Topics in Early Universe Cosmology

Johanna Karouby

Topics in Early Universe Cosmology

Johanna Karouby

Doctor of Philosophy

Physics Department

McGill University
Montreal, Quebec
June 2012

A thesis submitted in partial fulfilment
of the requirements for the degree of Doctor of Philosophy

Abstract

The study of the Early Universe raises some of the most fundamental questions in theoretical physics. This thesis explores three main aspects of early universe cosmology.

The first part discusses alternatives to the Big Bang scenario which is the current paradigm of cosmology. Namely, it discusses bouncing universe models where the initial Big Bang singularity is replaced by a finite size universe.

After reviewing the necessary cosmology background in the introduction, we show a specific model of a bouncing universe that contains additional "Lee-Wick fields", partners to the standard fields. In particular we prove that a Lee-Wick matter bounce is unstable when one adds radiation to matter.

In the second part of this thesis, we consider particle production via parametric resonance during preheating, at the end of cosmological inflation. Specifically, we prove that in the case of a speed-limited inflaton, non-canonical kinetic terms used to described any effective Lagrangian do not enhance particle production.

Finally, the last topic involves topological defects during the Quantum Chromodynamics phase transition. Namely, we study cosmic strings coming from pion fields present in the Standard Model of particle physics and find a mechanism to stabilize them. We show how a thermal bath of photons reduces the effective vacuum manifold to a circle and thus allows the presence of topologically stable pion strings.

Résumé

L'étude de l'Univers primordial adresse quelques-unes des questions les plus fondamentales de la physique théorique. Cette thèse a pour objet l'exploration de trois aspects principaux de la cosmologie primordiale.

Dans un premier temps, nous discutons d'une alternative au paradigme scientique qu'est le modèle du Big Bang. À savoir, nous explorons un model d'univers à rebond qui évite la singularité initiale du Big Bang. Nous commencerons dans l'introduction par revoir les éléments de base nécessaires à la compréhension de la cosmologie. À la suite de quoi, nous montrerons un modèle spécifique d'Univers à rebond contenant des champs additionnels particuliers en complément des champs présents habituellement. Ces nouveaux champs proviennent de ce qui s'appelle le modèle "Lee-Wick" de la physique des particules. En particulier, nous prouvons qu'un univers à rebond dans ce contexte est instable lorsque l'on ajoute une composante de radiation en plus de la matière.

Dans la seconde partie, nous considérons la production de particules via un phénomène de résonance paramétrique durant la phase de "préchauffement", à la fin de l'inflation cosmologique. Plus précisément, nous prouvons que dans le cas où l'inflaton a une limite de vitesse, les termes cinétiques non-canoniques décrivant n'importe quel Lagrangien effectif n'améliorent pas la production de particules.

Finalement, le dernier sujet abordé concerne les défauts topologiques pendant la transition de phase de la chromodynamique quantique. À savoir, nous étudions les cordes cosmiques provenant des champs de pions presents dans le modèle standard de la physique des particules et trouvons un méchanisme pour les stabiliser. Nous prouvons alors qu'un bain thermique de photons en contact avec ces cordes réduit la variété du vide à un cercle. Cela a pour effet d'autoriser la présence de "cordes pioniques" topologiquement stables.

Acknowledgements

My graduate studies were full of surprises, from the beginning to the end. Little did I know at the end of my exchange year at McGill University as an undergraduate student, that I would be finishing my doctorate program five years later. I must thank Canada, and McGill University, for giving me the opportunity for such a nice and rich experience. My initial plan was not to go to McGill, nor was I supposed to stay more than a year as an exchange undergraduate student.

I am indebted to my supervisor, Robert Brandenberger, for giving me the opportunity to take up a last summer internship as an undergraduate student, and most importantly to have accepted me as a Master's student, immediately after. In addition, I would like to thank him for his guidance, his great patience and giving me the opportunity to attend so many physics schools and conferences worldwide.

I must thank all of my collaborators and colleagues for helpful discussions, comments and suggestions during my graduates studies. I am also extremely grateful to the National Science and Engineering Research foundation of Canada (NSERC) for its financial support during part of my doctorate.
It is also a pleasure to thank all the physics friends I spent time with during conferences. I really enjoyed their nice companies, specially in China and in Corsica.

In addition, I wish to thank all my physics friends from McGill : Paul Franche, Aaron Vincent, Xue Wei, Nima Lashkari, Loison Hoi, Alisha Wissanji, Michael Pagano, Vincent Poitras and many others. I would like to especially thank Aaron Vincent for proofreading the introduction and the conclusion of this thesis. Special thanks also to the members of office 306 with whom we established some memorable traditions. I hope these traditions will remain alive even when the last of the 3 founders has left. Moreover, I would like to thank some of the “guys”of the physics department for their strong arms, for the many times I had to move out of my apartment. I am also grateful for the technical support I received the numerous times I had technical problems with my computer. I thank Elisa Gouvea for the nice discussions we had about physics and life. I am thankful to all the anthropologists and physicists from my 4 year soccer and Ultimate Frisbee teams with whom I played year-long, indoors and outdoors. I had, and am still having, great pleasure playing.

I would like also to thank all my non-students friends in Montreal, especially Nadia and all the “Friday night crowd”. They taught me a lot about life and I wish to thank each of them for their moral support and making me smile so often!

Finally, I would like to thank all of my family : my parents, Josyane and Philippe, my brother, Jeremy and my sister, Jennifer. I wish to thank my grandmothers, my uncles and my cousins as well. Each time I went back to France, I had a very good time. In addition, I wish to thank my parents in advance for accepting my choice of career. I am glad they are making progress on the way to accept the fact that I am not and probably will never be an engineer.

I am thankful to my friends living in France, especially to Valerie and Bruno, for the good time I have each time I see them, whether it is in Toulouse or in Paris.

Last but not least, I sincerely thank my **friend**, Alexander Wiegand, for his strong and constant moral support during the last two years of my doctorate. I must also thank him for succeeding in reading my handwriting and for typing some of my notes in LaTeX!

Preface

Contributions of Authors

This thesis contains material previously published in Physical Review D :

• Chapter 2 is based on [20] which was written in collaboration with Robert Brandenberger. I performed most of the analytical computations and wrote a large fraction of the body of the text. I derived analytically the energy densities, the equation of state parameters, the equations of motions and solved them. I used the Green function method in order to prove that the occurrence of a bounce depends on the initial conditions.

• Chapter 3 was published as reference [21] as a follow-up to the previous item. It was written with Robert Brandenberger and Taotao Qiu. Taotao Qiu performed all the numerical analysis and we performed most of the analytical calculations. I was involved in the inception of the idea, the discussions, performed most of the computations and wrote a fraction of the body of the text.

• Chapter 4 is based on [22] which was written in collaboration with Aaron Vincent and Bret Underwood. They took care of the numerical part, and the writing. I was involved in the discussion of the approach, analytical calculations, part of the writing and derived results for the appendix.

• Chapter 5 was published as reference [23] which was written in collaboration with Robert Brandenberger. I have found out how to apply finite-temperature field theory methods to our problem. I performed most of the analytical computations and wrote a large fraction of the body of the text.

Preface

Statement of Originality

Chapter 2 and 3 explore bouncing universe models containing fields coming from the Lee-Wick standard model. It was known that these fields could give rise to a non-singular bounce in the matter sector. We showed that this is not the case anymore when we add up gauge fields to the system [20]. First, we derived the most general Lagrangian for the pure radiation case only, adding up an effective coupling with its Lee-Wick partner that was not used before. The second step was to couple radiation and matter. This was done in [21] where we proved that the bounce is not stable under the addition of radiation. We used the Born approximation to decompose each field into a background part and perturbation terms of first and second order.

Chapter 4 shows how particle production occurs during preheating in the case of a speed-limited inflaton for a large class of effective Lagrangians with non-canonical kinetic terms [22]. We proved that preheating with non-canonical kinetic terms is less efficient than with canonical kinetic terms. In other words, introducing non-canonical kinetic terms does not enhance particle production via parametric resonance.

Chapter 5 deals with another topic : cosmic strings which are linear topological defects. We worked with pion fields present in the Standard Model of particle physics and found a rigorous stabilization mechanism by a thermal bath [23] for the pion string. We used finite temperature field theory in order to compute the effective potential of the system and the method applies to other defects contained in the Standard Model as well.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

Fundamental relations exist between the two vast areas of physics which are Cosmology and Particle Physics. Their similar energy scale creates a bridge, for example, between micro-physics in particle colliders and the early universe. Assuming the system of laws that governs physics at all scales is the same for cosmology and particle physics, one may use Particle Physics models in order to better understand the early universe. This principle is a pillar of modern cosmology and is used to probe various epochs of the Universe : From the most recent like recombination and cosmological nucleosynthesis, to earlier times, like the Quantum Chromodynamics (QCD) and the Electroweak phase transitions. The very early universe was at energy above any energy attainable in collider experiments. Thus, understanding cosmology could provide some clues for high-energy physics and conversely. As for now, no one can even tell if the Big Bang singularity did occur and alternative models exist. For example, using exotic types of matter can give rise to a Big Bounce that avoids the initial Big Bang singularity. As in standard particle physics models, the larger the energy density of the Universe, the more symmetry it possesses. The Universe initially had a high energy density, cooled down, losing more and more symmetries. After the Planck era, the Grand Unification (GUT) epoch during which all the forces of matter would have been unified, had the largest symmetry group. At energies around 10^{15}GeV, 10^{-38}s after the Big Bang, the strong force decoupled.

At present, the symmetry group of the Standard Model of particle physics, SU(3)xSU(2)xU(1) describes 3 of the 4 known fundamental interactions present in the Universe (gravity aside) : The strong force, with SU(3) symmetry appears on two major scales : On the one hand, as a nuclear force that binds neutrons and protons together. On the other hand, at a much smaller scale, in quantum chromodynamics (QCD). It is carried by gluons and creates bound states of quarks, hadrons, like pions that will be discussed later.

Moreover, SU(2) and U(1) respectively describe the symmetry of the weak and electromagnetic force.

At the atomic scale, the strong force is about 100 times stronger than electromagnetism. Electromagnetism is much stronger than the weak force that operates only on the extremely short distance scales found in an atomic nucleus (10^{-18}m). The weakest of all the forces, gravity, is not included in the Standard Model of Particle Physics but is described by General Relativity.

Understanding how all the particles of the Standard Model were created, challenges cosmologists. The current paradigm of early universe cosmology postulates a period of almost exponential expansion of the Universe, called cosmological inflation, before phase transitions giving rise to known forces begin. At very early times, this period of inflation would be followed by a period of intense reheating of the Universe during which all the matter would be created.

One of the most important consequences of the cooling down of the Universe and its loss of symmetry would be phase transitions. Each time a symmetry is broken, a phase transition occurs in the Early Universe and as a consequence, topological defects like cosmic strings, domain walls, or monopoles could form.

In condensed matter laboratories the creation of topological defects is widely observed and by analogy, the Universe should also contain some of them. Topological defects, if formed in the early Universe, would have some cosmological implications like, for example, gravitational wave emission from cosmic strings and could contribute to structure formation as well [1].

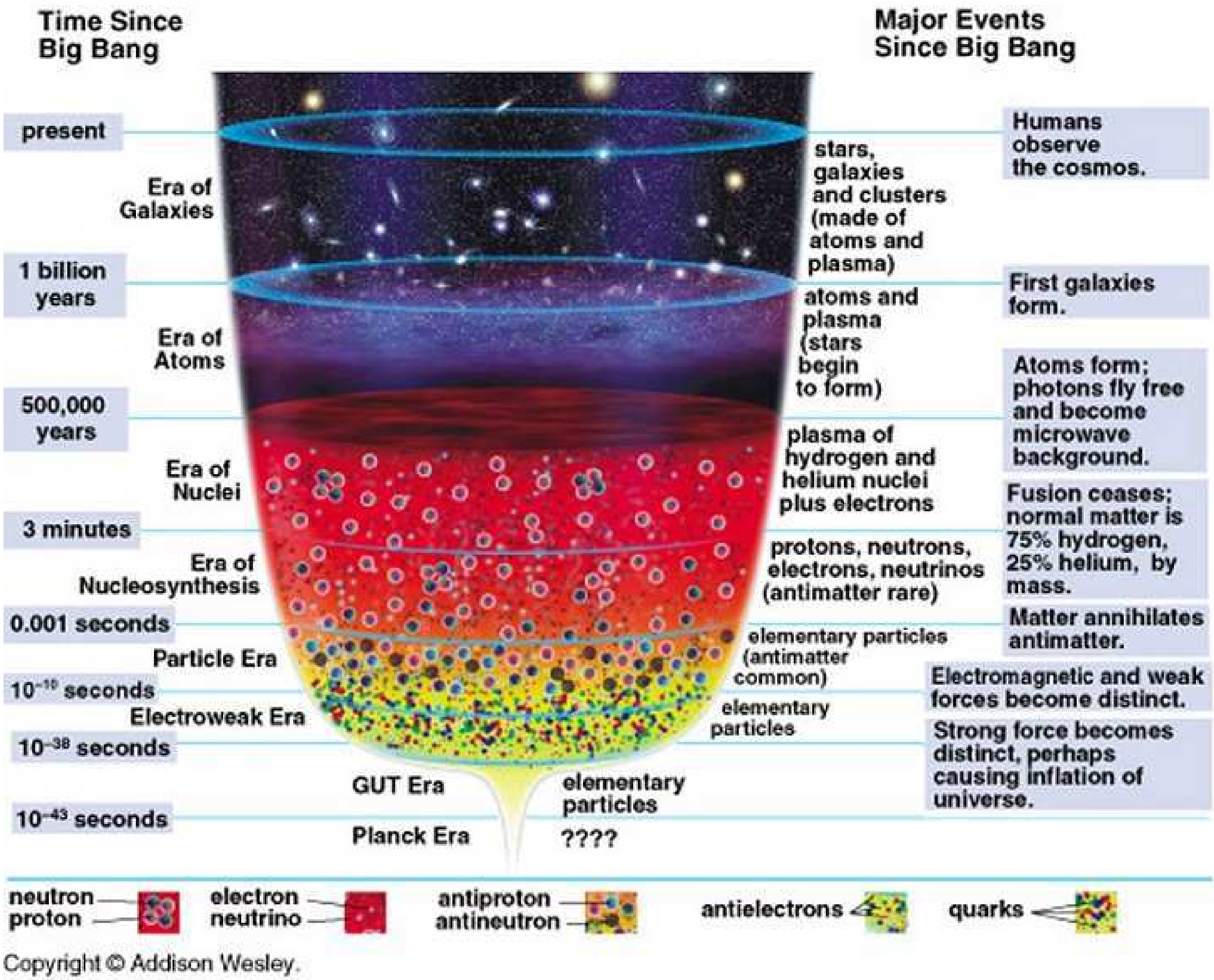

Figure 1.1: Timeline of Standard Big Bang Cosmology. The Universe went through several epochs, starting with a Big Bang at extremely high-energy density, greater than corresponding to $E_{Planck} = 10^{19} GeV$. It undergoes various phase transitions while losing more and more symmetries with time. A possible phase of inflation happens at very early times and elementary particles are created. Much later, growth of large structures like galaxies occurs. Figure from [2].

1.0.1 Overview

This thesis explores various aspects of the early Universe. First, the birth of the Universe still remains mysterious. In an attempt to find an alternative to the standard Big Bang model which has an initial singularity, cyclic and bouncing models appeared a long time ago. In the 1920s, even Albert Einstein considered the possibility of a cyclic model for the Universe as an (everlasting) alternative to the model of an expanding universe. In Chapter 2 and 3, a model of bouncing Universe, in which instead of a big bang the Universe contracts and re-expands at some point known as the bouncing point, is shown in detail. In order for a bouncing Universe to exist, one requires some specific particle physics content. In the case studied, the existence of a bounce is based on the presence of unconventional fields, namely the Lee-Wick fields.

A second important question is how all the matter observed today has been created. The standard lore is that a period of intense reheating, during which fast particle production occurs, takes place after a period of inflation. Preheating, a specific way of creating particles due to parametric resonance is derived for a certain class of models in Chapter 4.

Last but not least, one can study the QCD phase transition and the formation of topological defects, such as cosmic strings, that may occur at that time. In particular, the presence of a thermal bath of photons seem to stabilize these strings. This suggests that strings could have some cosmological imprint. Chapter 5 illustrates this point.

After a short review of basic aspects of cosmology, bouncing Universes, preheating, topological defects, QCD phase transition and finite-temperature field-theory are briefly described in sections 1.2-1.4.

1.1 Early Universe Cosmology

The Early Universe is defined as the epoch between the Big Bang, around 14 billions years ago and the decoupling time at around 380,000 years after the Big Bang, when photons were emitted from the last scattering surface. Current cosmological observations of the Cosmic Microwave Background (CMB) give a picture of the sky at that time. One of the great successes of inflationary cosmology is to have made precise predictions about CMB anisotropies. However, in spite of this phenomenological success, inflationary cosmology is not without its conceptual problems. These problems motivate the search for alternative proposals for the evolution of the early universe and for the generation of structure like galaxies. These alternatives must be consistent with current observations, and must make predictions distinguishable from signatures coming from inflation [3]. Roughly speaking, the Universe can be split into three different epochs (see fig.1.2 for more details). Close to the Big Bang time (or the bouncing point in the case of a bouncing Universe), $t \sim 0s$, the Universe was filled with a fluid behaving like radiation or ultra-relativistic matter. The temperature was so high that the kinetic energy dominated over the rest energy : $T \gg m_i$, where m_i is the rest mass of the species present at that time. The Universe consisted of photons, neutrinos, electrons, and other massive particles with very high kinetic energy. After some cooling, some massive particles decayed and others survived (protons, neutrons, electrons) whose masses eventually dominated over the radiation components (photons, neutrinos) when the energy density of matter and radiation became approximately equal. During the second phase, pressure-less matter or dust with equation of state $w = 0$, contributed the most to the energy content in what was called the matter dominated era. Finally, the Universe started to become dominated by dark energy and still is in this phase today. Its acceleration is due to some constant energy density parametrized by the cosmological constant, Λ, present in the Einstein equations. The value of important cosmological parameters during the three major epochs of the Universe is summarized in Table 1.1.

Epoch	Time (after the Big Bang)	Density (kg/m^3)	Temperature (K)	Main Events
	0 s	∞	∞	
	Radiation era			
Planck				Unknown physics; quantum gravity.
	10^{-43} s	10^{95}	10^{32}	
GUT*				Strong, weak, and electromagnetic forces unified.
	10^{-35} s	10^{75}	10^{27}	
Quark				Strong force frozen out. Heavy and light particles all in thermal equilibrium. Electroweak force freezes out at 10^{15} K.
	10^{-4} s	10^{16}	10^{12}	
Lepton				Only low-mass particles still in thermal equilibrium; neutrinos decouple at 10^{10} K.
	10^{2} s	10^{4}	10^{9}	
Nuclear				Deuterium and helium formed by fusion of protons and neutrons during first 1000 s.
	5×10^4 yr (2×10^{12} s)	6×10^{-16}	16,000	
	5×10^4 yr (2×10^{12} s)	6×10^{-16}	16,000	
Atomic	***Matter era***			Matter begins to dominate; atoms form; electromagnetic radiation decouples.
	2×10^8 yr (6×10^{15} s)	10^{-22}	60	
Galactic				Large-scale structure forms; first stars and quasars shine; galaxies form and grow.
	3×10^9 yr (10^{17} s)	2×10^{-25}	10	
Stellar	***Dark Energy era***			Galaxies merge and evolve; star formation peaks. Dark energy begins to dominate.
	$>10^{10}$ yr (3×10^{17} s)	3×10^{-27}	3	

Figure 1.2: The three main phases of standard cosmology : radiation dominated era, matter dominated era and dark-energy (or cosmological constant dominated) era. Table adapted from [4].

The spacetime deformations are encoded in the metric. For a flat, homogeneous and isotropic Universe, one uses the Friedmann−Lemaître−Robertson−Walker

(FLRW) metric :

$$ds^2 = -dt^2 + a(t)^2 dx^2$$

where $a(t)$ is the scale factor representing the way the size of the Universe increases. The metric can conveniently be written: $g_{\mu\nu} = diag(-1, a(t)^2, a(t)^2, a(t)^2)$.

The redshift, z, is commonly used as an alternative to describe cosmological time : $1 + z = \frac{a_0}{a}$, where a_0, the scale factor observed today, is conventionally set to 1. The conformal time, $\eta = \int \frac{dt}{a(t)}$, conveniently simplifies the FLRW metric,

$$ds^2 = a(\eta)(-d\eta^2 + dx^2).$$

A physical interpretation of η is the one of a clock that slows down as the Universe expands. In cosmology, energy sources often have an equation of state of the form $p = w\rho$ where p is the pressure density of the fluid, ρ its energy density and w is called the equation of state parameter.

Table 1.1: Behavior of different types of fluids when one of them dominates the total energy of the Universe.

Era	Radiation	Matter	Vacuum, Λ
$a(t)$	$t^{\frac{1}{2}}$	$t^{\frac{2}{3}}$	e^{Ht}
$a(\eta)$	η	η^2	$\frac{-1}{\eta}$
$\rho(a)$	a^{-4}	a^{-3}	constant
w	$\frac{1}{3}$	0	-1

The Hubble parameter, $H = \frac{\dot{a}}{a}$, represents the rate of expansion of the Universe. In 2010, the best fit value was $H_0 = 71.0 \pm 2.55\ (km/s)/Mpc$ based on WMAP data alone. A more recent 2011 estimate of the Hubble constant, using a new infrared camera on the Hubble Space Telescope (HST) to measure the distance and redshift for a collection of astronomical objects, gives $H_0 = 73.8 \pm 2.4\ (km/s)/Mpc$ [5]. In order to factor out the expansion of the Universe, one can use co-moving coordinates : $x_{com} = \frac{x}{a(t)}$.

In order for two regions of space to be causally connected at an early time and solve what is called the horizon problem, the co-moving Hubble radius, $(a(t)H)^{-1}$, must have decreased in the past. A period of rapid inflation at the very beginning of the Universe or a bouncing scenario satisfy this requirement.

Basics of General Relativity

General Relativity is the underlying theory used to derive major results in cosmology. Its main equations, the Einstein equations are written in units where $G = c = 1$ as

$$G_{\mu\nu} = 8\pi T_{\mu\nu} - g_{\mu\nu}\Lambda \tag{1.1}$$

where Λ is the cosmological constant and is responsible for the current accelerated expansion of the Universe. The Einstein tensor, $G_{\mu\nu}$, represents the deformation of spacetime with respect to Minkowski spacetime while the right-hand side describes the energy content. $T_{\mu\nu}$, the stress-energy tensor, is related to the matter action, S_m, through

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-det g_{\mu\nu}}}\frac{\delta S_m}{\delta g_{\mu\nu}}$$

Thus, a change in the field content immediately implies a change in the "shape" of the Universe. For the simplest case of a homogeneous and isotropic Universe, the FLRW Universe homogeneously grows with time at a rate given by the Hubble parameter, H. In standard cosmology, one considers the energy of the Universe as a perfect fluid. If the fluid is isotropic in some frame and the Universe is isotropic as well in another frame, then the fluid is at rest in co-moving coordinates.

As a result the energy density takes the form,

$$T^{\mu}_{\nu} = diag(-\rho, p, p, p).$$

For example, in the radiation case, the trace of the stress energy tensor vanishes and thus, $-\rho + 3p = 0$ which yields $w = \frac{p}{\rho} = \frac{1}{3}$.

Friedmann equations

As the Universe expands, the change in the energy $E = a^3\rho$ in a co-moving volume a^3 is given by $dE = -pd(a^3)$ like in standard thermodynamics. This yields

$$a\frac{d\rho}{da} = -3(\rho + p).$$

According to General Relativity, this expression derived from standard thermodynamics remains valid, if ρ denotes the energy density, T_{00}, instead of just the mass density. Differentiating with time leads to $\dot{\rho} = -3H(\rho + p)$. The T_{00} component of the Einstein equation (1.1) for the FLRW metric gives

$$\dot{H} = -H^2 - \frac{4\pi G}{3}(\rho + 3p) \quad \text{or} \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.2}$$

where G is the universal gravitational constant. Alternatively, one can use the Planck mass defined as : $M_{Pl} = \sqrt{\frac{\hbar c}{G}}$. By isotropy there is only one other equation from the $\mu\nu = ij$ component in (1.1) which combined with (1.2) gives

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \tag{1.3}$$

where K is the curvature parameter (which is set to zero in the flat space and in the following). These two equations, known as the Friedmann equations, are widely used in cosmology.

Cosmological inflation

In standard Big Bang cosmology, inflation is a period of intense expansion of the Universe at very early times, starting at around 10^{-34} seconds after the Big Bang. The temperature during inflation would drop by a factor of at least 100 000 (fig.1.2). In order to solve the *horizon problem* , inflation should last at least

60 e-folds. As a consequence, it would also flatten the Universe and solve the *flatness problem*. Inflation would also inflate away any unwanted exotic relics such as magnetic monopoles. Major cosmology problems are described at the end of this section.

The easiest way to create a period of almost exponentially expanding universe is to introduce a single scalar field, the inflaton, in FLRW spacetime [6].
The inflaton field slowly "rolls down" a potential as shown in fig.1.6 in accordance with the classical equation of motion :

$$\ddot{\varphi} + 3\,H\,\dot{\varphi} + \frac{dV}{d\varphi} = 0 \;\; \text{or for a } m^2\phi^2 \text{ potential,} \;\; \ddot{\varphi} + 3\,H\,\dot{\varphi} + m^2\,\varphi = 0. \tag{1.4}$$

Together with Friedmann equation in flat space, $H^2 = \frac{8\pi G}{3}\rho$, the system can be solved for the scale factor $a(t)$.

In order for inflation to occur through slow-roll, the kinetic energy must be much smaller than the potential energy : $\dot{\phi}^2 \ll V(\phi)$.
Moreover, the second derivative must be small enough for the inflationary process to last long enough : $\ddot{\phi} \ll |3H\dot{\phi}|,\; |V'|$ where ' denotes the derivative with respect to ϕ.

These two constraints can be rewritten using the so-called slow-roll parameters

$$\epsilon = \frac{M_{Pl}^2}{2}(\frac{V'}{V})^2 \;\ll 1 \quad \text{and} \quad \eta = \mathrm{M}_{\mathrm{Pl}}^2\frac{\mathrm{V}''}{\mathrm{V}} \ll 1.$$

For the simple $m^2\phi^2$ potential, $\epsilon = \eta = \frac{2M_{Pl}^2}{\phi^2}$ and the duration of the inflationary phase in e-folds is :

$$N = \int_{t_i}^{t_f} H dt \sim \frac{1}{M_{Pl}^2}\int_{\phi_f}^{\phi_i}\frac{V}{V'}d\phi \sim \frac{\phi_i^2}{4M_{Pl}^2} - \frac{1}{2}$$

where index i and f respectively refers to the beginning and end of inflation.

In order for inflation to last long enough, by 60 e-folds, ϕ needs to be greater than $16M_{Pl}$. At the end of inflation, the potential energy is converted into a thermalized gas of radiation of matter during reheating. The inflaton field then decays and fills the Universe with particles from the Standard Model.
Chapter 4 illustrates a particular way to reheat the Universe through parametric resonance, a process called preheating.

Decoupling of species

In Cosmology, present particles are thermal relics that left equilibrium when they decoupled from a thermal bath. As an example, today, normal matter has decoupled from a thermal bath of photons. As the Universe expands, the number density n_i, of particles of species i, decreases. Thus, the possibility that particles interact with each other decreases and so does the interaction rate, $\Gamma = n_i\langle\sigma v\rangle$ where σ is the thermally averaged cross-section, v the relative velocity and $\langle\sigma v\rangle$ is the total annihilation cross-section. Different species have different interaction rates with the photon fluid and as a consequence decouple at different epochs [7]. This is illustrated on fig.1.3.

A higher interaction rate implies a longer period of equilibrium with the thermal bath even when the particles becomes non-relativistic.

The expansion of the Universe (which is governed by the Hubble parameter H) dilutes the particles and Γ decreases with time. When, $\Gamma < H(t)$, particle species decouple from a thermal bath and "freeze-out" to the value they had at the decoupling time. Particles observed today are thermal relics that have decoupled from the photon fluid at earlier times.

Usually, observables in cosmology are computed using quantum field theory at zero temperature : there is no thermal equilibrium for the fields representing various energy components. At some point in the early Universe, including during

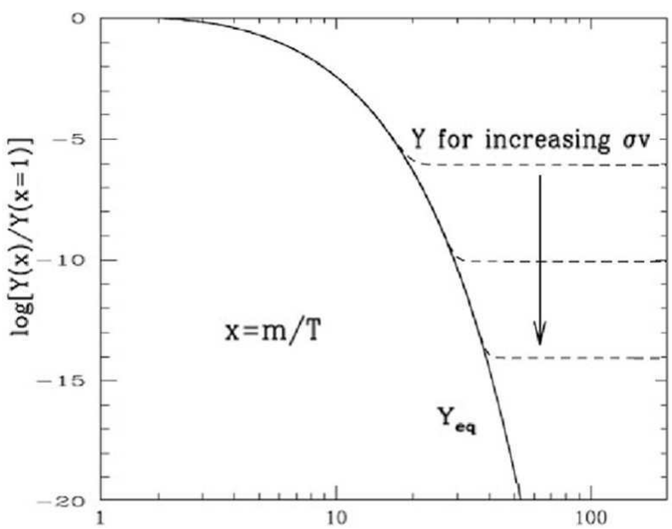

Figure 1.3: Freeze-out of a massive particle species. The solid line represents the thermal abundance and the dashed one, the actual one. The evolution of $Y(x)/Y(x=1)$ versus $x = m/T$ is shown where $Y = n/s$, n is the number density and s, the total entropy density of the Universe[7].

the Quark-hadron phase transition, the Universe was filled with pions, protons and neutrons, all in thermal equilibrium with photons. Chapter 5 shows the effects of a photon thermal bath on out-of-equilibrium fields present in the Standard Model of particle physics, namely the pions. Treating photons in a thermal state requires the use of finite-temperature field theory which is briefly described in section 1.4.2.

Problems of standard cosmology

Many problem and puzzles arise in standard big bang cosmology. To give a brief overview, some of them are listed in the following together with possible solutions.

The singularity problem

The Big Bang singularity marks the beginning of the time in conventional cosmology. At that point, time begins, a large amount of energy is contained in a Universe of size zero, so that energy density diverges. General Relativity breaks down at that point and no-one knows what should replace it.

An alternative to the big bang model like the Big Bounce discussed in chapters 2 and 3 solves this problem by avoiding the initial singularity.

The horizon problem

The Cosmological Microwave Background (CMB) shows a very homogeneous Universe whereas it is made of causally disconnected region at an earlier time. Logically, the fact that different regions have the exact same temperature means that they were in causal contact earlier. What is the mechanism that made these regions in causal contacts ?

The flatness problem

The Universe is currently at this peculiar time when the ratio of the actual density to its critical value (for flat space), Ω, is very close to 1.

The critical value for the energy density corresponds to the energy density for a flat Universe ($K = 0$): $\rho_{crit} = \frac{8\pi G}{3H^2}$. Thus, the ratio of the actual density to its critical value becomes : $\Omega = \frac{\rho}{\rho_{crit}} = \frac{8\pi G}{3H^2}\rho$. The question one can ask is, why is the Universe so flat whereas any tiny deviation from this very specific configuration would grow with the expansion of the Universe? In other word, why is the value of Ω so close to 1 whereas $\Omega = 1$ is an unstable fixed point?

The last two problems can be solved by cosmological inflation. In the context of inflation, one can study the way particles are created at its end. This is called reheating and will be studied in chapter 4.

The exotic relics problem

During the early universe phase transitions occurred and could give rise to topological defects, such as monopoles, strings or domain walls. However there have not yet been any observations of such objects.

Topological defects, cosmic strings in particular, are studied in the context of the QCD phase transition in chapter 5.

The dark matter problem

Most of the matter in the Universe is made of unknown matter, called Dark Matter. Particle physics models are trying to find out what is this mysterious matter.

The cosmological constant problem

The cosmological constant, Λ, is 120 orders of magnitude too small compared to what is expected from quantum field theory, assuming that the quantum vacuum is equivalent to the cosmological constant if you used the Planck energy as a cutoff scale. Supersymmetry would reduce this discrepancy to 60 orders of magnitude.

1.2 Alternative to Standard Big Bang Cosmology

The above picture describes what is called "Standard Big Bang Cosmology". Though this is the current paradigm of cosmology, some alternatives exist. One of the main problems with the Big Bang theory is that at the moment of the Big Bang, there is a singularity of zero volume and infinite energy. Models avoiding such a Big Bang singularity like the cycling Universe or the "Big Bounce" solve this problem. In addition to resolving the Big Bang singularity, a Big Bounce can also avoid the transplanckian problem of inflationary cosmology : perturbations observed today were inside the Planck volume at very early times, where General Relativity breaks down. However, in the bouncing Universe case, perturbations are never smaller than Planck length as shown on fig.1.5.

Two conditions are required to get a non-singular matter bounce :

At the bouncing point, the Hubble parameter vanishes, $\mathcal{H}$=0. The Friedmann equation (1.3) implies that the total energy density in the Universe vanishes at the bouncing point : $\Sigma\rho_i = 0$ where i indicates different components of the energy density like matter or radiation for example.

The second condition requires the time derivative of the Hubble parameter to be

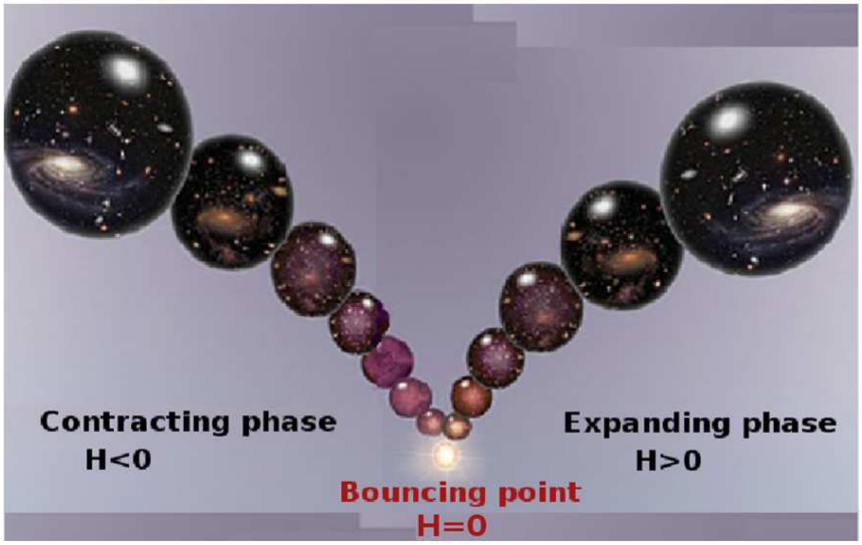

Figure 1.4: Bouncing Universe. Instead of a Big Bang origin of time, the Universe starts by contracting, reaches a minimal radius after which it starts expanding again as is observed today.

positive :

$$\frac{dH}{dt} > 0 \rightarrow \qquad \rho + p < 0 \rightarrow \qquad w = \frac{p}{\rho} < -1. \tag{1.5}$$

This violates the null-energy condition. Indeed, non-singular bounces may be investigated using effective field theory techniques, introducing matter fields which violate the null-energy condition and can lead to instabilities [8]. The null energy condition stipulates that $\rho + p > 0$ or equivalently $w > -1$. In some simple cases a connection between violation of the NEC and the presence of instabilities in the system has already been established [9]. Ghost fields (or equivalently Phantom fields) which have opposite sign kinetic terms, violate the NEC. NEC-violating models, plagued by ghost instabilities, can nevertheless be used as toy models to give rise to a non-singular bouncing Universe.

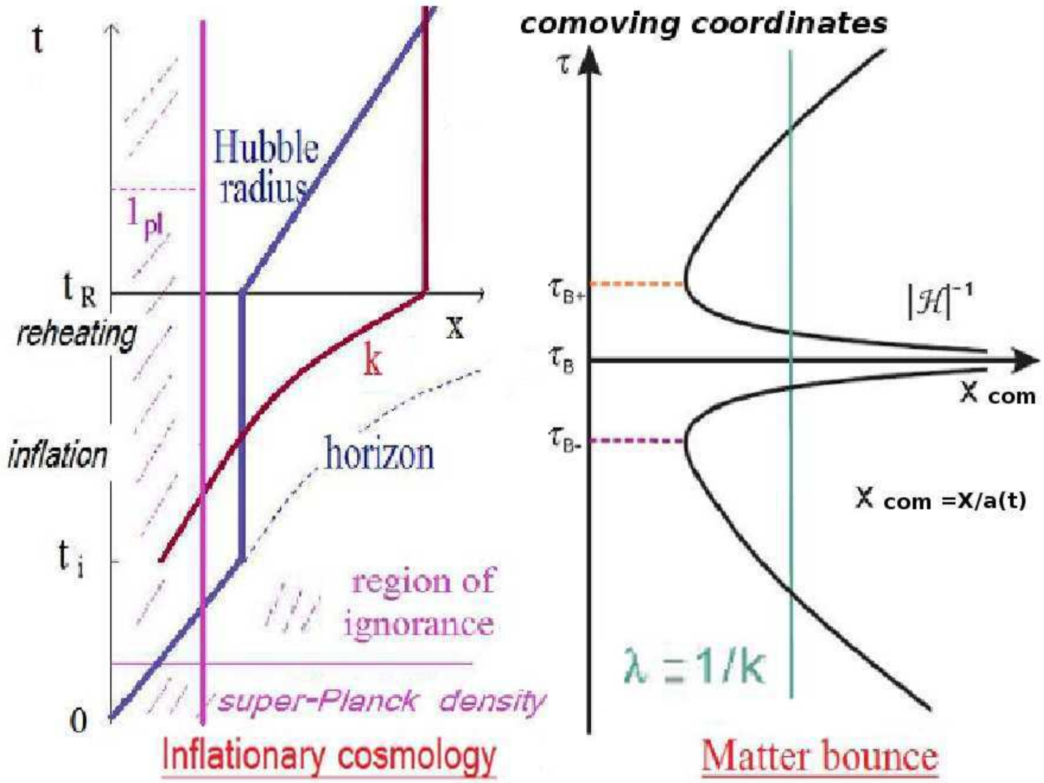

Figure 1.5: Evolution of cosmological perturbations in the inflationary Big Bang and in the Bounce model. Left: Space-time sketch of inflationary Big Bang scenario.The solid line labeled k(t) is the physical length of a fixed co-moving mode, k. Right: Spacetime sketch in the matter bounce scenario in conformal time, τ, and co-moving coordinates, $X_{com} = \frac{X}{a(t)}$. Perturbations never go below the Planck length. The vertical line indicates the wavelength for one mode. $|\mathcal{H}|^{-1}$ denotes the co-moving Hubble radius.

The Lee-Wick Bounce

One subset of these models, coming from a specific field theory construction is called the Lee-Wick (LW) Standard Model [24]. It introduces a new partner-field, the Lee-Wick field, to each field present in the initial Lagrangian. The Lee-Wick extension consists in adding higher-derivative quadratic terms in the Lagrangian. As a result, it contains ghosts and is non-unitary. Ghosts are scary because they violate unitarity and create instabilities in the theory. However, unitarity can be saved at the price of sacrificing microcausality.

A simple model to describe the Lee-Wick extension uses a single self-interacting scalar field, $\hat{\phi}$, with a higher-derivative term. The Lagrangian density is

$$\mathcal{L}_{hd} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2M^2}(\partial^2\hat{\phi})^2 - \frac{1}{2}m^2\hat{\phi}^2, \tag{1.6}$$

so the propagator of $\hat{\phi}$ is

$$\hat{D}(p) = \frac{i}{p^2 - p^4/M^2 - m^2} \ . \tag{1.7}$$

For $M \gg m$, this propagator has 2 poles at, $p^2 \simeq m^2$ and at $p^2 \simeq M^2$.
In order to describe this extra degree of freedom, one can introduce an auxiliary scalar field, $\tilde{\phi}$. The above Lagrangian is then equivalent to

$$\mathcal{L} = \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{2}m^2\hat{\phi}^2 - \tilde{\phi}\partial^2\hat{\phi} + \frac{1}{2}M^2\tilde{\phi}^2 \tag{1.8}$$

when replacing $\tilde{\phi}$ from L with its equation of motion. Alternatively, a functional integration over the field $\tilde{\phi}$ gives a Gaussian integral which yields the Lagrangian (1.6).
Defining $\phi = \hat{\phi} + \tilde{\phi}$, the Lagrangian 1.8 becomes, after integrating by parts,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}M^2\tilde{\phi}^2 - \frac{1}{2}m^2(\phi - \tilde{\phi})^2 \tag{1.9}$$

It is clear that there are two kinds of scalar fields: a normal scalar field ϕ and a new field $\tilde{\phi}$, called the Lee-Wick field. The sign of the quadratic term in the Lagrangian of the Lee-Wick field is opposite to the usual sign so one may worry about stability of the theory, even at the classical level.
For a very massive $\tilde{\phi}$ the propagator is given by

$$\tilde{D}(p) = \frac{-i}{p^2 - M^2}. \tag{1.10}$$

The LW field is associated with a non-positive definite norm on the Hilbert space, as indicated by the unusual sign of its propagator. Consequently, if this state were to be stable, unitarity of the S-matrix would be violated. However, as emphasized by Lee and Wick [11], unitarity is preserved provided that $\tilde{\phi}$ decays. This occurs in the theory described by 1.9 because $\tilde{\phi}$ is heavy and can decay to two ϕ-particles.

The mixing term $\frac{1}{2}m^2(\phi - \tilde{\phi})^2$ can be diagonalized by rotating the fields into new fields, ϕ' and $\tilde{\phi}'$.

$$\begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \phi' \\ \tilde{\phi}' \end{pmatrix}. \tag{1.11}$$

This transformation diagonalizes the Lagrangian if

$$\tanh 2\theta = \frac{-2m^2/M^2}{1 - 2m^2/M^2}. \tag{1.12}$$

A solution for the angle θ exists provided $M > 2m$.

From this transformation, one can easily redefine new fields and their corresponding new masses.

The Lagrangian describing the system becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2}m'^2 \phi'^2 - \frac{1}{2}\partial_\mu \tilde{\phi}' \partial^\mu \tilde{\phi}' + \frac{1}{2}M'^2 \tilde{\phi}'^2, \tag{1.13}$$

where m' and M' are linear combinations of m and M, corresponding to the masses of the diagonalized fields.

However, for this Lagrangian, the vacuum is unstable to quantum particle production [12] and thus this Lagrangian can only be considered as a low-energy effective field theory.

The higher derivative Lagrangian in the gauge sector is

$$\mathcal{L}_{hd} = -\frac{1}{2}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \frac{1}{M_A^2}\left(\partial^\mu \hat{F}_{\mu\nu}\right)\left(\partial^\lambda \hat{F}_\lambda{}^\nu\right), \tag{1.14}$$

where $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$. Introducing an auxiliary massive gauge bosons $\tilde{A}$, the higher derivative can be eliminated in the same way as in the scalar field case. Each gauge boson is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - M_A^2 \tilde{A}_\mu \tilde{A}^\mu + 2\hat{F}_{\mu\nu}\hat{\partial}^\mu \tilde{A}^\nu, \tag{1.15}$$

In order to diagonalize the kinetic terms, shifted fields defined by $\hat{A}_\mu = A_\mu + \tilde{A}_\mu$ are introduced. The Lagrangian for the pure radiation sector becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{M_A^2}{2}\tilde{A}_\mu \tilde{A}^\mu. \tag{1.16}$$

Chapter 3 and 4 explore the possibility of a bounce in the presence of radiation when a Lee-Wick massive gauge-field, partner of the standard photon field but with opposite kinetic terms, is introduced.

1.3 Particle production

Reheating is the phase creating all the matter in the Universe at the end of inflation. During slow roll inflation, the exponential growth of the Universe is due to a scalar field, the inflaton, slowly rolling down its potential.

To model inflation and reheating it is standard to use a potential that has a flat part and possesses a well as well:

_ Initially, during the inflation time, the inflaton slowly rolls down on the flat part;

_ At the end of inflation, the field oscillates at the bottom of the well, reheating the Universe and creating particles by transferring energy from the inflaton ϕ to an hypothetical scalar particle used to model matter in the Universe, the reheaton, χ.

Preheating would be the first stage of reheating and an extremely efficient way to reheat the Universe.

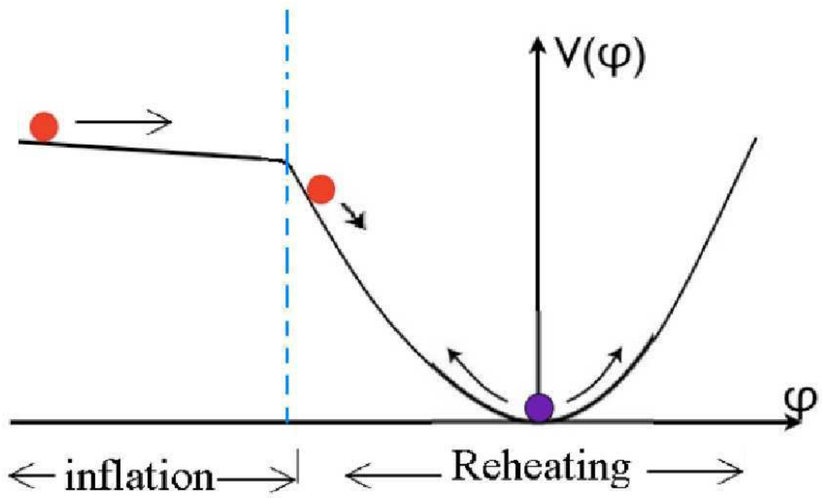

Figure 1.6: Inflaton slowly rolling down the potential before reheating the Universe by oscillating at the bottom of the well.

The way the field oscillates during preheating is similar to a child oscillating on a swing. The oscillatory motion of the center of gravity of the child would correspond to the motion of the inflaton at the bottom of the potential well. The amplitude of motion of the swing is analogous to the amplitude of the newly created field responsible for reheating the Universe, the reheaton.

The solution for each Fourier mode of the reheaton, χ_k, either diverges exponentially or has a fixed amplitude and just picks up a phase in the complex plane. This results in the presence of instability bands where the amplitude of the new particle grows exponentially and thus the number of χ-particles as well. In addition, there are also stability bands where preheating does not occur. The classical background value for χ is at the minimum of the χ effective potential, $V_{eff}(\chi) = \left(\frac{1}{2}m_\chi^2 + g^2\phi^2\right)\chi^2$, namely $\chi = 0$. When expanding the quantum fluctuations of χ :

$$\chi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\left(a_k\chi_k(t)e^{-i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger \delta\chi_k^*(t)e^{i\mathbf{k}\cdot\mathbf{x}}\right), \tag{1.17}$$

where $a_k, a_k^\dagger$ are the creation and annihilation operators, one obtains the following equation of motion for each reheaton mode χ_k :

$$\ddot{\chi}_k(t) + \left(k^2 + m_\chi^2 + g^2\phi(t)^2\right)\chi_k(t) = 0. \tag{1.18}$$

Mathematically, this system is described by Floquet Theory and Hill's equations, whose simplest examples are Mathieu equations.

Floquet Theory

In order to more readily use the tools of Floquet theory, one can simplify and rewrite the equation of motion in the standard form of Hill's equation:

$$\chi_k''(\tau) + \left[A_k(\tau) + q(\tau)f(\tau)\right]\chi_k(\tau) = 0. \tag{1.19}$$

$f(\tau)$ is a π-periodic function of the new time coordinate $\tau = t\frac{\pi}{T}$. The parameters A_k and q are time-dependent only when one considers the expansion of the Universe. Finally, the definitions of A_k and q impose the bound $A_k \geq q$.

Floquet's theorem (see [29]) states that solutions to (1.19) are of the form,

$$\chi_k(\tau) = e^{\mu_k\tau}g(\tau) + e^{-\mu_k\tau}g_2(\tau) \tag{1.20}$$

where $g(\tau)$ and $g_2(\tau)$ are periodic functions with period T; and μ_k, called the *Floquet exponent* or *characteristic exponent*, is complex. Clearly, when μ_k has a positive real part χ_k grows exponentially : this is the parametric resonance effect.

In order to find solutions of the form (1.20), $g(\tau)$ can be written as a Fourier expansion so that (1.20) reads

$$\chi_k(\tau) = e^{\mu_k\tau}\sum_{n=1}^{\infty} b_n \cos n\tau = e^{\mu_k\tau}\sum_{n=-\infty}^{\infty} c_n e^{in\tau} = \sum_{n=-\infty}^{\infty} c_n e^{(\mu_k+in)\tau}. \tag{1.21}$$

One can plug this Fourier series back into the equation of motion (1.19) to derive a recursion relation for the coefficients c_n in terms of μ_k, A_k, q.

The vanishing of the determinant, $\Delta(\mu_k, A_k, q)$, defined by the system of equation of motion for each c_n, defines the characteristic exponent

$$\mu_k = -\frac{i}{2\pi} \cos^{-1} \left[1 + \Delta(0, A_k, q)(\cos(2\pi A_k) - 1)\right] . \tag{1.22}$$

$\mu_k = \mu_k(A_k, q)$ is an implicit function A_k and q.

In non-expanding space, the evolution for the field χ for each mode k is set by

$$\ddot{\chi}(t) + [k^2 + M_\chi(t)^2]\chi(t) = 0$$

where $M_\chi(t) = g^2\phi^2(t)$. As an example, for the standard potential $V(\phi) = m^2\phi^2$ one just obtains a Matthieu equation : $\phi(t) = \Phi sin(mt)$ where m is the mass of the inflaton field and $q = \frac{4g^2\Phi}{m^2}$.

The limit $q \ll 1$ is the *narrow resonance* limit, so named because resonance occurs in bands whose size are proportional to some power of q.
The opposite one is called broad resonance : $q \gg 1$. There are two ways to understand the resonance in the broad regime : violation of adiabaticity of the evolution of the field χ, and mapping the problem to a quantum mechanical scattering problem. Each of them is examined in chapter 4.

1.4 Topological defects at finite temperature

1.4.1 Topological defects

Topological defects are commonly formed in laboratories and seen in condensed matter systems during phase transition. In cosmology, these defects could play an important role and contribute to structure formation [1]. Physically, defects can be seen as zones of trapped energy. Point-like defects called monopoles are predicted to have a magnetic charge but have not yet been observed. One dimensional and 2-dimensional defects respectively form cosmic strings and domain walls. There also exist more complicated delocalized topological defects that are unstable to collapse called textures. Cosmologically, it is known that only strings and textures are acceptable observationally even though none have been observed yet.

In chapter 5, the focus is on cosmic strings, more precisely on strings coming from fields present in the Standard Model of particle physics. Strings then arise from spontaneous symmetry breaking occurring when a scalar field, usually called the Higgs field, takes on its vacuum expectation value.

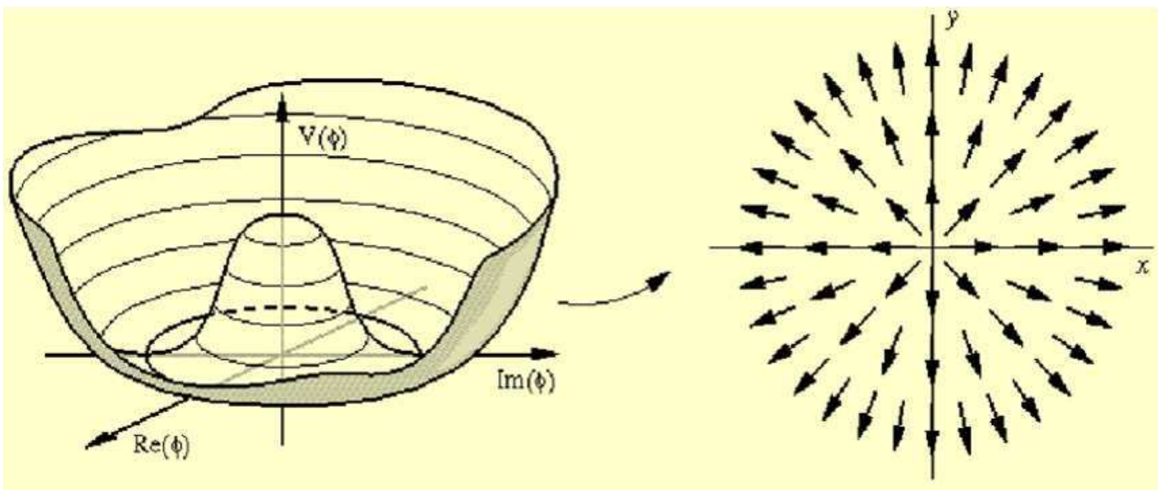

Figure 1.7: Left: An example of a symmetry-breaking potential, the Mexican hat-potential. Right: corresponding phase configuration in real space. The string would be localized at the origin and would extend in the z-direction. Figure from [15].

A simple symmetry breaking potential, the Mexican-hat potential, can give rise to strings through symmetry breaking. By looking at fig. 1.7 on the right panel, one can see by continuity that the field value has to vanish at the origin of the arrows: $\phi = 0$. The potential energy is then maximized and trapped along the z-axis and the string is extending in the z-direction. The constant quantity corresponding to the value of the field that minimizes the potential, is called the vacuum expectation value (VEV). The configurations minimizing the energy form a manifold, the vacuum manifold, $\mathcal{M}$. In the case of the Mexican-hat potential, $\mathcal{M}$

is just a circle : $\mathcal{M}$=S^1 and the VEV is $\langle\phi\rangle = e^{i\theta}v$ where v is the radius of the circle and θ the angle in the complex plane. The angle θ taken by the VEV and thus the value of the minimum will depend on the region of space. Topological defects corresponds to boundaries between regions with different choices of minima.

In particular, there is a non-trivial winding of the phase around a string as shown on fig.1.8.

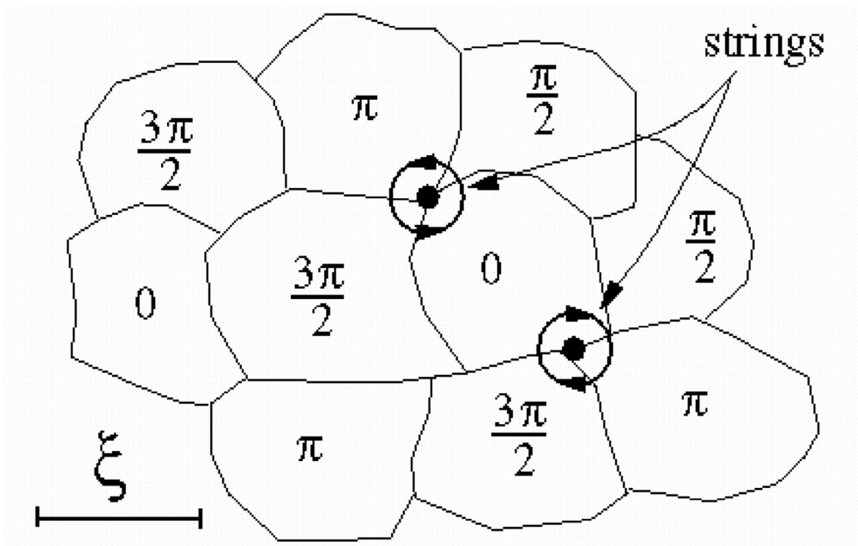

Figure 1.8: Non-trivial winding around strings. A string will be present only when there is a continuous winding between regions of different phases. The correlation length, ξ, describes the distance over which phases are uncorrelated. Figure from [15].

The Kibble mechanism

The Kibble mechanism demonstrates that the existence of defects structures depends on the topology of the vacuum manifold, $\mathcal{M}$, the space of minimum energy density configurations (ground states). Cosmic strings arise from the symmetry breaking from a group G(for example the group of rotations about the origin in the complex plane, U(1)) down to a subgroup H. The space of ground states, the vacuum manifold is such that $\mathcal{M}$=G/H. Topological defects exist provided that the right homotopy group is non-trivial : $\Pi_n(\mathcal{M}) \neq I$ with $n = 0, 1, 2$ for topological domain walls, strings and monopoles respectively. One-dimensional topological defects are formed when $\mathcal{M}$ is such that the first homotopy group, $\Pi_1(\mathcal{M})$, is non trivial [1]. This is equivalent to saying that $\mathcal{M}$ is not simply connected (it contains unshrinkable loops).

The Mexican hat potential is a good example of a U(1) symmetry breaking. In this case, the minima of energy lie on the circle making up the vacuum manifold : $\mathcal{M} = S^1$ as shown in fig.1.7.

1.4.2 QCD Phase transition in the early Universe

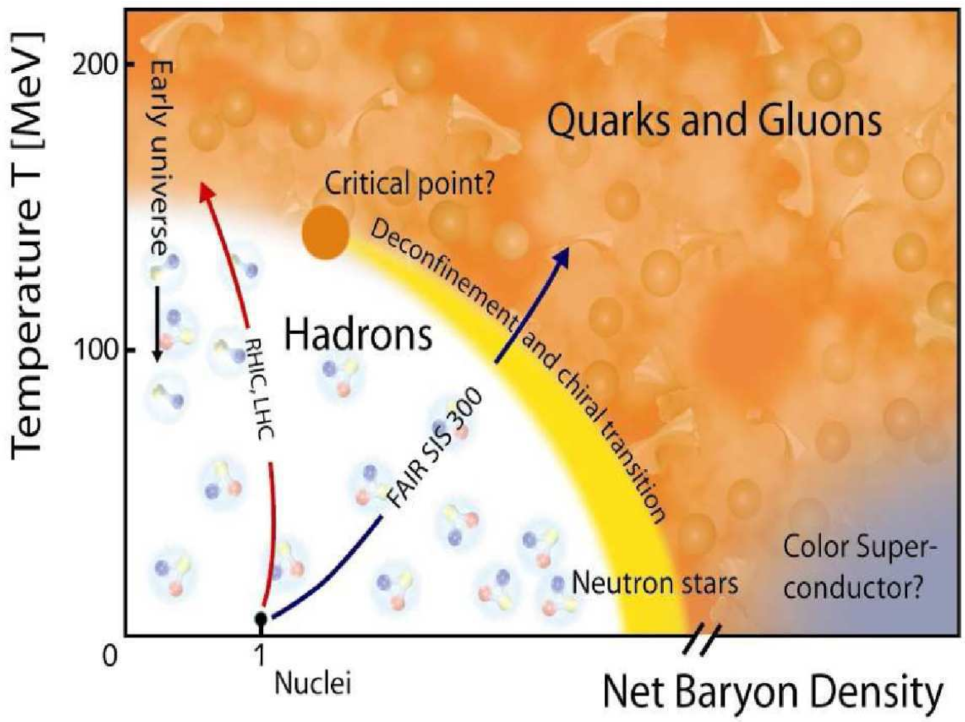

Figure 1.9: QCD phase transition in the Universe [16]

Approximately 10 microseconds after the Big Bang a phase transition from the quark-gluon plasma to a hadron gas is expected to have taken place at a temperature of about $T_{QCD} \sim 150 - 200\text{MeV}$. During the Quantum Chromodynamics (QCD) phase transition, composites of quarks called hadrons are formed. In particular, bound states of one quark and one anti-quark called mesons are created. Below this critical temperature, the physics of hadrons can be well described by a linear sigma model of four scalar fields which, three of them representing the pion triplet and one representing the σ-field.

Phase transition

A phase transition can be seen as a non-analyticity in the equation of state or in p(T) which describes the evolution of the pressure in function of the temperature. The transition is of first order if dp/dT has a discontinuity. If not, it is of higher-order [15]. Phase transitions are characterized by the order parameter. It is normally a quantity which is zero in one phase, usually above the critical temperature, and non-zero in the other. In case of a symmetry breaking potential, the order parameter of the system is given by the expectation value of the Higgs field, $\langle\phi\rangle$. On fig.1.10, one can see the jump in $\langle\phi\rangle$ occurring at the critical temperature.

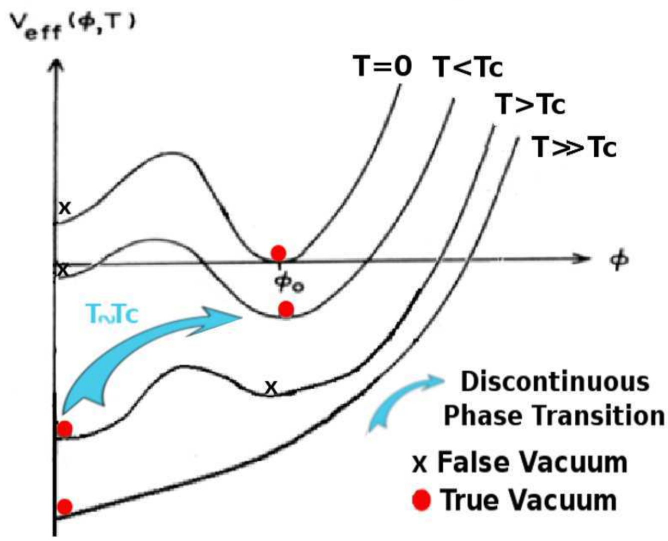

Figure 1.10: Example of the evolution of the potential shape at different temperature for a 1st order phase transition. In a 1st order phase transition the value of the Higgs field VEV, $\langle\phi\rangle$, changes discontinuisly close to the critical temperature, Tc .

If p(T) is continuous everywhere but the physical properties of the Universe change qualitatively within a short temperature interval, the transition is called a crossover. A simple example of crossover corresponds to the water to vapor transition at high-pressure. The current consensus on the QCD phase transition is that it is a crossover.

Thermal field theory

In order to model the effects of a thermal bath on a system, one can use finite-temperature field theory to compute physical observables [16]. One approach is to make the time imaginary and wrapped on itself with a period $\beta = 1/k_b T$, where k_b is the Boltzmann constant.

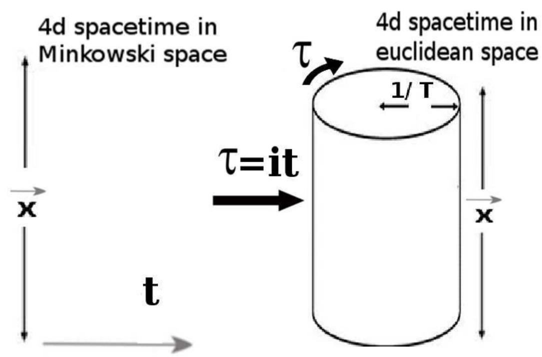

Figure 1.11: When the time is Euclideanized and made periodic, the spacetime can be seen as a cylinder of radius proportional to the inverse temperature : $r = \frac{1}{2\pi k_b T}$ and of infinite height.

As shown in fig.1.11, the new time variable, $\tau = it$, becomes compactified and runs from 0 to β instead of $-\infty$ to $+\infty$. As a result, spacetime becomes Euclidean: the metric goes from Minkowski, with signature (-, +,+,+)to a metric describing an Euclidean geometry (+,+,+,+). The Lagrangian density, $\mathcal{L}$, and the action, S, becomes Euclidean as well. The Euclidean action, S_E, is the integral of the Euclidean Lagrangian density, $\mathcal{L}_E$:

$$S_E = \int_0^\beta d\tau \int d^3x \, \mathcal{L}_E.$$

When computing a physical observable in thermal equilibrium, there are strong links with statistical physics. The partition function of the system, $\mathcal{Z} = \mathcal{Z}(\mathcal{V}, \mathcal{T}, \mu)$, can be used to describe the system as shown in fig.1.12.

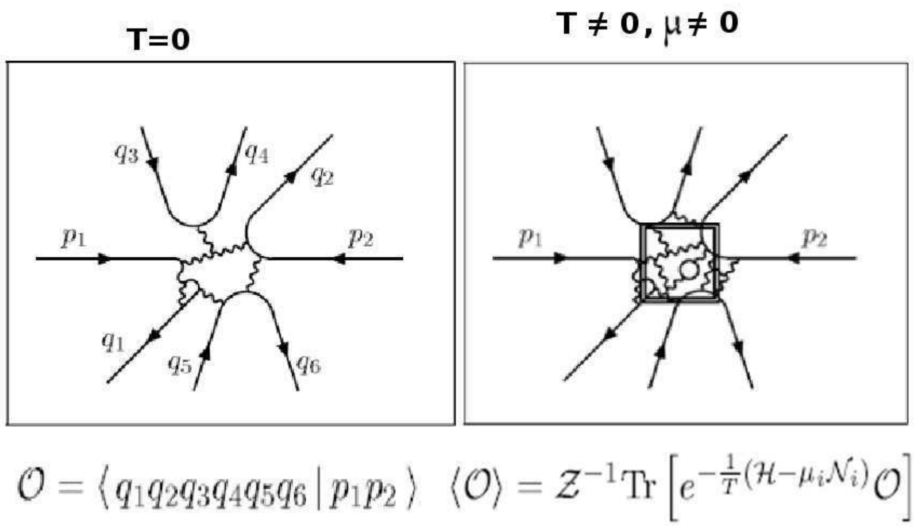

Figure 1.12: Particle scattering and simplification of the system at finite-temperature. Left: Complicated system at zero temperature. Right: At finite temperature and chemical potential, interactions within the box represent an ergodic system that thermalizes. Figure from [15].

$\mathcal{Z}$, the grand canonical partition function, is the trace of the density matrix of the system $\hat{\rho}$: $\mathcal{Z} = Tr\hat{\rho}$ where $\hat{\rho} = e^{-(H - \mu\hat{N})/T}$. H is the Hamiltonian, T is the temperature, μ the chemical potential and $\hat{N}$ is any conserved number operator. As in statistical physics, many physical observable can be defined from the partition function:

$$P = T\frac{\partial ln\mathcal{Z}}{\partial V}, \quad N = T\frac{\partial ln\mathcal{Z}}{\partial \mu}, \quad S = \frac{\partial T ln\mathcal{Z}}{\partial T}, \quad E = -PV + TS + \mu N \quad (1.23)$$

where P is the pressure, S the entropy and E, the energy of the system. As shown in fig.1.12, physical observables $\langle O \rangle$, can be computed via

$$\langle O \rangle = \frac{Tr[O\hat{\rho}]}{\mathcal{Z}}. \tag{1.24}$$

For a simple theory with boson fields ϕ and Euclidean action $S_E(\phi)$, the partition function is given by the path integral

$$Z = \int [d\phi] \exp[-S_E(\phi)]$$

where

$$S_E(\phi) = \int_0^\beta d\tau \int d^3x\, \mathcal{L}_E(\phi),$$

and the field ϕ satisfies periodic boundary conditions in the (imaginary) time direction $\phi(\tau = 0, x) = \phi(\tau = \beta, x)$. The quantum field theory may also involve fermions, in which case, fermion fields $\psi(\tau, x)$ instead satisfy anti-periodic boundary conditions : $\psi(\tau = 0, x) = -\psi(\tau = \beta, x)$.

In order to study more complicated systems describing the Universe it is useful to use the notion of effective potential. Usually, it can be defined by a Legendre transform via [19] :

$$Z[j] \equiv \exp\{iW[j]\} \tag{1.25}$$

and the effective action $\Gamma[\overline{\phi}]$ as the Legendre transform of (1.25)

$$\Gamma[\overline{\phi}] = W[j] - \int d^4x \frac{\delta W[j]}{\delta j(x)} j(x) \tag{1.26}$$

where $\overline{\phi}(x) = \frac{\delta W[j]}{\delta j(x)}$.

Factoring out an overall factor of spacetime volume, the effective potential, $V_{\text{eff}}(\phi_c)$, can be defined as,

$$\Gamma[\phi_c] = -\int d^4x = -\frac{V}{T} V_{\text{eff}}(\phi_c), \tag{1.27}$$

where ϕ_c is a constant.

In chapter 5, the effective potential coming from a thermal bath of photons acting on out-of-equilibrium fields is computed. In that case, finite-temperature field theory techniques apply only to the gauge field representing the photon field since it is the only field in thermal equilibrium. In order to compute the effective potential of the system, one can use the background field method and decompose each gauge field into a constant part and a perturbative part : $A_\mu = \bar{A}_\mu + A'_\mu$ where $\partial_\mu \bar{A}_\mu \partial^\mu \bar{A}^\mu = 0$. This method is further developed in chapter 5.

The linear sigma model

The linear sigma model is a simple toy model model where the symmetry breaking occurs when the sigma field takes on its vacuum expectation value and gives rise to a triplet of massless pions $\vec{\pi} = (\pi^0, \pi^+, \pi^-)$. Two are charged, π^+ and π^- and one is neutral, π_0. Table 1.2 shows some basic properties of the particles coming from the fields involved in the linear-sigma model.

Table 1.2: The Pion Triplet and its chiral partner

Pion	Anti-particle	Quark content	Rest mass MeV/c^2	Charge	Mean Lifetime in s
π^+	π^-	$u\bar{d}$	139.570 18(35)	+1 e	$2.6033 \pm 0.0005 \times 10^{-8}$
π^-	π^+	$\bar{u}d$	139.570 18(35)	-1 e	$2.6033 \pm 0.0005 \times 10^{-8}$
π^0	π^0	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$	134.976 6 (6)	0 e	$8.4 \pm 0.4 \times 10^{-17}$
Chiral partner σ : *Hypothetical*		$c_1(u\bar{u}+d\bar{d})+c_2 s\bar{s}$	600 (400-1200)	0 e	$6.6 - 11 \times 10^{-25}$

The following Lagrangian describes the system :

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - \eta^2)^2 \,, \tag{1.28}$$

where η^2 is the ground state expectation value of $\sigma^2 + \vec{\pi}^2$.

The potential term, $\frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - \eta^2)^2$ is invariant under the rotation of the four fields, σ,π^+, π^- and π^0. Thus, the symmetry of the vacuum manifold is $O(4)$ and the vacuum manifold is a 3-sphere : $\mathcal{M}= S^3$. According to the Kibble mechanism described above, the system does not admit any topologically stable strings since $\Pi_1(S^3) = 1$. By effectively reducing the vacuum manifold to S^1, it may be possible to obtain strings. Having an electric charge, the charged pions fields are coupled to electromagnetism and the Lagrangian can be promoted to a Lagrangian with covariant derivatives, electromagnetism becoming turned on.

In chapter 5, an effective potential is derived from the extra term coming from the coupling between the charged pions and gauge fields in thermal equilibrium. This gives rise to an “effective vacuum manifold” which admits strings.

REFERENCES

[1] R. H. Brandenberger, "Topological defects and structure formation," Int. J. Mod. Phys. A **9**, 2117 (1994) [arXiv:astro-ph/9310041].

R. H. Brandenberger and N. Turok, "Fluctuations From Cosmic Strings And The Microwave Background," Phys. Rev. D **33**, 2182 (1986).

H. Sato, " Galaxy Formation by Cosmic Strings," Prog. Theor. Phys. 75,1342 (1986)

A. Stebbins, "Cosmic strings and cold matter, " Astrophysical Journal, vol. 303, April 15, 1986, p. L21-L25.

[2] "The Physics of the Universe",
www.physicsoftheuniverse.com/topics_bigbang_timeline.html

[3] R. H. Brandenberger, PoS **ICFI2010**, 001 (2010) [arXiv:1103.2271 [astro-ph.CO]].

[4] Eric Chaisson, Steve McMillan, "Astronomy Today" (6th Edition), Modification of table 27.1, *Pearson Addison-Wesley, 2008*

[5] C. Ferguson, Alexei V. Filippenko, Saurabh W. Jha, Weidong Li, Ryan Chornock .
" A 3 percent Solution: Determination of the Hubble Constant With the Hubble Space Telescope and Wide Field Camera". The Astrophysics Journal : ApJ 730 119 (1 April 2011)
doi:10.1088/0004-637X/730/2/119

[6] S. M. Carroll, "Spacetime and geometry: An introduction to general relativity," *San Francisco, USA: Addison-Wesley (2004) 513 p*

[7] E. W. Kolb and M. S. Turner, "The Early Universe. REPRINTS", *Pearson Addison-Wesley, 1990*

[8] S. Dubovsky, T. Gregoire, A. Nicolis and R. Rattazzi, JHEP **0603**, 025 (2006) [arXiv:hep-th/0512260].

[9] S. D. H. Hsu, A. Jenkins and M. B. Wise, Phys. Lett. B **597**, 270 (2004) [arXiv:astro-ph/0406043].

[10] B. Grinstein, D. O'Connell and M. B. Wise, "The Lee-Wick standard model," Phys. Rev. D **77**, 025012 (2008) [arXiv:0704.1845 [hep-ph]].

[11] T. D. Lee and G. C. Wick, "Negative Metric and the Unitarity of the S Matrix," Nucl. Phys. B **9**, 209 (1969);
T. D. Lee and G. C. Wick, "Finite Theory of Quantum Electrodynamics," Phys. Rev. D **2**, 1033 (1970).

[12] J. M. Cline, S. Jeon and G. D. Moore, Phys. Rev. D **70**, 043543 (2004) [hep-ph/0311312].

M. B. Wise, "The Lee Wick Standard Model," arXiv:0908.3872 [hep-ph].

[13] A. Vilenkin and E. P. S. Shellard, "Cosmic Strings and Other Topological Defects," Cambridge University Press, 1994.

[14] W. Magnus and S. Winkler, "Hill's Equation," *Dover Publications, 1979.*

[15] "Cosmic strings and other defects"
www.damtp.cam.ac.uk/user/gr/public/cs_top.html

[16] GSI *http://www.gsi.de/forschung/fair_experiments/CBM/1intro_e.html*

[17] M. Laine, "Finite temperature field theory - with applications to cosmology," *Prepared for ICTP Summer School on Astroparticle Physics and Cosmology, Trieste, Italy, 17 Jun - 5 Jul 2002*

[18] J. I. Kapusta and C. Gale, "Finite-temperature field theory: Principles and applications," *Cambridge, UK: Univ. Pr. (2006) 428 p*

[19] R. Jackiw, "Functional evaluation of the effective potential," Phys. Rev. D **9**, 1686 (1974).

[20] J. Karouby and R. Brandenberger, Phys. Rev. D **82**, 063532 (2010)

[21] J. Karouby, T. Qiu and R. Brandenberger, Phys. Rev. D **84**, 043505 (2011) [arXiv:1104.3193 [hep-th]].

[22] J. Karouby, B. Underwood and A. C. Vincent, Phys. Rev. D **84**, 043528 (2011)

[23] J. Karouby and R. Brandenberger, Phys. Rev. D **85**, 107702 (2012)

Chapter 2

Radiation Bounce from the Lee-Wick Construction ?

The following is extracted from the article **"A Radiation Bounce from the Lee-Wick Construction?"** published in collaboration with Robert Brandenberger in $Phys.Rev.D\mathbf{82}, 063532$ (2010).

Abstract It was recently realized that matter modeled by the scalar field sector of the Lee-Wick Standard Model yields, in the context of a homogeneous and isotropic cosmological background, a bouncing cosmology. However, bouncing cosmologies induced by pressure-less matter are in general unstable to the addition of relativistic matter (i.e. radiation). Here we study the possibility of obtaining a bouncing cosmology if we add not only radiation, but also its Lee-Wick partner, to the matter sector. We find that, in general, no bounce occurs. The only way to obtain a bounce is to choose initial conditions with very special phases of the radiation field and its Lee-Wick partner.

2.1 Introduction

The inflationary scenario [1] is the current paradigm of early universe cosmology. It addresses some of the problems which the previous paradigm, the Standard Big Bang model, could not address, and it gave rise to the first theory of cosmological structure formation based on fundamental physics [2] whose predictions were later confirmed by the precision observations of the cosmic microwave background. Inflationary models, however, are faced with serious conceptual problems (see e.g. [3]), among which the singularity problem and the "Trans-Planckian" problem for fluctuations. In the context of General Relativity as the theory of space-time, it has been shown [4] that inflationary models have a singularity in the past and therefore cannot yield a complete theory of the early universe. The "Trans-Planckian" problem for fluctuations [3, 5] relates to the fact that in inflationary models, the wavelengths of perturbation modes which are observed today were smaller than the Planck scale in the early periods of inflation, and were thus in the "short wavelength zone of ignorance" in which we cannot trust the theory which is being used to track the fluctuations. In fact, in [5] it is shown that the predictions for observations are in fact rather sensitive to the physics assumed in this zone of ignorance. These conceptual problems of inflationary cosmology form one of the motivations for considering possible alternatives to inflation.

One of the alternative scenarios to inflation is the "matter bounce" paradigm (see e.g. [6, 7] for introductory expositions). In this scenario it is assumed that the Universe undergoes a nonsingular cosmological bounce. Time runs from $-\infty$ to $+\infty$. The time coordinate can always be adjusted such that the bouncing point is at time $t = 0$. The Hubble radius $H(t)^{-1}$ is the scale which separates wavelengths on which microphysics dominates (sub-Hubble) from those where matter forces are frozen out (super-Hubble). If the contracting and expanding phases far away from the bouncing point are described by General Relativity and we consider matter with pressure density $p > -\rho/3$ (where ρ is the energy density), then it follows that scales which are currently observed exited the Hubble radius at some point during the contracting phase. As was realized in [5, 9, 10], if the curvature fluctuations start out early in the contracting phase on sub-Hubble scales in their vacuum state, then the growth of the perturbations on super-Hubble scales during the period of contraction leads to a scale-invariant spectrum of curvature fluctuations on super-Hubble scales before the bounce. Detailed analyses of the evolution of cosmological fluctuations through the nonsingular bounce performed in the context of specific bouncing models (see e.g. [11, 12, 13]) show that the spectrum of curvature fluctuations is unchanged during the bounce on wavelengths which are large compared to the bounce time, a result which agrees with what is obtained by applying the Hwang-Vishniac matching conditions [14, 15] to connect perturbations across a spacelike "matching" hypersurface between a contracting and an expanding Friedmann universe.

By construction, a bouncing cosmology is nonsingular. In such a model, the wavelength of fluctuations which are being probed in current observations always remains far larger than typical microphysical scales. If the energy density at the bouncing point is set by the scale of particle physics Grand Unification, then the physical wavelength corresponding to the current Hubble radius is about 1mm, to quote just one number. Hence, the fluctuations remain in the regime controlled by the infrared limit of the theory, far from the trans-Planckian zone of ignorance.

The challenge is to obtain a bouncing cosmology. One must either give up General Relativity as the theory of space-time, or else one must invoke a new form of matter which violates some of the "usual" energy conditions (see [16] for a discussion of the assumptions underlying the singularity theorems of General Relativity).

For a recent review on how bouncing cosmologies can be obtained see [17]. We here mention but a few recent attempts. Introducing higher derivative gravity terms can lead to nonsingular cosmologies, as in the "nonsingular universe construction" of [18]. Similarly, the ghost-free higher derivative action of [19] leads to a bouncing cosmological background. Horava-Lifshitz gravity also leads to a bouncing cosmology provided that the spatial curvature does not vanish [20]. Bouncing cosmologies may also arise from quantum gravity, as e.g. in loop quantum cosmology (see e.g. [21] for a recent review). If we maintain General Relativity as the theory of space and time, then one can obtain a bounce by introducing new forms of matter such as "quintom" matter [22]. In this case, in addition to the matter sector with regular sign kinetic action, there is a new sector (a "ghost" sector) which has an opposite sign kinetic action.

Several decades ago, Lee and Wick [23] introduced a field theory construction which involves degrees of freedom with opposite sign kinetic terms. The Lee-Wick model aims at stabilizing the Higgs mass against quadratically divergent terms and is interesting to particle physicists since it can address the "hierarchy problem". The Lee-Wick construction was recently resurrected and extended to yield a "Lee-Wick Standard Model" [24, 25]. The Lee-Wick model can thus potentially provide a framework for obtaining a bouncing cosmology. In [26], the Higgs sector of the Lee-Wick Standard Model was analyzed and it was shown that, indeed, a bouncing cosmology emerges. However, the scalar field Lee-Wick bounce is unstable against the addition of regular radiation to the matter sector (as will be explained in Section 2 of this paper). Since we know that there is radiation in the Universe, one may then worry whether the Lee-Wick bounce can be realized at all. However, to be consistent with the philosophy of the Lee-Wick construction, Lee-Wick radiation terms with opposite sign kinetic actions must be added. In this paper we address the question whether, in this context, a cosmological bounce can be achieved. We find that unless the phases of the two fields are chosen in a very special way then no bounce will occur.

The outline of this paper is as follows: in the next section we briefly review the philosophy behind the Lee-Wick model and discuss why the scalar sector of the Lee-Wick model taken alone would yield a bouncing cosmology. In Section 3 we introduce the Lagrangian for Lee-Wick electromagnetism and derive the expression for the energy density. In order to study the cosmological implications of our action, we need to know how plane waves of the Lee-Wick partner of the radiation field evolves. This is the focus of Section 4. After understanding how regular and Lee-Wick radiation evolve, we can then study under which conditions a bouncing cosmology might result.

2.2 Review of the Lee-Wick Model and the Scalar Lee-Wick Bounce

We will review the Lee-Wick model and the Lee-Wick bounce in the simple case of a single scalar field $\hat{\phi}$. The hypothesis of Lee and Wick [23] was to add an extra scalar degree of freedom designed to cancel the quadratic divergences in scattering matrix elements. Originally, the new degree of freedom was introduced by adding a higher derivative term of the form $(\partial^2\hat{\phi})^2$ to the action, yielding a higher order differential equation and hence a new degree of freedom. It is, however, simpler to isolate the new degree of freedom by introducing an auxiliary scalar field $\tilde{\phi}$ and redefining the "physical" field to be ϕ (see [24]). After doing this and after a field rotation the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}M^2\tilde{\phi}^2 - \frac{1}{2}m^2(\phi-\tilde{\phi})^2 - V(\phi-\tilde{\phi})\,, \tag{2.1}$$

where $M \gg m$ is the mass scale of the new degree of freedom, and V is the original potential which after the field redefinition depends on both fields.

The field $\tilde{\phi}$ is called the Lee-Wick partner of ϕ. It has the opposite sign kinetic Lagrangian and the opposite sign of the mass square term. Hence, without any coupling to other fields or to gravity the evolution of $\tilde{\phi}$ would be stable and consist of oscillations about $\tilde{\phi} = 0$. However, in the presence of any coupling of $\tilde{\phi}$ with other fields there are serious potential instability and unitarity problems [27, 28, 29, 30]. Ways to make the theory consistent were discussed many years ago in [31] and more recently in [32] in the case of interest in the current paper, namely Lee-Wick electromagnetism. In [32], a proposal for a ultraviolet (UV) complete theory of Quantum Electrodynamics via the Lee-Wick construction was made. It was argued that the presence of ghost poles in virtual state propagators and the loss of microcausality do not necessary mean that causality is violated at macroscopic scales. This would be the case if the Lee-Wick particles decayed fast enough [33].

The Lee-Wick model has been resurrected in [24] with the goal of studying signatures of this alternative model to supersymmetry in LHC experiments. For some projects to try to test experimentally the predictions of the Lee-Wick model see e.g. [35, 34].

Let us now review [26] how a nonsingular bouncing cosmology can emerge from the scalar sector of a Lee-Wick model. In fact, for this to happen no coupling between these fields is required, and hence we will assume $V = 0$ in the following discussion. We take initial conditions at some initial time in which both the scalar field ϕ and its Lee-Wick partner $\tilde{\phi}$ are both oscillating about their ground states, and that the positive energy density of ϕ exceeds the absolute value of the negative energy density of $\tilde{\phi}$, i.e. we start in a phase dominated by regular matter.

We assume that the Universe is contracting with a Hubble rate dictated by the Friedmann equations.

Initially both fields are oscillating and their energy densities both scale as $a^{-3}(t)$, where $a(t)$ is the cosmic scale factor. Since $M \gg m$ while the energy density of $\tilde{\phi}$ is smaller than that of ϕ, the amplitude $\tilde{\mathcal{A}}$ of $\tilde{\phi}$ must be much smaller than the amplitude $\mathcal{A}$ of ϕ. During the initial period of contraction, both amplitudes increase at the same rate. At some point, however, $\tilde{A}$ becomes comparable to m_{pl}, the four dimensional Planck mass. As we know from the dynamics of chaotic inflation [36], at super-Planckian field values ϕ will cease to oscillate - instead, it will enter a "slow-climb" regime, the time reverse of the inflationary slow-roll phase. During this phase, the energy density of ϕ increases only slightly. However, $\tilde{\phi}$ continues to oscillate and its energy density increases in amplitude exponentially (still proportional to a^{-3}). The energy in $\tilde{\phi}$ (i.e. its absolute value) will hence rapidly catch up with that of ϕ. When this happens, H will vanish. Since the kinetic energy of $\tilde{\phi}$ overwhelms that of ϕ, $\dot{H} > 0$ and thus a nonsingular bounce will occur [26], the Universe will begin to expand.

The matter bounce in the Lee-Wick scalar field model was analyzed in detail in [26]. In particular, it was verified explicitly that initial vacuum fluctuations on sub-Hubble scales in the contracting period develop into a scale-invariant spectrum of curvature fluctuations on super-Hubble scales after the bounce. A distinctive prediction of this scenario is the shape and amplitude of the three-point function, the 'bispectrum" [37].

However, the scalar field Lee-Wick bounce in unstable towards the addition of radiation before the bounce [1] : Since the energy density in radiation scales as a^{-4} it becomes more important than that of $\tilde{\phi}$ as the Universe decreases in size, and will hence destabilize the bounce. Can the addition of a Lee-Wick partner to regular radiation help restore the bounce? This is the question we ask in this work. We will follow the same type of reasoning as above, but for the case of radiation: we now introduce a Lee-Wick gauge field, the partner of the standard one, which will initially be dominant. We use the Lagrangian for a U(1) Lee-Wick gauge boson (see [24]) to which we add a coupling term between the normal and the Lee-Wick field in order to allow the energy to flow from one component to the other. Our goal is to see if we can get a bouncing universe using this setup.

2.3 The Model

We will consider the radiation sector of Lee-Wick quantum electrodynamics and will start with a higher derivative Lagrangian [24] for a $U(1)$ gauge field A_μ of the form:

$$\mathcal{L}_{hd} \, = \, -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \frac{1}{2M_A^2}\mathcal{D}^\mu \hat{F}_{\mu\nu}\mathcal{D}^\lambda \hat{F}^\nu_\lambda \, , \tag{2.2}$$

where $F_{\mu\nu}$ is the field strength tensor associated with A_μ and $\mathcal{D}$ denotes the covariant derivative. Note the sign difference in the second term compared to [24] : This will prevent the appearance of a tachyonic massive Lee-Wick (L-W) gauge boson. The mass M_A corresponds to the mass of the new physics in the model. To solve

---

[1] The Lee-Wick matter bounce is also unstable against the addition of anisotropic stress in the initial conditions. This is a well-known problem for bouncing cosmologies which we will not further address in this paper.

the Hierarchy Problem of the Standard Model, this mass should be of the order of 1TeV.

The higher derivative terms in the above Lagrangian lead to an extra propagating mode. We can isolate it using the usual Lee-Wick construction by introducing a new field $\tilde{A}$ ($\hat{A} = A + \tilde{A}$) called Lee-Wick partner, which depends on derivatives of the original field and adjusting the gauge fields such that the kinetic term of the Lagrangian becomes diagonal in A_μ and $\tilde{A}_\mu$. We find that the propagator for the $\tilde{A}_a$ field has pole at $p^2 = M_A^2$ and has an opposite sign compared to the normal one. Thus, it is a ghost field (with the associated problems of instability and nonunitarity mentioned in the previous section). The Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu} - \tilde{F_{\mu\nu}}\tilde{F^{\mu\nu}}) + cF_{\mu\nu}\tilde{F^{\mu\nu}} + \frac{M_A^2}{2}\tilde{A}_a\tilde{A}^a \,. \tag{2.3}$$

We have added a coupling term, with coupling constant c, in order to allow the energy density to be able to flow from the normal field to the Lee-Wick field. Since the Lee-Wick sector is not observed in experiments today, we choose the two fields to be weakly coupled. In the case when the coupling constant is equal to zero, M_A is the mass of the L-W gauge field.

Note that the $U(1)$ gauge invariance of electromagnetism is broken by the addition of the Lee-Wick sector. In addition to the problem of ghosts, this is another serious potential problem for the model which we are currently investigating. Given that gauge invariance is violated, we need to justify our choice of the coupling between the two fields. We have used gauge invariance and power counting renormalizability to pick out the term we have added to the Lagrangian in order to describe the coupling. If the entire Lagrangian were gauge-invariant, this would clearly be the correct procedure. In the presence of a symmetry breaking term which is very small (for large values of M_A) we can use gauge invariance of the low energy terms in the action to justify neglecting small symmetry breaking coupling terms if we are interested in energy transfer between the two fields which should be operational already at low energies.

As our initial conditions in a contracting universe, we imagine that the usual radiation field dominates the energy-momentum tensor. This implies that we must set the initial amplitude of $\tilde{A}_\mu$ to be very small compared to that of the regular gauge field A_μ. In this case, then if M_A is large enough compared to the experimental energy scale, we would not expect to see the ghost radiation field in experiments.

The energy-momentum tensor following from (2.3) is

$$\begin{aligned} T_{\mu\nu} &= -\tfrac{1}{4}g_{\mu\nu}(F_{\lambda\sigma}F^{\lambda\sigma} - \tilde{F_{\lambda\sigma}}\tilde{F^{\lambda\sigma}} - 4cF_{\lambda\sigma}\tilde{F^{\lambda\sigma}}) + F_\mu{}^\lambda F_{\nu\lambda} - \tilde{F_\mu}^\lambda \tilde{F_{\nu\lambda}} \\ &\qquad +\tfrac{1}{2}g_{\mu\nu}{M_A}^2\tilde{A}_a\tilde{A}^a - M_A^2\tilde{A}_\mu\tilde{A}_\nu - 4cF_{\mu\lambda}\tilde{F_\nu^\lambda} \end{aligned} \tag{2.4}$$

and its trace is, contrary to the case of pure radiation, nonzero:

$$T^\mu_\mu = M_A^2\tilde{A}_a\tilde{A}^a\,. \tag{2.5}$$

Using the Friedmann metric given by (2.16), the energy density is equal to:

$$T_{00} = \frac{1}{4}(F^2 - \tilde{F}^2) - cF_{\lambda\sigma}\tilde{F}^{\lambda\sigma} + F_0{}^\lambda F_{0\lambda} - \tilde{F_0}^\lambda \tilde{F_{0\lambda}} - M_A^2(\frac{\tilde{A}^2}{2} + \tilde{A_0}^2) - 4cF_{0\lambda}\tilde{F_0^\lambda}\,. \tag{2.6}$$

We can split this into three different terms, the contribution of normal radiation,

$$\rho_A = \frac{1}{4}(F^2 + F_0{}^\lambda F_{0\lambda})\,, \tag{2.7}$$

the contribution from Lee-Wick radiation,

$$\rho_{\tilde{A}} = -\frac{1}{4}(\tilde{F}^2 + \tilde{F_0}^\lambda \tilde{F_{0\lambda}}) - M_A^2(\frac{\tilde{A}^2}{2} + \tilde{A_0}^2)\,, \tag{2.8}$$

and the term coming from the mixing between the two fields,

$$\rho_{A-\tilde{A}} = -c(F_{\lambda\sigma}\tilde{F^{\lambda\sigma}} + 4F_{0\lambda}\tilde{F_0^{\lambda}})\,. \tag{2.9}$$

The equation of state is like that of radiation but with an additional term proportional to the mass of the Lee-Wick gauge field:

$$w \equiv \frac{p}{\rho} = \frac{\rho}{3\rho} + \frac{T^{\mu}_{\mu}}{3\rho} = \frac{1}{3} + \frac{M_A^2\tilde{A}_a\tilde{A}^a}{3T_{00}}. \tag{2.10}$$

We note that this expression is valid only when the total energy density is nonzero, and thus it would not be valid at the bouncing point if there were a bounce.

We can actually define three different equation of state parameters, one for each type of energy:

$$\begin{aligned} w_A &= w_{A-\tilde{A}} = \frac{1}{3} \text{ and} \\ w_{\tilde{A}} &= \frac{1}{3} + \frac{M_A^2\tilde{A}_a\tilde{A}^a}{\rho_{\tilde{A}}}\,, \end{aligned} \tag{2.11}$$

the last of which is nonconstant in time. The equation of state parameter for the coupling term is the same as the one for normal radiation since the trace of the coupling energy-momentum tensor vanishes.

Our goal is to see under which conditions the above matter Lagrangian leads to a cosmological bounce. We will initially turn off the coupling between the two fields (i.e. set $c = 0$), derive the solutions of the equations of motion for both fields, and study what scaling with the cosmological scale factor $a(t)$ these solutions imply for the three contributions to the energy density discussed above. We find that - unlike what happens for the scalar field Lee-Wick model of [26] - there is no mechanism which leads to a faster increase in the energy density of the Lee-Wick partner field than that of the original radiation field. Thus, a bounce can only occur if there is a mechanism which drains energy from the original gauge field sector to the Lee-Wick partner field. It is for this reason that we have introduced a direct coupling term between the two fields in our Lagrangian. We will then study the effects of the coupling between the two fields, working in Fourier space and making use of the Green function method. We find that the sign of the energy transfer depends not only on the sign of the coupling coefficient c, but also on the phases of the oscillations of the two fields. Averaging over the phases, we find no net energy transfer, and hence there can be no cosmological bounce.

As initial conditions we choose a state in the contracting phase in which the regular radiation field is in thermal equilibrium at some initial time t_i. Since we want to start with a state which looks like the time reflection of the state we are currently in, we assume that the energy density in $\tilde{A}_\mu$ is initially subdominant. We, however, do assume that $\tilde{A}_\mu$ has excitations for modes with wave-number comparable to the initial temperature.

In the absence of coupling between the two fields, the distribution of A_μ would remain thermal, with a temperature T which blue-shifts as the Universe contracts. The corresponding energy density would scale as a^{-4}. The presence of coupling will lead to a departure from thermal equilibrium. We will assume, however, that $a(t)$ continues to scale like $\sqrt{t}$, the scale factor of radiation. If there were a bounce, this approximation would fail at some point sufficiently close to the bounce time.

2.4 Equations of Motion

The equations of motion obtained from varying the Lagrangian with respect to A_μ and $\tilde{A}_\mu$ are:

$$\begin{aligned} \partial_\mu(F^{\mu\nu} - 2c\tilde{F^{\mu\nu}}) + 3H(F^{0\nu} - 2c\tilde{F^{0\nu}}) &= 0 && (2.12) \\ -M_A^2\tilde{A}^\nu + \partial_\mu(\tilde{F^{\mu\nu}} + 2cF^{\mu\nu}) + 3H(\tilde{F^{0\nu}} + 2cF^{0\nu}) &= 0\,. && (2.13) \end{aligned}$$

Combining them, we find that the L-W field will act as a source term for the normal field:

$$\partial_\mu F^{\mu\nu} + 3HF^{0\nu} \;=\; \frac{2cM_A^2}{1+4c^2}\tilde{A}^\nu \tag{2.14}$$

but that the L-W field is decoupled from the normal one and therefore only depends on the initial conditions:

$$\partial_\mu \tilde{F^{\mu\nu}} + 3H\tilde{F^{0\nu}} - \frac{M_A^2}{1+4c^2}\tilde{A}^\nu \;=\; 0\,. \tag{2.15}$$

From this last equation, we can also read off the new mass which the Lee-Wick partner field obtains in the presence of coupling: $M_A' = \frac{M_A}{\sqrt{1+4c^2}}$, which is about the same as M_A at weak coupling. We can notice that at very strong coupling, the L-W gauge field becomes massless and therefore would evolve like a normal photon.

We will consider a homogeneous and isotropic universe with metric

$$ds^2 \;=\; -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]\,, \tag{2.16}$$

where t is physical time, x, y and z are the three spatial comoving coordinates, and we have for notational simplicity assumed that the Universe is spatially flat.

Since the equations of motion are linear, we can work in Fourier space, i.e. with plane wave solutions. There will be no coupling between different plane waves. For simplicity, we focus on waves propagating along the z-axis with the same wave number, k, for the Lee-Wick and the normal gauge field. We work in the real basis of Fourier modes $cos(kz)$ and $\sin(kz)$.

Without loss of generality we can restrict attention to one polarization mode which we take to be the electric field in the x direction and the magnetic field in the y direction. In this case, the only nonzero components of the field strength tensors are : $F^{01}, F^{13}, \tilde{F^{01}}$ and $\tilde{F^{13}}$. Using the temporal gauge where : $\tilde{A}_0 = A_0 = 0$, we find that only the first component of the gauge fields are nonzero, and we can make the ansatz

$$A_1(k,t) \;=\; f(t)cos(kz) \;\text{ and }\; \tilde{A}_1(k,t) \;=\; g(t)cos(kz) \tag{2.17}$$

or equivalently

$$A^1(k,t) \;=\; a(t)^{-2}f(t)cos(kz) \;\text{ and }\; \tilde{A}^1(k,t) \;=\; a(t)^{-2}g(t)cos(kz)\,. \tag{2.18}$$

From (2.14) and (2.15),we obtain two linear second order differential equations with a damping term for the coefficient functions $f(t)$ and $g(t)$:

$$\ddot{f}(t) + H\dot{f}(t) + [\frac{k}{a(t)}]^2 f(t) \;=\; -\frac{2c}{1+4c^2}M_A^2 g(t) \tag{2.19}$$

$$\ddot{g}(t) + H\dot{g}(t) + [[\frac{k}{a(t)}]^2 + \frac{M_A^2}{1+4c^2}]g(t) \;=\; 0\,. \tag{2.20}$$

For $\frac{k}{a} \ll \frac{M_A}{\sqrt{1+4c^2}}$, the L-W field behaves as a harmonic oscillator with angular frequency $\frac{M_A}{\sqrt{1+4c^2}}$. As a consequence of the cosmological dynamics the oscillator undergoes damping (in an expanding universe) or antidamping (in the case of interest to us, that of a contracting universe). The regular radiation field satisfies the equation of a driven oscillator, again subject to cosmological damping or anti-damping. Notice that(2.19) has a particular solution $f_p(t) = 2cg(t)$. The driving term can lead to energy transfer between the regular radiation field and its L-W partner. In the following we wish to study if the energy transfer is able to drain enough energy from the regular radiation field to enable a bounce to occur.

To solve these equations for any H(t), it is easier to use the conformal time $\eta = \int \frac{dt}{a}$ and to make things clearer, we introduce new functions u and v such that $u(\eta) = f(\eta)$ and $v(\eta) = g(\eta)$. Equations (2.19) and (2.20) can thus be rewritten as:

$$u''(\eta) + k^2 u(\eta) = -a(t)^2 \frac{2c}{1+4c^2} M_A^2 v(\eta) \tag{2.21}$$

$$v''(\eta) + [k^2 + a(t)^2 \frac{M_A^2}{1+4c^2}] v(\eta) = 0\,, \tag{2.22}$$

where $'$ denotes the derivative with respect to η. For a radiation-dominated universe, we have $a(\eta) = \eta$.

From (2.21) we see that in the absence of coupling we get simple oscillations in conformal time with frequency k for the normal gauge field. For the L-W field we get oscillations in conformal time, with a time dependant frequency $\tilde{k} = \sqrt{k^2 + a(t)^2 \frac{M_A^2}{1+4c^2}}$. In physical time these correspond to :

$$\begin{aligned} f(t) &= C cos(2\sqrt{t}k + \phi) \\ g(t) &= \frac{\alpha}{t^{\frac{1}{4}}} WhM(\frac{-ik^2\sqrt{1+4c^2}}{2M_A}, \frac{1}{4}, \frac{2iM_A t}{\sqrt{1+4c^2}}) + \frac{\beta}{t^{\frac{1}{4}}} WhW(\frac{-ik^2\sqrt{1+4c^2}}{2M_A}, \frac{1}{4}, \frac{2iM_A t}{\sqrt{1+4c^2}}) \end{aligned} \tag{2.23}$$

where WhM and WhW are the Whittaker functions (see e.g. [38]), α and β are constants characterizing the phase of $g(t)$ and ϕ is the phase of $f(t)$.

Before discussing the solutions of these equations we must specify our initial conditions. We consider a contracting phase dominated by regular radiation. Since we have in mind an initial state which looks like the time reverse of a state in the early radiation phase of our expanding cosmology, we will start at some time t_i in thermal equilibrium with a temperature much smaller than the Planck scale. The occupation numbers of the Fourier modes of the regular radiation field are hence given by the thermal distribution, with the peak wave-number being set by the temperature and hence much larger than the Hubble rate. We are thus considering modes inside the Hubble radius. Since we are interested in studying the possibility of obtaining a bounce, we will work at temperature higher than the mass M_A.

We assume that the energy density of the L-W radiation field is subdominant at the initial time t_i. The most conservative assumption is that the distribution of wave-numbers is also peaked at the initial temperature. These assumptions will allow us to pick out the limiting cases of the solutions of the above equations (to be discussed in the following section) which are relevant for us.

2.5 Solutions

2.5.1 Solutions for the Lee-Wick field

Depending on whether the physical wave-number is larger or smaller than the mass of the L-W gauge field, $M'_A = \frac{M_A}{\sqrt{1+4c^2}}$, we get different behaviors for the solution g. Since we are interested in exploring the solutions at high densities, close to the hypothetical bouncing point, we will assume that the temperature is larger than the mass L-W field. We will focus on wave-numbers close to the peak of the thermal distribution function, and hence $k/a > M'_A$. In this limit, the solutions for the L-W gauge field will simply be oscillating in conformal time with frequency k:

$$g(t) \, = \, \tilde{C} cos(\eta k) \, = \, \tilde{C} cos(2\sqrt{t}k) \, , \tag{2.24}$$

where we have used the scaling of $a(t)$ of a radiation-dominated universe to express the conformal time η in terms of physical time t, and where $\tilde{C}$ is a constant amplitude.

The normal gauge field satisfies a harmonic oscillator equation with a driving term with which the L-W field acts on it. The strength of the driving term is proportional to the coupling constant c in the Lagrangian. The general solution of the inhomogeneous equation for u is the general solution of the homogeneous equation plus a particular solution of the inhomogeneous equation whose amplitude is proportional to c and which can be determined using the Green function method (see later). The homogeneous solution for u is oscillating with frequency k.

For large wavelength, i.e. $\frac{k}{\eta} \ll M'_A$ the solutions for g behave like a combination of Bessel functions.

$$g(t) \;=\; \alpha\; t^{\frac{1}{4}} J(\frac{1}{4}, \frac{M_A t}{\sqrt{1+4c^2}}) + \beta\; t^{\frac{1}{4}} Y(\frac{1}{4}, \frac{M_A t}{\sqrt{1+4c^2}}) \,, \tag{2.25}$$

where α and β are constants that can be determined using the initial conditions and J and Y are, respectively, the Bessel functions of order $\frac{1}{4}$ of the first and the second kind.

A more physical way of understanding the behavior is to rewrite the solutions in the asymptotic limits. For large values of t and for $M'_A t \gg |\frac{1}{16} - 1|$, the L-W gauge field oscillates with a frequency corresponding to the mass of the L-W gauge field, M'_A. Indeed, in this case :

$$J(\frac{1}{4}, M'_A t) \;\;\approx\;\; \sqrt{\frac{2}{\pi M'_A t}} \cos\left(M'_A t - \frac{3\pi}{8}\right) \tag{2.26}$$

$$Y(\frac{1}{4}, M'_A t) \;\;\approx\;\; \sqrt{\frac{2}{\pi M'_A t}} \sin\left(M'_A t - \frac{3\pi}{8}\right) \tag{2.27}$$

Therefore, in this limit the L-W gauge field scales like $g(t) \propto t^{-1/4} \sim a(t)^{-1/2}$ when we are in a radiation-dominated period, which we are during a certain time since the initial state is dominated by regular radiation.

To better understand the behavior of the solutions in the small k limit and at large times, we can rewrite the solution using powers of the scale factor. The two independent solutions are

$$g(t) \;\approx\; a(t)^{-\frac{1}{2}} e^{\pm \int \frac{1}{2}\sqrt{H(t)^2 - \frac{4M_A^2}{1+4c^2}}dt} \,, \tag{2.28}$$

though this expression is valid only when the square root term in the exponential is approximately constant. Choosing the initial time t_i such that $\frac{M_A^2}{1+4c^2} \gg H(t_i)^2$ we see that this inequality stays valid only a finite period of time since $H(t_i)$ increases with time in a radiation phase of a contracting universe. We immediately get $g(t) \propto a(t)^{-1/2} cos(\frac{M_A}{\sqrt{1+4c^2}} t)$ which is in agreement with the behavior we found using asymptotic values of the Bessel functions.

In the opposite case, when t is close to 0 (and we are still considering large wave-numbers), the asymptotic forms of the Bessel functions of first and second kind scale as a power of t:

$$t^{\frac{1}{4}} J(\frac{1}{4}, \frac{M_A t}{\sqrt{1+4c^2}}) = \frac{2^{\frac{5}{4}} (\frac{M_A}{\sqrt{1+4c^2}})^{1/4} \Gamma(\frac{3}{4}) \sqrt{t}}{\pi} + o(t^2) \tag{2.29}$$

$$t^{\frac{1}{4}} Y(\frac{1}{4}, \frac{M_A t}{\sqrt{1+4c^2}}) = \frac{-2^{\frac{3}{4}}}{(\frac{M_A}{\sqrt{1+4c^2}})^{1/4} \Gamma(\frac{3}{4})} + \frac{2^{\frac{3}{4}} (\frac{M_A}{\sqrt{1+4c^2}})^{1/4} \Gamma(\frac{3}{4}) \sqrt{t}}{\pi} + \frac{1}{3} \frac{2^{\frac{3}{4}} M_A^2 t^2}{(\frac{M_A}{\sqrt{1+4c^2}})^{\frac{1}{4}} (1+4c^2) \Gamma(\frac{3}{4})} + \mathcal{O}(t^2)\,.$$

If we choose the amplitude of the two Bessel functions to be equal and opposite in (2.25), we get a cancellation of the square root term in g(t) and thus the L-W gauge field scales as $g(t) \approx C_3 - C_4 t^2 + o(t^2)$. In the general case we get $g(t) \approx C_3 + C_5 \sqrt{t}$ where C_3 and C_5 are constants.

Note that the closer we get to $t = 0$, less and less modes will satisfy the condition $k \ll |\eta| \frac{M_A}{\sqrt{1+4c^2}}$. Instead, they will evolve into the large wave-number regime discussed at the beginning of this subsection. They will oscillate and behave exactly as normal radiation.

We note that since g(t) is just oscillating , its effect on the normal field will decrease with time in a contracting phase as the source will scale as $a(t)^2 \sim t$ in a radiation-dominated era and time runs from $-\infty$ to 0 in the contracting phase.

2.5.2 Scaling of the Energy Densities

The energy densities for each type of radiation can be rewritten in terms of f, g and their derivatives for each mode k by averaging $< cos(kz)^2 >$ over the z-direction:

$$\rho_A(t,k) = \frac{1}{4a^2}[(\frac{k}{a})^2 f(t)^2 + \dot{f}(t)^2] \tag{2.30}$$

$$\rho_{\tilde{A}}(t,k) = -\frac{1}{4a^2}[((\frac{k}{a})^2 + \frac{M_{\tilde{A}}^2}{2})g(t)^2 + \dot{g}(t)^2] \tag{2.31}$$

$$\rho_{A-\tilde{A}}(t,k) = -\frac{c}{a^2}[(\frac{k}{a})^2 f(t)g(t) + \dot{f}(t)\dot{g}(t)]\,. \tag{2.32}$$

Rewriting this in term of conformal time, η, we get :

$$\rho_A(\eta,k) = \frac{1}{4a(\eta)^4}[u'(\eta)^2 + k^2 u(\eta)^2] \tag{2.33}$$

$$\rho_{\tilde{A}}(\eta,k) = \frac{-1}{4a(\eta)^4}[v'(\eta)^2 + [k^2 + \frac{M_{\tilde{A}}^2}{2}a(\eta)^2]v(\eta)^2] \tag{2.34}$$

$$\rho_{A-\tilde{A}}(\eta,k) = \frac{-c}{a(\eta)^4}[u'(\eta)v'(\eta) + k^2 u(\eta)v(\eta)]\,. \tag{2.35}$$

In the absence of coupling between the two fields the solutions for u correspond to undamped oscillations. Hence, the energy density of the regular radiation field scales as a^{-4} as we know it must. The contribution of all short wavelength modes to the L-W energy density also scales as a^{-4} since for these modes v is oscillating with constant amplitude. The coefficient is negative as expected for a ghost field. The third energy density, that due to interactions, also scales as a^{-4} for short wavelengths.

The contribution of long wavelength modes to the energy density of the L-W field and to the interaction energy density scale as a^{-p} with a power p which is smaller than 4. For large times, the power p is 3 in the energy density for the L-W field, i.e. a scaling like that of nonrelativistic matter. Close to $t = 0$ the power changes to $p = 2$. This can be seen most clearly from (2.31) and from the scalings of $g(t)$ derived earlier.

Hence, we conclude that in the absence of coupling between the two fields (i.e. for $c = 0$), the energy density in the regular radiation field will dominate throughout the contracting phase if it initially dominates, and hence no cosmological bounce will occur. In fact, for temperatures $T < M_A^{'}$, modes of v with values of k close to the peak of the thermal distribution scale as matter. Hence, the ratio of the energy density in the L-W field to the energy density in the regular radiation field decreases which renders it even more difficult to obtain a bounce. Once $T > M_A^{'}$, the energy densities in both fields scale as radiation.

2.5.3 Solution for the Regular Radiation Field

We now consider the evolution of the regular radiation field in the presence of a nonvanishing coupling with the L-W radiation field. Our starting point is the set of equations of motion (2.21) and (2.22). From (2.22) it follows that the ghost field v evolves independently. In turn, it influences the evolution of the regular radiation field u as a source term. We expect the coupling constant c to be small.

First, we show that the correction to the energy density in the presence of nonvanishing coupling is very small, namely of order c^2. We observe that if we turn on the coupling, the following is a solution of (2.21):

$$u(\eta)_{c\neq 0} \;=\; u(\eta)_{c=0} + 2cv(\eta)\,. \tag{2.36}$$

Inserting this into $\rho_A(k,\eta)$ (see (2.33)) yields

$$\rho_{A\ c\neq 0} \;=\; \rho_{A\ c=0} - 4c^2(\rho_{\tilde{A}} + \frac{M_A^2}{4}a(\eta)^{-2}v(\eta)^2) + \frac{c}{a(\eta)^4}[u'(\eta)v'(\eta) + k^2 u(\eta)v(\eta)] \tag{2.37}$$

Note that $\rho_{\tilde{A}}$ and v stay the same when we turn the coupling on. We also have a change in the expression for the coupling term in the energy density since it also depends on u:

$$\rho_{A-\tilde{A}\ c\neq 0} \;=\; -\frac{c}{a(\eta)^4}[u'(\eta)v'(\eta)+k^2u(\eta)v(\eta)] - \frac{2c^2}{a(\eta)^4}[v'(\eta)^2+k^2v(\eta)^2]\,. \quad (2.38)$$

The total energy density when the coupling is turned on is

$$\rho_{tot\ c\neq 0} \;=\; \rho_{A\ c\neq 0}+\rho_{\tilde{A}}+\rho_{A-\tilde{A}} \;=\; \rho_{A\ c=0}+(1+4c^2)\rho_{\tilde{A}} - c^2M_{\tilde{A}}^2 a(\eta)^{-2}v(\eta)^2 \quad (2.39)$$

This looks very much like the total energy we had before adding any coupling ($\rho_{tot\ c=0} = \rho_{A\ c=0}+\rho_{A-\tilde{A}}$) but with two correction terms of order c^2. Both correction terms appear to decrease the total energy density (recall that $\rho_{\tilde{A}}$ is negative). The second correction term (the last term in (2.39), however, increases less fast in a contracting background than the other terms, and the first correction term corresponds to a small time-independent renormalization of the energy density in the L-W field. Thus, it appears that if the energy density of the regular radiation field dominates initially, then it will forever and no bounce will occur. In the following we will confirm this conclusion by means of an analysis which compares solutions with and without coupling with the same initial conditions.

The evolution of u in the presence of the coupling with v can be determined using the Green function method. The general solution $u(\eta)$ of (2.21) is the sum of the solution $u_0(\eta)$ of the homogeneous equation which solves the same initial conditions as u and the particular solution $\delta u(\eta)$ with vanishing initial conditions. The particular solution is given by

$$\delta u(\eta) \;=\; u_1(\eta)\int_{\eta_I}^{\eta} d\eta^{'}\epsilon(\eta^{'})u_2(\eta^{'})s(\eta^{'}) - u_2(\eta)\int_{\eta_I}^{\eta} d\eta^{'}\epsilon(\eta^{'})u_1(\eta^{'})s(\eta^{'})\,, \quad (2.40)$$

where u_1 and u_2 are two independent solutions of the homogeneous equation, η_I is the initial conformal time ,

$\epsilon(\eta)$ is the Wronskian

$$\epsilon(\eta) \;=\; \left(u_1^{'}u_2 - u_2^{'}u_1\right)^{-1}, \tag{2.41}$$

and $s(\eta)$ is the source inhomogeneity

$$s(\eta) \;=\; -a^2\frac{2c}{1+4c^2}M_A^2 v(\eta)\,. \tag{2.42}$$

In our case, the solutions of the homogeneous equation are $u_1(\eta) = cos(k\eta)$ and $u_2(\eta) = sin(k\eta)$ and the Wronskian is $\epsilon(\eta) = -1/k$.

Since it is less hard to imagine a bounce once the energy densities in both fields scale as radiation, and since to study the possibility of a bounce it is important to investigate the dynamics at very high temperatures when the bulk of the Fourier modes of both fields scale as radiation, we will consider in the following Fourier modes for which v is oscillating.

We will now show that the sign of the energy transfer between the two fields depends on the relative phase between the oscillations of $u_0(\eta)$ and $v(\eta)$. We are interested in conformal time scales long compared to the oscillation time k^{-1} but short compared to the cosmological time. Hence, we can approximate the scale factor in (3.82) by a constant. A simple calculation then shows that if we choose phases for which $v(\eta) = v_0 sin(k\eta)$ and $u_0 = \mathcal{A}cos(k\eta)$ then

$$u(\eta) \;\simeq\; \big(\mathcal{A} - \frac{cv_0}{1+4c^2}\frac{M_A^2}{4k}(\eta-\eta_I)\big)cos(k\eta)\,. \tag{2.43}$$

For a coupling constant $c > 0$ this choice of phase hence leads to draining of energy density from the regular radiation field. On the other hand, the phase choice $v(\eta) = v_0 cos(k\eta)$ and $u_0(\eta) = \mathcal{A}sin(k\eta)$ leads to

$$u(\eta) \;\simeq\; \big(\mathcal{A} + \frac{cv_0}{1+4c^2}\frac{M_A^2}{4k}(\eta-\eta_I)\big)sin(k\eta) \tag{2.44}$$

and hence to a relative increase in the energy density of the regular radiation field.

We need to consider the full phase space of Fourier modes. Even if we only consider modes with fixed value of k given by the peak of the thermal distribution, we must sum over the different angles. Since there is no reason why the phases for different Fourier modes should be the same, we must take the expectation value of the energy transfer averaged over all possible choices of phases. This average obviously vanishes. Hence, we conclude that without unnatural fine tuning of phases it is not possible to obtain the required draining of the energy density from u to v.

2.6 Conclusions and Discussion

If the scalar field sector of the Lee-Wick Standard Model is coupled to Einstein gravity, then - in the absence of anisotropic stress - it is known that a bouncing cosmology can be realized. Since the energy density in radiation increases at a faster rate in a contracting universe compared to that of nonrelativistic matter, the cosmological bounce is unstable to the addition of radiation to the initial conditions early in the contracting phase. However, one may entertain the hope that the presence of the ghost radiation which is present in the Lee-Wick model might allow a bounce to occur in analogy to how the presence of ghost scalar field matter is responsible for the bounce in the scalar field Lee-Wick model.

For a Lee-Wick radiation bounce to occur, either the energy density of the ghost radiation would have to increase faster intrinsically than that of regular radiation, or there would have to be a mechanism which drains energy density from the regular radiation sector to the ghost sector. We have shown that neither happens, unless the initial phases of regular and ghost radiation are tuned in a very special way. Thus, we have shown that in the Lee-Wick Standard Model, the presence of radiation prevents a cosmological bounce from occurring.

The methods we have used in this paper could be applied to other proposals to obtain a bouncing cosmology by modifying the matter sector. Rather generically, one needs to worry whether any given proposal is robust towards the addition of radiative matter. The stability can be studied using the methods we have developed. Whether a channel to effectively drain energy density from radiation to ghost matter will exist may depend rather sensitively on the specific model. Here, we have shown that in the Lee-Wick Standard Model this does not happen. The same Green function method could be used to study the energy transfer in other models.

Cosmologies in which the bounce is induced by extra terms in the gravitational sector such as in the "nonsingular universe construction" [18], the model of [19] or the Horava-Lifshitz bounce [20] are more likely to be robust against the addition of matter. Specifically, the constructions of [18, 19] are based on theories which are asymptotically free in the sense that at high curvatures the coupling of any kind of matter to gravity goes to zero. This means that a bounce will not be effected by adding radiative matter. In Horava-Lifshitz gravity, there are higher spatial derivative gravitational terms which act as ghost matter scaling as a^{-4} and a^{-6}. The latter are present if we go beyond the "detailed balance case" and we allow for spatial curvature. In this case, once again radiative matter can be added without preventing a cosmological bounce.

Acknowledgments

We would like to thank Wang Yi for useful discussions. This research is supported in part by an NSERC Discovery Grant, by funds from the Canada Research Chair Program, and by a Killam Research Fellowship to RB.

REFERENCES

[1] Guth AH, "The Inflationary Universe: A Possible Solution To The Horizon And Flatness Problems," Phys. Rev. D **23**, 347 (1981).

[2] V. Mukhanov and G. Chibisov, "Quantum Fluctuation And Nonsingular Universe. (In Russian)," JETP Lett. **33**, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. **33**, 549 (1981)].

[3] R. H. Brandenberger, "Inflationary cosmology: Progress and problems," arXiv:hep-ph/9910410.

[4] A. Borde and A. Vilenkin, "Eternal inflation and the initial singularity," Phys. Rev. Lett. **72**, 3305 (1994) [arXiv:gr-qc/9312022].

[5] R. H. Brandenberger and J. Martin, Mod. Phys. Lett. A **16**, 999 (2001), [arXiv:astro-ph/0005432];
J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001), [arXiv:hep-th/0005209].

[6] R. H. Brandenberger, "Alternatives to Cosmological Inflation," arXiv:0902.4731 [hep-th].

[7] R. H. Brandenberger, "Cosmology of the Very Early Universe," arXiv:1003.1745 [hep-th].

[8] D. Wands, "Duality invariance of cosmological perturbation spectra," Phys. Rev. D **60**, 023507 (1999) [arXiv:gr-qc/9809062].

[9] F. Finelli and R. Brandenberger, "On the generation of a scale-invariant spectrum of adiabatic fluctuations in cosmological models with a contracting phase," Phys. Rev. D **65**, 103522 (2002) [arXiv:hep-th/0112249].

[10] L. E. Allen and D. Wands, "Cosmological perturbations through a simple bounce," Phys. Rev. D **70**, 063515 (2004) [arXiv:astro-ph/0404441].

[11] R. Brandenberger, H. Firouzjahi and O. Saremi, "Cosmological Perturbations on a Bouncing Brane," JCAP **0711**, 028 (2007) [arXiv:0707.4181 [hep-th]].

[12] S. Alexander, T. Biswas and R. H. Brandenberger, "On the Transfer of Adiabatic Fluctuations through a Nonsingular Cosmological Bounce," arXiv:0707.4679 [hep-th].

[13] Y. F. Cai, T. Qiu, R. Brandenberger, Y. S. Piao and X. Zhang, "On Perturbations of Quintom Bounce," JCAP **0803**, 013 (2008) [arXiv:0711.2187 [hep-th]]; Y. F. Cai and X. Zhang, "Evolution of Metric Perturbations in Quintom Bounce model," arXiv:0808.2551 [astro-ph].

[14] J. c. Hwang and E. T. Vishniac, "Gauge-invariant joining conditions for cosmological perturbations," Astrophys. J. **382**, 363 (1991).

[15] N. Deruelle and V. F. Mukhanov, "On matching conditions for cosmological perturbations," Phys. Rev. D **52**, 5549 (1995) [arXiv:gr-qc/9503050].

[16] S. W. Hawking and G. F. R. Ellis, "The Large scale structure of space-time," *Cambridge University Press, Cambridge, 1973*

[17] M. Novello and S. E. P. Bergliaffa, "Bouncing Cosmologies," Phys. Rept. **463**, 127 (2008) [arXiv:0802.1634 [astro-ph]].

[18] R. H. Brandenberger, V. F. Mukhanov and A. Sornborger, "A Cosmological theory without singularities," Phys. Rev. D **48**, 1629 (1993) [arXiv:gr-qc/9303001].

[19] T. Biswas, A. Mazumdar and W. Siegel, "Bouncing universes in string-inspired gravity," JCAP **0603**, 009 (2006) [arXiv:hep-th/0508194].

[20] R. Brandenberger, "Matter Bounce in Horava-Lifshitz Cosmology," Phys. Rev. D **80**, 043516 (2009) [arXiv:0904.2835 [hep-th]].

[21] M. Bojowald, "Loop quantum cosmology," Living Rev. Rel. **11**, 4 (2008).

[22] B. Feng, X. L. Wang and X. M. Zhang, "Dark Energy Constraints from the Cosmic Age and Supernova," Phys. Lett. B **607**, 35 (2005) [arXiv:astro-ph/0404224];
B. Feng, M. Li, Y. S. Piao and X. Zhang, "Oscillating quintom and the recurrent universe," Phys. Lett. B **634**, 101 (2006) [arXiv:astro-ph/0407432].

[23] T. D. Lee and G. C. Wick, "Negative Metric and the Unitarity of the S Matrix," Nucl. Phys. B **9**, 209 (1969);
T. D. Lee and G. C. Wick, "Finite Theory of Quantum Electrodynamics," Phys. Rev. D **2**, 1033 (1970).

[24] B. Grinstein, D. O'Connell and M. B. Wise, "The Lee-Wick standard model," Phys. Rev. D **77**, 025012 (2008) [arXiv:0704.1845 [hep-ph]].

[25] M. B. Wise, "The Lee Wick Standard Model," arXiv:0908.3872 [hep-ph].

[26] Y. F. Cai, T. t. Qiu, R. Brandenberger and X. m. Zhang, "Nonsingular Cosmology with a Scale-Invariant Spectrum of Cosmological Perturbations from Lee-Wick Theory," Phys. Rev. D **80**, 023511 (2009)

[27] D. G. Boulware and D. J. Gross, "Lee-Wick Indefinite Metric Quantization: A Functional Integral Approach," Nucl. Phys. B **233**, 1 (1984).

[28] N. Nakanishi, "Lorentz noninvariance of the complex-ghost relativistic field theory," Phys. Rev. D **3**, 811 (1971);
N. Nakanishi, "Remarks on the complex-ghost relativistic field theory," Phys. Rev. D **3**, 3235 (1971).

[29] A. M. Gleeson, R. J. Moore, H. Rechenberg and E. C. G. Sudarshan, "Analyticity, covariance, and unitarity in indefinite-metric quantum field theories," Phys. Rev. D **4**, 2242 (1971).

[30] J. M. Cline, S. Jeon and G. D. Moore, "The phantom menaced: Constraints on low-energy effective ghosts," Phys. Rev. D **70**, 043543 (2004) [arXiv:hep-ph/0311312].

[31] R. E. Cutkosky, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, "A non-analytic S-matrix," Nucl. Phys. B **12**, 281 (1969).

[32] A. van Tonder, "Unitarity, Lorentz invariance and causality in Lee-Wick theories: An asymptotically safe completion of QED," arXiv:0810.1928 [hep-th].

[33] B. Fornal, B. Grinstein and M. B. Wise, "Lee-Wick Theories at High Temperature," Phys. Lett. B **674**, 330 (2009) [arXiv:0902.1585 [hep-th]].

[34] R. S. Chivukula, A. Farzinnia, R. Foadi and E. H. Simmons, "Custodial Isospin Violation in the Lee-Wick Standard Model," arXiv:1002.0343 [hep-ph].

[35] E. Alvarez, L. Da Rold, C. Schat and A. Szynkman, "Vertex Displacements for Acausal Particles: Testing the Lee-Wick Standard Model at the LHC," JHEP **0910**, 023 (2009) [arXiv:0908.2446 [hep-ph]].

[36] A. Linde, "Chaotic Inflation," Phys. Lett. B **129**, 177 (1983).

[37] Y. F. Cai, W. Xue, R. Brandenberger and X. Zhang, "Non-Gaussianity in a Matter Bounce," JCAP **0905**, 011 (2009) [arXiv:0903.0631 [astro-ph.CO]].

[38] M. Abramowitz and I. Stegun (eds.) "Handbook of Mathematical Functions" (Dover, New York, 1964).

The Lee-Wick scenario in the pure radiation case, when only the gauge field and its Lee-Wick partner are present, allows a bouncing universe under very specific conditions. In the pure matter scenario, it has been shown that a bounce can occur. Mixing these two cases can shed light on the presence of a bounce when both radiation and matter coexist. The following chapter aims at understanding the energy transfer between radiation, that scales like a^{-4} *in an expanding Universe, and matter that scales like* a^{-3}*. It also discusses the conditions under which a bouncing universe occurs.*

Chapter 3

On the Instability of the Lee-Wick Bounce

The following is an extract from the article **"On the Instability of the Lee-Wick Bounce,"** published in collaboration with Taotao Qiu and Robert Brandenberger in *Phys. Rev. D* **84**, 043505 (2011).

Abstract It was recently realized [11] that a model constructed from a Lee-Wick type scalar field theory yields, at the level of homogeneous and isotropic background cosmology, a bouncing cosmology. However, bouncing cosmologies induced by pressureless matter are in general unstable to the addition of relativistic matter (i.e. radiation). Here we study the possibility of obtaining a bouncing cosmology if we add radiation coupled to the Lee-Wick scalar field. This coupling in principle would allow the energy to flow from radiation to matter, thus providing a drain for the radiation energy. However, we find that it takes an extremely unlikely fine-tuning of the initial phases of the field configurations for a sufficient amount of radiative energy to flow into matter. For general initial conditions, the evolution leads to a singularity rather than a smooth bounce.

3.1 Introduction

Both Standard [1] and Inflationary Cosmology [2] suffer from the initial singularity problem and hence cannot yield complete descriptions of the very early Universe. If one were able to construct a nonsingular bouncing cosmology, this problem would obviously disappear. However, in order to have a chance to obtain such a nonsingular cosmology, one must either go beyond Einstein gravity as a theory of space-time (see e.g. [3] for an early construction), or else one must make use of matter which violates the "null energy condition" (see [4] for a review of both types of approaches).

Interest in nonsingular bouncing cosmologies has increased with the realization that they can lead to alternatives to inflationary cosmology as a theory for the origin of structure in the Universe. A specific scenario which can arise at the level of homogeneous and isotropic cosmology is the "matter bounce" paradigm which is based on the realization [5, 6] that vacuum fluctuations which exit the Hubble radius during a matter-dominated contracting phase evolve into a scale-invariant spectrum of curvature perturbations on super-Hubble scales before the bounce. The key point is that the curvature fluctuation variable ζ grows on super-Hubble scales in a contracting phase, whereas it is constant on these large scales in an expanding phase. Since long wavelength modes exit the Hubble radius earlier than short wavelength ones, they grow for a longer period of time. This provides a mechanism for reddening the initial vacuum spectrum. It turns out that a matter-dominated contracting phase provides the specific boost in the power of long wavelength modes which is required in order to transform a vacuum spectrum into a scale-invariant one. Studies in the case of various nonsingular bounce models [7] have shown that on wavelengths long compared to the duration of the bounce phase, the spectrum of fluctuations is virtually unchanged during the bounce. Thus, a scale-invariant spectrum of curvature fluctuations survives on super-Hubble scales at late times.

Provided that the bounce can occur at energy scales much below the Planck scale, nonsingular cosmologies solve a key conceptual problem from which inflationary cosmology suffers, namely the "Trans-Planckian" problem for fluctuations [8, 9]: If the period of inflationary expansion of space lasts for more than $70H^{-1}$, where H is the Hubble expansion rate during inflation (in order to solve the key cosmological mysteries it was designed to explain, inflation has to last at least $50H^{-1}$), then the physical wavelengths of even the largest-scale fluctuation modes we see today will be even smaller than the Planck length at the beginning of inflation and thus in the "zone of ignorance" where the physics on which inflation and the theory of cosmological perturbations are based, namely Einstein gravity coupled to semiclassical field theory matter, will break down. In contrast, in a nonsingular bouncing cosmology the wavelength of modes which are currently probed by cosmological observations is never much smaller than 1mm (the physical wavelength of the mode which corresponds to our current Hubble radius evaluated when the temperature of the Universe was 10^{16}GeV) and hence many orders of magnitude larger than the Planck length. Thus, the fluctuations never enter the "trans-Planckian zone of ignorance" of sub-Planck-length wavelengths.

Possibly the simplest realization of the matter bounce scenario is the "quintom bounce" model [10] and is obtained by considering the matter sector to contain two scalar fields, one of them (the "ghost field") having the "wrong" sign of the kinetic action. The potential of the ghost scalar field also has the opposite sign to that of regular scalar fields such that in the absence of interactions, the ghost field has a classically stable minimum. As has been noticed in [11], such a quintom bounce model also arises from the scalar field sector of the "Lee-Wick" (LW) Lagrangian [12] which contains higher derivatives terms.

The quintom and Lee-Wick bouncing cosmologies are obtained in the following way [10, 11]: We begin in the contracting phase with both the regular and the ghost scalar field oscillating homogeneously in space about their respective vacua. We assume that the energy density is dominated by the regular matter field, and that hence the total energy density is positive. Once the amplitude of the regular scalar field exceeds the Planck scale, the field oscillations will freeze out and a slow-climb phase will begin during which the energy density of the field only grows slowly (this is the time reverse of the slow-roll phase in scalar field-driven inflation). However, the ghost field continues to oscillate and its energy density (which is negative) continues to grow in absolute value. Hence, the total energy density drops to zero, at which point the bounce occurs, as has been studied both analytically and numerically in the above-mentioned works. Note that the energy density in this bounce model scales as matter until the regular scalar field freezes out.

A major problem of bouncing cosmologies realized with matter which scales as a^{-3} as a function of the scale factor $a(t)$ is the potential instability of the homogeneous and isotropic background against the effects of radiation (which scales as a^{-4} and anisotropic stress which scales as a^{-6} [1] . If we simply add a noninteracting radiation component to the two scalar field system, then unless the initial energy density in radiation is tuned to be extremely small, then the radiation component will become dominant long before the bounce can arise, and will prevent the energy density in the ghost field from ever being able to become important, resulting in

[1] One of the major advantages of the ekpyrotic bouncing scenario [13] is that the contracting phase is stable against such effects.

a Big Crunch singularity. Similarly, unless the initial energy density in anisotropic stress is very small, it will come to dominate the energy density of the Universe long before the bounce is expected. The anisotropies will destabilize the homogeneous background cosmology, and will prevent a bounce. Note that at the quantum level, there is an additional severe problem for bounce models obtained with matter fields with ghostlike kinetic terms, namely the quantum instability of the vacuum (see e.g. [14]).

In this paper we will focus on the radiation instability problem. For the purpose of this discussion we will simply assume that anisotropic stress is absent. In a recent paper, two of us studied the possibility that a bounce could arise if radiation is supplemented with Lee-Wick radiation [15]. However, we showed that this hope is not realized: the addition of Lee-Wick radiation does not prevent the Big Crunch singularity from occurring. In the presence of radiation, the only hope to obtain a bounce is to introduce a coupling between radiation and ghost scalar field matter which could effectively drain energy density from the radiation field and prevent the energy density of radiation from becoming dominant. Here we study this possibility. However, at least for the specific Lagrangian which we consider, we find that a bounce only emerges for highly fine-tuned phases of the fields and their velocities in the initial conditions.

The paper is organized as follows: In Section II, we introduce the model we study, namely the scalar field sector of Lee-Wick theory coupled to radiation, and write down the general equations of motion. In Section III we set up the equations of motion linearized about the bounce background, treating the entire radiation field as an inhomogeneous fluctuation. In particular, we study the different terms which contribute to the energy-momentum tensor and identify those which could assist in obtaining a nonsingular bounce. In Section IV we study the solutions of the perturbed equations of motion, and in Section V we analyze the evolution of the different terms in the energy-momentum tensor, identifying the conditions which would be required in order to obtain a nonsingular bounce. We have also evolved the general equations of motion for the two inhomogeneous scalar field configurations and the classical inhomogeneous radiation field in the homogeneous background cosmology. Section VI summarizes some of the numerical results. Both the analytical and numerical results confirm that we need unnatural fine-tuning of the initial conditions in order to obtain a nonsingular bounce. In the final section we offer some conclusions and discussion.

3.2 The Model

The Lee-Wick scalar field model coupled to electromagnetic radiation is given by the following Lagrangian:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2M^2}(\partial^2\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \\ &\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - f(\phi, \partial^2\phi, F_{\mu\nu}F^{\mu\nu})\ , \end{aligned} \tag{3.1}$$

where m is the mass of the scalar field ϕ, and $V(\phi)$ is its potential. Here we adopt the convention that

$$ds^2 \ = \ -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{3.2}$$

where $a(t)$ is the scale factor of the Universe. Since it is a higher derivative Lagrangian in ϕ, the scalar field sector contains an extra degree of freedom with the "wrong" sign kinetic term and with a mass set by the scale M. We choose $m \leq M \leq m_{Pl}$, where m_{Pl} is the Planck mass, since we want the regular scalar field to dominate at low energies, but at the same time we do not want to worry about quantum gravity effects. The second line of the Lagrangian (3.1) contains the kinetic term of the radiation as well as the coupling term, where we assumed both for the sake of generality and because of foresight that the radiation field couples not only to the scalar field ϕ itself, but also to the higher derivative term. The electromagnetic tensor, $F_{\mu\nu}$, is related to the radiation field A_μ through the usual definition

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu \,, \tag{3.3}$$

where ∇_μ is the covariant derivative.

It is convenient to extract the extra degree of freedom as a separate scalar field. To do this, we use the field redefinitions

$$\begin{aligned} \phi &\equiv \phi_1 - \phi_2 \,, \\ \phi_2 &\equiv \partial^2\phi/M^2 \,. \end{aligned} \tag{3.4}$$

The Lagrangian (3.1) then takes on a simpler form:

$$\begin{aligned} \mathcal{L} \;=\; & -\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}M^2\phi_2^2 \\ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - f(\phi_1, \phi_2, F_{\mu\nu}F^{\mu\nu}) \;, \end{aligned} \tag{3.5}$$

where we have chosen the potential to be zero. In this new form, the Lagrangian describes two massive scalar fields with one of them (i.e., ϕ_2) behaving like a"ghost", and both of them coupled to the radiation field.

The coupling term $f(\phi_1, \phi_2, F_{\mu\nu}F^{\mu\nu})$ should in principle be arbitrary, however, in this paper we will take a specific form for convenience. The form will be:

$$f(\phi_1, \phi_2, F_{\mu\nu}F^{\mu\nu}) \;=\; -\frac{1}{4}(c\phi_1^2 + d\phi_2^2)F_{\mu\nu}F^{\mu\nu} \;, \tag{3.6}$$

where c and d are coupling constants which have mass dimension -2. The interaction terms are nonrenormalizable. To make sure that such terms could be thought of as arising from an effective field theory which is consistent at the bounce, we must make sure that the coefficients are chosen such that the contribution of the interaction term to the Lagrangian density is smaller than that of the other terms. This must be true even at energy densities at which the bounce occurs in the pure scalar field model. It is easy to see that this condition will be satisfied if the coefficients c and d are both of the order m_{pl}^{-2}.

It is the purpose of this paper to study the effects which these coupling terms have on the dynamics of the system. We know that in the absence of coupling, i.e. when $c = d = 0$, a bounce will only occur if the initial radiation energy density is tuned to a very small value compared to the scalar field energy density. This is because the positive definite energy density of radiation will scale as a^{-4} which is faster than that of the scalar fields, in particular the ghost scalar field. Generically, it will dominate the energy of the Universe after some amount of contraction, it will prevent the ghost scalar field energy density from catching up and will thus prevent a bounce, leading to a Big Crunch singularity instead. With nonvanishing values of c and d, however, the scalars are in principle able to drain energy from the radiation.

From the Lagrangian (3.5), one can obtain the stress-energy tensor $T_{\mu\nu}$ by varying the action with respect to the metric $g^{\mu\nu}$. In the hydrodynamical limit, we can take $T_{\mu\nu}$ to be of the form of $diag\{\rho, a^2(t)p_1, a^2(t)p_2, a^2(t)p_3\}$ where ρ and p are energy density and pressure, respectively. As a result of the variation, we obtain the following form of the stress-energy tensor:

$$
\begin{aligned}
T_{\mu\nu} &= g_{\mu\nu}\mathcal{L} + \partial_\mu\phi_1\partial_\nu\phi_1 - \partial_\mu\phi_2\partial_\nu\phi_2 + (1 - c\phi_1^2 - d\phi_2^2)F_{\mu\lambda}F_\nu^\lambda \ , \\
&= g_{\mu\nu}\Big[\frac{\dot{\phi}_1^2}{2} - \frac{1}{2a^2}\partial_i\phi_1\partial_i\phi_1 - \frac{1}{2}m^2\phi_1^2 - \frac{\dot{\phi}_2^2}{2} + \frac{1}{2a^2}\partial_i\phi_2\partial_i\phi_2 + \frac{1}{2}M^2\phi_2^2 - \frac{1}{4}(1 - c\phi_1^2 - d\phi_2^2)F^2\Big] \\
&\quad + \partial_\mu\phi_1\partial_\nu\phi_1 - \partial_\mu\phi_2\partial_\nu\phi_2 + (1 - c\phi_1^2 - d\phi_2^2)F_{\mu\lambda}F_\nu^\lambda \,, \qquad (3.7)
\end{aligned}
$$

where $F^2 = F_\mu\nu F^\mu\nu$. Since we will be studying the contribution of plane wave perturbations of the scalar fields and we will treat radiation as a superposition of waves, we kept the space-derivative terms.

By varying the Lagrangian with respect to the matter fields ϕ_1, ϕ_2 and A_μ, we also get the equations of motion for all three fields:

$$
\Box\phi_1 - (m^2 - \frac{c}{2}F^2)\phi_1 = 0 \ , \qquad (3.8)
$$

$$
\Box\phi_2 - (M^2 + \frac{d}{2}F^2)\phi_2 = 0 \ , \qquad (3.9)
$$

$$
\begin{aligned}
&(1 - c\phi_1^2 - d\phi_2^2)(\partial_\nu F^{\mu\nu} + 3HF^{\mu 0}) \\
&-2(c\phi_1\partial_\nu\phi_1 + d\phi_2\partial_\nu\phi_2)F^{\mu\nu} = 0 \qquad (3.10)
\end{aligned}
$$

which will be analyzed in detail in the rest of the paper.

3.3 Dynamics

Since the equations of motion are nonlinear, we cannot work in Fourier space, and use plane wave solutions. However, we are interested in how initially small amounts of radiation build up and possibly transfer their energy to scalar field fluctuations. We treat radiation as a superposition of fluctuations. Therefore it makes sense to linearize our equations about the homogeneous scalar field background. Thus, we make the following ansatz for the scalar fields:

$$\phi_1(t,z) = \phi_1^{(0)}(t) + \epsilon\phi_1^{(1)}(t,z) + \epsilon^2\phi_1^{(2)}(t) \tag{3.11}$$

$$\phi_2(t,z) = \phi_2^{(0)}(t) + \epsilon\phi_2^{(1)}(t,z) + \epsilon^2\phi_2^{(2)}(t) \ , \tag{3.12}$$

where the expansion parameter ϵ is taken to be much smaller than 1 [2] . The first term on the right hand side of each line, i.e. $\phi_{1,2}^{(0)}(t)$ correspond to the background fields, the terms $\phi_{1,2}^{(1)}(t,x)$ are the fluctuations, and the second order terms $\phi_{1,2}^{(2)}(t)$ describe the back-reaction of the fluctuations on the background and can be computed from the leading second order corrections (averaged over space) of the equations of motion [3] .

To simplify the analysis, we describe radiation in terms of plane waves in a fixed direction (which we take to be the z direction). Without loss of generality we can restrict attention to one polarization mode which we take to be the electric field in the x direction and the magnetic field in the y direction. In this case, the only nonzero components of the field strength tensor are F^{01} and F^{13}. Using the temporal gauge where $A_0 = 0$, we find that only the first component of the gauge

[2] The expansion parameter ϵ should be viewed as parameterizing the initial ratio of radiation energy to background scalar field energy. Thus, the leading contribution of the radiation field is first order in ϵ. Via the coupling terms in the Lagrangian with coefficients c and d, the linear radiation field induces linear scalar field inhomogeneities $\phi_1^{(1)}$ and $\phi_2^{(1)}$. These corrections will contain a further suppression factor since c and d are small coefficients. Similarly, the same coupling terms in the Lagrangian will lead to a perturbation $\delta\gamma$ of the rescaled radiation field γ which is of linear order in ϵ but suppressed by factors of c and d.

[3] By taking the scalar product of the second order equations with a fixed plane wave (instead of averaging over space) one could also compute the back-reaction of the fluctuations on the inhomogeneous modes.

field is nonzero. For a single wavelength fluctuation we can make the ansatz

$$A_1(k,t) \;=\; f(t)cos(kz) \equiv \gamma(k,t) \;, \tag{3.13}$$

or, equivalently,

$$A^1(k,t) \;=\; a(t)^{-2}\gamma(k,t) \;. \tag{3.14}$$

Since in the linearized equations of motion the Fourier modes are independent, we can consider $\phi_1^{(1)}$ and $\phi_2^{(1)}$ also to be plane waves propagating in z direction, so they depend only on z and t.

With Eqs. (3.11-3.14) in hand, we can write down the energy densities of the various fields at each order in perturbation theory.

3.3.1 The stress-energy tensor

First of all, we insert the above perturbative ansatz for the fields into the stress-energy tensor of the system. From the general expression (3.7) for $T_{\mu\nu}$ we get

$$\begin{aligned} T_{\mu\nu} \;&=\; g_{\mu\nu}\Big[\frac{\dot{\phi}_1^2}{2} - \frac{1}{2a^2}\partial_z\phi_1\partial_z\phi_1 - \frac{1}{2}m^2\phi_1^2 - \frac{\dot{\phi}_2^2}{2} + \frac{1}{2a^2}\partial_z\phi_2\partial_z\phi_2 + \frac{1}{2}M^2\phi_2^2 \\ &\quad -\frac{1}{4}(1-c\phi_1^2-d\phi_2^2)F^2\Big] + \partial_\mu\phi_1\partial_\nu\phi_1 - \partial_\mu\phi_2\partial_\nu\phi_2 + (1-c\phi_1^2-d\phi_2^2)F_{\mu\lambda}F_\nu^\lambda \;. \end{aligned} \tag{3.15}$$

The 00 component of Eq. (3.15) denotes the energy density of the system

$$\rho = \frac{1}{2}(\dot{\phi}_1^2 + \frac{k^2}{a^2}\phi_1^2 + m^2\phi_1^2) - \frac{1}{2}(\dot{\phi}_2^2 + \frac{k^2}{a^2}\phi_2^2 + M^2\phi_2^2) + (1-c\phi_1^2-d\phi_2^2)(\frac{F^2}{4} + F_{0\lambda}F_0^\lambda) \;, \tag{3.16}$$

so at each level in perturbation theory, we have

$$\rho^{(0)} = \frac{1}{2}(\dot{\phi_1^{(0)}}^2 + m^2\phi_1^{(0)2}) - \frac{1}{2}(\dot{\phi_2^{(0)}}^2 + M^2\phi_2^{(0)2}) \ , \tag{3.17}$$

$$\rho^{(1)} = (\dot{\phi}_1^{(0)}\dot{\phi}_1^{(1)} + m^2\phi_1^{(0)}\phi_1^{(1)}) - (\dot{\phi}_2^{(0)}\dot{\phi}_2^{(1)} + M^2\phi_2^{(0)}\phi_2^{(1)}) \ , \tag{3.18}$$

$$\begin{aligned}
\rho^{(2)} &= \frac{1}{2}(\dot{\phi_1^{(1)}}^2 + \dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)} + \frac{k^2}{a^2}\phi_1^{(1)2} + m^2\phi_1^{(1)2} + m^2\phi_1^{(0)}\phi_1^{(2)}) - \frac{1}{2}(\dot{\phi_2^{(1)}}^2 + \dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)} \\
&\quad + \frac{k^2}{a^2}\phi_2^{(1)2} + M^2\phi_2^{(1)2} + M^2\phi_2^{(0)}\phi_2^{(2)}) + (1 - c\phi_1^{(0)2} - d\phi_2^{(0)2})(\frac{F^2}{4} + F_{0\lambda}F_0^{\lambda}) \ , \\
&= \frac{1}{2}(\dot{\phi_1^{(1)}}^2 + \dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)} + \frac{k^2}{a^2}\phi_1^{(1)2} + m^2\phi_1^{(1)2} + m^2\phi_1^{(0)}\phi_1^{(2)}) \\
&\quad - \frac{1}{2}(\dot{\phi_2^{(1)}}^2 + \dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)} + \frac{k^2}{a^2}\phi_2^{(1)2} + M^2\phi_2^{(1)2} + M^2\phi_2^{(0)}\phi_2^{(2)}) \\
&\quad + (1 - c\phi_1^{(0)2} - d\phi_2^{(0)2})(\frac{k^2}{2a^4}\gamma^2 + \frac{\dot{\gamma}^2}{2a^2}) \ .
\end{aligned} \tag{3.19}$$

We can similarly obtain the pressure of the system from the ii components of Eq. (3.15). Note that due to the anisotropy in $T_{\mu\nu}$ caused by the gauge field as well as by the anisotropic fluctuations of the scalar fields, the pressures in the three directions are no longer identical. The pressure in each direction can be written as

$$\begin{aligned}
p_i &= \frac{1}{2}(\dot{\phi}_1^2 - \frac{k^2}{a^2}\phi_1^2 - m^2\phi_1^2) - \frac{1}{2}(\dot{\phi}_2^2 - \frac{k^2}{a^2}\phi_2^2 - M^2\phi_2^2) \\
&\quad - (1 - c\phi_1^2 - d\phi_2^2)(\frac{F^2}{4} - \frac{F_{i\lambda}F_i^{\lambda}}{a^2}) + \frac{\partial_i\phi_1\partial_i\phi_1}{a^2} - \frac{\partial_i\phi_2\partial_i\phi_2}{a^2} \ ,
\end{aligned} \tag{3.20}$$

with no summation over the index i. From this formula, we can see that at both zeroth and first order, the pressure is isotropic:

$$p_i^{(0)} = \frac{1}{2}(\dot{\phi_1^{(0)}}^2 - m^2\phi_1^{(0)2}) - \frac{1}{2}(\dot{\phi_2^{(0)}}^2 - M^2\phi_2^{(0)2}) \ , \tag{3.21}$$

$$p_i^{(1)} = (\dot{\phi}_1^{(0)}\dot{\phi}_1^{(1)} - m^2\phi_1^{(0)}\phi_1^{(1)}) - (\dot{\phi}_2^{(0)}\dot{\phi}_2^{(1)} - M^2\phi_2^{(0)}\phi_2^{(1)}) \ , \tag{3.22}$$

while the second order pressure for each direction gives

$$p_i^{(2)} = \frac{1}{2}(\dot{\phi_1^{(1)}}^2 + \dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)} - \frac{k^2}{a^2}\phi_1^{(1)2} + m^2\phi_1^{(1)^2} + m^2\phi_1^{(0)}\phi_1^{(2)}) - \tag{3.23}$$

$$\frac{1}{2}(\dot{\phi_2^{(1)}}^2 + \dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)} - \frac{k^2}{a^2}\phi_2^{(1)2} + M^2\phi_2^{(1)^2} + M^2\phi_2^{(0)}\phi_2^{(2)}) \tag{3.24}$$

$$+\frac{\partial_i\phi_1\partial_i\phi_1}{a^2} - \frac{\partial_i\phi_2\partial_i\phi_2}{a^2} - (1 - c\phi_1^{(0)^2} - d\phi_2^{(0)2})(\frac{F^2}{4} - \frac{F_{i\lambda}F_i^\lambda}{a^2}) ,$$

where $i = 1, 2, 3$.

We can thus obtain every component of $p_i^{(2)}$:

$$p_1^{(2)} = \frac{1}{2}(\dot{\phi_1^{(1)}}^2 + \dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)} - \frac{k^2}{a^2}\phi_1^{(1)2} + m^2\phi_1^{(1)^2} + m^2\phi_1^{(0)}\phi_1^{(2)}) - \frac{1}{2}(\dot{\phi_2^{(1)}}^2 + \dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)} - \frac{k^2}{a^2}\phi_2^{(1)2}$$

$$+M^2\phi_2^{(1)^2} + M^2\phi_2^{(0)}\phi_2^{(2)}) + (1 - c\phi_1^{(0)^2} - d\phi_2^{(0)^2})(\frac{k^2}{2a^4}\gamma^2 - \frac{\dot{\gamma}^2}{2a^2}) , \tag{3.25}$$

$$p_2^{(2)} = \frac{1}{2}(\dot{\phi_1^{(1)}}^2 + \dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)} - \frac{k^2}{a^2}\phi_1^{(1)2} + m^2\phi_1^{(1)^2} + m^2\phi_1^{(0)}\phi_1^{(2)}) - \frac{1}{2}(\dot{\phi_2^{(1)}}^2 + \dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)} - \frac{k^2}{a^2}\phi_2^{(1)2}$$

$$+M^2\phi_2^{(1)^2} + M^2\phi_2^{(0)}\phi_2^{(2)}) - (1 - c\phi_1^{(0)^2} - d\phi_2^{(0)^2})(\frac{k^2}{2a^4}\gamma^2 - \frac{\dot{\gamma}^2}{2a^2}) , \tag{3.26}$$

$$p_3^{(2)} = \frac{1}{2}(\dot{\phi_1^{(1)}}^2 + \dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)} - \frac{k^2}{a^2}\phi_1^{(1)2} + m^2\phi_1^{(1)^2} + m^2\phi_1^{(0)}\phi_1^{(2)}) - \frac{1}{2}(\dot{\phi_2^{(1)}}^2 + \dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)} - \frac{k^2}{a^2}\phi_2^{(1)2}$$

$$+M^2\phi_2^{(1)^2} + M^2\phi_2^{(0)}\phi_2^{(2)}) + (1 - c\phi_1^{(0)^2} - d\phi_2^{(0)^2})(\frac{k^2}{2a^4}\gamma^2 + \frac{\dot{\gamma}^2}{2a^2}) , \tag{3.27}$$

and the average is

$$p_{eff}^{(2)} = \frac{p_1^{(2)} + p_2^{(2)} + p_3^{(2)}}{3} \tag{3.28}$$

$$= \frac{1}{2}(\dot{\phi_1^{(1)}}^2 + \dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)} - \frac{k^2}{a^2}\phi_1^{(1)2} + m^2\phi_1^{(1)^2} + m^2\phi_1^{(0)}\phi_1^{(2)}) - \frac{1}{2}(\dot{\phi_2^{(1)}}^2 + \dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)} - \frac{k^2}{a^2}\phi_2^{(1)2}$$

$$+M^2\phi_2^{(1)^2} + M^2\phi_2^{(0)}\phi_2^{(2)}) + (1 - c\phi_1^{(0)^2} - d\phi_2^{(0)^2})(\frac{k^2}{6a^4}\gamma^2 + \frac{\dot{\gamma}^2}{6a^2}) . \tag{3.29}$$

From the above, we can also see that in order to analyze the behavior of the energy density up to second order, we need to know the evolution of scalar fields up to second order as well as that of the gauge field up to first order, while the behavior of the gauge field to second order is not required.

It will be useful in the following to separate the contributions to the energy density and pressure in a different way, namely,

i) the contribution from the background homogeneous part of the scalar fields,

$$\rho_\phi^h = \rho_\phi^{(0)} \qquad p_\phi^h = p_\phi^{(0)} \ , \tag{3.30}$$

ii) that of the scalar field perturbations (in slight abuse of notation we call this the "inhomogeneous" term),

$$\rho_\phi^{inh} = \epsilon\rho_\phi^{(1)} + \epsilon^2\rho_\phi^{(2)} \qquad p_\phi^{inh} = \epsilon p_\phi^{(1)} + \epsilon^2 p_\phi^{(2)} \ , \tag{3.31}$$

iii) the contribution of the gauge field,

$$\rho_g = \frac{1}{2a^2}(\dot\gamma^2 + \frac{k^2}{a^2}\gamma^2) \ , \tag{3.32}$$

$$p_g = \frac{1}{6a^2}(\dot\gamma^2 + \frac{k^2}{a^2}\gamma^2) \ , \tag{3.33}$$

(iv) the contribution of the coupling term,

$$\rho_c = -\frac{(c{\phi_1^{(0)}}^2 + d{\phi_2^{(0)}}^2)}{8a^2}(\dot\gamma^2 + \frac{k^2}{a^2}\gamma^2) = -\Phi\rho_g \ , \tag{3.34}$$

$$p_c = -\frac{(c{\phi_1^{(0)}}^2 + d{\phi_2^{(0)}}^2)}{24a^2}(\dot\gamma^2 + \frac{k^2}{a^2}\gamma^2) = -\Phi p_g \ , \tag{3.35}$$

where in the last equation we define Φ to be the quadratic combination of the two fields:

$$\Phi = (c{\phi_1^{(0)}}^2 + d{\phi_2^{(0)}}^2)/2 \, . \tag{3.36}$$

From the above we can deduce the equation of state parameter for each part:

$$
\begin{aligned}
w_\phi^h &= \frac{\dot{\phi}_1^{(0)^2} - m^2\phi_1^{(0)2} - \dot{\phi}_2^{(0)^2} + M^2\phi_2^{(0)2}}{\dot{\phi}_1^{(0)^2} + m^2\phi_1^{(0)2} - \dot{\phi}_2^{(0)^2} - M^2\phi_2^{(0)2}} \ , && (3.37)\\
w_\phi^{inh} &= \epsilon\left(\frac{p^{(1)}}{\rho^{(1)}} - \frac{p^{(0)}\rho^{(1)}}{\rho^{(0)2}}\right) + \epsilon^2\left(\frac{p^{(2)}}{\rho^{(0)}} - \frac{p^{(0)}\rho^{(2)}}{\rho^{(0)2}} - \frac{p^{(1)}\rho^{(1)}}{\rho^{(0)2}}\right) \ , && (3.38)\\
w_g &= w_c = \frac{1}{3} \ . && (3.39)
\end{aligned}
$$

From the equations above, we see that for positive values of the constants c and d, the coupling of the scalar field with the gauge field will give rise to a contribution ρ_c to the energy density which has the same equation of state but opposite sign to that of the gauge field. Therefore, the coupling can help drain energy from the gauge field. It is because of this mechanism that we might hope to achieve a cosmological bounce in the presence of radiation. A first indication on whether a bounce might occur can be obtained by considering the scaling of each contribution to the energy density as a function of the scale factor $a(t)$. To find these scalings, we need the time dependence of the linear and quadratic contributions to each field. Therefore, we need to solve the matter field equations of motion. In the following subsection, we present the equations for the fields at each order, while the solutions and detailed analysis will be performed in the next sections.

3.3.2 Equations of Motion

Keeping in mind the ansätze for A_μ, ϕ_1 and ϕ_2, their equations of motion at each order can be obtained from (3.8), (3.9) and (3.10):

a) At zeroth order,

$$
\begin{cases}
\ddot{\phi}_1^{(0)} + 3H\dot{\phi}_1^{(0)} + m^2\phi_1^{(0)} = 0 \ , \\
\\
\ddot{\phi}_2^{(0)} + 3H\dot{\phi}_2^{(0)} + M^2\phi_2^{(0)} = 0 \ .
\end{cases} \tag{3.40}
$$

Note that there is no equation at this order for A_μ because it is of first order in ϵ.

b) At first order,

$$\begin{cases} \ddot{\phi}_1^{(1)} + 3H\dot{\phi}_1^{(1)} + (\frac{k^2}{a^2} + m^2)\phi_1^{(1)} = 0 \ , \\ \\ \ddot{\phi}_2^{(1)} + 3H\dot{\phi}_2^{(1)} + (\frac{k^2}{a^2} + M^2)\phi_2^{(1)} = 0 \ , \\ \\ (1 - c{\phi_1^{(0)}}^2 - d{\phi_2^{(0)}}^2)(\partial_\nu F^{\mu\nu} + 3HF^{\mu 0}) \\ -2(c\phi_1^{(0)}\partial_\nu\phi_1^{(0)} + d\phi_2^{(0)}\partial_\nu\phi_2^{(0)})F^{\mu\nu} = 0 \ , \end{cases} \tag{3.41}$$

Making use of Eqs. (3.13) and (3.14), the equation for the gauge field can also be rewritten as

$$(1 - c{\phi_1^{(0)}}^2 - d{\phi_2^{(0)}}^2)(\ddot{\gamma} + H\dot{\gamma} + \frac{k^2}{a^2}\gamma) - 2(c\phi_1^{(0)}\dot{\phi}_1^{(0)} + d\phi_2^{(0)}\dot{\phi}_2^{(0)})\dot{\gamma} = 0 \tag{3.42}$$

c) At second order,

$$\begin{cases} \ddot{\phi}_1^{(2)} + m^2\phi_1^{(2)} - \frac{c}{2} < F_{\mu\nu}F^{\mu\nu} > \phi_1^{(0)} = 0 \ , \\ \\ \ddot{\phi}_2^{(2)} + M^2\phi_2^{(2)} + \frac{d}{2} < F_{\mu\nu}F^{\mu\nu} > \phi_2^{(0)} = 0 \ . \end{cases} \tag{3.43}$$

Here, pointed parentheses indicate spatial averaging (since we are only focusing on the zero mode of the second order field fluctuations). We also neglected the effect of Hubble friction since it does not give an important contribution for the second order fluctuations. There is a second reason for neglecting the effect of Hubble friction: in order for the energy transfer from radiation to scalar fields to be effective in draining enough energy from the radiation field to prevent a Big Crunch singularity, the time scale of the draining process must be shorter than the Hubble time. Hence, it is self-consistent to neglect terms that induce changes only on longer time scales.

3.4 The general solution

In this section we will solve the equations of motion (3.40), (3.41) and (3.43) to see if and how a bounce will happen.

It is usually useful to perform the analysis in the conformal frame where the conformal time $\eta \equiv \int a^{-1}(t)dt$ is used rather than the cosmic time. Additionally, to extract the dependence on the scale factor, it is convenient to use the following two variables:

$$u_1(\eta) \equiv a(\eta)\phi_1(\eta), \quad u_2(\eta) \equiv a(\eta)\phi_2(\eta) \ . \tag{3.44}$$

Hereafter, we will use $u_j^{(i)}(i = 0, 1, 2,\ j = 1, 2)$ to denote the i-th order perturbation of the j-th scalar field. Moreover, for simplicity, we can parameterize the scale factor $a(t)$ as

$$a(\eta) = a_0 t^p = a_0|\eta|^{\frac{p}{1-p}} \ , \tag{3.45}$$

with

$$p = \frac{2}{3(1+w)} \ , \tag{3.46}$$

where a_0 and w are the initial value of the scale factor and the equation of state of the Universe, respectively. This is a self-consistent assumption when w is nearly a constant. The evolution of w in our case will be shown numerically in Section VI.

3.4.1 Solutions for $\phi_1^{(0)}$ and $\phi_2^{(0)}$

Using the parametrization (3.45), the equations of motion at zeroth order of the two scalar fields become:

$$\begin{cases} u_1^{(0)\prime\prime} + (a_0^2 m^2 \eta^{\frac{2p}{1-p}} - \frac{p(2p-1)}{(1-p)^2\eta^2})u_1^{(0)} = 0 \ , \\ \\ u_2^{(0)\prime\prime} + (a_0^2 M^2 \eta^{\frac{2p}{1-p}} - \frac{p(2p-1)}{(1-p)^2\eta^2})u_2^{(0)} = 0 \ , \end{cases} \tag{3.47}$$

where a prime denotes the derivative with respect to conformal time η. Their solutions are:

$$u_1^{(0)} \sim \sqrt{|\eta|} H_{\pm\frac{1-3p}{2}}\big((1-p)am|\eta|\big) \ , \tag{3.48}$$

$$u_2^{(0)} \sim \sqrt{|\eta|} H_{\pm\frac{1-3p}{2}}\big((1-p)aM|\eta|\big) \ , \tag{3.49}$$

where $H_{\pm\frac{1-3p}{2}}$ represents the $(\pm\frac{1-3p}{2})$-th order Hankel function. Far away or close to the bounce, i.e. for $a|\eta| \gg m^{-1}, M^{-1}$ and $a|\eta| \ll m^{-1}, M^{-1}$, respectively, the approximate solutions are:

1)Oscillations for large values of the scale factor $a|\eta| \gg m^{-1}, M^{-1}$:

$$u_1^{(0)} \sim |\eta|^{\frac{p}{2(p-1)}} \sqrt{\frac{2}{(1-p)\pi a_0 m}} \cos\big((1-p)a_0 m|\eta|^{\frac{1}{1-p}} + \theta_1^{(0)}\big) \ , \tag{3.50}$$

$$u_2^{(0)} \sim |\eta|^{\frac{p}{2(p-1)}} \sqrt{\frac{2}{(1-p)\pi a_0 M}} \cos\big((1-p)a_0 M|\eta|^{\frac{1}{1-p}} + \theta_2^{(0)}\big) \ , \tag{3.51}$$

(where θ_1 and θ_2 are phases set by the initial conditions). In terms of the non-rescaled fields ϕ_i one obtains damped (or antidamped) oscillations (depending on whether we are in an expanding or a contracting period)

$$\phi_1^{(0)} \sim |\eta|^{\frac{3p}{2(p-1)}} \sqrt{\frac{2}{(1-p)\pi a_0^{\frac{3}{2}} m}} \cos\big((1-p)a_0 m|\eta|^{\frac{1}{1-p}} + \theta_1^{(0)}\big) \ , \tag{3.52}$$

$$\phi_2^{(0)} \sim |\eta|^{\frac{3p}{2(p-1)}} \sqrt{\frac{2}{(1-p)\pi a_0^{\frac{3}{2}} M}} \cos\big((1-p)a_0 M|\eta|^{\frac{1}{1-p}} + \theta_2^{(0)}\big) \ . \tag{3.53}$$

2)"Frozen" evolution for small values of the scale factor $a|\eta| \ll m^{-1}, M^{-1}$:

$$u_1^{(0)} \sim \frac{\big((1-p)a_0 m\big)^{\frac{1-3p}{2}} |\eta|^{\frac{1-2p}{1-p}}}{\Gamma(\frac{3(1-p)}{2})} + \frac{\big((1-p)a_0 m\big)^{-\frac{1-3p}{2}} |\eta|^{\frac{p}{1-p}}}{\Gamma(\frac{1+3p}{2})} \ , \tag{3.54}$$

$$u_2^{(0)} \sim \frac{\big((1-p)a_0 M\big)^{\frac{1-3p}{2}} |\eta|^{\frac{1-2p}{1-p}}}{\Gamma(\frac{3(1-p)}{2})} + \frac{\big((1-p)a_0 M\big)^{-\frac{1-3p}{2}} |\eta|^{\frac{p}{1-p}}}{\Gamma(\frac{1+3p}{2})} \ , \tag{3.55}$$

from which it follows that the nonrescaled fields ϕ_i evolve as

$$\phi_1^{(0)} \sim \frac{\big((1-p)a_0 m\big)^{\frac{1-3p}{2}}|\eta|^{\frac{1-3p}{1-p}}}{a_0\Gamma(\frac{3(1-p)}{2})} + \frac{\big((1-p)a_0 m\big)^{\frac{3p-1}{2}}}{a_0\Gamma(\frac{1+3p}{2})}\ , \tag{3.56}$$

$$\phi_2^{(0)} \sim \frac{\big((1-p)a_0 M\big)^{\frac{1-3p}{2}}|\eta|^{\frac{1-3p}{1-p}}}{a_0\Gamma(\frac{3(1-p)}{2})} + \frac{\big((1-p)a_0 M\big)^{\frac{3p-1}{2}}}{a_0\Gamma(\frac{1+3p}{2})}\ , \tag{3.57}$$

from which we can see that the last term of $\phi_i^{(0)}$ is a constant mode while the first term is a varying one. Depending on the value of p (or equivalently w) the varying mode could be growing (for $p > 1/3$ or $-1 < w < 1$), in which case it becomes dominant, or decaying (for $p < 1/3$ or for $w > 1$ or $w < -1$), in which case it becomes subdominant. We can usually neglect the decaying part of the fields.

3.4.2 Solutions for $\phi_1^{(1)}$ and $\phi_2^{(1)}$

Following the steps performed in the last subsection, we can also get the solutions for the first order components of the scalar fields. Using the equations (3.41) for the first order perturbations we obtain the following equations of motion for $u_1^{(1)}$ and $u_2^{(1)}$:

$$\begin{cases} u_1^{(1)''} + (k^2 + a_0^2 m^2 \eta^{\frac{2p}{1-p}} - \frac{p(2p-1)}{(1-p)^2\eta^2})u_1^{(1)} = 0\ , \\ \\ u_2^{(1)''} + (k^2 + a_0^2 M^2 \eta^{\frac{2p}{1-p}} - \frac{p(2p-1)}{(1-p)^2\eta^2})u_2^{(1)} = 0\ . \end{cases} \tag{3.58}$$

Depending on the value of k, we obtain different approximation solutions. For wavenumbers large compared both to the Hubble radius and to the mass term, we obtain oscillatory solutions with fixed amplitude.

Considering now modes which are still sub-Hubble (i.e. $k|\eta| > 1$) but for which the mass term dominates over the contribution of the field tension (i.e. the term involving k), we can neglect both the k^2 term and the term involving $\frac{p(2p-1)}{(1-p)^2\eta^2}$. The simplified equation for these modes is

$$\begin{cases} u_1^{(1)''} + a_0^2 m^2 \eta^{\frac{2p}{1-p}} u_1^{(1)} = 0 \ , \\ \\ u_2^{(1)''} + a_0^2 M^2 \eta^{\frac{2p}{1-p}} u_2^{(1)} = 0 \ , \end{cases} \tag{3.59}$$

whose solutions are

$$\begin{aligned} u_1^{(1)} &\sim \sqrt{|\eta|} H_{\frac{1-p}{2}}\big((1-p)a_0 m|\eta|\big) \\ &\sim |\eta|^{\frac{p}{2(p-1)}} \sqrt{\frac{2}{(1-p)\pi m a_0}} \cos\big((1-p)a_0 m|\eta| + \theta_1^{(1)}\big) \ , \end{aligned} \tag{3.60}$$

$$\begin{aligned} u_2^{(1)} &\sim \sqrt{|\eta|} H_{\frac{1-p}{2}}\big((1-p)a_0 M|\eta|\big) \\ &\sim |\eta|^{\frac{p}{2(p-1)}} \sqrt{\frac{2}{(1-p)\pi M a_0}} \cos\big((1-p)a_0 M|\eta| + \theta_2^{(1)}\big) \ . \end{aligned} \tag{3.61}$$

For modes outside the Hubble radius ($k\eta \ll 1$), we have

$$\begin{cases} u_1^{(1)''} + (a_0^2 m^2 \eta^{\frac{2p}{1-p}} - \frac{p(2p-1)}{(1-p)^2\eta^2}) u_1^{(1)} = 0 \ , \\ \\ u_2^{(1)''} + (a_0^2 M^2 \eta^{\frac{2p}{1-p}} - \frac{p(2p-1)}{(1-p)^2\eta^2}) u_2^{(1)} = 0 \ , \end{cases} \tag{3.62}$$

which have the same form as Eq. (3.47) so their solution will be the same as given in Eqs. (3.54) and (3.55). Note that on scales larger than the Hubble radius we cannot neglect the metric fluctuations. However, the typical time scale associated with the growth of metric fluctuations is the Hubble time scale, and we are looking for effects on shorter time scales, as already mentioned. Hence, the neglect of metric fluctuations is justified.

We have thus seen that the first order solutions for the scalar fields scale the same way with $|\eta|$ as the zeroth order solution. This is because in the small $|\eta|$ region where the $a''(t)/a(t)$ term dominates over the other ones, the equations for first order and zeroth order modes are almost the same. Thus, unless the energy density in the $u_i^{(1)}$ modes dominates at the initial time, it will never dominate over the background contribution from the $u_i^{(0)}$ terms. Thus we can conclude that the first order fluctuations of scalar fields will not prevent the bounce.

3.4.3 Solution for the gauge field γ

In this section, we will analyze the gauge field γ which is also considered to be of first order. The equation (3.42) can directly be transformed to conformal frame as:

$$\gamma'' + k^2\gamma - \frac{2(c\phi_1^{(0)}\phi_1^{(0)\prime} + d\phi_2^{(0)}\phi_2^{(0)\prime})}{1 - c{\phi_1^{(0)}}^2 - d{\phi_2^{(0)}}^2}\gamma' = 0 \ . \tag{3.63}$$

Since the coefficients c and d are small, we can take the last term to be a source term. In a first order Born approximation, we can write the total solution as

$$\gamma \simeq \gamma_0 + \delta\gamma \,, \tag{3.64}$$

where γ_0 is the solution for the homogeneous equation obtained by setting $c = d = 0$, while $\delta\gamma$ is the leading correction term obtained by inserting γ_0 into the source term (the last term in (3.63)).

The zeroth order (homogeneous) equation is easily solved and gives

$$\gamma_0 \sim \cos(k|\eta| + \theta_\gamma) \ . \tag{3.65}$$

For the first order equation, it is convenient to define

$$P(\eta) \equiv -\frac{2(c\phi_1^{(0)}\phi_1^{(0)\prime} + d\phi_2^{(0)}\phi_2^{(0)\prime})}{1 - c{\phi_1^{(0)}}^2 - d{\phi_2^{(0)}}^2} \,, \tag{3.66}$$

so that the equation becomes

$$\delta\gamma'' + k^2\delta\gamma + P(\eta)\gamma_0' = 0 \ , \tag{3.67}$$

where we neglected the small term $P(\eta)\delta\gamma'$. Inserting the solution of γ_0 (3.65), we get the following equation for $\delta\gamma$:

$$\delta\gamma'' + k^2\delta\gamma = -P(\eta)\gamma_0' \ . \tag{3.68}$$

We are interested in the scaling of $\delta\gamma$ as a function of time. For this purpose, we need to work out the scaling in time of the source term in (3.68). Since the solutions for $\phi_1^{(0)}$ and $\phi_2^{(0)}$ scale differently in time in the two time intervals discussed in Subsection (3.4.1), it is necessary to analyze these two intervals separately.

For times obeying $a|\eta| \gg m^{-1}, M^{-1}$, then by differentiating (3.52) and (3.53) with respect to η we have:

$$\begin{aligned} \phi_1^{(0)\prime} &\sim |\eta|^{\frac{p+2}{2(p-1)}}\sqrt{\frac{2}{(1-p)\pi a_0^{\frac{3}{2}}m}} \qquad (3.69)\\ &\times\big[-\frac{3p}{p-1}\cos\big((1-p)a_0 m|\eta|^{\frac{1}{1-p}}+\theta_1^{(0)}\big) + a_0 m|\eta|^{\frac{1}{1-p}}\sin\big((1-p)a_0 m|\eta|^{\frac{1}{1-p}}+\theta_1^{(0)}\big)\big]\ , \\ \phi_2^{(0)\prime} &\sim |\eta|^{\frac{p+2}{2(p-1)}}\sqrt{\frac{2}{(1-p)\pi a_0^{\frac{3}{2}}M}} \qquad (3.70)\\ &\times\big[-\frac{3p}{p-1}\cos\big((1-p)a_0 M|\eta|^{\frac{1}{1-p}}+\theta_1^{(0)}\big) + a_0 M|\eta|^{\frac{1}{1-p}}\sin\big((1-p)a_0 M|\eta|^{\frac{1}{1-p}}+\theta_1^{(0)}\big)\big]\ . \end{aligned}$$

Note that $|\eta|^{\frac{1}{1-p}} \sim t$ is a decaying mode in the contracting phase and thus the last terms inside the square brackets in the above formulae can be neglected compared to the first ones. Since

$$P(\eta) = -\frac{2(c\phi_1^{(0)}\phi_1^{(0)\prime} + d\phi_2^{(0)}\phi_2^{(0)\prime})}{1 - c{\phi_1^{(0)}}^2 - d{\phi_2^{(0)}}^2} \approx -2(c\phi_1^{(0)}\phi_1^{(0)\prime} + d\phi_2^{(0)}\phi_2^{(0)\prime})\,, \tag{3.71}$$

then combining all these results we get

$$\delta\gamma \sim C_1|\eta|^{\frac{1-4p}{1-p}} \ . \tag{3.72}$$

For $a|\eta| \ll m^{-1}, M^{-1}$, then differentiating (3.56) and (3.57) with respect to η we obtain

$$\phi_1^{(0)'} \sim \frac{1-3p}{1-p}\frac{\left((1-p)a_0 m\right)^{\frac{1-3p}{2}}|\eta|^{\frac{-2p}{1-p}}}{a_0\Gamma(\frac{3(1-p)}{2})} \ , \tag{3.73}$$

$$\phi_2^{(0)'} \sim \frac{1-3p}{1-p}\frac{\left((1-p)a_0 M\right)^{\frac{1-3p}{2}}|\eta|^{\frac{-2p}{1-p}}}{a_0\Gamma(\frac{3(1-p)}{2})} \ , \tag{3.74}$$

when $p > 1/3$ and

$$\phi_1^{(0)'} \sim \phi_2^{(0)'} \approx 0 \tag{3.75}$$

when $p < 1/3$. Then we can solve Equation (3.68) to get:

$$\begin{aligned} \delta\gamma &\sim C_2|\eta|^{\frac{3-7p}{1-p}} \ , \quad p > \frac{1}{3} \\ \delta\gamma &\sim C_3\cos(k|\eta| + \theta_{\delta\gamma}) \ . \quad p < \frac{1}{3} \end{aligned} \tag{3.76}$$

In the above expressions for $\delta\gamma$, C_1, C_2 and C_3 are complicated prefactors in front of the η-dependent terms.

In summary, we see that the interactions give only a subleading correction $\delta\gamma$ to γ_0.

3.4.4 Solutions for $\phi_1^{(2)}$ and $\phi_2^{(2)}$

Finally, let us consider the homogeneous component of the second order fluctuations of the scalars, namely, $\phi_1^{(2)}$ and $\phi_2^{(2)}$. If we only consider the energy density up to second order, these second order field perturbations give a contribution through their coupling to the background fields. In the following we find the solutions of (3.43) and study the effects of the induced terms in the stress-energy tensor on a possible bounce.

Given the solution for the gauge field γ obtained in the last subsection, it is easy to rewrite Eqs. (3.43) as:

$$\left\{\begin{array}{c} u_1^{(2)''} + a^2 m^2 u_1^{(2)} = c\frac{(k^2\gamma^2 - \gamma'^2)}{a^2} u_1^{(0)} \ , \\ \\ u_2^{(2)''} + a^2 M^2 u_2^{(2)} = -d\frac{(k^2\gamma^2 - \gamma'^2)}{a^2} u_2^{(0)} \ , \end{array}\right. \tag{3.77}$$

where we made use of the fact that

$$F_{\mu\nu}F^{\mu\nu} = 2(k^2\gamma^2 - a^2\dot{\gamma}^2)/a^4 \,. \tag{3.78}$$

Equation (3.77) has the same form as the zeroth order equation but with a small source term generated by the interaction with the gauge field. This equation can be solved using the Born approximation (details are given in the Appendix). The general solution is the sum of the general solution of the homogeneous solution plus the solution including the source which has vanishing initial data. The inhomogeneous term is suppressed by the coupling constants c and d compared to the homogeneous solution, but, as shown in the Appendix, it scales as a high power of η^{-1}. Via the coupling to the background scalar fields, the above second order terms enter into the expression for the energy density to second order. The signs of the corresponding terms in the energy density are indefinite in the sense that they depend on the phases of the initial field configurations. Since it is these terms that dominate the energy density near the bounce, we find that whether a bounce occurs or not depends sensitively on the phases in the initial conditions, and that in fact in the case of many plane wave modes initially excited, a bounce requires very special phase correlations.

3.5 Evolution of the components of the energy density

In the previous section we have solved all of the field equations up to second order in the amplitude of the fluctuations. We have found the scaling in time of each field at each order. Now we are ready to look at how all of the terms in the expression for the energy density $\rho^{(0)}$, $\rho^{(1)}$ and $\rho^{(2)}$ at various orders in perturbation theory (namely, Eqs. (3.17)-(3.19)) scale in time. This analysis is straightforward but very important if we are to determine whether a bounce is possible, since in four space-time dimensional classical Einstein Gravity with flat spatial sections a bounce can only happen when the negative terms in the energy density catch up to the positive contributions [10].

In the following we give a table of how each term contained in ρ scales with time as the background cosmology bouncing point (the bounce which is achieved in the absence of radiation and scalar field inhomogeneities) is approached. We will identify the terms which dominate in this limit. This will give us a good indication under which conditions a bounce can occur. The tables are structured as follows: the first line "Terms", indicates which term we are considering, the next set of lines "Behavior" gives the scaling in time in the various limits and in the two relevant ranges of the parameter p which indicates the equation of state, and the last line gives the sign with which the term contributes to the energy density. Note that we focus on the growing mode solution to each field (which is constant for small η in the case $p < 1/3$). We give separate tables for terms of zeroth, first and second order in ϵ.

a) For terms contained in $\rho^{(0)}$:

Table 3.1: Behavior of background terms for the energy density

Terms	${\dot{\phi}_1^{(0)}}^2$	$m^2{\phi_1^{(0)}}^2$	$-{\dot{\phi}_2^{(0)}}^2$	$-M^2{\phi_2^{(0)}}^2$
Behavior	$a^{-3-\frac{2}{p}}\big(a\lvert\eta\rvert \gg m^{-1}\big)$	$a^{-3}\big(a\lvert\eta\rvert \gg m^{-1}\big)$	$a^{-3-\frac{2}{p}}\big(a\lvert\eta\rvert \gg m^{-1}\big)$	$a^{-3}\big(a\lvert\eta\rvert \gg M^{-1}\big)$
	$a^{-6}\begin{pmatrix} a\lvert\eta\rvert \ll m^{-1} \\ p > \frac{1}{3}\end{pmatrix}$	$a^{-6+\frac{2}{p}}\begin{pmatrix} a\lvert\eta\rvert \ll m^{-1} \\ p > \frac{1}{3}\end{pmatrix}$	$a^{-6}\begin{pmatrix} a\lvert\eta\rvert \ll m^{-1} \\ p > \frac{1}{3}\end{pmatrix}$	$a^{-6+\frac{2}{p}}\begin{pmatrix} a\lvert\eta\rvert \ll M^{-1} \\ p > \frac{1}{3}\end{pmatrix}$
	$0\begin{pmatrix} a\lvert\eta\rvert \ll m^{-1} \\ p < \frac{1}{3}\end{pmatrix}$	$a^{0}\begin{pmatrix} a\lvert\eta\rvert \ll m^{-1} \\ p < \frac{1}{3}\end{pmatrix}$	$0\begin{pmatrix} a\lvert\eta\rvert \ll m^{-1} \\ p < \frac{1}{3}\end{pmatrix}$	$a^{0}\begin{pmatrix} a\lvert\eta\rvert \ll M^{-1} \\ p < \frac{1}{3}\end{pmatrix}$
Sign	Positive Definite	Positive Definite	Negative Definite	Negative Definite

b) For terms contained in $\rho^{(1)}$:

Table 3.2: Behavior of terms of order 1 in the energy density

Terms	$\dot{\phi}_1^{(0)}\dot{\phi}_1^{(1)}$	$\phi_1^{(0)}\phi_1^{(1)}$
Behavior	$a^{-3-\frac{2}{p}}\big(\lvert\eta\rvert \gg Max\{k^{-1},(am)^{-1}\}\big)$	$a^{-3}\big(\lvert\eta\rvert \gg Max\{k^{-1},(am)^{-1}\}\big)$
	$a^{-\frac{9}{2}-\frac{1}{p}}\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(am)^{-1}] \\ p > \frac{1}{3}\end{pmatrix}$	$a^{-\frac{9}{2}+\frac{1}{p}}\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(am)^{-1}] \\ p > \frac{1}{3}\end{pmatrix}$
	$a^{-6}\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(am)^{-1}\} \\ p > \frac{1}{3}\end{pmatrix}$	$a^{-6+\frac{2}{p}}\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(am)^{-1}\} \\ p > \frac{1}{3}\end{pmatrix}$
	$0\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(am)^{-1}] \\ p < \frac{1}{3}\end{pmatrix}$	$a^{-\frac{3}{2}}\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(am)^{-1}] \\ p < \frac{1}{3}\end{pmatrix}$
	$0\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(am)^{-1}\} \\ p < \frac{1}{3}\end{pmatrix}$	$a^{0}\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(am)^{-1}\} \\ p < \frac{1}{3}\end{pmatrix}$
Sign	Indefinite	Indefinite

Terms	$-\phi_2^{(0)}\phi_2^{(1)}$	$-\dot{\phi}_2^{(0)}\dot{\phi}_2^{(1)}$
Behavior	$a^{-3}\big(\lvert\eta\rvert \gg Max\{k^{-1},(aM)^{-1}\}\big)$	$a^{-3-\frac{2}{p}}\big(\lvert\eta\rvert \gg Max\{k^{-1},(aM)^{-1}\}\big)$
	$a^{-\frac{9}{2}+\frac{1}{p}}\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(aM)^{-1}] \\ p > \frac{1}{3}\end{pmatrix}$	$a^{-\frac{9}{2}-\frac{1}{p}}\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(aM)^{-1}] \\ p > \frac{1}{3}\end{pmatrix}$
	$a^{-6+\frac{2}{p}}\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(aM)^{-1}\} \\ p > \frac{1}{3}\end{pmatrix}$	$a^{-6}\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(aM)^{-1}\} \\ p > \frac{1}{3}\end{pmatrix}$
	$a^{-\frac{3}{2}}\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(aM)^{-1}] \\ p < \frac{1}{3}\end{pmatrix}$	$0\begin{pmatrix} \lvert\eta\rvert \in [k^{-1},(aM)^{-1}] \\ p < \frac{1}{3}\end{pmatrix}$
	$a^{0}\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(aM)^{-1}\} \\ p < \frac{1}{3}\end{pmatrix}$	$0\begin{pmatrix} \lvert\eta\rvert \ll Min\{k^{-1},(aM)^{-1}\} \\ p < \frac{1}{3}\end{pmatrix}$
Sign	Indefinite	Indefinite

These terms, however, all vanish if the energy density is defined by spatial averaging.

c) For terms contained in $\rho^{(2)}$:

Table 3.3: Behavior of terms of order 2 in the energy density

Terms	${\dot{\phi}_1^{(1)}}^2$	$\dot{\phi}_1^{(0)}\dot{\phi}_1^{(2)}$	$\frac{k^2}{a^2}{\phi_1^{(1)}}^2$	$m^2{\phi_1^{(1)}}^2$
Behavior	$a^{-3-\frac{2}{p}}\left(k\lvert\eta\rvert \gg 1\right)$	$a^{-7-\frac{1}{p}}\left(a\lvert\eta\rvert \gg m^{-1}\right)$	$a^{-5}\left(k\lvert\eta\rvert \gg 1\right)$	$a^{-3}\left(k\lvert\eta\rvert \gg 1\right)$
	$a^{-6}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p > \frac{1}{3}\end{matrix}\right)$	$a^{-\frac{17}{2}}\left(\begin{matrix} a\lvert\eta\rvert \ll m^{-1} \\ p > \frac{1}{3}\end{matrix}\right)$	$a^{-8+\frac{2}{p}}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p > \frac{1}{3}\end{matrix}\right)$	$a^{-6+\frac{2}{p}}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p > \frac{1}{3}\end{matrix}\right)$
	$0\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p < \frac{1}{3}\end{matrix}\right)$	$0\left(\begin{matrix} a\lvert\eta\rvert \ll m^{-1} \\ p < \frac{1}{3}\end{matrix}\right)$	$a^{-2}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p < \frac{1}{3}\end{matrix}\right)$	$a^{0}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p < \frac{1}{3}\end{matrix}\right)$
Sign	Positive Definite	Indefinite	Positive Definite	Positive Definite

Terms	$-{\dot{\phi}_2^{(1)}}^2$	$-\dot{\phi}_2^{(0)}\dot{\phi}_2^{(2)}$	$\frac{k^2}{a^2}{\phi_2^{(1)}}^2$	$-M^2{\phi_2^{(1)}}^2$
Behavior	$a^{-3-\frac{2}{p}}\left(k\lvert\eta\rvert \gg 1\right)$	$a^{-7-\frac{1}{p}}\left(a\lvert\eta\rvert \gg M^{-1}\right)$	$a^{-5}\left(k\lvert\eta\rvert \gg 1\right)$	$a^{-3}\left(k\lvert\eta\rvert \gg 1\right)$
	$a^{-6}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p > \frac{1}{3}\end{matrix}\right)$	$a^{-\frac{17}{2}}\left(\begin{matrix} a\lvert\eta\rvert \ll M^{-1} \\ p > \frac{1}{3}\end{matrix}\right)$	$a^{-8+\frac{2}{p}}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p > \frac{1}{3}\end{matrix}\right)$	$a^{-6+\frac{2}{p}}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p > \frac{1}{3}\end{matrix}\right)$
	$0\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p < \frac{1}{3}\end{matrix}\right)$	$0\left(\begin{matrix} a\lvert\eta\rvert \ll M^{-1} \\ p < \frac{1}{3}\end{matrix}\right)$	$a^{-2}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p < \frac{1}{3}\end{matrix}\right)$	$a^{0}\left(\begin{matrix} k\lvert\eta\rvert \ll 1 \\ p < \frac{1}{3}\end{matrix}\right)$
Sign	Negative Definite	Indefinite	Negative Definite	Negative Definite

Terms	$m^2\phi_1^{(0)}\phi_1^{(2)}$	$-M^2\phi_2^{(0)}\phi_2^{(2)}$
Behavior	$a^{-7+\frac{1}{p}}\left(a\lvert\eta\rvert \gg m^{-1}\right)$	$a^{-7+\frac{1}{p}}\left(a\lvert\eta\rvert \gg M^{-1}\right)$
	$a^{-\frac{17}{2}+\frac{2}{p}}\left(\begin{matrix} a\lvert\eta\rvert \ll m^{-1} \\ p > \frac{1}{3}\end{matrix}\right)$	$a^{-\frac{17}{2}+\frac{2}{p}}\left(\begin{matrix} a\lvert\eta\rvert \ll M^{-1} \\ p > \frac{1}{3}\end{matrix}\right)$
	$a^{-\frac{11}{2}+\frac{1}{p}}\left(\begin{matrix} a\lvert\eta\rvert \ll m^{-1} \\ p < \frac{1}{3}\end{matrix}\right)$	$a^{-\frac{11}{2}+\frac{1}{p}}\left(\begin{matrix} a\lvert\eta\rvert \ll M^{-1} \\ p < \frac{1}{3}\end{matrix}\right)$
Sign	Indefinite	Indefinite

These terms do not vanish upon spatial averaging.

Table 3.4: Behavior of terms of order 2, including a gauge field component in the energy density

Terms	$a^{-4}k^2\gamma_0^2 + a^{-2}\dot{\gamma}_0^2$	$(-c{\phi_1^{(0)}}^2 - d{\phi_2^{(0)}}^2) \times (a^{-4}k^2\gamma_0^2 + a^{-2}\dot{\gamma}_0^2)$	$a^{-4}k^2\gamma_0\delta\gamma$	$a^{-2}\dot{\gamma}\dot{\delta\gamma}$
Behavior	a^{-4}	$a^{-7}\big(a\lvert\eta\rvert \gg m^{-1}\big)$ $a^{-10+\frac{2}{p}}\big(a\lvert\eta\rvert \ll m^{-1},\ p > \frac{1}{3}\big)$ $a^{-4}\big(a\lvert\eta\rvert \ll m^{-1},\ p < \frac{1}{3}\big)$	$a^{-8+\frac{1}{p}}\big(a\lvert\eta\rvert \gg m^{-1}\big)$ $a^{-11+\frac{3}{p}}\big(a\lvert\eta\rvert \ll m^{-1},\ p > \frac{1}{3}\big)$ $a^{-4}\big(a\lvert\eta\rvert \ll m^{-1},\ p < \frac{1}{3}\big)$	$a^{-7}\big(a\lvert\eta\rvert \gg m^{-1}\big)$ $a^{-10+\frac{2}{p}}\big(a\lvert\eta\rvert \ll m^{-1},\ p > \frac{1}{3}\big)$ $a^{-4}\big(a\lvert\eta\rvert \ll m^{-1},\ p < \frac{1}{3}\big)$
Sign	Positive Definite	Indefinite (Depending only on c and d)	Indefinite	Indefinite

Note that we have expressed the time dependence in terms of the dependence on the scale factor $a(t)$. At this stage, we only need to focus on the exponent of the power-law scaling. The more negative the power is, the more rapidly the term grows in a contracting phase (since $a(t)$ is decreasing with time).

As mentioned earlier, the conditions for a bounce to occur in four space-time dimensional classical Einstein gravity with flat spatial sections is that the total energy density reaches zero during the contracting phase. Thus, there needs to be a negative definite term which starts out small but grows faster than the positive definite terms due to the regular scalar field and regular radiation. In the absence of radiation and scalar field inhomogeneities, it is the contribution to the energy density of the ghost field ϕ_2 which plays this role.

From the table we see that there are three kinds of terms: positive definite, negative definite and indefinite ones. The first set contains the kinetic and potential terms of the normal scalar as well as the free energy density of the gauge field, the second set is made up of the kinetic and potential terms of the ghost scalar, while the third set contains terms which arise due to the coupling terms between scalars or between scalars and gauge fields.

Looking first at the terms which are independent of the coupling term between the fields, we see from the first line of the "Behavior" set of lines in the third table that, indeed, in the presence of radiation the energy density in radiation grows faster than that in the two scalar fields, thus preventing a bounce. In the presence of coupling between the fields, however, there are terms which scale with a larger negative power of $a(t)$. The signs of some of them, however, depend on the initial phases for the linear fields γ, $\phi_1^{(1)}$ and $\phi_2^{(1)}$.

Note that the signs of the scalar coupling terms are determined by the evolution of each field and thus are hard to be identified in a general analysis. The same is true for the gauge coupling terms (the last two in the third table). However, the coupling terms between the scalar fields and the gauge field (the third to last in the third table) can be made negative/positive definite easily by setting the signs of the coefficients c and d to be both positive/negative.

It is reasonable to assume that the contracting phase begins with the regular scalar field dominating the energy density, and that the contribution of the Lee-Wick scalar is much smaller. For single Fourier mode initial conditions of the radiation field, this can be achieved with the appropriate choice of the initial phase (see Example 1 in the following section containing our numerical results). However, for multiple initial radiation Fourier modes excited any initial phase difference between the modes will produce a contribution with the wrong sign and will thus prevent a bounce (see Example 3 in the following section). In the presence of an infinite set of modes, the phase correlations required to obtain a bounce thus appear to have negligible measure in initial condition space. Thus, even in the presence of coupling between scalar fields and radiation, the Lee-Wick bounce is unstable.

The bounce, if it exists, will happen at a time which can be chosen to be $t = 0$. Its duration (the time interval lasting from the time the Hubble radius stops decreasing in the contracting phase until when it starts expanding in the post-bounce phase) will be denoted by Δt. Since the various components of the energy density scale with different powers of $a(t)$, it is clear that the duration of the bounce will be shorter or equal to the Hubble radius $H_{\rm max}^{-1}$ (which gives the time scale on which the ratios of energy densities in different components change) at the beginning of the bounce phase. For the background bounce model, we have $H_{\rm max} \sim m$.

There are two kinds of bounces according to the duration of the bounce phase i) If the period $\Delta t \simeq m^{-1}$, the bounce will go from the time

$$t_{B^-} \sim -\frac{(\Delta t)}{2} \sim -\frac{1}{2m} \tag{3.79}$$

to the time

$$t_{B^+} \sim \frac{(\Delta t)}{2} \sim \frac{1}{2m} \tag{3.80}$$

with a low speed. We call this a "slow bounce". In this case, the Universe will enter the bounce period at the critical time $t_c \sim m$, and only the $a|\eta| \gg m^{-1}$ approximate solutions of the previous tables will be applicable and not ones for the interval $a|\eta| \ll m^{-1}$. ii) If the period $\Delta t \ll m^{-1}$, the bounce will happen in a very short time with very fast speed. This can be called the "fast bounce". In this case, the Universe evolves from the far past ($-t_i$ with $|t_i| \gg 1$) to $t = 0$, passing through the point $t_c \sim m$, then entering into the region $|t| \lesssim m$ before finally reaching the bouncing point. In this case, both of the two approximate solutions of the field evolution will be applied.

Let us now consider the necessary conditions for a bounce (as we have indicated above and will see from the numerical analysis, these conditions are not sufficient

- in addition to the conditions which follow, appropriate correlations in the initial phases are required). We start in the region of time $a|\eta| \gg m^{-1}$. We study the conditions required to have the terms that might give a bounce grow relative to the other terms during this phase. If the conditions are not satisfied, or the bounce does not happen even if the conditions are satisfied, then a bounce may still occure in the $a|\eta| \ll m^{-1}$ region. The conditions for the terms in the energy which could compensate the positive radiation contribution to become dominant are then studied. If these conditions are not satisfied, either, then a bounce is impossible.

A necessary condition for a bounce to be possible requires the growth rate of one of the indefinite sign terms in the third table above exceed all that of all of the positive definite terms. In the $a|\eta| \gg m^{-1}$ region, this requires $8 - 1/p > 5$, which equivalently constrains the equation of state parameter w to be in the range $w < 1$. If this condition is satisfied in this region, then a slow bounce may happen depending on the choice of the initial phases.

If the condition is not satisfied in the $a|\eta| \gg m^{-1}$ region, the Universe may evolve into the $a|\eta| \ll m^{-1}$ region, in which the evolution of the fields are different, and new constraints on p and w will arise if a bounce is to be possible. Following the above logic, we find that the conditions under which a bounce might happen are much looser, namely $w > -7/6$.

To summarize this section: we have identified necessary conditions for a bounce to occur. Whether one actually does occur even if the conditions are satisfied depends on the initial phases of the fields. This must be studied numerically. In the following section we will give one example of specially chosen phases for which a bounce is possible. However, when we look at a more general choice of phases, the bounce will not occur.

3.6 Numerical Results

In order to support the analysis in the last section, we performed numerical calculations. Such numerical work is necessary because our analytical analysis is only approximate. In particular, we worked in perturbation theory up to order second order in ϵ. In addition, even in cases where our analytical analysis would indicate the possibility of a bounce, the perturbative analysis will break down near the bouncing point, and there is no assurance that the trends seen in the perturbative analysis will persist.

We have numerically solved the full nonlinear equations of motion for the matter fields in the presence of a homogeneous expanding background cosmology. The homogeneous cosmology is obtained numerically by solving the first Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho\,, \tag{3.81}$$

where G is Newton's gravitational constant (related to the Planck mass used earlier), and ρ is the total energy density, averaged over space.

Figures 3.1-3.6 are two groups of numerical results with different parameters. In both cases we choose the initial energy density of the gauge fields to be larger than that of the Lee-Wick scalar, but less than that of the normal scalar. These initial conditions correspond to the situation we are interested in, namely starting in a matter-dominated contracting phase in the presence of some radiation which is subdominant. Fig. 3.1, 3.2 and 3.3 show an example with parameters $c > 0$ and $d > 0$. We choose initial conditions in which a single Fourier mode fluctuation is excited, and in which the phases are chosen as indicated in the figure caption. For these initial phases, we obtain a bounce. In Fig. 3.1, we see that the equation of state w begins with a value slightly larger than 0, and then evolves to some nearly fixed value. For the case of our initial condition choice, it appears to be $w \simeq -0.6$, in the region where the bounce is allowed to happen. At the bouncing point, the equation of state will drop to $-\infty$, while after the bounce, the equation of state will rise again to $w \simeq 0.6$. Fig. 3.2 is the plot of the scale factor in this case which shows explicitly the occurrence of the bounce. Fig. 3.3 gives a comparison

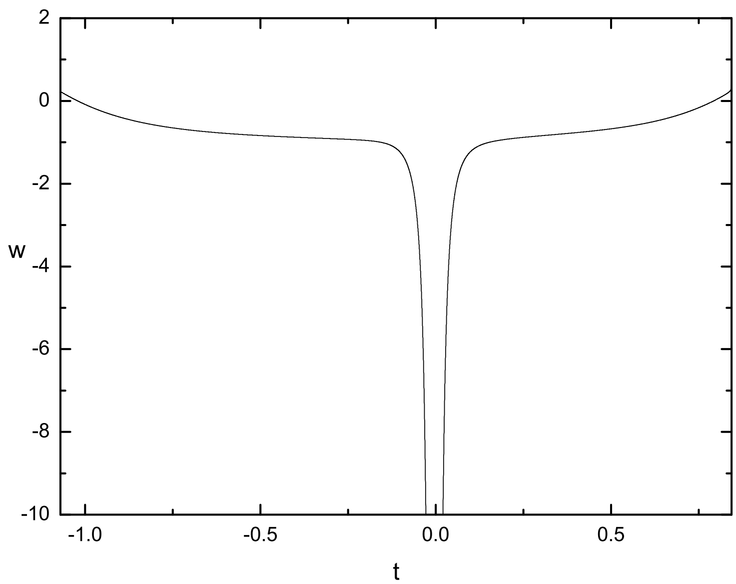

Figure 3.1: The evolution of the equation of state w with respect to cosmic time t (horizontal axis), in the first simulation, a simulation with only a single Fourier mode excited and phases chosen as indicated below. We see that w drops to $-\infty$, indicating that there is a bounce. The background fields are plotted in dimensionless units by normalizing by the mass $m_{rec} = 10^{-6} m_{Pl}$ while the time axis is displayed in units of m_{rec}^{-1}. The mass parameters m and M were chosen to be $m = 5m_{rec}$ and $M = 10m_{rec}$. The initial conditions were chosen to be $\gamma_i \simeq -1.85 \times 10^5 m_{rec}$, $\dot{\gamma}_i \simeq 7.35 \times 10^6 m_{rec}^2$, $\phi_{1i} \simeq 1.015 \times 10^5 m_{rec}$, $\dot{\phi}_{1i} \simeq 6.39 \times 10^5 m_{rec}^2$, $\phi_{2i} \simeq 2.54 \times 10^2 m_{rec}$, $\dot{\phi}_{2i} \simeq -4.96 \times 10^3 m_{rec}^2$. The coefficients c and d are chosen to be $c = 10^{-10} M_{rec}^2$ and $d = 10^{-10} M_{rec}^2$. The wavenumber is $k \simeq 0.01 hMpc^{-1}$.

of the energy densities of some components during the process. Initially, we set the energy density of the gauge field γ to be between the normal scalar and Lee-Wick scalar. When the evolution of the Universe enters into a region with nearly

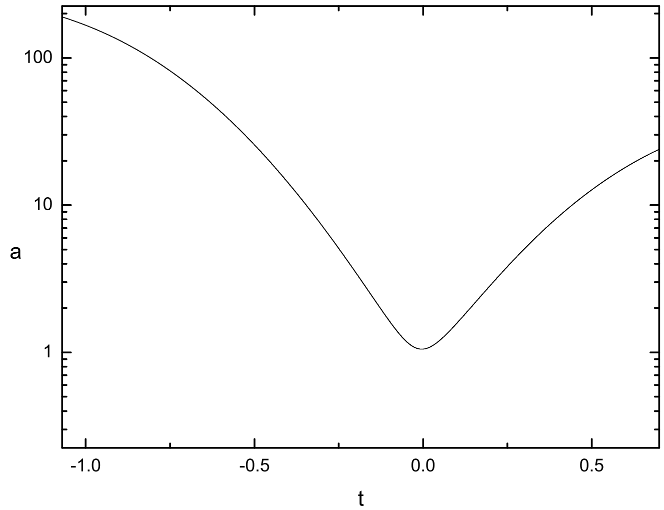

Figure 3.2: The scale factor of the Universe in the same simulation that leads to the evolution of the equation of state shown in Fig. 3.1. From the plot we see that the bounce happens at $t = 0$.

constant w, the gauge coupling component of energy density ρ_c will grow very fast. It is negative and thus enables the negative part of the energy density to catch up with the positive one, thus allowing the bounce to happen. For the inhomogeneous fluctuation, we choose the wavenumber to be $k \simeq 0.01hMpc^{-1}$ which corresponds to a scale which is observable by CMB and LSS experiments.

Figs. 3.4, 3.5 and 3.6 give the corresponding results in the case when we choose $c > 0$ while $d < 0$ (with all initial conditions identical). This case seems dangerous because the contribution of the Lee-Wick scalar to the fluctuation terms could lead to an instability. However, as we have mentioned before, since the effects of the Lee-Wick scalar are less than that of the normal scalar, it is still possible for the bounce to happen. Fig. 3.4 shows the equation of state of the system. We can see that the evolution of w is about the same as that in Fig. 3.1, since the change of the sign of d does not alter the result too much. Fig. 3.5 is the behavior of scale

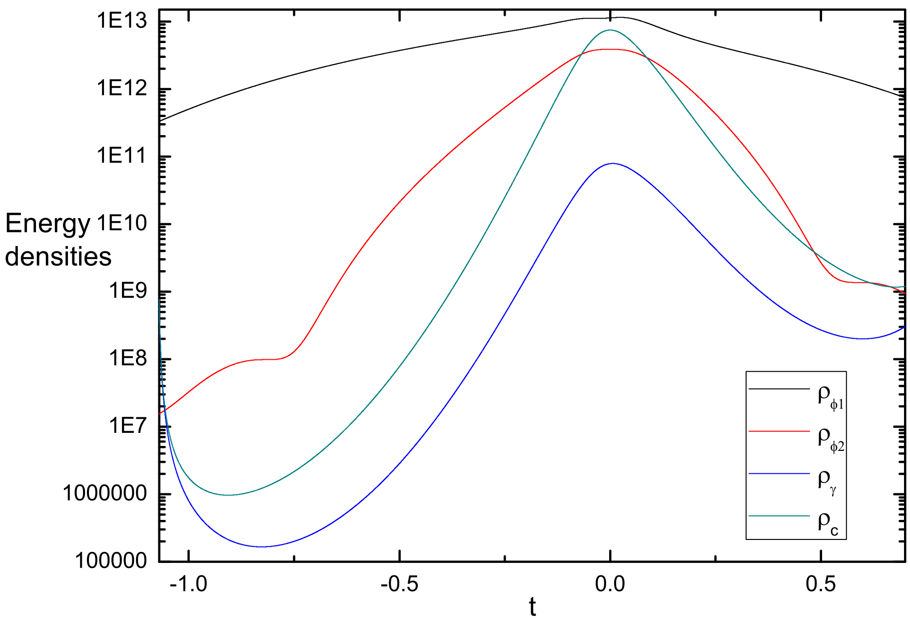

Figure 3.3: Energy densities of ϕ_1, ϕ_2 and γ in the system with parameters chosen as in Fig. 3.1. The curves from top to bottom are ρ_{ϕ_1} (black), ρ_{ϕ_2} (red), ρ_c (dark cyan) and ρ_γ (blue), respectively. The variables are also normalized with the mass scale $m_{rec} = 10^{-6} m_{Pl}$.

factor in this case while Fig. 3.6 gives the comparison of the energy densities of all components.

A change in the phase of the initial radiation field velocity will not change the results (if we keep the other initial conditions fixed). On the other hand, if we flip the sign of the initial velocity of one of the two scalar fields, then the sign of the dominant contribution to the energy density as we approach the bounce will flip and this will prevent a bounce. If we use initial conditions containing two excited Fourier modes, then we obtain a bounce only if the signs of the initial field velocities are both chosen as in the first run whose results are shown here. Different phases

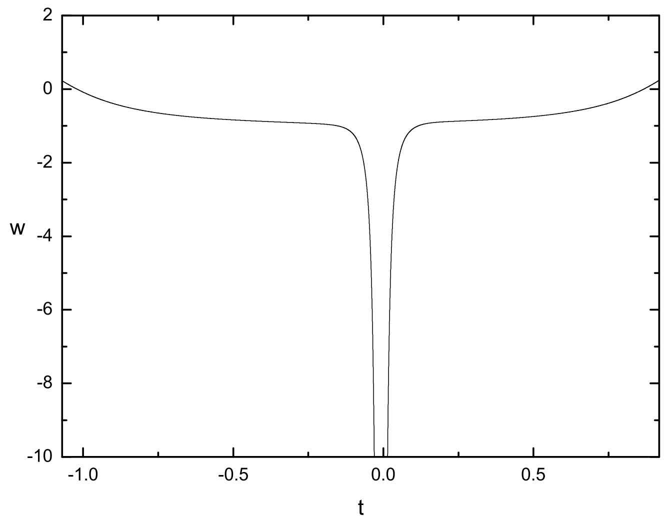

Figure 3.4: The evolution of the equation of state w as a function of cosmic time t (horizontal axis). The behavior that w drops to $-\infty$ indicating that a bounce takes place. The background fields are plotted in dimensionless units by normalizing by the mass $m_{rec} = 10^{-6} m_{Pl}$ while the time axis is displayed in units of m_{rec}^{-1}. The mass parameters m and M were chosen to be $m = 5m_{rec}$ and $M = 10m_{rec}$. The initial conditions were chosen to be $\gamma_i \simeq -1.85 \times 10^5 m_{rec}$, $\dot{\gamma}_i \simeq 7.35 \times 10^6 m_{rec}^2$, $\phi_{1i} \simeq 1.015 \times 10^5 m_{rec}$, $\dot{\phi}_{1i} \simeq 6.39 \times 10^5 m_{rec}^2$, $\phi_{2i} \simeq 2.54 \times 10^2 m_{rec}$, $\dot{\phi}_{2i} \simeq -4.96 \times 10^3 m_{rec}^2$. The coefficients c and d are chosen to be $c = 10^{-10} M_{rec}^2$ and $d = -10^{-10} M_{rec}^2$. The wavenumber $k \simeq 0.01 hMpc^{-1}$.

for the scalar field velocities of the two modes destroys the possibility of obtaining a bounce.

Figures 3.7, 3.8 and 3.9 show the results for the equation of state parameter w, the Hubble parameter H and the contribution of the various components to the total ρ in the case of a simulation in which two Fourier modes are excited, with velocities of both scalar fields having opposite signs from those in the previous example. As is obvious, a Big Crunch singularity occurs.

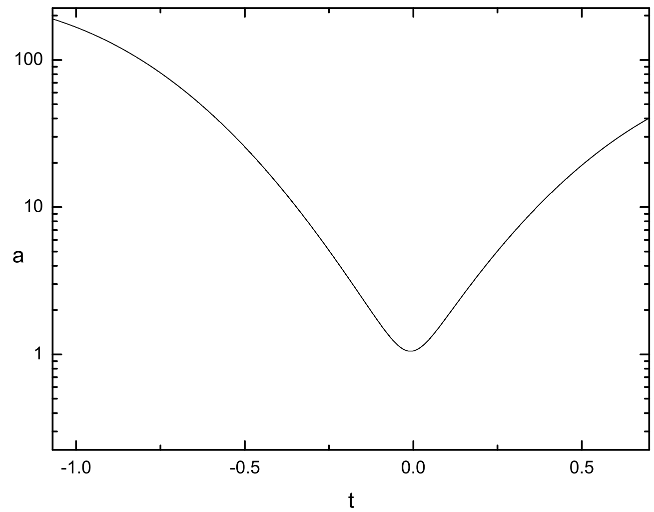

Figure 3.5: The scale factor of the Universe driven by the system with parameters chosen as in Fig. 3.4. From the plot we see that the bounce happens at $t = 0$.

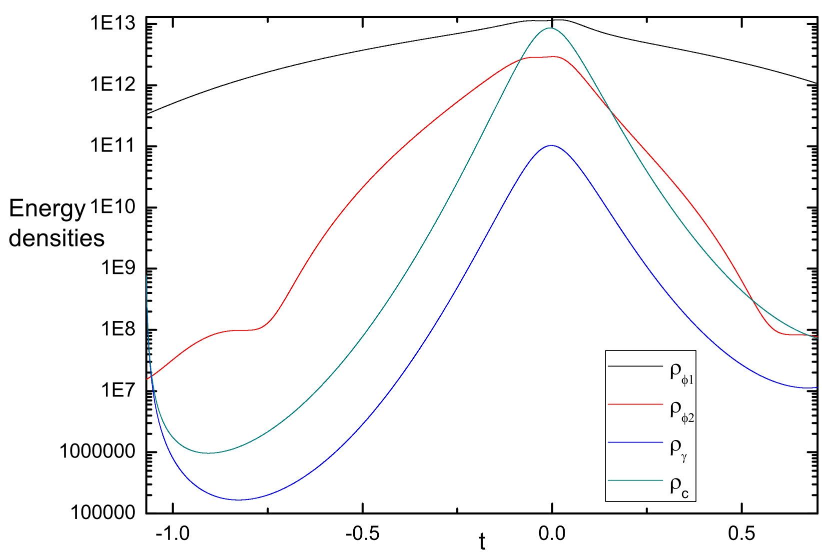

Figure 3.6: Energy densities of ϕ_1, ϕ_2 and γ in the system with parameters chosen as in Fig. 3.4. The curves from top to bottom are ρ_{ϕ_1} (black), ρ_{ϕ_2} (red), ρ_c (dark cyan) and ρ_γ (blue), respectively. The variables are also normalized with the mass scale $m_{rec} = 10^{-6} m_{Pl}$.

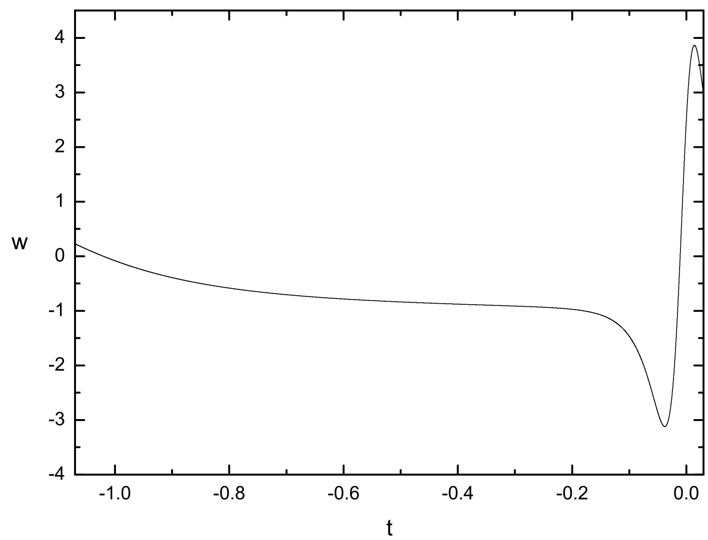

Figure 3.7: Evolution of the equation of state w as a function of cosmic time t (horizontal axis). The background fields are plotted in dimensionless units by normalizing by the mass $m_{rec} = 10^{-6}m_{Pl}$ while the time axis is displayed in units of m_{rec}^{-1}. The mass parameters m and M were chosen to be $m = 5m_{rec}$ and $M = 10m_{rec}$. This plot is the evolution of the system with two Fourier modes combined together. The one is of which the wavenumber $k \simeq 0.01hMpc^{-1}$ with initial conditions $\gamma_i \simeq -1.85\times10^5 m_{rec}$, $\dot{\gamma}_i \simeq 7.35\times10^6 m_{rec}^2$, $\phi_{1i} \simeq 1.015\times10^5 m_{rec}$, $\dot{\phi}_{1i} \simeq 6.39\times10^5 m_{rec}^2$, $\phi_{2i} \simeq 2.54\times10^2 m_{rec}$, $\dot{\phi}_{2i} \simeq -4.96\times10^3 m_{rec}^2$, which if taken alone will give the bounce as has been shown in the previous example. The other is of which the wavenumber $k \simeq 0.04hMpc^{-1}$ with initial conditions of the same initial values of the fields but the opposite signs of the scalar field velocity. From this plot we can see that the combination of the two Fourier mode will (generally) cause w blow up, thus preventing the bounce. This means that the bounce requires special fine-tuning of the initial phases for each Fourier mode. The coefficients c and d are chosen to be $c = 10^{-10}M_{rec}^2$ and $d = -10^{-10}M_{rec}^2$.

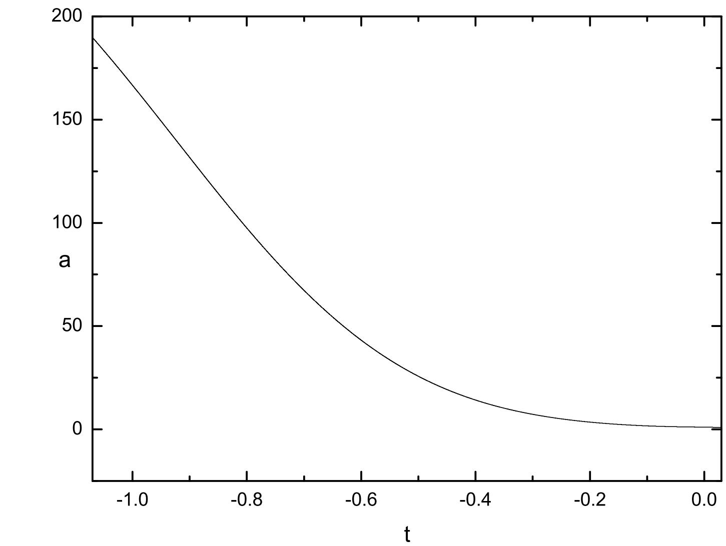

Figure 3.8: The Hubble constant of the Universe driven by the system with parameters chosen as in Fig. 3.7. From the plot we see that there is a singularity at $t = 0$.

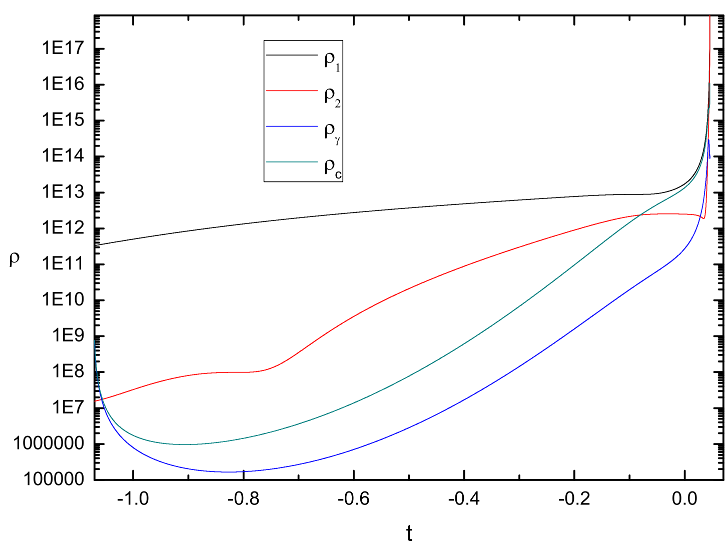

Figure 3.9: Energy densities of ϕ_1, ϕ_2 and γ in the system with parameters chosen as in Fig. 3.7. The curves from top to bottom are: ρ_{ϕ_1} (black), ρ_{ϕ_2} (red), ρ_c (dark cyan) and ρ_γ (blue), respectively. The variables are also normalized with the mass scale $m_{rec} = 10^{-6} m_{Pl}$.

3.7 Conclusions and Discussion

In this paper we analyzed in detail the possibility of obtaining a cosmological bounce in a model which corresponds to the scalar field sector of the Lee-Wick theory coupled to relativistic radiation. It is known that the scalar field sector of the Lee-Wick theory in the absence of other fields can yield a cosmological bounce [11]. In fact, the Universe will scale as nonrelativistic matter with $< w >\simeq 0$ both before and after the bounce. Thus, this model is a possible realization of the "matter bounce" scenario. However, this background is unstable to the introduction of radiation since in the contracting phase the growth of energy density in radiation will exceed that of matter and will lead to a Big Crunch singularity As has been shown in previous work [15], the introduction of a Lee-Wick partner to radiation does not prevent this instability. In this paper, we introduced an interaction between the radiation field and the scalar fields. The interaction could help drain energy from the radiation field to the Lee-Wick scalar, and thus could prevent the radiation from growing too fast to destroy the bounce.

We analyzed the equations describing the evolution of the three matter fields (regular scalar field, its Lee-Wick partner and the radiation field) on a cosmological background both analytically and numerically. Our analytical analysis was perturbative and made use of the second order using Born approximation. The expansion parameter is set by the initial amplitude of the gauge field. We solved the equations of motion for each field at each order, and obtained their approximations in different cases. We compared their contributions to the total energy density, and derived necessary conditions for a bounce to happen. To support our analysis, we also performed numerical calculations.

Specifically, we investigated initial conditions in which one or two Fourier modes of the radiation field and the scalar field fluctuations are excited. We found special initial conditions which indeed lead to a nonsingular bounce. Changing the sign of the initial scalar field velocity will destroy the bounce solution. In the presence of two Fourier modes, we found that a bounce requires identical initial phases for the two modes. For general initial conditions, we conjecture that the measure of such initial conditions which lead to a bounce is very small. We thus find that the addition of coupling terms between the scalar fields and radiation cannot save the Lee-Wick bounce background from the instability problem with respect to the addition of radiation (nor, for that matter, with respect to scalar field fluctuations). The instability problem with respect to anisotropic stress will be even worse.

We have studied a particular form of the coupling between the two scalar fields and radiation. We believe, however, that our conclusions - namely that the coupling cannot drain energy sufficiently fast from the radiation phase to prevent a singularity - will hold for more general couplings. The reason is that in the coupling terms can both turn radiation energy into scalar field energy and conversely turn scalar field energy into radiation. As in the example studied in this paper, it will require a fine-tuning of the phases in the initial conditions to prevent the channel generating radiation to be effective.

Acknowledgments

The work at McGill is supported in part by an NSERC Discovery Grant and by funds from the Canada Research Chair program. RB is recipient of a Killam Research Fellowship. The work at CYCU is funded in parts by the National Science Council and by the National Center for Theoretical Sciences.

Appendix : Green's function determination of $u_i^{(2)}$

The solution for the second order scalar field correction $u_i^{(2)}$ can be determined using the Green function method. The general solution of (3.77) is the sum of the solution $u_0(\eta)$ of the homogeneous equation which solves the same initial conditions as u and the particular solution $\delta u(\eta)$ which vanishes at time η_I. The particular solution is given by

$$\delta u(\eta) \;=\; u_a(\eta)\int_{\eta_I}^{\eta} d\eta'\epsilon(\eta')u_b(\eta')s(\eta') - u_b(\eta)\int_{\eta_I}^{\eta} d\eta'\epsilon(\eta')u_a(\eta')s(\eta')\ , \tag{3.82}$$

where u_1 and u_2 are two independent solutions of the homogeneous equation, $\epsilon(\eta)$ is the Wronskian:

$$\epsilon(\eta) \;=\; \left(u_a^{'}u_b - u_b^{'}u_a\right)^{-1}, \tag{3.83}$$

and $s(\eta)$ is the source inhomogeneity.

Recall from the main text that the second order field correction terms satisfy the equations:

$$\left\{\begin{array}{c} u_1^{(2)''} + a^2m^2u_1^{(2)} = c\frac{(k^2\gamma^2-\gamma'^2)}{a^2}u_1^{(0)}\ , \\ \\ u_2^{(2)''} + a^2M^2u_2^{(2)} = -d\frac{(k^2\gamma^2-\gamma'^2)}{a^2}u_2^{(0)}\ , \end{array}\right. \tag{3.84}$$

We will demonstrate the analysis for the case of $u_1^{(2)}$. Let us consider evolution for a short interval of time starting at some initial time η_I. Then, we can neglect the expansion of the Universe in the equation of motion and take $a(\eta) = a(\eta_I)$. We are then interested in how the result scales in η_I. Using this trick, the solutions of the homogeneous equation can be taken to be

$$\begin{aligned} u_a(\eta) &= \cos(\omega_m\eta) \\ u_b(\eta) &= \sin(\omega_m\eta) \end{aligned} \tag{3.85}$$

and the Wronskian is

$$\epsilon(\eta) = -\frac{1}{\omega_m} \text{ where } \omega_m = \sqrt{a^2m^2}\,. \tag{3.86}$$

Using the result for the background γ from the main text, the source term becomes

$$\begin{aligned} s_\gamma(\eta) &= c\frac{(k^2\gamma^2 - \gamma'^2)}{a^2} u_1^{(0)} \\ &\sim a^{-2}k^2|\eta|^{\frac{p}{2(p-1)}} \times \cos(2k|\eta| + 2\theta_\gamma)\cos((1-p)am|\eta| + \theta_1^{(0)}) \ , \end{aligned}$$

since $\gamma_0 \sim \cos(k|\eta| + \theta_\gamma)$.

Combining these results we obtain

$$\begin{aligned} u_1^{(2)} &\sim -\cos(\omega_m\eta)\int_{\eta_I}^{\eta}\frac{d\eta k^2}{a^3(t)m}|\eta|^{\frac{p}{2(p-1)}} \times \sin(\omega_m\eta)\cos(2k|\eta| + 2\theta_\gamma)\cos((1-p)am|\eta| + \theta_1^{(0)}) \\ &\quad + \sin(\omega_m\eta)\int_{\eta_I}^{\eta}\frac{d\eta k^2}{a^3(t)m}|\eta|^{\frac{p}{2(p-1)}} \times \cos(\omega_m\eta)\cos(2k|\eta| + 2\theta_\gamma)\sin((1-p)am|\eta| + \theta_1^{(0)}) \ , \\ u_2^{(2)} &\sim -\cos(\omega_M\eta)\int_{\eta_I}^{\eta}\frac{d\eta k^2}{a^3(t)M}|\eta|^{\frac{p}{2(p-1)}} \times \sin(\omega_M\eta)\cos(2k|\eta| + 2\theta_\gamma)\cos((1-p)aM|\eta| + \theta_1^{(0)}) \\ &\quad + \sin(\omega_M\eta)\int_{\eta_I}^{\eta}\frac{d\eta k^2}{a^3(t)M}|\eta|^{\frac{p}{2(p-1)}} \times \cos(\omega_M\eta)\cos(2k|\eta| + 2\theta_\gamma)\sin((1-p)aM|\eta| + \theta_1^{(0)}) \ . \end{aligned}$$

Note that if we only care about their scalings with respect to conformal time η or scale factor $a(t)$, the above solutions can be reduced to

$$u_{1,2}^{(2)} \propto |\eta|^{\frac{9p-2}{2(p-1)}} \propto a^{-\frac{9}{2}+\frac{1}{p}} \ , \tag{3.87}$$

and for the case of a matter-dominated era where $p = 2/3$, it is straightforward to show that

$$u_{1,2}^{(2)} \propto |\eta|^{-6} \propto a^{-3} \ . \tag{3.88}$$

REFERENCES

[1] S. W. Hawking, and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, England, 1973; A. Borde and A. Vilenkin, Phys. Rev. Lett. **72**, 3305 (1994).

[2] A. Borde and A. Vilenkin, "Eternal Inflation And The Initial Singularity," Phys. Rev. Lett. **72**, 3305 (1994) [arXiv:gr-qc/9312022].

[3] V. F. Mukhanov and R. H. Brandenberger, "A Nonsingular Universe," Phys. Rev. Lett. **68**, 1969 (1992);
R. H. Brandenberger, V. F. Mukhanov and A. Sornborger, "A Cosmological theory without singularities," Phys. Rev. D **48**, 1629 (1993) [arXiv:gr-qc/9303001].

[4] M. Novello and S. E. P. Bergliaffa, "Bouncing Cosmologies," Phys. Rept. **463**, 127 (2008) [arXiv:0802.1634 [astro-ph]].

[5] D. Wands, "Duality invariance of cosmological perturbation spectra," Phys. Rev. D **60**, 023507 (1999) [arXiv:gr-qc/9809062].

[6] F. Finelli and R. Brandenberger, "On the generation of a scale-invariant spectrum of adiabatic fluctuations in cosmological models with a contracting phase," Phys. Rev. D **65**, 103522 (2002) [arXiv:hep-th/0112249].

[7] P. Peter and N. Pinto-Neto, "Primordial perturbations in a nonsingular bouncing Universe model," Phys. Rev. D **66**, 063509 (2002) [arXiv:hep-th/0203013];
F. Finelli, "Study of a class of four dimensional nonsingular cosmological bounces," JCAP **0310**, 011 (2003) [arXiv:hep-th/0307068];
L. E. Allen and D. Wands, "Cosmological perturbations through a simple bounce," Phys. Rev. D **70**, 063515 (2004) [arXiv:astro-ph/0404441];
R. Brandenberger, H. Firouzjahi and O. Saremi, "Cosmological Perturbations on a Bouncing Brane," JCAP **0711**, 028 (2007) [arXiv:0707.4181 [hep-th]];
S. Alexander, T. Biswas and R. H. Brandenberger, "On the Transfer of Adiabatic Fluctuations through a Nonsingular Cosmological Bounce," arXiv:0707.4679 [hep-th];
F. Finelli, P. Peter and N. Pinto-Neto, "Spectra of primordial fluctuations in two-perfect-fluid regular bounces," Phys. Rev. D **77**, 103508 (2008) [arXiv:0709.3074 [gr-qc]];
A. Cardoso and D. Wands, "Generalised perturbation equations in bouncing

cosmologies," Phys. Rev. D **77**, 123538 (2008) [arXiv:0801.1667 [hep-th]];
Y. F. Cai, T. Qiu, R. Brandenberger, Y. S. Piao and X. Zhang, "On Perturbations of Quintom Bounce," JCAP **0803**, 013 (2008) [arXiv:0711.2187 [hep-th]];
X. Gao, Y. Wang, W. Xue and R. Brandenberger, "Fluctuations in a Hořava-Lifshitz Bouncing Cosmology," JCAP **1002**, 020 (2010) [arXiv:0911.3196 [hep-th]];
T. Qiu and K. C. Yang, "Perturbations in Matter Bounce with Non-minimal Coupling," JCAP **1011**, 012 (2010) [arXiv:1007.2571 [astro-ph.CO]];
C. Lin, R. H. Brandenberger and L. P. Levasseur, "A Matter Bounce By Means of Ghost Condensation," arXiv:1007.2654 [hep-th].

[8] R. H. Brandenberger, "Inflationary cosmology: Progress and problems," arXiv:hep-ph/9910410.

[9] J. Martin and R. H. Brandenberger, "The TransPlanckian problem of inflationary cosmology," Phys. Rev. D **63**, 123501 (2001), [arXiv:hep-th/0005209];
R. H. Brandenberger and J. Martin, "The robustness of inflation to changes in super-Planck-scale physics," Mod. Phys. Lett. A **16**, 999 (2001) [arXiv:astro-ph/0005432].

[10] Y. F. Cai, T. Qiu, Y. S. Piao, M. Li and X. Zhang, "Bouncing Universe with Quintom Matter," JHEP **0710**, 071 (2007) [arXiv:0704.1090 [gr-qc]].

[11] Y. F. Cai, T. t. Qiu, R. Brandenberger and X. m. Zhang, "A Nonsingular Cosmology with a Scale-Invariant Spectrum of Cosmological Perturbations from Lee-Wick Theory," Phys. Rev. D **80** (2009) 023511 [arXiv:0810.4677 [hep-th]].

[12] T. D. Lee and G. C. Wick, "Negative Metric and the Unitarity of the S Matrix," Nucl. Phys. B **9**, 209 (1969);
T. D. Lee and G. C. Wick, "Finite Theory of Quantum Electrodynamics," Phys. Rev. D **2**, 1033 (1970);
B. Grinstein, D. O'Connell and M. B. Wise, "The Lee-Wick standard model," Phys. Rev. D **77**, 025012 (2008) [arXiv:0704.1845 [hep-ph]].

[13] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, "The ekpyrotic Universe: Colliding branes and the origin of the hot big bang," Phys. Rev. D **64**, 123522 (2001) [arXiv:hep-th/0103239].

[14] J. M. Cline, S. Jeon and G. D. Moore, "The phantom menaced: Constraints on low-energy effective ghosts," Phys. Rev. D **70**, 043543 (2004) [arXiv:hep-ph/0311312].

[15] J. Karouby and R. Brandenberger, "A Radiation Bounce from the Lee-Wick Construction?," Phys. Rev. D **82**, 063532 (2010) [arXiv:1004.4947 [hep-th]].

Bouncing universes represent an alternative to the current paradigm of standard cosmology, inflation. One very important question that inflationary cosmology tries to answer concerns the creation of all the matter observed today. This chapter explores one of the mechanisms responsible for particle production at the end of inflation, during reheating. In particular, it studies preheating, a mechanism of fast particle production via parametric resonance, in models where the particle driving inflation, the inflaton, reaches its speed limit.

Chapter 4

Particle production during preheating

The following is extracted from the article **"Preheating with the Brakes On: The Effects of a Speed Limit"** published in collaboration with Aaron Vincent and Bret Underwood in *Phys. Rev. D* **84**, 043528 (2011).

Abstract We study preheating in models where the inflaton has a non-canonical kinetic term, containing powers of the usual kinetic energy. The inflaton field oscillating about its potential minimum acts as a driving force for particle production through parametric resonance. Non-canonical kinetic terms can impose a speed limit on the motion of the inflaton, modifying the oscillating inflaton profile. This has two important effects: it turns a smooth sinusoidal profile into a sharp sawtooth, enhancing resonance, and it lengthens the period of oscillations, suppressing resonance. We show that the second effect dominates over the first, so that preheating with a non-canonical inflaton field is less efficient than with canonical kinetic terms, and that the expansion of the Universe suppresses resonance even further.

4.1 Introduction

After a sufficiently long period of inflation the Universe would be cold and devoid of observable matter. The energy responsible for driving inflation is trapped in the (nearly) homogeneous inflaton field ϕ. In order for observable matter to emerge from the post-inflationary Universe, the inflaton field must couple to additional degrees of freedom in a way that the inflationary energy is dumped into observable matter through a process known as reheating [1, 2, 3]. For the purpose of reheating, the inflaton can couple to other scalar fields, fermions, or gauge fields.

If the inflaton couples to bosonic fields, such as other scalar fields, novel condensation effects can take place. In particular, because there is no exclusion principle, the inflaton field can transfer a large amount of energy to the reheating field χ (the "reheaton") in a process that is far from equilibrium. Such enhancements result from non-linear resonance effects due to the interaction between the inflaton and reheaton. For example, for the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}g^2\phi^2\chi^2 \,, \tag{4.1}$$

the equation of motion for large-scale modes of χ (neglecting the expansion of the Universe) becomes that of a harmonic oscillator with a time-dependent frequency,

$$\ddot{\chi}(t) + \left(k^2 + g^2\phi(t)^2\right)\chi(t) = 0 \tag{4.2}$$

where k is the comoving wavenumber of the χ field and $\phi = \phi(t)$ is the time-dependent background solution for the inflaton. At the end of single field inflation the inflaton oscillates about the minimum of its potential, so that the time-dependent frequency $\omega(t)^2 = k^2 + g^2\phi(t)^2$ oscillates with time. It is well-known that a harmonic oscillator with an oscillating time-dependent frequency can exhibit resonance effects, where the amplitude $\chi(t)$ of the oscillator grows exponentially with time. This post-inflationary exponential growth of certain long-wavelength modes of χ is dubbed "preheating" [4, 5, 6, 7, 3, 8], and has been studied in a variety of different contexts. See [9, 10, 11] for some reviews of this extensive literature. Most importantly for this paper, the majority of these studies (with the exception of [12]) have focused only on quadratic kinetic terms for the effective theory of reheating.

This approach towards inflationary and reheating model building neglects, however, the fact that these simple Lagrangians are really effective field theory (EFT) descriptions, only valid at sufficiently low energies. In particular, these Lagrangians should be understood as having been obtained by integrating out physics above some scale Λ at which new physics (such as new fields or new interactions) become important. The effects of physics above this energy can be parameterized in the EFT through non-renormalizable operators suppressed by powers of the scale of new physics,

$$\mathcal{L}_{eff} = \mathcal{L}_0 + \sum_{n>4} c_n \frac{\mathcal{O}_n}{\Lambda^{n-4}} \,. \tag{4.3}$$

As an example, consider the two-field Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\rho)^2 + \frac{\rho}{\Lambda}(\partial\phi)^2 - \frac{1}{2}\Lambda^2\rho^2 - V(\phi)\,, \tag{4.4}$$

with ϕ the inflaton field and ρ some heavy field with mass Λ. For energies below Λ, we can integrate out ρ at the classical (tree) level to obtain the effective Lagrangian [13, 14],

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial\phi)^2 + \frac{(\partial\phi)^4}{\Lambda^4} - V(\phi)\,. \tag{4.5}$$

The effective theory now contains a new contribution to the kinetic part of the action for the inflaton. More generally, one can consider as a low energy EFT a

Lagrangian for the inflaton of the form[1]

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(X, \phi)\,, \tag{4.6}$$

where $X \equiv -\frac{1}{2}(\partial\phi)^2$. Inflation with Lagrangians of this type can have novel features, such as a speed limit on the motion of the homogeneous inflaton and a sound speed of perturbations less than one $c_s^2 \leq 1$, that have important implications for inflationary perturbations and models [16, 17, 18, 19, 20].

It is the post-inflationary dynamics of (4.6) that is most interesting to us here. In particular, we will couple the non-canonical inflaton in (4.6) to a canonical reheaton field through a quartic interaction:

$$\mathcal{L}_{pre} = \mathcal{L}_{eff}(X, \phi) + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}g^2\phi^2\chi^2 - \frac{1}{2}m_\chi^2\chi^2\,. \tag{4.7}$$

The equation of motion for the reheaton is still of the form (4.2). However, since the resonance arising from the time-dependent harmonic oscillator (4.2) is a non-linear effect, it is sensitive to the precise profile of the inflaton $\phi(t)$ as it oscillates about its potential minimum. The profile of the oscillation is in turn sensitive to the non-canonical kinetic terms in (4.7). Thus, the modified dynamics from non-canonical Lagrangians can play an important role in the physics of preheating.

In this paper, we will discuss the implications of an inflaton sector with non-canonical kinetic terms for preheating. We first discuss in Section 4.2 the dynamics

---

[1] This is, of course, not the most general effective theory of the form (4.3). In particular, we have omitted terms involving higher derivatives. This can be done self-consistently as long as the higher derivatives are small for the physics we are interested in, which we will argue is the case. See also [13] for further analysis of the validity of the truncation (4.6) in the context of inflation. It is also worthwhile to note that certain Lagrangians of the form (4.6) are protected against corrections by powerful non-linear symmetries [15].

of a non-canonical inflaton oscillating about its minimum. A fairly generic feature is the existence of a speed limit on the motion of the inflaton, restricting how fast it can move in field space. When the oscillating inflaton saturates this speed limit, its profile is no longer sinusoidal but instead takes a saw-tooth form. In Section 4.3 we show how the equation of motion for reheaton perturbations can be recast into a form of Hill's equation, for which well-known techniques exist for finding resonance bands. Two competing effects lead to modifications of the standard theory of preheating: non-canonical kinetic terms lead to a "sharper" inflaton profile, enhancing resonance, while the period of oscillation is lengthened due to the speed limit, suppressing resonance. The net result is that the latter effect dominates, so that the resonance for non-canonical kinetic terms is *less efficient* than its canonical counterpart. After illustrating the effects of the expansion of the Universe, we summarize our results in Section 4.4. The Appendix contains further details about methods for computing the properties of parametric resonance.

4.2 Non-Canonical Kinetic Terms

Let us first focus on the implications that non-canonical kinetic terms of the form (4.6) have on the motion of the inflaton field oscillating about its minimum. Inflationary Lagrangians of this form with no potential energy have been proposed as an alternative to potential-dominated inflation, and are dubbed "k-inflation" [16]. More generally, however, we expect the inflationary sector to have both kinetic and potential energy; in this case, the effective Lagrangian $\mathcal{L}_{eff}(X, \phi)$ can be seen as an extension of standard slow-roll inflationary models with non-canonical kinetic terms. These non-canonical Lagrangians have a number of interesting properties that make them attractive to inflationary model building and phenomenology. On the model-building side, the non-canonical kinetic terms significantly modify the dynamics of the inflaton so that the speed of the inflaton can remain small even when rolling down a steep potential [18, 13]. In many cases, this results in a "speed limit" for the inflaton, namely a maximum for the speed of the homogeneous mode of the inflaton. This is a great advantage to model building, since then it is less necessary to fine-tune the inflationary potential to have very flat regions supporting slow roll inflation. Inflation with non-canonical kinetic terms can also lead to interesting signatures in the CMB, such as observable non-gaussianities [17, 19, 20].

We will choose to work with inflationary Lagrangians of the "separable" form (discussed in more detail in [13, 21]):

$$\mathcal{L}_{eff}(X,\phi) = \sum_{n\geq 0} c_n \frac{X^{n+1}}{\Lambda^{4n}} - V(\phi) = p(X) - V(\phi)\,, \tag{4.8}$$

where we have written $p(X)$ as a power series expansion in X/Λ^4, with Λ some UV energy scale. Since Λ is typically the mass scale of some heavy sector we have integrated out, we require it to be larger than the mass of the inflaton $\Lambda \gg m_\phi$ in order for such an effective field theory perspective to make sense. As a further restriction, we will take $c_0 = 1$, so that for small $X/\Lambda^4 \ll 1$, the Lagrangian reduces to the usual canonical Lagrangian. In order for the power series expansion to make sense, the series must have some non-zero radius of convergence R so that the series converges in the domain of convergence $X/\Lambda^4 \in [0, R)$, with $R \leq 1$ (it is not necessary that the series itself converge at the boundary). Finally, we will also require that the first and second derivatives of the power series are positive $\frac{\partial p}{\partial X}, \frac{\partial^2 p}{\partial X^2} > 0$ so as to guarantee that we satisfy the null energy condition and have subluminal propagation of perturbations [13, 22, 23, 24].

In a FLRW background

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \tag{4.9}$$

the scale factor $a(t)$ is driven by the energy density of the homogeneous inflaton $\phi(t)$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2}\left[2X\frac{\partial p}{\partial X} - p(X) + V(\phi)\right] , \tag{4.10}$$

where now $X = \frac{1}{2}\dot{\phi}^2$. The equation of motion for $\phi(t)$ in this background becomes,

$$\ddot{\phi} + 3H\dot{\phi}c_s^2 + \frac{\partial V}{\partial \phi}\frac{c_s^2}{\partial p/\partial X} = 0 \tag{4.11}$$

where the sound speed c_s^2, defined by

$$c_s^2 = \left(1 + 2X\frac{\partial^2 p/\partial X^2}{\partial p/\partial X}\right)^{-1} , \tag{4.12}$$

is so called because it also appears as the effective speed of perturbations of ϕ about this background. We will ignore the expansion of the Universe for now, dropping the Hubble friction term in (4.11), and will return to the effects of expansion of the Universe in Section 4.3.3.

For simplicity, we will take the potential $V(\phi)$ of the inflaton to only consist of a mass term $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$. This could be the entire inflaton potential, as in chaotic inflation, or just the form of the potential near its minimum. The difference is not particularly important, as we are primarily concerned with the phase when the homogeneous inflaton is oscillating about its minimum, where the quadratic form of the potential will be sufficient. For a scalar field with a canonical kinetic term in this potential, the equation of motion is that of a simple harmonic oscillator: the inflaton oscillates sinusoidally $\phi(t) = \Phi\sin(m_\phi t)$.

The behavior of a non-canonical kinetic term, however, can be qualitatively different. As with the canonical case, the potential provides a force that accelerates the inflaton. However, now the effective force $\frac{\partial V}{\partial \phi}\frac{c_s^2}{\partial p/\partial X}$ is also a function of the speed of the inflaton. Recall that the kinetic term $p(X)$ is a series which is only defined within a finite radius of convergence, $\frac{1}{2}\frac{\dot{\phi}^2}{\Lambda^4} \leq R$. As the inflaton speed approaches the radius of convergence $|\dot{\phi}| \to \sqrt{2R}\Lambda^2$ the series $p(X)$ or its derivatives $\partial p/\partial X, \partial^2 p/\partial X^2$ may converge or diverge, depending on the precise form of the series chosen. Notice that if the second derivative of the series $\partial^2 p/\partial X^2$ diverges faster than the first derivative $\partial p/\partial X$ at the boundary of the domain of convergence, so that $c_s^2 \to \frac{1}{2X}\frac{\partial p/\partial X}{\partial^2 p/\partial X^2} \to 0$, then the effective force vanishes. Said another way, as the force from the potential increases the speed of the inflaton, the non-canonical kinetic dynamics modify the *effective* force felt by the inflaton so that the effective force, and thus the acceleration of the inflaton, vanish as the inflaton approaches the radius of convergence. Thus, Lagrangians for which $\partial^2 p/\partial X^2$ diverges as $\dot{\phi}$ approaches the radius of convergence have a *speed limit*, such that $|\dot{\phi}| \leq \dot{\phi}_{max} = \sqrt{2R}\Lambda^2$. Certainly, as the inflaton approaches the speed limit, we are approaching the boundary of validity for the EFT (4.8). Thus, it is advantageous to have a symmetry that protects the form of the Lagrangian against further corrections as this threshold is approached [15]. Further, requiring perturbivity of the inflaton perturbations places a bound on the minimal sound speed c_s^2, thus restricting how close to the boundary of the EFT we can go [25, 26, 27]. Fortunately, there is a window where both the system is perturbative and the inflaton is close to its speed limit, so that the effects we are interested in below are still present.

For potentials that are sufficiently steep, the inflaton quickly attains its speed limit and stays there until it reaches the other side of the potential, where it decelerates and changes direction, as shown in Figure 4.1. This has two important consequences that will play a very important role in preheating:

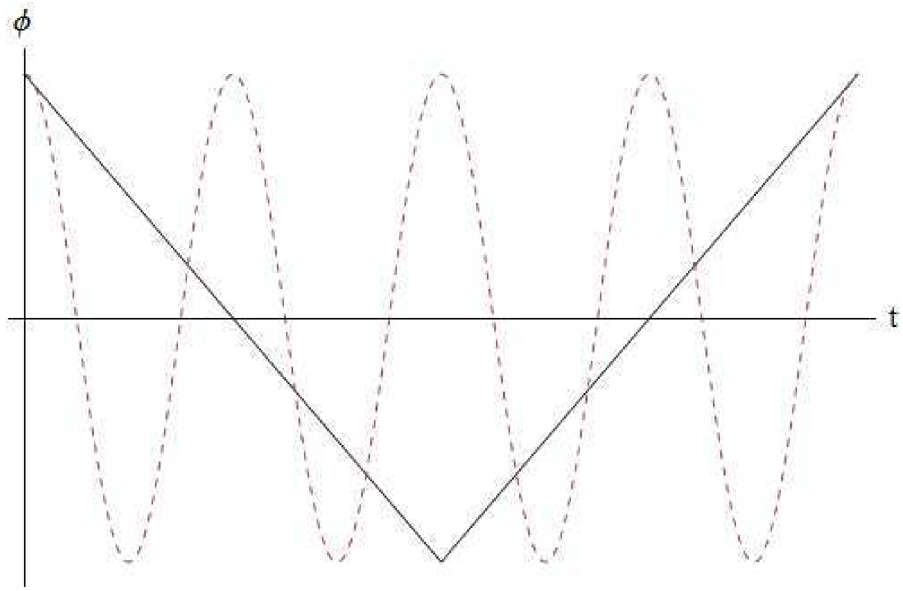

Figure 4.1: The motion of the homogeneous inflaton $\phi(t)$ about the minimum of the (quadratic) potential differs depending on whether the kinetic terms are canonical or non-canonical. For canonical kinetic terms the motion is sinusoidal. For non-canonical kinetic terms the motion approaches a saw-tooth with a much longer period as the inflaton saturates its speed limit. This form of the inflaton profile is universal for Lagrangians that have a speed limit, except for a very small region near the turning point which is unimportant for preheating.

(a) The profile becomes sharper, with a smooth sinusoidal profile turning into a saw-tooth profile.

(b) The speed limit slows the inflaton down, lengthening the period.

These two effects can be clearly seen in Figure 4.1. As mentioned earlier, the form of the potential about the minimum is in principle unrelated to the form of the potential during inflation. This implies that the behavior of the inflaton oscillating about its potential minimum can be dominated by the non-canonical kinetic terms, independent of whether inflation itself is dominated by these terms.

When the inflaton saturates the speed limit its profile along one of the legs is approximately linear in time[2] :

$$\phi(t) \approx \sqrt{2R}\,\Lambda^2(t - t_j)\,, \tag{4.13}$$

where t_j is the time when the inflaton crosses zero. For small Λ this linear approximation of the inflaton is valid up to the turning point $|\phi| \approx \Phi$, so the period of oscillation is

$$T_{NCR} = \frac{4\Phi}{\sqrt{2R}\,\Lambda^2}\,, \tag{4.14}$$

with $t \in [-\frac{T_{NCR}}{4}, \frac{T_{NCR}}{4}]$. In order for the non-canonical kinetic terms to be important, this must be much larger than the canonical period of oscillation $T_{CR} = 2\pi/m_\phi$ (otherwise the inflaton would not reach the speed limit during oscillation), so

$$T_{NCR} \gg T_{CR} \quad \Rightarrow \quad \Lambda \ll \left(\frac{2}{R}\right)^{1/4} \sqrt{\frac{\Phi m_\phi}{\pi}}\,. \tag{4.15}$$

This provides a precise condition for the small Λ limit where the system is very non-canonical. Together with the requirement that the effective field theory description make sense $\Lambda \gg m_\phi$, we have the following regime for Λ where both the EFT description makes sense and the speed limit is saturated:

$$1 \ll \frac{\Lambda}{m_\phi} \ll \left(\frac{2}{R}\right)^{1/4} \sqrt{\frac{\Phi}{\pi m_\phi}}\,. \tag{4.16}$$

Clearly, for too small of an amplitude Φ of the initial oscillation this cannot be satisfied. Since the expansion of the Universe and the transfer of energy from

---

[2] This implies that the acceleration is approximately zero, so higher derivative corrections to the effective Lagrangian (4.8) are very small, as discussed in footnote 1.

the inflationary to the reheating sector cause the amplitude of oscillation to decrease over time, eventually the inflaton no longer saturates the speed limit and the inflaton profile behavior returns to the canonical limit.

Perhaps the most well-known non-canonical Lagrangian that leads to a speed-limit for the inflaton is that of the DBI Lagrangian [18] (with constant warp factor, see [28]):

$$p(X) = -\Lambda^4 \left[\sqrt{1 - \frac{2X}{\Lambda^4}} - 1 \right] . \tag{4.17}$$

This Lagrangian arises by considering the motion of a space-filling D3-brane in a compact space, with ϕ taking the role of a transverse coordinate of the D3-brane. In this case the speed limit has a nice geometrical interpretation - it is just the effective speed of light for motion the extra dimensions (the effective speed of light is not necessarily unity if the metric on the internal space has non-trivial warping). We can also represent (4.17) as a power series representation in powers of X/Λ^4 as in (4.8) with a radius of convergence $R = 1/2$. But the condition for obtaining the speed limit behavior can be satisfied by a much larger set of Lagrangians, not just the DBI Lagrangian. For example, Lagrangians of the form [13],

$$p(X) = -\Lambda^4 \left[\left((1 - \frac{1}{R} \frac{X}{\Lambda^4} \right)^R - 1 \right] , \tag{4.18}$$

with $R < 1$ (which includes (4.17) for $R = 1/2$), or

$$p(X) = -\Lambda^4 \left[\log \left(1 - \frac{X}{\Lambda^4} \right) - 1 \right] \tag{4.19}$$

are all of the form (4.8) (with radii of convergence $R, 1$ respectively) and all lead to speed-limiting behavior for ϕ. Importantly, though, the details of preheating driven by such fields will be insensitive to these different choices of Lagrangians - as long as there is a speed limit, the profile $\phi(t)$ will be that of the solid line in Figure 4.1.

4.3 Preheating with Non-Canonical Inflation

4.3.1 Floquet Theory of Resonance

In the previous section we described the effects of non-canonical kinetic terms on the profile $\phi(t)$ of an inflaton oscillating about the minimum of its potential, ignoring the coupling of the inflaton to the reheaton field. Now, we will consider the impact on preheating of the coupling between the inflaton and reheaton sectors as in (4.7), with the inflationary Lagrangian given by (4.8),

$$\mathcal{L}_{pre} = p(X) - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{2}g^2\phi^2\chi^2\,, \tag{4.20}$$

where we will assume that $p(X)$ gives rise to a speed limit.

While the inflaton is dominated by its spatially homogeneous mode $\phi = \phi(t)$, which is oscillating with period T, the reheaton is assumed to have a vanishing background vacuum expectation value (VEV) $\langle\chi\rangle = 0$. Thus, we will consider fluctuations of the reheaton about the vacuum, which may be decomposed into a set of Fourier modes

$$\chi = \delta\hat{\chi}(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3}\left(e^{i\vec{k}\cdot\vec{x}}\delta\chi_k(t) + e^{-i\vec{k}\cdot\vec{x}}\delta\chi_k^*(t)\right). \tag{4.21}$$

Ignoring for now the expansion of the Universe, the equation of motion for the reheaton fluctuations is

$$\delta\ddot{\chi}_k(t) + \left(K + g^2\phi(t)^2\right)\delta\chi_k(t) = 0 \tag{4.22}$$

where $K \equiv k^2 + m_\chi^2$. This is easily recast into a driven harmonic oscillator known as Hill's equation [29], after a redefinition to a dimensionless time coordinate $\tau = 4\pi(t - t_0)/T$ (a prime denotes a derivative with respect to τ):

$$\delta\chi_k(\tau)'' + [A_k + qF(\tau)]\,\delta\chi_k(\tau) = 0, \tag{4.23}$$

where $F(\tau)$ is a 2π-periodic function symmetric about $\tau = 0$, satisfying $\int_{-\pi}^{\pi} F(\tau) d\tau = 0$. Floquet's theorem (see [29]) states that solutions to (4.23) have the form

$$\delta\chi_k(\tau) = e^{\tilde{\mu}_k \tau} g(\tau) + e^{-\tilde{\mu}_k \tau} g_2(\tau) \tag{4.24}$$

where $g(\tau), g_2(\tau)$ are oscillating solutions, and the Floquet growth exponent $\tilde{\mu}_k$ depends on A_k and q and is complex in general. For certain ranges of A_k and q, known as resonance bands, the real part of the Floquet growth exponent is non-zero[3] leading to exponentially growing solutions:

$$\delta\chi_k(\tau) \sim e^{\tilde{\mu}_k \tau} . \tag{4.25}$$

In order to compare models of preheating with different oscillating profiles $\phi(t)$ it is more convenient to parameterize the growth exponent in terms of *physical time* t:

$$\delta\chi_k(t) = e^{\mu_k t} \tag{4.26}$$

where $\mu_k = \tilde{\mu}_k \frac{2\pi}{T}$, with T the period of oscillation of the inflaton. The relation between the two growth exponents is that $\tilde{\mu}_k$ represents the growth *per oscillation* of the inflaton, while μ_k represents that growth *per unit of physical time*, which takes into account effects on the overall growth due to changes in the period. The physical growth exponent μ_k is the appropriate quantity to use to evaluate the rate of growth compared to the rate of expansion of the Universe. Particle production is tracked by the number density n_k of particles, constructed as the energy of a mode divided by the effective mass $m_{eff,k}^2 = k^2 + m_\chi^2 + g^2\phi(t)^2$, and also scales with

[3] Without loss of generality, we will take the real part of the Floquet exponent to be positive.

the Floquet exponent as

$$n_k = \frac{m_{eff,k}}{2} \left(\frac{|\delta \dot{\chi}_k(t)|^2}{m_{eff,k}^2} + |\delta \chi_k(t)|^2 \right) \sim e^{2\mu_k t} \,. \tag{4.27}$$

For any given oscillating profile $\phi(t)$ for the inflaton, with corresponding $F(\tau)$ in (4.23), it is possible to numerically determine the resonance bands; the procedure is outlined in Appendix 4.4. We distinguish two opposite regimes:

1) **CR**: Canonical Reheating, where the reheaton is coupled to an inflaton with a canonical kinetic term (or equivalently where $\Lambda^2 >> \dot{\phi}$ at all times so that the inflaton *effectively* behaves canonically), $p(X) \approx X$.
2) **NCR**: Non-Canonical Reheating, where the reheaton is coupled to an inflaton with a non-canonical kinetic term $p(X)$, such that the inflaton speed approaches the speed limit $\dot{\phi} \simeq \sqrt{2R}\,\Lambda^2$ as it oscillates about its minimum.

First, let us write the equation of motion for the reheaton fluctuations $\delta\chi_k(t)$ in the form (4.23) for the CR case. As discussed in the previous section, an inflaton oscillating in a quadratic potential with a canonical kinetic term has a sinusoidal profile $\phi(t) = \Phi \sin(m_\phi t)$. The reheaton equation of motion can then be recast into the form of a Hill equation (4.23) with the identifications:

$$A_k = \frac{2K + g^2\Phi^2}{8m_\phi^2}, \quad q_{CR} = \frac{g^2\Phi^2}{8m_\phi^2}, \tag{4.28}$$

$$\tau = 2m_\phi t, \quad F(\tau) = \cos\tau \tag{4.29}$$

Hill's equation in this form is more commonly known as the Mathieu equation, and the resonance bands, plotted in (K, Φ) space in Figure 4.2, take the familiar form from previous studies of preheating [6, 3].

In the opposite limit (NCR), when the inflaton has a non-canonical kinetic term with a speed limit $|\dot{\phi}|_{max} = \sqrt{2R}\,\Lambda^2$, the profile $\phi(t)$ becomes a saw-tooth, as in Figure 4.1. Along one of the "legs" of this profile, the inflaton is linear in time as in (4.13), so that the reheaton equation of motion becomes:

$$\delta\ddot{\chi}_k(t) + \left(K + 2Rg^2\Lambda^4 t^2\right)\delta\chi_k(t) = 0\,. \tag{4.30}$$

This can also be rewritten in the form of a Hill's equation (4.23) with the identifications:

$$\begin{aligned} A_k &= \frac{K\Phi^2}{2R\pi^2\Lambda^4} + \frac{1}{3}\frac{g^2\Phi^4}{2R\pi^2\Lambda^4}, \quad q_{NCR} = \frac{g^2\Phi^4}{2R\pi^2\Lambda^4}, \\ \tau &= \frac{\sqrt{2R}\,\Lambda^2\pi}{\Phi}t, \quad F(\tau) = \frac{\tau^2}{\pi^2} - \frac{1}{3}. \end{aligned} \tag{4.31}$$

The resonance bands for this form of Hill's equation are shown in Figure 4.2. The two regimes can easily be connected numerically.

There are two main regimes of interest of (4.23) for the resonance bands: narrow resonance $q \ll 1$ and broad resonance $q \gg 1$. For $g^2\Phi^2/m_\phi^2 \ll 1$, the CR scenario is in the narrow resonance regime. When non-canonical kinetic terms are important, however, the parameter q_{NCR} in the corresponding Hill's equation (4.31) is enhanced relative to the canonical case

$$q_{NCR} = \frac{g^2}{2R\pi^2}\left(\frac{\Phi}{m_\phi}\right)^2 \frac{m_\phi^2\Phi^2}{\Lambda^4} = q_{CR}\frac{4}{R\pi^2}\frac{m_\phi^2\Phi^2}{\Lambda^4}. \tag{4.32}$$

The enhancement factor $m_\phi^2\Phi^2/\Lambda^4 \gg 1$ is large in order for the inflaton to saturate the speed limit, so unless q_{CR} is correspondingly small, when the CR scenario is in the narrow resonance regime the NCR scenario is in broad resonance. This has important physical implications, since then not only is the growth per period $\tilde{\mu}_k$ larger for non-canonical kinetic terms, but there is also growth over a much larger range of scales. We will see in the next section, however, that this enhancement is overwhelmed by suppression of particle production due to the lengthening of the period.

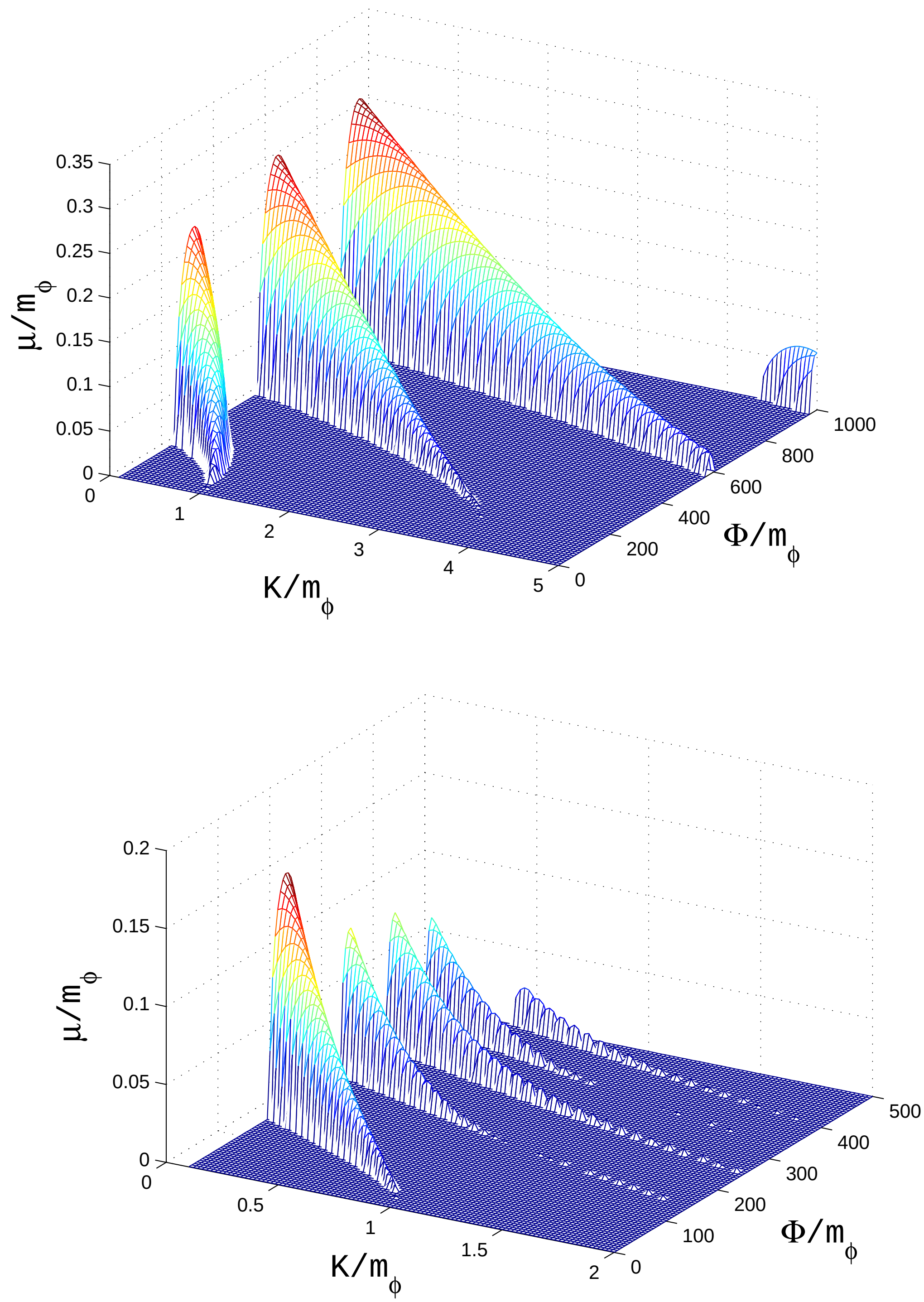

Figure 4.2: Resonance bands for reheaton perturbations as a function of the scale K and initial amplitude of the oscillating inflaton Φ with $g = 0.005$ in the case when the reheaton is coupled to an inflaton with canonical (CR scenario, up) and non-canonical (NCR scenario, down) kinetic terms.

The resonance due to (4.22) depends on the physical wavenumber of the fluctuation k; most particle production occurs when the effective mass $K + g^2\phi^2(t)$ vanishes. This implies that the resonance is most efficient at large scales $K \sim 0$, as can be seen in Figure 4.2. In practice, however, we cannot work on scales larger than the Hubble radius while neglecting metric perturbations, so $k > H$. In addition, a non-zero bare mass m_χ for the reheaton also keeps the effective mass from vanishing. But as long as we work on sufficiently large scales (and with a sufficiently large initial amplitude) so that $K \ll g^2\Phi^2$, the maximum of resonance at $K = 0$ will be a good approximation for the maximum resonance at large scales.

Figure 4.3 displays the growth exponent μ_k as a function of the initial amplitude Φ of the inflaton for large scales (namely $K = 0$). Several features are evident: for decreasing Λ, more resonance bands become accessible, because the system enters the broad resonance regime; the maxiumum size of the growth exponent in the first resonance band decreases as Λ decreases, reflecting the lengthening of the period; and the heights of the resonance bands for small Λ decrease with increasing Φ, also due to the lengthening of the period for large initial amplitudes (note that the period for canonical kinetic terms is independent of the amplitude).

This effect is not limited to the DBI case: Figure 4.4 compares the Floquet exponent for several Lagrangians of the forms (4.18) and (4.19). While the behaviors differ slightly as one moves away from the canonical case, they converge again in the regime where inflaton oscillations saturate the speed limit. The black dotted line in Figure 4.4 shows the Floquet exponent when the period lengthening is not taken into account: as the inflaton trajectory approaches the saw-tooth shape, energy is injected into the reheaton field over a longer amount of time, allowing for more particle production. This saturates when the slope $\dot{\phi}$ approaches the constant speed limit for most of the period. This enhancement is clearly subdominant, however,

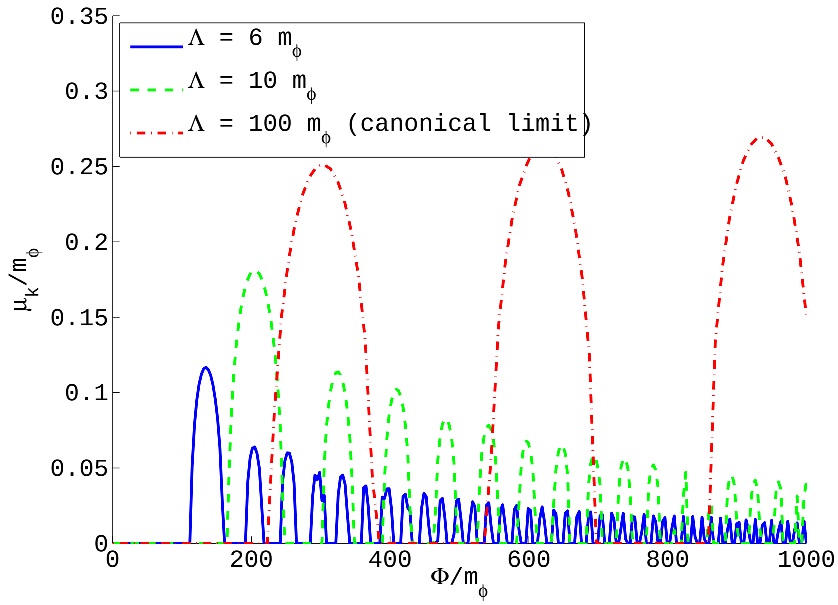

Figure 4.3: Floquet growth exponent μ_0 per unit of physical time for $K = 0$ for three different values of Λ in DBI inflaton-driven preheating. Lowering the speed limit Λ^2 of the inflaton greatly reduces the strength of parametric resonance because the period of oscillation (4.14) increases as $1/\Lambda^2$. The additional suppression of the resonance due to the dependence of the period on the amplitude of inflaton oscillations Φ is also evident.

when compared with the suppression of particle production from the elongation of the period itself.

We close this section by comparing our results with previous explorations in the literature. Ref. [12] found an expression for small $\dot{\phi}/\Lambda^2$ for the growth exponent μ_k in the case of a DBI inflaton by perturbing the canonical equation of motion. Up to order $\dot{\phi}/\Lambda^2$ and for $K = 0$, this corresponds to:

$$\mu_k \simeq \sqrt{\left(\frac{\theta_2}{2}\right)^2 - \left(\theta_0^{1/2} - 1\right)^2} \tag{4.33}$$

with

$$\theta_0 \equiv \frac{g^2\Phi^2}{2m^2}\left(1 + \frac{9\Phi^2 m_\phi^2}{32\Lambda^4}\right), \qquad \theta_2 \equiv \frac{g^2\Phi^2}{4m^2}\left(1 + \frac{3\Phi^2 m_\phi^2}{8\Lambda^4}\right). \tag{4.34}$$

This result is illustrated in the right panel of Fig. 4.4. For $m_\phi/\Lambda < 1$ it provides a good approximation of the gain in particle production one would expect from DBI inflation were it not for the lengthening of the period, which the authors of [12] correctly identified as "DBI friction" but did not quantify. We are interested in the opposite limit, however, where the non-canonical kinetic terms are more than just a small perturbation.

4.3.2 Non-Canonical Limit

In the non-canonical limit, the equation of motion for the reheaton perturbations (4.30) can be written in a simple and suggestive form by making a redefinition to a different dimensionless time coordinate than we considered earlier:

$$\tau = \left(g\sqrt{2R}\Lambda^2\right)^{1/2} t = \left(2Rg^2\right)^{1/4} \Lambda\, t\,. \tag{4.35}$$

The reheaton equation of motion then becomes

$$\frac{d^2\chi_k(\tau)}{d\tau^2} + \left[\kappa^2 + \tau^2\right] \chi_k(\tau) = 0\,, \tag{4.36}$$

where $\kappa^2 \equiv \frac{k^2+m_\chi^2}{\sqrt{2R}\,g\Lambda^2}$. The time range of the new time variable is $-\frac{\Delta\tau}{2} \leq \tau \leq \frac{\Delta\tau}{2}$ where $\Delta\tau = T_\Lambda(2g^2R)^{1/4}\Lambda = (8g^2/R)^{1/4}(\Phi/\Lambda)$.

The benefit of making this redefinition is that the problem of broad resonance can be mapped to the problem of scattering of a particle with energy κ^2 off of an inverted parabolic potential with an effective Schrödinger equation [6, 3]:

$$\frac{d^2\chi_k(\tau)}{d\tau^2} + (\kappa^2 - V_{eff}(\tau))\chi_k(\tau) = 0\,. \tag{4.37}$$

This scattering problem can be solved using standard techniques (see Appendix 4.4), leading to the result:

$$\tilde{\mu}_k = \frac{1}{2\pi} \ln\left[\frac{1+|R_k|^2|}{|D_k|^2} + 2\frac{|R_k|}{|D_k|^2}\cos\theta_{tot}\right] \tag{4.38}$$

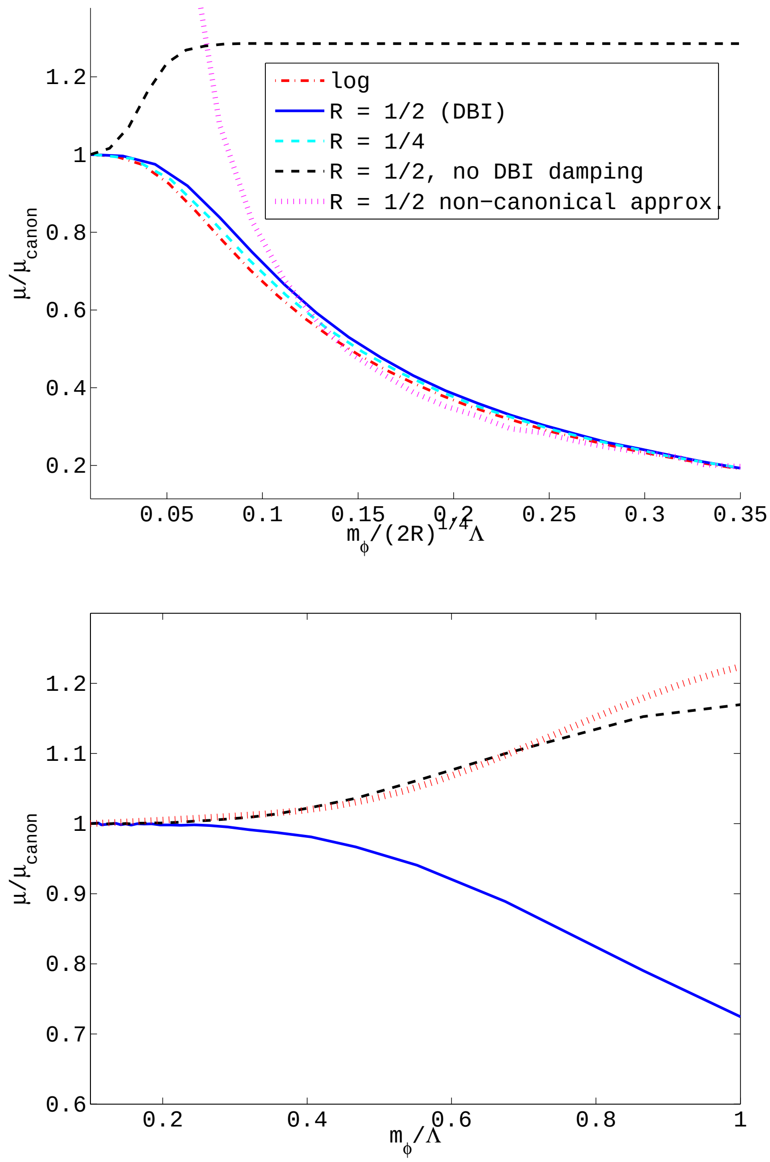

Figure 4.4: Floquet exponent μ of the first resonance band for $K = 0$ for $g = 0.005$ (up) and $g = 0.5$ (down). $\Lambda >> m_\phi$ corresponds to the canonical case CR. Parameter R in the left-hand panel refers to (4.18), and log refers to (4.19). In both panels the black dashed line is the growth of particle production per period, whereas the blue solid line is the true growth of particle production per physical unit of time. Additionally, in the left panel the magenta dotted line is the analytical result (4.41), showing good agreement with the numerical results in the regime where the speed limit is important. In the right panel we have also included the approximate result from [12] (dotted red line, (4.33)), which did not include the suppression from the speed limit. As discussed in Section 4.2, the effective theory breaks down for $m_\phi > \Lambda$ and results should therefore not be trusted beyond this.

in terms of the reflection and transmission amplitudes R_k, D_k and the phase of the reheaton mode θ_{tot}. For an inverted parabolic potential, the reflection and transmission coefficients are known, so the average growth index simplifies to be [3]:

$$\tilde{\mu}_k = \frac{1}{2\pi} \ln \left[1 + 2e^{-\pi\kappa^2} - 2\sin\theta_{tot} e^{-\frac{\pi}{2}\kappa^2} \sqrt{1 + e^{-\pi\kappa^2}} \right] . \tag{4.39}$$

For large scales $\kappa^2 \approx 0$ this has a maximum/average value of $\tilde{\mu}_k \approx (0.28, 0.175)$ [3].

The average growth index $\tilde{\mu}_k \approx 0.175$ only gives the typical growth over one period of oscillation for broad resonance, and is the same regardless of whether the inflaton dynamics are canonical or non-canonical. In particular, it does not depend on the type of non-canonical Lagrangian used - *any* non-canonical Lagrangian that leads to a speed limit for the oscillating inflaton will have the same average growth factor over one period $\tilde{\mu}_k$. The reason the average growth index does not depend on whether the system is canonical or non-canonical is because we are working in the broad resonance limit, where only the slope of the inflaton profile (in dimensionless coordinates) at the point where the inflaton crosses zero matters.

However, as discussed before, the true comparison of the particle production between the canonical and non-canonical models should be the growth μ_k over some fixed *physical* time period Δt

$$n_k \propto e^{2\tilde{\mu}_k \frac{2\pi}{T}\Delta t} \sim e^{2\mu_k \Delta t} , \tag{4.40}$$

where T is the period, so that the true growth index is the average growth index over one period divided by the length of the period, $\mu_k = \tilde{\mu}_k 2\pi/T$. This leads to the main result of this section: since the period for an oscillating inflaton with non-canonical dynamics is much longer than the period for an oscillating inflation with canonical dynamics $T_{NCR} \gg T_{CR}$, we have *much more growth in a canonical system* over some fixed physical time Δt, $(\mu_k)_{CR} \gg (\mu_k)_{NCR}$. In particular, the ratio between the two should fall off as the ratio of their periods,

$$\frac{(\mu_k)_{NCR}}{(\mu_k)_{CR}} = \sqrt{\frac{R}{2}}\frac{\pi\Lambda^2}{\Phi m_\phi} \ll 1\,, \tag{4.41}$$

for decreasing Λ at $\kappa = 0$. Indeed, this is precisely what is found using numerical techniques, as shown in Figure 4.4.

Finally, we comment on the possiblity for non-canonical kinetic terms to enhance resonance by turning a putative CR system in narrow resonance into a NCR system with broad resonance. A CR system in narrow resonance with $q_{CR} \ll 1$ has $(\mu_k)_{CR} \sim q_{CR} m_\phi$ [3]. As discussed in Section 4.3.1, the same system in NCR has $q_{NCR} \sim q_{CR} f^2$, with $f = \Phi m_\phi/(\sqrt{2R}\Lambda^2) \gg 1$. But as just discussed the growth exponent in broad resonance for NCR is a fixed number divided by this enhancement factor, $(\mu_k)_{NCR} \sim 0.1 m_\phi/f$. Thus, in order for the NCR system to be in broad resonance $q_{NCR} \gtrsim 1$, we have that $(\mu_k)_{NCR} \lesssim 0.1 m_\phi q_{CR}^{1/2}$, representing an enhancement from CR narrow resonance by a factor of $0.1/q_{CR}^{1/2}$. While this enhancement can be large if the original CR system is very far in the narrow resonance regime, the growth factor is still quite suppressed.

4.3.3 Expansion of the Universe

Up to now we have ignored the expansion of the Universe. Including expansion has several effects on preheating which have been well-studied for CR [6, 3, 8]:

i) The amplitude of inflaton oscillations decreases due to Hubble damping inversely proportional to time $\Phi(t) \sim t^{-1}$.

ii) The parameters of Hill's equation (A_k, q) become time dependent. In particular, $q_{CR} \sim t^{-2}$ due to the time dependence of the inflaton amplitude.

iii) The physical wavenumber $k_{phys} = k/a(t)$ redshifts with the expansion, so each comoving mode only spends a finite amount of time in a resonance band.

iv) The resonance is *stochastic* [3].

v) Preheating is ineffective when the rate of particle production drops below the expansion rate, e.g. when $\mu_k < H$. For narrow resonance this condition cannot be satisfied, thus particle production can only occur in the broad resonance regime.

We will now make some simple estimates of how these effects will change for NCR.

First, the Hubble friction term may be removed from the inflaton equation of motion (4.11) by rescaling the inflaton field by powers of the scale factor $\phi(t) = \Phi_0\tilde{\phi}(t)/a(t)^{3/2c_s^2}$, so that $\tilde{\phi}(t)$ is a fixed unit-amplitude oscillating function, and $\Phi(t) = \Phi_0/a(t)^{3/2c_s^2}$ is the time-dependent amplitude. When the period of oscillation of the inflaton is much smaller than the timescale for the expansion of the Universe, the energy density can be averaged over several oscillations $\rho_\phi \sim m_\phi^2\phi^2 \sim a(t)^{-3/c_s^2}$. The scale factor thus grows as $a(t) \sim t^{2/3c_s^2}$, so that the amplitude falls is inversely proportional to time $\Phi(t) \sim \Phi_0/t$ as with a canonical kinetic term. The decaying inflaton profile in an expanding background is shown in Figure 4.5. It is worth noting that in the far non-canonical limit the period of the inflaton can be larger than the Hubble time due to the period lengthening effect. When this occurs, Hubble friction damps a significant amount of the amplitude in the first oscillation, quickly bringing the system back to the previous case.

For a fixed cutoff scale Λ, the condition for the oscillating inflaton to saturate the speed limit (4.16) is violated once the amplitude of oscillation drops below some critical amplitude. The number of oscillations of the inflaton scales with time $N \sim t/T$, so that the time-dependent amplitude drops below this critical amplitude after $N \sim m_\phi\Phi_0/\Lambda^2$ oscillations, where Φ_0 is the initial amplitude. Thus the inflaton no longer saturates the speed limit after $N \sim m_\phi\Phi_0/\Lambda^2$ oscillations, and behaves canonically after this time.

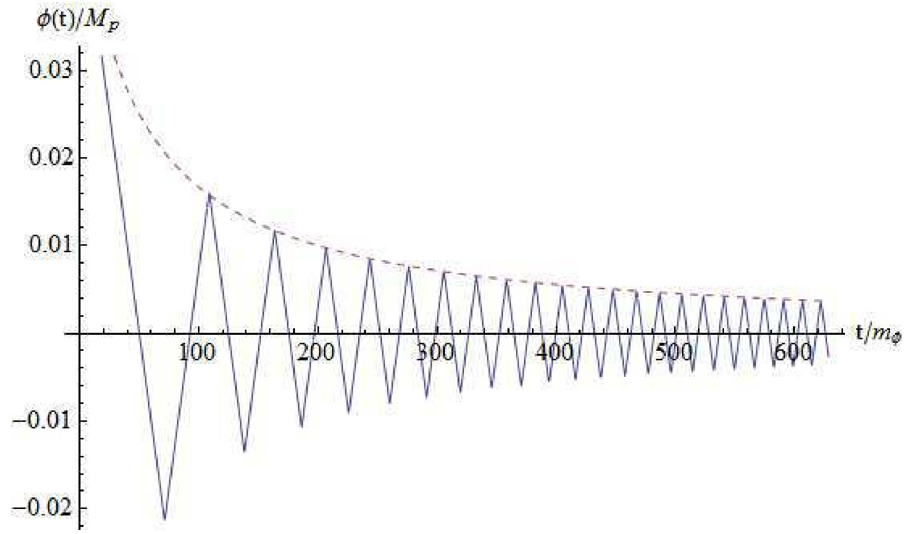

Figure 4.5: Expansion of the Universe causes the amplitude of the inflaton to decay inversely proportional to time, as shown here for $\Lambda = 10\, m_\phi$ and $m_\phi = 10^{-5} M_p$ with an initial amplitude of $\Phi_0 = 0.05 M_p$. The dashed line indicates the t^{-1} behavior of the decaying amplitude.

Another important effect is the dependence of the Hill's equation parameters (4.31) on time. Since q_{NCR} depends on the fourth power of the amplitude, it quickly becomes small at a much faster rate $q_{NCR} \sim t^{-4}$ than it would in CR, $q_{CR} \sim t^{-2}$. In particular, after only $N \sim g^{1/2}\Phi_0/\Lambda$ oscillations, the narrow resonance regime where $q_{NCR} \lesssim 1$ is attained. As noted above, narrow resonance with CR is inefficient for an expanding background. Since we argued in the previous section that particle production for NCR is suppressed compared to CR, narrow resonance for NCR is also inefficient in an expanding background. This means that after $N \sim g^{1/2}\Phi_0/\Lambda$ oscillations the resonance shuts off due to expansion of the Universe. The other effects of an expanding Universe listed above for CR (iii,iv) do not appear to be qualitatively different for NCR, so we will not comment further on them.

In summary, including expansion of the Universe for NCR leads to a decaying amplitude of inflaton oscillations inversely proportional to time $\Phi(t) \sim t^{-1}$. After many oscillations the amplitude decreases so much that either: (a) the inflaton ceases to saturate the speed limit during oscillation, effectively becoming canonical; or (b) the Hill's equation parameter $q_{NCR}(t)$ becomes so small that particle production cannot compete with the expansion of the Universe, shutting off the resonance. Which outcome dominates depends on the relative magnitudes of m_ϕ/Λ and $g^{1/2}$; when the former dominates, outcome (a) occurs first, and *vice-versa*.

4.4 Conclusion

Preheating in the post-inflationary Universe is the explosive production of particles far from thermal equilibrium that occurs as the inflaton field oscillates about its potential minimum. We have examined preheating for an inflaton sector that has non-canonical kinetic terms arising as an effective theory $\mathcal{L}_{eff}(X, \phi)$, for $X = \frac{1}{2}(\partial\phi)^2$, which may arise from the existence of new physics at some energy scale $\Lambda > m_\phi$. Effective theories of this type can give rise to a speed limit for the motion of the inflaton ϕ, as in DBI inflation [18]. In addition to having important implications for inflationary model building, the speed limit plays an important role in modifying the nature of preheating in the post-inflationary Universe. In particular, as the non-canonical terms become important, non-canonical preheating departs significantly from the canonical case via three main effects:

i) The sinusoidal inflaton profile becomes a saw-tooth, elongating the fraction of the inflaton period in which significant particle production may occur, and moves the system from narrow to broad resonance.

ii) Effect *i)* is offset by an elongation of the inflaton oscillation period by a factor $f = \sqrt{\frac{2}{R}\frac{\Phi m_\phi}{\pi\Lambda^2}} \gg 1$. This suppresses the amount of χ particle production *per unit time* by $1/f$.

iii) Effect *ii)* affects the competition between χ production and Hubble expansion, making preheating even less efficient in an expanding Universe.

In general, then, preheating when the inflation has non-canonical kinetic terms is *less efficient* than with a purely canonical kinetic term. This implies that if preheating is to be important at all in the early Universe, then the UV scale of new physics that couples kinetically to the inflaton must be sufficiently high that the effective non-canonical kinetic terms are negligible.

We have only focused on non-canonical kinetic terms for the inflaton sector in this paper. Certainly, however, due to the non-linear nature of the parametric resonance, it would be interesting to study how non-canonical kinetic terms for the reheating sector would affect the physics of preheating. We leave this to future work.

Acknowledgments

We would like to thank Robert Brandenberger and Tomislav Prokopec for useful conversations. B.U is supported in part by NSERC, an IPP (Institute of Particle Physics, Canada) Postdoctoral Fellowship, and by a Lorne Trottier Fellowship at McGill University. ACV was supported by FQRNT (Quebec) and NSERC (Canada).

Appendix A: Computing the Floquet exponent

There exists a straightforward method of computing the Floquet exponent μ_k as a function of the parameters A_k and q, which has been extensively covered in the literature (e.g. [29, 30, 31]).

We start with Hill's equation of the form (4.23). The periodic function $F(\tau)$ may be decomposed:

$$F(\tau) = \sum_{n=-M}^{M} d_n e^{in\tau}, \tag{4.42}$$

with A_k defined such that:

$$\int_{-\pi}^{\pi} F(\tau)d\tau = 0. \tag{4.43}$$

Floquet's theorem (see [29]) states that solutions to (4.23) are of the form,

$$\chi_k(\tau) = e^{\tilde{\mu}_k \tau} g(\tau) + e^{-\tilde{\mu}_k \tau} g_2(\tau) \tag{4.44}$$

where $g(\tau)$ and $g_2(\tau)$ are periodic functions with period T, and $\tilde{\mu}_k$, called the *Floquet exponent* or *characteristic exponent*, is complex. Clearly when $\tilde{\mu}_k$ has a non-zero real part we have exponential growth of χ_k — this is the parametric resonance effect. Without loss of generality, we will take the real part of $\tilde{\mu}_k$ to be positive, and so we will drop $g_2(\tau)$ since its coefficient is exponentially decreasing.

In order to find solutions of the form (4.44), we first Fourier expand $g(\tau)$

$$\chi_k(\tau) = \sum_{n=-\infty}^{\infty} c_n e^{(\tilde{\mu}_k + in)\tau} \tag{4.45}$$

and plug this Fourier series back into the equation of motion (4.23) to derive a recursion relation for the coefficients c_n in terms of $\tilde{\mu}_k, A_k, q, d_n$:

$$c_n + \frac{q \sum_{m=-M}^{M} d_m c_{n-m}}{((\tilde{\mu}_k + in)^2 + A_k)} = 0 \qquad \forall n \in (-\infty, \infty)\,. \tag{4.46}$$

These recursion relations define a matrix problem

$$B(\tilde{\mu}_k, A_k, q, d_m) \begin{pmatrix} c_{-n} \\ \vdots \\ c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = 0 \tag{4.47}$$

with $n \to \infty$, where the elements B_{rs} of the (infinite) matrix B are given by:

$$B_{rs} = \begin{cases} 1 & \text{if } r = s \\ \frac{q d_{r-s}}{(\tilde{\mu}_k + ir)^2 + A_k} & r \neq s \end{cases} . \tag{4.48}$$

The matrix problem (4.47) implies that the (infinite) matrix B is singular, so it must have vanishing determinant

$$\Delta(\tilde{\mu}_k, A_k, q, d_m) = |B_{rs}| = 0 \,. \tag{4.49}$$

The vanishing of the determinant (4.49) defines an implicit function $\tilde{\mu}_k = \tilde{\mu}_k(A_k, q, d_m)$. In practice, we perform a Fourier transform of the inflaton profile written in terms of $F(\tau)$ numerically, and evaluate the $M \times M$ determinant (4.49); a matrix size $M \sim 100$ is sufficient for convergence of the first tens of resonance bands.

Let us briefly comment on the differences in solving for the growth exponent $\tilde{\mu}_k$ using this method for canonical and non-canonical kinetic terms. Clearly, $F(\tau)$ for non-canonical kinetic terms (4.31) expanded in Fourier modes involves many terms, as opposed to the single Fourier mode for canonical kinetic terms. Thus, one difference between resonance with canonical and non-canonical kinetic terms is the inclusion of more terms in the Fourier expansion of the profile. Naively, then, there are more modes in the driving force with which the χ_k fields may resonate. This is the feature arising from the "sharpening" of the profile. However, the other important effect of the non-canonical kinetic terms is the extreme lengthening of the period of oscillation, thus suppressing the resonance in physical time.

Appendix B: Scattering from a Parabolic Potential

In many cases of interest, the equation for perturbations of the reheaton χ_k:

$$\ddot{\chi}_k(t) + \left[k^2 + m_\chi^2 + g^2\phi^2\right]\chi_k(t) = 0, \tag{4.50}$$

can be converted into a Schrödinger scattering problem. In this appendix, we will outline the equivalence between these two perspectives; see [6] for more on the correspondence. We will imagine that $\phi(t)$ has a large slope in some region for some finite amount of time, as shown in Figure 4.6. Linearizing about this point, we have

$$\phi \approx \phi_0 + \lambda(t - t_i)\,. \tag{4.51}$$

This linear approximation is valid roughly for some time interval $2\Delta t$, i.e. for $t_i - \Delta t \leq t \leq t_i + \Delta t$. Outside this interval, the field χ must behave adiabatically: its potential is slowly-varying and can therefore be approximated by a WKB approach. The equation of motion around the point of adiabaticity violation becomes:

$$\ddot{\chi}_k(t) + \left[k^2 + m_\chi^2 + g^2(\phi_0 + \lambda(t - t_i))^2\right]\chi_k(t) = 0\,. \tag{4.52}$$

We can map this problem into a Schrödinger-like equation by redefining the time variable $\tau' \equiv \left(\frac{g}{\lambda}\right)^{1/2}(\phi_0 + \lambda(t - t_i))$, so that the equation of motion is written:

$$\frac{d^2\chi_k(\tau')}{d\tau'^2} + \left[\kappa^2 + \tau'^2\right]\chi_k(\tau') = 0. \tag{4.53}$$

We have defined $\kappa^2 \equiv \frac{k^2+m_\chi^2}{\lambda g}$. The time range of the new variable is $\tau_i' - \Delta\tau' \leq \tau' \leq \tau_i' + \Delta\tau'$, where $\tau_i' = \phi_0(g/\lambda)^{1/2}$ and $\Delta\tau' = (\lambda g)^{1/2}\Delta t$. The transformed equation

(4.53) resembles a Schrödinger equation with the effective potential:

$$V_{eff} = \begin{cases} -(\tau_i' - \Delta\tau')^2 = -V_0 = \text{const} & \tau' < \tau_i' - \Delta\tau' \text{ (Region I)} \\ -\tau'^2 & \tau_i' - \Delta\tau' \leq \tau' \leq \tau_i' + \Delta\tau' \text{ (Region II)} \\ -(\tau_i' + \Delta\tau')^2 = -V_1 = \text{const} & \tau' > \tau_i' + \Delta\tau' \text{ (Region III)} \end{cases} \tag{4.54}$$

The setup of this effective scattering problem is shown in Figure 4.6.

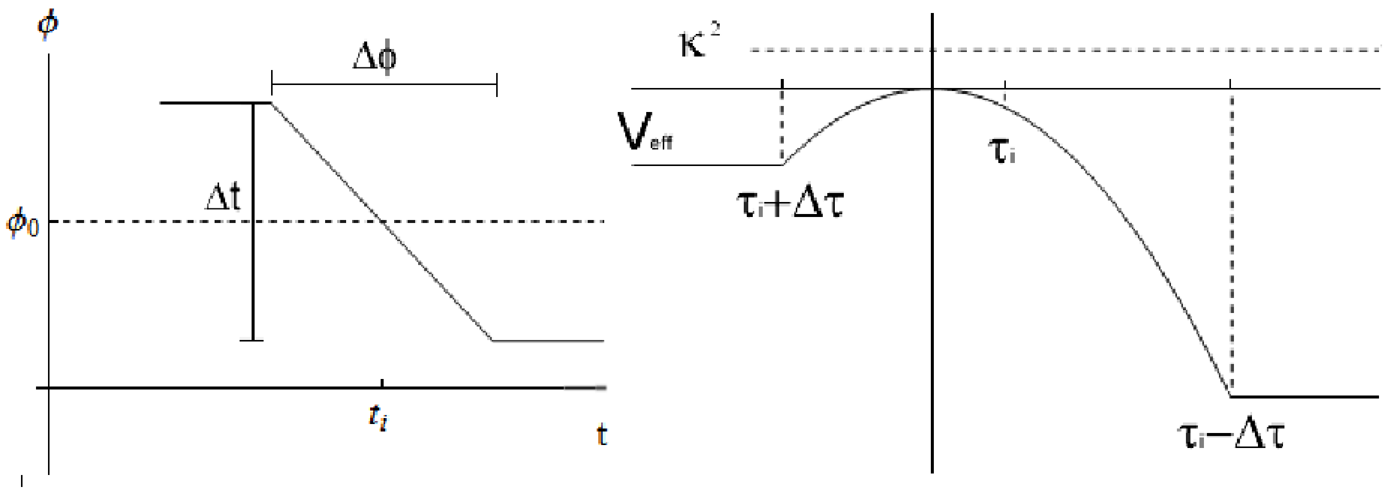

Figure 4.6: Left: Linearizing $\phi(t) \approx \phi_0 + \lambda(t - t_i)$ about some point (t_i, ϕ_0). Right: The associated effective scattering problem.

The solution in each region is therefore known:

$$\chi_k = \begin{cases} A_1 e^{-ik_1\tau'} + B_1 e^{+ik_1\tau'} & \text{(Region I)} \\ A_2 D_\nu\left((1+i)\tau'\right) + B_2 D_{\bar{\nu}}\left((-1+i)\tau'\right) & \text{(Region II)} \\ A_3 e^{-ik_2\tau'} + B_3 e^{+ik_2\tau'} & \text{(Region III)} \end{cases} \tag{4.55}$$

where $k_1 \equiv \sqrt{\kappa^2 + V_0^2}$, $k_2 = \sqrt{\kappa^2 + V_1^2}$, $\nu \equiv \frac{1}{2}i(i - \kappa^2)$, and the functions $D_\nu(x)$ are *parabolic cylinder functions*.

In the language of scattering matrices, the ingoing/outgoing waves are parameterized as

$$\begin{pmatrix} \alpha_k^{j+1} e^{-i\theta_j^k} \\ \beta_k^{j+1} e^{i\theta_j^k} \end{pmatrix} = \begin{pmatrix} \frac{1}{D_k} & \frac{R_k^*}{D_k^*} \\ \frac{R_k}{D_k} & \frac{1}{D_k^*} \end{pmatrix} \begin{pmatrix} \alpha_k^j e^{-i\theta_k^j} \\ \beta_k^j e^{i\theta_k^j} \end{pmatrix} \tag{4.56}$$

where D_k, R_k are the transmission and reflection coefficients, the α_k, β_k are normalized such that $|\alpha_k|^2 - |\beta_k|^2 = 1$ with the "particle number" defined as $n_k = |\beta_k|^2$, and $\theta_k^j \equiv \int_0^T \sqrt{k^2 + m_\chi^2 + g^2\phi^2 dt}$ is the accumulated WKB phase up to the time of scattering. Apart from an overall normalization, it should be clear that the α^j's correspond to the A_j's and the β^j's to the B_j's above.

From (4.56) we can write

$$\beta_k^{j+1} = \alpha_k^j e^{-2i\theta_k^j} \frac{R_k}{D_k} + \beta_k^j \frac{1}{D_k} \tag{4.57}$$

which we can use to find the final number of particles n_k^{j+1} in terms of the original number of particles n_k^j (together with the normalization condition on the α):

$$\begin{aligned} n_k^{j+1} &= \left|\frac{R_k}{D_k}\right|^2 + \frac{1 + |R_k|^2}{|D_k|^2} n_k^j \\ &\quad + 2\frac{|R_k|}{|D_k|^2} \cos\theta_{tot} \sqrt{(1 + n_k^j) n_k^j} \end{aligned} \tag{4.58}$$

where $\theta_{tot} \equiv \arg\alpha_k^j - \arg\beta_k^j - \arg R_k - 2\theta_k^j$ is the total accumulated phase. In the large n_k^j limit, we can drop the first term and simplify the last term, so that we have

$$n_k^{j+1} = \left[\frac{1 + |R_k|^2}{|D_k|^2} + 2\frac{|R_k|}{|D_k|^2} \cos\theta_{tot}\right] n_k^j \,. \tag{4.59}$$

The average growth exponent over one period is defined as

$$n_k^{j+1} = e^{2\tilde{\mu}_k \frac{\pi}{T} \Delta t} n_k^j = e^{2\pi\tilde{\mu}_k} n_k^j \tag{4.60}$$

for $\Delta t = T$ over one period. Thus, the growth exponent is,

$$\tilde{\mu}_k = \frac{1}{2\pi} \log \left[\frac{1 + |R_k|^2}{|D_k|^2} + 2\frac{|R_k|}{|D_k|^2} \cos\theta_{tot} \right]. \tag{4.61}$$

Averaging over many possible initial phases, we can consider θ_{tot} as a random variable; thus, we will take $\cos\theta_{tot} \sim 0$ on average.

In order to compute the scattering amplitudes R_k, D_k, one needs to solve the continuity equations across the boundaries of Regions I, II and III for the wave-functions (4.55). In general, it is not possible to solve these equations analytically. However, for special cases the equations can in fact be solved.

The first limit we will take is $|\tau_i'| \gg 1$; in the original setup, this corresponds to taking $\phi_0\sqrt{g}/\sqrt{\lambda}$ to be large, e.g. the region of linearization takes place very far from $\phi = 0$. In this limit the effective potential becomes essentially a step function:

$$V_{eff} = \begin{cases} -V_0 & \tau' < \tau_i' \text{ (Region I)} \\ -V_1 & \tau' > \tau_i' \text{ (Region II)} \end{cases} \tag{4.62}$$

where $|V_0 - V_1| = \sqrt{g/\lambda}|\Delta\phi|$, with $\Delta\phi$ the change in the inflaton over the time period Δt, and $|V_0| \sim |V_1| \gg 1$. In order for the step function approximation to be a valid approximation to the inverted quadratic effective potential, we need that the change over the step function be small, e.g. $\frac{|V_0 - V_1|}{V_1} \ll 1$. The transmission and reflection coefficients can easily be computed

$$\begin{aligned} D_k^2 &= \frac{4(\kappa^2 + V_0)(\kappa^2 + V_1)}{(2\kappa^2 + V_0 + V_1)^2}; \\ R_k^2 &= \frac{(V_0 - V_1)^2}{(2\kappa^2 + V_0 + V_1)^2}; \end{aligned} \tag{4.63}$$

so that the growth exponent becomes

$$\tilde{\mu}_k = \frac{1}{2\pi} \ln \left[\frac{1 + (V_0 - V_1)^2}{4\sqrt{(\kappa^2 + V_0)(\kappa^2 + V_1)}} \right]. \tag{4.64}$$

With the requirements above that the size of the step be small, we have $D_k \approx 1$ and $R_k \approx 0$, so that $\tilde{\mu}_k \sim \ln(1) \sim 0$. This is expected - if the inflation has a very large offset, the effective mass of the reheaton $m_{eff}^2 = k^2 + m_\chi^2 + g^2\phi_0^2$ is large, so there will be very little particle production.

Alternatively, in the limit $\tau_i' = 0$, the scattering potential becomes symmetric about $\tau = 0$. For $\Delta\tau \gg 1$ the scattering amplitudes are known [6]:

$$D_k = \frac{e^{i\varphi_k}}{\sqrt{1+e^{-\pi\kappa^2}}}; \tag{4.65}$$

$$R_k = -\frac{ie^{i\varphi_k}}{\sqrt{1+e^{\pi\kappa^2}}}; \tag{4.66}$$

where the angle φ_k is

$$\varphi_k = \arg\Gamma\left(\frac{1+i\kappa^2}{2}\right) + \frac{\kappa^2}{2}\left(1 + \ln\frac{2}{\kappa^2}\right) . \tag{4.67}$$

The implications of scattering in this case is considered in more detail in the main text.

REFERENCES

[1] A. Dolgov and A. D. Linde, Phys.Lett. **B116**, 329 (1982).

[2] L. Abbott, E. Farhi, and M. B. Wise, Phys.Lett. **B117**, 29 (1982).

[3] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys.Rev. **D56**, 3258 (1997), hep-ph/9704452.

[4] A. Dolgov and D. Kirilova, Sov.J.Nucl.Phys. **51**, 172 (1990).

[5] J. H. Traschen and R. H. Brandenberger, Phys.Rev. **D42**, 2491 (1990).

[6] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994), hep-th/9405187.

[7] Y. Shtanov, J. H. Traschen, and R. H. Brandenberger, Phys. Rev. **D51**, 5438 (1995), hep-ph/9407247.

[8] I. Zlatev, G. Huey, and P. J. Steinhardt, Phys. Rev. **D57**, 2152 (1998), astro-ph/9709006.

[9] B. A. Bassett, S. Tsujikawa, and D. Wands, Rev. Mod. Phys. **78**, 537 (2006), astro-ph/0507632.

[10] L. Kofman, Lect.Notes Phys. **738**, 55 (2008).

[11] R. Allahverdi, R. Brandenberger, F.-Y. Cyr-Racine, and A. Mazumdar, Ann.Rev.Nucl.Part.Sci. **60**, 27 (2010), 1001.2600.

[12] N. Bouatta, A.-C. Davis, R. H. Ribeiro, and D. Seery (2010), 1005.2425.

[13] P. Franche, R. Gwyn, B. Underwood, and A. Wissanji, Phys.Rev. **D81**, 123526 (2010a), 0912.1857.

[14] A. J. Tolley and M. Wyman (2009), 0910.1853.

[15] C. de Rham and A. J. Tolley, JCAP **1005**, 015 (2010), 1003.5917.

[16] C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, Phys. Lett. **B458**, 209 (1999), hep-th/9904075.

[17] J. Garriga and V. F. Mukhanov, Phys. Lett. **B458**, 219 (1999), hep-th/9904176.

[18] E. Silverstein and D. Tong, Phys. Rev. **D70**, 103505 (2004), hep-th/0310221.

[19] M. Alishahiha, E. Silverstein, and D. Tong, Phys. Rev. **D70**, 123505 (2004), hep-th/0404084.

[20] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, JCAP **0701**, 002 (2007), hep-th/0605045.

[21] P. Franche, R. Gwyn, B. Underwood, and A. Wissanji, Phys.Rev. **D82**, 063528 (2010b), 1002.2639.

[22] R. Bean, D. J. H. Chung, and G. Geshnizjani, Phys. Rev. **D78**, 023517 (2008), 0801.0742.

[23] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi, JHEP **10**, 014 (2006), hep-th/0602178.

[24] E. Babichev, V. Mukhanov, and A. Vikman, JHEP **02**, 101 (2008), 0708.0561.

[25] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, JHEP **03**, 014 (2008), 0709.0293.

[26] L. Leblond and S. Shandera, JCAP **0808**, 007 (2008), 0802.2290.

[27] S. Shandera, Phys. Rev. **D79**, 123518 (2009), 0812.0818.

[28] E. Pajer, JCAP **0804**, 031 (2008), 0802.2916.

[29] W. Magnus and S. Winkler, *Hill's Equation* (Dover Publications, 1979).

[30] T. Charters, A. Nunes, and J. P. Mimoso, Phys. Rev. **D78**, 083539 (2008), 0807.4805.

[31] J. Lachapelle and R. H. Brandenberger, JCAP **0904**, 020 (2009), 0808.0936.

The last three chapters dealt with very early universe cosmology before the age of the Universe reaches 10 microseconds. As the Universe cools more and more symmetries are broken during phases transitions. Those phase transitions are associated with the creation of topological defects. After reheating, 10 microseconds after the Big Bang, hadrons including pions are created during the QCD phase transition. At that time, pion strings may appear but are topologically unstable. This chapter makes use of finite-temperature field-theory in order to find stabilization mechanisms for a possible pion string in a thermal bath of photons.

Chapter 5

Effects of a Thermal Bath of Photons on Embedded String Stability

The following is an extract from the article **"Effects of a Thermal Bath of Photons on Embedded String Stability"** published in collaboration with Robert Brandenberger in *Phys. Rev.* D**85**, 107702 (2012).

Abstract We compute the corrections of thermal photons on the effective potential for the linear sigma model of QCD. Since we are interested in temperatures lower than the confinement temperature, we consider the scalar fields to be out of equilibrium. Two of the scalar field are uncharged while the other two are charged under the U(1) gauge symmetry of electromagnetism. We find that the induced thermal terms in the effective potential can stabilize the embedded pion string, a string configuration which is unstable in the vacuum. Our results are applicable in a more general context and demonstrate that embedded string configurations arising in a wider class of field theories can be stabilized by thermal effects. Another well-known example of an embedded string which can be stabilized by thermal effects is

the electroweak Z-string. We discuss the general criteria for thermal stabilization of embedded defects.

5.1 Introduction

Topological defects can play an important role in early universe cosmology (see e.g. [1, 2, 3] for overviews). On one hand, particle physics models which yield defects such as domain walls which have problematic and unobserved effects can be ruled out. On the other hand, topological defects may help explain certain cosmological observations. They could contribute to structure formation or generate primordial magnetic fields which are coherent on cosmological scales [4].

The Standard Model of particle physics does not give rise to topological defects which are stable in the vacuum. On the other hand, it is possible to construct stringlike configurations which would be topological defects if certain of the fields were constrained to vanish. If they are not, then the defects are unstable in the vacuum. Such defects are called "embedded defects" (see [5] for an overview). Two prime examples of such embedded defects are the pion string arising in the low-energy linear sigma model of QCD [6], and the electroweak Z-string [7] arising in the electroweak theory.

Both the pion string and the electroweak Z-string arise in models with two complex scalar fields, one of them uncharged with respect to the $U(1)$ gauge field of electromagnetism, the second charged. In terms of real fields, we have four real scalar fields $\phi_i, i = 0, .., 3$ with a bare potential of the form

$$V(\phi) \,=\, \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 , \tag{5.1}$$

where $\phi^2 \,=\, \sum_{i=0}^{3} \phi_i^2$. In both physical examples, two of the fields (ϕ_0 and ϕ_3) are uncharged whereas the two others are charged. Since the vacuum manifold of this theory is $\mathcal{M} = S^3$, there are no stable topological defects, only Π_3 defects which in cosmology are called textures [8].

However, the presence of an external electromagnetic field breaks the symmetry since only two of the fields couple to the photon field. Interactions with the photon field lift the potential in the charged field directions, leading to a reduced symmetry group G which is

$$G \;=\; U(1)_{global} \times U(1)_{gauge} \tag{5.2}$$

instead of $O(4)$. The vacuum manifold becomes a circle $\mathcal{M} = S^1$ corresponding to

$$\phi_1 = \phi_2 = 0, \qquad \phi_0^2 + \phi_3^2 = \eta^2 \,. \tag{5.3}$$

It is possible to construct embedded cosmic string solutions which are topological cosmic strings of the reduced theory with $\phi_1 = \phi_2 = 0$. The field configuration of such a string (centered at the origin of planar coordinates and extended along the z-axis) takes the form

$$\Phi(\rho, \theta) \;=\; f(\rho)\eta e^{i\theta} \,, \tag{5.4}$$

where the complex electrically neutral field is $\Phi = \phi_0 + i\phi_3$. $f(\rho)$ is a function which interpolates between $f(0) = 0$ and $f(\rho) = 1$ for $\rho \to \infty$ with a width which is of the order $\lambda^{-1/2}\eta^{-1}$. In the above, ρ and θ are the polar coordinates in the plane perpendicular to the z-axis.

In [9], a plasma stabilization mechanism for the pion string and the electroweak Z-string was proposed. The argument was based on interpreting the terms in the covariant derivative which couple the charged scalar field $\pi^+ = (1/\sqrt{2})(\phi_1 + i\phi_2)$ to the gauge field as a term which, if the gauge field is in thermal equilibrium, will add a term proportional to $\delta V \sim e^2T^2|\pi^+|^2$, to the effective potential of the scalar field sector. This lifts the potential in the direction of the charged scalar fields, leaving us with a reduced vacuum manifold given by (5.3).

In this paper we put this suggested stabilization mechanism on a firmer foundation, focusing on the example of the pion string. The setting of our analysis is the following: we are interested in temperatures below the chiral symmetry breaking transition. Hence, the scalar fields are out of thermal equilibrium. However, the photon field is in thermal equilibrium. This is not the usual setting for finite temperature quantum field theory (since not all fields are in thermal equilibrium) and hence non-standard techniques are required.

We compute the effective potential for the scalar fields obtained by integrating out the gauge field, taking it to be in thermal equilibrium. To do so we use a functional integral in which the time domain is Euclidean and ranges from 0 to β, where β is the inverse temperature. We find that the resulting scalar field effective potential has a broken symmetry and a vacuum manifold given by (5.3). There is hence an energetic barrier which has to be overcome to destroy a pion string.

5.2 The Pion String

The cosmological context of this work is Standard Big Bang Cosmology. At about 10 microseconds after the Big Bang a phase transition from the quark-gluon plasma to a hadron gas is expected to have taken place at a temperature of about $T_{QCD} \sim 150 - 200\text{MeV}$. Below this critical temperature, the physics of hadrons can be well described by a linear sigma model of four scalar fields which we collectively denote by ϕ, three of them representing the pions. If we make the standard assumption that the relevant bare quark masses vanish, i.e. $m_u = m_d = 0$, where the subscripts stand for the up and down quark, respectively, we know that the effective action (in vacuum) will have a $SU(2)$ symmetry which is spontaneously broken by a potential $V(\phi)$. This spontaneous symmetry breaking leads to a mass for the fourth scalar field, the so-called sigma field.

In the setup described above, the "vacuum manifold" $\mathcal{M}$, i.e. the set of field configurations which minimize the potential, is nontrivial and takes the form of a 3-sphere S^3. There are no stable topological defects associated with this symmetry breaking, only global textures which are not stable. In particular, since the first homotopy group of the vacuum manifold is trivial, i.e. $\Pi_1(\mathcal{M}) = 1$, there are no stable cosmic strings.

However, as discussed in [6], it is possible to construct embedded strings for which the charged pion fields are set to zero, and the two neutral fields (the neutral pion and the sigma fields) form a cosmic string configuration. This is the so-called "pion string".

Since two of the four scalar fields are charged (the two charged pion fields) while the remaining two are neutral, turning on the electromagnetic field will destroy the $O(4)$ symmetry and will break it down to $U(1)_{global} \times U(1)_{local}$. The second factor corresponds to rotations of the charged complex scalar field, the first to a rotation of the neutral one. The pion string can be viewed as the cosmic string configuration associated with the first $U(1)$.

As a toy model for the analytical study of the stabilization of embedded defects by plasma effects we consider the chiral limit of the QCD linear sigma model, involving the sigma field σ and the pion triplet $\vec{\pi} = (\pi^0, \pi^1, \pi^2)$, given by the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - \eta^2)^2 \,, \tag{5.5}$$

where η^2 is the ground state expectation value of $\sigma^2 + \vec{\pi}^2$. In the following, we denote the potential in (5.5) by V_0.

Two of the scalar fields, the σ and π_0, are electrically neutral, the other two are charged. Introducing the coupling to electromagnetism, it is convenient to write the bosonic sector $\mathcal{L}$ of the resulting Lagrangian in terms of the complex scalar fields

$$\pi^+ = \frac{1}{\sqrt{2}}(\pi^1 + i\pi^2), \; \pi^- = \frac{1}{\sqrt{2}}(\pi^1 - i\pi^2) . \tag{5.6}$$

According to the minimal coupling prescription we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0 + D^+_\mu\pi^+D^{\mu-}\pi^- - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V_0 ,$$

where $D^+_\mu = \partial_\mu + ieA_\mu$, $D^-_\mu = \partial_\mu - ieA_\mu$.

Effective pion-photon interactions appear through the covariant derivative. They break the $O(4)$ symmetry which the Lagrangian would have in the absence of the gauge field.

In the following we work in terms of the two complex scalar fields

$$\begin{aligned} \Phi &= \sigma + i\pi_0 \\ \pi_c &= \pi_1 + i\pi_2 , \end{aligned} \tag{5.7}$$

the first of which is electrically neutral, the second charged.

The minimum of the potential can be obtained for $\langle\pi_c\rangle = 0$ and $\langle\Phi\rangle = \eta$ and electromagnetism is unbroken. In that case, the vacuum manifold S^3 reduces to S^1 and some string configurations exist. They are not topologically stable since they can unwind by exciting the charged fields, i.e. $\langle\pi_c\rangle \neq 0$. This string solution is the "pion string". The field Φ vanishes in the center of the string (the charged field vanishes everywhere) and this implies that there is trapped potential energy along the string.

Note that if we distort the field configuration to have $\langle\pi_c\rangle \neq 0$, then the U(1) of electromagnetism gets broken and there is a magnetic flux of $\frac{2\pi}{e}$ in the core of the string. To see how this flux may arise, consider as starting point the pion string configuration with π_c vanishing everywhere. To see the effect of the charged field, let us now consider exciting a constant π_c in the core of the string and see how the potential energy evolves in this case. We easily find that as long as $|\pi_c| \ll \eta$, we get an increase in the potential energy in the core of the string:

$$V(\phi = 0, \pi_c) \simeq V_0(0,0) + \sqrt{\lambda V_0}|\pi_c|^2 \tag{5.8}$$

and the lowest potential energy configuration is obtained for $\pi_c = 0$. Based on potential energy arguments alone we would infer that we could get a stable string. However, the kinetic and gradient energies lead to an instability of the vacuum pion string configuration.

We want to consider the effect of a photon plasma on the stability of the pion string. Here we consider an ultrarelativistic plasma of photons. The effective Lagrangian that describes such a plasma is [10] :

$$\mathcal{L}_{eff} = \mathcal{L}_{QED} + \mathcal{L}_{\gamma} \tag{5.9}$$

We do not consider the electrons in the Lagrangian for QED, but simply take

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \tag{5.10}$$

where $F_{\mu\nu}$ is the field strength associated with the electomagnetic 4-potential A_μ.

The plasma terms in the effective Lagrangian (5.9) are

$$\mathcal{L}_{\gamma} = \frac{3}{4} m_\gamma^2 \, F_{\mu\alpha} \left\langle \frac{K^\alpha K^\beta}{(K \cdot \partial)^2} \right\rangle F^{\mu}{}_{\beta} , \tag{5.11}$$

where K^α is the four-vector which represents the momentum of the hard field in the loop, and m_γ is the thermal photon mass which is given by

$$m_\gamma^2 \;=\; \frac{e^2}{9}\left(T^2 + \frac{3}{\pi^2}\mu^2\right) \;, \tag{5.12}$$

and μ is the quark chemical potential.

What really matters is not the plasma behavior but rather its influence on a charged scalar field. We will use the "Hard Thermal Loop" formalism, that is we work in the limit where the plasma temperature is much higher than any momentum or mass scale in the problem. In our case we have the following for a scalar field ϕ in the fundamental representation of the gauge group [11]:

$$\mathcal{L}_s \;=\; \frac{3}{4}\, m_s^2\, \phi^\dagger \left\langle \frac{D^2}{(K\cdot D)^2} \right\rangle \phi \tag{5.13}$$

This reduces to

$$\mathcal{L}_s = m_s^2 \phi^\dagger \phi \tag{5.14}$$

where K is the moment of the hard field in the loop and $m_s = \frac{eT}{2}$ is the thermal scalar mass (see [12] for details).

Applying the above to our charged field π_c, the effective Lagrangian becomes

$$\begin{aligned} \mathcal{L} \;&=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0 \\ &\quad + D_\mu^+\pi^+ D^{\mu -}\pi^- - V_0 - \frac{e^2T^2}{4}\pi^+\pi^- \,. \end{aligned} \tag{5.15}$$

This gives rise to a new effective potential while retaining the gauge invariance of the Lagrangian:

$$V_{eff} \;=\; \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - \eta^2)^2 + \frac{e^2T^2}{4}\pi^+\pi_- \,, \tag{5.16}$$

where $\frac{e^2T^2}{4}\pi^+\pi^- \;=\; \frac{e^2T^2}{8}(\pi_1^2 + \pi_2^2) = \frac{e^2T^2}{8}|\pi_c|^2 \,.$

Based on potential energy considerations, the induced terms in the effective potential should lead to the stabilization of pion strings.

5.3 Effective potential computation

An improved way to study the stability of the pion string in the presence of a thermal bath is to determine the effective potential of the scalar fields (which are out of thermal equilibrium) in the presence of a thermal bath of photons. This effective potential can be obtained by computing the finite temperature functional integral over the gauge field, treating the scalar fields as external out-of-equilibrium classical ones. They are out of thermal equilibrium since their masses are heavy compared to the temperature if we are below the critical temperature. Note also that string configurations are out-of-equilibrium states below the Ginsburg temperature (which is the temperature slightly lower than the critical temperature when the defect network freezes out during the symmetry breaking phase transition [1, 2, 3]).

We make use of the imaginary time formalism [13] of thermal field theory (see e.g. [14] for a review). We work in Euclidean space-time: $t \to i\tau$ and $\tau : 0 \to T^{-1} = \beta$.

The starting point is the action

$$\begin{aligned} S[A^\mu, \Phi, \pi_c] &= \int d^4x[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \tfrac{\lambda}{4}(|\Phi|^2 + |\pi_c|^2 - \eta^2)^2] \\ &\quad + \int_0^\beta d\tau \int d^3x[D_\mu^+\pi^+ D^{\mu-}\pi^- - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}] \end{aligned} \tag{5.17}$$

where the space integral runs from $-\infty$ to $+\infty$.

In the imaginary time formalism of thermal field theory, the integration over four-momenta is carried out in Euclidean space with $k_0 = ik_4$, this means that the transition from zero temperature field theory is obtained via $\int \frac{d^4k}{(2\pi)^4} \to i\int \frac{d^4k_E}{(2\pi)^4}$.

Next, we recall that boson energies take discrete values, namely $k_4 = \omega_n = 2n\pi T$ with n an integer, and thus

$$\int \frac{d^4k_E}{(2\pi)^4} \to T\sum_n \int \frac{d^3k}{(2\pi)^3}. \tag{5.18}$$

We use this Matsubara mode decomposition for the gauge field only, because it is the only field in thermal equilibrium.

The standard definition for the effective potential is based on the Legendre transform of the generating functional (see [14, 15] for reviews). However, the finite temperature effective potential with the scalar fields viewed as classical background fields can also be defined as

$$\begin{aligned} Z[T] &= \int \mathcal{D}\Phi\mathcal{D}\pi_c\mathcal{D}A^\mu e^{-S[A^\mu,\Phi,\pi_c]} \\ &= \int \mathcal{D}\Phi\mathcal{D}\pi_c e^{-S[\Phi,\pi_c]}e^{-\frac{V_{eff}(\Phi,\pi_c)V}{T}} \end{aligned} \tag{5.19}$$

where $S[\Phi, \pi_c]$ is the gauge field independent part of the (non-Euclidean) action, V is the volume of the system and $\int d\tau d^3\mathbf{x} = \frac{\mathbf{V}}{\mathbf{T}}$.

To evaluate the total partition function $Z[T]$ of the system we work in the covariant Feynman gauge following the procedure reviewed e.g. in [16], according to which the two unphysical degrees of freedom of A^μ that correpond to the longitudinal and timelike photons are cancelled by the ghost and antighost fields c and $\bar{c}$:

$$\begin{aligned} &Z[T] = \int \mathcal{D}\Phi\mathcal{D}\pi_c\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}A_\mu e^{-S[\Phi,\pi_c]} \\ &\times e^{-\int_0^\beta d\tau \int d^3x\ \bar{c}(-\partial^2-e^2|\pi_c|^2)c} e^{-\int_0^\beta d\tau \int d^3x\ \frac{1}{2}A_\mu(\partial^2+e^2|\pi_c|^2)A_\mu} \end{aligned} \tag{5.20}$$

Here the summation of $A_\mu A_\mu$ is in Euclidean space since $A_0 \to iA_0$. We can see from above that the gauge field obtains an effective mass equal to $m_{eff} = e|\pi_c|$. Now we simply evaluate the Gaussian integration over the gauge field and the ghost fields.

$$
\begin{aligned}
Z[T] &= \int \mathcal{D}\Phi\mathcal{D}\pi_c e^{-S[\Phi,\pi_c]} \\
&\times e^{2\frac{1}{2}Tr[\ln(\omega_n^2+\mathbf{k}^2+m_{eff}^2)]} e^{-4\frac{1}{2}Tr[\ln(\omega_n^2+\mathbf{k}^2+m_{eff}^2)]} \\
Z[T] &= \int \mathcal{D}\Phi\mathcal{D}\pi_c e^{-S[\Phi,\pi_c]} e^{-Tr[\ln(\omega_n^2+\mathbf{k}^2+m_{eff}^2)]}
\end{aligned}
\tag{5.21}
$$

Comparing (5.21) with the definition (5.19) of the effective potential we find

$$
\begin{aligned}
V_{eff}(\Phi,\pi_c,T) &= V_0 + \lim_{V\to\infty}\frac{T}{V}\sum_{n\in\mathbb{Z}}\ln(\omega_n^2+\mathbf{k}^2+m_{eff}^2) + \text{cst} \\
&= \frac{\lambda}{4}(|\Phi|^2+|\pi_c|^2-\eta^2)^2 + 2\int\frac{d^3k}{(2\pi)^3}[\frac{\omega}{2} + T\ln(1-e^{-\frac{\omega}{T}})]
\end{aligned}
\tag{5.22}
$$

where $\omega = \sqrt{\mathbf{k}^2+m_{eff}^2}$ and $V_0 = \frac{\lambda}{4}(|\Phi|^2+|\pi_c|^2-\eta^2)^2$. The thermal part, $J(m_{eff},T) = \int\frac{d^3k}{(2\pi)^3}T\ln(1-e^{-\frac{\omega}{T}})$, admits a high-temperature expansion :

$$
\begin{aligned}
J(m_{eff},T) &= \frac{T}{2}\sum_n\int\frac{d^3k}{(2\pi)^3}\ln(\omega_n^2+\mathbf{k}^2+m_{eff}^2) \\
&\simeq -\frac{\pi^2T^4}{90} + \frac{m_{eff}^2T^2}{24} - \frac{m_{eff}^3T}{12\pi} \\
&\quad -\frac{m_{eff}^4}{32\pi^2}\left[\ln\left(\frac{m_{eff}e^{\gamma_E}}{4\pi T}\right) - \frac{3}{4}\right] + \mathcal{O}(\frac{m_{eff}^6}{T^2})\,.
\end{aligned}
\tag{5.23}
$$

The zero temperature part, $J_0(m)$, is UV divergent :

$$
\begin{aligned}
J_0(m) &= \int\frac{d^3p}{(2\pi)^3}\frac{\omega}{2}\bigg|_{\omega=\sqrt{p^2+m^2}} \\
&= -\frac{m^4\mu^{-2\epsilon}}{64\pi^2}\left[\frac{1}{\epsilon} + \ln\frac{\overline{\mu}^2}{m^2} + \frac{3}{2} + O(\epsilon)\right]\,.
\end{aligned}
\tag{5.24}
$$

The renormalized value of this integral has been obtained using the $\overline{MS}$ renormalization parameter $\bar{\mu}$. $d = 3-2\epsilon$ is the dimension of the momentum integral. Considering renormalization of coupling constants as well, will give $O(\hbar)$ corrections to the potential, but should not change the topology of the vacuum manifold.

For example, if the vacuum manifold is a circle it could become a titled circle but since these are small effects, this will not affect the presence of a topological defects.

At high-temperatures we can truncate the series in (5.23) and get

$$\begin{aligned} V_{eff}(\Phi,\pi_c,T) \;=\; & \frac{\lambda}{4}(|\Phi|^2+|\pi_c|^2-\eta^2)^2-\frac{\pi^2T^4}{45}+\frac{e^2|\pi_c|^2T^2}{12} \\ & -\frac{e^3|\pi_c|^3T}{6\pi}-\frac{e^4|\pi_c|^4}{16\pi^2}\left[\ln\left(\frac{e|\pi_c|e^{\gamma_E}}{4\pi T}\right)-\frac{3}{4}\right] \end{aligned}$$

where we neglect terms of order $\frac{e^6|\pi_c|^6}{T^2}$. This potential is also the approximate potential close to the (0,0) point of the (Φ,π_c)-plane. For consistency with the hard thermal loop approximation, we can check that, when the gauge field takes on an effective mass, m_{eff}, it has 3 polarizations instead of 2. This leads to the appearance of an overall factor of $\frac{3}{2}$ for the thermal part of the effective potential. The term quadratic in temperature then becomes $m_{eff}^2T^2/8$ as it appears in (5.16).

The above computation shows that the effects of the photon plasma create an energy barrier which lifts the scalar field effective potential in direction of the charged fields. In order to minimize this effective potential, the charged fields must go to zero. The "effective" vacuum manifold $\mathcal{M}$ is now no longer S^3 but rather $\mathcal{M} = S^1$ which has nontrivial first homotopy group and hence admits stable cosmic string solutions which are precisely the pion strings discussed earlier.

5.4 Conclusions

We have studied the effective potential for the scalar fields of the low-energy effective sigma model of QCD in the presence of a thermal bath of photons. We have shown that the plasma effects lift the potential in direction of the charged pion fields, and lead to an effective vacuum manifold which admits cosmic string solutions, the pion strings.

Our analysis puts the stabilization mechanism of [9] on a firmer footing. It shows that if pion strings form, they will be stabilized by plasma effects, at least at a classical level. The stability of pion strings to quantum processes remains to be studied.

The analysis in this paper applies not only to pion strings, but equally to the corresponding embedded strings in the electroweak theory, the Z-string. Thus, below the confinement scale the Standard Model of particle physics admits two types of classically stable embedded strings.

Our arguments, however, are more general and apply to many theories beyond the Standard Model. Given a theory with a multicomponent scalar field order parameter with one set of components which are neutral, and a second set which are charged - neutral and charged being with respect to the fields excited in the plasma. Then topological defects of the theory with vanishing charged order parameters become embedded defects of the full theory with the property that they are stabilized in the early Universe.

Stabilized embedded defects can have many applications in cosmology. For example, stabilized pion strings provide an explanation for the origin and large-scale coherence of cosmological magnetic fields [4]. On the other hand, stabilized embedded domain walls would lead to an overclosure problem. Hence, theories admitting those types of embedded defects would be ruled out by cosmological considerations.

Acknowledgments

We wish to thank Aleksi Kurkela and Guy Moore for useful discussions. This work is supported in part by a NSERC Discovery Grant to RB and by funds from the Canada Research Chair program. JK is supported by an NSERC Postgraduate Scholarship.

REFERENCES

[1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other Topological Defects* (Cambridge Univ. Press, Cambridge, 1994).

[2] M. B. Hindmarsh and T. W. B. Kibble, "Cosmic strings," Rept. Prog. Phys. **58**, 477 (1995) [arXiv:hep-ph/9411342].

[3] R. H. Brandenberger, "Topological defects and structure formation," Int. J. Mod. Phys. A **9**, 2117 (1994) [arXiv:astro-ph/9310041].

[4] R. H. Brandenberger and X. -m. Zhang, "Anomalous global strings and primordial magnetic fields," Phys. Rev. D **59**, 081301 (1999) [hep-ph/9808306].

[5] A. Achucarro and T. Vachaspati, "Semilocal and electroweak strings," Phys. Rept. **327**, 347 (2000) [Phys. Rept. **327**, 427 (2000)] [arXiv:hep-ph/9904229].

[6] X. Zhang, T. Huang and R. H. Brandenberger, "Pion and eta strings," Phys. Rev. D **58**, 027702 (1998) [hep-ph/9711452].

[7] T. Vachaspati, "Electroweak strings," Nucl. Phys. B **397**, 648 (1993).

[8] N. Turok, "Global Texture as the Origin of Cosmic Structure," Phys. Rev. Lett. **63**, 2625 (1989).

[9] M. Nagasawa and R. H. Brandenberger, "Stabilization of embedded defects by plasma effects," Phys. Lett. B **467**, 205 (1999) [hep-ph/9904261];
M. Nagasawa and R. Brandenberger, "Stabilization of the electroweak Z string in the early universe," Phys. Rev. D **67**, 043504 (2003) [hep-ph/0207246].

[10] E. Braaten, "Effective field theory for plasmas at all temperatures and densities," Can. J. Phys. **71**, 215 (1993) [arXiv:hep-ph/9303261].

[11] E. Braaten and R. D. Pisarski, "Simple effective Lagrangian for hard thermal loops," Phys. Rev. D **45**, 1827 (1992).

[12] U. Kraemmer, A. K. Rebhan and H. Schulz, "Hot scalar electrodynamics as a toy model for hot QCD," arXiv:hep-ph/9505307.

[13] L. Dolan and R. Jackiw, "Symmetry Behavior at Finite Temperature," Phys. Rev. D **9**, 3320 (1974).

[14] R. H. Brandenberger, “Quantum Field Theory Methods And Inflationary Universe Models,” Rev. Mod. Phys. **57**, 1 (1985).

[15] M. Laine, “Finite temperature field theory - with applications to cosmology,” *Prepared for ICTP Summer School on Astroparticle Physics and Cosmology, Trieste, Italy, 17 Jun - 5 Jul 2002*

[16] J. I. Kapusta and C. Gale, “Finite-temperature field theory: Principles and applications,” *Cambridge, UK: Univ. Pr. (2006) 428 p*

Chapter 6

Conclusion

Early Universe cosmology involves many epochs and most of them are not fully understood. Even the standard Big Bang model taken all the way back to the initial singularity has yet to be proved.

Inflation, the current paradigm of standard cosmology, has made impressive predictions for cosmic microwave background observations. However it suffers from several conceptual problems, in particular the singularity problem at $t = 0$. A Big Bounce replacing the Big Bang would avoid the initial singularity but requires infinite time in the past, so no origin of time. A non-singular bounce requires a new form of matter, with negative kinetic energy which may imply ghost instabilities. In this thesis, another instability, the instability of the contracting phase due to radiation, has been discussed. Many non-singular bouncing models are plagued by those instabilities and any type of matter involved in those models must be phenomenologically viable. None of the non-singular bouncing models are, as of today, as observationally satisfying as the current paradigm of inflation. Other models involving string theory, like the Ekpyrotic scenario, are trying to provide a valid alternative to inflation.

Returning to inflationary cosmology, problems like particle production at the end of inflation have not been solved, and the hypothetical scalar field driving inflation, the inflaton, has never been observed. This may be the Higgs boson, which, if produced in the lab, would be the first scalar particle ever discovered. Chapter 4 considered specific models where the inflaton has a speed limit, and studied how particle production occurred via parametric resonance. It proved that preheating when the inflation has non-canonical kinetic terms is less efficient than with a purely canonical kinetic term.

Another puzzle in cosmology is the absence of observed topological defects. The Universe being a system that is monotonically cooling, it should undergo some phase transitions during which defects formation should be ubiquitous. Chapter 5 showed a stabilization mechanism for defects existing in the Standard Model of particle physics, namely the pion strings. The method uses a thermal bath to stabilize those defects. If stable, the latter may have important consequences in cosmology.

The way the Universe is modeled today involves many uncertainties and simplifications. Indeed, toy models are widely used to describe more complicated physics. Many problems have yet to be solved in order to have a theory of the early Universe fully consistent with the Standard Model of particle physics and with current observations.