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présentée par

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Graphes & Marches aléatoires

Thèse soutenue à Rennes le 6 juillet 2012

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Résumé en français

Dans son article original [Tur38] de 1937, A. Turing formalise les notions de calculabilité et complexité en définissant ce qu'on nomme aujourd'hui machine de Turing. Après ce travail de Turing, des généralisations de la machine de Turing ont été données si bien que l'on peut distinguer quatre types de machines de Turing : les machines déterministes, non déterministes, probabilistes et quantiques (voir les définitions de la section 1.1). En terme de complexité temporelle, on peut montrer qu'une machine de Turing probabiliste calcule au moins autant de fonctions partielles qu'une machine de Turing déterministes (*i.e.* $\mathbf{P} \subset \mathbf{BPP}$) alors que la relation avec les machines non déterministes n'est pas claire. D'un point de vue mathématique, une machine de Turing probabiliste est une marche aléatoire sur un graphe dirigé localement fini (section 1.3).

0.1 Marches aléatoires et graphes dirigés

Un *graphe dirigé* est la donnée d'un quadruplet $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ où

- $\mathbb{G}^0$ est un ensemble dénombrable qui représente les *points* ou *noeuds* du graphe ;
- $\mathbb{G}^1$ est un ensemble dénombrable représentant les *arêtes* du graphe ;
- $r, s : \mathbb{G}^1 \to \mathbb{G}^0$ sont deux applications, appelées respectivement *cible* et *source*.

En général, nous nous restreindrons aux graphes dits *simples*, c'est à dire les graphes dirigés sans *arêtes multiples* — si $\alpha, \beta \in \mathbb{G}^1$ sont telles que $s(\alpha) = s(\beta)$ et $r(\alpha) = r(\beta)$ alors $\alpha = \beta$. L'absence d'arêtes multiples implique en particulier que l'application $(r, s) : \mathbb{G}^1 \to \mathbb{G}^0 \times \mathbb{G}^0 : \alpha \mapsto (r(\alpha), s(\alpha))$ est injective de sorte que l'ensemble des arêtes $\mathbb{G}^1$ peut être vu comme un sous-ensemble de $\mathbb{G}^0 \times \mathbb{G}^0$. Dans ce contexte les applications r et s sont superflues — r et s sont alors respectivement les projections canoniques sur la seconde et première coordonnées. Les *boucles* — *i.e.* les arêtes α telles que $s(\alpha) = r(\alpha)$ — seront en général proscrites.

Il sera également supposé que les graphes considérés sont *localement finis*, *i.e.* pour $x \in \mathbb{G}^0$, le *degré sortant* et *entrant*

$$\mathsf{deg}^-(x) = \mathsf{card}\ \{\alpha \in \mathbb{G}^1 : s(\alpha) = x\} < \infty \text{ et } \mathsf{deg}^+(x) < \infty.$$

Un graphe simple est non dirigé si et seulement si l'ensemble $\mathbb{G}^1$ est symétrique — *i.e.* $(x,y) \in \mathbb{G}^1 \iff (y,x) \in \mathbb{G}^1$. Dans ce cas, le degré sortant et le degré entrant sont égaux et on note alors

$$\mathsf{deg}(x) = \mathsf{deg}^-(x) = \mathsf{deg}^+(x)$$

le *degré* de x.

Dans ce contexte, une *marche aléatoire* sur un graphe $\mathbb{G}$ est une chaîne de Markov adaptée à la structure de graphe, c'est à dire une suite de variables aléatoires $(M_n)_{n\geq 0}$ à valeurs dans $\mathbb{G}^0$ telles que pour $n \geq 1$

$$\mathbf{P}(M_n = y | M_{n-1} = x) = \mathbf{P}(M_1 = y | M_0 = x) > 0 \text{ si et seulement si } (x,y) \in \mathbb{G}^1.$$

En particulier, la *marche aléatoire simple* sur $\mathbb{G}$ satisfait

$$\mathbf{P}(M_n = y | M_{n-1} = x) = \frac{1}{\mathsf{deg}^-(x)}.$$

D'une manière très générale, l'étude des marches aléatoires sur des graphes consiste à établir des connexions entre des propriétés de nature combinatoire ou algébrique et des propriétés stochastiques.

Si Γ est un groupe de type fini et $\mathcal{S}$ un ensemble de générateurs symétrique (c'est à dire, $s \in \mathcal{S} \iff s^{-1} \in \mathcal{S}$), le *graphe de Cayley* de Γ par rapport à $\mathcal{S}$, noté $\mathsf{Cayley}(\Gamma, \mathcal{S})$ est le graphe non dirigé simple $\mathbb{G}$ dont l'ensemble des points $\mathbb{G}^0 = \Gamma$ et l'ensemble des arêtes $\mathbb{G}^1$ est défini par

$$\mathbb{G}^1 = \{(x,y) \in \mathbb{G}^0 \times \mathbb{G}^0 : x^{-1}y \in \mathcal{S}\}.$$

Étudier une marche aléatoire sur un graphe de Cayley d'un groupe revient à se donner une probabilité μ de support $\mathcal{S}$ et d'étudier la chaîne de Markov $(M_n)_{n\geq 0}$ à valeurs dans Γ de transition

$$\mathbf{P}(M_n = y | M_{n-1} = x) = \mu(x^{-1}y).$$

On parle alors de marches aléatoires sur un groupe[1]. La littérature regorge de résultats établissant des connexions entre propriétés purement algébriques et propriétés stochastiques.

L'un des premiers exemples illustrant ce type de liens est dû à Pólya qui a considéré des marches aléatoires sur le groupe $\mathbb{Z}^N$ ([Pól21]). Ce résultat établit en particulier que la marche aléatoire est récurrente ou transiente selon que $N \leq 2$ ou $N \geq 3$. Plus surprenant, il existe une sorte de réciproque à ce théorème qui a été montré dans [Var86] en s'appuyant sur des résultats plus anciens tels [Bas72, Gui73, Gui70].

Théorème 0.1.1 ([Woe00]). *Soient Γ un groupe infini de type fini. Alors, il existe une mesure de probabilité μ sur Γ telle que la marche aléatoire associée est récurrente si et seulement si Γ contient un sous-groupe d'indice fini isomorphe à $\mathbb{Z}$ ou $\mathbb{Z}^2$.*

[1] On adopte ici sans le dire la convention de marche aléatoire à droite. Une marche aléatoire à gauche satisfait la propriété $\mathbf{P}(M_{n+1} = y | M_n = x) = \mu(yx^{-1})$.

Malheureusement, si les groupes de type fini donnent de nombreux exemples de graphes non dirigés, il serait appréciable d'établir de tels résultats de rigidité pour des graphes plus généraux. L'un des angles d'attaque consiste à remarquer qu'on peut associer à un graphe dirigé — resp. non dirigé — une structure de *semi-groupoide* — resp. *groupoide* — qui généralise la notion de groupe de sorte qu'une marche aléatoire sur de tels graphes se traduit en une marche aléatoire sur un semi-groupoide ou un groupoide. Cette identification est développée dans le chapitre 4.

0.2 Graphes contraints

Si le cadre théorique est assez bien établi, il existe assez peu de résultats sur les marches aléatoires sur des groupoides ou des semi-groupoides. C'est pourquoi, il était naturel de s'intéresser à des exemples. L'une des notions centrales dans les pages suivantes est celle de *graphe contraint*.

Considérons à nouveau un groupe de type fini Γ et donnons nous un ensemble de générateurs $\mathcal{S}$ que l'on suppose symétrique. Une *contrainte* est une fonction $f : \Gamma \times \mathcal{S} \to \{0, 1\}$. A cette contrainte, il est possible d'associer un *graphe contraint* (qui n'est pas nécessairement canonique). Soit $g_0 \in \Gamma$ un germe, on note $\mathbb{G}_0^0 = \{g_0\}$, puis on définit $\mathbb{G}_1^1$ par

$$\mathbb{G}_1^1 = \bigcup_{s \in \mathcal{S}} \{(g, gs) : g \in \mathbb{G}_0^0, f(g, s) = 1\},$$

et $\mathbb{G}_1^0$ par

$$\mathbb{G}_1^0 = \mathbb{G}_0^0 \cup \left[\bigcup_{s \in \mathcal{S}} \{gs : g \in \mathbb{G}_0^0, f(g, s) = 1\} \right].$$

On peut alors définir, bien que ce ne soit essentiel puisque l'on est en train de construire un graphe intrinsèquement simple, les applications cible et source partielles $r^{(1)}, s^{(1)} : \mathbb{G}_1^1 \to \mathbb{G}_1^0$: pour un élément $(g, h) \in \mathbb{G}_1^1$, l'application source est définie par $s^{(1)}((g, h)) = g$ et la cible par $r^{(1)}((g, h)) = h$. Ainsi, le quadruplet $(\mathbb{G}_1^0, \mathbb{G}_1^1, r^{(1)}, s^{(1)})$ est le graphe partiel contraint d'ordre 1. Pour les graphes partiels contraints d'ordre supérieur, on procède par induction. Soit $n \geq 1$, on définit $\mathbb{G}_{n+1}^1$ by

$$\mathbb{G}_{n+1}^1 = \bigcup_{s \in \mathcal{S}} \{(g, gs) : g \in \mathbb{G}_n^0, f(g, s) = 1\},$$

et $\mathbb{G}_{n+1}^0$ par

$$\mathbb{G}_{n+1}^0 = \mathbb{G}_n^0 \cup \left[\bigcup_{s \in \mathcal{S}} \{gs : g \in \mathbb{G}_n^0, f(g, s) = 1\} \right].$$

Enfin, les fonctions $r^{(n+1)}, s^{(n+1)} : \mathbb{G}_{n+1}^1 \to \mathbb{G}_{n+1}^0$ sont données pour $(g, h) \in \mathbb{G}_{n+1}^1$ par $r^{(n+1)}((g, h)) = h$ et $s^{(n+1)}((g, h)) = g$.

Par conséquent, cette procédure définit une famille de graphes contraints partiels

$$\{(\mathbb{G}_n^0, \mathbb{G}_n^1, r^{(n)}, s^{(n)})\}_{n\geq 1}$$

qui dépend du germe choisi. On définit alors le graphe contraint complet par

- $\mathbb{G}^0 = \lim_{n\to\infty} \mathbb{G}_n^0 = \bigcup_{n\geq 1} \mathbb{G}_n^0$,
- $\mathbb{G}^1 = \lim_{n\to\infty} \mathbb{G}_n^1 = \bigcup_{n\geq 1} \mathbb{G}_n^1$, et
- les applications $r, s : \mathbb{G}^1 \to \mathbb{G}^0$ sont définies de sorte que leurs restrictions à chaque ensemble $\mathbb{G}_n^1$ coincident avec les applications $r^{(n)}$ et $s^{(n)}$ respectivement.

Remarquons que le graphe de Cayley, $\mathsf{Cayley}(\Gamma, \mathcal{S})$, est le graphe contraint de contrainte f constante égale à 1. D'autre part, si la procédure ci-dessus assure la connexité du graphe contraint, il se peut que celui-ci soit fini. Quoiqu'il en soit, cette procédure fournit un grand nombre d'exemples et l'objectif sera de comprendre comment les propriétés stochastiques des marches aléatoires sur le groupe Γ s'étendent ou ne s'étendent pas aux marches aléatoires sur le graphe contraint.

0.2.1 Sous-graphe contraint du graphe de Cayley de $\mathbb{Z}^2$

Le premier exemple considéré est donné par un sous-graphe contraint du graphe de Cayley de $\mathbb{Z}^2$ — par rapport aux générateurs standards. Autrement dit, avec les notations de la précédente section, $\Gamma = \mathbb{Z}^2$, l'ensemble des générateurs $\mathcal{S} = \{\pm\varepsilon_i : i = 1, 2\}$ les générateurs standard de $\mathbb{Z}^2$ et la contrainte $f : \mathbb{Z}^2 \times \mathcal{S} \to \{0, 1\}$ est donnée pour $(z, s) \in \mathbb{Z}^2 \times \mathcal{S}$ par

$$f(z,s) = \begin{cases} 1 & \text{si } s = \pm\varepsilon_2, \text{ ou } s = \varepsilon_1 \text{ et } \langle z, \varepsilon_2\rangle > 0, \text{ ou } s = -\varepsilon_1 \text{ et } \langle z, \varepsilon_2\rangle < 0 \\ 0 & \text{sinon,} \end{cases}$$

où $\langle \cdot, \cdot \rangle$ est le produit scalaire usuel de $\mathbb{R}^2$ dans la base canonique $\{\varepsilon_1, \varepsilon_2\}$. Ce graphe est représenté sur la figure 1.

Comme tout graphe dirigé, l'ensemble des chemins de longueur finie peut-être muni d'une structure de semi-groupoide. D'autre part, cet exemple de graphe est moins artificiel qu'il ne paraît de prime abord puisqu'un graphe similaire a été étudié dans [MDM80] pour modéliser des diffusions en milieu poreux.

La marche aléatoire simple sur cet exemple de graphe est transiente (voir [CP03]). Une question naturelle est de se demander si il existe des fonctions harmoniques — positives ou bornées — non constantes. Cela amène à étudier ce qu'on appelle les *frontières de Martin et de Poisson*.

La notion de frontière de Martin trouve son origine dans un article de Martin [Mar41] et est étroitement liée à la notion de fonction harmonique positive du fait de l'existence d'un isomorphisme entre l'espace des fonctions harmoniques positives et la frontière de Martin (voir chapitre 2, paragraphe 2.1). Considérant le noyau de Green défini par

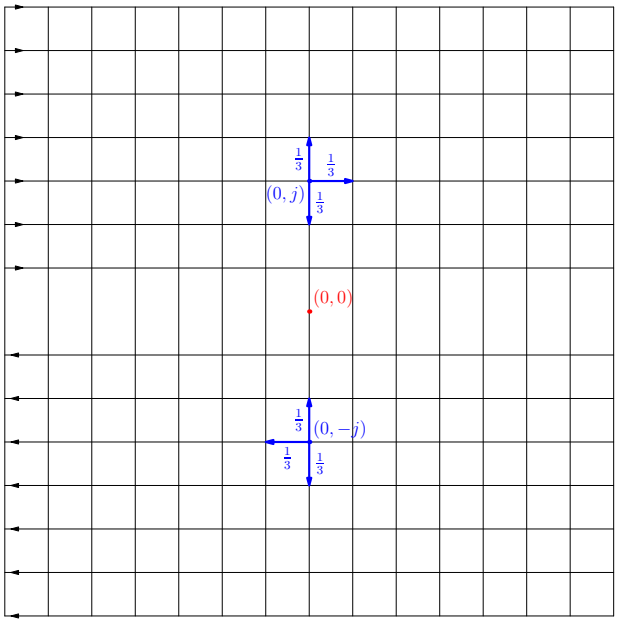


FIGURE 1: Le graphe contraint avec deux demi-plans orientés.

$G(x, y) = \sum_{n\geq 0} \mathbf{P}(M_n = y|M_0 = x)$, on peut remarquer que la fonction $x \to G(x, y)$ est surharmonique et même harmonique sauf au point y. Intuitivement, si on fait tendre y vers un point ξ à l'infini, en un sens à définir, la fonction de Green devient harmonique. Dans ce contexte, on travaille généralement avec une renormalisation de la fonction de Green appelée noyau de Martin et défini par

$$K(x, y) = \frac{G(x, y)}{G(o, y)},$$

pour $x, y \in \mathbb{G}^0$ et où $o \in \mathbb{G}^0$ est un point base fixé une fois pour toute. La compactification de Martin consiste à étendre l'espace d'états $\mathbb{G}^0$ en un espace compact $\widehat{\mathbb{G}}^0$, dont $\mathbb{G}^0$ est un sous-ensemble discret et dense, de sorte que chaque fonction $\mathbb{G}^0 \ni y \to K(x, y) \in \mathbb{R}$ s'étende par continuité en une unique fonction, encore notée $K(x, \cdot)$ et définie pour $\xi \in \widehat{\mathbb{G}}^0 \setminus \mathbb{G}^0$ par

$$K(x, \xi) = \lim_{y\to\xi} K(x, y),$$

où $y \to \xi$ est relative à la topologie de $\widehat{\mathbb{G}}^0$. De plus cette compactification est maximale, en ce sens que si $\xi \neq \eta$, alors les fonctions $K(\cdot, \xi)$ et $K(\cdot, \eta)$ sont distinctes. La frontière de Martin est l'espace topologique $\partial\mathbb{G}^0 = \widehat{\mathbb{G}}^0 \setminus \mathbb{G}^0$. Enfin, au moins dans le cas localement fini, les fonctions $x \to K(x, \xi)$ sont harmoniques positives pour tout $\xi \in \partial\mathbb{G}^0$.

La frontière de Poisson, quant à elle, a été historiquement définie comme sous espace mesurable de la frontière de Martin et caractérise l'espace des fonctions harmoniques bornées. En effet, la compactification de Martin, en définissant une topologie, permet d'établir un

théorème de convergence presque sûre de la marche aléatoire vers une variable aléatoire à valeurs dans la frontière de Martin $\partial\mathbb{G}^0$. Très grossièrement, le support de cette variable aléatoire limite est la frontière de Poisson.

Cependant, la frontière de Poisson peut-être également définie, de manière indépendante, comme un objet de la catégorie des espaces mesurés (voir [Kai92]). Dans l'annexe B, on montre que la frontière de Poisson est un quotient de la frontière queue qui, elle-même, est une limite inductive dans la catégorie des espaces mesurés. Aussi, la frontière queue peut-être vue, en un certain sens, comme l'espace dual à l'espace des trajectoires qui, dans le langage des catégories, est la limite projective d'un système projectif d'espaces mesurés.

Dans le contexte des marches aléatoires sur des groupes — ou sur des espaces homogènes — il existe un certain nombre de critères pour décider de la trivialité de la frontière de Poisson (voir par exemple [Aze70, Bab06, KV83, KW02, Kai00, Mar66]). La description de la frontière de Martin est souvent un problème délicat. D'une manière générale, cela demande d'estimer finement la fonction de Green. Ces estimés peuvent provenir de la géométrie du graphe sous-jacent (pour les arbres homogènes, une frontière géométrique naturelle existe, voir [Woe95]), ou de théorèmes limites locaux (c'est la méthode employée pour la détermination de la frontière de Martin des marches aléatoires sur $\mathbb{Z}^N$, voir [NS66, Uch98]), ou encore d'une action de groupe laissant invariant le noyau de Martin (voir par exemple [Ras10, KR11] où sont considérées des marches aléatoires sur un quart de plan), et même de principe de grandes déviations (voir [IMS94, IR08]).

Dans la suite, on note $(M_n)_{n\geq 0}$ la marche aléatoire simple sur le graphe $\mathbb{G}$, en particulier c'est une chaîne de Markov à valeurs dans $\mathbb{G}^0$. On définit alors par récurrence la suite de temps d'arrêts $(\tau_n)_{n\geq 0}$ par $\tau_0 = 0$ et pour $n \geq 0$

$$\tau_{n+1} = \inf\{t \geq \tau_n + 1 : M_n^{(2)} = 0\}$$

où $M_n = (M_n^{(1)}, M_n^{(2)})$. Il est alors facile de voir que $\mathbf{P}(\tau_n < \infty | M_0 = x) = 1$ pour tout $n \geq 0$ et $x \in \mathbb{G}^0$. La suite de variables aléatoires $(M_{\tau_n})_{n\geq 0}$ est elle-même une chaîne de Markov. Celle-ci reste confinée dans l'équateur — *i.e.* l'ensemble $\mathbb{Z} \times \{0\} \subset \mathbb{G}^0$ — sauf peut-être au temps 0. Il est aussi assez clair, du fait des symétries de $\mathbb{G}$, que les variables aléatoires $(Z_n)_{n\geq 1}$ à valeurs entières définies par $Z_n = M^{(1)}_{\tau_{n+1}} - M^{(1)}_{\tau_n}$ sont indépendantes et identiquement distribuées, autrement dit le processus $(M^{(1)}_{\tau_n})_{n\geq 1}$ est une marche aléatoire sur $\mathbb{Z}$. Par ailleurs, pour les mêmes raisons de symétrie du graphe $\mathbb{G}$, les variables aléatoires Z_n sont symétriques, c'est à dire, Z_n et $-Z_n$ sont de même loi. Ainsi, $(M^{(1)}_{\tau_n})_{n\geq 1}$ est une marche aléatoire symétrique. En fait, il est montré dans [CP03] que c'est une marche aléatoire à sauts non-bornés. Ensuite, la transformée de Fourier de la loi de saut est calculée et il est déduit par un argument standard la transience de cette marche aléatoire. De fait, la marche initiale est également transiente. La marche aléatoire $(M_n)_{n\geq}$ sera appelée marche *initiale* et la marche $(M_{\tau_n})_{n\geq 0}$, marche *induite*.

Dans ce manuscrit, il est montré que la frontière de Martin de la marche aléatoire sur le graphe contraint avec deux demi-plans orienté (*cf.* figure 1) est triviale. Ce résultat a fait l'objet d'une note aux *Comptes Rendus de l'académie des sciences* — *cf.* [dL11] — et les détails sa démonstration sont présentés dans l'annexe A. On rappelle ici l'énoncé du résultat.

Théorème 0.2.1. *Les frontières de Martin de la marche induite et de la marche originale sont triviales, en particulier, toutes les fonctions harmoniques positives sont constantes.*

Ce théorème repose sur des estimations fines du noyau de Green. Plus précisément, si on note $\eta_{s,t}(y)$ le temps local de $(M_n)_{n\geq 0}$ en $y \in \mathbb{G}^0$, *i.e.*

$$\eta_{s,t}(y) = \sum_{n=s}^{t-1} \mathbf{1}_y(M_n),$$

avec la convention $\sum_\emptyset = 0$, on montre que le noyau de Martin se décompose comme suit :

$$K(x,y) = \begin{cases} \frac{G(x,y)}{G(0,y)} & \text{si } x, y \in \mathbb{Z} \times \{0\} \\ \\ \frac{\mathbf{E}(\eta_{0,\tau_1}(y)|M_0=x)}{G(0,y)} + \sum_{z\in\mathbb{Z}\times\{0\}} \nu_x(z)K(z,y) & \text{sinon,} \end{cases}$$

où $\nu_x(z) = \mathbf{P}(M_{\tau_1} = z|M_0 = x)$.

Le théorème central dans la preuve de la trivialité de la frontière de Martin est le suivant.

Théorème 0.2.2. *Soient $z = (z_1, 0) \in \mathbb{Z} \times \{0\}$ et $y = (y_1, y_2) \in \mathbb{G}^0$. Alors la fonction de Green est donnée par*

$$G(z,y) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{it(y_1-z_1)} \frac{g(r(t))^{|y_2|}}{1-\phi(t)} dt,$$

où

$$r(t) = \tfrac{1}{3-2e^{it}}, \quad g(x) = \tfrac{1-\sqrt{1-x^2}}{x}, \text{ et } \quad \phi(t) = \mathsf{Re}\ r(t)^{-1} g(r(t)).$$

De plus, pour tout $\lambda \in \mathbb{R} \cup \{\pm\infty\}$, il existe une constante $s(\lambda) > 0$ telle que

1. *$G(z,y) \sim s(\pm\infty)|y_1 - z_1|^{-1/2}$, lorsque $y_1 y_2^{-2} \to \pm\infty$,*
2. *$G(z,y) \sim s(\lambda)|y_2|^{-1}$, lorsque $y_1 y_2^{-2} \to \lambda$.*

L'une des conséquences de la trivialité de la frontière de Martin est qu'il n'est pas possible de discriminer le comportement asymptotique des trajectoires de la marche. En réalité, par un argument de type Borel-Cantelli, on peut montrer que la marche induite est infiniment souvent à droite et à gauche de 0, aussi, il n'y a pas de directions priviligiées. Dans le cadre des marches aléatoires sur $\mathbb{Z}^N$ on peut montrer que la trivialité de la frontière de Martin dépend du drift[2]. En effet, si celui-ci est nul, alors la frontière est triviale. Cependant, si le drift n'est pas nul, la frontière de Martin est homéomorphe à la sphère de dimension $N-1$. Il est fort probable que ce changement régime ait aussi lieu lorsqu'on considère des marches aléatoires sur $\mathbb{G}$ plus générales, cependant, il est difficile d'avoir une idée de la forme de la frontière de Martin (est-ce un cercle ?). Ces marches plus générales n'ont pas été considérées du point de vue difficile de la frontière de Martin. Il est cependant

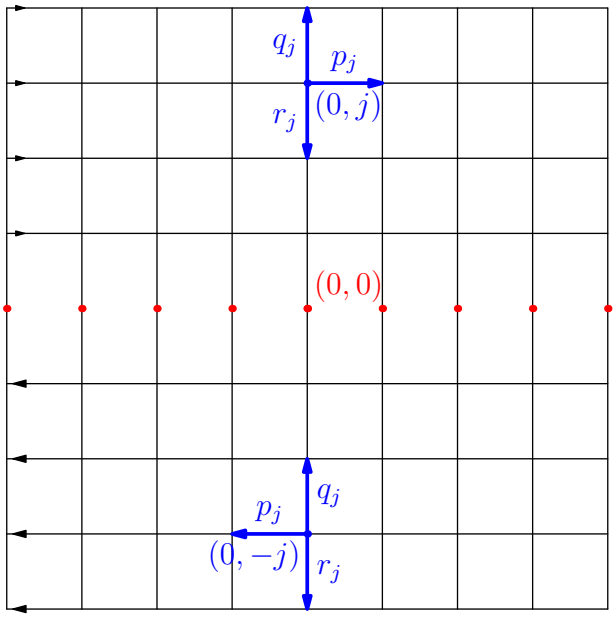


FIGURE 2: Le graphe contraint avec deux demi-plans orientés et un drift non constant.

souvent plus facile d'étudier la frontière de Poisson. Dans le cas de la marche aléatoire simple sur $\mathbb{G}$ la trivialité de la frontière de Poisson peut-être montrée de manière directe (*cf.* paragraphe 2.2.2 du chapitre 2).

Plus précisément, si on se donne $(p_y)_{y\in\mathbb{Z}}$ une suite de réels de $[0, 1)$ et $(q_y)_{y\in\mathbb{Z}}$ une autre suite de réels strictement positifs tels que $q_y < 1 - p_y$ pour tout $y \in \mathbb{Z}$; alors, sur le graphe $\mathbb{G}$ on peut considérer la marche aléatoire dont la probabilité, partant du point $(x, y) \in \mathbb{G}^0$, de se déplacer vers le haut est q_y, vers le bas est $r_y = 1 - p_y - q_y$ et horizontalement est p_y — *cf.* la figure 2.

Proposition 0.2.3. *La frontière de Poisson de la marche aléatoire sur $\mathbb{G}$ dont les transitions sont définies au paragraphe précédant est isomorphe (en tant qu'espace mesuré) à la frontière de Poisson de la marche aléatoire sur $\mathbb{Z}$ dont les probabilitées de transition sont données pour $u, v \in \mathbb{Z}$ par*

$$P(u, v) = \begin{cases} p_u & si\ u = v, \\ q_u & si\ v = u + 1, \\ 1 - p_u - q_u & si\ v = u - 1, \\ 0 & sinon. \end{cases}$$

[2]Il faut ajouter, en toute rigueur, une condition sur les moments de la loi de saut, cf. 2.1.3

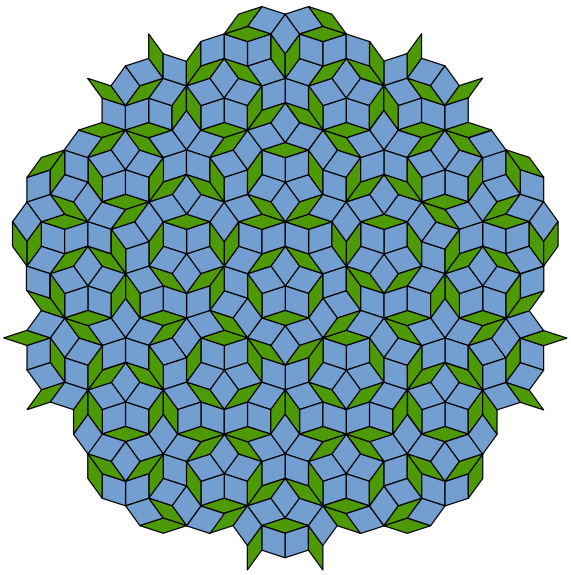


FIGURE 3: Le troisième pavage de Penrose avec deux prototuiles, figure tirée de [Wik12].

0.2.2 Sous-graphes contraints du graphe de Cayley de $\mathbb{Z}^N$ par la méthode de coupe et projection

Dans un second temps, nous considérons une famille d'exemples de graphes non dirigés et apériodiques obtenus grâce à la méthode de coupe et projection — voir par exemple [ODK88, KD86]. Le célèbre troisième pavage de Penrose (*cf.* la figure 3), défini au début des années 70, est un des premiers exemples de telles structures apériodiques. Depuis les années 1980 et la découverte des quasi-crystaux, les structures apériodiques ont été largement étudiées dans la littérature.

Les graphes considérés ici sont obtenus en pavant l'espace vectoriel réel $\mathbb{R}^d$ à l'aide de la méthode de *coupe et projection*. Plus précisément, on considère E un sous-espace vectoriel de $\mathbb{R}^N$ de dimension d appelé *espace réel*, puis on note $E_{\mathsf{int}} = E^{\perp}$ le supplémentaire orthogonal de E, appelé *espace interne*. Soit K le cube unité de $\mathbb{Z}^N \subset \mathbb{R}^N$. Une arête du graphe de Cayley de $\mathbb{Z}^N$ — selon l'ensemble des générateurs usuels $\mathcal{S}$ — est acceptée et projetée orthogonalement sur E si elle peut être translatée par un vecteur de E dans le cube $K+t$, $t \in E_{\mathsf{int}}$ fixé. Sous certaines hypothèses que l'on ne précise pas ici (mais qui sont toutefois génériques), cette méthode permet de construire un pavage $\mathcal{T}_t$ de l'espace E dont les tuiles sont les projections des faces d-dimensionelles du cube unité N-dimensionel. De plus, en fonction des espaces E et E_{int}, ces pavages sont périodiques, quasi-périodiques ou apériodiques — le groupe de translation du pavage est donné par $E \cap \mathbb{Z}^N$. De tels pavages définissent naturellement des graphes que l'on appellera *graphes de coupe et projection* dont les points et les arêtes sont respectivement les sommets et les côtés des tuiles définissant le pavage. Ce type de graphe est ainsi un sous-graphe contraint du graphe de Cayley

de $\mathbb{Z}^N$ — par rapport aux générateurs usuels. La contrainte dans ce cas est la fonction $f_t : \mathbb{Z}^N \times \mathcal{S} \to \{0,1\}$ donnée par

$$f_t(z,s) = \mathbf{1}_{\mathscr{K}_t \times \mathscr{K}_t}(z, z+s) \text{ for } (z,s) \in \mathbb{Z}^N \times \mathcal{S},$$

où $\mathscr{K}_t$ est l'ensemble $K + E + t$ appelé, pour des raisons évidentes, *bande de sélection*. Enfin, par construction, le graphe induit par un pavage est connexe, localement fini, non dirigé, simple et sans boucles.

Le troisième pavage de Penrose a été initialement défini à l'aide de règles d'assemblages des tuiles (voir le paragraphe 3.1.1 du chapitre 3). Ces règles d'assemblages assurent l'existence et la quasi-périodicité du pavage. Cependant, ce pavage peut être également construit par la méthode de coupe et projection (*cf.* [ODK88]) en considérant l'espace $E \subset \mathbb{R}^5$ engendré par les vecteurs v_1 et v_2 suivants

$$v_1 = (1, \cos(2\pi/5), -\cos(\pi/5), -\cos(\pi/5), \cos(2\pi/5)),$$

et,

$$v_2 = (0, \sin(2\pi/5), \sin(\pi/5), -\sin(\pi/5), -\sin(2\pi/5)).$$

Le pavage icosahédral de $\mathbb{R}^3$ est un autre exemple intéressant de pavages apériodiques du fait de son lien étroit avec les quasi-crystaux. Les quasi-crystaux ont été découverts expérimentalement par Shechtman en 1982 en étudiant les motifs de diffraction d'un alliage de Al-Mn. Ces motifs présentaient en effet une symétrie d'ordre 10 qui va à l'encontre de la théorie de la crystallographie. La description théorique de ce phénomène a été précisée dans l'article original [SBGC84] de D. Shechtman, I. Blech et J.W. Cahn. En outre, Shechtman s'est vu récompensé de ce travail par le prix Nobel de Chimie en 2011. En ce qui nous concerne, il se trouve que le pavage icosahédral de $\mathbb{R}^3$ modélise parfaitement cet alliage (*cf.* aussi [KD86]).

Le pavage icosahédral de $\mathbb{R}^3$ est obtenu à l'aide de la méthode de coupe et projection en considérant l'espace vectoriel $E \subset \mathbb{R}^6$ de dimension 3 défini comme étant l'image de la projection π donnée par sa matrice dans la base canonique de $\mathbb{R}^6$ par

$$\pi = \frac{1}{2\sqrt{5}} \begin{pmatrix} \sqrt{5} & 1 & -1 & -1 & 1 & 1 \\ 1 & \sqrt{5} & 1 & -1 & -1 & 1 \\ -1 & 1 & \sqrt{5} & 1 & -1 & 1 \\ -1 & -1 & 1 & \sqrt{5} & 1 & 1 \\ -1 & -1 & -1 & 1 & \sqrt{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & \sqrt{5} \end{pmatrix},$$

Sous certaines conditions techniques génériques qui seront précisées en temps voulu, on montre le théorème suivant (*cf.* le théorème 3.1.4 du paragraphe 3.1.4 au chapitre 3) :

Théorème 0.2.4. *Génériquement, la marche aléatoire simple sur le graphe de coupe et projection est récurrent si* $\dim E \leq 2$ *et transiente si* $\dim E \geq 3$.

Ce résultat est très similaire au théorème de Pólya, cependant, contrairement aux marches aléatoires sur $\mathbb{Z}^N$, le calcul des probabilités de retour à l'origine n'est plus explicite. Ainsi, la preuve de la transience requiert l'établissement d'inégalités isopérimétriques d-dimensionelles (établies, cependant, non pas sur le graphe initial, mais sur une puissance assez grande de celui-ci), alors que la récurrence est obtenue à l'aide d'estimées sur la borne inférieure de la probabilité de retour à l'origine. Cependant, l'établissement d'inégalités isopérimétriques est plus fort que l'application d'un critère de type Nash-Williams puisqu'on obtient en plus des estimées de la décroissance du noyau de la chaleur. Enfin, notons que ce théorème ne se limite pas aux marches aléatoires simples et peut être généralisé à des marches réversibles plus générales par des arguments standards — voir [Anc90] par exemple.

0.2.3 Groupoides et semi-groupoides

Comme cela a déjà été évoqué, une marche aléatoire sur un graphe dirigé — respectivement non dirigé — peut-être vue comme une marche aléatoire sur un semi-groupoide — respectivement un groupoide — de la même façon qu'une marche aléatoire sur le graphe de Cayley d'un groupe est vue comme une marche aléatoire sur un groupe. Les marches aléatoires sur les groupoides et semi-groupoides sont cependant assez peu considérées dans la littérature — *cf.* [Kai05, Ren80] pour les quelques références dont on dispose.

À la lumière des exemples considérés, on peut espérer qu'un certain nombre de propriétés valables dans le contexte des groupes se transfèrent dans le contexte des groupoides alors que dans le cas des semi-groupoides l'étude des marches aléatoires semblent plutôt difficile. L'une des raisons à cette différence est que dans le cadre des marches aléatoires sur un groupoide, on peut souvent espérer que celles-ci soient réversibles — en particulier, une marche aléatoire simple sur un graphe non dirigé, simple et localement fini est toujours réversible. Nous avons alors à disposition toutes les techniques de l'analyse fonctionnelle dans les espaces de Hilbert. C'est en particulier ce qui est employé dans le théorème énoncé dans le paragraphe précédant. Dans le cadre des graphes intrinsèquement dirigés, les marches aléatoires ne sont plus des chaînes de Markov réversibles.

Dans le chapitre 4, outre l'introduction des notions de groupoide et de semi-groupoide, il est aussi introduit cette notion de chaîne de Markov réversible. En particulier, la preuve du théorème 1 de [Var85] est reproduite ce qui permet de mettre en exergue le lien étroit entre l'opérateur de Laplace-Beltrami et l'opérateur de Markov ainsi que les propriétés puissantes qui en découlent.

Chapter 1

Turing machines, Markov chains, directed graphs

In this chapter, we start with the introduction of the notion of Turing machines. Then, before connecting random walks on directed graphs and Turing machines, we give notation and definitions related to Markov chains and directed graphs.

1.1 Definition of Turing machine

In this section, we define the notion of *Turing machines*. Turing machines were introduced in [Tur38] in 1937 by Alan Turing. Generally speaking, Turing machines provide with a theoretical model of computations so that we can define notions of decidability and computability — Does there exist an algorithm which solves a given problem in finite time or in finitely many computations ? — or as complexity — how many computations are needed to solve a given problem ?

We can distinguish four classes of Turing machines :

- *deterministic* Turing machines (DTM),
- *non deterministic* Turing machines (NTM),
- *probabilistic* Turing machines (PTM),
- *quantum* Turing machines (QTM).

We could not omit the latter class of quantum Turing machines in this introductive chapter since they have a priviliged place in the literature during the last decades. However, quantum Turing machines will not be discussed in details in the following work. For a description of QTM in terms of notions introduced here see [Pé11].

1.1.1 Deterministic Turing machine and the P class

There are several variants of deterministic Turing machines; all of them are equivalent in the sense that a problem is solvable by one variant if and only if it is solvable by any other variant within essentially the same amount of time. We suggest the following.

Definition 1.1.1. A *deterministic Turing machine* is a quadruple (A, S, u, s_0) where

1. A is a finite, non empty set of symbols, called *alphabet*, containing a particular symbol called the *blank symbol* and denoted by $\sharp$; we set $A_\flat = A \setminus \{\sharp\}$,
2. S is a finite non empty set of *states* ; we assume that S is partitioned into two non empty sets S_i and S_f called respectively *internal states* and *final states*,
3. $D = \{L, R\} \equiv \{-1, 1\}$ is the *displacement set*,
4. $u : A \times S \mapsto A \times S \times D$ is the *transition function*, and
5. $s_0 \in S_i$ the *initial state* of the machine.

Let $M_0 = (A, S, u, s_0)$ be a deterministic Turing machine with $A = \{0, 1, \sharp\}$, $S = S_i \cup S_f$ where $S_i = \{\mathsf{go}\}$, $S_f = \{\mathsf{halt}\}$, and transition function $u(a, s) = (a', s', d)$ defined by the following table :

a	s	a'	s'	d
0	go	0	go	L
1	go	1	go	L
$\sharp$	go	$\sharp$	halt	R

If the program, described by this Turing machine, starts with the head over any non-blank symbol of the input string, it ends with the head over the leftmost non-blank symbol while the string of symbols remains unchanged.

If W is a finite set, we denote by $W^* = \bigcup_{n\geq 0} W^n$ the set of *words* of finite length on the alphabet W — $W^0 = \{\epsilon\}$ and the element $\epsilon \in W^0$ is called the *empty word*. If $w \in W^*$, then by definition there exists $n \geq 0$ such that $w \in W^n$. We denote by $|w|$ the length of w and $|w| = n$.

For $\alpha \in A_\flat^*$, we denote by $\bar{\alpha} \in A^\infty$ the completion of the work α by blanks, namely $\bar{\alpha} = (\alpha_1, \cdots, \alpha_{|\alpha|}, \sharp, \sharp, \cdots)$.

Deterministic Turing machine as a dynamical system

Considering the example of Turing machine M_0, we can, without loss of generality, always assume that the machine starts at the first symbol of the input string $\alpha \in A_\flat^*$. Starting from $(\alpha, s_0, h_0 = 1)$, successive applications of the transition function u induce a dynamical system on $\mathbb{X} := A^* \times S \times \mathbb{Z}$. A configuration is an instantaneous description of the word written on the tape, the internal state of the machine and the position of the head, *i.e.* an element of $\mathbb{X}$.

Let $\tau_\alpha = \inf\{n \geq 1 : s_n \in S_f\}$. The program starting from initial configuration $(\alpha, s_0, h_0 = 1)$ stops running if $\tau_\alpha < \infty$, it never halts when $\tau_\alpha = \infty$. While $1 \leq n < \tau_\alpha$, the sequence $(\alpha^{(n)}, s_n, h_n)_{n<\tau_\alpha}$ is defined by updates of single characters; if, for $0 \leq n < \tau_\alpha$, we have $u(\alpha^{(n)}_{h_n}, s_n) = (a', s', d)$, then $(\alpha^{(n+1)}, s_{n+1}, h_{n+1})$ is defined by

$$\begin{aligned} s_{n+1} &= s' \\ h_{n+1} &= h_n + d \\ \alpha^{(n+1)} &= (\alpha^{(n)}_1, \cdots, \alpha^{(n)}_{h_{n-1}}, a', \alpha^{(n)}_{h_{n+1}}, \cdots, \alpha^{(n)}_{|\alpha^{(n)}|}). \end{aligned}$$

If the machine halts at some finite instant, the output is obtained by reading the tape from the leftmost non blank character to right until the first blank character. The sequence of words $(\alpha^{(n)})_n$ is termed a *computational path* or *computational history* starting from α.

Computable functions and decidable predicates

Every deterministic Turing machine M computes a specific partial function $\phi_M : A_\flat^* \to A_\flat^*$. Since the value of $\phi_M(\alpha)$ remains undetermined when the machine M does not halt, the function ϕ_M is termed partial, in general the domain $\mathsf{Dom}(\phi_M)$ of ϕ_M is a strict subset of $A_\flat^*$.

Definition 1.1.2 (Computable function)**.** A partial function $f : A_\flat^* \to A_\flat^*$ is called *computable* if there exists a deterministic Turing machine M such that $\phi_M = f$. In such a case, the function f is said to be computed by the machine M.

An application of the Cantor's diagonalisation yields the existence of non-computable functions.

Definition 1.1.3 (Predicate, Language)**.** A *predicate* $\mathscr{P}$ is a function taking Boolean values 0 or 1. A language $\mathscr{L}$ over an alphabet A is a subset of $A_\flat^*$.

Thus, for predicates $\mathscr{P}$ with $\mathsf{Dom}(\mathscr{P}) = A_\flat^*$, the set $\{\alpha \in A_\flat^* : \mathscr{P}(\alpha) = 1\}$ is a language. Hence, predicates are in bijection with languages.

Definition 1.1.4 (Decidability)**.** A predicate $\mathscr{P} : A_\flat^* \to \{0, 1\}$ is *decidable* if the function $\mathscr{P}$ is computable.

Let $\mathscr{P}$ be a predicate and $\mathscr{L}$ the corresponding language. The predicate is decidable if there exists a deterministic Turing machine such that for every word α, the machine halts after a finite number of steps and

- if $\alpha \in \mathscr{L}$, then the machine halts returning 1, and
- if $\alpha \notin \mathscr{L}$, then the machine halts returning 0.

Complexity classes of a DTM

Definition 1.1.5 (Space and time complexity)**.** Let M be a deterministic Turing machine and $s_M, t_M : \mathbb{N} \to \mathbb{R}_+$ be given functions. If for every $\alpha \in A_\flat^*$, the machine stops after visiting at most $s(|\alpha|)$ cells, we say that it works in *computational space* s_M. We say it works in *computational time* t_M if $\tau_\alpha \leq t_M(|\alpha|)$.

Computability of a function does not mean effective computability since the computing algorithm can require too much time or space. We say that $r : \mathbb{N} \to \mathbb{R}_+$ is of *polynomial growth* if there exist constants $C, \kappa > 0$ such that $r(n) \leq Cn^\kappa$, for large n. We write symbolically $r(n) = \mathsf{poly}(n)$.

Definition 1.1.6 (**P** class)**.** The *complexity class* **P** consists of all languages $\mathscr{L}$ whose predicates $\mathscr{P}$ are *decidable in polynomial time*, *i.e.* for every $\mathscr{L}$ in the class, there exist a deterministic Turing machine M such that $\phi_M = \mathscr{P}$ and $t_M(|\alpha|) = \mathsf{poly}(|\alpha|)$ for all $\alpha \in A_\flat^*$.

Similarly, we can define the class **PSPACE** of languages whose predicates are *decidable in polynomial space*.

Remark 1.1.7. We choose to define **P** and **PSPACE** as classes of complexity of languages, but it is obvious that the same can be done for partial functions although it can be shown that the obtained classes do not define a new notion of complexity. That is why, they will be still denoted by **P** and **PSPACE**.

It is obvious that **P** $\subset$ **PSPACE** because in a polynomial time, we can only visit a polynomial number of cells. Note that it is conjectured that **P** $\neq$ **PSPACE** which shows that this theory is far from being completely understood.

1.1.2 Non-deterministic Turing machine and the NP class

Definition 1.1.8. A *non-deterministic Turing machine* is a quadruple (A, S, u, s_0) where A, S, and s_0 are defined as in definition 1.1.1 and u is now a multivalued function, *i.e.* there are r different branches u_i for $i = 1, \cdots, r$ and $u_i : A \times S \mapsto A \times S \times D$. For every pair $(a, s) \in A \times S$ there are different possible outputs $(a_i^{'}, s_i^{'}, d_i)_{i=1,\cdots,r}$ and the choice of a particular branch can be done in a non-deterministic way.

The NP class

A computational path for a non deterministic Turing machine is determined by a choice of one legal transition at every step. Different steps are possible for the same input. Notice that NTM do not serve as models of practical devices but rather as logical tools for the formulation of problems. As we have done for the deterministic Turing machines, we give the class of languages which are computed by NTMs.

Definition 1.1.9 (**NP** class)**.** A language $\mathscr{L}$ (or its predicate $\mathscr{P}$) belongs to the **NP** class if there exists a non deterministic Turing machine such that

- if $\alpha \in \mathscr{L}$ for some $\alpha \in A_\flat^*$, then there exists a computational path with $\tau_\alpha \leq \mathsf{poly}(|\alpha|)$ returning 1,
- if $\alpha \notin \mathscr{L}$ for some $\alpha \in A_\flat^*$, then there exists no computational path with this property.

Remark 1.1.10. The definition of the **NPSPACE** class is obvious.

Remark 1.1.11. Note the specificity of non deterministic machine; with this kind of machine, we can not decide if a given word *is not* in the language. In terms of formal language theory, such a language is said to be *recursive*. A language $\mathscr{L} \in \mathbf{NP}$ such that $\mathscr{L}^\complement \in \mathbf{NP}$ is termed *recursively enumerable*.

Remark 1.1.12 (which can not be avoided). We see immediately that $\mathbf{P} \subset \mathbf{NP}$, but a famous problem consists in deciding whether $\mathbf{P} = \mathbf{NP}$ or $\mathbf{P} \neq \mathbf{NP}$.

1.1.3 Probabilistic Turing machine and the BPP class

We denote by $\tilde{\mathbb{R}}$ the set of real numbers computable by a deterministic Turing machine within accuracy 2^{-n} in $\mathsf{poly}(n)$ time.

Definition 1.1.13. A *probabilistic Turing machine* is a quintuple $(A, S, u, \mathbf{p}, s_0)$ where A, S, u, and s_0 are as in definition 1.1.8 while $\mathbf{p} = (p_1, \cdots, p_r) \in \tilde{\mathbb{R}}_+$, with $\sum_{i=1}^r p_i = 1$ is a probability vector on the set of branches of u. All branches correspond to legal actions ; at each step, the branch i is chosen with probability p_i, independently of previous choices.

The BPP class

Each $\alpha \in A_\flat^*$ generates a family of computational paths. The local probability structure on the transition functions induces a natural probability structure on the computational path space. The evolution of the machine is a Markov process (see section 1.3) with the state space $A_\flat^* \times S \times \mathbb{Z}$ and stochastic evolution kernel determined by the local probability structure $\mathbf{p}$. Hence any input gives a set of possible outputs each of them being assigned a probability of occurrence.

Definition 1.1.14 (**BPP** class)**.** Let $\epsilon \in (0, 1/2)$. A predicate $\mathscr{P}$ (hence a language $\mathscr{L}$) belongs to the **BPP** class if there exists a probabilistic Turing machine M such that for any $\alpha \in A_\flat^*$, $\tau_\alpha \leq \mathsf{poly}(|\alpha|)$ and

- if $\alpha \in \mathscr{L}$, then $\mathbf{P}(\mathscr{P}(\alpha) = 1) \geq 1 - \epsilon$, and
- if $\alpha \notin \mathscr{L}$, then $\mathbf{P}(\mathscr{P}(\alpha) = 1) \leq \epsilon$.

Remark 1.1.15. Repeating the computation of the machine M on the same input α, it can be shown that the definition of the class **BPP** does not depend on the specific chosen ϵ.

1.2 Directed graphs and random walks

In the next section we will show how a probabilistic Turing machine can be connected to a random walk on a directed graph. The main purpose of this section is to introduce notation and definitions related to directed graphs and Markov chains.

1.2.1 Directed graphs

A *directed graph* (or di-graph for short) $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ is the quadruple of a denumerable set $\mathbb{G}^0$ of *vertices*, a denumerable set $\mathbb{G}^1$ of directed *edges* and a pair of *range* and *source* functions, denoted respectively r and s, *i.e.* mappings $r, s : \mathbb{G}^1 \to \mathbb{G}^0$. In the sequel, we only consider graphs without loops (*i.e.* not containing edges $\alpha \in \mathbb{G}^1$ such that $r(\alpha) = s(\alpha)$) and without multiple edges (*i.e* if α and β are edges verifying simultaneously $s(\alpha) = s(\beta)$ and $r(\alpha) = r(\beta)$ then $\alpha = \beta$, in other words, the compound map $(s, r) : \mathbb{G}^1 \to \mathbb{G}^0 \times \mathbb{G}^0$ is injective). With these restrictions in force, $\mathbb{G}^1$ can be identified with a particular subset of $\mathbb{G}^0 \times \mathbb{G}^0$ and the functions r and s become superfluous because they are trivial *i.e* $s((x, y)) = x$ and $r((x, y)) = y$. The corresponding directed graph is then termed *simple*.

Remark 1.2.1. Although often used interchangeably in common language, *directedness* and *orientation* denote distinct notions in graph theory : directedness is a property encoded into the set $\mathbb{G}^1$ of allowed edges; orientation is an assignment of plus or minus sign to every edge (viewed as the set — not the ordered pair — of its endpoints). On defining a map $\iota : \mathbb{G}^1 \to \mathbb{G}^0 \times \mathbb{G}^0$ by $\mathbb{G}^1 \ni \alpha = (x, y) \mapsto \iota(\alpha) = (y, x) \in \mathbb{G}^0 \times \mathbb{G}^0$ (this map reverts the orientation of an edge), we observe that for an oriented but undirected graph, the range of ι is $\mathbb{G}^1$; for a directed graph, the range of ι can contain elements in $\mathbb{G}^0 \times \mathbb{G}^0 \setminus \mathbb{G}^1$. In both cases, the map ι is involutive. An undirected graph can be viewed as a directed one such that if $\alpha := (x, y) \in \mathbb{G}^1$ then $\iota(\alpha) = (y, x) \in \mathbb{G}^1$, *i.e.* the set of edges $\mathbb{G}^1$ is a symmetric subset of the Cartesian product $\mathbb{G}^0 \times \mathbb{G}^0$.

We also define, for each vertex $y \in \mathbb{G}^0$, its *inwards degree* by

$$\mathsf{deg}^+(y) = \mathsf{card}\{\alpha \in \mathbb{G}^1 : r(\alpha) = y\},$$

and its *outwards degree* by

$$\mathsf{deg}^-(y) = \mathsf{card}\{\alpha \in \mathbf{G}^1 : s(\alpha) = y\}.$$

If the graph is undirected, then $\mathsf{deg}^+(y) = \mathsf{deg}^-(y) = \mathsf{deg}(y)$ and we simply say the *degree* of y.

The graph $\mathbb{G}$ is said to be *connected* — or *transitive* — if for any vertices $x, y \in \mathbb{G}^0$ there exists a finite sequence $\alpha = (\alpha_1, \cdots, \alpha_k)$ of composable edges $\alpha_i \in \mathbb{G}^1$, for $i = 1, \cdots, k$, $k \in \mathbb{N}$, with $s(\alpha_1) = x$ and $r(\alpha_k) = y$, such that $r(\alpha_i) = s(\alpha_{i+1}) \in \mathbb{G}^0$, for all $i = 1, \cdots, k-1$. The above sequence α is called a *path* of length $k = |\alpha|$ from x to y, the set of all paths of length k is denoted by $\mathbb{G}^k$.

Remark 1.2.2. Notice that $\mathbb{G}^k$ is the set of paths composed from k composable edges, in general *strictly* contained into the Cartesian product $\times_{l=1}^{k} \mathbb{G}^1$.

We will always suppose the graphs to be connected. A graph will be said to be *genuinely directed* if there exist vertices $x, y \in \mathbb{G}^0$ such that $(x, y) \in \mathbb{G}^1$ but $(y, x) \notin \mathbb{G}^1$.

A directed graph is said to be *locally finite* if for all $y \in \mathbb{G}^0$ the inwards and outwards degree are finite :

$$\mathsf{deg}^+(y) < \infty \text{ and } \mathsf{deg}^-(y) < \infty \text{ for all } y \in \mathbb{G}^0.$$

It is said to have *bounded geometry* if it is uniformly locally finite, namely

$$\sup_{y \in \mathbb{G}^0} \mathsf{deg}^+(y) < \infty \text{ and } \sup_{y \in \mathbb{G}^0} \mathsf{deg}^-(y) < \infty.$$

From a combinatorial point of view, a directed graph with unbounded geometry can be considered as a pathological object in the sense that it does not describe the dynamics of any Turing machine.

In chapter 4, we explain in details how a directed graph (resp. undirected) is naturally endowed with a semi-groupoid (resp. groupoid) structure.

1.2.2 Markov chain : notation and definitions

Markov operator, transition matrix and canonical path space

Let X be a denumerable set. We denote by $\ell^\infty(X)$ the space of real bounded functions on X.

Definition 1.2.3. A linear operator $P : \ell^\infty(X) \to \ell^\infty(X)$ is called *Markov* if

- $Pf \geq 0$ whenever $f \geq 0$, and
- $P\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is the constant function equal to 1.

We denote by P^n the nth power of operator P with the convention $P^0 = \mathsf{id}$. We can also denote for all $n \geq 0$ and $x, y \in X$

$$P^n(x, y) = P^n \mathbf{1}_{\{y\}}(x).$$

Formally, $P = (P(x, y))_{x,y \in X}$ is also called a *transition matrix*. Obviously, we have

$$Pf(x) = \sum_{y \in X} P(x, y) f(y) \text{ for all } x \in X.$$

A non-negative operator P which satisfies the condition $P\mathbf{1} = \mathbf{1}$ is termed *stochastic*. Some authors consider *sub-stochastic* transition matrices, *i.e.* those satisfying $\sum_{y \in X} P(x, y) \leq 1$, but such a transition matrix can be easily extended (by extending the space X) to a stochastic one. If not explicitly specified, the transition matrices will always be assumed to be stochastic.

If μ is a probability measure on X we will use the standard notation μP for the probability measure on X defined by the relation

$$\mu P(y) = \sum_{x \in X} \mu(x) P(x, y).$$

Definition 1.2.4 (Markov chain)**.** A triple (X, P, θ), where P is a Markov operator acting on $\ell^\infty(X)$, and θ a probability measure on X, is called a *Markov chain* on the state space X with *transition operator* P and *initial distribution* θ.

In the sequel, we will often choose a specific point $o \in X$ and set $\theta = \delta_o$ where δ_o denote the Dirac mass at point o. Then, the notation (X, P, θ) is often abbreviated as (X, P).

Denoting by

- X^∞ the *path space*, *i.e.* the set defined as $X^\infty := \{(y_n)_{n\geq 0} : y_n \in X\}$, and
- $\mathcal{X}^\infty$ the σ-algebra generated by cylinders,

we endow the pair $(X^\infty, \mathcal{X}^\infty)$ with the canonical probability measure $\mathbf{P}^\theta$ induced by the Markov operator P and the initial distribution θ. If $\theta = \delta_x$ we write $\mathbf{P}^x$ instead of $\mathbf{P}^{\delta_x}$. We will denote by $(M_n)_{n\geq 0}$ the X-valued sequence of random variables of law $\mathbf{P}^\theta$.

Basic assumptions on Markov chains

A Markov chain (X, P, θ) is *irreducible* if for all $x, y \in X$ there exists $n \geq 0$ such that $P^n(x, y) > 0$. Without loss of generality, we will always assume that the Markov chain is irreducible.

We denote by $c_0(X)$ the subspace of $\ell^\infty(X)$ of functions that are finitely supported. A Markov operator is said to have *finite range* if $Pf \in c_0(X)$ whenever $f \in c_0(X)$. This assumption will be usually satisfied, however, sometimes it can be a restriction so that it will be specified when needed.

General properties of a Markov chain

Suppose that (X, P, θ) is irreducible, then (X, P, θ) is said to be *recurrent* if and only if for some — hence for all — $x \in X$, $\mathbf{P}^x(\exists n \geq 0 : M_n = x) = 1$. A Markov chain is said to be *transient* if and only if for some — hence for all — $x \in X$, $\mathbf{P}^x(\exists n \geq 0 : M_n = x) < 1$. By definition, an irreducible Markov chain is either recurrent or transient. The property of being recurrent or transient is termed the *type* of the Markov chain.

We denote by G the *Green function* defined for $x, y \in X$ by

$$G(x, y) = \sum_{n\geq 0} P^n(x, y).$$

A well known criterion related to the type of an irreducible Markov chain in terms of Green function is given by

1. $G(x, y) = \infty$ for all (for some) $x, y \in X \iff (X, P, \theta)$ recurrent,
2. $G(x, y) < \infty$ for all (for some) $x, y \in X \iff (X, P, \theta)$ transient.

1.2.3 Markov chain adapted to a graph structure

In this section, we aim at describing the relationship between the combinatorial notion of graph and the stochastic notion of Markov chain.

Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ be a connected locally finite directed graph. A simple way to define a Markov chain (X, P) adapted to $\mathbb{G}$ is to set $X = \mathbb{G}^0$ and define

$$P(x,y) = \begin{cases} \frac{1}{\mathsf{deg}^-(x)} \text{ if } (x,y) \in \mathbb{G}^1, \\ 0 \text{ otherwise.} \end{cases}$$

This Markov chain is called the *simple random walk* on the graph $\mathbb{G}$. The graph being locally finite and connected, such a Markov operator is irreducible and has finite range.

More generally, we can consider a Markov operator P such that for each $x \in \mathbb{G}^0$

$$\begin{cases} P(x,y) \in (0,1] \text{ if } (x,y) \in \mathbb{G}^1, \\ P(x,y) = 0 \text{ otherwise,} \end{cases}$$

with the additional property $\sum_{y \in \mathbb{G}^0} P(x,y) = 1$. Such a Markov chain is then called a *random walk* on the graph $\mathbb{G}$. The properties of irreducibility and having finite range are still satisfied.

Conversely, if we are given a Markov chain on a denumerable set X, we can construct a directed graph for which the Markov chain is adapted. More specifically, we set $\mathbb{G}^0 = X$ and $\mathbb{G}^1$ is the subset of $\mathbb{G}^0 \times \mathbb{G}^0$ such that $\mathbb{G}^1 \ni \alpha := (x,y)$ if and only if $P(x,y) > 0$. If the Markov operator P is irreducible and has finite range, then the corresponding graph, denoted by $\mathbb{G}(P)$, is connected and locally finite.

1.3 Random walks on di-graphs induced by a PTM

Contrary to the traditional questions arising from the computation theory, we are merely interested in the dynamics of a Turing machine and the amount of the specific information produced by them. Therefore, we suppose that the set of final states is empty so that such a Turing machine never halt.

We only show how, with a probabilistic Turing machine, we associate a random walk on a directed graph (the dynamical system induced by a deterministic Turing machine is a slight adaption of the one introduced in section 1.1.1).

Let $\mathbb{G}^0 = A_\flat^* \times S \times \mathbb{Z}$ be the set of vertices. The set of edges $\mathbb{G}^1$ is defined as a subset of $\mathbb{G}^0 \times \mathbb{G}^0$ by the condition $(x,y) \in \mathbb{G}^1$, where $\mathbb{G}^0 \ni x := (\alpha, s, d)$ and $\mathbb{G}^0 \ni y := (\alpha', s', d')$, is an edge if and only if there exists $i \in \{1, \cdots, r\}$ such that $u_i(\alpha_d, s) = (\tilde{a}, \tilde{s}, \tilde{d})$ and

$$\begin{aligned} s' &= \tilde{s} \\ d' &= d + \tilde{d} \\ \alpha' &= (\alpha_1, \cdots, \alpha_{d-1}, \tilde{a}, \alpha_{d+1}, \cdots, \alpha_{|\alpha|}). \end{aligned}$$

We can define the range and source functions $r, s : \mathbb{G}^1 \to \mathbb{G}^0$ by $r((x,y)) = y$ and $s((x,y)) = x$ for all $(x,y) \in \mathbb{G}^1$. Finally, the directed graph, denoted by $\mathbb{G}(M) = (\mathbb{G}^0, \mathbb{G}^1, r, s)$, is termed the directed graph induced by the probabilistic Turing machine.

The graph $\mathbb{G}(M)$ is connected if $r \geq 1$, *i.e.* the transition function consists of at least one branch, and is locally finite because the transition function has a finite number of branches.

The local probability structure on the transition functions gives rise to a Markov operator. More precisely, $P(x, y) = \mathbf{p}_i$ if the branch i can be chosen with probability $\mathbf{p}_i$.

As a conclusion, we can study probabilistic Turing machines from the point of view of random walks on directed graphs.

Chapter 2

Martin boundary of a directed graph

The following chapter is devoted to the determination of the boundaries associated with random walks on an example of directed graph. This directed graph can be regarded as a constrained subgraph of the Cayley graph of $\mathbb{Z}^2$. In a first part, we define the Martin and the Poisson boundaries and recall some well known examples for which the boundaries are completely described. In a second part, the main result of this chapter is presented without the details of computations which are postponed to appendix A.

2.1 Boundaries of a Markov operator

The Poisson and Martin boundaries are intimately related. The first can be defined as a pure measure theoretical object, many criteria exist to decide of its triviality (see for example [KV83, KW02] in the case of random walks on groups or homogeneous spaces and [Kai92] for more general criteria).

The Martin boundary is defined with the help of a compactification (see for instance [Saw97]), and that is why the latter is a more geometric construction. Alternatively, the Poisson boundary can be seen as a measure subspace of the Martin boundary. Thus, in case the Poisson boundary is not trivial, the Martin boundary gives an interesting geometric insight (see [Saw97]).

2.1.1 Martin boundary

Geometry of the space of surperharmonic functions

In this section we always do the usual assumption of irreducibility of the Markov chain (X, P).

A function $f \in \ell^\infty(X)$ is *superharmonic* if $Pf \leq f$ and *harmonic* if $Pf = f$. We denote by $\mathcal{S}^+$ the set of non-negative superharmonic functions and by $\mathcal{H}^+$ the subset of non-negative harmonic functions. It is well known that $\mathcal{S}^+$ is a convex *cone* with vertex $\mathbf{0}$, *i.e.* it is convex and if $u \in \mathcal{S}^+ \setminus \{0\}$, then the whole half-line $\{au : a \geq 0\}$ is contained in $\mathcal{S}^+$.

A *base* of a cone with vertex $\bar{v}$ is a subset $\mathcal{B}$ such that each element of the cone different from $\bar{v}$ can be uniquely written as $\bar{v} + a(u - \bar{v})$ with $a > 0$ and $u \in \mathcal{B}$. Let us fix a base point $o \in X$, then the set

$$\mathcal{B} = \{u \in \mathcal{S}^+ : u(o) = 1\}$$

is a base of the cone $\mathcal{S}^+$.

Finally, the set $\mathcal{S}^+$ can be endowed with the topology of pointwise convergence as a subset of the space of all real functions.

Theorem 2.1.1. *Let (X, P) be an irreducible Markov chain. Then,*

1. *$\mathcal{S}^+$ is closed and $\mathcal{B}$ is compact in the topology of pointwise convergence.*
2. *If P has finite range, then $\mathcal{H}^+$ is closed.*

The hypothesis of finite range is sufficient but not necessary. Nevertheless, note that the subset $\mathcal{H}^\infty_+$ of bounded positive harmonic functions is always closed in $\mathcal{S}^+$ by dominated convergence.

Since $\mathcal{B}$ is a base of the cone $\mathcal{S}^+$, a description of the base yields a description of the cone. Moreover, the base considered is convex and compact so that we only need to describe the set $\mathsf{ext}(\mathcal{B})$ of *extremal points* of $\mathcal{B}$. Recall that extremal points are elements of $\mathcal{B}$ which cannot be written as a convex combination $au_1 + (1 - a)u_2$ with $0 < a < 1$ of distinct elements u_1 and u_2 of $\mathcal{B}$.

It is easy to check that the function $x \mapsto G(x, y)$ is positive superharmonic for all y and strictly superharmonic at the point y

$$\begin{aligned}\sum_{z\in X} P(x,z)G(z,y) &= \sum_{n\geq 0}\sum_{z\in X} P(x,z)P^n(z,y)\\ &= \sum_{n\geq 0} P^{n+1}(x,y)\\ &= G(x,y) - \delta_x(y) \leq G(x,y)\end{aligned}$$

However, there is no reason for the function $x \mapsto G(x, y)$ to belong to $\mathcal{B}$. Hence, we normalize it.

Definition 2.1.2. 1. The *Martin kernel* is defined by

$$K(x,y) = \frac{G(x,y)}{G(o,y)}$$

for $x, y \in X$.

2. A function $h \in \mathcal{H}^+$ is called *minimal* if
   - $h(o) = 1$, and
   - if $h_1 \in \mathcal{H}^+$ and $h \geq h_1$ in each point, then $\frac{h_1}{h}$ is constant.

Theorem 2.1.3. *If (X, P) is transient, then the extremal elements of $\mathcal{B}$ are the Martin kernels and the minimal harmonic functions :*

$$\mathsf{ext}(\mathcal{B}) = \{K(\cdot, y) : y \in X\} \cup \{h \in \mathcal{H}^+ : h \text{ is minimal }\}$$

Let us give two criteria to better understand the notion of minimal harmonic function. Let h be a positive, non-zero superharmonic function. We can define the so-called Doob's h-process (see [Doo59]), it is the Markov chain with transition operator P_h defined by

$$P_h(x, y) = \frac{P(x, y)h(y)}{h(x)}.$$

The following simple properties can be easily checked.

1. The operator P_h is stochastic if and only if h is a positive, non-zero harmonic function;

2. a function u is superharmonic — resp. harmonic — with respect to P if and only if $\bar{u} = u/h$ is superharmonic — resp. harmonic — with respect to P_h.

We recall that $\mathcal{H}^\infty = \mathcal{H}^\infty(X, P)$ denote the space of bounded harmonic functions.

Lemma 2.1.4. *The space $\mathcal{H}^\infty$ is trivial — contains only constant functions — if and only if the constant harmonic function $\mathbf{1}$ is minimal.*

The following corollary provides with a characterization of minimal harmonic functions in terms of bounded harmonic functions of the Doob's h-process.

Corollary 2.1.5. *A function $h \in \mathcal{B}^+$ is minimal if and only if the space of bounded harmonic functions with respect to P_h is trivial.*

According to Choquet representation theory, if $x \in K$, where K is a metrizable compact convex subset of a topological locally convex vector space, then there exists a probability measure ν on the set $\mathsf{ext}(K)$ of extremal points such that

$$x = \int_{\mathsf{ext}(K)} c\nu(dc).$$

Nevertheless, the measure ν is not unique in general if K is not a simplex — see for example [Phe66]. Therefore, we would need to prove that a certain subset of the base $\mathcal{B}$ is a simplex and we would obtain an integral representation of non negative superharmonic functions. However, in this approach, the stochastic meaning of the Martin boundary is missed. That is why many authors usually prefer to prove the integral representation theorem in terms of Martin compactification which yields an extra result related to the convergence of the Markov chain to the boundary.

The Martin compactification

Given the countably infinite set X, by a compactification of X we mean a compact topological Haussdorff space $\hat{X}$ containing X such that

- the set X is dense in $\hat{X}$, and
- the induced topology on X is discrete in $\hat{X}$.

The set $\partial X = \hat{X} \setminus X$ is called the *boundary*. The set X can be compactified in many equivalent ways. Let us give one of them.

Theorem 2.1.6. *Let $\mathcal{F}$ be a denumerable family of real bounded functions on X. Then there exists a unique, up to homeomorphism, compactification $\hat{X} = X_{\mathcal{F}}$ of X such that*

1. *every function $f \in \mathcal{F}$ extends to a continuous function on $\hat{X}$, and*
2. *the family $\mathcal{F}$ separates the boundary points : if $\xi, \eta \in \hat{X} \setminus X$ are distinct, then there exists a function $f \in \mathcal{F}$ with $f(\xi) \neq f(\eta)$.*

The proof of this well known theorem is of general interest, that is why we will give its main steps.

Proof. First, let us prove the existence of such a compactification. For $x \in X$, we write $\mathbf{1}_x$ for the indicator function of the point x. We add all those indicator functions to $\mathcal{F}$, setting

$$\mathcal{F}^* = \mathcal{F} \cup \{\mathbf{1}_x : x \in X\}$$

For each $f \in \mathcal{F}^*$, there is a constant C_f such that $|f(x)| \leq C_f$ for all $x \in X$. Consider the topological product space

$$\Pi_{\mathcal{F}} = \prod_{f \in \mathcal{F}^*} [-C_f, C_f] = \{\phi : \mathcal{F}^* \mapsto \mathbb{R} : \phi(f) \in [-C_f, C_f] \text{ for all } f \in \mathcal{F}^*\}$$

The natural topology is the one of pointwise convergence. A neighborhood base at $\phi \in \Pi_{\mathcal{F}}$ is given by finite intersections of sets of the form $\{\psi \in \Pi_{\mathcal{F}} : |\psi(f) - \phi(f)| < \varepsilon\}$, as $f \in \mathcal{F}^*$ and $\varepsilon > 0$ vary.

We can embed X into $\Pi_{\mathcal{F}}$ via the map

$$X \ni x \mapsto i(x) = \phi_x \in \Pi_{\mathcal{F}}, \text{ where } \phi_x(f) = f(x) \text{ for } f \in \mathcal{F}^*.$$

If x, y are two distinct points of X, then $\phi_x(\mathbf{1}_x) = 1 \neq 0 = \phi_y(\mathbf{1}_x)$. Therefore, i is injective. Moreover, the neighborhood $\{\psi \in \Pi_{\mathcal{F}} : |\psi(\mathbf{1}_x) - \phi_x(\mathbf{1}_x)| < 1\}$ of $i(x) = \phi_x$ contains none of the functions ϕ_y for $y \neq x$. This means that $i(X)$, endowed with the induced topology, is a discrete subset of $\Pi_{\mathcal{F}}$. Thus we may identify X with $i(X)$.

Now we define $\hat{X} = X_{\mathcal{F}}$ as the closure of X in the compact topological space $\Pi_{\mathcal{F}}$. Hence, X is a dense discrete subset of $\hat{X}$ which is the type of compactification we were looking for.

Each $\xi \in \hat{X} \setminus X = \partial X$ is a function $\mathcal{F}^* \mapsto \mathbb{R}$ with $|\xi(f)| \leq C_f$. By the construction of $\hat{X}$, there must be a sequence (x_n) of distinct points in X that converges to ξ, that is, $f(x_n) = \phi_{x_n}(f) \to \xi(f)$ for every $f \in \mathcal{F}^*$. Let us define $f(\xi) = \xi(f)$. Observe that since $\phi_{x_n}(\mathbf{1}_x) = 0$ when $x_n \neq x$, then we have $\mathbf{1}_x(\xi) = \xi(\mathbf{1}_x) = 0$ for every x.

If (x_n) is an arbitrary sequence in X which converges to ξ in the topology of $\hat{X}$, then for each $f \in \mathcal{F}$ one has

$$f(x_n) = \phi_{x_n}(f) \to \xi(f) = f(\xi)$$

thus, f becomes a continuous function on $\hat{X}$. Finally, $\mathcal{F}$ separates the points of ∂X : if $\xi, \eta \in \partial X$, then they are also distinct in their original definition as functions on $\mathcal{F}^*$. Hence there is $f \in \mathcal{F}^*$ such that $\xi(f) \neq \eta(f)$. Since $\xi(\mathbf{1}_x) = \eta(\mathbf{1}_x) = 0$ for every x, we must have $f \in \mathcal{F}$. Thus, $f(\xi) \neq f(\eta)$.

The uniqueness part of the proof is classical and can be found in [Woe09]. The identity $\tau : X \mapsto X$ extends continuously to a map from $\hat{X} \mapsto \tilde{X}$, and we have to check that τ is actually an homeomorphism. □

A different usual construction of the Martin compactification is given by the completion with respect to a certain metric which depends on the family $\mathcal{F}^*$ — see [Saw97] or again [Woe09].

Definition 2.1.7 (Martin compactification)**.** Let (X, P) be an irreducible Markov chain. The *Martin compactification* of X with respect to P is defined as $\hat{X}(P) = \hat{X}_{\mathcal{F}}$, where $\mathcal{F}$ is the family of functions

$$\mathcal{F} = \{K(x, \cdot) : x \in X\}$$

The *Martin boundary* $\partial X = \hat{X}(P) \setminus X$ is the boundary of the compactification.

Lemma 2.1.8. *If (X, P) is transient and $\xi \in \partial X$, then $K(\cdot, \xi)$ is a positive superharmonic function. Moreover, if P has finite range, then the function $K(\cdot, \xi)$ is harmonic.*

Proof. We say that a sequence (y_n) tends to infinity if for all finite subset $A \subset X$, there exists an integer $N \geq 0$ such that for all $n \geq N$, $y_n \notin A$. By construction, there exists a sequence (y_n) in X, tending to ∞ such that $K(\cdot, y_n) \to K(\cdot, \xi)$ pointwise in X. Thus, $K(\cdot, \xi)$ is the pointwise limit of superharmonic functions $K(\cdot, y_n)$ and consequently a superharmonic function.

Now, in the case where P is of finite range

$$PK(x, y_n) = \sum_{y : P(x,y) > 0} P(x, y) K(y, y_n) = K(x, y_n) - \frac{\delta_x(y_n)}{K(o, y_n)}$$

If the summation is finite, it can be exchanged with the limit as $n \to \infty$. Since, $y_n \neq x$ for all but finitely many n, we get that $PK(x, \xi) = K(x, \xi)$. □

Convergence to the boundary

The interesting meaning of the Martin boundary is contained in the following theorem. As a compact metric space, $\hat{X}(P)$ carries a natural σ-algebra, namely the Borel σ-algebra. Speaking of a $\hat{X}(P)$-valued random variable, we mean a measurable function from the path space $(X^\infty, \mathcal{X}^\infty, \mathbf{P}^\theta)$ to $\hat{X}(P)$.

Theorem 2.1.9 (Convergence to boundary)**.** *If $(Z_n)_{n\geq 0}$ is a X-valued transient Markov chain, with Markov operator P, then there exists a random variable Z_∞ taking its values in ∂X such that for each $x \in X$,*

$$\mathbf{P}^x(\lim_{n\to\infty} Z_n = Z_\infty) = 1,$$

where the limit has to be understood as a limit in the topology of $\hat{X}(P)$.

The Poisson-Martin integral representation theorem

From the convergence theorem, we can construct a probability measure on the boundary ∂X. For all measurable subsets B of $\hat{X}$ we define

$$\nu_x(B) = \mathbf{P}^x(Z_\infty \in B).$$

Thus, by definition, if $f : \hat{X} \to \mathbb{R}$ is a ν_x-integrable function, then

$$\mathbf{E}^x(f(Z_\infty)) = \int_{\hat{X}} f(\xi)\nu_x(d\xi).$$

Actually, we have the following finer result.

Theorem 2.1.10. *The measure ν_x is absolutely continuous with respect to ν_o and (a version of) its Radon-Nikodym derivative is given by $\frac{d\nu_x}{d\nu_o} = K(x, \cdot)$. Namely, if $B \subset \hat{X}$ is a Borel set then,*

$$\nu_x(B) = \int_B K(x, \xi)\nu_o(d\xi).$$

We have now arrived at the point where we can give the second main theorem of the Martin boundary theory, after the one dealing with the convergence to the boundary.

Theorem 2.1.11 (Poisson-Martin representation formula)**.** *Let (X, P) be irreducible and transient, with Martin compactification $\hat{X}$ and Martin boundary ∂X. Then for all $h \in \mathcal{S}^+(X, P)$ there exists a Borel measure ν^h on $\hat{X}$ such that*

$$h(x) = \int_{\hat{X}} K(x, \xi)\nu^h(d\xi) \text{ for every } x \in X.$$

If h is harmonic then $\mathsf{supp}(\nu^h) \subset \partial X$.

We refer the reader to [Woe09] for a proof of this fact. Nonetheless, let us say that in the proof of this theorem a natural choice for the measure ν^h in the integral representation appears : for a Borel set $B \subset \hat{X}$

$$\nu^h(B) = h(o)\mathbf{P}_h^o(Z_\infty \in B) \tag{2.1}$$

where $\mathbf{P}_h^o$ is the probability measure on the path space induced by the h-process.

Theorem 2.1.12. *Let h be a minimal harmonic function. Then there is a point $\xi \in \partial X$ such that the unique measure ν on $\hat{X}$ which gives rise to an integral representation $h = \int_{\hat{X}} K(\cdot, \eta)\nu(d\eta)$ is the Dirac mass $\nu = \delta_\xi$.*

We define the *minimal Martin boundary* $\partial X_{\mathsf{min}}$ as the set of all $\xi \in \partial X$ such that $K(\cdot, \xi)$ is a minimal harmonic function. From theorem 2.1.12, we know that every minimal harmonic function arises in this way. It can be shown that the minimal Martin boundary is a Borel set of $\hat{X}$. By now, we can give an integral representation theorem with uniqueness of the representative measure.

Theorem 2.1.13. *If $h \in \mathcal{S}^+$, then the unique measure ν on $\hat{X}$ such that*

$$\nu(\partial X \setminus \partial X_{\mathsf{min}}) = 0$$

and

$$h(x) = \int_{\hat{X}} K(x, \xi)\nu(d\xi) \text{ for all } x \in X$$

is given by $\nu = \nu^h$ defined by equation (2.1).

2.1.2 Poisson boundary

We give in this section the definition of the Poisson boundary regarded as a measurable subspace of the Martin boundary. The Poisson boundary is a theoretical measure object and it can be constructed without the help of the Martin boundary (see [Kai92] for example). In appendix B, the Poisson boundary will be defined as the space of ergodic components of a dynamical system on a measure space arising as an inductive limit of measure spaces.

Poisson transform

If ν is a Borel measure on ∂X, then

$$h = \int_{\partial X} K(\cdot, \xi)\nu(d\xi)$$

defines a non-negative harmonic function. This relies on the monotone convergence theorem and the fact that $K(\cdot, \xi)$ is harmonic. Moreover, by theorem 2.1.11, if $u \in \mathcal{S}^+$, then we have

$$u(x) = \sum_{y \in X} K(x, y)\nu^u(y) + \int_{\partial X_{\mathsf{min}}} K(x, \xi)\nu^u(d\xi)$$

Set $g(x) = \sum_{x \in Y} K(x,y)\nu^u(y)$ and $h(x) = \int_{\partial X_{\min}} K(x,\xi)\nu^u(d\xi)$. Then as we have just observed, h is harmonic, and $\nu^h = \nu^u_{|\partial X}$ by theorem 2.1.13.

It can be shown that $g = Gf$ with $f = u - Pu$ so that we obtain the so-called *Riesz decomposition* of superharmonic function $u = Gf + h$ with extra informations on the harmonic part.

In the definition we gave of a Markov operator, it is said that $P\mathbf{1} = \mathbf{1}$. Hence, the constant function $\mathbf{1}$ is harmonic. Setting $B = \hat{X}$ in theorem 2.1.10, we see that the measure on $\hat{X}$ which gives rise to the integral representation of $\mathbf{1}$ in theorem 2.1.13, is the measure ν_o.

We consider the space $\mathbb{L}^1(\partial X, \nu_o)$ of ν_o-integrable functions on ∂X. Let $\phi \in \mathbb{L}^1(\partial X, \nu_o)$, then the Poisson integral of ϕ is the function

$$h(x) = \int_{\partial X} K(x,\xi)\phi(\xi)\nu_o(d\xi) = \int_{\partial X} \phi(\xi)\nu_x(d\xi) = \mathbf{E}^x(\phi(Z_\infty)), x \in X.$$

It is easy to see that h is a bounded harmonic function if ϕ is supposed to be bounded. Conversely, the following holds.

Theorem 2.1.14. *Every bounded harmonic function is the Poisson integral of a bounded measurable function on ∂X, in symbols*

$$\mathcal{H}^\infty \cong \mathbb{L}^\infty(\partial X, \nu_o).$$

Convergence to the Poisson boundary

We finish this section with a convergence theorem which gives the stochastic meaning of the Poisson boundary. This theorem has a more general version in terms of asymptotic σ-algebra.

Theorem 2.1.15 (Probabilistic Fatou theorem)**.** *Let $\phi \in \mathbb{L}^\infty(\partial X, \nu_o)$ and h be its Poisson integral, then*

$$\lim_{n\to\infty} h(Z_n) = \phi(Z_\infty), \nu_o\text{-almost surely.}$$

2.1.3 Examples : the cases of $\mathbb{Z}^N$ and free groups

An important part of the results related to boundaries has been given for random walks on groups of finite type.

Random walks on the lattice $\mathbb{Z}^N$

One of the first example of computation of the Martin boundary concerns random walks on the lattice $\mathbb{Z}^N$. In [NS66], Ney and Spitzer describe the Martin boundary in the case of a Markov operator with a finite first moment condition. Also, they assume that the drift is non zero.

Let μ be a probability measure on $\mathbb{Z}^N$, we can define for all $x, y \in \mathbb{Z}^N$ the transition operator by $P(x,y) = \mu(y-x)$. In the sequel we assume that P is irreducible. If μ satisfies

$$\sum_{x\in\mathbb{Z}^N} |x|\mu(x) = \sum_{x\in\mathbb{Z}^N} |x|P(0,x) < \infty,$$

where $|x|$ denotes the Euclidean distance of x from the origin, we can define the drift by

$$v = \sum_{x\in\mathbb{Z}^N} x\mu(x) \in \mathbb{Z}^N.$$

Finally, we define on $\mathbb{Z}^N$ a real function ϕ by

$$\phi(u) = \sum_{x\in\mathbb{Z}^N} P(0,x)e^{\langle x,u\rangle},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product. We define the two following sets

$$D = \{u \in \mathbb{Z}^N : \phi(u) \leq 1\} \text{ and } \partial D = \{u \in \mathbb{Z}^N : \phi(u) = 1\}.$$

For their proof, Ney and Spitzer need the additional assumption (♣); *every point of ∂D has a neighborhood in which ϕ is finite.* Under this assumption, it can be shown that the set D is compact and convex, the gradient

$$\nabla\phi(u) = \sum_{x\in\mathbb{Z}^N} xe^{\langle u,x\rangle}P(0,x)$$

exists everywhere on D and does not vanish on its boundary ∂D. Furthermore, the map

$$u \to \frac{\nabla\phi(u)}{|\nabla\phi(u)|}$$

determines a homeomorphism between ∂D and ∂S, where ∂S is the $(N-1)$-dimensional unit sphere. More precisely, the mapping

$$x \to \frac{x}{1+|x|}$$

maps $\mathbb{Z}^N$ on a countable subset S of the N-dimensional unit ball, moreover $\mathbb{Z}^N$ and S are homeomorphic — for the discrete topology. We complete the set S with respect to the Euclidean metric, and the lattice $\mathbb{Z}^N$ with respect to the metric

$$\rho(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|.$$

Then, the completion $\mathbf{Z}^N = \mathbb{Z}^N \cup \partial\mathbb{Z}^N$ is homeomorphic to $\bar{S}$, and $\partial\mathbb{Z}^N$ is homeomorphic to ∂S. The main theorem of [NS66] is given below under the conditions of irreducibility, non zero drift and (♣).

Theorem 2.1.16. *For $x \in \mathbb{Z}^N$, define $f_x : \mathbf{Z}^N \to \mathbb{R}$ by*

$$\begin{cases} f_x(y) = \frac{G(x,y)}{G(0,y)} \text{ for } y \in \mathbb{Z}^N, \text{ and} \\ \\ f_x(\xi) = e^{\langle \alpha(\xi), x \rangle} \text{ for } \xi \in \partial\mathbb{Z}^N, \end{cases}$$

where $\alpha : \partial\mathbb{Z}^N \to \partial D$ is the homeomorphism defined as the composition of the two homeomorphisms $\partial\mathbb{Z}^N \leftrightarrow \partial S$ and $\partial S \leftrightarrow \partial D$ defined in the above discussion. Then the function f is continuous on $\mathbf{Z}^N$.

Moreover, the family of function $\{f_x : x \in \mathbb{Z}^N\}$ obviously separates the points of the boundary $\partial\mathbb{Z}^N$. In other words, the Martin boundary of such a random walk is homeomorphic to a $(N-1)$-dimensional Euclidean sphere. Furthermore, every point $\xi \in \partial\mathbb{Z}^N$ are clearly minimal, consequently, the minimal Martin boundary is equal to the whole Martin boundary (and hence homeomorphic to the $(N-1)$-dimensional spheres). In the proof of this theorem, Ney and Spitzer establish a *local limit theorem* which gives rise to fine estimates of the Green function.

For $N = 3$ and μ a probability measure with a finite second moment, the case of zero drift is solved in [Spi76], part 26.1. In this case, the Martin boundary is trivial. The generalisation for $N \geq 4$ is immediate under the condition of a probability measure with finite moment of even order $2m > N - 2$. For $N = 1$ or $N = 2$, the Martin boundary is obvious because the random walk is recurrent.

In [Uch98], precise estimates of the Green function are given. Those estimates depend on the dimension and moment conditions.

Finally, the Poisson boundary is trivial in all cases, see for instance [CD60].

The free group

In the sequel, we denote by $\mathbf{F}_d$ the free group on d generators. In the context of free groups the description of the Martin boundary is obvious. The more general situation of homogeneous trees is studied in [Woe95]. The description of the Martin boundary of a homogeneous tree T involves a transitive action of a closed subgroup Γ, namely the group of all automorphisms (self-isometries) $\mathsf{AUT}(T)$ of the metric space (T, d) with d the usual graph metric on T. In this context, a random walk on T can be seen as the realisation of a random walk on Γ. The description of the Martin boundary in the context of free groups has the advantage to reveal the essential ideas without involving tedious details.

Let $\mathbf{F}_d = \langle a_1^{\pm 1}, \cdots, a_d^{\pm 1} \rangle$ be the free group with d generators. For every vertex $x, y \in \mathbf{F}_d$, there is a unique *geodesic segment* $\overline{xy}$. We denote by d the usual graph metric. A *ray* is an infinite reduced sequence $R = (x_i)_{i \geq 0}$ of successive neighbours. Two rays R and R' are said to be equivalent if they differ only by finitely many vertices. An *end* of the tree $\mathbf{F}_d$ is an equivalent class of rays. We denote by $\mathcal{E}$ the set of ends, and we write $\widehat{\mathbf{F}}_d = \mathbf{F}_d \cup \mathcal{E}$. For every $e \in \mathcal{E}$ and $x \in \mathbf{F}_d$, there exists a unique ray $\overline{xe}$ starting at x which represents e.

Let us fix a reference vertex $o \in \mathbf{F}_d$. If $\xi, \eta \in \widehat{\mathbf{F}}_d$, then the *confluent* $c = c(\xi, \eta)$ is the last common vertex on $\overline{o\xi}$ and $\overline{o\eta}$, unless $\xi = \eta \in \mathcal{E}$, in which case we set $c(\xi, \eta) = \xi$. We

set

$$(\xi|\eta) = \begin{cases} d(o,c), & \text{if } c \in \mathbf{F}_d, \text{ and} \\ \infty, & \text{otherwise.} \end{cases}$$

Noting that $(\xi|\eta) \geq \min\{(\xi|\zeta), (\zeta|\eta)\}$ for all $\xi, \zeta, \eta \in \widehat{\mathbf{F}}_d$, we define the metric ρ on $\widehat{\mathbf{F}}_d$ by

$$\rho(\xi,\eta) = \begin{cases} e^{-(\xi|\eta)}, & \text{if } \xi \neq \eta, \\ 0, & \text{if } \xi = \eta. \end{cases}$$

Thus $\widehat{\mathbf{F}}_d$ becomes a totally disconnected compact space with $\mathbf{F}_d$ open, dense and discrete.

Let μ be a probability measure with a finite support generating $\mathbf{F}_d$ as a semigroup. We can define the transition operator P by $P(x,y) = \mu(x^{-1}y)$ for $x, y \in \mathbf{F}_d$. We set $\tau_y = \inf\{n \geq 0 : M_n = y\}$ and define $\Pi(x,y) = \mathbf{P}^x(\tau_y < \infty)$, for $x, y \in \mathbf{F}_d$, the probability to hit y, starting from x, in finite time. For a group, the Martin kernel is given by

$$K(x,y) = \frac{G(x,y)}{G(o,y)} = \frac{\Pi(x,y)}{\Pi(o,y)}.$$

This equality is also available in the case of a transitive action of a group on a metric space preserving the Markov operator.

Moreover, in the particular case of free groups, we observe that for any $\xi \in \mathcal{E}$

$$\frac{\Pi(x,y_n)}{\Pi(o,y_n)} = \frac{\Pi(x,\xi)}{\Pi(o,\xi)} \frac{\Pi(\xi,y_n)}{\Pi(\xi,y_n)}.$$

This property comes from the 0-hyperbolicity of the free group $\mathbf{F}_d$. As a consequence of this formula, we have the following theorem.

Theorem 2.1.17. *For all $x \in \mathbf{F}_d$ the function $K(x,\cdot)$ extends continuously to a function on $\widehat{\mathbf{F}}_d$. Moreover, for $\xi \neq \eta \in \mathcal{E}$,* i.e. *such that $\rho(\xi,\eta) > 0$. there exists two points $x_1, x_2 \in \mathbf{F}_d$ such that $K(x_1,\xi) \neq K(x_2,\eta)$. In other word, the Martin boundary is homeomorphic to $\mathcal{E}$.*

In this example, we have seen how the group structure and the hyperbolicity simplify the description of the Martin boundary. In the context of free groups, the Poisson boundary is isomorphic to the Martin boundary. We refer the reader to [Kai00] for a general study of the Poisson boundary in the case of the hyperbolic groups.

2.2 Boundaries of the simple random walk on a di-graph

In [CP03], transience and recurrence are studied for the simple random walk on various types of partially horizontally oriented regular lattices. In this paragraph we aim at going further in the transient case by determining the Martin boundary of such a random walk.

2.2.1 Definition of the graph $\mathbb{H}$

We consider two dimensional lattices, i.e $\mathbb{G}^0 = \mathbb{Z}^2$ and $\mathbb{G}^1$ is a subset of nearest neighborhoods in $\mathbb{Z}^2$. We decompose $\mathbb{G}^0 = \mathbb{G}^0_1 \times \mathbb{G}^0_2$ into horizontal and vertical directions. More precisely, if $\mathbf{v} \in \mathbb{G}^0 = \mathbb{Z}^2$, then $\mathbf{v} = (v_1, v_2)$ with $v_i \in \mathbb{G}^0_i$ the usual coordinates in $\mathbb{Z}^2$. Let $\epsilon = (\epsilon_y)_{y \in \mathbb{G}^0_2}$ be a $\{-1, 0, 1\}$-valued sequence of variables.

Definition 2.2.1. We call ϵ-horizontally oriented lattice $\mathbf{G} = (\mathbf{G}, \epsilon)$, the directed graph with vertex set $\mathbb{G}^0 = \mathbb{Z}^2$ and edge set $\mathbb{G}^1$ with the condition $\alpha = (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$ if and only if one of the following holds

1. either $v_1 = u_1$ and $v_2 = u_2 \pm 1$
2. or $v_2 = u_2$ and $v_1 = u_1 + \epsilon_{u_2}$

Note that $\mathbf{G}$ is connected if and only if 1 and -1 are both in the range of ϵ.

Let ϵ be the sequence defined by $\epsilon_0 = 0$ and $\epsilon_y = \mathsf{sgn}(y)$ where sgn is the sign function, then, we denote by $\mathbb{H}$ the ϵ-graph induced.

This definition is due to [CP03], nevertheless we should replace this example in the context of constrained subgraphs of the Cayley graph of $\mathbb{Z}^2$. Let $\mathcal{S} = \{\pm\varepsilon_1, \pm\varepsilon_2\}$ be the canonical set of generators of $\mathbb{Z}^2$. Let f be a constraint of the Cayley graph of $\mathbb{Z}^2$, that is a function $f : \mathbb{Z}^2 \times \mathcal{S} \to \{0, 1\}$. In the example considered here, the constraint f is defined for $(z, s) \in \mathbb{Z}^2 \times \mathcal{S}$ as follows

$$f(z,s) = \begin{cases} 1 & \text{if } s = \pm\varepsilon_2, \\ & \text{or } s = \varepsilon_1 \text{ and } \langle z, \varepsilon_2 \rangle > 0 \\ & \text{or } s = -\varepsilon_1 \text{ and } \langle z, \varepsilon_2 \rangle < 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^2$ with respect to the canonical basis $(\varepsilon_1, \varepsilon_2)$. The constrained subgraph is the directed graph $(\mathbb{G}^0, \mathbb{G}^1, r, s)$ where

- $\mathbb{G}^0 = \mathbb{Z}^2$,
- $\mathbb{G}^1 = \{(z, z') \in \mathbb{G}^0 \times \mathbb{G}^0 : f(z, z' - z) = 1\}$,
- the source and range functions $s, r : \mathbb{G}^1 \to \mathbb{G}^0$ are defined such that $s((z, z')) = z$ and $r((z, z')) = z'$ for all $(z, z') \in \mathbb{G}^1$.

As a directed graph, the set of paths of finite lenght can be endowed with a semi-groupoid structure (see chapter 4).

2.2.2 Poisson boundary

The case of the simple random walk

The proof of the following proposition is an adaptation of ideas involved in the proof of the triviality of the Poisson boundary of random walks on Abelian group due to Choquet

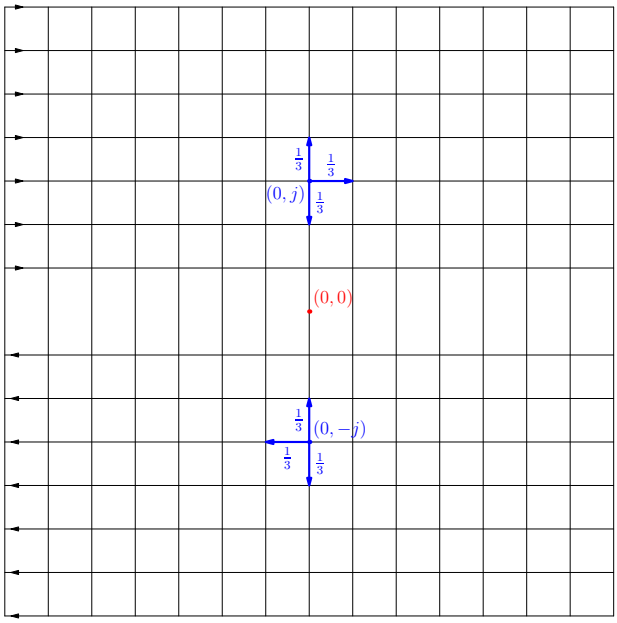


Figure 2.1: The half plane one-way lattice $\mathbb{H}$

and Deny (see [CD60]) or more specifically we will adapt the proof of theorem T1, chapter VI, in [Spi76].

Proposition 2.2.2. *The Poisson boundary of the simple random walk on* $\mathbb{H}$ *is trivial,* i.e *all bounded harmonic functions are constant.*

Elementary proof. Let h be a bounded harmonic function and $a = (\alpha, 0)$ a vector of $\mathbb{H}$. We set $g(x) = h(x) - h(x - a)$, then g is obviously harmonic

$$Pg(x) = h(x) - \sum_{y \in \mathbb{H}} P(x, y)h(y - a).$$

Thus, setting $z = y - a$, substituting in the sum, and noting that $P(x, z + a) = P(x - a, z)$ because P is invariant by horizontal translations, we get

$$Pg(x) = h(x) - \sum_{z \in \mathbb{H}} P(x - a, z)h(z) = h(x) - h(x - a) = g(x).$$

Now let $\sup_{x \in \mathbb{H}} g(x) = M < \infty$, choose a sequence x_n of points in $\mathbb{H}$ such that

$$\lim_{n \to \infty} g(x_n) = M,$$

and let

$$g_n(x) = g(x + x_n).$$

Since g is bounded, one can select a subsequence $x_n^{(1)}$ from the sequence x_n such that, for a certain $x = x_1$

$$\lim_{n\to\infty} g(x_1 + x_n^{(1)}) \text{ exists.}$$

However, we can do better since the vertex set of $\mathbb{H}$ is an Abelian discrete group. We can take a subsequence $x_n^{(2)}$ of the sequence $x_n^{(1)}$ such that $g(x + x_n^{(2)})$ has a limit at $x = x_1$ and also at $x = x_2$. This process can be continued. By the Cantor's diagonalisation principle, $\mathbb{H}$ being countable, there exists a subsequence n_l of positive integers and a real function g^* on $\mathbb{H}$ such that

$$\lim_{l\to\infty} g_{n_l}(x) = g^*(x)$$

for every $x \in \mathbb{Z}$. Moreover, it is obvious that

$$g^*(0) = M, \text{ and } g^*(x) \leq M \text{ for all } x \in \mathbb{H}.$$

Furthermore, the function g^* is harmonic by dominated convergence.

Recall that the simple random walk on $\mathbb{H}$ is irreducible because the graph is connected. Thus, applying the maximum principle to the harmonic function implies that $g^* \equiv g^*(0) = M$.

Let r be any positive integer and $\epsilon > 0$, we can find an integer n large enough such that

$$g_n(a) > M - \epsilon \ ; \ g_n(2a) > M - \epsilon \ ; \ \cdots \ ; \ g_n(ra) > M - \epsilon.$$

Going back to the definition of g_n and adding those r inequalities, we obtain

$$h(ra + x_n) - h(x_n) > r(M - \epsilon)$$

for n large enough. We can show that M can not be strictly positive. Indeed, if it was, the integer r could have been chosen so large that $r(M - \epsilon)$ exceeds the least upper bound of h. Therefore, it follows $g(x) \leq M \leq 0$ and $h(x) \leq h(x - a)$. Obviously, we can do the same reasoning for $-h$ and we would have $h(x) \geq h(x - a)$.

Setting $\tilde{h}(y) = h(x_0, y)$ for some x_0, we show that the bounded harmonic function $\tilde{h}$ is constant by the maximum principle.

□

Reducible random walk

In this paragraph, we give an example of reducible random walk for which the Poisson boundary is not trivial. In this example, the state space is the upper half plane $X = \mathbb{Z}\times\mathbb{Z}_+$, and we denote by (e_1, e_2) the canonical basis of $\mathbb{Z}^2$, *i.e.* $e_1 = (1, 0)$ and $e_2 = (0, 1)$. In addition, we set $X_- = \mathbb{Z} \times \{0\}$. Let $x \notin X_-$, the Markov operator is given for $y \in X$ by

$$P(x, y) = \begin{cases} q \text{ if } y = x + e_2, \\ p \text{ if } y = x + e_1, \\ r \text{ if } y = x - e_2, \text{ and} \\ 0, \text{ otherwise,} \end{cases}$$

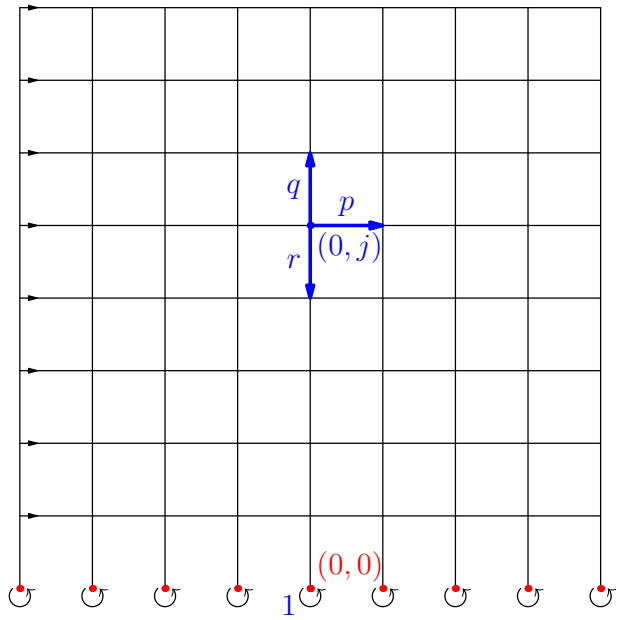


Figure 2.2: A reducible example with a non trivial Poisson boundary.

where $p, q, r \in (0,1)$ are such that $p+q+r=1$. If $x \in X_-$ then we set $P(x,x)=1$ and $P(x,y)=0$ for $y \neq x$. This is summarized in the figure 2.2.

Let $h : X \to \mathbb{R}$ be of the form $h(x) = e^{\langle x,a\rangle}$ where $a \in \mathbb{R}^2$ and $\langle \cdot, \cdot \rangle$ is the usual scalar product induced by $\mathbb{R}^2$. We will show that we can find $a \in \mathbb{R}^2$ such that h is a bounded harmonic function. Assuming $x \notin X_-$, we compute

$$Ph(x) = e^{\langle x,a\rangle}(qe^{a_2} + pe^{a_1} + re^{-a_2})$$

if we set $a = (a_1, a_2) \in \mathbb{R}^2$. Notice that $Ph(x) = h(x)$ for all function h and all $x \in X_-$. Consequently, h is harmonic if and only if $qe^{a_2} + pe^{a_1} + re^{-a_2} = 1$. Moreover, h is supposed to be bounded so that a_1 is zero and a_2 is non positive. Solving $qe^{2a_2} + (p-1)e^{a_2} + r = 0$, we find two solutions $a_2 = 0$ or $a_2 = \log \frac{r}{q}$. As a consequence, we have exhibited a family of harmonic functions $h(x) = \left(\frac{r}{q}\right)^{x_2}$ which are bounded if $q > r$ and unbounded if $q < r$.

The case of random walk on $\mathbb{H}$ with a drift

Looking at the proof of proposition 2.2.2, we observe that the crucial property is the translation invariance of the operator which allow to consider the simpler problem of the determination of the bounded harmonic functions associated with a specific random walk on $\mathbb{Z}$.

Let $(p_y)_{y\in\mathbb{Z}}$ be a sequence of real numbers in $[0,1)$ and let $(q_y)_{y\in\mathbb{Z}}$ be a sequence of positive real numbers such that $q_y < 1 - p_y$ for all $y \in \mathbb{Z}$. We suppose that, for $(x,y) \in \mathbb{H}$, the random walk can move horizontally with probability p_y, move up with probability q_y

and move down with probability $1 - p_y - q_y$ (figure 2.3). Bearing in mind what we have noticed, the following theorem does not require a proof.

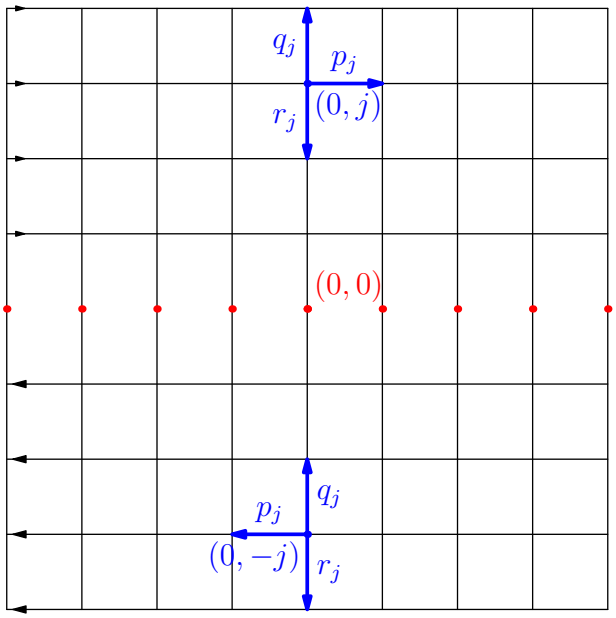


Figure 2.3: The half plane one-way lattice $\mathbb{H}$ with a non constant drift.

Theorem 2.2.3. *The Poisson boundary of the random walk with transition probabilities defined as above is isomorphic to the Poisson boundary of the random walk whose transition operator is defined for $x, y \in \mathbb{Z}$ by*

$$P(x,y) = \begin{cases} p_x & \textit{if } y{=}x, \\ q_x & \textit{if } y{=}x{+}1, \\ 1 - p_x - q_x = r_x & \textit{if } y{=}x{\text{-}}1, \\ 0 & \textit{otherwise.} \end{cases}$$

Consider the lattice $\mathbb{H}$ with the sequences of probabilities $(p_y)_{y\in\mathbb{Z}}$ and $(q_y)_{y\in\mathbb{H}}$ such that $p_0 = 0$, $p_y \equiv p$ for all $y \in \mathbb{Z}^*$, $q_0 = 1/2$, $q_y \equiv q > \frac{1-p}{2}$ for $y > 0$, and $q_y = 1 - p - q$ for $y < 0$. A simple computation shows that the Poisson boundary of the corresponding random walk on $\mathbb{Z}$ is trivial, hence, the Poisson boundary of the original Markov chain is also trivial. Consequently, the non triviality in the half-plane example is essentially due to the reducibility which artificially adds boundary points.

In our context, the orientation ϵ has been fixed once for all. However, it can be chosen randomly. If $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}}$ is a sequence of independent random variables it is shown in [CP03] that the corresponding simple random walk on $(\mathbf{G}, \epsilon)$ is transient for almost all ϵ. This result has been generalized in [GPLN08] for a random sequence ϵ for which ϵ_y is equal to 1 with probability f_y and -1 with probability $1 - f_y$ where $(f_y)_{y\in\mathbb{Z}}$ is a sequence

of stationary random variables satisfying $\mathbf{E}(f_0(1-f_0))^{-1/2} < \infty$. Finally, the case of a stationary sequence ϵ with decorrelation conditions is considered in [Pèn09] and, also, the corresponding simple random walk is shown to be transient. In those situations, the Poisson boundary remains obviously trivial (for all orientations) since, for all $y \in \mathbb{Z}$, $q_y \equiv p_y = \frac{1}{3}$ so that it defines a Markov operator on $\mathbb{Z}$ invariant by the natural $\mathbb{Z}$ action.

2.2.3 Triviality of the Martin boundary of $\mathbb{H}$

This section is devoted to the study of the Martin boundary. In the case of the simple random walk on $\mathbb{H}$ it is shown to be trivial.

Denote by $(M_n)_{n\geq 0}$ the simple random walk on the directed graph $\mathbb{G}$. Let $(\tau_n)_{n\geq 0}$ be a sequence of stopping times defined inductively by $\tau_0 = 0$ and

$$\tau_{n+1} = \inf\{t \geq \tau_n + 1 : M_t^{(2)} = 0\}$$

where $M_n = (M_n^{(1)}, M_n^{(2)})$, and we have for all $x \in \mathbb{H}$, $\mathbf{P}^x(\tau_n < \infty) = 1$.

The sequence of random variables $(M_{\tau_n})_{n\geq 1}$ is itself a Markov chain which can be seen as a random walk on $\mathbb{Z}$ with an infinite range Markov operator. It will be referred to as the *induced Markov chain* or the *embedded Markov chain*.

Theorem 2.2.4. *The Martin boundary of the embedded and original Markov chain $(M_n)_{n\geq 0}$ are trivial.*

The details of the proofs of these theorems are postponed to the appendix A. However, we give here their skeleton.

We denote by $\eta_{s,t}(y)$, for $s, t \geq 0$ and $y \in \mathbb{H}$, the local time of $(M_n)_{n\geq 0}$ in y, *i.e.*

$$\eta_{s,t}(y) = \sum_{n=s}^{t-1} \mathbf{1}_{M_n = y},$$

with the convention $\sum_\emptyset = 0$. Then, the Martin kernel can be rewritten in

$$K(x,y) = \begin{cases} \frac{G(x,y)}{G(0,y)} & \text{if } x, y \in \mathbb{Z} \times \{0\}; \\ \frac{\mathbf{E}^x(\eta_{0,\tau_1}(y))}{G(0,y)} + \sum_{z\in\mathbb{Z}\times\{0\}} \nu_x(z) K(z,y) & \text{otherwise} \end{cases} \tag{2.2}$$

where $\nu_x(z) = \mathbf{P}^x(M_{\tau_1} = z)$.

Proposition 2.2.5. *Let $z = (z_1, 0) \in \mathbb{Z} \times \{0\}$ and $y = (y_1, y_2) \in \mathbb{H}$. Then the Green function is given by the integral*

$$G(z,y) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{it(y_1 - z_1)} \frac{g(r(t))^{|y_2|}}{1 - \phi(t)} dt$$

where

$$r(t) = \tfrac{1}{3-2e^{it}}, \quad g(x) = \tfrac{1-\sqrt{1-x^2}}{x}, \quad \text{and} \quad \phi(t) = \mathsf{Re}\ r(t)^{-1} g(r(t)).$$

Moreover, if we suppose that $y_1 y_2^{-2}$ converges to $\lambda \in \mathbb{R} \cup \{\pm\infty\}$, then, for all λ, there exists a constant $s(\lambda)$ such that

(i) $G(z,y) \sim s(\pm\infty)|y_1 - z_1|^{-1/2}$ *when* $y_1 y_2^{-2}$ *goes to* $\pm\infty$ *;*

(ii) $G(z,y) \sim s(\lambda)|y_2|^{-1}$ *when* $y_1 y_2^{-2}$ *converges to a real* λ*.*

The first term of the second equation in 2.2 is shown to vanish whenever $y \in \mathbb{H}$ goes to infinity and we can prove that we can take the limit under the sum. Hence, the proposition 2.2.5 also implies the theorem 2.2.4.

Chapter 3

Cut-and-project scheme

At the begining considered as a mathematical game, the Penrose tilings have proven to be a very rich subject of study with many applications in different areas of mathematics and physics. Our point of view is different, since, for us, the third Penrose tiling and more generally those tilings obtained with the help of the cut-and-project sheme, provide with examples of constrained subgraph of the Cayley graph of $\mathbb{Z}^N$.

3.1 Quasi-periodic tilings and random walks

In this section, we introduce concisely the Penrose tilings of the Euclidean plane. The second part of the section consists of the reproduction of a submitted paper. In this paper, the Penrose and icosahedral tilings are constructed with the help of the cut-and-project scheme. Finally, it follows the statement related to the type of the simple random walk on the cut-and-project graph with its proof.

3.1.1 The Penrose approach

A first example of aperiodic tiling

Historically, the first quasi-periodic tiling constructed by Penrose in 1974 consisted of six prototiles

- 3 pentagons,
- 1 star,
- 1 boat, and
- 1 rhomb.

and a set of matching rules. These matching rules insure that the tiling is quasi-periodic and distinguish the 3 pentagons (see figure 3.1).

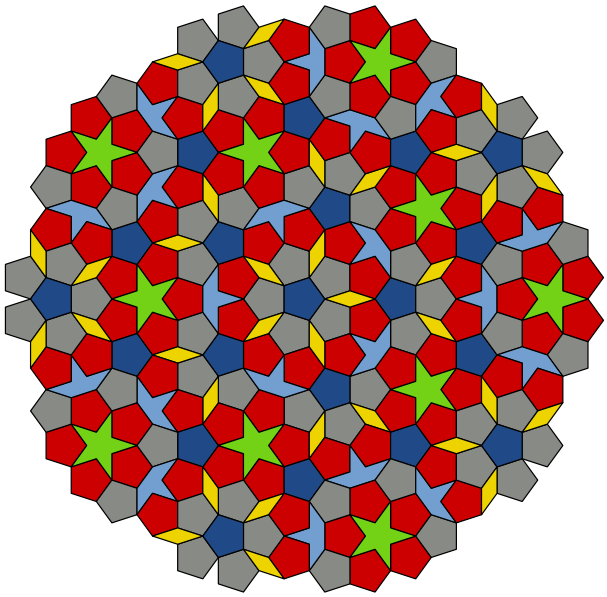


Figure 3.1: The first Penrose tiling with six prototiles : 3 pentagons, 1 star, 1 boat and 1 rhomb, figure from [Wik12].

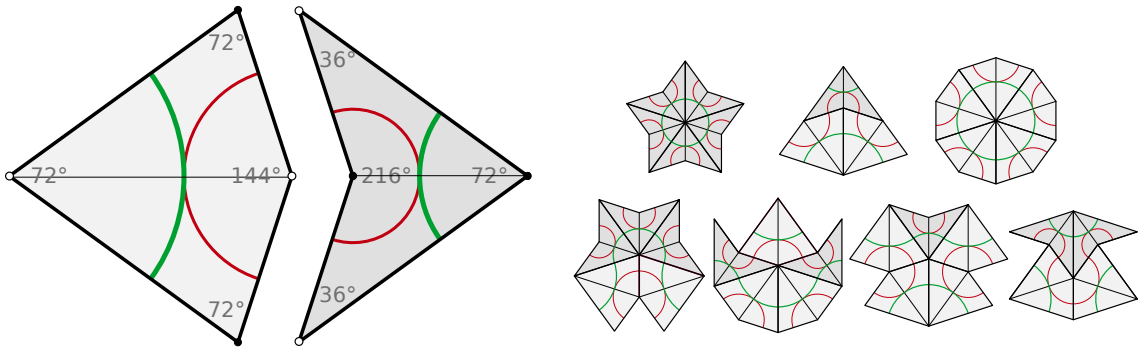
The kite and dart tiling

Penrose's second tiling uses two quadrilaterals called "kite" and "dart" which can be combined to form a rhomb. However, the matching rules prohibit such a combination (see figures 3.2a and 3.2b).

The tiling with rhombs

The most interesting Penrose tiling is the one consisting of two prototiles — a thin and a thick rhombs — with matching rules given in figure 3.3. The thin rhombs has four corners with two type of angles : the close one of measure $2\pi/5$ and the wide one of measure $3\pi/5$. For the thick rhombs, the closer angle measures $\pi/5$ whereas the wider one is equal to $4\pi/5$. This gives rise to a family of tilings whose one is given in figure 3.4. The Penrose's third tiling is fundamental because of its simplicity — only two prototiles — and can be constructed with the help of the cut-and-project scheme we define in the next paragraph.

The end of this section is devoted to the cut-and-project scheme, the construction of the third Penrose and icosahedral tilings with the help of this scheme and the statement related to the type of the simple random walk on the cut-and-project graph induced by a tiling. This part is the reproduction of a submitted paper.



(a) The kite and the dart with matching rules. (b) The seven patterns defined by matching rules.

Figure 3.2: The kite and dart tiling, figures from [Wik12]

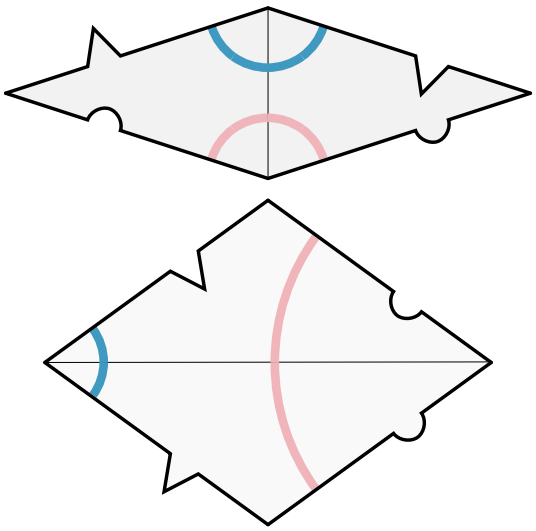


Figure 3.3: The two rhombs involved in the Penrose's third tiling with their matching rules, figure from [Wik12]

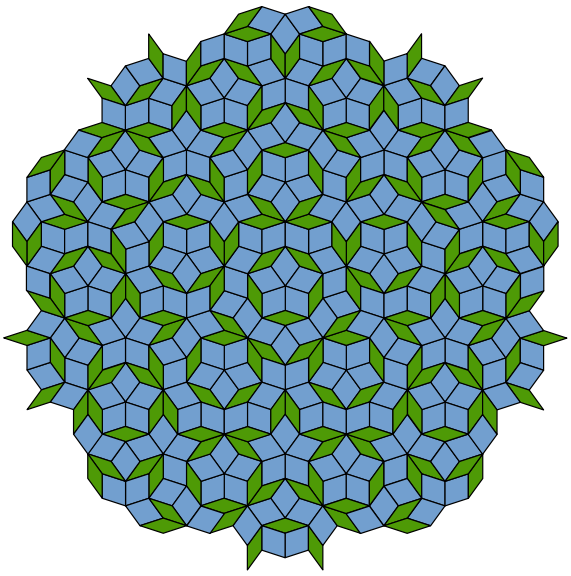


Figure 3.4: The third Penrose tiling consists of only 2 types of rhombs, figure from [Wik12].

3.1.2 Introduction and motivations

Starting with the theorem of Pólya stating that the simple random walk on the integer lattice $\mathbb{Z}^d$ is recurrent if and only if $d \leq 2$ — and transient otherwise — the case of random walks on finitely generated groups has been intensively studied in the literature. However, many questions are still open concerning random walks on weaker algebraic structures like groupoids or semi-groupoids. Being less rigid than groups, there is no reason *a priori* that those structures carry the same theory of random walks. For instance, in [CP03], the simple random walk on an example of sub-semigroupoid on vertices of $\mathbb{Z}^2$ is shown to be transient.

Roughly speaking, it is obvious that undirected graphs (resp. directed graphs) naturally carry a groupoid structure (resp. a semi-groupoid structure). As important results on the relationship between combinatorial (or algebraic) and stochastic properties, we might probably cite a statement due to [Dod84] on triangulation of surfaces and another theorem on circle packing which can be found in [Woe00]. The first one asserts that if, for the triangulation of a surface, the degree $\mathsf{deg}(x)$ of each vertex is such that $7 \leq \mathsf{deg}(x) \leq M < \infty$ for some integer M, then the simple random walk is transient. Actually, such a property on the coordination of the vertices implies that, as a metric space, the considered graph has negative curvature. Besides, for a circle packing of the plane, the dual graph — which is a triangulation — carries a recurrent simple random walk if the coordination of each vertex is lower than 6. The class of examples presented in this paper extend those results in at least two ways. First, in the 2-dimensional case, a tiling is no longer a triangulation but merely a quadrangulation. Secondly, it will be obvious in the next few lines that the

coordination of vertices can be made arbitrarily large by increasing the dimension of the internal space without observing a transition in the type of the random walk.

The class of graphs (or groupoids) considered in this paper are obtained by tiling the standard real vector space $\mathbb{R}^d$ with the help of the cut-and-project scheme. More precisely, let E be a d-dimensional vector subspace of $\mathbb{R}^N$, named the *real space*, and set $E_{\mathsf{int}} = E^\perp$ the orthogonal complement of E, called the *internal space*. Let K be the unit cube in $\mathbb{Z}^N$. An edge in the Cayley graph of $\mathbb{Z}^N$ is accepted and projected on E (orthogonally) if it can be translated by a vector of E in the unit cube $K+t$, $t \in E_{\mathsf{int}}$. Under suitable assumptions this method gives rise to a family of tilings $\mathcal{T}_t$ of the space E whose prototiles are the projections of the d-dimensional facets of the N-dimensional unit cube K. Moreover, depending on the configurations of the space E and E_{int} those tilings will be periodic, quasi-periodic or aperiodic — the group of translations is given by $E \cap \mathbb{Z}^N$. Such a tiling naturally defines a connected graph embedded in the space $\mathbb{R}^d$, called *the cut-and-project graph* — the vertex and edge sets are respectively the sets of vertices and sides defining the tiles. An important example which can be constructed using the cut-and-project scheme (see section 3.1.3 for the details or [ODK88] for the original statement) is the Penrose's third tiling of $\mathbb{R}^2$ with two type of rhombs — thin and thick — which has been initially defined by Penrose using matching rules forcing the tiling to be aperiodic. Another interesting example is the icosahedral tiling of $\mathbb{R}^3$ (see section 3.1.3 for its definition) because of its connection with quasi-crystals. The quasi-crystals have been discovered by Shechtman in 1982 observing that the diffraction patterns of an alloy of Al-Mn has a 10-fold symmetry which contradicts the classic theory of crystallography. The theoretical description of this discovery can be found in the seminal paper [SBGC84] of D. Shechtman, I. Blech and J. W. Cahn for which Shechtman has been awarded the Nobel prize in Chemistry in 2011. The icosahedral tiling appears to be the mathematical description of this alloy (see also [KD86]).

Avoiding technical assumptions, which are generic, the main theorem of this paper can be written as follows (see theorem 3.1.4 in section 3.1.4 for further details).

Theorem 3.1.1. *Generically, the simple random walk on the cut-and-project graph is recurrent if* $\mathsf{dim}\ E \leq 2$ *and transient if* $\mathsf{dim}\ E \geq 3$.

Given a tiling, we could also consider the *dual graph*; each tile is represented by a vertex and there is an edge between two tiles if they share a side. This approach is developed in [Tel10] for the Penrose tiling. In this example, the dual graph is shown to be quasi-isometric to $\mathbb{Z}^2$. However, the quasi-isometry is only established for the specific Penrose tiling and the result is not surprising since the vertex degree is constant ($\equiv 4$). Actually, the main result of Telcs is the establishment of an invariance principle for the so-called Penrose lattice. The latter is almost the same as the dual graph except that vertices are centers of rhombs of the Penrose tiling and the metric considered inherits the Euclidean metric of $\mathbb{R}^2$. Starting with the results of Telcs, we can provide with an elementary proof of recurrence on the dual graph of the Penrose tiling and the Penrose lattice. Instead, we concentrate on direct lattices.

The first section is devoted to the basic notation and definitions related to tilings and the description of the cut-and-project method. In the second section, we state the main result of this paper precising the assumptions hidden behind the term "generically". In the fourth section, we present the theorem of Schlottmann and comment on its application to prove our main result. The main result of this paper and the proof are given in section 3.1.4 and 3.1.6 respectively.

3.1.3 Tilings and cut-and-project scheme

We start with the description of the cut-and-project scheme to tile the real line. We consider the standard integer lattice $\mathbb{Z}^2$ of $\mathbb{R}^2$. Let E be an irrational line in $\mathbb{R}^2$, *i.e.* satisfying $E \cap \mathbb{Z}^N = \{0\}$ and E_{int} be the line orthogonal to E. We denote by K the unit square in $\mathbb{R}^2$. Thus, the translation of K along E defines a strip (see figure 3.5).

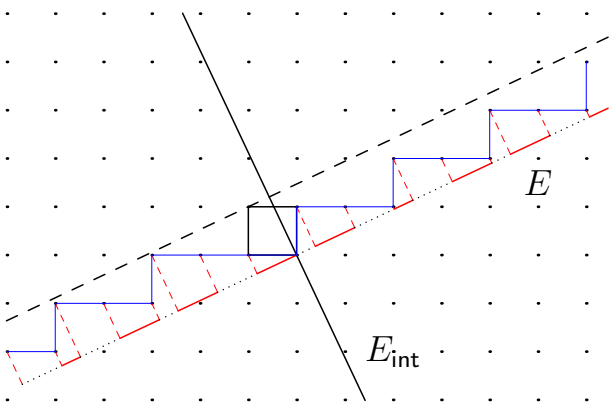


Figure 3.5: Quasi-periodic tiling of the real line within the cut-and-project scheme.

Consequently, we obtain a tiling of the space E with two types of segments (short and long). The short and long segments correspond to the projections of vertical and horizontal sides of the unit square which are entirely contained in the strip. Actually, it can be noticed that there is an ambiguity in the example of the figure 3.5 since two opposite sides of a the same unit square are completely contained in the strip so that we have to choose which one we project. However, the strip (or the square K) can be translated by a vector $t \in E_{\mathsf{int}}$ in such a way that there is no ambiguity. And since the projection of the lattice $\mathbb{Z}^2$ on the internal space is countable, there is no ambiguity for all but countably many $t \in E_{\mathsf{int}}$. Such a non-ambiguous $t \in E_{\mathsf{int}}$ will be called *generic* for obvious reasons.

Finally, in a non-ambiguous case, we observe that there is a unique broken line which is completely contained in the strip. This is the theorem of [ODK88], recalled at the end of this section, for the case of dimension 1.

Definition 3.1.2. A subset F of $\mathbb{R}^d$ is termed *regular* — for the usual topology of $\mathbb{R}^d$ — if it is bounded, has a non-empty interior $\mathsf{Int}(F)$, and is such that its closure $\mathsf{Clos}(F)$ coincides with $\mathsf{Clos}(\mathsf{Int}(F))$ and its interior $\mathsf{Int}(F)$ equals $\mathsf{Int}(\mathsf{Clos}(F))$.

Two regular subsets F_1 and F_2 are termed *congruent* if $F_1 = F_2$ up to a translation. The property of congruence induces an equivalence relation on the set of tiles; an equivalence class is termed a *prototile*.

Let $\mathcal{T}$ be a set of regular subsets of $\mathbb{R}^d$, we denote by $\mathcal{P}$ the corresponding set of prototiles, *i.e.* the factor set of $\mathcal{T}$ with respect the equivalence relation of congruence.

Definition 3.1.3. A denumerable set $\mathcal{T} = \{F_i\}_{i\in I}$ of regular subsets is a *tiling* of $\mathbb{R}^d$ if

- the corresponding set $\mathcal{P}$ of prototiles is finite,
- $\mathbb{R}^d = \bigcup_{i\in I} F_i$, and
- $\mathsf{interior}(F_i) \cap \mathsf{interior}(F_j)$ if $i \neq j \in I$.

If $\mathcal{T}$ is a tiling, then a regular set $F \in \mathcal{T}$ is called a *tile*.

Let E be a d-dimensional subspace of $\mathbb{R}^N$, and E_{int} be its orthogonal supplement in $\mathbb{R}^N$. The spaces E and E_{int} are termed respectively *real space* and *internal space*. We will denote by π and π_{int} the canonical projections from $\mathbb{R}^N = E \oplus E_{\mathsf{int}}$ to E and from $E \oplus E_{\mathsf{int}}$ to E_{int}. Thus we have the following

$$E \xleftarrow{\ \pi\ } E \oplus E_{\mathsf{int}} \xrightarrow{\ \pi_{\mathsf{int}}\ } E_{\mathsf{int}}\ .$$

We denote by K the unit cube in $\mathbb{Z}^N$, namely

$$K = \left\{ \sum_{i=1}^{N} \alpha_i \varepsilon_i : 0 \leq \alpha_i \leq 1 \right\},$$

where $(\varepsilon_1, \cdots, \varepsilon_N)$ is the canonical basis of $\mathbb{R}^N$.

Let p be an integer $0 \leq p \leq N$ and $M_p = \{I = (i_1, \cdots, i_p) \subset \{1, \cdots, N\}\}$ be the set of index sets with p elements. The p-facets of the unit cube are indexed by M_p as follows :

$$K_I = \left\{ \sum_{i\in I} \alpha_i \varepsilon_i : \alpha_i \in [0,1] \right\} \text{ for all } I \in M_p \text{ and } p > 0,$$

and $K_\emptyset = \{0\}$. Obviously, the unit cube K admits the decomposition $K = K_I + K_{I^\complement}$.

We assume that the configuration $\mathbb{R}^N = E \oplus E^\perp$ is *non degenerated*, *i.e.* for any $I = \{i_1, \cdots, i_d\} \in M_d$, the system $\{\pi(\varepsilon_i), i \in I\}$ is of rank d and the system $\{\pi_{\mathsf{int}}(\varepsilon_i), i \in I^\complement\}$ is of rank $N - d$. Actually, the two systems of vectors are simultaneously maximal or not maximal since $E_{\mathsf{int}} = E^\perp$. Moreover, maximality is a generic property and under this condition the d-facets of the unit cube are isomorphic to their projections on E by π, and also, the $(N-d)$-facets are isomorphic to their projections on π_{int}. We will denote by D_I the projection $\pi(K_I)$ of the d-facet related to $I \in M_d$. According to [ODK88] the set

$$\mathcal{T}_t = \{x + D_I : x = \pi(\xi), \pi_{\mathsf{int}}(\xi) \in \pi_{\mathsf{int}}(K_{I^\complement} + t), I \in M_d\}$$

is a tiling for almost every $t \in E_{\mathsf{int}}$ whose group of translations is given by $E \cap \mathbb{Z}^N$. Moreover, $\mathcal{T}_t$ is the projection of the unique d-dimensional faceted manifold entirely contained in the strip $K + E + t$ — for almost every $t \in E_{\mathsf{int}}$.

The Penrose's third tiling is obtained by the cut-and-project method if we consider the real space E in $\mathbb{R}^5$ spanned by the two following vectors (see [ODK88])

$$v_1 = (1, \cos(2\pi/5), -\cos(\pi/5), -\cos(\pi/5), \cos(2\pi/5)),$$

and,

$$v_2 = (0, \sin(2\pi/5), \sin(\pi/5), -\sin(\pi/5), -\sin(2\pi/5)).$$

For the icosahedral tiling of $\mathbb{R}^3$ — see [KD86] —, the vector subspaces E and E_{int} of $\mathbb{R}^6$ are defined with the help of projectors (which are both given here for the sake of simplicity)

$$\pi = \frac{1}{2\sqrt{5}} \begin{pmatrix} \sqrt{5} & 1 & -1 & -1 & 1 & 1 \\ 1 & \sqrt{5} & 1 & -1 & -1 & 1 \\ -1 & 1 & \sqrt{5} & 1 & -1 & 1 \\ -1 & -1 & 1 & \sqrt{5} & 1 & 1 \\ -1 & -1 & -1 & 1 & \sqrt{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & \sqrt{5} \end{pmatrix},$$

and

$$\pi_{\mathsf{int}} = \frac{1}{2\sqrt{5}} \begin{pmatrix} \sqrt{5} & -1 & 1 & 1 & -1 & -1 \\ -1 & \sqrt{5} & -1 & 1 & 1 & -1 \\ 1 & -1 & \sqrt{5} & -1 & 1 & -1 \\ 1 & 1 & -1 & \sqrt{5} & -1 & -1 \\ -1 & 1 & 1 & -1 & \sqrt{5} & -1 \\ -1 & -1 & -1 & -1 & -1 & \sqrt{5} \end{pmatrix}.$$

Consider a situation for which the set $\mathcal{T}_t$ is a tiling of some d-dimensional vector space E. We can define the *cut-and-project graph* as the projection on E along E_{int} of a certain constrained subgraph of the Cayley graph of the integer lattice $\mathbb{Z}^N$. Let us denote by $\mathscr{K}_t = K + E + t$ the strip parametrized by a vector $t \in E_{\mathsf{int}}$. Also, we denote by $\mathcal{S} = \{\pm\varepsilon_i : 1 \leq i \leq N\}$ the canonical set of generators of $\mathbb{Z}^N$. A constraint is a function $f : \mathbb{Z}^N \times \mathcal{S} \to \{0, 1\}$. In our context, we consider the constraint f defined for $(v, s) \in \mathbb{Z}^N \times \mathcal{S}$ by $f(v, s) = \mathbf{1}_{\mathscr{K}_t \times \mathscr{K}_t}(v, v + s)$.

Let $\xi_0 \in \mathscr{K}_t$ and denote by $\tilde{\mathbb{G}}_0^0 = \{\xi_0\}$. We define the set $\tilde{\mathbb{G}}_1^1$ by

$$\tilde{\mathbb{G}}_1^1 = \bigcup_{i=1}^{N} \bigcup_{q_i \in \{-1,1\}} \{(\xi, \xi + q_i\varepsilon_i) : \xi \in \tilde{\mathbb{G}}_0^0, f(\xi, q_i\varepsilon_i) = 1\},$$

and the set $\tilde{\mathbb{G}}_1^0$ by

$$\tilde{\mathbb{G}}_1^0 = \tilde{\mathbb{G}}_0^0 \cup \left[\bigcup_{i=1}^{N} \bigcup_{q_i \in \{-1,1\}} \{\xi + q_i\varepsilon_i : \xi \in \tilde{\mathbb{G}}_0^0, f(\xi, q_i\varepsilon_i) = 1\} \right].$$

One can easily define the range and source functions $\tilde{r}^{(1)}, \tilde{s}^{(1)} : \tilde{\mathbb{G}}_1^1 \to \tilde{\mathbb{G}}_1^0$: for an element $(\xi, \eta) \in \tilde{\mathbb{G}}_1^1$, the source function is defined by $s((\xi, \eta)) = \xi$ whereas the range function is given by $r((\xi, \eta)) = \eta$. Thus, the quadruple $(\tilde{\mathbb{G}}_1^0, \tilde{\mathbb{G}}_1^1, \tilde{r}^{(1)}, \tilde{s}^{(1)})$ is the partial constrained subgraph of order 1. For higher order partial constrained subgraphs, we proceed by induction. Let $n \geq 1$, we define $\tilde{\mathbb{G}}_{n+1}^1$ by

$$\tilde{\mathbb{G}}_{n+1}^1 = \bigcup_{i=1}^{N} \bigcup_{q_i \in \{-1,1\}} \{(\xi, \xi + q_i \varepsilon_i) : \xi \in \tilde{\mathbb{G}}_n^0, f(\xi, q_i \varepsilon_i) = 1\},$$

and $\tilde{\mathbb{G}}_{n+1}^0$ by

$$\tilde{\mathbb{G}}_{n+1}^0 = \tilde{\mathbb{G}}_n^0 \cup \left[\bigcup_{i=1}^{N} \bigcup_{q_i \in \{-1,1\}} \{\xi + q_i \varepsilon_i : \xi \in \tilde{\mathbb{G}}_n^0, f(\xi, q_i \varepsilon_i) = 1\} \right].$$

Finally, the functions $\tilde{r}^{(n+1)}, \tilde{s}^{(n+1)} : \tilde{\mathbb{G}}_{n+1}^1 \to \tilde{\mathbb{G}}_{n+1}^0$ are given for $(\xi, \eta) \in \tilde{\mathbb{G}}_{n+1}^1$ by $\tilde{r}^{(n+1)}((\xi, \eta)) = \eta$ and $\tilde{s}^{(n+1)}((\xi, \eta)) = \xi$.

As a consequence, this procedure defines a family of partial constrained subgraphs $\{(\tilde{\mathbb{G}}_n^0, \tilde{\mathbb{G}}_n^1, \tilde{r}^{(n)}, \tilde{s}^{(n)})\}_{n \geq 1}$. Thus, the complete constrained subgraph is naturally defined as follows

- $\tilde{\mathbb{G}}^0 = \lim_{n \to \infty} \tilde{\mathbb{G}}_n^0 = \bigcup_{n \geq 1} \tilde{\mathbb{G}}_n^0$,
- $\tilde{\mathbb{G}}^1 = \lim_{n \to \infty} \tilde{\mathbb{G}}_n^1 = \bigcup_{n \geq 1} \tilde{\mathbb{G}}_n^1$, and
- the functions $\tilde{r}, \tilde{s} : \tilde{\mathbb{G}}^1 \to \tilde{\mathbb{G}}^0$ are defined such that their restrictions to the set $\tilde{\mathbb{G}}_n^1$ are equal to the maps $\tilde{r}^{(n)}, \tilde{s}^{(n)}$ respectively.

Denoting by $\tilde{\mathbb{G}}^k$ the set of paths of length k, that is the set of sequences $(g_k, \cdots, g_1) \in (\tilde{\mathbb{G}}^1)^k$ satisfying $r(g_i) = s(g_{i+1})$ for $i = 1, \cdots, k-1$, we set $\tilde{\mathbb{G}}^* = \bigcup_{k \geq 0} \tilde{\mathbb{G}}^k$. The undirectedness of $\tilde{\mathbb{G}}$ implies that $(\eta, \xi) \in \tilde{\mathbb{G}}^1$ if and only if $(\xi, \eta) \in \tilde{\mathbb{G}}^1$ so that we can define the inverse of an edge $(\xi, \eta) \in \tilde{\mathbb{G}}^1$, denoted by $(\xi, \eta)^{-1}$, and defined by $(\xi, \eta)^{-1} = (\eta, \xi)$. Moreover, the path $((\xi, \eta), (\xi, \eta)^{-1})$ can be reduced to a path of length 0, and more precisely it is defined to be equal to $\xi \in \tilde{\mathbb{G}}^0$. More generally, a reducible path of arbitrary length has at least two consecutive edges α and β such that $\alpha = \beta^{-1}$ — or, of course, $\beta = \alpha^{-1}$. A path is said to be irreducible if we can not find such two consecutive edges. We will say that two paths are equivalent if after all possible reductions they are equal. This equivalence relation will be denoted by the symbol $\vdash$.

The factor set $\mathbf{G} = \tilde{\mathbb{G}}^* / \vdash$ is naturally endowed with a groupoid structure. Indeed, the range and source functions naturally extend to the set $\mathbf{G}$ since $r((g_k, \cdots, g_1)) = r(g_k)$ and $s((g_k, \cdots, g_1)) = s(g_1)$. Thus, we can define the set of composable pairs $\mathbf{G}^2$ by

$$\mathbf{G}^2 = \{(g, h) \in \mathbf{G} \times \mathbf{G} : r(h) = s(g)\}$$

and endow the set $\mathbf{G}$ with a product map $\mathbf{G}^2 \to \mathbf{G}$ which maps (g, h) on gh where gh is the composition[1] of g with h — with possible reductions. The inverse map $\mathbf{G} \to \mathbf{G}$ maps a path $(g_k, \cdots, g_1)$ to the reversed path $(g_1^{-1}, \cdots, g_k^{-1})$. It is easy to check that the space of units $\mathbf{G}^0$, and the domain and co-domain functions of the groupoid coincide with the set $\tilde{\mathbb{G}}^0$, the source and range functions $\tilde{r}, \tilde{s}$ of the graph.

The cut-and-project graph is given by the quadruple $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ where

- $\mathbb{G}^0 = \pi(\tilde{\mathbb{G}}^0)$,
- $\mathbb{G}^1 = \pi(\tilde{\mathbb{G}}^1)$,
- the functions $r, s : \mathbb{G}^1 \to \mathbb{G}^0$ are defined such that the relations $r(\pi(g)) = \pi(\tilde{r}(g))$ and $s(\pi(g)) = \pi(\tilde{s}(g))$ hold for all $g \in \tilde{\mathbb{G}}^1$. Such functions r and s are well defined because $\mathcal{T}_t$ is a tiling of the space E.

Obviously, by projecting the complete subgraph, we do not alter the groupoid structure — actually, being linear, the map π is a groupoid homomorphism.

3.1.4 Main result

The simple random walk on the graph $\mathbb{G}$ is the Markov chain $(M_n)_{n\geq 0}$ whose state space is the set $\mathbb{G}^0$ and the transition operator is defined for all $x, y \in \mathbb{G}^0$ by

$$P(x,y) = \begin{cases} P(x,y) = \frac{1}{\mathsf{deg}(x)} \text{ if } (x,y) \in \mathbb{G}^1, \\ 0 \text{ otherwise,} \end{cases}$$

where $\mathsf{deg}(x)$ denotes the degree of point $x \in \mathbb{G}^0$, *i.e* $\mathsf{deg}(x) = \mathsf{card}\{g \in \mathbb{G}^1 : s(g) = x\}$.

Theorem 3.1.4. *Assume that $\mathbb{R}^N = E \oplus E_{\mathsf{int}} = E \oplus E^{\perp}$, $\Lambda = \mathbb{Z}^N$, and that the non-degeneracy hypothesis is fulfilled. Consider the simple random walk $(M_n)_{n\geq 0}$ on the cut-and-project graph $\mathbb{G}$ induced by the tiling $\mathcal{T}_t$ for a generic $t \in E_{\mathsf{int}}$. If $E_{\mathsf{int}} \cap \mathbb{Z}^N = \{0\}$, then the following dichotomy holds,*

- *if* $\mathsf{dim}\ E \leq 2$ *then* $(M_n)_{n\geq 0}$ *is recurrent,*
- *if* $\mathsf{dim}\ E \geq 3$ *then* $(M_n)_{n\geq 0}$ *is transient.*

Even though the theorem is stated for simple random walk, it can be trivially extended to strongly reversible random walks, uniformly irreducible, and with bounded range (see [Anc90] for instance).

In the example of the icosahedral tiling, it is easy to check that $E_{\mathsf{int}} \cap \mathbb{Z}^6 = \{0\}$ and that the non-degeneracy hypothesis is fulfilled. Thus, the theorem applies and for all generic $t \in E_{\mathsf{int}}$ the simple random walk is transient. Unfortunately, it is obvious that the vector

[1] The elements g and h are seen here as morphisms and the product is the composition of g with h. Alternatively, we may see g and h as letters and the product would be the concatenation of the two letters. In this case, the set of composable pairs should be defined by $\mathbf{G}^2 = \{(g,h) \in \mathbf{G} : r(g) = s(h)\}$.

$(1,1,1,1,1)$ belongs to E_{int} in the configuration considered for the Penrose's third tiling so that we might ask for weaker hypothesis.

Note that the approach, involving Fourier transforms, usually used on $\mathbb{Z}^d$ to show that the simple random walk is recurrent or transient is no longer available in our context. Indeed, let $\mathcal{S} = \{\pm\varepsilon_i : i = 1, \cdots, N\}$ be the usual set of generators of $\mathbb{Z}^N$. If $\mathbf{s} = (s_1, \cdots, s_n) \in \mathcal{S}^n$ we denote by $\mathbf{x}(\mathbf{s}, x_0)$ the trajectory $(x_0, x_0 + s_1, \cdots, x_0 + s_1 + \cdots + s_n)$. Then, the characteristic function of the simple random walk $(M_n)_{n\geq 0}$ is given by

$$\mathbf{E}^{x_0}(e^{i\langle t, M_n\rangle}) = \sum_{\mathbf{s}\in\mathcal{S}^n} \mathsf{p}(\mathbf{x}(\mathbf{s}, x_0))\mathbf{1}_{(\mathscr{K}_t)^{n+1}}(\mathbf{x}(\mathbf{s}, x_0))e^{i\langle t, \pi(x_0)+\sum_{i=1}^n \pi(s_i)\rangle},$$

where $\mathsf{p}(\mathbf{x}(\mathbf{s}, x_0))$ is the weight of the trajectory $\mathbf{x}(\mathbf{s}, x_0)$ which is given by

$$\mathsf{p}(\mathbf{x}(\mathbf{s}, x_0)) = \prod_{i=1}^n \frac{1}{\mathsf{deg}(x_0 + \sum_{l=1}^{i-1} s_l)}$$

In the case of the projection of points of $\mathbb{Z}^2$ on an irrational line E, the tiling $\mathcal{T}_t$ is the projection of the only broken line entirely contained in the strip $\mathscr{K}_t$. Consequently, the degree of each point is 2 and the weight p of a trajectory of lenght n is simply given by 2^{-n}. Thus, by inverse Fourier transform, we can compute the return probability

$$P^{2n}(x_0, x_0) = \begin{cases} \frac{1}{2^{2n}} \sum_{\mathbf{s}\in\mathcal{S}^n} \mathbf{1}_{(\mathscr{K}_t)^{n+1}}(\mathbf{x}(\mathbf{s}, x_0)), \text{ if } \sum_{i=1}^{2n} \pi(s_i) = 0. \\ \\ 0, \text{ otherwise.} \end{cases}$$

Therefore, it is easy to conclude that $P^{2n}(x_0, x_0) = \frac{1}{2^{2n}}\binom{2n}{n} \sim Cn^{-1/2}$ for some $C > 0$.

In higher dimension, the degree of each point is no longer constant and the number of trajectories returning to x_0 is no longer so easy to compute. That is why in the sequel we use other technics to estimate the return probability. These estimates are essentially a consequence of a theorem of Schlottman (see [Sch98]) and isoperimetric inequalities ([Var85, Ger88]) for the statement related to transience and lower bounds on the operator P (see [LP95]) for the statement involving recurrence.

In the next section, we introduce the notion of model sets and the theorem of Schlottmann which describes the distribution of the points of such a model set.

3.1.5 Model sets and uniform distribution

The notion of model sets involved in the theorem of Schlottmann appears in a slightly different context. We still denote by E a d-dimensional vector subspace of the real standard vector space $\mathbb{R}^N$, and by E_{int} the orthogonal complement of E. We also denote by π and π_{int} the canonical projections on E and E_{int} respectively. However, we need not to restrict ourself to the integer lattice $\mathbb{Z}^N$.

Definition 3.1.5. A subset $\Lambda \subset E \oplus E_{\mathsf{int}}$ is called a *lattice* if Λ is a discrete Abelian subgroup of $E \oplus E_{\mathsf{int}}$ such that there exists a compact $K \subset E \oplus E_{\mathsf{int}}$ satisfying $\Lambda + K = E \oplus E_{\mathsf{int}}$ (Λ is said relatively dense).

In the context of model sets, it is usually assumed that $\pi_{\mathsf{int}}(\Lambda)$ is dense in E_{int}, and that π restricted to Λ is bijective on its image $\pi(\Lambda)$. Under such assumptions, we will say that the spaces E and E_{int} are in a *standard configuration*.

We denote by μ and μ_{int} the Lebesgue measure on E and E_{int} respectively. A *window* is a bounded subset W of E_{int} which is the closure of its interior with zero measure boundary (with respect to the Lebesgue measure μ_{int}). The set $L = \{\pi(x) : x \in \Lambda, \pi_{\mathsf{int}}(x) \in W\}$ is termed a *regular cut-and-project* set (or a regular *model* set). If the real space E and the internal space E_{int} are in a standard configuration, it can be shown that a regular cut-and-project set is a Delone set (see [Moo00] for instance).

Definition 3.1.6. A subset Λ of $\mathbb{R}^N$ is a *Delone* (Delaunay) set if the following holds

1. Λ is *relatively dense* : there exists a non empty open set $\mathcal{O}$ such that, for any $v \in E$, $v + \mathcal{O}$ contains a point of Λ ;

2. Λ is *uniformly discrete* : there exists a non empty open set $\mathcal{O}'$ such that, for any $v \in E$, $v + \mathcal{O}'$ contains at most one point of Λ.

Fixing a basis of E, we denote by $\|\cdot\|_p$ the standard p-norm on E, namely for $x \in E$,

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}.$$

In addition, the p-metric induced by the p-norm is denoted by d_p. We denote by $B_p(x, r)$ the ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$, and by $\partial B_p(x, r)$ the corresponding sphere. We simply write B and ∂B, without subscripts, if the choice of a specific metric is irrelevant in the involved result.

The version given here can be found in [Sch98], but a similar statement is shown in [Hof98].

Theorem 3.1.7. *Let $E \cong \mathbb{R}^d$, $E_{\mathsf{int}} \cong \mathbb{R}^{(N-d)}$. Let Λ be a lattice in $E \oplus E_{\mathsf{int}}$. Assume that the space E and E_{int} are in a standard configuration. Then, uniformly in $t \in E \oplus E_{\mathsf{int}}$,*

$$\lim_{r\to\infty} \frac{\mathsf{card}(\Lambda \cap (t + B(0, r) + W))}{\mu(B(0, r))} = \frac{\mu_{\mathsf{int}}(W)}{\mu \otimes \mu_{\mathsf{int}}(\tilde{\Lambda})}$$

where $\tilde{\Lambda}$ is a fundamental domain of Λ.

Combining arguments of [Hof98], we can deduce that when Λ is supposed to be the standard integer lattice $\mathbb{Z}^N$, the density of $\pi_{\mathsf{int}}(\mathbb{Z}^N)$ in E_{int} can be substituted by the simpler condition $\mathbb{Z}^N \cap E_{\mathsf{int}} = \{0\}$, which is itself equivalent to the injectivity of π restricted to the integer lattice $\mathbb{Z}^N$. Moreover, the window we consider in the sequel is given by $W = W_t = \pi_{\mathsf{int}}(K + t)$, for $t \in E_{\mathsf{int}}$.

3.1.6 Isoperimetric inequalities, reversible random walks

A Markov chain $(M_n)_{n\geq 0}$ on an undirected graph $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ is reversible if there exists a measure — the total conductance — $m : \mathbb{G}^0 \mapsto (0, \infty)$ such that

$$m(x)P(x, y) = m(y)P(y, x) \tag{3.1}$$

for all $x, y \in \mathbb{G}^0$. Since the graph is undirected, the degree $(\mathsf{deg}(x))_{x\in\mathbb{G}^0}$ is a measure satisfying (3.1) for the simple random walk. We view m indiscernably as a measure or as a combinatorial object.

Let us denote by $d_{\mathbb{G}}$ the usual *graph metric* on the cut-and-project graph $\mathbb{G}$ and by $B_{\mathbb{G}}(x, n)$ the ball of radius n centered at x, *i.e.*

$$B_{\mathbb{G}}(x, n) = \{y \in \mathbb{G}^0 : d_{\mathbb{G}}(x, y) \leq n\}.$$

For a finite subset $A \subset \mathbb{G}^0$ we will denote by ∂A the boundary of A defined as

$$\partial A = \{x \in A : \exists y \notin A \text{ with } d_{\mathbb{G}}(x, y) = 1\}.$$

The boundary of a ball $B_{\mathbb{G}}(x, n)$ will be denoted by $\partial B_{\mathbb{G}}(x, n)$. The *growth function* of $(\mathbb{G}, P)$ at the point x is given by $V_P(x, n) = m(B_{\mathbb{G}}(x, n))$. We set

$$V_P(n) = \inf_{x\in\mathbb{G}^0} V_P(x, n).$$

We omit the index P when the operator P defines the simple random walk.

To prove the statement on recurrence of theorem 3.1.4 we will use the following which can be found in [Woe00], but see [LP95] for the original result.

Theorem 3.1.8. *Suppose that $V_P(x, n) \leq m(x)\mathfrak{v}(n)$ where $\mathfrak{v} : \mathbb{N} \mapsto [2, \infty)$ is increasing and such that the function $n \mapsto \frac{n^2}{\log \mathfrak{v}(n)^2}$ is increasing and unbounded. Then*

$$\frac{P^{2n}(x, x)}{m(x)} \geq \frac{1}{3V(x, \mathfrak{m}(6n))}$$

where $\mathfrak{m}(n) = \min\{k : n \leq \frac{k^2}{\log \mathfrak{v}(k)^2}\}$.

Note that the graph metric of the Cayley graph of $\Lambda = \mathbb{Z}^n$ is nothing but the metric d_Λ induced by the 1-norm on $\mathbb{R}^N$ in the canonical basis which is denoted in the sequel $\|\cdot\|_\Lambda$. This allows us to compare the graph metrics $d_{\mathbb{G}}$ and d_Λ.

Lemma 3.1.9. *For all $x, y \in \mathbb{G}^0$,*

$$d_\Lambda(\xi, \eta) = d_{\mathbb{G}}(x, y),$$

where $(\xi, \eta) \in \Lambda^2$ is the unique pair in the strip $\mathscr{K}_t = K + E + t$, for $t \in E_{\mathsf{int}}$, such that $\pi(\xi) = x$ and $\pi(\eta) = y$.

This lemma states that a geodesic path in the graph can not be the projection of a non geodesic path of the lattice Λ.

Proof. This lemma is a direct consequence of the fact, due to [ODK88], that the tiling $\mathcal{T}_t$ is the projection of a unique d-dimensional faceted manifold entirely contained in the strip $K + E + t$. □

Consequently, the theorem of Schlottmann, [Sch98], with lemma 3.1.9 yields the following ball growth estimates.

Proposition 3.1.10. *Under the assumptions of theorem 3.1.4, the following estimate is satisfied for all $x \in \mathbb{G}^0$*

$$K^{-1} l^d \leq \mathsf{card} B_{\mathbb{G}}(x, l) \leq K l^d$$

for a constant $K > 1$ independent of $x \in \mathbb{G}^0$.

Proof. Let $x, y \in \mathbb{G}^0$ and let $(\xi, \eta)^2 \in (\Lambda \cap \mathscr{K})^2$ be the unique pair of points such that $\pi(\xi) = x$ and $\pi(\eta) = y$. On one hand, we obtain

$$d_2(x, y) \leq \|\pi\| d_\Lambda(\xi, \eta)$$

where $\|\pi\|$ is the matrix norm defined by

$$\|\pi\| = \sup_{y \in E \oplus E_{\mathsf{int}} : \|y\|_\Lambda \leq 1} \frac{\|\pi(y)\|_2}{\|y\|_\Lambda}.$$

On the other hand, there exist $u, v \in W \subset E_{\mathsf{int}}$ such that $\xi = x + u$ and $\eta = y + u$ (and obviously these u, v are uniquely determined). Thus, we get the following obvious inequality :

$$d_\Lambda(\xi, \eta) = \|\xi - \eta\|_\Lambda \leq \|x - y\|_\Lambda + \|u - v\|_\Lambda \leq c_0 \|x - y\|_2 + \mathsf{diam}(W).$$

Consequently, by lemma 3.1.9, we get

$$\|\pi\|^{-1} d_2(x, y) \leq d_{\mathbb{G}}(x, y) \leq c_0 d_2(x, y) + \mathsf{diam}(W).$$

Applying theorem 3.1.7 and remarking that

$$B_2(x, c_0^{-1}(n - \mathsf{diam}(W))) \subset B_{\mathbb{G}}(x, n) \subset B_2(x, n\|\pi\|),$$

we get the inequality of the proposition. □

Denote by a the conductance defined by $a(x, y) = m(x)P(x, y)$. For every $f \in c_0(\mathbb{G}^0)$ — the space of functions with compact support on $\mathbb{G}^0$ — we define

$$\|f\|_D^2 = \frac{1}{2} \sum_{x, y \in \mathbb{G}^0} a(x, y) |f(x) - f(y)|^2$$

the Dirichlet norm of f. Recall the following.

Theorem 3.1.11 (Theorem 1 of [Var85]). *Let $\alpha \geq 2$, if for every $f \in c_0(\mathbb{G}^0)$*

$$\|f\|_r \leq C\|f\|_D \text{ where } r = \frac{2\alpha}{\alpha - 2}, \tag{3.2}$$

$C > 0$ is independent of f, and $\|\cdot\|_r$ is the standard norm in $\ell^r(\mathbb{G}^0, m)$. Then we have,

$$\sup_{x,y\in\mathbb{G}^0} \frac{P^n(x,y)}{m(y)} = \mathscr{O}(n^{-\alpha/2}).$$

The isoperimetric inequality does not imply, in full generality, the inequality of theorem 3.1.11. However, for a function $f \in c_0(\mathbb{G}^0)$, the Sobolev norm is defined by

$$\|f\|_S = \sum_{x,y\in\mathbb{G}^0} a(x,y)|f(x) - f(y)|,$$

and according to proprosition of section 3 of [Var85], we have the following.

Proposition 3.1.12. *Let $\alpha > 2$ and assume that there exists $C > 0$ such that for all $f \in c_0(\mathbb{G}^0)$ the Sobolev inequality holds*

$$\|f\|_{\alpha/(\alpha-1)} \leq C\|f\|_S.$$

Then, there exists $C' > 0$ such that for all $f \in c_0(\mathbb{G}^0)$ the following holds

$$\|f\|_{2\alpha/(\alpha-2)} \leq C'\|f\|_D.$$

It is a matter of fact that d-dimensional isoperimetric inequality is equivalent to a Sobolev inequality with $\alpha = d$ (see [Woe00], proposition (4.3), p. 40). Because of technical difficulties, we will not be able to prove a d-dimensional isoperimetric inequality for the initial graph but only for its k-fuzz. Nonetheless, the k-fuzz construction leaving the type of the simple random walk invariant, the conclusion will be immediate.

If $\mathbb{G}$ is a graph, the k-fuzz of $\mathbb{G}$, denoted by $\mathsf{Fuzz}_k(\mathbb{G})$, has the same set of vertices as $\mathbb{G}$ and (x,y) is an edge in $\mathsf{Fuzz}_k(\mathbb{G})$ if and only if $1 \leq d_{\mathbb{G}}(x,y) \leq k$. We note ρ the graph metric on $\mathsf{Fuzz}_k(\mathbb{G})$. It is well known that the balls in the two graphs can be compared as well as the spheres, namely

$$B_\rho(x,n) = B_{\mathbb{G}}(x,kn) \text{ and } \partial B_\rho(x,n) = \bigcup_{l=kn-k+1}^{kn} \partial B_{\mathbb{G}}(x,l).$$

Proposition 3.1.13. *Let $d = \dim E$. The k-fuzz $\mathsf{Fuzz}_k(\mathbb{G})$ satisfies a d-dimensional isoperimetric inequality for k large enough,* i.e.

$$\mathsf{card} B_\rho(x,n) \leq K \mathsf{card} \partial B_\rho(x,n)^{d/(d-1)}$$

for some $K > 0$.

Proof. According to proposition 3.1.10 there exist $C_-, C_+ > 0$ such that for all $n \geq 1$

$$C_-(kn)^d \leq \mathsf{card} B_\rho(x,n) \leq C_+(kn)^d.$$

Hence, we need a lower bound of $\mathsf{card}\partial B_\rho(x,n)$, namely, we have to show that

$$\mathsf{card}\partial B_\rho(x,n) \geq \kappa(kn)^{d-1}.$$

By lemma 3.1.9, and from the proof of 3.1.10, we get

$$\|\pi\|^{-1} d_2(x,y) \leq d_{\mathbb{G}}(x,y) \leq c_0 d_2(x,y) + \mathsf{diam}(W).$$

Consequently, a point $y \in \partial B_\rho(x,n)$ satisfies

$$c_0^{-1}(kn - k + 1 - \mathsf{diam}(W)) \leq d_2(x,y) \leq \|\pi\| kn,$$

and in terms of balls we get

$$B_2(x, \|\pi\| kn) \setminus B_2(x, c_0^{-1}(kn - k + 1 - \mathsf{diam}(W))) \subset \partial B_\rho(x,n).$$

Since $c_0\|\pi\| \geq 1$, it is obvious that for any $k \geq 1$

$$B_2(x, c_0^{-1}kn) \setminus B_2(x, c_0^{-1}kn - c_0^{-1}(k - 1 + \mathsf{diam}(W))) \subset \partial B_\rho(x,n).$$

In the sequel, we need to adapt the proof of proposition 2.1 in [Sch98]. Setting $r = c_0^{-1}kn$ and $w = w(k) = c_0^{-1}(k - 1 + \mathsf{diam}(W))$, and defining

$$N(r,w,x,W) = \frac{\mathsf{card}(\Lambda \cap (B_2(x,r) \setminus B_2(x,r-w)) + W))}{\mu(B_2(x,r) \setminus B_2(x,r-w))},$$

we want to show that the inequality

$$\alpha\mu_{\mathsf{int}}(W) \leq N(r,w,x,W) \leq (1-\alpha)\mu_{\mathsf{int}}(W) \tag{3.3}$$

holds for some $\alpha > 0$. Obviously, we have,

$$\begin{aligned} N(r,w,x,W) = \beta_r &\frac{\mathsf{card}(\Lambda \cap (B_2(x,r) + W))}{\mu(B_2(x,r))} \\ &+ (1-\beta_r)\frac{\mathsf{card}(\Lambda \cap (B_2(x,r-w) + W))}{\mu(B_2(x,r-w))}, \end{aligned}$$

where $\beta_r = \frac{\mu(B_2(x,r))}{\mu(B_2(x,r)\setminus B_2(x,r-w))}$. Consequently, we can majorize

$$\begin{aligned} |N(r,w,x,W) - \mu_{\mathsf{int}}(W)| \leq \beta_r &\left| \frac{\mathsf{card}(\Lambda \cap (B_2(x,r) + W))}{\mu(B_2(x,r))} - \mu_{\mathsf{int}}(W) \right| \\ &+ (\beta_r - 1)\left| \frac{\mathsf{card}(\Lambda \cap (B_2(x,r-w) + W))}{\mu(B_2(x,r-w))} - \mu_{\mathsf{int}}(W) \right|. \end{aligned}$$

It follows from relations (3.20) and (3.23) in [Sch98] that

$$|N(r,w,x,W)-\mu_{\mathsf{int}}(W)| \leq \mu_{\mathsf{int}}(W)\left[\beta_r \frac{\mu\{B_2(x,r+\delta+\epsilon)\setminus B_2(x,r-\delta-\epsilon)\}}{\mu(B_2(x,r))} + (\beta_r-1)\frac{\mu\{B_2(x,r-w+\delta+\epsilon)\setminus B_2(x,r-w-\delta-\epsilon)\}}{\mu(B_2(x,r-w))}\right],$$

where $\delta > 0$ depends on the window W and $\epsilon > 0$ only depends on the lattice Λ. Obviously, for r large enough, there exists $\kappa_0 > 0$ such that $\beta_r \leq \kappa_0 \frac{r}{dw}$ and $\kappa_1 > 0$ such that

$$\beta_r \frac{\mu\{B_2(x,r+\delta+\epsilon)\setminus B_2(x,r-\delta-\epsilon)\}}{\mu(B_2(x,r))} \leq \kappa_1 \frac{\delta+\epsilon}{w(k)}.$$

Since w can be made arbitrarily large with $k \geq 1$, the quantity $\kappa_1 \frac{\delta+\epsilon}{w}$ can be made strictly smaller than 1, and we conclude that

$$|N(r,w,x,W) - \mu_{\mathsf{int}}(W)| \leq (1-\alpha)\mu_{\mathsf{int}}(W),$$

for some $\alpha = \alpha(w) = \alpha(k) > 0$. Consequently, the following holds for large enough k

$$\mathsf{card}\partial B_\rho(x,n) \geq \mathsf{card}(\Lambda \cap (B_2(x,r)\setminus B_2(x,r-w)) + W)) \geq \kappa(kn)^{d-1},$$

and the k-fuzz $\mathsf{Fuzz}_k(\mathbb{G})$ satisfies a d-dimensional isoperimetric inequality. □

Proof of theorem 3.1.4. Assume that $\mathsf{dim}\ E = 2$. Then the proposition 3.1.10 together with the theorem 3.1.8 applied for $\mathfrak{v}(n) = n^2$, and remarking that $1 \leq \mathsf{deg}(x) \leq 2N$ for all $x \in \mathbb{G}^0$, imply that

$$P^{2n}(x,x) \geq \frac{C_0}{\mathfrak{m}(6n)^2} \geq \frac{C_0}{n(\log n)(\log\log n)}.$$

Recurrence obviously follows.

Let $d = \mathsf{dim}\ E$ and suppose that $d \geq 3$, then the k-fuzz graph $\mathbb{G}^k$ satisfies a d-dimensional isoperimetric inequality by proposition 3.1.13. Then, according to theorem of [Var85], it satisfies a Dirichlet inequality with parameter $\alpha = d$, thus theorem 3.1.11 implies the transience of the simple random walk on the k-fuzz $\mathsf{Fuzz}_k(\mathbb{G})$, so, on $\mathbb{G}$ as well (see [Woe00]). □

3.1.7 Some open problems

An interesting problem would be to study the average degree of vertices which would explain why the simple random walk is recurrent for the 2-dimensional case.

Another problem concerns the estimate of the return probability. In the case of Pólya's random walks, up to a multiplicative constant, the return probability is equivalent to $n^{-d/2}$ where d is the dimension of the lattice $\mathbb{Z}^d$. As long as we are interested in the type of the random walk the rough estimates of this paper are sufficient. However, a first step in the computation of the Martin boundary — for $d \geq 3$ — should be to get an asymptotic equivalent of this return probability instead of the mere qualitative behavior "as $\mathscr{O}(n^{-d/2})$" we got here.

3.2 Circle packing and triangulations

In the previous section, we proved that simple random walks on a class of quasi-periodic graphs in the Euclidean plane are recurrent. By now, we want to discuss two theorems concerning simple random walks on planar graphs. The first one states that the simple random walk on the adjacency graph of a circle packing such that the degree $\mathsf{deg}(x)$ of each vertex satisfying $\mathsf{deg}(x) \leq 6$ is recurrent. The second one tells that the simple random walk on a triangulation of a surface such that $7 \leq \mathsf{deg}(x) \leq d < \infty$ is transient.

In the example of random walks on quasi-periodic graphs, the condition on the degree is not fulfilled (for some vertex $\mathsf{deg}(x) \leq 6$ and for other $\mathsf{deg}(x) \geq 7$) but is still uniformly bounded.

A *circle packing* in the plane is a collection $\{C_x\}$ of closed topological disks with pairwise disjoint interiors. With a circle packing, we can associate a graph, termed the *adjacency graph* of the packing; the vertex set consists of the center of the disks and there is an edge between x and y if $C_x \cap C_y \neq \emptyset$.

A surface is *triangulated* if it can be covered by a countable collection $\{T_x\}$ of triangles — *i.e.* 3-simplices — with pairwise disjoint interiors. A graph is naturally induced by a triangulation; its set of vertices is composed of the corners of the triangles and the edges are their sides. Such a graph is termed a *triangulation*. It is obvious that the adjacency graph of a packing of the plane is a triangulation.

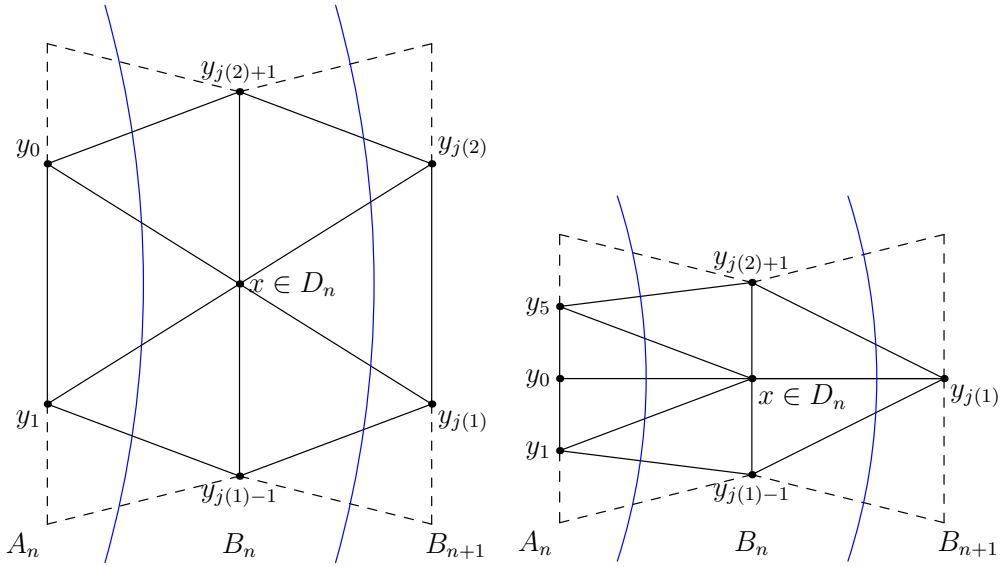
Proposition 3.2.1. *If the degree of any vertices is lower than 6, then the simple random walk on the adjacency graph of a circle packing of the plane is recurrent.*

Proof. Let $A_0 = \{x_0\}$ for some $x_0 \in \mathbb{G}^0$. We define inductively an increasing sequence of subgraphs: $A_{n+1} = A_n \cup B_n$, where $B_n = \{x \in A_n : d(x, A_n) = 1\}$. We decompose $B_n = C_n \cup D_n$ where C_n consists of those vertices having precisely one neighbour in A_n, D_n is the set of the remaining vertices.

Let $x \in B_n$ be a vertex having some neighbour in $\mathbb{G}^0 \setminus A_{n+1}$. By definition, this vertex x has also a neighbour in A_n. We denote by $y_0, \ldots, y_{m-1}, y_m = y_0$ the $m \leq 6$ neighbours of x in cyclic order with the convention that $y_0 \in A_n$. Let $j(1)$ and $j(2)$ be the minimal and maximal index such that $y_j \notin A_{n+1}$. Since $\mathbb{G}$ is the adjacency graph of a circle packing, hence a triangulation, the vertices $y_{j(1)-1}$ and $y_{j(2)+1}$ are in B_n (figure 3.6a). Those vertices are obviously different from y_0 so that $2 \leq j(1) \leq j(2) \leq m-2$. Also, $y_{j(1)}$ and $y_{j(2)}$ lie in D_{n+1}. Then, we need to consider two cases :

(i) $x \in D_n$. If $j(1) < j(2)$ then x has two neighbours in A_n, two neighbours in B_n and two in D_{n+1} (figure 3.6a). If $j(1) = j(2)$ then $y_{j(1)}$ is the only neighbour of x outside of A_{n+1}, and it lies in D_{n+1} (figure 3.6b).

(ii) $x \in C_n$. Similarly, we conclude that x has three neighbours in $\mathbb{G} \setminus A_{n+1}$, two of them are in D_{n+1} and one of them is in C_{n+1}, two neighbours in B_n each of then are in D_n, and one in A_n (figure 3.7).



(a) $x \in D_n$ and $j(1) < j(2)$. (b) $x \in D_n$ and $j(1) = j(2)$.

Figure 3.6: *Case 1 :* the vertex x has many neighbours in A_n.

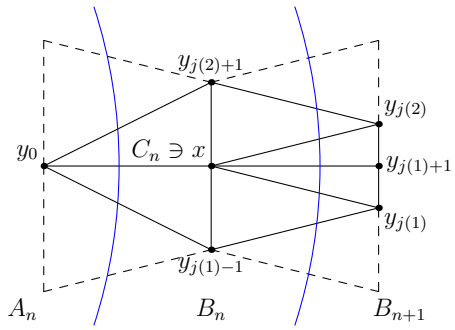


Figure 3.7: *Case 2 :* the vertex x has only one neighbour in A_n, *i.e.* $x \in C_n$.

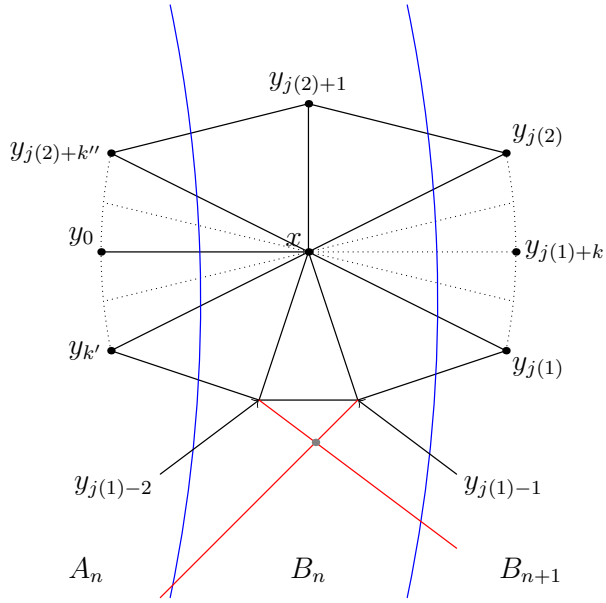


Figure 3.8: In a triangulation, a vertex x has exactly two neighbours in B_n.

Combining the two cases, we infer that the number of edges between A_{n+1} and B_{n+1} satisfies $\partial A_{n+1} \leq 2\mathsf{card}D_n + 3\mathsf{card}C_n$ and that

$$\mathsf{card}C_{n+1} \leq \mathsf{card}C_n, \text{ and } 2\mathsf{card}D_{n+1} \leq 2\mathsf{card}D_n + 3\mathsf{card}C_n.$$

Thus, we obtain $\partial A_{n+1} \leq 2\mathsf{card}D_0 + 3n\mathsf{card}C_0$. Also, it is obvious that $\mathsf{card}A_n = \mathscr{O}(n^2)$ so that we can apply the theorem 3.1.8 of [LP95] and the recurrence follows. □

Actually, in the proof given in [Woe00], he concludes by shortening each finite sets A_0 and B_n for $n \geq 0$ to a single point and applies the criterion of Nash-Williams.

According to [Dod84], if, for a triangulation, there exists an integer d such that for each vertex x, $7 \leq \mathsf{deg}(x) \leq d$, then the graph satisfies an isoperimetric inequality, hence the simple random walk is transient. We provide with a purely combinatorial proof of the result of Dodziuk, partially inspired by the proof above.

Proposition 3.2.2. *Let $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ be a triangulation. Assume that $7 \leq \mathsf{deg}(x) \leq d < \infty$ for all $x \in \mathbb{G}^0$, then the graph $\mathbb{G}$ satisfies a strong isoperimetric inequality. In addition, the simple random walk on $\mathbb{G}$ is transient.*

Proof. We define inductively an increasing sequence of subsets A_n of $\mathbb{G}^0$ by $A_0 = \{x_0\}$ for some $x_0 \in \mathbb{G}^0$ and $A_{n+1} = B_n \cup A_n$ where $B_n = \{y \in \mathbb{G}^0 : d(y, A_n) = 1\}$.

Fix $n \geq 1$ and let $x \in B_n$. Obviously, x has at least one neighbour in A_n. Denote by $y_0, \ldots, y_{l-1}, y_l = y_0$ the l neighbours of x ($l \leq d$). Without loss of generality, we can assume that $y_0 \in A_n$. Suppose for a while that x has at least one neighbour in B_{n+1}, then we claim that x has exactly two neighbours in B_n. Indeed, let us denote by $j(1)$ and $j(2)$

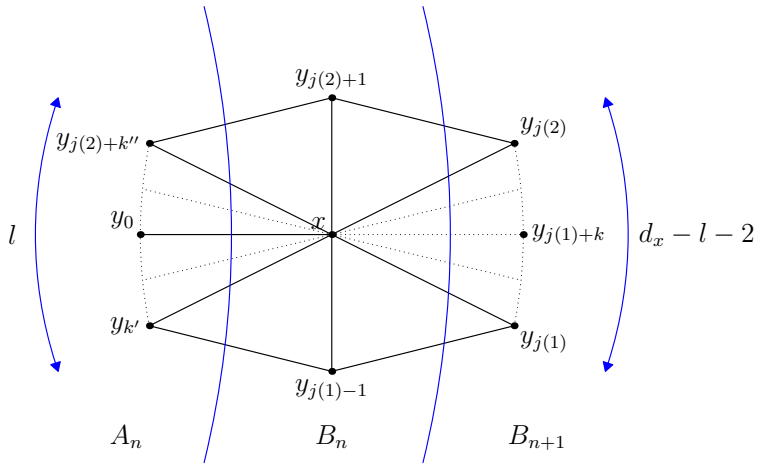


Figure 3.9: Distribution of the neighbours of x in A_n and B_{n+1}.

the minimal and maximal index such that $y_j \notin A_{n+1}$. By definition, $y_{j(1)-1}$ and $y_{j(2)+1}$ are in A_{n+1} and if they were in A_n it would imply that a point of A_n and a point of A_{n+1}^c have a common neighbour (namely, the vertex x) in B_n without being neighbours which is impossible because the graph $\mathbb{G}$ is a triangulation. We can see also that x has no other neighbour in B_n. Indeed, suppose for example that $y_{j(1)-1}$ and $y_{j(1)-2}$ are in B_n, then since $y_{j(1)-2}$ is a neighbour of x and x has a neighbour in B_{n+1}, the first has a neighbour in B_{n+1} — once again because $\mathbb{G}$ is a triangulation. Moreover, by construction $y_{j(1)-2}$ has a neighbour in A_n. This situation is impossible in a triangulation (see figure 3.8). Obviously, the same contradiction arises if we suppose $y_{j(2)+1}$ and $y_{j(2)+2}$ both in B_n. Consequently, the vertex x has k neighbours in A_n ($1 \leq k \leq l-3$) and $l-k-2$ neighbours in $\mathbb{G}^0 \setminus A_{n+1}$, this is summarized in figure 3.9.

We say that x is of type T_i, $i = 1, \ldots, l$, if it has i neighbours in B_{n-1}. Obviously, if x is of type T_i, then x has $l-i-2$ neighbours in B_{n+1}. Similarly, we say that a vertex $z \in B_{n+1}$ is of type T_i if it has i neighbours in B_n. We can compute that if y is of type T_i then $l-i-4$ of its neighbours in B_{n+1} are of type T_1, one is of type T_p, and the last is of type T_q. This gives rise to a finite graph whose the adjacency matrix $R_l = (r_{i,j})_{1 \leq i,j \leq l}$ by

$$\begin{cases} r_{1,j} &= l-j-4 \wedge 0 \text{ for } j = 1, \ldots, l \\ r_{i,p} &= 1 \\ r_{i,q} &= 1 \\ r_{i,j} &= 0 \text{ otherwise,} \end{cases}$$

where $a \wedge b$ is the minimum of a and b. Let R^+ and R^- be $(d \times d)$-dimensional matrices defined by

$$R^+ = \begin{pmatrix} d-5 & d-6 & \cdots & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1 \end{pmatrix},$$

and,

$$R^- = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Setting $S_n = (S_n^i)_{i=1,\dots,d}$ the column vector whose i^{th} coordinate is the number of points of type T_i in B_n, we obviously have

$$R^- S_n \leq S_{n+1} \leq R^+ S_n,$$

Indeed, for $i \neq 1$, we need to count only one of the T_i neighbours z of y since z has itself as a neighbour the point $y_{j(1)-1}$ or $y_{j(2)+1}$.

Analysing the spectra of R^+ and R^-, we conclude that the spheres are increasing exponentially fast and hence the balls are increasing exponentially fast as well. Consequently, the graph satisfies a strong isoperimetric inequality and the simple random walk is transient.

Returning to the hypothesis telling that x has at least one neighbour in B_{n+1}, it suffices to remark that if such points existed in a triangulation then an unbounded connected part of the plane would not be covered by the triangulation. □

The *local curvature* at point $x \in \mathbb{G}^0$ of an undirected graph is given by the integer $6-\mathsf{deg}(x)$. Let $(K_n)_{n\geq 0}$ be an increasing sequence of subsets of $\mathbb{G}^0$ such that $\bigcup_{n\geq 0} K_n = \mathbb{G}^0$. We define the *mean curvature* as the limit

$$\lim_{N\to\infty} 6 - \frac{1}{\mathsf{card} K_n} \sum_{x\in K_n} \mathsf{deg}(x).$$

Rewriting the proposition 3.2.1, we see that positive local curvature, *i.e.* $6-\mathsf{deg}(x) \geq 0$, for all vertices, hence the mean curvature is positive. On the contrary the proposition 3.2.2, the local curvature at each vertex is supposed to be negative, hence globally the mean curvature of the graph is negative as well.

Chapter 4

Groupoids and semi-groupoids

With a di-graph we can associate a semi-groupoid on the set of finite paths, as well as with an undirected graph we associate a groupoid. The purpose of this chapter is to explain this connection. This chapter aims at defining, furthermore, the notion of random walks on a groupoid and a semi-groupoid and make explicit the link with the "adapted" Markov chain evoked in the first chapter.

4.1 Groupoids and graphs

4.1.1 Groupoids

The notion of groupoids is an extension of the notion of groups in the sense that a groupoid is a set endowed with a binary operation which is not defined for all pairs of elements. Let us give the definition of [Ren80].

Definition 4.1.1 (Groupoid)**.** A *groupoid* is a set $\mathbf{G}$ endowed with a product map $\mathbf{G}^2 \ni (x, y) \to xy \in \mathbf{G}$ where $\mathbf{G}^2$ is a subset of $\mathbf{G} \times \mathbf{G}$ called the set of *composable pairs*, and an inverse map $\mathbf{G} \ni x \to x^{-1} \in \mathbf{G}$ such that the following relations are satisfied :

1. $(x^{-1})^{-1} = x$,
2. $(x, y), (y, z) \in \mathbf{G}^2 \Rightarrow (xy, z), (x, yz) \in \mathbf{G}^2$ and $(xy)z = x(yz)$,
3. $(x^{-1}, x) \in \mathbf{G}^2$ and if $(x, y) \in \mathbf{G}^2$ then $x^{-1}(xy) = y$,
4. $(x, x^{-1}) \in \mathbf{G}^2$ and if $(z, x) \in \mathbf{G}^2$ then $(zx)x^{-1} = z$.

For $x \in \mathbf{G}$, the source of x is $s(x) = x^{-1}x$ and its range is $r(x) = xx^{-1}$. We denote $\mathbf{G}^0$ the space of units defined by $\mathbf{G}^0 = r(\mathbf{G}) = s(\mathbf{G})$.

The following proposition links the notion of groupoid to the theory of categories. This could seem anecdotal, however, with this proposition, the connection with the undirected graphs become obvious and furthermore it highlights the essential ingredients we will need for semi-groupoids.

Proposition 4.1.2. *A groupoid is a small category in which homomorphisms are isomorphisms.*

A small category comes with an identification of the set of objects as a subset of morphisms by associating with any object A the identity morphism from A to A. This gives rise to a map $\epsilon : \mathbf{G}^0 \ni u \to \epsilon_u \in \mathbf{G}$ which associate with the unit u the identity ϵ_u of u — u seen as an object. This map satisfies the relation $s(\epsilon_u) = r(\epsilon_u) = u$ for all $u \in \mathbf{G}^0$.

4.1.2 Measure groupoids

We refer the reader to [Kai05] for the definition of a measurable groupoid and a Haar system. The point of view of [Ren80] is slightly different since the notion of topological groupoids is introduced and a continuity condition is required for the left Haar system.

Definition 4.1.3 (measurable groupoid)**.** A *measurable groupoid* consists of a groupoid $\mathbf{G}$ and a measurable σ-algebra on $\mathbf{G}$ such that

1. $\mathbf{G} \ni x \to x^{-1} \in \mathbf{G}$ is measurable, and
2. $\mathbf{G}^2 \ni (x, y) \to xy \in \mathbf{G}$ is measurable where $\mathbf{G}^2$ is a measurable set for the product σ-algebra on $\mathbf{G} \times \mathbf{G}$.

In the following definition, we denote by $\mathcal{M}^+(\mathbf{G})$ the space of non-negative measurable functions and by $\mathbf{G}^x$ and $\mathbf{G}_x$ the fibers $r^{-1}(x)$ and $s^{-1}(x)$ respectively. In this context, we can define the notion of left Haar system of a measurable groupoid.

Definition 4.1.4 (Haar system)**.** A *left Haar measure system* consists of a collection of measures $\{\lambda^u : u \in \mathbf{G}^0\}$ such that

1. the support $\mathsf{supp}\ \lambda^u$ of the measure λ^u is contained in the fiber $\mathbf{G}^u$ for all $u \in \mathbf{G}^0$,
2. for all $f \in \mathcal{M}^+(\mathbf{G})$, the map $u \mapsto \int_{\mathbf{G}} f(y)\lambda^u(dy)$ is measurable, and
3. for all $x \in \mathbf{G}$ and all $f \in \mathcal{M}^+(\mathbf{G})$, the following holds

$$\int_{\mathbf{G}} f(xy)\lambda^{s(x)}(dy) = \int_{\mathbf{G}} f(y)\lambda^{r(x)}(dy).$$

A Haar measure system need not exist in full generality. However, if $\mathbf{G}$ is a denumerable set we can check that the counting measures on the fibers $\mathbf{G}^x$ of r define a left Haar measure system. Indeed, define for $u \in \mathbf{G}^0$ the measure λ^u by

$$\lambda^u = \sum_{h \in \mathbf{G}^u} \delta_h,$$

and let $f \in \mathcal{M}^+(\mathbf{G})$. Noting that the map $\mathbf{G}^{s(g)} \ni h \to gh \in \mathbf{G}^{r(g)}$ is a bijection for all $g \in \mathbf{G}$, we obtain

$$\int_{\mathbf{G}} f(gy)\lambda^{s(g)}(dy) = \sum_{h \in \mathbf{G}^{s(g)}} f(gh) = \sum_{h \in \mathbf{G}^{r(g)}} f(h) = \int_{\mathbf{G}} f(y)\lambda^{r(g)}(dy).$$

Let ν be a measure on the space of units $\mathbf{G}^0$. We denote by $\lambda \star \nu$ the measure on $\mathbf{G}$ defined by

$$\lambda \star \nu = \int_{\mathbf{G}^0} \lambda^x \nu(dx).$$

The measure $\lambda \star \nu$ is said to be quasi-invariant if it is quasi-invariant by the map $\mathbf{G} \ni g \to g^{-1} \in \mathbf{G}$. A groupoid $\mathbf{G}$ endowed with a quasi-invariant measure $\lambda \star \nu$ is called a measure groupoid.

For a locally compact group G, setting $\mathbf{G} = G$ and $\mathbf{G}^2 = \mathbf{G} \times \mathbf{G}$, it follows immediately that G is a groupoid for which the maps r and s are trivial — $r(x) = s(x) = e$ for all $x \in \mathbf{G}$ where e is the neutral element of the group G — and consequently, the space units is reduced to $\{e\}$. Thus, a left Haar measure system in the case of locally compact groups consists of a collection $\{\lambda^e\}$ containing exactly one measure supported by the unique fiber carried by the neutral element which is precisely the group G. Moreover, the relation 3 in the definition corresponds to the fact that the measure λ^e is invariant by left translation, that is, it is a Haar measure, or more precisely, the Haar measure on G is the measure $\lambda \star \nu$ with $\nu = \delta_e$ the only probability measure on the singleton $\mathbf{G}^0 = \{e\}$.

Definition 4.1.5. A *system of transition probabilities* consists of a collection $\{\pi^g\}_{g \in \mathbf{G}}$ of probability measures on $\mathbf{G}$. We say that such a system is *invariant* if

1. $\mathsf{supp}\ \pi^g \subset \mathbf{G}^{s(g)}$ for all $g \in \mathbf{G}$, and
2. for all $(g', g) \in \mathbf{G}^2$, the following relation holds

$$\pi^{g'g} = g'\pi^g$$

where $g'\pi^g$ is the measure on $\mathbf{G}$ defined by

$$\int f(y)g'\pi^g(dy) = \int f(g'y)\pi^g(dy)$$

for all $f \in \mathcal{M}^+(\mathbf{G})$.

Proposition 4.1.6. *Every system $\{\mu^x : x \in \mathbf{G}^0\}$ of probability measures on the fibers $\mathbf{G}^x$, $x \in \mathbf{G}^0$, can be extended to a system of transition probabilities invariant on $\mathbf{G}$.*

Proof. Let $\{\mu^x : x \in \mathbf{G}^0\}$ be a system of probability measures on the fibers $\mathbf{G}^x$. We define the system of transition probabilities by

$$\pi^g = g\mu^{s(g)},$$

for all $g \in \mathbf{G}$. Thus, the measure π^g is defined for all $f \in \mathcal{M}^+(\mathbf{G})$ by the formula

$$\int f(y)\pi^g(dy) = \int f(y)g\mu^{s(g)}(dy) = \int f(gy)\mu^{s(g)}(dy).$$

The support of the measure π^g is, by definition, a subset of the fiber $\mathbf{G}^{s(g)}$ for all $g \in \mathbf{G}$. We need to check that the system $\{\pi^g : g \in \mathbf{G}\}$ is invariant. Let $f \in \mathcal{M}^+(\mathbf{G})$, and compute for $(g', g) \in \mathbf{G}^2$

$$\int f(y)\pi^{g'g}(dy) = \int f(y)g'g\mu^{s(g'g)}(dy) = \int f(g'y)dg\mu^{s(g)}(dy) = \int f(y)g'\pi^g(dy)$$

since, $s(g'g) = s(g)$. □

Coming back to the example of groups, a system of probability measures $\{\mu^u : u \in \mathbf{G}^0\}$ actually consists of exactly one measure μ whose support is contained in the unique fiber of the neutral element e, *i.e.* μ is a probability measure on G. Thus, we can define the transition probability $\pi^g = g\mu$ and we have for all $h \in G$

$$\pi^g(h) = \mu(g^{-1}h),$$

and the latter defines a Markov operator invariant under the left action of the group on itself, that is a right random walk on G.

Let $\mathbf{G}$ be groupoid with a left Haar measure system $\{\lambda^x : x \in \mathbf{G}^0\}$ and ν a measure on $\mathbf{G}^0$. With a system $\mu = \{\mu^x : x \in \mathbf{G}^0\}$ of probability measures concentrated on the fibers, we associate the system of transition probabilities $\{\pi^g : g \in \mathbf{G}\}$ invariant with respect to the groupoid. In this context, a Markov operator is the operator $\phi_\mu : \mathbf{L}^\infty(\mathbf{G}, \lambda \star \nu) \to \mathbf{L}^\infty(\mathbf{G}, \lambda \star \nu)$ defined by

$$\phi_\mu(f)(h) = \int f(g)\pi^h(dg) = \int f(g)h\mu^{s(h)}(dg) = \int f(hg)\mu^{s(h)}(dg)$$

for all $h \in \mathbf{G}$ and $f \in \mathbf{L}^\infty(\mathbf{G}, \lambda \star \nu)$.

4.1.3 The groupoid of an undirected graph

Let $(\mathbb{G}^0, \mathbb{G}^1, r, s)$ be an undirected graph and recall the notation $\mathbb{G}^k$ for the set of paths of length k, for $k \geq 0$. Denote by $\mathbf{G}$ the set of all finite paths up to reductions (see section 3.1.3 of chapter 3), namely $\mathbf{G} = \bigcup_{k\geq 0} \mathbb{G}^k / \vdash$. The range and source maps initially defined on $\mathbb{G}^1$ can be extended to the whole set $\mathbf{G}$. Indeed, let $g = (g_k, \cdots, g_1)$ be a path of length $k \geq 2$, then we can set $r(g) = r(g_k)$ and $s(g) = s(g_1)$. Thus, we can define the set of composable pairs $\mathbf{G}^2$ by

$$\mathbf{G}^2 = \{(g, h) \in \mathbf{G} : r(h) = s(g)\}$$

and the product map $\mathbf{G}^2 \ni (g, h) \mapsto gh$ where gh is the concatenation of the path h with g. More precisely, if $h = (h_k, \cdots, h_1)$ and $g = (g_l, \cdots, g_1)$ for some $l, k \geq 0$, then gh is

the path $gh = (g_l, \cdots, g_1, h_k, \cdots, h_1)$. We adopt here the convention of left concatenation because of the closed connection of groupoids with categories turning the finite paths of the graph in morphisms so that the left concatenation corresponds to the composition of morphisms. Furthermore, we can check that domain and co-domain functions of the groupoid $\mathbf{G}$ coincide with the range and source function, also the space of units $\mathbf{G}^0$ coincides with the set of vertices $\mathbb{G}^0$.

Consider a discrete group G and a probability measure μ on G. It is well-known that the right random walk with respect to μ is the Markov chain whose Markov operator $(P(g,h))_{g,h\in\mathbb{G}}$ is given by

$$P(g,h) = \mu(g^{-1}h).$$

This Markov operator is obviously invariant under the left action of G on itself, that is for all g, h, k the relation $P(gh, gk) = P(h, k)$ holds.

Analogously, we may define the left random walk with respect to μ, that is the Markov chain whose Markov operator is given by $p(g,h) = \mu(hg^{-1})$, for all $g, h \in G$, and this time such a Markov operator is invariant under the right action of G on itself.

Of course, considering the left or the right random walks is not essential since we can map the first one to the second one. Dealing with groupoids, we have a notion of left and right random walk, but we have also the notion of product as composition or product as concatenation. The two point of views are equivalent but the definition will be different — the system of probability measures $(\mu_x)_{x\in\mathbf{G}^0}$ may be supported by the fiber of r or the fiber of s depending on the case we are considering, for instance.

The definitions given in the last section are valid for a groupoid with a product as a composition of morphisms, and a right random walk — and, consequently, an invariance property for a "left action".

Definition 4.1.7. Let $\mathbf{G}$ be a groupoid, a $\mathbf{G}$-space is a set X endowed with a projection map $p : X \to \mathbf{G}^0$ and an action :

$$\{(g,x) \in \mathbf{G} \times X : s(g) = p(x)\} \ni (g,x) \mapsto gx \in X,$$

satisfying, when defined, the following relations:

1. $p(gx) = r(g)$,
2. $\epsilon_{p(x)}x = x$, and
3. $g(hx) = (gh)x$.

As a matter of fact, there is a natural action of a groupoid $\mathbf{G}$ on $\mathbf{G}$ with its range function $r : \mathbf{G} \to \mathbf{G}^0$ as projection map and action given by the left product. Nevertheless, there is also a natural action of the groupoid $\mathbf{G}$ on the space of units $\mathbf{G}^0$. The projection map is simply the identity map and the action is given for $(g,x) \in \mathbf{G} \times \mathbf{G}^0$ such that $s(g) = p(x) = x$ by $(g,x) \mapsto gx = r(g)$.

Recalling that the space of units $\mathbf{G}^0$ coincide with the set of vertices $\mathbb{G}^0$ in the context of the groupoids of undirected graphs, we conclude that a random walk on the groupoid $\mathbf{G}$ can be carried on a random walk on the initial graph $\mathbb{G}$.

4.2 Semi-groupoids and directed graphs

In the first section of this chapter, we have seen how, with an undirected graph, a groupoid can be associated. However, this does not cover the more general case of directed graph. The semi-groupoid appears as the modelling of such di-graph. Actually, the universe of directed graphs is a little smaller than that one of semi-groupoids.

4.2.1 Semi-groupoids

Let us give the definition of a semi-groupoid as in [Exe11]

Definition 4.2.1. A semigroupoid is a triple $(\mathbf{G}, \mathbf{G}^2, \cdot)$ where $\mathbf{G}$ is a set, $\mathbf{G}^2$ is a subset of $\mathbf{G} \times \mathbf{G}$, and

$$\cdot : \mathbf{G}^2 \to \mathbf{G}$$

is a binary operation which is associative in the following sense : if $f, g, h \in \mathbf{G}$ are such that either

(i) $(f, g) \in \mathbf{G}^2$ and $(g, h) \in \mathbf{G}^2$, or

(ii) $(f, g) \in \mathbf{G}^2$ and $(fg, h) \in \mathbf{G}^2$, or

(iii) $(g, h) \in \mathbf{G}^2$ and $(f, gh) \in \mathbf{G}^2$,

then all of $(f, g), (g, h), (fg, h)$ and (f, gh) lie in $\mathbf{G}^2$, and

$$(fg)h = f(gh).$$

The semi-groupoid structure is closely related to the theory of categories as in the case of groupoids even though the class of semi-groupoids can not be exhaustively described with categories (see [Exe11] for an example). However, since we are interested in the semi-groupoid structure of a directed graph, we are, actually, studying semi-groupoids induced by a small category.

Proposition 4.2.2. *To a small category corresponds a semi-groupoid with* $\mathbf{G}$ *the set of all morphisms,* $\mathbf{G}^2$ *the subset of* $\mathbf{G} \times \mathbf{G}$ *of composable morphisms, and the product map given by the composition of morphisms.*

If the semi-groupoid $\mathbf{G}$ is a category, then a morphism $f \in \mathbf{G}$ naturally comes with two objects A and B such that f is a morphism from A to B. Thus, the definitions of the range and source functions became obvious and will still be denoted by r and s respectively. In addition, we will denote by $\mathbf{G}^0$ the union of images of $\mathbf{G}$ by r and s, and the set $\mathbf{G}^0$ will be named the space of units of the semi-groupoid. In the sequel, we restrict ourselves to semi-groupoids which are small categories.

Note that, the notion of reduction in the set of paths of finite length were introduced in the case of undirected graphs to suit the definition of groupoids : all elements are invertible

and there is a notion of cancelation on the right and the left (see properties 3 and 4 of definition 4.1.1). In the case of directed graphs, we can also define a partially defined inverse map, however, this is useless regarding the definition of a semi-groupoid (roughly speacking, in the context of semi-groupoids, there is only a notion of associativity).

4.2.2 Measurable semi-groupoid

Definition 4.2.3 (measurable semi-groupoid)**.** A *measurable semi-groupoid* consists of a semi-groupoid $\mathbf{G}$ and a measurable σ-algebra on $\mathbf{G}$ such that

(i) $\mathbf{G}^2 \ni (x, y) \to xy \in \mathbf{G}$ is measurable where $\mathbf{G}^2$ is endowed with the σ-algebra induced by the product measurable σ-algebra on $\mathbf{G} \times \mathbf{G}$, and

(ii) the space of units $\mathbf{G}^0$ is endowed with a σ-algebra such the maps $r, s : \mathbf{G} \to \mathbf{G}^0$ are measurable.

The notion of left Haar measure system can be defined the same way as in the groupoid context. However, since the map $\mathbf{G}^{s(g)} \ni h \mapsto gh \in \mathbf{G}^{r(g)}$ is no longer a bijection, we can not insure the existence of such a measure system even in the case of a denumerable set $\mathbf{G}$.

Nevertheless, the definition 4.1.5 of transition probability systems and the proposition 4.1.6 are still valid in the context of semi-groupoids. Analgously to undirected graphs, the set of paths of finite length of a directed graph can be endowed with a semi-groupoid structure. Also, we can define the notion of $\mathbf{G}$-spaces and a semi-groupoid naturally acts on its space of units $\mathbf{G}^0$ in such a way that the random walk on $\mathbf{G}$ induces a random walk on the initial directed graph.

4.3 Reversible random walks

4.3.1 Definitions and notations

We consider in the sequel an undirected graph $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ which is supposed to be simple. Let $a : \mathbb{G}^1 \to \mathbb{R}_+$ be a σ-finite symmetric measure on the set of edges $\mathbb{G}^1$ that is for all $\alpha \in \mathbb{G}^1$ we have that $a(\alpha) = a(\alpha^{-1})$. This measure a gives rise to a σ-finite measure m on the set of nodes $\mathbb{G}^0$, called the *total conductance*. More precisely, we define m by the formula

$$m(x) = \sum_{\alpha : s(\alpha) = x} a(\alpha).$$

Since the graph $\mathbb{G}$ is assumed to be simple, we can identify the set of edges $\mathbb{G}^1$ to a symmetric subset of $\mathbb{G}^0 \times \mathbb{G}^0$. As a consequence, each edge α is uniquely determined by a pair of nodes (x, y). Thus, we can rewrite $a(\alpha) = a(x, y) = a(y, x) = a(\alpha^{-1})$ and $m(x) = \sum_{y \in \mathbb{G}^0} a(x, y)$.

We denote by $(M_n)_{n\geq 0}$ the $\mathbb{G}^0$-valued Markov chains whose transition probabilities $(P(x,y))_{x,y\in\mathbb{G}^0}$ are given for all $x,y\in\mathbb{G}^0$ by

$$P(x,y)=\frac{a(x,y)}{m(x)}. \tag{4.1}$$

Such a Markov chain is said to be *reversible*.

In the sequel, we will need to consider the notion of vertex and edge boundaries. Let A be a subset of $\mathbb{G}^0$, its vertex boundary is the set

$$\partial_0 A=\{y\in A^{\complement}: a(x,y)>0 \text{ for some } x\in A\},$$

and its edge boundary the set

$$\partial_1 A=\{\alpha\in\mathbb{G}^1:\ \text{either } r(\alpha)\in A, s(\alpha)\notin A \text{ or } r(\alpha)\notin A, s(\alpha)\in A\}.$$

For a function $f:\mathbb{G}^0\to\mathbb{R}$ we denote by $\nabla f:\mathbb{G}^1\to\mathbb{R}$ the gradient of f, defined for all $\alpha\in\mathbb{G}^1$, by

$$\nabla f(\alpha)=f(r(\alpha))-f(s(\alpha)).$$

We denote by $\Delta f:\mathbb{G}^0\to\mathbb{R}$ the operator defined for all $x\in\mathbb{G}^0$ by

$$\Delta f(x)=\frac{1}{m(x)}\sum_{\alpha:s(\alpha)=x} a(x,r(\alpha))\nabla f(\alpha).$$

We define the Laplacian $\mathscr{L}$ of f as the opposite of Δ, namely $\mathscr{L}=-\Delta$.

Now, it is easy to see that $\Delta f(x)=Pf(x)-f(x)=(P-\mathsf{id})f(x)$ where Pf is the real function defined for all $x\in\mathbb{G}^0$ by $Pf(x)=\sum_{y\in\mathbb{G}^0}P(x,y)f(y)$.

We denote by $\ell^2(\mathbb{G}^0,m,\mathbb{R})$ the space of m-squared summable real functions. This space turns into a Hilbert space if endowed with the inner product $(\cdot,\cdot)$, defined for all $f,g\in\ell^2(\mathbb{G}^0,m,\mathbb{R})$, by

$$(f,g)=\sum_{x\in\mathbb{G}^0} f(x)g(x)m(x).$$

The main goal of these considerations is to show a theorem appearing in [Var85] and to highlight the role played by the reversibility of the Markov chain.

First, we denote by $c_0(\mathbb{G}^0)$ the space of compactly supported real functions on $\mathbb{G}^0$. For a function $f\in\mathbb{G}^0$, we consider the quasi-norm of Dirichlet of f denoted by $\|f\|_D$ and defined by

$$\|f\|_D=\frac{1}{2}\sum_{\alpha\in\mathbb{G}^1} a(\alpha)|f(r(\alpha))-f(s(\alpha))|^2.$$

Also, we denote by $\|\cdot\|_r$ the standard norm on the space $\ell^r(\mathbb{G}^0,m,\mathbb{R})$ defined for all $f\in\ell^r(\mathbb{G}^0,m,\mathbb{R})$ by

$$\|f\|_r=\left(\sum_{x\in\mathbb{G}^0}|f(x)|^r m(x)\right)^{1/r}.$$

Finally, we introduce on the space $\ell^2(\mathbb{G}^1, a, \mathbb{R})$ the inner product $\langle \cdot, \cdot \rangle$ defined for all $\phi, \psi \in \ell^2(\mathbb{G}^1, a, \mathbb{R})$ by

$$\langle \phi, \psi \rangle = \frac{1}{2} \sum_{\alpha \in \mathbb{G}^1} \phi(\alpha)\psi(\alpha)a(\alpha).$$

We can remark that the Dirichlet quasi-norm is nothing but the norm of ∇f for the norm induced by the inner product $\langle \cdot, \cdot \rangle$.

Theorem 4.3.1. *Let P be a reversible Markov operator and let $n \geq 2$. We suppose that*

$$\|f\|_r \leq C\|f\|_D \text{ for all } f \in c_0(\mathbb{G}^0) \text{ where } r = \frac{2n}{n-2},$$

and $C > 0$ is independent of f. Then we have,

$$\sup_{x,y \in \mathbb{G}^0} \frac{P^l(x,y)}{m(y)} = \mathscr{O}(l^{-n/2}).$$

Actually the converse is also true — for $n > 2$ nevertheless — and we refer to [Var85] for further details. We can also recall the following theorem which can be found in [Woe00]. A pair $(\mathbb{G}, a)$, where $\mathbb{G}$ is an undirected graph and a a symmetric σ-finite measure on $\mathbb{G}^1$, is named a *weighted graph*. A Markov operator is canonically associated with a weighted graph by formula (4.1). Recall that a weighted graph satisfies a strong isoperimetric inequality if there exists a constant $\kappa > 0$ such that $m(A) \leq \kappa a(\partial_1 A)$ for all finite subsets $A \subset \mathbb{G}^0$. Finally, we denote by $\rho = \rho(P)$ the spectral radius of P defined by

$$\rho(P) = \limsup_{n \to \infty} P^n(x,y)^{1/n} \in (0,1].$$

The latter does not depend on $x, y \in \mathbb{G}^0$ in the irreducible case.

Theorem 4.3.2. *Let $(\mathbb{G}, a)$ be a weighted graph. Then the following statements are equivalent.*

1. *$(\mathbb{G}, a)$ satisfies a strong isoperimetric inequality.*

2. *(Dirichlet inequality) There is $\bar{\kappa} > 0$ such that*

$$\|f\|_2^2 \leq \bar{\kappa}\|f\|_D \text{ for every } f \in c_0(\mathbb{G}^0).$$

3. *The spectral radius $\rho(P)$ is strictly smaller than 1.*

4. *The Green kernel defines a bounded linear operator on $\ell^2(\mathbb{G}^0, m, \mathbb{R})$ by $Gf(x) = \sum_{y \in \mathbb{G}^0} G(x,y)f(y)$.*

4.3.2 Preliminary results

We begin this paragraph by proving the Green's formula, that is the following theorem.

Theorem 4.3.3 (Green's formula)**.** *Let $\mathbb{G}$ be a locally finite graph without isolated points. Consider a finite subset $F \subset \mathbb{G}^0$. Then, we have*

$$\sum_{x\in A} m(x)g(x)\Delta f(x) = -\frac{1}{2}\sum_{x,y\in A} a(x,y)\nabla f(x,y)\nabla g(x,y) + \sum_{x\in A, y\in A^{\complement}} a(x,y)\nabla f(x,y)g(x).$$

Proof. Let us compute the left hand side,

$$\begin{aligned}\sum_{x\in A} m(x)g(x)\Delta f(x) &= \sum_{x\in A} m(x)\left(\frac{1}{m(x)}\sum_{y\in\mathbb{G}^0}(f(y)-f(x))a(x,y)\right)g(x)\\ &= \sum_{x\in A}\sum_{y\in\mathbb{G}^0} a(x,y)(f(y)-f(x))g(x)\\ &= \sum_{x\in A}\sum_{y\in A} a(x,y)(f(y)-f(x))g(x)\\ &+ \sum_{x\in A}\sum_{y\notin A} a(x,y)\nabla f(x,y)g(x).\end{aligned}$$

By symmetry of a and exchanging the variables x and y, we get for the first term

$$\begin{aligned}\sum_{x,y\in A} a(x,y)(f(y)-f(x))g(x) &= \sum_{x,y\in A} a(y,x)(f(x)-f(y))g(y)\\ &= -\sum_{x,y\in A} a(x,y)(f(y)-f(x))g(y)\\ &= -\frac{1}{2}\sum_{x,y\in A} a(x,y)(f(y)-f(x))(g(y)-g(x))\end{aligned}$$

and the theorem follows. □

Definition 4.3.4 (Rayleigh quotient)**.** Let $f \in \ell^2(\mathbb{G}^0, m, \mathbb{R})$, we define the Rayleigh quotient by

$$\begin{aligned}R(f) = \frac{(\mathscr{L}f, f)}{(f,f)} &= \frac{-(\Delta f, f)}{(f,f)}\\ &= \frac{1}{2}\frac{\sum_{x,y\in\mathbb{G}^0} a(x,y)(\nabla f(x,y))^2}{\|f\|_2^2}\end{aligned}$$

Let F be a finite subset of $\mathbb{G}^0$ and denote by C_F the set of functions with support in F. Then, we can define the Laplacian restricted to C_F by

$$\mathscr{L}_F f(x) = f(x) - \sum_{x,y\in\mathbb{G}^0} P(x,y)f(y).$$

Notice that $\mathscr{L}_F$ *is not* the Laplacian of the subgraph of $\mathbb{G}^0$ having as vertex set F.

Lemma 4.3.5. *Let $f, g \in C_F$. Then, the following holds*

$$(\mathscr{L}_F f, g) = \frac{1}{2} \sum_{x,y\in\cup_1(F)} a(x,y)\nabla f(x,y)\nabla g(x,y),$$

where $\cup_r(F) = \bigcup_{x\in F} B(x,r)$.

Proof. By the Green's theorem, we compute

$$\begin{aligned}(\mathscr{L}_F f, g) &= \sum_{x\in\cup_1(F)} \mathscr{L} f(x)g(x)m(x) \\ &= \frac{1}{2} \sum_{x,y\in\cup_1(F)} a(x,y)\nabla f(x,y)\nabla g(x,y) \\ &- \sum_{x\in A, y\in A^{\complement}} \nabla f(x,y)g(y)a(x,y)\end{aligned}$$

and it suffices to remark that $g \in C_F$ so that the second term vanishes. □

Obviously, the operator — actually the matrix since C_F is finite dimensional — $\mathscr{L}_F$ is symmetric on C_F hence its spectrum is real and finite. The spectrum of $\mathscr{L}_F$ is denoted by

$$\mathsf{spec}\ \mathscr{L}_F = \{\lambda_1, \cdots, \lambda_{|F|}\}.$$

Moreover, it is easy to see that

$$\lambda_1(F) = \inf_{f\in C_F\setminus\{0\}} R(f) = \inf_{f\in C_F\setminus\{0\}} \frac{1}{2}\frac{\sum_{x,y\in\mathbb{G}^0} a(x,y)(\nabla f(x,y))^2}{\sum_{x\in\mathbb{G}^0} m(x)f(x)^2}.$$

Theorem 4.3.6. *Let F be a non empty finite subset of $\mathbb{G}^0$. Then, we have*

1. $0 < \lambda_1(F) \leq 1$,
2. $\lambda_1 + \lambda_{|F|}(F) \leq 2$, *also* $\mathsf{spec}\ \mathscr{L}_F \subset [\lambda_1(F), 2-\lambda_1(F)] \subset (0,2)$,
3. $\lambda_1(F)$ *decreases when F increases.*

Proof. We start with the first assertion. Since $\lambda_1(F) = R(f) \geq 0$, it suffices to show that $\lambda_1(F) \neq 0$. In order to prove this, we suppose that $\lambda_1(F) = 0$ and consider an eigenfunction f associated with λ_1. Then we must have $\nabla f(x,y) = 0$ for all $x, y \in \cup_1(F)$ with $a(x,y) > 0$. Thus for such vertices we have $f(x) = f(y)$.

Fix $x \in F$. Since F is finite, there exists a path $(x_0, \cdots, x_n)$ — *i.e.* a sequence of points satisfying $a(x_i, x_{i+1}) > 0$ for $0 \leq i \leq n-1$ — such that

$$x = x_0, \cdots, x_{n-1} \in F, x_n \notin F.$$

Consequently, we conclude that

$$f(x_0) = \cdots = f(x_{n-1}) = f(x_n) = 0$$

since $x_n \notin F$ and $f \in C_F$. Thus, $f(x) = 0$, but x was chosen arbitrarily in F so that $f \equiv 0$ on F.

As a conclusion, f can not be an eigenfunction, which implies that $\lambda_1(F) > 0$.

In order to show that $\lambda_1(F) \leq 1$, we observe that

$$\mathsf{trace}(\mathscr{L}_F) = \lambda_1(F) + \cdots + \lambda_{|F|}(F) = \sum (e_k, \mathscr{L}_F e_k),$$

where (e_k) is an arbitrary basis of $\ell^2(\mathbb{G}^0, m, \mathbb{R}) \cap C_F$. We can choose, for instance $e_x = \mathbf{1}_{\{x\}}$ and compute

$$\begin{aligned}
\mathsf{trace}(\mathscr{L}_F) &= \sum_{x \in F} (e_x, \mathscr{L}_F e_x) \\
&= \sum_{x \in F} \sum_{y,z \in \mathbb{G}^0} e_x(y)(\mathsf{id} - P)(y,z) e_x(z) \\
&= \sum_{x \in F} 1 - P(x,x) \leq |F|.
\end{aligned}$$

Furthermore, we have the following obvious minoration

$$\mathsf{trace}(\mathscr{L}_F) = \lambda_1(F) + \cdots + \lambda_{|F|}(F) \geq |F| \lambda_1(F),$$

and consequently, $\lambda_1(F) \leq 1$.

For the second assertion, we consider now that f is an eigenfunction corresponding to the eigenvalue $\lambda_{|F|}(F)$. We still have $\lambda_{|F|}(F) = R(f)$, and obviously

$$\lambda_1(F) \leq R(|f|) = \frac{1}{2} \frac{\sum_{x,y \in \mathbb{G}^0} a(x,y)(\nabla |f|(x,y))^2}{\sum_{x \in \mathbb{G}^0} m(x) f(x)^2}.$$

Now, we can use the following estimate

$$\begin{aligned}
(\nabla f(x,y))^2 + (\nabla |f|(x,y))^2 &= (f(y) - f(x))^2 + (|f(y)| - |f(x)|)^2 \\
&\leq 2(f(x)^2 + f(y)^2),
\end{aligned}$$

we deduce

$$\lambda_1(F) + \lambda_{|F|}(F) \leq \frac{\sum_{x,y\in\mathbb{G}^0} a(x,y)(f(x)^2 + f(y)^2)}{\sum_{x\in\mathbb{G}^0} m(x)f(x)^2} = 2.$$

The last assertion is obvious since F is increasing, then C_F is increasing so that the infimum $\inf_{f\in C_F\setminus\{0\}} R(f)$ is decreasing so is the eigenvalue $\lambda_1(F)$. □

4.3.3 Cheeger's inequality, isoperimetric inequalities and estimate of the Laplacian's first eigenvalue

The Laplacian's first eigenvalue can be estimated with the help of the Cheeger's constant. Recall that for a subset F of $\mathbb{G}^0$, we denote by $\partial_1 F$ the edge boundary, namely

$$\partial_1 F = \{\alpha \in \mathbb{G}^1 : s(\alpha) \in F, r(\alpha) \notin F\}.$$

Moreover, for a set of edges E, we write $a(E) = \sum_{\alpha\in E} a(\alpha)$.

Definition 4.3.7 (Cheeger's constant)**.** Let F be a finite subset of $\mathbb{G}^0$, the Cheeger's constant is defined by

$$h(F) = \inf_{\emptyset\neq S\subset F} \frac{a(\partial_1 S)}{m(S)},$$

i.e h is the largest constant such that $a(\partial_1 S) \geq h(F)m(S)$ for all non empty subsets S of F.

The Laplacian's first eigenvalue is connected with the Cheeger's constant with the help of the Cheeger's inequality.

Theorem 4.3.8 (Cheeger's inequality)**.** *All finite subsets F of $\mathbb{G}^0$ satisfy the following inequality*

$$\lambda_1(F) \geq \frac{1}{2}h(F)^2.$$

Before we prove this theorem, we need to show the following lemma.

Lemma 4.3.9. *Let $f \in C_F$ be non negative. Then,*

$$\sum_{\alpha\in\mathbb{G}^1} a(\alpha)|\nabla f(\alpha)| \geq h(F) \sum_{x\in\mathbb{G}^0} m(x)f(x).$$

Proof. Let $t \in \mathbb{R}_+$ and set for $f \in C_F$

$$S_t = \{x \in \mathbb{G}^0 : f(x) > t\} = \{x \in F : f(x) > t\}.$$

For $\alpha = (x, y)$, we set $I_\alpha = [f(x), f(y)) \subset \mathbb{R}$ assuming $f(x) \leq f(y)$, else we exchange x and y.

Thus, the set $\partial_1 S_t$ contains the edges $\alpha = (x, y)$ such that $f(x) \leq t < f(y)$, that is $t \in I_\alpha$. We can compute

$$a(\partial_1 S_t) = \sum_{\alpha \in \partial_1 S_t} a(\alpha) = \sum_{\alpha \in \mathbb{G}^1} a(\alpha) \mathbf{1}_{I_\alpha}(t),$$

and,

$$\int_{\mathbb{R}_+} a(\partial_1 S_t) dt = \sum_{\alpha \in \mathbb{G}^1} a(\alpha) \mathsf{Leb}(I_\alpha) = \sum_{\alpha \in \mathbb{G}^1} a(\alpha) |\nabla f(\alpha)|.$$

Hence the left hand side in the inequality of the lemma is nothing but $\int_{\mathbb{R}_+} a(\partial_1 S_t) dt$. Since $S_t \subset F$, and by definition of the Cheeger's constant, we obtain for all $t \geq 0$

$$a(\partial_1 S_t) \geq h(F) m(S_t),$$

so that it follows

$$\int_{t \geq 0} a(\partial_1 S_t) dt \geq h(F) \int_{t \geq 0} m(S_t) dt.$$

Now, let us compute the righ hand side integral,

$$\begin{aligned} \int_{t \geq 0} m(S_t) dt &= \int_{t \geq 0} \sum_{x \in S_t} m(x) dt \\ &= \int_{\mathbb{R}} \sum_{x \in \mathbb{G}^0} m(x) \mathbf{1}_{[0, f(x))}(t) dt \\ &= \sum_{x \in \mathbb{G}^0} m(x) f(x). \end{aligned}$$

and the lemma is proved. □

Proof. Proof of theorem 4.3.8

Let f be an eigenfunction of $\mathscr{L}_F$ corresponding to $\lambda_1(F)$. Then, $\lambda_1(F)$ can be written

$$\lambda_1(F) = \frac{\sum_{\alpha \in \mathbb{G}^1} |\nabla f(\alpha)|^2 a(\alpha)}{\sum_{x \in \mathbb{G}^0} f(x)^2 m(x)}.$$

It suffices to prove that $\sum_{\alpha \in \mathbb{G}^1} |\nabla f(\alpha)|^2 a(\alpha) \geq \frac{h(F)^2}{2} \sum_{x \in \mathbb{G}^0} f(x)^2 m(x)$.

Replacing f by f^2 in lemma 4.3.9, we get

$$\sum_{\alpha \in \mathbb{G}^1} a(\alpha) |\nabla f^2(\alpha)|^2 \geq h(F) \sum_{x \in \mathbb{G}^0} m(x) f(x)^2.$$

At this step, we have to estimate the left hand side using Cauchy-Schwarz inequality

$$\begin{aligned}
\sum_{\alpha\in\mathbb{G}^1} a(\alpha)|\nabla f^2(\alpha)| &= \frac{1}{2}\sum_{x,y\in\mathbb{G}^0} a(x,y)|f(y)^2 - f(x)^2| \\
&= \frac{1}{2}\sum_{x,y\in\mathbb{G}^0} a(x,y)|f(y)-f(x)||f(x)+f(y)| \\
&= \frac{1}{2}\left(\sum_{x,y\in\mathbb{G}^0} a(x,y)(f(y)-f(x))^2\right)^{1/2} \\
&\qquad \left(\sum_{x,y\in\mathbb{G}^0} a(x,y)(f(y)+f(x))^2\right)^{1/2} \\
&= \frac{1}{2}\left(2\sum_{\alpha\in\mathbb{G}^1} a(\alpha)|\nabla f(\alpha)|^2\right)^{1/2}\left(4\sum_{x\in\mathbb{G}^0} m(x)f(x)^2\right)^{1/2} \\
&= \left(2\sum_{\alpha\in\mathbb{G}^1} a(\alpha)|\nabla f(\alpha)|^2 \sum_{x\in\mathbb{G}^0} m(x)f(x)^2\right)^{1/2}.
\end{aligned}$$

Hence,

$$h(F)\|f\|_2^2 \le \|f\|_2\left(2\sum_{\alpha\in\mathbb{G}^1} a(\alpha)\|\nabla f(\alpha)\|^2\right)^{1/2},$$

and we get

$$\sum_{\alpha\in\mathbb{G}^1} a(\alpha)\|\nabla f(\alpha)|^2 \ge \frac{h(F)^2}{2}\|f\|_2^2$$

and the Cheeger's inequality follows. □

Definition 4.3.10 (Isoperimetric inequality)**.** A graph $(\mathbb{G}, a)$ satisfies IS_Φ if for all non empty finite subsets F of $\mathbb{G}^0$, we have

$$a(\partial_1 F) \ge \Phi(m(F)),$$

where Φ is a non negative function well defined for all $s \ge \inf_{x\in\mathbb{G}^0} m(x)$.

The Cheeger's inequality can be improved with the following theorem.

Theorem 4.3.11. *Assume $(\mathbb{G}, a)$ satisfies IS_Φ with a function Φ such that $\frac{\Phi(s)}{s}$ is decreasing with $s \ge 0$. Then, for all non empty finite subsets F of $\mathbb{G}^0$ we have*

$$\lambda_1(F) \ge \Lambda(m(F))$$

with $\Lambda(s) = \frac{1}{2}\left(\frac{\Phi(s)}{s}\right)^2$.

This theorem states that the isoperimetric inequality implies the Faber-Krahn inequality — see definition below — with function $\Lambda = \frac{1}{2}\left(\frac{\Phi(s)}{s}\right)^2$.

Definition 4.3.12 (Faber-Krahn inequality)**.** A graph $(\mathbb{G}, a)$ satisfies a Faber-Krahn inequality with Λ, (FK_Λ), if for all non empty finite subsets F of $\mathbb{G}^0$ we have $\lambda_1(F) \geq \Lambda(m(F))$.

Proof. Proof of theorem 4.3.11

The isoperimetric inequality IS_Φ implies that for all non empty subsets S of F we have

$$a(\partial_1 S) \geq \Phi(m(S)) = \frac{\Phi(m(S))}{m(S)} m(S) \geq \frac{\Phi(m(F))}{m(F)} m(S).$$

since $\frac{\Phi(s)}{s}$ is assumed to be decreasing. Consequently, we deduce

$$h(F) \geq \frac{\Phi(m(F))}{m(F)}.$$

Finally, the theorem 4.3.8 implies

$$\lambda_1(F) \geq \frac{1}{2} h(F)^2 \geq \frac{1}{2}\left(\frac{\Phi(m(F))}{m(F)}\right)^2 = \Lambda(m(F)).$$

□

4.3.4 Upper bound for the heat kernels

We assume now that the measure a satisfies $1 \leq a(\alpha) \leq M$ for some $M \geq 1$ and all $\alpha \in \mathbb{G}^1$. We assume also that the graph has bounded geometry, that is $\mathsf{deg}(x) \leq D$ for some D.

Lemma 4.3.13. *Let F be a finite subset of $\mathbb{G}^0$. Then, there exists a constant $c_0 > 0$ such that $m(\cup_1(F)) \leq c_0 m(F)$.*

We recall that $\cup_r(F) = \{y \in \mathbb{G}^0 : d(x, F) \leq r\}$.

Proof. We start with the equality $m(x) = \sum_{y \in \mathbb{G}^0} a(x, y)$, and remark that $1 \leq m(x) \leq D$ so that $|F| \leq m(F) \leq MD|F|$ where M is the constant appearing in the assumptions above.

Furthermore, we have that $|B(x, 1)| \leq D + 1$ and it follows that

$$|\cup_1(F)| \leq \sum_{x \in F} |B(x, 1)| \leq (D + 1)|F|.$$

Finally, we obtain $m(\cup_1(F)) \leq MD(D + 1)m(F) = c_0 m(F)$. □

The following theorem is equivalent to the theorem of Varopoulos introduced at the begining of this section. Recall that $p_n(\cdot,\cdot)$ denotes the heat kernel defined for $x, y \in \mathbb{G}^0$ by

$$p_n(x,y) = \frac{P^n(x,y)}{m(y)}.$$

Theorem 4.3.14. *Assume $(\mathbb{G}, a)$ satisfies the conditions recalled above and FK_Λ with $\Lambda(s) = cs^{-1/\beta}$ for some constant $c > 0$. Then,*

$$p_n(x,y) \leq cn^{-\beta}.$$

Proof. The proof will be split in several steps. If $f \in c_0(\mathbb{G}^0)$, then $\mathscr{L}f$ and Pf are also in $c_0(\mathbb{G}^0)$ since $\mathsf{supp}(Pf) \subset \cup_1(\mathsf{supp}(f))$. Moreover, the inner product (f,g) is well defined when $f, g \in c_0(\mathbb{G}^0)$.

Fix $z \in \mathbb{G}^0$ and set $f_n(x) = p_n(x,z)$. It is easy to see that $f_{n+1}(x) = Pf_n(x)$. Also, if we set $b_n = (f_n, f_n)$, we can compute

$$b_n = \sum_{x \in \mathbb{G}^0} p_n(x,z)p_n(x,z)m(x) = p_{2n}(z,z).$$

The strategy of the proof is to show that b_n is decreasing and estimate the differences $b_n - b_{n+1}$ so that we can obtain an upper bound of b_n and deduce an estimate of $p_n(x,y)$. Note that $b_n - b_{n+1} = (f_n, f_n) - (Pf_n, Pf_n)$.

Step 1: We start by proving that $(Pf, 1) = (f, 1)$ for $f \in c_0(\mathbb{G}^0)$.

We observe that $(f,1) = \sum_{x \in \mathbb{G}^0} f(x)m(x)$, and from Green's theorem 4.3.3 applied on $F = \cup_1(\mathsf{supp}(f))$, we compute

$$\begin{aligned}
(f,1) - (Pf,1) = (\mathscr{L}f, 1) &= \sum_{x \in F} \mathscr{L}f(x)1m(x) \\
&= \frac{1}{2}\sum_{x,y \in F} \nabla f(x,y)\nabla 1(x,y)a(x,y) \\
&\quad - \sum_{x \in F}\sum_{y \notin F} \nabla f(x,y)a(x,y).
\end{aligned}$$

The first sum vanishes because the gradient $\nabla 1(x,y) = 0$, whereas the second one vanishes too because $x \in \cup_1(\mathsf{supp}(f))$ and $y \notin \cup_1(\mathsf{supp}(f))$ for y such that $a(x,y) > 0$. This implies that if $x \notin \mathsf{supp}(f)$ then $\nabla f(x,y) = 0$.

Step 2: Consider the functional $Q(f,g) = (f,g) - (Pf, Pg)$ which is well defined for $f, g \in c_0(\mathbb{G}^0)$. We write simply $Q(f) = Q(f,f)$ for the diagonal.

Let F be a finite subset and f a function with $\cup_1(\mathsf{supp}(f)) \subset F$, then it can be shown that $Q(f) \geq \lambda_1(F)(f,f)$. Indeed, we clearly have that $\mathsf{supp}(f) \subset F$ so that $Pf = P_F f$ where $P_F = \mathsf{id} - \mathscr{L}_F$. We set $\mu_1 = 1 - \lambda_1(F)$. Thus, we obtain $\mathsf{spec}\ P_F \subset [-\mu_1, \mu_1]$ and $\|P_F\| \leq \mu_1$. In addition, we compute

$$\begin{aligned}
Q(f) &= (f,f) - (Pf, Pf) \\
&\geq \|f\|_2^2 - \mu_1^2\|f\|_2^2 = (1-\mu_1)(1+\mu_1)\|f\|_2^2 \\
&\geq \lambda_1\|f\|_2^2
\end{aligned}$$

and the assertion is proved.

Step 3: Let $g, h \in c_0(\mathbb{G}^0)$ and c a real constant such that

- $g \geq 0$ on $\mathbb{G}^0$,
- $h = c$ on $\mathsf{supp}\ g$,
- $h \leq c$ on $(\mathsf{supp}\ g)^{\complement}$.

Then, we can show that $Q(h, g) \geq 0$. In fact, we compute

$$Q(h, g) = (h, g) - (P^2 h, g) = (h - P^2 h, g).$$

Moreover, P^2 is a Markov kernel reversible with respect to m and P^2 induces weights $a_*(x, y) = m(x)P^2(x, y)$ and a measure $m_*(x) = \sum_{y \in \mathbb{G}^0} a_*(x, y) = m(x)$. We also denote $\mathscr{L}_* = \mathsf{id} - P^2$ the corresponding Laplacian on $(\mathbb{G}, a_*)$. Then, set $F = \mathsf{supp}\ g$ and compute with the help of the Green's theorem

$$\begin{aligned}(\mathscr{L}_* h, g) &= \sum_{x \in \mathbb{G}^0} \mathscr{L}_* h(x) g(x) m(x) \\ &= \frac{1}{2} \sum_{x, f \in F} \nabla h(x, y) \nabla g(x, y) a_*(x, y) \\ &\quad - \sum_{x \in F} \sum_{x \notin F} \nabla h(x, y) g(x) a_*(x, y).\end{aligned}$$

Then, the first sum is zero because $h = c$ on F so that $\nabla h = 0$ whereas the second one is non positive since $\nabla h(x, y) = h(y) - h(x)$, $y \notin F$ — thus, $h(y) \leq c$ — and $x \in F$ — thus, $h(x) = c$. We conclude

$$(\mathscr{L}_* h, g) = (h - P^2 h, g) = Q(h, g) \geq 0.$$

Step 4: Let $f \in c_0(\mathbb{G}^0)$ and $c \geq 0$. Then we obtain that $Q((f - c)_+) \leq Q(f)$ where $(f - c)_+$ denotes the non negative part of $(f - c)$. Actually, it suffices to set $g = (f - c)_+$, $h = f - g$, and the bilinearity of Q implies that $Q(f) = Q(h + g) = Q(g) + Q(h) + 2Q(h, g)$. By definition, g is a non negative function, thus it remains to check that the function h satisfies the conditions of step 3 :

- if $x \in \mathsf{supp}\ g$, $g(x) = f(x) - c > 0$ and $h(x) = c$ by definition of h;
- if $x \notin \mathsf{supp}\ g$, $g(x) = 0$ and $f(x) \leq c$, thus, $h(x) = f(x) - g(x) \leq c$.

By step 3, we get $Q(h, g) \geq 0$ and by step 2, $Q(h) \geq 0$ and it follows that $Q(f) \geq Q(g)$.

Step 5: Let $f \in c_0(\mathbb{G}^0)$ non negative. For all $s \geq 0$, define $F_s = \cup_1(\mathsf{supp}\ (f - s)_+)$. Then, $Q(f) \geq \lambda_1(F_s)((f, f) - 2s(f, 1))$. Indeed, setting $g = (f - s)_+$, we deduce by step 4 and step 2 that

$$Q(f) \geq Q(g) \geq \lambda_1(F_s)(g, g).$$

In addition, $g^2 \geq f^2 - 2sf$ because if $f \geq s$, $g = f - s$ and if follows $g^2 = f^2 - 2sf + s^2 \geq f^2 - 2sf$. In the other case, $f < s$, $g = 0$ and it follows $f^2 - 2sf = (f - 2s)f \leq 0 = g^2$. Finally, integrating the inequality with respect to the measure m, the following holds

$$(g, g) \geq (f, f) - 2s(f, 1).$$

Note that for $s = 0$, the statement of step 5 is equivalent to the statement of step 2. Furthermore, for $s = \frac{1}{4}\frac{(f,f)}{(f,1)}$, we obtain the following estimate

$$Q(f) \geq \frac{1}{2}\lambda_1(F_s)\|f\|_2^2.$$

Step 6: Let $(f_n)_{n\geq 0}$ be a sequence of non negative and compactly supported real functions such that $(f_0, 1) = 1$ and $f_{n+1} = Pf_n$. Setting $b_n = (f_n, f_n)$, we will show by induction that $b_n - b_{n+1} \geq c' b_n^{1+1/\beta}$ for some $\beta > 0$ and $c' = \frac{1}{8}c(4c_0)^{-1/\beta}$.

Since $f_0 \in c_0(\mathbb{G}^0)$ and $(f_0, 1) = 1$, one has $f_n \in c_0(\mathbb{G}^0)$ and $(f_n, 1) = 1$. In fact, we compute

$$(f_n, 1) = \sum_{x\in\mathbb{G}^0} m(x)P^n f_0(x) = \sum_{x\in\mathbb{G}^0} m(x)f_0(x) = (f_0, 1).$$

The estimate of step 5 for $s = \frac{1}{4}\frac{(f_n,f_n)}{(f_n,1)} = \frac{b_n}{4}$ gives us

$$Q(f_n) = b_n - b_{n+1} \geq \frac{1}{2}\lambda_1(F_s)b_n,$$

where $F_s = \cup_1(\mathsf{supp}(f_n - s)_+)$. Additionally, we can estimate by the Markov inequality the measure of F_s as follows

$$m(\mathsf{supp}(f_n - s)_+) \leq \frac{1}{s}\sum_{x\in\mathbb{G}^0} m(x)f_n(x) = \frac{1}{s}(f_n, 1) = \frac{1}{s}.$$

By lemma 4.3.13, we get $m(F_s) \leq \frac{c_0}{s} = \frac{4c_0}{b_n}$. From the FK_Λ inequality with $\Lambda(s) = cs^{-1/\beta}$ assumed in the theorem we are proving, we can write

$$\lambda_1(F_s) \geq cm(F_s)^{-1/\beta} \geq c(4c_0)^{-1/\beta}b_n^{1/\beta}.$$

As a conclusion, we get that

$$Q(f_n) = b_n - b_{n+1} = c' b_n^{1+1/\beta}.$$

where c' is the real constant in the theorem.

Step 7: If $(b_n)_{n\geq 0}$ is a sequence of positive real numbers satisfying $b_n - b_{n+1} \geq c' b_n^{1+1/\beta}$, then $b_n \leq \kappa n^{-\beta}$ with $\kappa = \left(\frac{\beta}{c'}\right)^\beta$.

For $\delta \geq 0$, $x > y > 0$, the mean value theorem implies that

$$y^{-\delta} - x^{-\delta} \geq \frac{\delta(x - y)}{x^{\delta+1}}.$$

Thus, applying to $\delta = 1/\beta$, we obtain

$$\begin{aligned} b_{n+1}^{-1/\beta} - b_n^{-1/\beta} &\geq \frac{b_n - b_{n+1}}{\beta b_{n+1}^{1+1/\beta}} \\ &\geq \frac{c'}{\beta} \frac{b_n^{1+1/\beta}}{b_n^{1+1/\beta}} = \frac{c'}{\beta}. \end{aligned}$$

Summing the left and side, we obtain

$$\sum_{k=0}^{n-1} (b_{k+1}^{-1/\beta} - b_k^{-1/\beta}) = b_n^{-1/\beta} - b_0^{-1/\beta} \geq \frac{c'}{\beta} n.$$

Hence, we easily check that $b_n \leq \kappa n^{-\beta}$.

Now we can finish the proof of the theorem. Fix a vertex $z \in \mathbb{G}^0$ and set $f_0 = \frac{1}{m(z)} \mathbf{1}_{\{z\}} \in c_0(\mathbb{G}^0)$. Obviously, one has $(f_0, 1) = 1$. Defining inductively $(f_n)_{n \geq 0}$ by $f_{n+1} = P f_n$ we can show that $f_n(x) = p_n(x, z)$. Indeed, we have

$$f_1(x) = P f_0(x) = \sum_{y \in \mathbb{G}^0} P(x, y) f_0(y) = \frac{P(x, z)}{m(z)} = p_1(x, z),$$

hence, the assertion is true for $n = 1$. Assume that the assertion is true for $n \geq 1$ and compute

$$\begin{aligned} f_{n+1}(x) = P f_n(x) &= \sum_{y \in \mathbb{G}^0} P(x, y) f_n(y) \\ &= \sum_{y \in \mathbb{G}^0} p_1(x, y) p_n(y, z) m(y) = p_{n+1}(x, z). \end{aligned}$$

I follows from step 6 that

$$b_n = (f_n, f_n) = p_{2n}(z, z) \leq c' n^{-\beta},$$

for all $z \in \mathbb{G}^0$.

Furthermore, by the lemma 4.3.15 below, we can show for $x, y \in \mathbb{G}^0$ that

$$p_{k+l}(x, y) \leq (p_{2k}(x, x) p_{2l}(y, y))^{1/2} \leq c'(kl)^{-\beta/2}.$$

For any integer $n \geq 2$ we decompose $n = k + l$ with $l = k$ if n is even and $l = k + 1$ otherwise. In both cases, the integers l and k satisfy $l \geq k \geq \frac{n-1}{2} \geq \frac{n}{4}$ and consequently $p_n(x, y) \leq c_1 n^{-\beta}$. For $n = 1$, $p_1(x, y) = \frac{P(x,y)}{m(y)}$ is bounded since m is supposed to be bounded from below.

□

Lemma 4.3.15. *For all $x, y \in \mathbb{G}^0$, the heat kernel satisfies the following inequality*

$$p_{n+m}(x, y) \leq (p_{2n}(x, x) p_{2m}(y, y))^{1/2}.$$

Proof. Using Cauchy-Schwartz and the reversibility, we get

$$\begin{aligned}
p_{n+m}(x,y) &= \sum_{z_i n \mathbb{G}^0} p_n(x,z)p_m(z,y)m(z)^{1/2}m(z)^{1/2} \\
&= \sum_{z\in\mathbb{G}^0} m(z)^{1/2}p_n(x,z)m(z)^{1/2}p_m(y,z) \\
&\leq \left(\sum_{z\in\mathbb{G}^0} m(z)p_n(x,z)^2\right)^{1/2}\left(\sum_{z\in\mathbb{G}^0} m(z)p_m(y,z)^2\right)^{1/2} \\
&= p_{2n}(x,x)^{1/2}p_{2m}(y,y)^{1/2}.
\end{aligned}$$

□

4.3.5 Conclusions

In the last section, we reproduce the proof of the theorem [Var85] to highlight the crucial property of reversibility — we should also cite [GT01]. One of the main reasons there are strong results in the context of reversible Markov chains is that the Laplace-Beltrami operator is closely related to the Markov operator and the Green function — $\mathscr{L} = \mathsf{id} - P$. Moreover, as an operator of the Hilbert space $\ell^2(\mathbb{G}^0, m, \mathbb{R})$, the Laplacian is self-adjoint so that its spectral theory is quite well known. More precisely, one can estimate the first eigenvalue with the help of the Faber-Krahn inequality which is an extension of isoperimetric inequality. In the case of non reversible random walks, very few result are known, nevertheless let us state a comparaison theorem for non-reversible Markov chain — see [Woe00].

Theorem 4.3.16. *Let P be an irreducible Markov operator with excessive measure ν and let Q be reversible with conductance a and total conductance m. Assume that*

(i) $\sup_{x\in\mathbb{G}^0} \frac{m(x)}{\nu(x)} < \infty$, *and*

(ii) there is $\varepsilon_0 > 0$ such that $P \geq \varepsilon_0 Q$ elementwise.

Then the recurrence P implies the recurrence of Q.

This result only deals with weak non reversible pertubations of reversible Markov chains. This theorem is useless, for instance, for genuinely directed graph like those studied in chapter 2. Furthermore, it does not give any informations in the case of transience of P or recurrence of Q.

In the case of genuinely directed graphs — or semi-groupoids — the Markov operator is no longer expressible in terms of Laplace operator and the previous theorem can never be applied so that the situation is even worse.

Appendix A

Di-graph $\mathbb{H}$: triviality of the Martin boundary

In this appendix, we give the details of the computations of the Martin kernel and conclude with the triviality of the Martin boundary of the simple random walk $(M_n)_{n\geq 0}$ on the directed graph $\mathbb{H}$.

In first section, the characteristic function of the induced random walk $(M_{\tau_n})_{n\geq 0}$ is given and sharp estimates of the Green function are obtained from a fine analysis of the singularity of the characteristic function.

In second section, the Martin kernel of the original chain is shown to be related to the Martin kernel of the induced chain. The symmetries of the graph $\mathbb{H}$ give rise to a closed formula of the Martin kernel. A fine analysis of this closed formula furnishes a sufficiently good estimate to deduce the triviality of the boundary.

A.1 Martin boundary of the induced Markov chain

A.1.1 Characteristic function of the embedded chain $(M_{\tau_n})_{n\geq 0}$

We start this paragraph by computing the characteristic function of the chain (M_{τ_n}). Provided that $M_{\tau_0} = M_0 \in \mathbb{Z} \times \{0\}$ — we denote in the sequel the latter set $\mathbb{H}_0$, this Markov chain can be regarded as a random walk on $\mathbb{Z}$, that is a sum of independent and identically distributed random variables. The triviality of its Martin boundary is not obvious because the jumps are not integrable.

Crucial ideas to compute the law of the chain (M_{τ_n}) are essentially those presented in [CP03]. Let us recall the notation.

Definition A.1.1. Let $(\psi_n)_{n\geq 0}$ be a sequence of independent, identically distributed, $\{-1, 1\}$-valued symmetric Bernoulli's variables and

$$Y_n = Y_0 + \sum_{k=1}^{n} \psi_k$$

for all $n \geq 1$ with $Y_0 = M_0^{(2)}$. Denote by

$$\eta_n(y) = \sum_{k=0}^{n} 1_{\{Y_k = y\}}$$

Definition A.1.2. Let $(\sigma_n)_{n\geq 0}$ be a sequence of stopping times defined by induction by $\sigma_0 = 0$ and

$$\sigma_{n+1} = \inf\{n \geq \sigma_n + 1 : Y_n = 0\}, \text{ for } n \geq 0.$$

More precisely, σ_n is the nth return time to the origin of a simple symmetric random walk on $\mathbb{Z}$.

Definition A.1.3. Let $(\xi_n^{(y)})_{n\geq 1, y\in\mathbb{V}_2}$ be a doubly infinite sequence of independent identically distributed $\mathbb{N}$-valued geometric random variables of parameters p and $q = 1 - p$. Let

$$X_n = \sum_{y\in\mathbb{V}_2} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}, n \in \mathbb{N}$$

Moreover, we denote $|X_n|$ the quantity $\sum_{y\in\mathbb{V}_2} |\epsilon_y| \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}, n \in \mathbb{N}$ which represent the total horizontal displacement.

Denote by T_n the time

$$T_n = n + \sum_{y\in\mathbb{V}_2} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$$

with the convention that the sum $\sum_i$ vanishes whenever $\eta_{n-1}(y) = 0$. Then

$$M_{T_n} = (X_n, Y_n)$$

Recall that τ_n denote the nth return to 0 of the vertical projection of the M_n's. One has the following.

Proposition A.1.4. *The law of M_{τ_n} is uniquely determined by the law of X_{σ_1},* i.e. *its characteristic function is given by*

$$\mathbf{E}^0(e^{i\langle t, M_{\tau_n}\rangle}) = \mathbf{E}^0(e^{it_1 X_{\sigma_1}})^n.$$

We denote by ϕ the characteristic function of X_{σ_1} with starting point 0. It is given by

$$\phi(t) = \mathbf{E}^0[\exp(itX_{\sigma_1})] = \mathsf{Re}\ r(t)^{-1} g(r(t))$$

where the functions g and r are defined by the formulae

$$g(x) = \frac{1 - \sqrt{1 - x^2}}{x} \text{ and } r(t) = \frac{p}{1 - qe^{it}}.$$

Proof. It is a matter of fact that $\tau_n = \sigma_n + \sum_{i=1}^n |X_{\sigma_n}| = T_{\sigma_n}$. Then,

$$\mathbf{E}^0(e^{i\langle t, M_{\tau_1}\rangle}) = \mathbf{E}^0(e^{it_1 X_{\sigma_1}})$$

We compute the law of X_{σ_1}. Denote by $\flat$ the vector $(0,1)$ and factorize by the first step of the random walk, thus

$$\begin{aligned}\mathbf{E}^0(e^{itX_{\sigma_1}}) &= \frac{1}{2}\left[\mathbf{E}^\flat(\exp(itX_{\sigma_1-1})) + \mathbf{E}^{-\flat}(\exp(itX_{\sigma_1-1}))\right] \\ &= \frac{1}{2}\left[\mathbf{E}^\flat(\exp(itX_{\sigma_1-1})) + \mathbf{E}^\flat(\exp(-itX_{\sigma_1-1}))\right]\end{aligned}$$

As a consequence, we only need to compute the following characteristic function

$$\begin{aligned}\mathbf{E}^\flat[\exp(itX_{\sigma_1-1})] &= \mathbf{E}^\flat\mathbf{E}^\flat[\exp(itX_{\sigma_1-1})|Y] \\ &= \mathbf{E}^\flat\left[\prod_{y\in\mathbb{Z}}\prod_{i=1}^{\eta_{\sigma_1-1}} \mathbf{E}^\flat[\exp(it\xi_i^{(y)})]\right] \\ &= \mathbf{E}^\flat[r(t)^{\sigma_1-1}]\end{aligned}$$

where r is the characteristic function of the $\xi_i^{(y)}$'s which are i.i.d, geometric random variables, so that r is given by

$$r(t) = \frac{p}{1-qe^{it}}$$

Therefore, we get a closed formula for the characteristic function of X_{σ_1-1}

$$\mathbf{E}^\flat[\exp(itX_{\sigma_1-1})] = \frac{g(r(t))}{r(t)},$$

where g is given by $g(x) = \mathbf{E}^\flat[x^{\sigma_1}]$ and satisfies the quadratic relation

$$g(x) = \frac{x}{2}(1+g(x)^2),$$

so that $g(x) = \frac{1-\sqrt{1-x^2}}{x}$. □

Remark A.1.5. In the example of di-graph $\mathbb{H}$, $p = \frac{1}{3} = 1-q$. Undoubtedly, all the following computations are still valid if p is chosen differently as soon as the drift vector remains parallel to the horizontal axis (if not, the stopping time τ_1 is no longer finite almost surely).

A.1.2 Estimation of the Green function

By inverse Fourier transform, we find a closed formula for the Green function of the induced random walk, namely

$$G(x,y) = \pi^{-1}\int_0^\pi \frac{\cos((y-x)t)}{1-\phi(t)}dt$$

and we want to get an asymptotic equivalent as $y \to \infty$. It appears that the function $[1-\phi]^{-1}$ has an integrable singularity for $t = 0$. The fruitful idea is to separate this singularity from the regular part of the function.

Proposition A.1.6. *There exists two analytic functions a, b in a neighborhood of 0 such that*

$$\frac{1}{1-\phi(t)} = \frac{c}{\sqrt{|t|}} + \sqrt{|t|}a(t) + b(t)$$

The proof of this proposition is postponed to section A.3. Having this decomposition in mind, a simple computation yields a precise estimate of the integral involved in the formula of the Green function.

Proposition A.1.7. *Denote by γ the function defined by*

$$\gamma(x) = \int_0^\pi \frac{\cos(xt)}{1-\phi(t)} dt.$$

Then, the limit of $\sqrt{x}\gamma(x)$ as $x \to \infty$ exists and is non zero.

Proof. Denote by R_a and R_b the convergence radii of a and b and choose $\epsilon > 0$ such that $\epsilon < R_a \wedge R_b$, then

$$\gamma(x) = \int_0^\pi \frac{\cos(xt)}{1-\phi(t)} dt = \int_0^\epsilon \frac{\cos(xt)}{1-\phi(t)} dt + \int_\epsilon^\pi \frac{\cos(xt)}{1-\phi(t)} dt$$

The second terms behaves like $\mathscr{O}\left(\frac{1}{x}\right)$ at infinity because on (ϵ, π) the function $\frac{1}{1-\phi}$ is infinitely continuously differentiable.

Because of the proposition A.1.6, the first integral term can be split in three parts $\gamma_0, \gamma_1, \gamma_2$. Then,

$$\gamma_0(x) = c \int_0^\epsilon \frac{\cos(xt)}{\sqrt{t}} dt,$$

and setting $u = xt$ we get

$$\gamma_0(x) = \frac{c}{x} \int_0^{\epsilon x} \sqrt{x} \frac{\cos(u)}{\sqrt{u}} du.$$

The latter is a convergent integral so that, when $x \to \infty$, $\gamma_0(x) \sim \frac{c'}{\sqrt{x}}$ with

$$c' = c \int_0^\infty \frac{\cos(u)}{\sqrt{u}} du$$

Secondly, $\gamma_2(x)$ behaves like $\mathscr{O}\left(\frac{1}{x}\right)$ at infinity. Indeed,

$$\gamma_2(x) = \int_0^\epsilon \cos(xt) b(t) dt$$

and b is infinitely continuously differentiable.

Finally, it remains to estimate the last term which is

$$\gamma_1(x) = \int_0^\epsilon \cos(xt) \sqrt{t} a(t) dt.$$

We may integrate by part,

$$\gamma_1(x) = \left[\sqrt{t}a(t)\frac{\sin(tx)}{x}\right]_0^\epsilon - \frac{1}{x}\int_0^\epsilon \left[\frac{a(t)}{2\sqrt{t}} + \sqrt{t}a'(t)\right]\sin(tx)dt$$

and it follows that γ_1 behaves like $\mathscr{O}\left(\frac{1}{x}\right)$ and the proposition is proved. □

Finally, we give the proof of the first part of theorem 2.2.4.

Proof of the first part of theorem 2.2.4. If we denote by G_0 the Green kernel of the Markov chain $(M_{\tau_n})_{n\geq 0}$ then we get for all $x, y \in \mathbb{Z} \times \{0\}$

$$G_0(x, y) = \gamma(y - x)$$

so that the Martin kernel is given by

$$K_0(x, y) = \frac{G_0(x, y)}{G_0(0, y)} = \frac{\gamma(y - x)}{\gamma(y)}$$

By proposition A.1.7, we have $\gamma(y) \sim \frac{c}{\sqrt{|y|}}$, consequently, for all unbounded sequences $(y_k)_{k\geq 0}$ of points of $\mathbb{Z}$, the limit of $K(x, y_k)$ is equal to 1 as k goes to infinity. Therefore, the Martin compactification is the one point compactification. □

A.2 Martin boundary of the original Markov chain

In this section, we will prove the triviality of the Martin boundary of the original Markov chain $(M_n)_{n\geq 0}$.

Denote by ν_x the probability, supported by $\mathbb{H}_0 = \mathbb{Z} \times \{0\}$, defined by

$$\nu_x(z) = \mathbf{P}^x(M_{\tau_1} = z).$$

Then, strong Markov property implies the following,

$$K(x, y) = \frac{\mathbf{E}^x(\eta_{0,\tau_1}(y))}{G(0, y)} + \sum_{z\in X_0} \nu_x(z)K(z, y) \tag{A.1}$$

for $x, y \in \mathbb{H}$.

In section A.2.1, we show — corollary A.2.7 — that the second term in equation A.1 goes to 1 as $|y|$ goes to infinity for all $x \in \mathbb{H}$, whereas in section A.2.2 the first term will be shown to vanish as $|y|$ goes to infinity.

A.2.1 Martin kernel conditioned by the first return time to $\mathbb{H}_0$

Martin kernel centered on $\mathbb{H}_0$ and Fourier transform

We first express the Martin kernel $K(z,y)$ in terms of Fourier transform for $z \in \mathbb{H}_0$.

Proposition A.2.1. *Let $z \in \mathbb{H}_0$ and $y \in \mathbb{H}$, then the Martin kernel is given by*

$$K(z,y) = \frac{\int_{-\pi}^{\pi} e^{ity_1 - itz \frac{g(r(t))^{|y_2|}}{1-\phi(t)}} dt}{\int_{-\pi}^{\pi} e^{ity_1 \frac{g(r(t))^{|y_2|}}{1-\phi(t)}} dt}$$

where g is given by

$$g(x) = \frac{1-\sqrt{1-x^2}}{x}$$

and r is given by

$$r(t) = \frac{1}{3-2e^{it}}.$$

Proof. If $y = (y_1, y_2) \in \mathbb{H}$ then we will denote by $\bar{y}$ the vector $\bar{y} = (y_1, -y_2)$. Using the geometry of the lattice $\mathbb{H}$, it is easy to see that

$$G(z,y) = G(\bar{y}, z) = \sum_{w \in \mathbb{H}_0} \nu_{\bar{y}}(w) G_0(w,z)$$

and

$$G(0,y) = G(\bar{y}, 0) = \sum_{w \in \mathbb{H}_0} \nu_{\bar{y}}(w) G_0(w,0),$$

for $z \in \mathbb{H}_0$ and $y \in \mathbb{H}$.

Consequently, using the translation invariance of G_0 and applying the substitution $v = w - z$ in the first sum, we get

$$K(z,y) = \frac{\sum_{v \in \mathbb{H}_0} \nu_{\bar{y}-z}(v) G_0(v,0)}{\sum_{v \in \mathbb{H}_0} \nu_{\bar{y}}(v) G_0(v,0)}.$$

Recall that $\nu_y(v) = \mathbf{P}^y(M_{\tau_1} = v) = \mathbf{P}^{(0,y_2)}(M_{\tau_1} = v - y_1)$, thus we can assume that $y = (0, y_2)$ and compute,

$$\nu_y(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itv + ity_1} \phi^{y_2}(t) dt$$

where ϕ^{y_2} is given by

$$\phi^{y_2}(t) = \mathbf{E}^{y_2}(e^{itX_{\sigma_1}}) = g(r(t))^{|y_2|},$$

and this comes from a simple modification of the computations of the proof of the proposition A.1.4. Then, let us compute the sum

$$\sum_{v \in \mathbb{H}_0} \nu_{\bar{y}-z} G_0(0,v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi^{-y_2}(t) e^{ity_1 - itz} \sum_{v \in \mathbb{H}_0} e^{itv} G_0(0,v) dt$$

and the summation is the Fourier series of the function $[1-\phi(t)]^{-1}$ computed in the section A.1.

As a consequence, we have to estimate the rate of convergence of the integral

$$\int_{-\pi}^{\pi} e^{ity_1-itz}\frac{\phi^{y_2}(t)}{1-\phi(t)}dt \tag{A.2}$$

when $y=(y_1,y_2)$ goes to infinity, that is when $|y_1|$ or $|y_2|$ goes to infinity. □

Sharp estimates of the Martin kernel centered on $\mathbb{H}_0$

In the spirit of section A.1.2, we first compute — see section A.3 — an analytic decomposition of the characteristic function of the Green function (centered on $\mathbb{H}_0$).

Proposition A.2.2. *The function $g\circ r$ can be decomposed in a neighborhood of 0 as follows*

$$g(r(t)) = 1-2\sqrt{|t|}e^{\mathrm{sgn}(t)i\frac{\pi}{4}} - \sqrt{|t|}\alpha(t)-\beta(t),$$

where α and β are analytic functions in a neighborhood of 0, satisfying $\alpha(0)=\beta(0)=0$.

We will estimate the rate of convergence of the integral (A.2). This rate depends on the relative rate of escape to infinity of y_1 with respect to y_2. It is straightforward to show that there are two cases depending on the ratio $\frac{y_1}{y_2^2}$:

- $\lim \frac{y_1}{y_2^2} = \lambda \in \mathbb{R}$
- $\lim \frac{y_1}{y_2^2} = \pm\infty$

The first case will be proved in proposition A.2.3 whereas the last one will be handled in proposition A.2.5.

Proposition A.2.3. *Assume that (y_1,y_2) goes to infinity in such a way that $\lim y_1y_2^{-2} = \lambda \in \mathbb{R}$. Then the sequence*

$$\left(|y_2|\int_{-\pi}^{\pi} e^{ity_1-itz}\frac{\phi^{y_2}(t)}{1-\phi(t)}dt\right)_{(y_1,y_2)\in\mathbb{Z}^2}$$

converges to a non zero constant.

Proof. Let n be a positive integer and set $m=y_1-z$, we begin to estimate the difference

$$D(t) = \frac{\phi^n(tn^{-2})}{n(1-\phi(tn^{-2}))} - Q(t)$$

where Q is given by

$$Q(t) = \frac{c\exp\{-2e^{\mathrm{sgn}(t)i\frac{\pi}{4}}\sqrt{|t|}\}}{\sqrt{|t|}}$$

where sgn is the function sign and c is the constant involved in the proposition A.1.6.

Let $\epsilon > 0$ be sufficiently small so that the decompositions in propositions A.2.2 and in A.1.6 are satisfied. Then for $|tn^{-2}| < \epsilon$ we have

$$\begin{aligned}\frac{\phi^n(tn^{-2})}{n(1-\phi(tn^{-2}))} - Q(t) = \exp\Big\{ & n\log(1 - 2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}\sqrt{|tn^{-2}|} \\ & -\sqrt{|tn^{-2}|}\alpha(tn^{-2}) - \beta(tn^{-2}))\Big\} \\ & \frac{1}{n}\left[\frac{c}{\sqrt{|tn^{-2}|}} + \sqrt{|tn^{-2}|}a(tn^{-2}) + b(tn^{-2})\right] - Q(t)\end{aligned}$$

Since $|tn^{-2}| < \epsilon$ and the quantity $x_n(t)$, defined by

$$x_n(t) = 2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}\sqrt{|tn^{-2}|} + \sqrt{|tn^{-2}|}\alpha(tn^{-2}) + b(tn^2),$$

goes to 0 as $|tn^{-2}|$ goes to 0, developing the log yields

$$\begin{aligned}D(t) = \exp\left\{-2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}\sqrt{|t|}\right\}\exp\left\{-\sqrt{|t|}\alpha(tn^{-2}) - n\beta(tn^{-2})\right\} \\ e^{nx_n(t)\epsilon(x_n(t))}\frac{c}{\sqrt{|t|}}\left[1 + \frac{|t|a(tn^{-2})}{cn^2} + \frac{\sqrt{|t|}b(tn^{-2})}{cn}\right] - Q(t)\end{aligned}$$

Now, we can factorize by Q

$$\begin{aligned}D(t) = Q(t)\Big\{ & \exp\left(-\sqrt{|t|}\alpha(tn^{-2}) - n\beta(tn^{-2}) + nx_n(t)\epsilon(x_n(t))\right) - 1 \\ & + \left[\frac{|t|a(tn^{-2})}{cn^2} + \sqrt{|t|}\frac{b(tn^{-2})}{cn}\right] \\ & \exp\left(-\sqrt{|t|}\alpha(tn^{-2}) - n\beta(tn^{-2}) + nx_n(t)\epsilon(x_n(t))\right)\Big\},\end{aligned}$$

and take modulus,

$$\begin{aligned}|D(t)| \le |Q(t)|\left|\exp\left\{-\sqrt{|t|}\alpha(tn^{-2}) - n\beta(tn^{-2}) + nx_n(t)\epsilon(x_n(t))\right\} - 1\right| \\ + |Q(t)|\left|\frac{|t|a(tn^{-2})}{cn^2} + \frac{\sqrt{|t|}b(tn^{-2})}{cn}\right| \\ \left|\exp\left\{-\sqrt{|t|}\alpha(tn^{-2}) - n\beta(tn^{-2}) + nx_n(t)\epsilon(x_n(t))\right\}\right|.\end{aligned}$$

As a consequence, we have that

$$\begin{aligned}\left|\frac{|t|a(tn^{-2})}{cn^2} + \frac{\sqrt{|t|}b(tn^{-2})}{c}\right| &= \sqrt{\frac{|t|}{n^2}}\left|\sqrt{\frac{|t|}{n^2}}\frac{a(tn^{-2})}{c} + \sqrt{\frac{n^2}{|t|}}\frac{b(tn^{-2})}{c}\right| \\ &\le N_\epsilon\sqrt{\frac{|t|}{n^2}}\end{aligned}$$

because the function $\rho(x) : x \mapsto \frac{\sqrt{(x)}}{c}a(x) + \frac{b(x)}{c\sqrt{x}}$ goes to 0 as x goes to 0. The dependence to ϵ of N_ϵ is not so strong, we actually have uniformity — due to the continuity of the function ρ in the neighborhood of 0 — in the sense that there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ we have $N_\epsilon < N_{\epsilon_0}$. This uniformity will be interesting in the sequel.

Using the following estimate,

$$|e^{a+ib} - 1| \leq e^a|b| + |e^a - 1|$$

we have, for any $a \in \mathbb{R}$,

$$|e^a - 1| = |a| \left| \sum_{n\geq 1} \frac{a^{n-1}}{n!} \right| \leq |a| \sum_{n\geq 1} \frac{|a|^{n-1}}{n!}. \tag{A.3}$$

Denoting by $\Upsilon(tn^{-2})$ the quantity

$$\Upsilon(tn^{-2}) = \alpha(tn^{-2}) + \frac{n}{\sqrt{|t|}}\beta(tn^{-2}) + \frac{n}{\sqrt{|t|}}x_n(t)\epsilon(x_n(t)).$$

The function $x \mapsto \alpha(x) + \frac{\beta(x)}{\sqrt{|x|}}$ is continuous at $x = 0$ and

$$|nx_n(t)\epsilon(x_n(t))| \leq \sqrt{|t|} \left| 2e^{\mathsf{sgn}(t)i\frac{\pi}{4}} + \alpha(tn^{-2}) + \frac{n}{\sqrt{t}}\beta(tn^{-2}) \right| |\epsilon(x_n(t))|$$

but the function $\tilde{\rho} : x \mapsto 2e^{\mathsf{sgn}(t)i\frac{\pi}{4}} + \alpha(x^2) + \frac{\beta(x)}{\sqrt{|x|}}$ is bounded so that

$$|nx_n(t)\epsilon(x_n(t))| \leq M\sqrt{|t|}K_\epsilon$$

where K_ϵ comes from the fact that $\epsilon(x_n(t))$ goes to 0 as $|tn^{-2}|$ goes to 0, so that $|\epsilon(x_n(t))| \leq K_\epsilon$. Summarising, $\Upsilon(tn^{-2})$ can be made arbitrarily small as $|tn^{-2}|$ goes to zero, namely $|\Upsilon(tn^{-2})| \leq L_\epsilon$. Thus,

$$|e^{-\sqrt{|t|}\Upsilon(tn^{-2})} - 1| \leq e^{-\sqrt{|t|}\mathsf{Re}\Upsilon(tn^{-2})}|\mathsf{Im}\sqrt{|t|}\Upsilon(tn^{-2})| + |e^{-\sqrt{|t|}\mathsf{Re}\Upsilon(tn^{-2})} - 1|$$

then, the first quantity is obviously majorized by

$$e^{-\sqrt{|t|}\mathsf{Re}\Upsilon(tn^{-2})}|\sqrt{|t|}\mathsf{Im}\Upsilon(tn^{-2})| \leq e^{L_\epsilon\sqrt{|t|}}\sqrt{|t|}L_\epsilon \tag{A.4}$$

whereas for the second quantity, we use the estimate (A.3) and we get

$$|e^{-\sqrt{|t|}\mathsf{Re}\Upsilon(tn^{-2})} - 1| \leq \sqrt{|t|}L_\epsilon \left| \sum_{k\geq 1} \frac{L_\epsilon^{k-1}|t|^{\frac{k-1}{2}}}{k!} \right|. \tag{A.5}$$

Finally, it is obvious that $|e^z| \leq e^{|z|}$ for any complex number z, so that the following estimate holds

$$|D(t)| \leq |Q(t)| \left\{ e^{M\sqrt{|t|}K_\epsilon} N_{\epsilon_0} \sqrt{\frac{|t|}{n^2}} + e^{L_\epsilon\sqrt{|t|}} \sqrt{|t|} L_\epsilon + L_\epsilon \sqrt{|t|} \left| \sum_{k\geq 1} \frac{L_\epsilon^{k-1} |t|^{\frac{k-1}{2}}}{k!} \right| \right\}. \tag{A.6}$$

Coming back to the proof of the proposition, we consider the first case, that is we suppose that mn^{-2} converges to a real number, and we fix a $\delta > 0$ such that the decomposition in A.2.2 and A.1.6 are satisfied. Then we can split

$$\begin{aligned} n \int_{-\pi}^{\pi} e^{itm} \frac{\phi^n(t)}{1-\phi(t)} dt &= n \int_{-\delta}^{\delta} e^{itm} \frac{\phi^n(t)}{1-\phi(t)} dt + n \int_{|t|>\delta} e^{itm} \frac{\phi^n(t)}{1-\phi(t)} dt \\ &= I_1(m,n,\delta) + I_2(m,n,\delta). \end{aligned}$$

Let us consider first, the term $I_1(m,n,\delta)$, then setting $t = un^{-2}$ and decomposing as follows, we get

$$\begin{aligned} n \int_{-\delta}^{\delta} e^{itm} \frac{\phi^n(t)}{1-\phi(t)} dt &= \int_{-n^2\delta}^{n^2\delta} e^{iumn^{-2}} \frac{\phi^n(un^{-2})}{n(1-\phi(un^{-2}))} du \\ &= \int_{-n^2\delta}^{n^2\delta} e^{iumn^{-2}} \left[\frac{\phi^n(un^{-2})}{n(1-\phi(un^{-2}))} - \frac{\exp\{-2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} \sqrt{|u|}\}}{\sqrt{|u|}} \right] du \\ &+ \int_{-\infty}^{\infty} e^{iumn^{-2}} \frac{\exp\{-2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} \sqrt{|u|}\}}{\sqrt{|u|}} du \\ &- \int_{|u|>n^2\delta} e^{iumn^{-2}} \frac{\exp\{-2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} \sqrt{|u|}\}}{\sqrt{|u|}} du. \\ &= I_3(m,n,\delta) + I_4(m,n) + I_5(m,n,\delta) \end{aligned}$$

It is easy to see that the term $I_5(m,n,\delta)$ converges to 0 as n goes to infinity at the rate $\mathscr{O}(e^{-\sqrt{\frac{\pi}{2}}n})$ as the tail of the integral of an integrable function.

Applying the dominated convergence theorem to the term $I_4(m,n)$ implies that it converges to

$$\int_{-\infty}^{\infty} e^{iu\lambda} \frac{\exp\{-2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} \sqrt{|u|}\}}{\sqrt{|u|}} du = s(\lambda)$$

which is a non zero constant for all λ.

Finally, it remains to show that the term $I_3(m,n,\delta)$ goes to 0. Using the estimate (A.6), we get

$$\begin{aligned} \left| \int_{-n^2\delta}^{n^2\delta} e^{iumn^{-2}} \frac{\phi^n(un^{-2})}{n(1-\phi(un^{-2}))} du \right| \leq \int_{-n^2\delta}^{n^2\delta} |Q(t)| \Bigg\{ & e^{M\sqrt{|t|}K_\epsilon} N_{\epsilon_0} \sqrt{\frac{|t|}{n^2}} \\ & e^{L_\epsilon\sqrt{|t|}} \sqrt{|t|} L_\epsilon + L_\epsilon \sqrt{|t|} \sum_{k\geq 1} \frac{L_\epsilon^{k-1} |t|^{\frac{k-1}{2}}}{k!} \Bigg\} dt. \end{aligned}$$

At this step, we have to choose $\epsilon > 0$ such that the decompositions A.2.2 and A.1.6 hold and such that $MK_\epsilon < \frac{\sqrt{2}}{4}$ and $L_\epsilon \leq \frac{\sqrt{2}}{2}$ so that the left hand-side integral is majorized by

$$\int_{-n^2\epsilon}^{n^2\epsilon} \frac{c}{n} N_{\epsilon_0} e^{-\frac{\sqrt{2}}{4}\sqrt{|t|}} + cL_\epsilon e^{-\frac{\sqrt{2}}{2}\sqrt{|t|}} + cL_\epsilon e^{-\sqrt{2}2\sqrt{|t|}} \sum_{k\geq 1} \frac{\sqrt{2}^{k-1}|t|^{\frac{k-1}{2}}}{4^{k-1}k!} dt$$
$$= I_6(n,\epsilon) + I_7(n,\epsilon) + I_8(n,\epsilon).$$

Then, the quantity $I_6(n,\epsilon)$ goes to 0 as n goes to infinity, the quantity $I_7(n,\epsilon)$ can be made arbitrarily small, namely it behaves like a $\mathscr{O}(L_\epsilon)$, and setting $t = \frac{u^2}{2}$, $I_8(n,\epsilon)$ becomes

$$2L_\epsilon \int_0^{n\sqrt{2\epsilon}} e^{-u} \sum_{k\geq 1} \frac{\sqrt{2}^{k-1}|u|^k}{8^{k-1}k!} dt.$$

Then, exchanging sum and integral, we get the majoration

$$2L_\epsilon \sum_{k\geq 1} \frac{\sqrt{2}^{k-1}}{8^{k-1}k!} \int_0^\infty e^{-u} u^k du.$$

But the latter integral is nothing but $(k+1)!$ thus the quantity $I_8(n,\epsilon)$ behaves like $\mathscr{O}(L_\epsilon)$ and as consequence it can be made arbitrarily small.

Finally, the term $I_2(m,n,\delta)$ goes to zero geometrically, and the proposition is proved. □

The following lemma is a refinement of a well known result on Fourier series.

Lemma A.2.4. *Let (f_n) be a sequence of 2π-periodic α-Hölder real functions with Hölder constant Kn and $0 < \alpha \leq 1$. Then for any $\epsilon > 0$, we have the estimate*

$$\left| \int_{-\epsilon}^{\epsilon} f_n(t) e^{itm} dt \right| \leq \frac{Ln}{1 + |m|^\alpha}$$

for all $n, m \in \mathbb{Z}$.

Proof. It is well known that

$$\int_{-\pi}^{\pi} f_n(t) e^{itm} dt = \int_0^{2\pi} \frac{1}{m} \sum_{k=0}^{m-1} \left[f_n\left(\frac{t}{m} + \frac{2k\pi}{m}\right) - f_n\left(\frac{2k\pi}{m}\right) \right] e^{it} dt.$$

Thus, we get that $\int_{-\epsilon}^{\epsilon} f_n(y) e^{itm} dt$ is given by

$$\int_0^{2\pi} \frac{1}{m} \sum_{k=0}^{m-1} \left[f_n\left(\frac{t}{m} + \frac{2k\pi}{m}\right) 1_{\Lambda_\epsilon}\left(\frac{t}{m} + \frac{2k\pi}{m}\right) - f_n\left(\frac{2k\pi}{m}\right) 1_{\Lambda_\epsilon}\left(\frac{2k\pi}{m}\right) \right] e^{it} dt$$

where $\Lambda_\epsilon = [\pi - \epsilon; \pi + \epsilon]$.

Then, the regularity of f_n gives us that for any x, y

$$\begin{aligned}|f_n(x)1_A(x) - f_n(y)1_A(y)| &\leq |f_n(x) - f_n(y)| + |f_n(y)||1_A(x) - 1_A(y)| \\ &\leq Kn|x-y|^\alpha + M|1_A(x) - 1_A(y)|.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left|\int_0^{2\pi} \frac{1}{m}\sum_{k=0}^{m-1}\left[f_n\left(\frac{t}{m}+\frac{2k\pi}{m}\right)1_{\Lambda_\epsilon}\left(\frac{t}{m}+\frac{2k\pi}{m}\right) - f_n\left(\frac{2k\pi}{m}\right)1_{\Lambda_\epsilon}\left(\frac{2k\pi}{m}\right)\right]e^{it}dt\right| \\ &\leq \int_0^{2\pi}\frac{1}{m}\sum_{k=0}^{m-1} Kn\left|\frac{t}{m}\right|^\alpha dt \\ &+ \int_0^{2\pi}\frac{1}{m}\sum_{k=0}^{n-1} M\left|1_{\Lambda_\epsilon}\left(\frac{t}{m}+\frac{2k\pi}{m}\right) - 1_{\Lambda_\epsilon}\left(\frac{2k\pi}{m}\right)\right| dt \\ &= J_1(m,\epsilon) + J_2(m,\epsilon).\end{aligned}$$

It is obvious that the quantity $J_1(m,\epsilon)$ is majorized by

$$\int_0^{2\pi}\frac{1}{m}\sum_{k=0}^{m-1} Kn\left|\frac{t}{n}\right|^\alpha dt \leq K'\frac{n}{|m|^\alpha}$$

For the quantity $J_2(m,\epsilon)$, we only have to observe that the difference of indicator function is non zero for only two integers k, and, in that case, the difference is obviously bounded so that

$$\int_0^{2\pi}\frac{1}{m}\sum_{k=0}^{n-1} M\left|1_{\Lambda_\epsilon}\left(\frac{t}{m}+\frac{2k\pi}{m}\right) - 1_{\Lambda_\epsilon}\left(\frac{2k\pi}{m}\right)\right| dt \leq \frac{2M}{m}.$$

Therefore the lemma is proved. □

Proposition A.2.5. *The sequence*

$$\left(\sqrt{|y_1|}\int_{-\pi}^{\pi} e^{ity_1 - itz}\frac{\phi^{y_2}(t)}{1-\phi(t)}dt\right)_{(y_1,y_2)\in\mathbb{Z}^2}$$

converges to a non zero constant as $\frac{y_1}{y_2^2}$ *goes to infinity.*

Proof. Like in the previous proposition, set $n = y_2$ and $m = y_1 - z$ for short. Thus, we want to estimate the integral

$$\int_{-\pi}^{\pi} e^{itm}\frac{g(r(t))^n}{1-\phi(t)}dt.$$

Choose $\delta > 0$ so that the decompositions in propositions A.2.2 and A.1.6 are satisfied and split the integral,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{itm}\frac{g(r(t))^n}{1-\phi(t)}dt &= \int_{-\delta}^{\delta} e^{itm}\frac{g(r(t))^n}{1-\phi(t)}dt + \int_{|t|>\delta} e^{itm}\frac{g(r(t))^n}{1-\phi(t)}dt \\ &= I_1(m,n,\delta) + I_2(m,n,\delta).\end{aligned}$$

The function $t \mapsto g(r(t))^n[1-\phi(t)]^{-1}$ being continuously differentiable on the set $\{|t| > \delta\}$, integrating by parts, we see that $I_2(m,n,\delta)$ goes to 0 like $\mathscr{O}(\frac{n}{m})$ *i.e.* like $o\left(\frac{1}{\sqrt{|m|}}\right)$.

Let us deal with the quantity $I_1(m,n,\delta)$, then we can write,

$$\begin{aligned}\frac{g(r(t))^n}{1-\phi(t)} =& \frac{g(r(t))^n - (1-2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}\sqrt{|t|}-it)^n}{1-\phi(t)} \\ &+ \frac{(1-2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}\sqrt{|t|}-it)^n - 1}{1-\phi(t)} + \frac{1}{1-\phi(t)} \\ &= R_1(n,t) + R_2(n,t) + R_3(n,t).\end{aligned}$$

We already know that the integral of the function $R_3(n,t)$

$$\int_{-\delta}^{\delta} e^{itm}[1-\phi(t)]^{-1}dt$$

is equivalent to the sequence $(c'|m|^{-1/2})_m$ as $|m|$ goes to infinity. Consider the function $R_1(n,t)$, then we can show it is Lipschitz with Lipschitz constant depending linearly on n. Let us denote by q the function,

$$q(t) = 1 - 2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}\sqrt{|t|} - it.$$

Then, we split

$$\begin{aligned}\frac{g(r(t))^n - q(t)^n}{1-\phi(t)} &= c\frac{g(r(t))^n - q(t)^n}{|t|^{1/2}} \\ &+ \left[g(r(t))^n - q(t)^n\right]|t|^{1/2}a(t) \\ &+ \left[g(r(t))^n - q(t)^n\right]b(t).\end{aligned}$$

Actually, if the first quantity is continuously differentiable, the two other quantities are also continuously differentiable because they are obviously smoother. Let us compute the derivative of the first function.

$$\begin{aligned}\frac{d}{dt}\frac{g(r(t))^n - q(t)^n}{|t|^{1/2}} &= \frac{d}{dt}\left(\alpha(t) + \frac{\beta(t)}{\sqrt{|t|}}\right)\sum_{k=0}^{n-1} g(t)^k q(t)^{n-k} \\ &= \left(\alpha'(t) + \frac{\beta'(t)}{\sqrt{|t|}} - \frac{\beta(t)}{2|t|^{3/2}}\right)\sum_{k=0}^{n-1} g(r(t))^k q(t)^{n-k} \\ &+ \left(\alpha(t) + \frac{\beta(t)}{\sqrt{|t|}}\right)\sum_{k=0}^{n-1}\left[kg'(t)g(r(t))^{k-1}q(t)^{n-k}\right. \\ &\left.+g(r(t))^k(n-k)q'(t)q(t)^{n-k-1}\right].\end{aligned}$$

The sums are estimated as follows,

$$\left|\sum_{k=0}^{n-1} g(r(t))^k q(t)^{n-k}\right| \leq n$$

and

$$\left|\sum_{k=0}^{n-1} kg'(r(t))g(r(t))^{k-1}q(t)^{n-k} + g(r(t))^k(n-k)q'(t)q(t)^{n-k-1}\right| \leq \frac{Mn}{\sqrt{|t|}}$$

where M is a upper bound of $|\sqrt{|t|}g'(r(t))|$ and $|\sqrt{|t|}q'(t)|$ in a neighborhood of 0.

Since, $\alpha(0) = \beta(0) = \beta'(0) = 0$, the function $t \mapsto (\alpha'(t) + \beta'(t)|t|^{-1/2} - \beta(t)2^{-1}|t|^{-3/2})$ is continuous, therefore bounded. Moreover, the function $t \mapsto (\alpha(t) + \beta(t)|t|^{-1/2})$ is a $\mathscr{O}(\sqrt{|t|})$. Finally, we have the following estimate of the derivative,

$$\left|\frac{d}{dt}\frac{g(r(t))^n - q(t)^n}{|t|^{1/2}}\right| \leq Mn$$

and this implies that the function $R_1(n,t)$ is Lipschitz with Lipschitz constant Mn.

By lemma A.2.4, there exists a constant K such that

$$\sqrt{|m|}\left|\int_{-\delta}^{\delta} \frac{g(r(t))^n - q(t)^n}{1-\phi(t)} e^{itm} dt\right| \leq \frac{Kn}{\sqrt{|m|}}$$

so that the integral goes to 0 when $\frac{\sqrt{|m|}}{n}$ goes to infinity.

It remains to estimate the integral of the function $R_2(n,t)$, namely

$$\int_{-\delta}^{\delta} \frac{q(t)^n - 1}{1-\phi(t)} e^{itm} dt$$

which can be split into

$$\begin{aligned}\int_{-\delta}^{\delta} c\frac{q(t)^n - 1}{|t|^{1/2}} e^{itm} dt + \int_{-\delta}^{\delta} (q(t)^n - 1)|t|^{1/2} a(t) e^{itm} dt + \int_{-\delta}^{\delta} (q(t)^n - 1) b(t) e^{itm} dt \\ = I_3(m,n,\delta) + I_4(m,n,\delta) + I_5(m,n,\delta)\end{aligned}$$

Considering the integral $I_3(m,n,\delta)$, factorizing the quantity $q(t)^n - 1$, and integrating by parts, we get

$$\begin{aligned}&\sqrt{|m|}\int_{-\delta}^{\delta} c\frac{q(t)^n - 1}{|t|^{1/2}} e^{itm} dt \\ &= -n\sqrt{|m|}\int_{-\delta}^{\delta} (2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} + i\sqrt{|t|})\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} + i\sqrt{|t|})^k e^{itm} dt \\ &= \frac{-n}{i\sqrt{|m|}}\left[(2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} + i\sqrt{|t|})\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} + i\sqrt{|t|})^k e^{itm}\right]_{-\epsilon}^{\epsilon} \\ &+ \frac{n}{i\sqrt{|m|}}\int_{-\epsilon}^{\epsilon}\frac{d}{dt}\left[(2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} + i\sqrt{|t|})\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2e^{\mathsf{sgn}(t) i\frac{\pi}{4}} + i\sqrt{|t|})^k\right] e^{itm} dt.\end{aligned}$$

The first quantity in the bracket is obviously bounded, so that the first term goes to 0 as $\frac{\sqrt{|m|}}{n}$ goes to infinity. Consequently, it only remains to show that the derivative involved in the integral is integrable. Let us compute it,

$$\begin{aligned}
&\frac{d}{dt}\left[(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})^k\right]\\
&=\frac{i}{4\sqrt{|t|}}\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})^k\\
&+(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})\frac{1}{n}\sum_{k=0}^{n-1}k(-1)^k(\frac{e^{\mathsf{sgn}(t)i\frac{\pi}{4}}}{\sqrt{|t|}}+i)(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})^{k-1}.
\end{aligned}$$

The Cesàro sum

$$\frac{1}{n}\sum_{k=0}^{n-1}k(-1)^k(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})^{k-1}$$

converges to 0 as n goes to infinity (hence is bounded). Thus, we get the estimate

$$\begin{aligned}
&\left|\frac{d}{dt}\left[(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|})^k\right]\right|\\
&\leq\frac{1}{4\sqrt{|t|}}+K\left|2e^{\mathsf{sgn}(t)i\frac{\pi}{4}}+i\sqrt{|t|}\right|\left|\frac{e^{\mathsf{sgn}(t)i\frac{\pi}{4}}}{\sqrt{|t|}}+i\right|
\end{aligned}$$

and the latter is integrable.

Quantities $I_4(m,n,\delta)$ and $I_5(m,n,\delta)$ can be estimated in the same way and the proposition is proved.

□

Asymptotics of the Martin kernel conditioned by first return time in $\mathbb{H}_0$

From proposition A.2.1, A.2.3 and A.2.5, we get the following corollaries.

Corollary A.2.6. *Let $z\in\mathbb{H}_0$, then we have*

$$\lim_{|y|\to\infty}K(z,y)=1.$$

Corollary A.2.6 implies the following.

Corollary A.2.7. *Let $x\in\mathbb{H}$, then*

$$\lim_{|y|\to\infty}\sum_{z\in\mathbb{H}_0}\nu_x(z)K(z,y)=1$$

Proof. From Corollary A.2.6 we have that, for any $z \in \mathbb{H}_0$,

$$\lim_{|y|\to\infty} K(z,y) = 1.$$

The sum $\sum_{(z,0)\in\mathbb{H}_0} \nu_x(z)K(z,y)$ is given by

$$\sum_{(z,0)\in\mathbb{H}_0} \nu_x(z)K(z,y) = \frac{\int_{-\pi}^{\pi} e^{ity_1}\phi^{y_2}(t)(1-\phi(t))^{-1}\sum \nu_x(z)e^{-itz}dt}{\int_{-\pi}^{\pi} e^{ity_1}\phi^{y_2}(1-\phi(t))^{-1}dt}.$$

Noting that the probability $\nu_x(z) = \nu_{-x}(-z)$, the following equality holds

$$\sum_{(z,0)\in\mathbb{H}_0} \nu_x(z)e^{-itz} = \sum_{(z,0)\in\mathbb{H}_0} \nu_{-x}(z)e^{itz}.$$

The latter is the characteristic function of ν_{-x} which is given (see the proof of theorem A.2.1) by

$$e^{-itx_1}\phi^{x_2}(t).$$

Replacing in the integral, we obtain

$$\sum_{(z,0)\in\mathbb{H}_0} \nu_x(z)K(z,y) = \frac{\int_{-\pi}^{\pi} e^{it(y_1-x_1)}\phi^{|y_2|+|x_2|}(t)(1-\phi(t))^{-1}dt}{\int_{-\pi}^{\pi} e^{ity_1}\phi^{y_2}(1-\phi(t))^{-1}dt}$$

and using the estimates of proposition A.2.3 and A.2.5, one has the announced convergence. □

A.2.2 Behavior before first return time

Recall the equation (A.1) holding for $x, y \in \mathbb{H}$,

$$K(x,y) = \frac{\mathbf{E}^x(\eta_{0,\tau_1}(y))}{G(0,y)} + \sum_{z\in\mathbb{H}_0} \nu_x(z)K(z,y).$$

It remains to show that the first term in this equation tends to zero.

Assume that $x_2, y_2 \geq 0$ and $y_1 \geq x_1$ and let us fix our notation. We will define by $s^\flat_{y_i}$ for $i = 1, 2$ the following stopping time,

$$s^\flat_{y_i} = \inf\{n \geq \flat : M_n^{(i)} = y_i, \forall k \leq n : M_n^{(i)} \neq 0\}, \text{ with}, \flat \in \{0,1\}.$$

Then, we will denote by $g_u(y)$ the probability

$$g_u(y) = \mathbf{P}^{(y_1,u)}(s^0_{y_2} < \infty | M^{(1)}_{\tau_1} \geq y_1).$$

Finally, the quantity h_y will denote the probability

$$h_y = \mathbf{P}^{(y_1,y)}(s^1_y < \infty | M^{(1)}_{\tau_1} \geq y_1).$$

Proposition A.2.8. *The quantity* $\mathbf{E}^x(\eta_{0,\tau_1}(y))$ *is given by*

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = \frac{1}{(1-h_{y_2})^2}\sum_{u\geq 0}\mu_x(u)g_u(y_2)$$

where μ_x *is defined by*

$$\mu_x(u) = \mathbf{P}^x(M_{s^0_{y_1}} = u, M^{(1)}_{\tau_1} \geq y_1).$$

Proof. As a matter of fact, we have that

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = \sum_{k\geq 0} k\mathbf{P}^x(\eta_{0,\tau_1} = k).$$

On conditioning by the event $\{s^0_{y_1} < \infty\}$, which is equal to the event $\{M^{(1)}_{\tau_1} \geq y_1\}$, we get

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = \mathbf{P}^x(M^{(1)}_{\tau_1} \geq y_1)\sum_{k\geq 0} k\mathbf{P}^x(\eta_{0,\tau_1}(y) = k|M^{(1)}_{\tau_1} \geq y_1).$$

By strong Markov property and observing that $s^0_{y_1}$ is finite on the event $\{M^{(1)}_{\tau_1} \geq y_1\}$, we get

$$\mathbf{P}^x(\eta_{0,\tau_1}(y) = k|M^{(1)}_{\tau_1} \geq y_1) = \sum_{u\geq 0}\mathbf{P}^x(M^{(2)}_{s^0_{y_1}} = u|M^{(1)}_{\tau_1} \geq y_1)$$
$$\mathbf{P}^{(y_1,u)}(\eta_{0,\tau_1}(y) = k|M^{(1)}_{\tau_1} \geq y_1).$$

Then, it is easy to see that

$$\mathbf{P}^{(y_1,u)}(\eta_{0,\tau_1}(y) = k|M^{(1)}_{\tau_1} \geq y_1) = g_u(y_2)h^{k-1}_{y_2}.$$

Finally, we get

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = \sum_{k\geq 0} k\sum_{u\geq 0}\mu_x(u)g_u(y_2)h^{k-1}_{y_2}$$

and grouping all terms, we obtain

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = \frac{1}{1-h_{y_2}}\sum_{u\geq 0}\mu_x(u)g_u(y_2),$$

proving thus the proposition. □

It is easy to get an upper bound for the probability h_{y_2} because at the site (y_1, y_2) it is possible to never come back with probability at least $1/3$ so that the quantity $(1-h_{y_2})^{-2}$ does not play any role in the asymptotics of the mean $\mathbf{E}^x(\eta_{0,\tau_1}(y))$.

Proposition A.2.9. *For any* $u \geq 0$*, the quantity* $g_u(y)$ *decreases exponentially fast to* 0 *as* y *goes to* ∞*.*

Proof. Recall that

$$g_u(y_2) = \mathbf{P}^{(y_1,u)}(s^0_{y_2} < \infty | M^{(1)}_{\tau_1} \geq y_1).$$

Actually we can majorize g_u by

$$g_u(y_2) \leq \mathbf{P}^{(y_1,u)}(\exists n \geq 0 : M^{(2)}_n = y_2 | M^{(1)}_{\tau_1} \geq y_1) = p_u(y_2).$$

Then, we can identify this probability with the probability to reach y_2 from u in the model of a simple random walk on $\mathbb{Z}$ with a cemetery attached to each site, where the random walk can die with probability 1/3.

If we replace the cemetery by binary trees, then the probability p_u satisfies

$$p_u(y_2) \leq F(u, y_2)$$

where $F(u, y_2)$ is the probability to hit y_2 from u in a homogeneous tree of degree 3. By the lemma (1.24), found on p.9 of [Woe09], we get $F(u, y_2) = 2^{-d(u,y_2)}$ where d is the usual graph metric in the tree. Thus, $g_u(y_2)$ decreases exponentially fast to 0.

□

Proposition A.2.10. *The quantity $\sum_{u\geq 0} \mu_x(u) g_u(y_2)$ behaves like $o(|y_2|^{-1})$ whenever $\frac{y_1}{y_2^2}$ converges to a finite limit and like $o(|y_1|^{-\frac{1}{2}})$ in other cases, namely when $\frac{y_1}{y_2^2}$ goes to $\pm\infty$.*

Before giving the proof of this fact, let us introduce some notation. We will denote by $(S_n)_{n\geq 0}$ the simple symmetric random walk on $\mathbb{Z}$. Recall that the characteristic function of (S_n) starting from z is given by

$$\mathbf{E}^z(e^{itS_n}) = e^{itz}(e^{it(S_1-S_0)})^n = e^{itz}(\cos(t))^n$$

On the set $\mathbb{N}$ we define the following Markov chain $(Z_n)_{n\geq 0}$ by its Markov operator $q : \mathbb{N} \times \mathbb{N} \mapsto [0, 1]$ by

$$q(x, y) = \begin{cases} \frac{2}{3} & y = x \geq 1 \\ \frac{1}{3} & y = x - 1, x \geq 1 \\ 1 & x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

On introducing the stopping time

$$T = \inf\{n \geq 0 : Z_n = 0\},$$

it is easy to compute its generating function.

Lemma A.2.11. *The generation function of T is given for any $h \geq 0$ by*

$$\mathbf{E}^h(x^T) = \left(\frac{x}{3 - 2x}\right)^h.$$

We can now prove the proposition A.2.10.

Proof. We can show that

$$\mu_x(u) = \mathbf{P}^x(M^{(2)}_{s^0_{y_1}} = u, M^{(1)}_{\tau_1} \geq y_1) = \sum_{m\geq 0} \mathbf{P}^{x_2}(S_m = u : S_k \neq 0, k \leq m) \\ \mathbf{P}^1(T = m + (y_1 - x_1)).$$

Then, by the mirroring principle, we have that

$$\mathbf{P}^{x_2}(S_m = u : S_k \neq 0, k \leq m) = \mathbf{P}^{x_2}(S_m = u) - \mathbf{P}^{-x_2}(S_m = u).$$

Thus,

$$\begin{aligned}\mu_x(u) &= \sum_{m\geq 0} \mathbf{P}^{x_2}(S_m = u)\mathbf{P}^1(T = m + (y_1 - x_1)) \\ &\quad - \sum_{m\geq 0} \mathbf{P}^{-x_2}(S_m = u)\mathbf{P}^1(T = m + (y_1 - x_1)) \\ &= \Sigma_1(x, y, u) + \Sigma_2(x, y, u).\end{aligned}$$

Then, let us compute the sum $\Sigma_1(x, y, u)$,

$$\begin{aligned}\Sigma_1(x, y, u) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{itx_2} \sum_{m\geq 0} (\cos(t))^m \mathbf{P}^1(T = m + (y_1 - x_1))e^{-itu}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} e^{itx_2-itu}dt\end{aligned} \tag{A.7}$$

where $F(x) = \mathbf{E}^1(x^T)$ is the generating function of T. Whereas the sum $\Sigma_2(x, y_1, u)$ is given by

$$\Sigma_2(x, y, u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} e^{-itx_2-itu}du. \tag{A.8}$$

As a consequence,

$$\mu_x(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} 2i\sin(tx_2)e^{-itu}dt.$$

Now, from proposition A.2.9, we get that

$$\sum_{u\geq 0} \mu_x(u)g_{y_2}(u) \leq \sum_{u\geq 0} \mu_x(u)2^{-|y_2-u|}$$

Split the sum

$$\begin{aligned}\sum_{u\geq 0} \mu_x(u)2^{-|y_2-u|} &= \sum_{u=0}^{y_2-1} \mu_x(u)2^{-(y_2-u)} \\ &\quad + \sum_{u=y_2}^{\infty} \mu_x(u)2^{-(u-y_2)} \\ &= \Sigma_3(x, y) + \Sigma_4(x, y),\end{aligned}$$

and, injecting (A.7) and (A.8), sums $\Sigma_3(x,y)$ and $\Sigma_4(x,y)$ become

$$\Sigma_3(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} 2i\sin(tx_2)\sum_{u=0}^{y_2-1} e^{-itu}2^{u-y_2}dt. \tag{A.9}$$

The geometric sum can be simplified by observing that

$$\sum_{u=0}^{y_2-1} e^{-itu}2^{u-y_2} = 2^{-y_2}\frac{(2e^{-it})^{y_2}-1}{2e^{-it}-1}$$

hence, the sum (A.9) becomes

$$I_1(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} 2i\sin(tx_2)2^{-y_2}\frac{(2e^{-it})^{y_2}-1}{2e^{-it}-1}dt. \tag{A.10}$$

Similarly,

$$\Sigma_4(x,y) = \frac{1}{2\pi}\int_{-2\pi}^{\pi} F(\cos(t))^{y_1-x_1} 2i\sin(tx_2)\sum_{u=y_2}^{\infty} e^{-itu}2^{-(u-y_2)}dt, \tag{A.11}$$

so that simplifying the geometric sum

$$\sum_{u=y_2}^{\infty} e^{-itu}2^{-(u-y_2)} = e^{-ity_2}(1-\frac{e^{-it}}{2})^{-1}$$

integral (A.11) becomes

$$I_2(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} 2i\sin(tx_2)e^{-ity_2}\frac{2}{2-e^{-it}}dt. \tag{A.12}$$

At this step, it remains to study the rate of convergence of $I_1(x,y)$ and $I_2(x,y)$. We have to distinguish two cases depending on the way that (y_1,y_2) goes to infinity :

- y_1 remains bounded ;
- $\lim \frac{y_2^2}{y_1} = \lambda$ for $\lambda \in \mathbb{R}\cup\{\pm\infty\}$ and y_1 is unbounded.

Let us handle the first case, and assume that y_1 is bounded. The function F has a unique singularity for $x = \frac{3}{2}$ so that $F(\cos(\cdot))$ is infinitely continuously differentiable for $|t| \leq \pi$. As a consequence of lemma A.2.4, the quantity $I_2(x,y)$ decreases like $\mathscr{O}\left(\frac{y_1^k}{y_2^k}\right)$ for arbitrary $k \geq 0$, *i.e.* like $\mathscr{O}\left(\frac{1}{y_2^k}\right)$ because y_1 is supposed to be bounded. For the quantity $I_1(x,y)$, we have the following

$$\begin{aligned}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} 2i\sin(tx_2)2^{-y_2}\frac{(2e^{-it})^{y_2}-1}{2e^{-it}-1}dt \\ = \int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1}\frac{2i\sin(tx_2)}{2e^{-it}-1}e^{-ity_2}dt \\ - 2^{-y_2}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1}\frac{2i\sin(tx_2)}{2e^{-it}-1}dt\end{aligned}$$

Then, on one hand, the first term goes to 0 as $\mathscr{O}(\frac{1}{y_2^k})$ by lemma A.2.4 — by virtue of the same arguments as for the quantity $I_2(x,y)$ — and on the other hand, the second term goes obviously exponentially fast to 0. Summarising, if y_1 remains bounded we have that

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = \mathscr{O}\left(\frac{1}{y_2^k}\right)$$

where k is non negative and can be arbitrarily large.

Let us deal with the second case, and suppose that y_1 is unbounded. Rewriting the quantity $I_2(x,y)$ by setting $t = \frac{u}{\sqrt{|y_1|}}$, we get

$$\begin{aligned}
&\frac{x_2}{y_1} 2i \int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1} \frac{y_1}{x_2} \sin(x_2 t) \frac{2}{2-e^{-it}} e^{-ity_2} dt \\
&= \frac{x_2}{y_1} 2i \int_{-\pi\sqrt{|y_1|}}^{\pi\sqrt{|y_1|}} F\left(\cos\frac{t}{\sqrt{|y_1|}}\right)^{y_1-x_1} \frac{\sqrt{|y_1|}}{x_2} \sin\left(\frac{x_2 t}{\sqrt{|y_1|}}\right) \frac{2e^{-i\frac{ty_2}{\sqrt{|y_1|}}}}{2-e^{-i\frac{t}{\sqrt{|y_1|}}}} dt.
\end{aligned} \tag{A.13}$$

Therefore,

$$\begin{aligned}
F\left(\cos\frac{t}{\sqrt{|y_1}}\right)^{y_1-x_1} &= \exp\left\{-\frac{3}{2}\frac{y_1-x_1}{y_1}t^2 + \frac{y_1-x_1}{y_1}t^2\epsilon\left(\frac{t^2}{y_1}\right)\right\} \\
&\longrightarrow e^{-\frac{3}{2}t^2} \text{ as } \frac{t^2}{|y_1|} \to 0,
\end{aligned}$$

implying the following pointwise convergence,

$$F\left(\cos\frac{t}{\sqrt{|y_1|}}\right)^{y_1-x_1} \frac{\sqrt{|y_1|}}{x_2} \sin\left(\frac{x_2 t}{\sqrt{|y_1|}}\right) \frac{2}{2-e^{-i\frac{t}{\sqrt{|y_1|}}}} \longrightarrow e^{-\frac{3}{2}t^2} t$$

as $\frac{t^2}{y_1} \to 0$. Let $\epsilon_0 > 0$ such that $\left|\frac{t^2}{y_1}\right| < \epsilon_0$, *i.e.* $\left|\epsilon\left(\frac{t^2}{y_1}\right)\right| \leq \frac{3}{4}$. Then we get the domination

$$\begin{aligned}
&\left|F\left(\cos\frac{t}{\sqrt{|y_1|}}\right)^{y_1-x_1} \frac{\sqrt{|y_1|}}{x_2} \sin\left(\frac{x_2 t}{\sqrt{|y_1|}}\right) \frac{2}{2-e^{-i\frac{t}{\sqrt{|y_1|}}}}\right| \\
&\leq 2Me^{-\frac{3}{2}t^2}|t|e^{\left|\frac{y_1-x_1}{y_1}\right|\left|t^2\right|\left|\epsilon\left(\frac{t^2}{y_1}\right)\right|} \\
&\leq 2Me^{-\frac{3}{8}t^2}|t|.
\end{aligned}$$

Consequently, we can split the integral (A.13) as follows

$$\begin{aligned}
&\frac{x_2}{y_1}2i\int_{-\pi\sqrt{|y_1|}}^{\pi\sqrt{|y_1|}} F\left(\cos\frac{t}{\sqrt{|y_1|}}\right)^{y_1-x_1}\frac{\sqrt{|y_1|}}{x_2}\sin\left(\frac{x_2t}{\sqrt{|y_1|}}\right)\frac{2e^{-i\frac{ty_2}{\sqrt{|y_1|}}}}{2-e^{-i\frac{t}{\sqrt{|y_1|}}}}dt\\
&=\frac{x_2}{y_1}2i\int_{-\epsilon_0\sqrt{|y_1|}}^{\epsilon_0\sqrt{|y_1|}} F\left(\cos\frac{t}{\sqrt{|y_1|}}\right)^{y_1-x_1}\frac{\sqrt{|y_1|}}{x_2}\sin\left(\frac{x_2t}{\sqrt{|y_1|}}\right)\frac{2e^{-i\frac{ty_2}{\sqrt{|y_1|}}}}{2-e^{-i\frac{t}{\sqrt{|y_1|}}}}dt\\
&+\frac{x_2}{y_1}2i\int_{|t|>\sqrt{|y_1|}\epsilon_0} F\left(\cos\frac{t}{\sqrt{|y_1|}}\right)^{y_1-x_1}\frac{\sqrt{|y_1|}}{x_2}\sin\left(\frac{x_2t}{\sqrt{|y_1|}}\right)\frac{2e^{-i\frac{ty_2}{\sqrt{|y_1|}}}}{2-e^{-i\frac{t}{\sqrt{|y_1|}}}}dt\\
&=I_3(x,y)+I_4(x,y).
\end{aligned}$$

The integral $I_3(x,y)$ converges by Lebesgue convergence to the integral

$$\int_{-\infty}^{\infty} e^{-\frac{3}{2}t^2}te^{-it\lambda}dt$$

with $\lambda=\lim\frac{y_2}{\sqrt{|y_1|}}$. And this integral can be easily computed,

$$\int_{-\infty}^{\infty} e^{-\frac{3}{2}t^2}te^{-it\lambda}dt=\frac{i\lambda}{3}\int_{-\infty}^{\infty} e^{-\frac{3}{2}t^2}e^{-it\lambda}dt=\frac{i\lambda}{3}\sqrt{\frac{2\pi}{3}}e^{-\frac{\lambda^2}{6}}.$$

Then substituting λ by the ratio $\frac{y_2^2}{y_1}$ the quantity (12) becomes

$$-\frac{2}{3}\frac{x_2}{y_1}\sqrt{\frac{2\pi}{3}}\frac{y_2}{\sqrt{|y_1|}}e^{-\frac{1}{6}\frac{y_2^2}{y_1}}.$$

We conclude that,

- if $\frac{y_2^2}{y_1}$ goes to 0, then $I_3(x,y)$ behaves like $o\left(\frac{1}{\sqrt{|y_1|}}\right)$;
- if $\frac{y_2^2}{y_1}$ goes to $\pm\infty$, $I_3(x,y)$ behaves like $o\left(\frac{1}{|y_2|}\right)$;
- finally, if $\frac{y_2^2}{y_1}$ converges to λ non zero real, then $I_3(x,y)$ behaves again like $o(\frac{1}{|y_2|})$.

Integrating by parts gives us the following estimate of $I_4(x,y)$,

$$\begin{aligned}
&\left|\frac{x_2}{y_1}2i\int_{|t|>\sqrt{|y_1|}\epsilon_0} F\left(\cos\frac{t}{\sqrt{|y_1|}}\right)^{y_1-x_1}\frac{\sqrt{|y_1|}}{x_2}\sin\left(\frac{x_2t}{\sqrt{|y_1|}}\right)\frac{2e^{-i\frac{ty_2}{\sqrt{|y_1|}}}}{2-e^{-i\frac{t}{\sqrt{|y_1|}}}}dt\right|\\
&\leq\frac{My_1}{y_2}L^{y_1-x_1}
\end{aligned}$$

because,

$$\sup_{|t|>\epsilon_0}\left|\frac{d}{dt}F(\cos(t))\right|<1.$$

As a consequence, the quantity $I_4(x,y)$ behaves like

- $o\left(\frac{1}{|y_2|}\right)$ if $\frac{y_1}{y_2^2}$ converges to a finite limit with y_1 unbounded.
- $o\left(\frac{1}{\sqrt{|y_1|}}\right)$ if $\frac{y_1}{y_2^2}$ goes to $\mathsf{sgn}(t)\infty$.

Turning to the quantity $I_1(x,y)$, we note that

$$\begin{aligned}
I_1(x,y) &= \int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1}2i\sin(tx_2)2^{-y_2}\frac{(2e^{-it})^{y_2}-1}{2e^{-it}-1}dt\\
&= \int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1}2i\sin(tx_2)\frac{e^{-ity_2}-1}{2e^{-it}-1}dt\\
&\quad - 2^{-y_2}\int_{-\pi}^{\pi} F(\cos(t))^{y_1-x_1}2i\sin(tx_2)\frac{1}{2e^{-it}-1}dt\\
&= I_5(x,y)+I_6(x,y).
\end{aligned}$$

The quantity $I_5(x,y)$ can be estimated along the same lines as the quantity $I_2(x,y)$ is whereas the quantity $I_6(x,y)$ behaves like $o\left(\frac{1}{|y_2|}\right)$ in the case where $\frac{y_1}{y_2^2}$ converges to finite limit. It remains to show that $I_6(x,y)$ behaves like $o\left(\frac{1}{\sqrt{|y_1|}}\right)$ in the case $\frac{y_1}{y_2^2}$ goes to infinity. We can estimate the integral $I_6(x,y)$ in the same manner as it has been done for the quantity $I_2(x,y)$ in the case of y_1 unbounded and y_2 fixed. □

Obviously, by symmetry, all these estimations can be made in the case $x_2, y_2 \leq 0$ and $y_1 \leq x_1$. And as soon as, $x_2y_2<0$ then the mean $\mathbf{E}^x(\eta_{0,\tau_1}(y))$ is zero, therefore we get the following.

Corollary A.2.12. *The quantity*

$$\frac{\mathbf{E}^x(\eta_{0,\tau_1}(y))}{G(0,y)}$$

in equation (A.1) goes to 0 *when* $|y|$ *goes to infinity.*

Proof. By propositions A.2.3, A.2.5 and A.2.10, we have that

- $G(o,y)$ is equivalent to $\left(\frac{c}{\sqrt{|y_1|}}\right)$ if $\frac{y_1}{y_2^2}$ goes to infinity ;
- $G(0,y)$ is equivalent to $\left(\frac{c'}{|y_2|}\right)$ if $\frac{y_1}{y_2^2}$ converges to a finite limit.

In the first case, the quantity

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = o\left(\frac{1}{\sqrt{|y_1|}}\right)$$

and in the second case,

$$\mathbf{E}^x(\eta_{0,\tau_1}(y)) = o\left(\frac{1}{|y_2|}\right).$$

Then, obviously, the ratio involved in the corollary converges to 0 in any direction as $|y|$ goes to infinity. □

Proof of the second part of theorem 2.2.4. Since for all $x \in \mathbb{H}$, $K(x, y_k)$ has no other limit point than 1 for all unbounded sequence (y_k) then, the Martin boundary is trivial. □

A.3 Proofs of analytic decompositions

Lemma A.3.1. *The function ϕ is given by*

$$\begin{aligned}\phi(t) &= \frac{1}{p^2}(1 - 2q\cos(t) + q^2\cos(2t)) \\ &- \Bigg[\sqrt{1 - 2q\cos(t) + q^2}\left((p^{-1}-1)^2 - \frac{2q}{p}(p^{-1}-1)\cos(t) + \frac{q^2}{p^2}\right)^{\frac{1}{4}} \\ &\left((p^{-1}+1)^2 - \frac{2q}{p}(p^{-1}+1)\cos(t) + \frac{q^2}{p^2}\right)^{\frac{1}{4}} \cos\Bigg[\arctan\left(\frac{-q\sin(t)}{1-q\cos(t)}\right) \\ &+\frac{1}{2}\arctan\left(\frac{-\sin(t)}{1-\cos(t)}\right) + \frac{1}{2}\arctan\left(\frac{-q\sin(t)}{1+p-q\cos(t)}\right)\Bigg]\Bigg].\end{aligned}$$

Furthermore, in the case of the simple random walk we have $p = 1/3 = 1 - q$, so that

$$\begin{aligned}\phi(t) &= (9 - 12\cos(t) + 4\cos(2t)) \\ &- \Bigg[\frac{\sqrt{13 - 12\cos(t)}}{3} 8^{1/4}\left((1-\cos(t))^{\frac{1}{4}} 4^{1/4}(5 - 4\cos(t))^{\frac{1}{4}}\right. \\ &\cos\Bigg[\arctan\left(\frac{-2\sin(t)}{3-2\cos(t)}\right) + \frac{1}{2}\arctan\left(\frac{-\sin(t)}{1-\cos(t)}\right) \\ &+\frac{1}{2}\arctan\left(\frac{-\sin(t)}{2-\cos(t)}\right)\Bigg]\Bigg].\end{aligned} \tag{A.14}$$

Proof. Denote by z the complex number $z = 1 - qe^{it}$, then we get

$$\phi(t) = \mathsf{Re}\frac{z^2}{p^2} - \frac{z}{p}\sqrt{\frac{z^2}{p^2} - 1}$$

A simple computation gives us that $\mathsf{Re}\frac{z^2}{p^2} = \frac{1}{p^2}(1 - 2q\cos(t) + q^2\cos(2t))$. It remains to make explicit the term with the square root. Start by expanding in polar form,

$$\frac{z}{p}\sqrt{\frac{z^2}{p^2} - 1} = \frac{z}{p}\sqrt{\frac{z}{p} - 1}\sqrt{\frac{z}{p} + 1},$$

then, we have for the modulus of z,

$$|z|^2 = 1 - 2q\cos(t) + q^2,$$

and for its argument

$$\mathsf{Arg}\left(\frac{z}{p}\right) = \arctan\left(\frac{-q\sin(t)}{1 - q\cos(t)}\right).$$

For the modulus and argument of $\frac{z}{p} - 1$

$$\left|\frac{z}{p} - 1\right|^2 = \left|\frac{1}{p} - 1 - \frac{q}{p}e^{it}\right|^2 = (p^{-1} - 1)^2 - \frac{2q}{p}(p^{-1} - 1)\cos(t) + \frac{q^2}{p^2},$$

and

$$\mathsf{Arg}\left(\frac{z}{p} - 1\right) = \arctan\left(\frac{-\sin(t)}{1 - \cos(t)}\right).$$

Finally, we have for $\frac{z}{p} + 1$

$$\left|\frac{z}{p} + 1\right|^2 = \left|\frac{1}{p} + 1 - \frac{q}{p}e^{it}\right|^2 = (p^{-1} + 1)^2 - \frac{2q}{p}(p^{-1} + 1)\cos(t) + \frac{q^2}{p^2},$$

and

$$\mathsf{Arg}\left(\frac{z}{p} + 1\right) = \arctan\left(\frac{-q\sin(t)}{1 + p - q\cos(t)}\right).$$

□

Proof of proposition A.1.6. It is easy to show that

$$\frac{-\sin(t)}{1 - \cos(t)} = -\frac{2}{t}(1 + A_0(t))$$

and that the power series of arctan in the neighborhood of $-\infty$ and $+\infty$ and is given by

$$\arctan(v) = \pm\frac{\pi}{2} - \sum_{n\geq 0}(-1)^n\frac{1}{(2n+1)v^{2n+1}}$$

and the $\pm$ depends on the fact that v is in the neighborhood of $\pm\infty$. Consequently, it gives

$$\arctan\left(\frac{-\sin(t)}{1 - \cos(t)}\right) = \mathsf{sgn}(t)\frac{\pi}{2} - \frac{t}{2}(1 - A_1(t))$$

with A_1 analytic such that $A_1(0) = 0$.

The functions $t \mapsto \arctan\left(\frac{-2\sin(t)}{3-2\cos(t)}\right)$ and $t \mapsto \arctan\left(\frac{-\sin(t)}{2-\cos(t)}\right)$ are analytic in a neighborhood of 0 and vanish for $t = 0$. Thus, the expansion in a power series of the cosine in equation A.14 is given by $\frac{\sqrt{2}}{2}(1 + A_2(t))$ where A_2 is analytic and $A_2(0) = 0$.

The only remaining problematic term is $(1-\cos(t))^{1/4}$ which can rewritten as $\sqrt{|t|}A_3(t)$ with A_3 a power series around 0.

Summarizing, there exists two analytic functions $A_4(t)$ and $A_5(t)$ with $A_4(0) \neq 0$ and $A_5(0) = 0$ such that

$$\phi(t) = 1 - \sqrt{|t|}A_4(t) - A_5(t)$$

and the proposition A.1.6 easily follows. □

Proof of proposition A.2.2. We already know that $g(r(t))$ is given by

$$g(r(t)) = \frac{1 - \sqrt{1 - r(t)^2}}{r(t)} = \frac{1}{r(t)} - \sqrt{\frac{1}{r(t)^2} - 1}.$$

The first term is very easy to decompose

$$\frac{1}{r(t)} = 3 - 2e^{it} = 1 + 2(1 - e^{it}) = 1 - \beta(t)$$

where β is given by $\beta(t) = 2\sum_{n\geq 1} \frac{(it)^n}{n!}$.

The second term with the square root requires a finer analysis. First we have to express the argument of the square in polar form.

$$\frac{1}{r(t)^2} - 1 = (3 - 2e^{it})^2 - 1 = 4(2 - e^{it})(1 - e^{it})$$

Then, we compute the square of the modulus,

$$\left|\frac{1}{r(t)^2} - 1\right|^2 = 32(5 - 4\cos(t))(1 - \cos(t))$$

Thus, the square root of the modulus is given by

$$\sqrt{\left|\frac{1}{r(t)^2} - 1\right|} = 2\sqrt{|t|}(1 + A_0(t))$$

where $A_0(t)$ is an analytic function satisfying $A_0(0) = 0$.

Let us now decompose the argument of the complex function $r(t)^{-2} - 1$,

$$\arg\left(\frac{1}{r(t)^2} - 1\right) = \arctan\frac{-\sin(t)}{2 - \cos(t)} + \arctan\frac{-\sin(t)}{1 - \cos(t)}.$$

The first term $\arctan\frac{-\sin(t)}{2-\cos(t)}$ is analytic as the composition of two analytic functions.

For the second term, we compute as in the proof of the proposition A.1.6

$$\arctan\left(\frac{-\sin(t)}{1-\cos(t)}\right) = \mathsf{sgn}(t)\frac{\pi}{2} - \frac{t}{2}(1 - A_1(t))$$

with A_1 analytic such that $A_1(0) = 0$.

Finally we get the following decomposition,

$$\sqrt{\frac{1}{r(t)^2} - 1} = \sqrt{|t|}(1 + A_0(t))e^{i\mathsf{sgn}(t)\frac{\pi}{4}}A_7(t)$$

with $A_0(0) = 0$ and $A_2(0) = 1$ and letting $\alpha(t) = A_0(t)A_2(t)e^{\mathsf{sgn}(t)i\frac{\pi}{4}}$, the proposition is proved. $\square$

Appendix B

Poisson boundary : the category point of view

B.1 Measurable partitions in Lebesgue spaces

Let $(X, \mathcal{X}, \mu)$ be a measure space. A collection $\alpha = (A_i)_{i \in I}$ of pairwise disjoint measurable subsets of X is a partition if their union covers X. Subsets which are unions of elements of the partition α are called α-sets.

Let α be a partition of X, and denote by X/α the factor set and by $\pi : X \to X/\alpha$ the canonical map which sends every point $x \in X$ on its equivalence class. Endow the set X/α with the σ-algebra $\mathcal{X}_\alpha$ generated by π, i.e the smallest one for which the map becomes measurable. We may define the measure μ_α on this σ-algebra as the image of the measure μ by the map π, namely $\mu_\alpha = \mu \circ \pi^{-1}$. Thus, the space $(X/\alpha, \mathcal{X}_\alpha, \mu_\alpha)$ is a measure space and the map π is measurable and measure preserving. Note that a subset $\Lambda \subset X/\alpha$ is $\mathcal{X}_\alpha$-measurable if and only if

$$\bigcup_{y \in \Lambda} \pi^{-1} y$$

is $\mathcal{X}$-measurable and $\pi^{-1}y$ is an element of the partition α.

Suppose we are given a countable family $S = (S_n)$ of measurable sets $S_n \in \mathcal{X}$. For every sequence $\omega = (\omega_n) \in \{0,1\}^\infty$, let $S_n(\omega_n) = S_n$ if $\omega_n = 1$ and $S_n(\omega_n) = X \setminus S_n$ if $\omega_n = 0$. Let us consider the set $\bigcap_{n=0}^\infty S_n(\omega)$. It is clear that the obtained sets (we take into account only nonempty ones) form a partition, which is denoted by $\alpha(S)$.

A partition α is called measurable if it has the form $\alpha = \alpha(S)$ for some at most countable family S of $\mathcal{X}$-measurable sets.

We shall say that the measure space $(X, \mathcal{X}, \mu)$ has a countable basis $\{B_n\}$ if

1. the sets $B_n \in \mathcal{X}$ separate the points in X, that is, for two distinct points x and y there exists B_n such that either $x \in B_n$ and $y \notin B_n$ or $x \notin B_n$ and $y \in B_n$,

2. the completion of $\sigma(B_n, n \geq 0)$ with respect to μ coincides with the completion of $\mathcal{X}$ with respect to μ, i.e $\sigma(B_n, n \geq 0)_\mu = \mathcal{X}_\mu$.

As above, consider a sequence $\omega = (\omega_n) \in \{0,1\}^\infty$, and define

$$E_\omega = \bigcap_{n\geq 0} B_n(\omega_n).$$

The basis $\{B_n\}$ separate the points so that each E_ω contains at most one point. The space X is said to be complete if each E_ω is non-empty.

Definition B.1.1. The measure space $(X, \mathcal{X}, \mu)$ is called Lebesgue if it is isomorphic mod 0 to a measure space $(X', \mathcal{X}', \mu')$ with a countable basis with respect to which it is complete.

Countable Cartesian products of Lebesgue spaces are Lebesgue, and the factor space of a Lebesgue space with respect to a measurable partition is a Lebesgue space. More details about Lebesgue space are given in section 9.4 of [Bog07]. As examples of Lebesgue space, there are Polish spaces with a Radon measure and countable spaces as shown below.

Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. For each integer n we denote by $\omega = (\omega_k)$ its binary expansion

$$n = \sum_{k=0}^{\infty} \omega_k 2^k$$

and define the set $C_k = \{\omega : \omega_k = 1\}$. Then, (C_k) is a countable basis which separates the integers. This basis is furthermore complete, indeed

$$\bigcap_{k\geq 0} C_k(\omega_k) = \{\omega\}.$$

Thus, $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is a Lebesgue space. Since every countable measure space $(X, \mathcal{P}(X), \mu)$ is isomorphic to the integers it is a Lebesgue space.

Let α be an arbitrary partition of the Lebesgue space $(X, \mathcal{X}, \mu)$. Let us suppose that by introducing certain measures μ_A, the elements A of this partition themselves are turned into measure spaces. We say that the system $\{\mu_A\}$ is a system of regular conditional measures with respect to α if

1. μ_A is a Lebesgue measure for μ_α-a.e point $A \in X/\alpha$,
2. for every measurable set $B \subset X$,
   (a) the set $B \cap A$ is measurable in its space A for μ_α-a.e $A \in X/\alpha$,
   (b) $\mu_A(B \cap A)$ is a measurable function of the point $A \in X/\alpha$,
   (c) and,
   $$\mu(B) = \int_{X/\alpha} \mu_\alpha(dA)\mu_A(B \cap A).$$

A partition α of a Lebesgue space is measurable if and only if there exists a system of regular conditional measures, and in such a case, it is essentially unique (see [Roh49]). A partition in a countable space is at most countable, consequently such partitions are measurable.

Finally, denote by Z the set of partitions of a set X, and let $\alpha, \beta \in Z$ then α is said to be finer than β if and only if for every element $A \in \alpha$ there exists an element $B \in \beta$ such that $A \subset B$. This fact is denoted by $\beta \preceq \alpha$. Thus, $(Z, \preceq)$ becomes a partially ordered set. For this partial order, there exists a coarsest partition given by the partition consisting of one element, the set X itself, and a finest partition, the partition into individual points of X. For any subfamily Y of partitions, it is well known that its greatest lower bound, denoted by $\bigwedge_{\alpha \in Y} \alpha$, and its least upper bound, denoted by $\bigvee_{\alpha \in Y} \alpha$, exist. Thus, the set Z is endowed with a complete lattice structure.

Denote by Z_X the collection of measurable partitions of the Lebesgue space $(X, \mathcal{X}, \mu)$. For sure, Z_X is a subset of Z and we can define a partial order as previously. However, generally speaking, Z_X is not a substructure of the structure Z. More precisely, the least upper bound of an at most countable subfamily of partitions always exists, but the greatest lower bound of even two partitions is not always measurable. That is why we have to introduce the notion of identical mod 0 partition. Two measurable partitions α and β are said identical mod 0 if we can find a subset X' of full measure such that restricted to this set, α and β are equal. Denote by $\bar{Z}_X$ the set of classes of measurable partition identical mod 0, then endowed with the partial order $\preceq$ (which is naturally extended to $\bar{Z}_X$), the partially ordered set $(\bar{Z}_X, \preceq)$ is a complete lattice (see [Roh49]).

From now, we will consider only measurable partitions mod 0.

B.2 Tail boundary of Markov chains

B.2.1 Tail boundary in the category of measure spaces

Let $(X, \mathcal{X}, m)$ be a measure space. A linear operator $P : \mathbb{L}^\infty(X, m) \to \mathbb{L}^\infty(X, m)$ is called Markov if $P1 = 1$ and $Pf \geq 0$ as soon as $f \geq 0$. Let θ be a probability measure on X supposed absolutely continuous with respect to m.

Suppose that the path space $(X^\infty, \mathcal{X}^{\otimes \mathbb{N}}, \mathbf{P}^\theta)$ is a Lebesgue space and denote by α_k the time k-coordinate partition so that two paths y and y' belong to the same element of the partition α_k iff $y_k = y'_k$; denote by $\alpha_k^n = \bigvee_{i=k}^n \alpha_i$ the supremum of the α_i (for $\preceq$). Here, n is allowed to be ∞. Partitions α_k are not always measurable, even in a Lebesgue space. In the sequel we will always assume that the partition α_k is measurable for each $k \geq 0$. Note that this property is satisfied in the case of a countable set X with the counting measure m.

Denote by $(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty})$, the factor space of X^∞ with respect to α_n^∞ and by pr_n the factor map. Note that the measure $\mu_{\alpha_n^\infty}$ depends on the initial distribution θ and thus we should write $\mu^\theta_{\alpha_n^\infty}$. However, for the construction of the tail boundary, the distribution θ is fixed once for all. For the Poisson boundary, we will consider different initial distributions

so that the notation will make it explicit.

The sequence of partitions $(\alpha_n^\infty)_{n\geq 0}$ is decreasing, thus, for all $m \geq n$, if two trajectories $x, y \in X^\infty$ are α_n^∞-equivalent, then they are α_m^∞-equivalent. Consequently, we can define a map $\pi_n^m : X^\infty/\alpha_n^\infty \to X^\infty/\alpha_m^\infty$ verifying the relation $\pi_n^m \circ \mathsf{pr}_n = \mathsf{pr}_m$. Moreover, this map is unique, if not, there exists a map $\tilde{\pi}_n^m$ (which verifies the previous relation) and a point $y_0 \in X^\infty/\alpha_n^\infty$ such that $\tilde{\pi}_n^m(y_0) \neq \pi_n^m(y_0)$. But, this implies that $\mathsf{pr}_m^{-1}(\tilde{\pi}_n^m y_0)$ and $\mathsf{pr}_m^{-1}(\pi_n^m y_0)$ are two distinct elements of the partition α_m^∞, which is a contradiction.

The map π_n^m is measurable, indeed for each measurable subset $\Lambda \subset X^\infty/\alpha_m^\infty$,

$$\mathsf{pr}_n^{-1}((\pi_n^m)^{-1}\Lambda) = (\pi_n^m \circ \mathsf{pr}_n)^{-1}\Lambda = \mathsf{pr}_m^{-1}\Lambda$$

and we conclude by noting that the two σ-algebras $\mathcal{X}_{\alpha_n^\infty}$ and $\mathcal{X}_{\alpha_m^\infty}$ are respectively generated by the maps pr_n and pr_m.

Remark that the map π_n^m is well defined. If ξ is a partition identical to α_n^∞ mod 0 then the corresponding map π_ξ is equal almost surely to π_n^m.

We may check that $(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \pi_n^m)_{n\leq m}$ is an inductive system in the category Meas of measurable spaces with measurable mappings as morphisms.

Denote by α^∞ the infimum of the α_n^∞ : $\alpha^\infty = \bigwedge_{n\geq 0} \alpha_n^\infty$, and denote by $(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu)$ the factor space of X^∞ with respect to α^∞ and by pr_∞ the factor map.

In the same way we have defined the map π_n^m, we can define a map $\pi_n : X^\infty/\alpha_n^\infty \to X^\infty/\alpha^\infty$ verifying $\pi_n \circ \mathsf{pr}_n = \mathsf{pr}_\infty$, indeed, again two trajectories x, y which are α_n^∞-equivalent are α^∞-equivalent because $\alpha^\infty \preceq \alpha_n^\infty$.

Lemma B.2.1. *For $m \geq n$, we have $\pi_n = \pi_m \circ \pi_n^m$.*

Proof. The relation is obvious because two α_n^∞-equivalent trajectories are α_m^∞-equivalent and two α_m^∞-equivalent trajectories are α^∞-equivalent. □

Lemma B.2.2. *The map π_n is measurable.*

Proof. It is the same proof as for the measurability of π_n^m replacing m by ∞. □

As a consequence,

Lemma B.2.3. *The σ-algebra $\mathcal{X}_{\alpha^\infty}$ and $\sigma(\pi_n, n \geq 0)$ are equal.*

Proof. The measurability of π_n for all n implies that $\mathcal{X}_{\alpha^\infty}$ is a sub-σ-algebra of $\sigma(\pi_n, n \geq 0)$.

Conversely, let $\Lambda \in \sigma(\pi_n, n \geq 0)$, then for each n, $\pi_n^{-1}\Lambda \in \mathcal{X}_{\alpha_n^\infty}$, that is $\mathsf{pr}_n^{-1}(\pi_n^{-1}\Lambda) = \mathsf{pr}_\infty^{-1}\Lambda \in \mathcal{X}^{\otimes\mathbb{N}}$. Therefore, Λ is $\mathcal{X}_{\alpha^\infty}$-measurable. □

Consider the subset $S \subset \prod_{n\geq 0} X^\infty/\alpha_n^\infty$ of admissible sequences, i.e such that $\pi_n^m(a_{\geq n}) = a_{\geq m}$. By definition, each point $a_{\geq n}$ corresponds to an element of the partition $A_n \in \alpha_n^\infty$, namely $\mathsf{pr}_n^{-1} a_{\geq n}$. To an admissible sequence $(a_{\geq n})_{n\geq 0}$ corresponds a sequence of subsets $(A_n)_{n\geq 0}$ with $A_n \in \alpha_n^\infty$, furthermore, this sequence A_n is increasing because partitions α_n^∞ are decreasing. We will say that an admissible sequence $(a_{\geq n})_{n\geq 0}$ converges to a point $a \in X^\infty/\alpha^\infty$ if the corresponding increasing sequence of subsets $(A_n)_{n\geq 0}$ is such that $A = \bigcup_{n\geq 0} A_n$ where $A \in \alpha^\infty$ corresponds to the point a.

Lemma B.2.4. *Let A be an element of α^∞. Then there exists an increasing sequence $(A_n)_{n\geq 0}$ of subsets of X^∞ verifying $A_n \in \alpha_n^\infty$ and $A = \bigcup_{n\geq 0} A_n$.*

Proof. The infimum α^∞ verifies by definition that for all decompositions β such that $\alpha^\infty \preceq \beta$ there exists an integer n such that $\alpha^\infty \preceq \alpha_n^\infty \preceq \beta$. Let A be an element of α^∞, then for all subsets $B \subset A$, one defines the decomposition

$$\beta_B = \alpha^\infty \vee \alpha(B) = \{C \cap B : C \in \alpha^\infty\} \cup \{C \cap (A \setminus B) : C \in \alpha^\infty\}.$$

Obviously, β_B is finer than α^∞, so that there exist an integer n and an element D_n of α_n^∞ such that $B \subset D_n \subset A$. Moreover, we may inductively define a sequence $D_{n+1}, D_{n+2}, \ldots$ with $D_k \in \alpha_k^\infty$: D_{n+1} is the unique element of α_{n+1}^∞ containing D_n, and so on. Consequently, we obtain

$$B \subset \bigcup_{k\geq n} D_k \subset A.$$

□

Thus, for each point $a \in X^\infty/\alpha^\infty$, there exist an admissible sequence $(a_{\geq n})_{n\geq 0} \in S$ which converges to a. The following lemma says that if two admissible sequences converge to the same point $a \in X^\infty/\alpha^\infty$ then they are eventually equal.

Lemma B.2.5. *Assume that $(a_{\geq n})_{n\geq 0}, (b_{\geq n})_{n\geq 0} \in S$ converge to the same point $a \in X^\infty/\alpha^\infty$, then there exists an integer n_0 such that $a_{\geq n_0} = b_{\geq n_0}$.*

Proof. Let $(a_{\geq n})_{n\geq 0}$ and $(b_{\geq n})_{n\geq 0}$ be two admissible sequences converging to the same point $a \in X^\infty/\alpha^\infty$. Denote by $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ the corresponding increasing sequences of subsets such that

$$A = \bigcup_{n\geq 0} A_n = \bigcup_{n\geq 0} B_n$$

where $A \in \alpha^\infty$ corresponds to the point a. We claim that there exists an integer n, such that $A_n = B_n$.

By contradiction, suppose that for all n, $A_n \neq B_n$, then

$$\bigcup_{n\geq 0} (A_n \cap B_n) = \emptyset$$

because A_n and B_n are distinct elements of the same partition.

Now, we have obviously that

$$\bigcup_{n\geq 0} (A_n \cap B_n) \subset \left(\bigcup_{n\geq 0} A_n\right) \cap \left(\bigcup_{n\geq 0} B_n\right)$$

and the converse inclusion holds as well. Indeed, let $u \in \left(\bigcup_{n\geq 0} X_n\right) \cap \left(\bigcup_{n\geq 0} Y_n\right)$ then, there exists n_0 and n_1 such that $u \in A_{n_0}$ and $u \in B_{n_1}$. If $n_0 \leq n_1$ then $A_{n_0} \subset A_{n_1}$ so that $u \in A_{n_1} \cap B_{n_1}$. So $u \in \bigcup_{n\geq 0}(X_n \cap Y_n)$. Clearly, if $n_1 \leq n_0$, we can do the converse.

Consequently, we get a contradiction

$$A = \left(\bigcup_{n\geq 0} A_n\right) \cap \left(\bigcup_{n\geq 0} B_n\right) = \emptyset.$$

□

We shall say that two admissible sequences are equivalent if and only if they are eventually equal. Denote by $\sim$ this relation, and by $S/\sim$ the factor set of S with respect to $\sim$. Then, lemma B.2.4 and B.2.5 tell us that $S/\sim$ and X^∞/α^∞ are in bijection.

Theorem B.2.6. *Let $(Y, \mathcal{Y})$ be a measurable space, and for $n \geq 0$, $\phi_n : X^\infty/\alpha_n^\infty \to Y$ be a sequence of measurable maps verifying $\phi_m \circ \pi_n^m = \phi_n$ for all $m \geq n$. Then, there exists a unique measurable map $\phi : X^\infty/\alpha^\infty \to Y$ such that $\phi_n = \phi \circ \pi_n$.*

Proof. Define the map $\psi : S \to Y$ by $\psi((a_{\geq n})_{n\geq 0}) = \phi_0(a_{\geq 0})$ where $(a_{\geq n})_{n\geq 0}$ is an admissible sequence. We claim that the map ψ can be factorized in a map $\bar{\psi} : S/\sim \to Y$. Indeed, let a, b be two $\sim$-equivalent admissible sequences, then there exist an integer n_0 such that $a_{\geq n_0} = b_{\geq n_0}$. Thus,

$$\begin{aligned} \phi_{n_0}(a_{\geq n_0}) &= \phi_{n_0}(b_{\geq n_0}) \\ \phi_{n_0}(\pi_0^{n_0} a_{\geq 0}) &= \phi_{n_0}(\pi_0^{n_0} b_{\geq 0}) \\ \phi_0(a_{\geq 0}) &= \phi_0(b_{\geq 0}). \end{aligned}$$

Since the sets $S/\sim$ and X^∞/α^∞ are in bijection, the map $\bar{\psi}$ naturally defines a map $\phi : X^\infty/\alpha^\infty \to Y$.

Let $a_{\geq 0}$ be a point of X^∞/α_0^∞, then the sequence $(\pi_0^n a_{\geq 0})_{n\geq 0}$ is of course admissible and

$$\phi \circ \pi_0(a_{\geq 0}) = \bar{\psi}((\pi_0^n a_{\geq 0})_{n\geq 0}) = \phi_0(a_{\geq 0}).$$

Now, let $a_{\geq n} \in X^\infty/\alpha_n^\infty$ and choose $a_{\geq 0} \in X^\infty/\alpha_0^\infty$ such that $\pi_0^n a_{\geq 0} = a_{\geq n}$. Then,

$$\phi \circ \pi_n(a_{\geq n}) = \phi \circ \pi_0(a_{\geq 0}) = \phi_0(a_{\geq 0}) = \phi_n(\pi_0^n a_{\geq 0}) = \phi_n(a_{\geq n}),$$

thus, $\phi \circ \pi_n = \phi_n$.

Actually, what we have just showed is that the inductive limit in the category Set of sets with maps as morphisms of the inductive system $(X^\infty/\alpha_n^\infty, \pi_n^m)_{n\leq m}$ is identified with $(X^\infty/\alpha^\infty, (\pi_n)_{n\geq 0})$.

By the surjectivity of π_n, the map ϕ is unique, and its measurability is a consequence of the lemma B.2.3. □

All those results imply that $(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, (\pi_n)_{n\geq 0})$ is the inductive limit in the category Meas of the system $(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \pi_n^m)_{m\geq n}$, in symbols

$$(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, (\pi_n)_{n\geq 0}) = \mathsf{Meas} - \varinjlim_{n\in\mathbb{N}} (X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \pi_n^m).$$

Let $\mu_{\alpha_n^\infty}$ be the measure on X^∞/α_n^∞ defined as above. For a measurable set Λ of X^∞/α_m^∞, we have the relation $\mu_{\alpha_m^\infty}(\Lambda) = \mu_{\alpha_n^\infty} \circ (\pi_n^m)^{-1}(\Lambda)$. Indeed,

$$\mu_{\alpha_n^\infty} \circ (\pi_n^m)^{-1}\Lambda = \mathbf{P}_\theta \circ \mathsf{pr}_n^{-1} \circ (\pi_n^m)^{-1}\Lambda = \mathbf{P}_\theta \circ \mathsf{pr}_m^{-1}\Lambda = \mu_{\alpha_m^\infty}\Lambda.$$

For all measurable subsets Λ of X^∞/α^∞, the sequence $(\mu_{\alpha_n^\infty} \circ \pi_n^{-1}\Lambda)_{n\geq 0}$ is obviously constant and equal to $\mu(\Lambda)$, indeed

$$\mu_{\alpha_n^\infty} \circ \pi_n^{-1}\Lambda = \mathbf{P}_\theta \circ \mathsf{pr}_n^{-1} \circ \pi_n^{-1}\Lambda = \mathbf{P}_\theta \circ \mathsf{pr}_\infty^{-1}\Lambda = \mu(A).$$

Endowed with μ, X^∞/α^∞ is a measure space, and we only have to check that

$$(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu, (\pi_n)_{n\geq 0})$$

is the inductive limit of the inductive system $(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty}, \pi_n^m)_{m\geq n}$ in the category of measure spaces with measurable and measure preserving maps as morphisms, denoted by M. Regarding what has been already done, we have to check that $\pi_n = \pi_m \circ \pi_n^m$ and show a similar version of theorem B.2.6.

Clearly, the following equality holds for all measurable subsets Λ of X^∞/α^∞

$$\mu_{\alpha_n^\infty} \circ \pi_n^{-1}\Lambda = \mu_{\alpha_n^\infty} \circ (\pi_n^m)^{-1} \circ \pi_m^{-1}\Lambda = \mu_{\alpha_m^\infty} \circ \pi_m^{-1}\Lambda.$$

Thus, $\pi_n = \pi_m \circ \pi_n^m$.

Theorem B.2.7. *Let $(Y, \mathcal{Y}, \nu)$ be a measure space, and for $n \geq 0$, $\phi_n : X^\infty/\alpha_n^\infty \to Y$ be a sequence of measurable and measure preserving maps verifying $\phi_m \circ \pi_n^m = \phi_n$ for all $m \geq n$. Then there exists a unique measurable and measure preserving map $\phi : X^\infty/\alpha^\infty \to Y$ such that $\phi_n = \phi \circ \pi_n$.*

Proof. From theorem B.2.6, we have the existence of a unique map ϕ, so that we only have to show that ϕ is measure preserving, that is

$$\mu \circ \phi^{-1} = \nu.$$

It is a matter of fact that $\mu_{\alpha_n^\infty} \circ \phi_n^{-1} = \nu$, thus it follows that $\mu_{\alpha_n^\infty} \circ \pi_n^{-1} \circ \phi^{-1} = \nu$. By definition, the left hand side of this equality is nothing else than $\mu \circ \phi^{-1}$. □

$$\begin{tikzcd}
X^\infty/\alpha_n^\infty \arrow[rr, "\pi_n^m"] \arrow[dr, "\pi_n"] \arrow[ddr, bend right, "\phi_n"'] & & X^\infty/\alpha_m^\infty \arrow[dl, "\pi_m"'] \arrow[ddl, bend left, "\phi_m"] \\
& X^\infty/\alpha^\infty \arrow[d, dashed, "\exists!\phi"] & \\
& Y &
\end{tikzcd}$$

B.2.2 The tail boundary in the category of Banach spaces

In some sense, we can get the same results by "duality" for random variables. Let $\mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty})$, $p \in [1, +\infty]$, be the space of integrable real random variables. Defining $\rho_m^n(g_m) = g_m \circ \pi_n^m$, we check that this map is well defined because it does not depend on the choice of the representative g_m, and that

$$(\mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty}), \rho_m^n)_{n \leq m}$$

is a projective system in the category Ban of Banach spaces with linear isometries as morphisms.

Our goal is to describe the projective limit of this system in terms of the inductive limit we studied in the last section.

Consider the Banach space $\mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu)$, and define

$$\rho_n : \mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu) \to \mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty})$$

by $\rho_n(X) = g \circ \pi_n$. The map ρ_n is a linear isometry of Banach spaces verifying $\rho_n \circ \rho_m^n = \rho_m$ for all $n \leq m$. Our aim is to show that the space $\mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu)$ is in fact the projective limit of the above projective system.

In a general setup, let $(B_n, h_m^n)_{n \leq m}$ be a projective system in the category Ban_1 of Banach spaces with linear contractions as morphisms. It is well known (see [CLM79] for more details) that this projective system always admits a projective limit. Furthermore, it is identified with the closed subset of $\Pi_{n \geq 0} B_n$ (for the sup norm $\|x\| = \sup_{n \geq 0} \|x_n\|_{B_n}$) consisting of all sequences $x = (x_n)_{n \geq 0}$ such that $h_m^n(x_m) = x_n$. Denoting this subset by L, and defining h_n by $h_n(x) = x_n$, we get

$$(L, h_n) = \mathsf{Ban}_1 - \varprojlim_{n \in \mathbb{N}} (B_n, h_m^n)_{n \leq m}.$$

Note that L is effectively a closed subset, because it may be written in the form

$$L = \bigcap_{n \leq m} \mathsf{Ker}(h_n - h_m^n \circ h_m).$$

Furthermore, the relation $h_m^n \circ h_m = h_n$ holds by definition.

For a given family $l_n : Z \to B_n$ of linear contractions satisfying $h_m^n \circ l_n = l_m$ for all $n \leq m$, we consider the map $\Pi l_n : Z \to \Pi B_n$. It is clear that its image belongs to L, so that it can be uniquely factored over L with the stated properties.

If we consider, now, that $(B_n, h_m^n)_{n \leq m}$ is a projective system in the category of Banach spaces with linear isometries as morphisms, then (L, h_n) is again the projective limit of this system. Indeed, it is obvious that the h_n are linear isometries. Moreover, if the l_n are assumed to be linear isometries, it is also the case for Πl_n.

From now, we assume that $B_n = \mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty})$, $\rho_m^n = h_m^n$, $Z = \mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu)$, and $l_n = \rho_n$. Then there exists a linear isometry

$$\rho : \mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu) \to L$$

such that $\rho_n = h_n \circ \rho$. We aim to show that the linear isometry ρ is in fact an isomorphism, that is, a linear bijective map such that the inverse map ρ^{-1} is itself a linear isometry. But in fact, if ρ is bijective, then ρ^{-1} is a linear isometry. On another hand, ρ is injective as a linear isometry, so that we only have to show the surjectivity.

Let $(g_n)_{n\geq 0} \in L$ and let $\tilde{g}_n \in \mathcal{L}^p(X^\infty/\alpha_n^\infty, \mathcal{A}_n, \mu_{\alpha_n^\infty})$ be a version of g_n. The lemma 3.1 of [Mac89] — see also the lemma 4.1 of [Mac95] — says that there exists an element $\tilde{g} \in \mathcal{L}^p(X^\infty/\alpha^\infty, \mathcal{A}, \mu)$ such that $\tilde{g} \circ \pi_n = \tilde{g}_n$ and $\|\tilde{g}\| = \sup_{n\geq 0} \|\tilde{g}_n\|$. Let $\tilde{g}'_n$ be another version of g_n, then $\tilde{g}' = \tilde{g}$ μ-a.e. Indeed, denote by N the set

$$N = \{x \in X^\infty/\alpha^\infty : \tilde{g}(x) \neq \tilde{g}'(x)\}$$

by N_n the set

$$N_n = \{x \in X^\infty/\alpha_n^\infty : \tilde{g}_n(x) \neq \tilde{g}'_n\}$$

and let $x \in N$, thus $\tilde{g}_(x) \neq \tilde{g}'(x)$ and for all integer n and all points $y \in \pi_n^{-1}x$ we have $\tilde{g} \circ \pi_n(y) \neq \tilde{g}' \circ \pi_n(y)$, that is $\tilde{g}_n(y) \neq \tilde{g}'(y)$. Consequently, the set $\pi_n^{-1}x$ is a subset of N_n and it follows for all integers n that $\pi_n^{-1}(N) \subset N_n$. Thus, we get for all n

$$\mu_{\alpha_n^\infty}(\pi_n^{-1}N) = \mu(N) \leq \mu_{\alpha_n^\infty}(N_n) = 0$$

We conclude that for all sequences $(g_n)_{n\geq 0} \in L$ there exists an element $g \in \mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{A}, \mu)$ such that $\rho(g) = (g_n)_{n\geq 0}$ because $\rho_n(g) = g_n$ for all n.

Thus, we have the following result,

Theorem B.2.8.
Let $(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty}, \pi_n^m)_{n\leq m}$ be an inductive system and $(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu, (\pi_n)_{n\geq 0})$ its inductive limit defined in the last section. Then

$$(\mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu), (\rho_n)_{n\geq 0}) \cong \varprojlim_{n\in\mathbb{N}} (\mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty}), \rho_m^n)_{n\leq m}.$$

$$\begin{tikzcd}
\mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty}) & & \mathbb{L}^p(X^\infty/\alpha_m^\infty, \mathcal{X}_{\alpha_m^\infty}, \mu_{\alpha_m^\infty}) \arrow[ll, "\rho_m^n"'] \\
& L \arrow[ul, "h_n"] \arrow[ur, "h_m"'] & \\
& \mathbb{L}^p(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu) \arrow[u, dashed, "\exists!\rho"] \arrow[uul, "\rho_n"] \arrow[uur, "\rho_m"'] &
\end{tikzcd}$$

We can remark that the range of ρ_n^0 is isometrically isomorphic to the subspace $E_n = \{f \in \mathbb{L}^p(X^\infty/\alpha_0^\infty, \mathcal{X}_{\alpha_0^\infty}, \mu_{\alpha_0^\infty}) : f \text{ is } \sigma(\pi_0^n) - \text{measurable}\}$.

$$\begin{tikzcd}
\mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty}) \arrow[r, "\rho_n^0"] & E_n \subset \mathbb{L}^p(X^\infty/\alpha_0^\infty, \mathcal{X}_{\alpha_0^\infty}, \mu_{\alpha_0^\infty}) \arrow[d, "\mathbf{E}(\cdot|\pi_0^{n+1})"] \\
\mathbb{L}^p(X^\infty/\alpha_{n+1}^\infty, \mathcal{X}_{\alpha_{n+1}^\infty}, \mu_{\alpha_{n+1}^\infty}) \arrow[u, "\rho_{n+1}^n"] \arrow[r, "\rho_{n+1}^0"'] & E_{n+1} \subset \mathbb{L}^p(X^\infty/\alpha_0^\infty, \mathcal{X}_{\alpha_0^\infty}, \mu_{\alpha_0^\infty})
\end{tikzcd}$$

Indeed, for all $g \in \mathbb{L}^p(X^\infty/\alpha_{n+1}, \mathcal{X}_{\alpha_{n+1}^\infty}, \mu_{\alpha_{n+1}^\infty})$, the following equality holds by $\sigma(\pi_0^{n+1})$-measurability of $\rho_{n+1}^0(g)$:

$$\mathbf{E}(\rho_n^0 \circ \rho_{n+1}^n(g)|\pi_0^{n+1}) = \mathbf{E}(\rho_{n+1}^0(g)|\pi_0^{n+1}) = \rho_{n+1}^0(g).$$

Consider a function $g \in \mathbb{L}^p(X^\infty/\alpha_0^\infty, \mathcal{X}_{\alpha_0^\infty}, \mu_{\alpha_0^\infty})$, then the sequence $(\mathbf{E}(g|\pi_0^n))_{n\geq 0}$ is a backward martingale. It converges almost surely to the integrable function $\mathbf{E}(g|\pi_0^\infty)$.

Let $f : X^\infty/\alpha_{n+1}^\infty \to \mathbb{R}$ be a bounded measurable function, and $g : X^\infty/\alpha_n^\infty \to \mathbb{R}$. Since the state space X is Lebesgue and the partitions are assumed to be measurable, there exists a system of regular conditional probabilities $\mu_{\alpha_n^\infty}^t$ and

$$\int_{X^\infty/\alpha_n^\infty} f \circ \pi_n^{n+1}(x) g(x) d\mu_{\alpha_n^\infty}(x) = \int_{X^\infty/\alpha_{n+1}^\infty} f(t) \underbrace{\int_{X^\infty/\alpha_n^\infty} g(x) d\mu_{\alpha_n^\infty}^t(x)}_{Q_n g(t)} d\mu_{\alpha_{n+1}^\infty}(t)$$

which gives rise to an operator

$$Q_n : \mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty}) \to \mathbb{L}^p(X^\infty/\alpha_{n+1}^\infty, \mathcal{X}_{\alpha_{n+1}^\infty}, \mu_{\alpha_{n+1}^\infty})$$

defined for $g \in \mathbb{L}^p(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu_{\alpha_n^\infty})$ by

$$Q_n g(t) = \int_{X^\infty/\alpha_n\infty} g(x) d\mu_{\alpha_n^\infty}^t(x).$$

In the next section we will show that this operator can be approximated.

B.2.3 Approximation of the operator Q_n

If we consider a function $g \in \mathbb{L}^p(X^\infty/\alpha_0^\infty, \mu_{\alpha_0^\infty})$ the function

$$Q_k \cdots Q_0 g$$

belongs to the space $\mathbb{L}^p(X^\infty/\alpha_n^\infty, \mu_{\alpha_n^\infty})$. Note that, if $g \in \mathbb{L}^p(\mathbb{X}^\infty/\alpha_n^\infty, \mu_{\alpha_n^\infty})$, it is obvious that

$$Q_n g(t) = \mathbf{E}_{\mu_{\alpha_n^\infty}}(g|\pi_n^{n+1} = t) \text{ for } \mu_{\alpha_{n+1}^\infty} - \text{almost all } t \in X^\infty/\alpha_{n+1}^\infty.$$

In the next paragraph, we present some results (which can be found in [Rao05]) which allow approximations of such conditional expectation.

Differentiation, conditioning and computation

Let us recall the general definition of conditional expectation. Let (Ω, Σ, μ) be measure space. For us, μ will be a probability, but the following results can be generalized to any σ-finite measure.

Suppose that $f : \Omega \mapsto \mathbb{R}$ is an integrable function on Ω and consider any sub-σ-algebra $\mathcal{B} \subset \Sigma$. Then the mapping $(A \in \mathcal{B})$

$$\nu_f : A \mapsto \int_A f d\mu$$

is a signed measure on $\mathcal{B}$ which vanishes on μ-null sets of $\mathcal{B}$, so that ν_f is absolutely continuous with respect to $\mu_{\mathcal{B}}$, the restriction of the measure μ on the σ-algebra $\mathcal{B}$. Hence by the Radon-Nikodym theorem, there exists a $\mu_{\mathcal{B}}$-unique integrable function $\tilde{f}$ on $(\Omega, \mathcal{B}, \mu_{\mathcal{B}})$ such that for all $A \in \mathcal{B}$

$$\nu_f(A) = \int_A \tilde{f} d\mu_{\mathcal{B}}$$

The function $\tilde{f}$ is called the conditional expectation of f with respect to $\mathcal{B}$, often denoted by $\mathbf{E}(f|\mathcal{B})$. Thus, the conditional expectation $\mathbf{E}(f|\mathcal{B})$ is nothing but the Radon-Nikodým's derivative of the measure ν_f with respect to $\mu_{\mathcal{B}}$, namely

$$\mathbf{E}(f|\mathcal{B}) = \frac{d\nu_f}{d\mu_{\mathcal{B}}} \ \mu_{\mathcal{B}} - \text{a.e}$$

Definition B.2.9. Let (Ω, Σ, μ) be a measure space and $A \subset \Omega$ be a subset.

1. For each $\omega \in A$, let $\{B_i^\omega, i \in I\} \subset \Sigma$ be a (not necessarily countable) family with a directed index set I, such that $B_i^\omega \rightharpoonup \omega$ in the Moore-Smith sense (i.e, it is a converging net, or $\mu^*(B_i^\omega \setminus \{\omega\}) \to 0$) and that for each co-final sequence J of I, $B_i^\omega \rightharpoonup \omega$ also. (Here co-final means for each $i \in I$, there is a $j \in J$ such that $j > i$.) Let $\mathcal{D}$ be the collection of all $\{B_i^\omega : 0 < \mu(B_i^\omega) < \infty, i \in I, \omega \in A\}$. Then $\mathcal{D}$ is called a differentiation or a derivation basis on A.

2. A converging net $\{B_i^\omega, i \in I\}$ is called contracting to ω, if there is an $i_0 \in I$ such that $\omega \in B_i^\omega$ for all $i > i_0$.

Let ν be a signed measure on Σ and $\{B_i^\omega, i \in I\}$ an ω-converging sequence for $\omega \in A \subset \Omega$, then we define the upper and the lower derivatives of ν relative to μ at $\omega \in A$ by

$$(D^*\nu)(\omega) = \sup\{\limsup_i \frac{\nu(B_i^\omega)}{\mu(B_i^\omega)} : \text{all nets } B_i^\omega \rightharpoonup \omega\}$$

and

$$(D_*\nu)(\omega) = \inf\{\liminf_i \frac{\nu(B_i^\omega)}{\mu(B_i^\omega)} : \text{all nets } B_i^\omega \rightharpoonup \omega\}$$

Note that sup and inf can be dropped when the B_i^ω-sequences are subsequences of a universal sequence.

Definition B.2.10. Let $\mathcal{D}$ be a differentiation basis of a subset $A \subset \Omega$. Then

1. a family $\mathcal{F} \subset \mathcal{D}$ is a fine covering of A if for each $\omega \in A$ there is some $\{B_i^\omega, i \in J\} \subset \mathcal{F}$ converging to ω.

2. If $\mathcal{D}$ is a differentiation basis in (Ω, Σ, μ) for $A \subset \Omega$ of finite outer measure, then $\mathcal{D}$ is said to satisfy the Vitali property relative to μ, provided for any fine covering $\mathcal{F} \subset \mathcal{D}$ of A and $\epsilon > 0$ and a measurable cover $\tilde{A}$ of A, there is at most a countable collection $\mathcal{C} \subset \mathcal{F}$ such that the following two conditions hold

   (a) $\mu(\tilde{A} \triangle V) = 0$ where $V = \bigcup_{B \in \mathcal{C}} B$

   (b) if $\phi_{\mathcal{C}}(\omega)$ is the number of sets of $\mathcal{C}$ to which ω belongs, then

   $$\int_V (\phi_{\mathcal{C}} - 1) d\mu(\omega) < \epsilon \tag{B.1}$$

   If the members of $\mathcal{C}$ are disjoint (or their pairwise intersection is μ-null so that B.1 is true for any $\epsilon > 0$) the corresponding $\mathcal{D}$ is said to have the strong Vitali property (modulo μ-null sets).

To have an alternative view of the above property, we state the following result due to R. de Possel.

Theorem B.2.11. *Let (Ω, Σ, μ) be a measure space and $\mathcal{D} \subset \Sigma$ be a derivative basis. Then $\mathcal{D}$ has a Vitali property iff for any $A \in \Sigma$ and $\omega \in A$ there exists an ω-converging net $\{B_i^\omega, i \in I\} \subset \mathcal{D}$ such that*

$$\lim_i \frac{\mu(B_i^\omega \cap A)}{\mu(B_i^\omega)} = \chi_A(\omega) \ , \ a.a\ \omega$$

The following theorem gives us an approximation result of the Radon-Nikodym derivative.

Theorem B.2.12. *Let (Ω, Σ) be measurable space. Let μ, ν be two finite measures on the σ-algebra Σ, and suppose ν absolutely continuous with respect to μ. If $\mathcal{D} \subset \Sigma$ is a derivation basis on Ω satisfying the Vitali property relative to both ν and μ, then*

$$(D\nu)(\omega) = \frac{d\nu}{d\mu} \ , \ \mu - a.e$$

Approximation of the operator Q_n for Markov chains on a denumerable state spaces

In the following, we assume that X is a countable set and the Markov operator P has a finite range.

As we have seen above, the following relation holds $\mu_{\alpha_n^\infty|\sigma(\pi_n^{n+1})}$-a.e

$$\mathbb{E}(g|\pi_n^{n+1}) = \frac{d\nu_g}{d\mu_{\alpha_n^\infty|\sigma(\pi_n^{n+1})}}$$

where ν_g is defined for all $A \in \sigma(\pi_n^{n+1})$ by

$$\nu_g(A) = \int_A g d\mu_{\alpha_n^\infty}$$

Fix $x \in X^\infty/\alpha_{n+1}^\infty$ and denote by B_i^x the subset

$$B_i^x = \{y \in X^\infty : y_{n+1} = x_{n+1}, \cdots, y_{n+i} = x_{n+i}\}$$

Then, it is clear that $\mu^*_{\alpha_n^\infty}(B_i^x \setminus \{x\}) \to 0$ as $i \to \infty$ and for all subsequences also. We will define the derivation basis $\mathcal{D}$ for some $A \subset X^\infty/\alpha_n^\infty$ by

$$\mathcal{D} = \{B_i^x : 0 < \mu_{\alpha_n^\infty}(B_i^x) < \infty, i \geq 1, x \in A\}$$

We have to show that $\mathcal{D}$ has the Vitali property. Let $\mathcal{F} \subset \mathcal{D}$ be a fine covering of A. Then, for each $x \in A$, there exists $\{B_j^x, j \in J\} \subset \mathcal{F}$ converging to x. Denote by A_i the set

$$A_i = \{x : \inf\{j : B_j^x \in \mathcal{F}\} = i\}$$

then the collection of sets A_i is a partition of A. Since, for each i there is at most a finite number of distinct B_i^x, A_i is a union of finite members of $\mathcal{F}$, namely

$$A_i = \bigcup B_{ij}$$

with $B_{ij} \in \mathcal{F}$. Thus we can set for $\mathcal{C}$ the collection of $B_{i,j}$. Then $\mathcal{C}$ is a countable partition, so that $\mathcal{D}$ has the strong Vitali property.

Note that, in our case, the Vitali property is easily checked because the support of measures $P(x, .)$ is a finite set for all x.

Thus, by the theorem of the previous section, we get $\mu_{\alpha_n^\infty|\sigma(\pi_n^{n+1})}$-a.e

$$\mathbf{E}(g|\pi_n^{n+1}) = D\nu_g$$

and moreover for $\mu_{\alpha_n^\infty|\sigma(\pi_n^{n+1})}$-a.e $t \in X^\infty/\alpha_{n+1}^\infty$

$$\mathbf{E}(g|\pi_n^{n+1} = t) = Q_n g(t) = D\nu_g(t)$$

Fix an $x \in X^\infty/\alpha_{n+1}^\infty$ and consider the linear form L_x on $\mathbb{L}^\infty(X^\infty/\alpha_n^\infty, \mu_{\alpha_n^\infty})$ defined by

$$L_x g = \lim_i \frac{\int_{B_i^x} g d\mu_{\alpha_n^\infty}}{\mu_{\alpha_n^\infty|\sigma(\pi_n^{n+1})}(B_i^x)}$$

The linear form L_x is continuous and its operator norm is $\|L_x\| = 1$.

For each function $g \in \mathbb{L}^\infty(X^\infty/\alpha_n^\infty, \mu_{\alpha_n^\infty})$ we can find a sequence of functions $g_j \in \mathbb{L}^\infty(X^\infty/\alpha_n^\infty, \mu_{\alpha_n^\infty})$ depending only on the first $j+1$ coordinates and converging almost surely and in $\mathbb{L}^\infty$ to g. Since the linear form L_x is continuous we can exchange the order of taking limits, namely

$$\lim_i \lim_j \frac{\int_{B_i^x} g_j d\mu_{\alpha_n^\infty}}{\mu_{\alpha_n^\infty}(B_i^x)} = \lim_j \lim_i \frac{\int_{B_i^x} g_j d\mu_{\alpha_n^\infty}}{\mu_{\alpha_n^\infty}(B_i^x)}$$

Now, recalling that

$$B_i^x = \{y \in X^\infty : y_{n+1} = x_{n+1}, \cdots, y_{n+1} = x_{n+i}\},$$

we can compute for $i \geq j$

$$\begin{aligned}\frac{\int_{B_i^x} g_j d\mu_{\alpha_n^\infty}}{\mu_{\alpha_n^\infty}(B_i^x)} &= \frac{\sum_{u\in X} g_j(u, x_{n+1}, \cdots, x_{n+j})\theta P^n(u)P(u, x_{n+1})\dots P(x_{n+i-1}, x_{n+i})}{\theta P^{n+1}(x_{n+1})\dots P(x_{n+i-1}, x_{n+i})} \\ &= \sum_{u\in X} g_j(u, x_{n+1}, \cdots, x_{n+j})\frac{\theta P^n(u)P(u, x_{n+1})}{\theta P^{n+1}(x_{n+1})}\end{aligned}$$

Therefore, the latter expression is independent of i, and we get

$$Q_n g(t) = \sum_{u\in X} g(u,t)\frac{\theta P^n(u)P(u, t_{n+1})}{\theta P^{n+1}(t_{n+1})}$$

for $\mu_{\alpha_{n+1}^\infty}$-almost all $t \in X^\infty/\alpha_{n+1}^\infty$.

Theorem B.2.13. *Let (X, P, θ) be a Markov chain on a denumerable state space X whose Markov operator has finite range. Then for $\mu_{\alpha_{n+1}^\infty}$-almost all $t \in X^\infty/\alpha_{n+1}^\infty$ $(n \geq 0)$, the operator Q_n is given for $g \in \mathbb{L}^\infty(X^\infty/\alpha_n^\infty, \mu_{\alpha_n^\infty})$ by*

$$Q_n g(t) = \sum_{u\in X} g(u,t)\frac{\theta P^n(u)P(u, t_{n+1})}{\theta P^{n+1}(t_{n+1})}$$

B.3 Poisson boundary

In the section B.2.1, we defined the tail boundary $(X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu^\theta, (\pi_n)_{n\geq 0})$ of a Markov chain (X, P, θ) as the inductive limit in the category Meas of measure spaces of the system $(X^\infty/\alpha_n^\infty, \mathcal{X}_{\alpha_n^\infty}, \mu^\theta_{\alpha_n^\infty}, (\pi_n^m)_{m\geq n})$. The tail boundary depends obviously on the Markov operator P but also on the initial distribution θ. The tail boundary distinguishes the asymptotic behavior of the trajectories. However, this boundary is not time invariant, *i.e.* it distinguishes shifted trajectories. In order to obtain time independent informations on the asymptotic behavior of the Markov chain (X, P, θ), we need to introduce the notion of Poisson boundary.

Let us recall the notation. We consider a X-valued Markov chain of Markov operator P and initial distribution θ on X. We can endow the path space X^∞ with the product σ-algebra $\mathcal{X}$. Thus we can define the probability measure $\mathbf{P}^\theta$ on the σ-algebra $\mathcal{X}$ associated with the Markov operator P and the probability measure θ.

The path space $(X^\infty, \mathcal{X})$ comes with the natural measurable action of the time shift $\tilde{S} : X^\infty \to X^\infty$ defined for $x \in X^\infty$ by $(\tilde{S}x)_i = x_{i+1}$ for $i \geq 0$. The point is that the time shift on the path space can not be extended directly to the tail boundary. The main difficulty comes from the fact that the partition α^∞ is defined modulo $\mathbf{P}^\theta$-null sets so that the map $\tilde{S}$ induces a map

$$S : (X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu^\theta) \to (X^\infty/\alpha^\infty, \mathcal{X}_{\alpha^\infty}, \mu^{\theta P}),$$

and from the measure theoretic point of view the two spaces are not isomorphic in general. This difficulty will be dropped by considering a larger measure space — namely the space $\mathbf{E}$, see below — containing each tail boundary for each probability measure θ.

We write $\mu \prec \nu$ if μ is absolutly continuous with respect to ν. Indeed, if $\theta_1 \prec \theta_2 \prec m$ are two probability measures on the state space X, then $\mathbf{P}^{\theta_1} \prec \mathbf{P}^{\theta_2}$ so that there is a natural embedding

$$(X^\infty/\alpha^\infty, \mu^{\theta_1}) \hookrightarrow (X^\infty/\alpha^\infty, \mu^{\theta_2})$$

in such a way that $\mu^{\theta_1} \prec \mu^{\theta_2}$.

Let us denote by $\mu_{\nu,n}$ the measure on the tail boundary of the Markov chain with an initial distribution $\nu \prec \theta P^n$. Such measures are called harmonic measures. Because of the natural embedding above, we can assume that all harmonic measures $\mu_{\nu,n}$ with $\nu \prec m$ are defined on a universal space $(\mathbf{E}, \mathcal{E})$ — the σ-alebgra $\mathcal{E}$ is generated by the family of σ-algebras of tail boundaries corresponding to each $\nu \prec m$. We denote by $[\mu_m]$ the class (two measures are equivalent if they are mutually absolutely continuous) of harmonic measures $\mu_{\nu,0}$ for $\nu \sim m$, and by $[\tilde{\mu}_m]$ the minimal measure class dominating all harmonic measures $\mu_{\nu,n}$, $\nu \prec m$ and $n \geq 0$. The measure space $(\mathbf{E}, \mathcal{E}, [\tilde{\mu}_m])$ is named the tail boundary of the Markov operator P (the initial distribution is no longer taken in consideration).

Thus, the time shift $\tilde{S}$ naturally induces a invertible action on the tail boundary of the Markov operator P (the inverse image of a tail set — measurable subset of $\mathbf{E}$ — is itself a tail set) denoted by S. Furthermore, we obviously have

$$\mu_{\nu,n} \circ S^{-1} = \mu_{\nu,n-1} = \mu_{\nu P,n}.$$

In particular, $[\tilde{\mu}_m] \circ S^{-1} = [\tilde{\mu}_m]$ so that the measure $[\tilde{\mu}_n]$ is quasi-invariant. In this context, the theorem of ergodic decomposition holds and we can define the Poisson boundary.

Theorem B.3.1. *There exists a Lebesgue space (Y, ξ) and a family of probability measures $\{\lambda^y\}_{y \in Y}$ on $\mathbf{E}$ such that*

- *for ξ-a.e. $y \in Y$, $\lambda^y(\cdot)$ is a probability on $\mathbf{E}$,*
- *for all measurable subsets $B \subset \mathbf{E}$, the map $y \mapsto \lambda^x(B)$ is measurable,*

- *for all measurable subsets $B \subset \mathbf{E}$, the following decomposition holds*

$$\lambda(B) = \int_Y \lambda^y(B)\xi(dy)$$

- *for ξ-a.e. $y \in Y$, λ^y is a quasi-invariant ergodic measure under the time shift.*

Definition B.3.2. The measure space (Y, ξ) is called the Poisson boundary of the Markov operator P. For an initial distribution $\theta \prec m$ the measure space (Y, ξ^θ), where ξ^θ is the image measure on Y of μ^θ, is named the Poisson boundary of the chain (X, P, θ)

We refer the reader to [Kai92] for further results on the Poisson boundary. In particular, it is shown in this paper that the Poisson boundary is isometrically isomorphic to the space of bounded harmonic functions, *i.e.* real bounded functions $f : X \to \mathbb{R}$ such that $Pf = f$. This theorem is actually the time independent version of a more general theorem etablishing an isometric isomorphism between the tail boundary and the space of bounded harmonic sequences, that is the space of sequences of functions $f = (f_n)_{n\geq 0}$, $f_n \in \mathbb{L}^\infty(X, \theta P^n)$ satisfying the conditions $f_n = Pf_{n+1}$ for all $n \geq 0$ with the norm $\|f\| = \sup_{n\geq 0} \|f_n\|_{\mathbb{L}^\infty(X,\theta P^n)}$. This theorem is the consequence of the convergence of a certain bounded backward martingale which is, in our context, the back the bounded backward martingale $(\mathbb{E}^\theta(f|\pi_n^\infty))_{n\geq 0}$ defined in section B.2.2. This theorem also has as a consequence the so-called Poisson formula which is undoubtly better known in the context of harmonic analysis and groups.

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RÉSUMÉ : L'étude des marches aléatoires fait apparaître des connexions entre leurs propriétés algébriques, géométriques ou encore combinatoires et leurs propriétés stochastiques. Le premier exemple de telles connexions est donné par le théorème de Pólya concernant les marches aléatoires aux plus proches voisins sur le groupe $\mathbb{Z}^N$. Si les marches aléatoires sur les groupes — ou sur des espaces homogènes — fournissent beaucoup d'exemples, il serait apréciable d'obtenir de tels résultats de rigidité sur des structures algébriques plus faibles telles celles de semi-groupoide ou de groupoide. Dans cette thèse il est considéré un exemple de semi-groupoide et un exemple de groupoide, tous les deux sont définis à partir de sous-graphes contraints du graphe de Cayley d'un groupe — le premier graphe est dirigé alors que le second ne l'est pas. Pour ce premier exemple, on précise un résultat de Campanino et Petritis — ils ont montré que la marche aléatoire simple était transiente pour cet exemple de graphe dirigé — en déterminant la frontière de Martin associée à cette marche et établissant sa trivialité. Dans le second exemple apparaissant dans ce manuscrit, on considère des pavages quasi-périodiques de l'espace euclidien obtenus à l'aide de la méthode de coupe et projection. Nous considérons la marche aléatoire simple le long des arêtes des polytopes constituant le pavage, et nous répondons à la question du type de celle-ci, c'est-à-dire nous déterminons si elle est récurrente ou transiente. Nous montrons ce résultat en établissant des inégalités isopérimétriques. Cette stratégie permet d'obtenir des estimées de la vitesse de décroissance du noyau de la chaleur, ce que n'aurait pas permis l'utilisation d'un critère de type Nash-Williams.

ABSTRACT: The study of random walks demonstrates connections between their algebraic, combinatorial, geometric and stochastic properties. The first example of such a connection was given in a theorem of Pólya dealing with nearest neighbourhood random walks on the group $\mathbb{Z}^N$. Random walks on groups provide with many examples, however it should be interesting to have such rigid results in the case of weaker algebraic structures such that semigroupoids and groupoids. In this thesis, one example of semigroupoid and one example of groupoid are considered; they are both defined as constrained subgraphs of the Cayley graph of a group — the first one is genuinely directed contrary to the second one which is undirected. For this first example, it has been shown by Campanino and Petritis that the simple random walk is transient. Here, we refine this statement by determining the Martin boundary of this process and show its triviality. In the second example, we consider quasi-periodic tilings of the Euclidean spaces obtained with the help of the cut-and-project scheme. We have considered the simple random walk along the sides of the polytopes constituting the tiling and answered the question of its type, *i.e.* we determined whether the random walk is recurrent or transient. This result is a consequence of isoperimetric inequalities. This strategy allow us to obtain estimates of the rate of convergence of the heat kernel which could not have been done with the help of the Nash-Williams criterion.