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Oscillateurs couplés, désordre et synchronisation

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Docteur de l'Université Paris VI - Pierre et Marie Curie

Discipline : Mathématiques

présentée par

Eric Luçon

Oscillateurs couplés, Désordre et Synchronisation

---

dirigée par **Giambattista Giacomin** et **Lorenzo Zambotti**

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À la mémoire de mon grand-père, Roger Burban.

"parce que Babylone n'est autre chose
qu'un infini jeu de hasard",
J.-L. Borges - Fictions

"J'étais très mauvais en sciences",
Eugène Ionesco - L'impromptu de l'Alma.

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Oscillateurs couplés, désordre et synchronisation.

Résumé

Dans cette thèse, nous étudions le modèle de synchronisation de Kuramoto et plus généralement des systèmes de diffusions interagissant en champ moyen, en présence d'un aléa supplémentaire appelé désordre. La motivation principale en est l'étude du comportement du système en grande population, pour une réalisation fixée du désordre (modèle quenched).

Ce document, outre l'introduction, comporte quatre chapitres. Le premier s'intéresse à la convergence de la mesure empirique du système d'oscillateurs vers une mesure déterministe solution d'un système d'équations aux dérivées partielles non linéaires couplées (équation de McKean-Vlasov). Cette convergence est prouvée indirectement via un principe de grandes déviations dans le cas averaged et directement dans le cas quenched, sous des hypothèses plus faibles sur le désordre.

Le deuxième chapitre est issu d'un travail en commun avec Giambattista Giacomin et Christophe Poquet et concerne la régularité des solutions de l'EDP limite ainsi que la stabilité de ses solutions stationnaires synchronisées dans le cas d'un désordre faible.

Les deux derniers chapitres étudient l'influence du désordre sur une population d'oscillateurs de taille finie et illustrent des problématiques observées dans la littérature physique. Nous prouvons dans le troisième chapitre un théorème central limite quenched associé à la loi des grands nombres précédente : on montre que le processus de fluctuations quenched converge, en un sens faible, vers la solution d'une EDPS linéaire. Le dernier chapitre étudie le comportement en temps long de cette EDPS, illustrant le fait que les fluctuations dans le modèle de Kuramoto ne sont pas auto-moyennantes.

Mots-clefs

Synchronisation, modèle de Kuramoto, processus de diffusion interagissants, mécanique statistique, systèmes désordonnés, systèmes hors-équilibre, grandes déviations, EDP non linéaires, équations aux dérivées partielles stochastiques.

Coupled oscillators, disorder and synchronization.

Abstract

In this thesis, we study the synchronization Kuramoto model and more generally systems of mean-field interacting diffusions on the circle, in the presence of another source of randomness, called disorder. The main motivation of this work is to study the large-population behavior of the system, for a fixed realization of the disorder (quenched model).

This document contains, after the introduction, four chapters. The first one addresses the convergence of the empirical measure of the system of oscillators to a deterministic measure that solves a system of coupled, nonlinear partial differential equations (McKean-Vlasov equation). This convergence is indirectly proved through a large deviation principle in the averaged case and through a direct proof in the quenched case, under weaker assumptions on the disorder.

The second chapter is part of a joint work with Giambattista Giacomin and Christophe Poquet and concerns the regularity of solutions of the limiting PDE as well as the stability of its nontrivial synchronized solution in the case of weak disorder.

The last two chapters tackle the issue of the influence of the disorder on finite-size populations of rotators and illustrate problematics already observed in physical literature. We prove in the third chapter a central limit theorem associated to the previous law of large numbers: the quenched fluctuation process is shown to converge, in a weak sense, to the solution of a linear SPDE. We study in the last chapter the large-time behavior of this solution, illustrating the fact that fluctuations in the Kuramoto model are not self-averaging.

Keywords

Synchronization, Kuramoto model, interacting diffusion processes, statistical mechanics, disordered systems, non-equilibrium systems, large deviations, nonlinear PDE, stochastic partial differential equation.

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Chapter 1

Introduction

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1 Content of the thesis

This PhD thesis focuses on the study of the Kuramoto model, firstly introduced in the 70's by Yoshiki Kuramoto ([53]) in order to capture the main mechanisms responsible for the phenomenon of *synchronization* in biological or physical systems. The Kuramoto model has been extensively studied since, mostly in the physical literature. The main advantage of this model, which is a particular case of mean-field interacting particles in random media, is its simplicity and tractability. However many interesting questions remain unanswered.

A great part of this work, although motivated by the Kuramoto model, is also applicable for more general models of diffusions in random media. Hence, most of the results presented here will be stated in this general framework. We shall focus on the microscopic system as well as on its large size limit, the main underlying issue being about the influence of the disorder on the behavior of the system as the number of particles increases. More particularly, the main motivation for this work is to try to answer to the following question:

Does the Kuramoto model behave similarly when one averages with respect to every possibilities of the disorder (*averaged model*) or when one fixes a typical realization of the disorder (*quenched model*)?

The main material of this work consists in three articles, each of them accounting for one chapter of the thesis:

- The first article [55], published in *Electronic Journal of Probability*, studies the issue of the convergence and fluctuations of the empirical measure in a quenched set-up. In particular, it gives an explicit formulation of the fluctuation process in the limit of large number of particles, as a solution to an Ornstein-Uhlenbeck equation in infinite dimension. It corresponds mostly to Chapter 4 (the question of the law of large numbers may also be found in Chapter 2),
- The second [41] (submitted to *Journal of Dynamics and Differential Equations*) is a joint work with Giambattista Giacomin and Christophe Poquet. It addresses mostly the question of both linear and nonlinear stability of synchronization in the continuous version of the Kuramoto model, at least in the case where the disorder is small, along with a number of explicit estimates on the size of the disorder. It is the object of Chapter 3.
- The third [56] (submitted to *Journal of Functional Analysis*) concerns the analysis of the long-time behavior of the fluctuation process found in Chapter 4. The main conclusion is that, at the scale of fluctuations, self-averaging no longer holds: for finite size systems, the fluctuations of the disorder compete with the fluctuations of the particles and make the macroscopic system rotate. This is treated in Chapter 5.

This introduction has two main purposes. The first part gives a brief presentation of the physical background of the Kuramoto synchronization model. Although motivated by the Kuramoto model as it was originally defined [1], this work also applies to more general models of diffusions in random environment. After a brief review of the existing literature on the subject, we introduce the objects of interest that will be studied here, that is the empirical measures of the rotators.

In connection with the Kuramoto model, the notion of synchronization in the limit of large number of particles and its stability will be addressed in this work and will be

[1]. that is with a sine interaction, see (1.1).

related to the nontrivial stationary solutions of a class of nonlinear and nonlocal partial differential equations, the McKean-Vlasov equations.

The second part of this introduction is devoted to a precise statement of the results of the four chapters of this thesis.

2 A short story of the Kuramoto model

The phenomenon of synchronization is omnipresent in physical as well as in biological contexts. Even though it is nearly impossible to properly account for the literature and the models proposed for this phenomenon, we can at least cite several major examples where synchronization plays a significant role: collective behavior of populations of insects (crickets/circadas chirping, fireflies flashing) [18, 69, 94], neuronal networks and epileptic seizures [2], audience clapping [66], arrays of lasers [52], circadian rhythms [92], pacemaker heart cells [61], etc.

All of those phenomena concern large families of coupled oscillators that present a coherent collective behavior. While a precise description of each of the different instances in which synchronization emerges demands specific, possibly very complex, models, the Kuramoto model has emerged as capturing some of the fundamental aspects of synchronization.

We refer to [2, 83] and references therein for further aspects of synchronization and the role of Kuramoto description in those models.

2.1 The model and generalizations

2.1.1 Synchronization of heterogeneous oscillators

The model we study here is the Kuramoto model *including white noise forces* (see [2, § III.]). The principle is the following: we consider a large family of oscillators (or rotators) living in the one dimensional sphere $\mathbf{S} := \mathbf{R}/2\pi\mathbf{Z}$, within a mean-field interaction (see Figure 1.1). This interaction is perturbed by thermal noise, represented by a family of independent and identically distributed Brownian motions.

Moreover, we make the assumption that the rotators are similar but not necessarily identical. Thus, the model captures the fact that each rotator has a tendency to obey to its own natural frequency which may differ from one rotator to another. Those frequencies are chosen at random and independently for each rotator; hence, this supplementary source of randomness will be considered as a *disorder*.

More precisely, we consider for all $N \geqslant 1$, the solutions $(\theta_1, \dots, \theta_N)$ to the following system of coupled stochastic differential equations:

$$\forall i = 1, \dots, N,\ \theta_{i,t} = \xi_i - \frac{K}{N} \sum_{j=1}^{N} \int_0^t \sin(\theta_{i,s} - \theta_{j,s})\, \mathrm{d}s + \omega_i t + \sigma B_{i,t}, \quad (t \in [0, T]) \tag{1.1}$$

where

1. $T > 0$ is a finite (but arbitrary) time horizon,
2. $(B_i)_{i=1,\dots,N}$ is a sequence of N standard independent Brownian motions (which stands for the thermal noise in the system),
3. $\sigma > 0$ is the intensity of the thermal noise,

[2]. We refer in particular here to the thesis of G. Wainrib [93] for a discussion on the link between Kuramoto model and neuronal models.

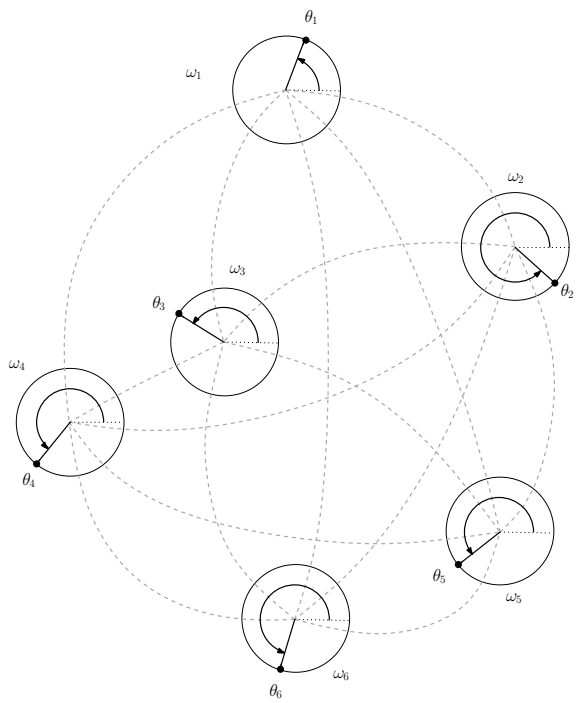

Figure 1.1: A configuration of the Kuramoto model for $N = 6$.

4. $(\omega_i)_{i=1,\dots,N}$ is a family of independent identically distributed real random variables of law μ. In what follows, we will refer to μ as *the law of the disorder*. Each ω_i represents the *intrinsic frequency* for the oscillator θ_i,
5. $K > 0$ is the coupling strength.

Remark 1.1. Since we will not be concerned in this work with the issue of the behavior of the system as $\sigma \searrow 0$, we will assume, with no loss of generality, that $\sigma := 1$ in the following.

Since the θ_i are meant to be angles in $[0, 2\pi)$ (describing the position of each rotator on the circle $\mathbf{S}$) we will rather consider $(\theta_i \mod 2\pi)$ (and with a small abuse of notations, θ_i and $(\theta_i \mod 2\pi)$ will be most of the time identified). We suppose in addition that the rotators are initially independently and identically distributed according to some law $\gamma(\,\mathrm{d}\theta)$ on $\mathbf{S}$:

$$(\xi_i)_{i \geqslant 1} \sim \gamma^{\otimes \mathbf{N}} \tag{1.2}$$

This, with evolution (1.1), defines a proper diffusion on $\mathbf{S}^N$.

Remark 1.2. Note that the first two terms of the right hand side (r.h.s.) of (1.1) are antagonist: the mean-field interaction term $-\frac{K}{N}\sum_{i=1}^N \sin(\cdot - \cdot)$ tends to make the oscillators rotate together, whereas the term of disorder $\omega_j t$ tends to force each oscillator to obey to its own frequency. The intuition for this model (which can be made rigorous at least in simple instances, see Chapter 3) is that it exhibits a phase transition: if K is lower than a critical value K_c, the interaction is not strong enough to ensure synchronization; on the contrary, if $K \geqslant K_c$, the rotators θ_j will tend to synchronize.

2.1.2 Averaged model vs Quenched model

We stress the fact that there are in this model *two* sources of randomness:

1. The randomness that chooses once and for all the initial frequencies $(\omega_1, \dots, \omega_N)$ at the beginning of the experiment,
2. The randomness that comes from the Brownian motions (thermal noise).

In particular, we will denote as $\mathbb{P}$ the law of the sequence of the disorder and as $\mathbf{P} := W^{\otimes \mathbf{N}}$ the law of the sequence of i.i.d. Brownian motions. The corresponding expectations will be denoted as $\mathbb{E}$ and $\mathbf{E}$ respectively. The Lebesgue measure on $\mathbf{S}$ will be denoted as λ.

Since the aim of the thesis is to study the influence of the randomness coming from the disorder on the behavior of (1.1), we will sometimes write $\theta_j^{(\omega)}$ in place of θ_j, so as to specify the dependence of the rotators on the initial choice of the disorder. Here, the notation (ω) stands for the *whole sequence of frequencies*,

$$(\omega) := (\omega_i)_{i \geqslant 1}, \tag{1.3}$$

whereas we will use the notation $\underline{\omega}$ for the finite sequence of N frequencies

$$\underline{\omega} := (\omega_1, \dots, \omega_N). \tag{1.4}$$

Analogous notations will be used for a finite sequence of rotators $\underline{\theta} := (\theta_1, \dots, \theta_N)$. Hence, using the terminology of *disordered models*, one can consider the behavior of the system (1.1) in at least two ways:

1. The *averaged* Kuramoto model: the behavior of the rotators is here considered under the *joint law* of both Brownian motions and disorder; in other words, in this model we average with respect to every typical choice of the disorder.
2. The *quenched* Kuramoto model: here, one *fixes* a typical choice of the sequence (ω) and considers the law of the rotators under the law of the Brownian motions only.

Of course, as far as the biological modeling is concerned, the quenched model is more interesting and this aspect will be the main point of interest in the thesis.

In the more general set-up of disordered models [16], the question of the difference between the quenched behavior and the averaged one is crucial and has received much interest in recent years. Reviewing the vast literature on this subject is hopeless, but one could cite at least the case of the disordered Ising model, pinning models [40], random walks in random environment [96], or random polymers [31].

2.1.3 Symmetries in the Kuramoto model

From (1.1), it is straightforward to notice that the Kuramoto model presents the following symmetries:

- *Rotation invariance*: for fixed (ω), if $\left(\theta_i^{(\omega)}\right)_{1 \leqslant i \leqslant N}$ satisfies (1.1), then for any constant $\theta_0 \in \mathbf{S}$, $\left(\theta_i^{(\omega)} + \theta_0\right)_{1 \leqslant i \leqslant N}$ also satisfies (1.1).
- *Even symmetry*: if one fixes a sequence of disorder (ω), then the law of any solution $\left(\theta_i^{(\omega)}\right)_{1 \leqslant i \leqslant N}$ to (1.1) under the law of the Brownian motions is the same as the law of $\left(-\theta_i^{(-\omega)}\right)_{1 \leqslant i \leqslant N}$.

Note that all theses symmetries find their counterpart in the continuous model (see Remark 1.8, p. 24). In that sense, the reader will not be surprised to encounter in various places of this work functions that share analogous symmetries (in particular functions $(\theta, \omega) \in \mathbf{S} \times \mathbf{R} \mapsto f(\theta, \omega)$ such that $f(\theta, \omega) = f(-\theta, -\omega))$ [3].

Furthermore, an important remark is that one can always assume that the distribution of the disorder μ (which is a probability on $\mathbf{R}$) is centered

$$\mathbb{E}(\omega) := \int_{\mathbf{R}} \omega \, \mathrm{d}\mu(\omega) = 0.$$

In fact, if this is not the case, we can map the model (1.1) to a model with $\mathbb{E}(\omega) = 0$ by putting ourselves on the frame that rotates with speed $\mathbb{E}(\omega)$, that is if we consider the Kuramoto diffusion (1.1) with a drift $(\theta_{i,t} - \mathbb{E}(\omega)t)_{i=1,\dots,N}$.

2.1.4 Generalizations of the Kuramoto model

Numerous generalizations of the Kuramoto model have been proposed and discussed, mostly in the physical literature. It is not possible to review the entirety of the existing articles and we refer to [2] for a global review on the subject.

Previous works: Several works concern the Kuramoto model (1.1) in the case $\sigma = 0$ (as well as in the limit as $\sigma \searrow 0$). In particular, the questions of partial synchronization *vs.* synchronization in the $N \to \infty$-limit for the case $\sigma = 0$ are discussed in [84, 85] as well as in [6] on the basis on numerical simulations. The question of the stability of the incoherent stationary solution is discussed in [87].

In the case where $\sigma > 0$, among the physical literature that exists on the subject, it is worth to mention reference [86] where the question of the stability (as $N \to \infty$) of the incoherent solution is addressed, in the case of a unimodal distribution of the disorder. The dynamical properties of the more complex case of multi-modal distributions is discussed in [13, 1, 14].

One generalization of the Kuramoto model is to see what happens beyond the mean-field case, that is when we add geometry to the system so that the interaction strength K actually depends on the position of the oscillators. This has been also addressed in the physical literature, see [84, 85, 77] for example.

Another way to go beyond the mean-field case is to introduce time-delays in the interaction. We refer to [65, 95, 21] on this subject.

The Kuramoto model, as it is stated in (1.1), has been the subject of several works in the mathematical literature: one should in particular mention the article of Dai Pra and den Hollander [27], who studied averaged large deviations and central limit theorems for this model, as well as the PhD Thesis of Francesca Collet [23] (see also [24]) which tackles the issue of critical fluctuations for the disordered Kuramoto model.

We mention the article of Bertini, Giacomin, Pakdaman [9] as well as [42] which concern the dynamics of the Kuramoto model, in the case where there is no disorder. Another model of synchronization on a lattice in the presence of an additional potential is studied in [7, 8].

[3]. See e.g. Remark 1.8 or Lemma 5.12, p. 145.

The active rotators model: An interesting extension of the Kuramoto model is the *active rotators model* (see e.g. [81, 78]). This model concerns similar interacting rotators in the presence of an additional potential U:

$$\forall i = 1, \dots, N,\ \theta_{i,t} = \xi_i - \frac{K}{N} \sum_{j=1}^{N} \int_0^t \left(\sin(\theta_{i,s} - \theta_{j,s}) + U(\theta_{i,s}, \omega_i)\right) \mathrm{d}s + B_{i,t}, \quad (t \in [0,T]) \tag{1.5}$$

where the potential U is supposed regular and bounded. The dynamical properties for this model tend to be fairly more complicated than the model studied here. In this extent, a recent result [43] tackles the issue of stability of synchronization for the active rotators, in absence of disorder.

2.2 The non-disordered case

Before going into the details of our model, we focus in this paragraph on the case where the disorder is absent, that is $\mu = \delta_0$. This case has been studied in details by Bertini, Giacomin and Pakdaman in [9]. In this case, (1.1) becomes:

$$\forall i = 1, \dots, N,\ \theta_{i,t} = \xi_i - \frac{K}{N} \sum_{j=1}^{N} \int_0^t \sin(\theta_{i,s} - \theta_{j,s}) \, \mathrm{d}s + B_{i,t}, \quad (t \in [0,T]). \tag{1.6}$$

2.2.1 Mean-field interacting diffusions

Here, one can see (1.6) as a particular case of systems of interacting diffusions, which interact through their empirical measure

$$t \mapsto \nu_{N,t} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\theta_{j,t}}. \tag{1.7}$$

Such models of weakly interacting diffusions have extensively been studied in their general formulation

$$\forall i = 1, \dots, N,\ \theta_{i,t} = \xi_i + \int_0^t b[\theta_{i,s}, \nu_{N,s}] \, \mathrm{d}s + \int_0^t \sigma[\theta_{i,s}, \nu_{N,s}] \, \mathrm{d}B_{i,s}, \quad (t \in [0,T]), \tag{1.8}$$

where $b[\cdot,\cdot]$ and $\sigma[\cdot,\cdot]$ are smooth coefficients. A vast literature exists on the subject, we refer in particular to the works of Gärtner [38], Oeslchläger [67], McKean [59] and Sznitman [89]. More precisely, such models of particles are deeply linked with problematics coming from propagation of chaos properties ([48, 88]) or Euler scheme approximations for partial differential equations (e.g. [15, 57, 90, 91]).

In particular, it is now well known (see e.g. [38, 67, 49]) that, under mild hypothesis on the coefficients b and σ, the empirical measure ν_N (1.7) converges weakly (as a process) to a deterministic measure-valued process $t \mapsto \nu_t$ that is absolutely continuous w.r.t. Lebesgue measure and whose density $t \mapsto q_t$ is a classical solution to the McKean-Vlasov equation (where a is the square of σ):

$$\partial_t q_t(\theta) = \frac{1}{2} \partial_\theta^2 \left(a[\theta, \nu_t] q_t(\theta)\right) - \partial_\theta \left(q_t(\theta) b[\theta, \nu_t]\right), \quad (t \in (0,T]). \tag{1.9}$$

Many works have gone beyond this law of large numbers and have studied the corresponding central limit theorem (see e.g. [88, 49, 80, 36]).

2.2.2 The case of the sine-model

The case of the Kuramoto model is a particular case of § 2.2.1 for

$$b[\theta, m] := -K \int_{\mathbf{S}} \sin(\theta - \theta') m(\,\mathrm{d}\theta'), \text{ and } \sigma[\theta, m] := 1.$$

The system (1.6) presents the major advantage of being reversible. Indeed, it is easily seen that the system (1.6) is reversible under the Gibbs measure

$$\mu_{N,K}(\,\mathrm{d}\underline{\theta}) := \frac{1}{Z_{N,K}} \exp\left(-2K H_N(\underline{\theta})\right) \,\mathrm{d}\underline{\theta}, \tag{1.10}$$

where $Z_{N,K}$ is the normalization constant and the Hamiltonian H_N is given by

$$H_N(\underline{\theta}) := -\frac{1}{2N} \sum_{j=1}^{N} \sum_{i=1}^{N} \cos\left(\theta_j - \theta_i\right). \tag{1.11}$$

The measure $\mu_{K,N}$ is the Gibbs measure of a classical statistical mechanics model: the mean field spin XY model with single spin state space $\mathbf{S}$, i.e. mean field plane rotators.

Moreover, it is proved in [9, Prop. 1.2], that (1.6) is more or less the only case [4] where this system is reversible. In particular, if the distribution of the disorder is non-degenerate, the system (1.1) is not reversible and we come into the domain of non-equilibrium statistical mechanics.

As far as the system (1.6) is concerned, the McKean-Vlasov equation (1.9) reduces to (recall that $\sigma = 1$)

$$\partial_t q_t(\theta) \;=\; \frac{1}{2} \partial_\theta^2 q_t(\theta) - \partial_\theta \Big(q_t(\theta)(J * q_t(\theta)) \Big), \tag{1.12}$$

where $J(\theta) = -K \sin(\theta)$ and $*$ is the convolution. An important observation is that (1.12) is of gradient flow type, (e.g. [68] and references therein). These properties have been exploited in [9] in order to extract a number of properties of (1.12) and notably the linear stability of synchronization in the non-disordered case. Those results will be crucial for our work and we will refer to them when required.

Note also that a recent work [42] addresses the question of the global dynamics between incoherent and synchronized solutions in the non-disordered case.

2.3 Our model

Although we will have constantly in mind the disordered Kuramoto model defined in (1.1), a number of results presented in the thesis are also valid for models more general than (1.1). To be precise, we consider the following system of stochastic differential equations:

$$\forall i = 1, \ldots, N, \; \theta_{i,t} = \xi_i + \frac{1}{N} \sum_{j=1}^{N} \int_0^t b(\theta_{i,s}, \theta_{j,s}, \omega_i, \omega_j) \,\mathrm{d}s + \int_0^t c(\theta_{i,s}, \omega_i) \,\mathrm{d}s + B_{i,t}, \; (t \in [0, T]), \tag{1.13}$$

where b and c are regular coefficients. One should see (1.13) as a disordered version of (1.8), and the rotators θ_i as diffusions in $\mathbf{S}$ in a random environment.

[4]. among the class of mean-field models with disorder, where the sine interaction is replaced by a general function $h(\cdot)$; see [9, Prop. 1.2] for a precise statement.

Remark 1.3 (The *sine-model*)**.** In the particular case where

$$b(\theta, \theta', \omega, \omega') = K \sin(\theta' - \theta) \text{ and } c(\theta, \omega) = \omega, \tag{1.14}$$

we retrieve the original Kuramoto model (1.1). By opposition to the general case (1.13), the case (1.1) with a sine interaction will be referred to as *the sine-model*.

2.3.1 The empirical measure and flow

As in the non-disordered case § 2.2.1, it appears that the relevant object for the study of the $N \to \infty$ limit of (1.13) is the empirical measure of the oscillators $\underline{\theta} := (\theta_1, \dots, \theta_N)$.

The empirical measure L_N: if one chooses an time horizon $T > 0$ (fixed but arbitrary), one can consider each diffusion θ_i ($1 \leqslant i \leqslant N$) as a random element in the space $\mathcal{C}([0,T], \mathbf{S})$ of continuous functions from $[0,T]$ to $\mathbf{S}$. For any choice of the disorder $\underline{\omega} := (\omega_1, \dots, \omega_N)$, one may consider the empirical measure on both oscillators and disorder (also referred to in [27] as *double-layer empirical measure*).

Definition 1.4 (Empirical measure)**.** For a fixed choice of rotators $\underline{\theta} = (\theta_1, \dots, \theta_N)$ and disorder $(\omega_1, \dots, \omega_N)$, we define the empirical measure

$$L_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{(\theta_j, \omega_j)}, \tag{1.15}$$

where $\delta_{(\theta,\omega)}$ is the Dirac measure in $(\theta, \omega) \in \mathcal{C}([0,T], \mathbf{S}) \times \mathbf{R}$.

Thus, L_N is a random element in the set $\mathcal{M}_1(\mathcal{C}([0,T], \mathbf{S}) \times \mathbf{R})$ of probability measures on $\mathcal{C}([0,T], \mathbf{S}) \times \mathbf{R}$. When necessary, the notations $L_N^{(\omega)}$ and $L_N[\underline{\theta}, \underline{\omega}]$ will also be used in order to specify the random dependence of L_N in the sequence of disorder $(\omega) := (\omega_i)_{i \geqslant 1}$ or in the oscillators $\underline{\theta}$.

The empirical flow ν_N: Another relevant object for the study of the large N limit of the system is the empirical flow derived from the empirical measure L_N through the continuous map $Q \in \mathcal{M}_1(\mathcal{C}([0,T], \mathbf{S}) \times \mathbf{R}) \mapsto (Q_t)_{t \in [0,T]} \in \mathcal{C}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$ where for all $t \in [0,T]$, $Q_t \in \mathcal{M}_1(\mathbf{S} \times \mathbf{R})$ is the marginal of Q at time t:

$$Q_t(A \times B) := Q((\theta, \omega), \theta_t \in A, \omega \in B).$$

The image of the empirical measure L_N is then given by the following definition

Definition 1.5 (Empirical flow)**.** For a fixed choice of rotators $\underline{\theta} = (\theta_1, \dots, \theta_N)$ and disorder $(\omega_1, \dots, \omega_N)$, we define the empirical flow as the family of probabilities on $\mathbf{S} \times \mathbf{R}$ $(\nu_{N,t})_{t \in [0,T], N \geqslant 1}$:

$$\forall t \in [0,T], \ \nu_{N,t} := \frac{1}{N} \sum_{j=1}^{N} \delta_{(\theta_{j,t}, \omega_j)}, \tag{1.16}$$

where $\delta_{(\theta,\omega)}$ is the Dirac measure in $(\theta, \omega) \in \mathbf{S} \times \mathbf{R}$.

Here, ν_N will be considered as a random process in $\mathcal{C}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$ when $\mathcal{M}_1(\mathbf{S} \times \mathbf{R})$ is endowed with the topology of weak convergence. Again, we will make use when required of the notation $\nu_N^{(\omega)}$ so as to insist on the fact that ν_N effectively depends on the random choice of the sequence of disorder $(\omega) = (\omega_i)_{i \geqslant 1}$.

3 Large deviations and convergences of the empirical measures

The purpose of Chapter 2 is to establish rigorously a law of large numbers for the empirical measures L_N and ν_N (analogous to the convergence discussed in the non-disordered case § 2.2.1) and to make explicit the corresponding limit as a solution to a McKean-Vlasov evolution, similar to the one found in (1.9).

More concretely, we are concerned here with the issue of the convergence as $N \to \infty$ of the random objects L_N and ν_N defined in (1.15) and (1.16) respectively, to a deterministic time-dependent measure $(\nu_t(\,\mathrm{d}\theta,\,\mathrm{d}\omega))_{t\in[0,T]}$, weak solution of the deterministic nonlinear partial differential McKean-Vlasov equation:

$$\int_{\mathbf{S}\times\mathbf{R}} \varphi \,\mathrm{d}\nu_t = \int_{\mathbf{S}\times\mathbf{R}} \varphi \,\mathrm{d}\theta\,\mathrm{d}\gamma + \int_0^t \mathrm{d}s \int_{\mathbf{S}\times\mathbf{R}} \left(\frac{1}{2}\varphi'' + \varphi'(b[\cdot,\cdot,\nu_s] + c)\right) \mathrm{d}\nu_s, \tag{1.17}$$

for every $(\theta,\omega) \mapsto \varphi(\theta,\omega)$ continuous bounded on $\mathbf{S}\times\mathbf{R}$, twice differentiable w.r.t. the first variable θ, with bounded derivatives and where

$$b[\theta,\omega,m] = \int b(\theta,\theta',\omega,\omega')\,\mathrm{d}m(\theta',\omega').$$

Note here that the notation φ' stands for the derivative of φ w.r.t. the first variable θ. This convergence has been heuristically obtained in the physical literature, see [2] and references therein. To our knowledge, the first rigorous proof of this convergence has been given by Dai Pra and den Hollander in [27], in the case where the interaction is governed by an Hamiltonian. The main restriction in [27] is that it only concerns the case where the disorder remains bounded. The techniques involved in [27] come from large deviation theory in the *averaged model*. In particular, [27] does not tackle the issue of the *quenched* convergence of the empirical measures.

The purpose of Chapter 2 is threefold:

3.1 Uniqueness in McKean-Vlasov equation (1.17)

The first important result of Chapter 2 concerns the plausible limit of the empirical measures defined in (1.17). Proposition 2.20 in Chapter 2 effectively proves that there is uniqueness of any possible solution to the weak formulation of McKean-Vlasov equation (1.17). This can be done by generalizing a work by Oelschläger [67] who proved similar uniqueness for McKean-Vlasov equation, for interacting diffusions without disorder.

Since we know that any possible limit point of L_N or ν_N is in fact unique, we are now left with proving the existence of convergent subsequences for both $(L_N)_{N\geqslant 1}$ and $(\nu_N)_{N\geqslant 1}$. We follow here two strategies:

3.2 Averaged large deviation principle:

The first part of Chapter 2 concerns a generalization of the result of Dai Pra and den Hollander to the case where the disorder is no longer bounded. Namely, we prove a large deviation principle for the empirical measure L_N in the case where the interaction is given by a random Hamiltonian under the assumption that the distribution of the disorder μ admits exponential moments.

3.2.1 Main hypothesis

We give ourselves two functions $(u,\omega,\omega') \mapsto f(u,\omega,\omega')$ and $(\theta,\omega) \mapsto g(\theta,\omega)$ such that
- f is of class $\mathcal{C}^2$ w.r.t. its first variable u and is bounded (uniformly in ω,ω') on $\mathbf{S}\times\mathbf{R}^2$ as well as its derivatives,
- for all $u \in \mathbf{S}$, $(\omega,\omega') \in \mathbf{R}^2$, $f(u,\omega,\omega') = f(-u,\omega',\omega)$,
- g is of class $\mathcal{C}^2$ w.r.t. its first variable θ such that $\partial_\theta^2 g$ is uniformly bounded in (θ,ω).

We suppose now that the coefficients b and c in (1.13) have the following form:

$$b(\theta,\theta',\omega,\omega') = -\partial_u f(\theta'-\theta,\omega,\omega'),$$
$$c(\theta,\omega) = -\partial_\theta g(\theta,\omega).$$

This situation corresponds exactly to the case where the interaction (1.13) is governed by a random Hamiltonian:

$$\forall i = 1,\dots,N, \quad \mathrm{d}\theta_{i,t} = -\partial_{\theta_i} H_N(\underline{\theta},\underline{\omega})\,\mathrm{d}t + \mathrm{d}B_{i,t},$$

where

$$H_N(\underline{\theta},\underline{\omega}) := \frac{1}{2N}\sum_{i,j=1}^{N} f(\theta_j-\theta_i,\omega_i,\omega_j) + \sum_{i=1}^{N} g(\theta_i,\omega_i).$$

Remark 1.6. Note that, contrarily to [27], we do not suppose any uniform boundedness in ω for g and $\partial_\theta g$, since we have in mind the *sine-model* where $g(\theta,\omega) = -\theta\omega$.

Additionally, we make the following hypothesis on the distribution of the disorder:

$$\forall t > 0, \int_{\mathbf{R}} e^{t|\omega|}\,\mathrm{d}\mu(\omega) < \infty.$$

3.2.2 Averaged convergence of L_N

Under these hypotheses, the first main result of Chapter 2 is an averaged large deviation principle for the empirical measure L_N:

Theorem (see Theorem 2.11)**.** *The law of the empirical measure L_N under the joint law of oscillators and disorder, satisfies a strong large deviation principle in $\mathcal{M}_1(\mathcal{C}([0,T],\mathbf{S})\times\mathbf{R})$, at speed N, for the topology of weak convergence, governed by a good rate function $Q \mapsto \mathcal{G}(Q)$, whose zeros $Q^*(\,\mathrm{d}\theta,\,\mathrm{d}\omega)$ satisfy the weak McKean-Vlasov equation* (1.17)*. In particular, L_N converges weakly to the unique solution ν to* (1.17)*.*

The proof of Theorem 2.11 uses similar arguments to the ones used in [27], in particular standard techniques from large deviation theory, including Varadhan's lemma and Sanov's theorem, which are recalled at the beginning of Chapter 2. The major difference with [27] is that we do not suppose that disorder remains bounded. In particular, bounding the disorder entails some technical complications and requires the use of techniques derived from the notion of *exponential approximations* of measures, developed in [29, p. 131].

Once again, we stress the fact that this is an *averaged* result, in the sense that the convergence of L_N is understood with respect to the joint law of Brownian motions and disorder. It is nonetheless possible to derive from the previous theorem a *quenched* convergence result using Borel-Cantelli techniques. We refer to Proposition 2.13 for a precise statement.

3.3 Quenched convergence of the empirical flow

More importantly than the fact that the previous result is stated in the framework of the *averaged model*, the hypothesis made about the law of the disorder is quite restrictive (existence of exponential moments of μ). One could hope to prove a more direct convergence result which would be true in the *quenched* model as well as under less restrictive assumptions on the law of the disorder. Note also that the following result gets rid of the assumption that the interaction is governed by an Hamiltonian.

The rest of Chapter 2 is dedicated to the statement of a direct proof of convergence of the empirical flow ν_N defined in (1.16); we insist on the fact that the convergence is here considered when the sequence of disorder is fixed (*quenched model*).

3.3.1 Main hypothesis

Let us consider (1.13) in the case where the coefficients b and c satisfy the following hypotheses:

- $(\theta, \theta', \omega, \omega') \mapsto b(\theta, \theta', \omega, \omega')$ is bounded, Lipschitz-continuous,
- $(\theta, \omega) \mapsto c(\theta, \omega)$ is Lipschitz-continuous w.r.t. its first variable, but not necessarily bounded [5],
- $\omega \mapsto \sup_{\theta \in \mathbf{S}} |c(\theta, \omega)|$ is continuous.

The Lipschitz bounds for b and c are supposed to be uniform in ω.

The disorder (ω) is assumed to be a sequence of identically distributed random variables (but not necessarily independent), such that the law of each ω_i is μ. We suppose also that for $\mathbb{P}$-almost every sequence (ω),

$$\frac{1}{N} \sum_{i=1}^{N} \sup_{\theta \in \mathbf{S}} |c(\theta, \omega_i)| \to_{N \to \infty} \int \sup_{\theta \in \mathbf{S}} |c(\theta, \omega)| \, \mathrm{d}\mu(\omega) < \infty.$$

We suppose also a convergence of the initial empirical measure, for the topology of weak convergence:

$$\text{For } \mathbb{P} - a.e. \ (\omega), \ \nu_{N,0}^{(\omega)} \overset{N \to \infty}{\longrightarrow} \nu_0, \quad \textit{in law, in } \mathcal{M}_1(\mathbf{S} \times \mathbf{R}).$$

3.3.2 Quenched convergence of the empirical flow

Under the previous hypotheses, we prove in Chapter 2 the following result

Theorem (see Theorem 2.16)**.** *For $\mathbb{P}$-almost every sequence (ω), the random variable $\nu_N^{(\omega)}$ defined in* (1.16) *is tight in the space $\mathbf{D}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$ of càdlàg functions in $\mathcal{M}_1(\mathbf{S} \times \mathbf{R})$, and each accumulation point ν is the solution of the weak McKean-Vlasov equation* (1.17)*. Moreover, under the same hypotheses, the law of ν_N under the joint law of the oscillators and disorder (averaged model) is also tight in the same space and its accumulations points are also solutions to* (1.17)*.*

In particular, we have both quenched and averaged convergence of $(\nu_N)_{N \geqslant 1}$ towards the unique solution ν to (1.17)*.*

[5]. recall the *sine-model*, where $c(\theta, \omega) = \omega$.

3.3.3 Self-averaging phenomenon

This result illustrates in particular the *self-averaging* character of the diffusion (1.13) which we study here. Due to the mean-field character of the system, its large N-limit (at least at the scale of the law of large numbers) is deterministic and independent of any initial choice of the disorder (ω).

A similar question for the dependence in the disorder at the scale of the central limit theorem associated to this convergence seems much harder to treat and will be the main subject of discussion of Sections 5 and 6.

4 Stability of synchronization in the continuous model

The main conclusion of Chapter 2 is that McKean-Vlasov evolution (1.17) is the proper object for the study the large N-limit of the system (1.13).

The purpose of Chapter 3 is to study more precisely this nonlinear equation and to make the link between the phenomenon of synchronization and the existence of nontrivial solutions to (1.17) in the case of the *sine-model*.

The main material of Chapter 3 is taken from a joint work [41] with Giambattista Giacomin and Christophe Poquet.

4.1 Regularity of the McKean-Vlasov semigroup

The first result of Chapter 3 concerns regularity properties of evolution (1.17) and in particular the regularizing character of its semigroup.

In that sense, in addition to the assumptions made in § 3.3, we suppose that

- for all ω, ω', $(\theta, \theta') \mapsto b(\theta, \theta', \omega, \omega')$ is of class $\mathcal{C}^\infty$ with bounded derivatives,
- for all ω, $\theta \mapsto c(\theta, \omega)$ is of class $\mathcal{C}^\infty$, with derivatives uniformly bounded on every $\mathbf{S} \times [-M, M]$, for all $M > 0$.

More precisely, the following proposition states that that for any measure-valued initial condition, the solution $t \mapsto \nu_t$ of (1.17) is in fact regular in time and space and that its density $q_t(\theta, \omega)$ (w.r.t. to the Lebesgue measure) is the strong solution of a system of coupled nonlinear and nonlocal partial differential equations which will be referred to as the *strong formulation* of McKean-Vlasov evolution:

Proposition (see Proposition 3.1)**.** *For every initial condition $\nu_0(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = \nu_0^\omega(\,\mathrm{d}\theta)\mu(\,\mathrm{d}\omega)$ on $\mathbf{S} \times \mathbf{R}$, for all $T > 0$, the unique solution ν to* (1.17) *in $\mathcal{C}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$ has the following properties: for all $t > 0$, ν_t is absolutely continuous with respect to $\lambda \otimes \mu$ and for μ-a.e. $\omega \in Supp(\mu)$, its density $(t, \theta, \omega) \mapsto q_t(\theta, \omega)$ is strictly positive on $(0, T] \times \mathbf{S}$, is $\mathcal{C}^\infty$ in (t, θ) and solves the following Fokker-Planck equation* (1.18)*:*

$$\forall \omega \in \mathrm{Supp}(\mu),\ \forall \theta \in \mathbf{S},$$
$$\partial_t q_t(\theta, \omega) = \frac{1}{2}\partial_\theta^2 q_t(\theta, \omega) - \partial_\theta \left\{ q_t(\theta, \omega) \left(\int_{\mathbf{S}\times\mathbf{R}} b(\theta, \theta', \omega, \omega') q_t(\theta', \omega')\,\mathrm{d}\theta'\,\mathrm{d}\mu(\omega') + c(\theta, \omega) \right) \right\}. \tag{1.18}$$

Remark 1.7. Note that it is also possible to state a result of regularity with respect to the disorder (see Proposition 3.5), provided we assume some initial regularity of the initial condition [6].

[6]. Since the evolution (1.18) is degenerate with respect to the component ω, there is no point in hoping that the evolution will be regularizing in ω: we need to assume that the initial condition is itself regular in ω.

Let us be explicit on the main properties of (1.18):

1. It has quadratic nonlinearity,
2. It is nonlocal,
3. If the support of μ is infinite (e.g. $\mu = \mathcal{N}(0,1)$), (1.18) is in fact equivalent to an infinite number of coupled equations; indeed, in order to know $q(\cdot,\omega)$ for one $\omega \in \mathrm{Supp}(\mu)$, it is necessary to know $q(\cdot,\omega')$ for every $\omega' \in \mathrm{Supp}(\mu)$.

To fix ideas, one may for example consider the simple binary case where $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. Then (1.18) reduces to two equations (one for $+1$, the other for -1) which are coupled via the averaged probability measure $\frac{1}{2}(q_t(\theta,+1) + q_t(\theta,-1))\,\mathrm{d}\theta$. But for more general situations ($\mu = \mathcal{N}(0,1)$ say) this would consist of an infinite number of coupled equations.

The case of the sine-model: In the particular case of the *sine-model* case, (1.18) becomes

$$\partial_t q_t(\theta,\omega) \;=\; \frac{1}{2}\partial_\theta^2 q_t(\theta,\omega) - \partial_\theta\Big(q_t(\theta,\omega)(\langle J * q_t\rangle_\mu(\theta) + \omega)\Big), \tag{1.19}$$

where

$$\forall\theta \in \mathbf{S}, \quad J(\theta) = -K\sin(\theta),$$

and where $*$ denotes the convolution and $\langle\cdot\rangle_\mu$ is a notation for the integration with respect to μ, so that $\langle J * u\rangle_\mu(\theta) = \int_{\mathbf{R}}\int_{\mathbf{S}} J(\theta')u(\theta-\theta',\omega)\,\mathrm{d}\theta'\,\mathrm{d}\mu(\omega)$ is the convolution of J and u, averaged with respect to the disorder.

Remark 1.8 (Symmetries in the continuous model)**.** Note that the symmetries mentioned in § 2.1.3 also holds for the continuous model:

- *Rotation invariance:* if $q_t(\theta,\omega)$ solves (1.19) so does $q_t(\cdot+\theta_0,\omega)$ for any constant $\theta_0 \in \mathbf{S}$,
- *Even symmetry:* if $q_t(-\theta,-\omega)|_{t=0} = q_t(\theta,\omega)|_{t=0}$, then it is true for all $t > 0$.

In particular, the stationary solutions of (1.19) will share these symmetries, see (3.15), p. 77.

4.2 Phase transition in synchronization

The rest of Chapter 3 concerns the particular case of the *sine-model*.

The first step towards making rigorous mathematical sense of the meaning of *synchronization* in the Kuramoto model was carried out by Sakaguchi [75] (see also [2], [30] and references therein). It is now well understood that crucial features of evolution (3.1) are captured in the *sine-model* by order parameters $r_t \geqslant 0$ and $\psi_t \in \mathbf{S}$ defined by:

$$r_t e^{i\psi_t} = \int_{\mathbf{S}\times\mathbf{R}} e^{i\theta} q_t(\theta,\omega)\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega). \tag{1.20}$$

The quantity r_t captures in fact the degree of synchronization of a solution (the profile $q_t \equiv \frac{1}{2\pi}$ for example corresponds to $r_t = 0$ and represents a total lack of synchronization) and ψ_t identifies the center of synchronization: this is true and rather intuitive for unimodal profiles. In this framework, it turns out that synchronization reads in the existence of nontrivial stationary solutions q to the evolution (1.19), which can be parameterized by the (stationary) order parameter r defined by the stationary version of (1.20). What makes the computations particularly tractable in the *sine-model* is that the nonlinearity in (1.19) can be explicitly written in terms of the order parameter r (at least when μ is symmetric): if $b(\theta,\theta',\omega,\omega') = K\sin(\theta'-\theta)$, then $\int b(\theta,\theta',\omega,\omega')\,\mathrm{d}\nu_s(\theta',\omega')$ is exactly equal

to $-Kr_s \sin(\theta)$. This allows to express any stationary solution q of (1.19) in terms of the order parameter r, which itself depends on q through (1.20). Thus, one has to see (1.20) as a *fixed-point relation* for which any solution r provides a stationary solution [7] $q(\cdot, \cdot, r)$ to (1.19). We refer to § 3.2, p. 92 for details.

In that sense, the continuous Kuramoto evolution (1.19) is the prototype of a dynamics that exhibits a *phase transition*: at an intuitive level, if the interaction strength K is not large enough, then the rotators do not interact sufficiently, $r = 0$ is the only solution to (1.20) and the system is not synchronized, whereas if K is large enough, nontrivial solutions appear (synchronization). The easiest illustration of such a phase transition is given by the *non-disordered Kuramoto model* (1.12):

Proposition (see [75] and Proposition 3.8)**.** *The non-disordered* sine-model *presents a phase transition: if $K > 0$ is the coupling strength in the* sine-model*, then one can distinguish between two regimes:*

1. *The case $K \leqslant K_c := 1$: there is a unique stationary solution to* (1.12) *called* incoherent solution
$$q(\cdot) \equiv \frac{1}{2\pi}. \tag{1.21}$$

2. *The case $K > K_c$: the incoherent solution $\frac{1}{2\pi}$ coexists with a whole circle of* synchronized solutions
$$\{q(\cdot + \theta_0);\, \theta_0 \in \mathbf{S}\}\,, \tag{1.22}$$
for q given by
$$\forall \theta \in \mathbf{S}, \quad q_0(\theta) = \frac{1}{Z_0} e^{2Kr_0 \cos(\theta)},$$
where $Z_0 = \int_{\mathbf{S}} e^{2Kr_0 \cos(\theta)}\, \mathrm{d}\theta$ is the normalization constant and r_0 satisfies (1.20)*.*

For nontrivial disorder, an analogous parameterization holds (see Proposition 3.10) although the phase portrait gets much more complicated even for simples instances of disorder (in particular for some distributions μ, two distinct circles of synchronized solutions may coexist, see Proposition 3.11) and many questions about the nature of the phase transition remain unfortunately unanswered. Nevertheless, one result of Chapter 3 shows that the same phase transition occurs, at least in the case where the disorder is small. We refer in particular to Proposition 3.13 for a precise statement. The simplest illustration of the difficulty of solving the fixed-point relation (1.20) is the binary case $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$. We detail (although not rigorously) in § 2.2.3, p. 79 the variety of what may happen even in this simple case.

4.3 Stability of synchronization

The main goal of Chapter 3 is to tackle the issue of the stability of nontrivial stationary solutions to McKean-Vlasov evolution (1.19), in the case of a non-degenerate disorder.

As far as the incoherent solution $\frac{1}{2\pi}$ is concerned, it has been proved by Strogatz and Mirollo [86] that it is linearly stable before synchronization and unstable after. We refer to Proposition 3.22 for a precise statement.

[7]. and hence a whole circle of stationary thanks to the rotation invariance.

4.3.1 The non-disordered case

The purpose of the work by Bertini, Giacomin and Pakdaman [9] is precisely to answer the question of the linear stability of the synchronized solution in the non-disordered model (1.12). Namely, if L_{q_0} is the linear operator resulting from the linearization of evolution (1.12) around its unique [8] synchronized stationary solution q_0,

$$L_{q_0}u := \frac{1}{2}\partial_\theta^2 u - \partial_\theta \left\{ q_0(J*u) + u(J*q_0) \right\}, \tag{1.23}$$

then it is proved in [9] that L_{q_0} is essentially self-adjoint in a certain space $\mathbf{H}$ of distributions [9], which spectrum lies in the negative part of the real line. What is more, explicit estimates on the spectral gap $\lambda_K(L_{q_0})$ are proved (see Proposition 3.23 for a precise statement).

4.3.2 The case where the disorder is small

The main result of Chapter 3 is contained in Theorem 3.27 and is a disordered version of the result of [9]. In the case where the disorder (seen as a perturbation) is small and using techniques from functional analysis (perturbations of self-adjoint operators and analytic semigroups), Theorem 3.27 states that in a disordered version $\mathbf{H}_\mu$ of the space $\mathbf{H}$ used in § 4.3.1 and for a sufficiently small disorder, there exists a unique circle of synchronized stationary solutions to (1.19) and this circle is both linearly and nonlinearly stable. More precisely, we make the assumption that the support of μ is small:

$$\exists \delta > 0,\ \mathrm{Supp}(\mu) \subseteq [-\delta, \delta].$$

The object of interest here is the linear operator L_q^δ arising from the linearization of the evolution (1.19) around its unique [8] nontrivial stationary solution q (see Proposition 3.13):

$$L_q^\delta u(\theta,\omega) := \frac{1}{2}\partial_\theta^2 u(\theta,\omega) - \partial_\theta \left(u(\theta,\omega) \left(\langle J*q\rangle_\mu(\theta) + \omega \right) + q(\theta,\omega)\langle J*u\rangle_\mu(\theta) \right), \tag{1.24}$$

with domain

$$\mathcal{D} := \left\{ u \in C^2(\mathbf{S}\times\mathbf{R});\ \int_{\mathbf{S}} u(\theta,\omega)\,\mathrm{d}\theta = 0 \ \text{ for all } \omega \right\}. \tag{1.25}$$

Then one has

Theorem (see Theorem 3.27)**.** *The operator L_q^δ has the following spectral properties: 0 is a simple eigenvalue for L_q^δ, with eigenspace spanned by $(\theta,\omega)\mapsto q'(\theta,\omega)$. Moreover, for all $K>1$, $\rho\in(0,1)$, $\alpha\in(0,\pi/2)$, there exists $\delta_\star = \delta_\star(K,\rho,\alpha)$ such that for all $0\leqslant\delta\leqslant\delta_\star$, the following is true:*

- *L_q^δ is closable in $\mathbf{H}_\mu$;*
- *The spectrum of L_q^δ lies in a cone C_α with vertex 0 and angle α*

$$C_\alpha := \left\{ \lambda\in\mathbb{C};\ \frac{\pi}{2}+\alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2}-\alpha \right\} \subseteq \{ z\in\mathbf{C};\ \Re(z)\leqslant 0\}\ ;$$

- *There exists $\alpha'\in(0,\frac{\pi}{2})$ such that L_q^δ is the infinitesimal generator of an analytic semigroup defined on a sector $\{\lambda\in\mathbb{C},\ |\arg(\lambda)|<\alpha'\}$;*

[8]. Up to rotation.
[9]. For a precise definition of the space $\mathbf{H}$, see § 2.3, p. 82.

– *The distance between* 0 *and the rest of the spectrum is strictly positive and is at least equal to* $\rho\lambda_K(L_{q_0})$*, where* $\lambda_K(L_{q_0})$ *is the spectral gap found in the non-disordered case in § 4.3.1.*

Moreover, we are able to find asymptotic estimates (in terms of K) of the size of the disorder $\delta_\star$ as $K \searrow 1$ and $K \to \infty$ (see Proposition 3.28). In particular, those estimates seemingly sharp as $K \searrow 1$, are not accurate as $K \to \infty$, although one would intuitively believe that the stronger the coupling, the easier it is for the oscillators to remain synchronized. In that sense, the study of the $K \to \infty$-case seems to require alternative methods to the ones developed in this thesis.

Note that a different approach developed by Giacomin, Pakdaman, Pellegrin and Poquet [43] allows to tackle this issue of stability of synchronization and can be generalized to the case where μ is not symmetric, as well as for more general models (e.g. active rotators, see [76]). This approach relies on the notion of *stable normal hyperbolic manifold* (see [79]) and the fact that this structure is robust under perturbations. We will only mention this result (§ 2.4.4, p. 89) and we refer to [41] for precise statements and proofs.

5 Quenched fluctuations of the empirical measure

The purpose of Chapters 2 and 3 was to study the continuous Kuramoto model that arises in the $N \to \infty$-limit. We focus in the next chapter on the finite size effects on synchronization of the Kuramoto model.

The material for Chapter 4 is taken from an article [55] published in *Electronic Journal of Probability.*

5.1 Motivations

As already pointed out in Paragraph 3.3.3 of this introduction, a crucial aspect of the quenched convergence result, which is a law of large numbers, is that it shows the *self-averaging* character of this limit: every typical disorder configuration leads to the same deterministic evolution (1.18) as $N \to \infty$.

However, it seems quite clear even at a superficial level that if we consider the central limit theorem associated to this convergence, self-averaging does not hold since the fluctuations of the disorder compete with the dynamical fluctuations. This leads to a remarkable phenomenon (pointed out e.g. in [6, § 10.2, p. 47] on the basis of numerical simulations): even if the distribution μ is symmetric, the fluctuations of a fixed chosen sample of the disorder makes it *not symmetric* and thus the center of the synchronization of the system *slowly* (i.e. with a speed of order $1/\sqrt{N}$) rotates in one direction and with a speed that depends randomly on the sample of the disorder (Fig. 1.2 and 1.3).

To fix ideas one may think of the simple case where $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$: in a finite sample $(\omega_1, \dots, \omega_N)$ of spins in $\{\pm 1\}^N$, there may be a majority of positive spins (equal to $+1$) with respect to the negative spins (equal to -1), so that the positive spins have a tendency to induce a global rotation in their sense.

This non self-averaging phenomenon can be tackled in the *sine-model* by computing the finite-size order parameters (Fig. 1.3):

$$r_{N,t} e^{i\psi_{N,t}} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_{j,t}} = \int_{\mathbf{S}\times\mathbf{R}} e^{i\theta} \, \mathrm{d}\nu_{N,t}(\theta, \omega), \tag{1.26}$$

where ν_N is the empirical measure of the system defined in (1.16).

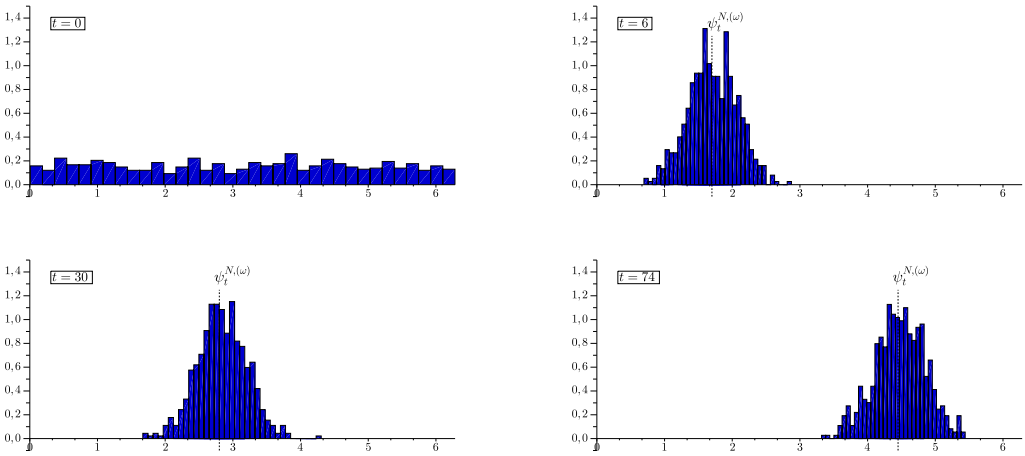

Figure 1.2: We plot here the evolution of the marginal on $\mathbf{S}$ of $\nu_N^{(\omega)}$ for $N = 600$ oscillators in the *sine-model* ($\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, $K = 6$). The oscillators are initially chosen independently and uniformly on $[0, 2\pi]$ independently of the disorder. First the dynamics leads to synchronization of the oscillators ($t = 6$) to a profile which is close to a nontrivial stationary solution of McKean-Vlasov equation. Secondly, we observe that the center $\psi_{N,t}^{(\omega)}$ of this density moves to the right with an approximately constant speed; what is more, this speed of fluctuation turns out to be sample-dependent (see Fig. 1.3).

Remark 1.9. Note that the parameters $(r_{N,t}, \psi_{N,t})$ are the microscopic equivalents of the macroscopic order parameters (r_t, ψ_t) defined in (1.20). In this chapter, we will also use the notation $\left(r_{N,t}^{(\omega)}, \psi_{N,t}^{(\omega)}\right)$ in order to specify the dependence of these objects in the quenched disorder (ω).

In particular, the order parameter $\psi_{N,t}^{(\omega)}$ captures the position of the center of synchronization for finite N (see Figure 1.2). It is seen on Figure 1.3 that $t \mapsto \psi_{N,t}^{(\omega)}$ has an approximately linear behavior whose slope depends on the choice of the disorder.

Since the order parameters $(r_{N,t}, \psi_{N,t})$ are actually functions of the empirical flow ν_N (1.16), it suffices to have a fluctuation result for the empirical flow ν_N in order to obtain a fluctuation result for $r_{N,t}$ and $\psi_{N,t}$. As a step towards understanding this non self-averaging phenomenon, the goal of Chapter 4 is to establish a fluctuation theorem for ν_N around the McKean-Vlasov limit in a quenched set-up (see Theorem 4.4) and to discuss the influence of the disorder on the longtime dynamics of the limit fluctuation process.

5.2 Previous results

Results of fluctuations for the empirical measure of diffusions in $\mathbf{R}$ or in $\mathbf{S}$ have been proved in different contexts. In particular, there exists a vast literature on the subject for diffusions in absence of disorder. We refer in particular to the works of Fernandez and Méléard [36] and Hitsuda and Mitoma [47], who studied fluctuations for empirical measures in the case without disorder. Note also that a large deviation principle is also proved in [19].

As far as disordered models are concerned, the study of fluctuations of the empirical measure in an *averaged set-up* has been the subject of several works. In particular, a

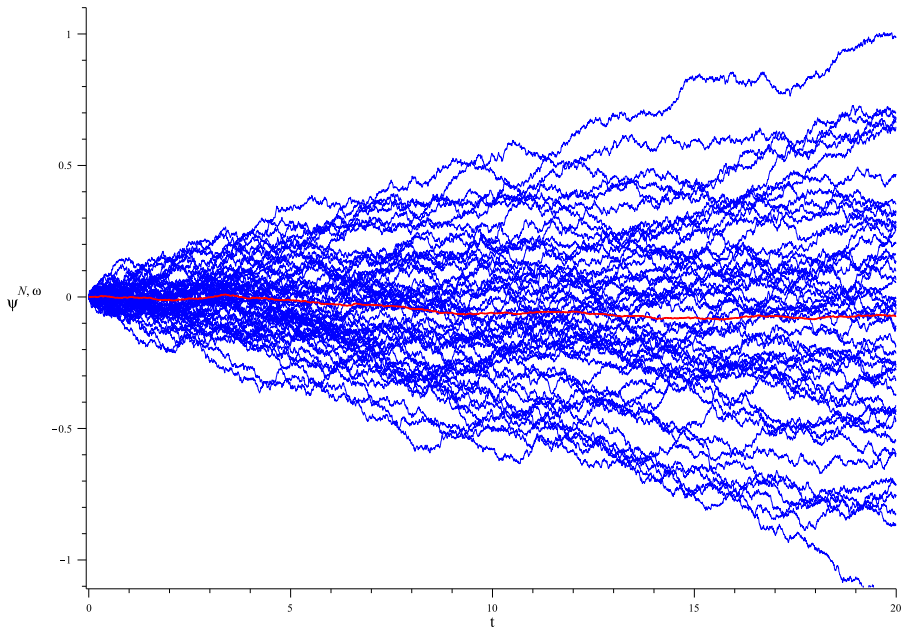

Figure 1.3: Trajectories of the center of synchronization $\psi_N^{(\omega)}$ in the *sine-model* for different realizations of the disorder ($\mu = \frac{1}{2}(\delta_{-0.5} + \delta_{0.5})$, $K = 4$, $N = 400$). We observe here the non self-averaging phenomenon: direction and speed of the center depend on the choice of the initial N-sample of the disorder. Moreover, these simulations are compatible with speeds of order $1/\sqrt{N}$. The red trajectory corresponds to the case where we average w.r.t. the disorder: non self-averaging does not hold in the averaged model.

central limit theorem for the empirical measure in the averaged Kuramoto model can be found in [27], applying techniques introduced by Bolthausen [12]. A fluctuation theorem may also be found in [25] for a model of social interaction here again in an averaged set-up. We also mention a recent work [24] that concerns the behavior of the fluctuations in the Kuramoto model near criticality.

But the point is that an *averaged* fluctuation result is not relevant for our purpose. Indeed, if we average with respect to every typical possibilities of the disorder, then *on average*, there are as many positive initial frequencies as negative frequencies. In other words, (see the red trajectory in Figure 1.3) since we get rid of the initial asymmetry between the positive frequencies w.r.t. the negative ones, the self-averaging phenomenon disappears: there is no more sample-dependent rotation of the whole system in the averaged situation. What we need here is really a *quenched* fluctuation result, that is for a fixed choice of a sample of initial frequencies $(\omega_1, \dots, \omega_N)$ [10].

The result we prove concerns the behavior as $N \to \infty$ of the *quenched* fluctuation process of the empirical flow ν_N (1.16) around its limit ν (1.17)

$$\eta_N^{(\omega)} := \sqrt{N}\left(\nu_N^{(\omega)} - \nu\right).$$

More precisely, what we prove is the quenched convergence of η_N, seen as a continuous process in the Schwartz space $\mathcal{S}'$ of tempered distributions on $\mathbf{S} \times \mathbf{R}$ to the solution $t \mapsto \eta_t$ of an Ornstein-Uhlenbeck process (1.27). We insist on the fact that the limit process η

[10]. which is *a priori* asymmetric in general, at least when we do not artificially remove the asymmetry by imposing for each sampled frequency its exact opposite, see Remark 1.11, p. 34 and [6, § 10.2, p. 47].

is explicit, in the sense that the linear operator governing its evolution is explicitly given and deterministic. The quenched convergence is here understood as a weak convergence *in law w.r.t. the disorder* and is more technically involved than the convergence in the averaged system.

5.2.1 Main hypotheses

In addition to the hypothesis made in § 3.3, we make the following assumptions about b and c (where $\mathcal{D}_p$ is the set of all differential operators of the form $\partial_u^k \partial_\omega^l$ with $k+l \leqslant p$):

$$\begin{cases} b \in \mathcal{C}_b^\infty(\mathbf{S}\times\mathbf{R}), \quad c \in \mathcal{C}^\infty(\mathbf{S}\times\mathbf{R}), \\ \exists \alpha > 0,\ \sup\limits_{D\in\mathcal{D}_6} \displaystyle\int_{\mathbf{R}} \frac{\sup_{\theta\in\mathbf{S}} |Dc(\theta,\omega)|^2}{1+|\omega|^{2\alpha}}\,\mathrm{d}\omega < \infty, \end{cases}$$

Furthermore, we make the following assumption about the law of the disorder (α is defined as above):

$$\text{the } (\omega_j) \text{ are i.i.d. and} \quad \int_{\mathbf{R}} |\omega|^{4\alpha}\,\mathrm{d}\mu(\omega) < \infty.$$

5.2.2 Quenched fluctuations of the empirical flow

We are now in position to state the main result: if we wanted to mimic the quenched convergence result in § 3.3.2, we would say that for almost every choice of the sequence of disorder $(\omega) \in \mathrm{Supp}(\mu)^{\mathbf{N}}$, the process $\eta_N^{(\omega)}$ converges in law to some process η. This is not true and we have to introduce a weaker notion of quenched convergence. More precisely, for fixed (ω), we may consider $\mathcal{H}_N^{(\omega)}$, the law of the process $\eta_N^{(\omega)}$; $\mathcal{H}_N^{(\omega)}$ belongs to $\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))$, where $\mathcal{S}'$ is the usual Schwartz space of tempered distributions on $\mathbf{S}\times\mathbf{R}$. We are here interested in the convergence of the random variable $(\omega) \mapsto \mathcal{H}_N^{(\omega)}$ in the set $\mathcal{M}_1(\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}')))$:

Theorem (see Theorem 4.4)**.** *The sequence $(\omega) \mapsto \mathcal{H}_N^{(\omega)}$ converges in law in the set $\mathcal{M}_1(\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}')))$ to the random variable $\omega \mapsto \mathcal{H}^\omega \in \mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))$, where for all ω, $\mathcal{H}^\omega$ is the law of the process η^ω solution in $\mathcal{S}'$ of the following Ornstein-Uhlenbeck equation:*

$$\eta_t^\omega = X^\omega + \int_0^t \mathcal{L}_s^* \eta_s^\omega\,\mathrm{d}s + W_t, \tag{1.27}$$

where, $\mathcal{L}_s^*$ is an explicit second order linear differential operator and for all fixed ω, X^ω is a non-centered Gaussian process [11] *with explicit covariance and with nontrivial mean value $C(\omega)$. As a random variable in ω, $\omega \mapsto C(\omega)$ is a Gaussian process* [11] *with explicit covariance. Moreover, W is independent on the initial value X.*

This weak notion of convergence (in law, *in law with respect to the disorder*) is due to the fact that we work in a quenched model and thus, we do not integrate w.r.t. the disorder.

Remark 1.10. Note that a similar *averaged* fluctuation theorem also holds (see Theorem 4.7): in particular, we retrieve the averaged central limit theorem found for the empirical measure found in [27, Th. 4, p. 744].

[11]. indexed by functions φ on $\mathbf{S}\times\mathbf{R}$.

An important remark is that in the evolution (1.27), the linear operator $\mathcal{L}_s^*$ is deterministic; the only dependence in ω lies in the initial condition X^ω, through its nontrivial means $C(\omega)$. As we will see in Chapter 4 (see also § 6.5.2 in this introduction), the Gaussian process $C(\cdot)$ is a random object that captures in law the initial asymmetry of the disorder (see (1.36)).

Understanding how the deterministic operator $\mathcal{L}_s^*$ propagates the initial dependence in ω on the whole trajectory is the object of Chapter 5. In that sense, it requires a precise understanding of the spectral properties of $\mathcal{L}_s^*$, which appears to be deeply linked to the linear operator in McKean-Vlasov equation (1.18) linearized around its nontrivial stationary solution.

The rest of Chapter 4 is to devoted to the derivation of quenched fluctuation results for the order parameters in the *sine-model* $r_{N,t}$ and $\psi_{N,t}$ defined in (1.26) (see Propositions 4.9 and 4.11).

6 Non self-averaging phenomenon at the scale of fluctuations

The purpose of Chapter 5 is to understand the asymptotic behavior of the fluctuation process η that we found in Chapter 4.

6.1 Non self-averaging for the limit fluctuation process

A natural question is the following: does the self-averaging phenomenon noticed in the microscopic system also hold for the limit fluctuation process η defined in (1.27)? At least on the basis on numerical simulations (see Figure 1.4), the answer is positive: numerical computations of the trajectories of the limit process of fluctuations $t \mapsto \eta_t$ clearly show a non self-averaging phenomenon analogous to the one observed in Figure 1.3: η_t^ω not only depends on ω through its initial condition X^ω, but also for all positive time $t > 0$.

The fact that we are able to find an explicit formulation for the limit process η in terms of a solution of an Ornstein-Uhlenbeck equation in infinite dimension (Theorem 4.4) allows to have a good understanding of the dynamical properties of η_t as $t \to \infty$. The study of the non self-averaging phenomenon for the process η_t as $t \to \infty$ in the *sine-model* is deeply linked with the spectral properties of the operator $\mathcal{L}_t^*$ which happens to coincide with the linearized Kuramoto operator L_{q_t} around the solution $t \mapsto q_t$ to (1.19):

$$\forall \omega \in \mathrm{Supp}(\mu), \forall \theta \in \mathbf{S},$$
$$\mathcal{L}_t^* h(\theta,\omega) = L_{q_t} h(\theta,\omega) = \frac{1}{2}\partial_\theta^2 h(\theta,\omega) - \partial_\theta \left(h(\theta,\omega) \left(\langle J * q_t \rangle_\mu(\theta) + \omega \right) + q_t(\theta,\omega) \langle J * h \rangle_\mu(\theta) \right). \tag{1.28}$$

The spectral analysis of L_{q_t} enables to study the dependence of $t \mapsto \eta_t$ on the mean value $C(\omega)$ of its initial condition η_0^ω. It is worth to note that $C(\omega)$ precisely captures the fluctuations of the initial disorder (see § 6.5.2). Thus, the key point of this chapter is to understand how different initial conditions in evolution (1.27) may lead to distinct approximately linear trajectories of the fluctuation process. Hence, one has to see (1.27) as a continuous time evolution that models in law the behavior of the fluctuations of the system as $N \to \infty$, where the initial fluctuations of the disorder lies in the non-trivial mean value of its initial condition.

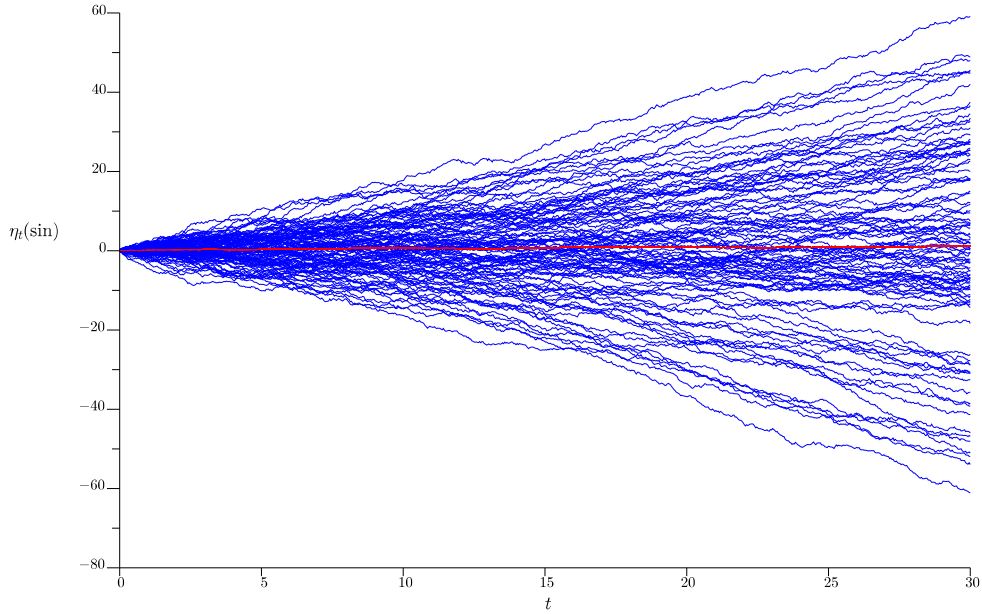

Figure 1.4: We plot here the evolution of the process $\eta_t(\sin)$, for different realizations of ω; the trajectories are sample-dependent and compatible with the ones observed in Fig 1.3.

6.2 Main hypothesis

For the rest of this chapter, we consider the case where

$$\mu = \frac{1}{2}\left(\delta_{-\omega_0} + \delta_{\omega_0}\right), \tag{1.29}$$

where $\omega_0 > 0$ is a fixed parameter. This assumption appears to be quite restrictive, but the generalization of the results presented here to other distributions μ does not seem straightforward.

We also restrict ourselves to the stationary case, that is where $q|_{t=0} = q_t$ is equal for all $t > 0$ to the synchronized (non-trivial) stationary solution q of evolution (1.19). In this case, the object of interest is the stationary version of (1.28), that is

$$Lh(\theta,\omega) := \frac{1}{2}\partial_\theta^2 h(\theta,\omega) - \partial_\theta\left(h(\theta,\omega)\left(\langle J * q\rangle_\mu(\theta) + \omega\right) + q(\theta,\omega)\langle J * h\rangle_\mu(\theta)\right). \tag{1.30}$$

The domain $\mathcal{D}$ of the operator L is here given by:

$$\mathcal{D} := \left\{h(\theta,\omega);\ \forall\omega, \theta \mapsto h(\theta,\omega) \in \mathcal{C}^2(\mathbf{S}),\ \int_{\mathbf{S}\times\mathbf{R}} h(\theta,\omega)\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega) = 0\right\}. \tag{1.31}$$

We have already encountered the operator L in the different context of the stability on synchronization in Section 4 of this introduction. Note that the domain (1.31) chosen here differs from the one (1.25) chosen in Section 4. Indeed, the key point is to understand that evolution (1.27) does not live in domain (1.25) since the fluctuations of the disorder are precisely captured by the fact that η has a non-trivial mean-value $C(\omega)$ for fixed ω.

This choice of domain entails technical complications but the main ideas remain the same as the ones used in Chapter 3: the general theory of perturbations of self-adjoint operators ([51]) and analytic semigroup of operators ([70]) as well as usual techniques about SPDEs in Hilbert spaces ([26]). We make use here of the precise knowledge we have about the same linear operator L_{q_0} in the case without disorder (see [9]).

6.3 Non self-averaging phenomenon for the operator L and existence of a Jordan block

The linear trajectories that depend on the initial condition as observed in Figure 1.4 are reminiscent of an analogous deterministic finite-dimensional example: let us consider the 2-dimensional evolution $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = L \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, for $L = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then it is trivial to see that the solutions of this system are linear in time: $\frac{x(t)}{t} \to y_0$ as $t \to \infty$. This can be easily generalized for matrices L $n \times n$ with spectrum in $\{\lambda \in \mathbf{C};\ \Re(\lambda) \leqslant 0\}$ such that the Jordan decomposition of L admits a $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$-block; this is precisely equivalent to the existence of x and y such that $Lx = 0$ and $Ly = x$.

The purpose of the first theorem of Chapter 5 is to prove an analogous existence of a Jordan block for the operator L:

Theorem (see Theorem 5.7)**.** *For any fixed $\omega_0 > 0$, if q is the stationary solution in* (1.19)*, then*

$$Lq' = 0. \tag{1.32}$$

Moreover, there exists $p \in \mathcal{D}$ such that

$$\forall \theta \in \mathbf{S}, \forall \omega \in \mathrm{Supp}(\mu), \qquad Lp(\theta, \omega) = q'(\theta, \omega). \tag{1.33}$$

In particular, the characteristic space of L in 0 *is* at least *of dimension* 2*.*

Theorem 5.7 is based on a priori estimates on the Dirichlet form associated to the operator L and an extension of Lax-Milgram Theorem.

6.4 Spectral properties of L and position of the spectrum

The second goal of Chapter 5 is to prove that, at least for small disorder, L generates an analytic semi-group of operators for an appropriate Sobolev norm, with spectrum confined in the part of the complex plane with negative real part:

Theorem (see Theorem 5.9)**.** *There exists a space H of distributions such that the operator $(L, \mathcal{D})$ is densely defined, closable, its closed extension having compact resolvent. In particular, its spectrum consists of isolated eigenvalues with finite multiplicities.*

Moreover, for all $K > 1$, for all $\alpha \in (0, \frac{\pi}{2})$, there exists $\omega_\star = \omega_\star(K, \alpha) > 0$ such that, for all $0 < \omega_0 < \omega_\star$, the following is true:

- *The spectrum of L lies in a cone C_α with vertex* 0 *and angle α*

$$C_\alpha := \left\{ \lambda \in \mathbb{C};\ \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha \right\} \subseteq \{z \in \mathbf{C};\ \Re(z) \leqslant 0\}\ ;$$

- *There exists $\alpha' \in (0, \frac{\pi}{2})$ such that L is the infinitesimal generator of an analytic semi-group defined on a sector $\Delta_{\alpha'} := \{\lambda \in \mathbb{C},\ |\arg(\lambda)| < \alpha'\}$;*
- *the dimension of the characteristic space in* 0 *is exactly* 2*, spanned by q' and p, where p is defined in Theorem 5.7,*
- *the eigenvalue* 0 *is at a positive distance from the rest of the spectrum.*

6.5 Long time evolution of fluctuations SPDE

6.5.1 Non self-averaging for the fluctuation process

Putting Theorem 5.7 and Theorem 5.9 together, it is easy to see that evolution (1.27) presents a linear behavior, at least where the noise term W is absent. Using standard techniques of analysis of stochastic partial differential equations ([26]), it is possible to prove that adding the disorder does not change much to the linear behavior of the fluctuation process.

Recall in particular Theorem 4.4 in § 5.2.2: the initial condition η_0^ω in (1.27) has a nontrivial mean-value $C(\omega)$; as a function of ω, C is a Gaussian process with explicit covariance.

Theorem (see Theorem 5.3)**.** *Under the hypothesis of Theorem 5.9, there exists a unique weak solution η_t to* (1.27) *in H. Moreover, η satisfies the following asymptotic linear behavior: for each realization $C(\omega)$ of the mean value of the initial condition η_0^ω*

$$\frac{\eta_t^\omega}{t} \xrightarrow[t\to\infty]{\text{in law}} v(\omega)q'. \tag{1.34}$$

Moreover, as a function of ω, $\omega \mapsto v(\omega)$ is a real random variable with Gaussian distribution and with variance

$$\sigma_v^2 := \left(2\int_{\mathbf{S}} p_+(\theta)\,\mathrm{d}\theta\right)^{-2}, \tag{1.35}$$

where $p_+(\theta) := p(\theta, \omega_0)$ is defined by (1.33)*.*

6.5.2 Initial asymmetry of the disorder

Let us give an intuition of the random variable v: as seen in Chapter 5, v may be seen as $\int_{\mathbf{S}} C_+$ where C_+ is the limit in law of the microscopic process $C_{N,+}$ (indexed by functions $\varphi : \mathbf{S} \to \mathbf{R}$) defined by

$$\forall\varphi,\ C_{N,+}(\varphi) := \left(\int_{\mathbf{S}} \varphi(\cdot)\,\mathrm{d}\gamma\right)\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\mathbb{1}_{(\omega_i=\omega_0)} - \frac{1}{2}\right) := \left(\int_{\mathbf{S}} \varphi(\cdot)\,\mathrm{d}\gamma\right)\frac{\alpha_N}{\sqrt{N}}. \tag{1.36}$$

Here, α_N is exactly the (centered) number of frequencies among $(\omega_1, \dots, \omega_N)$ that are positive, so that $C_{N,+}$ captures the lack of symmetry of the initial chosen disorder: if $\alpha_N = 0$, there are as many positive frequencies as negative frequencies, whereas the case $\alpha_N > 0$ (resp. $\alpha_N < 0$) represents the case of an asymmetry in favor of positive (resp. negative) frequencies.

In the case of i.i.d. frequencies such as in the hypotheses of Theorem 5.3, one easily retrieves the variance σ_v^2 defined in (5.10). Note that we believe that $\frac{1}{\int_{\mathbf{S}} p_+}$ is in fact equal to $-\omega_0$; this fact is derived from non-rigorous computations and verified by numerical simulations.

Remark 1.11. In [6, § 10.2, p. 47], Balmforth and Sassi found on the basis of numerical experiments that if we get rid artificially of the asymmetry between positive frequencies and negative frequencies, the disorder-induced rotation disappears. More precisely, the simulations carried out in [6] are ruled by the following principle: consider a sample of $2N$ rotators ($N \geqslant 1$) and pick up at random the frequencies ω_i for exactly N rotators. For each of those N frequencies ω_i, attribute to one the remaining N rotators the exact

opposite frequency $-\omega_i$. For this choice of frequencies, there is no rotation, no matter how the first N frequencies are sampled.

We stress the fact that we retrieve this phenomenon in Theorem 5.3 in the case where the first N rotators are i.i.d. samples of $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$, since in that case α_{2N} is equally zero for all $N \geqslant 1$ and the consequent limit speed v is equally zero in this case.

6.6 Perspectives

6.6.1 Possible generalizations

Moreover, one could hope to generalize the previous results in at least two directions. Firstly, we restrict ourselves to the binary case $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$. Note that the proof of Theorem 5.7 concerning the existence of a Jordan block (although written in this particular case for the reader's convenience) is not specific to this restrictive case: one could easily rewrite the same proof in the case of more general distributions μ (even with unbounded support), satisfying appropriate integrability conditions in 0 and in ∞.

The main restriction on μ concerns Theorem 5.9: the hypothesis $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$ is critical for its proof. Secondly, there are some deeper considerations which may prove possible generalizations of Theorem 5.9 to be difficult: indeed, the key argument of the proof is based on the fact that perturbing a finite dimensional kernel of an operator A by a sufficiently small perturbation B leads to a kernel for the operator $A + B$ with the same finite dimension. But for distributions more general than binary distributions $\frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$, the kernel of L is likely to become of infinite dimension, so that those perturbation arguments cannot be applied.

Secondly, Theorem 5.9 is only proved for small disorder ω_0 whereas one would expect it to be true even for large disorder. It is indeed natural to believe that the non self-averaging phenomenon not only holds for large disorder but would even be more noticeable in that case. However, since Theorem 5.9 relies on perturbation arguments, to prove similar results for large ω_0 would seem to require alternative methods.

6.6.2 Other possible time scales

More importantly, Theorem 5.3 gives an expression of the speed at which the limit of the fluctuation process η rotates. Since in this thesis, we are only considering time scales of order $O(1)$ (that is for arbitrary but fixed time horizon T), this may not be the right speed of rotation for the finite-size system: on larger time-scales (or order $O(\sqrt{N})$ say), there may be accumulations of fluctuations which may lead to a different expression of the speed of rotation. Hence, to understand this phenomenon properly, an interesting result would be the study of the rescaled empirical process $(\nu_{\sqrt{N}t})_{t,N}$, as N and t are large. This has not been carried out in the thesis and would be a natural perspective for this work.

Chapter 2

Convergences of the empirical measure

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1 Introduction

1.1 The Kuramoto model

In this chapter, we address the issue of the behavior as $N \to \infty$ of the empirical measure L_N and flow ν_N defined in (1.15) and (1.16) respectively, in the general case

where the interaction (rewritten here for the ease of exposition) is given by (1.13):

$$\forall i = 1, \dots, N,\ \theta_{i,t} = \xi_i + \frac{1}{N}\sum_{j=1}^{N}\int_0^t b(\theta_{i,s}, \theta_{j,s}, \omega_i, \omega_j)\,\mathrm{d}s + \int_0^t c(\theta_{i,s}, \omega_i)\,\mathrm{d}s + B_{i,t},\ (t \in [0,T]), \tag{2.1}$$

where the initial conditions ξ^i are independent and identically distributed with law γ, and independent of the Brownian motion $(B) = (B_i)_{i \geqslant 1}$, and where b (resp. c) is a smooth function, 2π-periodic w.r.t. the two first (resp. first) variables. The disorder $(\omega) = (\omega_i)_{i \geqslant 1}$ is a realization of i.i.d. random variables with law μ.

Note in particular that the results of this chapter are not specific to the *sine-model* (1.1). All that follows will be carried out for general interaction functions b and c, although we will have constantly in mind the *sine-model* (see in particular Remark 2.8 and 2.14). Note also that the dependence of the results stated here on the fact that the diffusion (2.1) live on the *compact* space $\mathbf{S}^N$ is marginal and could also be stated in the non-compact case of diffusions in $\mathbf{R}^d$ for some $d \geqslant 1$ (see in particular Remarks 2.17 and 2.25).

1.2 Results and plan of the chapter

The main purpose of this chapter is to show the convergence, as $N \to \infty$, (heuristically obtained in the physical literature, see [2] and references therein) of the empirical measure L_N and the empirical flow ν_N under certain assumption on the distribution of the disorder μ to a time-dependent measure $(\,\mathrm{d}\nu_t(\theta,\omega))_{t\in[0,T]}$, whose density w.r.t. Lebesgue measure at time t, $q_t(\theta,\omega)$ is the solution of a system of deterministic coupled nonlinear partial differential McKean-Vlasov equation (recall (1.17)).

To that purpose, we follow here two strategies:

1.2.1 Averaged large deviation principle:

The first aim of this chapter is to prove a large deviation principle (LDP) for the empirical measure L_N and the empirical flow ν_N in the context of the *averaged model* (see § 2.3). Theorem 2.11 concerns a LDP for L_N in the case where the interaction is given by a random Hamiltonian under the assumption that the distribution of the disorder μ admits exponential moments. The proof of Theorem 2.11, (exposed in Section 3) relies on usual techniques from large deviation theory, whose main results are recalled in § 2.1.

The case where the distribution of the disorder is of compact support is derived from the work of Dai Pra and den Hollander, [27] who proved an analogous large deviation principle for diffusions in $\mathbf{R}$ in the case of a bounded Hamiltonian. We present in Section 3.2 the techniques involved in the study of the compact support; the fact that we work in $\mathbf{S}$ instead in $\mathbf{R}$ entails some technical differences [1] from the work of Dai Pra and den Hollander but the main ideas remain the same. The key argument of the proof is the comparison between the i.i.d. case where the interaction term has been removed and the case with the whole interaction, through the calculation of a Radon-Nykodym derivative (§ 3.1). Thus, one can see the law of the empirical measure as an exponential transformation of the i.i.d. case without interaction (§ 3.2) and Sanov's theorem and Varadhan's lemma leads to conclusion.

But, when the disorder is not bounded, the functional involved in Varadhan's lemma is not bounded nor even well-defined. This case requires a specific proof (see § 3.3). The idea here is to apply the results of § 3.2 to the distribution $\mu(\cdot|[-M,M])$, that is the

[1]. See in particular Remark 2.25 in Section 3.

law μ conditioned on an arbitrary compact interval $[-M, M]$, and then make $M \to \infty$. The techniques used here are derived from the notion of *exponential approximations* of measures, developed in [29].

1.3 Quenched convergence of the empirical flow

One of the drawbacks of the previous results is that they are stated in the framework of the *averaged model*, (although one can derive from the large deviation estimates a *quenched convergence* result, using Borel-Cantelli techniques, see Proposition 2.13). More importantly, one could hope to prove a similar convergence result, under less restrictive assumptions on the law of the disorder than the existence of exponentials moments for μ, required for the proof of Theorem 2.11.

The rest of the chapter is dedicated to state a direct proof of convergence (Section 4) of the empirical flow: the purpose of Theorem 2.16 is to show the convergence of the sequence $(\nu_N)_{N \geqslant 1}$, under weaker assumption on μ ($\int_{\mathbf{R}} |\omega| \, \mathrm{d}\mu(\omega) < \infty$), via a tightness argument in the space $\mathbf{D}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$ of càdlàg functions in $\mathcal{M}_1(\mathbf{S} \times \mathbf{R})$. We insist on the fact that the convergence is here considered when the sequence of disorder is fixed (*quenched model*).

As already mentioned in the introduction, this result illustrates in particular the *self-averaging* character of the diffusions we study here. Due to the mean-field character of the system, its large N-limit (at least at the scale of the law of large numbers) is independent of any typical choice of the initial choice of the disorder (ω). We will go back to this phenomenon when we will come to the study of the fluctuations around this deterministic limit in Chapter 4.

1.4 McKean-Vlasov equation

In the large deviation set-up as well as in the quenched convergence result, the main conclusion is that both L_N and ν_N concentrates around the solution of a deterministic system of partial differential equations (McKean-Vlasov equation (2.11)). We conclude this chapter by stating a result (Proposition 2.20) of uniqueness of any possible solution to this system, which effectively proves the convergence of L_N or ν_N to this solution.

The McKean-Vlasov equation (2.11) will be our main object of interest in Chapter 3.

2 Setting and notations

2.1 Preliminaries on large deviation theory

We recall here the main definitions and results on large deviation theory that will be of constant use for the proof of the large deviation principle of the empirical measure L_N (Theorem 2.11) which is the first main result of this chapter. We refer to classical references on the subject, e.g. [20] or [29].

Let $\mathcal{X}$ be a regular topological space ([29, § 4.1, p. 116]) endowed with a regular σ-field $\mathcal{B}$.

2.1.1 Large deviation principle

Definition 2.1 (Rate function, [29], § 1.2, p. 4)**.** A rate function I is a lower semi-continuous mapping $I : \mathcal{X} \to [0, \infty]$ such that the level set $\Psi_I(\alpha) := \{x \in \mathcal{X};\, I(x) \leqslant \alpha\}$ is

closed for all $\alpha \in [0, \infty)$. A *good* rate function is a rate function for which all the level sets $\Psi_I(\alpha)$ are compact subset of $\mathcal{X}$.

Let $(\rho_N)_{N \geqslant 1}$ be a sequence of probability measures on $(\mathcal{X}, \mathcal{B})$.

Definition 2.2 (Large deviation principle)**.** The sequence $(\rho_N)_{N \geqslant 1}$ satisfies the large deviation principle in $\mathcal{X}$, at speed N, with rate function I if for all $A \in \mathcal{B}$,

$$-\inf_{x \in \mathring{A}} I(x) \leqslant \liminf_{N \to \infty} \frac{1}{N} \ln \rho_N(A) \leqslant \limsup_{N \to \infty} \frac{1}{N} \ln \rho_N(A) \leqslant -\inf_{x \in \bar{A}} I(x).$$

Proposition 2.3 (Contraction principle, [29], Th. 4.2.1, p. 126)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be regular topological spaces, $f : \mathcal{X} \to \mathcal{Y}$ a continuous function and $I : \mathcal{X} \to [0, \infty]$ a good rate function. For each $y \in \mathcal{Y}$ define*

$$I'(y) := \inf \{I(x);\, x \in \mathcal{X},\ y = f(x)\}.$$

Then I' is a good rate function on $\mathcal{Y}$ and if a sequence of probabilities (ρ_N) on $\mathcal{X}$ satisfies a LDP for the good rate function I, then the sequence of probabilities $(\rho_N \circ f^{-1})$ satisfies a LDP for the good rate function I'.

Proposition 2.4 (Varadhan's lemma, [29], Th. 4.3.1, p. 137)**.** *If $(\rho_N)_{N \geqslant 1}$ satisfies a large deviation principle with speed N, governed by the good rate function I, then for any bounded continuous function $F : \mathcal{X} \to \mathbf{R}$, we have*

$$\lim_{N \to \infty} \frac{1}{N} \ln \int_{\mathcal{X}} e^{NF(x)} \, \mathrm{d}\rho_N(x) = \sup_{x \in X} (F(x) - I(x)).$$

Proposition 2.5 ([29], p. 140)**.** *If $(\rho_N)_{N \geqslant 1}$ satisfies a large deviation principle with speed N, governed by the good rate function I, and $F : \mathcal{X} \to \mathbf{R}$ is a bounded continuous function, then the sequence $(\tau_N)_{N \geqslant 1}$ defined by:*

$$\forall A \in \mathcal{B}, \tau_N(A) := \frac{\int_A e^{NF} \, \mathrm{d}\rho_N}{\int_{\mathcal{X}} e^{NF} \, \mathrm{d}\rho_N},$$

satisfies a large deviation principle with speed N, governed by the good rate J:

$$J : x \mapsto I(x) - F(x) - \inf_{y \in X} (I(y) - F(y)).$$

2.1.2 Sanov's Theorem

Let Σ a Polish space and endow $\mathcal{X} := \mathcal{M}_1(\Sigma)$ with the τ-topology (that is the coarsest topology on $\mathcal{M}_1(\Sigma)$ such that the evaluations $\nu \in \mathcal{M}_1(\Sigma) \mapsto \int \varphi \, \mathrm{d}\nu$ are continuous, φ being measurable and bounded) and with the cylinder σ-field $\mathcal{B}^*$ (that is the smallest σ-field that makes the previous evaluations measurable).

Define also on $\mathcal{X}$ the relative entropy of two elements ν, $\tilde{\nu}$:

Definition 2.6. We define the *relative entropy* of $\nu \in \mathcal{M}_1(\Sigma)$ with respect to $\tilde{\nu} \in \mathcal{M}_1(\Sigma)$ by

$$\mathcal{H}(\nu|\tilde{\nu}) := \begin{cases} \int_\Sigma \frac{\mathrm{d}\nu}{\mathrm{d}\tilde{\nu}} \ln \left(\frac{\mathrm{d}\nu}{\mathrm{d}\tilde{\nu}} \right) \mathrm{d}\tilde{\nu} & \text{if } \nu \ll \tilde{\nu}, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

where in the case where ν is absolutely continuous w.r.t. $\tilde{\nu}$ ($\nu \ll \tilde{\nu}$), $\frac{\mathrm{d}\nu}{\mathrm{d}\tilde{\nu}}$ stands for the Radon-Nykodym derivative of ν w.r.t. $\tilde{\nu}$.

We are in position to state Sanov's Theorem concerning empirical measures of i.i.d. random variables in Σ:

Proposition 2.7 (Sanov's theorem, [29], Th.6.2.10, p. 263)**.** *If $(Y_i)_{i \geqslant 1}$ is an i.i.d. sequence in Σ with law p_Y, then the empirical measures $L_N^Y := \frac{1}{N}\sum_{i=1}^N \delta_{Y_i} \in \mathcal{M}_1(\Sigma)$ satisfy a large deviation principle in $\mathcal{M}_1(\Sigma)$ in the τ-topology (and also in the weak topology) at speed N, for the good rate function $\mathcal{H}(\cdot|p_Y)$.*

2.2 Notations

Let Y Polish space, with a regular σ-field $\mathcal{B}$. In this chapter, a constant use will be made of the following notations:

- $\mathcal{M}_1(Y)$: the set of probability measures on Y,
- $(\mathcal{M}_1(Y), \tau)$: $\mathcal{M}_1(Y)$ endowed with the τ-topology, namely the coarsest topology on $\mathcal{M}_1(Y)$ such that the evaluations $\mu \mapsto \int \varphi \, \mathrm{d}\mu$ are continuous, φ being measurable and bounded,
- $(\mathcal{M}_1(Y), w)$: $\mathcal{M}_1(Y)$ endowed with the topology of weak convergence, namely the coarsest topology on $\mathcal{M}_1(Y)$ such that the evaluations $\mu \mapsto \int \varphi \, \mathrm{d}\mu$ are continuous, where φ are bounded continuous,
- $(\mathcal{M}_1(Y), v)$: $\mathcal{M}_1(Y)$ endowed with the topology of vague convergence, namely the coarsest topology on $\mathcal{M}_1(Y)$ such that the evaluations $\mu \mapsto \int \varphi \, \mathrm{d}\mu$ are continuous, where φ are continuous with compact support.

Note that we have the following comparison of the previous topologies

$$v \subseteq w \subseteq \tau.$$

We will also denote the set of finite measures on Y by $\mathcal{M}_F(Y)$. What is more, we will use the following:

- if X is a metric space, $\mathcal{B}_X$ will be its Borel σ-field,
- $\mathcal{C}_b(X)$ (resp. $\mathcal{C}_b^p(X)$, $p = 1, \dots, \infty$), the set of bounded continuous functions (resp. bounded continuous with bounded continuous derivatives up to order p) on X, (X will be often $\mathbf{S} \times \mathbf{R}$),
- $\mathcal{C}_c(X)$ (resp. $\mathcal{C}_c^p(X)$, $p = 1, \dots, \infty$), the set of continuous functions with compact support (resp. continuous with compact support with continuous derivatives up to order p) on X,
- $\mathbf{D}([0,T], X)$, the set of right-continuous with left limits functions with values on X, endowed with the Skorokhod topology,

Throughout this chapter, Σ and Σ_N will be the shorthands for

$$\Sigma := \mathcal{C}([0,T], \mathbf{S}) \times \mathbf{R},$$
$$\Sigma_N := \mathcal{C}([0,T], \mathbf{S}^N) \times \mathbf{R}^N.$$

We will use C as a constant which may change from a line to another.

2.3 Averaged large deviation principle in the case of Hamiltonian interaction

The main goal of this part is to state a large deviation principle for the empirical measure L_N defined in (1.15) in the case where the Kuramoto interaction is given by a random

Hamiltonian. Namely, we define, for $\underline{\theta} = (\theta_1, \dots, \theta_N) \in \mathbf{S}^N$, and $\underline{\omega} = (\omega_1, \dots, \omega_N) \in \mathbf{R}^N$,

$$H_N(\underline{\theta}, \underline{\omega}) := \frac{1}{2N} \sum_{i,j=1}^{N} f(\theta_j - \theta_i, \omega_i, \omega_j) + \sum_{i=1}^{N} g(\theta_i, \omega_i), \tag{2.3}$$

$$:= H_N^f(\underline{\theta}, \underline{\omega}) + H_N^g(\underline{\theta}, \underline{\omega}). \tag{2.4}$$

where f and g satisfy the following assumptions:

- $(\theta, \omega, \omega') \mapsto f(\theta, \omega, \omega')$ is a bounded continuous function on $\mathbf{S} \times \mathbf{R}^2$, twice differentiable w.r.t. the first component θ with bounded derivatives (with bounds uniform in ω, ω'),
- $(\theta, \omega) \mapsto g(\theta, \omega)$ is a continuous function on $\mathbf{R} \times \mathbf{R}$, twice differentiable w.r.t. the first component, such that $(\theta, \omega) \mapsto g''(\theta, \omega)$ is bounded on $\mathbf{R} \times \mathbf{R}$,
- Even symmetry of f:

$$\forall (\theta, \omega, \omega') \in \mathbf{S} \times \mathbf{R}^2, \quad f(\theta, \omega, \omega') = f(-\theta, \omega', \omega). \tag{2.5}$$

The hypothesis made on μ for the large deviation result (Theorem 2.11) is the following:

$$\forall t > 0, \ \int_{\mathbf{R}} e^{t|\omega|} \, \mathrm{d}\mu(\omega) < \infty. \tag{2.6}$$

In that case, the Kuramoto dynamics is given by the system of N stochastic differential equations:

$$\forall i = 1, \dots, N, \quad \mathrm{d}\theta_{i,t} = -\partial_{\theta_i} H_N(\underline{\theta}, \underline{\omega}) \, \mathrm{d}t + \mathrm{d}B_{i,t}, \tag{2.7}$$

Remark 2.8. Note that the *sine-model* (1.1) corresponds to the choice of $f(\theta, \omega, \omega') = -K \cos(\theta)$ and $g(\theta, \omega) = -\theta\omega$. In particular, it is important to notice that we do not assume that g is bounded *uniformly in* ω, since the *sine-model* precisely concerns a function g that is not uniformly bounded for general distribution of the disorder.

Remark 2.9. Although the natural space in where to consider rotators in the Kuramoto model is $\mathbf{S}$ (or $\mathbf{S}^N$), the solution to (2.7) lives, strictly speaking, in $\mathbf{R}^N$. In particular, we point out the fact that in the *sine-model*, one cannot make sense of $g(\theta, \omega) = -\theta\omega$ as an element of $\mathcal{C}(\mathbf{S} \times \mathbf{R})$. To avoid such artificial difficulties, it will be more convenient to treat firstly the diffusion (2.7) as an element of $\mathbf{R}^N$ and then project on $\mathbf{S}^N$. But the results presented here are valid in the compact case $\mathbf{S}$ as well as in the case $\mathbf{R}$.

Remark 2.10. Note that the symmetry assumption (2.5) is not restrictive at all since in the general case, one can always replace an arbitrary function f by $(\theta, \omega, \omega') \mapsto \frac{1}{2}\left(f(\theta, \omega, \omega') + f(-\theta, \omega', \omega)\right)$ without changing the Hamiltonian (2.3).

The aim of Section 3 is to prove a large deviation principle for the empirical measure L_N (recall that $\Sigma = \mathcal{C}([0, T], \mathbf{S}) \times \mathbf{R}$):

Theorem 2.11 (Averaged LDP)**.** *Under* (2.6)*, the law of the empirical measure L_N under the joint law of oscillators and disorder, satisfies a strong large deviation principle in $\mathcal{M}_1(\Sigma)$, at speed N, for the topology of weak convergence, governed by a good rate function $Q \mapsto \mathcal{G}(Q)$, whose zeros $Q^*(\,\mathrm{d}\theta, \mathrm{d}\omega) \in \mathcal{M}_1(\Sigma)$ satisfies the following weak McKean-Vlasov equation (where $(\theta, \omega) \mapsto \varphi(\theta, \omega)$ is any continuous bounded on $\mathbf{S} \times \mathbf{R}$, twice differentiable w.r.t. the first variable θ, with bounded derivatives):*

$$\begin{aligned} \int_{\mathbf{S}\times\mathbf{R}} \varphi \, \mathrm{d}Q_t^* = \int_{\mathbf{S}\times\mathbf{R}} \varphi \, \mathrm{d}Q_0^* + \int_0^t \mathrm{d}s \Bigg\{ \frac{1}{2} \int_{\mathbf{S}\times\mathbf{R}} \varphi'' \, \mathrm{d}Q_s^* \\ + \int_{\mathbf{S}\times\mathbf{R}} \varphi' \left(\int_{\mathbf{S}\times\mathbf{R}} f'(\theta' - \cdot, \cdot, \omega') \, \mathrm{d}Q_s^*(\theta', \omega') - g' \right) \mathrm{d}Q_s^* \Bigg\}. \end{aligned} \tag{2.8}$$

Equation (2.8) is a disordered version of a weak formulation of a McKean-Vlasov equation, that is a system of coupled nonlinear and nonlocal partial differential equations, which will be studied extensively in Chapter 3. As already mentioned in the introduction, McKean-Vlasov equations are well-known mathematical objects that naturally appear as deterministic limits of mean-field models (recall § 2.2.1, p. 17).

In § 2.5, we will see in fact that any solution to (2.8) is unique, so that the rate function $\mathcal{G}(\cdot)$ has a unique zero $Q*$, which solves (2.8). In particular, a standard result of large deviation shows the following proposition:

Proposition 2.12 (Averaged convergence of L_N)**.** *Under* (2.6)*, the law of the empirical measure L_N under the joint law of oscillators and disorder converges weakly to δ_{Q^*} where Q^* is the unique solution of McKean-Vlasov equation* (2.8)*.*

The previous result is averaged. With a bit more work, using Borel Cantelli techniques, one can derive from the large deviation principle a quenched convergence result:

Proposition 2.13 (Quenched convergence of L_N)**.** *Under* (2.6)*, for $\mathbb{P} = \mu^{\otimes \mathbf{N}}$-almost-every choice of a sequence of disorder (ω), the law of the empirical measure L_N under the law of oscillators converges weakly to δ_{Q^*} where Q^* is the unique solution of McKean-Vlasov equation* (2.8)*.*

2.4 Quenched convergence of the empirical flow and self-averaging phenomenon

2.4.1 The model

Theorem 2.11 states an *averaged* large deviation principle, that is to say when the law of the empirical measure L_N is considered *under the joint law of both oscillators and disorder.* The model studied now, which is more interesting as far as the biological applications are concerned, is *quenched*: for a fixed realization of the disorder $(\omega) = (\omega_i)_{i \geqslant 1}$, do we have the convergence of the empirical flow $\nu_N^{(\omega)}$? The answer to this question is affirmative and will be established under weaker assumptions on the moments of the disorder than the exponential moments previously supposed in (2.6).

Moreover, we no longer assume that the interaction between the rotators derives from a Hamiltonian: the quenched convergence result is establish in the case of the random interaction (2.1) which is more general than the Hamiltonian case (2.7).

Remark 2.14. This situation is more general than the framework for the large deviation results in § 2.3: indeed, if one chooses $b(\theta, \theta', \omega, \omega') = -f'(\theta' - \theta, \omega, \omega')$ and $c(\theta, \omega) = -g'(\theta, \omega)$, one retrieves the hypothesis made in § 2.3.

Once again, instead of considering θ_i as elements of $\mathbf{R}$, we will consider their projection on $\mathbf{S}$. For simplicity, we will keep the same notation θ_i for this projection [2].

2.4.2 Main results

Main hypothesis: We consider here the general case where $(\theta, \theta', \omega, \omega') \mapsto b(\theta, \theta', \omega, \omega')$ is bounded, Lipschitz-continuous, and 2π-periodic w.r.t. the two first variables. $(\theta, \omega) \mapsto c(\theta, \omega)$ is assumed to be Lipschitz-continuous w.r.t. its first variable, but not necessarily bounded (see the *sine-model*, where $c(\theta, \omega) = \omega$). We also suppose that the function

[2]. See Remark 2.17 for possible generalizations to the non-compact case.

$\omega \mapsto S(\omega) := \sup_{\theta\in\mathbf{S}} |c(\theta,\omega)|$ is continuous (this is in particular true if c is uniformly continuous w.r.t. to both variables (θ,ω), and obvious in the *sine-model* where $S(\omega) = |\omega|$). The Lipschitz bounds for b and c are supposed to be uniform in ω.

The disorder (ω) is assumed to be a sequence of identically distributed random variables (but not necessarily independent), such that the law of each ω_i is μ. We suppose that the sequence (ω) satisfies the following property: for $\mathbb{P}$-almost every sequence (ω),

$$\frac{1}{N}\sum_{i=1}^{N} \sup_{\theta\in\mathbf{S}} |c(\theta,\omega_i)| \to_{N\to\infty} \int \sup_{\theta\in\mathbf{S}} |c(\theta,\omega)| \,\mathrm{d}\mu(\omega) < \infty. \tag{2.9}$$

We make the following hypothesis on the initial empirical measure:

$$\text{For } \mathbb{P}-a.e.\ (\omega),\ \nu_{N,0}^{(\omega)} \overset{N\to\infty}{\longrightarrow} \nu_0, \quad \textit{in law, in } (\mathcal{M}_1(\mathbf{S}\times\mathbf{R}), w). \tag{2.10}$$

Note that, under (2.10), the second marginal of ν_0 is necessarily μ.

Remark 2.15. The required hypotheses about the disorder and the initial conditions are weaker than for the large deviation principle:

- the (identically distributed) variables (ω_i) need not be independent: we simply need a law of large numbers only concerning the function $\omega \mapsto \sup_{\mathbf{S}} |c(\cdot,\omega)|$; the hypothesis (2.9) is verified, for example, in the case of i.i.d. random variables, or in the case of an ergodic stationary Markov process.
- Condition (2.9) is weaker than (2.6) on page 42; for the *sine-model*, (2.9) reduces to $[\int_{\mathbf{R}} |\omega| \,\mathrm{d}\mu(\omega) < \infty]$,
- the initial values need not be independent, we only assume a convergence of their empirical measure.

In Section 4, we show the following:

Theorem 2.16. *Under the hypothesis* (2.9) *and* (2.10)*, for* $\mathbb{P}$*-almost every sequence* (ω)*, the random variable* $\nu_N^{(\omega)}$ *is tight in the space* $\mathbf{D}([0,T],(\mathcal{M}_1(\mathbf{S}\times\mathbf{R}),w))$*, and each accumulation point* ν *is the solution of the following weak equation (for every* $(\theta,\omega)\mapsto\varphi(\theta,\omega)$ *continuous bounded on* $\mathbf{S}\times\mathbf{R}$*, twice differentiable w.r.t. the first variable* θ*, with bounded derivatives):*

$$\int_{\mathbf{S}\times\mathbf{R}} \varphi \,\mathrm{d}\nu_t = \int_{\mathbf{S}\times\mathbf{R}} \varphi \,\mathrm{d}\nu_0 + \int_0^t \mathrm{d}s \int_{\mathbf{S}\times\mathbf{R}} \left(\frac{1}{2}\varphi'' + \varphi'(b[\cdot,\cdot,\nu_s] + c)\right) \mathrm{d}\nu_s, \tag{2.11}$$

where

$$b[\theta,\omega,m] = \int b(\theta,\theta',\omega,\omega') \,\mathrm{d}m(\theta',\omega').$$

Moreover, under the same hypotheses, the law of ν_N *under the joint law of the oscillators and disorder (averaged model) is also tight in the same space and its accumulations points are also solutions to* (2.11)*.*

Remark 2.17 (Generalization to the non-compact case)**.** The assumption that the state variables are in $\mathbf{S}$, although motivated by the Kuramoto model, is not absolutely essential: Theorem 2.16 still holds in the non-compact case (e.g. when $\theta\in\mathbf{S}$ is replaced by $x\in\mathbf{R}^d$), under the additional assumptions of boundedness of $x \mapsto |c(x,\omega)|$ and the first finite moment of the initial condition: $\int_{\mathbf{R}} |x| \,\mathrm{d}\gamma(x) < \infty$.

2.4.3 Self-averaging phenomenon

Equation (2.11) is the analogue, in a more general set-up, of the weak McKean-Vlasov equation established in (2.8). As already said in § 2.3, we will prove in fact that such solution to (2.11) is unique. Since a tight family of measures with a unique accumulation point actually converges, the following proposition is straightforward:

Proposition 2.18 (Quenched convergence of ν_N)**.** *Under the hypothesis* (2.10) *and* (2.9)*, for* $\mathbb{P}$*-almost every sequence* (ω)*, the random variable* $\nu_N^{(\omega)}$ *converges in the space of càdlàg trajectories* $\mathbf{D}([0,T],(\mathcal{M}_1(\mathbf{S}\times\mathbf{R}),w))$ *to the unique solution* ν *of* (2.11)*.*

Remark 2.19. We insist on the fact that the above convergence is true for almost every realization of the disorder (ω): each typical choice of the sequence of frequencies $(\omega_i)_{i\geqslant 1}$ for the oscillators $(\theta_i)_{i\geqslant 1}$ leads to the same evolution in the large N limit, and this evolution *does not depend on this choice of initial frequencies.*

2.5 Uniqueness in McKean-Vlasov Equation

The last main result of this chapter concerns uniqueness of the solution to the weak formulations of McKean-Vlasov equation (2.8) and (2.11). Since the set-up in § 2.4 is more general than the one studied in § 2.3 (recall Remark 2.14), we place ourselves in the framework of § 2.4.

Proposition 2.20 (Uniqueness in McKean-Vlasov equation)**.** *Fix any initial condition* $\nu_0\in\mathcal{M}_1(\mathbf{S}\times\mathbf{R})$ *with second marginal (on the disorder* $\omega\in\mathbf{R}$*) equal to* μ*. Then there is a unique solution* $\nu\in\mathbf{D}([0,T],(\mathcal{M}_1(\mathbf{S}\times\mathbf{R}),w))$ *to the weak McKean-Vlasov equation* (2.11)*.*

Remark 2.21. As already mentioned, an immediate corollary of Proposition 2.20 is, in the case of the large deviation principle (§ 2.3), the averaged convergence of the empirical measure L_N (Proposition 2.12) and in the quenched set-up (§ 2.4), the quenched convergence of the empirical flow ν_N (Proposition 2.18).

The proof of Proposition 2.20 is a generalization of a work by Oelschläger [67] who proved similar uniqueness for McKean-Vlasov equation, for interacting diffusions without disorder. Another proof of uniqueness can be found in [27] (via heat kernel estimations under some regularity assumptions on the initial condition) or in [38] via a martingale argument.

3 Averaged large deviation principle of the empirical measure

We now turn to the proof of the large deviation result, that is Theorem 2.11.

Equation (2.7) can be rewritten in a more compact form, (recall that we suppose for a moment that $\underline{\theta}\in\mathbf{R}^N$ instead of $\mathbf{S}^N$):

$$\mathrm{d}\underline{\theta_t} = -\nabla H_N(\underline{\theta_t},\underline{\omega})\,\mathrm{d}t + \mathrm{d}\underline{B_t}, \tag{2.12}$$

where $\underline{B}$ stands for $(B_1,\dots,B_N)$. We endow (2.12) with the following initial condition:

$$\underline{\theta_0}\sim\gamma^N, \tag{2.13}$$

where γ is a probability law on $\mathbf{R}$.

Under the assumptions made on f and g, the system (2.12)–(2.13) has a unique (strong) solution. For fixed $\underline{\omega}$, we denote by $P_N^{\underline{\omega}} \in \mathcal{M}_1(\mathcal{C}([0,T],\mathbf{R}^N))$ the (unique) law of any solution $\underline{\theta}$ to (2.12)–(2.13).

3.1 Expression of Radon-Nykodym derivative

The idea of the proof is to compare the diffusion *with interaction* (2.12) with the case where the interaction term $-\nabla H_N^f \,\mathrm{d}t$ (recall (2.4)) is removed. Namely, we introduce the diffusion without interaction (2.14):

$$\mathrm{d}\underline{\theta_t} = -\nabla H_N^g(\underline{\theta_t},\underline{\omega})\,\mathrm{d}t + \mathrm{d}\underline{B_t}. \tag{2.14}$$

From the definition (2.4) of H_N^g, it is straightforward to see that the solution to (2.14) $\underline{\theta} = (\theta_1,\dots,\theta_N)$ consists of *independent*[3] diffusions θ_i each of them solution to $\mathrm{d}\theta_i = -g'(\theta_i,\omega_i)\,\mathrm{d}t + \mathrm{d}B_{i,t}$. We denote $W_N^{\underline{\omega}} = W^{\omega_1}\times\cdots\times W^{\omega_N}$, the law of any solution to (2.14).

Although for a *fixed* choice of $\underline{\omega}$, the rotators $(\theta_i)_{1\leqslant i\leqslant N}$ in (2.14) are not identically distributed, if we consider the sequence of *couples* $(\theta_i,\omega_i)_{1\leqslant i\leqslant N}$, it is easily seen from (2.14) that it is indeed a sequence of independent and identically distributed random variables in $\mathcal{C}([0,T],\mathbf{R})\times\mathbf{R}$ with law $(W^\omega\otimes\mu)$ defined by:

$$(W^\omega\otimes\mu)(A,B) := \int_B \mathrm{d}\mu(\omega)W^\omega(A), \tag{2.15}$$

where A is a Borel set of $\mathcal{C}([0,T],\mathbf{R})$ and B is a Borel set of $\mathbf{R}$. In that case, Sanov's theorem (Proposition 2.7) provides us with a large deviation principle for the empirical measure governed by the entropy $\mathcal{H}(\cdot|W^\omega\otimes\mu)$.

With whole interaction, we are able via Girsanov's theorem to express the law of the empirical measure as an exponential transformation of the law without interaction. Varadhan's lemma (Proposition 2.4) and Proposition 2.5 lead to conclusion.

3.1.1 Application of Girsanov's theorem

We need first to compute the Radon-Nykodym derivative between the case with both interaction and disorder (2.12) and the case without interaction (2.14):

Lemma 2.22. *For a given disorder $\underline{\omega}$ and a fixed sequence of Brownian motions $\underline{B}$, if $\mathcal{F}_t = \sigma(\underline{B_s}, s\leqslant t)$ is the filtration generated by the Brownian motion $\underline{B}$, then*

$$\begin{aligned}\frac{\mathrm{d}P_N^{\underline{\omega}}}{\mathrm{d}W_N^{\underline{\omega}}}_{|\mathcal{F}_t} &= \exp\Bigg(H_N^f(\underline{\theta_0},\underline{\omega}) - H_N^f(\underline{\theta_t},\underline{\omega}) + \int_0^t \nabla H_N^f(\underline{\theta_s},\underline{\omega}).\nabla H_N^g(\underline{\theta_s},\underline{\omega})\,\mathrm{d}s \\ &\quad + \frac{1}{2}\sum_{i=1}^N \int_0^t \partial^2_{\theta_i} H_N^f(\underline{\theta_s},\underline{\omega})\,\mathrm{d}s - \frac{1}{2}\int_0^t \left\|\nabla H_N^f(\underline{\theta_s},\underline{\omega})\right\|^2 \mathrm{d}s\Bigg).\end{aligned} \tag{2.16}$$

Remark 2.23. Note that the Radon-Nykodym derivative is not the same as the one calculated in [27] (see Remark 2.25).

[3]. but not identically distributed since the ω_i are different.

Proof of Lemma 2.22. Let us fix $\underline{\omega}$ and $\underline{B}$ and consider $\underline{\theta}$ the unique solution to (2.14). For all $0 \leqslant t \leqslant T$, set

$$M_t := \int_0^t -\nabla H_N^f(\underline{\theta_s}).\,\mathrm{d}\underline{B_s}.$$

Thanks to the assumptions made on f, we see that the Novikov Condition ([50, Cor. 5.13, p. 199]) is satisfied so that $\mathcal{E}(M)_t := \exp\left(M_t - \frac{1}{2}\langle M\,,\,M\rangle_t\right)$ is a martingale. We can define $Q \in \mathcal{M}_1(\mathcal{C}([0,T],\mathbf{R}^N))$ by $\frac{\mathrm{d}Q}{\mathrm{d}W_N^{\underline{\omega}}|_{\mathcal{F}_t}} := \exp\left(M_t - \frac{1}{2}\langle M\,,\,M\rangle_t\right)$. Applying Girsanov's theorem (cf. [50, Th. 5.1, p. 191]), we have under Q

$$\underline{\tilde{B}_t} := \underline{B_t} + \int_0^t \nabla H_N^f(\underline{\theta_s})\,\mathrm{d}s$$

is a Brownian motion. Consequently, $(\underline{\theta}, \underline{\tilde{B}})$ is a weak solution to (2.12). By uniqueness in law, $Q = P_N^{\underline{\omega}}$.

It thus remains to compute $\mathcal{E}(M)_t$ in terms of the Hamiltonian H_N. Considering that, under $W_N^{\underline{\omega}}$, $\mathrm{d}\underline{B_t} = \mathrm{d}\underline{\theta_t} + \nabla H_N^g(\underline{\theta_t}, \underline{\omega})\,\mathrm{d}t$:

$$\begin{aligned}\frac{\mathrm{d}P_N^{\underline{\omega}}}{\mathrm{d}W_N^{\underline{\omega}}|_{\mathcal{F}_t}} &= \exp\left(-\int_0^t \nabla H_N^f(\underline{\theta_s}).\,\mathrm{d}\underline{B_s} - \frac{1}{2}\langle M\,,\,M\rangle_t\right),\\ &= \exp\left(-\int_0^t \nabla H_N^f(\underline{\theta_s}).\,\mathrm{d}\underline{\theta_s} - \int_0^t \nabla H_N^f(\underline{\theta_s}).\nabla H_N^g(\underline{\theta_s}, \underline{\omega})\,\mathrm{d}s - \frac{1}{2}\langle M\,,\,M\rangle_t\right).\end{aligned}$$

Applying Ito's formula to the function H_N, we get:

$$\int_0^t \nabla H_N^f(\underline{\theta_s}, \underline{\omega}).\,\mathrm{d}\underline{\theta_s} = H_N^f(\theta_t, \underline{\omega}) - H_N^f(\theta_0, \underline{\omega}) - \frac{1}{2}\sum_{i=1}^N \int_0^t \partial_{\theta_i}^2 H_N^f(\underline{\theta_s}, \underline{\omega})\,\mathrm{d}s.$$

Finally, a calculation shows that:

$$\langle M\,,\,M\rangle_t = \int_0^t \left\| \nabla H_N^f(\underline{\theta_s}) \right\|^2 \mathrm{d}s.$$

That concludes the proof of Lemma 2.22. □

3.1.2 Expression in terms of the empirical measure L_N

We can express the previous Radon-Nykodym derivative in terms of the empirical measure L_N defined in (1.15):

Proposition 2.24.

$$\frac{\mathrm{d}P_N^{\underline{\omega}}}{\mathrm{d}W_N^{\underline{\omega}}} = \exp\left(N\mathcal{J}(L_N) - \mathcal{K}(L_N)\right), \tag{2.17}$$

where, for $Q \in \mathcal{M}_1(\mathcal{C}([0,T],\mathbf{R}) \times \mathbf{R})$,

$$\mathcal{K}(Q) := \frac{1}{2}\int \mathrm{d}Q(\theta,\omega) f''(0,\omega,\omega), \tag{2.18}$$

$$\mathcal{J}(Q) = \mathcal{J}_1(Q) + \mathcal{J}_2(Q) + \mathcal{J}_3(Q) + \mathcal{J}_4(Q), \tag{2.19}$$

where,

$$\begin{aligned}
\mathcal{J}_1(Q) &= \frac{1}{2}\int \mathrm{d}Q(\theta,\omega)\int \mathrm{d}Q(\theta,\omega')\left[f(\theta_0-\theta_0',\omega,\omega')-f(\theta_T-\theta_T',\omega,\omega')\right],\\
\mathcal{J}_2(Q) &= \int_0^T \mathrm{d}s\int \mathrm{d}Q(\theta,\omega)\int \mathrm{d}Q(\theta',\omega')f'(\theta_s-\theta_s',\omega,\omega')g'(\theta_s,\omega),\\
\mathcal{J}_3(Q) &= \frac{1}{2}\int_0^T \mathrm{d}s\int \mathrm{d}Q(\theta,\omega)\int \mathrm{d}Q(\theta',\omega')f''(\theta_s-\theta_s',\omega,\omega'),\\
\mathcal{J}_4(Q) &= -\frac{1}{2}\int_0^T \mathrm{d}s\int \mathrm{d}Q(\theta,\omega)\left(\int \mathrm{d}Q(\theta',\omega')f'(\theta_s-\theta_s',\omega,\omega')\right)^2.
\end{aligned}$$

Proof of Proposition 2.24. A straightforward calculation yields:

$$\begin{aligned}
\forall i\in\{1,\dots,N\},\quad \partial_{\theta_i}H_N^f &= \frac{1}{N}\sum_{j=1}^N f'(\theta_i-\theta_j,\omega_i,\omega_j),\\
\partial_{\theta_i}H_N^g &= g'(\theta_i,\omega_i),
\end{aligned}$$

and,

$$\forall i\in\{1,\dots,N\},\quad \partial_{\theta_i}^2H_N^f = \frac{1}{N}\sum_{j=1}^N f''(\theta_i-\theta_j,\omega_i,\omega_j)-\frac{1}{N}f''(0,\omega_i,\omega_i).$$

Proposition 2.24 is then simply a reformulation of (2.16) in terms of the empirical measure. □

3.1.3 The Radon-Nykodym derivative on S

Proposition 2.24 is stated in the context of rotators θ_i in $\mathbf{R}$. The aim of this paragraph is to show that we can derive the same expression as (2.17) for the projections of the rotators on $\mathbf{S}$. In that purpose, let us introduce some temporary notations: we define $\tilde{\mathcal{J}}$ (resp. $\tilde{\mathcal{K}}$), functional on $\mathcal{M}_1(\mathcal{C}([0,T],\mathbf{S})\times\mathbf{R})$, having the same expression than $\mathcal{J}$ (resp. $\mathcal{K}$) defined on $\mathcal{M}_1(\mathcal{C}([0,T],\mathbf{R})\times\mathbf{R})$ in Prop. 2.24. Moreover, let us temporarily make the distinction between the empirical measure $L_N\in\mathcal{M}_1(\mathcal{C}([0,T],\mathbf{R})\times\mathbf{R})$ for the rotators (θ_i) in $\mathbf{R}$ and $\widetilde{L}_N\in\mathcal{M}_1(\mathcal{C}([0,T],\mathbf{S})\times\mathbf{R})$ for their projections $(\tilde{\theta}_i)$ on $\mathbf{S}$.

Thanks to (2.19) and (2.18), it is easy to see that $\mathcal{J}(L_N)=\tilde{\mathcal{J}}(\widetilde{L}_N)$ and $\mathcal{K}(L_N)=\tilde{\mathcal{K}}(\widetilde{L}_N)$. Consequently, it is immediate to see that the same expression as (2.17) is valid for the laws $\tilde{P}_N^{\underline{\omega}}$ and $\tilde{W}_N^{\underline{\omega}}$ for the oscillators $(\tilde{\theta}_i)$ in $\mathbf{S}$:

$$\frac{\mathrm{d}\tilde{P}_N^{\underline{\omega}}}{\mathrm{d}\tilde{W}_N^{\underline{\omega}}} = \exp\left(N\tilde{\mathcal{J}}(\widetilde{L}_N)-\tilde{\mathcal{K}}(\widetilde{L}_N)\right).$$

From now, for simplicity of notations, we will drop the tilde notations, considering all oscillators as elements of $\mathbf{S}$.

Remark 2.25. We could wonder why we haven't compared $P_N^{\underline{\omega}}$ with the law W_N of N independent Brownian motions (*that is the system* $\mathrm{d}\underline{\theta_t}=\mathrm{d}\underline{B_t}$, *i.e. the system* (2.12) *where both the interaction term* $-\nabla H_N^f$ *and the disorder term* $-\nabla H_N^g$ *have been removed*), as it is done in the case of bounded functions f and g in [27]. The same calculation

gives rise to another functional $\mathcal{I}(\cdot)$ in place of $\mathcal{J}(\cdot)$, which can be written as $\mathcal{I}(Q) = \mathcal{I}_1(Q) + \mathcal{I}_2(Q) + \mathcal{I}_3(Q)$, where,

$$\begin{aligned}
\mathcal{I}_1(Q) &= -\frac{1}{2}\int_0^T \mathrm{d}s \int \mathrm{d}Q(\theta,\omega)\left(\int \mathrm{d}Q(\theta',\omega')f'(\theta'_s-\theta_s,\omega,\omega') + g'(\theta_s,\omega)\right)^2,\\
\mathcal{I}_2(Q) &= \frac{1}{2}\int_0^T \mathrm{d}s \int \mathrm{d}Q(\theta,\omega)\left(\int \mathrm{d}Q(\theta',\omega')f''(\theta'_s-\theta_s,\omega,\omega')\right) + g''(\theta_s,\omega),\\
\mathcal{I}_3(Q) &= \int \mathrm{d}Q(\theta,\omega)\Big\{-\frac{1}{2}\int \mathrm{d}Q(\theta',\omega')\left[f(\theta_T-\theta'_T,\omega,\omega') - f(x_0-\theta'_0,\omega,\omega')\right]\\
&\qquad - \left(g(\theta_T,\omega) - g(\theta_0,\omega)\right)\Big\}.
\end{aligned}$$

Recall that we have in mind the case of the *sine-model* where $g(\theta,\omega) = -\theta\omega$. In that case, even if the disorder is bounded, the term $g(\theta_T,\omega) - g(\theta_0,\omega)$ in the expression of $\mathcal{I}_3(Q)$ may not be bounded in θ (in the model in $\mathbf{R}$), and furthermore we do not have $\mathcal{I}(L_N) = \tilde{\mathcal{I}}(\tilde{L}_N)$. The choice of comparing (2.12) with (2.14) gets around such minor difficulties.

3.2 Averaged large deviation principle: the case μ with compact support

We are concerned here with proving Theorem 2.11 in the particular case where the distribution μ is with compact support, namely:

$$\exists M > 0, \quad \mathrm{Supp}(\mu) \subseteq [-M, M]. \tag{2.20}$$

In this case, the expression $\int_0^t \nabla H_N(\underline{\theta_s}).\nabla H_N^g(\underline{\theta_s},\underline{\omega})\,\mathrm{d}s$ in (2.16) can be replaced by

$$\int_0^t \nabla H_N(\underline{\theta_s}).\nabla H_N^g(\underline{\theta_s},\underline{\omega})\mathbb{1}_{\underline{\omega}\in[-M,M]^N}\,\mathrm{d}s.$$

Then $\mathcal{J}_2(Q)$ in Proposition 2.24 can be replaced by:

$$\mathcal{J}_2(Q) = \int_0^T \mathrm{d}s \int \mathrm{d}Q(\theta,\omega)\int \mathrm{d}Q(\theta',\omega')f(\theta_s-\theta'_s,\omega,\omega')g'(\theta_s,\omega)\mathbb{1}_{\omega\in[-M,M]}.$$

The functional $\mathcal{J}(\cdot)$ satisfies the following properties:

Lemma 2.26. *$\mathcal{J} : \mathcal{M}_1(\Sigma) \to \mathbf{R}$ is bounded and continuous, when $\mathcal{M}_1(\Sigma)$ is endowed with the τ-topology.*

Proof of Lemma 2.26. It is clear thanks to the assumptions made on the functions f and g: every function integrated in the expression of $\mathcal{J}(\cdot)$ is measurable and bounded. □

3.2.1 Sanov's theorem and Varadhan's lemma

We are now in position to go from the i.i.d. case without interaction (2.14) to the whole case with interaction (2.12). Recall that $\Sigma = \mathcal{C}([0,T],\mathbf{S})\times\mathbf{R}$ and $\Sigma_N = \mathcal{C}([0,T],\mathbf{S}^N)\times\mathbf{R}^N$. We have the following measurable function:

$$\begin{array}{rccc} L_N: & \Sigma_N & \to & \mathcal{M}_1(\Sigma)\\ & (\underline{\theta},\underline{\omega}) & \mapsto & L_N[\underline{\theta},\underline{\omega}], \end{array}$$

where $\mathcal{M}_1(\Sigma)$ is endowed with the τ-topology. The σ- field on the set $\mathcal{M}_1(\Sigma)$ is the cylinder σ-field $\mathcal{B}^*$, namely the smallest σ-field that the evaluations $\mu \mapsto \int \varphi\,\mathrm{d}\mu$ are measurable.

The set of trajectories Σ_N can be equipped with several probability measures:

The case without interaction: let us recall that for all N, if $\underline{\theta}$ satisfies the system without interaction (2.14), then the law of $(\underline{\theta}, \underline{\omega})$ is a product probability measure $(W^\omega \otimes \mu)^N$ where $W^\omega \otimes \mu$ (recall (2.15)) is the law of one couple (θ, ω) where $\omega \sim \mu$ and θ is the solution of $\mathrm{d}\theta_t = -g'(\theta_t, \omega)\,\mathrm{d}t + \mathrm{d}B_t$.

The case with interaction: if $\underline{\theta}$ solves the whole system with interaction (2.12), then the law of the couple $(\underline{\theta}, \underline{\omega})$ is given by $P_N^{\underline{\omega}} \otimes \mu^N$ defined by:

$$P_N^{\underline{\omega}} \otimes \mu^N(A \times B) := \int_B \mu^N(\,\mathrm{d}\underline{\omega}) P_N^{\underline{\omega}}(A),$$

where A is a Borel set of $\mathcal{C}([0,T], \mathbf{S}^N)$ and B is a Borel set of $\mathbf{R}^N$.

Definition 2.27. We will denote by $\mathbf{W}_N$ the law of the empirical measure L_N under the law without interaction $(W^\omega \otimes \mu)^N$ and by $\mathbf{P}_N$ the law of the empirical measure L_N under the law with interaction $P_N^{\underline{\omega}} \otimes \mu^N$. Note that both $\mathbf{W}_N$ and $\mathbf{P}_N$ are probability measures on $\mathcal{M}_1(\Sigma)$.

The first proposition of this section is then:

Proposition 2.28. *Under* (2.20), $\mathbf{P}_N$ *satisfies a principle of large deviation in* $\mathcal{M}_1(\Sigma)$, *for the* τ*-topology, with speed* N, *for the good rate function* $\mathcal{G}(\cdot)$ *defined on* $\mathcal{M}_1(\Sigma)$ *by:*

$$\mathcal{G} : Q \mapsto \mathcal{H}(Q|W^\omega \otimes \mu) - \mathcal{J}(Q), \tag{2.21}$$

where $\mathcal{H}$ *is the relative entropy.*

Proof of Proposition 2.28. For all $C \in \mathcal{B}^*$,

$$\begin{aligned}
\mathbf{P}_N(C) &= \int_{\Sigma_N} \mathbb{1}_{\{(\underline{\theta},\underline{\omega});\, L_N[\underline{\theta},\underline{\omega}] \in C\}}\, \mathrm{d}P_N^{\underline{\omega}}(\underline{\theta})\, \mathrm{d}\mu^N(\underline{\omega}), \\
&= \int_{\Sigma_N} \mathbb{1}_{\{(\underline{\theta},\underline{\omega});\, L_N[\underline{\theta},\underline{\omega}] \in C\}} e^{N\mathcal{J}(L_N[\underline{\theta},\underline{\omega}]) - \mathcal{K}(L_N[\underline{\theta},\underline{\omega}])}\, \mathrm{d}\left(W_N^{\underline{\omega}} \otimes \mu^N\right)(\underline{\theta}, \underline{\omega}), \qquad (2.22) \\
&= \int \mathbb{1}_{\{Q \in C\}} e^{N\mathcal{J}(Q) - \mathcal{K}(Q)}\, \mathrm{d}\mathbf{W}_N(Q), \qquad (2.23)
\end{aligned}$$

where we used (2.17) in (2.22).

The key point is that under $(W^\omega \otimes \mu)^N$, the couples $(\theta_i, \omega_i)_{1 \leqslant i \leqslant N}$ are independent and identically distributed. Applying Sanov's theorem (Proposition 2.7), we obtain that $(\mathbf{W}_N)_{N \geqslant 1}$ satisfies a large deviation principle, at speed N, governed by the good rate function $Q \mapsto \mathcal{H}(Q|W^\omega \otimes \mu)$, where $\mathcal{H}$ is the relative entropy (see (2.2)):

$$\begin{aligned}
\forall A \in \mathcal{B}^*, \quad -\inf\left\{\mathcal{H}(Q|W^\omega \otimes \mu);\, Q \in \mathring{A}\right\} &\leqslant \liminf_{N\to\infty} \frac{1}{N} \ln \mathbf{W}_N(A) \leqslant \limsup_{N\to\infty} \frac{1}{N} \ln \mathbf{W}_N(A) \\
&\leqslant -\inf\left\{\mathcal{H}(Q|W^\omega \otimes \mu);\, Q \in \bar{A}\right\}.
\end{aligned}$$

The idea here is to derive a large deviation principle for $(\mathbf{P}_N)_{N \geqslant 1}$, knowing that $\mathbf{P}_N$ is an exponential transformation of $\mathbf{W}_N$ (recall (2.23)). Indeed, using (2.23), for all $A \in \mathcal{B}^*$,

$$\mathbf{P}_N(A) = \frac{\int_A e^{N\mathcal{J}(\cdot) - \mathcal{K}(\cdot)}\, \mathrm{d}\mathbf{W}_N}{\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot) - \mathcal{K}(\cdot)}\, \mathrm{d}\mathbf{W}_N}.$$

The assumption made on f ensures that the linear functional $\mathcal{K}(\cdot)$ defined in (2.18) is bounded on $\mathcal{M}_1(\Sigma)$:

$$\forall Q \in \mathcal{M}_1(\Sigma), \quad |\mathcal{K}(Q)| \leqslant \frac{\| f'' \|_\infty}{2}.$$

Consequently, for all $A \in \mathcal{B}^*$,

$$e^{-\| f'' \|_\infty} \frac{\int_A e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N}{\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N} \leqslant \mathbf{P}_N(A) \leqslant e^{\| f'' \|_\infty} \frac{\int_A e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N}{\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N},$$

so that

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \ln \left(\frac{\int_A e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N}{\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N} \right) &\leqslant \liminf_{N\to\infty} \frac{1}{N} \ln \mathbf{P}_N(A) \leqslant \limsup_{N\to\infty} \frac{1}{N} \ln \mathbf{P}_N(A) \\ &\leqslant \limsup_{N\to\infty} \frac{1}{N} \ln \left(\frac{\int_A e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N}{\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N} \right). \end{aligned} \tag{2.24}$$

We are now in position to apply Proposition 2.5: since the sequence of probabilities $(\mathbf{W}_N)_{N \geqslant 1}$ in $\mathcal{M}_1(\Sigma)$ satisfies a large deviation principle at speed N for the good rate function $Q \mapsto \mathcal{H}(Q|W^\omega \otimes \mu)$, then the sequence of probabilities $A \mapsto \frac{\int_A e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N}{\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N}$ satisfies a large deviation principle in $\mathcal{M}_1(\Sigma)$ for the good rate function

$$Q \mapsto \mathcal{H}(Q|W^\omega \otimes \mu) - \mathcal{J}(Q) - \inf_{\mathcal{M}_1(\Sigma)} \left(\mathcal{H}(\cdot|W^\omega \otimes \mu) - \mathcal{J}(\cdot) \right).$$

Thanks to (2.24), this is also true for the sequence $(\mathbf{P}_N)_{N \geqslant 1}$.

It remains to show that $\inf_{\mathcal{M}_1(\Sigma)} (\mathcal{H}(\cdot|W^\omega \otimes \mu) - \mathcal{J}(\cdot)) = 0$: indeed, since $\mathbf{P}_N$ is a probability,

$$\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot) - \mathcal{H}(\cdot)} \, \mathrm{d}\mathbf{W}_N = 1.$$

So, we have for all $N \geqslant 1$:

$$0 = \frac{1}{N} \ln \left(\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot) - \mathcal{K}(\cdot)} \, \mathrm{d}\mathbf{W}_N \right) \leqslant \frac{\| f'' \|_\infty}{2N} + \frac{1}{N} \ln \left(\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N \right).$$

Using Varadhan's Lemma (Proposition 2.4), we obtain, in the limit as $N \to \infty$:

$$0 \leqslant \sup_{\mathcal{M}_1(\Sigma)} \left(\mathcal{J}(\cdot) - \mathcal{H}(\cdot|W^\omega \otimes \mu) \right).$$

Bounding $\mathcal{K}(\cdot)$ by below, we have in the same way

$$0 = \frac{1}{N} \ln \left(\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot) - \mathcal{K}(\cdot)} \, \mathrm{d}\mathbf{W}_N \right) \geqslant -\frac{\| f'' \|_\infty}{2N} + \frac{1}{N} \ln \left(\int_{\mathcal{M}_1(\Sigma)} e^{N\mathcal{J}(\cdot)} \, \mathrm{d}\mathbf{W}_N \right).$$

Consequently, $\sup_{\mathcal{M}_1(\Sigma)} \left(\mathcal{J}(\cdot) - \mathcal{H}(\cdot|W^\omega \otimes \mu) \right) \leqslant 0$, and Proposition 2.28 is proved. $\square$

3.2.2 A simpler expression of the rate function

The expression of the rate function $\mathcal{G}(\cdot)$ in Proposition 2.28 is rather complicated. We introduce an auxiliary SDE to simplify this expression:

For fixed $q \in \mathcal{M}_1(\mathbf{S} \times \mathbf{R})$ and $\omega \in \mathbf{R}$, let us denote by $\beta^{q,\omega}$ the function defined on $\mathbf{S}$ by:

$$\forall \theta \in \mathbf{S}, \quad \beta^{q,\omega}(\theta) = \int f'(\theta' - \theta, \omega, \omega')\, \mathrm{d}q(\theta', \omega').$$

Then, for fixed $Q \in \mathcal{M}_1(\Sigma)$, we consider $P^{\omega,Q}$ the law of the projection on $\mathbf{S}$ of the unique (strong) solution to the following SDE:

$$\mathrm{d}\theta_t = \beta^{Q_t,\omega}(\theta_t)\, \mathrm{d}t - g'(\theta_t, \omega)\, \mathrm{d}t + \mathrm{d}b_t;\ x_0 \sim \gamma, \tag{2.25}$$

where,

$$Q_t(A \times B) = Q\{(\theta, \omega), \theta_t \in A, \omega \in B\}.$$

Recall (2.15): for all $\omega \in \mathrm{Supp}(\mu)$, W^ω is the law of (the projection on $\mathbf{S}$ of) the solution to $\mathrm{d}\theta_t = -g'(\theta, \omega)\, \mathrm{d}t + \mathrm{d}b_t$. The following proposition provides a simpler expression of the rate function $\mathcal{G}(\cdot)$ defined in (2.21):

Proposition 2.29. *For all $Q \in \mathcal{M}_1(\Sigma)$,*

$$\forall Q \in \mathcal{M}_1(\Sigma), \quad \mathcal{J}(Q) = \int \ln\left(\frac{\mathrm{d}P^{\omega,Q}}{\mathrm{d}W^\omega}(\theta)\right) \mathrm{d}Q(\theta, \omega), \tag{2.26}$$

and,

$$\forall Q \in \mathcal{M}_1(\Sigma), \quad \mathcal{G}(Q) = \mathcal{H}(Q|P^Q), \tag{2.27}$$

where $P^Q \in \mathcal{M}_1(\Sigma)$ is given by:

$$\mathrm{d}P^Q(\theta, \omega) := \mathrm{d}\mu(\omega)\, \mathrm{d}P^{\omega,Q}(\theta). \tag{2.28}$$

Proof of Proposition 2.29. Applying again Girsanov's theorem, we have:

$$\ln\left(\frac{\mathrm{d}P^{\omega,Q}}{\mathrm{d}W^\omega}\right) = -\frac{1}{2}\int_0^t \mathrm{d}s \left(\beta^{Q_s,\omega}(\theta_s)\right)^2 + \int_0^t \beta^{Q_s,\omega}(\theta_s)\, \mathrm{d}b_s.$$

The first term integrated over Q gives rise to the term $\mathcal{J}_4(Q)$ in Proposition 2.24. For the second term, we have, considering that under W^ω, $\mathrm{d}b_t = \mathrm{d}\theta_t + g'(\theta_t, \omega)\mathbb{1}_{\omega \in [-M,M]}\, \mathrm{d}t$,

$$\int_0^t \beta^{Q_s,\omega}(\theta_s)\, \mathrm{d}b_s = \int_0^t \beta^{Q_s,\omega}(\theta_s)\, \mathrm{d}\theta_s + \int_0^t \beta^{Q_s,\omega}(\theta_s) g'(\theta_s, \omega)\mathbb{1}_{\omega \in [-M,M]}\, \mathrm{d}s.$$

The last term integrated over Q gives rise to the term $\mathcal{J}_2(Q)$ in Proposition 2.24.

It remains to consider the term $\int_0^t \beta^{Q_s,\omega}(\theta_s)\, \mathrm{d}\theta_s$. We have successively:

$$\begin{aligned}\int_0^t \beta^{Q_s,\omega}(\theta_s)\, \mathrm{d}\theta_s &= \int_0^t \int \mathrm{d}Q(\theta, \omega) \int \mathrm{d}Q(\theta', \omega') f'(\theta'_s - \theta_s, \omega, \omega')\, \mathrm{d}\theta_s, \\ &= \frac{1}{2}\int_0^t \int \mathrm{d}Q(\theta, \omega) \int \mathrm{d}Q(\theta', \omega') f'(\theta'_s - \theta_s, \omega, \omega')[\,\mathrm{d}\theta_s - \mathrm{d}\theta'_s].\end{aligned}$$

Applying Ito's formula to the semi-martingale (θ, θ'), we have,

$$\begin{aligned}\int_0^t \beta^{Q_s,\omega}(\theta_s)\, \mathrm{d}\theta_s = &\frac{1}{2}\int \mathrm{d}Q(\theta, \omega) \int \mathrm{d}Q(\theta', \omega') \left[f(\theta_0 - \theta'_0, \omega, \omega') - f(\theta_t - \theta'_t, \omega, \omega')\right] \\ &+ \frac{1}{2}\int \mathrm{d}Q(\theta, \omega) \int \mathrm{d}Q(\theta', \omega') \left[\int_0^t f''(\theta_s - \theta'_s, \omega, \omega')\, \mathrm{d}s\right].\end{aligned}$$

The last equality gives the terms $\mathcal{J}_1(Q)$ and $\mathcal{J}_3(Q)$ and proves equality (2.26).

We finish by proving equality (2.27). For all $Q \in \mathcal{M}_1(\Sigma)$, we have, using (2.21) and (2.26), for $Q \ll W^\omega \otimes \mu$:

$$\mathcal{G}(Q) = \int \mathrm{d}Q \ln\left(\frac{\mathrm{d}Q}{\mathrm{d}(W^\omega \otimes \mu)}\right) + \ln\left(\frac{\mathrm{d}(W^\omega \otimes \mu)}{\mathrm{d}P^Q}\right) = \int \mathrm{d}Q \ln\left(\frac{\mathrm{d}Q}{\mathrm{d}P^Q}\right) = \mathcal{H}(Q|P^Q).$$

If Q is not absolutely continuous w.r.t. $W^\omega \otimes \mu$, Q is not absolutely continuous w.r.t. P^Q since $P^Q \sim W^\omega \otimes \mu$. Proposition 2.29 follows. □

At this point, we know that the sequence $(P_N)_{N \geqslant 1}$ concentrates on the zeros of the rate function $\mathcal{G}(\cdot)$ defined in (2.27). Before characterizing such zeros (see § 3.4), we generalize in the next paragraph the large deviation principle in the case of unbounded disorder.

3.3 Averaged large deviation principle: the general case

We want to prove Theorem 2.11 under the assumption (2.6): the idea here is to apply the previous section in the case where the disorder is conditioned to remain in a compact interval $[-M, M]$. Doing so, the previous section gives a large deviation principle for the law of the empirical measure $(L_N)_{N \geqslant 1}$ for each $M > 0$. The main point of this section is to make $M \to \infty$ and derive a large deviation principle in the case where the distribution of the disorder μ satisfies (2.6). To do so, we define the following probabilities μ_M (at least well defined for large M):

Definition 2.30. For all $M > 0$, for all A Borel set of $\mathbf{R}$,

$$\mu_M(A) := \frac{\mu(A \cap [-M, M])}{\mu([-M, M])},$$

and for all $N \geqslant 1$,

$$\mu_M^N := (\mu_M)^N,$$

the law of N i.i.d. random variables with law μ, conditioned to remain in $[-M, M]$.

Let $\mathbf{P}_N^M$ be the law of L_N when $\underline{\omega}$ is distributed under μ_M^N:

$$\forall A \in \mathcal{B}^*, \quad \mathbf{P}_N^M(A) = \int_{\Sigma_N} \mathbb{1}_{\{(\underline{\theta},\underline{\omega});\, L_N[\underline{\theta},\underline{\omega}] \in A\}} P_N^{\underline{\omega}}(\,\mathrm{d}\underline{\theta}) \mu_M^N(\,\mathrm{d}\underline{\omega}).$$

Since $\mathrm{Supp}(\mu_M) \subseteq [-M, M]$, we can apply the previous part to obtain:

Proposition 2.31. *For all $M > 0$, $(\mathbf{P}_N^M)_{N \geqslant 1}$ satisfies a large deviation principle in $\mathcal{M}_1(\Sigma)$, at speed N, for the τ-topology, with the good rate function $\mathcal{G}_M(\cdot)$:*

$$\forall Q \in \mathcal{M}_1(\Sigma), \quad \mathcal{G}_M(Q) = \mathcal{H}(Q|P_M^Q),$$

where $\mathrm{d}P_M^Q(\theta, \omega) := \mathrm{d}\mu_M(\omega)\, \mathrm{d}P^{\omega,Q}(\theta)$.

3.3.1 Exponentially good approximations

The main difficulty of this part is to derive a large deviation principle for $(\mathbf{P}_N)_{N \geqslant 1}$ from the sequence of large deviation principles verified by $(\mathbf{P}_N^M)_{N \geqslant 1}$ for each $M \geqslant 1$: this can be done by using the notion of *exponentially good approximations of measures*, developed in [29, § 4.2].

The space $(\mathcal{M}_1(\Sigma), w)$ endowed with the topology of weak convergence (recall the definition of w in 2.2) is a metrizable space, for the following metric d (see [32, Th. 12, p. 262]):

$$\forall \nu, \tilde{\nu} \in \mathcal{M}_1(\Sigma), \quad d(\nu, \tilde{\nu}) := \sup \left\{ \left| \int \varphi \, \mathrm{d}\nu - \int \varphi \, \mathrm{d}\tilde{\nu} \right| ; \varphi \in BL_1(\Sigma) \right\}, \tag{2.29}$$

where $BL_1(\Sigma)$ is the set of all bounded Lipschitz continuous functions on $\Sigma = \mathcal{C}([0,T], \mathbf{S}) \times \mathbf{R}$, of Lipschitz norm bounded by 1. The topology w is, by definition, coarser than the τ-topology. Applying the contraction principle ([29, Th. 4.2.1 p. 126]), Proposition 2.31 is also true for the weak convergence topology.

Following [29, p. 131], we show that $\mathbf{P}_N^M$ are exponentially good approximations of $\mathbf{P}_N$:

Proposition 2.32 (Exponentially good approximations)**.** *For all $\delta > 0$, we define:*

$$\Gamma_\delta = \{(\nu, \tilde{\nu}) \in \mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma);\, d(\nu, \tilde{\nu}) > \delta\}.$$

There exists a coupling $\mathbf{Q}_{N,M}$, probability on $\mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma)$, such that the marginals of $\mathbf{Q}_{N,M}$ are $\mathbf{P}_N$ and $\mathbf{P}_N^M$ and which verifies:

$$\forall \delta > 0, \quad \lim_{M \to \infty} \limsup_{N \to \infty} \frac{1}{N} \ln \mathbf{Q}_{N,M}(\Gamma_\delta) = -\infty.$$

The large deviation principle stated in Theorem 2.11 is then a consequence of Proposition 2.32 and the following proposition (see [29, Th. 4.2.16, p. 131]):

Proposition 2.33. *The sequence $(\mathbf{P}_N)_{N \geqslant 1}$ satisfies a* weak *large deviation principle in $\mathcal{M}_1(\Sigma)$, at speed N, in the weak topology, with the rate function:*

$$\tilde{\mathcal{G}}(Q) := \sup_{\delta > 0} \liminf_{M \to \infty} \inf_{R \in B(Q,\delta)} \mathcal{H}(R | P_M^R). \tag{2.30}$$

The functional $\tilde{\mathcal{G}}(\cdot)$ is a good rate function and coincides with $\mathcal{G}(\cdot)$ defined in (2.27)*. Moreover, for every closed F in the weak topology in $\mathcal{M}_1(\Sigma)$*

$$\inf_{Q \in F} \mathcal{H}(Q | P^Q) \leqslant \limsup_{M \to \infty} \inf_{Q \in F} \mathcal{H}(Q | P_M^Q),$$

so that $(\mathbf{P}_N)_{N \geqslant 1}$ satisfies the strong *large deviation principle for the good rate function $\mathcal{G}(\cdot)$.*

Provided we characterize the zeros of the rate function $\mathcal{G}(\cdot)$ (see § 3.4), with all these properties (whose proofs are given at the end of § 3.3), Theorem 2.11 will be proved.

3.3.2 Proof of the exponential approximations

Definition of the coupling $\mathbf{Q}_{N,M}$: The aim of this paragraph is to build a coupling on $\mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma)$, where the corresponding marginals are the law of the empirical measures of processes in $\Sigma_N = \mathcal{C}([0,T], \mathbf{S}^N) \times \mathbf{R}^N$, $(\underline{\theta}, \underline{\omega})$ and $(\underline{\tilde{\theta}}, \underline{\tilde{\omega}})$ which have the same initial values, are driven by the same Brownian motion, and where one disorder is the restriction on $[-M, M]$ of the other.

More concretely, following [54, Th. 6.3], we know that there exists a measurable application $\mathcal{P}$:

$$\begin{array}{rccc} \mathcal{P}: & \mathbf{S}^N \times \mathcal{C}([0,T], \mathbf{R}^N) \times \mathbf{R}^N & \to & \mathcal{C}([0,T], \mathbf{S}^N) \\ & (\underline{\xi}, \underline{B}, \underline{\omega}) & \mapsto & \mathcal{P}\left[\underline{\xi}, \underline{B}, \underline{\omega}\right], \end{array}$$

such that for all choice of the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, and for all choice of a $\mathcal{F}_t$-adapted Brownian Motion $\underline{B}$, and for all initial $\underline{\xi}$ which is $\mathcal{F}_0$ measurable, the process $\mathcal{P}\left[\underline{\xi}, \underline{B}, \underline{\omega}\right]$ is (the projection on $\mathbf{S}$ of) the only solution to the SDE (2.12), with disorder $\underline{\omega}$.

Let us choose one Brownian motion $\underline{B}$ in $\mathbf{R}^N$, one sequence of i.i.d random variables $(\xi_i)_{1 \leqslant i \leqslant N}$ with law γ, independent of $\underline{B}$, and one sequence of i.i.d. real random variables $(U_i)_{1 \leqslant i \leqslant N}$ with uniform distribution on $[0,1]$, independent of $(\underline{B}, \underline{\xi})$. If we denote by $t \mapsto F_\mu(t) := \mu(]-\infty,\ t])$ (resp. $t \mapsto F_{\mu_M}(t) := \mu_M(]-\infty,\ t])$), the cumulative distribution function of the law μ (resp. μ_M) and by $s \mapsto F_\mu^{-1}(s) := \inf\{t;\ F_\mu(t) \geqslant s\}$ (resp $F_{\mu_M}^{-1}$), its pseudo-inverse function. We consider the following random variable, having values in $\left(\mathcal{C}([0,T], \mathbf{S}^N) \times \mathbf{R}^N\right)^2$:

$$\left(\underline{B}, \underline{\xi}, \underline{U})\right) \mapsto \left((\underline{\theta}, \underline{\omega}), \left(\underline{\tilde{\theta}}, \underline{\tilde{\omega}}\right)\right),$$

where

$$(\underline{\theta}, \underline{\omega}) := \left(\mathcal{P}\left[\underline{B}, \underline{\xi}, F_\mu^{-1}(U_i)\right], F_\mu^{-1}(U_i)\right),$$

and

$$\left(\underline{\tilde{\theta}}, \underline{\tilde{\omega}}\right) := \left(\mathcal{P}\left[\underline{B}, \underline{\xi}, F_{\mu_M}^{-1}(U_i)\right], F_{\mu_M}^{-1}(U_i)\right).$$

We denote by $Q_{N,M}$ the law of the processes $\left((\underline{\theta}, \underline{\omega}), (\underline{\tilde{\theta}}, \underline{\tilde{\omega}})\right) \in \Sigma_N \times \Sigma_N$ and by $\mathbf{Q}_{N,M}$, probability on $\mathcal{M}_1(\Sigma) \times \mathcal{M}_1(\Sigma)$ the law of the corresponding couple of empirical measures $(L_N[\underline{\theta}, \underline{\omega}], L_N[\underline{\tilde{\theta}}, \underline{\tilde{\omega}}])$. It is quite clear, by construction, that the marginals of $\mathbf{Q}_{N,M}$ are $\mathbf{P}_N$ and $\mathbf{P}_N^M$.

Proof of Proposition 2.32: In order to prove Proposition 2.32, we need the two following lemmas 2.34 and 2.35:

Lemma 2.34. *For all $\underline{\theta}$, $\underline{\tilde{\theta}}$ in $\mathcal{C}([0,T], \mathbf{S}^N)$ and $\underline{\omega}$, $\underline{\tilde{\omega}}$ in $\mathbf{R}^N$,*

$$d\left(L_N[\underline{\theta}, \underline{\omega}], L_N[\underline{\tilde{\theta}}, \underline{\tilde{\omega}}]\right) \leqslant \frac{1}{N} \sum_{i=1}^N \left(\left\| \theta_i - \tilde{\theta}_i \right\|_\infty + |\omega_i - \tilde{\omega}_i|\right).$$

Proof of Lemma 2.34. Let φ be a bounded Lipschitz continuous function on Σ, such that $\| \varphi \|_L \leqslant 1$:

$$\left|\varphi(\theta, \omega) - \varphi(\tilde{\theta}, \tilde{\omega})\right| \leqslant \left(\left\| \theta - \tilde{\theta} \right\|_\infty + |\omega - \tilde{\omega}|\right).$$

Then,

$$\begin{aligned}\left|\int \varphi \,\mathrm{d}L_N[\underline{\theta},\underline{\omega}] - \int \varphi \,\mathrm{d}L_N[\underline{\tilde{\theta}},\underline{\tilde{\omega}}]\right| &= \frac{1}{N}\left|\sum_{i=1}^N \Big(\varphi(\theta_i,\omega_i) - \varphi(\tilde{\theta}_i,\tilde{\omega}_i)\Big)\right|,\\ &\leqslant \frac{1}{N}\sum_{i=1}^N \Big|\varphi(\theta_i,\omega_i) - \varphi(\tilde{\theta}_i,\tilde{\omega}_i)\Big|,\\ &\leqslant \frac{1}{N}\sum_{i=1}^N \Big(\Big\|\,\theta_i - \tilde{\theta}_i\,\Big\|_\infty + |\omega_i - \tilde{\omega}_i|\Big).\end{aligned}$$

Taking the supremum over all φ, we have the result. □

The following lemma states a control on the processes $\underline{\theta}$ and $\underline{\tilde{\theta}}$ under the coupling $Q_{N,M}$:

Lemma 2.35. *The marginals $\mu^N \otimes P_N^{\underline{\omega}}$ and $\mu_M^N \otimes P_N^{\underline{\tilde{\omega}}}$ of the coupling $Q_{N,M}$ defined in § 3.3.2 satisfy the following property: there exists a constant A, such that, for all $N \in \mathbf{N}$, $N \geqslant 1$, for μ^N almost every $\underline{\omega}$ and $\underline{\tilde{\omega}}$,*

$$\frac{1}{N}\int \sum_{i=1}^N \Big\|\,\theta_i - \tilde{\theta}_i\,\Big\|_\infty P_N^{\underline{\omega}} \otimes P_N^{\underline{\tilde{\omega}}}(\,\mathrm{d}\underline{\theta},\,\mathrm{d}\underline{\tilde{\theta}}) < A\left(\frac{1}{N}\sum_{i=1}^N |\omega_i - \tilde{\omega}_i|\right).$$

Proof of Lemma 2.35. For fixed $\underline{\omega}$ and $\underline{\tilde{\omega}}$, let $\underline{\theta}$ and $\underline{\tilde{\theta}}$ sampled according to the coupling $Q_{N,M}$ that is (the projection on $\mathbf{S}$ of) two solutions to (2.7), with disorder respectively $\underline{\omega}$ and $\underline{\tilde{\omega}}$, with the same initial condition $\underline{\xi}$ and driven by the same Brownian motion $\underline{B}$. Then, for all $i \in \{1,\dots,N\}$, for all $s \in [0,T]$, for some constant $C > 0$

$$\begin{aligned}\Big|\theta_{i,s} - \tilde{\theta}_{i,s}\Big| &\leqslant \frac{1}{N}\sum_{j=1}^N \int_0^s \Big|f'(\theta_{j,u} - \theta_{i,u},\omega_i,\omega_j) - f'(\tilde{\theta}_{j,u} - \tilde{\theta}_{i,u},\omega_i,\omega_j)\Big|\\ &\quad + \Big|g'(\theta_{i,u},\omega_i) - g'(\tilde{\theta}_{i,u},\tilde{\omega}_i)\Big|\,\mathrm{d}u,\\ &\leqslant \frac{C}{N}\sum_{j=1}^N \int_0^s \Big|\theta_{i,u} - \tilde{\theta}_{i,u}\Big| + \Big|\theta_{j,u} - \tilde{\theta}_{j,u}\Big|\,\mathrm{d}u + |\omega_i - \tilde{\omega}_i|\,s,\end{aligned}$$

So, for all $t \in [0,T]$,

$$\frac{1}{N}\sum_{i=1}^N \sup_{s\leqslant t}\Big|\theta_{i,s} - \tilde{\theta}_{i,s}\Big| \leqslant \frac{2C}{N}\sum_{i=1}^N \int_0^t \sup_{v\leqslant u}\Big|\theta_{i,v} - \tilde{\theta}_{i,v}\Big|\,\mathrm{d}u + \frac{1}{N}\sum_{i=1}^N |\omega_i - \tilde{\omega}_i| t.$$

If we take expectation (w.r.t. the Brownian motion) in the last inequality, and if we introduce

$$\psi_t := \frac{1}{N}\sum_{i=1}^N \mathbf{E}\left[\sup_{s\leqslant t}\Big|\theta_{i,s} - \tilde{\theta}_{i,s}\Big|\right],$$

we have:

$$\psi_t \leqslant 2C\int_0^t \psi(u)\,\mathrm{d}u + \frac{1}{N}\sum_{i=1}^N |\omega_i - \tilde{\omega}_i| t.$$

Applying Gronwall's lemma, we obtain, for some constant $C > 0$:

$$\forall t \in [0,T], \quad \psi_t \leqslant C \left(\frac{1}{N} \sum_{i=1}^{N} |\omega_i - \tilde{\omega}_i| \right).$$

The result follows. □

We are now in position to prove the result concerning exponentially good approximations:

Proof of Proposition 2.32. For all $\delta > 0$,

$$\mathbf{Q}_{N,M}(\Gamma_\delta) = Q_{N,M}((\underline{\theta},\underline{\omega}),(\underline{\tilde{\theta}},\underline{\tilde{\omega}}),\ d\left(L_N[\underline{\theta},\underline{\omega}], L_N[\underline{\tilde{\theta}},\underline{\tilde{\omega}}]\right) > \delta),$$

where we recall that $Q_{N,M}$ is the law of the coupled processes built in § 3.3.2. So, by Lemma 2.34,

$$\begin{aligned}\mathbf{Q}_{N,M}(\Gamma_\delta) \leqslant\ & Q_{N,M}\left((\underline{\theta},\underline{\omega}),(\underline{\tilde{\theta}},\underline{\tilde{\omega}}),\ \frac{1}{N}\sum_{i=1}^{N}\left\| \theta_i - \tilde{\theta}_i \right\|_\infty > \frac{\delta}{2}\right) \\ & + Q_{N,M}\left((\underline{\theta},\underline{\omega}),(\underline{\tilde{\theta}},\underline{\tilde{\omega}}),\ \frac{1}{N}\sum_{i=1}^{N}|\omega_i - \tilde{\omega}_i| > \frac{\delta}{2}\right).\end{aligned}$$

Considering Lemma 2.35, it suffices to prove the following:

$$\forall \delta > 0, \lim_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N} \ln Q_{N,M}\left(\frac{1}{N}\sum_{i=1}^{N}|\omega_i - \tilde{\omega}_i| > \delta\right) = -\infty. \tag{2.31}$$

Let us prove (2.31): recall that under $Q_{N,M}$, for each $1 \leqslant i \leqslant N$, $\omega_i = F_\mu^{-1}(U_i)$ and $\tilde{\omega}_i = F_{\mu_M}^{-1}(U_i)$ where U_i is uniformly distributed on $[0,1]$ so that $\omega_i \sim \mu$ and $\tilde{\omega}_i \sim \mu_M$. Let us fix some $t > 0$. A straightforward calculation leads to:

$$\limsup_{N\to\infty} \frac{1}{N} \ln Q_{N,M}\left(\frac{1}{N}\sum_{i=1}^{N}|\omega_i - \tilde{\omega}_i| > \delta\right) \leqslant \ln \mathbb{E}_{[0,1]}\left(e^{t|F_\mu^{-1}(U) - F_{\mu_M}^{-1}(U)|}\right) - t\delta,$$

where the expectation $\mathbb{E}_{[0,1]}(\cdot)$ is taken w.r.t. the uniform law on $[0,1]$. Then,

$$\begin{aligned}\mathbb{E}_{[0,1]}\left(e^{t|F_\mu^{-1}(U) - F_{\mu_M}^{-1}(U)|}\right) \leqslant\ & \mathbb{E}_{[0,1]}\left(e^{2t|F_\mu^{-1}(U) - F_{\mu_M}^{-1}(U)|\mathbb{1}_{|F_\mu^{-1}(U)| \leqslant M}}\right)^{\frac{1}{2}} \\ & \times \mathbb{E}_{[0,1]}\left(e^{2t|F_\mu^{-1}(U) - F_{\mu_M}^{-1}(U)|\mathbb{1}_{|F_\mu^{-1}(U)| > M}}\right)^{\frac{1}{2}}, \\ \leqslant\ & e^{t\sup_s\left(|F_\mu^{-1}(s) - F_{\mu_M}^{-1}(s)|\mathbb{1}_{|F_\mu^{-1}(s)| \leqslant M}\right)} \\ & \times \mathbb{E}_{[0,1]}\left(e^{4t|F_\mu^{-1}(U)|\mathbb{1}_{|F_\mu^{-1}(U)| > M}}\right)^{\frac{1}{2}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}_{[0,1]}\left(e^{4t|F_\mu^{-1}(U)|\mathbb{1}_{|F_\mu^{-1}(U)| > M}}\right) &= \mathbb{E}_\mu\left(e^{4t|\omega|\mathbb{1}_{|\omega| > M}}\right), \\ &= \mathbb{E}_\mu\left(e^{4t|\omega|}\mathbb{1}_{|\omega| > M}\right) + \mathbb{P}_\mu\left(|\omega| \leqslant M\right).\end{aligned}$$

Consequently,

$$\limsup_{N\to\infty} \frac{1}{N} \ln Q_{N,M}\left(\frac{1}{N}\sum_{i=1}^{N}|\omega_i - \tilde{\omega}_i| > \delta\right) \leqslant t \sup_s \left(|F_\mu^{-1}(s) - F_{\mu_M}^{-1}(s)|\mathbb{1}_{|F_\mu^{-1}(s)|\leqslant M}\right) + \frac{1}{2}\ln\left(\mathbb{E}_\mu\left(e^{4t|\omega|}\mathbb{1}_{|\omega|>M}\right) + \mathbb{P}_\mu\left(|\omega|\leqslant M\right)\right) - t\delta. \tag{2.32}$$

A short calculation shows that $\sup_{u\in[-M,M]}|F_\mu(u) - F_{\mu_M}(u)| \to_{M\to\infty} 0$, and so does $\sup_s |F_\mu^{-1}(s) - F_{\mu_M}^{-1}(s)|\mathbb{1}_{F_\mu^{-1}(s)\in[-M,M]}$. Using the existence of exponential moments of μ (recall (2.6)), we easily see that the second term of (2.32) goes to 0 as $M\to\infty$ so that

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{N}\ln Q_{N,M}\left(\frac{1}{N}\sum_{i=1}^{N}|\omega_i - \tilde{\omega}_i| > \delta\right) \leqslant -t\delta. \tag{2.33}$$

Since $t>0$ is arbitrary, making $t\to\infty$ in (2.33) gives the result (2.31). Proposition 2.32 is proved. □

3.3.3 Identification of the rate function

At this point, Proposition 2.33 states a *weak* large deviation principle for the rate function $\tilde{\mathcal{G}}(\cdot)$ defined by (2.30). The next step is to identify $\tilde{\mathcal{G}}(\cdot)$ with the rate function $\mathcal{G} : Q \mapsto \mathcal{H}(Q|P^Q)$ found in § 3.2 in the case where μ is of compact support. This is the purpose of the two following propositions:

Proposition 2.36. *For all $Q\in\mathcal{M}_1(\Sigma)$,*

$$\tilde{\mathcal{G}}(Q) \leqslant \mathcal{H}(Q|P^Q). \tag{2.34}$$

Proof of Proposition 2.36. We only need to consider the case $Q \ll P^Q$. For each $Q\in\mathcal{M}_1(\Sigma)$, we denote by Q_2 the marginal w.r.t. ω and we write:

$$\mathrm{d}Q(\theta,\omega) = \mathrm{d}Q_2(\omega)\otimes\mathrm{d}Q^\omega(\theta).$$

If $Q\ll P^Q$, then (recall $\mathrm{d}P^Q(\theta,\omega) = \mathrm{d}\mu(\omega)\otimes\mathrm{d}P^\omega(\theta)$),

$$Q_2 \ll \mu, \quad \text{and,} \quad Q^\omega \ll P^{Q,\omega}, \quad \text{for } \mu \text{ a.e. } \omega.$$

In particular, there exists a nonnegative measurable function h such that $\mathrm{d}Q_2(\omega) = h(\omega)\,\mathrm{d}\mu(\omega)$. We define, for all $M>0$, $R = R_M \in \mathcal{M}_1(\Sigma)$ by:

$$\mathrm{d}R_M(\theta,\omega) := \mathrm{d}R_{M,2}(\omega)\otimes\mathrm{d}Q^\omega(\theta),$$

where,

$$\mathrm{d}R_{M,2}(\omega) := h(\omega)\,\mathrm{d}\mu_M(\omega).$$

For such a R_M, we have:

- $\mathcal{G}_M(R_M) = \mathcal{H}(R_M|P_M^{R_M}) < \infty$: indeed, for Q_2 almost every ω, we easily see (via Girsanov's theorem) that $P^{Q,\omega}\sim P^{R_M,\omega}$. Therefore, for all A Borel set of $\mathcal{C}([0,T],\mathbf{R})$, if $P^{R_M,\omega}(A)=0$ then, $P^{Q,\omega}(A)=0$, so, $R_M^\omega(A) = Q^\omega(A) = 0$. So, $R_M^\omega \ll P^{R_M,\omega}$. By definition of $R_{M,2}$, we also have $R_{M,2}\ll\mu_M$. As a conclusion, $R_M \ll P_M^{R_M}$, and a calculation shows:

$$\mathcal{G}_M(R_M) = \mathcal{H}(R_M|P_M^{R_M}) = \mathcal{H}(R_{M,2}|\mu_M) + \int \mathrm{d}R_{M,2}(\omega)\mathcal{H}(Q^\omega|P^{R_M,\omega}) < \infty. \tag{2.35}$$

– There exists M_0 such that, for all $M \geqslant M_0$, $R_M \in B(Q,\delta)$: indeed, for $\varphi \in BL_1(\Sigma)$, we have successively:

$$\begin{aligned} A_M(\varphi) &:= \left| \int \varphi(\theta,\omega)\,\mathrm{d}R_M(\theta,\omega) - \int \varphi(\theta,\omega)\,\mathrm{d}Q(\theta,\omega) \right|, \\ &= \left| \int \varphi(\theta,\omega) h(\omega) \left(\frac{\mathbb{1}_{[-M,M]}(\omega)}{\mu([-M,M])} - 1 \right) \mathrm{d}Q^\omega(\theta)\,\mathrm{d}\mu(\omega) \right|, \\ &\leqslant \int h(\omega) \left| \frac{\mathbb{1}_{[-M,M]}(\omega)}{\mu([-M,M])} - 1 \right| \mathrm{d}\mu(\omega). \end{aligned}$$

Let $M_0 > 0$ such that, for all $M \geqslant M_0$,

$$\int h(\omega) \left| \frac{\mathbb{1}_{[-M,M]}(\omega)}{\int h \mathbb{1}_{[-M,M]}\,\mathrm{d}\mu} - 1 \right| \mathrm{d}\mu(\omega) < \delta.$$

For all such (fixed) M,

$$A_M(\varphi) < \delta.$$

Taking the supremum on every $\varphi \in BL_1(\Sigma)$, we have the result.

– We have the following equality:

$$\liminf_{M\to\infty} \mathcal{H}(R_M|P^{R_M}) \leqslant \mathcal{H}(Q|P^Q). \tag{2.36}$$

Indeed, one has for all $M \geqslant 1$

$$\begin{aligned} \mathcal{H}(R_{M,2}|\mu_M) + e^{-1} &= \int_{\mathbf{R}} \left(\ln(h(\omega))h(\omega) + e^{-1} \right) \mathrm{d}\mu_M(\omega), \\ &= \int_{\mathbf{R}} \left(\ln(h(\omega))h(\omega) + e^{-1} \right) \frac{\mathbb{1}_{[-M,M]}(\omega)}{\mu[-M,M]}\,\mathrm{d}\mu(\omega), \\ &\leqslant \frac{1}{\mu[-M,M]} \int_{\mathbf{R}} \left(\ln(h(\omega))h(\omega) + e^{-1} \right) \mathrm{d}\mu(\omega), \end{aligned} \tag{2.37}$$

where we used in (2.37) the fact that $x \mapsto x\ln(x) + e^{-1} \geqslant 0$ on $[0,+\infty)$. Consequently, one obtains

$$\liminf_{M\to\infty} \mathcal{H}(R_{M,2}|\mu_M) \leqslant \mathcal{H}(Q_2|\mu). \tag{2.38}$$

Moreover, we have for all $M \geqslant 1$

$$\int_{\mathbf{R}} \mathrm{d}R_{M,2}(\omega)\mathcal{H}(Q^\omega|P^{R_M,\omega}) = \int_{\mathbf{R}} h(\omega)\mathcal{H}(Q^\omega|P^{R_M,\omega})\,\mathrm{d}\mu_M(\omega), \tag{2.39}$$

$$\leqslant \frac{1}{\mu[-M,M]} \int_{\mathbf{R}} h(\omega)\mathcal{H}(Q^\omega|P^{R_M,\omega})\,\mathrm{d}\mu(\omega), \tag{2.40}$$

since $h(\omega)\mathcal{H}(Q^\omega|P^{R_M,\omega}) \geqslant 0$ for all $\omega \in \mathrm{Supp}(\mu)$. But,

$$\mathcal{H}(Q^\omega|P^{R_M,\omega}) = \int_{\mathbf{R}} \mathrm{d}Q^\omega(\theta) \ln\left(\frac{\mathrm{d}Q^\omega}{\mathrm{d}P^{R_M,\omega}} \right), \tag{2.41}$$

$$= \mathcal{H}(Q^\omega|W^\omega) - \int_{\mathbf{R}} \mathrm{d}Q^\omega(\theta) \ln\left(\frac{\mathrm{d}P^{R_M,\omega}}{\mathrm{d}W^\omega} \right). \tag{2.42}$$

A calculation on the last term of (2.42), analogous to the proof of Proposition 2.29 (based on Girsanov's theorem) easily shows that for every $\omega \in \mathrm{Supp}(\mu)$, $M \mapsto$

$\mathcal{H}(Q^\omega|P^{R_M,\omega})$ converges to $\mathcal{H}(Q^\omega|P^{Q,\omega})$ as $M \to \infty$. Moreover, the assumptions made on f and g (see § 2.3) and the integrability assumption on μ ensure that there exists a μ-integrable function $H(\cdot)$ such that for all $M \geqslant 1$, $\mathcal{H}(Q^\omega|P^{R_M,\omega}) \leqslant H(\omega)$. Applying dominated convergence theorem to (2.40), we obtain

$$\liminf_{M\to\infty} \int_{\mathbf{R}} \mathrm{d}R_{M,2}(\omega)\mathcal{H}(Q^\omega|P^{R_M,\omega}) \leqslant \int_{\mathbf{R}} \mathrm{d}Q_2(\omega)\mathcal{H}(Q^\omega|P^{Q,\omega}). \tag{2.43}$$

Using (2.38) and (2.43) in (2.35) gives the desired result (2.36).

We are now in position to prove (2.34): for each $M \geqslant M_0$, we consider $R = R_M$ previously defined. We have for all $M \geqslant M_0$,

$$\inf_{R\in B(Q,\delta)} \mathcal{H}(R|P_M^R) \leqslant \mathcal{H}(R_M|P_M^{R_M}),$$

So,

$$\liminf_{M\to\infty} \inf_{R\in B(Q,\delta)} \mathcal{H}(R|P_M^R) \leqslant \liminf_{M\to\infty} \mathcal{H}(R_M|P_M^{R_M}) \leqslant \mathcal{H}(Q|P^Q).$$

Taking the supremum on $\delta > 0$ in the last inequality, we have the result. □

Proposition 2.37. *For all $Q \in \mathcal{M}_1(\Sigma)$,*

$$\tilde{\mathcal{G}}(Q) \geqslant \mathcal{H}(Q|P^Q). \tag{2.44}$$

Proof of Proposition 2.37. – The function $Q \mapsto \mathcal{H}(Q|P^Q)$ is lower semi-continuous w.r.t. the topology of weak convergence: indeed, for all $M > 0$, the contraction principle says in particular that $Q \mapsto \mathcal{H}(Q|P_M^Q)$ is lower semi-continuous w.r.t. the topology of weak convergence. Thus, for all ω, $Q \mapsto \mathcal{H}(Q^\omega|P^{\omega,Q})$ is lower semi-continuous (it corresponds to the case $\mu := \delta_\omega$). Consequently, for all Q, and $Q_n \to Q$, we have successively:

$$\begin{aligned}\int \mathrm{d}Q_2(\omega)\mathcal{H}(Q^\omega|P^{Q,\omega}) &\leqslant \int \mathrm{d}Q_2(\omega) \liminf_{n\to\infty} \mathcal{H}(Q_n^\omega|P^{Q_n,\omega}), \quad (Q \mapsto \mathcal{H}(Q^\omega|P^{\omega,Q}) \text{ l.s-c.}),\\ &\leqslant \liminf_{n\to\infty} \int \mathrm{d}Q_{n,2}(\omega)\mathcal{H}(Q_n^\omega|P^{Q_n,\omega}).\end{aligned}$$

We used Fatou's lemma for varying measures (see [74]), at the second line.

So, $Q \mapsto \int \mathrm{d}Q_2(\omega)\mathcal{H}(Q^\omega|P^{Q,\omega})$ is sequentially lower semi-continuous and so lower semi-continuous, as the weak convergence topology is metrizable. But, for all μ, $Q \mapsto \mathcal{H}(Q_2|\mu)$ is lower semi-continuous w.r.t. the weak topology (Contraction principle). Following (2.35), we obtain the result.

– Proof of (2.44): for all $\varepsilon > 0$, for all $Q \in \mathcal{M}_1(\Sigma)$, by lower semi-continuity, there exists $\delta > 0$ such that, for all $R \in B(Q,\delta)$,

$$\mathcal{H}(R|P^R) \geqslant \min(\mathcal{H}(Q|P^Q) - \varepsilon, \frac{1}{\varepsilon}).$$

But,

$$\mathcal{H}(R|P^R) = \mathcal{H}(R_2|\mu) + \int \mathrm{d}R_2(\omega)\mathcal{H}(R^\omega|P^{R,\omega}).$$

Let $M > 0$ and R such that $R_2 \ll \mu_M$: then $\frac{\mathrm{d}R_2}{\mathrm{d}\mu} = \frac{1}{\mu([-M,M])}\frac{\mathrm{d}R_2}{\mathrm{d}\mu_M}$ and $\mathcal{H}(R_2|\mu) = -\ln\mu([-M,M]) + \mathcal{H}(R_2|\mu_M)$. So, for all $R \in B(Q,\delta)$ such that $R_2 << \mu_M$, we have:

$$\mathcal{H}(R|P_M^R) \geqslant \min(\mathcal{H}(Q|P^Q) - \varepsilon, \frac{1}{\varepsilon}) + \ln\mu([-M,M]). \tag{2.45}$$

(2.45) is still true for all $R \in B(Q,\delta)$ such that R_2 is not absolutely continuous w.r.t. μ_M. Thus,

$$\inf_{R\in B(Q,\delta)} \mathcal{H}(R|P_M^R) \geqslant \min(\mathcal{H}(Q|P^Q) - \varepsilon, \frac{1}{\varepsilon}) + \ln \mu([-M,M]). \tag{2.46}$$

Taking $\liminf_{M\to\infty}$ on both sides of (2.46), we get the result. □

3.3.4 Coercivity of I in the general case

Let us now prove that the rate function $Q \mapsto \mathcal{H}(Q|P^Q)$ is coercive:

Proposition 2.38. *In the general case, the function $Q \mapsto \mathcal{H}(Q|P^Q)$ is coercive.*

Proof of Proposition 2.38. Let $\alpha > 0$. We want to show that the set of measures $K_\alpha := \left\{R \in \mathcal{M}_1(\Sigma);\ \mathcal{H}(R|P^R) \leqslant \alpha\right\}$ is compact. We only have to show the sequential compactness: let $(R_p)_{p\geqslant 1}$ a sequence of elements of K_α; we show that there exists some converging subsequence in K_α.

For all M, let $\psi_M : \mathbf{R} \to \mathbf{R}$ such that ψ_M is piecewise linear, $\psi_M \equiv 1$ on $[-M+1, M-1]$, $\phi_M \equiv 0$ on $[-M,M]^c$.

For $R \in K_\alpha$, we define R^M as:

$$\forall A, B,\ R^M(A\otimes B) := \int_B \int_A \psi_M(\omega)\,\mathrm{d}R^\omega(\theta)\,\mathrm{d}R_2(\omega).$$

R^M belongs to $\mathcal{M}_{\leqslant 1}(\Sigma)$, the set of positive measures on $\Sigma = \mathcal{C}([0,T],\mathbf{S}) \times \mathbf{R}$ of mass lower or equal than 1. $\mathcal{M}_{\leqslant 1}$ is compact for the weak topology and for all $M > 0$, the function $R \in \mathcal{M}_{\leqslant 1} \mapsto \mathcal{H}(R|P_M^R)$ is coercive. So, for all $\alpha > 0$, the set $\tilde{K}_\alpha^M := \left\{R \in \mathcal{M}_{\leqslant 1};\ \mathcal{H}(R|P_M^R) \leqslant \alpha\right\}$ is compact w.r.t. the weak topology.

For all $M > 0$, $R_2^M \ll \mu_M$ and $\frac{\mathrm{d}R_2^M}{\mathrm{d}\mu_M} = \psi_M \mu([-M,M]) \frac{\mathrm{d}R_2}{\mathrm{d}\mu}$. Since $\psi_M \leqslant 1$, for all $R \in K_\alpha$, $R^M \in \tilde{K}_\alpha^M$.

For any sequence $(R_p)_p \in K_\alpha$, we consider the double sequence $(R_p^M)_{M,p}$. For all M, the sequence $(R_p^M)_p$ is in a compact. By a diagonal argument, there exists an extraction ϕ and a sequence of $\mathcal{M}_{\leqslant 1}$, $(\mathrm{d}R_*^M)(\theta,\omega) = \mathrm{d}R_{2,*}^M(\omega) \otimes \mathrm{d}R_*^{M,\omega}(\theta)$, such that, for all $M > 0$, $R_{\phi(p)}^M$ converges weakly to R_*^M as $p \to \infty$. Since $R_*^M \in \tilde{K}_\alpha^M$, $R_{2,*}^M \ll \mu$ and there exists an nonnegative function g_M such that $\mathrm{d}R_{2,*}^M(\omega) = g_M(\omega)\,\mathrm{d}\mu(\omega)$. Considering that $R_{\phi(p)}^M(\varphi) = R_{\phi(p)}^{M+1}(\varphi)$ for all $\varphi(\theta,\omega) := l(\theta)k(\omega)$, with $\mathrm{Supp}(k) \subseteq [-M,M]$, we see that $g_{M+1|[-M,M]} = g_M$. Thus, one can define g such that $g_{|[-M,M]} = g_M$. Furthermore, we see that, for μ-almost every ω, $\mathrm{d}R_*^{M,\omega}(\theta) = \mathrm{d}R_*^{M+1,\omega}(\theta)$ and thus, does not depend on M (and the common value is denoted as R_*^ω). We define $\mathrm{d}R_*(\theta,\omega) := g(\omega)\,\mathrm{d}\mu(\omega) \otimes \mathrm{d}R_*^\omega(\theta)$.

We easily see that $R_{\phi(p)}(\varphi)$ converges to $R_*(\varphi)$, for all functions $\varphi(\theta,\omega) = l(\theta) \times k(\omega)$, k with compact support. But, we notice that the set $\{R_2;\ R \in K_\alpha\}$ is uniformly absolutely continuous w.r.t. μ (see [20, Prop. 23.4]):

$$\forall \varepsilon > 0, \exists \delta > 0, \forall A \in \mathcal{B}(\mathbf{R}), (\mu(A) < \delta \Rightarrow \forall R \in K_\alpha, R_2(A) < \varepsilon)\,.$$

In particular, the sequence $(R_{2,\phi(p)})_p$ is tight. So, we have the weak convergence of $\left(R_{\varphi(p)}\right)_{p\geqslant 1}$ to $R_* \in \mathcal{M}_1$, which is the result. □

Following [29, Th. 4.2.16 (b), p. 131], we conclude this section by proving that the *weak* large deviation principle proved before is in fact *strong*:

Lemma 2.39. *For every closed set F, we have:*

$$\inf_{Q\in F} \mathcal{H}(Q|P^Q) \leqslant \limsup_{M\to\infty} \inf_{Q\in F} \mathcal{H}(Q|P^Q_M).$$

Proof of Lemma 2.39. Let F be closed for the weak topology, and $M > 0$. Let $Q \in F$ such that $Q_2 \ll \mu_M$. Then,

$$\mathcal{H}(Q|P^Q) = \mathcal{H}(Q|P^Q_M) + \ln \mu([-M, M]).$$

So,

$$\mathcal{H}(Q|P^Q_M) \geqslant \inf_{Q\in F} \mathcal{H}(Q|P^Q) - \ln \mu([-M, M]),$$

and, the previous inequality is still true if $Q_2 \not\ll \mu_M$. So,

$$\inf_{Q\in F} \mathcal{H}(Q|P^Q_M) \geqslant \inf_{Q\in F} \mathcal{H}(Q|P^Q) - \ln \mu([-M, M]).$$

The result follows. □

3.4 Zeros of the rate function $\mathcal{G}(\cdot)$

The proof of Theorem 2.11 will be completed when we characterize the zeros of the rate function $Q \mapsto \mathcal{G}(Q) = \mathcal{H}(Q|P^Q)$ that governs the large deviation of $(P_N)_{N\geqslant 1}$:

Proposition 2.40. *Any zero of the rate function $\mathcal{G}(\cdot)$ satisfies the weak formulation of McKean-Vlasov equation* (2.8).

Proof of Proposition 2.40. Any $Q \in \mathcal{M}_1(\Sigma)$ is a zero of $\mathcal{G}(Q) = \mathcal{H}(Q|P^Q)$ if and only if $Q = P^Q$. Recall (2.28): $\mathrm{d}P^Q(\theta, \omega) = \mathrm{d}\mu(\omega)\,\mathrm{d}P^{\omega,Q}(\theta)$, where $P^{\omega,Q}$ is the law of the diffusion (2.25). Consequently, the second marginal with respect to the disorder of Q is μ and, conditionally on ω its marginal on $\mathbf{S}$ is the law of the diffusion (2.25). But the weak formulation (2.8) is simply the Fokker-Planck equation associated to the diffusion (2.25). That concludes the proof of Proposition 2.40. □

3.5 From averaged large deviation to quenched convergence

The purpose of this paragraph is to prove Proposition 2.13, that is to derive a quenched convergence result from the exponential estimates on the averaged convergence of the law of L_N towards δ_{Q^*} obtained from the LDP proved in Theorem 2.11.

Proof of Proposition 2.13. Let us fix some $\delta > 0$ and consider

$$\alpha(\delta) := \inf \{\mathcal{G}(Q);\, Q \in B(Q^*, \delta)^c\},$$

where $\mathcal{G}(\cdot)$ is the good rate function defined in (2.27) that controls the LDP for the averaged law of L_N (recall Theorem 2.11) and $B(Q^*, \delta)$ is the open ball of radius δ for the distance defined in (2.29). Since $\mathcal{G}(\cdot)$ is a good rate function, it is routine to show that $\alpha(\delta) > 0$. Then a consequence of Theorem 2.11 is

$$\limsup_{N\to\infty} \frac{1}{N} \ln \mathbf{P}_N(B(Q^*, \delta)^c) \leqslant -\alpha(\delta) < 0.$$

In particular, there exists $N_0 = N_0(\delta)$ such that for all $N \geqslant N_0$

$$\mathbf{P}_N(B(Q^*, \delta)^c) \leqslant e^{-N\frac{\alpha(\delta)}{2}}. \tag{2.47}$$

But

$$\mathbf{P}_N(B(Q^*,\delta)^c) = \mathbb{E}\left(\mathbf{E}\left(\mathbb{1}_{\{(\underline{\theta},\underline{\omega});\, L_N[\underline{\theta},\underline{\omega}]\in B(Q^*,\delta)^c\}}\right)\right), \tag{2.48}$$

$$= \mathbb{E}\left(\mathbf{P}_N^{\underline{\omega}}(B(Q^*,\delta)^c)\right), \tag{2.49}$$

where $\mathbf{P}_N^{\underline{\omega}}(B(Q^*,\delta)^c) = \mathbf{E}\left(\mathbb{1}_{\{(\underline{\theta},\underline{\omega});\, L_N[\underline{\theta},\underline{\omega}]\in B(Q^*,\delta)^c\}}\right)$ is the probability (conditioned on the disorder (ω)) that L_N is outside the ball $B(Q^*,\delta)$ *under the law of the Brownian motion only*. Consequently, one obtains from (2.47), (2.49) and from Markov inequality that

$$\mathbb{P}\left(\mathbf{P}_N^{\underline{\omega}}(B(Q^*,\delta)^c) \geqslant e^{-N\frac{\alpha(\delta)}{4}}\right) \leqslant e^{-N\frac{\alpha(\delta)}{4}}. \tag{2.50}$$

In particular,

$$\sum_{N\geqslant 1} \mathbb{P}\left(\mathbf{P}_N^{\underline{\omega}}(B(Q^*,\delta)^c) \geqslant e^{-N\frac{\alpha(\delta)}{4}}\right) < \infty. \tag{2.51}$$

Applying Borel-Cantelli lemma ([11, Th. 4.3, p. 59]) we obtain that for $\mathbb{P}-$almost every sequence (ω),

$$\exists N_1 = N_1(\delta) \geqslant 1, \forall N \geqslant N_1, \quad \mathbf{P}_N^{\underline{\omega}}(B(Q^*,\delta)^c) < e^{-N\frac{\alpha(\delta)}{4}}. \tag{2.52}$$

Note that the assertion "for $\mathbb{P}-$almost every sequence (ω)" actually depends on $\delta > 0$. If we restrict to a countable set Δ of values of $\delta > 0$, there exists a set of sequences (ω) with $\mathbb{P}$-probability 1, such that for all $\delta \in \Delta$ (2.52) is true. For such (ω), for all $\delta \in \Delta$, (2.52) gives: $\mathbf{P}_N^{\underline{\omega}}(B(Q^*,\delta)^c) \to 0$ as $N \to \infty$, which proves Proposition 2.13. □

4 Quenched convergence of the empirical flow

In this section we prove Theorem 2.16. Reformulating (2.1) in terms of $\nu_N^{(\omega)}$, we have:

$$\forall i = 1,\dots N,\ \forall t \in [0,T],\ \theta_{i,t} = \xi_i + \int_0^t b[\theta_{i,s},\omega_i,\nu_{N,s}]\,\mathrm{d}s + \int_0^t c(\theta_{i,s},\omega_i)\,\mathrm{d}s + B_{i,t}, \tag{2.53}$$

where we recall that $b[\theta,\omega,m] := \int b(\theta,\theta',\omega,\omega')\,\mathrm{d}m(\theta',\omega')$.

4.1 Outline of the proof

The idea of the proof of Theorem 2.16 is the following: we show the tightness of the sequence $(\nu_N^{(\omega)})$ firstly in $\mathbf{D}([0,T],(\mathcal{M}_1,v))$ where $\mathcal{M}_1$ is endowed with the topology of vague convergence (recall Notations in § 2.2), which is quite simple since $\mathcal{C}_c(\mathbf{S}\times\mathbf{R})$ is separable. Then by an argument of boundedness of the second marginal of any accumulation point, thanks to (2.9), we show the tightness in $\mathbf{D}([0,T],(\mathcal{M}_1,w))$, when $\mathcal{M}_1$ is endowed with the weak convergence topology.

We decompose the proof of Theorem 2.16 into several steps:

1. Tightness of $\mathcal{L}(\nu_N^{(\omega)})$ in $\mathbf{D}([0,T],(\mathcal{M}_1,v))$,
2. Equation verified by any accumulation point,
3. Characterization of the marginals of any limit,
4. Tightness in $\mathbf{D}([0,T],(\mathcal{M}_1,w))$.

4.2 Equation verified by $\nu_N^{(\omega)}$

For $\varphi \in \mathcal{C}_b^2(\mathbf{S} \times \mathbf{R})$, we denote by φ', φ'' the first and second derivative of φ with respect to the first variable. Moreover, if $m \in \mathcal{M}_1(\mathbf{S} \times \mathbf{R})$, then $\langle m\,,\,\varphi\rangle$ stands for $\int_{\mathbf{S}\times\mathbf{R}} \varphi(\theta,\omega)\,\mathrm{d}m(\theta,\omega)$.

Applying Ito's formula to (2.53), we obtain, for all $\varphi \in \mathcal{C}_b^2(\mathbf{S} \times \mathbf{R})$,

$$\left\langle \nu_{N,t}^{(\omega)}\,,\,\varphi\right\rangle = \left\langle \nu_{N,0}^{(\omega)}\,,\,\varphi\right\rangle + \int_0^t \mathrm{d}s \left\{ \frac{1}{2}\left\langle \nu_{N,s}^{(\omega)}\,,\,\varphi''\right\rangle + \left\langle \nu_{N,s}^{(\omega)}\,,\,\varphi'(b[\cdot,\cdot,\nu_{N,s}^{(\omega)}] + c)\right\rangle \right\} + M_{N,t}^{(\omega)}(\varphi), \tag{2.54}$$

where

$$M_{N,t}^{(\omega)}(\varphi) := \frac{1}{N}\sum_{j=1}^N \int_0^t \varphi'(\theta_j,\omega_j)\,\mathrm{d}B_{j,s} \tag{2.55}$$

is a square-integrable martingale (recall that φ' is bounded).

4.3 Tightness of $\mathcal{L}(\nu_N^{(\omega)})$ for the vague topology $\mathbf{D}([0,T],(\mathcal{M}_1,v))$

The space $\mathcal{C}_c(\mathbf{S}\times\mathbf{R})$ is separable: let $(\varphi_k)_{k\geqslant 1}$ (elements of $\mathcal{C}^\infty(\mathbf{S}\times\mathbf{R})$) a dense sequence in $\mathcal{C}_c(\mathbf{S}\times\mathbf{R})$, and let $\varphi_0 \equiv 1$. We define $\Omega := \mathbf{D}([0,T],(\mathcal{M}_1,v))$ and the applications Π_φ, $\varphi \in \mathcal{C}_c(\mathbf{S}\times\mathbf{R})$ by:

$$\begin{array}{rccc} \Pi_\varphi : & \Omega & \to & \mathbf{D}([0,T],\mathbf{R}) \\ & m & \mapsto & \langle m\,,\,\varphi\rangle\,. \end{array}$$

Let $(P_N)_{N\geqslant 1}$ a sequence of probabilities on Ω. We recall the following result:

Lemma 2.41 ([73], Th. 2.1, p. 56)**.** *If for all $k \geqslant 0$, the sequence $(P_N \circ \Pi_{\varphi_k}^{-1})_{N\geqslant 1}$ is tight in $\mathcal{M}_1(\mathbf{D}([0,T],\mathbf{R}))$, then the sequence $(P_N)_{N\geqslant 1}$ is tight in $\mathcal{M}_1(\mathbf{D}([0,T],(\mathcal{M}_1,v)))$.*

Hence, it suffices to have a criterion for tightness in $\mathbf{D}([0,T],\mathbf{R})$. Let $(X_t^N)_{N\geqslant 1}$ be a sequence of processes in $\mathbf{D}([0,T],\mathbf{R})$ and $(\mathcal{F}_t^N)_{N\geqslant 1}$ a sequence of filtrations such that X^N is $\mathcal{F}^N$-adapted. Let $\phi^N = \{\text{stopping times for } \mathcal{F}^N\}$. We have (cf. Billingsley [11]):

Lemma 2.42 (Aldous' criterion)**.** *If the following holds,*

1. *$\mathcal{L}\left(\sup_{t\leqslant T}\left|X_t^N\right|\right)_{N\geqslant 1}$ is tight,*
2. *For all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that*

$$\limsup_N \sup_{S,S'\in\phi^N;S\leqslant S'\leqslant (S+\delta)\wedge T} \mathbf{P}\left(\left|X_S^N - X_{S'}^N\right| > \eta\right) \leqslant \varepsilon,$$

then $\mathcal{L}(X^N)_{N\geqslant 1}$ is tight.

Proposition 2.43. *The sequence $\mathcal{L}(\nu_N^{(\omega)})$ is tight in $\mathbf{D}([0,T],(\mathcal{M}_1,v))$.*

Proof of Proposition 2.43. For all $\varepsilon > 0$, for all $k \geqslant 1$ (the case $k = 0$ is straightforward),

$$\mathbf{P}\left(\sup_{t\leqslant T}\left|\left\langle \nu_{N,t}^{(\omega)}\,,\,\varphi_k\right\rangle\right| > \frac{1}{\varepsilon}\right) \leqslant \varepsilon \,\|\,\varphi_k\,\|_\infty\, \mathbf{E}\left[\sup_{t\leqslant T}\left|\underbrace{\left\langle \nu_{N,t}^{(\omega)}\,,\,1\right\rangle}_{=1}\right|\right], \quad \text{(Markov Inequality)}.$$

The tightness of $\mathcal{L}\left(\sup_{t\leqslant T}\left|\left\langle \nu_{N,t}^{(\omega)}\,,\,\varphi_k\right\rangle\right|\right)_{N\geqslant 1}$ for all $k \geqslant 0$ follows.

For all $k \geqslant 1$, one can rewrite (2.54) (for $\varphi = \varphi_k$) into the sum of a process of bounded variations $A^{(\omega)}_{N,t}(\varphi_k)$ and the martingale part $M^{(\omega)}_{N,t}(\varphi_k)$ defined in (2.55):

$$\left\langle \nu^{(\omega)}_{N,t}\,,\,\varphi_k \right\rangle = \left\langle \nu^{(\omega)}_{N,0}\,,\,\varphi_k \right\rangle + A^{(\omega)}_{N,t}(\varphi_k) + M^{(\omega)}_{N,t}(\varphi_k),$$

Then it suffices to verify Lemma 2.42, (2) for A and M separately. For all $\varepsilon > 0$ and $\eta > 0$, for all stopping times $S, S' \in \phi^N; S \leqslant S' \leqslant (S+\delta) \wedge T$, we have:

$$\begin{aligned} a_N &:= \mathbf{P}\left(\left|A^{(\omega)}_{N,S'}(\varphi_k) - A^{(\omega)}_{N,S}(\varphi_k)\right| > \eta\right), \\ &\leqslant \frac{1}{\eta}\mathbf{E}\left[\int_S^{S'} \mathrm{d}s \left|\left\langle \nu^{(\omega)}_{N,s}\,,\,\varphi_k'(b[\cdot,\cdot,\nu^{(\omega)}_{N,s}] + c)\right\rangle\right|\right] + \frac{1}{\eta}\mathbf{E}\left[\frac{1}{2}\int_S^{S'} \mathrm{d}s \left|\left\langle \nu^{(\omega)}_{N,s}\,,\,\varphi_k''\right\rangle\right|\right], \\ &\leqslant \frac{C}{\eta}\mathbf{E}\left[S' - S\right] \leqslant \varepsilon, \quad \text{for } \delta \text{ sufficiently small,} \end{aligned} \tag{2.56}$$

where we used in (2.56) that φ_k is of compact support for $k \geqslant 1$; in particular the function $(\theta,\omega) \mapsto \varphi_k'(\theta,\omega)c(\theta,\omega)$ is bounded. Furthermore,

$$\begin{aligned} \mathbf{P}\left(\left|M^{(\omega)}_{N,S'}(\varphi_k) - M^{(\omega)}_{N,S}(\varphi_k)\right| > \eta\right) &= \mathbf{P}\left(\left|M^{(\omega)}_{N,S'}(\varphi_k) - M^{(\omega)}_{N,S}(\varphi_k)\right|^2 > \eta^2\right), \\ &\leqslant \frac{1}{\eta^2}\mathbf{E}\left[\left|M^{(\omega)}_{N,S'}(\varphi_k) - M^{(\omega)}_{N,S}(\varphi_k)\right|^2\right], \\ &\leqslant \frac{1}{(N\eta)^2}\mathbf{E}\left[\sum_{i=1}^N \int_S^{S'} \varphi_k'^2(\theta_{i,s},\omega_i)\,\mathrm{d}s\right] \leqslant \frac{C}{N\eta^2}\delta, \end{aligned}$$

for some constant $C > 0$ depending of the supremum norm of φ_k'. So Aldous' criterion is verified for both processes A and M. Applying Lemma 2.41, Proposition 2.43 is proved. □

At this point, $\mathcal{L}(\nu^{(\omega)}_N)$ is tight in $\mathbf{D}([0,T],(\mathcal{M}_1,v))$.

4.4 Equation satisfied by any accumulation point in $\mathbf{D}([0,T],(\mathcal{M}_1,v))$

Using hypothesis (2.10), it is easy to show that the following equation is satisfied for every accumulation point ν, for every $\varphi \in \mathcal{C}^2_c(\mathbf{S}\times\mathbf{R})$ (we use here that $\mathbf{S}$ is compact):

$$\langle \nu_t\,,\,\varphi\rangle = \langle \nu_0\,,\,\varphi\rangle + \int_0^t \mathrm{d}s\,\langle \nu_s\,,\,\varphi'(b[\cdot,\cdot,\nu_s] + c)\rangle + \frac{1}{2}\int_0^t \mathrm{d}s\,\langle \nu_s\,,\,\varphi''\rangle. \tag{2.57}$$

For any accumulation point ν, the following lemma gives a uniform bound for the second marginal of ν:

Lemma 2.44. *Let ν be an accumulation point of $(\nu^{(\omega)}_N)_N$ in $\mathbf{D}([0,T],(\mathcal{M}_1,v))$. For all $t \in [0,T]$, we define by $(\nu_{t,2})$ the second marginal of ν_t:*

$$\forall A \in \mathcal{B}(\mathbf{R}), \quad (\nu_{2,t})(A) = \int_{\mathbf{S}\times A} \mathrm{d}\nu_t(\theta,\omega).$$

Then, for all $t \in [0,T]$,

$$\int_{\mathbf{R}} \sup_{\theta\in\mathbf{S}} |c(\theta,\omega)|\,\mathrm{d}\nu_{2,t}(\omega) \leqslant \int_{\mathbf{R}} \sup_{\theta\in\mathbf{S}} |c(\theta,\omega)|\,\mathrm{d}\mu(\omega). \tag{2.58}$$

Proof of Lemma 2.44. Let ϕ be a $\mathcal{C}^2$ positive function on $\mathbf{R}$ such that $\phi \equiv 1$ on $[-1,1]$, $\phi \equiv 0$ on $[-2,2]$ and $\| \phi \|_\infty \leqslant 1$. Let,

$$\forall k \geqslant 1, \quad \phi_k := \omega \mapsto \phi\left(\frac{\omega}{k}\right).$$

Then $\phi_k \in \mathcal{C}_c^2(\mathbf{R})$ and $\phi_k(\omega) \to_{k\to\infty} 1$, for all $\omega \in \mathbf{R}$. We have also for all $\omega \in \mathbf{R}$, $|\phi_k(\omega)| \leqslant \| \phi \|_\infty$, $|\phi_k'(\omega)| \leqslant \| \phi' \|_\infty$, $|\phi_k''(\omega)| \leqslant \| \phi'' \|_\infty$. We have successively, denoting $S(\omega) := \sup_{\theta\in\mathbf{S}} |c(\theta,\omega)|$,

$$\begin{aligned}
\int_{\mathbf{S}\times\mathbf{R}} S(\omega')\,\mathrm{d}\nu_t(\theta,\omega') &= \int_{\mathbf{S}\times\mathbf{R}} \liminf_{k\to\infty} \phi_k(\omega')S(\omega')\,\mathrm{d}\nu_t(\theta,\omega'),\\
&\leqslant \liminf_{k\to\infty} \int_{\mathbf{S}\times\mathbf{R}} \phi_k(\omega')S(\omega')\,\mathrm{d}\nu_t(\theta,\omega'), \text{ (Fatou's lemma)},\\
&= \liminf_{k\to\infty} \lim_{N\to\infty} \int_{\mathbf{S}\times\mathbf{R}} \phi_k(\omega')S(\omega')\,\mathrm{d}\nu_{N,t}^{(\omega)}(\theta,\omega'), \qquad (2.59)\\
&\leqslant \lim_{N\to\infty} \int_{\mathbf{S}\times\mathbf{R}} S(\omega')\,\mathrm{d}\nu_{N,t}^{(\omega)}(\theta,\omega'), (\text{since } \| \phi \|_\infty \leqslant 1).
\end{aligned}$$

Equality (2.59) is true since $(\theta,\omega) \mapsto \phi_k(\omega)S(\omega)$ is of compact support in $\mathbf{S}\times\mathbf{R}$ (recall that S is supposed to be continuous by hypothesis).

But, by definition of $\nu_{N,t}^{(\omega)}$, and using the hypothesis (2.9) concerning μ, we have,

$$\lim_{N\to\infty} \int_{\mathbf{S}\times\mathbf{R}} S(\omega')\,\mathrm{d}\nu_{N,t}^{(\omega)}(\theta,\omega') = \int_{\mathbf{R}} S(\omega')\,\mathrm{d}\mu(\omega'). \qquad (2.60)$$

The result follows and Lemma 2.44 is proved. □

4.5 Tightness of $\mathcal{L}(\nu_N^{(\omega)})$ in the weak topology $\mathbf{D}([0,T],(\mathcal{M}_1,w))$

The purpose of this paragraph is to show the tightness of the sequence $\nu_N^{(\omega)}$ in the space $\mathbf{D}([0,T],(\mathcal{M}_1,w))$ that is when $\mathcal{M}_1$ is endowed with the topology of weak convergence. We have the following lemma (see [60]):

Lemma 2.45. *Let $(X_N)_{N\geqslant 1}$ be a sequence of processes in $\mathbf{D}([0,T],(\mathcal{M}_1,w))$ and X a process belonging to $\mathcal{C}([0,T],(\mathcal{M}_F,w))$. Then,*

$$X_N \xrightarrow{\mathcal{L}} X \Leftrightarrow \begin{cases} X_N \xrightarrow{\mathcal{L}} X & \text{in } \mathbf{D}([0,T],(\mathcal{M}_F,v)),\\ \langle X_N\,,\,1\rangle \xrightarrow{\mathcal{L}} \langle X\,,\,1\rangle & \text{in } \mathbf{D}([0,T],\mathbf{R}). \end{cases}$$

So, it suffices to show, for any accumulation point ν:

1. $\langle \nu\,,\,1\rangle = 1$: Equality (2.57) is true for all $\varphi \in \mathcal{C}_c^2(\mathbf{S}\times\mathbf{R})$, so in particular for $\varphi_k(\theta,\omega) := \phi_k(\omega)$ for the function $\phi(\cdot)$ defined in the proof of Lemma 2.44. Using inequality (2.58), we can apply dominated convergence theorem to (2.57). We then have $\langle \nu_t\,,\,1\rangle = 1$, for all $t\in[0,T]$. The fact that (2.57) is verified for all $\varphi \in \mathcal{C}_b^2(\mathbf{S}\times\mathbf{R})$ can be shown in the same way.
2. Continuity of the limit: For all $0 \leqslant s \leqslant t \leqslant T$, for all $\varphi \in \mathcal{C}_b^2(\mathbf{S}\times\mathbf{R})$,

$$\begin{aligned}
|\langle \nu_t\,,\,\varphi\rangle - \langle \nu_s\,,\,\varphi\rangle| \leqslant K\int_s^t |\langle \nu_u\,,\,\varphi' b[\cdot,\cdot,\nu_u]\rangle|\,\mathrm{d}u + \frac{1}{2}\int_s^t |\langle \nu_u\,,\,\varphi''\rangle|\,\mathrm{d}u\\
+ \int_s^t |\langle \nu_u\,,\,\varphi' c\rangle|\,\mathrm{d}u \leqslant C\times|t-s|\,, \quad \text{for some constant } C.
\end{aligned}$$

Noticing that we used again Lemma 2.44 to bound the last term, we have the result.

Remark 2.46. It is then easy to see that the second marginal (on the disorder) of any accumulation point is μ.

Consequently, $\mathcal{L}(\nu_N^{(\omega)})$ is tight in $\mathbf{D}([0,T],(\mathcal{M}_1,w))$ and any accumulation point ν satisfy (2.57). Theorem 2.16 is proved.

5 Uniqueness in McKean-Vlasov equation

The purpose of this section is to prove Proposition 2.20 concerning the uniqueness in McKean-Vlasov equation (2.11). We follow the lines of Oelschläger [67, Lemma 10, p. 474], who proved a similar result for diffusions without disorder.

We can rewrite (2.11) in a more compact way:

$$\forall \varphi \in \mathcal{C}_b^2(\mathbf{S}\times\mathbf{R}), 0 \leqslant t \leqslant T, \quad \langle \nu_t\,,\,\varphi\rangle = \langle \nu_0\,,\,\varphi\rangle + \int_0^t \langle \nu_s\,,\,L(\nu_s)(\varphi)\rangle\,\mathrm{d}s, \tag{2.61}$$

where,

$$\forall \theta \in \mathbf{S}, \forall \omega \in \mathbf{R}, \quad L(\nu)(\varphi)(\theta,\omega) := \frac{1}{2}\varphi''(\theta,\omega) + \varphi'(\theta,\omega)\left(b[\theta,\omega,\nu] + c(\theta,\omega)\right).$$

Let $t \mapsto \nu_t$ be any solution of (2.61). One can then introduce the following SDE:

$$\left\{\begin{array}{rcl} \mathrm{d}\xi_t &=& (b[\xi_t,\omega_t,\nu_t] + c(\xi_t,\omega_t))\,\mathrm{d}t + \mathrm{d}W_t, \quad (\xi_0,\omega_0)\sim\nu_0, \\ \mathrm{d}\omega_t &=& 0. \end{array}\right. \tag{2.62}$$

The system (2.62) has a unique (strong) solution $(\xi_t,\omega_t)_{t\in[0,T]} = (\xi_t,\omega_0)_{t\in[0,T]}$. The proof of uniqueness in (2.61) consists in two steps:

1. Let us prove firstly that for all $t\in[0,T]$, ν_t is equal to the law $\mathcal{L}_{(\xi_t,\omega_t)} \in \mathcal{M}_1(\mathbf{S}\times\mathbf{R})$ of the strong solution of system (2.62).

   Indeed, if one applies Ito's formula to (2.62), one obtains:

   $$\forall \varphi \in \mathcal{C}_b^2(\mathbf{S}\times\mathbf{R}), \forall t\in[0,T], \quad \left\langle \mathcal{L}_{(\xi_t,\omega_t)}\,,\,\varphi\right\rangle = \langle \nu_0\,,\,\varphi\rangle + \int_0^t \left\langle \mathcal{L}_{(\xi_s,\omega_s)}\,,\,L(\nu_s)(\varphi)\right\rangle \mathrm{d}s.$$

   In particular, both $(\mathcal{L}_{(\xi_t,\omega_t)})_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$, processes in $\mathbf{D}([0,T],\mathcal{M}_1(\mathbf{S}\times\mathbf{R}))$, solve the following equation:

   $$\forall \varphi \in \mathcal{C}^2(\mathbf{S}\times\mathbf{R}), \forall t\in[0,T], \quad \langle P_t\,,\,\varphi\rangle = \langle \nu_0\,,\,\varphi\rangle + \int_0^t \langle P_s\,,\,L(\nu_s)(\varphi)\rangle\,\mathrm{d}s. \tag{2.63}$$

   Let $(P_t)_{t\in[0,T]} \in \mathbf{D}([0,T],\mathcal{M}_1(\mathbf{S}\times\mathbf{R}))$ be any solution of the previous equation (2.63). Let $(t,\theta,\omega)\mapsto\varphi_t(\theta,\omega)\in\mathcal{C}_b([0,T]\times\mathbf{S}\times\mathbf{R})$ with one bounded continuous partial derivative with respect to t and bounded first and second partial derivatives

with respect to $\theta \in \mathbf{S}$. Then, for any $l \geqslant 1$, for $\delta = t/l$,

$$\begin{aligned}
\langle P_t\,,\,\varphi_t\rangle - \langle \nu_0\,,\,\varphi_0\rangle &= \sum_{k=0}^{l-1} \Big\langle P_{(k+1)\delta}\,,\,\varphi_{(k+1)\delta}\Big\rangle - \langle P_{k\delta}\,,\,\varphi_{k\delta}\rangle,\\
&= \sum_{k=0}^{l-1} \left\{ \Big\langle P_{(k+1)\delta}\,,\,\varphi_{(k+1)\delta}\Big\rangle - \left\langle P_{k\delta}\,,\,\varphi_{(k+1)\delta} - \int_{k\delta}^{(k+1)\delta} \partial_s\varphi_s\,\mathrm{d}s \right\rangle \right\}\\
&= \sum_{k=0}^{l-1} \left\{ \Big\langle P_{(k+1)\delta}\,,\,\varphi_{(k+1)\delta}\Big\rangle - \Big\langle P_{k\delta}\,,\,\varphi_{(k+1)\delta}\Big\rangle \right.\\
&\qquad\qquad \left. + \int_{k\delta}^{(k+1)\delta} \langle P_{k\delta}\,,\,\partial_s\varphi_s\rangle\,\mathrm{d}s \right\},\\
&= \sum_{k=0}^{l-1} \int_{k\delta}^{(k+1)\delta} \left\{ \Big\langle P_s\,,\,L(\nu_s)\varphi_{(k+1)\delta}\Big\rangle + \langle P_{k\delta}\,,\,\partial_s\varphi_s\rangle \right\} \mathrm{d}s, \quad (2.64)
\end{aligned}$$

where we used in (2.64) the fact that (recall (2.63)) for any $0 \leqslant u < v \leqslant T$ and function ψ,

$$\langle P_v\,,\,\psi\rangle - \langle P_u\,,\,\psi\rangle = \int_u^v \langle P_w\,,\,L(\nu_w)\psi\rangle\,\mathrm{d}w.$$

So,

$$\begin{aligned}
\langle P_t\,,\,\varphi_t\rangle - \langle \nu_0\,,\,\varphi_0\rangle &= \sum_{k=0}^{l-1} \left\{ \int_{k\delta}^{(k+1)\delta} \langle P_s\,,\,(L(\nu_s) + \partial_s)\,\varphi_s\rangle\,\mathrm{d}s \right.\\
&\quad \left. + \int_{k\delta}^{(k+1)\delta} \Big(\Big\langle P_s\,,\,L(\nu_s)(\varphi_{(k+1)\delta} - \varphi_s)\Big\rangle - \langle P_s - P_{k\delta}\,,\,\partial_s\varphi_s\rangle \Big)\,\mathrm{d}s \right\},\\
&= \int_0^t \langle P_s\,,\,(L(\nu_s) + \partial_s)\,\varphi_s\rangle\,\mathrm{d}s + r(\delta),
\end{aligned}$$

with $\lim_{\delta\to 0} r(\delta) = 0$, since $P \in \mathbf{D}([0,T], \mathcal{M}_1)$. At the limit, we have:

$$\langle P_t\,,\,\varphi_t\rangle = \langle \nu_0\,,\,\varphi_0\rangle + \int_0^t \langle P_s\,,\,(L(\nu_s) + \partial_s)\,\varphi_s\rangle\,\mathrm{d}s. \tag{2.65}$$

We can apply (2.65) to $\varphi^T(t,x,\omega) := \mathbf{E}\left[\varphi(\xi^{x,t}(T),\omega)\right]$, where $\xi^{x,t}$ is the solution of (2.62) with $\xi^{x,t}(t) = x$. This leads to:

$$P_t = \mathcal{L}_{(\xi_t,\omega_t)}.$$

The result follows.

2. We know that, necessarily, any solution $(\nu_t)_{t\in[0,T]}$ to (2.61) is such that $\nu_t = \mathcal{L}_{(\xi_t,\omega_t)}$ for all $t \in [0,T]$. Let us now prove that such process $(\xi_t,\omega_t)_{t\in[0,T]}$ is unique. In this way, ν will be uniquely defined as the law of the process (ξ,ω). For simplicity, we will denote as $(\mathcal{L}_t)_{t\in[0,T]}$ in place of $(\mathcal{L}_{(\xi_t,\omega_t)})_{t\in[0,T]}$ the law of this process.
   By the first part of the proof, we see that the solution of (2.62) is a solution of the following stochastic differential equation (2.66):

$$\mathrm{d}\xi_t = (b[\xi_t,\omega,\mathcal{L}_t] + c(\xi_t,\omega))\,\mathrm{d}t + \mathrm{d}W_t, \quad (\xi_0,\omega) \sim \nu_0. \tag{2.66}$$

Note that since the marginal of ν_0 on the disorder is μ by hypothesis, this is also true for $\mathcal{L}_t \in \mathcal{M}_1(\mathbf{S} \times \mathbf{R})$ for all $t \in [0, T]$; namely one can write:

$$\mathrm{d}\mathcal{L}_t(\theta, \omega) = \mathrm{d}\mu(\omega) \otimes \mathrm{d}\mathcal{L}_t^{\omega}(\theta), \tag{2.67}$$

where $\mathcal{L}_t^{\omega}$ is the law of ξ_t conditioned on ω.
It suffices to show that (2.66) has a unique solution. Let ξ^1 and ξ^2 be two solutions of (2.66), with $(\xi_0^1, \omega) = (\xi_0^2, \omega)$, a.s. and driven by the same Brownian motion W_t. The law of (ξ_i, ω), $i = 1, 2$, will denoted as $\mathcal{L}_i$. Then we have, for some constant $C > 0$ which may change from a line to another:

$$\begin{aligned}\mathbf{E}\left(\left|\xi_t^1 - \xi_t^2\right|^2\right) &\leqslant C \int_0^t \mathrm{d}s \mathbf{E}\left(\left|b[\xi_s^1, \omega, \mathcal{L}_{s,1}] - b[\xi_s^2, \omega, \mathcal{L}_{s,2}]\right|^2 + \left|c(\xi_t^1, \omega) - c(\xi_t^2, \omega)\right|^2\right), \\ &\leqslant C \int_0^t \left\| \mathcal{L}_{s,1}^{\omega} - \mathcal{L}_{s,2}^{\omega} \right\|_1^2 \mathrm{d}\mu(\omega)\, \mathrm{d}s,\end{aligned}$$

where $\| m \|_1 := \sup\{\langle m\,,\, \varphi\rangle\,;\, \varphi \in BL_1(\mathbf{S})\}$. But,

$$\begin{aligned}\left\| \mathcal{L}_{s,1}^{\omega} - \mathcal{L}_{s,2}^{\omega} \right\|_1^2 &\leqslant \left(\int\int \mathrm{d}\mathcal{L}(\xi^1, \xi^2)(\theta, \theta')|\theta - \theta'|\right)^2, \\ &= \mathbf{E}\left(\left|\xi_s^1 - \xi_s^2\right|\right)^2 \leqslant \mathbf{E}\left(\left|\xi_s^1 - \xi_s^2\right|^2\right).\end{aligned}$$

An application of Gronwall's Lemma shows that $\xi_t^1 = \xi_t^2$ almost surely (t fixed). The right-continuity of both processes imply that $\xi_t^1 = \xi_t^2$ for all $t \in [0, T]$, with probability 1. Uniqueness in (2.61) follows and Proposition 2.20 is proved.

Chapter 3

Regularity and stability properties of McKean-Vlasov evolution

The material of this chapter is taken from a joint work with Giambattista Giacomin and Christophe Poquet [41], which has been submitted for publication to *Journal of Dynamics and Differential Equations*.

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1 Introduction

The main conclusion of Chapter 2 is that the McKean-Vlasov evolution (2.11) (whose weak form is rewritten below for the clarity of exposition) is the relevant object for the study of the large N limit in the Kuramoto model.

Namely, for any functions $(\theta, \theta', \omega, \omega') \mapsto b(\theta, \theta', \omega, \omega')$ and $(\theta, \omega) \mapsto c(\theta, \omega)$ satisfying the regularity hypothesis made in Chapter 2 (§ 2.4.2), for any positive time $T > 0$, the weak formulation of McKean-Vlasov evolution, whose unknown ν belongs to $\mathcal{C}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$, is given by (for all $t \in [0,T]$, and $\varphi \in \mathcal{C}_b^2(\mathbf{S} \times \mathbf{R})$, the set of bounded continuous functions on $\mathbf{S} \times \mathbf{R}$ with two bounded continuous derivatives w.r.t. the first variable $\theta \in \mathbf{S}$),

$$\begin{aligned}\int \varphi(\theta,\omega)\,\mathrm{d}\nu_t(\theta,\omega) = \int \varphi(\theta,\omega)\,\mathrm{d}\nu_0(\theta,\omega) + \frac{1}{2}\int_0^t \int \varphi''(\theta,\omega)\,\mathrm{d}\nu_s(\theta,\omega)\,\mathrm{d}s \\ + \int_0^t \int \varphi'(\theta,\omega)\left(\int b(\theta,\theta',\omega,\omega')\,\mathrm{d}\nu_s(\theta',\omega') + c(\theta,\omega)\right)\mathrm{d}\nu_s(\theta,\omega)\,\mathrm{d}s,\end{aligned} \tag{3.1}$$

where the second marginal (w.r.t. to the disorder ω) of the initial condition $\nu_0(\,\mathrm{d}\theta,\,\mathrm{d}\omega)$ is $\mu(\,\mathrm{d}\omega)$ so that one can write

$$\nu_0(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = \nu_0^\omega(\,\mathrm{d}\theta)\mu(\,\mathrm{d}\omega)\,,$$

where ν_0^ω is a probability measure on $\mathbf{S}$, for μ-a.e. ω. We know (Proposition 2.20) that there exists a unique solution to (3.1). Note in particular that existence can be seen as a consequence of either the fact that a good rate function always has at least one zero (Theorem 2.11) or either from the tightness of the empirical flow in the quenched context (Theorem 2.16).

1.1 Regularity of the McKean-Vlasov semigroup

The first aim of this chapter is to establish regularity properties of evolution (3.1), proving in particular that its solution is in fact regular in time and space and that its density $q_t(\theta,\omega)$ (w.r.t. to the Lebesgue measure) is the strong solution of the *strong formulation of McKean-Vlasov evolution* defined in (1.18).

Note that this regularity result is again not specific to the *sine-model* and will be proved in whole generality. The rest of the chapter concerns the *sine-model*, that is, as already mentioned, in the particular case where $b(\theta,\theta',\omega,\omega') = K\sin(\theta'-\theta)$ and $c(\theta,\omega) = \omega$.

1.2 Phase transition in synchronization

The second goal of the chapter is to recall the mathematical background for the meaning of *synchronization* already set up by Sakaguchi in [75] in the case of the disordered *sine-model* and to discuss the existence of stationary solutions to (1.18) for nontrivial distributions of the disorder.

After making explicit the proper parameterization for the synchronized solutions of (1.18), based on the order parameters r_t and ψ_t mentioned in (1.20), we show the phase transition remains the same as the one that happens in the non-disordered case (see Propositions 3.10 and 3.13), at least when the disorder is small. Proposition 3.10 relies on perturbation argument of the non-disordered case, for which Pearce ([71]) has shown that the fixed-point function governing the phase transition is strictly concave.

In the case of arbitrary distributions of the disorder, the phase transition from incoherence to synchronization can become very complex, even for simple instances of μ. In particular, the case of binary distribution of the disorder $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$ which is an example where two distinct circles of synchronized solutions may coexists (Proposition 3.11) is discussed in details in § 2.2.3.

1.3 Stability of synchronization

The third and main goal of this chapter is to tackle the issue of the stability of nontrivial stationary solutions to McKean-Vlasov evolution (3.1).

Based on the work by Bertini, Giacomin and Pakdaman [9], who addressed the question of the linear stability of the synchronized solution in the non-disordered model, we prove an analogous result, at least in the case where the disorder (seen as a perturbation) is small: using techniques from functional analysis (perturbations of self-adjoint operators and analytic semigroups), the main result of this chapter (Theorem 3.27) states that for a sufficiently positive small disorder, there exists a unique circle of synchronized stationary solutions to (3.1) and that this circle is both linearly and nonlinearly stable.

We conclude this chapter by stating an alternative approach of stability of synchronization developed for the non-disordered *sine-model* by G. Giacomin, K. Pakdaman, X. Pellegrin and C. Poquet [43] which can be applied to our concern and presents the major advantage of being generalizable to the case where μ is not symmetric, as well as for more general models (e.g. active rotators, see [76]). This approach relies on the notion of *stable normal hyperbolic manifold* (see [79]) and the fact that this structure is robust under perturbations. We will only mention this result (§ 2.4.4) and we refer to [41] for precise statements and proofs.

2 Settings and main results

2.1 Strong formulation of McKean-Vlasov evolution and regularity of the semigroup

2.1.1 The general case

The first aim of this chapter is to establish regularity properties of the solution of the nonlinear equation (3.1).

In what follows, in addition to the hypothesis made on b and c in Chapter 2, § 2.4.2, we assume that

- for all ω, ω', $(\theta, \theta') \mapsto b(\theta, \theta', \omega, \omega')$ is of class $\mathcal{C}^\infty$ with bounded derivatives,
- for all ω, $\theta \mapsto c(\theta, \omega)$ is of class $\mathcal{C}^\infty$, with derivatives uniformly bounded on every $\mathbf{S} \times [-M, M]$, for all $M > 0$.

We come now to the statement of the regularity result:

Proposition 3.1. *For all probability measure $\nu_0(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = \nu_0^\omega(\,\mathrm{d}\theta)\mu(\,\mathrm{d}\omega)$ on $\mathbf{S} \times \mathbf{R}$, for all $T > 0$, there exists a unique solution ν to (3.1) in $\mathcal{C}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$ such that for all $\varphi \in \mathcal{C}(\mathbf{S} \times \mathbf{R})$,*

$$\lim_{t \searrow 0} \int_{\mathbf{S}\times\mathbf{R}} \varphi(\theta, \omega)\,\mathrm{d}\nu_t(\theta, \omega) = \int_{\mathbf{S}\times\mathbf{R}} \varphi(\theta, \omega)\,\mathrm{d}\nu_0^\omega(\theta)\,\mathrm{d}\mu(\omega).$$

Moreover, for all $t > 0$, ν_t is absolutely continuous with respect to $\lambda \otimes \mu$ and for μ-a.e. $\omega \in Supp(\mu)$, its density $(t, \theta, \omega) \mapsto p_t(\theta, \omega)$ is strictly positive on $(0, T] \times \mathbf{S}$, is $\mathcal{C}^\infty$ in (t, θ)

and solves the following Fokker-Planck equation (3.2)*:*

$$\partial_t p_t(\theta,\omega) = \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta \left\{ p_t(\theta,\omega) \left(\int_{\mathbf{S}\times\mathbf{R}} b(\theta,\theta',\omega,\omega') p_t(\theta',\omega')\,\mathrm{d}\theta'\,\mathrm{d}\mu(\omega') + c(\theta,\omega) \right) \right\}. \tag{3.2}$$

Remark 3.2. The existence and uniqueness of a solution to (3.2) in the space $\mathbf{L}^2(\lambda\otimes\mu)$ of square-integrable functions on $\mathbf{S}\times\mathbf{R}$ can be tackled using Banach fixed-point arguments (see [79, Section 4.7]), but one can obtain more regularity from the theory of fundamental solutions of parabolic equations (see [4] and [37]).

Remark 3.3. We retrieve here the *strong formulation of McKean-Vlasov evolution* (1.18) we mentioned in the main introduction of this thesis. We insist on the fact that (3.2) is indeed a (possibly) infinite system of *coupled* PDEs with quadratic nonlinearity (recall § 4.1).

We focus now on the regularity of the solution $p_t(\theta,\omega)$ of (3.2) with respect to the disorder ω. Since the evolution (3.2) is degenerate with respect to the component ω, in order to state some regularity result with respect to the disorder we need to start from a regular initial condition in ω: we assume here that the initial condition ν_0 is such that for all $\omega\in\mathrm{Supp}(\mu)$, $\nu_0^\omega(\,\mathrm{d}\theta)$ is absolutely continuous with respect to Lebesgue measure on $\mathbf{S}$: there exists a positive integrable function $\gamma(\cdot,\omega)$ of integral 1 on $\mathbf{S}$ such that $\nu_0^\omega(\,\mathrm{d}\theta) = \gamma(\theta,\omega)\,\mathrm{d}\theta$.

Remark 3.4. Note that this is a generalization of the hypothesis made on the initial condition of the oscillators in (2.13) where we supposed that the oscillators were initially i.i.d. realizations with law γ. We make here the slight abuse of notations in confusing here the initial γ with the density $\gamma(\cdot,\omega)$.

In this case, we have

Proposition 3.5 (Regularity w.r.t. the disorder)**.** *For every* $(t_0,\theta_0)\in(0,\infty)\times\mathbf{S}$*, for every* ω_0 *which is an accumulation point in* $\mathrm{Supp}(\mu)$ *such that the following holds*

$$\int_{\mathbf{S}} |\gamma(\theta,\omega)-\gamma(\theta,\omega_0)|\,\mathrm{d}\theta \to 0, \quad \text{as } \omega\to\omega_0\,,$$

then the solution p *of* (3.2) *defined on* $(0,\infty)\times\mathbf{S}\times\mathrm{Supp}(\mu)$ *is continuous at the point* (t_0,θ_0,ω_0)*.*

Propositions 3.1 and 3.5 are proved in § 3.1.

2.1.2 The case of the sine-model

For the sake of completeness, let us specify what happens in the particular case of the *sine-model* case (1.1):

Proposition 3.6 (Strong McKean-Vlasov evolution for the *sine-model*)**.** *In the case where* $b(\theta,\theta',\omega,\omega') = K\sin(\theta'-\theta)$ *(*$K>0$*) and* $c(\theta,\omega)=\omega$*, the Fokker-Planck equation* (3.2) *becomes:*

$$\partial_t p_t(\theta,\omega) \;=\; \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)(\langle J * p_t\rangle_\mu(\theta)+\omega)\Big), \tag{3.3}$$

where

$$\forall\theta\in\mathbf{S}, \quad J(\theta) = -K\sin(\theta), \tag{3.4}$$

and where $*$ *denotes the convolution and* $\langle\cdot\rangle_\mu$ *is a notation for the integration with respect to* μ*, so that* $\langle J*u\rangle_\mu(\theta) = \int_{\mathbf{R}}\int_{\mathbf{S}} J(\theta')u(\theta-\theta',\omega)\,\mathrm{d}\theta'\,\mathrm{d}\mu(\omega)$ *is the convolution of* J *and* u*, averaged with respect to the disorder.*

Of course, all that has been stated before remain valid in the particular case of the *sine-model*. Note in particular that q is $\mathcal{C}^\infty$ in (t,θ) in this case.

2.2 Stationary solutions and synchronization in the sine-model

In this paragraph, we restrict ourselves to the case of the *sine-model* (recall (1.1) and Remark 1.3, p. 19). In order to understand the phenomenon of synchronization properly, one has to make a rigorous computation of the stationary solutions of the McKean-Vlasov evolution (3.3).

We also recall Remark 1.8, p. 24: in view of the rotational invariance mentioned for the *sine-model* (recall also § 2.1.3, p. 15), if $(\theta,\omega)\mapsto q(\theta,\omega)$ is a stationary solution to (3.3), then for all $\theta_0\in\mathbf{S}$, $(\theta,\omega)\mapsto q(\theta+\theta_0,\omega)$ is also a stationary solution. In other terms, finding a nontrivial stationary solution to (3.3) is in fact equivalent to find *a whole circle of stationary solutions* indexed by $\theta_0\in\mathbf{S}$.

Remarkably ([75], see also [30]), all the stationary solutions to (3.3) can be written in a semi-explicit way at least when the distribution μ of the disorder is symmetric: more precisely, in [75], Sakaguchi established (in the case of a symmetric disorder) a one-to-one correspondence between the stationary solutions to (3.3) (up to rotation) and the solutions $r\in[0,1]$ to a fixed-point relation $r=\Psi^\mu(2Kr)$ where the fixed-point function $\Psi^\mu(\cdot)$ will be made explicit below. An immediate consequence is that the system exhibits a phase transition as the coupling strength K increases from 0 to ∞.

Before going into the details of the general case, we insist on the situation where there is no disorder ($\mu=\delta_0$), which has its own interest since its phase transition is particularly simple.

2.2.1 The non-disordered case

The Kuramoto evolution (3.3) in the case with no disorder ($\mu=\delta_0$) reduces to

$$\partial_t p_t(\theta) \;=\; \frac{1}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta\Big(p_t(\theta)(J*p_t(\theta))\Big). \tag{3.5}$$

Remark 3.7. The evolution (3.5) will be referred to, in the rest of this chapter as the *non-disordered PDE*.

In this case the fixed-point relation is given by

$$r_0 \;=\; \Psi_0(2Kr_0) \qquad \text{where} \qquad \Psi_0(x) \;:=\; \frac{I_1(x)}{I_0(x)}\,, \tag{3.6}$$

where we used standard notations for the modified Bessel functions

$$I_i(x) \;=\; \frac{1}{2\pi}\int_{\mathbf{S}}(\cos(\theta))^i\exp(x\cos(\theta))\,\mathrm{d}\theta \qquad i=0,1\,.$$

For each solution $r_0\in[0,1]$ of (3.6), the corresponding stationary solution (see Figure 3.1) will be given by

$$\forall\theta\in\mathbf{S},\quad q_0(\theta)=\frac{1}{Z_0}e^{2Kr_0\cos(\theta)}, \tag{3.7}$$

where $Z_0=\int_{\mathbf{S}}e^{2Kr_0\cos(\theta)}\,\mathrm{d}\theta$ is the normalization constant. It is easy to see that the mapping $\Psi_0(\cdot)$ is continuous, increasing, concave (see [71]) and such that $\Phi_0(0)=0$, $\lim_{x\to\infty}\Psi_0(x)=1$ and $\partial_x\Psi_0(x)|_{x=0}=\frac{1}{2}$ [1]. Consequently, the phase diagram in this case is simple and is summarized in the following proposition:

[1]. or equivalently the derivative of $\Psi_0(2K\cdot)$ at the origin is K.

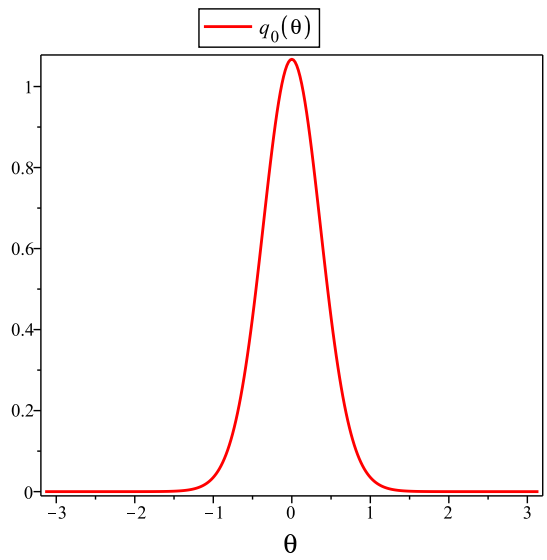


Figure 3.1: The plot of the synchronized solution q_0 for $K = 4$. Each stationary solution to (3.5) is given by a rotation of this solution by an arbitrary $\theta_0 \in \mathbf{S}$.

Proposition 3.8 (Phase transition in the non-disordered *sine-model*, see [75])**.** *The non-disordered* sine-model *presents a phase transition (see Figure 3.2): if $K > 0$ is the coupling strength in the* sine-model*, then one can distinguish between two regimes:*

1. *The case $K \leqslant K_c := 1$ (see Fig. 3.2a): then $r_0 = 0$ is the fixed-point in* (3.6) *and the corresponding stationary solution to* (3.5) *is the* incoherent solution

$$q(\cdot) \equiv \frac{1}{2\pi}. \tag{3.8}$$

2. *The case $K > K_c$ (see Fig. 3.2b): then there exists a unique nontrivial fixed-point $r_0 > 0$ in* (3.6) *and the incoherent solution $\frac{1}{2\pi}$ cohabits with a whole circle of* synchronized solutions

$$M_{rev} := \{q(\cdot + \theta_0);\, \theta_0 \in \mathbf{S}\}\,, \tag{3.9}$$

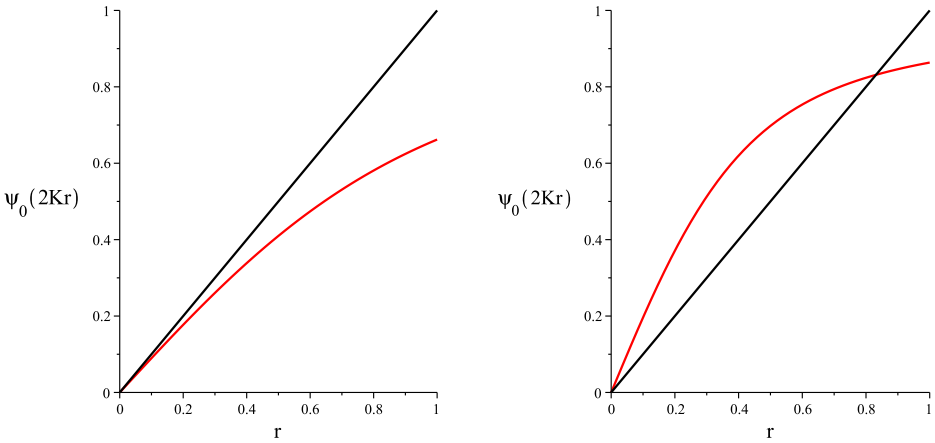
for q given by (3.7)*.*

Remark 3.9. In the notation M_{rev}, *rev* stands for *reversible.*

2.2.2 The symmetric case

We now turn to the study of general case, that is μ symmetric. Although an analogous parameterization of the stationary solutions to (3.3) exists, the phase transition appears to be fairly more complicated, even for simple instances of the distribution μ. The main complications come from the fact that the fixed-point function fails to be concave in general; in particular a number of questions regarding existence or uniqueness of nontrivial stationary solutions remain unfortunately open. After making the parameterization explicit in the general case, we will consider the example of the binary case which is particularly relevant as far as such difficulties are concerned.

As in the non-disordered case § 2.2.1, the following proposition holds:



(a) Function $\Psi_0(2K\cdot)$ for $K = 0.5$. (b) Function $\Psi_0(2K\cdot)$ for $K = 3.5$.

Figure 3.2: The plot of the fixed-point function $\Psi_0(2K\cdot)$ for two choices of K shows a phase transition at the critical point $K_c = 1$: for $K \leqslant 1$ (Fig. 3.2a) the incoherent value $r_0 = 0$ is the only fixed-point whereas for $K > K_c$ (Fig. 3.2b) a unique nontrivial synchronized solution $r_0 > 0$ appears.

Proposition 3.10. *There is a one-to-one correspondence between the stationary solutions to* (3.3) *(up to rotation), and the solutions $r \in [0,1]$ of the fixed-point equation:*

$$r \;=\; \Psi^\mu(2Kr), \quad \textit{where} \quad \Psi^\mu(x) \;:=\; \int_{\mathbf{R}} \frac{\int_{\mathbf{S}} \cos(\theta) S(\theta,\omega,x)\,\mathrm{d}\theta}{Z(\omega,x)}\,\mathrm{d}\mu(\omega)\,, \tag{3.10}$$

where

$$S(\theta,\omega,x) \;=\; e^{G(\theta,\omega,x)}\left[(1-e^{4\pi\omega})\int_0^\theta e^{-G(u,\omega,x)}\,\mathrm{d}u + e^{4\pi\omega}\int_0^{2\pi} e^{-G(u,\omega,x)}\,\mathrm{d}u\right], \tag{3.11}$$

for

$$\forall\theta\in\mathbf{S}, \omega\in\mathrm{Supp}(\mu), \forall x\in\mathbf{R}, \quad G(\theta,\omega,x) = x\cos(\theta) + 2\omega\theta \tag{3.12}$$

and where

$$Z(\omega,x) = \int_{\mathbf{S}} S(\theta,\omega,x)\,\mathrm{d}\theta \tag{3.13}$$

is the normalization constant.

For each solution r to (3.10)*, the corresponding stationary solutions (see Figure 3.3 are given by the circle*

$$M := \{q(\theta+\theta_0,\omega);\, \theta_0\in\mathbf{S}\}\,, \tag{3.14}$$

where

$$q(\theta,\omega) \;:=\; \frac{S(\theta,\omega,2Kr)}{Z(\omega,2Kr)}\,. \tag{3.15}$$

One can verify by direct calculations (see Proposition 3.35) that $\Psi^\mu(\cdot)$ is continuous with $\lim_{x\to\infty}\Psi^\mu(x) = 1$ such that,

$$\Psi^\mu(0) = 0, \quad \partial_x\Psi^\mu(0) = \frac{1}{2\tilde{K}}, \text{ for } \tilde{K} \;:=\; \left(\int_{\mathbf{R}} \frac{\mathrm{d}\mu(\omega)}{1+4\omega^2}\right)^{-1}, \tag{3.16}$$

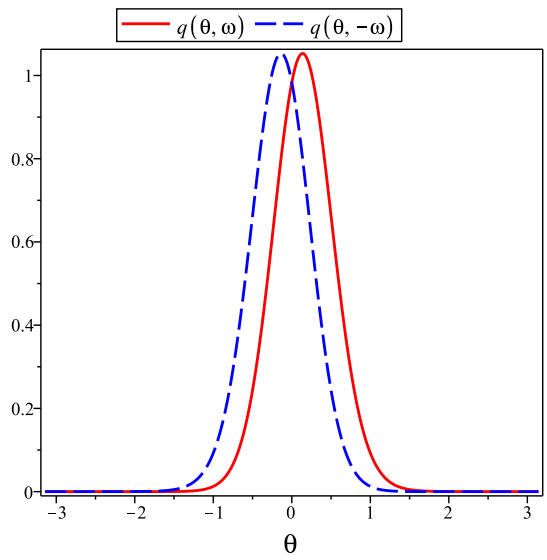


Figure 3.3: The plot of one synchronized solution $q(\cdot, \pm\omega)$ for $K = 4$ and $\omega = 0.5$. Each stationary solution to (3.3) is given by a rotation of this solution by an arbitrary $\theta_0 \in \mathbf{S}$.

and $\partial_x^2 \Psi^\mu(0) = 0$, and

$$\partial_x^3 \Psi^\mu(0) = -\frac{3}{4} D(\mu), \text{ with } D(\mu) = \int_{\mathbf{R}} \left(\frac{1}{2(1+\omega^2)} - \frac{8\omega^2}{(1+4\omega^2)^2} \right) \mathrm{d}\mu(\omega). \tag{3.17}$$

As in § 2.2.1, a series of remarks are in order:

1. As in the non-disordered case, $r = 0$ always solves (3.10) and this corresponds to the *incoherent solution* $q(\cdot,\cdot) \equiv \frac{1}{2\pi}$. It is the only one as long as K does not exceed critical value K_c which is in any case not larger than $\tilde{K}$ since the derivative of $\Psi^\mu(2K\cdot)$ at the origin is precisely given by $\frac{K}{\tilde{K}}$ so that it is larger than one if and only if $K > \widetilde{K}$, see Figure 3.6.
2. When (3.10) admits a fixed-point $r > 0$, and this is certainly the case if $K > \widetilde{K}$, a nontrivial stationary solution is present and in fact, by rotation symmetry, a circle of nontrivial stationary solutions. Such solutions correspond to a synchronization phenomenon, since the distribution of the phases is no longer trivial.

Despite the fact that the fixed-point function $\Psi^\mu(\cdot)$ is explicit (and may be seen as a perturbation of the non-disordered one (3.6)), the main difficulties arise in the precise study of its concavity on the interval $[0, 1]$. If one can have a precise knowledge about the concavity of $\Psi^\mu(\cdot)$ *in a neighborhood of the origin*, very little is known about its concavity *away from* 0.

The situation is all the more different from the non-disordered case given the fact that we know that strict concavity fails to hold, even for simple distributions of the disorder (see for example Figure 3.5 and Figure 3.6):

Proposition 3.11. *There exists probability measures μ for which $r \mapsto \Psi^\mu(2Kr)$ admits at least two distinct nontrivial fixed-point $r_1 > 0$ and $r_2 > 0$.*

Nevertheless, it is believed that the concavity holds in the case where μ is *unimodal*, although this conjecture still remains open:

Conjecture 3.12. *If μ is unimodal, that is if μ is absolutely continuous with respect to Lebesgue measure λ on $\mathbf{R}$ with density $\omega \mapsto \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(\omega)$ that is an even function of ω, strictly decreasing on $[0,\infty)$, then*

$$\Psi^{\mu}(\cdot) \text{ is strictly concave on } [0,1]. \tag{3.18}$$

Although we are not able to give an answer to Conjecture 3.12 in whole generality, one can prove (3.18) in the particular case where μ is *small*:

Proposition 3.13. *For all $M > 0$, $\alpha > 0$, $K_{\max} > 0$, there exists $\delta = \delta(M, \alpha, K_{\max}) > 0$ such for all probability measure μ which verifies*

$$\int_{\mathbf{R}} |\omega|^5 \,\mathrm{d}\mu(\omega) \leqslant M, \tag{3.19}$$

$$D(\mu) \geqslant \alpha, \tag{3.20}$$

$$\int_{\mathbf{R}} |\omega|^3 \,\mathrm{d}\mu(\omega) \leqslant \delta, \tag{3.21}$$

then (3.18) *is true.*

Remark 3.14. It is easily seen that the hypothesis of Proposition 3.13 are in particular true for measures μ with compact support in $[-\omega_0, \omega_0]$ for some small $\omega_0 > 0$.

Note that if μ is unimodal, then $D(\mu) > 0$. Since $\mu \mapsto D(\mu)$ is continuous for the weak topology on $\mathcal{M}_1(\mathbf{R})$, the hypothesis of Proposition 3.13 are also true (for some $\alpha > 0$) for any set of probabilities μ that is compact w.r.t. the weak topology, e.g. for a set of Gaussian measures with small, uniformly bounded variance.

2.2.3 An example: the binary case

As seen in the previous paragraph, the phase diagram appears to be fairly more complicated than in the case with no disorder. In particular the phase transition may *not* be given by the derivative at the origin of the fixed-point function. In that sense, the example of the binary case where $\mu = \frac{1}{2}\left(\delta_{-\omega_0} + \delta_{\omega_0}\right)$ for some $\omega_0 > 0$ is particularly illustrative.

Remark 3.15. We make here the slight abuse of notations by writing Ψ^{ω_0} instead of $\Psi^{\frac{1}{2}\left(\delta_{-\omega_0}+\delta_{\omega_0}\right)}$ for the fixed-point function (3.10) and $D(\omega_0)$ instead of $D(\frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0}))$ for the functional $D(\cdot)$ defined in (3.17).

Let us fix some $K > 1$ and make ω_0 vary from 0 to ∞. In this case, the behavior of the fixed-point function $r \mapsto \Psi^{\omega_0}(2Kr)$ is governed (at least in a neighborhood of the origin) by two parameters:

- the slope at the origin: $\partial_r \Psi^{\omega_0}(2K\cdot)|_{r=0} = \frac{K}{1+4\omega_0^2}$. In particular, this slope is greater than 1 if and only if $\omega_0 \leqslant \omega_s(K) := \frac{\sqrt{K-1}}{2}$,
- the third derivative at the origin, given in this case by:

$$\partial_r^3 \Psi^{\omega_0}(2K\cdot)|_{r=0} = -6K^3 D(\omega_0) = -6K^3 \left(\frac{1}{2(1+\omega_0^2)} - \frac{8\omega_0^2}{(1+4\omega_0^2)^2} \right).$$

  It is easy to see that $D(\omega_0)$ is positive whenever $0 \leqslant \omega_0 \leqslant \omega_c := \frac{1}{2\sqrt{2}}$ and negative for $\omega_0 \geqslant \omega_c$ (see Figure 3.4). As a consequence, if $0 \leqslant \omega_0 \leqslant \omega_c$ then $\Psi^{\omega_0}(2K\cdot)$ is locally concave near the origin whereas it is locally convex in a neighborhood of 0 in the case where $\omega_0 > \omega_c$.

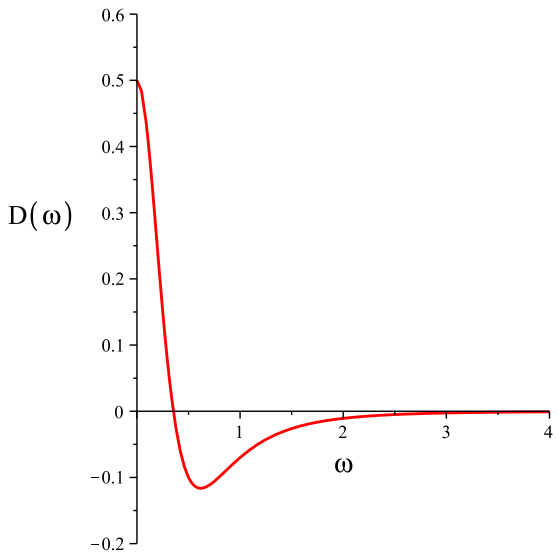


Figure 3.4: The plot of $\omega_0 \mapsto D(\omega_0)$.

For all $K > 1$, we also introduce the first value of the disorder ω_0 (denoted as $\omega_e(K)$) from which there is no longer any synchronized solution (or equivalently any nontrivial solution $r > 0$ solution to (3.10)):

$$\omega_e(K) := \sup \{\omega_0 > 0;\ \exists r > 0,\ r = \Psi^{\omega_0}(2Kr)\}.$$

Remark 3.16. In the notations $\omega_s(K)$, ω_c and $\omega_e(K)$, s stands for *slope*, c for *concavity* and e for *existence*. Note that ω_c does not depend on K.

The first result of this paragraph is the following:

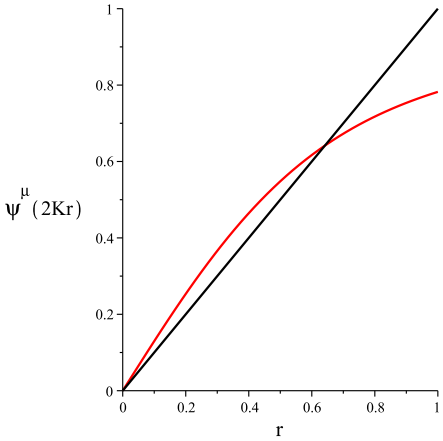
Proposition 3.17. *For all $K > 1$, the following inequality is true*

$$0 < \omega_s(K) \leqslant \omega_e(K) < \infty. \tag{3.22}$$

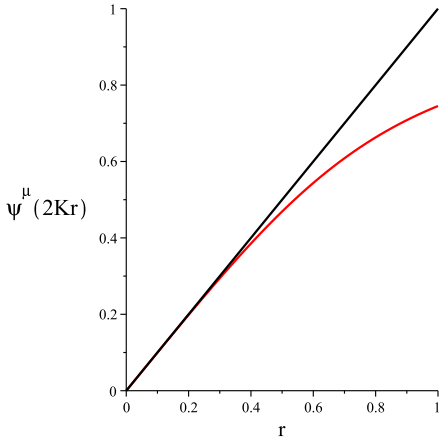
Moreover, determining whether the change (as ω_0 increases) of local concavity at the origin happens before or after the change of slope w.r.t. the critical value 1 allows to distinguish two cases:

The case $1 < K \leqslant \frac{3}{2}$, (Figure 3.5): this case corresponds to the inequality $\omega_s(K) \leqslant \omega_c$. In other terms, as ω_0 increases from 0 to ∞, the slope at the origin decreases from $K > 1$ (when $\omega_0 = 0$) to values lower than 1 (when $\omega_0 \geqslant \omega_s(K)$) with strict concavity in a neighborhood of the origin. Since $\omega_s(K) \leqslant \omega_c$, the change of (local) concavity only happens afterwards.

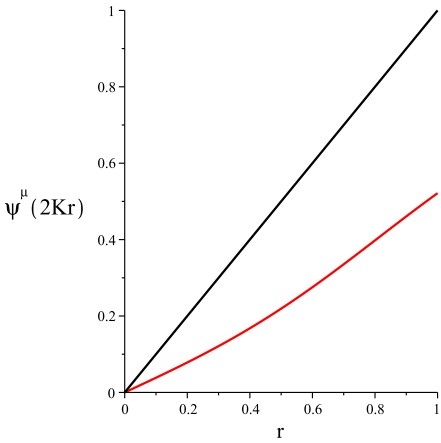
The intuition for this case (although not rigorously established) is the following: for $0 \leqslant \omega_0 \leqslant \omega_s(K)$, there is a unique nontrivial solution $r > 0$ and for $\omega_s(K) < \omega_0$ only the trivial solution $r = 0$ subsists. In particular we believe that $\omega_e(K) = \omega_s(K)$ in this case (see Figure 3.5). In order to make this conjecture rigorous, we would need to extrapolate the local study of the fixed-point function in 0 to the whole trajectory $r \in [0, 1] \mapsto \Psi^{\omega_0}(2Kr)$ which requires a deeper understanding of the Taylor's expansion of Ψ^{ω_0}.



(a) $\omega_0 = 0.1$.



(b) $\omega_0 = \omega_s(K) = 0.3$.



(c) $\omega_0 = 0.8$.

Figure 3.5: Plot of the fixed-point function $\Psi^{\omega_0}(2K\cdot)$ in the case where $K = 1.36$ for increasing values of ω_0: in Fig. 3.5a as well as in Fig. 3.5b, the fixed-point function is still concave (at least near the origin). However, it is no longer the case in Fig. 3.5c.

The case $\frac{3}{2} < K$, (Figure 3.6): on the contrary, this case corresponds to the inequality $\omega_c \leqslant \omega_s(K)$; namely, the change of local concavity at the origin happens *before* the change of slope w.r.t. the value 1. In particular, we are sure that in this case, there exists some ω_0 for which *two distinct synchronized solutions exist* (see Fig. 3.6c): Proposition 3.11 is proved.

The conjecture about the phase transition for this case (see Figure 3.6) is the following: for $0 < \omega_0 < \omega_s(K)$, there is a unique nontrivial fixed-point $r > 0$, for $\omega_s(K) < \omega_0 < \omega_e(K)$, two distinct nontrivial fixed-points $r_1 > 0$ and $r_2 > 0$ coexist, and for $\omega_e(K) < \omega$ only the trivial solution $r = 0$ subsists. Once again, proving this conjecture requires a deeper understanding of the fixed-point function Ψ^{ω_0}.

The rest of this chapter is devoted to tackle the issue of stability of synchronized solutions.

2.3 Weighted Sobolev spaces

The analysis of stability of synchronization will be carried out in the context of appropriate distributions spaces that we introduce here. These *weighted-Sobolev spaces* will be of constant use in this chapter, as well as in Chapter 5. We first focus on weighted-Sobolev spaces of functions $\theta \mapsto h(\theta)$ on $\mathbf{S}$ (§ 2.3.1) and then introduce the corresponding spaces for functions with disorder $(\theta, \omega) \mapsto h(\theta, \omega)$ on $\mathbf{S} \times \mathrm{Supp}(\mu)$ (§ 2.3.3):

2.3.1 Weighted Sobolev spaces:

For any bounded positive weight function $k(\cdot)$ on $\mathbf{S}$ such that $\int_{\mathbf{S}} k(\theta)\,\mathrm{d}\theta = 1$, we may consider the space $\mathbf{L}^2_k$ closure of $\{h \in \mathcal{C}(\mathbf{S});\ \int_{\mathbf{S}} h(\theta)\,\mathrm{d}\theta = 0\}$ w.r.t. the norm:

$$\| h \|_{2,k} := \left(\int_{\mathbf{S}} \frac{h^2(\theta)}{k(\theta)}\,\mathrm{d}\theta \right)^{\frac{1}{2}}. \tag{3.23}$$

Since h is with zero mean value, each of its primitives are 2π-periodic. In particular, we can consider $\mathbf{H}^{-1}_k$ the closure of $\{h \in \mathcal{C}(\mathbf{S});\ \int_{\mathbf{S}} h(\theta)\,\mathrm{d}\theta = 0\}$ with respect to the following weighted Sobolev norm:

$$\|h\|_{-1,k} := \left(\int_{\mathbf{S}} \frac{\mathcal{H}^2(\theta)}{k(\theta)}\,\mathrm{d}\theta \right)^{\frac{1}{2}}, \tag{3.24}$$

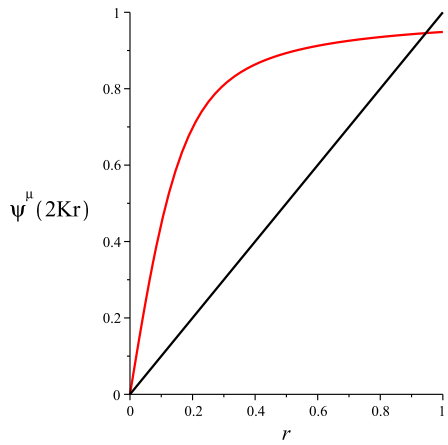
where $\mathcal{H}$ is the primitive of h on $\mathbf{S}$ such that $\int_{\mathbf{S}} \frac{\mathcal{H}}{k} = 0$.

Remark 3.18. In particular, we will make a constant use of the space $\mathbf{H}^{-1}_{q_0}$ (that is for $k(\cdot) = q_0(\cdot)$ where q_0 (see (3.7)) is the stationary solution of the non-disordered system) which is the natural space for the study of linear stability of synchronization in the non-disordered case (see § 2.4.2).

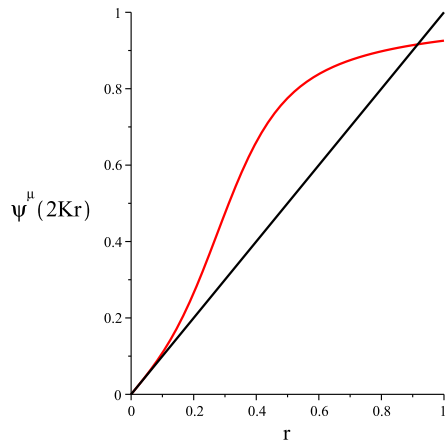
Remark 3.19. In the case of a constant weight $k(\cdot) \equiv \frac{1}{2\pi}$, we will write $(\mathbf{L}^2, \|\cdot\|_2)$ and $(\mathbf{H}^{-1}, \|\cdot\|_{-1})$ instead of $(\mathbf{L}^2_{\frac{1}{2\pi}}, \|\cdot\|_{2,\frac{1}{2\pi}})$ and $(\mathbf{H}^{-1}_{\frac{1}{2\pi}}, \|\cdot\|_{-1,\frac{1}{2\pi}})$.

2.3.2 Rigged Hilbert space

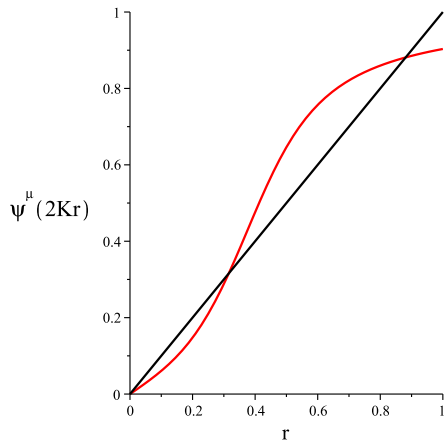
Note that one can understand the spaces $\mathbf{H}^{-1}_k$ as part of a *Gelfand-triple* construction (see [9, § 2.2] or [17, p. 81]): indeed, one can identify $\mathbf{H}^{-1}_k$ as the dual space V' of the space V closure of $\{v \in \mathcal{C}^1(\mathbf{S});\ \int_{\mathbf{S}} v = 0\}$ with respect to the norm $\|h\|_V :=$



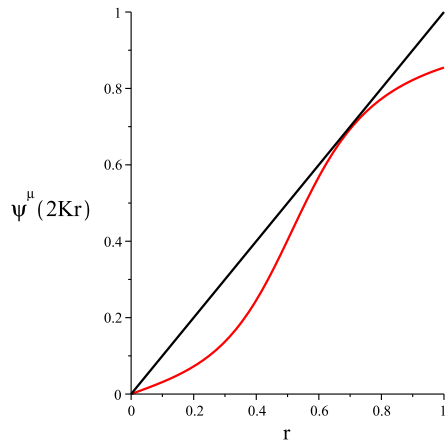
(a) $\omega_0 = 0$.



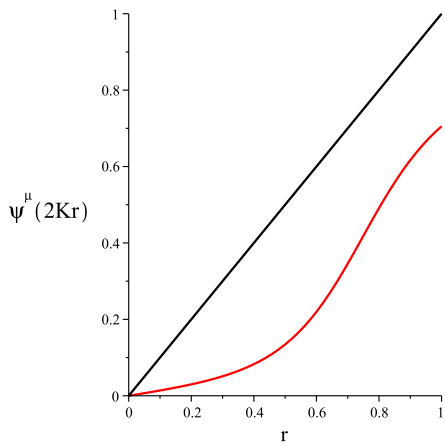
(b) $\omega_0 = \omega_s(K) = 1$.



(c) $\omega_0 = 1.4$.



(d) $\omega_0 = 1.98 \approx \omega_e(K)$.



(e) $\omega_0 = 3$.

Figure 3.6: Plot of the fixed-point function $\Psi^{\omega_0}(2K\cdot)$ in the case where $K = 5$ for increasing values of ω_0: until $\omega_s(K)$ there is only one nontrivial solution (Fig. 3.6a and Fig. 3.6b), whereas there is coexistence of two nontrivial fixed-points between $\omega_s(K)$ and $\omega_e(K)$ (Fig. 3.6c). If $\omega_0 \geqslant \omega_e(K)$ the synchronized solutions disappear (Fig. 3.6d and Fig. 3.6e).

$\left(\int_{\mathbf{S}} h'(\theta)^2 k(\theta)\,\mathrm{d}\theta\right)^{\frac{1}{2}}$. The pivot space is the usual $\mathbf{L}^2(\lambda)$ (endowed with the Hilbert norm $\|h\|_{\mathbf{L}^2} := \left(\int_{\mathbf{S}} h(\theta)^2\,\mathrm{d}\theta\right)^{\frac{1}{2}}$). In particular, one easily sees that the inclusion $V \subseteq \mathbf{L}^2(\lambda)$ is dense. Consequently, one can define $T : \mathbf{L}^2(\lambda) \to V'$ by setting $Th(v) = \int_{\mathbf{S}} h(\theta)v(\theta)\,\mathrm{d}\theta$. One can prove that T continuously injects $\mathbf{L}^2(\lambda)$ into V' and that $T(\mathbf{L}^2(\lambda))$ is dense into V' so that one can identify $h \in \mathbf{L}^2(\lambda)$ with $Th \in V'$. Then for $h \in \mathbf{L}^2(\lambda)$,

$$\|h\|_{V'} = \|Th\|_{V'} = \sup_{v\in V} \frac{\int_{\mathbf{S}} \mathcal{H}h'}{\|v\|_V} = \sqrt{\int_{\mathbf{S}} \frac{\mathcal{H}^2}{k}}\,, \tag{3.25}$$

where we used in (3.25) Cauchy-Schwarz inequality for the lower bound and chose $v' := \frac{\mathcal{H}}{k}$ for the upper bound. All that enables us to identify $\mathbf{H}_k^{-1}$ with V'.

2.3.3 Weighted Sobolev spaces (with disorder):

The natural space in for the study of synchronization stability is the space of functions $(\theta,\omega) \mapsto h(\theta,\omega)$ in the domain

$$\mathcal{D} := \left\{ h \in C^2(\mathbf{S}\times\mathbf{R});\ \int_{\mathbf{S}} h(\theta,\omega)\,\mathrm{d}\theta = 0 \ \text{ for all } \omega \right\}. \tag{3.26}$$

such that each component $h(\cdot,\omega)$ live in a certain $\mathbf{H}^{-1}_{k(\cdot,\omega)}$ for a weight $k(\cdot,\omega)$ (which may depends on the disorder $\omega \in \mathrm{Supp}(\mu)$). More precisely, for any family of positive weight functions $(k(\cdot,\omega))_{\omega\in\mathrm{Supp}(\mu)}$, we denote as $\mathbf{H}^{-1}_{\mu,k}$ the closure of $\mathcal{D}$ w.r.t. the norm:

$$\|h\|_{\mu,-1,k} := \left(\int_{\mathbf{R}} \|h(\cdot,\omega)\|^2_{-1,k(\cdot,\omega)}\,\mathrm{d}\mu(\omega)\right)^{\frac{1}{2}} = \left(\int_{\mathbf{S}\times\mathbf{R}} \frac{\mathcal{H}^2(\theta,\omega)}{k(\theta,\omega)}\,\mathrm{d}\theta\,\mathrm{d}\mu\right)^{\frac{1}{2}}. \tag{3.27}$$

We will also consider the analogous averaged weighted $\mathbf{L}^2$-spaces, that is, for any family of positive weights $(k(\cdot,\omega))_{\omega\in\mathrm{Supp}(\mu)}$, the space $\mathbf{L}^2_{\mu,k}$ given by the norm:

$$\|h\|_{\mu,2,k} := \left(\int_{\mathbf{R}}\int_{\mathbf{S}} \frac{h(\theta,\omega)^2}{k(\theta,\omega)}\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega)\right)^{\frac{1}{2}}. \tag{3.28}$$

Remark 3.20. Let us remark that for all bounded weights k_1 and k_2

$$\|u\|^2_{\mu,-1,k_1} \leqslant \frac{\|k_2\|_\infty}{\|k_1\|_\infty}\|u\|^2_{\mu,-1,k_2}\,,$$

so that all the norms we have introduced are equivalent, provided the corresponding weights are bounded.

Remark 3.21. In the particular case of $k(\cdot,\omega) \equiv \frac{1}{2\pi}$ for all $\omega \in \mathrm{Supp}(\mu)$, we will write $\mathbf{H}^{-1}_\mu := \mathbf{H}^{-1}_{\mu,\frac{1}{2\pi}}$ and the corresponding norm will be denoted as $\|\cdot\|_{\mu,-1}$. We will also write $(\mathbf{L}^2_\mu, \|\cdot\|_{\mu,2})$ instead of $(\mathbf{L}^2_{\mu,\frac{1}{2\pi}}, \|\cdot\|_{\mu,2,\frac{1}{2\pi}})$.

2.4 Stability of stationary solutions

We now to turn to the issue of stability of the stationary solutions in the *sine-model*.

2.4.1 The incoherent stationary solution

As far as the incoherent solution $\frac{1}{2\pi}$ is concerned, the subject of its local stability has already been studied by Strogatz and Mirollo in [86]:

Proposition 3.22 (Strogatz, Mirollo, [86])**.** *If μ is unimodal, then the incoherent solution $\frac{1}{2\pi}$ is linearly stable for $K \leqslant \tilde{K}$ and unstable for $K > \tilde{K}$, where $\tilde{K}$ is defined in* (3.16)*.*

The main purpose of this chapter is to treat the mode delicate issue of the stability properties of the nontrivial solutions (that is for $r > 0$). In the next paragraph, we consider the case where there is no disorder ($\mu = \delta_0$) which has been fully treated by Bertini, Giacomin, and Pakdaman [9], and will be of constant use for the rest of the chapter.

2.4.2 Stability of synchronization: the non-disordered case

Recent advances have been made concerning the dynamical properties of the non-disordered Kuramoto model, which (as already explained in the main introduction (§ 2.2.2, p. 2.2.2)) is reversible and whose corresponding Fokker-Planck PDE (3.5) is of gradient flow type. These properties have been exploited in [9] in order to prove the linear stability of the nontrivial stationary solution (3.7), with explicit spectral gap estimates.

Namely, if one linearizes the Kuramoto evolution (3.5) around its stationary solution q_0 (see (3.7)), one obtains

$$\forall t > 0, \quad \partial_t u_t = L_{q_0} u_t, \tag{3.29}$$

where L_{q_0} is the linear operator arising from the linearization of evolution (3.5):

$$L_{q_0} u := \frac{1}{2}\partial_\theta^2 u - \partial_\theta \left[q_0 (J * u) + u (J * q_0) \right], \tag{3.30}$$

with domain

$$\mathcal{D}_0 := \left\{ u \in \mathcal{C}^2(\mathbf{S});\ \int_{\mathbf{S}} u(\theta)\, \mathrm{d}\theta = 0 \right\}. \tag{3.31}$$

The main purpose of [9] was to prove that the linear dynamics (3.29) is exponentially stable in $\mathbf{H}_{q_0}^{-1}$, where $\mathbf{H}_{q_0}^{-1}$ is the weighted-Sobolev space introduced in Remark 3.18:

Proposition 3.23 (Bertini-Giacomin-Pakdaman)**.** *$(L_{q_0}, \mathcal{D}_0)$ is essentially self-adjoint in $\mathbf{H}_{q_0}^{-1}$. The spectrum of (the self-adjoint extension of) L_{q_0} is pure point lying in $(-\infty, 0)$; 0 is in the spectrum, with one-dimensional eigenspace (spanned by q_0'). Moreover, the distance $\lambda_K(L_{q_0})$ between the eigenvalue 0 and the rest of the spectrum is strictly positive.*

Remark 3.24. We will strongly rely on Proposition 3.23 for the study on the disordered case, considering it mostly as a perturbation of the case with no disorder.

What is more, the whole long-time dynamics of the nonlinear evolution (3.5) is addressed in [42], showing in particular the existence of a global attractor in both cases $K \leqslant 1$ and $K > 1$ and proving nonlinear stability of synchronized solutions for $K > 1$.

2.4.3 Stability of synchronization: the symmetric case

The main goal of the chapter is to address the question of the stability properties of synchronization in the disordered-Kuramoto model. We focus on the case where μ is symmetric. Recall (§ 2.1.3, p. 15) that we can always assume (by an appropriate change of variables) that the mean value $\mathbb{E}(\omega) = \int_{\mathbf{R}} \omega \, \mathrm{d}\mu(\omega)$ of the disorder is zero. Moreover we

rewrite here the natural frequencies as $\delta\omega$, with δ a nonnegative parameter. We assume moreover that

$$\mathrm{Supp}(\mu) \subseteq [-1,1]\,. \tag{3.32}$$

In this set-up, (3.3) becomes

$$\partial_t p_t^\delta(\theta,\omega) \;=\; \frac{1}{2}\partial_\theta^2 p_t^\delta(\theta,\omega) - \partial_\theta\Big(p_t^\delta(\theta,\omega)(\langle J * p_t^\delta\rangle_\mu(\theta) + \delta\omega)\Big)\,. \tag{3.33}$$

Note that this leads to (obvious) changes to the definitions of the stationary solutions (3.15) and the fixed-point formulation (3.10). We have introduced this parameterization because the results that we present are for small values of δ. Namely we are going to show that when μ is symmetric, $K > 1$ and $K > \widetilde{K}$, there exists $\delta_0 > 0$ such that for $\delta \in [0,\delta_0]$ there is, up to rotation symmetry, only one synchronized solution $(\theta,\omega) \mapsto q(\theta,\omega)$ to (3.33) and that it is (linearly and nonlinearly) stable. In particular, the main object of interest will be the linearized operator around this stationary solution:

$$L_q^\delta u(\theta,\omega) \;:=\; \frac{1}{2}\partial_\theta^2 u(\theta,\omega) - \partial_\theta\left(u(\theta,\omega)\left(\langle J * q\rangle_\mu(\theta) + \delta\omega\right) + q(\theta,\delta\omega)\langle J * u\rangle_\mu(\theta)\right), \tag{3.34}$$

with domain $\mathcal{D}$ given by (3.26).

The main result of this section (Theorem 3.27) concerns the spectral properties of the linear operator L_q^δ and is based on the rather good understanding that we have of the case without disorder, that is the *non-disordered PDE* mentioned in (3.5) and studied in § 2.4.3. Some of the results in [9] are recalled in the next section, but they are not directly applicable because the $\delta = 0$ case that corresponds to what interests us is rather

$$\partial_t p_t(\theta,\omega) \;=\; \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)(\langle J * p_t\rangle_\mu(\theta)\Big)\,, \tag{3.35}$$

In (3.35), the *natural frequencies* have no effective role beyond separating the various rotators into populations with given natural (ineffective) frequency that now are just labels. Hence from now on, evolution (3.35) will be called *labeled PDE*. But in order to set-up a proper perturbation procedure we need to control (3.35) and, in particular, we need (and establish) a spectral gap inequality for the evolution (3.35) linearized around the nontrivial solutions.

This spectral analysis is going to be central for our stability result, based on ad hoc estimates, made possible by the explicit expressions (3.10)-(3.15), and using results in the general theory of operators [70] and perturbation theory of self-adjoint operators [51].

This question of stability has been already repeatedly approached but looking at synchronized solutions as bifurcation from incoherence. The results are hence for K close to the critical value corresponding to the breakdown of linear stability of $1/2\pi$: one can find a detailed review of the vast literature on this issue in [2, Sec. III]. Our results are instead for arbitrary $K > 1$, but δ smaller than $\delta_0(K)$ and of course $\delta_0(K)$ vanishes as K approaches 1.

The labeled PDE: Let us now focus on the *labeled PDE* (3.35) and let us insist on the fact that we are interested in solutions such that $p_t^\delta(\cdot,\omega)$ is a probability density. Observe then that if $q(\theta,\omega)$ is a stationary solution of (3.35), we see (Proposition 3.1) that q is C^∞ with respect to θ and that $\langle q\rangle_\mu$ is a stationary solution for (3.5). So there exists $\psi \in \mathbf{S}$ such that $\langle q\rangle_\mu = q_\psi$ and a short computation leads to

$$\langle J * q\rangle_\mu(\theta) \;=\; -K\sin(\theta-\psi)\,,$$

and, since $\int_{\mathbf{S}} q(\theta,\omega)\,\mathrm{d}\theta = 1$ for almost all ω, we obtain that $q(\cdot,\omega) = q_\psi(\cdot)$ for almost all ω. In conclusion, with some abuse of notation, we can say the stationary solutions of (3.5) and (3.35) are the same: of course in the second case the function space includes the dependence on ω, so we choose a different notation, that is M_0 for the corresponding circle of nontrivial stationary solutions in place of M_{rev} defined in (3.9).

An important issue for us is the stability of M_0 (for its existence we are assuming $K > 1$) and for this we denote by A the linearized evolution operator of (3.35) around q_0

$$Au(\theta,\omega) := \frac{1}{2}\partial_\theta^2 u(\theta,\omega) - \partial_\theta\Big(q_0(\theta)\langle J * u\rangle_\mu(\theta) + u(\theta,\omega)(J * q_0)(\theta)\Big) \tag{3.36}$$

with the same domain (3.26) as L_q^δ.

The following result, which is to the *labeled PDE* (3.35) what Proposition 3.23 is to the *non-disordered PDE* (3.5), concerns the linear dynamics of the *labeled evolution* (3.35). Recall the definitions of the weighted-Sobolev in § 2.3.3:

Proposition 3.25. *A is essentially self-adjoint in $\mathbf{H}^{-1}_{\mu,q_0}$. Moreover the spectrum lies in $(-\infty,0]$, 0 is a simple eigenvalue, with eigenspace spanned by $\partial_\theta q_0$, and there is a spectral gap, that is the distance λ_K between 0 and the rest of the spectrum is positive.*

Proposition 3.25 is just technical and its proof will be omitted; we refer to [41] for details. It is worth to note that this result strongly relies on Proposition 3.23 (see [9, Th. 1.8]) and that the (lower) bound on the spectral gap λ_K that we obtain coincides with the quantity $\lambda(K)$ in [9, Th. 1.8] (this bound can be improved as explained in [9, Sec. 2.5] and sharp estimates on the spectral gap can be obtained in the limit $K \searrow 1$ and $K \nearrow \infty$).

Existence of synchronized solutions for small disorder: Recall that we restrict ourselves to the case in which the distribution of the disorder μ is symmetric. In this case we already know (see Proposition 3.10) that for K sufficiently large there is at least one whole circle of stationary solutions. Actually, using Proposition 3.13 we can show that for δ small there is just one circle, that we call M_δ, of nontrivial stationary solutions.

In order to be precise about this issue, we point out that (3.10)-(3.15) are written for (3.3) while we work rather with (3.33). The changes are obvious, but we introduce a notation for the analog of (3.10):

$$r_\delta = \Psi_\delta^\mu(2Kr_\delta), \qquad \text{where, } \Psi_\delta^\mu(x) := \int_{\mathbf{R}} \frac{\int_{\mathbf{S}} \cos(\theta) S(\theta,\delta\omega,x)\,\mathrm{d}\theta}{Z(\delta\omega,x)}\,\mathrm{d}\mu(\omega)\,.$$

A reformulation of Proposition 3.13 in this context gives the following

Lemma 3.26. *For all $K_{\max} > 0$, there exists $\delta_1 = \delta_1(K_{\max}) > 0$ such that, for all $0 < K < K_{\max}$ and all $\delta \leqslant \delta_1$ the function Ψ_δ^μ is strictly concave on $[0, 2K_{\max}]$. Therefore (3.10) has a unique positive solution $r_\delta = r_\delta(K,\mu)$. Moreover $\lim_{\delta\searrow 0} r_\delta = r_0$.*

Main result of stability: We place ourselves within the framework of Lemma 3.26, in the sense that δ is small enough to ensure the uniqueness of a nontrivial stationary solution (of course existence requires $K > \tilde{K}_\delta$ and this is implied by $K > 1$ if δ is sufficiently small). We prove a number of properties of the linear operator (3.34), saying notably that it has a simple eigenvalue at zero, that the rest of spectrum is at a positive distance from zero and it lies in a cone in that is in the negative complex half plane.

Theorem 3.27 (Linear stability for weak disorder)**.** *The operator L_q^δ has the following spectral properties: 0 is a simple eigenvalue for L_q^δ, with eigenspace spanned by $(\theta,\omega)\mapsto q'(\theta,\omega)$. Moreover, for all $K>1$, $\rho\in(0,1)$, $\alpha\in(0,\pi/2)$, there exists $\delta_2=\delta_2(K,\rho,\alpha)$ such that for all $0\leqslant\delta\leqslant\delta_2$, the following is true:*

- *L_q^δ is closable in $\mathbf{H}^{-1}_{\mu,q_0}$ and its closure has the same domain as the domain of the self-adjoint extension of A defined in* (3.36)*;*
- *The spectrum of L_q^δ lies in a cone C_α with vertex 0 and angle α*

$$C_\alpha := \left\{\lambda\in\mathbb{C};\ \frac{\pi}{2}+\alpha\leqslant\arg(\lambda)\leqslant\frac{3\pi}{2}-\alpha\right\}\subseteq\{z\in\mathbf{C};\ \Re(z)\leqslant 0\}\ ;$$

- *There exists $\alpha'\in(0,\frac{\pi}{2})$ such that L_q^δ is the infinitesimal generator of an analytic semigroup defined on a sector $\{\lambda\in\mathbb{C},\ |\arg(\lambda)|<\alpha'\}$;*
- *The distance between 0 and the rest of the spectrum is strictly positive and is at least equal to $\rho\lambda_K$, where λ_K is the spectral gap of the operator A.*

The results mentioned in Theorem 3.27 are qualitative but what we really prove are quantitative explicit estimates:

Proposition 3.28. *For all $K>1$, $\rho\in(0,1)$, $\alpha\in(0,\pi/2)$, the conclusions of Theorem 3.27 are true at least when $0\leqslant\delta\leqslant\delta_K^\star$, for some $\delta_K^\star$ satisfying the following equivalents (where C is a positive numerical constant):*

$$\delta_K^\star\sim C\max\left(\rho(1-\sin(\alpha)),\ell(\rho)\right)\lambda_K,\quad \text{as } K\searrow 1\,, \tag{3.37}$$

$$\delta_K^\star\sim C\max\left(\rho(1-\sin(\alpha)),\ell(\rho)\right)\frac{e^{-24K}}{K^2}\lambda_K,\quad \text{as } K\to\infty\,, \tag{3.38}$$

where

$$\ell(\rho):=\max\left(\frac{2}{\rho+1},\frac{2}{1-\rho}\right).$$

Remark 3.29. We refer to [9, § 2.5] concerning the asymptotics for the spectral gap λ_K (as $K\searrow 1$ and $K\to\infty$) that are used in Proposition 3.28.

Note that estimate (3.38) does not seem to be sharp as $K\to\infty$, since the intuition for the model is that the larger K is, the easier it is for the rotators to synchronize and the more stable stationary solutions should be. Hence, understanding the case where K is large seems to require alternative methods to the perturbations arguments used in this chapter.

Proofs of Theorem 3.27 and Proposition 3.28 may be found in Section 4.

Nonlinear stability: From the previous linear stability result, one can derive that in fact the circle of stationary solutions $M=\{q_{\theta_0}(\theta,\omega):=q(\theta+\theta_0,\omega);\ \theta_0\in\mathbf{S}\}$ (recall (3.14)) is *locally nonlinearly stable*. This can be done by using arguments introduced in [45, Chap. 5, Sec. 1] (see in particular [45, Ex. 6, p. 108]):

Proposition 3.30 (Nonlinear synchronization stability)**.** *We place ourselves under the hypothesis of Theorem 3.27. For any $\psi\in\mathbf{S}$, there exists $\varepsilon>0$ such that for any solution p to* (3.33) *with $\|p|_{t=0}-q_\psi\|_{\mu,2}\leqslant\varepsilon$, then $\|p_t-q_\psi\|_{\mu,2}\leqslant\varepsilon$ for all $t>0$ and there exists $\psi_\infty\in\mathbf{S}$ such that*

$$\forall\lambda<\lambda_K,\ \|p_t-q_{\psi+\psi_\infty}\|_{\mu,1}=O(e^{-\lambda t}).$$

Remark 3.31. Note that Proposition 3.30 is very similar to [42, Th. 4.8] which concerns the same nonlinear stability result around the stationary solution, in the case with no disorder. We refer to the proof of [42, Th. 4.8] for details.

2.4.4 Stability of synchronization: the asymmetric case

The question of stability of synchronization in the case where μ is no longer symmetric is much more delicate. In particular, there is no obvious way to parameterize the stationary solutions to McKean-Vlasov evolution (3.3) (see Remark 3.34) and the dynamics in the asymmetric case appears to be fairly more complicated than the dynamics in the symmetric case. One way to get around this difficulty is to directly perturb the nonlinear Kuramoto evolution (3.33) instead of perturbing the linearized evolution as in the symmetric case.

The purpose of this paragraph is to outline the method and results previously developed by Giacomin, Pakdaman, Pellegrin and Poquet ([43]) for non-disordered models which can be applied in [41] in order to establish the stability of synchronization in the asymmetric case.

The methods developed in [42] exploit the notion of *normally hyperbolic structure* [46, 79] of the manifold of stationary solutions of (3.35) and the robustness of such structures (like in [43]). In this context, the aim of Giacomin et al. [43] was to prove the existence and stability of invariant normally hyperbolic manifolds for interacting diffusions in absence of disorder.

It is worth to note that the normal hyperbolic manifold approach allows to treat cases that are substantially more general than the *sine-model* and notably the case of the *active rotators model*:

$$\partial_t p_t(\theta,\omega) \;=\; \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)(\langle J * p_t\rangle_\mu(\theta) + \delta U(\theta,\omega))\Big)\,, \tag{3.39}$$

which is a particular instance of (3.2) in the case where $b(\theta,\theta',\omega,\omega') = K\sin(\theta'-\theta)$ and $c(\theta,\omega) = \delta U(\theta,\omega)$.

The *active rotators model* (previously mentioned in the physics literature, see e.g. [81] and references therein) concerns similar interacting rotators in the presence of an additional potential. The dynamical properties for this model tend to be fairly more complicated than the Kuramoto model (1.1) studied here. In (3.39), each oscillator has its own nontrivial dynamics which may be very different from the dynamics of other oscillators: consider for example

$$U(\varphi,\omega) \;=\; b + \omega + a\sin(\varphi)\,, \quad a,b \in \mathbf{R}\,,$$

and μ uniform over $[-1,1]$. For $a \in (-1,1)$ there are some *active rotators* [81, 43] that in absence of noise and interaction ($\sigma = K = 0$) rotate (this happens if $|b+\omega| > |a|$ and of course the direction of rotation depends on the sign of $b+\omega$) and others that instead are stuck at a fixed-point (this happens if $|b+\omega| \le |a|$).

Main result of stability in the sine-model in the asymmetric case: From an intuitive point of view, if the distribution of the disorder is asymmetric, there are more rotators that are likely to rotate in one given direction than in the other direction; on the whole, the system has a tendency to rotate in this direction and rotating waves appear in evolution (3.33).

The approach of [41] makes this intuition rigorous by proving that there is a synchronization regime for $K > 1$ and δ small and showing the existence of a family of stables rotating solutions to (3.33). More precisely, the normal hyperbolic manifold approach allows to prove the existence of a solution $p_t^\delta(\theta,\omega)$ to (3.33) of the form $q(\theta - c_\mu(\delta)t)$, for which it can be shown that $c_\mu(\delta) = O(\delta^3)$ with explicit expression of $\lim_{\delta\searrow 0} c_\mu(\delta)/\delta^3$: this is a rotating wave (or limit-cycle) for the dynamical system (3.33) and its stability under perturbations is established.

For $C > 0$ and $M \subset \mathbf{L}^2_\mu$, we set $\mathcal{N}_{\mu,2}(M, C) := \{u : \text{there exists } v \in M \text{ such that } \| u - v \|_{\mu,2} \leq C\}$. In the statement below $q \in M_0$ is the element of the manifold such that $q(\cdot, \omega) = q_0(\cdot)$, cf. (3.7), with $r_0(K) > 0$ (hence $K > 1$).

The main result in [41] (which is not proved in this thesis) is the following:

Theorem 3.32. *For every $K > 1$, there exists $\delta_0 = \delta_0(K) > 0$ such that for $|\delta| \leq \delta_0$, there exists $\widetilde{q}_\delta \in \mathbf{L}^2_\mu$, satisfying $\| \widetilde{q}_\delta - q \|_{\mu,2} = O(\delta)$ and a value $c_\mu(\delta) \in \mathbf{R}$ such that if we set*

$$q_t^{(\psi)}(\theta, \omega) := \widetilde{q}_\delta(\theta - c_\mu(\delta)t - \psi)\,,$$

then $q_t^{(0)}$ solves (3.33)*. Moreover*

1. *the family of solutions $\{q_\cdot^{(\psi)}\}_\psi$ is stable in the sense that there exist two positive constants $\beta = \beta(K)$ and $C = C(K)$ such that if $p_0^\delta \in \mathcal{N}_{\mu,2}(M_0, \delta)$, and $\int_{\mathbf{S}} p_0^\delta(\theta, \omega)\,\mathrm{d}\theta = 0$, μ-a.s., then there exists $\psi_0 \in \mathbf{S}$ such that for all $t \geq 0$*

$$\left\| q_t^{(\psi_0)} - p_t^\delta \right\|_{\mu,2} \leq 2C \exp(-\beta t)\,.$$

2. *we have*

$$c_\mu(\delta) = \delta^3 \frac{\left\langle \omega \partial_\theta n^{(2)}\,,\, \partial_\theta q_0 \right\rangle_{\mu,-1,q_0}}{\langle \partial_\theta q_0\,,\, \partial_\theta q_0 \rangle_{-1,q_0}} + O(\delta^5)\,,$$

where $n^{(2)}$ is the unique solution of

$$An^{(2)} = \omega \partial_\theta n^{(1)} \quad \text{and} \quad \left\langle n^{(2)}\,,\, \partial_\theta q_0 \right\rangle_{\mu,-1,q_0} = 0\,,$$

and $n^{(1)}$ is the unique solution of

$$An^{(1)} = \omega\, \partial_\theta q_0 \quad \text{and} \quad \left\langle n^{(1)}\,,\, \partial_\theta q_0 \right\rangle_{\mu,-1,q_0} = 0\,.$$

In this thesis, we only state this result. The interested reader may refer to [41, Section 2] for proofs and further explicit estimates on the speed of rotation.

2.5 Organization of remainder of the chapter

The purpose of Section 3 is to establish the regularity results and the existence of stationary solutions of § 2.1. In Section 4, we prove the main result of linear stability (Theorem 3.27) along with a number of related quantitative estimates leading to Proposition 3.28.

3 Regularity and stationarity in McKean-Vlasov evolution

3.1 Regularity of the semigroup (proofs of Propositions 3.1 and 3.5)

The first aim of this part is to prove the regularity results stated in § 2.1. We begin with the proof of Proposition 3.1 concerning the strong formulation of McKean-Vlasov evolution and the regularity of its semigroup:

Proof of Proposition 3.1. Let us fix $T > 0$, $\omega \in \mathrm{Supp}(\mu)$ and $t \mapsto \nu_t$ the unique solution in $\mathcal{C}([0,T], \mathcal{M}_1(\mathbf{S} \times \mathbf{R}))$ to (3.1). Let us define

$$R(t,\theta,\omega) := \int_{\mathbf{S}\times\mathbf{R}} b(\theta,\theta',\omega,\omega')\,\mathrm{d}\nu_t(\theta',\omega') + c(\theta,\omega), \tag{3.40}$$

$$= b[\theta,\omega,\nu_t] + c(\theta,\omega). \tag{3.41}$$

Under the hypothesis made on b and c in § 2.1.1, for fixed $\omega \in \mathrm{Supp}(\mu)$, $R(\cdot,\cdot,\omega)$ is continuous in time and $\mathcal{C}^\infty$ in θ. Consider now the linear equation

$$\partial_t p_t(\theta,\omega) \;=\; \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\Big(p_t(\theta,\omega)R(t,\theta,\omega)\Big)\,, \tag{3.42}$$

such that for μ-a.e. ω, for all $\varphi \in \mathcal{C}(\mathbf{S})$,

$$\int_{\mathbf{S}} \varphi(\theta)p_t(\theta,\omega)\,\mathrm{d}\theta \overset{t\searrow 0}{\longrightarrow} \int_{\mathbf{S}} \varphi(\theta)\,\mathrm{d}\nu_0^\omega(\theta)\,. \tag{3.43}$$

Suppose for a moment that we have found a weak solution $p_t(\theta,\omega)$ to (3.42)-(3.43) such that for μ-a.e. ω, $p_t(\cdot,\omega)$ is strictly positive on $(0,T]\times\mathbf{S}$. In particular for such a solution p, the quantity $\int_{\mathbf{S}} p_t(\theta,\omega)\,\mathrm{d}\theta$ is conserved for $t > 0$, so that $p_t(\cdot,\omega)$ is indeed a probability density for all $t > 0$. Then both probability measures $\nu_t(\,\mathrm{d}\theta,\,\mathrm{d}\omega)$ and $p_t(\theta,\omega)\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega)$ solve, for all $\varphi \in \mathcal{C}_b^2(\mathbf{S}\times\mathbf{R})$

$$\int_{\mathbf{S}\times\mathbf{R}} \varphi(\theta,\omega)\,\mathrm{d}\nu_t(\theta,\omega) = \int_{\mathbf{S}\times\mathbf{R}} \varphi(\theta,\omega)\,\mathrm{d}\nu_0(\theta,\omega) + \frac{1}{2}\int_0^t \int_{\mathbf{S}\times\mathbf{R}} \varphi''(\theta,\omega)\,\mathrm{d}\nu_s(\theta,\omega)\,\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbf{S}\times\mathbf{R}} \varphi'(\theta,\omega)R(t,\theta,\omega)\,\mathrm{d}\nu_s(\theta,\omega)\,\mathrm{d}s. \tag{3.44}$$

Note that (3.44) is exactly a reformulation of (2.63) in Chapter 2, p. 67, which has been used in Proposition 2.20 to prove uniqueness in the weak formulation in McKean-Vlasov evolution. The key point of the proof of Proposition 2.20 was precisely to show that there is uniqueness in (3.44). Hence, by uniqueness in (3.44), $\nu_t(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = p_t(\theta,\omega)\,\mathrm{d}\theta\mu(\,\mathrm{d}\omega)$, which is the result. So it suffices to exhibit a weak solution $p_t(\theta,\omega)$ to (3.42) such that (3.43) is satisfied.

This fact can be deduced from standard results for uniform parabolic PDEs (see [4] and [37] for precise definitions). In particular, a usual result, which can be found in [4, §7 p. 658], states that (3.42) admits a fundamental solution $\Gamma(\theta,t;\theta',s,\omega)$ $(t > s)$, which is bounded above and below (see [4, Th.7, p. 661]):

$$\frac{1}{C\sqrt{t-s}}\exp\left(\frac{-C(\theta-\theta')^2}{\sqrt{t-s}}\right) \leqslant \Gamma(\theta,t;\theta',s,\omega) \leqslant \frac{C}{\sqrt{t-s}}\exp\left(\frac{-(\theta-\theta')^2}{C\sqrt{t-s}}\right). \tag{3.45}$$

Note that the constant $C > 0$ only depends on T and the *structure* of the linear operator in (3.42) (see [4, Th.7, p. 661] and [4, § 1, p. 615]). In particular, thanks to the assumptions made on b and c, this constant only depends on $M > 0$, whenever $\omega \in [-M, M]$.

Note that the proof given in [4] is done for $\theta \in \mathbf{R}$ but can be readily adapted to our case ($\theta \in \mathbf{S}$).

Moreover, thanks to Corollary 12.1, p. 690 in [4], the following expression of $p_t(\theta,\omega)$

$$p_t(\theta,\omega) = \int_{\mathbf{S}} \Gamma(\theta,t;\theta',0,\omega)\,\mathrm{d}\nu_0^\omega(\theta') \tag{3.46}$$

defines a weak solution of (3.42) on $(0,T] \times \mathbf{S}$ (namely a weak solution on $(\tau,T] \times \mathbf{S}$, for all $0 < \tau < T$) such that (3.43) is satisfied. The positivity and boundedness of $p_t(\cdot,\omega)$ for $t > 0$ is an easy consequence of (3.45). The smoothness of $p_\cdot(\cdot,\omega)$ on $(0,T] \times \mathbf{S}$ can be derived by standard bootstrap methods. □

Let us now focus on the regularity with respect to the disorder (Proposition 3.5):

Proof of Proposition 3.5. For any ω in the support of μ, let for all $t > 0$, $\theta \in \mathbf{S}$

$$u(t,\theta,\omega) := p_t(\theta,\omega) - p_t(\theta,\omega_0),$$

where $(p_t(\cdot,\cdot))_{t \geqslant 0}$ is the unique solution of (3.42). It is easy to see that u is a strong solution to the following PDE

$$\partial_t u(t,\theta,\omega) - \left[\frac{1}{2}\partial_\theta^2 u(t,\theta) - \partial_\theta \left(u(t,\theta)R(t,\theta,\omega_0)\right)\right] = \mathcal{R}(t,\theta,\omega),$$

where $\mathcal{R}(t,\theta,\omega) := \partial_\theta \left[p_t(\theta,\omega)\left(R(t,\theta,\omega) - R(t,\theta,\omega_0)\right)\right]$ and with initial condition (since $\nu_0^\omega(\,\mathrm{d}\theta) = \gamma(\theta,\omega)\,\mathrm{d}\theta$ for all ω)

$$u(t,\theta,\omega)|_{t\searrow 0} = \gamma(\theta,\omega) - \gamma(\theta,\omega_0). \tag{3.47}$$

Then applying [37, Th. 12 p. 25], $u(t,\theta,\omega)$ can be expressed as

$$\begin{aligned} u(t,\theta,\omega) = &\int_{\mathbf{S}} \Gamma(\theta,t;\theta',0,\omega_0)(\gamma(\theta',\omega) - \gamma(\theta',\omega_0))\,\mathrm{d}\theta' \\ &- \int_0^t \int_{\mathbf{S}} \Gamma(\theta,t;\theta',s,\omega_0)\mathcal{R}(s,\theta',\omega)\,\mathrm{d}\theta'\,\mathrm{d}s. \end{aligned} \tag{3.48}$$

For the first term of the r.h.s. of (3.48), we have

$$\left|\int_{\mathbf{S}} \Gamma(\theta,t;\theta',0,\omega_0)(\gamma(\theta,\omega) - \gamma(\theta,\omega_0))\,\mathrm{d}\theta'\right| \leqslant \frac{C}{\sqrt{t}} \int_{\mathbf{S}} |\gamma(\theta,\omega) - \gamma(\theta,\omega_0)|\,\mathrm{d}\theta',$$

which converges to 0, for fixed $t > 0$, as $\omega \to \omega_0$ by hypothesis (3.47).

Secondly, it is easy to see from the definition (3.46) of the density p and the estimates (3.45) and [37, Th.9 p. 263] concerning the fundamental solution Γ that both $p_t(\theta,\omega)$ and $\partial_\theta p_t(\theta,\omega)$ are bounded uniformly on $(t,\theta,\omega) \in [0,T] \times \mathbf{S} \times [-M,M]$ for $M > 0$ such that $|\omega_0| < M$. In particular, a standard result shows that for fixed (t,θ), the second term of the r.h.s. of (3.48) goes to 0 as $\omega \to \omega_0$. But then the joint continuity of p at (t_0,θ_0,ω_0) follows from (3.46) and uniform estimates on Γ (see [37, Th.9 p. 263]). □

3.2 Stationarity in McKean-Vlasov equation

3.2.1 A reminder of the construction of the fixed-point function (Proposition 3.10)

We recall here the basic steps of the proof of Sakaguchi [75] (see also [30]) about the calculation of the stationary solutions to (3.3),

$$0 = \frac{1}{2}\partial_\theta^2 q(\theta,\omega) - \partial_\theta\Big(q(\theta,\omega)(\langle J * q\rangle_\mu(\theta) + \omega)\Big) \tag{3.49}$$

Sketch of proof of Proposition 3.10. The stationary version of the order parameters defined in (1.20) are given by ($r > 0$, $\psi \in \mathbf{R}$):

$$re^{i\psi} = \int_{\mathbf{S}\times\mathbf{R}} e^{i\theta} q(\theta,\omega)\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega) \tag{3.50}$$

Rewriting (3.49) in terms of the parameters r and ψ, we have:

$$0 = \frac{1}{2}\partial_\theta^2 q(\theta,\omega) - \partial_\theta\left(q(\theta,\omega)(Kr\sin(\psi-\cdot)+\omega)\right). \tag{3.51}$$

So, the problem we have to solve is the following:

Problem 3.33. Find $(r,\psi,q) \in [0,+\infty)\times\mathbf{R}\times\mathcal{C}(\mathbf{S}\times\mathbf{R})$ which verifies the following conditions:

1. for μ almost every ω, $\theta \mapsto q(\theta,\omega)$ is defined on $\mathbf{S}$ (i.e. $q(\cdot,\omega)$ is a 2π-periodic function) of class $\mathcal{C}^2$, nonnegative, of integral 1 on $[0,2\pi]$.
2. for μ almost every ω, $q(\cdot,\omega)$ satisfies (3.51),
3. (r,ψ,q) satisfies (3.50).

The first observation comes from the rotational invariance of the *sine-model* (Remark 1.8, p. 24): namely, $(r,\psi,q(\cdot,\cdot))$ solves Problem 3.33 if and only if $(r,0,q(\cdot+\psi,\cdot))$ solves Problem 3.33. In particular, from now we can suppose $\psi = 0$ so that (3.51) becomes:

$$0 = \frac{1}{2}\partial_\theta^2 q(\theta,\omega) - \partial_\theta\left(q(\theta,\omega)(-Kr\sin(\cdot)+\omega)\right). \tag{3.52}$$

Integrating (3.52) twice, we obtain that there exist two constants (depending on ω), $C_1(\omega)$ and $C_2(\omega)$ such that for all $\theta \in \mathbf{S}$,

$$q(\theta,\omega,r) = e^{G(\theta,\omega,2Kr)}\left[C_1(\omega)\int_0^\theta e^{-G(u,\omega,2Kr)}\,\mathrm{d}u + C_2(\omega)\right],$$

where $G(\theta,\omega,x)$ is defined by (3.12). Imposing the conditions that for all ω, $\theta \mapsto q(\theta,\omega,r)$ is a probability density and that $q(\cdot,\omega,r)$ is 2π-periodic fixes the constants C_1 and C_2 to

$$C_1(\omega) = \frac{1-e^{4\pi\omega}}{Z(\omega,2Kr)},$$
$$C_2(\omega) = \frac{e^{4\pi\omega}\int_0^{2\pi}\exp(-G(u,\omega,2Kr))\,\mathrm{d}u}{Z(\omega,2Kr)},$$

where $Z(\omega,2Kr)$ is the normalization constant defined in (3.13). It is straightforward to see that we define in that way a nonnegative, 2π-periodic function of integral 1 on $[0,2\pi]$, such that $q(\theta,\omega,r) = q(-\theta,-\omega,r)$ for all θ,ω.

As a conclusion, $(r,0,q)$ is a solution to Problem 3.33 if and only if r verifies the fixed-point relation (3.50) with $\psi = 0$ which is equivalent to

$$\begin{cases} r &= \displaystyle\int_{\mathbf{S}\times\mathbf{R}} \cos(\theta)q(\theta,\omega,r)\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega) = \Psi^\mu(2Kr), \\ 0 &= \displaystyle\int_{\mathbf{S}\times\mathbf{R}} \sin(\theta)q(\theta,\omega,r)\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega). \end{cases} \tag{3.53}$$

In the case where μ is symmetric, thanks to the symmetry of q ($q(\theta,\omega) = q(-\theta,-\omega)$ for all θ,ω), it is easy to see that the second equation in (3.53) is always verified. The stationary solutions to (3.3) can then be parameterized by the solutions to the first equation of (3.53), that is exactly (3.10). □

Remark 3.34. In the case where μ is not symmetric, the second equation in (3.53) is nontrivial; hence the problem of characterizing the solutions to (3.53) is far more complicated than in the symmetric case. However, using techniques developed in [43] there is a way to get around this difficulty and to make sense of stability of synchronization in the asymmetric case (recall § 2.4.4).

3.2.2 Concavity for small disorder (proof of Proposition 3.13)

Throughout this paragraph, μ will be a symmetric measure on $\mathbf{R}$. The object of interest here is the fixed-point function Ψ^μ defined in (3.10). The first properties of Ψ^μ (see [30]) are straightforward:

Proposition 3.35 (First properties of Ψ^μ)**.** *The following properties are true:*

1. $x \mapsto \Psi^\mu(x)$ *is continuous,*
2. $\lim_{x\to\infty} \Psi^\mu(x) = 1$,
3. *Seen as a function from* $\mathbf{R}$ *to* $(-1,1)$, $x \mapsto \Psi^\mu(x)$ *is odd.*
4. Ψ^μ *satisfies the following Taylor's expansion in* x*:*

$$\Psi^\mu(x) = \frac{x}{2\tilde{K}} - x^3\frac{D(\mu)}{8} + O(x^5),$$

where, $\tilde{K}$ *is defined in* (3.16) *and* $D(\mu)$ *is defined in* (3.17)*.*

Proof of Proposition 3.35. Those basic properties of Ψ^μ (already proved in [30]) follow from direct calculations. We refer to [30, Chap. X, p. 118] for details. □

In the case where there is no disorder ($\mu = \delta_0$), Pearce proved (see [71, Lemma 4, p. 315]) in the apparently different context of classical XY-spin model (for a detailed discussion on the link with these models see [9] and [2, § IV.B.]) that the fixed-point function Ψ_0 defined in (3.6) is strictly concave. We rely on this result in order to prove the strict concavity of the fixed-point function in the general case Ψ^μ defined by (3.10). But since $\partial_x^2\Psi_0(0) = 0$, we must treat the case of the concavity near the origin as a separate case.

Concavity in a neighborhood of the origin: Let us first introduce some notations: for all $M > 0$, define

$$\mathcal{M}_M := \left\{\mu \in \mathcal{M}_1(\mathbf{R});\ \int_{\mathbf{R}} |\omega|^5 \,\mathrm{d}\mu(\omega) \leqslant M\right\},$$

and for all $\alpha > 0$

$$\mathcal{P}_\alpha := \{\mu \in \mathcal{M}_1(\mathbf{R});\ D(\mu) \geqslant \alpha\}.$$

Let us prove the following:

Proposition 3.36 (Concavity at the origin)**.** *For all* $M > 0$ *and* $\alpha > 0$*, there exists* $x_0 = x_0(M,\alpha) > 0$ *such that* Ψ^μ *is concave on* $[0, x_0]$ *for all* $\mu \in \mathcal{M}_M \cap \mathcal{P}_\alpha$*.*

Proof of Proposition 3.36. Making the rest in the Taylor's expansion (3.17) explicit gives:

$$\forall x > 0,\ \Psi''_\mu(x) = -L(\mu)\frac{3}{4}x + \int_0^x (x-t)\Psi_\mu^{(4)}(t)\,\mathrm{d}t.$$

A long calculation shows that, using the fact that $\mu \in \mathcal{M}_M$, $\sup_{t\in[0,2K_{\max}]} |\Psi_\mu^{(4)}(t)| \leqslant C$, for a certain constant C (depending on $K_{\max}$ and M). Consequently, $\frac{1}{x}\left|\int_0^x (x-t)\Psi_\mu^{(4)}(t)\,\mathrm{d}t\right|$ tend to 0 as $x \to 0$.

Furthermore, by definition of $\mathcal{P}_\alpha$, $D(\mu) > \alpha > 0$. If we choose $x_0 > 0$ such that for all $0 < x \leqslant x_0$, $\frac{1}{x}\left|\int_0^x (x-t)\Psi_\mu^{(4)}(t)\,\mathrm{d}t\right| < \frac{3\alpha}{4}$, then $\Psi''_\mu(x) < 0$ for all $0 < x \leqslant x_0$. □

Concavity away from the origin: We now turn to the concavity of Ψ^μ away from the origin.

Proposition 3.37 (Concavity away from 0)**.** *For all $x_0 > 0$, there exists some $\delta > 0$ such that, if μ satisfies*

$$\int_{\mathbf{R}} |\omega|^3 \,\mathrm{d}\mu(\omega) \leqslant \delta,$$

then Ψ^μ is strictly concave on $[x_0, 2K_{\max}]$.

Proof of Proposition 3.37. Let us use the fact that the fixed-point function without disorder Ψ_0 is strictly concave on $(0,\infty)$: for fixed $x_0 > 0$, there exists $\eta > 0$ such that for all $x \in [x_0, 2K_{\max}]$, $\Psi''_0(x) < -\eta$.

The conclusion of Proposition 3.37 will be straightforward provided we prove that there exists some $\delta > 0$ sufficiently small so that for any measure μ satisfying (3.21) (with this value of δ) the following holds:

$$\sup_{x\in[x_0,2K_{\max}]} \left|\partial_x^2\Psi^\mu(x) - \partial_x^2\Psi_0(x)\right| \leqslant \frac{\eta}{2}. \tag{3.54}$$

Let us prove (3.54). Let us recall the notations of Proposition 3.10 and denote as $\psi^\mu(\omega,x)$ the integrand in the expression of Ψ^μ in (3.10):

$$\forall \omega \in \mathrm{Supp}(\mu), \forall x \in \mathbf{R}, \quad \psi^\mu(\omega,x) := \frac{A(\omega,x)}{Z(\omega,x)} \tag{3.55}$$

where $A(\omega,x) = \int_{\mathbf{S}} \cos(\theta) S(\theta,\omega,x)$ for $S(\cdot)$ is given by (3.11) and $Z(\omega,x) = \int_{\mathbf{S}} S(\theta,\omega,x)\,\mathrm{d}\theta$ is the normalization constant. First notice that an easy change of variables in (3.11) gives a more compact formulation of $S(\cdot)$:

$$\forall \omega \in \mathrm{Supp}(\mu), \forall x \in \mathbf{R}, \quad S(\theta,\omega,x) = e^{G(\theta,\omega,x)} \int_{\theta-2\pi}^{\theta} e^{-G(u,\omega,x)}\,\mathrm{d}u.$$

Note also that one also has the following (rough) estimates on A and Z: there exists a constant $C > 0$, only depending on $K_{\max}$ such that for all $\omega \in \mathrm{Supp}(\mu)$, for all $x \in [0, 2K_{\max}]$,

$$\begin{aligned} \frac{1}{C}\frac{e^{4\pi\omega}-1}{4\pi\omega} \leqslant Z(\omega,x) \leqslant C\frac{e^{4\pi\omega}-1}{4\pi\omega}, \\ |A(\omega,x)| \leqslant C\frac{e^{4\pi\omega-1}}{4\pi\omega}, \end{aligned} \tag{3.56}$$

as well as for every $i \in \{0,1,2\}$

$$\begin{aligned} \max\left(\left|\partial_x^i A(\omega,x)\right|, \left|\partial_x^i Z(\omega,x)\right|\right) &\leqslant C\frac{e^{4\pi\omega}-1}{4\pi\omega}, \\ \left|\partial_x^i A(\omega,x) - \partial_x^i A(0,x)\right| &\leqslant C\left|\frac{e^{4\pi\omega}-1}{4\pi\omega} - 1\right|, \\ \left|\partial_x^i Z(\omega,x) - \partial_x^i Z(0,x)\right| &\leqslant C\left|\frac{e^{4\pi\omega}-1}{4\pi\omega} - 1\right|. \end{aligned}$$

Let us only prove (3.56), the proofs of the other inequalities being similar: indeed we have successively

$$\int_0^{2\pi} e^{G(\theta,\omega,x)} \int_{\theta-2\pi}^{\theta} e^{-2K_{\max}} e^{-2\omega u}\,\mathrm{d}u\,\mathrm{d}\theta \leqslant Z(\omega,x) \leqslant \int_0^{2\pi} e^{G(\theta,\omega,x)} \int_{\theta-2\pi}^{\theta} e^{2K_{\max}} e^{-2\omega u}\,\mathrm{d}u\,\mathrm{d}\theta,$$
$$e^{-2K_{\max}} \frac{e^{4\pi\omega}-1}{2\omega} \int_0^{2\pi} e^{x\cos(\theta)}\,\mathrm{d}\theta \leqslant Z(\omega,x) \leqslant e^{2K_{\max}} \frac{e^{4\pi\omega}-1}{2\omega} \int_0^{2\pi} e^{x\cos(\theta)}\,\mathrm{d}\theta,$$
$$4\pi^2 e^{-4K_{\max}} \frac{e^{4\pi\omega}-1}{4\pi\omega} \leqslant Z(\omega,x) \leqslant 4\pi^2 e^{4K_{\max}} \frac{e^{4\pi\omega}-1}{4\pi\omega}.$$

Let us now differentiate twice ψ_μ w.r.t. x (note that the notation $Z'(\omega,x)$ stands for the derivative w.r.t. x):

$$\partial_x^2 \psi^\mu(\omega,x) = \frac{A''(\omega,x)}{Z(\omega,x)} - 2\frac{A'(\omega,x)Z'(\omega,x)}{Z^2(\omega,x)} - \frac{A(\omega,x)Z''(\omega,x)}{Z^2(\omega,x)} + 2\frac{A(\omega,x)(Z'(\omega,x))^2}{Z^3(\omega,x)}. \tag{3.57}$$

Let us cope with the first term in (3.57):

$$\begin{aligned}
\left|\frac{A''(\omega,x)}{Z(\omega,x)} - \frac{A(0,x)''}{Z(0,x)}\right| &= \left|\frac{A''(\omega,x)Z(0,x) - A''(0,x)Z(\omega,x)}{Z(\omega,x)Z(0,x)}\right|, \\
&\leqslant \frac{1}{Z(\omega,x)Z(0,x)}\Big(\left|A''(\omega,x) - A''(0,x)\right| Z(0,x) \\
&\quad + \left|A''(0,x)\right| \left|Z(\omega,x) - Z(0,x)\right| \Big), \\
&\leqslant C\frac{4\pi\omega}{e^{4\pi\omega}-1}\left|\frac{e^{4\pi\omega}-1}{4\pi\omega} - 1\right| = C\left|1 - \frac{4\pi\omega}{e^{4\pi\omega}-1}\right| \leqslant 4\pi C|\omega|,
\end{aligned} \tag{3.58}$$

where we used in (3.58) the fact that for all $y \in \mathbf{R}$, $\left|1 - \frac{y}{e^y-1}\right| \leqslant |y|$. We conclude by studying the fourth term of (3.57), (the two remaining terms are similar):

$$\begin{aligned}
\left|\frac{A(\omega,x)Z'(\omega,x)^2}{Z(\omega,x)^3} - \frac{A(0,x)Z'(0,x)^2}{Z(0,x)^3}\right| &\leqslant \frac{\left|A(\omega,x)Z'(\omega,x)^2 - A(0,x)Z'(0,x)^2\right| Z(0,x)^3}{Z(\omega,x)^3 Z(0,x)^3} \\
&\quad + \frac{|A(0,x)|Z'(0,x)^2 \left|Z(\omega,x)^3 - Z(0,x)^3\right|}{Z(\omega,x)^3 Z(0,x)^3}, \\
&= (I) + (II).
\end{aligned}$$

We have, for an appropriate constant C,

$$\begin{aligned}
(I) &\leqslant \frac{\left|A(\omega,x) - A(0,x)\right| Z'(\omega,x)^2}{Z(\omega,x)^3 Z(0,x)^3} + \frac{\left|Z'(\omega,x)^2 - Z'(0,x)^2\right| A(0,x)^2}{Z(\omega,x)^3 Z(0,x)^3}, \\
&\leqslant C\left|\frac{e^{4\pi\omega}-1}{4\pi\omega} - 1\right| \left(\frac{e^{4\pi\omega}-1}{4\pi\omega}\right)^{-1} + C\left(1 + \frac{e^{4\pi\omega}-1}{4\pi\omega}\right)\left|\frac{e^{4\pi\omega}-1}{4\pi\omega} - 1\right| \left(\frac{e^{4\pi\omega}-1}{4\pi\omega}\right)^{-3}, \\
&= C\left|1 - \frac{4\pi\omega}{e^{4\pi\omega}-1}\right| \left(1 + \frac{4\pi\omega}{e^{4\pi\omega}-1} + \left(\frac{4\pi\omega}{e^{4\pi\omega}-1}\right)^2\right), \\
&\leqslant C|\omega|\left(1 + |\omega| + |\omega|^2\right).
\end{aligned}$$

And,

$$\begin{aligned}
(II) &\leqslant C\frac{\left|Z(\omega,x)^3 - Z(0,x)^3\right|}{Z(\omega,x)^3},\\
&\leqslant \frac{C\left|Z(\omega,x)-Z(0,x)\right|\left(Z(\omega,x)^2+Z(0,x)Z(\omega,x)+Z(0,x)^2\right)}{Z(\omega,x)^3},\\
&\leqslant C\left|\frac{e^{4\pi\omega}-1}{4\pi\omega}-1\right|\left(\frac{e^{4\pi\omega}-1}{4\pi\omega}\right)^{-3}\left(\left(\frac{e^{4\pi\omega}-1}{4\pi\omega}\right)^2+\frac{e^{4\pi\omega}-1}{4\pi\omega}+1\right),\\
&= C\left|1-\frac{4\pi\omega}{e^{4\pi\omega}-1}\right|\left(1+\frac{4\pi\omega}{e^{4\pi\omega}-1}+\left(\frac{4\pi\omega}{e^{4\pi\omega}-1}\right)^2\right),\\
&\leqslant C|\omega|\left(1+|\omega|+|\omega|^2\right).
\end{aligned}$$

Integrating over μ those inequalities, we easily see that $\sup_{x\in[0,2K_{\max}]}\left|\partial_x^2\Psi^\mu(x)-\partial_x^2\Psi_0(x)\right|$ is small if $\int_{\mathbf{R}}|\omega|^3\,\mathrm{d}\mu(\omega)$ is small, so that (3.54) is verified for small $\delta>0$. Proposition 3.37 is proved. □

3.2.3 The binary case

In this paragraph, we prove Proposition 3.17. Some estimates in (3.22) are straightforward: the fact that $0<\omega_s(K)$ is direct for $K>1$ and the fact that $\omega_s(K)\leqslant\omega_e(K)$ comes from the continuity of Ψ^{ω_0} and the fact that $\lim_{x\to\infty}\Psi^{\omega_0}(x)=1$.

The main difficulty lies in proving the fact that $\omega_e(K)<\infty$. This is a direct consequence of following convergence:

$$\forall K>1,\ \sup_{r\in[0,1]}\left|\Psi^{\omega_0}(2Kr)\right|\to_{\omega_0\to\infty}0. \tag{3.59}$$

In order to prove (3.59), let us recall a usual result:

Lemma 3.38 (Lagrange Method)**.** *Let f be a function $\mathbf{R}\to\mathbf{R}$, $f\in\mathcal{C}^3(\mathbf{R})$ such as f, f' and f'' are bounded on every compact of $\mathbf{R}$. Then, for all $a<b$, as $\omega\to\infty$*

$$\int_a^b e^{-2\omega x}f(x)\,\mathrm{d}x = e^{-2\omega a}\left(\frac{f(a)}{2\omega}+\frac{f'(a)}{4\omega^2}+\frac{f''(a)}{8\omega^3}\right)+o\left(\frac{e^{-2\omega a}}{\omega^3}\right),$$

where the rest is controlled by the supremum norms of f and of its derivatives.

Using Lemma 3.38, one can estimate the asymptotic behavior of

$$\Psi^{\omega_0}(2Kr) = \frac{\int_{\mathbf{S}}\cos(\theta)S(\theta,\omega_0,2Kr)\,\mathrm{d}\theta}{Z(\omega_0,2Kr)} = \frac{A(\omega_0,2Kr)}{Z(\omega_0,2Kr)},$$

for A and Z already used in (3.55). A (long) calculation based on Lemma 3.38 shows the following:

$$A(\omega_0,2Kr) = e^{4\pi\omega_0}\left(\frac{1}{2\omega_0^2}+o\left(\frac{1}{\omega_0^2}\right)\right),\ \text{as } \omega_0\to\infty \tag{3.60}$$

$$Z(\omega_0,2Kr) = e^{4\pi\omega_0}\left(\frac{\pi}{\omega_0}+o\left(\frac{1}{\omega_0^2}\right)\right),\ \text{as } \omega_0\to\infty \tag{3.61}$$

where the rests are controlled uniformly in $r\in[0,1]$. Consequently, the convergence (3.59) is easy to derive from (3.60), (3.61) and the definition of Ψ^{ω_0}. □

Let us now concentrate on the issue of stability of synchronization.

4 Stability of synchronization in the symmetric case

The purpose of this section is to prove Theorem 3.27 along with a number of explicit estimates. The main idea of the proof of Theorem 3.27 is to consider the evolution with disorder (3.33) as a perturbation of the *labeled evolution* (3.35). In other terms, if the disorder is small, one may see the operator L_q^δ defined in (3.34) as a perturbation of the self-adjoint operator A defined in (3.36) for which we know (Proposition 3.25) that it is self-adjoint in a proper Sobolev space introduced in § 2.3. The basic tools used here come from the theory of perturbation of self-adjoint operators (see [51]) and of perturbation of analytic semigroups (see [70]).

We first begin with a brief outline of the proof of Proposition 3.25 concerning the linear stability in the *labeled* model (3.35). As already said, Proposition 3.25 is just of technical interest and is the exact analogous to Proposition 3.23 for the non-disordered model (3.5) and strongly relies on the proof of the reversible case given in [9]. The interested reader may find more details in [41, § 3.3].

4.1 Linear stability in the labeled evolution (proof of Proposition 3.25)

The key argument is that the *labeled* operator A defined in (3.36) is symmetric [2] for the scalar product $\langle \cdot\,,\,\cdot\rangle_{\mu,-1,q_0}$ (recall its definition in (3.27)). Namely, a short computation shows that for u and v in $\mathcal{D}$ (recall the definition of the domain $\mathcal{D}$ in (3.26)),

$$\langle v\,,\,Au\rangle_{\mu,-1,q_0} = -\frac{1}{2}\int_{\mathbf{S}\times\mathbf{R}} \frac{uv}{q_0}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \int_{\mathbf{S}\times\mathbf{R}^2} v(\tilde{J}*u)\,\mathrm{d}\lambda\,\mathrm{d}(\mu\otimes\mu)\,, \tag{3.62}$$

where $\tilde{J}(\theta) = K\cos(\theta)$. As in [9], the first result concerns an estimate on the Dirichlet form associated to A (in comparison, see [9, Prop. 2.3]):

Proposition 3.39. *For all $K > 1$, there exists $c_K \in (0, 1/2)$ such that for all $u \in \mathbf{L}^2_\mu$ such that for almost every ω, $\int_{\mathbf{S}} u(\cdot,\omega) = 0$*

$$\mathcal{E}_\mu(u) := -\langle Au\,,\,u\rangle_{\mu,-1,q_0} \geqslant c_K\,\langle u - P_{\mu,2}u\,,\,u - P_{\mu,2}u\rangle_{\mu,2,q_0}\,,$$

where $P_{\mu,2}$ is the projection in $\mathbf{L}^2_\mu$ along q_0':

$$P_{\mu,2}u = \frac{\langle u\,,\,q_0'\rangle_{\mu,2,q_0}}{\langle q_0'\,,\,q_0'\rangle_{2,q_0}} q_0'\,.$$

The two following lemmas [3] compare the scalar products $\langle \cdot\,,\,\cdot\rangle_{\mu,2,q_0}$ and $\langle \cdot\,,\,\cdot\rangle_{\mu,-1,q_0}$:

We define the projection in $\mathbf{H}^{-1}_{\mu,q_0}$ along q_0':

$$P_{\mu,-1}u = \frac{\langle u\,,\,q_0'\rangle_{\mu,-1,q_0}}{\langle q_0'\,,\,q_0'\rangle_{-1,q_0}} q_0'\,.$$

Lemma 3.40. *For every $K > 1$ there exists a constant $C = C(K) > 0$ such that for $u \in \mathbf{L}^2_\mu$ such that $\int_{\mathbf{S}} u = 0$, for almost every ω*

$$\langle u - P_{\mu,2}u\,,\,u - P_{\mu,2}u\rangle_{\mu,2,q_0} \geqslant C\,\langle u - P_{\mu,-1}u\,,\,u - P_{\mu,-1}u\rangle_{\mu,-1,q_0}\,.$$

[2]. In the same way that the reversible operator L_{q_0} is symmetric for $\langle \cdot\,,\,\cdot\rangle_{-1,q_0}$, see [9, Eq. (2.14)].

[3]. Their counterparts in the reversible case are Lemmas 2.4 and 2.5 in [9].

Lemma 3.41. *For every $K > 1$ there exists $c = c(K) > 0$ such that for $u \in \mathbf{L}^2_\mu$ such that $\int_{\mathbf{S}} u = 0$ for almost every ω and*

$$\langle u\,,\,u\rangle_{\mu,-1,q_0} \;\geqslant\; c\,\langle P_{\mu,2}u\,,\,P_{\mu,2}u\rangle_{\mu,2,q_0}\;.$$

We are now in position to prove Proposition 3.25:

Sketch of proof of Proposition 3.25. Of course Proposition 3.39 and Lemma 3.40 imply directly the spectral gap for the operator A mentioned in Proposition 3.25:

$$-\langle Au\,,\,u\rangle_{\mu,-1,q_0} \;\geqslant\; c_K C\,\langle u - P_{\mu,-1}u\,,\,u - P_{\mu,-1}u\rangle_{\mu,-1,q_0} \qquad \text{for all } u \in \mathbf{H}^{-1}_{\mu,q_0}\,. \tag{3.63}$$

We now prove the self-adjoint property of A. It is sufficient to prove that the range of $1 - A$ is dense in $\mathbf{H}^{-1}_{\mu,q_0}$ (see [17, p. 113]). For $u, v \in \mathcal{D}$, we have

$$\begin{aligned}\langle v\,,\,(1-A)u\rangle_{\mu,-1,q_0} \;=\; &-\int_{\mathbf{R}}\int_{\mathbf{S}} v(\theta,\omega)\left(\int_0^\theta \frac{\mathcal{U}}{q_0}\right)\mathrm{d}\theta\,\mathrm{d}\mu + \frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{S}} \frac{vu}{q_0}\,\mathrm{d}\theta\,\mathrm{d}\mu \\ &- \int_{\mathbf{R}}\int_{\mathbf{S}^2} v(\theta,\omega)\tilde{J} * u(\theta,\omega')\,\mathrm{d}\theta\,\mathrm{d}(\mu\otimes\mu)\,. \end{aligned}\tag{3.64}$$

The right side of this expression is still defined for $u, v \in \mathbf{L}^2_\mu$ and there exists $c > 0$ such that

$$\langle v\,,\,(1-A)u\rangle_{\mu,-1,q_0} \;\leqslant\; c\,\|\,u\,\|_{\mu,2}\,\|\,v\,\|_{\mu,2}\;,$$

Furthermore from (3.63) and Lemma 3.41 we have

$$\langle u\,,\,(1-A)u\rangle_{\mu,-1,q_0} \;\geqslant\; \frac{1}{c}\,\|\,u\,\|^2_{\mu,2}\;.$$

So the bilinear form $(u, v) \mapsto \langle v\,,\,(1-A)u\rangle_{\mu,-1,q_0}$ is continuous and coercive on $\mathbf{L}^2_\mu \times \mathbf{L}^2_\mu$. If $f \in \mathbf{H}^{-1}_{\mu,q_0}$, the linear form $v \mapsto \langle v\,,\,f\rangle_{\mu,-1,q_0}$ is continuous on $\mathbf{L}^2_\mu$, so that Lax-Milgram Theorem gives the existence of a unique $u \in \mathbf{L}^2_\mu$ such that for all $v \in \mathbf{L}^2_\mu$

$$\langle v\,,\,(1-A)u\rangle_{\mu,-1,q_0} \;=\; \langle v\,,\,f\rangle_{\mu,-1,q_0}\;.$$

Since

$$\langle v\,,\,f\rangle_{\mu,-1,q_0} \;=\; -\int_{\mathbf{R}}\int_{\mathbf{S}} v(\theta,\omega)\left(\int_0^\theta \frac{\mathcal{F}}{q_0}\right)\mathrm{d}\theta\,\mathrm{d}\mu\,,$$

from (3.64) we obtain that for almost θ and ω

$$-\int_0^\theta \frac{\mathcal{U}(\theta',\omega)}{q_0(\theta')}\,\mathrm{d}\theta' + \frac{u(\theta,\omega)}{2q_0(\theta)} - \int_{\mathbf{R}}\left(\tilde{J} * u\right)(\theta,\omega)\,\mathrm{d}\mu \;=\; -\int_0^\theta \frac{\mathcal{F}(\theta',\omega)}{q_0(\theta')}\,\mathrm{d}\theta'\,.$$

So it is clear that if f is continuous with respect to θ, then u has a version C^2 with respect to θ. Thus $u \in \mathcal{D}$ and applying $\partial_\theta(q_0(\theta)\partial_\theta\cdot)$ to the both sides of this last expression, we get $(1-A)u = f$. Since this kind of functions f is dense in $\mathbf{H}^{-1}_{\mu,q_0}$, we can conclude that the range of $1-A$ is dense, and that A is essentially self-adjoint. This completes the proof of Proposition 3.25. □

In the next paragraph, we turn to the study of the linear stability of the system with disorder (3.33):

4.2 Decomposition of the operator L_q^δ

In what follows, $K > 1$ and $r_0 = \Psi_0(2Kr_0) > 0$ are fixed. In order to study the spectral properties of the operator L_q^δ for general distribution of disorder, we decompose L_q^δ in (3.34) into the sum of the self-adjoint operator A defined in (3.36) and a perturbation B which will be considered to be small w.r.t. A, namely:

$$Bu(\theta,\omega) := -\partial_\theta \left(u(\theta,\omega)\langle J * \varepsilon(q)\rangle_\mu + \varepsilon(q)(\theta,\omega,\delta)\langle J * u\rangle_\mu(\theta) + \delta\omega u(\theta,\omega)\right), \tag{3.65}$$

where

$$\varepsilon(q) := (\theta,\omega,\delta) \mapsto q(\theta,\delta\omega) - q_0(\theta), \tag{3.66}$$

is the difference between the stationary solution with disorder and the one without disorder.

Remark 3.42. Note that, since the whole operator L_q^δ is no longer self-adjoint nor symmetric, its spectrum need not be real. In that extent, one has to deal in this section with the complexified versions of the scalar products defined in § 2.3. Thus, we will assume for the rest of this section that we work with complex versions of these scalar products. The properties of the operator A (Proposition 3.25) are obviously still valid, since A is symmetric and real.

We will also use the following standard notations: for an operator F, we will denote by $\rho(F)$ the set of all complex numbers λ for which $\lambda - F$ is invertible, and by $R(\lambda, F) := (\lambda - F)^{-1}$, $\lambda \in \rho(F)$ the resolvent of F. The spectrum of F will be denoted as $\sigma(F)$.

Before beginning our perturbation procedure, the first proposition concerns the analytic semigroup generated by the self-adjoint operator A.

Proposition 3.43. *The (extension of the) operator A is the infinitesimal generator of a strongly continuous semigroup of contractions $T_A(t)$ on $\mathbf{H}^{-1}_{\mu,q_0}$.*

Moreover, for every $0 < \alpha < \frac{\pi}{2}$ this semigroup can be extended to an analytic semigroup $T_A(z)$ defined on $\Delta_\alpha := \{z \in \mathbb{C};\, |\arg(z)| < \alpha\}$.

We recall here the result we use concerning analytic extensions of strongly continuous semigroups. Its proof can be found in [70, Th 5.2, p. 61].

Proposition 3.44. *Let $T(t)$ a uniformly bounded strongly continuous semigroup, whose infinitesimal generator F is such that $0 \in \rho(F)$ and let $\alpha \in (0, \frac{\pi}{2})$. The following statements are equivalent:*

1. *$T(t)$ can be extended to an analytic semigroup in the sector*

$$\Delta_\alpha := \{\lambda \in \mathbf{C};\, |\arg(\lambda)| < \alpha\},$$

and $\|T(z)\|$ is uniformly bounded in every closed sub-sector $\bar{\Delta}'_\alpha$, $\alpha' < \alpha$, of Δ_α,

2. *There exists $M > 0$ such that*

$$\rho(F) \supset \Sigma = \left\{\lambda \in \mathbf{C};\, |\arg(\lambda)| < \frac{\pi}{2} + \alpha\right\} \cup \{0\}, \tag{3.67}$$

and

$$\|R(\lambda, F)\| \leqslant \frac{M}{|\lambda|}, \quad \lambda \in \Sigma, \lambda \neq 0. \tag{3.68}$$

Proof of Proposition 3.43. The proof in § 4.1 of Proposition 3.25 concerning the self-adjointness of A in $\mathbf{H}^{-1}_{\mu,q_0}$ shows that A satisfies the hypothesis of Lumer-Phillips Theorem (see [70, Th 4.3, p. 14]): A is the infinitesimal generator of a C_0 semigroup of contractions denoted by $T_A(t)$.

The rest of the proof is devoted to show the existence of an analytic extension of this semigroup in a proper sector. We follow here the lines of the proof of Th 5.2, p. 61-62, in [70], but with explicit estimates on the resolvent, in order to quantify properly the appropriate size of the perturbation.

Let us first replace the operator A by a small perturbation: for all $\varepsilon > 0$, let $A_\varepsilon := A - \varepsilon$, so that 0 belongs to $\rho(A_\varepsilon)$. The operator A_ε has the following properties: as A, it is self-adjoint and generates a strongly continuous semigroup of operators (which is $T_{A,\varepsilon}(t) = T_A(t)e^{-\varepsilon t}$).

Since A is self-adjoint, it is easy to see that

$$\forall \lambda \in \mathbf{C} \setminus \mathbf{R}, \| R(\lambda, A_\varepsilon) \|_{\mu,-1,q_0} \leqslant \frac{1}{|\Im(\lambda)|}, \tag{3.69}$$

and since the spectrum of A is negative, for every $\lambda \in \mathbf{C}$ such that $\Re(\lambda) > 0$,

$$\| R(\lambda, A_\varepsilon) \|_{\mu,-1,q_0} \leqslant \frac{1}{|\lambda|}. \tag{3.70}$$

For any $\alpha \in (0, \frac{\pi}{2})$, let

$$\Sigma_\alpha := \left\{ \lambda \in \mathbb{C};\ |\arg(\lambda)| < \frac{\pi}{2} + \alpha \right\}. \tag{3.71}$$

Let us prove that for $\lambda \in \Sigma_\alpha$,

$$\| R(\lambda, A_\varepsilon) \|_{\mu,-1,q_0} \leqslant \frac{1}{1-\sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{3.72}$$

Note that (3.72) is clear from (3.69) and (3.70) when λ is such that $\Re(\lambda) \geqslant 0$.

Let us consider $\sigma > 0, \tau \in \mathbf{R}$ to be chosen appropriately later.

Let us write the following Taylor expansion for $R(\lambda, A_\varepsilon)$ around $\sigma + i\tau$ (at least well defined in a neighborhood of $\sigma + i\tau$ since $\sigma > 0$):

$$R(\lambda, A_\varepsilon) = \sum_{n=0}^{\infty} R(\sigma + i\tau, A_\varepsilon)^{n+1}((\sigma + i\tau) - \lambda)^n. \tag{3.73}$$

From now, we fix $\lambda \in \Sigma_\alpha$ with $\Re(\lambda) < 0$. This series $R(\lambda, A_\varepsilon)$ is well defined in λ if one can choose σ, τ and $k \in (0,1)$ such that $\| R(\sigma + i\tau, A_\varepsilon) \|_{\mu,-1,q_0} |\lambda - (\sigma + i\tau)| \leqslant k < 1$. In particular, using (3.69), it suffices to have $|\lambda - (\sigma + i\tau)| \leqslant k|\tau|$ and since $\sigma > 0$ is arbitrary, it suffices to find $k \in (0,1)$ and τ with $|\lambda - i\tau| \leqslant k|\tau|$ to obtain the convergence of (3.73). For this $\lambda \in \Sigma_\alpha$ with $\Re(\lambda) < 0$, let us define λ' and τ as in Figure 3.7. Then, $|\lambda - i\tau| \leqslant |\lambda' - i\tau| = \sin(\alpha)|\tau|$ with $\sin(\alpha) \in (0,1)$. So the series converges for $\lambda \in \Sigma_\alpha$ and one has, using again (3.69),

$$\| R(\lambda, A_\varepsilon) \|_{\mu,-1,q_0} \leqslant \frac{1}{(1-\sin(\alpha))|\tau|} \leqslant \frac{1}{1-\sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{3.74}$$

The fact that $T_{A,\varepsilon}(t)$ can be extended to an analytic semigroup $T_{A,\varepsilon}(z)$ on the domain Δ_α is a simple application of (3.74) and Proposition 3.44, with $M := \frac{1}{1-\sin(\alpha)}$.

Let us then define $\tilde{T_A}(z) := e^{\varepsilon z} T_{A,\varepsilon}(z)$, for $z \in \Delta_\alpha$ so that $\tilde{T_A}$ is an analytic extension of T_A (an argument of analyticity shows that $\tilde{T_A}$ does not depend on ε). □

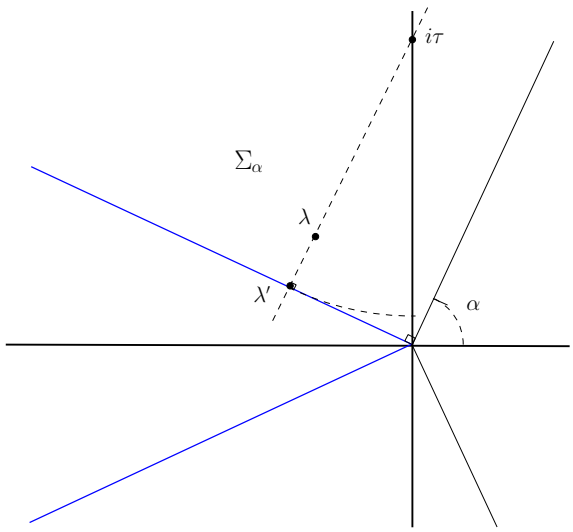


Figure 3.7: The set Σ_α.

Remark 3.45. Note that estimate (3.72) is also valid in the limit as $\varepsilon \to 0$: for all $\alpha \in (0, \frac{\pi}{2})$, $\lambda \in \Sigma_\alpha$,

$$\| R(\lambda, A) \|_{\mu,-1,q_0} \leqslant \frac{1}{1-\sin(\alpha)} \cdot \frac{1}{|\lambda|} . \tag{3.75}$$

4.3 Spectral properties of $L_q^\delta = A + B$

In this part, we show that if the perturbation B is small enough with respect to A, one has the same spectral properties for $L_q^\delta = A + B$ as for A. In this extent, we recall that μ is of compact support in $[-1, 1]$, and the disorder is rescaled by $\delta > 0$.

Proposition 3.46.

The operator B is A-bounded, in the sense that there exist explicit constants $a_{K,\delta}$ and $b_{K,\delta}$, depending on K and δ such that for all u in the domain of (the closure of) A

$$\| Bu \|_{\mu,-1,q_0} \leqslant a_{K,\delta} \| u \|_{\mu,-1,q_0} + b_{K,\delta} \| Au \|_{\mu,-1,q_0} . \tag{3.76}$$

Moreover, for fixed $K > 1$, $a_{K,\delta} = O(\delta)$ and $b_{K,\delta} = O(\delta)$, as $\delta \to 0$.

The latter proposition is based on the fact that the difference $\varepsilon(q)(\theta, \omega, \delta) = q(\theta, \delta\omega) - q_0(\theta)$ in (3.66) is small if the scale parameter δ tend to 0:

Lemma 3.47. *For $\delta > 0$, let us define*

$$\| \varepsilon(q) \|_\infty := \sup_{\substack{\theta \in \mathbf{S}, |\omega| \leqslant 1 \\ 0 < u < \delta}} |\varepsilon(q)(\theta, \omega, u)| .$$

Then for all $K > 1$, $\| \varepsilon(q) \|_\infty = O(\delta)$, as $\delta \to 0$. More precisely, for $K > 1$, $\delta > 0$, the following inequality holds:

$$\| \varepsilon(q) \|_\infty \leqslant \varepsilon_{K,\delta} ,$$

where the constant $\varepsilon_{K,\delta}$ can be chosen explicitly in terms of K and δ:

$$\varepsilon_{K,\delta} := \frac{\delta}{\pi} e^{8\pi\delta} \left(2 + 3e^{4\pi\delta}\right) e^{14K\bar{r}_\delta} \left(1 + 2\pi e^{2K\bar{r}_\delta}\right) , \tag{3.77}$$

where we recall that $\bar{r}_\delta = \max(r_0, r_\delta)$.

Proof of Lemma 3.47. Recall that the disordered stationary solution q (3.15) is given by

$$q(\theta,\delta\omega) \;:=\; \frac{S(\theta,\delta\omega,2Kr_\delta)}{Z(\delta\omega,2Kr_\delta)},$$

where $S(\theta,\omega,x)$ is defined in (3.11) and that the non-disordered one (3.7) is given by $q_0(\theta) = \frac{S(\theta,0,2Kr_0)}{Z(0,2Kr_0)} = \frac{e^{2Kr_0\cos(\theta)}}{\int_{\mathbf{S}} e^{2Kr_0\cos(\theta)}\,\mathrm{d}\theta}$. Since $q(\theta,\delta\omega) = q(-\theta,-\delta\omega)$, it suffices to consider the case $\delta\omega > 0$. A simple computation shows that

$$Z(\delta\omega,2Kr_\delta) \;\geqslant\; 4\pi^2 e^{-4Kr_\delta} e^{-4\pi\delta}\,, \tag{3.78}$$

and that

$$|S(\theta,0)| \;\leqslant\; 2\pi e^{4Kr_0}\,. \tag{3.79}$$

Using $|q(\theta,\delta\omega)-q_0(\theta)| \leqslant \frac{1}{Z(\delta\omega)Z(0)}\left(Z(0)|S(\theta,\delta\omega)-S(\theta,0)| + |S(\theta,0)||Z(0)-Z(\delta\omega)|\right)$, one has to deal with, successively:

– for fixed $\theta\in\mathbf{S}$, $|S(\theta,\delta\omega)-S(\theta,0)| \leqslant \delta\cdot\sup_{|\omega|\leqslant 1}|\frac{\mathrm{d}}{\mathrm{d}\omega}S(\theta,\delta\omega)|$. A long calculation shows that the latter expression $|\frac{\mathrm{d}}{\mathrm{d}\omega}S(\theta,\delta\omega)|$ can be bounded above by the constant $8\pi^2 e^{4Kr_\delta}e^{4\pi\delta}\left(2+3e^{4\pi\delta}\right)$, that is,

$$|S(\theta,\delta\omega)-S(\theta,0)| \;\leqslant\; \delta 8\pi^2 e^{4Kr_\delta}e^{4\pi\delta}\left(2+3e^{4\pi\delta}\right)\,. \tag{3.80}$$

– Using $|Z(\delta\omega)-Z(0)| = |\int_{\mathbf{S}}(S(\theta,\delta\omega)-S(\theta,0))\,\mathrm{d}\theta|$ and (3.80), one has directly:

$$|Z(\delta\omega)-Z(0)| \;\leqslant\; \delta 16\pi^3 e^{4Kr_\delta}e^{4\pi\delta}\left(2+3e^{4\pi\delta}\right)\,. \tag{3.81}$$

Putting together (3.78), (3.79), (3.80) and (3.81), one obtains the result. □

We are now in position to prove the A-boundedness of B:

Proof of Proposition 3.46. B is A-bounded: let us fix a u in the domain of the closure of A. Then we have $\|\,Bu\,\|_{\mu,-1,q_0} = \|\,\mathcal{B}u\,\|_{2,q_0,\mu}$, where $\mathcal{B}u$ is the appropriate primitive of Bu, namely:

$$\begin{aligned}\mathcal{B}u(\theta,\omega) \;:=\; & -\left(u(\theta,\omega)\langle J*\varepsilon(q)\rangle_\mu + \varepsilon(q)(\theta,\omega,\delta)\langle J*u\rangle_\mu(\theta) + \delta\omega u(\theta,\omega)\right)\\ & + \left(\int_{\mathbf{S}}\frac{1}{q_0}\right)^{-1}\left(\int_{\mathbf{S}}\frac{u(\theta,\omega)\langle J*\varepsilon(q)\rangle_\mu + \varepsilon(q)(\theta,\omega,\delta)\langle J*u\rangle_\mu(\theta) + \delta\omega u(\theta,\omega)}{q_0(\theta)}\,\mathrm{d}\theta\right).\end{aligned}$$

One can easily shows that there exists a constant $c^{(1)}_{K,\delta}$, depending only on $K>1$ and $\delta>0$ such that:

$$\|\,Bu\,\|_{\mu,-1,q_0} \;\leqslant\; c^{(1)}_{K,\delta}\,\|\,u\,\|_{2,q_0,\mu}\,. \tag{3.82}$$

Indeed, an easy calculation shows that $|\langle J*\varepsilon(q)\rangle_\mu| \leqslant 4K\,\|\,\varepsilon(q)\,\|_\infty$ and that

$$\begin{aligned}|\langle J*u\rangle_\mu(\cdot)| \;&\leqslant\; K\left(\int_{\mathbf{S}}\sin(\cdot-\varphi)^2 q_0(\varphi)\,\mathrm{d}\varphi\right)^{\frac12}\|\,u\,\|_{2,q_0,\mu}\\ &\leqslant K\left(\int_{\mathbf{S}} q_0(\varphi)\,\mathrm{d}\varphi\right)^{\frac12}\|\,u\,\|_{2,q_0,\mu} = K\,\|\,u\,\|_{2,q_0,\mu}\,.\end{aligned}$$

So we have for all θ, ω (recall that Z_0 is the normalization constant in (3.7)):

$$\begin{aligned} |\mathcal{B}u(\theta,\omega)| \leqslant & \ (4K \| \varepsilon(q) \|_\infty + \delta|\omega|)\, |u| + 2K \| \varepsilon(q) \|_\infty \| u \|_{2,q_0,\mu} \\ & + Z_0^{-1} (4K \| \varepsilon(q) \|_\infty + \delta|\omega|) \left(\int_{\mathbf{S}} \frac{|u|^2}{q_0} \right)^{\frac{1}{2}} . \end{aligned}$$

Hence, inequality (3.82) is true for the following choice of $c^{(1)}_{K,\delta}$ (recall that $\varepsilon_{K,\delta}$ is defined in (3.77)):

$$c^{(1)}_{K,\delta} := \left(6\,(4K\varepsilon_{K,\delta} + \delta)^2 + 12K^2 Z_0^2 \varepsilon^2_{K,\delta} \right)^{\frac{1}{2}} . \tag{3.83}$$

Remark 3.48. Note that, thanks to Lemma 3.47, one has that $c^{(1)}_{K,\delta} = O(\delta)$ as $\delta \to 0$.

In order to complete the proof of the inequality (3.76), it suffices to prove that there exist constants $c^{(2)}_K$ and $c^{(3)}_K$, only depending on K such that, for all u:

$$\| u \|_{2,q_0,\mu} \leqslant c^{(2)}_K \| Au \|_{\mu,-1,q_0} + c^{(3)}_K \| u \|_{\mu,-1,q_0} . \tag{3.84}$$

The rest of this first of the proof is devoted to find explicit expressions of $c^{(2)}_K$ and $c^{(3)}_K$, and is based on a usual interpolation argument.

For all integer $n > 1$, one can compute the linear operator $f \mapsto f'$ in terms of a sum of two integral operators, namely:

$$f' = I_n(f'') + J_n(f) , \tag{3.85}$$

where $I_n : f \mapsto \int_0^{2\pi} i_n(\theta,\varphi) f(\varphi)\, \mathrm{d}\varphi$ (resp. $J_n : f \mapsto \int_0^{2\pi} j_n(\theta,\varphi) f(\varphi)\, \mathrm{d}\varphi$) is the integral operator whose kernel $i_n(\theta,\varphi)$ (resp. $j_n(\theta,\varphi)$) is defined by:

$$\begin{cases} i_n(\theta,\varphi) := \frac{\varphi^{n+1}}{2\pi\theta^n} , & j_n(\theta,\varphi) := -\frac{n(n+1)\varphi^{n-1}}{2\pi\theta^n} , & 0 \leqslant \varphi < \theta \leqslant 2\pi , \\ i_n(\theta,\varphi) := \frac{-(2\pi-\varphi)^{n+1}}{2\pi(2\pi-\theta)^n} , & j_n(\theta,\varphi) := \frac{n(n+1)(2\pi-\varphi)^{n-1}}{2\pi(2\pi-\theta)^n} , & 0 \leqslant \theta < \varphi \leqslant 2\pi . \end{cases} \tag{3.86}$$

Equality (3.85) can be easily verified by integrations by parts. Since,

$$\begin{cases} \int_0^{2\pi} |i_n(\theta,\varphi)|\, \mathrm{d}\varphi \leqslant \frac{2\pi}{n+2}, & \int_0^{2\pi} |i_n(\theta,\varphi)|\, \mathrm{d}\theta \leqslant \frac{2\pi}{n-1} , \\ \int_0^{2\pi} |j_n(\theta,\varphi)|\, \mathrm{d}\varphi \leqslant \frac{n+1}{\pi} , & \int_0^{2\pi} |j_n(\theta,\varphi)|\, \mathrm{d}\theta \leqslant \frac{n(n+1)}{\pi(n-1)} , \end{cases} \tag{3.87}$$

we see (cf. [51, p. 143-144]) that I_n and J_n are bounded operators on $\mathbf{L}^2(\mathbf{S})$, namely:

$$\|I_n\| \leqslant \frac{2\pi}{n-1}, \quad \|J_n\| \leqslant \frac{n(n+1)}{\pi(n-1)} .$$

So, applying relation (3.85) for $f = \mathcal{U}$ we get, for μ-almost every ω:

$$\left(\int_{\mathbf{S}} |u(\theta,\omega)|^2\, \mathrm{d}\theta \right)^{\frac{1}{2}} \leqslant \frac{2\pi}{n-1} \left(\int_{\mathbf{S}} |u'(\theta,\omega)|^2\, \mathrm{d}\theta \right)^{\frac{1}{2}} + \frac{n(n+1)}{\pi(n-1)} \left(\int_{\mathbf{S}} |\mathcal{U}(\theta,\omega)|^2\, \mathrm{d}\theta \right)^{\frac{1}{2}} .$$

This gives

$$\| u \|_{2,\mu} \leqslant \frac{2\pi}{n-1} \| u' \|_{2,\mu} + \frac{n(n+1)}{\pi(n-1)} \| \mathcal{U} \|_{2,\mu} . \tag{3.88}$$

Since $\| \mathcal{U} \|_{2,q_0,\mu} = \| u \|_{\mu,-1,q_0}$, it only remains to control $\| u' \|_{2,q_0,\mu}$ with $\| Au \|_{\mu,-1,q_0}$: like for the beginning of this proof for the operator B, we have $\| Au \|_{\mu,-1,q_0} = \| \mathcal{A}u \|_{2,q_0,\mu}$, where $\mathcal{A}u$ is the appropriate primitive of Au:

$$\begin{aligned}\mathcal{A}u(\theta,\omega) &:= \frac{1}{2}u'(\theta,\omega) - (u(\theta,\omega)(J*q_0) + q_0(\theta)\langle J*u\rangle_\mu(\theta)) \\ &\quad + \left(\int_{\mathbf{S}} \frac{1}{q_0}\right)^{-1} \left(\int_{\mathbf{S}} \left\{ \frac{u(\theta,\omega)(J*q_0)}{q_0(\theta)} + \frac{1}{2}u(\theta,\omega)\partial_\theta\left(\frac{1}{q_0(\theta)}\right)\right\} \mathrm{d}\theta\right).\end{aligned}$$

Using inequalities $|\langle J*u\rangle|_\mu(\cdot) \leqslant K\sqrt{\pi}\,\| u \|_{2,\mu}$, and $\int_{\mathbf{S}} \frac{|u(\cdot,\omega)|}{q_0} \leqslant Z_0^{\frac{1}{2}} e^{Kr_0} \left(\int_{\mathbf{S}} |u(\cdot,\omega)^2|\right)^{\frac{1}{2}}$, an easy calculation shows that:

$$|u'(\cdot,\omega)| \leqslant 2|\mathcal{A}u(\cdot,\omega)| + 2Kr_0|u(\cdot,\omega)| + 2\sqrt{\pi}Kq_0(\cdot)\,\| u \|_{2,\mu} + \frac{4Kr_0}{Z_0^{\frac{1}{2}}} e^{Kr_0}\left(\int_{\mathbf{S}} |u(\cdot,\omega)^2|\right)^{\frac{1}{2}},$$

and thus,

$$\| u' \|_{2,\mu} \leqslant 4\,\| \mathcal{A}u \|_{2,\mu} + 4K\left(r_0^2 + \pi Z_0^{-1}e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} \| u \|_{2,\mu}\,, \tag{3.89}$$

and by putting (3.88) and (3.89) together we obtain

$$\begin{aligned}\| u \|_{2,\mu} &\leqslant \frac{8\pi}{n-1}\,\| \mathcal{A}u \|_{2,\mu} + \frac{2\pi}{n-1}4K\left(r_0^2 + \pi Z_0^{-1}e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} \| u \|_{2,\mu} \\ &\quad + \frac{n(n+1)}{\pi(n-1)}\,\| u \|_{\mu,-1,q_0}\,.\end{aligned}$$

Let us choose the integer $n = \left\lfloor 16\pi K\left(r_0^2 + \pi Z_0^{-1}e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} + 1\right\rfloor$ so that

$$\frac{2\pi}{n-1}4K\left(r_0^2 + \pi Z_0^{-1}e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} \leqslant \frac{1}{2}\,.$$

In this case, we obtain:

$$\begin{aligned}\| u \|_{2,q_0,\mu} &\leqslant \frac{e^{2Kr_0}}{4K\left(r_0^2 + \pi Z_0^{-1}e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}}}\,\| Au \|_{\mu,-1,q_0} \\ &\quad + \frac{e^{2Kr_0}\left(16K\left(r_0^2 + \pi Z_0^{-1}e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}} + 3\right)^2}{16\pi^2 K\left(r_0^2 + \pi Z_0^{-1}e^{2Kr_0}(1+8r_0^2)\right)^{\frac{1}{2}}}\,\| u \|_{\mu,-1,q_0}\,,\end{aligned} \tag{3.90}$$

which is precisely the inequality (3.84) we wanted to prove. Inequalities (3.82) and (3.84) give the result, for $a_{K,\delta} := c^{(1)}_{K,\delta} \cdot c^{(3)}_K$ and $b_{K,\delta} := c^{(1)}_{K,\delta} \cdot c^{(2)}_K$. □

Proposition 3.49. *For all $K > 1$, there exists $\delta_3(K) > 0$ such that for all $0 < \delta \leqslant \delta_3(K)$, the operator L^δ_q is closable. In that case, its closure has the same domain as the closure of A.*

Proof. Let us choose $\delta_3(K) > 0$ so that

$$b_{K,\delta_3(K)} < 1 \tag{3.91}$$

where $b_{K,\delta}$ is the constant introduced in (3.76), then, for all $0 < \delta \leqslant \delta_3(K)$, the operator B is A-bounded with A-bound strictly lower than 1. The result is then a consequence of Th. IV-1.1, p. 190 in [51]. □

4.4 The spectrum of L_q^δ

We divide our study into two parts: the determination of the position of the spectrum within a sector and its position near 0.

4.4.1 Position of the spectrum away from 0

We prove mainly that the perturbed operator L_q^δ still generates an analytic semigroup of operators on an appropriate sector. An immediate corollary is the fact that the spectrum lies in a cone whose vertex is zero.

We know (Proposition 3.43) that for all $0 < \alpha < \frac{\pi}{2}$, A generates an analytic semigroup of operators on $\Delta_\alpha := \{\lambda \in \mathbf{C};\, |\arg(\lambda)| < \alpha\}$.

Proposition 3.50. *For all $K > 1$, $0 < \alpha < \frac{\pi}{2}$ and $\varepsilon > 0$, there exists $\delta_4 > 0$ (depending on α, K and ε) such that for all $0 < \delta < \delta_4$, the spectrum of $L_q^\delta = A + B$ lies within $\Theta_{\varepsilon,\alpha} := \left\{\lambda \in \mathbf{C};\, \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha\right\} \cup \{\lambda \in \mathbf{C};\, |\lambda| \leqslant \varepsilon\}$. Moreover, there exists $\alpha' \in (0, \frac{\pi}{2})$ such that the operator L_q^δ still generates an analytic semigroup on $\Delta_{\alpha'}$.*

Proof of Proposition 3.50. Let $0 < \alpha < \frac{\pi}{2}$ be fixed. Following (3.76) and using (3.75), one can easily deduce an estimate on the bounded operator $BR(\lambda, A)$, for $\lambda \in \Sigma_\alpha$:

$$\begin{aligned}
\| BR(\lambda, A)u \|_{\mu,-1,q_0} &\leqslant a_{K,\delta} \| R(\lambda, A)u \|_{\mu,-1,q_0} + b_{K,\delta} \| AR(\lambda, A)u \|_{\mu,-1,q_0} \\
&\leqslant a_{K,\delta} \frac{1}{(1-\sin(\alpha))|\lambda|} \| u \|_{\mu,-1,q_0} \\
&\quad + b_{K,\delta} \left(1 + \frac{1}{1-\sin(\alpha)}\right) \| u \|_{\mu,-1,q_0} .
\end{aligned}$$

Let us fix $\varepsilon > 0$ and choose δ so that:

$$\max\left(4b_{K,\delta}\left(\frac{1}{1-\sin(\alpha)} + 1\right), \frac{4a_{K,\delta}}{(1-\sin(\alpha))\varepsilon}\right) \leqslant 1 . \tag{3.92}$$

Then, for $\lambda \in \Sigma_\alpha$ such that $|\lambda| > \varepsilon \geqslant \frac{4a_{K,\delta}}{1-\sin(\alpha)}$, we have

$$\| BR(\lambda, A)u \|_{\mu,-1,q_0} \leqslant \frac{1}{2} \| u \|_{\mu,-1,q_0} .$$

In particular, $1 - BR(\lambda, A)$ is invertible with $\left\| (1 - BR(\lambda, A))^{-1} \right\|_{\mu,-1,q_0} \leqslant 2$. A direct calculation shows that

$$(\lambda - (A + B))^{-1} = R(\lambda, A)\,(1 - BR(\lambda, A))^{-1} .$$

One deduces the following estimates on the resolvent: for $\lambda \in \Sigma_\alpha$, $|\lambda| > \varepsilon$,

$$\left\| R(\lambda, L_q^\delta) \right\|_{\mu,-1,q_0} \leqslant \frac{2}{(1-\sin(\alpha))|\lambda|} . \tag{3.93}$$

Estimate (3.93) has two consequences: firstly, one deduces immediately that the spectrum $\sigma(L_q^\delta)$ of L_q^δ is contained in $\Theta_{\varepsilon,\alpha}$:

$$\sigma(L_q^\delta) \subseteq \left\{\lambda \in \mathbf{C};\, \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha\right\} \cup \{\lambda \in \mathbf{C};\, |\lambda| \leqslant \varepsilon\} . \tag{3.94}$$

Secondly, (3.93) entails that L_q^δ generates an analytic semigroup of operators on an appropriate sector. Indeed, if one denotes by $L_{q,\varepsilon}^\delta := L_q^\delta - \varepsilon$, one deduces from (3.94) that $0 \in \rho(L_{q,2\varepsilon}^\delta)$ and that for all $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ (in particular, $|\lambda| < |\lambda + 2\varepsilon|$)

$$\begin{aligned} \left\| R(\lambda, L_{q,2\varepsilon}^\delta) \right\|_{\mu,-1,q_0} &= \left\| R(\lambda + 2\varepsilon, L_q^\delta) \right\|_{\mu,-1,q_0} \leqslant \frac{2}{(1-\sin(\alpha))|\lambda + 2\varepsilon|}, \\ &\leqslant \frac{2}{(1-\sin(\alpha))|\lambda|}. \end{aligned} \tag{3.95}$$

Hence, using the same arguments of Taylor expansion as in the proof of Proposition 3.43 and applying Proposition 3.44, one easily sees that $L_{q,2\varepsilon}^\delta$ generates an analytic semigroup in a (a priori) smaller sector $\Delta_{\alpha'}$, where $\alpha' \in (0, \frac{\pi}{2})$ can be chosen as $\alpha' := \frac{1}{2}\arctan\left(\frac{1-\sin(\alpha)}{2}\right)$. But if $L_{q,2\varepsilon}^\delta$ generates an analytic semigroup, so does L_q^δ. □

4.4.2 Position of the spectrum near 0

Let us apply Proposition 3.50 for fixed $K > 1$, $\alpha \in (0, \frac{\pi}{2})$, $\rho \in (0,1)$ and $\varepsilon := \rho\lambda_K$, where we recall that λ_K is the spectral gap between the eigenvalue 0 for the non-perturbed operator A and the rest of the spectrum $\sigma(A) \setminus \{0\}$. Let $\Theta_{\varepsilon,\alpha}^+ := \{\lambda \in \Theta_{\varepsilon,\alpha};\ \Re(\lambda) \geqslant 0\}$ be the subset of $\Theta_{\varepsilon,\alpha}$ which lies in the positive part of the complex plane (see Fig. 3.8). In order to show the linear stability, one has to make sure that one can choose a perturbation B small enough so that no eigenvalue of $A + B$ remains in the small set $\Theta_{\varepsilon,\alpha}^+$.

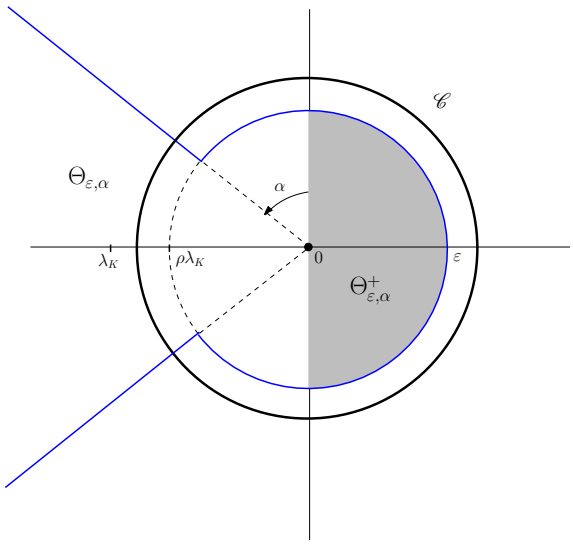


Figure 3.8: The set $\Theta_{\varepsilon,\alpha}$.

Since $\lambda_K > 0$, one can separate 0 from the rest of the spectrum of A by a circle $\mathscr{C}$ centered in 0 with radius $(\frac{\rho+1}{2})\lambda_K$. The appropriate choice of ε ensures that the interior of the disk delimited by $\mathscr{C}$ contains $\Theta_{\varepsilon,\alpha}^+$ (see Figure 3.8).

The main argument is the following: by construction of $\mathscr{C}$, 0 is the only eigenvalue (with multiplicity 1) of the non-perturbed operator A lying in the interior of $\mathscr{C}$. A principle of local continuity of eigenvalues shows that, while adding a sufficiently small perturbation B to A, the interior of $\mathscr{C}$ still contains exactly one eigenvalue (which is *a priori* close but not equal to 0) with the same multiplicity.

But we already know that for the perturbed operator $L_q^\delta = A + B$, 0 is always an eigenvalue (since $L_q^\delta q' = 0$). One can therefore conclude that, by uniqueness, 0 is the only element of the spectrum of L_q^δ within the interior of $\mathscr{C}$, and is an eigenvalue with multiplicity 1. In particular, there is no element of the spectrum in the positive part of the complex plane.

In order to quantify the appropriate size of the perturbation B, one has to have explicit estimates on the resolvent $R(\lambda, A)$ on the circle $\mathscr{C}$.

Lemma 3.51. *There exists some explicit constant $c_{\mathscr{C}} = c_{\mathscr{C}}(K, \rho)$ such that for all $\lambda \in \mathscr{C}$,*

$$\| R(\lambda, A) \|_{\mu,-1,q_0} \leqslant c_{\mathscr{C}} \,, \tag{3.96}$$

$$\| AR(\lambda, A) \|_{\mu,-1,q_0} \leqslant 1 + \left(\frac{1+\rho}{2}\right) \lambda_K \cdot c_{\mathscr{C}} \,. \tag{3.97}$$

One can choose $c_{\mathscr{C}}$ as $\frac{1}{\lambda_K} \max\left(\frac{2}{\rho+1}, \frac{2}{1-\rho}\right) := \frac{\ell(\rho)}{\lambda_K}$.

Proof of Lemma 3.51. Applying the spectral theorem (see [33, Th. 3, p. 1192]) to the essentially self-adjoint operator A, there exists a spectral measure E vanishing on the complementary of the spectrum of A such that $A = \int_{\mathbf{R}} \lambda \, \mathrm{d}E(\lambda)$. In that extent, one has for any $\zeta \in \mathscr{C}$

$$R(\zeta, A) = \int_{\mathbf{R}} \frac{\mathrm{d}E(\lambda)}{\lambda - \zeta} \,.$$

In particular, for $\zeta \in \mathscr{C}$

$$\| R(\zeta, A) \|_{\mu,-1,q_0} \leqslant \sup_{\lambda \in \sigma(A)} \frac{1}{|\lambda - \zeta|} \leqslant \frac{\ell(\rho)}{\lambda_K} \,.$$

The estimation (3.97) is straightforward. □

We are now in position to apply our argument of local continuity of eigenvalues: Following [51, Th III-6.17, p. 178], there exists a decomposition of the operator A according to $\mathbf{H}_{\mu,q_0}^{-1} = \mathbf{H}_0 \oplus \mathbf{H}'$ (in the sense that $A\mathbf{H}_0 \subset \mathbf{H}_0$, $A\mathbf{H}' \subset \mathbf{H}'$ and $P\mathcal{D}(A) \subset \mathcal{D}(A)$, where P is the projection on $\mathbf{H}_0$ along $\mathbf{H}'$) in such a way that A restricted to $\mathbf{H}_0$ has spectrum $\{0\}$ and A restricted to $\mathbf{H}'$ has spectrum $\sigma(A) \setminus \{0\}$.

Let us note that the dimension of $\mathbf{H}_0$ is 1, since the characteristic space of A in the eigenvalue 0 is reduced to its kernel which is of dimension 1.

Then, applying [51, Th. IV-3.18, p. 214], and using Proposition 3.46, we find that if one chooses $\delta > 0$, such that

$$\sup_{\lambda \in \mathscr{C}} \left(a_{K,\delta} \| R(\lambda, A) \|_{\mu,-1,q_0} + b_{K,\delta} \| AR(\lambda, A) \|_{\mu,-1,q_0} \right) < 1, \tag{3.98}$$

then the perturbed operator L_q^δ is likewise decomposed according to $\mathbf{H}_{\mu,q_0}^{-1} = \tilde{\mathbf{H}}_0 \oplus \tilde{\mathbf{H}}'$, in such a way that $\dim(\mathbf{H}_0) = \dim(\tilde{\mathbf{H}}_0) = 1$, and that the spectrum of L_q^δ is again separated in two parts by $\mathscr{C}$. But we already know that the characteristic space of the perturbed operator L_q^δ according to the eigenvalue 0 is, at least, of dimension 1 (since $L_q^\delta q' = 0$). We can conclude, that for such an $\delta > 0$, 0 is the only eigenvalue in $\mathscr{C}$ and that $\dim(\tilde{\mathbf{H}}_0) = 1$.

Applying Lemma 3.51, we see that condition (3.98) is satisfied if we choose $\delta > 0$ so that:

$$a_{K,\delta} c_{\mathscr{C}} + b_{K,\delta} \left(1 + \left(\frac{1+\rho}{2}\right) \lambda_K c_{\mathscr{C}} \right) < 1. \tag{3.99}$$

In particular, in that case, the spectrum of L_q^δ is contained in

$$\left\{\lambda \in \mathbb{C};\ \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha\right\} \subseteq \{z \in \mathbf{C};\ \Re(z) \leqslant 0\}\ .$$

Finally, the following proposition sums-up the sufficient conditions on δ for the conclusions of Theorem 3.27 to be satisfied:

Proposition 3.52. *Recall the definitions of $a_{K,\delta}$ and $b_{K,\delta}$ in Proposition 3.46. If $\delta > 0$ satisfies the following conditions*

$$\begin{aligned} b_{K,\delta} &\leqslant 1\,, \\ 4b_{K,\delta}\left(\frac{1}{1-\sin(\alpha)} + 1\right) &\leqslant 1\,, \\ \frac{4a_{K,\delta}}{\rho\lambda_K\,(1-\sin(\alpha))} &\leqslant 1\,, \\ a_{K,\delta}\frac{\ell(\rho)}{\lambda_K} + b_{K,\delta}\left(1 + \left(\frac{1+\rho}{2}\right)\ell(\rho)\right) &< 1\,. \end{aligned} \tag{3.100}$$

the conclusions of Theorem 3.27 are true.

Proof. One has simply to sum-up conditions (3.91), (3.92) with $\varepsilon = \rho\lambda_K$ and (3.99). The following estimate (3.101) can be obtained by (long) estimations on the coefficients $a_{K,\delta}$ and $b_{K,\delta}$. □

Remark 3.53. The conditions in Proposition 3.52 can be simplified. For example one can exhibit an explicit constant c such that if δ satisfies

$$\begin{aligned} \delta e^{12\pi\delta} \leqslant\ & c e^{-20K\bar{r}_\delta} \max\Bigg(1, \left(\frac{1-\sin(\alpha)}{2-\sin(\alpha)}\right), \frac{\rho\lambda_K(1-\sin(\alpha))e^{-4K\bar{r}_\delta}}{K^2}, \\ & \frac{\lambda_K}{K^2 e^{4K\bar{r}_\delta}\ell(\rho) + \lambda_K\left(1 + \left(\frac{1+\rho}{2}\right)\ell(\rho)\right)}\Bigg) \end{aligned} \tag{3.101}$$

the conditions in (3.100) are fulfilled. Explicit estimates on the spectral gap λ_K can be found in [9, Sec. 2.5]. Proposition 3.28 is an direct consequence of (3.101).

Chapter 4

Quenched fluctuations of the empirical measure

The material for this chapter is taken from an article [55] published in *Electronic Journal of Probability*.

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1 Introduction and motivations

1.1 Disordered-induced rotation and fluctuations of the order parameters

In Chapter 2, we proved the quenched convergence of the empirical flow ν_N to the solution of McKean-Vlasov equation specifically studied in Chapter 3 in the case of the *sine-model*.

The purpose of this chapter is to address the issue of the fluctuations of the empirical flow around its McKean-Vlasov limit; thus the main result of this chapter (Theorem 4.4) concerns a Central Limit Theorem in a quenched set-up (namely the quenched fluctuations of ν_N around its limit ν).

Before stating our result, let us briefly recall our motivations: as already pointed out in Remark 2.19, p. 45, a crucial aspect of the quenched convergence result, which is a law of large numbers, is that it shows the *self-averaging* character of this limit: every typical disorder configuration leads as $N \to \infty$ to the same deterministic evolution (3.1).

However, we pointed out in the main introduction (recall Section 5, p. 5) that for a finite sample of oscillators the fluctuations of the disorder make the whole system rotate in a given direction at a speed of order $1/\sqrt{N}$ which depends on the initial choice of the disorder (recall in particular Figures 1.2 and 1.3). In other words, the self-averaging phenomenon does not hold at the scale of fluctuations.

As already mentioned in the main introduction, this non self-averaging phenomenon can be tackled in the *sine-model* by computing the finite-size order parameters r_N and ψ_N:

$$r_{N,t}e^{i\psi_{N,t}} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_{j,t}} = \int_{\mathbf{S}\times\mathbf{R}} e^{i\theta}\,\mathrm{d}\nu_{N,t}(\theta,\omega). \tag{4.1}$$

Thus the main motivation of this chapter is to derive a quenched fluctuation result for r_N and ψ_N. Since the latter parameters are actually functions of the empirical flow ν_N, we will prove a quenched fluctuation result (Theorem 4.4) for the empirical flow ν_N around its McKean-Vlasov limit. We insist on the fact that we are not interested here in an averaged fluctuation result: averaging with respect to the disorder makes the disorder-induced rotation disappear.

Although the main motivation of this chapter is the study of the *sine-model*, all of the results we prove here are valid in the general framework of diffusions in random environment (2.1); therefore, we will place ourselves under the hypotheses of Chapter 2.

1.2 Quenched fluctuations of the empirical flow

The result we prove concerns the behavior as $N \to \infty$ of the *quenched* fluctuation process of the empirical flow ν_N (1.16) around its McKean-Vlasov limit (3.1)

$$\eta_N^{(\omega)} := \sqrt{N}\left(\nu_N^{(\omega)} - \nu\right).$$

More precisely, what we prove is the quenched convergence of η_N, seen as a continuous process in the Schwartz space $\mathcal{S}'$ of tempered distributions on $\mathbf{S}\times\mathbf{R}$ to the solution $t \mapsto \eta_t$ of an Ornstein-Uhlenbeck process (4.9). In particular, we insist on the fact that the limit process η is explicit, in the sense that the linear operator governing its evolution is explicitly given and deterministic. The quenched convergence is here understood as a weak convergence *in law w.r.t. the disorder* and is more technically involved than the

convergence in the averaged system. We refer to Section 3 for detailed statements and definitions.

This chapter is organized as follows: Section 2 introduces the model and the main results. In Section 3, the quenched Central Limit Theorem is proved. The last section 4 applies the fluctuation result to the behavior of the order parameters in the *sine-model*.

2 Notations and main results

2.1 Notations

We recall the necessary notations that will be used in the chapter.

- if X is a metric space, $\mathcal{B}_X$ will be its Borel σ-field,
- $\mathcal{C}_b(X)$ (resp. $\mathcal{C}_b^p(X)$, $p = 1, \dots, \infty$), the set of bounded continuous functions (resp. bounded continuous with bounded continuous derivatives up to order p) on X, (X will be often $\mathbf{S} \times \mathbf{R}$),
- $\mathcal{C}_c(X)$ (resp. $\mathcal{C}_c^p(X)$, $p = 1, \dots, \infty$), the set of continuous functions with compact support (resp. continuous with compact support with continuous derivatives up to order p) on X,
- $\mathbf{D}([0,T], X)$, the set of right-continuous with left limits functions with values on X, endowed with the Skorokhod topology,
- $\mathcal{M}_1(Y)$, the set of probability measures on Y (Y topological space, with a regular σ-field $\mathcal{B}$),
- $\mathcal{M}_F(Y)$, the set of finite measures on Y,
- $(\mathcal{M}_1(Y), w)$: $\mathcal{M}_1(Y)$ endowed with the topology of weak convergence, namely the coarsest topology on $\mathcal{M}_1(Y)$ such that the evaluations $\nu \mapsto \int f \, \mathrm{d}\nu$ are continuous, where f are bounded continuous,
- $(\mathcal{M}_1(Y), v)$: $\mathcal{M}_1(Y)$ endowed with the topology of vague convergence, namely the coarsest topology on $\mathcal{M}_1(Y)$ such that the evaluations $\nu \mapsto \int f \, \mathrm{d}\nu$ are continuous, where f are continuous with compact support.

We will use C as a constant which may change from a line to another.

2.2 The model

We place ourselves within the general framework of § 2.4 in Chapter 2, that is where the interaction (rewritten below for the sake of clarity) is governed by functions b and c that satisfy the regularity assumptions made in Chap. 2, § 2.4.2:

For $i = 1, \dots, N$, for $T > 0$, for all $t \in [0, T]$,

$$\theta_{i,t} = \xi_i + \frac{1}{N} \sum_{j=1}^{N} \int_0^t b(\theta_{i,s}, \theta_{j,s}, \omega_i, \omega_j) \, \mathrm{d}s + \int_0^t c(\theta_{i,s}, \omega_i) \, \mathrm{d}s + B_{i,t}, \tag{4.2}$$

where the initial conditions ξ_i are independent and identically distributed with law γ, and independent of the Brownian motion $(B) = (B_i)_{i \geqslant 1}$. The disorder $(\omega) = (\omega_i)_{i \geqslant 1}$ is a realization of i.i.d. random variables with law μ.

As before, we will denote as $\mathbf{P}$ the law of the sequence of Brownian Motions and as $\mathbb{P}$ the law of the sequence of the disorder. The corresponding expectations will be denoted as $\mathbf{E}$ and $\mathbb{E}$ respectively.

2.3 The fluctuation process

The main object of interest of this chapter is the fluctuation process $\eta_N^{(\omega)}$ of $\nu_N^{(\omega)}$ around its limit ν (recall (3.1)):

Definition 4.1. For all $t \leqslant T$, for fixed $(\omega) \in \mathbf{R}^{\mathbf{N}}$, we define:

$$\eta_{N,t}^{(\omega)} = \sqrt{N}\left(\nu_{N,t}^{(\omega)} - \nu_t\right).$$

Once again the superscript (ω) in $\eta_{N,t}^{(\omega)}$ is here to recall that the fluctuations depend on the initial choice of the quenched disorder (ω). Note that for for fixed N, t and (ω), $\eta_{N,t}^{(\omega)}$ is a signed measure on $\mathbf{S} \times \mathbf{R}$. Thus, a convenient way to consider the process $t \mapsto \eta_{N,t}^{(\omega)}$ will be to see it as an element of the larger set $\mathcal{C}([0,T], \mathcal{S}'(\mathbf{S} \times \mathbf{R}))$ of continuous processes with values in the set $\mathcal{S}'(\mathbf{S} \times \mathbf{R})$ of tempered distributions on $\mathbf{S} \times \mathbf{R}$.

2.4 Main result: quenched fluctuations of the empirical flow

Let us turn to the statement of the main result of the chapter: Theorem 4.4.

Theorem 2.16 says that for $\mathbb{P}$-almost every realization (ω) of the disorder, we have the convergence of $\nu_N^{(\omega)}$ towards ν, which is a law of large numbers. We are now interested in the corresponding Central Limit Theorem associated to this convergence, namely, for a *fixed* realization of the disorder (ω), in the asymptotic behavior, as $N \to \infty$ of the fluctuation field $\eta_N^{(\omega)}$ defined in (4.1).

The ideas used here are inspired by the work of B. Fernandez and S. Méléard ([36]) who studied the convergence of the similar fluctuations in the case without disorder. We will rely on the ideas introduced in [36] although adding a quenched disorder will make statements and proofs more technically demanding and will in particular require to introduce a weaker notion of convergence.

2.4.1 Hypotheses

In addition to the hypothesis made in § 2.4.2, p. 43, we make the following assumptions about b and c (where $\mathcal{D}_p$ is the set of all differential operators of the form $\partial_u^k \partial_\omega^l$ with $k + l \leqslant p$):

$$\begin{cases} b \in \mathcal{C}_b^\infty(\mathbf{S}^2 \times \mathbf{R}^2), \quad c \in \mathcal{C}^\infty(\mathbf{S} \times \mathbf{R}), \\ \exists \alpha > 0, \sup\limits_{D \in \mathcal{D}_6} \displaystyle\int_{\mathbf{R}} \frac{\sup_{\theta \in \mathbf{S}} |Dc(\theta, \omega)|^2}{1 + |\omega|^{2\alpha}} \, \mathrm{d}\omega < \infty, \end{cases} \tag{4.3}$$

Furthermore, we make the following assumption about the law of the disorder (α is defined in (4.3)):

$$\text{the } (\omega_j) \text{ are i.i.d. and} \quad \int_{\mathbf{R}} |\omega|^{4\alpha} \, \mathrm{d}\mu(\omega) < \infty. \tag{4.4}$$

Remark 4.2. The regularity hypothesis about b and c can be weakened (namely $b \in \mathcal{C}_b^n(\mathbf{S}^2 \times \mathbf{R}^2)$ and $c \in \mathcal{C}^m(\mathbf{S} \times \mathbf{R})$ for sufficiently large n and m) but we have kept here $m = n = \infty$ for the sake of clarity.

Remark 4.3. In the case of the *sine-model*, Hypothesis (4.3) is satisfied with $\alpha = 2$ for example.

2.4.2 The main theorem

In order to state the fluctuation theorem, we need some further notations: for all $s \in [0,T]$, let $\mathcal{L}_s$ be the second order differential operator defined by

$$\mathcal{L}_s(\varphi)(\theta,\omega) := \frac{1}{2}\varphi''(\theta,\omega) + \varphi'(\theta,\omega)(b[\theta,\omega,\nu_s] + c(\theta,\omega)) + \langle \nu_s\,,\, \varphi'(\cdot,\cdot)b(\cdot,\theta,\cdot,\omega)\rangle\,. \quad (4.5)$$

Let W the Gaussian process with covariance (where φ_1 and φ_2 are regular functions on $\mathbf{S}\times\mathbf{R}$):

$$\mathbf{E}(W_t(\varphi_1)W_s(\varphi_2)) = \int_0^{s\wedge t} \langle \nu_u\,,\, \varphi_1'\varphi_2'\rangle\,\mathrm{d}u. \quad (4.6)$$

For all φ_1,φ_2 bounded and continuous on $\mathbf{S}\times\mathbf{R}$, let

$$\begin{aligned}\Gamma_1(\varphi_1,\varphi_2) &= \int_{\mathbf{R}} \mathrm{Cov}_\gamma\left(\varphi_1(\cdot\,,\omega),\varphi_2(\cdot\,,\omega)\right)\mathrm{d}\mu(\omega), \quad (4.7)\\ &= \int_{\mathbf{S}\times\mathbf{R}} \left(\varphi_1 - \int_{\mathbf{S}}\varphi_1(\cdot\,,\omega)\,\mathrm{d}\gamma\right)\left(\varphi_2 - \int_{\mathbf{S}}\varphi_2(\cdot\,,\omega)\,\mathrm{d}\gamma\right)\mathrm{d}\gamma\,\mathrm{d}\mu(\omega),\end{aligned}$$

and

$$\begin{aligned}\Gamma_2(\varphi_1,\varphi_2) &= \mathrm{Cov}_\mu\left(\int_{\mathbf{S}}\varphi_1\,\mathrm{d}\gamma, \int_{\mathbf{S}}\varphi_2\,\mathrm{d}\gamma\right), \quad (4.8)\\ &= \int_{\mathbf{R}}\left(\int_{\mathbf{S}}\varphi_1\,\mathrm{d}\gamma - \int_{\mathbf{S}\times\mathbf{R}}\varphi_1\,\mathrm{d}\gamma\,\mathrm{d}\mu\right)\left(\int_{\mathbf{S}}\varphi_2\,\mathrm{d}\gamma - \int_{\mathbf{S}\times\mathbf{R}}\varphi_2\,\mathrm{d}\gamma\,\mathrm{d}\mu\right)\mathrm{d}\mu.\end{aligned}$$

We are now in position to state the main result: Theorem 4.4, which is proved in Section 3.

For fixed (ω), we may consider $\mathcal{H}_N^{(\omega)}$, the law of the process $\eta_N^{(\omega)}$; $\mathcal{H}_N^{(\omega)}$ belongs to $\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))$, where $\mathcal{S}'$ is the usual Schwartz space of tempered distributions on $\mathbf{S}\times\mathbf{R}$. We are here interested in the law of the random variable $(\omega)\mapsto\mathcal{H}_N^{(\omega)}$ which is hence an element of $\mathcal{M}_1(\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}')))$.

Theorem 4.4 (Fluctuations in the quenched model)**.** *Under* (4.3), (4.4), *the sequence* $(\omega)\mapsto\mathcal{H}_N^{(\omega)}$ *converges in law in* $\mathcal{M}_1\left(\mathcal{M}_1\left(\mathcal{C}([0,T],\mathcal{S}')\right)\right)$ *to the random variable* $\omega\mapsto\mathcal{H}^\omega$, *where* $\mathcal{H}^\omega$, *element of* $\mathcal{M}_1\left(\mathcal{C}([0,T],\mathcal{S}')\right)$, *is the law of the process* η^ω *solution in* $\mathcal{S}'$ *of the following Ornstein-Uhlenbeck equation:*

$$\eta_t^\omega = X^\omega + \int_0^t \mathcal{L}_s^*\eta_s^\omega\,\mathrm{d}s + W_t, \quad (4.9)$$

where, $\mathcal{L}_s^*$ *is the formal adjoint operator of* $\mathcal{L}_s$ *defined in* (4.5) *and for all fixed* ω, X^ω *is a non-centered Gaussian process with covariance* Γ_1 *and with mean value* $C(\omega)$. *As a random variable in* ω, $\omega\mapsto C(\omega)$ *is a Gaussian process with covariance* Γ_2. *Moreover,* W *is independent on the initial value* X.

Remark 4.5. An important remark is that in the evolution (4.9), the linear operator $\mathcal{L}_s^*$ is deterministic; the only dependence in ω lies in the initial condition X^ω, through its nontrivial means $C(\omega)$. However, numerical simulations of trajectories of η^ω (see Fig. 1.4) clearly show a non self-averaging phenomenon, analogous to the one observed in Fig 1.3: η_t^ω not only depends on ω through its initial condition X^ω, but also for all positive time $t>0$.

Understanding how the deterministic operator $\mathcal{L}_s^*$ propagates the initial dependence in ω on the whole trajectory is the object of Chapter 5. In that sense, it requires a

precise understanding of the spectral properties of $\mathcal{L}_s^*$, which appears to be deeply linked to the linear operator in McKean-Vlasov equation (3.2) linearized around its nontrivial stationary solution.

Remark 4.6 (Generalization to the non-compact case)**.** As for the quenched convergence result (see Remark 2.17 in Chapter 2), it is possible to extend Theorem 4.4 to the (analogous but more technical) case where $\theta \in \mathbf{S}$ is replaced by $x \in \mathbf{R}^d$. To this purpose, one has to introduce an additional weight $(1+|x|^\alpha)^{-1}$ in the definition of the Sobolev norms in Section 3 and to suppose appropriate hypothesis concerning the first moments of the initial condition γ ($\int |x|^\beta \, \mathrm{d}\gamma(x) < \infty$ for a sufficiently large β).

2.4.3 Averaged fluctuations of the empirical flow

Even if the main purpose of this chapter is to prove the quenched result, it is worth to note that an easy byproduct of Theorem 4.4 is an averaged fluctuation result:

Theorem 4.7 (Fluctuations in the averaged model)**.** *Under* (4.3), (4.4)*, the law of the sequence* $(\eta_N)_{N \geqslant 1}$*, under the* joint law of the Brownian motions and disorder *converges in* $\mathcal{C}([0,T],\mathcal{S}')$ *to the law of the Orstein-Uhlenbeck process* η *solution in* $\mathcal{S}'$ *of the following equation:*

$$\eta_t = X + \int_0^t \mathcal{L}_s^* \eta_s \, \mathrm{d}s + W_t,$$

where X *is a* centered *Gaussian process with covariance*

$$C_{av}(\varphi) := \int_{\mathbf{S}\times\mathbf{R}} \left(\varphi(\theta,\omega) - \int_{\mathbf{S}\times\mathbf{R}} \varphi(\theta,\omega) \, \mathrm{d}\lambda(\theta) \, \mathrm{d}\mu(\omega) \right)^2 \mathrm{d}\lambda(\theta) \, \mathrm{d}\mu(\omega),$$

and where W *is the Gaussian process defined in Proposition 4.25.*

Remark 4.8. Note that one can see Theorem 4.7 as a generalization of the result established in [27, Th. 4, p. 744] in the case of an Hamiltonian interaction.

2.5 Fluctuations of the order parameters in the sine-model

We now apply Theorem 4.4 to the study of the order parameters in the *sine-model* defined in (4.1):

For given $N \geqslant 1$, $t \in [0,T]$ and disorder $(\omega) \in \mathbf{R}^{\mathbf{N}}$, let $r_{N,t}^{(\omega)} \geqslant 0$ and $\zeta_{N,t}^{(\omega)} \in \mathbf{S}$ such that

$$r_{N,t}^{(\omega)} \zeta_{N,t}^{(\omega)} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_{j,t}} = \int_{\mathbf{S}\times\mathbf{R}} e^{i\theta} \, \mathrm{d}\nu_{N,t}^{(\omega)}(\theta,\omega).$$

Proposition 4.9 (Convergence and fluctuations for $r_{N,t}^{(\omega)}$)**.** *We have the following:*

1. *Convergence of* $r_{N,t}^{(\omega)}$*: For* $\mathbb{P}$*-almost every realization of the disorder* (ω)*,* $r_N^{(\omega)}$ *converges in law to* $r \in \mathcal{C}([0,T],\mathbf{R})$ *defined by*

$$t \in [0,T] \mapsto r_t := \left(\langle \nu_t \, , \cos(\cdot) \rangle^2 + \langle \nu_t \, , \sin(\cdot) \rangle^2 \right)^{\frac{1}{2}}.$$

2. *If* $r_0 > 0$ *then*

$$\forall t \in [0,T], \quad r_t > 0. \tag{4.10}$$

3. *Fluctuations of* $r_{N,t}^{(\omega)}$ *around its limit: Let*

$$t \mapsto \mathcal{R}_{N,t}^{(\omega)} := \sqrt{N}\left(r_{N,t}^{(\omega)} - r_t\right) \tag{4.11}$$

be the fluctuation process. For fixed disorder (ω)*, let* $\mathfrak{R}_N^{(\omega)} \in \mathcal{M}_1(\mathcal{C}([0,T],\mathbf{R}))$ *be the law of* $\mathcal{R}_N^{(\omega)}$*. Then, under* (4.10)*, the random variable* $(\omega) \mapsto \mathfrak{R}_N^{(\omega)}$ *converges in law to the random variable* $\omega \mapsto \mathfrak{R}^\omega$*, where* $\mathfrak{R}^\omega$ *is the law of* $\mathcal{R}^\omega := \frac{1}{r}\left(\langle \nu\,,\cos(\cdot)\rangle \langle \eta^\omega\,,\cos(\cdot)\rangle + \langle \nu\,,\sin(\cdot)\rangle \langle \eta^\omega\,,\sin(\cdot)\rangle\right)$.

Remark 4.10. In simpler terms, this double convergence in law corresponds for example to the convergence in law of the corresponding characteristic functions (since the tightness is a direct consequence of the tightness of the process η); i.e. for $t_1,\dots,t_p \in [0,T]$ $(p \geqslant 1)$ the characteristic function of $\left(\mathcal{R}_{N,t_1}^{(\omega)},\dots,\mathcal{R}_{N,t_p}^{(\omega)}\right)$ for fixed (ω) converges in law, as a random variable in (ω), to the random characteristic function of $(\mathcal{R}_{t_1}^{\omega},\dots,\mathcal{R}_{t_p}^{\omega})$.

Proposition 4.11 (Convergence and fluctuations for $\zeta_N^{(\omega)}$)**.** *We have the following:*

1. *Convergence of* $\zeta_N^{(\omega)}$*: Under* (4.10)*, for* $\mathbb{P}$*-almost every realization of the disorder* (ω)*,* $\zeta_N^{(\omega)}$ *converges in law to* $\zeta : t \in [0,T] \mapsto \zeta_t := \frac{\langle \nu_t\,,e^{i\theta}\rangle}{r_t}$*,*
2. *Fluctuations of* $\zeta_N^{(\omega)}$ *around its limit: Let*

$$t \mapsto \mathcal{Z}_{N,t}^{(\omega)} := \sqrt{N}\left(\zeta_{N,t}^{(\omega)} - \zeta_t\right)$$

be the fluctuation process. For fixed disorder (ω)*, let* $\mathfrak{Z}_N^{(\omega)} \in \mathcal{M}_1(\mathcal{C}([0,T],\mathbf{R}))$ *be the law of* $\mathcal{Z}_N^{(\omega)}$*. Then, under* (4.10)*, the random variable* $(\omega) \mapsto \mathfrak{Z}_N^{(\omega)}$ *converges in law to the random variable* $\omega \mapsto \mathfrak{Z}^\omega$*, where* $\mathfrak{Z}^\omega$ *is the law of*

$$\mathcal{Z}^\omega := \frac{1}{r^2}\left(r\,\langle \eta^\omega\,,\cos(\cdot)\rangle + \left\langle \nu\,,e^{i\theta}\right\rangle \mathcal{R}^\omega\right).$$

In the *sine-model*, we have $\zeta_N^{(\omega)} = e^{i\psi_N^{(\omega)}}$ where $\psi_N^{(\omega)}$ is defined in (4.1) and is plotted in Fig. 1.3, p. 29. Some trajectories of the process $\mathcal{Z}^\omega$ are plotted in Fig. 1.4, p. 32.

This fluctuation result concerning the order parameters in the *sine-model* is proved in Section 4.

3 Proof of the fluctuation result

Now we turn to the proof of Theorem 4.4. To that purpose, we need to introduce some distribution spaces:

3.1 Distribution spaces

Let $\mathcal{S} := \mathcal{S}(\mathbf{S}\times\mathbf{R})$ be the usual Schwartz space of rapidly decreasing infinitely differentiable functions. Let $\mathcal{D}_p$ be the set of all differential operators of the form $\partial_\theta^k\partial_\omega^l$ with $k+l \leqslant p$. We know from Gelfand and Vilenkin [39] p. 82-84, that we can introduce on $\mathcal{S}$ a nuclear Fréchet topology by the system of seminorms $\|\cdot\|_p$, $p = 1,2,\dots$, defined by

$$\|\varphi\|_p^2 = \sum_{k=0}^{p}\int_{\mathbf{S}\times\mathbf{R}}(1+|\omega|^2)^{2p}\sum_{D\in\mathcal{D}_k}|D\varphi(\theta,\omega)|^2\,\mathrm{d}\theta\,\mathrm{d}\omega.$$

Let $\mathcal{S}'$ be the corresponding dual space of tempered distributions. Although, for the sake of simplicity, we will mainly consider $\eta_N^{(\omega)}$ as a process in $\mathcal{C}([0,T],\mathcal{S}')$, we need some more precise estimations to prove tightness and convergence. We need here the following norms (see [3]):

For every integer j, $\alpha \in \mathbf{R}^+$, we consider the space of all real functions φ defined on $\mathbf{S}\times\mathbf{R}$ with derivative up to order j such that

$$\| \varphi \|_{j,\alpha} := \left(\sum_{k_1+k_2 \leqslant j} \int_{\mathbf{S}\times\mathbf{R}} \frac{|\partial_\theta^{k_1}\partial_\omega^{k_2}\varphi(\theta,\omega)|^2}{1+|\omega|^{2\alpha}} \,\mathrm{d}\theta\,\mathrm{d}\omega \right)^{1/2} < \infty.$$

Let $W_0^{j,\alpha}$ be the completion of $\mathcal{C}_c^\infty(\mathbf{S}\times\mathbf{R})$ for this norm; $(W_0^{j,\alpha}, \|\cdot\|_{j,\alpha})$ is a Hilbert space. Let $W_0^{-j,\alpha}$ be its dual space.

Let $C^{j,\alpha}$ be the space of functions φ with continuous partial derivatives up to order j such that

$$\lim_{|\omega|\to\infty} \sup_{\theta\in\mathbf{S}} \frac{|\partial_\theta^{k_1}\partial_\omega^{k_2}\varphi(\theta,\omega)|}{1+|\omega|^\alpha} = 0, \text{ for all } k_1+k_2 \leqslant j,$$

with norm

$$\| \varphi \|_{C^{j,\alpha}} = \sum_{k_1+k_2 \leqslant j} \sup_{\theta\in\mathbf{S}} \sup_{\omega\in\mathbf{R}} \frac{|\partial_\theta^{k_1}\partial_\omega^{k_2}\varphi(\theta,\omega)|}{1+|\omega|^\alpha}.$$

We have the following embeddings:

$$W_0^{m+j,\alpha} \hookrightarrow C^{j,\alpha}, m>1, j\geqslant 0, \alpha \geqslant 0,$$

i.e. there exists some constant C such that

$$\| \varphi \|_{C^{j,\alpha}} \leqslant C \| \varphi \|_{m+j,\alpha}. \tag{4.12}$$

Moreover,

$$W_0^{m+j,\alpha} \hookrightarrow W_0^{j,\alpha+\beta}, m>1, j\geqslant 0, \alpha\geqslant 0, \beta>1.$$

Thus there exists some constant C such that

$$\| \varphi \|_{j,\alpha+\beta} \leqslant C \| \varphi \|_{m+j,\alpha}.$$

We then have the following dual continuous embedding:

$$W_0^{-j,\alpha+\beta} \hookrightarrow W_0^{-(m+j),\alpha}, \ m>1, \alpha\geqslant 0, \beta>1. \tag{4.13}$$

It is quite clear that $\mathcal{S} \hookrightarrow W_0^{j,\alpha}$ for any j and α, with a continuous injection.

We now prove some continuity of linear mappings in the corresponding spaces:

Lemma 4.12. *For every $\theta,\theta' \in \mathbf{S}$, $\omega\in\mathbf{R}$, for all α, the linear mappings $W_0^{3,\alpha} \to \mathbf{R}$ defined by*

$$D_{\theta,\theta',\omega}(\varphi) := \varphi(\theta,\omega) - \varphi(\theta',\omega); D_{\theta,\omega} := \varphi(\theta,\omega); H_{\theta,\omega} = \varphi'(\theta,\omega), \tag{4.14}$$

are continuous and

$$\| D_{\theta,\theta',\omega} \|_{-3,\alpha} \leqslant C|\theta-\theta'|\,(1+|\omega|^\alpha), \tag{4.15}$$

$$\| D_{\theta,\omega} \|_{-3,\alpha} \leqslant C\,(1+|\omega|^\alpha), \tag{4.16}$$

$$\| H_{\theta,\omega} \|_{-3,\alpha} \leqslant C\,(1+|\omega|^\alpha). \tag{4.17}$$

Proof of Lemma 4.12. Let φ be a function of class $\mathcal{C}^\infty$ with compact support on $\mathbf{S}\times\mathbf{R}$, then,

$$\begin{aligned}|\varphi(\theta,\omega)-\varphi(\theta',\omega)| &\leqslant |\theta-\theta'|\sup_u |\varphi'(u,\omega)|\,,\\ &\leqslant |\theta-\theta'|\,(1+|\omega|^\alpha)\sup_{u,\omega}\left(\frac{|\varphi'(u,\omega)|}{1+|\omega|^\alpha}\right),\\ &\leqslant |\theta-\theta'|\,(1+|\omega|^\alpha)\,\|\,\varphi\,\|_{1,\alpha}\,,\\ &\leqslant C|\theta-\theta'|\,(1+|\omega|^\alpha)\,\|\,\varphi\,\|_{3,\alpha}\,,\end{aligned}$$

following (4.12) with $j=1$ and $m=2>1$. Then, (4.15) follows from a density argument. (4.16) and (4.17) are proved in the same way. □

3.2 The non-linear process

The proof of Theorem 4.4 is based on the existence of the nonlinear process associated to McKean-Vlasov equation. Such existence has been studied by numerous authors (eg. Dawson [28], Jourdain-Méléard [49], Malrieu [57], Shiga-Tanaka [80], Sznitman [88], [89]) mostly in order to prove some propagation of chaos properties for systems without disorder. We consider the present similar case where disorder is present. Let us give some intuition of this process. One can replace the nonlinearity in McKean-Vlasov equation (3.2), p. 74, by an arbitrary measure $m(\,\mathrm{d}\theta,\,\mathrm{d}\omega)$:

$$\partial_t q_t(\theta,\omega)=\frac{1}{2}\partial_\theta^2 q_t(\theta,\omega)-\partial_\theta\left\{q_t(\theta,\omega)\left(b[\theta,\omega,m]+c(\theta,\omega)\right)\right\}.$$

In this particular case, it is usual to interpret $q_t(\cdot,\omega)$ as the time marginals of the following diffusion:

$$d\bar\theta_t=dB_t+b[\bar\theta_t,\omega,m]dt+c(\bar\theta_t,\omega)dt,\ \ \omega\sim\mu. \tag{4.18}$$

It is then natural to consider the following problem, where m is replaced by the proper process ν: on a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,B,\xi_0,Q)$, endowed with a Brownian motion B and with a $\mathcal{F}_0$ measurable random variable $\xi_0\sim\gamma$, we introduce the following system:

$$\left\{\begin{array}{lcl}\bar\theta_t &=& \xi_0+\int_0^t b[\bar\theta_s,\omega,\nu_s]\,\mathrm{d}s+\int_0^t c(\bar\theta_s,\omega)\,\mathrm{d}s+B_t,\\ \omega &\sim& \mu,\\ \nu_t &=& \mathcal{L}(\bar\theta_t,\omega),\forall t\in[0,T].\end{array}\right. \tag{4.19}$$

Proposition 4.13. *There is pathwise existence and uniqueness for Equation* (4.19).

Proof of Proposition 4.13. There exist several ways to prove Proposition 4.13.

One way is to recall that we already have encountered this nonlinear process where we proved uniqueness of the weak McKean-Vlasov formulation in Chapter 2, Proposition 2.20 (recall (2.62)). More precisely, we show in Chap. 2, § 5 that any solution $t\mapsto\nu_t$ to the weak formulation of McKean-Vlasov equation (2.11) solves (2.62) with $\nu_t=\mathcal{L}(\xi_t,\omega_t)$, which is exactly (4.19). In particular, since there exists a solution $t\mapsto\nu_t$ to (2.11), then there exists a process (θ_t^ω,ω) which satisfies (4.19). The pathwise uniqueness was precisely proved in the end of Chap. 2, § 5.

An alternative proof of Proposition 4.13 is based on arguments used by Sznitman [89] in a context of propagation of chaos properties (see in particular [89, Th 1.1, p. 172]). This proof, which has its own interest is developed in the following. The main idea consists in

using a Picard iteration in the space of probabilities on $\mathcal{C}([0,T],\mathbf{S})\times\mathbf{R}$ endowed with an appropriate Wasserstein metric.

Namely, let us consider the set $\mathcal{M}_\mu$ of probability measures on $\mathcal{C}([0,T],\mathbf{S})\times\mathbf{R}$ with second component (w.r.t. $\mathbf{R}$) is μ and endow this set with the Wasserstein metric

$$D_T(m_1,m_2) := \inf_{\substack{X^{(1)}\sim m_1\\ X^{(2)}\sim m_2}} \left\{ \mathbf{E}\left(\sup_{s\leqslant T} |X_s^{(1)} - X_s^{(2)}| \wedge 1 \right) \right\}. \tag{4.20}$$

In (4.20), the $X^{(i)}$ are understood as random variables on a certain probability space $(\Omega,\mathbf{P})$; nevertheless, note that the definition of (4.20) does not depend on the choice of this probability space. Formula (4.20) defines a complete metric on $\mathcal{M}_\mu$ that gives the topology of weak convergence.

Let us denote by $\Phi:\mathcal{M}_\mu\to\mathcal{M}_\mu$ the functional which maps any measure $m(\,\mathrm{d}\theta,\,\mathrm{d}\omega)\in\mathcal{M}_\mu$ to the law $\Phi(m)$ of (θ,ω) where $(\theta_t)_{0\leqslant t\leqslant T}$ is the only solution to (4.18). Note that $\bar m$ is a fixed point of Φ ($\Phi(\bar m)=\bar m$) if and only if the corresponding process $(\bar\theta,\omega)$ is a solution to (4.19). As in [89, Lemma 1.3], we prove the following

$$\forall t\leqslant T,\ D_t(\Phi(m_1),\Phi(m_2)) \leqslant C_T\int_0^t D_u(m_1,m_2)\,\mathrm{d}u. \tag{4.21}$$

If we prove (4.21), the proof of Proposition 4.13 will be finished since in that case, one can iterate this inequality and find

$$\forall k\geqslant 1,\ D_T(\Phi^{k+1}(m),\Phi^k(m)) \leqslant C_T^k\frac{T^k}{k!}D_T(\Phi(m),m),$$

which gives that $\left(\Phi^k(m)\right)_{k\geqslant 1}$ is a Cauchy sequence, and thus converges to some $\bar m$, fixed-point of Φ. The uniqueness of such a fixed-point comes also from (4.21) and Gronwall's lemma.

Let us now prove (4.21): for $X^{(1)}=(\theta_1,\omega_0)$ and $X^{(2)}=(\theta_2,\omega_0)$ solutions to (4.18) driven by the same Brownian motion, with the same initial condition, we have successively for all $0\leqslant t\leqslant T$,

$$\begin{aligned}|\theta_t^{(1)}-\theta_t^{(2)}| &\leqslant \int_0^t \left|b[\theta_s^{(1)},\omega_0,m_{1,s}]-b[\theta_s^{(2)},\omega_0,m_{2,s}]\right| + \left|c(\theta_s^{(1)},\omega_0)-c(\theta_s^{(2)},\omega_0)\right|\,\mathrm{d}s,\\ &\leqslant C\int_0^t \left|\theta_s^{(1)}-\theta_s^{(2)}\right|\wedge 1\,\mathrm{d}s + C\int_0^t \mathbf{E}|X_s^{(1)}-X_s^{(2)}|\wedge 1\,\mathrm{d}s.\end{aligned}$$

Consequently,

$$\sup_{s\leqslant t}|X_s^{(1)}-X_s^{(2)}| \leqslant C\int_0^t \left|X_s^{(1)}-X_s^{(2)}\right|\wedge 1\,\mathrm{d}s + C\int_0^t D_s(m_1,m_2)\,\mathrm{d}s.$$

Gronwall's lemma leads to conclusion. □

3.3 Fluctuations in the quenched model

The key argument of the proof of Theorem 4.4 is to estimate the error made in approximating (as $N\to\infty$) the rotators θ_i solutions to (4.2) with independent copies of the nonlinear process $\bar\theta_i$ solution to (4.19) when both processes have the same initial condition and are driven by the same Brownian motions (see (4.22)).

A major difference between this work and [36] is that, since in our quenched model, we only integrate w.r.t. Brownian motions *and not* w.r.t. the disorder, one has to deal with remaining terms [1]that would have disappeared in the *averaged model* (see Z_N in Proposition 4.14, to compare with [36, Lemma 3.2]). The main technical difficulty of Proposition 4.14 is to control the asymptotic behavior of such terms, see (4.23). As in [36], having proved Prop. 4.14, the key argument of the proof is a uniform estimation of the norm of the process $\eta_N^{(\omega)}$, see Propositions 4.15 and 4.19, based on the generalized stochastic differential equation verified by $\eta_N^{(\omega)}$, see (4.29).

3.3.1 Preliminary results

We consider here a fixed realization of the disorder $(\omega) = (\omega_1, \omega_2, \dots)$. On a common filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, (B^i)_{i \geqslant 1}, Q)$, endowed with a sequence of i.i.d. $\mathcal{F}_t$-adapted Brownian motions $(B_i)_{1 \leqslant i \leqslant N}$ and with a sequence of i.i.d. $\mathcal{F}_0$ measurable random variables (ξ_i) with law γ, we define as $(\theta_i)_{1 \leqslant i \leqslant N}$ the solution of (4.2), and as $(\bar{\theta}_i)_{1 \leqslant i \leqslant N}$ the solution to (4.19), with the same Brownian motion $(B_i)_{1 \leqslant i \leqslant N}$ and with the same initial value ξ_i.

The main technical proposition, from which every norm estimation of $\eta_N^{(\omega)}$ follows is the following:

Proposition 4.14.

$$\mathbf{E}\left[\sup_{t \leqslant T} \left|\theta_{i,t} - \bar{\theta}_{i,t}\right|^2\right] \leqslant C/N + Z_N(\omega_1, \dots, \omega_N), \tag{4.22}$$

where the random variable $(\omega) \mapsto Z_N(\omega)$ *is such that:*

$$\lim_{A \to \infty} \limsup_{N \to \infty} \mathbb{P}\left(N Z_N(\omega) > A\right) = 0. \tag{4.23}$$

The (rather technical) proof of Proposition 4.14 is postponed to the end of the chapter (see § 5). Once again, we stress the fact that the term Z_N would have disappeared in the averaged model.

The first norm estimation of the process $\eta_N^{(\omega)}$ (which will be used to prove tightness) is a direct consequence of Proposition 4.14 and of a Hilbertian argument:

Proposition 4.15. *Under the hypothesis* (4.4) *on* μ*, the process* $\eta_N^{(\omega)}$ *satisfies the following property: for all* $T > 0$*,*

$$\sup_{t \leqslant T} \mathbf{E}\left[\left\| \eta_{N,t}^{(\omega)} \right\|_{-3,2\alpha}^2\right] \leqslant A_N(\omega_1, \dots, \omega_N), \tag{4.24}$$

where

$$\lim_{A \to \infty} \limsup_{N \to \infty} \mathbb{P}\left(A_N > A\right) = 0.$$

Proof of Proposition 4.15. For all $\varphi \in W_0^{3,2\alpha}$, writing

$$\begin{aligned}\left\langle \eta_{N,t}^{(\omega)}, \varphi \right\rangle &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \left\{\varphi(\theta_{i,t}, \omega_i) - \varphi(\bar{\theta}_{i,t}, \omega_i)\right\} + \frac{1}{\sqrt{N}} \sum_{i=1}^N \left\{\varphi(\bar{\theta}_{i,t}, \omega_i) - \langle \nu_s, \varphi \rangle\right\}, \\ &=: S_{N,t}^{(\omega)}(\varphi) + T_{N,t}^{(\omega)}(\varphi),\end{aligned}$$

[1]. These remaining terms that are not bounded for fixed (ω) but in law w.r.t. (ω) explain why we had to introduce the weak notion of convergence for the statement of Theorem 4.4.

we have:

$$\left\langle \eta_{N,t}^{(\omega)}\,,\,\varphi\right\rangle^2 \leqslant 2\left(S_{N,t}^{(\omega)}(\varphi)^2 + T_{N,t}^{(\omega)}(\varphi)^2\right). \tag{4.25}$$

But, by convexity, (recall (4.14))

$$S_{N,t}^{(\omega)}(\varphi)^2 \leqslant \sum_{i=1}^{N} D^2_{\theta_{i,t},\bar\theta_{i,t},\omega_i}(\varphi).$$

Then, applying the latter equation to an orthonormal system $(\varphi_p)_{p\geqslant 1}$ in the Hilbert space $W_0^{3,2\alpha}$, summing on p, we have by Parseval's identity on the continuous functional $D_{\theta_{i,t},\bar\theta_{i,t},\omega_i}$,

$$\begin{aligned}
\mathbf{E}\left[\left\| S_{N,t}^{(\omega)}\right\|^2_{-3,2\alpha}\right] &\leqslant \mathbf{E}\left[\sum_{i=1}^{N}\left\| D_{\theta_{i,t},\bar\theta_{i,t},\omega_i}\right\|^2_{-3,2\alpha}\right],\\
&\leqslant C\sum_{i=1}^{N}(1+|\omega_i|^{4\alpha})\mathbf{E}\left[\left|\theta_{i,t}-\bar\theta_{i,t}\right|^2\right], &(4.26)\\
&\leqslant C\sum_{i=1}^{N}\left(1+|\omega_i|^{4\alpha}\right)\left(C/N + Z_N(\omega_1,\dots,\omega_N)\right), &(4.27)
\end{aligned}$$

where we used (4.15) in (4.26), and (4.22) in (4.27).

On the other hand,

$$\begin{aligned}
\mathbf{E}\left[T_{N,t}^{(\omega)}(\varphi)^2\right] &= \frac{1}{N}\mathbf{E}\left[\left\{\sum_{i=1}^{N}\left(\varphi(\bar\theta_{i,t},\omega_i)-\langle\nu_t\,,\,\varphi\rangle\right)\right\}^2\right],\\
&= \frac{1}{N}\mathbf{E}\left[\sum_{i=1}^{N}\left(\varphi(\bar\theta_{i,t},\omega_i)-\langle\nu_t\,,\,\varphi\rangle\right)^2\right]\\
&+ \frac{1}{N}\mathbf{E}\left[\sum_{i\neq j}(\varphi(\bar\theta_{i,t},\omega_i)-\langle\nu_t\,,\,\varphi\rangle)(\varphi(\bar\theta_{j,t},\omega_j)-\langle\nu_t\,,\,\varphi\rangle)\right],\\
&\leqslant \frac{2}{N}\mathbf{E}\left[\sum_{i=1}^{N}\left(\varphi(\bar\theta_{i,t},\omega_i)^2+\langle\nu_t\,,\,\varphi\rangle^2\right)\right] + \frac{1}{N}\sum_{i\neq j}G(\varphi)(\omega_i)G(\varphi)(\omega_j),\\
&\leqslant \frac{2}{N}\mathbf{E}\left[\sum_{i=1}^{N}\varphi(\bar\theta_{i,t},\omega_i)^2\right] + 2\langle\nu_t\,,\,\varphi\rangle^2 + \left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}G(\varphi)(\omega_i)\right)^2,
\end{aligned}$$

where $G(\varphi)(\omega) := \int\varphi(\theta,\omega_i)\,\mathrm{d}\nu_t^{\omega_i}(\theta) - \langle\nu_t\,,\,\varphi\rangle$. If we apply the same Hilbertian argument as for $S_N^{(\omega)}$, we see

$$\mathbf{E}\left[\left\| T_{N,t}^{(\omega)}\right\|^2_{-3,2\alpha}\right] \leqslant \frac{2C}{N}\mathbf{E}\left[\sum_{i=1}^{N}(1+|\omega_i|^{4\alpha})\right] + C + \left\|\varphi\mapsto\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}G(\varphi)(\omega_i)\right)\right\|^2_{-3,2\alpha}, \tag{4.28}$$

It is easy to see that the last term in (4.28) can be reformulated as $B_N(\omega_1,\dots,\omega_N)$, with the property that $\lim_{A\to\infty}\limsup_{N\to\infty}\mathbb{P}(B_N > A) = 0$. Combining (4.23), (4.25), (4.27) and (4.28), Proposition 4.15 is proved. □

3.3.2 Tightness of the fluctuation process

Applying Ito's formula to (4.2), we obtain, for all φ bounded function on $\mathbf{S}\times\mathbf{R}$, with two bounded derivatives w.r.t. θ, for every sequence (ω), for all $t\leqslant T$:

$$\left\langle \eta_{N,t}^{(\omega)}\,,\,\varphi\right\rangle = \left\langle \eta_{N,0}^{(\omega)}\,,\,\varphi\right\rangle + \int_0^t \left\langle \eta_{N,s}^{(\omega)}\,,\,\mathcal{L}_s^{\nu_N}(\varphi)\right\rangle \mathrm{d}s + M_{N,t}^{(\omega)}(\varphi), \tag{4.29}$$

where, for all $\theta\in\mathbf{S}$, $\omega\in\mathbf{R}$,

$$\mathcal{L}_s^{\nu_N}(\varphi)(\theta,\omega) = \frac{1}{2}\varphi''(\theta,\omega) + \varphi'(\theta,\omega)\left(b[\theta,\omega,\nu_{N,s}] + c(\theta,\omega)\right) + \left\langle \nu_s\,,\,\varphi'(\cdot,\cdot)b(\cdot,\theta,\cdot,\omega)\right\rangle, \tag{4.30}$$

and $M_{N,t}^{(\omega)}(\varphi)$ is a real continuous martingale with quadratic variation process

$$\left\langle M_N^{(\omega)}(\varphi)\right\rangle_t = \int_0^t \left\langle \nu_{N,s}^{(\omega)}\,,\,\varphi'(\theta,\omega)^2\right\rangle \mathrm{d}s.$$

Lemma 4.16. *For every N, the operator $\mathcal{L}_s^{\nu_N}$ defines a linear mapping from $\mathcal{S}$ into $\mathcal{S}$ and for all $\varphi\in\mathcal{S}$,*

$$\|\,\mathcal{L}_s^{\nu_N}(\varphi)\,\|_{3,2\alpha}^2 \leqslant C\,\|\,\varphi\,\|_{6,\alpha}^2\,.$$

Proof of Lemma 4.16. The terms $\frac{1}{2}\varphi''(\theta,\omega)$ and $\varphi'(\theta,\omega)b[\theta,\omega,\nu_{N,s}]$ in (4.30) clearly satisfy the lemma. We study the two remaining terms:

$$\begin{aligned}
\|\,\left\langle \nu_s\,,\,\varphi' b(\cdot,\theta,\cdot,\omega)\right\rangle\,\|_{3,2\alpha}^2 &= \sum_{k_1+k_2\leqslant 3}\int_{\mathbf{S}\times\mathbf{R}} \frac{\left\langle \nu_s\,,\,\varphi'\partial_\theta^{k_1}\partial_\omega^{k_2} b(\cdot,\theta,\cdot,\omega)\right\rangle^2}{1+|\omega|^{4\alpha}}\,\mathrm{d}\theta\,\mathrm{d}\omega,\\
&\leqslant C\int_{\mathbf{R}}\frac{1}{1+|\omega|^{4\alpha}}\,\mathrm{d}\omega\int_{\mathbf{S}\times\mathbf{R}}\varphi'(\theta,\omega)^2\,\mathrm{d}\nu_s(\theta,\omega),\\
&\leqslant C\,\|\,\varphi\,\|_{C^{3,\alpha}}^2\int_{\mathbf{R}}\frac{1}{1+|\omega|^{4\alpha}}\,\mathrm{d}\omega\int_{\mathbf{S}\times\mathbf{R}}(1+|\omega|^\alpha)^2\,\mathrm{d}\nu_s(\theta,\omega),\\
&\leqslant C\,\|\,\varphi\,\|_{6,\alpha}^2\int_{\mathbf{R}}\frac{1}{1+|\omega|^{4\alpha}}\,\mathrm{d}\omega\int_{\mathbf{R}}(1+|\omega|^\alpha)^2\,\mathrm{d}\mu(\omega).
\end{aligned}$$

And,

$$\|\,\varphi'(\theta,\omega)c(\theta,\omega)\,\|_{3,2\alpha}^2 = \sum_{k_1+k_2\leqslant 3}\int_{\mathbf{S}\times\mathbf{R}}\frac{\left(\partial_\theta^{k_1}\partial_\omega^{k_2}\left\{\varphi'(\theta,\omega)c(\theta,\omega)\right\}\right)^2}{1+|\omega|^{4\alpha}}\,\mathrm{d}\theta\,\mathrm{d}\omega.$$

Here, it suffices to estimate, for every differential operator $D_i=\partial_\theta^{u_i}\partial_\omega^{v_i}$, $i=1,2$ with $u_1+u_2+v_1+v_2\leqslant 3$, the following term:

$$\begin{aligned}
\int_{\mathbf{S}\times\mathbf{R}}\frac{|D_1\varphi'(\theta,\omega)D_2c(\theta,\omega)|^2}{1+|\omega|^{4\alpha}}\,\mathrm{d}\theta\,\mathrm{d}\omega &\leqslant \int_{\mathbf{S}\times\mathbf{R}}\frac{|D_1\varphi'(\theta,\omega)|^2}{(1+|\omega|^\alpha)^2}\frac{|D_2c(\theta,\omega)|^2(1+|\omega|^\alpha)^2}{1+|\omega|^{4\alpha}}\,\mathrm{d}\theta\,\mathrm{d}\omega,\\
&\leqslant C\,\|\,\varphi\,\|_{6,\alpha}^2\int_{\mathbf{R}}\frac{\sup_{\theta\in\mathbf{S}}|D_2c(\theta,\omega)|^2}{1+|\omega|^{2\alpha}}\,\mathrm{d}\omega.
\end{aligned}$$

The result follows from the assumptions made on c. □

For the tightness criterion used below, we need to ensure that the trajectories of the fluctuation process are almost surely continuous: in that purpose, we need some more precise evaluations than in Prop. 4.15.

Lemma 4.17. *The process* $\left(M_{N,t}^{(\omega)}\right)_{t\in[0,T]}$ *satisfies, for every* (ω)*, and for every* $T>0$*,*

$$\mathbf{E}\left[\sup_{t\leqslant T}\left\| M_{N,t}^{(\omega)}\right\|_{-3,2\alpha}^2\right] \leqslant \frac{C}{N}\sum_{i=1}^N\left(1+|\omega_i|^{4\alpha}\right).$$

Remark 4.18. In particular, a consequence of (4.4) is that, for $\mathbb{P}$-almost every sequence (ω),

$$\sup_N \mathbf{E}\left[\sup_{t\leqslant T}\left\| M_{N,t}^{(\omega)}\right\|_{-3,2\alpha}^2\right] \leqslant \sup_N \frac{C}{N}\sum_{i=1}^N\left(1+|\omega_i|^{4\alpha}\right) < \infty. \tag{4.31}$$

Proof of Lemma 4.17. Let $(\varphi_p)_{p\geqslant 1}$ a complete orthonormal system in $W_0^{3,2\alpha}$. For fixed N, by Doob's inequality, $\sum_{p\geqslant 1}\mathbf{E}\left[\sup_{t\leqslant T}\left(M_{N,t}^{(\omega)}(\varphi_p)\right)^2\right]$ is bounded by

$$\begin{aligned}
C\sum_{p\geqslant 1}\mathbf{E}\left[M_{N,T}^{(\omega)}(\varphi_p)^2\right] &= C\sum_{p\geqslant 1}\mathbf{E}\left[\int_0^T\left\langle \nu_{N,s}^{(\omega)},\, \varphi_p'(\theta,\omega)^2\right\rangle \mathrm{d}s\right],\\
&= \frac{1}{N}\sum_{i=1}^N\mathbf{E}\left[\int_0^T\sum_{p\geqslant 1}\varphi_p'(\theta_{i,s},\omega_i)^2\,\mathrm{d}s\right],\\
&= \frac{1}{N}\sum_{i=1}^N\mathbf{E}\left[\int_0^T\left\| H_{\theta_{i,s},\omega_i}\right\|_{3,2\alpha}^2\,\mathrm{d}s\right] \leqslant \frac{C}{N}\sum_{i=1}^N\left(1+|\omega_i|^{4\alpha}\right),
\end{aligned}$$

where we used (4.17) in the last inequality. That concludes the proof of Lemma 4.17. □

Proposition 4.19. *For every* N*, every* (ω)*,*

$$\mathbf{E}\left[\sup_{t\leqslant T}\left\| \eta_{N,t}^{(\omega)}\right\|_{-6,\alpha}^2\right] < C_N(\omega_1,\dots,\omega_N), \tag{4.32}$$

with

$$\lim_{A\to\infty}\limsup_{N\to\infty}\mathbb{P}\left(C_N>A\right)=0.$$

Proof of Proposition 4.19. Let (ψ_p) be a complete orthonormal system in $W_0^{6,\alpha}$ of $\mathcal{C}^\infty$ function on $\mathbf{S}\times\mathbf{R}$ with compact support. We prove the stronger result:

$$\mathbf{E}\left[\sum_{p\geqslant 1}\sup_{t\leqslant T}\left\langle \eta_{N,t}^{(\omega)},\, \psi_p\right\rangle^2\right] < \infty.$$

Indeed,

$$\left\langle \eta_{N,t}^{(\omega)},\, \psi_p\right\rangle^2 \leqslant C\left(\left\langle \eta_{N,0}^{(\omega)},\, \psi_p\right\rangle^2 + T\int_0^t\left\langle \eta_{N,s}^{(\omega)},\, \mathcal{L}_s^{\nu_N}(\psi_p)\right\rangle^2\,\mathrm{d}s + M_{N,t}^{(\omega)}(\psi_p)^2\right).$$

By Doob's inequality,

$$\begin{aligned}
\mathbf{E}\left[\sum_{p\geqslant 1}\sup_{t\leqslant T}\left\langle \eta_{N,t}^{(\omega)},\, \psi_p\right\rangle^2\right] \leqslant C\Bigg(&\mathbf{E}\left[\left\| \eta_{N,0}^{(\omega)}\right\|_{-6,\alpha}^2\right] + \mathbf{E}\int_0^T\sum_{p\geqslant 1}\left\langle \eta_{N,s}^{(\omega)},\, \mathcal{L}_s^{\nu_N}(\psi_p)\right\rangle^2\,\mathrm{d}s\\
&+\sum_{p\geqslant 1}\mathbf{E}\left[M_{N,T}^{(\omega)}(\psi_p)^2\right]\Bigg).
\end{aligned}$$

By Lemma 4.16, we have:

$$\left|\left\langle \eta_{N,s}^{(\omega)}\,,\,\mathcal{L}_s^{\nu_N}(\psi)\right\rangle\right| \leqslant C\left\|\,\eta_{N,s}^{(\omega)}\,\right\|_{-3,2\alpha}\|\,\psi\,\|_{6,\alpha}\,.$$

Then,

$$\begin{aligned}\mathbf{E}\left[\int_0^T\sum_{p\,\geqslant\,1}\left\langle \eta_{N,s}^{(\omega)}\,,\,\mathcal{L}_s^{\nu_N}(\psi_p)\right\rangle^2\,\mathrm{d}s\right] &\leqslant C^2\int_0^T\mathbf{E}\left[\left\|\,\eta_{N,s}^{(\omega)}\,\right\|_{-3,2\alpha}^2\right]\mathrm{d}s,\\ &\leqslant C^2T\sup_{s\,\leqslant\,T}\mathbf{E}\left[\left\|\,\eta_{N,s}^{(\omega)}\,\right\|_{-3,2\alpha}^2\right]\leqslant C^2TA_N,\end{aligned}$$

where A_N is defined in Proposition 4.15. The result follows. □

Proposition 4.20. *1. For every N, for $\mathbb{P}$-almost every (ω), the trajectories of the fluctuation process $\eta_N^{(\omega)}$ are almost surely continuous in $\mathcal{S}'$,*

2. For every N, for $\mathbb{P}$-almost every (ω), the trajectories of $M_N^{(\omega)}$ are almost surely continuous in $\mathcal{S}'$.

Proof of Proposition 4.20. We only prove for $M_N^{(\omega)}$, since, using Proposition 4.19, the proof is the same for $\eta_N^{(\omega)}$. Let (φ_p) be a complete orthonormal system in $W_0^{-3,2\alpha}$, then for every fixed N and (ω), we know from the proof of Lemma 4.17, that for all $\varepsilon>0$, there exists some $M_0>0$ such that

$$\sum_{p\,\geqslant\,M_0}\sup_{t\,\leqslant\,T}\left(M_{N,t}^{(\omega)}(\varphi_p)\right)^2<\frac{\varepsilon}{3},a.s.$$

Let (t_m) be a sequence in $[0,T]$ such that $t_m\to_{m\to\infty}t$.

$$\begin{aligned}\left\|\,M_{N,t_m}^{(\omega)}-M_{N,t}^{(\omega)}\,\right\|_{-3,2\alpha}^2&=\sum_{p\,\geqslant\,1}\left(M_{N,t_m}^{(\omega)}-M_{N,t}^{(\omega)}\right)^2(\varphi_p),\\ &\leqslant\sum_{p=1}^{M_0}\left(M_{N,t_m}^{(\omega)}-M_{N,t}^{(\omega)}\right)^2(\varphi_p)+\frac{2\varepsilon}{3}\leqslant\varepsilon,\end{aligned}$$

if t_m is sufficiently large. □

We are now in position to prove the tightness of the fluctuation process. Let us recall some notations: for fixed N and (ω), $\mathcal{H}_N^{(\omega)}$ is the law of the process $\eta_N^{(\omega)}$. Hence, $\mathcal{H}_N^{(\omega)}$ is an element of $\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))$, endowed with the topology of weak convergence and with $\mathcal{B}^*$, the smallest σ-algebra such that the evaluations $Q\mapsto\langle Q\,,\,f\rangle$ are measurable, f being measurable and bounded.

We will denote by Θ_N the law of the random variable $(\omega)\mapsto\mathcal{H}_N^{(\omega)}$. The main result of this part is the following:

Theorem 4.21 (Quenched tightness of the fluctuation process)**.** *The following statements are true:*

1. for $\mathbb{P}$-almost every sequence (ω), the law of the process $M_N^{(\omega)}$ is tight in the space $\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))$,

2. The law Θ_N of the sequence $(\omega)\mapsto\mathcal{H}_N^{(\omega)}$ is tight on $\mathcal{M}_1\left(\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))\right)$.

Before proving Theorem 4.21, we recall the following result and notations (cf. Mitoma [62], Th 3.1, p. 993):

Proposition 4.22 (Mitoma's criterion)**.** *Let $(P_N)_{N \geqslant 1}$ be a sequence of probability measures on $(\mathcal{C}_{\mathcal{S}'} := \mathcal{C}([0,T], \mathcal{S}'), \mathcal{B}_{C_{\mathcal{S}'}})$. For each $\varphi \in \mathcal{S}$, we denote by Π_φ the mapping of $\mathcal{C}_{\mathcal{S}'}$ to $\mathcal{C} := \mathcal{C}([0,T], \mathbf{R})$ defined by*

$$\Pi_\varphi : \psi(\cdot) \in \mathcal{C}_{\mathcal{S}'} \mapsto \langle \psi(\cdot)\,,\, \varphi \rangle \in \mathcal{C}.$$

Then, if for all $\varphi \in \mathcal{S}$, the sequence $(P_N \Pi_\varphi^{-1})_{N \geqslant 1}$ is tight in $\mathcal{C}$, the sequence $(P_N)_{N \geqslant 1}$ is tight in $\mathcal{C}_{\mathcal{S}'}$.

Remark 4.23. A closer look to the proof of Mitoma shows that it suffices to verify the tightness of $(P_N \Pi_\varphi^{-1})_{N \geqslant 1}$ for φ in a countable dense subset of the nuclear Fréchet space $(\mathcal{S}, \| \cdot \|_p\,, p \geqslant 1)$.

Thanks to Mitoma's result, it suffices to have a tightness criterion in $\mathbf{R}$. We recall here the usual result (cf. Billingsley [11]): A sequence of $(\Omega_N, \mathcal{F}_{N,t})_{N \geqslant 1}$-adapted processes $(Y_N)_{N \geqslant 1}$ with paths in $\mathcal{C}([0,T], \mathbf{R})$ is tight if both of the following conditions hold:

– Condition [T]: for all $t \leqslant T$ and $\delta > 0$, there exists $a > 0$ such that

$$\sup_N \mathbf{P}\left(|Y_{N,t}| > a\right) \leqslant \delta, \tag{$T_{t,\delta,a}$}$$

– Condition [A]: for all $\eta_1, \eta_2 > 0$, there exists $C > 0$ and N_0 such that for all $\mathcal{F}^N$-stopping times τ_N,

$$\sup_{N \geqslant N_0} \sup_{\theta \leqslant C} \mathbf{P}\left(|Y_{N,\tau_N} - Y_{N,\tau_N+\theta}| \geqslant \eta_2\right) \leqslant \eta_1. \tag{$A_{\eta_1,\eta_2,C}$}$$

Proof of Theorem 4.21. 1. Tightness of $(M_N^{(\omega)})_{N \geqslant 1}$: for a fixed realization of the disorder (ω), for fixed $\varphi \in \mathcal{S}$, we have:

– For all $t \in [0,T]$, for all $\delta > 0$, for all $a > 0$,

$$\begin{aligned}
\mathbf{P}\left(\left|M_{N,t}^{(\omega)}(\varphi)\right| > a\right) &\leqslant \frac{\mathbf{E}\left[\sup_{t \leqslant T}\left\{M_{N,t}^{(\omega)}(\varphi)^2\right\}\right]}{a^2}, \\
&\leqslant \frac{\mathbf{E}\left[\sup_{t \leqslant T}\left\| M_{N,t}^{(\omega)}\right\|_{-3,2\alpha}^2 \| \varphi \|_{3,2\alpha}^2\right]}{a^2}, \\
&\leqslant \frac{C \| \varphi \|_{3,2\alpha}^2}{a^2} \sup_N \frac{1}{N} \sum_{i=1}^N \left(1 + |\omega_i|^{4\alpha}\right), \text{(cf. (4.31))}, \\
&\leqslant \delta,
\end{aligned}$$

for a suitable $a > 0$ (depending on (ω)). Condition [T] is satisfied.

– Let us verify Condition [A]: For every $\varphi \in \mathcal{S}$, for every $\delta, \theta, \eta_1, \eta_2 > 0, \theta \leqslant \delta$, for

every stopping time τ_N,

$$\begin{aligned}
u_N &:= \mathbf{P}\left(|M_{N,\tau_N+\theta}(\varphi) - M_{N,\tau_N}(\varphi)| > \eta_2\right) \leqslant \frac{1}{\eta_2^2}\mathbf{E}\left[|M_{N,\tau_N+\theta}(\varphi) - M_{N,\tau_N}(\varphi)|^2\right], \\
&\leqslant \frac{1}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\left\langle \nu_{N,s}\,,\,\varphi'(\theta,\omega)^2\right\rangle \mathrm{d}s\right], \\
&\leqslant \|\,\varphi\,\|_{6,\alpha}^2\,\frac{1}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\int_{\mathbf{S}\times\mathbf{R}}\|\,H_{\theta,\omega}\,\|_{-6,\alpha}^2\,\mathrm{d}\nu_{N,s}\,\mathrm{d}s\right], \\
&\leqslant \|\,\varphi\,\|_{6,\alpha}^2\,\frac{C}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\frac{1}{N}\sum_{i=1}^N(1+|\omega_i|^{4\alpha})\,\mathrm{d}s\right], \text{ (cf. (4.13) and (4.17))}, \\
&\leqslant \|\,\varphi\,\|_{6,\alpha}^2\,\frac{C\delta}{\eta_2^2}\sup_N\left(\frac{1}{N}\sum_{i=1}^N(1+|\omega_i|^{4\alpha})\right).
\end{aligned}$$

This last term is lower or equal than η_1 for δ sufficiently small (depending on (ω)).

2. Tightness of $(\Theta_N)_{N\geqslant 1}$: we need to be more careful here, since the tightness is *in law w.r.t. the disorder*. Let $(\varphi_j)_{j\geqslant 1}$ be a countable family in the nuclear Fréchet space $\mathcal{S}$. Without any restriction, we can always suppose that $\|\,\phi_j\,\|_{6,\alpha} = 1$, for every $j \geqslant 1$. We define the following decreasing sequences (indexed by $J \geqslant 1$) of subsets of $\mathcal{M}_1\left(\mathcal{C}([0,T],\mathcal{S}')\right)$:

$$\begin{aligned}
K_1^\varepsilon(\varphi_1,\dots,\varphi_J) &:= \left\{P;\,\forall t, \forall 1\leqslant j\leqslant J,\ P\Pi_{\varphi_j}^{-1} \text{ satisfies } (T_{t,\delta,C_1})\right\}, \\
K_2^\varepsilon(\varphi_1,\dots,\varphi_J) &:= \left\{P;\,\forall 1\leqslant j\leqslant J, \forall \eta_1,\eta_2>0, P\Pi_{\varphi_j}^{-1} \text{ satisfies } (A_{\eta_1,\eta_2,C_2})\right\},
\end{aligned}$$

where $C_1 = C_1(\varepsilon,\delta)$, $C_2 = C_2(\varepsilon,\eta_1,\eta_2)$ will be precised later. By construction and by Mitoma's theorem (cf. Remark 4.23),

$$K^\varepsilon := \bigcap_J (K_1^\varepsilon(\varphi_1,\dots,\varphi_J)\cap K_2^\varepsilon(\varphi_1,\dots,\varphi_J))$$

is a relatively compact subset of $\mathcal{M}_1\left(\mathcal{C}([0,T],\mathcal{S}')\right)$. In order to prove tightness of (Θ_N), it is sufficient to prove that, for all $\varepsilon > 0$,

$$\forall i = 1,2,\ \limsup_N \Theta_N\left(\bigcup_J K_i^\varepsilon(\phi_1,\dots,\phi_J)^c\right) \leqslant \varepsilon. \tag{4.33}$$

For $\varepsilon > 0$, let $A = A(\varepsilon)$ such that $\liminf_{N\to\infty}\mathbb{P}\left(A_N \leqslant A\right) \geqslant 1-\varepsilon$, and

$$\liminf_{N\to\infty}\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^N(1+|\omega_i|^{4\alpha}) + A_N(\omega_1,\dots,\omega_N) \leqslant A\right) \geqslant 1-\varepsilon,$$

where A_N is the random variable defined in Proposition 4.15. We define the corresponding constants (for a sufficiently large constant C only dependent on b and c):

$$C_1(\varepsilon,\delta) := \sqrt{\frac{A(\varepsilon)}{\delta}}, \quad C_2(\varepsilon,\eta_1,\eta_2) := \frac{\eta_1\eta_2^2}{CA(\varepsilon)}.$$

Then,

$$\begin{aligned}\Theta_N(K_1^\varepsilon(\phi_1,\dots,\phi_J)) &\geqslant \mathbb{P}\left((\omega),\ \forall t,\ \forall 1\leqslant j\leqslant J,\ \forall\delta,\ \frac{\mathbf{E}\left[\left|\left\langle \eta_{N,t}^{(\omega)},\,\phi_j\right\rangle\right|^2\right]}{C_1(\delta,\varepsilon)^2}\leqslant\delta\right),\\ &\geqslant \mathbb{P}\left((\omega),\ \sup_{t\leqslant T}\mathbf{E}\left[\left\|\eta_{N,t}^{(\omega)}\right\|_{-6,\alpha}^2\right]\leqslant A\right),\ \text{(by definition of } C_1),\\ &\geqslant \mathbb{P}\left(A_N\leqslant A(\varepsilon)\right),\ \text{(cf. (4.13) and (4.24))}.\end{aligned}$$

Letting $J\to\infty$ in the latter inequality, we obtain:

$$\Theta_N(\bigcup_J K_1^\varepsilon(\phi_1,\dots,\phi_J)^c)\leqslant\mathbb{P}(A_N>A).$$

Taking on both sides $\limsup_{N\to\infty}$, we get the result.
Furthermore, for $\eta_2>0$, $0<\theta\leqslant C_2$ and $\tau_N\leqslant T$ a stopping time, for all $1\leqslant j\leqslant J$,

$$\begin{aligned}\mathbf{P}\left(\left|\int_{\tau_N}^{\tau_N+\theta}\left\langle\eta_{N,s}^{(\omega)},\,\mathcal{L}_s^{\nu_N}(\varphi_j)\right\rangle\,\mathrm{d}s\right|\geqslant\eta_2\right) &\leqslant\frac{1}{\eta_2^2}\mathbf{E}\left[\left|\int_{\tau_N}^{\tau_N+\theta}\left\langle\eta_{N,s}^{(\omega)},\,\mathcal{L}_s^{\nu_N}(\varphi_j)\right\rangle\,\mathrm{d}s\right|^2\right],\\ &\leqslant\frac{C_2}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\left|\left\langle\eta_{N,s}^{(\omega)},\,\mathcal{L}_s^{\nu_N}(\varphi_j)\right\rangle\right|^2\,\mathrm{d}s\right],\\ &\leqslant\frac{C_2}{\eta_2^2}\int_0^T\mathbf{E}\left|\left\langle\eta_{N,s}^{(\omega)},\,\mathcal{L}_s^{\nu_N}(\varphi_j)\right\rangle\right|^2\,\mathrm{d}s,\\ &\leqslant\frac{CC_2}{\eta_2^2}\int_0^T\mathbf{E}\left[\left\|\eta_{N,s}^{(\omega)}\right\|_{-3,2\alpha}^2\right]\,\mathrm{d}s,\\ &\leqslant\frac{CTC_2}{\eta_2^2}A_N,\ \text{(cf. (4.24))}.\end{aligned}$$

And,

$$\mathbf{P}\left(|M_{N,\tau_N+\theta}(\varphi_j)-M_{N,\tau_N}(\varphi_j)|>\eta_2\right)\leqslant\frac{CC_2}{\eta_2^2}\left(\frac{1}{N}\sum_{i=1}^N(1+|\omega_i|^{4\alpha})\right).$$

So, for all $j\geqslant 1$, by definition of C_2,

$$\mathbf{P}\left(\left|\eta_{N,\tau_N+\theta}^{(\omega)}(\varphi_j)-\eta_{N,\tau_N}^{(\omega)}(\varphi_j)\right|\geqslant\eta_2\right)\leqslant\frac{\eta_1}{A(\varepsilon)}\left(A_N+\frac{1}{N}\sum_{i=1}^N(1+|\omega_i|^{4\alpha})\right).$$

Consequently,

$$\Theta_N(K_2^\varepsilon(\phi_1,\dots,\varphi_J))\geqslant\mathbb{P}\left(A_N+\frac{1}{N}\sum_{i=1}^N(1+|\omega_i|^{4\alpha})>A(\varepsilon)\right).$$

Letting $J\to\infty$, we get $\limsup_N\Theta_N\left(\bigcup_J K_2^\varepsilon(\varphi_1,\dots,\varphi_J)^c\right)\leqslant\varepsilon$. Estimation (4.33) is proved.

That concludes the proof of Theorem 4.21. □

3.3.3 Identification of the limit

The proof of the fluctuation result will be complete when we identify any possible limit of $(\Theta_N)_{N \geqslant 1}$.

Proposition 4.24 (Identification of the initial value)**.** *The initial fluctuation process* $(\omega) \mapsto \mathcal{L}\left(\eta_{N,0}^{(\omega)}\right)$ *converges in law to the random variable* $\omega \mapsto \mathcal{L}(X^\omega)$*, where for all* ω*,* $X^\omega = C(\omega) + Y$*, with* Y *a centered Gaussian process with covariance* Γ_1*. Moreover* $\omega \mapsto C(\omega)$ *is a Gaussian process with covariance* Γ_2*, where* Γ_1 *and* Γ_2 *are defined in* (4.7) *and* (4.8)*.*

Proof of Proposition 4.24. For simplicity, we only identify here the law of $\left\langle \eta_{N,0}^{(\omega)}, \varphi \right\rangle$ for all φ. The same proof works for the law of finite-dimensional vectors, (where $p \geqslant 1$) $\left(\left\langle \eta_{N,0}^{(\omega)}, \varphi_1 \right\rangle, \dots, \left\langle \eta_{N,0}^{(\omega)}, \varphi_p \right\rangle\right)$. We write Γ_i for $\Gamma_i(\varphi, \varphi)$, $i = 1, 2$. One has:

$$\begin{aligned}
\left\langle \eta_{N,0}^{(\omega)}, \varphi \right\rangle &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(\varphi(\xi_i, \omega_i) - \int_{\mathbf{S}} \varphi(\theta, \omega_i) \, \mathrm{d}\gamma(\theta) \right) \\
&\quad + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(\int_{\mathbf{S}} \varphi(\theta, \omega_i) \, \mathrm{d}\gamma(\theta) - \langle \nu_0, \varphi \rangle \right), \\
&=: A_N^{(\omega)} + B_N^{(\omega)}.
\end{aligned}$$

It is easy to see that $B_N^{(\omega)}$ converges in law to $Z_2 \sim \mathcal{N}(0, \Gamma_2)$. Moreover, for $\mathbb{P}$-almost every (ω), $A_N^{(\omega)}$ converges in law to $Z_1 \sim \mathcal{N}(0, \Gamma_1)$ (see Billingsley [10], Th. 27.3 p. 362). That means that for all $u \in \mathbf{R}$, $\psi_{A_N}(u) := \mathbf{E}_\gamma\left(e^{iuA_N^{(\omega)}}\right)$ converges to $\psi_Y(u) := e^{-\frac{u^2}{2\Gamma_1}}$. But, then, for all $F \in \mathcal{C}_b(\mathbf{R})$,

$$\mathbb{E}\left[F\left(\mathbf{E}_\gamma\left[e^{iu\left\langle \eta_{N,0}^{(\omega)}, \varphi \right\rangle}\right]\right)\right] = \mathbb{E}\left[F\left(\mathbf{E}_\gamma\left[e^{iu(A_N^{(\omega)} + B_N^{(\omega)})}\right]\right)\right] = \mathbb{E}\left[F\left(e^{iuB_N^{(\omega)}} \psi_N(u)\right)\right].$$

Since $\psi_N(u)$ converges almost surely to a constant, the limit of the expression above exists (Slutsky's theorem [44, p. 318]) and is equal to $\mathbb{E}\left[F\left(e^{iuZ_2 - \frac{u^2}{2\Gamma_1}}\right)\right]$. □

Proposition 4.25 (Identification of the martingale part)**.** *For* $\mathbb{P}$*-almost every* (ω)*, the sequence* $(M_N^{(\omega)})_{N \geqslant 1}$ *converges in law in* $\mathcal{C}([0,T], \mathcal{S}')$ *to a Gaussian process* W *with covariance defined in* (4.6)*.*

Proof of Proposition 4.25. For fixed (ω), $(M_N^{(\omega)})_{N \geqslant 1}$ is a sequence of uniformly square-integrable continuous martingales (Remark 4.18), which is tight in $\mathcal{C}([0,T], \mathcal{S}')$. Let W_1 and W_2 be two accumulation points (continuous square-integrable martingales which *a priori* depend on (ω)) and $\left(M_{\phi(N)}^{(\omega)}\right)_{N \geqslant 1}$ and $\left(M_{\psi(N)}^{(\omega)}\right)_{N \geqslant 1}$ be two subsequences converging to W_1 and W_2, respectively. Note that we can suppose that $\phi(N) \leqslant \psi(N)$ for all N. For all $\varphi \in \mathcal{S}$, $\lim_{N \to \infty} \left\langle M_{\phi(N)}^{(\omega)}(\varphi), M_{\psi(N)}^{(\omega)}(\varphi) \right\rangle_t = \langle W_1(\varphi), W_2(\varphi) \rangle_t$, for all t, and

$$\left\langle M_{\phi(N)}^{(\omega)}(\varphi), M_{\psi(N)}^{(\omega)}(\varphi) \right\rangle_t = \int_0^t \left\langle \nu_{\phi(N),s}, (\varphi')^2 \right\rangle \mathrm{d}s.$$

We now have to identify the limit: we already know that for $\mathbb{P}$-almost every realization of the disorder (ω), $(\nu_N^{(\omega)})_{N \geqslant 1}$ converges in law to ν. But, the latter expression, seen

as a function of ν, is continuous. So $\langle W_1(\varphi)\,,\,W_2(\varphi)\rangle_t = \int_0^t \langle \nu_s\,,\,(\varphi')^2\rangle$. So $W_1 - W_2$ is a continuous square integrable martingale whose Doob-Meyer process is 0. So $W_1 = W_2$ and is characterized as the Gaussian process with covariance given in (4.6). The convergence follows. □

Proposition 4.26. *The Gaussian process W introduced in Proposition 4.25 is independent with the initial condition X introduced in Proposition 4.24.*

Proof of Proposition 4.26. We prove more: the triple (Y, C, W) is independent. For sake of simplicity, we only consider the case of $(Y(\varphi), C(\varphi), W_t(\varphi))$ for fixed t and φ.

Let us first recall some notations: let $A_N^{(\omega)}$, $B_N^{(\omega)}$ and $M_{N,t}^{(\omega)}(\varphi)$ be the random variables defined in the proof of Proposition 4.24 and 4.25 and let $\psi_{A_N}(u) := \mathbf{E}\left(e^{iuA_N^{(\omega)}}\right)$, $\psi_{B_N}(v) := \mathbf{E}\left(e^{ivB_N^{(\omega)}}\right)$, $\psi_{M_N}(w) := \mathbf{E}\left(e^{iwM_{N,t}^{(\omega)}(\varphi)}\right)$ be their characteristic functions $(u, v, w \in \mathbf{R})$. We know that, for almost every (ω), $\psi_{A_N}(u)$ converges to $\psi_Y(u) = e^{-\frac{u^2}{2\Gamma_1}}$ and that $\psi_{M_N}(w)$ converges to the deterministic function $\psi_W(w) := \mathbf{E}\left(e^{iwW_t(\varphi)}\right)$. But, if $\psi_C(v) = \mathbb{E}\left(e^{iwC}\right)$, then, for all $u, v, w \in \mathbf{R}$, using the independence of the Brownian motions with the initial conditions,

$$\mathbb{E}\left(\mathbf{E}\left(e^{iuA_N^{(\omega)}+ivB_N^{(\omega)}+iwM_{N,t}^{(\omega)}(\varphi)}\right) - e^{ivB_N^{(\omega)}}\psi_{A_N}(u)\psi_{M_N}(w)\right) = 0.$$

Using Slutsky's theorem, we see that any limit couple (Y, C, W) satisfies

$$\mathbb{E}\left(\mathbf{E}\left(e^{iuY+ivC+iwW_t(\varphi)}\right)\right) = \psi_Y(u)\psi_C(v)\psi_W(w).$$

which is the desired result. □

We now turn to the characterization of the accumulation points of the sequence $(\Theta_N)_{N\geqslant 1}$. We recall that the limit second order differential operator $\mathcal{L}_s$ is defined by

$$\mathcal{L}_s(\varphi)(\theta,\omega) := \frac{1}{2}\varphi''(\theta,\omega) + \varphi'(\theta,\omega)(b[\theta,\omega,\nu_s] + c(\theta,\omega)) + \langle \nu_s\,,\,\varphi'(\cdot,\cdot)b(\cdot,\theta,\cdot,\omega)\rangle\,.$$

As in Lemma 4.16, we can prove the following:

Lemma 4.27. *Assume* (4.3). *Then for every N, $s \in [0,T]$, (ω), the operator $\mathcal{L}_s$ and $\mathcal{L}_s^{\nu_N}$ are linear continuous from $\mathcal{S}$ to $\mathcal{S}$ and*

$$\|\,\mathcal{L}_s(\varphi)\,\|_{6,\alpha} \leqslant C\,\|\,\varphi\,\|_{8,\alpha}\,,$$

$$\|\,\mathcal{L}_s^{\nu_N}(\varphi)\,\|_{6,\alpha} \leqslant C\,\|\,\varphi\,\|_{8,\alpha}\,.$$

We are now in position to prove Theorem 4.4:

Proof of Theorem 4.4. Let Θ be an accumulation point of Θ_N. Thus, for a certain subsequence (which will be also denoted as N for notations purpose), the random variable $(\omega) \mapsto \mathcal{H}_N^{(\omega)}$ converges in law to a random variable $\mathcal{H}$ with values in $\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))$ with law Θ. Applying Skorohod's representation theorem, there exists some probability space $(\Omega^{(1)}, \mathbf{P}^{(1)}, \mathcal{F}^{(1)})$ and random variables defined on $\Omega^{(1)}$, $\omega_1 \mapsto H_N^{\omega_1}$ and $\omega_1 \mapsto H^{\omega_1}$ such that H_N has the same law as $(\omega) \mapsto \mathcal{H}_N^{(\omega)}$, H has the same law as $\mathcal{H}$, and for $\mathbf{P}^{(1)}$-almost every $\omega_1 \in \Omega^{(1)}$, $H_N^{\omega_1}$ converges to H^{ω_1} in $\mathcal{M}_1(\mathcal{C}([0,T],\mathcal{S}'))$.

An easy application of Proposition 4.19 and Borel-Cantelli's Lemma shows that $\mathbf{P}^{(1)}$-almost surely, $\mathbf{E}\left(\sup_{t\leqslant T}\|\eta_t^{\omega_1}\|_{-6,\alpha}\right)<\infty$. Then we know from Lemma 4.27 that the integral term $\int_0^t \mathcal{L}_s^*\eta_s^{\omega_1}\,\mathrm{d}s$ makes sense as a Bochner's integral in $W_0^{-8,\alpha}\subseteq\mathcal{S}'$.

Let $\eta_N^{\omega_1}$ with law $H_N^{\omega_1}$; $\eta_N^{\omega_1}$ converges in law to some η^{ω_1} with law H^{ω_1}. By uniqueness in law convergence, using Propositions 4.24 and 4.25, we see that $(\eta_0^{\omega_1},W)$ as the same law as (X^{ω_1},W). For fixed $\varphi\in\mathcal{S}$, we define F_φ from $\mathcal{C}([0,T],\mathcal{S}')$ into $\mathbf{R}$ by $F_\varphi(\gamma):=\langle\gamma_t\,,\,\varphi\rangle-\langle\gamma_0\,,\,\varphi\rangle-\int_0^t\langle\gamma_s\,,\,\mathcal{L}_s\varphi\rangle\,\mathrm{d}s$. The function F_φ is continuous and since $\eta_N^{\omega_1}$ converges in law to η^{ω_1}, the sequence $(F_\varphi(\eta_N^{\omega_1}))_{N\geqslant 1}$ converges in law to $F_\varphi(\eta^{\omega_1})$. To prove the theorem, it remains to show that the law of the term $\int_0^t\left\langle\eta_{N,s}^{\omega_1}\,,\,\mathcal{L}_s^{\nu_N}\varphi-\mathcal{L}_s\varphi\right\rangle\mathrm{d}s$ converges in law to 0. We show that there is convergence in probability: For all $\varepsilon>0$, for all $A>0$, using Proposition 4.19, Lemma 4.27, and Cauchy-Schwarz's inequality,

$$\begin{aligned}
U_{N,\varepsilon}&:=\mathbf{P}^{(1)}\left(\mathbf{E}\left[\int_0^t\left|\left\langle\eta_{N,s}^{\omega_1}\,,\,(\mathcal{L}_s^{\nu_N}-\mathcal{L}_s)(\varphi)\right\rangle\right|\mathrm{d}s\right]>\varepsilon\right),\\
&=\mathbb{P}\left(\mathbf{E}\left[\int_0^t\left|\left\langle\eta_{N,s}^{(\omega)}\,,\,(\mathcal{L}_s^{\nu_N}-\mathcal{L}_s)(\varphi)\right\rangle\right|\mathrm{d}s\right]>\varepsilon\right),\\
&\leqslant\mathbb{P}\left(\int_0^t\mathbf{E}\left[\left\|\eta_{N,s}^{(\omega)}\right\|_{-6,\alpha}^2\right]^{1/2}\mathbf{E}\left[\|(\mathcal{L}_s^{\nu_N}-\mathcal{L}_s)(\varphi)\|_{6,\alpha}^2\right]^{1/2}\mathrm{d}s>\varepsilon\right),\\
&\leqslant\mathbb{P}\left(C_N(\omega_1,\dots,\omega_N)^{1/2}\int_0^t\mathbf{E}\left[\|(\mathcal{L}_s^{\nu_N}-\mathcal{L}_s)(\varphi)\|_{6,\alpha}^2\right]^{1/2}\mathrm{d}s>\varepsilon\right)\ \text{(cf. Prop 4.19)},\\
&\leqslant\mathbb{P}\left(\int_0^t\mathbf{E}\left[\|(\mathcal{L}_s^{\nu_N}-\mathcal{L}_s)(\varphi)\|_{6,\alpha}^2\right]^{1/2}\mathrm{d}s>\frac{\varepsilon}{\sqrt{A}}\right)+\mathbb{P}\left(C_N>A\right).
\end{aligned}$$

Using (4.19), it suffices to prove that, for all $\varepsilon>0$,

$$\limsup_{N\to\infty}\mathbb{P}\left(\int_0^t\mathbf{E}\left[\|(\mathcal{L}_s^{\nu_N}-\mathcal{L}_s)(\varphi)\|_{6,\alpha}^2\right]^{1/2}\mathrm{d}s>\varepsilon\right)=0. \tag{4.34}$$

Indeed, for every $\varphi\in\mathcal{S}$,

$$\mathcal{U}_{N,s}(\varphi)(\theta,\omega):=(\mathcal{L}_s^{\nu_N}-\mathcal{L}_s)(\varphi)(\theta,\omega)=\varphi'(\theta,\omega)(b[\theta,\omega,\nu_{N,s}]-b[\theta,\omega,\nu_s]).$$

An analogous calculation as in Lemma 4.16 shows that, using Lipschitz assumptions on b, and Proposition 4.14:

$$\mathbf{E}\left[\left\|\sup_{s\leqslant t}\mathcal{U}_s^N(\varphi)\right\|_{6,\alpha}^2\right]\leqslant\|\varphi\|_{8,\alpha}^2\,(C/N+D_N(\omega_1,\dots,\omega_N)),$$

with the property that $\lim_{A\to\infty}\limsup_N\mathbb{P}(ND_N>A)=0$. Equation (4.34) is a direct consequence.

Since there is uniqueness in law in (4.9), Θ is perfectly defined, and thus, unique. The convergence follows. □

4 Proofs for the fluctuations of the order parameters

We end by the proofs of paragraph 2.5.

4.1 Proof of Proposition 4.9

1. This is straightforward since $r_N^{(\omega)} = \left|\int_{\mathbf{S}\times\mathbf{R}} e^{i\theta}\,\mathrm{d}\nu_N^{(\omega)}\right|$ and since for $\mathbb{P}$-almost every disorder (ω), $\nu_N^{(\omega)}$ converges weakly to ν.
2. The following sequences are well defined: $\forall k \geqslant 0$,

$$u_k(t) := \int_{\mathbf{S}\times\mathbf{R}} e^{-|\omega|}\omega^k \cos(\theta)\,\mathrm{d}\nu_t(\theta,\omega),$$
$$v_k(t) := \int_{\mathbf{S}\times\mathbf{R}} e^{-|\omega|}\omega^k \sin(\theta)\,\mathrm{d}\nu_t(\theta,\omega).$$

Let $E = (\ell_\infty(\mathbf{N}), \|\cdot\|_\infty)$ be the Banach space of real bounded sequences endowed with its usual $\|\cdot\|_\infty$ norm, ($\|u\|_\infty = \sup_{k\geqslant 0}|u_k|$). For all $t > 0$, let $\mathcal{A}_t : E\times E \to E\times E$, be the following linear operator (where (u,v) is a typical element of $E\times E$): For all $k \geqslant 0$

$$\begin{cases} \mathcal{A}_t(u,0)_k &= -\frac{1}{2}u_k - \alpha_k(t)v_0 + \beta_k(t)u_0 - Kv_{k+1}, \\ \mathcal{A}_t(0,v)_k &= -\frac{1}{2}v_k + \gamma_k(t)v_0 - \alpha_k(t)u_0 + Ku_{k+1}, \end{cases}$$

where,

$$\alpha_k(t) = \int_{\mathbf{S}\times\mathbf{R}} e^{-|\omega|}\omega^k \cos(\cdot)\sin(\cdot)\,\mathrm{d}\nu_t,$$
$$\beta_k(t) = \int_{\mathbf{S}\times\mathbf{R}} e^{-|\omega|}\omega^k \sin^2(\cdot)\,\mathrm{d}\nu_t,$$
$$\gamma_k(t) = \int_{\mathbf{S}\times\mathbf{R}} e^{-|\omega|}\omega^k \cos^2(\cdot)\,\mathrm{d}\nu_t.$$

$(t,u,v) \mapsto \mathcal{A}_t\cdot(u,v)$ is globally Lipschitz-continuous map from $[0,T]\times E\times E$ into $E\times E$ and one easily verifies considering (2.11) (in the case of the *sine-model*) and developing the sine interaction that $t \mapsto (u(t),v(t))$ satisfies in $E\times E$ the following linear inhomogeneous Cauchy Problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u(t),v(t)) &= \mathcal{A}_t\cdot(u(t),v(t)), \\ u_k(0) &= \int_{\mathbf{S}\times\mathbf{R}} e^{-|\omega|}\omega^k\cos(\cdot)\,\mathrm{d}\nu_0,\ \forall k\geqslant 0 \\ v_k(0) &= \int_{\mathbf{S}\times\mathbf{R}} e^{-|\omega|}\omega^k\sin(\cdot)\,\mathrm{d}\nu_0,\ \forall k\geqslant 0. \end{cases}$$

Let us suppose that there exists some $t_0 \in [0,T]$ such that $r_{t_0} = 0$, namely $u_0(t_0) = v_0(t_0) = 0$. Then, if $(\tilde{u},\tilde{v})$ is the constant function on $[0,T]$ such that for all $k \geqslant 0$, $\tilde{u}_k \equiv u_k(t_0), \tilde{v}_k \equiv v_k(t_0)$, then $(\tilde{u},\tilde{v})$ satisfy the same Cauchy Problem as (u,v) with initial condition at time t_0. By Cauchy-Lipschitz theorem, both functions coincide on $[0,T]$. In particular, u_0 and v_0 are always zero and thus $r \equiv 0$.
3. We suppose (4.10). A simple calculation shows that the fluctuation process $\mathcal{R}_N$ defined in (4.11) verifies for all $t \in [0,T]$,

$$\mathcal{R}_{N,t}^{(\omega)} = \frac{\left\langle \eta_{N,t}^{(\omega)}\,,\,\cos(\cdot)\right\rangle\left\langle \nu_{N,t}^{(\omega)}+\nu_t\,,\,\cos(\cdot)\right\rangle + \left\langle \eta_{N,t}^{(\omega)}\,,\,\sin(\cdot)\right\rangle\left\langle \nu_{N,t}^{(\omega)}+\nu_t\,,\,\sin(\cdot)\right\rangle}{r_{N,t}^{(\omega)} + r_t},$$
$$= \frac{\Re\left(\left\langle \eta_{N,t}^{(\omega)}\,,\,e^{i\theta}\right\rangle\overline{\left\langle \nu_{N,t}^{(\omega)}+\nu_t\,,\,e^{i\theta}\right\rangle}\right)}{\left|\left\langle \nu_{N,t}^{(\omega)}\,,\,e^{i\theta}\right\rangle\right| + r_t}.$$

Let $u_N^{(\omega)} := \left\langle \nu_N^{(\omega)}\,,\, e^{i\theta}\right\rangle$, $v_N^{(\omega)} := \left\langle \eta_N^{(\omega)}\,,\, e^{i\theta}\right\rangle$ and $u := \left\langle \nu\,,\, e^{i\theta}\right\rangle$, $v^{\omega} := \left\langle \eta^{\omega}\,,\, e^{i\theta}\right\rangle$ be their corresponding limits. The result follows if we prove the following property: the random variables $(\omega) \mapsto \mathcal{L}\left(u^{N,(\omega)}, v^{N,(\omega)}\right)$ converges in law to the random variable $\omega \mapsto \mathcal{L}\left(u, v^{\omega}\right)$. The tightness of this random variable follows from the convergence of both empirical measure and fluctuation process. As already said in Remark 4.10, it suffices to prove the convergence of the finite-dimensional marginals $(\underline{u}_{\underline{t}}^{N,(\omega)}, \underline{v}_{\underline{t}}^{N,(\omega)}) = \left((u_{t_1}^{N,(\omega)}, \ldots, u_{t_p}^{N,(\omega)}), (v_{t_1}^{N,(\omega)}, \ldots, v_{t_p}^{N,(\omega)})\right)$, for all element of $[0,T]$, $t_1, \ldots, t_p$, $p \geqslant 1$.

Since the limit of $(u_N^{(\omega)})_{N \geqslant 1}$ is a constant, this is mainly a consequence of Slutsky's theorem. But since this is a convergence *in law with respect to the disorder*, one has to adapt the proof. We prove the following: $\forall G \in \mathcal{C}_b^1(\mathbf{R})$, $\forall \underline{r} = (r_1, \ldots, r_p) \in \mathbf{R}^p$, $\forall \underline{s} = (s_1, \ldots, s_p) \in \mathbf{R}^p$,

$$\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{N,\underline{t}}^{(\omega)}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right)\right] \underset{N\to\infty}{\to} \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right],$$

where $\varphi_{(\underline{X},\underline{Y})}(\underline{r}, \underline{s}) = \mathbf{E}\left[e^{i\underline{r}\cdot\underline{X} + i\underline{s}\cdot\underline{Y}}\right]$ is the characteristic function of the couple $(\underline{X}, \underline{Y})$. Indeed, we have successively:

$$\begin{aligned}
a_N &:= \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{N,\underline{t}}^{(\omega)}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|, \\
&\leqslant \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{N,\underline{t}}^{(\omega)}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right)\right]\right| \\
&+ \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|, \\
&\leqslant C\mathbb{E}\left|\varphi_{(\underline{u}_{N,\underline{t}}^{(\omega)}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s}) - \varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right| \\
&+ \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|, \\
&\leqslant C\mathbb{E}\mathbf{E}\left|e^{i\underline{r}\cdot\underline{u}_{N,\underline{t}}^{(\omega)}} - e^{i\underline{r}\cdot\underline{u}_{\underline{t}}}\right| \\
&+ \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{N,\underline{t}}^{(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|.
\end{aligned}$$

But, we have $\mathbf{E}\left|e^{i\underline{r}_{\underline{t}}\cdot\underline{u}_{N,\underline{t}}^{(\omega)}} - e^{i\underline{r}\cdot\underline{u}_{\underline{t}}}\right| \leqslant \min\left(2, |\underline{r}||\underline{u}_{N,\underline{t}}^{(\omega)} - \underline{u}_{\underline{t}}|\right)$. So, for all $\varepsilon > 0$,

$$\mathbf{E}\left|e^{i\underline{r}\cdot\underline{u}_{N,\underline{t}}^{(\omega)}} - e^{i\underline{r}\cdot\underline{u}_{\underline{t}}}\right| \leqslant \varepsilon|\underline{r}| + 2\mathbf{P}\left(\left|\underline{u}_{N,\underline{t}}^{(\omega)} - \underline{u}_{\underline{t}}\right| > \varepsilon\right).$$

Taking $\limsup_{N\to\infty}$, and letting $\varepsilon \to 0$, we get $\lim a_N = 0$. The result follows. $\square$

4.2 Proof of Proposition 4.11

The proof of Proposition 4.11 is really similar to the previous one and thus, will be omitted. Note that it relies on the two following straightforward equalities:

$$\zeta_N^{(\omega)} = \frac{\left\langle \nu\,,\, e^{i\theta} \right\rangle}{r_N^{(\omega)}},$$
$$\sqrt{N}\left(\zeta_N^{(\omega)} - \zeta\right) = \frac{1}{r \cdot r_N^{(\omega)}}\left(r\left\langle \eta_N^{(\omega)}\,,\, e^{i\theta}\right\rangle + \left\langle \nu\,,\, e^{i\theta}\right\rangle \mathcal{R}_N^{(\omega)}\right).$$

5 Proof of Proposition 4.14

We prove the main technical result concerning the approximation (4.22) of the oscillators $(\theta_i)_{1 \leqslant i \leqslant N}$ in (4.2) by the nonlinear process $(\bar{\theta}_i)_{1 \leqslant i \leqslant N}$ defined in (4.19).

Thanks to the Lipschitz continuity of b and c, introducing $\bar{\nu}_N$ as the empirical measure corresponding to $(\bar{\theta}_i, \omega_i)$, we have, (inserting $b[\bar{\theta}_{i,s}, \omega_i, \nu_{N,s}] - b[\bar{\theta}_{i,s}, \omega_i, \bar{\nu}_{N,s}]$ in the b term),

$$\begin{aligned}
\mathbf{E}\left[\sup_{s \leqslant t}\left|\theta_{i,s} - \bar{\theta}_{i,s}\right|^2\right] &\leqslant C\left(\int_0^t \mathbf{E}\left[\left(b[\theta_{i,s}, \omega_i, \nu_{N,s}] - b[\bar{\theta}_{i,s}, \omega_i, \nu_s]\right)^2\right] \mathrm{d}s\right. \\
&\quad \left. + \int_0^t \mathbf{E}\left[\left(c(\theta_{i,s}, \omega_i) - c(\bar{\theta}_{i,s}, \omega_i)\right)^2\right] \mathrm{d}s\right), \\
&\leqslant C\left(2\int_0^t \mathbf{E}\left[\sup_{u \leqslant s}\left|\theta_{i,u} - \bar{\theta}_{i,u}\right|^2\right] \mathrm{d}s\right. \\
&\quad + \int_0^t \sup_{1 \leqslant j \leqslant N} \mathbf{E}\left[\sup_{u \leqslant s}\left|\bar{\theta}_{j,u} - \theta_{j,u}\right|^2\right] \mathrm{d}s \\
&\quad \left. + \int_0^t \mathbf{E}\left[\left(b[\bar{\theta}_{i,s}, \omega_i, \bar{\nu}_{N,s}] - b[\bar{\theta}_{i,s}, \omega_i, \nu_s]\right)^2\right] \mathrm{d}s\right).
\end{aligned}$$

Applying Gronwall's Lemma to $\sup_{1 \leqslant j \leqslant N} \mathbf{E}\left[\sup_{u \leqslant t}\left|\bar{\theta}_{j,u} - \theta_{j,u}\right|^2\right]$, it suffices to prove that for some random variable Z_N in $(\omega_1, \ldots, \omega_N)$:

$$\int_0^t \mathbf{E}\left[\left(b[\bar{\theta}_{i,s}, \omega_i, \bar{\nu}_{N,s}] - b[\bar{\theta}_{i,s}, \omega_i, \nu_s]\right)^2\right] \mathrm{d}s \leqslant C/N + Z_N(\omega_1, \ldots, \omega_N).$$

Indeed, for all $1 \leqslant i \leqslant N$:

$$u_{i,N} := \left(b[\bar{\theta}_{i,s}, \omega_i, \bar{\nu}_{N,s}] - b[\bar{\theta}_{i,s}, \omega_i, \nu_s]\right)^2 = \frac{1}{N^2}\left(\sum_{j=1}^N T(\bar{\theta}_i, \bar{\theta}_j)^2 + \sum_{k \neq l} T(\bar{\theta}_i, \bar{\theta}_k) T(\bar{\theta}_i, \bar{\theta}_l)\right),$$

where $T(\bar{\theta}_i, \bar{\theta}_j) := b(\bar{\theta}_{i,s}, \bar{\theta}_{j,s}, \omega_i, \omega_j) - \int_{\mathbf{S}\times\mathbf{R}} b(\bar{\theta}_{i,s}, \theta, \omega_i, \omega)\, \mathrm{d}\nu_s(\theta, \omega)$. Since b is bounded, we see that the first term is of order $(1/N)$. We only have to study the remaining term:

$$\mathbf{E}\left[\sum_{k \neq l} T(\bar{\theta}_i, \bar{\theta}_k) T(\bar{\theta}_i, \bar{\theta}_l)\right] \leqslant CN + \mathbf{E}\left[\sum_{\substack{k \neq i, l \neq i \\ k \neq l}} T(\bar{\theta}_i, \bar{\theta}_k) T(\bar{\theta}_i, \bar{\theta}_l)\right].$$

Since the $(\bar{\theta}_i)$ are independent, if we take conditional expectation w.r.t. $(\bar{\theta}_r, r \neq l)$ in the last term, we get:

$$\mathbf{E}\left[\sum_{\substack{k\neq i, l\neq i\\ k\neq l}} T(\bar{\theta}_i,\bar{\theta}_k)T(\bar{\theta}_i,\bar{\theta}_l)\right] = \mathbf{E}\left[\mathbf{E}\left[\sum T(\bar{\theta}_i,\bar{\theta}_k)T(\bar{\theta}_i,\bar{\theta}_l)\middle|\,\bar{\theta}_r, r\neq l\right]\right],$$

$$= \mathbf{E}\left[\sum_{\substack{k\neq i, l\neq i\\ k\neq l}} T(\bar{\theta}_i,\bar{\theta}_k)G_l(\bar{\theta}_i)\right] = \mathbf{E}\left[\sum_{\substack{k\neq i, l\neq i\\ k\neq l}} G_k(\bar{\theta}_i)G_l(\bar{\theta}_i)\right],$$

where $G_l(\bar{\theta}) = G(\bar{\theta},\omega_i,\omega_l) = \int_{\mathbf{S}} b(\bar{\theta},\theta',\omega_i,\omega_l)\,\mathrm{d}\nu_s^{\omega_l}(\theta') - \int_{\mathbf{S}\times\mathbf{R}} b(\bar{\theta},\theta',\omega_i,\omega)\,\mathrm{d}\nu_s(\theta',\omega)$. If one defines

$$Z_N(\omega_1,\ldots,\omega_N) := \frac{C}{N}\int_0^T \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{l=1}^N G(\bar{\theta}_{i,s},\omega_i,\omega_l)\right)^2\right]\mathrm{d}s,$$

in order to prove (4.23) it suffices to show that for some constant C,

$$\mathbb{E}\left[\int_0^T \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{l=1}^N G(\theta_{i,s},\omega_i,\omega_l)\right)^2\right]\mathrm{d}s\right] \leqslant C.$$

The rest of the proof is devoted to prove this last assertion: we have successively (setting $U_N(\bar{\theta}_{i,s},\underline{\omega}) := \frac{1}{\sqrt{N}}\sum_{l=1}^N G(\theta_{i,s},\omega_i,\omega_l)$)

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \mathbf{E}\left[U_N(\theta_{i,s},\underline{\omega})^2\right]\mathrm{d}s\right] &\leqslant \int_0^T \mathbf{E}\left[\mathbb{E}\left[U_N(\theta_{i,s},\underline{\omega})^2\right]\right]\mathrm{d}s,\\
&\leqslant \frac{1}{N}\int_0^T \mathbf{E}\left[\mathbb{E}\left[\sum_{k=1}^N\sum_{l=1}^N G(\theta_{i,s},\omega_i,\omega_k)G(\bar{\theta}_{i,s},\omega_i,\omega_l)\right]\right]\mathrm{d}s,\\
&\leqslant \frac{1}{N}\int_0^T \mathbf{E}\left[\mathbb{E}\left[\sum_{l=1}^N G(\bar{\theta}_{i,s},\omega_i,\omega_l)^2\right]\right]\mathrm{d}s + C\\
&+ \frac{1}{N}\int_0^T \mathbf{E}\left[\mathbb{E}\left[\sum_{l\neq k,\ l\neq i,\ k\neq i} G(\theta_{i,s},\omega_i,\omega_k)G(\bar{\theta}_{i,s},\omega_i,\omega_l)\right]\right]\mathrm{d}s.
\end{aligned}$$

The first term of the RHS of the last inequality is bounded, since b is bounded. But, if we condition w.r.t. ω_r for $r \neq i, r \neq k$, we see that the second term is zero. The result follows. □

Chapter 5

Non self-averaging fluctuations in the sine-model

The main material is taken from an article [56], submitted for publication to *Journal of Functional Analysis*.

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The purpose of this chapter is to address the issue of the long time behavior of the fluctuation process η found in Chapter 4, in the case of the *sine-model*.

1 Introduction

1.1 The fluctuation process

For the ease of exposition, we recall some notations already used in the previous chapters. In Chapter 4 we proved that the fluctuation process

$$\eta^{N,(\omega)} := \sqrt{N}\left(\nu^{N,(\omega)} - \nu\right)$$

converges (in a weak sense) to the unique solution in the space of continuous processes with values in $\mathcal{S}'(\mathbf{S}\times\mathbf{R})$ of the following stochastic partial differential equation:

$$\eta_t = \eta_0^\omega + \int_0^t \mathcal{L}_s^* \eta_s \,\mathrm{d}s + W_t, \tag{5.1}$$

where

- $\mathcal{S}'(\mathbf{S}\times\mathbf{R})$ is understood as the usual space of tempered distributions on $\mathbf{S}\times\mathbf{R}$,
- the variable ω captures the dependence on the disorder in the $N\to\infty$-limit,
- W is a Gaussian process with covariance

$$\mathbf{E}(W_t(\varphi_1)W_s(\varphi_2)) = \int_0^{s\wedge t}\int_{\mathbf{S}\times\mathbf{R}} \varphi_1'(\theta,\omega)\varphi_2'(\theta,\omega)q_u(\theta,\omega)\,\mathrm{d}\theta\mu(\,\mathrm{d}\omega)\,\mathrm{d}u. \tag{5.2}$$

- $\mathcal{L}_t^*$ is the formal adjoint of the second order differential operator:

$$\mathcal{L}_s(\varphi)(\theta,\omega) := \frac{1}{2}\varphi''(\theta,\omega) + \varphi'(\theta,\omega)(b[\theta,\omega,\nu_s] + c(\theta,\omega)) + \left\langle \nu_s\,,\, \varphi'(\cdot,\cdot)b(\cdot,\theta,\cdot,\omega)\right\rangle. \tag{5.3}$$

- For fixed ω, η_0^ω is a Gaussian process with given covariance and with *nontrivial mean value* $C(\omega)$. As a function in ω, $C(\omega)$ is itself a Gaussian process,
- the initial condition η_0^ω is independent with W.

For the rest of this chapter, we restrict ourselves to the case where

$$\mu = \frac{1}{2}\left(\delta_{-\omega_0} + \delta_{\omega_0}\right), \tag{5.4}$$

where $\omega_0 > 0$ is a fixed parameter. This assumption (5.4) on μ appears to be quite restrictive, but generalizing parts of the results we present here to more general distributions μ does not seem to be straightforward.

1.2 Microscopic interpretation of the initial condition

We insist on the fact that the only dependence in the disorder in (5.1) lies in the initial condition η_0^ω and that η_0^ω is for fixed ω, a Gaussian process on $\mathcal{S}'(\mathbf{S}\times\mathbf{R})$ with a non-trivial mean $C(\omega)$. As a function of ω, $C(\cdot)$ is itself a Gaussian process on $\mathcal{S}'(\mathbf{S}\times\mathbf{R})$ with covariance

$$\Gamma_C(\varphi_1,\varphi_2) = \mathrm{Cov}_\mu\left(\int_{\mathbf{S}}\varphi_1(\cdot,\omega)\,\mathrm{d}\gamma, \int_{\mathbf{S}}\varphi_2(\cdot,\omega)\,\mathrm{d}\gamma\right),$$

where we recall that γ is the initial law of the rotators and φ_1, φ_2 are functions from $\mathbf{S} \times \mathbf{R}$ to $\mathbf{R}$. What is important to understand here is that the random variable $C(\cdot)$ precisely captures the fluctuations of the disorder (recall § 6.5.2, p. 34 in the main introduction of the thesis). In particular, one easily sees from the covariance Γ_C that $C(\cdot) \equiv 0$ if $\mu = \delta_0$: fluctuations of the disorder lead to an initial condition η_0^ω in (5.1) with non-trivial mean-value $C(\omega)$.

Under the framework of (5.4), the test functions φ can be identified with *couples of functions* $(\psi_+, \psi-)$ where $\psi_\pm(\cdot) = \varphi(\cdot, \pm\omega_0)$. As we will see in Theorem 5.3, the relevant object is in fact the process $(C_+(\psi))$ (indexed by functions $\psi : \mathbf{S} \to \mathbf{R}$) that is the restriction of the process C to the component on $+\omega_0$:

$$\forall \psi : \mathbf{S} \to \mathbf{R},\ C_+(\psi) := C((\psi, 0)). \tag{5.5}$$

As already mentioned in the main introduction (recall § 6.5.2, p. 34) and as easily seen from the proof of Proposition 4.24, p.129, the process C_+ is the limit in law of the process $C_{N,+}(\psi) := \frac{\alpha_N}{\sqrt{N}} \left(\int_{\mathbf{S}} \psi(\cdot)\, \mathrm{d}\gamma\right)$, where α_N is the centered number of positive frequencies.

Hence, one has to see (5.1) as a continuous time evolution that models in law the behavior of the fluctuations of the system as $N \to \infty$, where the initial fluctuations of the disorder lies in the non-trivial mean value of its initial condition.

1.3 Disorder-induced rotation and non self-averaging phenomenon

As seen in the main introduction (Section 6), numerical simulations (Figure 1.4) of the process given by (5.1) show a non self-averaging phenomenon for the fluctuation process compatible with the one observed for the rotation of the finite system (Figure 1.3): the trajectories of $\eta^\omega(\sin)$ seem to have an approximately linear behavior, whose slope depends on the choice of ω (that is the choice of the initial condition). Hence, a way to capture the non self-averaging phenomenon for the finite system is to analyze the asymptotic behavior of the solution of (5.1) as t goes to ∞, and its dependence on the mean value of the initial condition η_0^ω. More explicitly, the key point of this chapter is to understand how different initial conditions in evolution (5.1) may lead to distinct approximately linear trajectories of the fluctuation process.

We tackle this issue via a spectral analysis of the evolution operator L_{q_t} defined in (1.28), p. 31 via arguments (analogous to those used in Chapter 3 from the general theory of perturbations of self-adjoint operators ([51]) and analytic semigroup of operators ([70]) and usual techniques about SPDEs in Hilbert spaces ([26]), using the precise knowledge we have about the same linear operator L_{q_0} in the case without disorder (see [9]).

1.4 Organization of the chapter

The chapter is organized as follows: in Section 2, we precise the main set-up and notations and state the main results. In particular, Theorem 5.7 and Theorem 5.9 deal with the spectral properties of the evolution operator L_{q_t} at least when the disorder is small. Secondly, we state the main result of this chapter: Theorem 5.3 establish the linear asymptotics of the fluctuation process solution of (5.1).

Section 3 is devoted to prove Theorem 5.7. In Section 4 we prove Theorem 5.9, whereas Theorem 5.3 is proved in Section 5.

2 Main definitions and results

We recall the following standard notations, already used in Chapter 3: for an operator F, we will denote by $\rho(F)$ the set of all complex numbers λ for which $\lambda - F$ is invertible, and by $R(\lambda, F) := (\lambda - F)^{-1}$, $\lambda \in \rho(F)$ the resolvent of F. The spectrum of F will be denoted as $\sigma(F)$.

2.1 The evolution operator $\mathcal{L}_t^*$

The main issue of this chapter is to prove asymptotic results for the solution of the SPDE (5.1) as $t \to \infty$; this fact is deeply linked to the spectral properties of the operator $\mathcal{L}_t^*$ which is the formal adjoint of (5.3).

An easy integration by part shows the following:

Proposition 5.1. *The operator $\mathcal{L}_t^*$ coincides at least on regular functions φ with L_{q_t} defined in* (1.28).

We will restrict ourselves to the stationary case, that is for $q|_{t=0} = q_t$ equal to the synchronized (non-trivial) stationary solution q (3.15), p. 77 of evolution (3.3), p. 74. In this case, the object of interest is the stationary version of (1.28), that is

$$Lh(\theta,\omega) := \frac{1}{2}\partial_\theta^2 h(\theta,\omega) - \partial_\theta \left(h(\theta,\omega) \left(\langle J * q \rangle_\mu(\theta) + \omega \right) + q(\theta,\omega) \langle J * h \rangle_\mu(\theta) \right) . \tag{5.6}$$

The domain $\mathcal{D}$ of the operator L is given by:

$$\mathcal{D} := \left\{ h(\theta,\omega);\ \forall \omega, \theta \mapsto h(\theta,\omega) \in \mathcal{C}^2(\mathbf{S}),\ \int_{\mathbf{S}\times\mathbf{R}} h(\theta,\omega)\, \mathrm{d}\theta \mu(\mathrm{d}\omega) = 0 \right\} . \tag{5.7}$$

Remark 5.2. We point out that the choice of the domain $\mathcal{D}$ of L is of prime importance for our study of evolution (5.1). We have already encountered the same operator L in the study of the linear stability of the stationary solution q (see Chapter 3) since the linearized evolution of (3.3) is precisely given by $\partial_t h_t = L h_t$. As seen in Chapter 3, the natural domain for this latter evolution is then

$$\left\{ h(\theta,\omega);\ \forall \omega, \theta \mapsto h(\theta,\omega) \in \mathcal{C}^2(\mathbf{S}),\ \forall \omega,\ \int_{\mathbf{S}} h(\theta,\omega)\, \mathrm{d}\theta = 0 \right\} . \tag{5.8}$$

Indeed for all ω, $q(\cdot,\omega)$ is a probability density on $\mathbf{S}$ so that perturbing by elements of domain (5.8) enables to remain within the set of functions with integral 1 on $\mathbf{S}$.

But here, the key point is to understand that evolution (5.1) does not live in domain (5.8) since the fluctuations of the disorder are precisely captured by the fact that η has a non-trivial mean-value $C(\omega)$ for fixed ω. We will henceforth work with the domain $\mathcal{D}$ defined in (5.7) instead of (5.8). This choice of domain (apart from being technically more demanding) is all the more relevant since we will see that the non self-averaging phenomenon holds in (5.7) and not in (5.8) (see Remark 5.8).

2.2 Long time evolution of fluctuations SPDE

We now turn to the main result of the chapter, which concerns the asymptotic behavior of the fluctuation process η defined in (5.1). We postpone here the proper definition of the space H to Remark 5.6.

Theorem 5.3. *Under the hypothesis of Theorem 5.9, there exists a unique weak solution η_t to* (5.1) *in H. Moreover, η satisfies the following linear behavior: for each realization $C(\omega)$ of the mean value of the initial condition η_0^ω*

$$\frac{\eta_t^\omega}{t} \xrightarrow[t\to\infty]{\text{in law}} v(\omega)q'. \tag{5.9}$$

Moreover, as a function of ω, $\omega \mapsto v(\omega)$ is a real random variable with Gaussian distribution and with variance

$$\sigma_v^2 := \left(2\int_{\mathbf{S}} p_+(\theta)\,\mathrm{d}\theta\right)^{-2}, \tag{5.10}$$

where $p_+(\theta) := p(\theta,\omega_0)$ is defined by (5.16).

Remark 5.4. Note that the random variable v found in Theorem 5.3 is precisely given by $v = \frac{\int_{\mathbf{S}} C_+}{\int_{\mathbf{S}} p_+}$ where the process C_+ is defined in (5.5).

Theorem 5.3 illustrates the fact that the fluctuations for the finite system are disorder dependent. The continuous time representation (5.1) of the discrete fluctuation process (1.1) does share the same non self-averaging behavior, as seen in Figure 1.4: the trajectories of the solution η_t of (5.1) have a linear behavior whose slope only depends on the initial condition η_0^ω.

The approach followed here is to make $N \to \infty$ and then make the time $t \to \infty$. In other terms, we concentrate here on the behavior of the system for bounded times, that is when the time horizon $T > 0$ is of order 1 w.r.t. the number of particles N. In particular, this work does not address the behavior of the system for larger time scales.

In that sense, understanding rigorously the disorder-induced rotation observed in Figure 1.3 would require to study the behavior of the empirical flow ν_N defined in (1.16) on time scales of order $\sqrt{N}$, which has not been carried out in this work.

Proof of Theorem 5.3 is decomposed in several steps: the first main goal of this chapter is to state a spectral decomposition of the operator L defined in (5.6), based on perturbation arguments from the non-disordered case $\mu = \delta_0$.

2.3 Distribution spaces

As in Chapter 3, we need to introduce *weighted Sobolev spaces* that we will use throughout this chapter. The fact that we work on a domain (5.7) of functions with nontrivial mean value requires to define distributions spaces slightly more general than the ones defined in Chap. 3, § 2.3, p.82, but the principle of construction remains the same. What is more, we will use the same notations as in Chapter 3.

We first focus on weighted-Sobolev spaces of functions $\theta \mapsto h(\theta)$ on $\mathbf{S}$ (§ 2.3.1) and then introduce the corresponding spaces for functions $(\theta,\omega) \mapsto h(\theta,\omega)$ on $\mathbf{S} \times \mathrm{Supp}(\mu)$ (§ 2.3.2):

2.3.1 Weighted Sobolev spaces:

For any bounded positive weight function $k(\cdot)$ on $\mathbf{S}$ such that $\int_{\mathbf{S}} k(\theta)\,\mathrm{d}\theta = 1$, we may consider the space $\mathbf{L}^2_k$ closure of $\mathcal{C}(\mathbf{S})$ w.r.t. the norm:

$$\| h \|_{2,k} := \left(\int_{\mathbf{S}} \frac{h^2(\theta)}{k(\theta)}\,\mathrm{d}\theta\right)^{\frac{1}{2}}. \tag{5.11}$$

The decomposition of h into the sum of Span(k) and its orthogonal supplementary in $\mathbf{L}^2_k$ may be written as:

$$h = \left(\int_{\mathbf{S}} h\right) \cdot k + h_0,$$

where $\int_{\mathbf{S}} h_0 = 0$. Since h_0 is with zero mean value, each of its primitives are 2π-periodic. In particular, we can consider $\mathbf{H}^{-1}_k$ the closure of $\mathcal{C}(\mathbf{S})$ with respect to the following weighted Sobolev norm:

$$\|h\|_{-1,k} := \left(\left(\int_{\mathbf{S}} h\right)^2 + \int_{\mathbf{S}} \frac{\mathcal{H}_0^2}{k}\right)^{\frac{1}{2}}, \tag{5.12}$$

where $\mathcal{H}_0$ is the primitive of h_0 on $\mathbf{S}$ such that $\int_{\mathbf{S}} \frac{\mathcal{H}_0}{k} = 0$.

Remark 5.5. In the case of a constant weight $k(\cdot) \equiv \frac{1}{2\pi}$, we will write $(\mathbf{L}^2, \|\cdot\|_2)$ and $(\mathbf{H}^{-1}, \|\cdot\|_{-1})$ instead of $(\mathbf{L}^2_{\frac{1}{2\pi}}, \|\cdot\|_{2,\frac{1}{2\pi}})$ and $(\mathbf{H}^{-1}_{\frac{1}{2\pi}}, \|\cdot\|_{-1,\frac{1}{2\pi}})$.

2.3.2 Weighted Sobolev spaces (with disorder):

The natural space in where to study the operator L is the space of functions h in $\mathcal{D}$ such that each component $h(\cdot,\omega)$ live in a certain $\mathbf{H}^{-1}_{k(\cdot,\omega)}$ for a weight $k(\cdot,\omega)$ (which may depends on the disorder $\omega \in \mathrm{Supp}(\mu)$). More precisely, for any family of positive weight functions $(k(\cdot,\omega))_{\omega\in\mathrm{Supp}(\mu)}$, we denote as $\mathbf{H}^{-1}_{\mu,k}$ the closure of $\mathcal{D}$ w.r.t. the norm:

$$\|h\|_{\mu,-1,k} := \left(\int_{\mathbf{R}} \|h(\cdot,\omega)\|^2_{-1,k(\cdot,\omega)} \mu(\,\mathrm{d}\omega)\right)^{\frac{1}{2}} = \left(\int_{\mathbf{R}} \left(\int_{\mathbf{S}} h\,\mathrm{d}\theta\right)^2 \mathrm{d}\mu + \int_{\mathbf{R}}\int_{\mathbf{S}} \frac{\mathcal{H}_0^2}{k}\,\mathrm{d}\theta\,\mathrm{d}\mu\right)^{\frac{1}{2}}. \tag{5.13}$$

We will also consider the analogous averaged weighted $\mathbf{L}^2$-spaces, that is, for any family of positive weights $(k(\cdot,\omega))_{\omega\in\mathrm{Supp}(\mu)}$, the space $\mathbf{L}^2_{\mu,k}$ given by the norm:

$$\|h\|_{\mu,2,k} := \left(\int_{\mathbf{R}}\int_{\mathbf{S}} \frac{h(\theta,\omega)^2}{k(\theta,\omega)}\,\mathrm{d}\theta\,\mathrm{d}\mu(\omega)\right)^{\frac{1}{2}}. \tag{5.14}$$

Remark 5.6. In the particular case of $k(\cdot,\omega) \equiv \frac{1}{2\pi}$ for all $\omega \in \mathrm{Supp}(\mu)$, we will write $H := \mathbf{H}^{-1}_{\mu,\frac{1}{2\pi}}$ and the corresponding norm will be denoted as $\|\cdot\|_H$. We will also write $(\mathbf{L}^2_\mu, \|\cdot\|_{\mu,2})$ instead of $(\mathbf{L}^2_{\mu,\frac{1}{2\pi}}, \|\cdot\|_{\mu,2,\frac{1}{2\pi}})$.

The main theorem concerning the operator L will be stated in H for the ease of exposition but its proof will require the introduction of weighted-Sobolev spaces $\mathbf{H}^{-1}_{\mu,k}$ for nontrivial weights k.

2.4 Non self-averaging phenomenon for the operator L and existence of a Jordan block

Understanding the linear behavior depending on the initial condition (recall Figure 1.4, p.32) of the trajectories of solutions of (5.1) is deeply linked with the spectral properties of the operator L at the origin 0, and more precisely to the existence of a Jordan block for the eigenvalue 0 (recall § 6.3, p. 33); indeed the first main result of the chapter is:

Theorem 5.7. *For any fixed $\omega_0 > 0$, if q is the stationary solution in* (3.15)*, then*

$$Lq' = 0. \tag{5.15}$$

Moreover, there exists $p \in \mathcal{D}$ such that

$$\forall \theta \in \mathbf{S}, \forall \omega \in \mathrm{Supp}(\mu), \qquad Lp(\theta, \omega) = q'(\theta, \omega). \tag{5.16}$$

In particular, the characteristic space of L in 0 *is* at least *of dimension* 2.

Remark 5.8. An important remark on Theorem 5.7 is that $p(\cdot, \omega)$ is with non-trivial mean value for all $\omega \in \mathrm{Supp}(\mu)$. In particular, we have reasons to believe (on the basis of numerical simulations as well as on non-rigorous calculations) that the following holds

$$\int_{\mathbf{S}} p(\theta, \omega_0)\, \mathrm{d}\theta = -\int_{\mathbf{S}} p(\theta, -\omega_0)\, \mathrm{d}\theta = -\frac{1}{\omega_0}. \tag{5.17}$$

In other terms, if we had chosen domain (5.8) instead of domain (5.7), we would not have found such p; the Jordan block $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in the matrix representation (5.20) of the operator L does not exists on the domain (5.8).

Theorem 5.7 is proved in Section 3.

2.5 Spectral properties of L and position of the spectrum

The second goal of this chapter is to prove that L generates an analytic semi-group of operators with spectrum confined in the part of the complex plane with negative real part:

Theorem 5.9. *In the Hilbert space H defined in Remark 5.6, the operator $(L, \mathcal{D})$ is densely defined, closable, its closed extension having compact resolvent. In particular, its spectrum consists of isolated eigenvalues with finite multiplicities.*

Moreover, for all $K > 1$, for all $\alpha \in (0, \frac{\pi}{2})$, for all $\rho \in (0, 1)$, there exists $\omega_\star = \omega_\star(K, \alpha, \rho) > 0$ such that, for all $0 < \omega_0 < \omega_\star$, the following is true:

- *The spectrum of L lies in a cone C_α with vertex* 0 *and angle α*

$$C_\alpha := \left\{ \lambda \in \mathbb{C};\ \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha \right\} \subseteq \{ z \in \mathbf{C};\ \Re(z) \leqslant 0 \}\ ;$$

- *There exists $\alpha' \in (0, \frac{\pi}{2})$ such that L is the infinitesimal generator of an analytic semi-group defined on a sector $\Delta_{\alpha'} := \{\lambda \in \mathbb{C},\ |\arg(\lambda)| < \alpha'\}$;*
- *the dimension of the characteristic space in* 0 *is exactly* 2*, spanned by q' and p, where p is defined in Theorem 5.7,*
- *the eigenvalue* 0 *is separated from the rest of the spectrum at a distance $\lambda_K(L) = \lambda(L, K, \rho)$ at least equal to $\rho \cdot \min\left(\lambda_K(L_{q_0}), \frac{1}{2} e^{-4Kr_0}\right)$, where L_{q_0} and r_0 are defined in* (3.6)*, p. 75 and* (3.30)*, p. 85 respectively.*

Remark 5.10. As a consequence of Theorem 5.9, there exists a decomposition of the space H into the direct sum (a priori not orthogonal, since L is not self-adjoint)

$$H = G_0 \oplus G_{<0}\,, \tag{5.18}$$

where G_0 is of dimension 2 (spanned by q' and p) such that the restriction of the operator L to G_0 has spectrum $\{0\}$ and the restriction of L to $G_{<0}$ has spectrum $\sigma(L) \setminus \{0\} \subseteq$

$\{\lambda \in \mathbf{C};\, \Re(\lambda) < 0\}$. We will denote as P_0 the corresponding projection on G_0 along to $G_{<0}$, and $P_{<0} = 1 - P_0$. In particular, there exist unique continuous linear forms $\ell_{q'}$ and ℓ_p such that for all $h \in H$

$$P_0 h := \ell_{q'}(h) q' + \ell_p(h) p. \tag{5.19}$$

To fix ideas, one may think of the following matrix representation for the operator L:

$$L = \underbrace{\begin{pmatrix} P_0\, LP_0 & P_0\, LP_{<0} \\ P_{<0}\, LP_0 & P_{<0}\, LP_{<0} \end{pmatrix}}_{\substack{G_0 = \\ \mathrm{Span}(q',p)}\quad G_{<0}} = \begin{array}{c} \left(\begin{array}{cc|ccc} 0 & 1 & & \ell_{q'}\,(LP_{<0}) & \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & P_{<0}\, LP_{<0} & \\ 0 & 0 & & & \end{array}\right) \\ \underbrace{}_{q'}\ \underbrace{}_{p}\qquad \underbrace{}_{G_{<0}} \end{array} \begin{array}{l} \}q' \\ \}p \\ \\ \\ \end{array} \left.\begin{array}{l} \\ \\ \end{array}\right\} G_0 \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} G_{<0} \tag{5.20}$$

Note that the second line in the matrix representation (5.20) of L is indeed equally zero since for all $h \in H$, Lh is of zero mean value on $\mathbf{S}$; in particular $\ell_p(Lh) = 0$ for all $h \in H$.

Remark 5.11. Note that any element $h = (h(\theta,\omega))_{\theta\in\mathbf{S},\omega\in\mathrm{Supp}(\mu)}$ can be identified in our binary case (5.4) with a couple $(h_+(\theta), h_-(\theta))_{\theta\in\mathbf{S}}$. Moreover, any $h \in H$ can be decomposed according to (5.18):

$$h = \ell_{q'}(h) q' + \ell_p(h) p + P_{<0}\, h.$$

Let us integrate the latter decomposition w.r.t. θ. Since $\int_{\mathbf{S}} Lu = 0$ for all $u \in \mathcal{D}$, we have $\int_{\mathbf{S}} P_{<0}\, h = 0$ so that one can actually find an explicit formulation for the functional ℓ_p:

$$\ell_p(h) = \frac{\int_{\mathbf{S}} h_+}{\int_{\mathbf{S}} p_+} = \frac{\int_{\mathbf{S}} h_-}{\int_{\mathbf{S}} p_-}. \tag{5.21}$$

The last equality in (5.21) is due to the fact that $\int_{\mathbf{S}} (h_+ + h_-) = \int_{\mathbf{S}} (p_+ + p_-) = 0$.

3 On the existence of a Jordan block for L (Proof of Theorem 5.7)

3.1 Proof of Theorem 5.7

The purpose of this section is to prove Theorem 5.7. The symmetry of the system (recall § 2.1.3, p. 15 and Remark 1.8, p. 24) leads to consider the set of distributions which are odd w.r.t. the variable $(\theta,\omega) \in \mathbf{S} \times \mathrm{Supp}(\mu)$:

$$\mathcal{O} := \{h;\, \forall (\theta,\omega) \in \mathbf{S} \times \mathrm{Supp}(\mu),\ h(-\theta,-\omega) = -h(\theta,\omega)\}\,.$$

We also denote by $\mathcal{N}$ the set of functions that with zero mean-value for all $\omega \in \mathrm{Supp}(\mu)$:

$$\mathcal{N} := \left\{ h \in \mathcal{D};\, \forall \omega \in \mathrm{Supp}(\mu),\, \int_{\mathbf{S}} h(\theta,\omega)\, \mathrm{d}\theta = 0 \right\}.$$

In Lemma 5.12 (whose proof is left to the reader), we sum-up the basic properties of the stationary solution q (recall (3.15), p. 77):

Lemma 5.12. *The following statements are true:*

1. $q' \in \mathcal{O}$,
2. *If* $h \in \mathcal{O}$ *then* $Lh \in \mathcal{O}$,
3. $q' \in \mathcal{N}$, *and for all* $h \in \mathcal{D}$, $Lh \in \mathcal{N}$,
4. *There exist some positive constants* c *and* C *such that for all* θ, $\omega \in \mathrm{Supp}(\mu)$, $0 < c \leqslant q(\theta, \omega) \leqslant C$,
5. *For all* $\theta \in \mathbf{S}$, $\omega \in \mathrm{Supp}(\mu)$,

$$\frac{1}{2}\partial_\theta q(\theta,\omega) = q(\theta,\omega)\left(\langle J * q\rangle_\mu + \omega\right) + \kappa(\omega)\,, \tag{5.22}$$

where $\kappa(\omega) = \frac{1-e^{4\pi\omega}}{2Z(\omega)}$.

Note in particular, that the fact that $q' \in \mathcal{O}$ can be seen as a consequence of Remark 1.8, p. 24. A direct calculation easily shows that q' is in the kernel of L (it corresponds to the rotation invariance of the problem). The rest of this section is devoted to prove the existence of an element $p \in \mathcal{D}$ such that $Lp = q'$.

We recall here the definition of the weighted Sobolev spaces introduced in § 2.3.2: we make use here of the space $\mathbf{H}^{-1}_{\mu,q}$ defined in (5.13) in the case of $k = q$ and of the space $\mathbf{L}^2_\mu$ defined in Remark 5.6. The main result is the following:

Proposition 5.13. *For every* $\omega_0 > 0$, *in the binary case* (5.4), *for every* $v \in \mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O}$ *(and in particular for* $v = q'$*), there exists some* $p \in \mathbf{L}^2_\mu \cap \mathcal{O}$ *such that*

$$\forall l \in \mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O},\ \langle Lp\,,\, l\rangle_{\mu,-1,q} = \langle v\,,\, l\rangle_{\mu,-1,q}\,. \tag{5.23}$$

Moreover, in the case $v = q'$, *any* p *that satisfies* (5.23) *is in fact a regular function (*$p(\cdot,\omega) \in \mathcal{C}^\infty(\mathbf{S})$ *for all* $\omega \in \mathrm{Supp}(\mu)$*) and is a classical solution to* (5.16).

Remark 5.14. The scope of Proposition 5.13 is more general than the restrictive case of a binary law $\mu = \frac{1}{2}\left(\delta_{-\omega_0} + \delta_{\omega_0}\right)$; indeed the following proof works for more general distributions μ, the only additional requirement being integrability conditions in 0 and $+\infty$ which are obviously satisfied in the case (5.4), see Remark 5.20.

Proof of Proposition 5.13 consists in several lemmas:

Lemma 5.15. *For* $h \in \mathcal{O} \cap \mathcal{D}$, $l \in \mathcal{O} \cap \mathcal{D}$, *let us introduce the Dirichlet form*

$$\mathcal{E}_L(h,l) := \langle Lh\,,\, l\rangle_{\mu,-1,q}\,.$$

$\mathcal{E}_L(\cdot,\cdot)$ *is well defined on* $\mathcal{D}(\mathcal{E}_L) := \left(\mathbf{L}^2_\mu \cap \mathcal{O}\right) \times \left(\mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O}\right)$ *and one can decompose* $\mathcal{E}_L(\cdot,\cdot)$ *into:*

$$\forall (h,l) \in \mathcal{D}(\mathcal{E}_L),\ \mathcal{E}_L(h,l) = \Gamma(h,l) + K\ell(h)\cdot\ell(l), \tag{5.24}$$

where $\Gamma(\cdot,\cdot)$, *bilinear form on* $\mathcal{D}(\mathcal{E}_L)$ *and* $\ell(\cdot)$ *linear form on* $\mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O}$, *are defined as follows:*

$$\forall (h,l) \in \mathcal{D}(\mathcal{E}_L),\ \Gamma(h,l) := -\frac{1}{2}\int_{\mathbf{S}\times\mathbf{R}} \frac{hl}{q}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \int_{\mathbf{S}\times\mathbf{R}} \kappa(\cdot)\frac{h\mathcal{L}}{q^2}\,\mathrm{d}\lambda\,\mathrm{d}\mu, \tag{5.25}$$

$$\forall l \in \mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O},\ \ell(l) := \int_{\mathbf{S}\times\mathbf{R}} l\sin(\cdot)\,\mathrm{d}\lambda\,\mathrm{d}\mu, \tag{5.26}$$

where κ *in* (5.25) *is defined in* (5.22).

Lemma 5.16. *For all continuous linear form f on $\mathbf{H}^{-1}_{\mu,q}\cap\mathcal{O}$, there exists some $p_1 \in \mathbf{L}^2_\mu\cap\mathcal{O}$ such that for $l \in \mathbf{H}^{-1}_{\mu,q}\cap\mathcal{O}$*

$$\Gamma(p_1, l) = f(l).$$

Lemma 5.17. *The linear form $\ell(\cdot)$ defined in* (5.26) *can be expressed as a scalar product on $\mathbf{H}^{-1}_{\mu,q}\cap\mathcal{O}$: there exists $p_2 \in \mathcal{D}\cap\mathcal{O}$, for all $l \in \mathbf{H}^{-1}_{\mu,q}\cap\mathcal{O}$*

$$\ell(l) = \langle Lp_2\,,\, l\rangle_{\mu,-1,q}\,.$$

Let us admit for a moment Lemmas 5.15, 5.16 and 5.17 and let us prove Proposition 5.13:

Proof of Proposition 5.13. Let v be a fixed element of $\mathbf{H}^{-1}_{\mu,q}\cap\mathcal{O}$. Applying Lemma 5.16 to the continuous linear form $f(l) = \langle v\,,\, l\rangle_{\mu,-1,q}$, there exists some $p_1 \in \mathbf{L}^2_\mu\cap\mathcal{O}$ such that $\Gamma(p_1, l) = \langle v\,,\, l\rangle_{\mu,-1,q}$, which gives using Lemma 5.15 and Lemma 5.17:

$$\begin{aligned}\langle v\,,\, l\rangle_{\mu,-1,q} &= \Gamma(p_1, l),\\ &= \langle Lp_1\,,\, l\rangle_{\mu,-1,q} - K\ell(p_1)\ell(l),\\ &= \langle Lp_1\,,\, l\rangle_{\mu,-1,q} - K\ell(p_1)\,\langle Lp_2\,,\, l\rangle_{\mu,-1,q}\,.\end{aligned}$$

We can conclude that the variational formula (5.23) is verified for the following choice of p:

$$p := p_1 - K\ell(p_1)p_2 \in \mathbf{L}^2_\mu\cap\mathcal{O}.$$

Let us prove now that such p is in fact a regular function in θ: since $p_2 \in \mathcal{D}$ is regular in θ, it suffices to prove that for all $\omega \in \mathrm{Supp}(\mu)$, $\theta \mapsto p_1(\theta,\omega)$ is $\mathcal{C}^2$ (in fact $\mathcal{C}^\infty$) in θ. We start from the definition of p_1:

$$\forall l \in \mathbf{H}^{-1}_{\mu,q}\cap\mathcal{O},\quad \Gamma(p_1, l) = \langle q'\,,\, l\rangle_{\mu,-1,q}\,.$$

Since this true for all $l \in \mathbf{H}^{-1}_{\mu,q}\cap\mathcal{O}$, thanks to the expression of Γ in (5.25), we obtain that for any fixed $\omega \in \mathrm{Supp}(\mu)$, for Lebesgue-almost every $\theta \in \mathbf{S}$:

$$\frac{1}{4}\frac{p_1(\theta,\omega)}{q(\theta,\omega)} = -\kappa(\omega)\left(\int_0^\theta \frac{p_1(u,\omega)}{q(u,\omega)^2}\,\mathrm{d}u\right) + \int_0^\theta \frac{Q(u,\omega)}{q(u,\omega)}\,\mathrm{d}u, \tag{5.27}$$

where $Q(\cdot,\omega)$ is the primitive of $q'(\cdot,\omega)$ such that $\int_{\mathbf{S}} \frac{Q(\cdot,\omega)}{q(\cdot,\omega)} = 0$. But, using that q is bounded and $\mathcal{C}^\infty$ in θ and that $p_1(\cdot,\omega) \in \mathbf{L}^2$, the primitive $\int_0^\theta \frac{p_1(u,\omega)}{q(u,\omega)^2}\,\mathrm{d}u$ as a $\mathcal{C}^1$ version. So, thanks to (5.27), p_1 has a $\mathcal{C}^1$ version. So, the right-hand side of (5.27) is a least $\mathcal{C}^2$, and so does p_1. The same repeated argument shows that p_1 is $\mathcal{C}^\infty$ in θ. That concludes the proof of Proposition 5.13. □

It remains to prove the three lemmas:

Proof of Lemma 5.15. Let us prove equality (5.24): since $\mathcal{L}$ is a primitive of l, one has

$$\frac{1}{2}\int_{\mathbf{S}} \frac{(\partial_\theta h)\mathcal{L}}{q} = -\frac{1}{2}\int_{\mathbf{S}} \frac{hl}{q} + \frac{1}{2}\int_{\mathbf{S}} \frac{h\mathcal{L}}{q^2}\partial_\theta q\,.$$

Using (5.22), for $\omega \in \mathrm{Supp}(\mu)$

$$\frac{1}{2}\int_{\mathbf{S}} \frac{(\partial_\theta h)\mathcal{L}}{q} = -\frac{1}{2}\int_{\mathbf{S}} \frac{hl}{q} + \int_{\mathbf{S}} \frac{h\mathcal{L}}{q}\left(\langle J * q\rangle_\mu(\cdot) + \omega\right) + \kappa(\omega)\int_{\mathbf{S}} \frac{h\mathcal{L}}{q^2}.$$

Thanks to the expression of Lh in (5.6), we obtain

$$\begin{aligned} B(h,l) &= -\frac{1}{2}\int_{\mathbf{S}\times\mathbf{R}} \frac{hl}{q} + \int_{\mathbf{S}\times\mathbf{R}} \kappa(\cdot)\frac{h}{q^2}\mathcal{L} - \int_{\mathbf{S}\times\mathbf{R}} \langle J * h\rangle_\mu \mathcal{L}, \\ &= \Gamma(h,l) - \int_{\mathbf{S}\times\mathbf{R}} \langle J * h\rangle_\mu \mathcal{L}. \end{aligned} \tag{5.28}$$

Lastly, integrating by parts the last term in (5.28) and expanding the cosine function (recall $J(\cdot) = -K\sin(\cdot)$), we obtain:

$$\begin{aligned} -\int_{\mathbf{S}\times\mathbf{R}} \langle J * h\rangle_\mu \mathcal{L} &= K\left(\int_{\mathbf{S}\times\mathbf{R}} \cos(\cdot) l \,\mathrm{d}\lambda\,\mathrm{d}\mu\right)\left(\int_{\mathbf{S}\times\mathbf{R}} \cos(\cdot) h \,\mathrm{d}\lambda\,\mathrm{d}\mu\right) \\ &\quad + K\left(\int_{\mathbf{S}\times\mathbf{R}} \sin(\cdot) l \,\mathrm{d}\lambda\,\mathrm{d}\mu\right)\left(\int_{\mathbf{S}\times\mathbf{R}} \sin(\cdot) h \,\mathrm{d}\lambda\,\mathrm{d}\mu\right). \end{aligned}$$

But, since $l \in \mathcal{O}$, the first term in the latter expression is zero. The result (5.24) follows. □

Proof of Lemma 5.16. In this proof, we use the following extension to Lax-Milgram Theorem (cf. [82, chap. III]):

Proposition 5.18 (Lions-Lax-Milgram)**.** *Let $\{\mathcal{H}, |.|\}$ be a Hilbert space and $\{\mathcal{G}, \|\,.\,\|\}$ a normed linear space. Suppose $\Gamma : \mathcal{H}\times\mathcal{G} \to \mathbf{R}$ is bilinear and that $\Gamma(\cdot,\varphi)$ is continuous for each $\varphi \in \mathcal{G}$. If there exists some constant $C > 0$ such that*

$$\inf_{\|\varphi\|=1} \sup_{|h|\leqslant 1} |\Gamma(h,\varphi)| \geqslant C > 0, \quad \textit{(weak coercivity)},$$

Then for each $f \in \mathcal{G}'$ there exists some $p \in \mathcal{H}$ such that $\Gamma(p,\varphi) = f(\varphi)$ for all $\varphi \in \mathcal{G}$.

The principle of the proof of Lemma 5.16 is to show that the bilinear function Γ defined in (5.25) satisfies Proposition 5.18 for

$$\mathcal{H} := \mathbf{L}^2_\mu \cap \mathcal{O}, \quad \text{endowed with} \quad \|\cdot\|_{\mu,2}, \tag{5.29}$$

$$\mathcal{G} := \mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O} \cap \mathbf{L}^\infty(\mathbf{S}\times\mathbf{R}), \quad \text{endowed with} \quad \|\cdot\|_{\mu,-1,q}. \tag{5.30}$$

Namely, we have the following:

1. For each $l \in \mathcal{G}$, $\Gamma(\cdot,l)$is continuous on $\mathbf{L}^2_\mu \cap \mathcal{O}$: indeed, for the first term of $\Gamma(h,l)$, we have

$$\left|\int_{\mathbf{S}\times\mathbf{R}} \frac{hl}{q}\,\mathrm{d}\lambda\,\mathrm{d}\mu\right| \leqslant C\,\|\,l\,\|_\infty \int_{\mathbf{S}\times\mathbf{R}} |h|\,\mathrm{d}\lambda\,\mathrm{d}\mu \leqslant C\,\|\,h\,\|_{\mu,2}.$$

   And for the second term, using the boundedness of q:

$$\int_{\mathbf{S}\times\mathbf{R}} \left|\kappa(\cdot)\frac{h}{q^2}\mathcal{L}\right| \,\mathrm{d}\lambda\,\mathrm{d}\mu \leqslant C\int_{\mathbf{S}\times\mathbf{R}} |h\mathcal{L}|\,\mathrm{d}\lambda\,\mathrm{d}\mu \leqslant C\,\|\,h\,\|_{\mu,2}\,\|\,l\,\|_{\mu,-1,q}.$$

2. Γ is weakly coercive: let us fix $l \in \mathcal{G}$ such that $\|\,l\,\|_{\mu,-1,q} = 1$.

   Let us choose $h = g\mathcal{L} \in \mathbf{L}^2_\mu \cap \mathcal{O}$, where for all $\omega \in \mathrm{Supp}(\mu)$, $g(\cdot,\omega)$ is a 2π-periodic function to be defined later. Then, by integration by parts in the equality (5.25)

$$\Gamma(h,l) = -\frac{1}{2}\int_{\mathbf{S}\times\mathbf{R}} \frac{g}{q} l\mathcal{L} + \int_{\mathbf{S}\times\mathbf{R}} \kappa(\cdot)\frac{g}{q^2}\mathcal{L}^2 = \int_{\mathbf{S}\times\mathbf{R}} \left\{\frac{1}{4}\partial_\theta\left(\frac{g}{q}\right) + \kappa(\cdot)\frac{g}{q^2}\right\}\mathcal{L}^2. \tag{5.31}$$

Consider now for fixed $\omega \in \mathrm{Supp}(\mu)$ the following first order ODE, with periodic boundary condition:

$$\frac{1}{4}\partial_\theta f(\cdot,\omega) + \kappa(\omega)\frac{f(\cdot,\omega)}{q(\cdot,\omega)} = \frac{1}{q(\cdot,\omega)}, \quad \text{with } f(0,\omega) = f(2\pi,\omega). \tag{5.32}$$

Then for any $\omega \in \mathrm{Supp}(\mu) \setminus \{0\}$, an explicit calculation (left to the reader) shows that there exists a unique solution to (5.32), $\theta \mapsto f(\theta,\omega)$.

Remark 5.19. In the case $\omega = 0$, (5.32) reduces to $\frac{1}{4}\partial_\theta f = \frac{1}{q_0}$ which is incompatible with the condition $f(0) = f(2\pi)$, since $\int_{\mathbf{S}} \frac{1}{q_0}\,\mathrm{d}\theta > 0$: there is no such 2π-periodic solution in the case $\omega = 0$.

Moreover, it is straightforward to see that $\| \int_{\mathbf{R}} |f(\cdot,\omega)|\,\mathrm{d}\mu \|_{\infty,\mathbf{S}} \leqslant C$, for some constant $C > 0$.

Remark 5.20. It is easy to see that $f(\cdot,\omega)$ is not bounded as $\omega \to 0$ and $\omega \to \infty$; thus, the same control on f for more general distributions μ requires additional integrability assumptions for μ.

If we choose h such that $h = g \cdot \mathcal{L}$ with $g(\cdot,\omega) = q(\cdot,\omega)f(\cdot,\omega)$, we have the following:
- By construction of f, using (5.32) in (5.31), $\Gamma(h,l) = \| l \|^2_{\mu,-1,q} = 1$,
- $\| h \|^2_{\mu,2} \leqslant C \int_{\mathbf{S}} f^2 \frac{\mathcal{L}^2}{q}\,\mathrm{d}\lambda \leqslant C$. So, $\sup_{\| h \|_{\mu,2,1} \leqslant 1} |\Gamma(h,l)| \geqslant \frac{1}{C}$, where C is independent of $l \in \mathcal{G}$ such that $\| l \|_{\mu,-1,q} = 1$.

Applying Proposition 5.18, we obtain the existence of some $p_1 \in \mathbf{L}^2_\mu \cap \mathcal{O}$ such that $\Gamma(p_1,l) = f(l)$, for all $l \in \mathcal{G}$. But by density, this is also true for $l \in \mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O}$. □

Proof of Lemma 5.17. Since there exists some constants $C, c > 0$ such that for all $\omega \in \mathrm{Supp}(\mu)$, $\theta \in \mathbf{S}$, $0 < c \leqslant q(\theta,\omega) \leqslant C$, ℓ is continuous on $\mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O}$ (as well as on $\mathbf{L}^2_\mu \cap \mathcal{O}$). More precisely, by Riesz theorem, there exists a unique $e \in \mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O}$ such that for all $l \in \mathbf{H}^{-1}_{\mu,q} \cap \mathcal{O}$, $\ell(l) = \langle e\,,\, l\rangle_{\mu,-1,q}$. One can be more explicit: a simple calculation shows that this $(\theta,\omega) \mapsto e(\theta,\omega)$ corresponds to the primitive $\mathcal{E}(\theta,\omega) = -q(\theta,\omega)\cos(\theta)$, that is:

$$\forall \theta \in \mathbf{S},\ \omega \in \mathrm{Supp}(\mu), \quad e(\theta,\omega) = -q'(\theta,\omega)\cos(\theta) + q(\theta,\omega)\sin(\theta). \tag{5.33}$$

Let us introduce the following function $p_2 \in \mathbf{L}^2_\mu \cap \mathcal{O}$:

$$p_2(\theta,\omega) = \frac{e^{-B(\theta,\omega)}}{1 - e^{4\pi\omega}} \int_{\mathbf{S}} e^{B(u,\omega)+4\pi\omega}\,\mathrm{d}u + \int_0^\theta e^{B(u,\omega)-B(\theta,\omega)}\,\mathrm{d}u,$$

for

$$B(\theta,\omega) = -2\left(Kr\left(\cos(\theta) - 1\right) + \omega\theta\right).$$

Then one readily verifies that Lp_2 is proportional to e. □

4 Global spectral properties of operator L (Proof of Theorem 5.9)

The purpose of this section is to prove Theorem 5.9.

The main ingredients of proof of Theorem 5.9 are similar to the techniques used in Chapter 3: the idea is to decompose the operator L defined by (5.6) on the domain $\mathcal{D}$ given by (5.7) into the sum of a self-adjoint operator A (in a weighted Sobolev space for appropriate weights) and a perturbation B which will be considered to be small w.r.t. A,

relying on the fact that the evolution operator L in the disordered Kuramoto model can be considered as a perturbation of the evolution operator L_{q_0} without disorder defined in (3.30) and already studied in [9].

Namely, it is straightforward to verify from (5.6) that L can be written as $L = A + B$ where, for all $h \in \mathcal{D}$, for all $\omega \in \mathrm{Supp}(\mu)$,

$$Ah(\theta,\omega) := \frac{1}{2}\partial_\theta^2 h(\theta,\omega) - \partial_\theta\Big(h(\theta,\omega)(J * q_0)(\theta) + q_0(\theta)\langle J * h\rangle_\mu\Big), \tag{5.34}$$

and,

$$Bh(\theta,\omega) := -\partial_\theta\Big(h(\theta,\omega)\{\langle J * (q - q_0)\rangle_\mu(\theta) + \omega\} + (q(\theta,\omega) - q_0(\theta))\langle J * h\rangle_\mu\Big). \tag{5.35}$$

Remark 5.21. We make the abuse of notations in using the same letter A for (5.34) and for the *labeled* operator (3.36), although the two operators have nothing in common. It is important to note that we cannot rely here on the knowledge we have about the *labeled* operator (3.36) since *we are not working on the same domain*: we work here with functions with *nontrivial* mean value (recall (5.7)) and the spectral analysis of (3.36) becomes much harder. In particular, the kernel of the *labeled* operator (3.36) is likely to be of infinite dimension in (5.7) for distribution μ with infinite support. This difficulty explains in particular the fact that we need to restrict in this chapter to the binary case (5.4).

We divide the proof of Theorem 5.9 into three parts: in § 4.1, we prove that A is essentially self-adjoint (and thus generates an analytic semigroup) in some weighted-Sobolev space (recall § 2.3) for an appropriate choice of weights. The purpose of § 4.2 is to establish precise control of the size of the perturbation B w.r.t. A. The third step of the proof, carried out in § 4.3 consists in deriving similar spectral properties for the operator $L = A + B$, especially the fact that the spectrum of L lies in the part of the complex plane with negative real part. Note that this section strongly relies on the fact that μ is a binary distribution (recall (5.4)).

4.1 Spectral properties of the operator A

In this paragraph, we prove mainly that A defined in (5.34) is essentially self-adjoint for a Sobolev norm that is equivalent to the norm $\|\cdot\|_H$ defined in § 2.3.2.

Since we are working in the domain $\mathcal{D}$ (recall (5.7)), the test functions h are such that $\int_{\mathbf{R}} h(\cdot,\omega)\,\mathrm{d}\mu = \frac{1}{2}\left(h(\cdot,+\omega_0) + h(\cdot,-\omega_0)\right)$ has zero mean value on $\mathbf{S}$. The idea of this paragraph is to reformulate the operator A in terms of the sum $\frac{1}{2}(h(\cdot,+\omega_0) + h(\cdot,-\omega_0))$ and the difference $\frac{1}{2}(h(\cdot,+\omega_0) - h(\cdot,-\omega_0))$; namely, we define the following 2×2 invertible matrix:

$$M := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and for $h \in \mathcal{D}$, let $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) := M \cdot h$, namely

$$\begin{cases} u(\cdot) &:= \frac{1}{2}(h(\cdot,+\omega_0) + h(\cdot,-\omega_0)), \\ v(\cdot) &:= \frac{1}{2}(h(\cdot,+\omega_0) - h(\cdot,-\omega_0)). \end{cases}$$

We are now able to define the following operator: $\tilde{A} := M \circ A \circ M^{-1}$, defined on the domain $\widetilde{\mathcal{D}}$

$$\widetilde{\mathcal{D}} := \left\{(u,v) \in \mathcal{C}^2(\mathbf{S}) \times \mathcal{C}^2(\mathbf{S});\ \int_{\mathbf{S}} u(\theta)\,\mathrm{d}\theta = 0\right\}, \tag{5.36}$$

given by

$$\forall (u,v) \in \widetilde{\mathcal{D}}, \quad \tilde{A}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \tilde{A}_1 u \\ \tilde{A}_2 v \end{pmatrix} := \begin{pmatrix} \frac{1}{2}\partial_{\theta^2} u - \partial_\theta \left[u(J * q_0) + q_0 (J * u) \right] \\ \frac{1}{2}\partial_{\theta^2} v - \partial_\theta \left[v(J * q_0) \right] \end{pmatrix}. \tag{5.37}$$

The operator $\tilde{A}$ is uncoupled w.r.t. variables u and v; consequently, in order to diagonalize $\tilde{A}$, it suffices to diagonalize both components of $\tilde{A}$, namely $\tilde{A}_1$ and $\tilde{A}_2$. This is the purpose of the two following propositions 5.22 and 5.23.

We make use here of the weighted Sobolev norms $\|\cdot\|_{-1,k}$ defined in (5.12) for different choices of $k(\cdot)$: an important remark is that the first component $\tilde{A}_1$ (with domain $\{u \in \mathcal{C}^2(\mathbf{S}),\ \int_{\mathbf{S}} u = 0\}$) is exactly equal to the McKean-Vlasov operator L_{q_0} with no disorder defined in (3.30). Following § 2.4.2, p.85 and [9], the natural space in where to study the operator $\tilde{A}_1$ is the space $\mathbf{H}^{-1}_{q_0}$ defined in (5.12), for the weight $k(\cdot) = q_0(\cdot)$ the non-disordered stationary solution (recall (3.7)). In this space, we have

Proposition 5.22. *In $\mathbf{H}^{-1}_{q_0}$, $\tilde{A}_1$ is essentially self-adjoint with compact resolvent and spectrum in the negative part of the real axis. 0 is a one-dimensional eigenvalue, spanned by q_0'. The spectral gap $\lambda_K(\tilde{A}_1)$ between 0 and the rest of the spectrum is strictly positive.*

Moreover, the self-adjoint extension of $\tilde{A}_1$ is the infinitesimal generator of a strongly continuous semi-group of contractions $\tilde{T}_1(t)$ on $\mathbf{H}^{-1}_{q_0}$. For every $0 < \alpha < \frac{\pi}{2}$, this semigroup can be extended to an analytic semigroup $\tilde{T}_1(z)$ defined on $\Delta_\alpha = \{\lambda;\ |\arg(\lambda)| < \alpha\}$ and one has the following estimate on its resolvent (where $\Sigma_\alpha = \{\lambda \in \mathbf{C};\ |\arg(\lambda)| < \frac{\pi}{2} + \alpha\} \cup \{0\}$):

$$\forall \alpha \in (0, \frac{\pi}{2}),\ \forall \lambda \in \Sigma_\alpha,\ \left\| R(\lambda, \tilde{A}_1) \right\|_{-1,q_0} \leqslant \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{5.38}$$

The second component $\tilde{A}_2$ can be seen as a second order ordinary differential operator, with domain $\{v \in \mathcal{C}^2(\mathbf{S})\}$. As we will see in § 4.1.2, the natural space in which to study $\tilde{A}_2$ is the space $\mathbf{H}^{-1}_w$, for the choice of the weight function $\theta \mapsto w(\theta) = \frac{e^{-\Phi(\theta)}}{\int_{\mathbf{S}} e^{-\Phi}}$, with

$$\Phi(\theta) := -2Kr_0 \cos(\theta). \tag{5.39}$$

Namely, we have

Proposition 5.23. *The operator $(\tilde{A}_2, \mathcal{C}^2(\mathbf{S}))$ is essentially self-adjoint in $\mathbf{H}^{-1}_w$ and has compact resolvent. Hence, its spectrum consists of isolated eigenvalues with finite multiplicities. The kernel of $\tilde{A}_2$ is of dimension 1, spanned by $w(\theta) = \frac{e^{-\Phi(\theta)}}{\int_{\mathbf{S}} e^{-\Phi}}$. Moreover, we have the following spectral gap estimation:*

$$\forall v \in \mathcal{C}^2(\mathbf{S}), \quad -\left\langle \tilde{A}_2 v\,,\, v \right\rangle_{-1,w} \geqslant \frac{e^{-4Kr_0}}{2} \left\| v - \left(\int_{\mathbf{S}} v \right) w \right\|_{-1,w}, \tag{5.40}$$

so that the spectrum of $\tilde{A}_2$ lies in the negative part of the real axis and the distance between 0 and the rest of the spectrum $\lambda_K(\tilde{A}_2)$ is at least equal to $\frac{e^{-4Kr_0}}{2}$. One also has explicit estimate on the resolvent of $\tilde{A}_2$:

$$\forall \alpha \in (0, \frac{\pi}{2}),\ \forall \lambda \in \Sigma_\alpha,\ \left\| R(\lambda, \tilde{A}_2) \right\|_{-1,w} \leqslant \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{5.41}$$

Putting things together, the natural norm for the operator $\tilde{A}$ is the Hilbert-norm, for $(u,v) \in \widetilde{\mathcal{D}}$: $\left(\| u \|_{-1,q_0}^2 + \| v \|_{-1,w}^2\right)^{\frac{1}{2}}$. But since $\tilde{A}$ is the conjugate of A through the invertible matrix M, to say that $\tilde{A}$ is essentially self-adjoint for the previous norm is equivalent to say that A is essentially self-adjoint for the corresponding conjugate norm:

$\forall h \in \mathcal{D}$,

$$\| h \|_{H_w} := \left(\left\| \frac{1}{2} \left(h(\cdot,+\omega_0) + h(\cdot,-\omega_0)\right) \right\|_{-1,q_0}^2 + \left\| \frac{1}{2} \left(h(\cdot,+\omega_0) - h(\cdot,-\omega_0)\right) \right\|_{-1,w}^2 \right)^{\frac{1}{2}}. \tag{5.42}$$

The results of § 4.1 can be summed-up in the following proposition, which is an easy consequence of Propositions 5.22 and 5.23:

Proposition 5.24. *For the norm $\| \cdot \|_{H_w}$ defined in* (5.42)*, the operator $(A, \mathcal{D})$ is essentially self-adjoint, with compact resolvent. In particular, the spectrum of (the self-adjoint extension of) A is pure-point, and consists of eigenvalues with finite multiplicities. Moreover it lies in the negative part of the real-axis and A is the infinitesimal generator of an analytic semigroup of operators $T_A(z)$ defined on a domain $\Delta_\alpha = \{z \in \mathbf{C};\ |\arg(z)| < \alpha\}$, for any $0 < \alpha < \frac{\pi}{2}$. One also has the following estimate about the resolvent of A:*

$$\forall \alpha \in (0, \frac{\pi}{2}),\ \forall \lambda \in \Sigma_\alpha,\ \| R(\lambda, A) \|_{H_w} \leqslant \frac{1}{1 - \sin(\alpha)} \cdot \frac{1}{|\lambda|}. \tag{5.43}$$

The kernel of A is of dimension 2, spanned by $\left\{ q_0' + \frac{e^{-\Phi}}{\int_{\mathbf{S}} e^{-\Phi}}, q_0' - \frac{e^{-\Phi}}{\int_{\mathbf{S}} e^{-\Phi}} \right\}$ and the eigenvalue 0 is separated from the rest of the spectrum with a distance

$$\lambda_K(A) := \min\left(\lambda_K(\tilde{A}_1), \lambda_K(\tilde{A}_2)\right),$$

where $\lambda_K(\tilde{A}_1)$ and $\lambda_K(\tilde{A}_2)$ are introduced in Propositions 5.22 and 5.23 respectively.

Remark 5.25. Note that this norm $\| \cdot \|_{H_w}$ is equivalent to the norm $\| \cdot \|_H$ defined in § 2.3.2, since the weights q_0 and w are bounded above and below. In particular, in the space H, the operator A (although no longer self-adjoint) still generates an analytic semigroup with the same spectrum and the same spectral gap.

The aim of paragraph § 4.1.1 (resp. § 4.1.2) is to prove Proposition 5.22 (resp. Proposition 5.23).

4.1.1 Spectral properties of $\tilde{A}_1$: proof of Proposition 5.22

Much about the operator $\tilde{A}_1$ has already been done since it exactly corresponds to the linear evolution operator L_{q_0} of the non-disordered Kuramoto model studied in [9]: we know from Proposition 3.23 that $\tilde{A}_1$ is essentially self-adjoint and dissipative in $\mathbf{H}_{q_0}^{-1}$. We refer to Chapter 3 for the results we used on extension of analytic semigroups (recall in particular Proposition 3.44). The fact that $\tilde{A}_1$ generates an analytic semigroup $\tilde{T}_1(t)$ in an appropriate sector can be proved in a very analogous way as what was done for the *labeled* operator in the proof of Proposition 3.43 so we skip the details of the proof of Proposition 5.22.

4.1.2 Spectral properties of $\tilde{A}_2$: proof of Proposition 5.23

$\tilde{A}_2$ may be written as

$$\tilde{A}_2 v = \frac{1}{2} v'' + \partial_\theta \left(v K r_0 \sin(\cdot) \right), \tag{5.44}$$

where $r_0 = \Psi_0(2Kr_0)$ (recall (3.6)).

One recognizes in $\tilde{A}_2$ a second order differential operator on $\mathcal{C}^2(\mathbf{S})$ which can be identified to a simple instance of a Fokker-Planck operator with a sine potential (see [72]). This operator can easily be seen, by integrations by parts in an appropriate weighted $\mathbf{L}^2$-space as a Sturm-Liouville operator operating on $\mathcal{C}^2$, 2π-periodic functions. A vast literature exists on the spectral properties of such operators (see [63], [33], [22], [64], [5] or [35]). But a $\mathbf{L}^2$-norm is not appropriate for the future study of the fluctuation equation (5.1); a look at the covariance structure of W (see (5.2)) shows that W naturally lives in a $\mathbf{H}^{-1}$-space instead of a $\mathbf{L}^2$-space.

An easy calculation shows that $\tilde{A}_2$ can be rewritten in terms of the weight function Φ defined in (5.39):

$$\tilde{A}_2 v = \frac{1}{2} \partial_\theta \left(e^{-\Phi} \partial_\theta \left(e^{\Phi} v \right) \right). \tag{5.45}$$

Let w be:

$$w(\theta) := \frac{e^{-\Phi(\theta)}}{\int_{\mathbf{S}} e^{-\Phi}}. \tag{5.46}$$

One directly sees from (5.45) that w lies in the kernel of $\tilde{A}_2$: $\tilde{A}_2 w = 0$.

Let us place ourselves in the framework of the weighted-Sobolev spaces (denoted as $\mathbf{L}^2_w$ and $\mathbf{H}^{-1}_w$) constructed in § 2.3.1, in the particular case of $k(\cdot) = w(\cdot)$. In particular, we decompose each $v \in \mathcal{C}^2$ into $v = (\int_{\mathbf{S}} v) \cdot w + v_0$, with the $\mathbf{L}^2$-norm $\| v \|_{2,w} = \left(\int_{\mathbf{S}} \frac{|v|^2}{w} \right)^{\frac{1}{2}}$ and the $\mathbf{H}^{-1}$-norm $\| v \|_{-1,w} = \left((\int_{\mathbf{S}} v)^2 + \int_{\mathbf{S}} \frac{|\mathcal{V}_0|^2}{w} \right)^{\frac{1}{2}}$. We denote by $\langle \cdot \, , \, \cdot \rangle_{-1,w}$ the associated scalar product.

We are now in position to prove Proposition 5.23:

– In $\mathbf{H}^{-1}_w$, the operator $(\tilde{A}_2, \mathcal{C}^2(\mathbf{S}))$ is formally symmetric: indeed, for u, $v \in \mathcal{C}^2(\mathbf{S})$, we have successively,

$$\begin{aligned} \left\langle \tilde{A}_2 u \, , \, v \right\rangle_{-1,w} &= \left(\int_{\mathbf{S}} e^{-\Phi} \right) \int_{\mathbf{S}} e^{\Phi} \left(\frac{1}{2} e^{-\Phi} \partial_\theta \left(e^{\Phi} u \right) \right) \mathcal{V}_0 = -\frac{1}{2} \left(\int_{\mathbf{S}} e^{-\Phi} \right) \int_{\mathbf{S}} e^{\Phi} u v_0, \\ &= -\frac{1}{2} \int_{\mathbf{S}} \frac{u_0 v_0}{w} - \frac{1}{2} \int_{\mathbf{S}} u \int_{\mathbf{S}} v_0 = -\frac{1}{2} \int_{\mathbf{S}} \frac{u_0 v_0}{w}. \end{aligned} \tag{5.47}$$

The latter expression being symmetric in u and v.

– Let us prove that $(\tilde{A}_2, \mathcal{C}^2(\mathbf{S}))$ is essentially self-adjoint: let $\mathcal{E}_2$ be the following Dirichlet form

$$\mathcal{E}_2(u, v) := \left\langle u \, , \, (1 - \tilde{A}_2) v \right\rangle_{-1,w} = \int_{\mathbf{S}} u \int_{\mathbf{S}} v + \int_{\mathbf{S}} \frac{\mathcal{V}_0 \mathcal{U}_0}{w} + \frac{1}{2} \int_{\mathbf{S}} \frac{u_0 v_0}{w}. \tag{5.48}$$

Then it is easy to see that $\mathcal{E}_2$ is a continuous bilinear form on $\mathbf{L}^2_w$ (thanks to Poincaré inequality). Moreover $\mathcal{E}_2$ is coercive, namely, for all $u \in \mathbf{L}^2_w$:

$$\begin{aligned} \mathcal{E}_2(u, u) &= \left(\int_{\mathbf{S}} u \right)^2 + \int_{\mathbf{S}} \frac{\mathcal{U}_0^2}{w} + \frac{1}{2} \int_{\mathbf{S}} \frac{u_0^2}{w}, \\ &\geqslant \left(\int_{\mathbf{S}} u \right)^2 + \frac{1}{2} \left\| u - \left(\int_{\mathbf{S}} u \right) w \right\|_{2,w}^2 \geqslant \frac{1}{2} \| u \|_{2,w}^2 . \end{aligned} \tag{5.49}$$

Since for all $f \in \mathbf{H}_w^{-1}$, the linear form $v \mapsto \langle v\,,\, f\rangle_{-1,w}$ is continuous on $\mathbf{L}_w^2$, an application of Lax-Milgram Theorem shows that for such an $f \in \mathbf{H}_w^{-1}$ there exists an unique $u \in \mathbf{L}_w^2$ such that for all $v \in \mathbf{L}_w^2$

$$\mathcal{E}_2(v,u) = \langle v\,,\, f\rangle_{-1,w}\,.$$

It is then easy to see that $\int_{\mathbf{S}} f = \int_{\mathbf{S}} u$ and that for almost every $\theta \in \mathbf{S}$,

$$\frac{1}{2}\frac{u_0(\theta)}{w(\theta)} = -\int_0^\theta \frac{\mathcal{F}_0}{w} + \int_0^\theta \frac{\mathcal{U}_0}{w}. \tag{5.50}$$

Since $u \in \mathbf{L}_w^2$, $\mathcal{U}_0$ admits a $\mathcal{C}^1$-version and if we assume that f is square-integrable, the same argument holds for the first term of the right-hand side of (5.50). So, if f is square integrable, u_0 admits a $\mathcal{C}^2$-version. To sum-up, if we suppose that f is continuous, there exists $u \in \mathcal{C}^2(\mathbf{S})$ such that, applying $\partial_\theta\left(e^{-\Phi}\partial_\theta(\cdot)\right)$ to (5.50):

$$\begin{aligned} f = f_0 + \left(\int_{\mathbf{S}} f\right) w &= -\frac{1}{2}\partial_\theta\left(e^{-\Phi}\partial_\theta\left(e^{\Phi}u_0\right)\right) + u_0 + p_w(u)w, \\ &= -\tilde{A}_2 u_0 + u = (1-\tilde{A}_2)u. \end{aligned} \tag{5.51}$$

But since those functions f are dense in $\mathbf{H}_w^{-1}$, we see that the range of $1-\tilde{A}_2$ is dense so that $\tilde{A}_2$ is essentially self-adjoint.
Secondly, the spectral gap estimation (5.40) holds: indeed, for every $u \in \mathcal{C}^2(\mathbf{S})$, we have using (5.47) and Poincaré inequality:

$$\begin{aligned} -\left\langle \tilde{A}_2 v\,,\, v\right\rangle_{-1,w} &= \frac{1}{2}\left(\int_{\mathbf{S}} e^{-\Phi}\right)\int_{\mathbf{S}} e^{\Phi} v_0^2, \\ &\geqslant \frac{1}{2}e^{-2Kr_0}\left(\int_{\mathbf{S}} e^{-\Phi}\right)\int_{\mathbf{S}} \mathcal{V}_0^2 \geqslant \frac{1}{2}e^{-4Kr_0}\left(\int_{\mathbf{S}} e^{-\Phi}\right)\int_{\mathbf{S}} e^{\Phi}\mathcal{V}_0^2, \\ &= \frac{1}{2}e^{-4Kr_0}\left\| v - \left(\int_{\mathbf{S}} v\right) w\right\|_{-1,w}^2. \end{aligned}$$

Moreover, $\tilde{A}_2$ has compact resolvent: it suffices to prove that $\lambda - \tilde{A}_2$ has compact resolvent for at least one value of λ. We prove it for $\lambda = 1$ which is indeed in the resolvent set, thanks to the beginning of this proof. For $u \in \mathbf{H}_w^{-1}$, let us consider $f := (1-\tilde{A}_2)^{-1}u$ so that $\left\langle f\,,\,(1-\tilde{A}_2)f\right\rangle_{-1,w} = \langle f\,,\,u\rangle_{-1,w}$. Using the coerciveness of $\mathcal{E}_2$, one has for some constant c, $c\,\|\,f\,\|_{2,w}^2 \leqslant \langle f\,,\,u\rangle_{-1,w} \leqslant \|\,f\,\|_{-1,w}\,\|\,u\,\|_{-1,w}$. Using the continuous injection of $\mathbf{L}_w^2$ into $\mathbf{H}_w^{-1}$ (say $\|\cdot\|_{-1,w} \leqslant C\,\|\cdot\|_{2,w}$, for some positive constant C), one has

$$\|\,f\,\|_{2,w} \leqslant \frac{C}{c}\,\|\,u\,\|_{-1,w}\,.$$

So $(1-\tilde{A}_2)^{-1}$ maps sequences that are bounded in $\mathbf{H}_w^{-1}$ into sequences that are bounded in $\mathbf{L}_w^2$. It remains then to prove that the injection of $\mathbf{L}_w^2$ into $\mathbf{H}_w^{-1}$ is compact. This is indeed true since for every $v \in \mathbf{H}_w^{-1}$, one has, by Cauchy-Schwartz inequality $|\mathcal{V}_0(\theta) - \mathcal{V}_0(\theta')| \leqslant C\,\|\,v_0\,\|_{2,w}\sqrt{|\theta-\theta'|} \leqslant C\,\|\,v\,\|_{2,w}\sqrt{|\theta-\theta'|}$. That means that, by Ascoli-Arzela Theorem that the sets $\left\{v \in \mathbf{H}_w^{-1};\ \|\,v\,\|_{2,w} \leqslant cst\right\}$ are relatively compact in $\mathcal{C}(\mathbf{S})$ and also in $\mathbf{L}_w^2$. That completes the proof.
The fact that $\tilde{A}_2$ generates an analytic semigroup $\tilde{T}_2(z)$ on the same sector Δ_α as well as estimation (5.41) can be derived in the same way as in § 4.1.1.
That concludes the proof of Proposition 5.24.

4.2 Control on the perturbation B

In order to derive spectral properties for the operator $L = A + B$, we need to have a precise estimation about the smallness of the perturbation B w.r.t. operator A studied in the previous paragraph § 4.1.

Remark 5.26. Note that for simplicity, we work now with the norm $\|\cdot\|_H$ (recall Remark 5.6); as already mentioned this norm is equivalent to the norm $\|\cdot\|_{H_w}$ used in § 4.1. Recall also the definition of the space $(\mathbf{L}^2, \|\cdot\|_2)$ defined in Remark 5.5 and of $(\mathbf{L}^2_\mu, \|\cdot\|_{\mu,2})$ defined in Remark 5.6.

Secondly, since the whole operator L is no longer self-adjoint nor symmetric, its spectrum need not be real. Thus, we will assume for the rest of this document that we work with the complexified versions of the scalar products defined previously in this chapter. The results concerning the operator A are obviously still valid.

The smallness of the perturbation B with respect to A can be quantified in terms of the difference $\| q(\cdot,\omega) - q_0(\cdot) \|_\infty$, $\omega \in \mathrm{Supp}(\mu)$. For the ease of exposition, we do not attempt to derive precise estimations in K of this difference $\| q(\cdot,\omega) - q_0(\cdot) \|_\infty$ (Lemma 5.27) and of coefficients $a(\omega_0)$ and $b(\omega_0)$ (Lemma 5.28); these bounds are similar to the ones calculated in [41]. We will denote as c a positive constant (depending on K) which may change from a line to another.

Lemma 5.27. *For $\omega > 0$ and $K > 1$, let us define*

$$\| q - q_0 \|_\infty := \sup_{\theta\in\mathbf{S},\, |u| \leqslant \omega} |q(\theta,u) - q_0(\theta)|.$$

Then $\| q - q_0 \|_\infty = O(\omega)$, as $\omega \to 0$.

Proof. This is clear since one can bound $\partial_\omega q(\theta,\omega)$ uniformly in (θ,ω), as $\omega \to 0$ (by a constant depending on K). □

The purpose of the following proposition is to prove the boundedness w.r.t. A of the perturbation B in terms of the size of the disorder:

Proposition 5.28. *The operator B is A-bounded in the sense that there exist positive constants $a(\omega_0) = a(\omega_0, K)$ and $b(\omega_0) = b(\omega_0, K)$ such that*

$$\forall h \in \mathcal{D}, \quad \| Bh \|_H \leqslant a(\omega_0) \| h \|_H + b(\omega_0) \| Ah \|_H , \tag{5.52}$$

and moreover, for fixed $K > 1$,

$$a(\omega_0) = O(\omega_0), \text{ and } b(\omega_0) = O(\omega_0), \quad \text{as } \omega_0 \to 0, \tag{5.53}$$

Proof of Proposition 5.28. Recall that in what follows, c is a constant (depending on K) that may change from a line to another. The proof consists in two steps:

1. There exist some constant α_{K,ω_0} such that for all $h \in \mathcal{D}$,

$$\| Bh \|_H \leqslant \alpha_{K,\omega_0} \| h \|_{\mu,2} , \tag{5.54}$$

Indeed, for given $h \in \mathcal{D}$, for all $\omega \in \mathrm{Supp}(\mu)$, we have $\|Bh(\cdot,\omega)\|_{-1,\frac{1}{2\pi}} = \| \mathcal{B}h(\cdot,\omega) \|_2$, where $\mathcal{B}h(\cdot,\omega)$ is the appropriate primitive of $Bh(\cdot,\omega)$ in $\mathbf{H}^{-1}_{\frac{1}{2\pi}}$, namely:

$$\begin{aligned}\mathcal{B}h := & -h\left(\langle J * (q - q_0)\rangle_\mu + \omega\right) - (q - q_0) \cdot \langle J * h\rangle_\mu \\ & + \int_{\mathbf{S}} h \cdot \left(\langle J * (q - q_0)\rangle_\mu + \omega\right) + \int_{\mathbf{S}} (q - q_0) \cdot \langle J * h\rangle_\mu .\end{aligned}$$

Using the boundedness of q_0 and the fact that $|(J*\varepsilon)| \leqslant 4K\|\varepsilon\|_\infty$ and $|\langle J * h\rangle_\mu| \leqslant \frac{K}{\sqrt{2}}\|\langle h\rangle_\mu\|_2$, it is easy to deduce that, for some constant $c>0$:

$$|\mathcal{B}h| \leqslant c(\|q-q_0\|_\infty+\omega_0)(|h|+\|\langle h\rangle_\mu\|_2). \tag{5.55}$$

Consequently,

$$\|Bh\|_H = \left(\langle\|\mathcal{B}h\|_2^2\rangle_\mu\right)^{\frac{1}{2}} \leqslant c(\|q-q_0\|_\infty+\omega_0)\|h\|_{\mu,2}, \tag{5.56}$$

so that (5.54) is satisfied for some coefficients which verify $\alpha_{K,\omega_0}=O(\omega_0)$ as $\omega_0\to 0$.

2. The second step of the proof is to control the $\mathbf{L}^2$-norm $\|h\|_{\mu,2}$ of h with the $\mathbf{H}^{-1}$-norms of Ah and h: namely we prove that there exist constants such that

$$\|h\|_{\mu,2} \leqslant \gamma_K\|Ah\|_H + \delta_K\|h\|_H. \tag{5.57}$$

The proof is based on a usual interpolation argument: for all integer $n>1$, for any $f\in\mathcal{C}^2(\mathbf{S})$, one has

$$\|f'\|_2^2 \leqslant \sqrt{n}\|f\|_2\frac{\|f''\|_2}{\sqrt{n}} \leqslant \frac{n}{2}\|f\|_2^2+\frac{\|f''\|_2^2}{2n}. \tag{5.58}$$

Let us use this interpolation relation (5.58) to derive (5.57): for all $h\in\mathcal{D}$, $\omega\in\mathrm{Supp}(\mu)$, one has

$$\|h(\cdot,\omega)\|_2^2 = \left(\int_{\mathbf{S}} h(\cdot,\omega)\right)^2 + \|h_0(\cdot,\omega)\|_2^2$$

Applying relation (5.58) with $f(\cdot)=\mathcal{H}_0(\cdot,\omega)$ we obtain

$$\|h(\cdot,\omega)\|_2^2 \leqslant \left|\int_{\mathbf{S}} h(\cdot,\omega)\right|^2 + \frac{n}{2}\|\mathcal{H}_0(\cdot,\omega)\|_2^2 + \frac{\|h'(\cdot,\omega)\|_2^2}{2n}, \tag{5.59}$$

where we used the fact that $h_0'(\cdot,\omega)=h'(\cdot,\omega)$. Integrating w.r.t. μ,

$$\|h\|_{\mu,2}^2 \leqslant \int_{\mathbf{R}}\left|\int_{\mathbf{S}} h\right|^2 \mathrm{d}\mu + \frac{n}{2}\|\mathcal{H}_0\|_{\mu,2}^2 + \frac{\|h'\|_{\mu,2}^2}{2n}. \tag{5.60}$$

But, as previously for the operator B, a simple calculation shows that for all $\omega\in\mathrm{Supp}(\mu)$, we have $\|Ah(\cdot,\omega)\|_{-1,\frac{1}{2\pi}} = \|\mathcal{A}h(\cdot,\omega)\|_2$, where $\mathcal{A}h$ is the appropriate primitive of Ah (recall (5.34)) in $\mathbf{H}^{-1}_{\frac{1}{2\pi}}$:

$$\mathcal{A}h = \frac{1}{2}h' - h(J*q_0) - q_0\langle J*h\rangle_\mu + \int_{\mathbf{S}}\left(h(J*q_0)+q_0\langle J*h\rangle_\mu\right), \tag{5.61}$$

so that, for some constant $c>0$

$$\|h'\|_{\mu,2}^2 \leqslant 12\|\mathcal{A}h\|_{\mu,2}^2 + c\|h\|_{\mu,2}^2. \tag{5.62}$$

Injecting this inequality in (5.60), one obtains, for some constant $c>0$

$$\|h\|_{\mu,2}^2 \leqslant \int_{\mathbf{R}}\left|\int_{\mathbf{S}} h\right|^2 \mathrm{d}\mu + \frac{n}{2}\|\mathcal{H}_0\|_{\mu,2}^2 + \frac{1}{2n}\left(12\|\mathcal{A}h\|_{\mu,2}^2 + c\|h\|_{\mu,2}^2\right). \tag{5.63}$$

Choosing $n > 1$ sufficiently large so that the coefficient in front of $\| h \|_{\mu,2}^2$ in the right-hand side of (5.63) is lower than $\frac{1}{2}$ leads to (for some constant $c > 0$):

$$\| h \|_{\mu,2}^2 \leqslant 2 \int_{\mathbf{R}} \left| \int_{\mathbf{S}} h \right|^2 \mathrm{d}\mu + c \| \mathcal{H}_0 \|_{\mu,2}^2 + c \| \mathcal{A}h \|_{\mu,2}^2 \leqslant c \| h \|_H^2 + c \| Ah \|_H^2 ,$$

which shows (5.57). Putting together estimates (5.54) and (5.57), we find the A-boundedness of B (5.52) with coefficients $a(\omega_0)$ and $b(\omega_0)$ which satisfy (5.53), thanks to Lemma 5.27. □

Proposition 5.29. *The operator B is A-compact, in the sense that for any sequence $(h_p)_{p \geqslant 0} \in \mathcal{D}^{\mathbf{N}}$ such that $\| h_p \|_H$ and $\| Ah_p \|_H$ are bounded, there exists a convergent subsequence for $(Bh_p)_{p \geqslant 1}$.*

Proof of Proposition 5.29. Let $(h_p)_{p \geqslant 0}$ a sequence in $\mathcal{D}$ such that $\| h_p \|_H$ and $\| Ah_p \|_H$ are bounded by a constant c. A closer look at the operator B defined in (5.35) and the definition of the norm $\| \cdot \|_H$ in (5.13) shows that it suffices to prove that there exists a subsequence (h_{p_k}) such that (h_{p_k}) converge in $\mathbf{L}_\mu^2$.

In particular, for all $p \geqslant 0$, $\| Ah_p \|_H \leqslant c$. Using this boundedness and (5.62), we have $\left\| h_p' \right\|_{\mu,2} \leqslant c + c \| h_p \|_{\mu,2}$, so that

$$\left\| h_p' \right\|_{\mu,2} \leqslant c + c \left(\int_{\mathbf{R}} \left| \int_{\mathbf{S}} h \right|^2 \mathrm{d}\mu + \| h_{0,p} \|_{\mu,2}^2 \right), \tag{5.64}$$

$$\leqslant c + c \left(\int_{\mathbf{R}} \left| \int_{\mathbf{S}} h \right|^2 \mathrm{d}\mu + \frac{n}{2} \| \mathcal{H}_{0,p} \|_{\mu,2}^2 + \frac{\left\| h_p' \right\|_{\mu,2}^2}{2n} \right), \tag{5.65}$$

where we used again (5.58) for $f = \mathcal{H}_{0,p}(\cdot, \omega)$. Choosing a sufficiently large $n > 1$ leads to $\left\| h_{0,p}' \right\|_{\mu,2} = \left\| h_p' \right\|_2 \leqslant c + c \| h_p \|_H \leqslant c$ for a constant c independent of $p \geqslant 0$. An easy application of Cauchy-Schwartz inequality leads to the inequality $|h_{0,p}(\theta, \omega) - h_{0,p}(\theta', \omega)| \leqslant \left\| h_{0,p}' \right\|_{\mu,2} \sqrt{|\theta - \theta'|}$, for all $\omega \in \{\pm\omega_0\}$. Since the functions $(h_{0,p})$ are such that $\int_{\mathbf{S}} h_{0,p} = 0$ for all $p \geqslant 0$, Ascoli-Arzela Theorem and the previous bound show the existence of a convergent subsequence (h_{0,p_k}) (for each $\omega \in \mathrm{Supp}(\mu)$) in the space of continuous functions on $\mathbf{S}$. In particular, this subsequence is convergent in $\mathbf{L}_\mu^2$ and is renamed $(h_{0,p})_{p \geqslant 0}$, with a slight abuse of notations.

The fact that $\int_{\mathbf{R}} | \int_{\mathbf{S}} h_p | \, \mathrm{d}\mu \leqslant c$ shows that one can extract a further subsequence of (h_p) which is also convergent in $\mathbf{L}_\mu^2$. This concludes the proof. □

4.3 Spectral properties of $L = A + B$

We are now in position to extent the spectral properties we obtained in § 4.1 for the operator A to the perturbed operator $L = A + B$, relying on the control we have the perturbation B (see § 4.2) and on the theory of perturbation of operators ([51]) and analytic semi-groups ([70]).

4.3.1 The spectrum of L is pure point

Proposition 5.30. *For all K, for all $\omega > 0$,*

1. *the operator $(L, \mathcal{D})$ is closable. In that case, its closure has the same domain as the closure of A,*
2. *the closure of L has compact resolvent. In particular, its spectrum is pure point.*

Remark 5.31. Note that Proposition 5.30 is valid without any assumption on the smallness of ω, since it relies on the relative compactness of B with respect to A (Prop. 5.29).

Proof of Proposition 5.30. It is a simple consequence of the relative compactness of B w.r.t. the self-adjoint operator A. The first assertion of Proposition 5.30 is a consequence of [51, Th. 1.11, p.194] and the second assertion can be found in [58, Lemma 3.6, p.17] for example. □

4.3.2 L generates an analytic operator

We prove mainly that the perturbed operator L generates an analytic semigroup of operators on a appropriate sector. An immediate corollary is the position of the spectrum in a cone whose vertex is zero.

We know (Proposition 5.24) that for all $0 < \alpha < \frac{\pi}{2}$, A generates a semigroup of operators on $\Delta_\alpha = \{z \in \mathbf{C};\ |\arg(z)| < \alpha\}$.

Proposition 5.32. *For all $K > 1$, $\varepsilon > 0$ and $0 < \alpha < \frac{\pi}{2}$, there exists $\omega_1 > 0$ (depending on α, K and ε) such that for all $0 < \omega < \omega_1$, the spectrum of L lies within the sector $\Theta_{\varepsilon,\alpha} := \left\{\lambda \in \mathbf{C};\ \frac{\pi}{2} + \alpha \leqslant \arg(\lambda) \leqslant \frac{3\pi}{2} - \alpha\right\} \cup \{\lambda \in \mathbf{C};\ |\lambda| \leqslant \varepsilon\}$. For such ω, L generates an analytic semigroup of operators on $\Delta_{\alpha'}$, for some $\alpha' \in (0, \frac{\pi}{2})$.*

Proof of Proposition 5.32. Let $0 < \alpha < \frac{\pi}{2}$ be fixed. Thanks to (5.43), there exists a constant $c > 0$ (which comes from the equivalence of the norms $\|\cdot\|_H$ and $\|\cdot\|_{H_w}$) such that for every $\lambda \in \Sigma_\alpha := \{\lambda \in \mathbf{C};\ |\arg(\lambda)| < \frac{\pi}{2} + \alpha\}$, we have the estimates:

$$\| R(\lambda, A) \|_H \leqslant \frac{c}{(1 - \sin(\alpha))|\lambda|} \quad \text{and,} \quad \| AR(\lambda, A) \|_H \leqslant 1 + \frac{c}{(1 - \sin(\alpha))}.$$

Then for $\lambda \in \Sigma_\alpha$, $h \in \mathcal{D}$:

$$\begin{aligned} \| BR(\lambda, A)h \|_H &\leqslant a(\omega) \| R(\lambda, A)h \|_H + b(\omega) \| AR(\lambda, A)h \|_H, \\ &\leqslant \left(a(\omega) \frac{c}{(1 - \sin(\alpha))|\lambda|} + b(\omega) \left(1 + \frac{c}{1 - \sin(\alpha)} \right) \right) \| h \|_H. \end{aligned}$$

Let us fix $\varepsilon > 0$ and choose ω_1 such that:

$$\max\left(\frac{4a(\omega_1)c}{\varepsilon(1 - \sin(\alpha))}, 4b(\omega_1) \left(1 + \frac{c}{1 - \sin(\alpha)} \right) \right) \leqslant 1.$$

For this choice of ω_1, for all $0 < \omega < \omega_1$, for any $\lambda \in \Sigma_\alpha$ such that $|\lambda| \geqslant \varepsilon \geqslant \frac{4a(\omega_1)c}{1 - \sin(\alpha)}$, we have $\| BR(\lambda, A)h \|_H \leqslant \frac{1}{2} \| h \|_H$, and thus the operator $1 - BR(\lambda, A)$ is invertible with $\| 1 - BR(\lambda, A) \|_H \leqslant 2$. Since it can easily be shown that

$$R(\lambda, A + B) = R(\lambda, A) \left(1 - BR(\lambda, A)\right)^{-1},$$

one easily deduces the following estimates about the resolvent of the perturbed operator $L = A + B$:

$$\forall \lambda \in \Sigma_\alpha,\ |\lambda| \geqslant \varepsilon,\ \| R(\lambda, L) \|_H \leqslant \frac{2}{(1 - \sin(\alpha))|\lambda|}. \tag{5.66}$$

The fact that the spectrum of L lies within $\Theta_{\varepsilon,\alpha}$ is a straightforward consequence of (5.66). Secondly, (5.66) entails that L generates an analytic semigroup of operators on an appropriate sector. Indeed, if one denotes by $L_\varepsilon := L - \varepsilon$, one deduces from (5.66) that $0 \in \rho(L_{2\varepsilon})$ and that for all $\lambda \in \mathbf{C}$ with $\Re(\lambda) > 0$ (in particular, $|\lambda| < |\lambda + 2\varepsilon|$)

$$\begin{aligned} \| R(\lambda, L_{2\varepsilon}) \|_H &= \| R(\lambda + 2\varepsilon, L) \|_H \leqslant \frac{2}{(1-\sin(\alpha))|\lambda + 2\varepsilon|}, \\ &\leqslant \frac{2}{(1-\sin(\alpha))|\lambda|}. \end{aligned} \tag{5.67}$$

Hence, using the same arguments of Taylor expansion as in the proof of Proposition 5.22 and applying Proposition 3.44, one easily sees that $L_{2\varepsilon}$ generates an analytic semigroup in a (a priori) smaller sector $\Delta_{\alpha'}$, where $\alpha' \in (0, \frac{\pi}{2})$ can be chosen as $\alpha' := \frac{1}{2}\arctan\left(\frac{1-\sin(\alpha)}{2}\right)$. But if $L_{2\varepsilon}$ generates an analytic semigroup, so does L. □

4.3.3 0 is an eigenvalue with multiplicity 2

Let us fix $K > 1$, $\alpha \in (0, \frac{\pi}{2})$, $\rho \in (0,1)$ and define $\varepsilon = \rho\lambda_K(A)$. Applying Proposition 3.50, we know that for small ω (depending on K, α, ρ), L generates an analytic semigroup on $\Theta_{\varepsilon,\alpha}$. Let $\Theta^+_{\varepsilon,\alpha} := \{\lambda \in \Theta_{\varepsilon,\alpha};\ \Re(\lambda) \geqslant 0\}$ be the subset of $\Theta_{\varepsilon,\alpha}$ which lies in the positive part of the complex plane.

The purpose of this paragraph is to show that one can choose a perturbation B small enough so that no non-zero eigenvalue of $A + B$ remains in the small set $\Theta^+_{\varepsilon,\alpha}$.

To do so, we proceed by an argument of local perturbation: we know that the distance $\lambda_K(A) = \min\left(\lambda_K(\tilde{A}_1), \lambda_K(\tilde{A}_2)\right)$ between the eigenvalue 0 and the rest of the spectrum of A is strictly positive. In particular, one can separate 0 from the rest of the spectrum of A by a circle $\mathscr{C}$ centered in 0 with radius $\left(\frac{\rho+1}{2}\right)\lambda_K(A)$. Note that the appropriate choice of ε made at the beginning of this paragraph ensures that the interior of $\mathscr{C}$ contains $\Theta^+_{\varepsilon,\alpha}$.

The main argument is the following: by construction of $\mathscr{C}$, 0 is the only eigenvalue (with multiplicity 2) of the non-perturbed operator A lying in the interior of $\mathscr{C}$. A principle of continuity of eigenvalues shows that, while adding a small enough perturbation B to A, the interior of $\mathscr{C}$ still contains either one eigenvalue with algebraic multiplicity 2 or two eigenvalues with multiplicity 1; those perturbed eigenvalues remain close but are *a priori* not equal to the initial eigenvalue 0 (see Figure 5.1).

But we already know that for the perturbed operator $L = A + B$, 0 is always an eigenvalue (since $Lq' = 0$ and $Lp = q'$, recall Th. 5.7). Therefore, the algebraic multiplicity of 0 for the operator L is *at least* 2. By uniqueness, one can conclude that 0 is the *only* element of the spectrum of L within $\mathscr{C}$, and is an eigenvalue with algebraic multiplicity *exactly* 2. In particular, there is no element of the spectrum in the positive part of the complex plane.

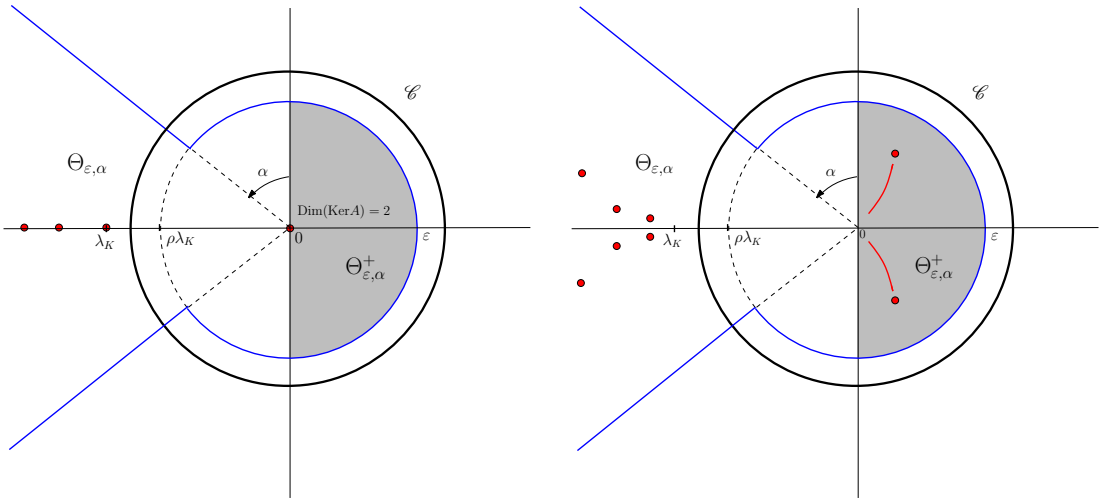
In order to make this argument precise, we need to quantify the appropriate size of the perturbation B, by explicit estimates on the resolvent $R(\lambda, A)$ on the circle $\mathscr{C}$:

Lemma 5.33. *There exists some explicit constant $c_{\mathscr{C}}(K)$ only dependent on K, such that for all $\lambda \in \mathscr{C}$,*

$$\| R(\lambda, A) \|_H \leqslant c_{\mathscr{C}}(K), \tag{5.68}$$

$$\| AR(\lambda, A) \|_H \leqslant 1 + \left(\frac{\rho+1}{2}\right) \cdot c_{\mathscr{C}}(K). \tag{5.69}$$

One can choose $c_{\mathscr{C}}(K)$ as $c_{\mathscr{C}}(K) = \frac{1}{\lambda_K(A)} \max\left(\frac{2}{\rho+1}, \frac{2}{1-\rho}\right)$.



(a) Position of the spectrum for the self-adjoint operator A

(b) Possible position of the spectrum for the operator L

Figure 5.1: The sets $\Theta_{\varepsilon,\alpha}$ and $\Theta^+_{\varepsilon,\alpha}$. Note that the two dimensional eigenvalue 0 for the operator A (Fig. 5.1a) may split in two single eigenvalues for the perturbed operator L (Fig. 5.1b). These eigenvalues are the only ones within the circle $\mathscr{C}$. But since we already know that the kernel of L is of dimension 2, 0 is still a double eigenvalue for L, by uniqueness.

Proof of Lemma 5.33. Applying the spectral theorem (see [33, Th. 3, p.1192]) to the essentially self-adjoint operator A, there exists a spectral measure E vanishing on the complementary of the spectrum of A such that $A = \int_{\mathbf{R}} \lambda \, \mathrm{d}E(\lambda)$. In that extent, one has for any $\zeta \in \mathscr{C}$

$$R(\zeta, A) \;=\; \int_{\mathbf{R}} \frac{\mathrm{d}E(\lambda)}{\lambda - \zeta}\,.$$

In particular, for $\zeta \in \mathscr{C}$

$$\| R(\zeta, A) \|_H \;\leqslant\; \sup_{\lambda \in \sigma(A)} \frac{1}{|\lambda - \zeta|} \leqslant \frac{\max\left(\frac{2}{\rho+1}, \frac{2}{1-\rho}\right)}{\lambda_K(A)}\,.$$

The estimation (5.69) is straightforward. □

We are now in position to apply our argument of local continuity of eigenvalues: following [51], Th III-6.17, p.178, there exists a decomposition of the operator A according to $H = F_0 \oplus F_{<0}$ (in the sense that $AF_0 \subset F_0$, $AF_{<0} \subset F_{<0}$ and $P_0\mathcal{D} \subset \mathcal{D}$, where P_0 is the projection on F_0 along $F_{<0}$) in such a way that A restricted to F_0 has spectrum $\{0\}$ and A restricted to $F_{<0}$ has spectrum $\sigma(A) \setminus \{0\} \subseteq \{\lambda \in \mathbf{C};\, \Re(\lambda) < 0\}$. Let us note that the dimension of F_0 is *exactly* 2, since the characteristic space of A for the eigenvalue 0 is reduced to its kernel which is of dimension 2 (see Prop. 5.24).

Then, applying [51], Th. IV-3.18, p.214 and using Proposition 5.28, we find that if one chooses $\omega_2 > 0$, such that

$$\sup_{\lambda \in \mathscr{C}} \left(a(\omega_2) \, \| R(\lambda, A) \|_H + b(\omega_2) \, \| AR(\lambda, A) \|_H \right) < 1, \tag{5.70}$$

then for all $0 < \omega < \omega_2$, the perturbed operator L is likewise decomposed according to $H = G_0 \oplus G_{<0}$, in such a way that $\dim(F_0) = \dim(G_0) = 2$, and that the spectrum of L is

again separated in two parts by $\mathscr{C}$. But thanks to Theorem 5.7, we already know that the characteristic space of the perturbed operator L according to the eigenvalue 0 is *at least* of dimension 2 (since $Lq' = 0$ and $Lp = q'$). We can conclude, that for such an $0 < \omega < \omega_2$, 0 is the only eigenvalue in $\mathscr{C}$ and that $\dim(G_0)$ is *exactly* 2.

Applying Lemma 5.33, we see that condition (5.70) is satisfied if we choose $\omega_2 > 0$ so that:

$$a(\omega_2)c_{\mathscr{C}}(K) + b(\omega_2)\left(1 + \left(\frac{\rho+1}{2}\right)c_{\mathscr{C}}(K)\right) < 1. \tag{5.71}$$

In particular, in that case, the spectrum of L is contained in $\{z \in \mathbf{C};\ \Re(z) \leqslant 0\}$.

Theorem 5.9 is proved.

5 Non self-averaging phenomenon for the fluctuation process (Proof of Theorem 5.3)

The purpose of this section is to prove Theorem 5.3.

In our framework, (recall that $\mu = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$), the solution η of evolution (5.1) in $\mathcal{S}'(\mathbf{S} \times \mathbf{R})$ acts on test functions φ of the form $\varphi = (\varphi(\cdot, +\omega_0), \varphi(\cdot, -\omega_0))$. In particular, one can understand η as an element of H by identifying η with $(\eta^{\omega_0}, \eta^{-\omega_0})$, where, for any smooth function $\psi : \mathbf{S} \to \mathbf{R}$, $\eta^{\omega_0}(\psi) := \eta(\psi, 0)$ and $\eta^{-\omega_0}(\psi) := \eta(0, \psi)$. Defining analogously $W^{\pm\omega_0}$ for the Wiener process W in (5.2), the object of interest is then

$$\forall t > 0,\ \eta_t = \eta_0 + \int_0^t L\eta \,\mathrm{d}s + W_t. \tag{5.72}$$

5.1 The noise W as a cylindrical Brownian Motion

We first focus on the regularity properties of the noise W: in the stationary case ($q_t = q_t|_{t=0} = q$ for all $t > 0$) the covariance defined in (5.2) becomes, for any regular functions on $\mathbf{S} \times \mathbf{R}$, φ_1 and φ_2, $s, t > 0$:

$$\mathbf{E}\left(W_s(\varphi_1)W_t(\varphi_2)\right) = (s \wedge t)\int_{\mathbf{R}}\int_{\mathbf{S}} \varphi_1'\varphi_2' q \,\mathrm{d}\lambda\,\mathrm{d}\mu. \tag{5.73}$$

Consequently, it is easily seen that $(W_t^{+\omega_0}, W_t^{-\omega_0})$ is a couple of two independent Gaussian processes with covariance (where $\psi_1, \psi_2 : \mathbf{S} \to \mathbf{R}$):

$$\forall \omega \in \{+\omega_0, -\omega_0\},\ \mathbf{E}\left(W_s^{\omega}(\psi_1)W_t^{\omega}(\psi_2)\right) = \frac{1}{2}(s \wedge t)\int_{\mathbf{S}} \psi_1'\psi_2' q(\cdot, \omega). \tag{5.74}$$

In what follows, we will denote by H_0 the closed subspace of H consisting of elements of H with zero mean-value; in particular the norm $\|\cdot\|_H$ defined in (5.13) coincides on H_0 with:

$$\forall h \in H_0,\ \|h\|_H = \left(2\pi\int_{\mathbf{R}}\int_{\mathbf{S}} \mathcal{H}^2\right)^{\frac{1}{2}}, \tag{5.75}$$

where we recall that $\mathcal{H}$ is the primitive of h such that $\int_{\mathbf{S}} \mathcal{H} = 0$. The purpose of this paragraph is to show that, following [26, p. 96], W has the same law as a Q-Wiener process in the Hilbert space H_0, for an appropriate bounded symmetric operator Q on H_0: indeed, if one denotes by X a Q-Wiener process on H_0, with the following definition of Q

$$\forall h \in H_0,\ Qh := \partial_\theta\left(q\mathcal{H}\right), \tag{5.76}$$

then one readily verifies that the Gaussian process $(W(\varphi))_\varphi$ has the same law as the process $(X(\varphi))_\varphi := \left(\left\langle X_t\,,\,\binom{\varphi''}{0}\right\rangle_H, \left\langle X_t\,,\,\binom{0}{\varphi''}\right\rangle_H\right)$.

Note that if the weight q had not been present in (5.74), it would have been straightforward to see that W is a Q-Wiener process on H_0, with $Q = I$. The fact that this weight is present entails some technical complications, but one has to consider the operator Q defined in (5.76) only as a perturbation of the case $Q = I$.

5.2 Existence and uniqueness of a weak solution to the fluctuation equation

We now turn to the existence and uniqueness of a weak solution of (5.72). We recall that any H-valued predictable process η_t, $t \in [0,T]$ is a *weak solution* of (5.72) if the trajectories of η are almost surely Bochner integrable and if for all $\varphi \in \mathcal{D}(L^*)$ and for all $t \in [0,T]$

$$\eta_t(\varphi) = \eta_0(\varphi) + \int_0^t \eta_s(L^*\varphi)\,\mathrm{d}s + W_t(\varphi). \tag{5.77}$$

Proposition 5.34. *Equation has a unique weak solution in H, given by the mild formulation*

$$\eta_t = T_L(t)\eta_0 + \int_0^t T_L(t-s)\,\mathrm{d}W_s, \quad t \in [0,T]. \tag{5.78}$$

In order to prove Proposition 5.34, one has to define properly the stochastic convolution $W_L(t) := \int_0^t T_L(t-s)\,\mathrm{d}W_s$. In this purpose, let use prove firstly that the inverse of A is of class trace:

Lemma 5.35. *The operator A^{-1} is of class trace in H. Equivalently, if $(\lambda_n^{(A)})_{n\geqslant 1}$ is the sequence of eigenvalues of the self-adjoint operator A, one has*

$$\sum_{n=1}^{+\infty} \frac{1}{\lambda_n^{(A)}} < +\infty. \tag{5.79}$$

Proof of Lemma 5.35. Since A and $\tilde{A} = M \circ A \circ M^{-1}$ (recall (5.37)) are conjugate, it suffices to prove (5.79) when A is replaced by $\tilde{A}$. The idea of the proof is that identity (5.79) is true when $\tilde{A} = (\tilde{A}_1, \tilde{A}_2)$ is replaced by $(-\Delta, -\Delta)$ and that $\tilde{A}$ is only a relatively-bounded perturbation of this case. More precisely, the proof relies on the following MinMax Principle [34, p. 1543]:

Proposition 5.36. *Let $(F, \mathcal{D}(F))$ a self-adjoint linear operator on a separable Hilbert space $\mathcal{H}$ such that F is positive, with compact resolvent. We denote by $\mathcal{S}^n$ the family of n-dimensional subspace of $\mathcal{H}$, and for $n \geqslant 1$ we let λ_n the number defined as follows*

$$\lambda_n := \sup_{G\in\mathcal{S}^n} \inf_{u\in(G\cap\mathcal{D}(F))\setminus\{0\}} \frac{\langle u\,,\,Fu\rangle_{\mathcal{H}}}{\langle u\,,\,u\rangle_{\mathcal{H}}}. \tag{5.80}$$

Then there exists a complete orthonormal system $(\psi_n)_{n\geqslant 1}$ such that

$$F\psi_n = \lambda_n\psi_n, \quad n \geqslant 1.$$

In other words, the sequence $(\lambda_n)_{n\geqslant 1}$ is the non-decreasing enumeration of the eigenvalues of F, each repeated a number of times equal to its multiplicity. Moreover, the sup *in* (5.80) *is attained for G equal to the span of $\{\psi_1, \ldots, \psi_n\}$.*

Let us apply Proposition 5.36 to $F = -\Delta$ with domain

$$\mathcal{D}(-\Delta) := \left\{ u;\, u \in \mathcal{C}^2(\mathbf{S}),\ \int_{\mathbf{S}} u(\theta)\,\mathrm{d}\theta = 0 \right\}, \tag{5.81}$$

in $\mathbf{H}^{-1}_{\frac{1}{2\pi}}$ (recall the definition of $\mathbf{H}^{-1}_{\frac{1}{2\pi}}$ in (5.12)) and let us denote by

$$\mathcal{E}_0(u,v) := \langle u\,,\, -\Delta v\rangle_{-1,\frac{1}{2\pi}} = 2\pi \int_{\mathbf{S}} uv$$

the Dirichlet form associated to $-\Delta$. Note that $\mathcal{E}_0$ is well defined on $\mathbf{L}^2 \supset \mathcal{D}(-\Delta)$. Then, denoting by $(\lambda_n^{(-\Delta)})_{n\geqslant 1}$ the sequence of eigenvalues associated to $-\Delta$ in $\mathbf{H}^{-1}_{\frac{1}{2\pi}}$:

$$\lambda_n^{(-\Delta)} = \sup_{G\in\mathcal{S}^n} \inf_{u\in(G\cap\mathcal{D}(-\Delta))\setminus\{0\}} \frac{\mathcal{E}_0(u,u)}{\langle u\,,\, u\rangle_{-1,\frac{1}{2\pi}}}.$$

Since the supremum is attained for $G = \{\psi_1,\dots,\psi_n\} \subseteq \mathbf{L}^2$, one has in fact:

$$\lambda_n^{(-\Delta)} = \sup_{G\in\mathcal{S}^n} \inf_{u\in(G\cap\mathbf{L}^2)\setminus\{0\}} \frac{\mathcal{E}_0(u,u)}{\langle u\,,\, u\rangle_{-1,\frac{1}{2\pi}}}.$$

Secondly, note that one does not change the result by considering $1 - A_1$ instead of $-A_1$. Hence, if ones denotes by $\mathcal{E}_1(u,v) := \langle u\,,\,(1-A_1)v\rangle_{-1,q_0}$ the Dirichlet form associated to $1 - A_1$, one deduces from [9, Eq.(2.47)] that $\mathcal{E}_1$ is well defined on $\mathbf{L}^2$ and that it is equivalent to $\mathcal{E}_0$: there exists a constant $C > 0$ such that

$$\forall u \in \mathbf{L}^2, \quad \frac{1}{C}\mathcal{E}_0(u,u) \leqslant \mathcal{E}_1(u,u) \leqslant C\mathcal{E}_0(u,u).$$

Then, using again Proposition 5.36,

$$\lambda_n^{(1-A_1)} = \sup_{G\in\mathcal{S}^n} \inf_{u\in(G\cap\mathbf{L}^2)\setminus\{0\}} \frac{\mathcal{E}_1(u,u)}{\langle u\,,\, u\rangle_{-1,q_0}}.$$

Since the norms $\|\cdot\|_{-1,\frac{1}{2\pi}}$ and $\|\cdot\|_{-1,q_0}$ are equivalent, one directly sees that there exist constants $c, C > 0$ such that, for all $n \geqslant 1$

$$c\lambda_n^{(-\Delta)} \leqslant \lambda_n^{(1-A_1)} \leqslant C\lambda_n^{(-\Delta)}. \tag{5.82}$$

One can prove similar bounds for A_2 in the Hilbert space $\mathbf{H}^{-1}_w$ in the same way: first notice that any eigenvector which corresponds to a non-zero eigenvalue of A_2 is necessarily with zero mean-value, so that it suffices to work on the domain $\{v \in \mathbf{L}^2,\ \int_{\mathbf{S}} v = 0\}$. It is then easy to deduce from (5.49) that both Dirichlet forms $\mathcal{E}_0$ and $\mathcal{E}_2$ (recall definition (5.48)) are equivalent on the subspace of $\mathbf{L}^2$ with zero mean-value. In particular using Proposition 5.36, one easily obtains similar bounds as (5.82) for A_2 and (5.79) follows. □

Then following the lines of [26], one can deduce from Lemma 5.35 the following:

Proposition 5.37 ([26], Prop. 5.25)**.** *For all $t > 0$, the linear operator*

$$\tilde{Q}_t := \int_0^t T_L(s) Q T_L(s)^*\,\mathrm{d}s$$

is of class trace.

Proof of Proposition 5.37. It is easy to see from the definition (5.35) of the perturbation B that B satisfies the following

– B is a continuous linear operator from $\mathbf{L}^2_\mu$ into H,

– there exists a constant $c > 0$ such that for all $h \in \mathcal{D}$, $\langle Bh\,,\,h\rangle_H \leqslant c\,\|\,h\,\|^2_H$.

Under this condition, for $\lambda > c$, for any $h \in \mathcal{D}$,

$$\left\langle \tilde{Q}_t(\lambda - L)^{\frac{1}{2}}h\,,\,(\lambda - L)^{\frac{1}{2}}h\right\rangle_H \leqslant C\,\|\,h\,\|^2_H\,.$$

Consequently, there exists a bounded linear operator $\tilde{G}_t$ such that

$$\tilde{Q}_t = (\lambda - L)^{-\frac{1}{2}}\tilde{G}_t(\lambda - L^\star)^{-\frac{1}{2}}.$$

So the linear operator

$$\widetilde{L}_t = -(-A)^{\frac{1}{2}}(\lambda - L)^{-\frac{1}{2}}\tilde{G}_t(\lambda - L^\star)^{-\frac{1}{2}}(-A)^{\frac{1}{2}}$$

is also bounded and $\operatorname{Tr}\tilde{Q}_t = \operatorname{Tr} A^{-1}\widetilde{L}_t < \infty$. □

Then an application of [26, Th. 5.2] shows that the stochastic convolution $W_L(\cdot)$ is well defined as a predictable process in H. The assumptions of [26, Th. 5.4] concerning the existence and uniqueness a weak solution of (5.72) are satisfied and Proposition 5.34 is proved.

5.3 Linear asymptotic behavior of the fluctuation process

We are in position to prove the main statement of Theorem 5.3, that is the asymptotic behavior (5.9) of the mild solution (5.78). We will make a constant use of Remark 5.10 about the spectral decomposition of the operator L (under the hypothesis of Theorems 5.7 and 5.9). We will refer in particular to the matrix representation (5.20) of L.

Note also that the continuous linear form $\ell_{q'}(\cdot)$ on H can be represented, by Riesz representation theorem, as a scalar product w.r.t. some vector $v \in H$: $\ell_{q'}(\cdot) = \langle v\,,\,\cdot\rangle_H$.

The convergence (5.9) is an easy consequence of Remark 5.11 and the following two propositions:

Proposition 5.38. *The stochastic convolution $W_L(t)$ satisfies the following linear behavior, as $t \to \infty$: $\frac{W_L(t)}{t} \to 0$, where the convergence is in law.*

Proposition 5.39. *For every initial condition η_0, $\frac{T_L(t)\eta_0}{t}$ converges, as $t \to \infty$, to $\ell_p(\eta_0)q'$.*

Let us now prove these two propositions:

Proof of Proposition 5.38. Recall that W is a Q-Wiener process in H_0, which can be decomposed into $H_0 = \operatorname{Span}(q') \oplus G_{<0}$. Note also that the restriction on H_0 of the projection P_0 defined on H by (5.19) coincides with $\ell_{q'}(\cdot)q'$. With a small abuse of notations, we will use the same notation P_0 for this restriction on H_0.

Let us decompose the stochastic convolution into $W_L(t) = \int_0^t T_L(t-s)P_0\,\mathrm{d}W_s + \int_0^t T_L(t-s)P_{<0}\,\mathrm{d}W_s$, and treat the two terms separately:

Let us consider the first term $\frac{1}{t}\int_0^t T_L(t-s)P_0\,\mathrm{d}W_s$. One has successively, using that $T_L(u)q' = q'$ for all $u > 0$

$$\frac{1}{t}\int_0^t T_L(t-s)P_0\,\mathrm{d}W_s = \frac{q'}{t}\int_0^t T_L(t-s)\ell_{q'}\,\mathrm{d}W_s, \tag{5.83}$$

$$= \frac{q'}{t}\ell_{q'}W_t = \frac{q'}{t}\,\langle v\,,\,W_t\rangle_H\,. \tag{5.84}$$

Thanks to the Q-Wiener structure of W (see § 5.1), one has

$$\mathbf{E}\left(\left\| \frac{1}{t}\int_0^t T_L(t-s)P_0\,\mathrm{d}W_s \right\|_H^2\right) = \frac{\| q' \|_H^2}{t^2}\mathbf{E}\left(|\langle v\,,\,W_t\rangle_H|^2\right) = \frac{\| q' \|_H^2}{t}\langle Qv\,,\,v\rangle_H\,,$$

which converges to 0 as $t \to \infty$.

As far as the second term $\int_0^t T_L(t-s)P_{<0}\,\mathrm{d}W_s$ is concerned, it is easy to see that it is the unique weak solution in H_0 of

$$w_t = \int_0^t Lw_s\,\mathrm{d}s + P_{<0}\,W_t. \tag{5.85}$$

Let us decompose evolution (5.85) along this decomposition $H_0 = \mathrm{Span}(q') \oplus G_{<0}$: writing $w_t = P_0\,w_t + P_{<0}\,w_t := y_t + z_t$, one has:

$$\left\{\begin{array}{lcl} z_t &=& \int_0^t P_0\,Ly_s\,\mathrm{d}s, \\ y_t &=& \int_0^t P_{<0}\,LP_{<0}\,y_s\,\mathrm{d}s + P_{<0}\,W_t. \end{array}\right.$$

Since the operator $P_{<0}\,LP_{<0}$ has its spectrum in the negative part of the complex plane with a strictly positive spectral gap $\lambda_K(L)$ and generates an semigroup of operators, it is immediate to see from the covariance estimates of stochastic convolutions (see [26, Th. 5.2, p.119]) that there exist some $t_0 > 0$ and a constant $c > 0$ such that for all $t \geqslant t_0$

$$\mathbf{E}\left(\| y_t \|_H\right)^2 \leqslant \mathbf{E}\left(\| y_t \|_H^2\right) \leqslant ce^{-\frac{\lambda_K(L)}{2}t} \tag{5.86}$$

Consequently, one has

$$\begin{aligned} \mathbf{E}\left(\| z_t \|_H\right) &\leqslant \int_0^t \mathbf{E}\left(\| P_0\,Ly_s \|_H\right)\,\mathrm{d}s = \| q' \|_H \int_0^t \mathbf{E}\left(|\ell_{q'}(Ly_s)|\right)\,\mathrm{d}s, \\ &= \int_0^t \mathbf{E}\left(|\langle v\,,\,Ly_s\rangle_H|\,\| q' \|_H\right)\,\mathrm{d}s, \\ &\leqslant \| q' \|_H \| L^*v \|_H \int_0^t \mathbf{E}\left(\| y_s \|_H\right)\,\mathrm{d}s, \\ &= \| q' \|_H \| L^*v \|_H \left(\int_0^{t_0} \mathbf{E}\left(\| y_s \|_H\right)\,\mathrm{d}s + \int_{t_0}^t \mathbf{E}\left(\| y_s \|_H\right)\,\mathrm{d}s\right). \end{aligned} \tag{5.87}$$

In particular, it is immediate from (5.86) and (5.87) to see that the following convergence holds:

$$\frac{\mathbf{E}\left(\| z_t \|_H\right)}{t} \to_{t\to\infty} 0. \tag{5.88}$$

Putting together (5.86) and (5.88), Proposition 5.38 is proved. □

Proof of Proposition 5.39. Let us fix an initial condition $\eta_0 \in H$. Then $X(t) := T_L(t)\eta_0$ is the unique solution in H of

$$X(t) = \eta_0 + \int_0^t LX_s\,\mathrm{d}s. \tag{5.89}$$

Decompose $X(t)$ along the direct sum $G_0 \oplus G_{<0}$, that is $X(t) = \alpha(t)q' + \beta(t)p + Y(t)$, with $Y(t) \in G_{<0}$. Then, projecting on q', p and $G_{<0}$ respectively (see (5.20)), one obtains that (5.89) is equivalent to

$$\forall t > 0, \quad \left\{\begin{array}{lcl} \alpha(t) &=& \ell_{q'}(\eta_0) + \int_0^t \left(\beta(s) + \ell_{q'}(LP_{<0}\,Y(s))\right)\,\mathrm{d}s, \\ \beta(t) &=& \ell_p(\eta_0), \\ Y(t) &=& P_{<0}\,\eta_0 + \int_0^t P_{<0}\,LP_{<0}\,Y(s)\,\mathrm{d}s. \end{array}\right. \tag{5.90}$$

Then, since $T_{P_{<0}LP_{<0}}(t)$ is a semigroup of contraction whose infinitesimal generator has a strictly positive spectral gap $\lambda_K(L)$, there exists in particular a constant $c > 0$ such that $Y(t) = T_{P_{<0}LP_{<0}}(t)P_{<0}\eta_0$ and $\| Y(t) \|_H \leqslant ce^{-\frac{\lambda_K(L)}{2}t}$ (in particular, $\frac{1}{t}\| Y(t) \|_H \to_{t\to\infty} 0$). Then, using representation of $\ell_{q'}(\cdot)$ in terms of a scalar product w.r.t. v,

$$\frac{\alpha(t)}{t} = \frac{\ell_{q'}(\eta_0)}{t} + \ell_p(\eta_0) + \frac{1}{t}\int_0^t \ell_{q'}(LP_{<0}Y(s))\,\mathrm{d}s, \tag{5.91}$$

$$= \frac{\ell_{q'}(\eta_0)}{t} + \ell_p(\eta_0) + \frac{1}{t}\int_0^t \langle P_{<0}^* L^* v\,,\, Y(s)\rangle_H\,\mathrm{d}s. \tag{5.92}$$

Using the previous exponential bound for $Y(s)$, it is easy to see that $\frac{\alpha(t)}{t}$ converges to $\ell_p(\eta_0)$ as $t \to \infty$. The result of Proposition 5.39 follows. □

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