

Quelques contributions à l'analyse mathématique et numérique d'équations cinétiques collisionnelles

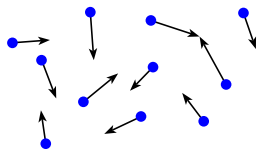
Soutenance de thèse de doctorat de
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21 Septembre 2012

Different Scales of Modeling

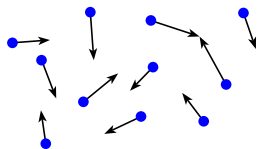
- **Microscopic:** Newton equations for N -particles system,



$$(x_i(t), v_i(t))_{1, \dots, N}$$

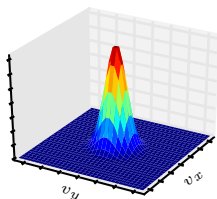
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- **Microscopic:** Newton equations for N -particles system,
- **Mesoscopic:** Boltzmann-like equations,



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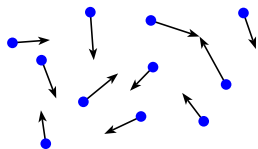
$$N \rightarrow \infty$$



$$f^E(t, x, v)$$

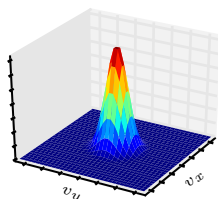
Different Scales of Modeling

- **Microscopic:** Newton equations for N -particles system,
- **Mesoscopic:** Boltzmann-like equations,
- **Macroscopic:** Euler/Navier-Stokes-like fluid equations.



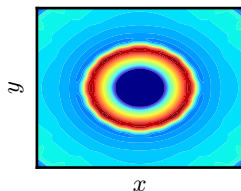
$$(x_i(t), v_i(t))_{1, \dots, N}$$

$$N \rightarrow \infty$$



$$f^\varepsilon(t, x, v)$$

$$\varepsilon \rightarrow 0$$



$$\rho(t, x), \mathbf{u}(t, x), E(t, x)$$

- 1 The Granular Gases Equation
 - Modeling
 - Mathematical Results
- 2 The Rescaling Velocity Method
 - Choice of the Scaling
 - Applications
- 3 Numerical Simulations

Boltzmann-like Kinetic Equations

General Scaled Equation

Study of a particle distribution function $f^\varepsilon(t, x, v)$, depending on time $t > 0$, space $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$, solution to

$$\begin{cases} \frac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon), \\ f^\varepsilon(0, x, v) = f_0(x, v), \end{cases} \quad (1)$$

where Q is the **collision operator**, describing the microscopic collision dynamic and ε is a scaling parameter.

- ε is usually the **Knudsen number**, ratio of the mean free path between collision by the typical length scale of the problem;
- Q only acts on the v variable;
- Boundary conditions in space are also needed to describe completely the problem.

Modeling Assumptions

Microscopic dynamics

- **Localized interactions:** particles interact only by contact (**collision**), at a given time t and a given position x ;
- **Diluted gases:** collisions occur between two particles at the same time (we neglect the collisions of three particles or more);
- **Boltzmann chaos assumption:** the velocity of two colliding particles are uncorrelated before collision.

The microscopic collision process is said to be

- **elastic** when the kinetic energy is conserved during a collision (this is for example the case for a perfect molecular gas);
- **inelastic** when a fraction of the kinetic is dissipated during a collision.

The Granular Gases Equation

The granular gases equation describes the behavior of a dilute gas of particles when the only interactions taken into account are binary **inelastic** collisions.



Applications

Pollen dissemination, avalanches, planetary rings^a, ...

^aKawai, Shida, *J. Phys. Soc. Japan* (1990)

The collision operator

$$Q_e(f)(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \left[\frac{f(v') f(v_*')}{e^2} - f(v) f(v_*) \right] B \, d\sigma \, dv_*,$$

where B is the **collision kernel**, $e \in [0, 1]$ the constant **restitution coefficient**, and (v', v_*') the pre-collisional velocities of two particles of post-collisional velocities (v, v_*) .

Features of the Model

If $e < 1$ (true inelasticity),

- Conservation of impulsions and dissipation of kinetic energy:

$$v' + v'_* = v + v_*,$$


$$\frac{|v'|^2}{2} + \frac{|v'_*|^2}{2} - \frac{|v|^2}{2} - \frac{|v_*|^2}{2} = -\frac{1-e^2}{4} |(v - v_*) \cdot \omega|^2 \leq 0.$$

- Collisional transform¹: for $\sigma \in \mathbb{S}^{d-1}$

$$\begin{cases} v' = \frac{v + v_*}{2} - \frac{1-e}{4e}(v - v_*) + \frac{1+e}{4e}|v - v_*|\sigma, \\ v'_* = \frac{v + v_*}{2} + \frac{1-e}{4e}(v - v_*) - \frac{1+e}{4e}|v - v_*|\sigma. \end{cases}$$

- Meaningful even in dimension 1: the collision process is given by

$$\{v', v'_*\} = \{v, v_*\} \quad \text{or} \quad \left\{ \frac{v + v_*}{2} \pm \frac{e}{2}(v - v_*) \right\}.$$

¹with Jacobian different from 1, contrary to the elastic case. 

The Inelastic Collision Operator

For two velocities $v, v_* \in \mathbb{R}^d$, we set $u := v - v_*$, $\hat{u} := \frac{u}{|u|}$; if ψ is a smooth test function, $\psi' := \psi(v')$, $\psi_* := \psi(v_*)$, $\psi'_* := \psi(v'_*)$.

- **Weak form** of the collision operator:

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} f_* f (\psi' + \psi'_* - \psi - \psi_*) B(|u|, \hat{u} \cdot \sigma, E(f)) d\sigma dv dv_*,$$

where $E(f) := 1/2 \int_{\mathbb{R}^d} f(v) |v|^2 dv$ is the **kinetic energy** and $B \geq 0$ the **collision kernel**.

- **Strong form**: if e is non-zero,

$$\mathcal{Q}_e(f, f)(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \left(\frac{1}{e^2} f' f'_* - f f_* \right) B(|u|, \hat{u} \cdot \sigma, E(f)) d\sigma dv_*.$$

Macroscopic Properties

Conservation of mass and momentum, dissipation of kinetic energy:

$$\int_{\mathbb{R}^d} \mathcal{Q}_e(f, f) \left(1, v, \frac{|v|^2}{2} \right) dv = (0, 0, -D(f)),$$

where $D(f)$ is the energy dissipation functional:

$$D(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f f_* \Delta(|v - v_*|, E(f)) dv dv_* \geq 0,$$

and $\Delta(|u|, E)$ is the **dissipation rate** (for $\cos \theta := \hat{u} \cdot \sigma$):

$$\Delta(|u|, E) := \frac{1 - e^2}{4} \int_{\mathbb{S}^{d-1}} |u \cdot \omega|^2 B(|u|, \cos \theta, E) d\omega \geq 0.$$

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We call **cooling process** (or Haff's Law^a) the asymptotic behavior of $E(f)(t)$. We say that there is **blow up** if $E(f)(t) \rightarrow 0$ when $t \rightarrow T_c$, $T_c < +\infty$ being the lifetime of the solution.

^aP.K. Haff, *J. Fluid Mech.* (1983)

A State of the Art

- Space homogeneous setting $\partial_t f = \mathcal{Q}_e(f, f)$
 - ▶ **Cauchy problem** with pseudo-Maxwellian molecules: Bobylev-Carrillo-Gamba (2000), Bobylev-Cercignani-Toscani (2003);
 - ▶ **Cauchy problem** with general kernels (including hard spheres): Toscani (2000), Mischler-Mouhot-Ricard (2006);
 - ▶ **Qualitative behavior** (with and without thermal bath): Bobylev-Gamba-Panferov (2004), Gamba-Panferov-Villani (2004);
 - ▶ **Cooling process** and asymptotic/self-similar behavior: Li-Toscani (2004), Mischler-Mouhot (2006), Alonso-Lods (2010);
 - ▶ **Stability** and convergence towards the (unique) self-similar solution: Mischler-Mouhot (2009).

- Full space-dependent equation $\partial_t f + v \cdot \nabla_x f = \mathcal{Q}_e(f, f)$
 - ▶ **Cauchy problem** near vacuum: Alonso (2009);
 - ▶ Investigation of the **hydrodynamic limit**: Toscani (2004), Carlen-Chow-Grigo (2010).

Anomalous Gases

$$\frac{\partial f}{\partial t} = E(f)^{-a} Q_e(f, f) \quad (2)$$

Theorem (TR, SIAM J. Math. Anal, 2012)

Let f be solution to (2), with initial condition $0 \leq f_{in} \in L^1_3 \cap L^p$ for $p > 1$. Then, we have

- **subcritical case:** if $0 \leq a < 1/2$, $\alpha = 1/(2a - 1) < 0$ and

$$(C_1 t + 1)^{2\alpha} \leq E(f)(t) \leq (C_2 t + 1)^{2\alpha}, \forall t > 0;$$

- **critical case:** if $a = 1/2$, $e^{-C_3 t} \leq E(f)(t) \leq e^{-C_4 t}$, $\forall t > 0$;

- **supercritical case:** if $a > 1/2$, $\alpha = 1/(2a - 1) > 0$ and

$$(-C_5 t + 1)^{2\alpha} \leq E(f)(t) \leq (-C_6 t + 1)^{2\alpha}, \forall t < T_c.$$

Moreover, there exists $V \in C^1(0, T_c)$ and $0 \leq G \in L^1_3$ such that $F(t, v) := V(t)^d G(V(t)v)$ is solution to (2).

Element of Proof

$$\frac{\partial f}{\partial t} = E(f)^{-a} Q_e(f, f)$$

- Introduction of the scaling

$$\begin{cases} f(t, v) = V(t)^d g(T(t), V(t)v), \\ V(0) = 1, \quad T(0) = 0, \quad \lim_{t \rightarrow T_c} T(t) = \lim_{t \rightarrow T_c} V(t) = +\infty. \end{cases}$$

- Choose V and T as

$$\begin{cases} V'(t) = E(f)(t)^{-a}, \\ T'(t)V(t) = V'(t). \end{cases}$$

- Then g is solution to

$$\frac{\partial g}{\partial s} + \nabla_w \cdot (wg) = Q_e(g, g). \quad (3)$$

- Study of the moment of (3).

Spectral Study of the Linearized Granular Gases Operator

- **Final goal:** Establish the non-linear stability of the solutions to the full equation

$$\frac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}_e(f^\varepsilon, f^\varepsilon) + \frac{1-e}{\varepsilon} \Delta_v f$$

under the (physical) hypothesis $1 - e \sim \varepsilon > 0$.

- **Our contribution:** Study of the spectrum in $L^1(\exp(a v^s) dv)$ for $0 < s < 1$ of the **linearized operator**

$$\mathcal{L}_{(e,\gamma)} h := 2\mathcal{Q}_e(h, F_e) + (1 - e)\Delta_v h - i(\gamma \cdot v) h,$$

where F_e is the unique (*non-explicit*) solution to

$$\mathcal{Q}_e(F_e, F_e) + (1 - e)\Delta_v F_e = 0, \quad \int_{\mathbb{R}^d} F_e(v) (1, v) dv = (1, 0).$$

- **Method used:** A generalization to the non-hilbertian L^1 setting of the classical work of Ellis and Pinsky about the fluid-dynamic limit of the linearized elastic Boltzmann equation.

Localization of the Spectrum

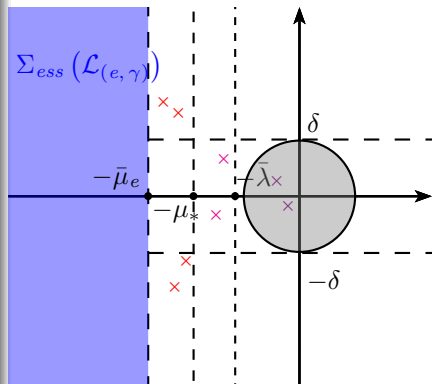
Theorem (TR)

There exists $e_* \in (0, 1)$ such that for $e \in (e_*, 1]$, there exists a constant $\bar{\mu}_e > 0$ such that the essential spectrum verifies

$$\Sigma_{\text{ess}}(\mathcal{L}(e, \gamma)) \subset \{\zeta \in \mathbb{C} : \Re \zeta \leq -\bar{\mu}_e\}.$$

The remaining part of the spectrum is composed of discrete eigenvalues, verifying for $\delta > 0$: there exist some constants $0 < \bar{\lambda} < \mu_* < \mu_e$ and a nonnegative number γ_0 such that for all $|\gamma| \leq \gamma_0$, if $\lambda \in \Sigma_d(\mathcal{L}(e, \gamma))$, then

- $\Re \lambda \geq -\mu_* \Rightarrow |\Im \lambda| \leq \delta$;
- $\Re \lambda \geq -\bar{\lambda}/2 \Rightarrow |\lambda| \leq \delta$.



Behavior of the Eigenvalues for Small Space Frequencies

Goal: finding a triple $(\lambda, \gamma, h) \in \mathbb{R}^d \times \mathbb{C} \times L^1(\exp(a v^s) dv)$ such that

$$(-i(\gamma \cdot v) + \mathcal{L}_e) h = \lambda h. \quad (4)$$

Theorem (TR)

There exist some open sets $U_1 \times U_2 \subset \mathbb{R} \times \mathbb{C}$, neighborhood of $(0, 0)$, and

$$\begin{cases} \lambda^{(j)} : U_1 \times (e_*, 1] \rightarrow U_2 & \forall j \in \{-1, \dots, d\}, \\ h^{(j)} : U_1 \times \mathbb{S}^{d-1} \times (e_*, 1] \rightarrow L^1(m^{-1}) & \forall j \in \{-1, \dots, d\}, \end{cases}$$

C^∞ functions verifying

- the triple $(\rho\omega, \lambda^{(j)}(\rho, e), h^{(j)}(\rho, \omega, e))$ is solution to the eigenvalue problem (4), for all $e \in (e_*, 1]$, $\rho \in U_1$, $\omega \in \mathbb{S}^{d-1}$, $j \in \{-1, \dots, d\}$;

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C^∞ functions verifying

- the eigenvalue $\lambda^{(j)} \in C^\infty(U_1 \times (e_*, 1])$ verifies

$$\begin{cases} \lambda^{(j)}(0, 1) = 0, & \forall j \in \{-1, \dots, d\}, \\ \frac{\partial \lambda^{(j)}}{\partial \rho}(0, 1) = j i \sqrt{1 + \frac{2}{d}}, & \forall j \in \{-1, 0, 1\}, \\ \frac{\partial \lambda^{(j)}}{\partial \rho}(0, 1) = 0, & \forall j \in \{2, \dots, d\}, \\ \frac{\partial \lambda^{(0)}}{\partial e}(0, 1) = -C < 0; \end{cases}$$

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C^∞ functions verifying

- if a triple $(\lambda, \rho\omega, h)$ is solution to the problem (4) for $(\rho, \lambda) \in U_1 \times U_2$, then necessarily $\lambda = \lambda^{(j)}$ for some $j \in \{-1, \dots, d\}$.

Elements of Proof

Localization result → A L^1 perturbative approach:

- Decomposition of the linearized operator as a sum of compact and Schrödinger-like operators;
- Application of a Banach variant of Weyl's Theorem about the stability of the essential spectrum;
- Use of space homogeneous coercivity and spectral gap estimates to obtain the behavior of the eigenvalues.

Taylor expansion → Generalization of Ellis & Pinsky's arguments:

- Reformulation of the eigenvalue problem, using bounded operators;
- Projection of the new problem onto the space of **elastic** collisional invariants;
- Finite dimensional resolution, taking advantage of both elastic and space homogeneous results.

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The Rescaling

This is a joint work with F. Filbet

→ **Goal:** to simulate numerically on a fixed velocity grid

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f), \quad \forall (t, x, v) \in [0; \infty) \times \Omega \times \mathbb{R}^d,$$

where Q preserves mass and momentum.

→ **Problem:** The action of Q on f may induce concentration (granular gases operators) or spreading (diffusive operators) phenomena, difficult to deal with numerically.

→ **Possible solution:** Introduce a new distribution g as

$$f(t, x, v) = \frac{1}{\omega(t, x)^d} g\left(t, x, \frac{v}{\omega(t, x)}\right),$$

with ω an accurate measure of the “support” or **scale** of the distribution $f(t, x, v)$ in velocity variables.

In the Space Homogeneous Setting

Let $f := f(t, v)$ solution to

$$\frac{\partial f}{\partial t} = \frac{1}{\varepsilon} \mathcal{Q}(f)$$

For a given $\omega > 0$, introduce $g := g(t, \xi)$ as

$$f(t, v) = \frac{1}{\omega(t)^d} g\left(t, \frac{v}{\omega(t)}\right)$$

Then, for $\tilde{\mathcal{Q}}(g) := \mathcal{Q}(f)$, g is solution to

$$\frac{\partial g}{\partial t} - \operatorname{div}_{\xi} \left(\frac{1}{\omega} \frac{\partial \omega}{\partial t} \xi g \right) = \frac{\omega^d}{\varepsilon} \tilde{\mathcal{Q}}(g)$$

Some possible choices for ω :

→ Diffusion operator² $\mathcal{Q}(f) = \Delta_v(f) \Rightarrow \omega(t) = (1+t)^{1/2}$;

→ Granular gases operator³ $\mathcal{Q}(f) = \mathcal{Q}_e(f) \Rightarrow \omega(t) = (1+t)^{-1}$.

²Carrillo, Toscani (1998)

³Haff (1983), Mischler-Mouhot (2006)

The Macroscopic Scaling

We set

$$f(t, x, v) = \frac{1}{\omega(t, x)^d} g\left(t, x, \frac{v}{\omega(t, x)}\right), \quad \omega = \sqrt{\frac{2E}{d\rho}},$$

where g is solution to

$$\frac{\partial g}{\partial t} + \operatorname{div}_x (\omega \xi g) - \operatorname{div}_\xi \left[\left(\frac{1}{\omega} \frac{\partial \omega}{\partial t} \xi + \xi \otimes \xi \nabla_x \omega \right) g \right] = \frac{\omega^d}{\varepsilon} \tilde{\mathcal{Q}}(g), \quad (5)$$

and (ρ, \mathbf{u}, E) given by the **independent system**

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \operatorname{div}_x (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) = 0, \\ \partial_t E + \operatorname{div}_x (\mathbf{u} (E + p)) = G(\rho, \mathbf{u}, E), \end{cases}$$

where G will be defined depending on the model.

Application to the Boltzmann Equation

For the Boltzmann operator \mathcal{Q}_B in the variable hard spheres case

$$B(|z|, \cos \theta) = |z|^\lambda b(\cos \theta):$$

→ $\mathcal{Q}_B(f) = \omega^{\lambda-d} \mathcal{Q}_B(g)$, which yields the kinetic equation for g

$$\frac{\partial g}{\partial t} + \operatorname{div}_x (\omega \xi g) - \operatorname{div}_\xi \left[\left(\frac{1}{\omega} \frac{\partial \omega}{\partial t} \xi + \xi \otimes \xi \nabla_x \omega \right) g \right] = \frac{\omega^\lambda}{\varepsilon} \mathcal{Q}_B(g).$$

→ The scaling function ω is given by the solution to the following calibration fluid equation

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \operatorname{div}_x (\rho \mathbf{u} \otimes \mathbf{u} + \rho T \mathbf{I}) = 0, \\ \partial_t E + \operatorname{div}_x (\mathbf{u} (E + \rho T)) = 0. \end{cases}$$

Application to the Granular Gases Equation

For the granular gases operator \mathcal{Q}_e with hard spheres kernel, we have:

→ $\mathcal{Q}_e(f) = \omega^{1-d} \mathcal{Q}_e(g)$, which yields the kinetic equation for g

$$\frac{\partial g}{\partial t} + \operatorname{div}_x (\omega \xi g) - \operatorname{div}_\xi \left[\left(\frac{1}{\omega} \frac{\partial \omega}{\partial t} \xi + \xi \otimes \xi \nabla_x \omega \right) g \right] = \frac{\omega}{\varepsilon} \mathcal{Q}_e(g).$$

→ The scaling function ω is given by the solution⁴ to the following calibration fluid equation

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}_x (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \operatorname{div}_x (\rho (\mathbf{u} \otimes \mathbf{u}) + \rho T \mathbf{I}) = 0, \\ \partial_t E + \operatorname{div}_x (\mathbf{u} (E + \rho T)) = -K_d \rho^2 T^{3/2}. \end{array} \right.$$

⁴This system can be computed e.g. using a **weak inelasticity** assumption

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Numerical schemes

Kinetic part:

$$\frac{\partial g}{\partial t} + \operatorname{div}_x (\omega \xi g) - \operatorname{div}_\xi \left[\left(\frac{1}{\omega} \frac{\partial \omega}{\partial t} \xi + \xi \otimes \xi \nabla_x \omega \right) g \right] = \frac{\omega^d}{\varepsilon} \tilde{Q}(g)$$

- Collision term Q : spectral scheme⁵ for Q_B and Q_e ;
- Free transport term $\operatorname{div}_x (\omega \xi g)$: finite volume Lax-Friedrichs method with Van Leer's flux limiter⁶;
- Drift term $\operatorname{div}_\xi (\dots)$: related to the treatment of the macroscopic system;

Macroscopic part:

$$\frac{\partial U}{\partial t} + \nabla_x \cdot F(U) = G(U), \quad U \in \mathbb{R}^{d+2} \left(\Rightarrow \omega = \sqrt{2 U_{d+1}/d U_0} \right)$$

- Reconstruction of the fluxes with a WENO-5 procedure⁷.

⁵Pareschi-Perthame (1998), Pareschi-Russo (2000), Filbet-Pareschi-Toscani (2005)

⁶Van Leer (1977)

⁷Shu (1999)

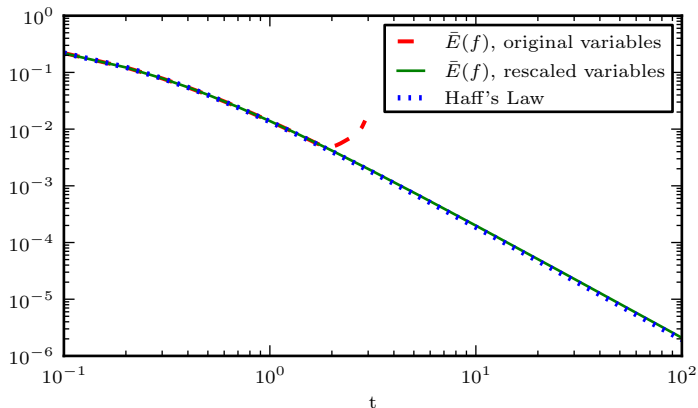
Convergence toward a Dirac Mass

Solution to the space homogeneous granular gases equation:

Classical (dots, $N_v = 128$) vs. rescaled variables (solid line, $N_v = 32$)

Large Time Behavior of the Granular Gases Equation

Cooling process: Evolution of the global kinetic energy ($d_x = d_v = 1$) in classical and rescaled variables

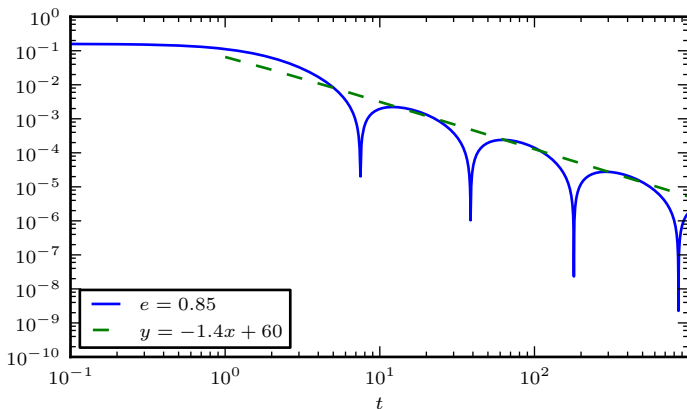


Trend to Equilibrium (Granular Gases Equation)

Evolution of $\|\rho_f(t) - 1\|_{L^1}$ (Periodic boundary conditions)

$$f_0(x, v) = \mathcal{M}_{\rho_0(x), 0, 1}(v), \quad \forall (x, v) \in [0, L] \times \mathbb{R},$$

with $\rho_0(x) = 1 + 0.1 \cos(\pi x)$, $L = 1$ and $\varepsilon = 1 - e = 0.01$

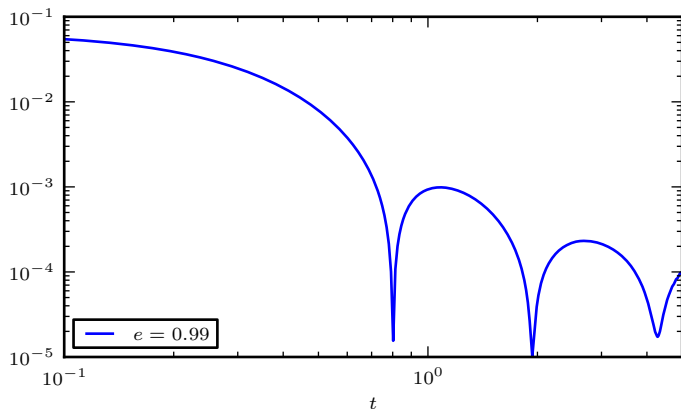


Trend to Equilibrium (Granular Gases Equation)

Evolution of $\|\rho_f(t) - 1\|_{L^1}$ (Specular boundary conditions)

$$f_0(x, v) = \mathcal{M}_{\rho_0(x), 0, 1}(v), \quad \forall (x, v) \in [0, L] \times \mathbb{R},$$

with $\rho_0(x) = 1 + 0.1 \cos(\pi x)$, $L = 1$ and $\varepsilon = 1 - e = 0.15$



Trend to Equilibrium (Granular Gases Equation)

Let us consider the scaling

$$\begin{cases} f(t, x, v) = V(t)^d h(s(t), x, V(t)v), \\ V(t) = 1 + (1 - e)t, \quad s(t) = \log(1 + (1 - e)t)/(1 - e). \end{cases}$$

The scaled distribution h is then solution to

$$\frac{\partial h}{\partial s} + \nu \cdot \nabla_x h + (1 - e) \nabla_\nu(\nu h) = \frac{Q_e(h, h)}{\varepsilon}$$

Trend to Equilibrium (Granular Gases Equation)

Let us consider the scaling

$$\begin{cases} f(t, x, v) = V(t)^d h(s(t), x, V(t)v), \\ V(t) = 1 + (1 - e)t, \quad s(t) = \log(1 + (1 - e)t)/(1 - e). \end{cases}$$

The scaled distribution h is then solution to

$$\frac{\partial h}{\partial s} + \nu \cdot \nabla_x h + (1 - e) \nabla_\nu(\nu h) = \frac{Q_e(h, h)}{\varepsilon}$$

When $t \rightarrow \infty$, the behavior of h is formally the one of $e^{\lambda_j t}$ with

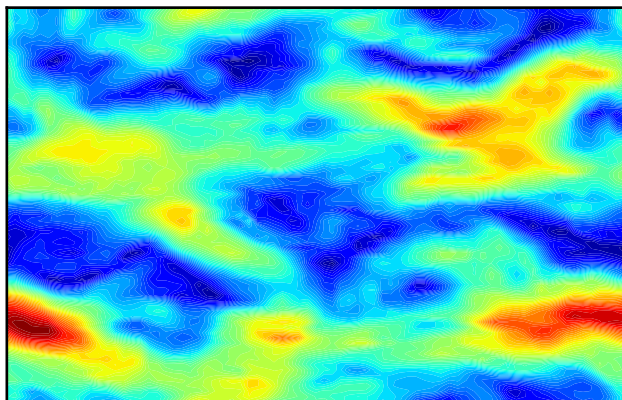
$$\lambda_j = i\eta \lambda_j^{(1)} + \eta^2 \lambda_j^{(2)} - (1 - e)C + \mathcal{O}(\eta^3) + \mathcal{O}((1 - e)^2).$$

For $\eta = 2\pi/L \Rightarrow \beta(1 - e)$, where β is the damping rate, will thus be constant:

Restitution coefficient e	Damping rate β	$-(1 - e)\beta$
0.8	-1.05	0.21
0.85	-1.4	0.21
0.9	-2.05	0.205
0.95	-4.1	0.205

Clustering in a Force Free Granular Gas

$$N_x = N_y = 75, N_{v_x} = N_{v_y} = 32, e = 0.8$$



Contour plot of the local Mach number $M(x) := |\mathbf{u}(\mathbf{x})|/C_s(\mathbf{x})$ where C_s is the local speed of sound.

Work in progress

- Expansion up to the second order in γ of the spectrum of the linearized granular gases operator \mathcal{Q}_e and application to the true non-linear problem;
- Development of spectral methods for the non cut-off Boltzmann equation (with L. Pareschi);
- Development of hybrid hydrodynamic/kinetic solvers (with F. Filbet);
- Application of the rescaling method to more complex problems (formation of planetary rings).