement $H \rtimes K$ est le produit cartésien de H et K muni de la loi de groupe suivante :

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1 \cdot f_{k_1}(h_2), k_1 \cdot k_2). \quad (7)$$

L'élément neutre $e_{H \rtimes_f K}$ de cette loi de composition est (e_H, e_K) , et l'inverse d'un élément (h, k) est défini par :

$$(h, k)^{-1} = (f_{k^{-1}}(h^{-1}), k^{-1}). \quad (8)$$

Si f est le morphisme trivial, i.e., $f_k(h) = h \quad \forall k \in K$, alors nous avons le produit direct.

Soit G un groupe et soient H et K des sous-groupes de G tels que H est distingué, $H \cap K = 1$ et $G = HK$. Alors G est isomorphe au produit semi-direct de H et K suivant l'automorphisme de conjugaison :

$$f_k(h) = khk^{-1}.$$

On dit dans ce cas que G est le **produit semi-direct interne** de H et K . Inversement, si G est un produit semi-direct externe de deux groupes H et K , on peut injecter H et K dans G par les applications ι_1 et ι_2 définies par $h \mapsto \iota_1(h) = (h, 1)$ et $k \mapsto \iota_2(k) = (1, k)$. On vérifie que G est le produit semi-direct interne de $\tilde{H} = \iota_1(H)$ et $\tilde{K} = \iota_2(K)$.

0.1.5 Le produit en couronne $C_r \wr S_n$

Soit r un entier positif. Posons $\zeta = e^{2i\pi/r}$. Le groupe cyclique d'ordre r noté C_r est le groupe engendré par ζ . Notons C_r^n le produit direct n -fois de C_r . On peut considérer C_r^n comme l'ensemble des applications ϵ de $[n]$ vers C_r qui à i associe $\epsilon(i) = \epsilon_i$. L'opération sur C_r^n est donc définie par :

$$\forall \epsilon, \epsilon' \in C_r^n, \quad (\epsilon \cdot \epsilon')(i) = \epsilon(i) \cdot \epsilon'(i). \quad (9)$$

Considérons le morphisme f de S_n vers $\text{Aut}(C_r^n)$ défini par :

$$\forall \sigma \in S_n, \forall \epsilon \in C_r^n, \quad f(\sigma)(\epsilon) = \epsilon \circ \sigma^{-1}. \quad (10)$$

Le produit en couronne $C_r \wr S_n$ de C_r et S_n est le produit semi-direct de C_r^n et S_n suivant le morphisme f ,

$$C_r \wr S_n = C_r^n \rtimes_f S_n.$$

La loi de groupe sur $C_r \wr S_n$ est donc définie par :

$$(\epsilon, \sigma) \cdot (\epsilon', \sigma') = ((\epsilon_1 \epsilon'_{\sigma^{-1}(1)}, \epsilon_2 \epsilon'_{\sigma^{-1}(2)}, \dots, \epsilon_n \epsilon'_{\sigma^{-1}(n)}), \sigma \circ \sigma'). \quad (11)$$

L'inverse d'un élément (ϵ, σ) est défini par

$$(\epsilon, \sigma)^{-1} = ((\epsilon \circ \sigma^{-1})^{-1}, \sigma^{-1}),$$

où $\epsilon^{-1}(i) = \frac{1}{\epsilon(i)}$ est l'inverse de ϵ_i dans C_r .

Exemple 2. Prenons $n = 4$; $r = 3$; et soient :

$$\epsilon = (1, \zeta^2, \zeta, \zeta), \sigma = 2413 \text{ et } \epsilon' = (\zeta, 1, 1, \zeta), \sigma' = 1342.$$

Alors nous avons :

$$(1, \zeta^2, \zeta, \zeta, 2413) \cdot (\zeta, 1, 1, \zeta, 1342) = (1, 1, \zeta^2, \zeta, 2134).$$

0.1.6 Le Groupe des permutations colorées $G(r, n)$

On appelle permutation colorée (r -colorée) d'ordre n toute permutation π de l'alphabet :

$$\Sigma_{r,n} := C_r \times [n] = \{\xi i \mid i \in [n], \xi \in C_r\}, \quad (12)$$

vérifiant la propriété :

$$\pi(\xi i) = \xi \pi(i) \quad \forall \xi \in C_r, i \in [n]. \quad (13)$$

Notons $G(r, n)$ le groupe des permutations r -colorées d'ordre n . Comme dans le cas du groupe hyperoctaédral, une permutation colorée π est complètement définie par sa restriction sur $[n]$:

$$\pi = [\pi_1, \pi_2, \dots, \pi_n].$$

Ainsi, le cardinal de $G(r, n)$ est égal à $r^n n!$. Pour les petites valeurs de r on utilise j barres au-dessus de i au lieu de $\zeta^j i$. Par exemple, la permutation $\zeta^2 1 \zeta^2 4 \zeta 3$ s'écrit tout simplement $\bar{2} 1 \bar{\bar{4}} \bar{3} \in G(3, 4)$.

Soit $\pi \in G(r, n)$ et $\pi(i) = \xi x$ avec $\xi \in C_r$ et $x \in [n]$ alors on note :

$$x = |\pi(i)| \quad \text{et} \quad \xi = \text{color}_\pi(x). \quad (14)$$

On note $|\pi|$ la permutation définie par $|\pi|(i) = |\pi(i)|$ pour tout $i \in [n]$. Pour l'exemple ci-dessus, $|\pi| = 2 1 4 3$ et $\text{color}_\pi(2) = \zeta$; $\text{color}_\pi(1) = 1$; $\text{color}_\pi(4) = \zeta^2$ et $\text{color}_\pi(3) = \zeta$.

Comme dans le groupe symétrique, une permutation colorée π peut être représentée en produit de cycles. Les cycles de π se construisent à partir des cycles de $|\pi|$, ensuite on rétablit les couleurs de chaque entier.

Exemple 3.

$$2143 = (12)(34) \implies \bar{2} 1 \bar{\bar{4}} \bar{3} = (1, \bar{2})(\bar{3}, \bar{\bar{4}}).$$

0.1.7 Isomorphisme entre $C_r \wr S_n$ et $G(r, n)$

Le groupe $G(r, n)$ des permutations r -colorées et le produit en couronne $C_r \wr S_n$ sont isomorphes par l'application :

$$(\epsilon, \sigma) \longmapsto [\epsilon_{\sigma(1)} \sigma_1, \epsilon_{\sigma(2)} \sigma_2, \dots, \epsilon_{\sigma(n)} \sigma_n].$$

Nous avons par exemple la correspondance suivante :

$$(1, \zeta, \zeta, \zeta^2, 2 1 4 3) \longleftrightarrow \bar{2} 1 \bar{\bar{4}} \bar{3}.$$

Convention : normalement la représentation de $\pi = [\xi_1\sigma_1, \dots, \xi_n\sigma_n]$ en tant qu'élément du produit en couronne est (ϵ, σ) avec $\epsilon_i = \xi_{\sigma^{-1}(i)}$, mais par convention, on écrit $\pi = (\xi, \sigma)$. Par exemple, on écrit

$$\bar{2}1\bar{4}\bar{3} = (\zeta, 1, \zeta^2, \zeta, 2143)$$

au lieu de $(1, \zeta, \zeta, \zeta^2, 2143)$.

Dans toute la suite, on utilisera la convention d'écriture :

$$[\xi_1\sigma_1, \dots, \xi_n\sigma_n] \equiv (\xi_1, \xi_2, \dots, \xi_n, \sigma). \quad (15)$$

Par conséquent, si $\pi = (\xi, |\pi|)$ alors $\xi(i)$ est la couleur de la i -ième lettre de π et non pas la couleur de l'entier i . La couleur de l'entier i dans l'écriture de π est notée $\text{color}_\pi(i)$.

Le produit en couronne s'identifie ainsi avec le groupe des permutations colorées.

Soit ζ un générateur du groupe cyclique C_r . Le groupe $G(r, n)$, ($r > 1$) est engendré par les générateurs :

$$s_0 = [\zeta, 2, \dots, n]; \quad s_i = [1, 2, \dots, i-1, i+1, i, i+2, \dots, n] \quad \forall i \in [n-1].$$

Ces générateurs sont liés par les relations :

$$\left\{ \begin{array}{ll} m_{0,0} = r & \text{si } r \text{ est impair et } \frac{r}{2} \text{ sinon;} \\ m_{0,1} = 2r; \\ m_{i,i} = 1 & \forall i \geq 1; \\ m_{i,i+1} = 3 & \forall i \in [n-1]; \\ m_{i,j} = 2 & \text{si } i, j \in [n] \text{ tels que } |i-j| \geq 2. \end{array} \right. \quad (16)$$

On remarque que seuls $G(1, n)$ et $G(2, n)$ sont des groupes de Coxeter parmi les groupes de permutations colorées.

0.1.8 Le groupe de réflexions complexes $G(r, p, n)$

Pour $\pi = (\epsilon, \sigma) \in G(r, n)$, on définit les statistiques “col” et “col_{*i*}” :

$$\text{col}_i \pi := \sum_{0 \leq j \leq r-1} j \chi(\epsilon_i = \zeta^j) \quad (1 \leq i \leq n); \quad (17)$$

$$\text{col } \pi := \sum_{1 \leq i \leq n} \text{col}_i \pi, \quad (18)$$

où $\chi(A) = 1$ si A est vraie et $\chi(A) = 0$ sinon. Par exemple, si $\pi = \bar{2}1\bar{4}\bar{3}$ alors $\text{col}_1(\pi) = \text{col}_4(\pi) = 1$; $\text{col}_2(\pi) = 0$ et $\text{col}_3(\pi) = 2$ donc $\text{col}(\pi) = 4$.

Soit p un entier tel que $p|r$. Le groupe de réflexions complexes $G(r, p, n)$ est le sous-groupe de $G(r, n)$ défini par :

$$G(r, p, n) = \{\pi \in G(r, n) \mid \text{col}(\pi) \equiv 0 \pmod{p}\}. \quad (19)$$

Pour $p = 1$, alors $G(r, 1, n)$ n'est autre que $G(r, n)$.

Pour $p = 2$ et $r = 2$, $G(2, 2, n) = D_n$, c'est le groupe des permutations signées paires. Les générateurs de D_n sont :

$$s_0 = [\bar{2}, \bar{1}, 3, \dots, n] \text{ et } \forall i \in [n-1], \quad s_i = [1, 2, \dots, i-1, i+1, i, i+2, \dots, n],$$

avec les relations :

$$\left\{ \begin{array}{l} m_{i,i} = 1 \quad \forall i \geq 0; \\ m_{0,1} = 2; \\ m_{0,2} = 3; \\ m_{i,i+1} = 3 \quad \forall i \in [n-1]; \\ m_{i,j} = 2 \quad \text{si } i, j \in [n] \text{ tels que } |i-j| \geq 2. \end{array} \right. \quad (20)$$

C'est donc un groupe de Coxeter de type D .

0.1.9 Relations d'ordre sur l'alphabet coloré

Comme nous l'avons signalé précédemment, le choix de la relation d'ordre sur l'alphabet r -coloré $\Sigma_{r,n}$ est très important pour pouvoir étendre certaines statistiques sur $G(r, n)$. Il y a $(nr)!$ possibilités d'ordonner $\Sigma_{r,n}$ mais généralement on utilise celles qui prolongent l'ordre naturel sur $[n]$. Nous allons donner les quelques relations d'ordre fréquemment utilisées. Tout d'abord nous définissons une relation d'ordre sur C_r .

Comme le groupe cyclique C_r est engendré par $\zeta = e^{\frac{2\pi}{r}i}$, pour tout élément ξ de C_r il existe un unique entier j tel que $0 \leq j \leq r-1$ et $\xi = \zeta^j$. On définit sur C_r la relation d'ordre suivante :

$$\zeta^a < \zeta^b \iff b < a, \quad 0 \leq a, b \leq r-1,$$

donc

$$1 > \zeta > \zeta^2 > \dots > \zeta^{r-1}.$$

Voici maintenant quelques relations d'ordre sur l'alphabet r -colorée $\Sigma_{r,n}$.

Définition 0.1 (ordre par valeur : *OV*).

$$\xi_1 x_1 < \xi_2 x_2 \iff [x_1 < x_2] \text{ ou } [(x_1 = x_2) \text{ et } (\xi_1 < \xi_2)]. \quad (21)$$

Dans $\Sigma_{3,4}$ par exemple

$$\bar{\bar{1}} < \bar{1} < \bar{1} < 1 < \bar{\bar{2}} < \bar{2} < \bar{2} < 2 < \bar{\bar{3}} < \bar{3} < \bar{3} < 3 < \bar{\bar{4}} < \bar{4} < \bar{4} < 4.$$

Définition 0.2 (ordre par couleur : *OC*).

$$\xi_1 x_1 < \xi_2 x_2 \iff [\xi_1 < \xi_2] \text{ ou } [(\xi_1 = \xi_2) \text{ et } (x_1 < x_2)]. \quad (22)$$

Dans $\Sigma_{3,4}$ on a par exemple :

$$\bar{\bar{1}} < \bar{\bar{2}} < \bar{\bar{3}} < \bar{\bar{4}} < \bar{1} < \bar{2} < \bar{3} < \bar{4} < 1 < 2 < 3 < 4.$$

Définition 0.3 (ordre mixte : *OM*).

$$\xi_1 x_1 < \xi_2 x_2 \iff \left[\begin{array}{ll} \xi_1 = \xi_2 = 1 & \text{et } x_1 < x_2, \\ \text{ou} \\ \xi_1 \neq 1 & \text{et } \xi_2 = 1, \\ \text{ou} \\ \xi_1 \neq 1 & \text{et } x_1 > x_2, \\ \text{ou} \\ \xi_1 < \xi_2 & \text{et } x_1 = x_2. \end{array} \right. \quad (23)$$

En particulier, pour $(r, n) = (3, 4)$ on a :

$$\bar{\bar{4}} < \bar{4} < \bar{4} < \bar{\bar{3}} < \bar{3} < \bar{3} < \bar{\bar{2}} < \bar{2} < \bar{2} < \bar{\bar{1}} < \bar{1} < \bar{1} < 1 < 2 < 3 < 4,$$

et pour $r = 2$, *OM* est :

$$\bar{n} < \overline{n-1} < \dots < \bar{2} < \bar{1} < 1 < 2 < \dots < n.$$

0.1.10 Quelques statistiques

Statistiques classiques

Soit $a = (a_1, a_2, \dots, a_n)$ où a_i est une lettre de l'alphabet coloré $\Sigma_{r,n}(12)$. On définit les statistiques suivantes

$$\begin{aligned} \text{des } a &= \sum_{i=1}^n \chi(a_i > a_{i+1}); & \text{exc } a &= \sum_{i=1}^n \chi(a_i > i); \\ \text{fix } a &= \sum_{i=1}^n \chi(a_i = i); & \text{maj } a &= \sum_{i=1}^{n-1} i \chi(a_i > a_{i+1}); \\ \text{inv } a &= \sum_{\substack{1 \leq i \leq n-1 \\ i < j \leq n}} \chi(a_i > a_j). \end{aligned}$$

Fonction longueur ℓ

Soit G un groupe et S un ensemble de générateurs, on définit la *fonction longueur* (par rapport à S) par

$$\ell(\pi) = \min\{k \in \mathbb{N} : \pi = s_1 s_2 \cdots s_k; s_i \in S\}, \forall \pi \in G.$$

La statistique “fonction longueur” coïncide avec le *nombre d’inversions* “inv” sur S_n . Soit flag-inversion “finv” la statistique sur B_n définie à l’aide de la relation d’ordre OM par :

$$\text{finv}(\sigma) = \text{inv}(\sigma) + \sum_{\substack{1 \leq i \leq n-1 \\ i < j \leq n}} \chi(\bar{a}_i > a_j), \forall \sigma \in B_n.$$

Alors, la “fonction longueur” coïncide avec “finv” sur B_n .

Le flag-major

Rappelons que le groupe des permutations colorées est engendré par $s_0 = [\zeta, 2, 3, \dots, n]$ et les transpositions $s_i = (i, i+1)$ pour tout $i \in [n-1]$. Considérons les générateurs t_i suivants :

$$t_i = \prod_{j=0}^i s_{i-j} \quad 0 \leq i \leq n-1. \quad (24)$$

Chaque élément π de $G(r, n)$ peut s’écrire de manière unique sous la forme

$$\pi = t_{n-1}^{k_{n-1}} t_{n-2}^{k_{n-2}} \cdots t_1^{k_1} t_0^{k_0}, \quad (25)$$

avec $0 \leq k_i < r(i+1)$ pour tout i variant de 0 à $n-1$. Adin et Roichman [1] ont défini l’extension flag-major “fmaj” sur $G(r, n)$ de la statistique indice majeur par :

$$\text{fmaj}(\pi) = \sum_{i=0}^{n-1} k_i. \quad (26)$$

Ils ont démontré que la statistique “fmaj” et “ r maj + col” sont identiques si “maj” est calculée par rapport à la relation d’ordre OC sur $\Sigma_{r,n}$:

$$\zeta^{r-1} 1 < \cdots < \zeta^{r-1} n < \cdots < \zeta 1 < \cdots < \zeta n < 1 < \cdots < n.$$

Ainsi nous avons la définition équivalente du flag-major sur $G(r, n)$:

$$\text{fmaj}(\pi) = r \text{ maj}(\pi) + \text{col}(\pi), \quad (27)$$

où col est la statistique définie dans (18). Cette définition s’étend aux groupes de réflexions complexes.

Quelques statistiques sur B_n

Dans le groupe hyperoctaédral particulièrement, on définit les ensembles, multi-ensembles et statistiques suivants pour chaque $\sigma \in B_n$.

$$\begin{aligned}
\text{Neg}(\sigma) &= \{i \in [n] : \sigma(i) < 0\}; \\
\text{Des}_A(\sigma) &= \{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}; \\
\text{Des}_B(\sigma) &= \{i \in [0, n-1] \mid \sigma_i > \sigma_{i+1}\} \text{ avec } \sigma(0) = 0; \\
\text{NDes}(\sigma) &= \text{Des}_A(\sigma) \uplus \{-\sigma(i) : i \in \text{Neg}(\sigma)\}; \\
\text{neg}(\sigma) &= \# \text{Neg}(\sigma); \\
\text{ndes}(\sigma) &= \# \text{NDes}(\sigma); \\
\text{des}_A(\sigma) &= \# \text{Des}_A(\sigma); \\
\text{des}_B(\sigma) &= \# \text{Des}_B(\sigma); \\
\text{nmaj}(\sigma) &= \sum_{i \in \text{NDes}(\sigma)} i; \\
\text{fdes}(\pi) &= 2\text{des}_A(\sigma) + \chi(\sigma(1) < 0).
\end{aligned}$$

Exemple 4. Si $\sigma = \bar{3}1\bar{6}2\bar{4}\bar{5} \in B_6$, alors avec la relation d'ordre OM nous avons :

$$\begin{aligned}
\text{Des}_A(\sigma) &= \{2, 4, 5\}; & \text{Des}_B(\sigma) &= \{0, 2, 4, 5\}; & \text{Neg}(\sigma) &= \{1, 3, 5, 6\}; \\
\text{NDes}(\sigma) &= \{2, 3, 4^2, 5^2, 6\}; & \text{neg}(\sigma) &= 4; & \text{ndes}(\sigma) &= 7; \\
\text{nmaj}(\sigma) &= 29; & \text{fdes}(\sigma) &= 7.
\end{aligned}$$

Les k -successions

Nous allons généraliser sur $G(r, n)$ les statistiques des valeurs de successions de saut k ou tout simplement k -succession. On distingue les successions linéaires et circulaires. Soit $\pi = (\varepsilon, |\pi|) \in G(r, n)$, $k > 0$ et $2 \leq i \leq n$. On dit que $\pi(i)$ est une k -succession linéaire (valeur de k -succession linéaire) de π si et seulement si

$$|\pi(i)| = |\pi(i-1)| + k \text{ et } \varepsilon(i) = \varepsilon(i-1). \quad (28)$$

Soit maintenant $k \geq 0$ et $i \in [n]$. On dit que $\pi(i)$ est une k -succession circulaire de π si et seulement si

$$|\pi(i)| = i + k \text{ et } \varepsilon(i) = 1. \quad (29)$$

On note respectivement $\mathcal{L}^k(\pi)$ et $\mathcal{C}^k(\pi)$ l'ensemble des valeurs de k -successions linéaires et circulaires, et $l^k(\pi) = \#\mathcal{L}^k(\pi)$, $c^k(\pi) = \#\mathcal{C}^k(\pi)$.

Exemple 5. Si $\pi = 1\bar{2}46\bar{3}\bar{5} \in G(3,6)$ alors nous avons :

$$\mathcal{L}^2(\pi) = \{6, \bar{5}\}; \quad \mathcal{C}^2(\pi) = \{6\}; \quad l^2(\pi) = 2; \quad c^2(\pi) = 1.$$

On remarque qu'un point fixe n'est autre qu'une 0-succession circulaire.

0.1.11 Les classes m -fixées des permutations

Soit $n > 0$, $m \in [n]$, et $\pi = (\varepsilon, |\pi|)$ une permutation colorée. Notons $\text{FIX}(\pi)$ l'ensemble des points fixes de π .

m -fixées-isolées

On dit que π est m -fixée-isolée si π vérifie les trois conditions suivantes :

- i) $\forall i \in [m], \text{color}_\pi(i) = 1$;
- ii) $\text{FIX}(\pi) \subseteq [m]$;
- iii) chaque cycle de $|\pi|$ contient au plus un entier inférieur ou égal à m .

Exemple 6. $\sigma = (1\bar{6}8)(2)(3\bar{9})(4)(\bar{5}\bar{7})$ est 4-fixée isolée dans $G(3,9)$.

m -fixées-croissantes

On dit que π est m -fixée-croissante si π vérifie les trois conditions suivantes :

- i) $\forall i \in [m], \varepsilon(i) = 1$;
- ii) $\text{FIX}(\pi) \subseteq [m]$;
- iii) $\pi(1) < \pi(2) < \dots < \pi(m)$.

Exemple 7. $\sigma = 1258\bar{3}9\bar{7}\bar{4}6$ est 4-fixée croissante dans $G(3,9)$.

0.1.12 q -notations

Notons respectivement $[n]_q$, $[n]_q!$ et $\begin{bmatrix} n \\ k \end{bmatrix}_q$ les q -analogues de n , $n!$ et du coefficient binomial :

$$[n]_q := \frac{1 - q^n}{1 - q}; \quad [n]_q! := \prod_{1 \leq i \leq n} [i]_q; \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n - k]_q!}.$$

Pour tout $n \geq 0$ on note

$$(x; q)_n = \begin{cases} (1-x)(1-xq) \cdots (1-xq^{n-1}) & \text{si } n > 0, \\ 1 & \text{si } n = 0. \end{cases}$$

Alors nous avons

$$[n]_q! = \frac{(q; q)_n}{(1-q)^n}.$$

Enfin on définit le coefficient q -multinomial :

$$\left[\begin{matrix} m \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q = \frac{(q; q)_m}{(q; q)_{m_1} (q; q)_{m_2} \cdots (q; q)_{m_s}}.$$

0.2 Préliminaires

Nous allons maintenant passer en revue quelques travaux prolongeant l'étude statistiques du groupe symétrique au groupe hyperoctaédral, aux groupes des permutations colorées et plus généralement aux groupes de réflexions complexes.

0.2.1 Sur un Théorème de MacMahon

Il y a presque cent ans (1915), MacMahon a montré l'équidistribution des statistiques "maj" (indice majeur) et "inv" (nombre d'inversion) sur l'ensemble $R_{\mathbf{m}}$ des réarrangements du mot $1^{m_1} 2^{m_2} \dots s^{m_s}$ où $m = m_1 + m_2 + \cdots + m_s$. Si $m_1 = m_2 = \cdots = m_s = 1$ alors $m = s$ et $R_{\mathbf{m}} = S_s$ le groupe symétrique d'ordre s .

Théorème 0.4 (MacMahon). *On a :*

$$\sum_{\pi \in R_{\mathbf{m}}} q^{\text{maj}(\pi)} = \sum_{\pi \in R_{\mathbf{m}}} q^{\text{inv}(\pi)} = \left[\begin{matrix} m \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q. \quad (30)$$

En particulier nous avons l'équidistribution sur S_n :

$$\sum_{\pi \in S_n} q^{\ell(\pi)} = \sum_{\pi \in S_n} q^{\text{maj}(\pi)} = [n]_q!. \quad (31)$$

En 1968, Foata [37] a donné une preuve combinatoire du Théorème 0.4 par la seconde transformation fondamentale φ . Dix ans plus tard, Foata et Schützenberger [38] ont établi un raffinement sur les *classes de descentes*

$$T_n(M) = \{\pi \in S_n \mid \text{Des}(\pi^{-1}) = M\}, \quad M \subseteq [n-1].$$

Théorème 0.5 (Foata-Schützenberger). *Pour tout $M \subseteq [n-1]$, on a :*

$$\sum_{\pi \in T_n(M)} q^{\ell(\pi)} = \sum_{\pi \in T_n(M)} q^{\text{maj}(\pi)}. \quad (32)$$

En 2001, Adin et Roichman [1] ont généralisé (31) sur le groupe hyperoctaédral en définissant l'extension flag-major “fmaj” de l'indice majeur [1, Théorème 2], puis en 2004, sur le groupe des permutations signées paires D_n [3, corollaire 5.5] avec une autre extension “fmaj_D” de l'indice majeur. Dans ce même article (Théorème 3.4), ils ont établi une extension sur B_n du raffinement de Foata-Schützenberger.

Théorème 0.6 (Adin-Roichman). *On a :*

$$\sum_{\pi \in B_n} q^{\ell_B(\pi)} = \sum_{\pi \in B_n} q^{\text{fmaj}(\pi)}; \quad (33)$$

$$\sum_{\pi \in D_n} q^{\ell_D(\sigma)} = \sum_{\pi \in D_n} q^{\text{fmaj}_D(\sigma)}, \quad (34)$$

où ℓ_B et ℓ_D sont les fonctions longueurs sur B_n et D_n . Et la statistique “fmaj_D” est définie par :

$$\text{fmaj}_D(\sigma) = \text{fmaj}(\tilde{\sigma}),$$

avec

$$\tilde{\sigma}(i) = \begin{cases} \sigma(i) & \text{si } i < n, \\ |\sigma(n)| & \text{si } i = n. \end{cases}$$

Voici une extension sur le groupe hyperoctaédral du théorème de Foata-Schützenberger. Soit $M \subseteq [0, n-1]$ et

$$T_n^B(M) = \{\pi \in B_n \mid \text{Des}_B(\pi^{-1}) = M\}$$

le B -analogue de $T_n(M)$.

Théorème 0.7 (Adin-Roichman). *On a :*

$$\sum_{\pi \in T_n^B(M)} q^{\ell_B(\pi)} = \sum_{\pi \in T_n^B(M)} q^{\text{fmaj}(\pi)}. \quad (35)$$

Pour les théorèmes 0.6 et théorème 0.7, les calculs sont faits par rapport à la relation d'ordre OM (voir la définition 0.3). Notons que la relation (33) reste vraie si on remplace la relation d'ordre par OC .

Récemment, Foata et Han [36] ont établi une extension de la seconde transformation fondamentale φ sur B_n et sur les mots signés. Rappelons que φ transforme la statistique “maj” en “inv” sur S_n et sur les mots ordinaires. Nous donnons dans cette thèse une extension de φ sur le produit en couronne $G(r, n)$ et sur les mots r -colorés.

En 2004, Haglund et al. [43] ont obtenu une expression pour le polynôme générateur de l’indice flag-majeur sur le produit en couronne $G(r, n)$. Cette formule généralise la relation (31).

Théorème 0.8 (Haglund-Loehr-Remmel). *On a :*

$$\sum_{\pi \in G(r, n)} q^{f_{\text{maj}}(\pi)} = \frac{(q^r; q^r)_n}{(1 - q)^n}. \quad (36)$$

Ce résultat ne dépend pas de la relation d’ordre choisie.

0.2.2 Sur une identité de Carlitz

En 1954, Carlitz [19] a établi la fameuse identité suivante :

$$\sum_{k \geq 0} [k + 1]_q^n t^k = \frac{A_n(t, q)}{(t; q)_{n+1}}, \quad (37)$$

où $A_n(t, q)$ est le q -polynôme eulerien. C’est en 1975 [18], qu’il a trouvé une interprétation combinatoire de $A_n(t, q)$ par le couple (des, maj).

$$A_n(t, q) = \sum_{\sigma \in S_n} t^{\text{des}(\sigma)} q^{\text{maj}(\sigma)}.$$

Nous avons la fonction génératrice :

$$\sum_{n \geq 0} \frac{A_n(t, q)}{(t; q)_{n+1}} \frac{u^n}{n!} = \sum_{k \geq 0} t^k \exp([k + 1]_q u). \quad (38)$$

Suite à la question posée par Foata en Juillet 2000, à propos de l’extension sur B_n de cette distribution conjointe, Adin Roichman [2], Chow et Gessel [25], Bagno et Biagioli [7], Biagioli et Zeng [9] ont généralisé l’identité de Carlitz sur le groupe hyperoctaédral, sur les groupes de réflexions complexes $G(r, p, n)$ et sur le produit en couronne $G(r, n)$. Les extensions de Adin et al. sur B_n sont faites en termes de *statistiques négatives* (ndes, nmaj) et en *statistiques flag* (fdes, fmaj) (voir sous-section 0.1.10).

Théorème 0.9 (Adin-Roichman). *Les couples $(\text{ndes}, \text{nmaj})$ et $(\text{fdes}, \text{fmaj})$ sont équidistribués sur B_n . Soit*

$$B_n^{(\text{fdes}, \text{fmaj})}(t, q) = \sum_{\pi \in B_n} t^{\text{ndes}(\pi)} q^{\text{nmaj}(\pi)} = \sum_{\pi \in B_n} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)}, \quad (39)$$

leur polynôme générateur commun. Alors

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{B_n^{(\text{fdes}, \text{fmaj})}(t, q)}{(1-t)(t^2 q^2; q^2)_n}. \quad (40)$$

Les calculs sont faits par rapport à la relation d'ordre OM. Mais pour $(\text{fdes}, \text{fmaj})$ le résultat reste vrai même si on remplace OM par OC.

En 2006, Bagno et Biagioli [7] ont trouvé une généralisation de l'identité (40) sur les groupes de réflexions complexes $G(r, p, n)$ en utilisant la relation d'ordre OC.

On étend sur $G(r, p, n)$ la statistique “fdes” par :

$$\text{fdes}(\pi) = r \text{des}(\pi) + \text{col}_1(\pi).$$

On note $G(r, p, n)^{(\text{fdes}, \text{fmaj})}(t, q)$ le polynôme générateur de $(\text{fdes}, \text{fmaj})$ sur $G(r, p, n)$. Soit

$$G(r, p, n)^{(\text{fdes}, \text{fmaj})}(t, q) = \sum_{\pi \in G(r, p, n)} t^{\text{fdes}(\pi)} q^{\text{fmaj}(\pi)}. \quad (41)$$

Théorème 0.10 (Bagno-Biagioli). *on a :*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{G(r, p, n)^{(\text{fdes}, \text{fmaj})}(t, q)}{(1-t)(t^r q^r; q^r)_{n-1} (1-t^d q^{nd})}, \quad (42)$$

où $d = \frac{r}{p}$.

La spécialisation pour $p = 1$ donne l'extension de l'identité de Carlitz sur $C_r \wr S_n$.

Parallèlement, un autre type d'extension sur B_n en termes de $(\text{des}_B, \text{fmaj})$ a été donné dans [25]. Posons

$$B_n^{(\text{des}_B, \text{fmaj})}(t, q) = \sum_{\pi \in B_n} t^{\text{des}_B \pi} q^{\text{fmaj}(\pi)}.$$

Théorème 0.11 (Chow-Gessel). *Avec la relation d'ordre OM, on a :*

$$\sum_{k \geq 0} [2k+1]_q^n t^k = \frac{B_n^{(\text{des}_B, \text{fmaj})}(t, q)}{(t; q^2)_{n+1}}, \quad (43)$$

et

$$\sum_{n \geq 0} \frac{B_n^{(\text{des}_B, \text{fmaj})}(t, q)}{(t; q^2)_{n+1}} \frac{u^n}{n!} = \sum_{k \geq 0} t^k \exp([2k+1]_q u). \quad (44)$$

Ce dernier résultat a été généralisé dans [9].

Soit des_G l'extension sur $G(r, n)$ de des_B définie par $\text{des}_G(\pi) = \#\text{Des}_G(\pi)$, où

$$\text{Des}_G(\pi) = \{i \in [0, n-1] \mid \pi_i > \pi_{i+1}\} \text{ avec } \pi(0) = 0 \quad \forall \pi \in G(r, n).$$

On pose :

$$G(r, n)^{(\text{des}_G, \text{fmaj})}(t, q) = \sum_{\pi \in G(r, n)} t^{\text{des}_G \pi} q^{\text{fmaj}(\pi)}.$$

Théorème 0.12 (Biagioli-Zeng). *Avec la relation d'ordre OM, on a :*

$$\sum_{k \geq 0} [rk+1]_q^n t^k = \frac{G(r, n)^{(\text{des}_G, \text{fmaj})}(t, q)}{(t; q^r)_{n+1}}. \quad (45)$$

Ainsi l'identité de Carlitz définie sur le groupe symétrique peut s'étendre de différentes manières sur d'autres groupes plus généraux.

0.2.3 Autre raffinement du polynôme eulérien

Considerons maintenant le raffinement suivant du polynôme eulérien :

$$P_n(t, x, y) = \sum_{\sigma \in S_n} t^{\text{exc}(\sigma)} x^{\text{fix}(\sigma)} y^{\text{cyc}(\sigma)}. \quad (46)$$

Rappelons que les statistiques “exc” et “des” sont équidistribuées sur le groupe symétrique. La première transformation fondamentale de Foata en fournit une preuve combinatoire. Soient $\text{fix}(\pi)$ et $\text{cyc}(\pi)$ les nombres de points fixes et de cycles d'une permutation π . Nous avons les formules pour les cas particuliers suivants (voir [8] et [44]) :

$$P_n(t, 1, -1) = -(t-1)^{n-1}, \quad (47)$$

et

$$P_n(t, 0, -1) = -t[n-1]_t. \quad (48)$$

A noter que Ksavrelof et Zeng [44] ont donné un raffinement de la relation (48) sur les dérangements.

En 2006, Bagno et Garber [8] ont trouvé des extensions de ces relations sur $G(r, n)$ et sur D_n . Notons respectivement $P_{r,n}(q, s, t)$ et $P_{D,n}(q, s, t)$ les extensions sur $G(r, n)$ et D_n de $P_n(q, s, t)$. Ils ont utilisé les deux relations d'ordre OV et OC sur $\Sigma_{r,n}$.

Théorème 0.13 (Bagno-Garber). *Pour tous $n \geq 1$, $r \geq 1$ on a :*

$$P_{r,n}(t, 1, -1) = -\frac{(t^r - 1)^n}{t - 1}, \quad (49)$$

et

$$P_{r,n}(t, 0, -1) = -t[r]_t^n [n - 1]_t. \quad (50)$$

A noter que ces identités sont vraies avec chacune des deux relations d'ordre citées ci-dessus. Mais pour l'extension de la relation (47) sur D_n , le résultat varie suivant la relation d'ordre utilisée. Pour éviter les confusions, on note $P_{D,n}^{OC}(t, x, y)$ (resp. $P_{D,n}^{OV}(t, x, y)$) le polynôme $P_{D,n}(t, x, y)$ calculé avec la relation d'ordre OC (resp. OV).

Théorème 0.14 (Bagno-Garber). *Pour $n \geq 1$, on a*

$$P_{D,n}^{OC}(t, 1, -1) = (1 - t^2)^{n-1}, \quad (51)$$

et

$$P_{D,n}^{OV}(t, 1, -1) = -\frac{1}{2}(t - 1)^{n-1}((1 + t)^n + (1 - t)^n). \quad (52)$$

0.2.4 Sur le nombre de dérangements

Rappelons qu'un *dérangement* d'ordre n est une permutation π de S_n qui n'a pas de point fixe i.e. $\pi(i) \neq i \quad \forall i \in [n]$. On note d_n le nombre de dérangements d'ordre n . Un *dérangement relatif* d'ordre n est une permutation de S_n qui n'a pas de 1-*succession circulaire* i.e. $\pi(i) \neq i + 1 \quad \forall i \in [n - 1]$. On note q_n le nombre de dérangements relatifs d'ordre n . La notion de dérangement peut être étendue à $G(r, n)$. Un dérangement r -coloré d'ordre n est une permutation $\pi \in G(r, n)$ telle que $\pi(i) \neq i \quad \forall i \in [n]$. On note $\mathcal{D}(r, n)$ l'ensemble des dérangements r -colorés. Des relations sur d_n sont bien connues, comme par exemple :

$$d_{n+1} = n(d_n + d_{n-1}), \quad (53)$$

$$d_n = nd_{n-1} + (-1)^n. \quad (54)$$

Dans cette thèse, nous allons généraliser ces relations sur $G(r, n)$. D'autre part, nous avons la relation suivante entre les nombres de dérangements et de dérangements relatifs :

$$q_n = d_n + d_{n-1}. \quad (55)$$

Chen [20] en 1996, Clarke et al. [27] en 1997 ont donné des preuves combinatoires de cette relation. Récemment, Chen et Zhang [21] ont généralisé cette relation dans le groupe hyperoctaédral par l'intermédiaire de la notion de *Signed Skew Derangement*. Dans le premier chapitre de cette thèse, nous généralisons cette relation en termes de *succession* dans $G(r, n)$.

Par ailleurs, en 1989, Wachs [54], a considéré le q -analogue du nombre de dérangements :

$$d_n(q) = \sum_{\sigma \in \mathcal{D}(1, n)} q^{\text{maj}(\sigma)},$$

et a démontré combinatoirement que :

$$d_n(q) = [n]_q! \sum_{k=0}^n \frac{(-1)^k q^{\binom{k}{2}}}{[k]_q!}. \quad (56)$$

En 2005, Chow [24] a défini une extension de ce q -dérangement sur B_n

$$d_n^B(q) = \sum_{\sigma \in \mathcal{D}(2, n)} q^{\text{fmaj} \sigma}. \quad (57)$$

et a démontré que

$$d_n^B(q) = [2]_q [4]_q \cdots [2n]_q \sum_{k=0}^n \frac{(-1)^k q^{2\binom{k}{2}}}{[2]_q [4]_q \cdots [2k]_q}. \quad (58)$$

0.2.5 Sur une identité de Gessel-Simion

Considérons l'identité de Gessel et Simion suivante [55] (voir aussi [40]) :

$$\sum_{\sigma \in S_n} \text{sign}(\sigma) q^{\text{maj}(\sigma)} = [n]_{\pm q}!, \quad (59)$$

où $\text{sign}(\sigma) = (-1)^{\ell(\sigma)}$ (ℓ la fonction longueur) et

$$[n]_{\pm q}! = \prod_{i=1}^n [i]_{(-1)^{i-1}q}.$$

Adin, Gessel et Roichman [4] ont donné un B -analogue de cette identité avec l'extension “fmaj”, de la statistique “maj” et en utilisant la relation d'ordre OC , ils ont établi l'extension suivante :

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{fmaj}(\sigma)} = [2]_{-q} [4]_{-q} \cdots [2n]_{(-1)^n q}. \quad (60)$$

En utilisant la relation OM avec la même statistique “fmaj”, Fire [40] a trouvé le résultat suivant :

$$\sum_{\sigma \in B_n} \text{sign}(\sigma) q^{\text{fmaj}(\sigma)} = (q; -1)_n [n]_{\pm q^2}!. \quad (61)$$

Ainsi, nous avons vu avec ces quelques exemples cités parmi tant d'autres, les difficultés rencontrées par les chercheurs pour les extensions des résultats classiques aux permutations colorées. Le problème se pose généralement sur la recherche d'une extension convenable de la statistique étudiée et de la relation d'ordre utilisée. Cette thèse a pour but de renforcer ces travaux actifs en prenant le tableau d'Euler comme sujet d'étude. Voici nos résultats principaux.

0.3 Résultats principaux

Le tableau d'Euler est une matrice triangulaire infinie $(g_n^m)_{n>1, 0 \leq m \leq n}$ définie par les relations suivantes :

$$\begin{cases} g_n^n = n! & (m = n); \\ g_n^m = g_n^{m+1} - g_{n-1}^m & (0 \leq m \leq n-1). \end{cases} \quad (62)$$

Comme $n!$ est le cardinal du groupe symétrique d'ordre n , le tableau d'Euler peut être considéré comme une matrice associée au groupe symétrique. La table 1 nous donne les premières valeurs des g_n^m .

$n \backslash m$	0	1	2	3	4	5
0	1					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
5	44	53	64	78	96	5!

TABLE 1 – Valeurs de g_n^m pour $0 \leq m \leq n \leq 5$.

Dumont et Randrianarivony [30] ont donné une interprétation du coefficient g_n^m en termes de *points fixes*.

Théorème 0.15 (Dumont-Randrianarivony). *L'entier g_n^m s'interprète comme le nombre de permutations de S_n dont les points fixes sont majorés par m .*

Tandis que Rakotondrajao a trouvé une interprétation en termes de statistique des *successions circulaires*.

Théorème 0.16 (Rakotondrajao). *L'entier g_n^m est le nombre de permutations de S_n sans m -succession circulaire.*

En particulier, les éléments de la première colonne correspondent aux nombres de *dérangements* dans S_n :

$$g_n^0 = d_n.$$

Dans les deux premiers chapitres de cette thèse, nous construisons un r -analogue $(g_{r,n}^m)_{n,m \geq 0}$ du tableau d'Euler en substituant la condition initiale $n!$ par $r^n n!$ pour que la matrice obtenue soit liée au produit en couronne $G(r, n)$ dont le cardinal est $r^n n!$.

$$\begin{cases} g_{r,n}^n = r^n n! & (m = n); \\ g_{r,n}^m = g_{r,n}^{m+1} - g_{r,n-1}^m & (0 \leq m \leq n-1). \end{cases} \quad (63)$$

Nous généralisons sur $G(r, n)$ les résultats sur le tableau d'Euler dans le groupe symétrique. Le théorème suivant nous donne une formule explicite.

Théorème 0.17. *Pour tous entiers $m \geq 0$, $r > 0$, $n > 0$, on a :*

$$g_{r,n+m}^m = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} r^{m+i} (m+i)!.$$

De plus, on peut calculer les fonctions génératrices exponentielles simple et double de $g_{r,n+m}^m$.

Théorème 0.18. *On a*

$$\sum_{n \geq 0} g_{r,n+m}^m \frac{u^n}{n!} = \frac{r^m m! \exp(-u)}{(1 - ru)^{m+1}},$$

et

$$\sum_{m,n \geq 0} g_{r,n+m}^m \frac{x^m u^n}{m! n!} = \frac{\exp(-u)}{1 - rx - ru}.$$

Nous pouvons étendre sur $G(r, n)$ les interprétations de Rakotondrajao et de Dumont-Randrianarivony.

Théorème 0.19. *Soient les entiers $r > 0, n \geq 0$ et $n \geq m \geq 0$. Alors, pour tout entier k tel que $0 \leq k \leq m$, l'entier $g_{r,n}^m$ est le nombre de permutations dans $G(r, n)$ dont les k -successions circulaires sont incluses dans $[m]$.*

Les spécialisations pour $(r = 1, k = 0)$ et $(r = 1, k = m)$ de ce théorème correspondent respectivement aux résultats des Théorème 0.15 et 0.16. Le cas particulier $m = 0$ et $k = 0$ nous donne quand à lui le r -analogue de la formule du nombre de *dérangements* suivant :

$$d_{r,n} = g_{r,n}^0 = n! \sum_{i=0}^n \frac{(-1)^i r^{n-i}}{i!}. \quad (64)$$

Plusieurs relations importantes sur le nombre de dérangements ont été établies sur le groupe symétrique. Rappelons la relation (55) :

$$q_n = d_n + d_{n-1}.$$

Dans le résultat suivant, nous généralisons cette relation en termes de k -*succession* dans le produit en couronne $G(r, n)$.

Théorème 0.20. *Soient n, k et m des entiers tels que $n \geq 1, k \geq 0$ et $m \geq 0$. Alors*

$$l_{r,n+1,m}^{k+1} = c_{r,n+1,m}^k + c_{r,n,m}^k - c_{r,n,m-1}^k,$$

où $c_{r,n,m}^k$ (resp. $l_{r,n,m}^k$) désigne le nombre des permutations dans $G(r, n)$ ayant m k -successions circulaires (resp. linéaires). On convient de poser $c_{r,n,-1}^k = 0$.

En particulier, on retrouve pour $(k, r) = (0, 1)$ le résultat de Clarke et al. dans [27], et pour $(k, r, m) = (0, 2, 0)$ le résultat principal de Chen et Zhang [21].

Notre preuve repose sur l'extension d'une variante de la première transformation fondamentale de Foata qui transforme les *successions circulaires* en *successions linéaires* (voir Chapitre 1, Théorème 1.2.3).

Par ailleurs, les coefficients $(g_{r,n}^m)_{n,m}$ sont divisibles par $r^n n!$. Posons :

$$d_{r,n}^m = \frac{g_{r,n}^m}{r^n m!}. \quad (65)$$

Pour tout entier positif r fixé, nous avons la nouvelle matrice de coefficients généraux $(d_{r,n}^m)_{n,m \geq 0}$ qui est définie par :

$$\begin{cases} d_{r,n}^n = 1 & (m = n); \\ d_{r,n}^m = r(m+1) d_{r,n}^{m+1} - d_{r,n-1}^m & (0 \leq m \leq n-1). \end{cases} \quad (66)$$

Dans son récent article [47], Rakotondrajao a étudié le cas $r = 1$ de cette matrice. Elle a démontré que le nombre $d_{1,n}^m$ s'interprète comme le nombre des *m-fixed point-permutations*. C'est la classe des permutations de S_n telles que les points fixes sont majorés par m et chaque cycle contient au plus un élément de $[m]$. Nous généralisons cette interprétation pour r quelconque. Nous avons trouvé deux classes de permutations colorées pour notre interprétation. La classe des *m-fixées-isolées* qui est une extension de *m-fixed-permutation* de Rakotondrajao et la classe des *m-fixées-croissantes* (voir sous section 0.1.11).

Théorème 0.21. *Pour tout entier $r > 0$ et $0 \leq m \leq n$, $d_{r,n}^m$ est le nombre des m-fixées-isolées dans $G(r, n)$.*

Théorème 0.22. *Pour tout entier $r > 0$ et $0 \leq m \leq n$, $d_{r,n}^m$ est le nombre des m-fixées-croissantes dans $G(r, n)$.*

Ces résultats montrent évidemment que l'ensemble des permutations *m-fixées-isolées* et *m-fixées croissantes* ont le même cardinal. Nous donnons une preuve combinatoire de cette propriété.

Nous terminons le deuxième chapitre par des démonstrations combinatoires des relations suivantes :

$$d_{r,n}^m = r(m+1) d_{r,n}^{m+1} - d_{r,n-1}^m, \quad (0 \leq m \leq n-1; \quad r > 0),$$

$$d_{r,n}^m + d_{r,n-2}^{m-1} = r n d_{r,n-1}^m, \quad (m \geq 1, \quad n \geq 2),$$

et la relation de récurrence de $G(r, n)$ -analogue du nombre de dérangements

$$d_{r,n}^0 = r n d_{r,n-1}^0 + (-1)^n \quad (n \geq 1),$$

où $d_{r,0}^0 = 1$, $d_{r,1}^0 = r - 1$ et $d_{r,1}^1 = 1$.

Dans le chapitre 3, nous étudions un (r, q, x) -analogue du tableau d'Euler. Plus précisément, nous allons étendre sur $G(r, n)$ le (q, x) -analogue étudié par Clarke, Han et Zeng [27] en 1997. Ainsi pour tous entiers positifs r et x , posons :

$$A_{r,n}(q, x) = [r]_q [2r]_q \cdots [nr]_q \sum_{k=0}^n \frac{(x-1)(x-q^r) \cdots (x-q^{r(k-1)})}{[r]_q [2r]_q \cdots [kr]_q}.$$

Définissons $\{g_{r,n}^m(q, x)\}_{m \geq n \geq 0}$ de la manière suivante :

$$\begin{cases} g_{r,n}^n(q, x) = A_{r,n}(q, x); \\ g_{r,n}^m(q, x) = g_{r,n}^{m+1}(q, x) - xq^{r(n-m-1)}g_{r,n-1}^m(q, x) \quad (0 \leq m \leq n-1). \end{cases}$$

La spécialisation $q = x = r = 1$ nous donne le tableau d'Euler. Nous trouvons la forme explicite :

Théorème 0.23. *Pour $n \geq m \geq 0$, on a :*

$$g_{r,n}^m(q, x) = \sum_{k=0}^{n-m} (-x)^k \begin{bmatrix} n-m \\ k \end{bmatrix}_{q^r} q^{r \binom{k}{2}} (n-k)!_{q^r} [r]_q^{n-k} \sum_{i=0}^{n-k} \prod_{j=0}^{i-1} (x - q^{rj}) \frac{i!_{q^r}}{[r]_q^i}.$$

Clarke et al. ont construit la nouvelle statistique mahonienne “maf” sur S_n et ils ont donné une interprétation combinatoire du polynôme $g_{1,n}^m(q, x)$ en termes de cette statistique “maf” et de la statistique “fix” des points fixes.

Pour pouvoir étendre sur $G(r, n)$ cette interprétation combinatoire, nous avons besoin de définir un r -analogue de la statistique “maf” que nous appellerons flag – maf, notée tout simplement “fmaf” :

$$\text{fmaf}(\pi) = r \left(\sum_{i \in \text{FIX}(\pi)} i - \sum_{1 \leq i \leq \text{fix}(\pi)} i \right) + \text{fmaj Der}(\pi),$$

où $\text{Der}(\pi)$ est la partie dérangée de π définie ci-dessous.

Soit $\pi = x_1 x_2 \cdots x_n \in G(r, n)$. Supposons que π a $n - m$ points fixes et que $y_1 y_2 \cdots y_m$ est le mot obtenu de π en supprimant les points fixes, avec $y_i = \epsilon_i |y_i|$. Notons rank la bijection croissante de $\{|y_1|, |y_2|, \dots, |y_m|\}$ sur $[m]$ et $z_i = \epsilon_i \text{rank}(|y_i|)$, alors $\text{Der}(\pi) = z_1 z_2 \cdots z_m$.

Remarque 8. *Dans toute la suite, les statistiques sont calculées par rapport à la relation d'ordre OC.*

Exemple 9. *Si $\pi = 1 \bar{7} 3 6 5 \bar{9} \bar{2} 8 4 \in G(3, 9)$, alors les points fixes de π sont 1, 3, 5, 8 et $y_1 y_2 \cdots y_5 = \bar{7} 6 \bar{9} \bar{2} 4$ donc $\text{Der}(\pi) = \bar{4} 3 \bar{5} \bar{1} 2$.*

Nous allons montrer que la bijection-clé de Clarke et al. dans [27] peut s'étendre facilement sur $G(r, n)$, nous permettant aussi de prouver le résultat suivant.

Théorème 0.24. *Les triplets de distributions $(\text{fmaf}, \text{exc}, \text{fix})$ et $(\text{fmaj}, \text{exc}, \text{fix})$ sont équadistribués sur $G(r, n)$. De plus*

$$\sum_{\sigma \in G(r, n)} q^{\text{fmaf } \sigma} x^{\text{fix } \sigma} = [r]_q [2r]_q \cdots [nr]_q \sum_{k=0}^n \frac{(x-1)(x-q^r) \cdots (x-q^{r(k-1)})}{[r]_q [2r]_q \cdots [kr]_q}.$$

En ce qui concerne l'interprétation combinatoire de $g_{r,n}^m(q, x)$ pour r quelconque, nous notons $G_m(r, n)$ l'ensemble des permutations r -colorées d'ordre n dont les points fixes sont minorés par $n - m + 1$. En particulier, $G_n(r, n) = G(r, n)$ et $\mathcal{D}(r, n) := G_0(r, n)$ est l'ensemble des *dérangements* r -colorés d'ordre n .

Théorème 0.25. *Pour tous entiers $r > 0$, $x \geq 0$, $n \geq m \geq 0$,*

$$g_{r,n}^m(q, x) = \sum_{\sigma \in G_m(r, n)} q^{\text{fmaf } \sigma} x^{\text{fix } \sigma}.$$

Pour $x = 1$ et $m = 0$, on en déduit la formule explicite des (r, q) -*dérangements* $d_{r,n}(q) := g_{r,n}^0(q, x)$ suivante.

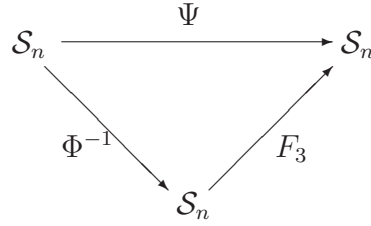
Corollaire 0.26. *On a :*

$$d_{r,n}(q) = \sum_{\sigma \in \mathcal{D}_{r,n}} q^{\text{fmaj } \sigma} = [r]_q [2r]_q \cdots [nr]_q \sum_{k=0}^n \frac{(-1)^k q^{r \binom{k}{2}}}{[r]_q [2r]_q \cdots [kr]_q}.$$

Par les spécialisations $r = 1$ et $r = 2$ de cette identité, on retrouve les relations (56) et (58) qui sont obtenues respectivement par Wachs [54] et Chow [24].

Tout récemment, Foata et Han [34] ont construit deux transformations F_3 et Φ telles que la composition $F_3 \circ \Phi^{-1}$ possède la même propriété que la bijection Ψ de Clarke et al. [27] qui transforme “maj” en “maf” sur S_n . Mais leur égalité reste un problème ouvert. Nous allons démontrer dans le chapitre quatre qu'effectivement les deux transformations de Foata et Han fournissent une factorisation de la bijection de Clarke et al. En d'autres termes, le diagramme de la figure 3 est commutatif.

Dans le dernier chapitre, nous donnons un r -analogue de la relation (30). Rappelons que les statistiques “inv” et *fonction longueur* ℓ sont identiques

FIGURE 3 – Factorisation de Ψ par les transformations de Foata-Han

sur S_n et suivant Brenti [15], la fonction longueur ℓ_B sur B_n peut s'exprimer par :

$$\ell_B(\pi) = \text{inv}(\pi) + \sum_{i \in \text{Neg}(\pi)} |\pi(i)|. \quad (67)$$

et cette expression coïncide avec la restriction de *flag inversion* “finv” sur B_n (voir [36]) où

$$\text{finv}(\pi) = \text{inv}(\pi) + \overline{\text{inv}}(\pi) + \text{neg}(\pi), \quad (68)$$

et

$$\overline{\text{inv}}(\pi) = \sum_{\substack{1 \leq i \leq n-1 \\ i < j \leq n}} \chi \left(\overline{\pi(i)} > \pi(j) \right).$$

Le Théorème 0.4 de MacMahon s'étend sur B_n .

$$\sum_{\pi \in B_n} q^{\text{finv}(\pi)} = \sum_{\pi \in B_n} q^{\text{fmaj}(\pi)} = \frac{(q^2; q^2)_n}{(q; q)_n} [n]_{q!}. \quad (69)$$

Foata et Han [36] ont donné une preuve combinatoire de cette dernière formule au moyen d'une extension de la seconde transformation fondamentale de Foata qui échange les statistiques “inv” et “maj” sur S_n et sur les mots [37]. La relation d'ordre utilisée est l' OM .

Dans le groupe des permutations colorées, avec la relation d'ordre OM , la statistique *fonction longueur* ℓ_G est obtenue par la formule suivante (voir [9, 50])

$$\ell_G(\pi) = \text{inv}(\pi) + \sum_{\text{col}_i(\pi) \neq 0} (|\pi(i)| + \text{col}_i(\pi) - 1). \quad (70)$$

Adin et Roichman [1] ont montré que les statistiques ℓ_G et “fmaj” ne sont plus équidistribuées sur $G(r, n)$ pour tout $r \geq 3$. Mais pour pouvoir donner un $G(r, n)$ -analogue du Théorème de MacMahon, nous avons besoin d'étendre sur $G(r, n)$ la statistique “inv”. Ainsi nous généralisons la statistique “finv” définie dans la relation (68) en :

$$\text{finv}(\pi) := \sum_{\substack{1 \leq i < j \leq m \\ \xi \in C_r}} \chi(\xi\pi(i) > \pi(j)) + \text{col}(\pi), \quad \pi \in G(r, n). \quad (71)$$

Nous allons montrer que cette extension flag-inversion est équidistribuée avec “fmaj” sur $G(r, n)$ et plus généralement sur les mots r -colorés.

Soit $m_1, m_2, \dots, m_s$ des entiers tels que $m_1 + m_2 + \dots + m_s = m$. Notons $G_{r, \mathbf{m}}$ l'ensemble des réarrangements du mot $1^{m_1} 2^{m_2} \dots s^{m_s}$ tels qu'une lettre $i \in [s]$ peut être remplacée par ξi pour tout $\xi \in C_r$.

Théorème 0.27. *On a :*

$$\sum_{w \in G_{r, \mathbf{m}}} q^{\text{fmaj } w} = \sum_{w \in G_{r, \mathbf{m}}} q^{\text{finv } w} = \frac{(q^r; q^r)_m}{(q; q)_m} \left[\begin{matrix} m \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q.$$

Remarque 10. *Notre preuve repose sur l'extension $\widehat{\Phi}$ sur $G(r, n)$ de la seconde transformation fondamentale Φ de Foata sur S_n dans [37]. Nous utilisons la relation d'ordre OC sur $\Sigma_{r, n}$.*

Chapitre 1

Euler's table for $G(r, n)$

This chapter is the first part of [31]. Euler's difference table associated to the sequence $\{n!\}$ leads naturally to the counting formula for the derangements. In this paper, we study Euler's difference table associated to the sequence $\{r^n n!\}$ and the generalized derangement problem. For the coefficients appearing in the later table, we will give the combinatorial interpretations in terms of two kinds of k -successions of the group $C_r \wr S_n$. In particular for $r = 1$, we recover the known results for the symmetric groups, while for $r = 2$ we obtain the corresponding results for the hyperoctahedral group.

1.1 Introduction

The *problème de rencontres* in classical combinatorics consists in counting permutations without fixed points (see [28, p. 9–12]). On the other hand, one finds in the works of Euler (see [30]) the following table of differences:

$$g_n^n = n! \quad \text{and} \quad g_n^m = g_n^{m+1} - g_{n-1}^m \quad (0 \leq m \leq n-1).$$

Clearly this table leads naturally to an explicit formula for g_n^0 , which corresponds to the number of derangements of $[n] = \{1, \dots, n\}$. Since $n!$ is the cardinality of the symmetric group of $[n]$, Euler's difference table can be considered to be an array associated to the symmetric group.

In the last two decades, much effort has been made to extend various enumerative results on symmetric groups to other Coxeter groups, the wreath product of a cyclic group with a symmetric group, and more generally to complex reflection groups. The reader is referred to [1, 3, 25, 24, 35, 43, 8, 16, 7] and the references cited there for the recent works in this direction. In this chapter we shall consider the *problème de rencontres* in the group $C_r \wr S_n$

$$\begin{cases} g_{r,n}^n = r^n n! & (m = n); \\ g_{r,n}^m = g_{r,n}^{m+1} - g_{r,n-1}^m & (0 \leq m \leq n-1). \end{cases} \quad (1.1)$$

$n \backslash m$	0	1	2	3	4	5	$n \backslash m$	0	1	2	3	4	5
0	1						0	1					
1	0	1!					1	1	$2^1 1!$				
2	1	1	2!				2	5	6	$2^2 2!$			
3	2	3	4	3!			3	29	34	40	$2^3 3!$		
4	9	11	14	18	4!		4	233	262	296	336	$2^4 4!$	
5	44	53	64	78	96	5!	5	2329	2562	2824	3120	3456	$2^5 5!$
	$(g_{1,n}^m)$							$(g_{2,n}^m)$					

is the cardinality of S_n and $g_{1,n}^0$ is the number of *derangements*, i.e., the fixed point free permutations in S_n . The combinatorial interpretation for the general coefficients $g_{1,n}^m$ was first studied by Dumont and Randrianarivony [30] and then by Clarke et al.[27]. More recently Rakotondrajao [47, 48] has given further combinatorial interpretations of these coefficients in terms of *k-successions* in symmetric groups. As $g_{2,n}^n = 2^n n!$ is the cardinality of the hyperoctahedral group B_n , Chow [24] has given a similar interpretation for $g_{2,n}^0$ in terms of derangements in the hyperoctahedral groups. It is not hard to see that the coefficient $g_{r,n}^m$ is divisible by $r^m m!$. This prompted us to introduce $d_{r,n}^m = g_{r,n}^m / r^m m!$. We derive then from (1.1) the following allied array $(d_{r,n}^m)_{n,m \geq 0}$:

$$\begin{cases} d_{r,n}^n = 1 & (m = n); \\ d_{r,n}^m = r(m+1)d_{r,n}^{m+1} - d_{r,n-1}^m & (0 \leq m \leq n-1). \end{cases} \quad (1.2)$$

The aim of this chapter is to study the coefficients $g_{r,n}^m$ and $d_{r,n}^m$ in the colored group $G(r, n)$, i.e., the wreath product of a cyclic group and a symmetric group.

$n \backslash m$	0	1	2	3	4	5	$n \backslash m$	0	1	2	3	4	5
0	1						0	1					
1	0	1					1	1	1				
2	1	1	1				2	5	3	1			
3	2	3	2	1			3	29	17	5	1		
4	9	11	7	3	1		4	233	131	37	7	1	
5	44	53	32	13	4	1	5	2329	1281	353	65	9	1
$(d_{1,n}^m)$							$(d_{2,n}^m)$						

Table 1.2: Values of $d_{r,n}^m$ for $0 \leq m \leq n \leq 5$ and $r = 1$ or 2 .

1.2 Main results

We first generalize the notion of k -succession introduced by Rakotondrajao [48] in the symmetric group to $G(r, n)$.

Defintion 11 (k -circular succession). *Given a permutation $\pi \in G(r, n)$ and a nonnegative integer k , the value $\pi(i)$ is a k -circular succession at position $i \in [n]$ if $\pi(i) = i + k$. In particular a 0-circular succession is also called fixed point.*

Remark. Some words are in order about the requirement $\pi(i) = i + k$ in this definition. The “wraparound” is not allowed, i.e., $i + k$ is not to be interpreted $\text{mod } n$, also $i + k$ needs to be uncolored, i.e., $i + k \in [n]$, in order to count as a k -circular succession.

Denote by $C^k(\pi)$ the set of k -successions circulaires of π and let $c^k(\pi) = \# C^k(\pi)$. In particular $FIX(\pi)$ denotes the set of fixed points of π . For example, for the permutation

$$\pi = \left(\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \bar{1} & 5 & \bar{9} & \bar{6} & 8 & \bar{7} & \bar{\bar{3}} & \bar{4} & \bar{2} \end{array} \right) \in G(4, 9),$$

the values 5 and 8 are the two 3-circular successions at positions 2 and 5. Thus $C^3(\pi) = \{5, 8\}$.

The following is our main result on the combinatorial interpretation of the coefficients $g_{r,n}^m$ in terms of k -circular successions.

Theorem 1.2.1. *For any integer k such that $0 \leq k \leq m$, the entry $g_{r,n}^m$ equals the number of permutations in $G(r, n)$ whose k -circular successions are included in $[m]$. In particular, by taking $k = 0$ and $k = m$, respectively, either of the following holds.*

- (i) The entry $g_{r,n}^m$ is the number of permutations in $G(r, n)$ whose fixed points are included in $[m]$.
- (ii) The entry $g_{r,n}^m$ is the number of permutations in $G(r, n)$ without m -circular succession.

For example, the permutations in $G(2, 2)$ whose fixed points are included in $[1]$ are:

$$21, \quad 1\bar{2}, \quad \bar{2}1, \quad 2\bar{1}, \quad \bar{1}\bar{2}, \quad \bar{2}\bar{1};$$

while those without 1-circular succession are:

$$12, \quad \bar{1}\bar{2}, \quad 1\bar{2}, \quad \bar{1}2, \quad \bar{2}1, \quad 2\bar{1}.$$

Note that Dumont and Randrianarivony [30] proved the $r = 1$ case of (i), while Rakotondrajao [48] proved the $r = 1$ case of (ii).

Let $c_{r,n,m}^k$ be the number of colored permutations in $G(r, n)$ with m k -circular successions.

Theorem 1.2.2. *Let n , k and m be integers such that $n \geq 1$, $k \geq 0$ and $m \geq 0$. Then*

$$c_{r,n+1,m}^{k+1} = c_{r,n+1,m}^k + c_{r,n,m}^k - c_{r,n,m-1}^k, \quad (1.3)$$

where $c_{r,n,-1}^k = 0$.

Defintion 12 (k -linear succession).

For $\pi = (\varepsilon, |\pi|) \in G(r, n)$, the value $|\pi(i)|$ ($2 \leq i \leq n$) is a k -linear succession ($k \geq 1$) of π at position i if

$$\begin{cases} |\pi(i)| &= |\pi(i-1)| + k, \\ \varepsilon(i) &= \varepsilon(i-1). \end{cases} \quad (1.4)$$

Denote by $\mathcal{L}^k(\pi)$ the set of k -linear successions of π and let $l^k(\pi) = \#\mathcal{L}^k(\pi)$. Let $l_{r,n,m}^k$ be the number of colored permutations in $G(r, n)$ with m k -linear successions. For example, 9 and 3 are the two 2-linear successions of the permutation $\pi = \bar{5}\bar{2}479\bar{1}\bar{3}\bar{8}6 \in G(4, 9)$.

Defintion 13 (Skew k -linear succession). For $\pi = (\varepsilon, \sigma) \in G(r, n)$, the value $\sigma(i)$ ($1 \leq i \leq n$) is a skew k -linear succession ($k \geq 1$) of π at position i if

$$\pi(i) = \pi(i-1) + k,$$

where, by convention, $\sigma(0) = 0$ and $\varepsilon(0) = 1$.

Denote by $\mathcal{L}^{*k}(\pi)$ the set of skew k -linear successions of π and $l^{*k}(\pi) = \#\mathcal{L}^{*k}(\pi)$. The number of permutations in $G(r, n)$ with m skew k -linear successions is $l_{r,n,m}^{*k}$. Obviously we have the following relation:

$$\mathcal{L}^{*k}(\pi) = \begin{cases} \mathcal{L}^k(\pi), & \text{if } \pi(1) \neq k; \\ \mathcal{L}^k(\pi) \cup \{k\}, & \text{otherwise.} \end{cases} \quad (1.5)$$

Let δ be the bijection from $G(r, n)$ onto itself defined by:

$$\pi = \pi_1 \pi_2 \cdots \pi_n \longmapsto \delta(\pi) = \pi_n \pi_1 \pi_2 \cdots \pi_{n-1}. \quad (1.6)$$

Theorem 1.2.3. *For any integer $k \geq 0$ there is a bijection Φ from $G(r, n)$ onto itself such that for $\pi \in G(r, n)$,*

$$\mathcal{C}^{k+1}(\pi) = \mathcal{L}^{k+1}(\Phi(\pi)), \quad (1.7)$$

and

$$\mathcal{C}^k(\delta(\pi)) = \mathcal{L}^{*(k+1)}(\Phi(\pi)). \quad (1.8)$$

Thanks to the transformation Φ the two statistics c^k and l^k are equidistributed on the group $G(r, n)$ for $k \geq 1$. So we can replace the left-hand sides of (2.1) by $l_{r,n+1,m}^{k+1}$ and derive the following interesting result.

Corollary 14. *Let n, k and m be integers such that $n \geq 1, k \geq 0$ and $m \geq 0$. Then*

$$l_{r,n+1,m}^{k+1} = c_{r,n+1,m}^k + c_{r,n,m}^k - c_{r,n,m-1}^k, \quad (1.9)$$

where $c_{r,n,-1}^k = 0$.

Our proof of the last two theorems is a generalization of that given by Clarke et al [27], where the $(k, r) = (0, 1)$ case of Corollary 14 is proved. Note that the $(k, r, m) = (0, 2, 0)$ case of (1.9) is the main result of a recent paper by Chen and Zhang [21].

In order to interpret the entry $d_{r,n}^m$ we need the following definition.

Defintion 15. *For $0 \leq m \leq n$, a permutation $\pi = (\varepsilon, |\pi|)$ in $G(r, n)$ is called m -increasing-fixed if it satisfies the following conditions:*

- i) $\forall i \in [m], \varepsilon(i) = 1$;
- ii) $\text{FIX}(\pi) \subseteq [m]$;
- iii) $\pi(1) < \pi(2) < \cdots < \pi(m)$.

Let $I_{r,n}^m$ be the set of m -increasing-fixed permutations in $G(r, n)$. For example,

$$I_{2,3}^2 = \{1\,2\,\bar{3}, \quad 1\,3\,2, \quad 1\,3\,\bar{2}, \quad 2\,3\,1, \quad 2\,3\,\bar{1}\}.$$

Theorem 1.2.4. *For $0 \leq m \leq n$, the entry $d_{r,n}^m$ equals the cardinality of $I_{r,n}^m$.*

Proof. Let $F_{r,n}^m$ be the set of permutations with fixed points included in $[m]$ in $G(r, n)$. By Theorem 1.2.1 the cardinality of $F_{r,n}^m$ equals $g_{r,n}^m$. We define a mapping $f : (\tau, \pi) \mapsto \tau \odot \pi$ from $G(r, m) \times F_{r,n}^m$ to $F_{r,n}^m$ as follows:

$$\tau \odot \pi = \pi(\tau^{-1}(1))\pi(\tau^{-1}(2)) \dots \pi(\tau^{-1}(m))\pi(m+1) \dots \pi(n).$$

Clearly f defines a group action of $G(r, m)$ on the set $F_{r,n}^m$. We can choose an element π in each orbit such that

$$\forall i \in [m], \quad \varepsilon(i) = 1 \quad \text{and} \quad \pi(1) < \pi(2) < \dots < \pi(m).$$

As the cardinality of the group $G(r, m)$ is $r^m m!$, we derive that the number of the orbits equals $g_{r,n}^m / r^m m!$. $\square$

Rakotondrajao [48] gave a different interpretation for $d_{r,n}^m$ when $r = 1$. We can generalize her result as in the following theorem.

Definition 16. *For $0 \leq m \leq n$, a permutation π in $G(r, n)$ is called m -isolated-fixed if it satisfies the following conditions:*

- i) $\forall i \in [m], \text{color}_\pi(i) = 1$;
- ii) $\text{FIX}(\pi) \subseteq [m]$;
- iii) *each cycle of π has at most one point in common with $[m]$.*

Let $D_{r,n}^m$ be the set of m -isolated-fixed permutations in $G(r, n)$. For example,

$$D_{2,3}^2 = \{(1)(2)(\bar{3}), (1, 3)(2), (1, \bar{3})(2), (1)(2, 3), (1)(2, \bar{3})\}.$$

Note that $\pi = \bar{3}\,1\,2 \notin D_{2,3}^2$ because 1 and 2 are in the same cycle.

Theorem 1.2.5. *For $0 \leq m \leq n$, the entry $d_{r,n}^m$ equals the cardinality of $D_{r,n}^m$.*

The rest of this chapter is organized as follows: The proofs of Theorems 1.2.1, 1.2.2, 1.2.3 and 1.2.5 will be given in Sections 3, 4, 5 and 6, respectively.

1.3 Proof of Theorem 1.2.1

Let m and k be integers such that $n \geq m \geq k \geq 0$. Denote by $G(r, n, m, k)$ the set of permutations in $G(r, n)$ whose k -circular successions are bounded by m and $s_{r,n}^m = \#G(r, n, m, k)$. We show that the sequence $(s_{r,n}^m)$ satisfies (1.1).

By definition, we have immediately $G(r, n, n, k) = G(r, n)$ and then $s_{r,n}^n = r^n n!$. Now, suppose $m < n$, then $G(r, n, m+1, k) \setminus G(r, n, m, k)$ is the set of permutations in $G(r, n, m+1, k)$ whose maximal k -circular succession is $m+1$. It remains to show that the cardinality of the latter set equals $s_{r,n-1}^m$. To this end, we define a simple bijection $\rho : \pi \mapsto \pi'$ from $G(r, n, m+1, k) \setminus G(r, n, m, k)$ to $G(r, n-1, m, k)$ as follows. Starting from any $\pi = \pi_1 \pi_2 \dots \pi_n$ in $G(r, n, m+1, k) \setminus G(r, n, m, k)$, we construct π' by deleting $\pi_{m+1-k} = m+1$ and replacing each letter π_i by $\pi_i - 1$ if $|\pi_i| > m+1$. Conversely, starting from $\pi' = \pi'_1 \pi'_2 \dots \pi'_{n-1}$ in $G(r, n-1, m, k)$, one can recover π by inserting $m+1$ between π'_{m-k} and π'_{m-k+1} and then replacing each letter π'_i by $\pi'_i + 1$ if $|\pi'_i| > m$. For example, if $\pi = 3\bar{9}5\bar{8}\bar{7}\bar{6}2\bar{1}4 \in G(3, 9, 5, 2)$, then $\pi' = 3\bar{8}\bar{7}\bar{6}\bar{5}2\bar{1}4 \in G(3, 8, 4, 2)$. Note that π' has a k -circular succession $j \geq m+1$ if and only if $j+1 \geq m+2$ is a k -circular succession of π . Therefore, the maximal k -circular succession of π is $m+1$ if and only if the k -circular successions of π' are bounded by m . This completes the proof.

Remark. The above argument does not explain why $g_{r,n}^m$ is independent from k ($0 \leq k \leq m$). We can provide such an argument as follows. Consider the following simple bijection d which consists in transforming $\pi = \pi_1 \pi_2 \pi_3 \dots \pi_n$ into $d(\pi) = \pi' = \pi_2 \pi_3 \dots \pi_n \pi_1$. Clearly the k -successions of π are bounded by m if and only if the $(k+1)$ -successions of π' are bounded by m . Hence, denoting by d^j the composition of j -times of d , the application of $d^{k_2-k_1}$ permits to pass from k_1 -successions to k_2 -successions if $k_1 < k_2$. In particular if we apply m times the mapping d to a permutation whose fixed points are bounded by m then we obtain a permutation without m -succession and vice versa.

1.4 Proof of Theorem 1.2.2

Let $S_n^k(x)$ be the counting polynomial of the statistic c^k on the group $G(r, n)$, i.e.,

$$S_n^k(x) = \sum_{\pi \in G(r,n)} x^{c^k(\pi)} = \sum_{m=0}^n c_{r,n,m}^k x^m. \quad (1.10)$$

Then (1.3) is equivalent to the following equation:

$$S_{n+1}^{k+1}(x) = S_{n+1}^k(x) + (1-x)S_n^k(x). \quad (1.11)$$

By (1.6) it is readily seen that

$$C^{k+1}(\pi) = \begin{cases} C^k(\delta(\pi)), & \text{if } \pi_n \neq k+1; \\ C^k(\delta(\pi)) \setminus \{k+1\}, & \text{otherwise.} \end{cases} \quad (1.12)$$

It follows that

$$\begin{aligned} S_{n+1}^{k+1}(x) &= \sum_{\substack{\pi \in G(r, n+1) \\ \pi(1)=k+1}} x^{c^k(\pi)-1} + \sum_{\substack{\pi \in G(r, n+1) \\ \pi(1) \neq k+1}} x^{c^k(\pi)} \\ &= \sum_{\substack{\pi \in G(r, n+1) \\ \pi(1)=k+1}} x^{c^k(\pi)-1} + \sum_{\pi \in G(r, n+1)} x^{c^k(\pi)} - \sum_{\substack{\pi \in G(r, n+1) \\ \pi(1)=k+1}} x^{c^k(\pi)}. \end{aligned} \quad (1.13)$$

For any $\pi \in G(r, n+1)$ such that $\pi(1) = k+1$ we can associate bijectively a permutation $\pi' \in G(r, n)$ such that $c^k(\pi) = c^k(\pi') + 1$ as follows: $\forall i \in [n]$,

$$\pi'(i) = \begin{cases} \pi(i+1), & \text{if } \pi(i+1) \leq k; \\ \pi(i+1) - 1, & \text{if } \pi(i+1) > k. \end{cases}$$

Therefore we can rewrite (1.13) as (1.11).

We can also derive Theorem 1.2.2 from Theorem 1.2.1. First we prove a lemma.

Lemma 1.1. *For $0 \leq k \leq n - m$ there holds*

$$c_{r,n,m}^k = \binom{n-k}{m} g_{r,n-m}^k. \quad (1.14)$$

Proof. To construct a permutation π in $G(r, n)$ with m k -circular successions we can first choose m positions $i_1, \dots, i_m$ of k -circular successions among the first $n - k$ ones and then construct a permutation π_0 of order $n - m$ without k -circular successions on the remaining $n - m$ positions, where and in what follows we shall assume that $i_1 < i_2 < \dots < i_m$. More precisely, there is a bijection $\theta : \pi \mapsto (I, \pi_0)$, where $I = \{i_1, \dots, i_m\}$, from the set of the colored permutations of order n with m k -successions to the product of the set of all m -subsets of $[n - k]$ and the set of colored permutations of order $n - m$ without k -circular successions. Denote by $G_{r,n,k,i}$ the set of all permutations in $G(r, n)$ whose maximal position of k -circular successions equals i . Define the mapping $R_i : \pi \mapsto \pi'$ from $G_{r,n,k,i}$ to $G(r, n - 1)$ such that the linear form of π' is obtained from $\pi = \pi_1 \dots \pi_n$ by removing the letter $(i + k)$ and

replacing each colored letter π_j by $\pi_j - 1$ if $|\pi_j| > i + k$. It is readily seen that the map R_i is a bijection and $c^k(\pi') = c^k(\pi) - 1$. Indeed it is easy to see that $j + k$ is a k -circular succession of π different of $i + k$ if and only if $j + k$ is a k -circular succession of π' . Hence, $\pi_0 = R_{i_1} \circ R_{i_2} \circ \cdots \circ R_{i_m}(\pi)$ is a colored permutation without k -circular succession in $G(r, n - m)$.

Conversely given a subset $I = \{i_1, i_2, \dots, i_m\}$ of $[n - k]$ and a colored permutation π_0 without k -circular succession in $G(r, n - m)$ we can construct

$$\pi = \theta^{-1}(I, \pi_0) = R_{i_m}^{-1} \circ R_{i_{m-1}}^{-1} \circ \cdots \circ R_{i_1}^{-1}(\pi_0),$$

where $R_i^{-1}(\pi')$ is obtained from $\pi' = \pi'_1 \dots \pi'_{n-1}$ by inserting the integer $(i + k)$ between π'_{i-1} and π'_i and replacing each colored letter π'_j by $\pi'_j + 1$ if $|\pi'_j| \geq i + k$. Therefore

$$c_{r,n,m}^k = \binom{n-k}{m} c_{r,n-m,0}^k. \quad (1.15)$$

By Theorem 1.2.1 (ii) we have $c_{r,n,0}^k = g_{r,n}^k$. Substituting this in (1.15) yields then (1.14). □

By (1.14) we see that (1.3) is equivalent to

$$\begin{aligned} \binom{n-k}{m} g_{r,n+1-m}^{k+1} &= \binom{n+1-k}{m} g_{r,n+1-m}^k \\ &\quad + \binom{n-k}{m} g_{r,n-m}^k - \binom{n-k}{m-1} g_{r,n+1-m}^k. \end{aligned}$$

Since $g_{r,n+1-m}^{k+1} - g_{r,n-m}^k = g_{r,n-m+1}^k$ by (1.1), we can rewrite the last equation as

$$\binom{n-k}{m} g_{r,n+1-m}^k = \binom{n+1-k}{m} g_{r,n+1-m}^k - \binom{n-k}{m-1} g_{r,n+1-m}^k,$$

which is obvious in view of the identity $\binom{n-k}{m} = \binom{n+1-k}{m} - \binom{n-k}{m-1}$. This completes the proof of Theorem 1.2.1.

1.5 Proof of Theorem 1.2.3.

There is a well-known bijection on the symmetric groups transforming the cyclic structure into linear structure (see [39] and [52, p. 17]). We need a variant of this transformation, say $\varphi : S_n \rightarrow S_n$, as follows.

Given a permutation $\sigma \in S_n$ written as a product of cycles, arrange the cycles in the decreasing order of their maximum elements from left to right with the maximum element at the end of each cycle. We then obtain $\varphi(\sigma)$ by erasing the parentheses. Conversely, starting from a permutation written in one-line form $\sigma' = a_1 a_2 \dots a_n$, find out the *right-to-left maxima* of σ' from right to left and decompose the word σ' into blocks by putting a bar at the right of each right-to-left maximum and construct a cycle of σ with each block.

For example, if $\sigma = (3, 1, 4, 6, 9)(5, 7, 8)(2) \in S_9$ then we have:

$$\varphi(\sigma) = 3\ 1\ 4\ 6\ 9\ 5\ 7\ 8\ 2.$$

Conversely, starting from $\sigma' = 3\ 1\ 4\ 6\ \mathbf{9}\ 5\ 7\ \mathbf{8}\ \mathbf{2}$, so the right-to-left maxima are 9, 8 and 2, then the decomposition into blocks is $3\ 1\ 4\ 6\ 9|5\ 7\ 8|2|$ and we recover σ by putting parentheses around each block.

Lemma 1.2. *For $k \geq 1$, the mapping φ transforms the k -successions circulaires to k -successions linéaires and vice versa.*

Proof. Indeed, an integer p is a k -succession circulaire of σ if and only if there is an integer $i \in [n]$ such that $\sigma(i) = i + k$, so i and $i + k$ are two consecutive letters in the one-line form of $\varphi(\sigma)$. Conversely if i and $i + k$ are two consecutive letters in the one-line form of a permutation τ then i cannot be a right-to-left maximum, so i and $i + k$ are in the same cycle of $\varphi^{-1}(\tau)$, say σ , and then $\sigma(i) = i + k$. $\square$

We now construct a bijection $\Phi : \pi \mapsto \pi'$ from $G(r, n)$ onto itself such that

$$C^k(\pi) = L^k(\pi') \quad (k \geq 1). \quad (1.16)$$

Let $\sigma = |\pi|$ and $\sigma' = |\pi'|$.

Bijection Φ : First define $\sigma' = \varphi(\sigma)$: Factorize σ as product of l disjoint cycles $C_1, \dots, C_l$. Suppose that h_i and g_i are, respectively, the length and greatest element of the cycle C_i ($1 \leq i \leq l$) such that $g_1 > g_2 > \dots > g_l$. Then

$$\sigma' = \sigma(g_1) \dots \sigma^{h_1-1}(g_1) g_1 \sigma(g_2) \dots \sigma^{h_2-1}(g_2) g_2 \dots \sigma(g_l) \dots \sigma^{h_l-1}(g_l) g_l. \quad (1.17)$$

Let $T_\sigma = \{\sigma(g_i), i \in [l]\}$. It remains to define $\text{color}_{\pi'}(\sigma^j(g_i))$ for all $i \in [l]$ and $1 \leq j \leq h_i$. We proceed by induction on j as follows: For each $i \in [l]$ let

$$\text{color}_{\pi'}(\sigma(g_i)) = \text{color}_\pi(\sigma(g_i)),$$

and for $j = 2, \dots, h_i$ define

$$\text{color}_{\pi'}(\sigma^j(g_i)) = \text{color}_{\pi'}(\sigma^{j-1}(g_i)) \cdot \text{color}_{\pi}(\sigma^j(g_i)). \quad (1.18)$$

It is easy to establish the inverse of Φ .

Bijection Φ^{-1} : Starting from π' we can recover $\sigma = |\pi|$ by applying φ^{-1} to σ' . Suppose σ' is given as in (1.17) and $g_1, \dots, g_l$ are the left-to-right-maxima. We then determine $\text{color}_{\pi}(\sigma^j(g_i))$ for all $i \in [l]$ and $1 \leq j \leq h_i$ as follows: For each $i \in [l]$ let

$$\text{color}_{\pi}(\sigma(g_i)) = \text{color}_{\pi'}(\sigma(g_i)),$$

and for $j = 2, \dots, h_i$ define

$$\text{color}_{\pi}(\sigma^j(g_i)) = \text{color}_{\pi'}(\sigma^j(g_i))(\text{color}_{\pi'}(\sigma^{j-1}(g_i)))^{-1}, \quad (1.19)$$

where $(\text{color}_{\pi}(i))^{-1}$ is the inverse of $\text{color}_{\pi}(i)$ in the cyclic group C_r .

Lemma 1.3. *For $k \geq 1$, the mapping Φ transforms a k -circular succession of π to a k -linear succession of π' and vice versa.*

Proof. By Lemma 1.2 we have the equivalence: $\sigma(i)$ is a k -circular succession of σ if and only if $\sigma(i)$ is a k -linear succession of σ' . It remains to verify that if $\sigma(i)$ is a k -circular succession of σ then

$$\text{color}_{\pi}(\sigma(i)) = 1 \Leftrightarrow \text{color}_{\pi'}(\sigma(i)) = \text{color}_{\pi'}(i). \quad (1.20)$$

Note that if $\sigma(i)$ is a k -circular succession of σ then i and $\sigma(i)$ must be in the same cycle and that $\sigma(i)$ cannot be in T_{σ} for, otherwise, i would be the greatest element of the cycle but this is impossible because $\sigma(i) = i + k$ ($k \geq 1$).

Now, assume that $\sigma(i)$ is a k -circular succession of σ .

(i) Suppose $\text{color}_{\pi}(\sigma(i)) = 1$. As $\sigma(i) \notin T_{\sigma}$ and $\sigma^{-1}(\sigma(i)) = i$, we have:

$$\text{color}_{\pi'}(\sigma(i)) = \text{color}_{\pi'}(i) \cdot \text{color}_{\pi}(\sigma(i)) = \text{color}_{\pi'}(i).$$

(ii) Suppose that $\text{color}_{\pi'}(\sigma(i)) = \text{color}_{\pi'}(i)$. As $\sigma(i) \notin T_{\sigma}$, we have

$$\text{color}_{\pi}(\sigma(i)) = \text{color}_{\pi'}(\sigma(i)) / \text{color}_{\pi'}(i) = 1.$$

Hence (1.20) is established. □

Obviously the above lemma is equivalent to (1.7). We obtain (1.8) by combining (1.7), (1.5) and (1.12).

We conclude this section with an example. Consider

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \bar{3} & 4 & \bar{9} & \bar{8} & 7 & \bar{5} & 6 & \bar{2} & \bar{1} \end{pmatrix} \in G(4, 9).$$

Factorizing $\sigma = |\pi|$ into cycles we get $\sigma = (1, 3, 9)(2, 4, 8)(6, 5, 7)$, then

$$\sigma' = 1 \ 3 \ 9 \ 2 \ 4 \ 8 \ 6 \ 5 \ 7 \quad \text{and} \quad T_\sigma = \{1, 2, 6\}.$$

The colors of $\sigma'(i)$ for $i \in [9]$ are computed as follows:

$$\begin{aligned} \text{color}_{\pi'}(1) &= \text{color}_\pi(1) = \zeta^2 & \text{for } 1 \in T_\sigma; \\ \text{color}_{\pi'}(3) &= \text{color}_{\pi'}(1) \cdot \text{color}_\pi(3) = \zeta^2 \cdot \zeta = \zeta^3 & \text{for } 3 \notin T_\sigma; \\ \text{color}_{\pi'}(9) &= \text{color}_{\pi'}(3) \cdot \text{color}_\pi(9) = \zeta^3 \cdot \zeta = 1 & \text{for } 9 \notin T_\sigma; \\ \text{color}_{\pi'}(2) &= \text{color}_{\pi'}(2) = \zeta^2 & \text{for } 2 \in T_\sigma; \\ \text{color}_{\pi'}(4) &= \text{color}_{\pi'}(2) \cdot \text{color}_\pi(4) = \zeta^2 \cdot 1 = \zeta^2 & \text{for } 4 \notin T_\sigma; \\ \text{color}_{\pi'}(8) &= \text{color}_{\pi'}(4) \cdot \text{color}_\pi(8) = \zeta^2 \cdot \zeta = \zeta^3 & \text{for } 8 \notin T_\sigma; \\ \text{color}_{\pi'}(6) &= \text{color}_\pi(6) = 1 & \text{for } 6 \in T_\sigma; \\ \text{color}_{\pi'}(5) &= \text{color}_{\pi'}(6) \cdot \text{color}_\pi(5) = 1 \cdot \zeta = \zeta & \text{for } 5 \notin T_\sigma; \\ \text{color}_{\pi'}(7) &= \text{color}_{\pi'}(5) \cdot \text{color}_\pi(7) = \zeta \cdot 1 = \zeta & \text{for } 7 \notin T_\sigma. \end{aligned}$$

Thus we have $\pi \mapsto \pi' = \bar{1} \ \bar{3} \ 9 \ \bar{2} \ \bar{4} \ \bar{8} \ 6 \ \bar{5} \ \bar{7}$. We have $C^2(\pi) = \{4, 7\} = L^{*2}(\pi')$.

Conversely, starting from π' , we can recover σ by φ^{-1} and the colors of $\sigma(i)$ ($i \in [n]$) by (1.19). As $\sigma = (139)(248)(657)$ and $T_\sigma = \{1, 2, 6\}$, we have, for example,

$$\text{color}_\pi(9) = \text{color}_{\pi'}(9) \cdot \text{color}_{\pi'}^{-1}(3) = 1 \cdot \zeta = \zeta$$

for $9 \notin T_\sigma$.

1.6 Proofs of Theorem 1.2.5

We shall give two proofs by using Theorems 1.2.4 and 1.2.1, respectively.

1.6.1 First Proof

We shall define a mapping $\varphi : \pi \mapsto \pi'$ from $D_{r,n}^m$ to $I_{r,n}^m$ in two steps. First we establish the correspondence $|\pi| \mapsto |\pi'|$ and then determine the color

transformation. Define the permutation $|\pi'| = |\pi'|(1) \dots |\pi'|(n)$ such that $|\pi'|(1) \dots |\pi'|(m)$ is the increasing rearrangement of $|\pi|(1), \dots, |\pi|(i_m)$ and $|\pi'|(m+1) \dots |\pi'|(n) = |\pi|(m+1) \dots |\pi|(n)$. Conversely, starting from $\pi' \in I_{r,n}^m$, for each $i \in [m]$ we construct the cycle of $|\pi|$ containing i by

$$(|\pi'|^{-s}(i), \dots, |\pi'|^{-2}(i), |\pi'|^{-1}(i), i),$$

where s is the smallest non negative integer such that $|\pi'|^{-s}(i) \in T$, where

$$T := \{|\pi'|(i), i \in [m]\},$$

and by convention $|\pi'|^0(i) = i$. In particular if $i \in [m] \cap T$, then $s = 0$ and i is a fixed point of $|\pi|$. The other cycles remain unaltered.

For example, for $\pi = (1)(2, \bar{7}, \bar{6})(3, \bar{5}, 9)(4)(\bar{8}) \in D_{3,9}^4$ (i.e., $n = 9$, $r = 3$, $m = 4$), we have $|\pi| = (1)(2, 7, 6)(3, 5, 9)(4)(8)$ and $|\pi'| = 145792683 = (1)(2476)(359)(8)$, so $T = \{1, 4, 5, 7\}$.

Now, we describe the color transformation. For each $i \in [m]$, since the letter i of π (m -isolated-fixed) as well as the letter $\pi(i)$ of π' (m -increasing-fixed) are uncolored, the transformation of the colors is obtained by exchanging the color of $|\pi|(i)$ and that of i , namely

$$\text{color}_{\pi'}(i) = \text{color}_{\pi}(|\pi|(i)),$$

and

$$\text{color}_{\pi}(i) = \text{color}_{\pi'}(|\pi|(i)) = 1 \quad \forall i \in [m];$$

the colors of other letters remain unaltered, i.e.,

$$\text{color}_{\pi'}(i) = \text{color}_{\pi}(i) \quad \forall i \in [n] \setminus (T \cup [m]).$$

Continuing the above example, we have:

$$\begin{aligned} \text{color}_{\pi'}(2) &= \text{color}_{\pi}(\pi(2)) = \text{color}_{\pi}(7) = \zeta^1; \quad \text{color}_{\pi'}(3) = \text{color}_{\pi}(\pi(3)) = \\ \text{color}_{\pi}(5) &= \zeta^2, \text{ hence } \pi' = 1 \ 4 \ 5 \ 7 \ 9 \ \bar{2} \ \bar{6} \ \bar{8} \ \bar{3}. \end{aligned}$$

1.6.2 Second proof

Let $G_{r,n}^m := G(r, n, m, 0)$ be the set of permutations in $G(r, n)$ whose fixed points are included in $[m]$.

Let $\pi = (\epsilon, \sigma)$ be a permutation in $G_{r,n}^m$, written as a product of disjoint cycles. For each $i \in [m]$, let s be the smallest integer ≥ 1 such that $\sigma^s(i) \in [m]$ and $w_{\pi}(i) = [\varepsilon_{\sigma(i)}\sigma(i)] \dots [\varepsilon_{\sigma^{s-1}(i)}\sigma^{s-1}(i)]$, where $\varepsilon = \epsilon \circ \sigma^{-1}$. Clearly $w_{\pi}(i) = \emptyset$ if $s = 1$. Let Ω_{π} be the product of cycles of π which have no common point with $\{i\zeta^j | i \in [m], 0 \leq j \leq r-1\}$ and π_m be the permutation

in $G(r, m)$ obtained from π by deleting the cycles in Ω_π and letters in $w_\pi(i)$ for $i \in [m]$. For example, if $\pi = (\bar{1} \bar{4} 7 \bar{3} 2 \bar{6} 5)(\bar{8})(\bar{9}) \in G_{3,9}^3$ then $\pi_3 = (\bar{1} \bar{3} 2)$,

$$w_\pi(1) = \bar{4} 7, \quad w_\pi(2) = \bar{6} 5, \quad w_\pi(3) = \emptyset \quad \text{and} \quad \Omega_\pi = (\bar{8})(\bar{9}).$$

Let $T(\pi) = (w_\pi(1), w_\pi(2), \dots, w_\pi(m), \Omega_\pi)$ and define the relation $\sim$ on $G_{r,n}^m$ by

$$\pi_1 \sim \pi_2 \Leftrightarrow T(\pi_1) = T(\pi_2).$$

Clearly this is an equivalence relation. To determine the equivalence class $\mathcal{C}_\pi$ of each permutation $\pi \in G_{r,n}^m$ we consider the mapping $\theta : (\tau, \pi) \mapsto \theta(\tau, \pi)$ from $G(r, m) \times G_{r,n}^m$ to $G_{r,n}^m$, where the cyclic factorization of $\theta(\tau, \pi)$ is obtained by inserting the word $w_\pi(i)$ after each letter $i\zeta^j$ appearing in a cycle of τ for each $i \in [m]$ and some $j : 0 \leq j \leq r-1$, and then add the cycles in Ω_π . For example, if $\pi = (\bar{1} \bar{4} 7 \bar{3} 2 \bar{6} 5)(\bar{8})(\bar{9}) \in G_{3,9}^3$ and $\tau = (\bar{1} \bar{2})(3) \in G(3, 3)$ then

$$\theta(\tau, \pi) = (\bar{1} \bar{4} 7 \bar{2} \bar{6} 5)(3)(\bar{8})(\bar{9}).$$

Clearly $\mathcal{C}_\pi = \{\theta(\tau, \pi) | \tau \in G(r, m)\}$. Indeed, by definition $\theta(\tau, \pi) \sim \pi$ for each $\tau \in G(r, m)$ and $\pi \in G_{r,n}^m$ and conversely, if $\pi' \sim \pi$ then $\pi' = \theta(\pi'_m, \pi)$ for $T(\pi') = T(\pi)$. Moreover, suppose $\theta(\tau, \pi) = \theta(\tau', \pi) = \pi'$ for $\tau, \tau' \in G(r, m)$, then $\tau = \tau' = \pi'_m$. Hence the cardinality of each equivalence class is $r^m m!$ and, by Theorem 1.2.1 the number of equivalence classes equals $d_{r,n}^m = g_{r,n}^m / r^m m!$. Choosing $\theta(\iota, \pi)$ as the representative of the class $\mathcal{C}_\pi$, where ι is the identity of $G(r, m)$, yields the desired result.

Chapitre 2

Generating functions and further results

This chapter is the second part of [31]. We give the generating function of the coefficients $g_{r,n}^m$'s and derive more recurrence relations for the coefficients $g_{r,n}^m$'s and $d_{r,n}^m$'s. In particular, we shall prove an explicit formula for the r -derangement numbers:

$$d_{r,n}^0 = g_{r,n}^0 = n! \sum_{i=0}^n \frac{(-1)^i r^{n-i}}{i!}, \quad (2.1)$$

which implies immediately the following recurrence relation:

$$d_{r,n}^0 = r n d_{r,n-1}^0 + (-1)^n \quad (n \geq 1). \quad (2.2)$$

Note that (2.2) is the r -version of a famous recurrence for derangements. Using the combinatorial interpretation for $g_{r,n}^m$ and $d_{r,n}^m$ it is possible to derive bijective proofs of these recurrence relations. However we will just give combinatorial proofs for (2.2) and two other recurrences by generalizing the combinatorial proofs of Rakotondrajao [48] for $r = 1$ case, and leave the others for the interested readers.

2.1 Generating functions

For any function $f : \mathbf{Z} \rightarrow \mathbb{C}$ introduce the difference operator:

$$\Delta f(n) = f(n) - f(n-1).$$

Then it is easy to see by induction on $N \geq 0$ that

$$\Delta^N f(n) = \sum_{i=0}^N (-1)^i \binom{N}{i} f(n-i) = \sum_{i=0}^N (-1)^{N-i} \binom{N}{i} f(n-N+i). \quad (2.3)$$

Proposition 17. *For $m \geq 0$ the following identities hold true:*

$$g_{r,n+m}^m = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} r^{m+i} (m+i)!, \quad (2.4)$$

$$\sum_{n \geq 0} g_{r,n+m}^m \frac{u^n}{n!} = \frac{r^m m! \exp(-u)}{(1 - ru)^{m+1}}, \quad (2.5)$$

$$\sum_{m,n \geq 0} g_{r,n+m}^m \frac{x^m}{m!} \frac{u^n}{n!} = \frac{\exp(-u)}{1 - rx - ru}. \quad (2.6)$$

Proof. Setting $f(n) = g_{r,n}^n$ then $g_{r,n+m}^{n+m-i} = \Delta^i f(n+m)$ for $i \geq 0$. It follows from (2.3) that

$$g_{r,n+m}^m = \Delta^n f(n+m) = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} r^{m+i} (m+i)!. \quad (2.7)$$

Multiplying the above identity by $u^n/n!$ and summing over $n \geq 0$ we obtain

$$\sum_{n \geq 0} g_{r,n+m}^m \frac{u^n}{n!} = r^m m! \sum_{n,i \geq 0} (-1)^{n-i} \binom{m+i}{i} \frac{r^i u^n}{(n-i)!}.$$

Shifting n to $n+i$ yields

$$\sum_{n \geq 0} g_{r,n+m}^m \frac{u^n}{n!} = r^m m! \left(\sum_{n \geq 0} (-1)^n \frac{u^n}{n!} \right) \cdot \left(\sum_{i \geq 0} \binom{m+i}{i} (ru)^i \right),$$

which is clearly equal to the right-hand side of (2.5). Finally multiplying (2.5) by $x^m/m!$ and summing over $m \geq 0$ yields (2.6). $\square$

Setting $m = 0$ in (2.4) yields immediately formula (2.1).

2.2 Some recurrence relations

2.2.1 Relations concerning $g_{r,n}^m$

Proposition 18. *For $r \geq 0$ and $0 \leq m \leq n$ there hold*

$$g_{r,n}^m = (rn - 1)g_{r,n-1}^m + r(n - m - 1)g_{r,n-2}^m \quad (n \geq 2); \quad (2.8)$$

$$g_{r,n}^m = r(n - m)g_{r,n-1}^m + rm g_{r,n-1}^{m-1} \quad (m \geq 1, n \geq 1); \quad (2.9)$$

$$g_{r,n}^m = rn g_{r,n-1}^m - rm g_{r,n-2}^{m-1} \quad (m \geq 1, n \geq 2); \quad (2.10)$$

where $g_{r,0}^0 = 1$, $g_{r,1}^0 = r - 1$ and $g_{r,1}^1 = r$.

Proof. Let $F(u)$ be the left-hand side of (2.5). Differentiating $F(u)$ and using the right-hand side of (2.5) we get

$$(1 - ru)F'(u) = [r(m + 1) - 1 + ru]F(u). \quad (2.11)$$

Equating the coefficients of $u^n/n!$ in (2.11) yields

$$g_{r,n+m+1}^m = [r(m + n + 1) - 1]g_{r,n+m}^m + rn g_{r,n+m-1}^m,$$

which gives (2.8) by shifting $n + m + 1$ to n .

Next, multiplying the two sides of (2.5) by $1 - ru$ gives

$$(1 - ru) \sum_{n \geq 0} g_{r,n+m}^m \frac{u^n}{n!} = \frac{r^m m! \exp(-u)}{(1 - ru)^m} = rm \sum_{n \geq 0} g_{r,n+m-1}^{m-1} \frac{u^n}{n!}. \quad (2.12)$$

Equating the coefficients of $u^n/n!$ yields

$$g_{r,n+m}^m - rn g_{r,n+m-1}^m = rm g_{r,n+m-1}^{m-1}, \quad (2.13)$$

which is (2.9) by shifting $n + m$ to n .

Finally, we derive (2.10) from (2.9) and (1.1):

$$\begin{aligned} g_{r,n}^m &= rn g_{r,n-1}^m - rm(g_{r,n-1}^m - g_{r,n-1}^{m-1}) \\ &= rn g_{r,n-1}^m - rm g_{r,n-2}^{m-1}. \end{aligned}$$

The proof is thus completed. $\square$

2.2.2 Relations concerning $d_{r,n}^m$

It is easy to convert the above relations for $g_{r,n}^m$ to those for $d_{r,n}^m$.

Proposition 19. *For $r \geq 0$ and $0 \leq m \leq n$ we have*

$$d_{r,n}^m = (rn - 1)d_{r,n-1}^m + r(n - m - 1)d_{r,n-2}^m \quad (n \geq 2); \quad (2.14)$$

$$d_{r,n}^m = r(n - m)d_{r,n-1}^m + d_{r,n-1}^{m-1} \quad (m \geq 1, n \geq 1); \quad (2.15)$$

$$d_{r,n}^m + d_{r,n-2}^{m-1} = rnd_{r,n-1}^m \quad (m \geq 1, n \geq 2), \quad (2.16)$$

where $d_{r,0}^0 = 1$, $d_{r,1}^0 = r - 1$ and $d_{r,1}^1 = 1$.

Proof. The equations (2.14), (2.15) and (2.16) follow directly from Proposition 18. $\square$

Using the combinatorial interpretation for $d_{r,n}^m$ in Theorem 1.2.5 we now give combinatorial interpretations of (1.2), (2.2) and (2.16) by generalizing the proofs of Rakotondrajao [48], which correspond to the $r = 1$ case.

2.3 Combinatorial proof of (1.2), (2.2), (2.16)

2.3.1 Combinatorial proof of (1.2)

We shall prove that the cardinality of $D_{r,n}^m$ satisfies the following recurrence:

$$d_{r,n}^n = 1 \quad \text{and} \quad d_{r,n}^{m-1} + d_{r,n-1}^{m-1} = rm d_{r,n}^m \quad (1 \leq m \leq n). \quad (2.17)$$

First, the identity permutation is the only n -isolated-fixed permutation in $G(r, n)$, so $d_{r,n}^n = 1$. To prove (2.17) we construct a bijection $\vartheta : \pi \mapsto (\epsilon, \alpha, \pi')$ from $D_{r,n-1}^{m-1} \cup D_{r,n}^{m-1}$ to $\mathcal{C}_r \times [m] \times D_{r,n}^m$ as follows:

Let $\sigma = |\pi|$ and factorize π into disjoint cycles.

1. If $\pi \in D_{r,n-1}^{m-1}$, then $\epsilon = 1$, $\alpha = m$, the cycles of π' are obtained from those of π by substituting $\zeta^j i$ by $\zeta^j i + 1$ if $i \geq m$ and then adding the cycle (m) .

2. If $\pi \in D_{r,n}^{m-1}$, let C_m be the cycle of σ containing m . Then $\epsilon = \text{color}_\pi(m)$ and α is the smallest integer in C_m ; let q be the smallest integer such that $\sigma^q(m) = \alpha$; then σ' is obtained from σ by deleting the letters $m, \sigma(m), \dots, \sigma^{q-1}(m)$ from C_m and creating a new cycle $(m \sigma(m) \dots \sigma^{q-1}(m))$. Finally, define the color of $i \in [n]$ in π' by

$$\text{color}_{\pi'}(i) = \begin{cases} \text{color}_\pi(i), & \text{if } i \neq m; \\ 1, & \text{if } i = m. \end{cases}$$

For example, let $r = 3$, $n = 9$ and $m = 6$. If $\pi = (\bar{1}\bar{6})(2)(3)(4)(5)(\bar{7}\bar{8}) \in D_{3,8}^5$, then

$$\epsilon = 1; \alpha = 6 \quad \text{and} \quad \pi' = (1\bar{7})(2)(3)(4)(5)(6)(\bar{8}\bar{9});$$

if $\pi = (1)(2\bar{9}\bar{6}\bar{8})(3)(4)(5)(\bar{7}) \in D_{3,9}^5$ then

$$\alpha = 2, \quad \epsilon = \zeta \quad \text{and} \quad \pi' = (1)(2\bar{9})(3)(4)(5)(\bar{7})(6\bar{8}).$$

It remains to show that ϑ is a bijection. Given (ϵ, α, π') let $\sigma' = |\pi'|$. We define the inverse $\vartheta^{-1} : (\epsilon, m, \pi') \mapsto \pi$ as follows:

1. If $\alpha = m$; $\epsilon = 1$ and $\pi'(m) = m$ then the cycles of π are obtained by deleting the cycle (m) and replacing $\zeta^j i$ by $\zeta^j i - 1$ if $i \geq m$ in π' .
2. If $\alpha = m$; $\epsilon = 1$ and $\pi'(m) \neq m$ then $\pi = \pi'$.
3. If $\alpha = m$; $\epsilon \neq 1$ then we get π from π' by replacing m by ϵm .
4. If $\alpha < m$; σ is obtained from σ' by removing the cycle which contains m and then inserting the word $m\sigma'(m)\sigma'^2(m)\dots\sigma'^{q-1}(m)$, where $\sigma'^q(m) = m$, in the cycle which contains the integer α just before the integer α . In other words, we define $m = \sigma(\sigma'^{-1}(\alpha))$, $\sigma^j(m) = \sigma'^j(m)$ for $1 \leq j \leq q-1$ and $\sigma^q(m) = \alpha$. Finally define $\text{color}_\pi(i) = \text{color}_{\pi'}(i)$ for $i \neq m$ and $\text{color}_\pi(m) = \epsilon$.

To see that this is indeed the inverse of ϑ we just note the following simple facts:

- If $\pi \in D_{r,n-1}^{m-1}$ then $\epsilon = 1$, $\alpha = m$ and $m \in \text{FIX}(\pi')$. We have $\vartheta(\pi) \in E_1 = \{(1, m, \pi'), m \in \text{FIX}(\pi'), \pi' \in D_{r,n}^m\}$.
- If $\pi \in D_{r,n}^{m-1} \cap D_{r,n}^m$ then $\epsilon = 1$, $\alpha = m$ and $m \notin \text{FIX}(\pi')$. In this case $\pi' = \pi$ we have $\vartheta(\pi) \in E_2 = \{(1, m, \pi'), m \notin \text{FIX}(\pi'), \pi' \in D_{r,n}^m\}$.

- If $m \notin \text{cycle of } \sigma \text{ containing } i < m$ and $\text{color}_\pi(m) \neq 1$ then $\alpha = m$, $\epsilon \neq 1$; and π' is obtained from π by just replacing ϵm by m . We have $\vartheta(\pi) \in E_3 = \{(\epsilon, m, \pi'), \epsilon \neq 1, \pi \in D_{r,n}^m\}$.
- If $m \in \text{cycle of } \sigma \text{ containing } i < m$ then $\alpha < m$. In this case the image π' is defined by the second case of the construction of ϑ . We have $\vartheta(\pi) \in E_4 = \mathcal{C}_r \times [m-1] \times D_{r,n}^m$.

Clearly $\{E_1, E_2, E_3, E_4\}$ is a partition of $D_{r,n}^m$.

2.3.2 Combinatorial proof of (2.2)

We shall prove the following version of (2.2):

$$\begin{aligned} rnd_{r,n-1}^0 - 1 &= d_{r,n}^0 && \text{if } n \text{ is odd,} \\ rnd_{r,n-1}^0 &= d_{r,n}^0 - 1 && \text{if } n \text{ is even.} \end{aligned}$$

Denote by $\mathcal{D}(r, n)$ the set of derangements in $G(r, n)$. Let $E_n = \emptyset$ if n is odd and $E_n = \{(1\ 2)(3\ 4) \cdots (n-1\ n)\}$ if n is even. Introduce also $F_n = \emptyset$ if n is even and $F_n = \{1\} \times \{n\} \times E_{n-1}$ if n is odd. We are going to define a mapping $\tau_r : (\varepsilon, k, \pi) \mapsto \pi'$ from $(\mathcal{C}_r \times [n] \times \mathcal{D}(r, n-1)) \setminus F_n$ to $\mathcal{D}(r, n) \setminus E_n$, which implies the above identities.

Factorize π into disjoint cycles. We construct the cyclic factorization of π' by distinguishing several cases and by giving an example in $G(4, 9)$ for each case.

Let $c(k)$ be the length of the cycle of π containing k and write $\widehat{k} = \text{color}_\pi(k) \cdot k$.

1. If $k < n$, we obtain π' by inserting εn just after $\widehat{k}$ in a cycle of π .

Example: $\varepsilon = \zeta^3$, $k = 3$, $\pi = (\bar{1}\ \bar{4}\ 2)(\bar{3})(\bar{5}\ \bar{6}\ 8\ \bar{7})$ then

$$\pi' = (\bar{1}\ \bar{4}\ 2)(\bar{9}\bar{3})(\bar{5}\ \bar{6}\ 8\ \bar{7}).$$

2. If $k = n$ and $\varepsilon \neq 1$, we obtain π' by creating the cycle (εn) .

Example:

$$\varepsilon = \zeta^3, k = 9, \pi = (\bar{1}\ \bar{4}\ 2)(\bar{3})(\bar{5}\ \bar{6}\ 8\ \bar{7}) \text{ then } \pi' = (\bar{1}\ \bar{4}\ 2)(\bar{3})(\bar{5}\ \bar{6}\ 8\ \bar{7})(\bar{9}).$$

3. Suppose $k = n$ and $\varepsilon = 1$. Let $p \geq 0$ be the smallest integer such that the transposition $(2p+1, 2p+2)$ is not a cycle of π . In all the examples of this part we take $p = 2$.

3.1 If $\text{color}_\pi(2p+1) = 1$ then

3.1.1 If $2p+2$ is a 1-circular succession of $|\pi|$ then π' is obtained by deleting $2p+1$ and creating the cycle $(n, 2p+1)$.

Example: $\pi = (1\ 2)(3\ 4)(5\ \bar{6})(\bar{7}\ \bar{8})$ then $\pi' = (1\ 2)(3\ 4)(\bar{6})(\bar{7}\ \bar{8})(9\ 5)$.

3.1.2 If $2p+2$ is not a 1-circular succession of $|\pi|$ then:

a) If $c(2p+1) = 2$ then π' is obtained by deleting the cycle $(2p+1, \pi(2p+1))$ and inserting $2p+1$ just before the letter $\widehat{2p+2}$ and creating the cycle $(\lambda n, |\pi|(2p+1))$ where $\lambda = \text{color}_\pi(|\pi|(2p+1))$.

Example: $\pi = (12)(34)(5\bar{8})(\bar{6}\bar{7})$ then $\pi' = (12)(34)(5\bar{6}\bar{7})(\bar{9}\bar{8})$.

b) If $c(2p+1) > 2$. Let $a = |\pi|^{-1}(2p+1)$ and $\xi = \text{color}_\pi(a)$ then π' is obtained by deleting the letter $\xi \cdot a$ and creating the cycle $(\xi n, a)$.

Example: $\pi = (12)(34)(5\bar{8}\bar{6}\bar{7})$ then $\pi' = (12)(34)(5\bar{8}\bar{6})(\bar{9}\bar{7})$.

3.2 If $\text{color}_\pi(2p+1) = \gamma \neq 1$ then

3.2.1 If $c(2p+1) = 1$ then π' is obtained by deleting the letter $\gamma \cdot (2p+1)$ and creating the cycle $(\gamma n, 2p+1)$.

Example: $\pi = (12)(34)(\bar{5})(\bar{6}\bar{8}\bar{7})$ then $\pi' = (12)(34)(\bar{9}\bar{5})(\bar{6}\bar{8}\bar{7})$.

3.2.2 If $c(2p+1) \neq 1$. Let $a = |\pi|^{-1}(2p+1)$ and $\gamma = \text{color}_\pi(a)$ then π' is obtained by deleting the letter $\gamma \cdot a$ and creating the cycle $(\gamma n, a)$.

Example: $\pi = (12)(34)(\bar{5}\bar{8}\bar{7}\bar{6})$ then $\pi' = (12)(34)(\bar{5}\bar{8}\bar{7})(\bar{9}\bar{6})$.

Here is the inverse algorithm of $\tau_r : \pi' \mapsto (\varepsilon, k, \pi)$. Denote by $c(n)$ the length of the cycle of π' containing n . In what follows we write $\rho = \text{color}_{\pi'}(n)$ and $\tilde{k} = \text{color}_{\pi'}(k) \cdot k$.

- If $c(n) \geq 3$ or $c(n) = 1$ or $c(n) = 2$ and $\text{color}_{\pi'}(|\pi'|(n)) \neq 1$ then $\varepsilon = \text{color}_{\pi'}(n)$ and $k = |\pi'|^{-1}(n)$ and we obtain π by deleting the letter $\tilde{n}$.
- If $c(n) = 2$ and $\text{color}_{\pi'}(|\pi'|(n)) = 1$ then $\varepsilon = 1$ and $k = n$. Let p be the smallest integer such that the transposition $(2p+1, 2p+2)$ is not a cycle of π' .
 - a. If $\pi'(n) = 2p+1$ and $\rho = 1$ then we delete the cycle containing n and insert the letter $2p+1$ just before the letter $\widehat{2p+2}$.

- b. If $\pi'(n) \neq 2p+1$ and $\text{color}_{\pi'}(2p+1) = 1$ and $|\pi'| (2p+1) = 2p+2$, we first delete $2p+1$ and the cycle containing n , then create the cycle $(\rho \cdot \pi'(n), 2p+1)$.
- c. If $|\pi'| (2p+1) \neq 2p+2$ and $\pi'(n) \neq 2p+1$ then we delete the cycle containing n and then insert the letter $\rho \cdot \pi'(n)$ before the letter $\widetilde{2p+1}$.
- d. If $\pi'(n) = 2p+1$ and $\rho \neq 1$ then we delete the cycle containing n and create the cycle containing the single letter $2p+1$ with the color ρ .

For example, the mapping $\tau_2 : (\mathcal{C}_2 \times [3] \times \mathcal{D}(2, 2)) \setminus F_3 \longrightarrow \mathcal{D}(2, 3) \setminus E_3$, where $E_3 = \emptyset$ and $F_3 = (1, 3, (1\ 2))$, is given in the following table.

$\pi \setminus (\varepsilon, k)$	(1, 1)	(1, 2)	(1, 3)	(ζ , 1)	(ζ , 2)	(ζ , 3)
(12)	(132)	(123)		($\bar{1}\bar{3}\bar{2}$)	($\bar{1}\bar{2}\bar{3}$)	(12)($\bar{3}$)
($\bar{1}\bar{2}$)	($\bar{1}\bar{3}\bar{2}$)	($\bar{1}\bar{2}\bar{3}$)	($\bar{1}$)(32)	($\bar{1}\bar{3}\bar{2}$)	($\bar{1}\bar{2}\bar{3}$)	($\bar{1}\bar{2}$)($\bar{3}$)
($\bar{1}\bar{2}$)	(13 $\bar{2}$)	($\bar{1}\bar{2}\bar{3}$)	(13)($\bar{2}$)	($\bar{1}\bar{3}\bar{2}$)	($\bar{1}\bar{2}\bar{3}$)	(12)($\bar{3}$)
($\bar{1}\bar{2}$)	($\bar{1}\bar{3}\bar{2}$)	($\bar{1}\bar{2}\bar{3}$)	($\bar{3}\bar{2}$)($\bar{1}$)	($\bar{1}\bar{3}\bar{2}$)	($\bar{1}\bar{2}\bar{3}$)	($\bar{1}\bar{2}$)($\bar{3}$)
($\bar{1}$)($\bar{2}$)	($\bar{1}\bar{3}$)($\bar{2}$)	($\bar{1}$)($\bar{2}\bar{3}$)	($\bar{3}\bar{1}$)($\bar{2}$)	($\bar{1}\bar{3}$)($\bar{2}$)	($\bar{1}$)($\bar{2}\bar{3}$)	($\bar{1}$)($\bar{2}$)($\bar{3}$)

2.3.3 Combinatorial proof of (2.16)

By Theorem 1.2.5 the coefficient $d_{r,n}^m$ equals the cardinality of $D_{r,n}^m$. We are going to establish a bijection $\Phi : (\rho, \alpha, \pi) \longmapsto \pi'$ from $\mathcal{C}_r \times [n] \times D_{r,n-1}^m$ to $D_{r,n}^m \cup D_{r,n-2}^{m-1}$.

Let $\sigma = |\pi|$ and $\sigma' = |\pi'|$. The cyclic factorization of π' is obtained from that of π as follows:

1. If $\alpha = n$, $\rho = 1$ and $1 \in \text{FIX}(\pi)$, we get $\pi' \in D_{r,n-2}^{m-1}$ by deleting the cycle (1) and decreasing all other letters by 1.
2. If $\alpha = n$ and $\rho \neq 1$ then we create the cycle (ρn) . In this case $\pi' \in D_{r,n}^m$ and the cycle containing n is of length 1 but n is not a fixed point of π' .
3. If $\alpha = n$, $\rho = 1$ and $1 \notin \text{FIX}(\pi)$, then we delete $\pi(1)$ from its cycle and create a new cycle $(\gamma n, \sigma(1))$ where $\gamma = \text{color}_{\pi}(\sigma(1))$. In this case $\pi'(n) > m$.

4. If $\alpha < n$ then we insert the letter ρn just before α .

To show that the mapping Φ is a bijection we construct its inverse as follows.

1. If $\pi' \in D_{r,n-2}^{m-1}$ then $\alpha = n$, $\rho = 1$ and π is obtained from π' by adding 1 to all letters and creating the cycle (1).
2. If $\pi' \in D_{r,n}^m$ and the cycle containing n is of length 1, then let $\alpha = n$, $\rho = \text{color}_{\pi'}(n)$ and π is obtained from π' by deleting the letter ρn .
3. If $\pi' \in D_{r,n}^m$ and the cycle containing n is of length 2 with $\pi'(n) > m$ then let $\alpha = n$, $\rho = 1$ and π is obtained from π' by deleting the letter n and inserting the letter $\gamma\sigma'(n)$ just after 1 where $\gamma = \text{color}_{\pi'}(n)$.
4. In all other cases, let $\alpha = \sigma'(n)$, $\rho = \text{color}_{\pi'}(n)$ and π is obtained from π' by just deleting the letter ρn .

For $n = 9; m = 4; r = 3$ we give some examples to illustrate the above bijection.

$$\pi' = (1\bar{\bar{5}})(2)(3\bar{6})(4\bar{7}) \in D_{r,n-2}^{m-1} \text{ then } \alpha = 9, \rho = 1 \text{ and}$$

$$\pi = (1)(2\bar{\bar{6}})(3)(4\bar{7})(5\bar{8}).$$

$$\pi' = (1\bar{\bar{5}})(2\bar{8})(3\bar{6})(4\bar{7})(\bar{9}) \in D_{r,n}^m \text{ then } \alpha = 9, \rho = \zeta^2 \text{ and}$$

$$\pi = (1\bar{\bar{5}})(2\bar{8})(3\bar{6})(4\bar{7}).$$

$$\pi' = (15)(2\bar{8})(3)(4\bar{7})(\bar{9}6) \in D_{r,n}^m \text{ then } \alpha = 9, \rho = 1 \text{ and}$$

$$\pi = (1\bar{\bar{6}}5)(2\bar{8})(3)(4\bar{7}).$$

$$\pi' = (1\bar{\bar{5}})(2\bar{8})(\bar{6})(4)(\bar{9}\bar{7}3) \in D_{r,n}^m \text{ then } \alpha = 7, \rho = \zeta^2 \text{ and}$$

$$\pi = (1\bar{\bar{5}})(2\bar{8})(\bar{6})(4)(\bar{7}3).$$

Chapitre 3

The q -Euler's table for $G(r, n)$

This is the first part of [32]. In 1997 Clarke et al. introduced a q -analogue of Euler's difference table for $n!$ using a key bijection Ψ on symmetric groups. In this chapter we extend their results to the wreath product of a cyclic group with the symmetric group. By generalizing their bijection Ψ we prove the equidistribution of the triple statistics (fix, exc, fmaj) and (fix, exc, fmaf) on wreath products, where fix, exc, fmaj and fmaf denote the number of fixed points, the number of excedances, the flag major index and the flag maf index, respectively. As a consequence we obtain a new mahonian statistic fmaf on wreath products.

3.1 Introduction

As said in chapter 0, many authors have been trying extending various enumerative results on symmetric groups to other classical reflection groups. In particular Adin and Roichman [1] introduced the flag major index fmaj on wreath products of a cyclic group with the symmetric group, and Haglund et al. [43] proved that

$$\sum_{\sigma \in G(r, n)} q^{\text{fmaj } \sigma} = [r]_q [2r]_q \cdots [nr]_q. \quad (3.1)$$

In this chapter we shall consider some natural generalizations of (3.1) by studying a wreath product analogue of Euler's q -difference table $\{g_{r, n}^m(q)\}_{m \geq n \geq 0}$ defined by the following recurrence:

$$\begin{cases} g_{r, n}^n(q) = [r]_q [2r]_q \cdots [nr]_q, \\ g_{r, n}^m(q) = g_{r, n}^{m+1}(q) - q^{r(n-m-1)} g_{r, n-1}^m(q) \end{cases} \quad (0 \leq m \leq n-1). \quad (3.2)$$

For example, when $r = 1$ and $r = 2$, the first values of $g_{r,n}^m(q)$ are given as follows:

- $r = 1$

$n \backslash m$	0	1	2	3
0	1			
1	0	1		
2	q	q	$1 + q$	
3	$q[2]_q$	$q[3]_q$	$q^3 + 2q^2 + q$	$(1 + q)(1 + q + q^2)$

- $r = 2$

$n \backslash m$	0	1	2	3
0	1			
1	q	$[2]_q$		
2	$q[4]_q + q^2$	$q[4]_q + q^2[2]_q$	$[2]_q[4]_q$	
3	$g_{2,3}^0$	$g_{2,3}^1$	$g_{2,3}^2$	$[2]_q[4]_q[6]_q$

where

$$\begin{aligned} g_{2,3}^2 &= q^9 + 3q^8 + 5q^7 + 7q^6 + 8q^5 + 7q^4 + 5q^3 + 3q^2 + q, \\ g_{2,3}^1 &= q^9 + 3q^8 + 5q^7 + 6q^6 + 6q^5 + 5q^4 + 4q^3 + 3q^2 + q, \\ g_{2,3}^0 &= q^9 + 2q^8 + 4q^7 + 4q^6 + 5q^5 + 5q^4 + 4q^3 + 3q^2 + q. \end{aligned}$$

It is remarkable that $g_{r,n}^m(q)$ are polynomials in q with non-negative integral coefficients. For $r = 1$, Clarke et al. [27] proved that the entry $g_{1,n}^m(q)$ is actually the generating function for a subset of S_n by the Mahonian statistic *maf*. Their proof is based on a key bijection Ψ on S_n transforming the statistic *maf* to the statistic *major index*. Our aim is to show that the results in [27] can be readily extended to wreath products. More precisely, we will find a combinatorial interpretation and an explicit formula for $g_{r,n}^m(q)$ by introducing a new mahonian statistic *fmaf* on the wreath products and by extending Clarke et al.'s bijection Ψ to the colored permutations.

3.2 Definitions and main results

We use the linear order “ OC ” on the alphabet $\Sigma_{r,n}$. Recall that $i \in [n]$ is a *fixed point* of $\sigma \in G(r, n)$ if $\sigma(i) = i$. Let $\text{FIX}(\sigma)$ be the set of fixed points of σ and $\text{fix } \sigma$ the cardinality of $\text{FIX}(\sigma)$. The colored permutation σ has a *descent* at $i \in \{1, 2, \dots, n-1\}$ if $\sigma(i) > \sigma(i+1)$ and i is called a *descent place* of σ . Let $\text{DES}(\sigma)$ be the set of descent places of σ . The

major index of σ , denoted by $\text{maj } \sigma$, is the sum of all the descent places of σ . If $y_1 y_2 \cdots y_m$ is the word obtained by deleting the fixed points of σ and $y_i = \varepsilon_i |y_i|$ ($1 \leq i \leq m$), writing $z_i = \varepsilon_i \text{rank}(|y_i|)$ with ‘rank’ being the increasing bijection from $\{|y_1|, |y_2|, \dots, |y_m|\}$ to $[m]$, then the *derangement part* of σ is defined to be

$$\text{Der}(\sigma) = z_1 z_2 \cdots z_m. \quad (3.3)$$

We now define the main eulerian and mahonian statistics on the wreath product $G(r, n)$. If $\sigma = x_1 \dots x_n \in G(r, n)$ then the statistics des_r , exc and col are defined by

$$\begin{aligned} \text{des}_r \sigma &= \sum_{i=1}^{n-1} \chi(x_i > x_{i+1}), & \text{exc } \sigma &= \sum_{i=1}^n \chi(x_i > i), \\ \text{col } \sigma &= \sum_{j=0}^{r-1} j \cdot |\text{COL}_j(\sigma)|, \end{aligned}$$

where $\text{COL}_j(\sigma) = \{i \in [n] : \frac{x_i}{|x_i|} = \zeta^j\}$ and $\chi(A) = 1$ if A is true and 0 otherwise. When $r = 1$ we shall write $\text{des} = \text{des}_r$. The maj and maf statistics are defined by

$$\text{maj } \sigma = \sum_{i=1}^{n-1} i \cdot \chi(x_i > x_{i+1}), \quad \text{and} \quad \text{maf } \sigma = \text{maj } \text{Der}(\sigma) + \sum_{j=1}^k (i_j - j),$$

where $\text{FIX}(\sigma) = \{i_1, i_2, \dots, i_k\}$. The *flag-maj* fmaj and *flag-maf* fmaf statistic are defined by

$$\text{fmaj } \sigma = r \cdot \text{maj } \sigma + \text{col}(\sigma), \quad \text{and} \quad \text{fmaf } \sigma = r \cdot \text{maf } \sigma + \text{col}(\sigma).$$

Remark. While the statistic fmaj was first introduced by Adin and Roichman [1], the statistic fmaf is new and reduces to maf of Clarke et al. [27] for $r = 1$.

Example. If $\sigma = \mathbf{1\bar{8}346\bar{2}7\bar{5}9} \in G(4, 9)$, then $\text{maj } \sigma = 1 + 5 + 7 = 13$, $\text{FIX } \sigma = \{1, 3, 4, 7, 9\}$ and $\text{Der}(\sigma) = \mathbf{\bar{4}3\bar{1}\bar{2}}$. Therefore $\text{maj } \text{Der}(\sigma) = 2$ and

$$\text{maf } \sigma = 2 + ((1 - 1) + (3 - 2) + (4 - 3) + (7 - 4) + (9 - 5)) = 11.$$

Since $\text{COL}_0(\sigma) = \{1, 3, 4, 5, 7, 9\}$, $\text{COL}_1(\sigma) = \{2, 8\}$, $\text{COL}_2(\sigma) = \{6\}$, $\text{COL}_3(\sigma) = \emptyset$, we have $\text{col}(\sigma) = 0 \times 6 + 1 \times 2 + 2 \times 1 = 4$. Finally,

$$\text{fmaj } \sigma = 4 \times 13 + 4 = 56, \quad \text{and} \quad \text{fmaf } \sigma = 4 \times 11 + 4 = 48.$$

Remark. Other notions of descents have also been considered previously, see [53].

We first show that the statistics $(\text{fix}, \text{fmaf})$ and $(\text{fix}, \text{fmaj})$ are equidistributed on $G(r, n)$ and their common distribution has an explicit formula.

Theorem 3.2.1. *The triple statistics $(\text{fmaf}, \text{exc}, \text{fix})$ and $(\text{fmaj}, \text{exc}, \text{fix})$ are equidistributed on $G(r, n)$. Moreover the common generating function*

$$\sum_{\sigma \in G(r, n)} q^{\text{fmaf } \sigma} x^{\text{fix } \sigma} = \sum_{\sigma \in G(r, n)} q^{\text{fmaj } \sigma} x^{\text{fix } \sigma} \quad (3.4)$$

has the explicit formula

$$g_{r, n}(q, x) := [r]_q [2r]_q \cdots [nr]_q \sum_{k=0}^n \frac{(x-1)(x-q^r) \cdots (x-q^{r(k-1)})}{[r]_q [2r]_q \cdots [kr]_q}. \quad (3.5)$$

Remark. When $r = 1$ Gessel and Reutenauer [42] first proved that the generating function of $(\text{fmaj}, \text{fix})$ on S_n is given by (3.5). For general r and $x = 1$ we recover Haglund et al's formula (3.1) for the generating function of fmaj on $G(r, n)$. For $x = 0$, we derive from (3.4) and (3.5) an explicit formula for the colored q -derangement number:

$$d_{r, n}(q) := \sum_{\sigma \in \mathcal{D}_{r, n}} q^{\text{fmaj } \sigma} = [r]_q [2r]_q \cdots [nr]_q \sum_{k=0}^n \frac{(-1)^k q^{r \binom{k}{2}}}{[r]_q [2r]_q \cdots [kr]_q}. \quad (3.6)$$

The $r = 1$ and $r = 2$ cases of (3.6) were first obtained by Gessel (unpublished) and Wachs [54], and Chow [24], respectively. Finally (3.6) yields immediately the following recurrence relation for the colored q -derangement numbers:

$$d_{r, n+1}(q) = [rn + r]_q d_{r, n}(q) + (-1)^{n+1} q^{r \binom{n+1}{2}}. \quad (3.7)$$

Introduce the q -shifted factorials

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \dots, \text{ or } \infty,$$

then the q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}, \quad n \geq m \geq 0.$$

Instead of the colored q -Euler table (3.2), as Clarke et al. [27], we can consider a more general triangle than (3.2) by taking $g_{r,n}^n(q, x) := g_{r,n}(q, x)$ as the diagonal coefficients and replace the recurrence relation in (3.2) by

$$g_{r,n}^m(q, x) = g_{r,n}^{m+1}(q, x) - xq^{r(n-m-1)}g_{r,n-1}^m(q, x) \quad (0 \leq m \leq n-1). \quad (3.8)$$

For $n \geq m \geq 0$, denote by $G_m(r, n)$ the set of permutations σ in $G(r, n)$ such that $\text{FIX}(\sigma) \subset \{n-m+1, \dots, n-1, n\}$, i.e., for all $i \leq n-m$, $\sigma(i) \neq i$. In particular we have $G_n(r, n) = G(r, n)$ and $\mathcal{D}(r, n) := G_0(r, n)$ is the set of colored derangements of order n . The following theorem gives a full description of $g_{r,n}^m(q, x)$, which generalizes the previous results in [27] for $r = 1$ and [31] for $q = 1$, respectively.

Theorem 3.2.2. *For $n \geq m \geq 0$ we have the following explicit formula:*

$$g_{r,n}^m(q, x) = \sum_{k=0}^{n-m} (-x)^k \begin{bmatrix} n-m \\ k \end{bmatrix}_{q^r} q^{r \binom{k}{2}} g_{r,n-k}(q, x) \quad (3.9)$$

and the combinatorial interpretation:

$$g_{r,n}^m(q, x) = \sum_{\sigma \in G_m(r, n)} q^{\text{fmaf } \sigma} x^{\text{fix } \sigma}. \quad (3.10)$$

Remark. The two statistics fmaf and fmaj are identical on $\mathcal{D}(r, n)$, equidistributed on $G(r, n)$, but not equidistributed on $G_m(r, n)$ for $0 < m < n$.

3.3 Proof of Theorem 3.2.1

We first recall Clarke et al.'s bijection Ψ , which will also be used in chapter 4, and then show that one can extend Ψ to $G(r, n)$ with the following property:

$$(\text{maf}, \text{exc}, \text{fix}, \text{col})\Psi(\sigma) = (\text{maj}, \text{exc}, \text{fix}, \text{col})\sigma, \quad \forall \sigma \in G(r, n). \quad (3.11)$$

3.3.1 Clarke et al.'s bijection Ψ

Let $\sigma = x_1 x_2 \dots x_n \in S_n$ with $x_0 = x_{n+1} = +\infty$. For $0 \leq i \leq n$, a pair $(i, i+1)$ of positions is the j -th slot of σ provided that $x_i \neq i$ and that σ has $i-j$ fixed points f such that $f < i$. Of course, the j -th slot is $(j, j+1)$ if σ is a derangement. Clearly we can insert a fixed point into the j -th slot and obtain the permutation

$$\langle \sigma, j \rangle = x'_1 x'_2 \dots x'_i (i+1) x'_{i+1} \dots x'_n, \quad (3.12)$$

where $x' = x$ if $x \leq i$ and $x' = x + 1$ if $x > i$ for all $x \in [n]$.

Now, if σ is a derangement in S_n and $(i_1, i_2, \dots, i_m)$ a sequence of integers such that $0 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n$, we can insert successively m fixed points in σ and obtain a permutation τ in S_{n+m} :

$$\tau = \langle \sigma, i_1, \dots, i_m \rangle = \langle \langle \sigma, i_1, \dots, i_{m-1} \rangle, i_m \rangle. \quad (3.13)$$

Note that $\text{Der}(\tau) = \sigma$ and the fixed points of the last permutation are $i_1 + 1, i_2 + 2, \dots, i_m + m$. Conversely, any permutation $\tau \in S_{m+n}$ with m fixed points can be written as (3.13) in one and only one way. Thus, if $S(\sigma, m)$ denotes the set of permutations in S_{n+m} with derangement part $\sigma \in \mathcal{D}_n$, then

$$S(\sigma, m) = \{ \langle \sigma, i_1, \dots, i_m \rangle \mid 0 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n \}.$$

Let $\sigma \in S_n$. The j -th slot $(i, i+1)$ of σ is said to be *green* if $\text{des}\langle \sigma, j \rangle = \text{des } \sigma$, *red* if $\text{des}\langle \sigma, j \rangle = \text{des } \sigma + 1$. We assign *values* from 0 to g to the green slots of σ from right to left, and values from $g+1$ to n to the red slots from left to right. Denote the value of the j -th slot by g_j . The bijection $\Psi : S(\sigma, m) \longrightarrow S(\sigma, m)$ is defined by induction on $m \geq 0$ as follows:

1. Ψ is the identity mapping on $S(\sigma, 0)$.
2. Define Ψ on $S(\sigma, 1)$ by

$$\Psi \langle \sigma, i \rangle = \langle \sigma, g_i \rangle. \quad (3.14)$$

3. Let $m > 1$ and suppose that Ψ has been defined on $S(\sigma, k)$ for $0 \leq k \leq m-1$. Consider $\tau = \langle \sigma, i_1, \dots, i_m \rangle$. Suppose that the i_m -th slot of σ is green. Then, if $\Psi \langle \sigma, i_1, \dots, i_{m-1} \rangle = \langle \sigma, j_2, \dots, j_m \rangle$, we define

$$\Psi(\tau) = \langle \sigma, g_{i_m}, j_2, \dots, j_m \rangle. \quad (3.15)$$

Suppose that the i_m -th slot of σ is red. Let k be the smallest positive integer such that $i_{m-k} < i_m$. Thus $i_{m-k} < i_{m-k+1} = \dots = i_m$. Then,

- if $k = m$ we define

$$\Psi(\tau) = \langle \sigma, \underbrace{g_{i_m} - i_m, \dots, g_{i_m} - i_m}_{m-1 \text{ terms}}, g_{i_m} \rangle. \quad (3.16)$$

- if $k < m$ and $\Psi \langle \sigma, i_1, \dots, i_{m-k} \rangle = \langle \sigma, j_1, \dots, j_{m-k} \rangle$, we define

$$\Psi(\tau) = \langle \sigma, \underbrace{g_{i_m} - i_m, \dots, g_{i_m} - i_m}_{k-1 \text{ terms}}, j_1 + 1, \dots, j_{m-k} + 1, g_{i_m} \rangle. \quad (3.17)$$

3.3.2 Generalization of Ψ to $G(r, n)$

To extend the insertion algorithm (3.13) to colored permutations we just need to modify equation (3.12) as follows:

$$\langle \sigma, j \rangle = x'_1 x'_2 \dots x'_i (i+1) x'_{i+1} \dots x'_n, \quad (3.18)$$

where $x' = x$ if $|x| \leq i$ and

$$x' = x + 1 := \varepsilon_x(|x| + 1), \quad (3.19)$$

if $|x| > i$ for all $x \in \Sigma_{r,n}$. Thus each colored permutation $\tau \in G(r, n)$ can be written as (3.13).

Example. Let $\tau = 1 \bar{9} 3 10 5 6 7 4 2 \bar{\bar{8}} \in G(3, 10)$. Then $\sigma := \text{Der}(\tau) = \bar{4} 5 2 1 \bar{\bar{3}} \in \mathcal{D}(3, 5)$ and $\tau = \langle \sigma, 0, 1, 2, 2, 2 \rangle$. Indeed,

$$\begin{aligned} \langle \sigma, 0 \rangle &= 1 \bar{5} 6 3 2 \bar{4}, \\ \langle \sigma, 0, 1 \rangle &= 1 \bar{6} 3 7 4 2 \bar{\bar{5}}, \\ \langle \sigma, 0, 1, 2 \rangle &= 1 \bar{7} 3 8 5 4 2 \bar{\bar{6}}, \\ \langle \sigma, 0, 1, 2, 2 \rangle &= 1 \bar{8} 3 9 5 6 4 2 \bar{\bar{7}}, \\ \langle \sigma, 0, 1, 2, 2, 2 \rangle &= 1 \bar{9} 3 10 5 6 7 4 2 \bar{\bar{8}}. \end{aligned}$$

So $\tau = \langle \sigma, 0, 1, 2, 2, 2 \rangle$.

As in symmetric group we say that the i -th slot ($0 \leq i \leq n$) of $\sigma \in G(r, n)$ is green if $\text{des}_r \langle \sigma, i \rangle = \text{des}_r \sigma$ and red if $\text{des}_r \langle \sigma, i \rangle = \text{des}_r \sigma + 1$. In the same way, we can assign a value to each slot of σ . Now, let σ be a derangement and $G(\sigma, m) := \{\tau \in G(r, n+m) \mid \text{Der}(\tau) = \sigma\}$. We can extend Ψ to a bijection on $G(\sigma, m)$. It follows that $\text{Der}(\Psi(\tau)) = \text{Der}(\tau)$ for any $\tau \in G(\sigma, m)$. Since $\text{exc } \tau = \text{exc}(\text{Der}(\tau))$ and $\text{col } \tau = \text{col}(\text{Der}(\tau))$, we have immediately

$$(\text{exc}, \text{fix}, \text{col})\Psi(\tau) = (\text{exc}, \text{fix}, \text{col})\tau.$$

It remains to verify $\text{maj } \tau = \text{maf}(\Psi(\tau))$. On symmetric groups, the proof of the latter equality is based on the following result.

Lemma 3.1 (Clarke et al.). *Let σ be a derangement in symmetric group and g_i the value of its i -th slot then*

$$\text{maj} \langle \sigma, i \rangle = \text{maj } \sigma + g_i. \quad (3.20)$$

Now, the substitution $x \rightarrow x'$ in (3.18) is compatible with the linear order (see (5.3)) on the alphabet $\Sigma_{r,n}$, namely

$$\forall a, b \in \Sigma_{r,n} \quad a < b \iff a' < b',$$

hence

$$\text{DES}(x'_1 x'_2 \dots x'_i) = \text{DES}(x_1 x_2 \dots x_i)$$

and

$$\text{DES}(x'_{i+1} \dots x'_n) = \text{DES}(x_{i+1} \dots x_n).$$

So the proof of Lemma 3.1 in [27] remains valid when we replace a derangement σ by a any colored derangement. Then we derive (3.11) as in [27].

Example.

Let $\tau = 1 \bar{9} 3 10 5 6 7 4 2 \bar{8} \in G(3, 10)$. We have $\text{DES}\tau = \{1, 4, 7, 8, 9\}$ and $\text{maj } \tau = 29$, $\text{Der}(\tau) = \sigma = \bar{4} 5 2 1 \bar{3}$; $\text{maj } \sigma = 9$. By the previous example, we can write $\tau = \langle \sigma, 0, 1, 2, 2, 2 \rangle$. Note that slots 1, 3, 4, 5 are *green*, while slots 0 and 2 are *red*. So the sequence of values of the slots is $(g_0, g_1, g_2, g_3, g_4, g_5) = (4, 3, 5, 2, 1, 0)$. The algorithm Ψ goes as follows:

- $\Psi\langle\sigma, 0\rangle = \langle\sigma, g_0\rangle = \langle\sigma, 4\rangle$;
- Since 1 is green, $\Psi\langle\sigma, 0, 1\rangle = \langle\sigma, g_1, 4\rangle = \langle\sigma, 3, 4\rangle$;
- Since 2 is red, $\Psi\langle\sigma, 0, 1, 2, 2, 2\rangle = \langle\sigma, g_2 - 2, g_2 - 2, 3 + 1, 4 + 1, g_2\rangle = \langle\sigma, 3, 3, 4, 5, 5\rangle$.

Thus $\Psi(\tau) = \bar{6} 8 2 4 5 1 7 \bar{3} 9 10$. Note that

$$\begin{aligned} \text{maf } \Psi(\tau) &= \text{maj } \sigma + 3 + 3 + 4 + 5 + 5 = 29 = \text{maj}(\tau), \\ \text{fmaj } \Psi(\tau) &= \text{fmaj}(\tau) = 3 \times 29 + 3 = 90. \end{aligned}$$

3.3.3 Proof of (3.5)

Recall Cauchy's q -binomial formula (cf. [6, p.26]):

$$\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} u^n = \frac{(au; q)_\infty}{(u; q)_\infty}.$$

In particular we have Euler's formula by taking $a = 0$:

$$e_q(u) := \sum_{n \geq 0} \frac{u^n}{(q; q)_n} = \frac{1}{(u; q)_\infty}. \quad (3.21)$$

Let $f_{r,n}(q, x) = \sum_{\sigma \in G(r,n)} x^{\text{fix } \sigma} q^{\text{fmaf } \sigma}$. By the insertion algorithm, we can write any permutation $\sigma \in G(r, n)$ with $n - k$ ($0 \leq k \leq n$) fixed points as $\sigma = \langle \pi, i_1, \dots, i_{n-k} \rangle$, where $\pi \in \mathcal{D}_{r,k}$ and $0 \leq i_1 \leq \dots \leq i_{n-k} \leq k$. Since $\text{fmaf } \sigma = \text{fmaj } \pi + r(i_1 + \dots + i_{n-k})$, we have

$$\begin{aligned} f_{r,n}(q, x) &= \sum_{k=0}^n x^{n-k} \sum_{\pi \in \mathcal{D}(r,k)} \sum_{\substack{\sigma \in G(r,n) \\ \text{Der}(\sigma) = \pi}} q^{\text{fmaf } \sigma} \\ &= \sum_{k=0}^n x^{n-k} \sum_{\pi \in \mathcal{D}(r,k)} q^{\text{fmaj } \pi} \sum_{0 \leq i_1 \leq \dots \leq i_{n-k} \leq k} q^{r(i_1 + \dots + i_{n-k})} \\ &= \sum_{k=0}^n x^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^r} d_{r,k}(q). \end{aligned}$$

Therefore

$$\sum_{n \geq 0} f_{r,n}(q, x) \frac{u^n}{(q^r; q^r)_n} = e_{q^r}(xu) \sum_{k=0}^{\infty} d_{r,k}(q) \frac{u^k}{(q^r; q^r)_k}. \quad (3.22)$$

Since $f_{r,n}(q, 1) = (1 - q)^{-n} (q^r; q^r)_n$ by (3.1) and (3.4), setting $x = 1$ in (3.22) yields then

$$\sum_{n \geq 0} d_{r,n}(q) \frac{u^n}{(q^r; q^r)_n} = \frac{1}{1 - u/(1 - q)} \frac{1}{e_{q^r}(u)}.$$

Substituting this back to (3.22) we obtain

$$\sum_{n \geq 0} f_{r,n}(q, x) \frac{u^n}{(q^r; q^r)_n} = \frac{1}{1 - u/(1 - q)} \frac{e_{q^r}(xu)}{e_{q^r}(u)}. \quad (3.23)$$

On the other hand, by (3.5) we have

$$\begin{aligned} \sum_{n \geq 0} g_{r,n}(q, x) \frac{u^n}{(q^r, q^r)_n} &= \sum_{n \geq 0} \sum_{i \geq 0} \frac{(x - 1)(x - q^r) \cdots (x - q^{r(i-1)})}{(q^r; q^r)_i} \frac{u^n}{(1 - q)^{n-i}} \\ &= \sum_{i \geq 0} \frac{(x^{-1}; q^r)_i}{(q^r; q^r)_i} (xu)^i \sum_{n \geq 0} \left(\frac{u}{1 - q} \right)^n \\ &= \frac{1}{1 - u/(1 - q)} \frac{(u; q^r)_{\infty}}{(xu; q^r)_{\infty}}, \end{aligned}$$

which is equal to the right-hand side of (3.23) by Euler's formula (3.21). It follows that $f_{r,n}(q, x) = g_{r,n}(q, x)$.

Remark. As $(1 - u)e_q(u) = e_q(qu)$, we can also write (3.23) as

$$\sum_{n \geq 0} f_{r,n}(q, x) \frac{u^n}{(q^r; q^r)_n} = \frac{(1 - q)e_{q^r}(xu)}{e_{q^r}(q^r u) - qe_{q^r}(u)}.$$

3.4 Proof of Theorem 3.2.2

To derive an explicit formula for $g_{r,n}^m(q)$ we give a more general formula, which is a variant of a result in [56, Th. 3] and may be also interesting in its own right.

Lemma 3.2. *Let $(a_{n,m})_{0 \leq m \leq n}$ be an array defined by*

$$\begin{cases} a_{0,m} = x_m & (m = n); \\ a_{n,m} = z_m a_{n-1,m+1} + y_n a_{n-1,m} & (0 \leq m \leq n-1). \end{cases} \quad (3.24)$$

If $e_i(y_1, y_2, \dots, y_n)$ denotes the i -th elementary symmetric polynomial of $y_1, \dots, y_n$, then

$$a_{n,m} = \sum_{k=0}^n x_{m+k} (z_m z_{m+1} \dots z_{m+k-1}) e_{n-k}(y_1, y_2, \dots, y_n). \quad (3.25)$$

Proof. The formula is obviously true for $n = 0$ and $n = 1$. Suppose that it is true until $n - 1$. Since $(1 + y_1 t)(1 + y_2 t) \dots (1 + y_n t) = \sum_{i=0}^n e_i(y_1, \dots, y_n) t^i$, we then have

$$\begin{aligned} a_{n,m} &= y_n a_{n-1,m} + z_m a_{n-1,m+1} \\ &= y_n \sum_{k=0}^{n-1} (z_m z_{m+1} \dots z_{m+k-1}) e_{n-1-k}(y_1, y_2, \dots, y_{n-1}) x_{m+k} \\ &\quad + z_m \sum_{k=0}^{n-1} (z_{m+1} z_{m+2} \dots z_{m+k}) e_{n-1-k}(y_1, y_2, \dots, y_{n-1}) x_{m+k+1} \\ &= y_n e_{n-1}(y_1, y_2, \dots, y_{n-1}) x_m \\ &\quad + \sum_{k=0}^{n-1} \left(\prod_{j=0}^k z_{m+j} \right) \\ &\quad (y_n e_{n-2-k}(y_1, y_2, \dots, y_{n-1}) + e_{n-1-k}(y_1, y_2, \dots, y_{n-1})) x_{m+1+k} \\ &= e_n(y_1, y_2, \dots, y_n) x_m + \sum_{k=1}^n \left(\prod_{j=0}^{k-1} z_{m+j} \right) e_{n-k}(y_1, y_2, \dots, y_n) x_{m+k}. \end{aligned}$$

This completes the proof. □

Now, specializing the array (3.24) with $x_m = g_{r,m}^m(q, x)$, $z_m = 1$ and $y_n = -xq^{r(n-1)}$, then the q -binomial formula $(1+t)(1+qt)\cdots(1+q^{n-1}t) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} t^k$ implies that

$$e_k(1, q, q^2, \dots, q^{n-1}) = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \quad (0 \leq k \leq n).$$

Applying (3.25) we get

$$a_{n,m} = g_{r,n+m}^m(q, x) = \sum_{k=0}^n (-x)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^r} q^{r\binom{n-k}{2}} g_{r,m+k}^{m+k}(q, x).$$

Shifting n by $n - m$ and then replacing k by $n - m - k$ yields (3.9).

Let $f_{r,n}^m(q, x) := \sum q^{\text{fmaf } \sigma} x^{\text{fix } \sigma}$ ($\sigma \in G_m(r, n)$) be the right-hand side of (3.10). Then $f_{r,n}^n(q, x) = g_{r,n}^n(q, x)$ by Theorem 3.2.1. For each fixed n we will show that $\{f_{r,n}^m(q, x)\}$ satisfies the recurrence relation (3.8). For $0 \leq m \leq n - 1$ define

$$E := G_{m+1}(r, n) \setminus G_m(r, n) = \{\sigma \in G_m(r, n) : \sigma(n - m) = n - m\}.$$

By (3.13), each permutation $\sigma \in E$ can be written as

$$\sigma = \langle \text{Der } \sigma, i_1 - 1, i_2 - 2, \dots, i_s - s \rangle,$$

where $i_1, \dots, i_s$ are the fixed points of σ arranged in increasing order ($i_1 = n - m$). Let $\sigma' = \langle \text{Der } \sigma, i_2 - 2, \dots, i_s - s \rangle$. Then the mapping $\sigma \mapsto \sigma'$ is a bijection from E to $G_m(r, n - 1)$ such that:

$$\text{fix } \sigma = \text{fix } \sigma' + 1,$$

and

$$\text{fmaf } \sigma = r((i_1 - 1) + (i_2 - 2) + \cdots + (i_s - s)) + \text{fmaj Der } \sigma.$$

It follows that $\text{fmaf } \sigma = \text{fmaf } \sigma' + r(n - m - 1)$. Hence

$$\begin{aligned} f_{r,n}^{m+1}(q, x) &= \sum_{\sigma \in G_m(r, n)} q^{\text{fmaf } \sigma} x^{\text{fix } \sigma} + \sum_{\sigma' \in G_m(r, n-1)} q^{\text{fmaf } \sigma' + r(n-m-1)} x^{\text{fix } \sigma} \\ &= f_{r,n}^m(q, x) + xq^{r(n-m-1)} f_{r,n-1}^m(q, x). \end{aligned}$$

This completes the proof of (3.10).

Chapitre 4

Factorization of Clarke et al.'s bijection

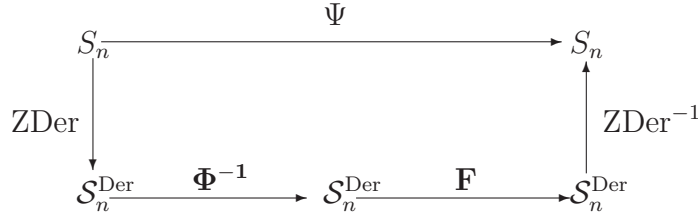
This chapter is the second part of [32]. Foata and Han [34] have recently constructed two new transformations on symmetric groups and *noticed* that the composition of their two transformations has some common property as Clarke et al.'s bijection Ψ in [27]. A natural question is then to ask whether these two algorithms are *identical*. Our aim of this chapter is to settle this open question.

4.1 Introduction

Let $0 \leq m \leq n$ and v be a nonempty word of length m on the alphabet $[n]$. Denote by $\text{Sh}(0^{n-m}v)$ the set of all *shuffles* of the words 0^{n-m} and v , that is, the set of all words it is possible to construct using $(n-m)$ 0's and the letters in v by preserving the order of all the letters in v . For any word $w = x_1 \dots x_n$ in $\text{Sh}(0^{n-m}v)$, we call *pillar* any non zero letter of w and write $\text{Pil}(w) := v$ and $\text{Zero } w := \{i : 1 \leq i \leq n, x_i = 0\}$. Let $\text{pil } w$ be the length of $\text{Pil } w$ and $\text{zero } w$ be the cardinality of $\text{Zero } w$. Clearly w is completely characterized by the pair $(\text{Zero } w, \text{Pil } w)$. Given a permutation $\sigma = \sigma(1)\sigma(2) \dots \sigma(n+m) \in S_{n+m}$, let $(j_1, j_2, \dots, j_m)$ be the increasing sequence of the integers k such that $\sigma(k) \neq k$ for $1 \leq k \leq n$. The word $w = x_1 x_2 \dots x_{n+m}$ derived from σ by replacing each fixed point by 0 and each other letter $\sigma(j_k)$ by $\text{rank}(\sigma(j_k))$ will be denoted by $\text{ZDer}(\sigma)$. For example, if $\sigma = \mathbf{1\ 8\ 3\ 4\ 6\ 2\ 7\ 5\ 9} \in S_9$ then $\text{ZDer}(\sigma) = 0\ 4\ 0\ 0\ 3\ 1\ 0\ 2\ 0$.

Let

$$\mathcal{S}_n^{\text{Der}} := \bigcup_{m,v} \text{Sh}(0^{n-m}v) \quad (0 \leq m \leq n, v \in \mathcal{D}_m).$$

Figure 4.1: The Foata-Han factorization of Clarke et al.'s bijection Ψ

It is obvious (see [34, Proposition 1.3]) that the map ZDer is a bijection from S_n to S_n^{Der} . Recently Foata and Han [34] have constructed two transformations Φ and $\mathbf{F} := \mathbf{F}_3$ on S_n^{Der} , apparently related to Ψ (see subsection 3.3.1). Our object is to show that their two bijections provide indeed a factorization of Clarke et al.'s bijection Ψ .

Theorem 4.1.1. *We have*

$$\Psi = \text{ZDer}^{-1} \circ \mathbf{F} \circ \Phi^{-1} \circ \text{ZDer}. \quad (4.1)$$

In other words, the diagram in Figure 1 is commutative.

4.2 Foata-Han's first transformation Φ

Let v be a derangement of order m and $w = x_1 x_2 \cdots x_n \in \text{Sh}(0^{n-m}v)$ ($0 \leq n \leq m$). Thus $v = x_{j_1} x_{j_2} \cdots x_{j_m}$, where $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. Recall that “rank” is the increasing bijection of $\{j_1, j_2, \dots, j_m\}$ onto the interval $[m]$. A positive letter x_k of w is said to be *excedent* (resp. *subexcedent*) if $x_k > \text{rank}(k)$ (resp. $x_k < \text{rank}(k)$). Accordingly, a letter is *non-subexcedent* if it is either equal to 0 or excedent.

We define n bijections ϕ_l ($1 \leq l \leq n$) from $\text{Sh}(0^{n-m}v)$ ($0 \leq n \leq m$) onto itself in the following manner: for each l such that $n - m + 1 \leq l \leq n$ let $\phi_l(w) := w$. When $1 \leq l \leq n - m$, let x_j denote the l -th letter of w , equal to 0, when w is read *from left to right*. Three cases are next considered (by convention, $x_0 = x_{n+1} = +\infty$):

- (1) x_{j-1}, x_{j+1} both non-subexcedent;
- (2) x_{j-1} non-subexcedent, x_{j+1} subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;
- (3) x_{j-1} subexcedent, x_{j+1} non-subexcedent; or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case (1) holds, let $\phi_l(w) := w$.

When case (2) holds, determine the *greatest* integer $k \geq j+1$ such that $x_{j+1} < x_{j+2} < \dots < x_k < \text{rank}(k)$ so that $w = x_1 \dots x_{j-1} 0 x_{j+1} \dots x_k x_{k+1} \dots x_n$ and define:

$$\phi_l(w) := x_1 \dots x_{j-1} x_{j+1} \dots x_k 0 x_{k+1} \dots x_n.$$

When case (3) holds, determine the *smallest* integer $i \leq j-1$ such that $\text{rank}(i) > x_i > x_{i+1} > \dots > x_{j-1}$ so that $w = x_1 \dots x_{i-1} x_i \dots x_{j-1} 0 x_{j+1} \dots x_n$ and define $\phi_l(w) := x_1 \dots x_{i-1} 0 x_i \dots x_{j-1} x_{j+1} \dots x_n$.

It is important to note that ϕ_l has no action on the 0's other than the l -th one. The mapping Φ is defined to be the composition product $\Phi := \phi_1 \phi_2 \dots \phi_{n-1} \phi_n$. To verify that Φ is bijective, Foata and Han introduce a class of bijections ψ_l , whose definitions are parallel to the definitions of the ϕ_l 's. Let $w = x_1 x_2 \dots x_n \in \text{Sh}(0^{n-m}v)$ ($0 \leq n \leq m$). We define n bijections ψ_l ($1 \leq l \leq n$) of $\text{Sh}(0^{n-m}v)$ onto itself in the following manner: For each l such that $n - m + 1 \leq l \leq n$ let $\psi_l(w) = w$. When $1 \leq l \leq n - m$, let x_j denote the l -th letter of w , equal to 0, when w is read from left to right. Consider the following three cases (remember $x_0 = x_{n+1} = +\infty$ by convention):

- (1') x_{j-1}, x_{j+1} both non-subexcedent;
- (2') x_{j-1} subexcedent, x_{j+1} non-subexcedent or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} > x_{j+1}$;
- (3') x_{j-1} non-subexcedent, x_{j+1} subexcedent or x_{j-1}, x_{j+1} both subexcedent with $x_{j-1} < x_{j+1}$.

When case (1') holds, let $\psi_l(w) := w$.

When case (2') holds, determine the smallest integer $k \leq j-1$ such that $x_k < \dots < x_{j-1} < \text{rank}(j-1)$ so that $w = x_1 \dots x_{k-1} x_k \dots x_{j-1} 0 x_{j+1} \dots x_n$ and define:

$$\psi_l(w) := x_1 \dots x_{i-1} 0 x_i \dots x_{j-1} x_{j+1} \dots x_n.$$

When case (3') holds, determine the greatest integer $k \geq j+1$ such that $\text{rank}(j+1) > x_{j+1} > \dots > x_k$ so that $w = x_1 \dots x_{j-1} 0 x_{j+1} \dots x_k x_{k+1} \dots x_n$ and define:

$$\psi_l(w) := x_1 \dots x_{j-1} x_{j+1} \dots x_k 0 x_{k+1} \dots x_n.$$

As shown in [34], the product $\Phi^{-1} := \psi_n \psi_{n-1} \dots \psi_1$ is the inverse bijection of Φ .

4.3 Foata-Han's second transformation $\mathbf{F}$

The bijection $\mathbf{F}$ maps each shuffle class $\text{Sh}(0^{n-m}v)$ with v an arbitrary word of length m ($0 \leq m \leq n$) onto itself. When $n = 1$ the unique element of the shuffle class is sent onto itself. Also let $\mathbf{F}(w) = w$ when $\text{des}(w) = 0$. Let $n \geq 2$ and assume that $\mathbf{F}(w')$ has been defined for all words w' with nonnegative letters, of length $n' \leq n - 1$. Let w be a word of length n such that $\text{des}(w) \geq 1$. We may write

$$w = w'a0^rb,$$

where $a \geq 1$, $b \geq 0$ and $r \geq 0$. Three cases are considered:

$$(1) \quad a \leq b; \quad (2) \quad a > b, \quad r \geq 1; \quad (3) \quad a > b, \quad r = 0.$$

In case (1) define $\mathbf{F}(w) = \mathbf{F}(w'a0^rb) := (\mathbf{F}(w'a0^r))b$. In case (2) we may write $\mathbf{F}(w'a0^r) = w''0$ and then define

$$\begin{aligned} \gamma \mathbf{F}(w'a0^r) &:= 0w''; \\ \mathbf{F}(w) = \mathbf{F}(w'a0^rb) &:= (\gamma \mathbf{F}(w'a0^r))b = 0w''b. \end{aligned}$$

In short, add one letter “0” to the left of $\mathbf{F}(w'a0^r)$, then delete the rightmost letter “0” and add b to the right. In case (3) remember that $r = 0$. Write

$$\mathbf{F}(w'a) = 0^{m_1}x_1v_10^{m_2}x_2v_2 \cdots 0^{m_k}x_kv_k,$$

where $m_1 \geq 0$, $m_2, \dots, m_k$ are all positive, then $x_1, x_2, \dots, x_k$ are positive letters and $v_1, v_2, \dots, v_k$ are words with positive letters, possibly empty. Then define:

$$\begin{aligned} \delta \mathbf{F}(w'a) &:= x_10^{m_1}v_1x_20^{m_2}v_2x_3 \cdots x_k0^{m_k}v_k; \\ \mathbf{F}(w) = \mathbf{F}(w'ab) &:= (\delta \mathbf{F}(w'a))b. \end{aligned}$$

In short, move each positive letter occurring just after a 0-factor of $\mathbf{F}(w'a)$ to the beginning of that 0-factor and add b to the right.

4.4 Proof of Theorem 4.1.1

We show that the composition $\text{ZDer}^{-1} \circ \mathbf{F} \circ \Phi^{-1} \circ \text{ZDer}$ satisfies the relations (3.14)–(3.17) characterizing the bijection Ψ . The proof is based on Lemmas 4.1, 4.3 and 4.4, which will be proved in Section 4.5.

Let $\tau = \langle \sigma, i_1, i_2, \dots, i_m \rangle \in \mathcal{S}_{n+m}$, where $\sigma = x_1 \dots x_n \in \mathcal{D}_n$ is the derangement part of τ . Since the positions of fixed points of τ are the same as that of zeros of $\text{ZDer}\tau$, we write

$$\text{ZDer}\tau = [\sigma, i_1, i_2, \dots, i_m] \in \text{Sh}(0^m \sigma).$$

Thus the k -th zero from left to right of $\text{ZDer}\tau$ is at the position $i_k + k$ for $k \leq m$ and i_k is the number of pillars at the left of the k -th zeros in $\text{ZDer}\tau$. Consequently, writing $i := i_m$ then

$$[x_1 x_2 \dots x_n, i_1, i_2, \dots, i_m] = w x_i 0^r x_{i+1} x_{i+2} \dots x_n, \quad (4.2)$$

where w is a word with $i - 1$ pillars and r the largest integer satisfying $i_{m+1-r} = i$. The integer r must be positive for the last zero is located just at the left of the $(i + 1)$ -th pillar.

Example. Let $\Omega = [3\ 1\ 4\ 5\ 2, 0, 0, 0, 1, 1, 2, 2, 2, 2]$. Then $i = 2$, $r = 4$, $w = 000300$ and

$$\Omega = w 1 0^4 4 5 2 = 000300\mathbf{1}0000452.$$

For brevity we introduce the following notations:

For any $t \geq 0$ and $I = (i_1, \dots, i_m) \in \mathbb{N}^m$, let $I + t = (i_1 + t, \dots, i_m + t)$, $t^{[m]} = (t, \dots, t) \in \mathbb{N}^m$. Moreover, for any $\omega \in \text{Sh}(0^{n-m}v)$, if $\mathbf{F}(\omega) = [\sigma, i_1, \dots, i_m]$, then we write

$$\mathbf{F}(\omega) = [\sigma, I_\omega] \quad \text{with} \quad I_\omega = (i_1, \dots, i_m).$$

Besides, for any non empty finite word w we denote, respectively, by $\mathbf{F}(w)$ and $\mathbf{L}(w)$ the first and last letter of w when w is read from left to right.

Lemma 4.1. *Let w_1, w_2 be two non empty words such that $\mathbf{L}(w_1) > 0$ and $\text{zero}(w_2) = 0$. Assume that $(\mathbf{L}(w_1), \mathbf{F}(w_2)) = (a, b)$ and $\text{pil}(w_1) = \nu$, $\text{des}(w_2) = t$. Let $\mu = w_1 0^r w_2$ with $r \geq 0$. Then*

$$\text{zero}(w_1) \neq 0 \implies I_\mu = \begin{cases} (t^{[r]}, I_{w_1} + t + 1), & \text{if } a > b, \\ I_{w_1} + t, & \text{if } a < b \text{ and } r = 0, \\ (t^{[r-1]}, I_{w_1} + t + 1, \nu + t), & \text{if } a < b \text{ and } r > 0; \end{cases} \quad (4.3)$$

and

$$\text{zero}(w_1) = 0 \implies I_\mu = \begin{cases} \emptyset, & \text{if } r = 0, \\ (t^{[r]}), & \text{if } r > 0 \text{ and } a > b, \\ (t^{[r-1]}, \nu + t), & \text{if } r > 0 \text{ and } a < b. \end{cases} \quad (4.4)$$

Let $\sigma = x_1x_2 \cdots x_n \in S_n$. The color of a *slot* $(i, i+1)$ of σ can be characterized (see [27]) as follows:

- The slot $(i, i+1)$ of σ is green if and only if one of the following conditions is satisfied:

$$(G_1) \quad x_i > x_{i+1} > i; \quad (G_2) \quad x_i < i < x_{i+1}; \quad (G_3) \quad i > x_i > x_{i+1}. \quad (4.5)$$

- The slot $(i, i+1)$ of σ is red if and only if one of the following conditions is satisfied:

$$(R_1) \quad i < x_i < x_{i+1}; \quad (R_2) \quad x_i > i \geq x_{i+1}; \quad (R_3) \quad x_i < x_{i+1} \leq i. \quad (4.6)$$

By convention $x_0 = x_{n+1} = +\infty$. If we denote by d_i the number of descents of $\langle \sigma, i \rangle$ at right of i (σ is a derangement) then Lemma 3.1 implies that $g_i = d_i$ if the i -th slot is green, and $g_i = d_i + i$ if the slot is red.

Lemma 4.2. *Let $\sigma = x_1x_2 \cdots x_n \in S_n$, let $(i, i+1)$ be the j -th slot of σ and $t_i = \text{des}(x_{i+1} \cdots x_n)$ ($1 \leq i \leq n$) then*

$$d_j = \begin{cases} t_i & \text{in the cases } G_1, G_2, \text{ or } R_1; \\ t_i + 1 & \text{in the cases } G_3, R_2, \text{ or } R_3. \end{cases}$$

Proof. Clearly, in the cases G_1, G_2, R_1 the value x_{i+1} is an exceedant of σ , so $i+1$ is not a descent place of $\langle \sigma, j \rangle$ while in the cases G_3, R_2 and R_3 the value x_{i+1} is a sub-exceedant, so $i+1$ is a descent place of $\langle \sigma, j \rangle$. $\square$

Let $\tau = \langle \sigma, i_1, i_2, \dots, i_m \rangle \in S_{n+m}$ and $\Omega = \text{ZDer}\tau = [\sigma, i_1, \dots, i_m]$. We distinguish two cases according to the color of the insertion slot.

4.4.1 The i_m -th slot of σ is green

Define

$$\omega = [\sigma, i_1, \dots, i_{m-1}], \quad \omega' = \Phi^{-1}(\omega) \quad \text{and} \quad \Omega' = \Phi^{-1}(\Omega).$$

We must check (3.14) and for $m > 1$,

$$\mathbf{F} \circ \Phi^{-1}(\omega) = [\sigma, j_2, \dots, j_m] \implies \mathbf{F} \circ \Phi^{-1}(\Omega) = [\sigma, g_{i_m}, j_2, \dots, j_m], \quad (4.7)$$

which is equivalent to

$$I_{\Omega'} = (g_{i_m}, I_{\omega'}). \quad (4.8)$$

Lemma 4.3. *We have the following factorizations:*

$$\omega' = w_1 0^r w_2 0^{r'} w_3 \quad \text{and} \quad \Omega' = w_1 0^{r+1} w_2 0^{r'} w_3 \quad (r, r' \geq 0),$$

where $w_1 \neq \emptyset$, $w_2 \neq \emptyset$ and w_3 are words on non negative integers. Moreover, if $(a, b, a', b') = (L(w_1), F(w_2), L(w_2), F(w_3))$, then the following properties hold true:

- i) $\text{zero}(w_2 w_3) = 0$, and $L(w_1) > 0$,
- ii) if $r = 0$, then $a > b$,
- iii) if $r' > 0$, then $r' = 1$, $a' < b'$ and $r = 0$,
- iv) $\text{des}(w_2 0^{r'} w_3) = g_{i_m}$.

Notice that if $r' = 1$ then $w_3 \neq \emptyset$ by iii). Let $w'_2 = w_2 w_3$. If $m = 1$ then $\omega = \omega' = \sigma$ and $\Omega = [\sigma, i_1]$. Hence $\text{zero}(\omega) = 0$ and $r = r' = 0$. Thus $\omega' = w_1 w'_2$ and $\Omega' = w_1 0 w'_2$, where $a > b = F(w'_2)$ and $\text{des}(w'_2) = g_{i_1}$. It follows then from (4.4) that

$$I_{\Omega'_1} = (g_{i_1}). \quad (4.9)$$

We now prove (4.8) for $m > 1$.

1) If $r' = 0$ then

$$\omega' = w_1 0^r w'_2 \quad \text{and} \quad \Omega' = w_1 0^{r+1} w'_2, \quad (4.10)$$

where $\text{des}(w'_2) = t = g_{i_m}$ and $\text{zero}(w'_2) = 0$. As $m > 1$ we cannot have $\text{zero}(w_1) = 0$ and $r = 0$ simultaneously. It remains to verify the following two cases:

- i) $\text{zero}(w_1) \neq 0$, $a > b$ or $r > 0$ and $a < b$,
- ii) $\text{zero}(w_1) = 0$, $r > 0$ and $a \neq b$.

Applying Lemma 4.1 to either case yields

$$I_{\Omega'} = (t, I_{\omega'}) = (g_{i_m}, I_{\omega'}). \quad (4.11)$$

2) If $r' = 1$, then $a' < b'$, $r = 0$ and $a > b$. Hence

$$\omega' = v 0 w_3 \quad \text{and} \quad \Omega' = V 0 w_3,$$

where $v = w_1 w_2$ and $V = w_1 0 w_2$. Let $t_2 = \text{des}(w_2)$, $t_3 = \text{des}(w_3)$ and $t = \text{des}(w_2 0 w_3) = g_{i_m}$, $\nu = \text{pil}(v) = \text{pil}(V)$. If $\text{zero}(w_1) \neq 0$, as $a > b$, by (4.3) we have

$$I_v = I_{w_1} + t_2 + 1 \quad \text{and} \quad I_V = (t_2, I_{w_1} + t_2 + 1) = (t_2, I_v). \quad (4.12)$$

Since $\text{zero}(v) > 0$, $\text{zero}(V) > 0$ and $a' < b'$, by (4.3) we have

$$I_{\omega'} = (I_v + t_3 + 1, \nu + t_3) \quad \text{and} \quad I_{\Omega'} = (I_V + t_3 + 1, \nu + t_3). \quad (4.13)$$

It follows that $I_{\Omega'} = (t_2 + t_3 + 1, I_{\omega'})$ and (as $a' < b'$) $t = t_2 + t_3 + 1$. Now, if $\text{zero}(w_1) = 0$ then, by (4.4),

$$I_{\omega'} = (\nu + t_3) \quad \text{and} \quad I_V = (t_2). \quad (4.14)$$

As $\text{zero}(V) > 0$, it follows from (4.13) and (4.14) that $I_{\Omega'} = (t, \nu + t_3) = (t, I_{\omega'})$.

4.4.2 The i_m -th slot of σ is red

Let k be the largest integer such that $i_{m-k+1} = i_m$. Define $\omega = [\sigma, i_1, \dots, i_{m-k}]$, $\omega' = \Phi^{-1}(\omega)$, $\Omega' = \Phi^{-1}(\Omega)$ and $\tilde{\Omega} = \psi_{m-k} \cdots \psi_1(\Omega)$. For $k = m$ the relation (3.16) is equivalent to

$$I_{\Omega'} = ((g_{i_m} - i_m)^{[m-1]}, g_{i_m}), \quad (4.15)$$

which corresponds to (3.14) when $m = 1$. For $k < m$ the relation (3.17) is equivalent to

$$I_{\Omega'} = ((g_{i_m} - i_m)^{[k-1]}, I_{\omega'} + 1, g_{i_m}). \quad (4.16)$$

We now verify (4.15) and (4.16). Recall that $\sigma = x_1 \dots x_n$. For convenience, we use write $i := i_m$ in what follows.

Assume that x_{i+1} is excedent.

This corresponds to the case (R_1) , i.e., x_i and x_{i+1} are both excedent and $x_i < x_{i+1}$. Hence, all the zeros of ω are at the left of x_i and all the zeros of ω' remains at the left of x_i . It follows that

$$\omega' = w_1 x_i x_{i+1} \cdots x_n \quad \text{and} \quad \tilde{\Omega} = w_1 x_i 0^k x_{i+1} \cdots x_n \quad (4.17)$$

Besides, as the map ψ_j is identity for $m - k + 1 \leq j \leq m$, we have $\Omega' = \tilde{\Omega}$. Let $t = \text{des}(x_{i+1} \cdots x_n)$. By Lemma 4.2 we have $t = g_i - i$.

If $\text{zero}(w_1) = 0$ then $m = k$. From (4.4) we derive $I_{\Omega'} = (t^{[k-1]}, i + t) = ((g_i - i)^{[m-1]}, g_i)$, which is (4.15). If $\text{zero}(w_1) \neq 0$, by (4.3) we have

$$I_{\omega'} = (I_{w_1 x_i} + t) \quad \text{and} \quad I_{\Omega'} = (t^{[k-1]}, I_{w_1 x_i} + t + 1, i + t),$$

which is precisely (4.16).

Assume that x_{i+1} is subexcedent.

We need the following result.

Lemma 4.4. *Let h be the largest integer such that $i + 1 > x_{i+1} > \cdots > x_h$ and $l = i + 1$ if x_i is excedent or the last zero of w' is located between x_i and x_{i+1} otherwise l be the smallest integer such that $x_l < \cdots < x_{i+1} < i + 1$ and that ω' does not contain zero at the right of x_l . Set $T = \text{des}(x_l \cdots x_n)$; $w_3 = x_{h+1} \cdots x_n$; $t' = \text{des}(w_3)$, then we have*

$$\omega' = w_1 x_{l-1} 0^r x_l \cdots x_h w_3, \quad (4.18)$$

$$\Omega' = w_1 x_{l-1} 0^{r+k-1} x_l \cdots x_h 0 w_3. \quad (4.19)$$

Moreover, the following identities hold true:

- i) ($r = 0$ and $x_{l-1} > x_l$) or ($r = 1$ and $x_{l-1} < x_l$),
- ii) $T + 1 = g_i - i$,
- iii) $h + t' = g_i$.

Let $t = \text{des}(x_l \cdots x_h)$. As $x_h < x_{h+1}$ we have $T = t + t'$. Write $\Omega' = V 0 w_3$ with

$$V = w_1 x_{l-1} 0^{r+k-1} x_l \cdots x_h.$$

- Suppose that $k = m$. Then $r = 0$, $\text{zero}(w_1 x_{l-1}) = 0$ and $x_{l-1} > x_l$.
 - If $k = 1$ then $\text{zero}(V) = 0$. By (4.4) we have $I_{\Omega'} = (h + t') = (g_i)$, which is equivalent to (3.14).
 - If $k > 1$ then $\text{zero}(V) > 0$. By (4.3) we have $I_{\Omega'} = (I_V + t' + 1, h + t')$, while (4.4) yields $I_V = t^{[k-1]}$. Hence $I_{\Omega'} = ((T + 1)^{[k-1]}, h + t')$, which is equivalent to (4.15).
- Suppose that $k < m$. There are three cases:
 - (a) $r = 0$, we have $\text{zero}(w_1) \neq 0$, $x_{l-1} > x_l$ and, by (4.3),

$$I_{\omega'} = (I_{w_1 x_{l-1}} + T + 1) \quad \text{and} \quad I_V = (t^{[k-1]}, I_{w_1 x_{l-1}} + t + 1).$$
 - (b) $r = 1$ and $\text{zero}(w_1) \neq 0$, we have $x_{l-1} < x_l$. By (4.3) $I_{\omega'} = (I_{w_1 x_{l-1}} + T + 1, l - 1 + T)$ and $I_V = (t^{[k-1]}, I_{w_1 x_{l-1}} + t + 1, l - 1 + t)$.
 - (c) $r = 1$ and $\text{zero}(w_1) = 0$, by (4.4) we have

$$I_{\omega'} = (l - 1 + T) \quad \text{and} \quad I_V = (t^{[k-1]}, l - 1 + t).$$

On the other hand, in any case, we have $\text{zero}(V) > 0$. By (4.3),

$$I_{\Omega'} = (I_V + t' + 1, h + t') = ((T + 1)^{[k-1]}, I_{\omega'} + 1, h + t'),$$

which is equivalent to (4.16).

Example. Let $\tau = 15342768 \in S_8$ then $\tau = \langle \sigma, 0, 1, 1, 4 \rangle$, where $\text{Der}\tau = \sigma = 2143$. So $\langle \sigma, 0 \rangle = 13254$; $\langle \sigma, 1 \rangle = 32154$; $\langle \sigma, 2 \rangle = 21354$; $\langle \sigma, 3 \rangle = 21543$. $\langle \sigma, 4 \rangle = 21435$; hence the slots 0, 2, 4 are green, while the slots 1 and 3 are red. Therefore $(g_0, \dots, g_4) = (2, 3, 1, 4, 0)$, and

- slot 0 is green $\implies \Psi\langle \sigma, 0 \rangle = \langle \sigma, g_0 \rangle = \langle \sigma, 2 \rangle$,
- slot 1 is red $\implies \Psi\langle \sigma, 0, 1, 1 \rangle = \langle \sigma, g_1 - 1, 2 + 1, g_1 \rangle = \langle \sigma, 2, 3, 3 \rangle$,
- slot 4 is green $\implies \Psi\langle \sigma, 0, 1, 1, 4 \rangle = \langle \sigma, g_4, 2, 3, 3 \rangle = \langle \sigma, 0, 2, 3, 3 \rangle$.

Thus $\Psi(\tau) = \langle \sigma, 0, 2, 3, 3 \rangle = 13248675$.

On the other hand, applying Φ^{-1} to $w := \text{ZDer}\tau = 02001430 \in \text{Sh}(0^4\sigma)$, as 2 and 4 are excedent, we get

Id		1	2	3	4	5	6	7	8	
w	=	0	2	0	0	1	4	3	0	
$\psi_1(w)$	=	0	2	0	0	1	4	3	0	Case (1')
$\psi_2(\psi_1(w))$	=	0	2	0	0	1	4	3	0	Case (1')
$\psi_3(\psi_2(\psi_1(w)))$	=	0	2	0	1	0	4	3	0	Case (3') with $k = 5$
$\psi_4(\psi_3(\psi_2(\psi_1(w))))$	=	0	2	0	1	0	4	0	3	Case (2') with $k = 7$.

Thus $\Phi^{-1}(w) = 02010403$. Now we apply $\mathbf{F}$ to $w = 02010403$:

$\mathbf{F}(02) = 02$	no descent
$\mathbf{F}(020) = \delta(02)0 = 200$	Case (3)
$\mathbf{F}(0201) = \gamma(200)1 = 0201$	Case (2)
$\mathbf{F}(02010) = \delta(0201)0 = 20100$	Case (3)
$\mathbf{F}(020104) = 201004$	Case (1)
$\mathbf{F}(0201040) = \delta(020104)0 = 2104000$	Case (3)
$\mathbf{F}(02010403) = \gamma(2104000)3 = 02104003$	Case (2).

So $\mathbf{F}(02010403) = [\sigma, 0, 2, 3, 3]$ and $\text{ZDer}^{-1}([\sigma, 0, 2, 3, 3]) = \langle \sigma, 0, 2, 3, 3 \rangle = 13248675$. Therefore (4.1) is checked.

4.5 Proofs of the three lemmas

4.5.1 Proof of Lemma 4.1

Recall that δ and Υ are the transformations used in the cases (b) and (c) of the algorithm **F**. Let α a word of length ν on the alphabet of positive integers and $b > 0$. Let $w = [\alpha, i_1, \dots, i_z] \in \text{Sh}(0^z\alpha)$. Note that

$$wb = [\alpha b, i_1, \dots, i_z], \quad w0 = [\alpha, i_1, \dots, i_z, \nu], \quad 0w = [\alpha, 0, i_1, \dots, i_z].$$

Hence, if $L(w) = 0$, i.e., $i_z = \nu$, then $\Upsilon[\alpha, i_1, \dots, i_z] = [\alpha, 0, i_1, \dots, i_{z-1}]$. On the other hand, if $x_1, \dots, x_n$ are positive integers, then

$$x_1 \cdots x_k 0^r x_{k+1} \cdots x_n = [x_1 \cdots x_n, k^{[r]}]$$

It follows that $\delta(w) = [\alpha, i_1 + 1, \dots, i_z + 1]$. Now, consider the word $\mu = w_1 0^r w_2$, where $\text{Pil}(\mu) = \sigma$, $\text{Pil}(w_1) = \alpha$ and $|\alpha| = \nu$. Since w_1 and w_2 are non empty we can write $w_1 = v_1 a$ and $w_2 = b v_2$. Therefore $\mu = v_1 a 0^r b v_2$.

If $\text{zero}(w_1) = z \neq 0$ and $a > b$ set $I_{w_1} = (l_1, \dots, l_z)$. Then, we have succesively $\mathbf{F}(v_1 a 0) = \delta(\mathbf{F}(v_1 a))0 = [\alpha, l_1 + 1, \dots, l_z + 1, \nu] \implies I_{v_1 a 0} = (I_{v_1 a} + 1, \nu)$, $\mathbf{F}(v_1 a 0 0) = \Upsilon(\mathbf{F}(v_1 a 0))0 = [\alpha, 0, l_1 + 1, \dots, l_z + 1, \nu] \implies I_{v_1 a 0^r} = (0^{[r-1]}, I_{v_1 a} + 1, \nu)$, and $\mathbf{F}(v_1 a 0^r b) = \Upsilon(\mathbf{F}(v_1 a 0^{[r]}))b = [\alpha b, 0^r, I_{w_1} + 1]$. Finally, as $\text{zero}(v_2) = 0$, we have $\mathbf{F}(\mu) = [\sigma, t^{[r]}, I_{w_1} + 1 + t]$, which corresponds to the first case of (4.3). The other cases can be proved similarly.

4.5.2 Proof of Lemma 4.3

Recall that $\Omega = [\sigma, i_1, \dots, i_m]$, where

$$\sigma = x_0 x_1 \cdots x_n x_{n+1}, \quad \omega = [\sigma, i_1, \dots, i_{m-1}], \quad \Omega' = \Phi^{-1}(\Omega), \quad \omega' = \Phi^{-1}(\omega).$$

Set $\tilde{\omega} = \psi_{m-2} \circ \cdots \circ \psi_1(\omega)$, $\tilde{\Omega} = \psi_{m-1} \circ \cdots \circ \psi_1(\Omega)$ and $i = i_m$. The last zero of ω is at the left of x_{i+1} so is the last zero of $\tilde{\omega}$ by definition of Φ^{-1} . So all $m - 2$ zeros of ω' are on the left of x_{i+1} and only the last can be on the left or on the right of x_{i+1} . Hence ω' is of the following form:

$$\omega' = v_1 x_i 0^\epsilon x_{i+1} x_{n+2} \cdots x_h 0^{r'} x_{h+1} \cdots x_n x_{n+1},$$

where $\epsilon \geq 0$, and $0 \leq r' \leq 1$. Keeping in mind the definition of Φ^{-1} , if x_{i+1} is excedent (the case of G_1 or G_2) then we have necessarily $r' = 0$ and If x_{i+1} is subexcedent (the case of G_3) then it is impossible for the last zero of ω' to be between x_i and x_{i+1} , otherwise when one applies to ω' the map ψ_{m-1}^{-1} , it corresponds to the case (2) therefore the last zero of $\tilde{\omega}$ would be on the right of x_{i+1} , that is impossible. So in case of G_3 we have necessarily $r' = 1$ or ($r' = 0$ and $\epsilon = 0$). We consider the following three cases:

- G_1 or (G_2 and $\epsilon > 0$). Since $r' = 0$, hence ω' and $\tilde{\omega}$ are of the following forms:

$$\omega' = v_1 x_i 0^\epsilon x_{i+1} \dots x_{n+1}, \quad \tilde{\Omega} = v_1 x_i 0^{\epsilon+1} x_{i+1} \dots x_{n+1}.$$

In this case ψ_m is the identity therefore $\Omega' = \tilde{\Omega}$ and then we have

$$r = \epsilon; \quad r' = 0 \quad w_1 = v_1 x_i; \quad w_2 = x_{i+1} \dots x_n x_{n+1}; \quad w_3 = \emptyset.$$

If $r = 0$ we have G_1 so $a > b$ and by Lemma 4.2, $\text{des}(w_2) = g_i$.

- (G_2 and $\epsilon = 0$) or (G_3 and $r' = 0$). In this case, let k be the smallest integer such that $x_k < \dots < x_i \leq i$ and ω' does not contain any zero on the right of x_k . Thus ω' and $\tilde{\Omega}$ are of the following forms:

$$\begin{aligned} \omega' &= v_1 x_{k-1} 0^r x_k \dots x_i x_{i+1} \dots x_{n+1}, \\ \tilde{\Omega} &= v_1 x_{k-1} 0^r x_k \dots x_i 0 x_{i+1} \dots x_{n+1}. \end{aligned}$$

Since $\Omega' = \psi_m(\tilde{\Omega})$ and the map ψ_m corresponds to the case (2'), we have

$$\Omega' = v_1 x_{k-1} 0^{r+1} x_k \dots x_i x_{i+1} \dots x_n x_{n+1}.$$

Moreover $r' = 0$ $w_1 = v_1 x_{k-1}$ and $w_2 = x_k \dots x_n x_{n+1}$; $w_3 = \emptyset$. As $\text{des}(x_k \dots x_i) = 0$, we have $\text{des}(w_2) = g_i$ by Lemma 4.2.

- (G_3 and $r' = 1$). In this case, both x_i and x_{i+1} must be subexcedances and $x_i > x_{i+1}$ and h must be the largest integer such that $i + 1 > x_{i+1} > \dots > x_h$ (by applying ϕ_{m-1} to $\tilde{\omega}$). Let k be the smallest integer such that $k > x_k > \dots > x_{i+1}$ and ω' contains only one zero on the right of x_k . Thus we have

$$\omega' = v_1 x_{k-1} 0^\alpha x_k \dots x_i x_{i+1} \dots x_h 0 x_{h+1} \dots x_{n+1}.$$

On the other hand, the last zero of $\tilde{\omega}$ is at the left of x_{i+1} , more precisely just at the left of x_k because $k > x_k > x_i > x_{i+1} > \dots > x_h$ by the map ϕ_{m-1} . Hence

$$\tilde{\omega} = v_1 x_{k-1} 0^{\alpha+1} x_k \dots x_i x_{i+1} \dots x_h x_{h+1} \dots x_{n+1}.$$

Set $\tilde{\tilde{\Omega}} = \psi_{m-2} \circ \dots \circ \psi_1(\Omega)$. Then

$$\tilde{\tilde{\Omega}} = [\tilde{\omega}, i] = v_1 x_{k-1} 0^{\alpha+1} x_k \dots x_i 0 x_{i+1} \dots x_h x_{h+1} \dots x_{n+1}.$$

When we apply ψ_{m-1} to $\tilde{\tilde{\Omega}}$, it corresponds to the case (3'), so

$$\tilde{\Omega} = \psi_{m-1}(\tilde{\tilde{\Omega}}) = v_1 x_{k-1} 0^\alpha x_k \cdots x_i 0 0 x_{i+1} \cdots x_h x_{h+1} \cdots x_{n+1}.$$

Similary we have

$$\Omega' = \psi_m(\tilde{\Omega}) = v x_{k-1} 0^\alpha x_k \cdots x_i 0 x_{i+1} \cdots x_h 0 x_{h+1} \cdots x_{n+1}.$$

In this situation we take $r = 0$ ($r' = 1$) and

$$w_1 = v_1 x_{k-1} 0^\alpha x_k \cdots x_i; \quad w_2 = x_{i+1} \cdots x_h; \quad w_3 = x_{h+1} \cdots x_{n+1}.$$

As $x_h < x_{h+1}$ we have $\text{des}(w_2 0 w_3) = \text{des}(w_2 w_3) + 1 = g_i$ by Lemma 4.2.

4.5.3 Proof of Lemma 4.4

Since the i -th slot of σ is red and x_{i+1} is subexcedent, we are in the situation of R_2 or R_3 . Recall that all the zeros of ω are located at the left of x_i so are all of $\tilde{\omega} = \psi_{m-k-1} \circ \cdots \circ \psi_1(\omega)$. We show that the last zero of ω' is at the left of x_{i+1} . In the R_2 case x_i is excedent and all zeros of ω' are on the left of x_i . In the R_3 case, suppose that the last zero of ω' is at the right of x_{i+1} , then, by applying the reverse mapping $\psi_{m-k}^{-1} = \phi_{m-k}$, the last zero of $\tilde{\omega}$ cannot be at the left of x_i because $x_i < x_{i+1}$. This is absurd. In order to show (4.18) and (4.19) set $\Omega_{m-k} = \tilde{\Omega}$ and for all j such that $m-k < j \leq m$ set $\Omega_j = \psi_j(\Omega_{j-1})$. There are two cases:

- If $l \leq i$ then

$$\omega' = w_1 x_{l-1} 0^r x_l \cdots x_i x_{i+1} \cdots x_h x_{h+1} w_3,$$

and

$$\tilde{\Omega} = w_1 x_{l-1} 0^r x_l \cdots x_i 0^k x_{i+1} \cdots x_h x_{h+1} w_3.$$

Noticing that $x_i < x_{i+1} < i+1$, so x_i is subexcedent and the application of ψ_j to Ω_{j-1} corresponds to case (2') for all $m-k < j < m$ and ψ_m corresponds to case (3'). Therefore

$$\Omega' = w_1 x_{l-1} 0^{r+k-1} x_l \cdots x_i x_{i+1} \cdots x_h 0 x_{h+1} w_3.$$

- If $l = i+1$, then $\omega' = w_1 x_i 0^r x_{i+1} \cdots x_h x_{h+1} w_3$ and $\tilde{\Omega} = w_1 x_i 0^{r+k} x_{i+1} \cdots x_h x_{h+1} w_3$. In this case, ψ_j corresponds to the case (1) for all $m-k < j < m$, and ψ_m corresponds to case (3'). So $\Omega' = w_1 x_i 0^{r+k-1} x_{i+1} \cdots x_h 0 x_{h+1} w_3$.

It remains to verify the three conditions of Lemma 4.4 in the above two cases. It is clear that $x_{l-1} > x_l$ if $r = 0$. Moreover, neither $r > 1$ nor ($r = 1$ and $x_{l-1} > x_l$) is possible because, otherwise, when we apply ϕ_{m-k} to ω' , it corresponds to case (2), so the last zero of $\tilde{\omega}$ would be at the right of x_{i+1} , but this is absurd. So the condition (i) is verified. Besides, as $\text{des}(x_l \cdots x_{i+1}) = 0$ we have $T = \text{des}(x_{i+1} \cdots x_n)$, and by Lemma 4.2, we derive the condition (ii). Finally, by definition of h we have $\text{des}(x_{i+1} \cdots x_h) = h - i - 1$, and $T = \text{des}(x_{i+1} \cdots x_n) = h - i - 1 + t'$. Thus $h + t' = g_i$.

Chapitre 5

Flag-inv and Flag-maj on colored words

This chapter is our article [33]. In [37], Dominique Foata constructed a map Φ , called *second fundamental transformation*, exchanging the integer-valued statistics *Inversion number* “inv” and *Major Index* “maj” on words whose letters are integers. Later, Foata and Han introduced the *flag-inversion number* “finv” and extended Φ on signed words and permutations, showing that the *flag major index* “fmaj” and “finv” were equidistributed. In this chapter we give an extension of Φ on *r-colored words*. By this extension, we show that the bistatistics (fmaj, des*) and (finv, col) are equidistributed, where “col” is the sum of color and “des*” a new statistic derived from “des”.

5.1 Introduction

The *second fundamental transformation*, denoted by Φ and described in the proceedings [37] by Foata, is defined on finite words whose letters are integers. If $\mathbf{m} = (m_1, \dots, m_s)$ is a sequence of nonnegative integers, let $R_{\mathbf{m}}$ be the set of all rearrangements $w = x_1 x_2 \dots x_m$ of the sequence $1^{m_1} 2^{m_2} \dots s^{m_s}$ where $m = m_1 + m_2 + \dots + m_s$. The transformation Φ maps each word w on another word $\Phi(w)$ and has the following properties:

1. $\text{maj } w = \text{inv } \Phi(w)$;
2. $\Phi(w)$ is a rearrangement of w and the restriction of Φ to $R_{\mathbf{m}}$ is a bijection of $R_{\mathbf{m}}$ onto itself.

Further properties were proved later on by Foata, Schützenberger [38] and Björner, Wachs [13], in particular, when the transformation is restricted to

act on the symmetric group S_s .

The purpose of this chapter is to extend the transformation Φ on r -colored words.

Let C_r be the r -cyclic group generated by $\zeta = e^{2i\pi/r}$. By r -colored word, we understand a pair (ε, x) where $\varepsilon \in (C_r)^m$ and x is a word of length m whose letters are nonnegative integers. For reasons which will appear, if $w := (\varepsilon, x)$ is a r -colored word where $\varepsilon = (\zeta^{k_1}, \zeta^{k_2}, \dots, \zeta^{k_m})$ and $x = x_1 x_2 \dots x_m$, we write $w := w_1 w_2 \dots w_m$ where $w_j = \zeta^{k_j} x_j$ ($1 \leq j \leq m$). For any $1 \leq j \leq m$, $\varepsilon_j = \zeta^{k_j}$ is called the *color* of w_j and k_j is the power of this color. For small value of r , we shall use k_j bars over x_j instead of $\zeta^{k_j} x_j$. For example, if $w = \zeta^2 3 \zeta^2 1 \zeta^0 4 \zeta 1 \zeta^2 3$, then we write $w = \bar{3}\bar{1}4\bar{1}\bar{3}$.

Any r -colored word can be considered as a finite word of the alphabet $\Sigma_r := \{\xi j; \xi \in C_r, j \geq 1\}$.

Let $w := w_1 w_2 \dots w_m := \varepsilon_1 x_1 \varepsilon_2 x_2 \dots \varepsilon_m x_m$ be a r -colored word, we note

$$\begin{aligned} |w_i| &:= x_i \quad (1 \leq i \leq m); \\ |w| &:= |w_1| |w_2| \dots |w_m|; \end{aligned} \tag{5.1}$$

and define the statistic *power-color* col by

$$\text{col}_i w := \sum_{0 \leq j \leq r-1} j \chi(\varepsilon_i = \zeta^j) \quad (1 \leq i \leq m); \tag{5.2}$$

$$\text{col } w := \sum_{1 \leq i \leq m} \text{col}_i w.$$

If $\mathbf{m} = (m_1, \dots, m_s)$ is a sequence of nonnegative integers such that $m_1 + \dots + m_s = m$, let $G_{r, \mathbf{m}}$ be the set of all r -colored words $w = w_1 w_2 \dots w_m$ such that $|w| \in R_{\mathbf{m}}$. The class $G_{r, \mathbf{m}}$ contains $r^m \binom{m}{m_1, m_2, \dots, m_s}$ r -colored words. When $m_1 = m_2 = \dots = m_s = 1$, the class $G_{r, \mathbf{m}}$ is the wreath product $C_r \wr S_s$ denoted by $G_{r, s}$. We define the order relation on Σ_r as follows:

$$\zeta^j i > \zeta^{j'} i' \iff [j < j'] \quad \text{or} \quad [(j = j') \quad \text{and} \quad (i > i')]. \tag{5.3}$$

The restriction of this order to the class of ordinary words (with nonnegative letters) is the usual order.

As in [36], the statistics “inv” and “maj” must be adapted to r -colored words and correspond to classical statistics when applied to ordinary words. Let

$$(\omega; q)_n := \begin{cases} 1 & \text{if } n = 0; \\ (1 - \omega)(1 - \omega q) \dots (1 - \omega q^{n-1}) & \text{if } n \geq 1; \end{cases}$$

denote the usual q -ascending factorial of ω and

$$\left[\begin{matrix} m_1 + m_2 + \cdots + m_s \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q := \frac{(q; q)_{m_1 + m_2 + \cdots + m_s}}{(q; q)_{m_1} (q; q)_{m_2} \cdots (q; q)_{m_s}}$$

be the q -multinomial coefficient.

With the order relation defined in (5.3), the natural extensions of the *flag-major index* “fmaj” and the *flag-inversion number* “finv” introduced by Foata and Han [36] on r -colored words are defined as follows: for all r -colored word $w := w_1 w_2 \cdots w_m$,

$$\text{fmaj } w := r \sum_{i=1}^{m-1} i \chi(w_i > w_{i+1}) + \text{col } w;$$

$$\text{finv } w := \sum_{\substack{1 \leq i < j \leq m \\ \xi \in C_r}} \chi(\xi w_i > w_j) + \text{col } w.$$

Foata and Han defined $(-q; q)_m \left[\begin{matrix} m \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q$

as a q -analog of $2^m \binom{m}{m_1, m_2, \dots, m_s}$.

By analogy, $\frac{(q^r; q^r)_m}{(q; q)_m} \left[\begin{matrix} m \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q$ is a natural q -analog of $r^m \binom{m}{m_1, m_2, \dots, m_s}$.

By induction on the length of w , we have the following identity:

$$\frac{(q^r; q^r)_m}{(q; q)_m} \left[\begin{matrix} m \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q = \sum_{w \in G_{r, \mathbf{m}}} q^{\text{finv } w}.$$

Proof. we consider the bijective transformation

$$\phi : G_{r, \mathbf{m}} \longrightarrow \{0, 1, \dots, r-1\} \times \bigcup_{k=1}^s G_{r, \mathbf{m} - \mathbf{1}_k},$$

$$w := w_1 w_2 \cdots w_m \longmapsto (\alpha, w') := (\text{col}_m w, w_1 w_2 \cdots w_{m-1}),$$

where $\mathbf{m} - \mathbf{1}_k = (m_1, m_2, \dots, m_{k-1}, m_k - 1, m_{k+1}, \dots, m_s)$.

We have, $k = |w_m|$, and

$$\begin{aligned} \text{finv } w &= \text{finv } w' + \alpha + \sum_{\substack{1 \leq i \leq m-1 \\ 0 \leq j \leq r-1}} \chi(\zeta^j |w_i| > w_m) \\ &= \text{finv } w' + m\alpha + (m_{k+1} + \cdots + m_s) \chi(k < s). \end{aligned}$$

So,

$$\sum_{w \in G_{r, \mathbf{m}}} q^{\text{finv } w} = \sum_{0 \leq s \leq r-1} q^{m_s} A(q) = \frac{1 - q^{rm}}{1 - q^m} A(q),$$

where

$$A(q) = \sum_{w' \in G_{r, \mathbf{m}-\mathbf{1}_s}} q^{\text{finv } w'} + \sum_{1 \leq k \leq s-1} q^{m_{k+1} + \dots + m_s} \sum_{w' \in G_{r, \mathbf{m}-\mathbf{1}_k}} q^{\text{finv } w'}.$$

By induction, for each $1 \leq k \leq s$,

$$\begin{aligned} \sum_{w' \in G_{r, \mathbf{m}-\mathbf{1}_k}} q^{\text{finv } w'} &= \frac{(q^r; q^r)_{m-1}}{(q; q)_{m-1}} \frac{(q; q)_{m-1}}{(q; q)_{m_1} \cdots (q; q)_{m_{k-1}} (q; q)_{m_k-1} (q; q)_{m_{k+1}} \cdots (q; q)_{m_s}} \\ &= \frac{(q^r; q^r)_{m-1} (1 - q^{m_k})}{(q; q)_m} \frac{(q; q)_m}{(q; q)_{m_1} \cdots (q; q)_{m_k} \cdots (q; q)_{m_s}}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{1 \leq k \leq s-1} q^{m_{k+1} + \dots + m_s} \sum_{w' \in G_{r, \mathbf{m}-\mathbf{1}_k}} q^{\text{finv } w'} &= \frac{(q^r; q^r)_{m-1}}{(q; q)_m} \frac{(q; q)_m}{(q; q)_{m_1} \cdots (q; q)_{m_k} \cdots (q; q)_{m_s}} \sum_{1 \leq k \leq s-1} q^{m_{k+1} + \dots + m_s} (1 - q^{m_k}) \\ &= \frac{(q^r; q^r)_{m-1} (q^{m_s} - q^m)}{(q; q)_m} \frac{(q; q)_m}{(q; q)_{m_1} \cdots (q; q)_{m_k} \cdots (q; q)_{m_s}}, \end{aligned}$$

and

$$\begin{aligned} \sum_{w \in G_{r, \mathbf{m}}} q^{\text{finv } w} &= \frac{1 - q^{rm}}{1 - q^m} \left(\frac{(q^r; q^r)_{m-1} (1 - q^{m_s})}{(q; q)_m} \frac{(q; q)_m}{(q; q)_{m_1} \cdots (q; q)_{m_k} \cdots (q; q)_{m_s}} \right. \\ &\quad \left. + \frac{(q^r; q^r)_{m-1} (q^{m_s} - q^m)}{(q; q)_m} \frac{(q; q)_m}{(q; q)_{m_1} \cdots (q; q)_{m_k} \cdots (q; q)_{m_s}} \right) \\ &= \frac{(q^r; q^r)_m}{(q; q)_m} \frac{(q; q)_m}{(q; q)_{m_1} \cdots (q; q)_{m_k} \cdots (q; q)_{m_s}} \\ &= \frac{(q^r; q^r)_m}{(q; q)_m} \left[\begin{matrix} m \\ m_1, m_2, \dots, m_s \end{matrix} \right]_q. \end{aligned}$$

□

We construct the extension $\widehat{\Phi}$ of the second fundamental transformation Φ on r -colored words in the next section. Define

$$\text{des}^* w = r \text{des } w - \text{des } |w| + \text{col}_1 w, \quad (5.4)$$

where $\text{des } w := \sum_{i=1}^{m-1} \chi(w_i > w_{i+1})$.

The main purpose of this chapter is to prove the following theorem.

Theorem 5.1.1. *The transformation $\widehat{\Phi}$ constructed in the section 5.2 has the following properties*

1. *For every r -colored word w , $(\text{fmaj}, \text{des}^*) w = (\text{finv}, \text{col}) \widehat{\Phi}(w)$;*
2. *The restriction of $\widehat{\Phi}$ to each class $G_{r,m}$ is a bijection of $G_{r,m}$ onto itself.*

Corollary 20. *For each $m = (m_1, m_2, \dots, m_s)$ the bistatistics $(\text{fmaj}, \text{des}^*)$ and $(\text{finv}, \text{col})$ are equidistributed on $G_{r,m}$.*

Example. Let's take the hyperoctahedral group of order 2.

w	1 2	$\bar{1}$ 2	1 $\bar{2}$	$\bar{1}$ $\bar{2}$	2 1	$\bar{2}$ 1	2 $\bar{1}$	$\bar{2}$ $\bar{1}$
$\text{fmaj } w$	0	1	3	2	2	1	3	4
$\text{des}^* w$	0	1	2	1	1	0	1	2
$\text{finv } w$	0	1	2	3	1	2	3	4
$\text{col } w$	0	1	1	2	0	1	1	2

Now consider the statistic Rfinv defined on hyperoctahedral group as follows:

$$\text{Rfinv } w = \text{inv } w + \sum_{i=1}^n |w_i| \chi(w_i < 0).$$

If on use the natural order relation on $[-n, n]$

$$-n < -(n-1) < \dots < -1 < 1 < \dots < (n-1) < n, \quad (5.5)$$

Brenti [15] derive that finv coincide with traditional *length function* ℓ and Adin and Roichman [1] proved that Rfinv and fmaj are equidistributed on hyperoctahedral group.

Back to order relation (5.3) on $[-n, n]$, ie

$$-1 < \dots < -(n-1) < -n < 1 < \dots < (n-1) < n,$$

one has

$$\ell \neq \text{Rfinv} \quad \text{and} \quad \text{finv} \neq \text{Rfinv}.$$

but we observe that Rfinv remain equidistributed with fmaj and we prove that its extension on wreath product is also mahonian. We have the following theorem.

Theorem 5.1.2. *The statistic Rfinv defined on wreath product $C_r \wr S_n$ by*

$$\text{Rfinv } w = \text{inv } w + \sum_{i=1}^n |w_i| \text{col}_i w \quad (5.6)$$

is mahonian.

By Haglund, Loehr and Remmel [43] we have the following corollary

Corollary 21. *We have:*

$$\sum_{\sigma \in G(r,n)} q^{\text{Rfinv } \sigma} = \sum_{\sigma \in G(r,n)} q^{\text{finv } \sigma} = \frac{(q^r; q^r)_n}{(1-q)^n}. \quad (5.7)$$

5.2 The construction of $\widehat{\Phi}$

Let us recall the second fundamental transformation Φ [37]. First, for each integer x , we recall the transformation γ_x . Let $w = x_1 x_2 \cdots x_m$ be a word with positive letters. The first (resp. last) letter x_1 (resp. x_m) is denoted by $F(w)$ (resp. $L(w)$). If $L(w) \leq x$ (resp. $L(w) > x$), w admits the unique factorisation :

$$(u_1 y_1, u_2 y_2, \dots, u_p y_p)$$

called its *x-right-to-left factorisation* having the following properties:

1. each y_i ($1 \leq i \leq p$) is a letter verifying $y_i \leq x$ (resp. $y_i > x$);
2. each u_i ($1 \leq i \leq p$) is a factor which is either empty or has all its letters greater than (resp. smaller than or equal to) x .

Then, the bijective transformation γ_x maps $w = u_1 y_1 u_2 y_2 \dots u_p y_p$ onto the word

$$\gamma_x(w) = y_1 u_1 y_2 u_2 \cdots y_p u_p.$$

Foata defined $\Phi(w)$ by induction on the length of w . If w has length one, then $\Phi(w) = w$. If it has more than one letter, write the word as vx where x is the last letter and define $\Phi(vx)$ to be the juxtaposition product

$$\Phi(vx) := \gamma_x(\Phi(v))x. \quad (5.8)$$

We now define $\widehat{\Phi}$ as follows.

For each word $u = x_1 x_2 \cdots x_m$ with nonnegative letters and each element

$\epsilon := (\epsilon_1, \dots, \epsilon_m)$ of $(\mathcal{C}_r)^m$, we note $\Psi_u(\epsilon)$ the element $\epsilon' = (\epsilon'_1, \dots, \epsilon'_m)$ of $(\mathcal{C}_r)^m$ defined as follows:

$$\begin{cases} \epsilon'_i = \frac{\epsilon_i}{\epsilon_{i+1}} \zeta^{-\chi(x_i > x_{i+1})} & \text{if } i < m; \\ \epsilon'_m = \epsilon_m. \end{cases} \quad (5.9)$$

Let $w := (\epsilon, u)$ be a r -colored word ($u = |w|$). Define

$$\widehat{\Phi}(w) = (\Psi_u(\epsilon), \Phi(u)). \quad (5.10)$$

Example. Let's take $r = 4$ and $w = \bar{3}\bar{1}4\bar{1}\bar{3}$. We have

$$w = ((\zeta^2, \zeta, 1, \zeta, \zeta^2), 31413).$$

By construction of Φ (relation (5.8)), we successively have

$$\begin{aligned} \Phi(3) &= 3, \\ \Phi(31) &= \gamma_1(\Phi(3))1 = \gamma_1(3)1 = 31, \\ \Phi(314) &= \gamma_4(\Phi(31))4 = \gamma_4(31)4 = 314, \\ \Phi(3141) &= \gamma_1(\Phi(314))1 = \gamma_1(314)1 = 3411, \\ \Phi(31413) &= \gamma_3(\Phi(3141))3 = \gamma_3(3411)3 = 31413. \end{aligned}$$

In the other hand,

$$\epsilon'_1 = \frac{\zeta^2}{\zeta} \zeta^{-1} = 1, \quad \epsilon'_2 = \frac{\zeta}{1} = \zeta, \quad \epsilon'_3 = \frac{1}{\zeta^1} \zeta^{-1} = \zeta^2, \quad \epsilon'_4 = \frac{\zeta^1}{\zeta^2} = \zeta^3, \quad \epsilon'_5 = \zeta^2.$$

Therefore

$$w' = \widehat{\Phi}(\bar{3}\bar{1}4\bar{1}\bar{3}) = 3\bar{1}\bar{4}\bar{1}\bar{3}.$$

We have

$$(\text{fmaj}, \text{des}^*)w = (\text{finv}, \text{col})w' = (34, 8).$$

5.3 Proof of Theorem 5.1.1

Lemma 5.1. *Let $w := w_1 w_2 \dots w_m$ be a r -colored word. With the notation in the relation (5.2), we have*

$$\text{finv } w = \text{inv } |w| + \sum_{i=1}^m i \text{col}_i w. \quad (5.11)$$

Proof. For all integers i, j, k such that $1 \leq i < j \leq m$ and $0 \leq k \leq r-1$, one has:

$$\chi(\zeta^k |w_i| > w_j) = \chi(k < \text{col}_j w) + \chi(k = \text{col}_j w) \chi(|w_i| > |w_j|).$$

Thus,

$$\text{finv } w = \sum_{j=1}^m (j-1) \text{col}_j w + \text{inv } |w| + \sum_{j=1}^m \text{col}_j w = \text{inv } |w| + \sum_{j=1}^m j \text{col}_j w. \quad \square$$

Now, let $w = (\epsilon, |w|)$ be a r -colored word of length m and $w' := (\epsilon', |w'|) := \widehat{\Phi}(w)$.

For each i such that $1 \leq i \leq m-1$,

$$\begin{aligned} - \text{ if } |w_i| \leq |w_{i+1}|, \text{ then } \epsilon'_i &= \frac{\epsilon_i}{\epsilon_{i+1}} = \zeta^{\text{col}_i w - \text{col}_{i+1} w}, \\ - \text{ if } |w_i| > |w_{i+1}|, \text{ then } \epsilon'_i &= \frac{\epsilon_i}{\epsilon_{i+1}} \zeta^{-1} = \zeta^{\text{col}_i w - \text{col}_{i+1} w - 1}. \end{aligned}$$

So,

$$\begin{aligned} \text{col}_i w' &= [\text{col}_i w - \text{col}_{i+1} w + r \chi(\text{col}_i w < \text{col}_{i+1} w)] \chi(|w_i| \leq |w_{i+1}|) \\ &\quad + [\text{col}_i w - \text{col}_{i+1} w - 1 + r \chi(\text{col}_i w \leq \text{col}_{i+1} w)] \chi(|w_i| > |w_{i+1}|) \\ &= \text{col}_i w - \text{col}_{i+1} w + r \chi(\text{col}_i w < \text{col}_{i+1} w) \\ &\quad + r \chi(\text{col}_i w = \text{col}_{i+1} w) \chi(|w_i| > |w_{i+1}|) - \chi(|w_i| > |w_{i+1}|). \end{aligned}$$

Thus, we have

$$\begin{aligned} \text{col } w' &= \sum_{i=1}^m \text{col}_i w' \\ &= \sum_{i=1}^m \text{col}_i w - \sum_{i=2}^m \text{col}_i w + r \sum_{i=1}^{m-1} [\chi(\text{col}_i w < \text{col}_{i+1} w) \\ &\quad + \chi(\text{col}_i w = \text{col}_{i+1} w) \chi(|w_i| > |w_{i+1}|)] - \sum_{i=1}^{m-1} \chi(|w_i| > |w_{i+1}|) \\ &= \text{col}_1 w + r \text{des } w - \text{des } |w| \\ &= \text{des}^* w, \end{aligned}$$

and,

$$\begin{aligned}
\text{finv } w' &= \text{inv } |w'| + \sum_{i=1}^m i \text{col}_i w' \\
&= \text{maj } |w| + \sum_{i=1}^m i \text{col}_i w - \sum_{i=1}^m (i-1) \text{col}_i w \\
&\quad + r \sum_{i=1}^{m-1} i [\chi(\text{col}_i w < \text{col}_{i+1} w) \\
&\quad + \chi(\text{col}_i w = \text{col}_{i+1} w) \chi(|w_i| > |w_{i+1}|)] - \sum_{i=1}^{m-1} i \chi(|w_i| > |w_{i+1}|) \\
&= \text{maj } |w| + \sum_{i=1}^m \text{col}_i w + r \sum_{i=1}^{m-1} i \chi(w_i > w_{i+1}) - \text{maj } |w| \\
&= r \sum_{i=1}^{m-1} i \chi(w_i > w_{i+1}) + \text{col } w \\
&= \text{fmaj } w.
\end{aligned}$$

Finally, $\widehat{\Phi}$ is a bijection of $G_{r,\mathbf{m}}$ onto itself. Indeed, let $w' := (\epsilon', u')$ be an element of $G_{r,\mathbf{m}}$. By the relation (5.9), if $w := (\epsilon, u)$ is an element of $G_{r,\mathbf{m}}$ such that $\widehat{\Phi}(w) = w'$, then $u = \Phi^{-1}(u')$, $\epsilon_m = \epsilon'_m$ and, for $i < m$,

$$\epsilon_i = \epsilon'_m \prod_{i \leq j \leq m-1} \epsilon'_j \zeta^{\chi(x_j > x_{j+1})},$$

where $u := x_1 x_2 \cdots x_m$.

5.4 Proof of Theorem 5.1.2

Consider the following transformations

▷ **Transformation ρ**

$$\rho : G(r, n) \longrightarrow G(r, n),$$

$$w = (\epsilon, |w|) \longmapsto \rho(w) = w' = (\epsilon', |w'|),$$

where

$$|w'| = |w|^{-1} \quad \text{and} \quad \epsilon'_i = \epsilon_{|w|^{-1}(i)}.$$

▷ **Transformation τ**

For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ in $[0, r-1]^n$, put $\Sigma_\alpha = \{\zeta^{\alpha_1}, \zeta^{\alpha_2}, \dots, \zeta^{\alpha_n}\}$ and let G_α be the class of r -colored permutations whose letters are in Σ_α , i.e.

$$G_\alpha = \{w = w_1 w_2 \cdots w_n \in G(r, n); \quad \text{col}_i w = \alpha_i, \quad \forall i \in [n]\}.$$

Note that $\#G_\alpha = n!$. We denote by I_α the increasing bijection from $[n]$ onto Σ_α and we define τ for each class G_α by $\tau(w) = w'_1 w'_2 \cdots w'_n$ where

$$w'_i = I_\alpha(|w_i|).$$

Lemma 5.2. *For any $w \in G(r, n)$,*

$$\text{finv } w = \text{Rfinv } \tau \circ \rho(w).$$

Proof of Lemma 5.2

Let $w = w_1 w_2 \cdots w_n \in G(r, n)$. Consider the following auxiliary statistics

$$\begin{aligned} \wp(w) &:= \sum_{i=1}^n i \text{col}_i w; \\ \mathfrak{S}(w) &:= \sum_{i=1}^n |w_i| \text{col}_i w; \\ |\text{inv } w| &:= \text{inv } |w|. \end{aligned}$$

It is easy to see that ρ is an involution preserving $|\text{inv } w|$ and transforming $\wp$ into $\mathfrak{S}$ and vice versa:

$$(|\text{inv } w|, \wp)w = (|\text{inv } w|, \mathfrak{S})\rho(w),$$

and τ preserves $\mathfrak{S}$ and transforms $|\text{inv } w|$ into $\text{inv } w$, i.e

$$(|\text{inv } w|, \mathfrak{S})w = (\text{inv } w, \mathfrak{S})\rho(w).$$

By Lemma 5.1

$$\begin{aligned} \text{finv } w &= |\text{inv } w| + \wp(w) \\ &= |\text{inv } w| + \mathfrak{S}(\rho(w)) = \text{inv } \tau \circ \rho(w) + \mathfrak{S} \tau \circ \rho(w) \\ &= \text{Rfinv } \tau \circ \rho(w). \end{aligned}$$

Example. Let $w = 5\bar{3}\bar{1}2\bar{4}$, $\text{finv } w = 18$; $\rho(w) = \bar{3}4\bar{2}\bar{5}1$. Let $\alpha = (0, 2, 1, 0, 1)$. I_α is defined as follows:

i	1	2	3	4	5
$I_\alpha(i)$	$\bar{2}$	$\bar{3}$	$\bar{5}$	1	4

So

$$\tau \circ \rho(w) = w' = \bar{5}1\bar{3}4\bar{2} \quad \text{and} \quad \text{Rfinv } w' = 18.$$

Chapitre 6

Conclusion et perspectives

Nous avons étudié dans le chapitre 3 le polynôme

$$g_{r,n}(q, x) = \sum_{\pi \in G(r,n)} q^{\text{fma} \pi} x^{\text{fix} \pi}.$$

Ici la notion de point fixe est limitée à la couleur neutre, c'est à dire que l'entier i est un point fixe de π si et seulement si $\pi(i) = i$. Mais on peut considérer également les points fixes colorés.

Pour chaque $i \in \{0, 1, \dots, r-1\}$, définissons la statistique fix_i suivante.

Définition 6.1. Soit $\pi \in G(r, n)$. On dit que l'entier $j \in [n]$ est un point fixe de couleur i si et seulement si

$$\pi(j) = \zeta^i j. \quad (6.1)$$

On note $\text{FIX}_i(\pi)$ l'ensemble des points fixes de couleur i de π et $\text{fix}_i(\pi)$ le cardinal de $\text{FIX}_i(\pi)$.

D'autre part, nous avons défini la statistique col comme somme des col_i , ($1 \leq i \leq n$) avec

$$\text{col}_i \pi := \sum_{0 \leq j \leq r-1} j \chi(\varepsilon_i = \zeta^j) \quad (1 \leq i \leq n); \quad (6.2)$$

où $\pi = (\epsilon, |\pi|)$.

Si nous considérons la statistique coul_i définie pour tout $i \in [0, r-1]$ par

$$\text{coul}_i \pi := \sum_{j=1}^n \chi(\varepsilon_j = \zeta^i), \quad (6.3)$$

alors la statistique col est obtenue par :

$$\text{col } \pi := \sum_{0 \leq i \leq r-1} i \text{ cou}_i \pi. \quad (6.4)$$

Notre perspective est d'étudier le raffinement de $g_{r,n}(q, x)$ suivant :

$$g_{r,n}(q, (x_i), (y_i)) = \sum_{\pi \in G(r,n)} q^{\text{fmaj } \pi} \prod_{i=0}^r x_i^{\text{fix}_i \pi} y_i^{\text{cou}_i \pi}. \quad (6.5)$$

Le cas particulier pour $r = 2$ est étudié par Foata Han dans [35] avec la relation d'ordre OM .

Si on note $B_n(q, x_0, x_1, z)$ le polynôme défini par l'identité :

$$\begin{aligned} \sum_{n \geq 0} B_n(q, x_0, x_1, z) \frac{u^n}{(q^2, q^2)_n} \\ = (1 - u \frac{1 + qz}{1 - q^2})^{-1} \times \frac{(u; q^2)_\infty}{(ux_0; q^2)_\infty} \frac{(-uqx_1z; q^2)_\infty}{(-uqz; q^2)_\infty}, \end{aligned}$$

alors Foata et Han ont démontré [35, Théorème 1.3] que

$$B_n(q, x_0, x_1, z) = \sum_{\pi \in G(2,n)} q^{\text{fmaj } \pi} x_0^{\text{fix}_0 \pi} x_1^{\text{fix}_1 \pi} z^{\text{cou}_1(\pi)}.$$

Cette perspective a pour but de voir si ce théorème est extensible pour r quelconque.

Notre deuxième perspective repose sur les problèmes de motifs. Ce domaine prend une place importante en combinatoire énumérative. Des résultats classiques peuvent être interprétés en terme de motifs. Par exemple, les nombres de partitions de $[n]$, de chemins de Dyck, de “non-overlapping partitions” de $[n]$, d'involutions, de chemins de Motzkin, peuvent être interprétés en terme de motifs (voir les résultats principaux de Anders Claesson [26]).

Le problème d'extension des motifs sur les permutations colorées n'est pas évident. Mansour [46] a donné des résultats sur les motifs colorés de longueur 2 et sur quelques cas particuliers pour les longueurs quelconques. Rappelons que si $\pi \in S_n$ et $\tau \in S_k$ ($k \leq n$), une occurrence de τ dans π est une suite

$$1 \leq i_1 < i_2 < \dots < i_k \leq n,$$

telle que

$$\tau = \rho(\pi_{i_1})\rho(\pi_{i_2}) \dots \rho(\pi_{i_k}),$$

où ρ est la bijection croissante de $[k]$ sur $\{\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}\}$.

Soit maintenant $\pi = (\epsilon, |\pi|) \in G(r, n)$ et $\tau = (\varepsilon, |\tau|) \in G(r, k)$. On dit que π contient le motif τ (pattern) s'il existe une suite $1 \leq i_1 < i_2 < \dots < i_k \leq n$ vérifiant les deux conditions suivantes :

- (i) Il existe une occurrence de $|\tau|$ dans $|\pi|$,
- (ii) $\epsilon_{i_j} = \varepsilon_j, \quad \forall j \in [k]$.

On dit que π évite le motif τ (τ -avoiding) si π ne contient pas τ . On note $G(r, n)(\tau)$ l'ensemble des τ -avoiding dans $G(r, n)$. Plus généralement, si T est un ensemble de motifs, soit $G(r, n)(T)$ l'ensemble des permutations colorées de $G(r, n)$ qui ne contient aucun motif dans T .

Il a été démontré dans [46] que :

$$\forall \tau \in G(r, 2), \quad |G(r, n)(\tau)| = \sum_{j=0}^n j!(r-1)^j \binom{n}{j}^2,$$

et que

$$|G(r, n)(T)| = n!(n+r-1)(r-1)^{n-1},$$

pour les ensembles T suivants :

- * $T = \{(1, 2), (2, 1)\}$ pour $r \geq 1$,
- * $T = \{(1, 2), (1, \bar{2})\}$ pour $r \geq 2$,
- * $T = \{(1, \bar{2}), (2, \bar{1})\}$ pour $r \geq 2$,
- * $T = \{(1, \bar{2}), (\bar{2}, 1)\}$ pour $r \geq 2$,
- * $T = \{(1, \bar{2}), (1, \bar{\bar{2}})\}$ pour $r \geq 3$.

De plus, si $2 \leq a \leq b$ et $r \geq b$ alors pour tout $n \geq 1$,

$$|G(r, n)(T)| = \sum_{i+j \leq n} \binom{n}{i, j, n-i-j}^2 (n-i-j)!(r-2)^{n-i-j},$$

pour les ensembles T suivants :

- * $T = \{(1, 2), (1\zeta^{a-1}, 2\zeta^{b-1})\}$,
- * $T = \{(1, 2), (2\zeta^{a-1}, 1\zeta^{b-1})\}$,

$$* \quad T = \{(1, \bar{2}), (\bar{1}, \bar{\bar{2}})\},$$

$$* \quad T = \{(1, \bar{2}), (\bar{\bar{2}}, \bar{1})\}.$$

Le motif $\tau = (\varepsilon, |\tau|) \in G_{r,k}$ est dit homogène si et seulement s'il existe un entier $u \in [0, r-1]$ tel que

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = \zeta^u.$$

Dans ce cas, on note $\tau = [|\tau|]_{(u)}$. Plus généralement, si $T \subset S_k$, on écrit $T_{(u)} = \{[\alpha]_{(u)} : \alpha \in T\}$. Alors Mansour a montré que pour tout u , $0 \leq u \leq r-1$, $T \subset S_k, n \geq 0$, on a :

$$|G(r, n)(T_{(u)})| = \sum_{j=0}^n j!(r-1)^j \binom{n}{j}^2 |S_{n-j}(T)|.$$

Il a également donné deux autres cas particuliers.

Notre perspective est de renforcer les résultats sur l'énumération des permutations colorées relative aux motifs colorés de longueur supérieure ou égale à 3.

Index

- Excedent, subexcedent, 68
- Factorisation d'une permutation en
 - produit de cycle, 5
- Flag-major, 12
- Fonction longueur, 1, 12
- Graphe de Coxeter, 3
- Groupe de Coxeter, 3
- Groupe de réflexions complexes, 9
- Groupe des permutations colorées,
 - 8
- Groupe des permutations signées paires,
 - 10
- Groupe hyperoctaédral, 6
- Groupe symétrique, 4
- Indice majeur, 1
- Inversion, 1
- k -successions, 13
- m -fixe croissantes , 14
- m -fixe isolée, 14
- Nombres de descentes, d'inversions,
 - 1
- pillar, 67
- Produit en couronne, 7
- Produit semi-direct, 6
- q -notations, 14
- Relations d'ordre sur l'alphabet coloré, 10
- slot, red slot, green slot, 59
- Statistique, 1
- Statistiques eulériennes, 1
- Statistiques mahoniennes, 1
- Tableau d'Euler, 2
- Transposition, 5

Index des notations

B_n , 6	$G(r, n)$, 8
$C_r \wr S_n$, 7	g_n^m , 2, 23
$\mathcal{C}^k(\pi)$, 14	$G_{r,m}$, 29
$c^k(\pi)$, 14	$g_{r,n}^m$, 24
χ , 10	$G(r, n, m, k)$, 37
col, 9	$g_{r,n}^m(q)$, 56
$\text{color}_\pi(x)$, 8	$G(r, p, n)$, 9
col_i , 9	
COL_j , 57	$H \rtimes K$, 6
$c_{r,n,m}^k$, 24	
Der, 26, 57	inv, 1
$\mathcal{D}(r, n)$, 21	$I_{r,n}^m$, 36
des, 1	I_ω , 71
DES, 56	
des*, 85	$\ell(\pi)$, 12
Des _A , 13	$\mathcal{L}^k(\pi)$, 14
des _A , 13	$l^k(\pi)$, 14
Des _B , 13	ℓ_B , 17
des _B , 13	ℓ_D , 17
Des _G , 19	ℓ_G , 29
des _G , 19	$l_{r,n,m}^k$, 24
D_n , 10	m-1 _k , 83
$D_{r,n}^m$, 36	maj, 1
$d_{r,n}^m$, 32	
fdes, 13	[n], 4
finv, 12, 29	NDes, 13
FIX, 14	ndes, 13
fmaf, 26	Neg, 13
fma _{jD} , 17	neg, 13
	nmaj, 13
$G_1, G_2, G_3, R_1, R_2, R_3$, 72	
	OC, 11
	OM, 11

OV , 11

Pil , 67

pil , 67

q -notations, 15

$rank$, 57

$Rfinv$, 85

$[\sigma, j]$, 71

$Sh(0^n v)$, 67

$\Sigma_{r,n}$, 8

$\langle \sigma, j \rangle$, 60

$t^{[m]}$, 71

T_σ , 40

$ZDer$, 67

$Zero$, 67

Bibliographie

- [1] R. M. Adin and Y. Roichman, The flag major index and group actions on polynomial rings, *European J. Combin.* 22 (2001), 431–446.
- [2] R. M. Adin, F. Brenti and Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, *Adv. in Appl. Math.*, 27 (2001), 210–224.
- [3] R. M. Adin, F. Brenti and Y. Roichman, Equi-distribution over descent classes of the hyperoctahedral group, *J. Combin. Theory Ser. A* 113 (2006), no. 6, 917–933.
- [4] R. M. Adin, I. M. Gessel and Y. Roichman, Signed mahonians, *J. Combin. Theory Ser. A* 109 (2005), 25–43.
- [5] T. Andreescu and Z. Feng, *A path to Combinatorics for Undergraduates*, Birkhäuser, Boston, 2004.
- [6] G. E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original.
- [7] E. Bagno and R. Biagioli, Colored-descent representations of complex reflection groups $G(r, p, n)$, *Israel J. Math.* 160 (2007), 317–347.
- [8] E. Bagno and D. Garber, On the excedance number of colored permutation groups, *Sém. Lothar. Combin.*, 53 (2004/06), Art. B53f, 17 pp. (electronic).
- [9] R. Biagioli and J. Zeng, Enumerating wreath products via Garsia-Gessel bijections, Arxiv : 0909.4009 v1 [math CO] 22 septembre 2009.
- [10] R. Biagioli and J. Zeng, On some analogues of Carlitz’s Identity for the hyperoctahedral Group Arxiv : 0909.3961 v1 [math CO] 22 septembre 2009.

- [11] R. Biagioli, Permutation Statistics and Polynomials on Coxeter Groups, Ph.D. Thesis, University of Rome “La Sapienza”, 2003.
- [12] R. Biagioli, Major and descent statistics for the even-signed permutation group, *Adv. in Appl. Math.*, vol 31, issue 1, (July 2003), 163-179.
- [13] A. Björner and M. L. Wachs, Permutation Statistics and Linear Extensions of Posets, *J. Combin. Theory Ser. A* 58 (1991), 85 – 114.
- [14] N. Bourbaki, Groupes et algèbre de Lie, Chap 4,5,6, Hermann, 1968.
- [15] F. Brenti, q-Eulerian Polynomials Arising from Coxeter Groups, *European J. Combin.* 15(1994), 417-441.
- [16] K. Briggs and J. Remmel, m -rook numbers and a generalization of a formula of Frobenius to $C_m \wr S_n$, *J. Combin. Theory Ser. A* 113 (2006), no. 6, 1138–1171.
- [17] R.A. Brualdi, *Introductory Combinatorics*, 2nd Ed., North-Holland, Amsterdam, 1992.
- [18] L. Carlitz, A combinatorial property of q-Eulerian numbers, *Amer. Math. Monthly* 82 (1975), 51-54.
- [19] L. Carlitz, q -Bernoulli and Eulerian numbers, *Trans. Amer. Math. soc.*, 76 (1954), 332-350.
- [20] W. Y. C. Chen, The skew, relative and classical derangments, *Discrete Math.*, 160 (1996), 235-239.
- [21] W. Chen, and J. Zhang, The skew and relative derangments of type B, *Electronic J. Combin.*, 14 (2007), #N24.
- [22] W. Y. C. Chen, Y. Gao and J. He, Labeled partitions with colored permutations, *Discrete Math.*, vol 309, (2009) issue 21, 6235-6244.
- [23] Y. Cherniavsky, Abstract Young representations for Coxeter groups of type D, *Adv. in Appl. Math.*, 41 (2008), no. 4, 561–583.
- [24] C. Chow , On derangement polynomials of type B, *Sém. Lothar. Combin.* 55 (2005/06), Art. B55b, 6 pp. (electronic).
- [25] C. Chow and Gessel (I.), On the descent numbers and major indices for the hyperoctahedral group, *Adv. in Appl. Math.*, 38 (2007), no. 3, 275–301.

- [26] A. Claesson, Generalized pattern avoidance, arxiv : 0011235 v1, [math.CO.] (22/11/2000).
- [27] R. Clarke, G. N. Han and J. Zeng, A combinatorial interpretation of the Seidel generation of q -derangement numbers, Ann. Comb. 1 (1997), no.4, 313–327.
- [28] L. Comtet, Analyse combinatoire, Tome second, Collection SUP, Presses Universitaires de France, 1970.
- [29] D. Dumont, Matrice d'Euler-Seidel, Sém. Lothar. Combin. 305c (1981).
- [30] D. Dumont et A. Randrianarivony, Dérangements et nombres de Genocchi, Discrete Math. 132 (1994), no. 1-3, 37–49.
- [31] H. Faliharimalala and J. Zeng, Derangements and Euler's difference table for $C_\ell \wr S_n$, Electronic J. Combinatorics, R65 of Volume 15(1), 2008.
- [32] H. Faliharimalala and J. Zeng, Fix-Euler-Mahonian statistics on wreath products, to appear in Adv. in Appl. Math.
- [33] H. Faliharimalala and A. Randrianarivony, Flag-major index and Flag-inversion number on colored words and wreath product, Sém. Lothar. Combin., 62 (2010), Art. B62C, 10 pp. (electronic).
- [34] D. Foata and G.N. Han, Fix-Mahonian Calculus, I : two transformations, Europ. J. Combinatorics, Volume 29, Issue 7(2008), 1721-1732.
- [35] D. Foata and G.N. Han, Signed words and permutations, IV ; fixed and fixed points, preprint, Israel J. Math., 163, no. 1, 2008, 217-240.
- [36] D. Foata and G.N. Han, Signed Words and permutations, I ; A fundamental transformation, Proc. Amer. Math. Soc., 135 (2007), n°1, 31-40.
- [37] D. Foata, On the Netto inversion number of a sequence, Proc. Amer. Math. Soc., 19 (1968), 236 – 240.
- [38] D. Foata and M.P. Schützenberger, Major Index and Inversion number of Permutations, Math. Nachr. 83 (1978), 143 – 159.
- [39] D. Foata and M.P. Schützenberger, Théorie géométrique des polynômes eulériens, Lecture Notes in Mathematics, Springer-Verlag, vol. 138, 1970.
- [40] M. Fire, Statistics on wreath products, arxiv : math /0409421 v2 [math.CO] 12 Nov 2005.

- [41] M. Geck and G. Malle, Reflection groups, in : *Handbook of Algebra*, vol. 4 (edited by M. Hazewinkel), Elsevier, Amsterdam, 2006, 337-383.
- [42] I. Gessel and C. Reutenauer, Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A*, vol. 64, 1993, 189–215.
- [43] J. Haglund, N. Loehr and J. B. Remmel, Statistics on wreath products, perfect matchings, and signed words, *European J. Combin.* 26 (2005), no. 6, 835–868.
- [44] G. Ksavelof and J. Zeng, Two involutions for signed excedance numbers *Sém. Lothar. Combin.*, 49 (2002/04), Art. B49e, 8 pp. (electronic).
- [45] P.A. MacMahon, *Combinatory analysis*, vol 1, Cambridge Univ. Press, London, 1915.
- [46] T. Mansour, Pattern avoidance in colored permutations, *Sém. Lothar. Combin.*, 46(2004), Art. B46g.
- [47] F. Rakotondrajao, k -fixed-points-permutations, *Integers*, electronic journal of combinatorial number theory, 7 (2007), #A36.
- [48] F. Rakotondrajao, On Euler’s difference table, *Proc. of FPSAC’07*, Tianjin, 2007.
- [49] A. Randrianarivony, Un q -tableau d’Euler, *European J. Combin.* 31 (2010), n°1, 64–71. 05A05 (05A15).
- [50] V. Reiner, Signed permutation statistics, *European J. Combin.* 14 (1993), no. 6, 553–567.
- [51] J. Riordan, *Introduction to combinatorial analysis*, originally published by Wiley, New York, 1958, and reprinted by Dover Publications, Inc., Mineola, NY, 2002.
- [52] R. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge Studies in Advanced Math. 49, 1997.
- [53] E. Steingrímsson, *Permutation Statistics of indexed and Poset Permutations*. MIT November 25, 1991.
- [54] M. Wachs, On q -derangement numbers, *Proc. Amer. Math. Soc.* 106 (1989), 273–278.

- [55] M. Wachs, An involution for signed Eulerian numbers, *Discrete Math.* 99 (1992), 59-62.
- [56] J. Zeng, The Akiyama-Tanigawa algorithm for Carlitz's q -Bernoulli numbers, *Integers : Electronic J. Combin. Number Theory* 6 (2006), #05.