Proper and weak-proper trees in edges-colored graphs and multigraphs

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Résumé

Dans la présente thèse nous étudions l'extraction d'arbres dans des graphes arêtes-coloriés. Nous nous concentrons sur la recherche d'arbres couvrants proprement arête-coloriés et faiblement arête-coloriés, notée PST et WST.

- Nous montrons que les versions d'optimisation de ces problèmes sont NP-Complete dans le cas général des graphes arêtes-coloriés, et nous proposons des algorithmes pour trouver ces arbres dans le cas des graphes arêtes-coloriés sans cycles proprement arêtes-coloriés. Nous donnons également quelques limites de nonapproximabilité.
- Nous proposons des conditions suffisantes pour l'existence de la PST dans des graphes arêtes-coloriés (pas forcément propre), en fonction de différents paramètres de graphes, tels que : nombre total de couleurs, la connectivité et le nombre d'arêtes incidentes de différentes couleurs pour un sommet.
- Nous nous intéressons aux chemins hamiltoniens proprement arêtes-coloriés dans le cas des multigraphes arêtes-coloriés. Ils présentent de l'intérêt pour notre étude, car ce sont également des arbres couvrants proprement arêtes-coloriés. Nous établissons des conditions suffisantes pour qu'un multigraphe contienne un chemin hamiltonien proprement arêtes-coloriés, en fonction de plusieurs paramètres tels que le nombre d'arêtes, le degré d'arêtes, etc.
- Puisque l'une des conditions suffisantes pour l'existence des arbres couvrants proprement arêtes-coloriés est la connectivité, nous prouvons plusieurs bornes supérieures pour le plus petit nombre de couleurs nécessaires pour la k-connectivité-propre. Nous énonons plusieurs conjectures pour les graphes généraux et bipartis, et on arrive á les prouver pour k = 1.

Mots clés : graphes arêtes-coloriés, algorithmes, chemin hamiltonien, arbres proprement arêtes-coloriées, arbres faiblement arêtes-coloriées, nonapproximabilité, coloration

Abstract

In this thesis, we investigate the extraction of trees from edge-colored graphs. We focus on finding trees with properties based on coloring. Namely, we deal with proper and weak proper spanning trees, denoted PST and WST.

- We show the optimization versions of these problems to be NP-hard in the general case of edge-colored graphs and we provide algorithms to find these trees in the case of edge-colored graphs without properly edge-colored cycles. We also provide some nonapproximability bounds.
- We investigate the existence of a PST in the case of edge-colored graphs under certain conditions on the graph, both structural and related to the coloration. We consider sufficient conditions that guarantee the existence of a PST in edge-colored (not necessarily proper) graphs with any number of colors. The conditions we consider are combinations of various parameters such as : total number of colors, number of vertices, connectivity and the number of incident edges of different colors to the vertices.
- We then consider properly edge-colored Hamiltonian paths in the edge-colored multigraphs, which are relevant to our study since they are also PST. We establish sufficient conditions for a multigraph to contain a proper edge-colored Hamiltonian path, depending on several parameters such as the number of edges, the degree of edges, etc.
- Since one of the sufficient conditions for the existence of proper spanning trees is connectivity, we prove several upper bounds for the smallest number of colors needed to color a graph such that it is k-proper-connected. We state several conjectures for general and bipartite graphs, and we prove them for k = 1.

Keywords : Edge-colored graphs, algorithms, Hamiltonian paths, proper tree, weak tree, nonapproximability, coloring

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Chapitre 1

Introduction

Graph Theory is an important research area in mathematics and computer science, studied for over two hundred years. Graphs are one of the prime objects of study in Discrete Mathematics, these mathematical structures model pairwise relations between objects. Thus, a graph is a set of nodes (or "vertices") and a set of edges, connecting pairs of these nodes. Graphs are among the most universal structures for modeling various relations and processes in scientific domains such as physics, chemistry, biology, economics, social systems, as well as in different areas of Computer Science such as databases, networks, etc.

Furthermore, many problems of practical interest can be represented by graphs. Consider the example of a city map presented in Figure 1. Imagine a pedestrian interested in finding the fastest way to go from point A to point B. How does he proceed ? He looks at the map and chooses the streets he should take in order to reach the desired destination. The same situation applies for a person who drives a car, however, the latter should take into account that some streets can be taken in a uni-directional way only.
Clearly, in this example the city map can be represented as a graph, the intersections of the streets being the nodes and the streets being the oriented edges between these nodes. Each oriented edge represents a one-way street. This kind of graphs are called *directed graphs*. And the problem of finding the best itinerary for reaching the point B starting from the point A can be modeled as a problem of finding the shortest path connecting the nodes A and B in the *directed graph*.

In general, graph theory is divided in two branches : directed graphs and undirected (simple) graphs. Undirected graphs were studied extensively, then the directed graphs came into focus due to their theoretical and practical importance. Note, that undirected graphs can be seen as a subclass of directed graphs, with each undirected edge represented by two oppositely-directed edges.
Nevertheless, even these two classes of graphs are not powerful enough to model all existing real-life constraints. For this reason new concepts in graph definitions are required. Going back to our above example, if one wants to differentiate highways from national roads, and main from secondary roads, a good idea would be to use colors. When the edges of graphs are colored,

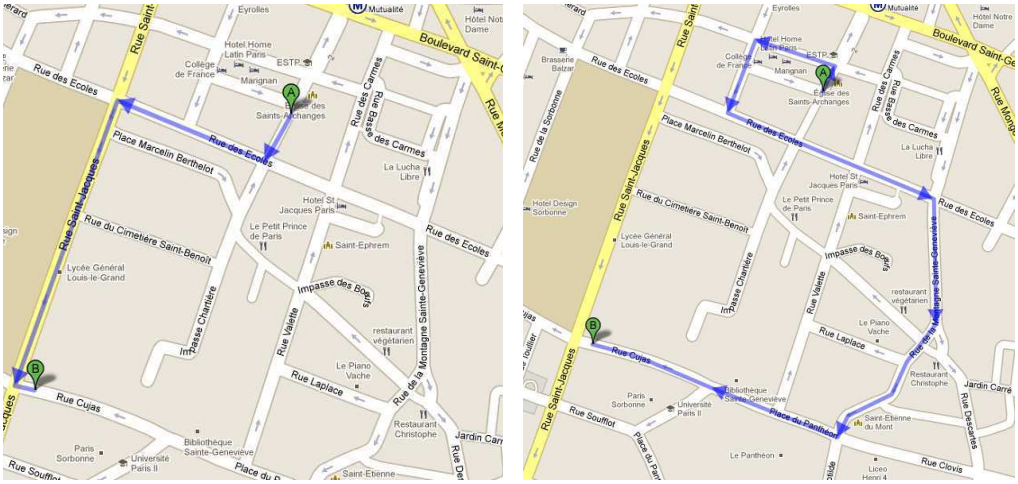

FIGURE 1.1 – The itinerary for a pedestrian and for a car

we talk about $edge-colored$ graphs, which generalize the directed graphs. Such generalization allows the formulation of a wide range of challenging problems, in addition to those existing in the case of usual graphs. Namely, suppose that we would also like to find an itinerary that uses highways exclusively. This problem corresponds to finding a monochromatic path of a given color between two given nodes.
Thus, one can observe that problems related to $edge-colored$ graphs often consist in finding (extracting) classical structures (subgraphs) such as paths, cycles and trees, with, in addition, certain constraints or properties imposed on colors.
Trivial properties of the trees that can be found in edge-colored graphs, are *monochromaticity* and *heterochromaticity*. The former consists in having all the edges colored in same color, the latter requires distinct colored edges. $Heterochromatic$ structures are also called *rainbow*, *multicolored*, $colorful$ and *polychromatic* by different authors. Results on subgraphs and trees with these properties can be found in [61, 21, 44, 66, 5, 26, 39], as well as a recent survey on heterochromatic (rainbow colored) graphs in [50]. These results are of interest for us, since rainbow colored graphs are also *proper* - a property introduced in the sequel.

The constraint that we focus on in this thesis, is that of *properness*. The notion of *properness* consists in having no two adjacent edges of the same color in the structure we want to find. In the literature, algorithms are proposed to check the existence of properly edge-colored subgraphs of a given type in an edge-colored graph. Corresponding results are presented in [14]. Subgraphs with this property are also studied in [4, 63, 16, 56, 55, 43, 8, 10]. In particular, Y. Manoussakis et al. [54] investigated the existence, counting and enumeration of properly colored pairwise internally disjoint paths with common endpoints. They also studied properly colored subgraphs including Hamiltonian paths and cycles, cycles with a given lower bound on their

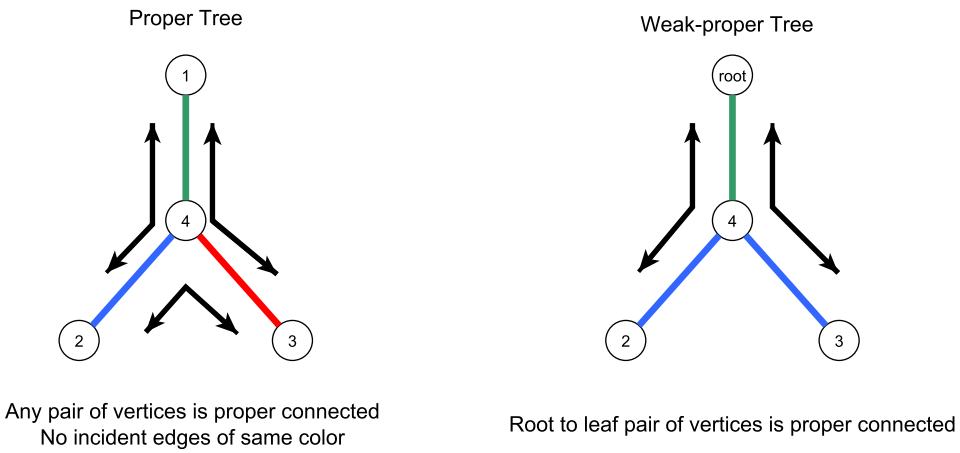

FIGURE 1.2 – Proper Tree and Weak-proper Tree

length, spanning trees, stars, and *cliques* in $edge-colored$ complete graphs.

In the present thesis we investigate the extraction of trees from edge-colored graphs. A tree is a connected acyclic subgraph. In graph theory there exist many important problems related to trees with different levels of difficulty. The difficulty depends on the properties of the tree we are trying to find. The properties can be related to the desired degree of each vertex, which is the number of edges incident to a vertex. For example, for a simple graph, it is difficult to find a bounded degree spanning tree [1] (see [40, 65] for more details). It is also difficult to find a spanning tree maximizing the number of vertices of degree one (which are also called leafs).
Since we are interested in $edge-colored$ graphs, we focus on finding trees with properties based on coloring. Namely, we deal with *proper* and *weak proper* spanning trees (see figure 1.2), denoted PST and WST. A *proper tree* is a proper subgraph, whose uncolored version is acyclic. A *proper spanning tree* is a spanning proper subgraph, whose uncolored version is connected and acyclic. A *weak proper* tree is *rooted*, the $root-to-leaf$ paths are *proper* and it's uncolored version is acyclic.

Let us see an example of application of *properly colored* structures in edge-colored graphs. Consider some points that send and receive information. Different frequencies are used for receiving and for sending.
Suppose now that certain points cannot send information directly to other points since there exists some natural barrier. Let us represent these points as graph vertices and the possibility to send and receive data between any two points as graph edges. Let us define a bijective function for the edges coloring : one color per frequency. Each vertex uses different frequencies for receiving and for sending operations, thus the corresponding edges have different colors. Various

1. A subgraph that spans all vertices of a graph is said to be a spanning subgraph, and a spanning tree is a subgraph in which each pair of vertices is connected by exactly one path.

problems emerge in this context :

(1) Suppose we would like to test if one point can send information to another one. This problem corresponds to finding a proper colored path between two vertices in the edge-colored graph.

(2) Suppose we would also like to know if two given vertices can communicate. To discover that, it is sufficient to find a proper colored cycle or a closed trail.

(3) If we would like to determine how many vertices can communicate in this network simultaneously, then we have to find a proper colored subgraph.
If we want to broadcast the information to a maximum number of vertices from a given vertex, we have to find a maximum weak-proper tree for a given vertex as a root, since each point can receive on one frequency and send on all the others.

(4) Another possible problem is to find a maximum number of vertices that will always receive the broadcasted information and any of these vertices can serve as broadcasting point. In this case the subgraph to find corresponds to a maximum proper tree.

In addition, application areas of finding properly-colored Hamiltonian paths and cycles in edge-colored graphs include genetics [31, 60], social sciences [28] and VLSI optimization [67].

After studying PST and WST in edge-colored graphs, we consider a similar problem in the case of edge-colored *multigraphs*. If we allowed more than one edges (but yet a finite number) between the same pair of vertices in a graph, the resulting structure is called a *multigraph*. Such edges are called parallel edges. If more than one edge is present between two vertices, then the edges can be of different colors or of the same color. In our case, we assume that parallel edges are of different colors. Clearly, multigraphs offer new kind of restrictions and possibilities. It is also clear that real life problems can be modeled more faithfully using such structures.
The class of edge-colored multigraphs generalizes that of directed graphs. Several other generalizations of directed graphs exist, namely : edge-colored digraphs, hypertournaments and star hypergraphs. However, it is the class of edge-colored multigraphs that has been given the main attention in graph theory. This because many concepts and results on directed graphs can be extended to edge-colored multigraphs and also because edge-colored multigraphs have several important applications. Significant results were obtained concerning these graphs [11, 10]. In this work we considered proper and weak proper spanning trees in edge-colored multigraphs.

This thesis is organized as follows. In **chapter two**, we give basic terminology and notations, as well as some notions of complexity, that are needed for the understanding of the results exposed in the next chapters.

In **chapter three**, we present an overview of the current state of the art of proper spanning

trees (PST) and weak proper spanning trees (WST) problems in edge-colored graphs.

In **chapter four**, we consider maximum proper edge-colored trees and maximum weak proper trees in edge-colored graphs. We consider these problems from the graph theoretic as well as from the algorithmic viewpoints. We prove their optimization versions to be NP-hard in general and provide algorithms for finding such trees in graphs without properly edge-colored cycles. We also derive some nonapproximability bounds.

In **chapter five**, we study the existence of PST in edge-colored graphs under certain assumptions on the graph, both structural and related to the coloring. We consider sufficient conditions that guarantee the existence of proper spanning trees in graphs with edges colored (not necessarily properly) with any number of colors. The conditions we consider are the combinations of various parameters of colored graphs such as : total number of colors, number of vertices, connectivity and number of differently colored edges incident to a vertex. These particular conditions are of interest since there exist no good characterizations for the existence of such colored trees in general edge-colored graphs.

In the **chapter six** we consider proper Hamiltonian paths in edge-colored multigraphs. A proper Hamiltonian path is a path containing all the vertices of the multigraph such that no two adjacent edges have the same color. Since proper Hamiltonian paths are also proper spanning trees, they are of interest for our study. We establish sufficient conditions for a multigraph to have a proper Hamiltonian path, depending on several parameters such as the number of edges, the rainbow degree, etc.

In **chapter seven**, we focus on a problem related to color connectivity. Since one of the sufficient conditions for proper spanning trees to exist is the connectivity, we prove several upper bounds for the smallest number of colors needed for k-proper-connectivity. We state several conjectures for general and bipartite graphs, and prove them for $k = 1$. Then, some algorithmic results are proposed.

Finally, in **chapter eight**, we present the conclusion of our work and some open problems for future research.

Chapitre 2

Notation and Terminology

Contents

In this chapter we introduce the main notions, necessary to the understanding of this thesis. We begin by the terminology and recall the common notions of the graph theory in general and of edge-colored graphs in particular. Then we introduce the notions of complexity and approximation.

2.1 Notation and terminology

We use the notation and terminology given by Jean-Claude Fournier in [37]. However, the reader is warned that there may arise some differences.

(Undirected) graph

An *(undirected) graph* $G = (V, E)$ is defined by two finite sets : a non-empty set V of elements called *vertices*, a set E (which may be empty) of elements called *edges*. Each edge e has two associated vertices, x and y, (not necessarily distinct), called its *endpoints*. The graphs considered in this thesis are finite. The vertex set V and the edge set E of a graph G can be denoted by $V(G)$ and $E(G)$, respectively. The number of vertices of a graph, is usually denoted by n. The number of edges of a graph, is denoted by m. An edge e is simply denoted by xy (or yx) instead of the customary mathematical notation $\{x, y\}$ when $x \neq y$.

Adjacent, incident, neighbors, loop, parallel edges

The endpoints x and y of an edge e are *adjacent*. An edge e is said to be *incident* to each of its endpoints. If the both the endpoints on an edge are same vertex then this edge is a *loop*. All adjacent vertices of a vertex x are the *neighbors* of this vertex. Edges e and e' is said to be *parallel* or *multiple edges* if they have the same endpoints. A *simple* graph is a graph without loops or multiple edges.

Independent set

An *independent set* or *stable set* is a set of vertices in a graph, no two of which are adjacent.

Underlying simple graph

We call *underlying simple graph*, of a non-simple graph G, the graph defined as follows : it has the same set of vertices as G and two vertices are joined by an edge if and only if they are different and joined by at least one edge in G.

Isomorphic

We say that a graph G=(V, E) is *isomorphic* to an graph H=(S, F) if there exists a bijection : $\phi : V \to S$ such that $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in F$.

Complete graph

A simple graph G where each two vertices are adjacent is called a *complete graph*. A complete graph is determined by the number n of its vertices and it is generally denoted by K_n. The edge set of a complete graph is equal to $\binom{n}{2} = \frac{n(n-1)}{2}$.

Subgraph

A *subgraph* $H = (S, F)$ of a graph $G = (V, E)$ is the graph where $S \subseteq V$ and $F \subseteq E$ and where each edge $e \in F$ has both its endpoints in S.

Vertex–induced subgraph

A *vertex-induced subgraph* H is *induced* by a set of vertices $S \subseteq V$, if it is a graph of the form $H = (S, F)$, where F is the set of the edges of E both of whose endpoints are in S.

Edge-induced subgraph

An *edge-induced subgraph* is *induced* by a subset of the edges $F \subseteq E$ of a graph $G = (V, E)$, if it is a graph of the form $H = (S, F)$, where S is the set of vertices that are their endpoints.

Spanning subgraph A subgraph $H = (S, F)$ of $G = (V, E)$ is called a *spanning subgraph* if $S = V$. It can be a spanning subgraph of set of edges F too.

$G - x$ Let $G = (V, E)$ then $G - S$, where $S \subset V$, is a subgraph of G induced by $V \setminus S$. In other words, it is a subgraph of G obtained by removing the vertex set S from G and the incident edges. Let $G = (V, E)$ then $G - F$, where $F \subset E$, is the subgraph of G induced by $E \setminus F$. In other words, it is the induced subgraph of G obtained by removing the edge set F from G. In this thesis we use $G - x$ instead of $G - \{x\}$ for $x \in S$ and $G - e$ instead of $G - e$ for $e \in F$.

Walk A *walk* of a graph $G = (V, E)$ is a sequence of the form :

$$(x_0, e_1, x_1, \ldots, e_k, x_k)$$

where $k \geq 0$, x_i are vertices of G, such that there exists an edge $e_{i+1} \in E$ for each $x_i x_{i+1}$, and x_0 and x_k are called the endpoints of the walk. We say that x_0 and x_k are *linked* by the walk.

Length The *length* of the walk in this case is k. A walk may have zero length, it consist of one vertex in this case. In the case when G is a simple graph, we define a walk by the sequence $(x_0, x_1, \cdots, x_k)$ of its vertices. A *subwalk* of a walk is a walk defined as a subsequence between two vertices of the initial walk.

Trail A *trail* is a walk where all the edges are distinct. In other words, when the walk *does not go twice* through the same edge.

Path A *path* is a walk where its vertices are all distinct.

Cycle A *cycle* is a closed path, ie $x_0 = x_k$. A cycle cannot have zero length. A loop is a cycle of length 1. In the case of simple graphs, a cycle may be defined by the sequence $(x_0, x_1, \cdots, x_0)$ of its vertices. A cycle is called *even* or *odd*, depending on whether its length is even or odd.

Degree The *degree* of a vertex x in a graph G is the number of edges in G incident to x. The degree is denoted by $d(x)$.

Isolated vertex A vertex of degree zero is called an *isolated* vertex.

Minimum degree δ The *minimum degree* of a graph G is the smallest degree of its vertices. The minimum degree is denoted by δ or $\delta(G)$. It is clear that any graph G, has at least one vertex with its degree equal to the minimum degree.

Maximum degree $\triangle$ — Similarly the *maximum degree* of G is the largest degree of its vertices. The *maximum degree* of G is denoted by $\triangle$ or $\triangle(G)$. Any graph G has at least one vertex with its degree equal to the maximum degree.

Regular graph – k-regular graph — A graph G is called *regular* if the degrees of its vertices are all equal. In this case, the graph is called a *k-regular graph*, if the common vertex degree is k.

Connected - disconnected graph — If every pair of vertices of a graph G are linked by a path in G, then we say that G is *connected*. If it is not the case, then we say that G is a *disconnected graph*.

Connected component — The maximal connected subgraphs of G are called its *components* or *connected components*. A disconnected graph has at least two connected components. Similary, a connected graph has only one connected component. Observe that the connected components of a graph are subgraphs with no common vertices and no common edges. It defines the decomposition of the graph into connected components. Observe that if a graph has a spanning connected subgraph, then it is itself connected.

Bipartite graph — A *bipartite* graph G, is any graph whose vertex set can be partitioned into two disjoint subsets, such that each edge has an endpoint in each subset. A bipartite graph is denoted by $G = (V, S, E)$, where V and S are the two subsets of vertices, and E is the set of edges. Is evident that $V \cup S$ is the set of vertices of G.

Complete bipartite graph — A *complete bipartite graph* is the bipartite graph $G = (V, S, E)$ where the set of its edges is $E = \{xy | x \in V, y \in S\}$, i.e. for each pair of a vertices one each from V and S is an edge in $E(G)$. We denote it by $K_{p,q}$, where p is the cardinality of V and q the cardinality of S.

Tree — A *tree* is a connected acyclic graph. Here, acyclic means a graph without a cycle. It follows that a tree is a simple graph. A path is a special type of tree.

Leaf — Any vertex in a tree of degree 1 is called a *leaf*.

Root, rooted tree — When we distinguish a special vertex in a tree and call it a root, we get a *rooted tree*.

Forest — A *forest* is an acyclic graph. The connected components of a forest are trees. Forests generalize trees.

Bridge	If a graph G has an edge e such that $G - e$ has one more connected component than G, then this edge e is called a *bridge* of G. If x and y are endpoints of a bridge e then in $G - e$, x and y are no longer linked by a path. In fact the edge e *separates* the vertices x and y. If G is connected then after removing e, $G - e$ is disconnected.
Spanning tree	A *spanning tree* of a graph G is a spanning subgraph of G which is a tree
Cut vertex	A *cut vertex* of a graph G is a vertex x such that $G - x$ has at least one more connected component than G. This idea leads to a classic and useful decomposition of graphs.
Block	A *block* of a graph G is a maximal connected subgraph without a cut vertex of G.
Block decomposition	The set of blocks of G constitutes the *block decomposition* of G.
Connectivity, $\kappa(G)$	The smallest number of vertices whose removal makes G disconnected or reduces G to one vertex is called *connectivity* of G. $\kappa(G)$ denotes the connectivity of G.
$k -$ *connected*	A graph G is $k - connected$ if $\kappa(G) \geq k$.
Edge connectivity	The *edge connectivity* $k'(G)$ of a graph G, with more than one vertex, is the smallest number of edges whose removal disconnects the graph. $\kappa'(G)$ denotes the *edge connectivity* of G.
(*Edge*) *cut*	Let B be a set of edges whose removal disconnects G, then the set of edges B is what we call an (*edge*) *cut* of G.
$k - edge -$ *connected*	A graph G is $k - edge - connected$ if $\kappa'(G) \geq k$.
Matching	A *matching* of a graph G is a set M of edges such that no two edges share a common endpoint.
Matched or saturated vertex	A vertex of the graph is said to be *matched* (*saturated*) by a matching M (or $M - saturated$), if it is an endpoint of an edge of M. Otherwise it is said to be *unmatched* (*unsaturated*) by M (or $M - unsaturated$).
Perfect matching	If every vertex of G is matched, then matching M is said to be $perfect$.

Maximal matching — A matching is said to be *maximal* if it is impossible to add an edge to it.

Maximum matching — A matching M is said to be *maximum* if it has the greatest possible number of edges. It is a matching with the greatest number of edges among all the possible maximal matchings.

Compatible (or alternating) path relative to a matching — For a given graph $G = (V, E)$ and a matching M of G, we call a path P, *compatible* (or *alternating*) relative to M, or an $M - compatible$ ($M - alternating$), if it is a path alternating between edges in M and $E \setminus M$.

Augmenting path for M — An M-alternating path is said to be an *augmenting path for M*, or an *M-augmenting path*, if its endvertices are unmatched.

Factor — A *factor* of a graph G is a spanning subgraph, i.e., a subgraph that has the same vertex set as G.

$k - factor$ — A $k - regular$ spanning subgraph is called a $k - factor$. Observe that $1 - factor$ is a graph H on V vertices and the edge set corespond to a perfect matching.

k-factorization — A *k-factorization* partitions the edges of the graph into disjoint k-factors.

Euler trail, tour, Eulerian graph — A *Euler trail* of a graph G is a trail containing all the edges of G. Is the trail that goes exactly once through each edge of G. A *Euler tour* of G is a closed Euler trail. We call a graph *Eulerian graph* if it has an Euler tour.

Hamilton cycle, Hamiltonian graph — A *Hamilton cycle* of a graph G is a cycle going through all the vertices of the graph. A graph G is called *Hamiltonian graph* if it has a Hamilton cycle.

Hamilton path — A *Hamilton path* of a graph G is a path going through all the vertices of a graph.

Complement of a graph — Let $G = (V, E)$, then we call a complement of a graph and denote them $\bar{G}$ the graph on V vertices and the edge set $[V]^2 \setminus E$. In other terms let $G = (V, E)$ and let F be the set of edges in K_V, then the complement of G is $\bar{G} = (V, F \setminus E)$.

Edge coloring An *edge coloring* of a graph $G = (V, E)$ is a map $c : E \to S$ with $c(e) \neq c(f)$ of any adjacent edges e and f. Let k be the minimum number of colors used for colorings G, called $k - edge - colorings$ of G. Then k is called the cromatic index or edge cromatic number, and its denoted by $\chi'(G)$.

2-edge-colored A graph is *2-edge colored* if each edge is colored either red or blue.

Proper or properly edge-colored An edge-colored graph is *properly edge-colored (or proper)* if any two adjacent edges are colored differently.

Proper path A path in an edge-colored graph is said to be *properly edge-colored* (or *proper*), if every two adjacent edges differ in color.

k-proper connected An edge-colored graph G is *k-proper connected* if any two vertices are connected by k internally pairwise vertex-disjoint proper paths.

k-proper connected, $pc_k(G)$ We define the *k-proper connection number* of a k-connected graph G, denoted by $pc_k(G)$, as the smallest number of colors that are needed in order to make G k-proper connected.

$X_c\{1, 2, \cdots, c\}$ We denote by $X_c\{1, 2, \cdots, c\}$ a set of given $c \geq 2$ colors.

G^c By G^c is denoted an edge-colored graph, so that each edge is colored with some color $i \in X_c$ and no two parallel edges joining the same pair of vertices have the same color.

$V(G^c), E(G^c)$ The vertex and edge-sets of G^c are denoted by $V(G^c)$ and $E(G^c)$, respectively. When no confusion arises, is used V, E instead of $V(G^c), E(G^c)$ respectively.

The order of G^c *The order of G^c* is the number n of its vertices.

$E^i(G^c)$ For a given color i, $E^i(G^c)$ denotes the set of edges of G^c on color i. When no confusion arises, E^i is used instead of $E^i(G^c)$.

K_n^c A edge-colored complete graphs, is denoted by K_n^c.

$N^i(x)$ By $N^i(x)$ We denote the neighbors of a vertex x in some color $i \in X_c$, ie the vertices adjacent to x with edges colored in color $i \in X_c$.

$N(x)$ The *neighborhood* of a vertex x, is denoted by $N(x)$. The neighborhood of a vertex x is defined as the union $N(x) = \cup_i N^i(x)$.

$N_H(x)$	The *neighborhood* of a vertex x in the subgraph H in some color $i \in X_c$, is denoted by $N_H^i(x)$, ie the vertices in H adjacent to x with edges colored in color $i \in X_c$.
$N_H^i(x)$	The *neighbors* of a vertex x in the subgraph H, is denoted by $N(x)$. The neighborhood of a vertex x is defined as the union $N(x) = \cup_i N^i(x)$.
$i-degree$, *color degree*, $d^i(x)$	The colored *i–degree* of x, denoted by $d^i(x)$ equals $\lvert N^i(x)\rvert$, i.e., the cardinality of $N^i(x)$.
xy, $c(xy)$	The edge between the vertices x and y is denoted by xy, and its color by $c(xy)$.
Rainbow degree, $rd(x)$	The *rainbow degree* of x, denoted by $rd(x)$ equals the maximum number of distinct colors present on edges with endvertex x.
$rd_H(x)$	The *rainbow degree* of x in the subgraph H, denoted by $rd_H(x)$, is equals to the maximum number of distinct colors presented on edges with endvertices x and the vertices in H.
Rainbow degree of a graph, $rd(G^c)$	The *rainbow degree of a graph* G^c, denoted by $rd(G^c)$ equals the minimum $rd(x), \forall x \in V(G^c)$.
H_{rs}	Let G^c be an *edge-colored* graph. The *rainbow spanning subgraph* of G^c, denoted by H_{rs}, is the subgraph that spann all vertices of G^c and all edges of H are colored by distinct colors. The $E(H) \subseteq E(G^c)$ and $X_H \subseteq X_{G^c}$.
Proper cycle	A *proper cycle* in G^c is a proper subgraph whose underlying non-colored graph is a cycle.
Acyclic edge-colored graph	An edge-colored graph is said to be *acyclic* if it does not contain proper cycles. Observe that in this case the underlying non-colored graph can contain cycles, but any properly colored subgraph is acyclic.
PT, PST, MPT	A *proper edge-colored tree* or simply a *proper tree*, denoted by PT, is a proper edge-colored subgraph whose underlying non-colored version is a tree. If the PT spans all the vertices of a G^c then this is a *proper edge-colored spanning tree* or simply a *proper spanning tree*, denoted by PST. For *maximum proper edge-colored tree* we use *maximum proper tree* as terminology, and denote it MPT.

WT, WST, MWT	A *weak edge-colored tree* or simply a *weak tree*, denoted WT, is an edge-colored subgraph whose underlying non-colred version is a rooted tree, with the property that all $root-to-leaf$ paths are properly edge colored. If the WT spans all vertices of a G^c then this is a *weak edge-colored spanning tree* or simply a *weak spanning tree* denoted WST. For *maximum weak edge-colored tree* we use *maximum weak tree* as terminology, and denote it MWT.
Heterochromatic, rainbow subgraph	A *heterochromatic subgraph* of a edge-colored graph G^c is a subgraph whose edges are colored with distinct colors, i.e. the color set $\|\chi_c\|$ is equal to $\|V(G^c)\| - 1$.
k-path-cycle subgraph	A *k-path-cycle subgraph* of a graph G^c is a subgraph of G consisting a disjoint collection of k paths and some cycles.

2.2 Notions and definition of Complexity

In this section we introduce the notion of complexity and some of the complexity classes, necessary for the understanding of the results of this thesis. The analysis of complexity of a problem allows to classify it along with other already identified problems, and constitutes one of the steps to its resolution. Thus, all computational problems are separated into complexity classes. We focus on two main classes : P and NP, these acronyms mean *polynomial* class and $non-deterministic\ polynomial$ class, respectively. The non-determinism represents our incapacity (which may be temporary) to find directly, that is without the help of a certificate, the right answer, in polynomial time.
After briefly presenting the concept of computational complexity, we introduce these two classes, P and NP, as well as the notion of optimization problem [1].

2.2.1 The notion of Complexity

The concept of time complexity corresponds to the time required to run a program. The running time depends on several factors, such as the size of the data to be processed, the data encoding, the machine that executes the program and others. It is clear that processing a graph of 10 vertices and 1000 will not take the same time. So, the size of the data to be processed is an important factor that directly influences the running time. In fact we can decide if the running time is satisfying or not in terms of the data that the program processes, by using the notion of a *complexity function*.

When processing the data, a program executes some tasks. These tasks take some processing time. In order to measure the running time, we use the number of “elementary” operations

1. We use the definitions and the examples from the book of Fournier [36]

performed by the program. The set of tasks taken together is called an algorithm. A complete formalization of the notion of an algorithm requires a model capable of describing an automatic process. Alan Turing in 1937, described the "Turing" machine or, as he called it, an "a(utomatic)-machine". The Turing machine represents a computing machine, reduced to its simplest expression. Each task can be done in some "elementary" operations on the "Turing" machine. The model of this machine is used by the complexity theory in order to formalize the concept of the algorithm. However, theoretical results on the complexity are independent of the model of the machine.
The efficiency of the algorithm that solves a given problem on the data of size n is measured by a function, representing the number of elementary operations it performs in order to process the data of this size. This function is called the run-time *complexity*[2]. This function is obtained by calculating the upper bound for the number of operations, necessary to execute the algorithm on the data of any possible size. In this case we talk about the *worst case complexity*. We are only interested in the order of magnitude of this function.
Therefore, the complexity is a good measure for algorithm's efficiency and is widely employed for categorizing the problems depending on their theoretical difficulty.

2.2.2 Class P

After this brief introduction, we go to the heart of complexity theory, relevant to this thesis, that is the polynomiality criterion.

As shown in Table 2.1, the size of the data to be processed is an important factor to take into consideration for the performance of the algorithm. When the input size increases the difference between a polynomial function and an exponential growth becomes manifest. The example shows the running time required for data of size $n = 10, 20, \ldots, 70$, for different complexity functions. The running times remain reasonable for polynomial functions n and n^2. In the case of the exponential function 2^n the time grows for $n = 70$ to 374 centuries, which is clearly not practical. It is well known that asymptotically an exponential function (with a positive base) has a faster growth than any power function.

This polynomial growth criterion of the complexity function was therefore introduced, in particular by J. Edmonds in 1965. This criterion is pertinent, because of its asymptotic nature and because of stability of this criterion relative to the composition of algorithms. Since the composition of polynomial functions is itself a polynomial function.

The classic notation of Landau is used to express the complexity of an algorithm. An algorithm is of complexity $O(f(n))$ when the number of elementary operations is upper bounded by

2. There exists a complexity in space, that measure the amount of space necessary for processing the data of a given size. However, usually we are not interested in this complexity, therefore, when saying "complexity", the run-time complexity is meant.

Size of problem	Complexity function		
	n	n^2	2^n
10	$0.01\mu s$	$0.1\mu s$	$1.024\mu s$
20	$0.02\mu s$	$0.4\mu s$	1.049 ms
30	$0.03\mu s$	$0.9\mu s$	1.074 s
40	$0.04\mu s$	$1.6\mu s$	18.3 minutes
50	$0.05\mu s$	$2.5\mu s$	13.0 days
60	$0.06\mu s$	$3.6\mu s$	36.6 years
70	$0.07\mu s$	$4.9\mu s$	374 centuries

TABLE 2.1 – Comparison of complexity functions

$f(n)$ multiplied by a constant, under the condition that the integer n is greater than a certain value. If the function f is polynomial, the algorithm is called *polynomial*. The problems that can be solved using these algorithms are also called *polynomial*. The set of all these problems define the *complexity class* denoted P.
A polynomial function behaves as its highest degree term, and taking into account the constant involved in O, $f(n)$ is replaced by an expression of the form $O(n^k)$. The case $k = 1$ is the best possible, since the algorithm needs at least the time to read the data. This is the case of linear algorithms.

2.2.3 Class NP

A problem is recognized as class P, when it can be solved algorithmically in polynomial time. The class P contains a lot of problems, such as finding shortest paths or a spanning tree in a simple graph. But there exist problems that cannot be solved polynomially.

Take the **Hamiltonian cycle problem** for instance. It consists in finding a cycle H in a simple graph G, that contains all the vertices of G, ie $V(H) = V(G)$. Now for a given graph G and a given cycle H, is it H **hamiltonian**, ie $V(H) = V(G)$? Such a problem with a "yes" or "no" answer is called a *decision problem*. One way to solve it is to find all the possible cycles in G, take the subset with cycles of vertex set equal to $V(G)$, and verify if H is one of these cycles. If, for a given H, $V(H) = V(G)$ then H is hamiltonian. Verifying the "yes" answer is easy.

The set of all decision problems where the "yes"-answers can be verified in polynomial time form the **NP class**. The NP class contains problems for which it is possible to check a "yes"-answer polynomially without being able to find polynomially this answer. Observe that the "no"-answers are not required to be easy (or polynomially) to check.

An algorithm that is used to check a "yes"-answer is called *verifier* or *checking-algorithm*. Therefore, another way to define the NP class, is to define the class of all the problems for which

there exist checking-algorithms that executes in polynomial time.

The input data for a problem is called an instance of a problem. In our example these are the graph G and the cycle H.
Formally, the class NP is defined as follows.

Definition 2.2.1. *Let π be a decision problem. Now π is in NP if there exists a polynomial algorithm A and a polynomial p such that for any instance x of A the answer is "yes" if and only if there is a datum y such that $|y| \leq p(|x|)$ and algorithm A applied to x associated with y gives the answer "yes" ($|x|$ designates the size of x, likewise for $|y|$). The algorithm A is the checking-algorithm and y is a certificate, for instance x of π. The certificate is necessarily of polynomial size compared to the size of the instance (the converse is not important). In this case such a certificate is said to be succinct.*

Since a polynomial algorithm can be a verifier for the problems of the class P, and the certificate can be empty, we can conclude that $P \subseteq NP$. To this point, it has not been proved that the inclusion is strict.

2.2.4 NP-complete problems

Since $P \subseteq NP$, it is important to know which problems can be solved in polynomial time. Since has not been unproved that $P = NP$, it is natural to look for problems that cannot be solved in polynomial time under the condition that $P \neq NP$. Problems, that can not be solved in polynomial time unless $P = NP$, are called $NP - complete$ problems.
In order to prove that a problem is $NP - complete$, one has to show that the problem is at least as hard as another problem from this class. To do this, the concept of *reduction* is used. To check if a problem is $NP - complete$, the reduction has to be polynomial.

Definition 2.2.2. *Polynomial reduction.*
A problem π_1 can be polynomially reduced to a problem π_2 if there is a polynomial algorithm, called a reduction-algorithm from π_1 to π_2, which calculates for each instance x_1 of π_1, an instance x_2 of π_2 such that the answer for x_1 is "yes" if and only if the answer for x_2 is "yes" (x_1 and x_2 have the same answer).

This is a very important and strong relation, since if there exists a polynomial algorithm solving the problem π_2, then a polynomial time algorithm can be designed that solves π_1.
So if we denote the polynomial reduction by $\leq_p$, then :

Definition 2.2.3. *NP-complete problem. A decision problem π_1 is NP-complete if π_1 is in NP, and $\forall \pi \in NP$, $\pi \leq_p \pi_1$.*

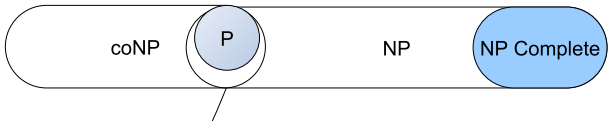

FIGURE 2.1 – Classes of complexity

A problem to which all the others can be polynomially reduced is, in a way, a "universal" problem. For the class NP such a problem is called NP-complete. The first NP-complete problem, found at the beginning of the 1970s, is a problem of logic, called a satisfiability problem denoted SAT.

2.2.5 Class coNP

For a problem of class NP, the "no"-answer may not have an obvious succinct certificate. For example, concerning the problem of existence of a Hamiltonian cycle in a graph : if it is easy to see how to "certify" the answer "yes" (by giving a Hamiltonian cycle), it is not obvious how to certify the answer "no". The Class $coNP$ is defined as the class of decision problems for which the "complement" problem, that is the problem set in order to invert the answers "yes" and "no", is in NP.

It is not established if $NP = coNP$.

Consider the Hamiltonian problem, that is in NP : "Given a graph G, is it Hamiltonian ?[3]". In order to verify the "yes" answer we provide a Hamiltonian cycle. The complementary problem is in coNP : "Given a graph G is it non-Hamiltonian ?". In order to verify the "no" answer we provide a Hamiltonian cycle.
A possible configuration of the previous classes is shown in Figure 2.1.

2.2.6 Best, worst and average case complexity

The deterministic sorting algorithm quicksort, solves the problem of sorting a list of integers given as the input. If the input is already sorted, the algorithm takes time $O(n\ logn)$ to sort. If the input is sorted in the reverse order, the algorithm takes $O(n^2)$ time to sort. If all possible permutations of the input list are likely, the average time for sorting is $O(n\ logn)$. The quicksort algorithm has the average performance of $O(n\ logn)$.
The best, worst and average case complexity refers to three different ways of measuring the time complexity. It is clear that some inputs of size n may be faster to solve than the others. For this reason, the following complexities are defined :

- **Best-case** complexity is the complexity of solving the problem for the best input of size n.
- **Worst-case** complexity is the complexity of solving the problem for the worst input of size n.

3. a Hamiltonian graph is a graph, having a Hamiltonian cycle

– **Average-case** complexity is the complexity of solving the problem on an average.

2.2.7 Approximability

To this point, we have seen what a decision problem is, now we introduce the **optimization problems**. An optimization problem is the problem of finding the *best* solution among a collection of possible solutions.
But how the best solution is defined ? For example, let G be a simple graph where x, y are two vertices of G. Consider the following decision problem : "Does G have a Hamiltonian path that starts and ends at x and y respectively ?". An optimization problem in this case would be : "For a given graph G and its vertices x, y, what is the longest path between these two vertices ?".
Another example of an optimization problem is : "Given G and two vertices x and y, what is the shortest path between these two vertices ?". Suppose the solution is 1 edge, which means that x and y are adjacent. Then the corresponding decision problem would be : "Given graph G and two vertices x and y, is there a path from x to y of length 2 or less ?"
As one can notice, for each optimization problem there exists a corresponding decision problem. And the best solution, called **optimal solution** OPT, consists in satisfying the *goal function*, i.e. minimum or maximum.

Definition 2.2.4. *An optimization problem denoted P is a quadruple (I, f, m, g) where*
– *I is a set of instances ;*
– *if $n \in I$, then $f(n)$ is the set of feasible solutions ;*
– *if, for given n, $s \in f(n)$ is a feasible solution, then $m(n, s)$ is the measure of s ;*
– *g is the goal function, either* **min** *or* **max**.

Using this definition we can say that the goal is to find an optimal solution for some instance, and we denote it $m(n, s) = g\{m(n, s')|s' \in f(n)\}$.
If an optimization problem π has an NP-complete decision version, then π is NP-hard. But if the corresponding decision problem is in NP, then π is also in *NP-optimization problems* class.
An **NP-optimization problem** (NPO) is an optimization problem with additional conditions :
– the size of every feasible solution $s \in f(n)$ is polynomially bounded in the size of the given instance n,
– the languages $\{ n \mid n \in I \}$ and $\{ (n, s) \mid s \in f(n) \}$ can be recognized in polynomial time, and
– m is polynomial-time computable.

When an optimization problem is NP-hard, no polynomial time algorithm exists that finds the best solution, unless $P = NP$.
Since finding the best solution is not easy, one may be interested in a solution close to the optimal. In practice, a solution that is nearly optimal may be good enough and may be much easier to find. Such a solution can be found by using *approximation algorithms*. When the solution given

by the approximation algorithm differs from the optimum only by a constant, such an algorithm is called an *absolute approximation algorithm*. We use the definition below, given by Korte [52].

Definition 2.2.5. *An* **absolute approximation algorithm** *for an optimization problem P is a polynomial-time algorithm A for P for which there exists a constant k such that*

$$|A(I) - OPT(I)| \leq k,$$

for all instances I of P.

Unfortunately, very few classical NP-hard optimization problems have an absolute approximation algorithm to solve them. In most cases one has to be satisfied with relative performance guarantees. Here we have to restrict ourselves to the problems with nonnegative weights.

Definition 2.2.6. *Let P be an optimization problem with nonnegative weights and $k \geq 1$. A* **k-factor approximation algorithm** *for P is a polynomial-time algorithm A for P, such that*

$$\tfrac{1}{k} OPT(I) \leq A(I) \leq kOPT(I)$$

for all instances I of P. We also say that A has **performance ratio k**.

The first inequality applies to maximization problems, the second one to minimization problems. Observe that for instances I where $OPT(I) = 0$ we require an exact solution. The 1-factor approximation algorithms are precisely the exact polynomial-time algorithms.

The set of NPO problems, that have an approximation algorithm and a positive real number r such that $k \leq r$, form the APX class.

2.2.8 Non-approximability

When it is NP-hard to approximate some specific optimization problems with a given approximation ratio, we talk about *non-approximabilty*. A basic tool for establishing such results is the so-called "gap-preserving reduction".

Let P_1 and P_2 be two optimization problems.

Depending on whether P_1 and P_2 are minimization or maximization problems, we have slightly different definitions.

Definition 2.2.7. *Let P_1 and P_2 be the maximization problems. A gap-preserving reduction from P_1 to P_2 is a polynomial time algorithm which, given an instance $x \in P_1$, produces an instance $y \in P_2$ such that :*

- *if $OPT(x) \geq h(x)$, then $OPT(y) \geq h'(y)$, and*
- *if $OPT(x) < g(|x|)h(x)$, then $OPT(y) < g'(|y|)h'(y)$*

for some functions $h(x), g(|x|), h'(y), g'(|y|)$ with $g(|x|), g'(|y|) \leq 1$.

Likewise, if P_1 is a minimization problem and P_2 is the maximization problem, we have :

Definition 2.2.8. *Let P_1 be a minimization problem and P_2 be a maximization problem. A gap-preserving reduction from P_1 to P_2 is a polynomial time algorithm which, given an instance $x \in P_1$, produces an instance $y \in P_2$ such that :*
- *if $OPT(x) \leq h(x)$, then $OPT(y) \geq h'(y)$, and*
- *if $OPT(x) > g(|x|)h(x)$, then $OPT(y) < g'(|y|)h'(y)$*

for some functions $h(x), g(|x|), h'(y), g'(|y|)$ with $g(|x|) \geq 1$ and $g'(|y|) \leq 1$.

If P_1 and P_2 are minimization problems, we have :

Definition 2.2.9. *Let P_1 and P_2 be minimization problems. A gap-preserving reduction from P_1 to P_2 is a polynomial time algorithm which, given an instance $x \in P_1$, produces an instance $y \in P_2$, such that :*
- *if $OPT(x) \leq h(x)$, then $OPT(y) \leq h'(y)$, and*
- *if $OPT(x) > g(|x|)h(x)$, then $OPT(y) > g'(|y|)h'(y)$*

for some functions $h(x), g(|x|), h'(y), g'(|y|)$ with $g(|x|), g'(|y|) \geq 1$.

Finally, if P_1 is a maximization problem and P_2 is a minimization problem, we have :

Definition 2.2.10. *Let P_1 be a maximization problem and P_2 be a minimization problem. A gap-preserving reduction from P_1 to P_2 is a polynomial time algorithm which, given an instance $x \in P_1$, produces an instance $y \in P_2$ such that :*
- *if $OPT(x) \geq h(x)$, then $OPT(y) \leq h'(y)$, and*
- *if $OPT(x) < g(|x|)h(x)$, then $OPT(y) > g'(|y|)h'(y)$*

for some functions $h(x), g(|x|), h'(y), g'(|y|)$ with $g(|x|) \leq 1$ and $g'(|y|) \geq 1$.

As one can see, by using P_1 we can found some bounds for approximation ratio of P_2.

2.2.9 Conclusion

In this chapter we introduced different concepts, used extensively in the next chapters. These notions and definitions are commonly used in graph theory and by a majority of researchers in edge-colored graphs and multigraphs.
Such diversity of tools and notions, used to study the edge-colored graphs, shows the richness and the complexity of the questions that are addressed by this field.

Chapitre 3

Related Work

Contents

In this chapter we present an overview of the prominent results in $edge-colored$ graphs obtained during the last 20 years. $Edge-colored$ graphs being a relatively new area of research in Graph Theory, the questions about the existence of the most simple structures such us paths, cycles, trees, connectivity, edge connectivity, matching as well as others problems, require more sophisticated techniques, compared to the traditional graphs. In this chapter we present the important results connected to our topic.

3.1 Proper edge-colored Hamiltonian cycles and paths

It is obvious that from a proper $edge-colored$ Hamiltonian cycle we can find a proper $edge-colored$ Hamiltonian path, and this path is also a proper spanning tree. So the results on properly edge-colored Hamiltonian cycles and paths are directly connected to our problem.

The problem of finding a properly edge-colored Hamiltonian cycle and path is NP-Complete for general graphs. It is NP-Complete for the properly edge-colored graphs as well. It is enough to color every edge with a different color and this graph will be properly edge-colored, however it will be Hamiltonian only if the initial graph was Hamiltonian. So the results that have been obtained to this day deal with special conditions for the $edge-colored$ graphs that have properly

edge-colored Hamiltonian cycles and paths. A properly edge-colored cycle and path is also called an *alternating* cycle and path.
The most natural way is to see what happens in $edge-colored$ complete graphs. Bollóbas and Erdos [18] studied the problem of finding a properly edge-colored Hamiltonian path in K_n^c, - a complete graph colored with c colors. Let $\triangle(G^c)$ denote the maximum degree in same color in G^c. The following conjecture is given in [18].

Conjecture 3.1.1. *[18] Every K_n^c with $\triangle(K_n^c) \leq \lfloor \frac{n}{2} \rfloor - 1$ contains an alternating Hamiltonian cycle.*

Clearly, such a graph is not properly edge-colored Hamiltonian if there is an odd number of vertices and $c = 2$.

Bollóbas and Erdos proved that if $\triangle(K_n^c) < \frac{n}{69}$, then it contains a properly edge-colored Hamiltonian cycle. This condition was improved by Chen and Daykin [25] to $\triangle(K_n^c) \leq \frac{n}{17}$ and later, by Shearer [64] to $\triangle(K_n^c) \leq \frac{n}{7}$ and again by Alon and Gutin in [8] stating that $\triangle(K_n^c) \leq (1 - 1\sqrt{2} - \epsilon)n$, which is the best improvement until today. It is shown that, for $\epsilon > 0$ and $n > n_0(\epsilon)$, any complete graph K on n vertices, whose edges are colored so that no vertex is incident to more than $(1 - 1\sqrt{2} - \epsilon)n$ edges of the same color, contains a properly edge-colored Hamilton cycle. Moreover, for every k between 3 and n any such K contains a properly edge-colored cycle of length k.

Theorem 3.1.2. *[8] For every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon)$ so that, for every $n > n_0$, every K_n^c satisfying*

$$\triangle(K_n^c) \leq (1 - 1\sqrt{2} - \epsilon)n \qquad [= (0.2928... - \epsilon)n]$$

contains alternating cycles of all lengths between 3 and n.

In the case where $n = 2m + 1$ they show that every edge of K_n^c is contained in a properly edge-colored cycle of each desired length between 4 and n (but not necessarily of length 3).

Theorem 3.1.3. *[8] For every $\epsilon > 0$ there exists an $m_0 = m_0(\epsilon)$ so thats for every $m > m_0$, every $K_{m,m}^c$ satisfying*

$$\triangle(K_{m,m}^c) \leq (1 - 1\sqrt{2} - \epsilon)m \qquad [= (0.2928... - \epsilon)m]$$

contains an alternating cycle of every even length between 4 and 2m.

Another interesting result was obtained by Barr [13].

Theorem 3.1.4. *Every K_n^c without monochromatic triangles contains an alternating Hamiltonian path.*

For a set $\mathscr{F}$ of connected graphs, a spanning subgraph F of a graph is called an $\mathscr{F}$-factor, if every component of F is isomorphic to a member of $\mathscr{F}$.
So one properly edge-colored cycle factor with one properly edge-colored cycle is a properly edge-colored Hamiltonian cycle. Thus finding the sufficient conditions for an $edge-colored$ graph to have a minimum properly edge-colored cycles factor can help in our quest (of finding PST and WST). M. Bánkfalvi and Z. Bánkfalvi [12] considered a similar problem for K_{2p}^2.

Theorem 3.1.5. *[12] Let K_{2p}^2 be a two edge-colored complete graph with vertex-set $V(K_{2p}^2) = \{x_1, x_2, ..., x_{2p}\}$. Assume that $d^r(x_1) \leq d^r(x_2) \leq ... \leq d^r(x_{2p})$. The graph K_{2p}^2 contains an alternating factor with a minimum number m of alternating cycles, if and only if there are m numbers k_i, $2 \leq k_i \leq p-2$, such that for each i, $i = 1, 2, ..., m$, we have :*

$$d^r(x_1) + d^r(x_2) + \cdots + d^r(x_{k_i}) + d^r(x_{2p}) + d^r(x_{2p-1}) + d^r(x_{2p-2}) + \cdots + d^r(x_{2p-k_i+1}) = k_i^2.$$

A.Benkouar, Y.Manoussakis, V.T. Paschos and R.Saad [15] propose an algorithm that can find the minimum properly edge-colored factor in K_n^2 in $O(n^{2.5})$ time. This algorithm is based on the algorithm of maximum matching in general graphs made by S.Even and O.Kariv [34].
The algorithm for finding the minimum properly edge-colored cycle factor works as follows. Take a maximum matching induced in one color graph K_n^2, and after in the second color. Now if neither of the matchings is perfect then, it is trivial that there are not properly edge-colored cycle factor. If they are perfects then the union of these two matchings give us the properly edge-colored Hamiltonian path.

Theorem 3.1.6. *[15] There exists an $O(n^{2.5})$ algorithm for finding Hamiltonian cycles in a 2-edge-colored complete graph K_n^2.*

Theorem 3.1.7. *[15]*

Any 2-edge-colored complete graph K_n^2 has a Hamiltonian path if and only if the graph K_n^2 has :
(i) an alternating factor, or
(ii) an "almost alternating factor", that is, a spanning subgraph which differs from a factor by a color of exactly one edge e. Or, finally
(iii) an odd number of vertices and, furthermore, K_n^2 has a red matching M_r and a blue one M_b, each one having the cardinality $\frac{(n-1)}{2}$.

E. Bampis, Y. Manoussakis, and I. Milis [9] gave a parallel algorithm for the problem of finding properly edge-colored Hamiltonian cycle, if any, in 2-edge-colored complete graphs. Their parallel solution uses a perfect matching algorithm putting the alternating Hamiltonian cycle problem into the RNC class. In addition, a sequential version of the parallel algorithm improves the computation time of the fastest known sequential algorithm for the alternating Hamiltonian cycle problem by a factor of $O(\sqrt{n})$.
These are results for complete graphs, but what about general $edge-colored$ graphs ? Can we

check whether or not there exists a properly edge-colored cycle in the given graph ?

Problem 3.1.8. *Given a c-edge-colored graph G^c, check whether G^c contains an alternating cycle.*

The first to study this problem were Grossman and Haggkvist [41]. Grossman and Haggkvist gave a characterization of two-edge-colored graphs, which have an alternating cycle. The case $c \geq 3$ was proved recently by Yeo [70] :

Theorem 3.1.9. *[41, 70] Let G be a c-edge-colored graph, $c \geq 2$, such that every vertex of G is incident to at least two edges of different colors. Then either G has a cut vertex separating colors, or G has an alternating cycle.*

There are three interesting corollaries of the Theorem 3.1.9 (for the case c = 2), all proved in [41].

Corollary 3.1.10. *Let M be a perfect matching in a graph G. If no edge of M is a cut edge of G, then G has a cycle whose edges are taken alternately from M and $G - M$.*

Corollary 3.1.11. *Let G be a 2-edge-colored Eulerian graph so that all monochromatic degrees are odd. Then G has an alternating cycle.*

Corollary 3.1.12. *Let G be a 2-edge-colored graph so that both red and blue subgraphs of G are regular and non-trivial. Then G has an alternating cycle.*

A $k - path - cycle$ subgraph of an edge-colored graph G^c is a subgraph of G^c consisting a disjoint collection of k paths and some cycles. If the number of cycles is zero we call it a $k - path$ subgraph. A cycle subgraph is a nonempty $0 - path - cycle$. subgraph. Feng et al. [35] proved the following :

Theorem 3.1.13. *[35] A K_n^c $(c \geq 2)$ has a properly edge-colored Hamiltonian path if and only if K_n^c contains a properly edge-colored spanning 1-path-cycle subgraph.*

Working on the following conjecture proposed by Abouelaoualim et al. [3], Gutin [42] proved that it is false. The conjecture is based on $color - degree$ constraint for the existence of long properly edge-colored cycles, and in his theorem Gutin proves that this constraint is not relevant.

Conjecture 3.1.14. *[3] Let G be a c-edge-colored undirected graph of order n with $\delta_{mon}(G) = d \geq 1$. Then G has a properly colored cycle of length at least $min\{n, cd\}$. Moreover, if $c > 2$, then G has a properly colored cycle of length at least $min\{n, cd + 1\}$.*

Theorem 3.1.15. *[42] For each $d \geq 1$ there is an edge-colored graph G with $\delta_{mon}(G) = d$ and with no properly colored cycle.*

In the absence of characterization of K_n^c with a properly edge-colored Hamiltonian cycle, sufficient conditions are of interest. Manoussakis, Spyratos, Tuza and Voigt [56] proved the following results.

Theorem 3.1.16. *[56] If $c \geq \frac{1}{2}(n-1)(n-2)+2$, then every K_n^c has a properly edge-colored Hamilton cycle.*

Theorem 3.1.17. *[56] If $c \geq \frac{1}{2}(n-3)(n-4)+2$, then there exists $n_0 = n_0(c)$ such that for all $n \geq n_0$, K_n^c has a properly edge-colored Hamiltonian cycle.*

Abouelaoualim et al. [4] studied the $s-t$ properly edge-colored longest paths in c-edge-colored graph graphs. They propose a polynomial time procedure for finding the longest properly edge-colored $s-t$ path in graphs with no properly edge-colored cycles.

Theorem 3.1.18. *[4] Assume that G^c has no properly edge-colored cycles. Then we can always find in polynomial time a longest properly edge-colored $s-t$ path or otherwise to conclude that such a path does not exist in G^c.*

Using same techniques they proved the following theorem that deal with a properly edge-colored path between two given vertices.

Theorem 3.1.19. *[4] Let G^c be a c-edge colored graph with no properly edge-colored closed trails and two vertices $s, t \in V(G^c)$. Then, we can always find in polynomial time a longest properly edge-colored $s-t$ trail in G^c, provided that one exists.*

Recently Abouelaoualim et al. [3] gave some sufficient degree conditions for the existence of properly edge-colored cycles and paths in edge-colored graphs, multigraphs and random graphs. In particular, they prove that an $edge-colored$ multigraph of order n of at least three colors and with minimum colored degree greater than or equal to $\lceil \frac{n+1}{2} \rceil$, has properly edge-colored cycles of all possible lengths, including Hamiltonian cycles. Longest properly edge-colored paths and Hamiltonian paths between given vertices are considered as well.

Theorem 3.1.20. *[3] Let G^c be a 2-edge colored graph such that for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1,2\}$. Then G^c has a properly edge-colored path of length at least $2d$.*

In the case where $c \geq 3$, the following corollary was proved.

Corollary 3.1.21. *[3] Let G^c be an c-edge-colored graph graph, $c \geq 3$. If, for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1,2,...,c\}$, then G^c has a properly edge-colored path of length $2\lfloor \frac{c}{2} \rfloor d$*

For $edge-colored$ multigraphs they give similar theorems. For the following theorem they give a construction of a graph H_1 that is used in the theorem. In fact, the property of the graph is that any longest properly edge-colored cycle in H_1 has a length d.

Theorem 3.1.22. *[3] Let G^c be an c-edge-colored graph multigraph, $c \geq 2$. Assume that for every vertex x, $d^i(x) \geq d \geq 1$, $i \in \{1, 2, ..., c\}$. Then G^c has either a properly-edge colored path of length at least $min\{n-1, 2d\}$ or otherwise a properly-edge colored cycle of length $d+1$, unless G^c is isomorphic to H1, in which case it has a cycle of length d.*

Using this theorem 3.1.22 they give some sufficient degree conditions for an edge-colored multigraph to have a properly edge colored Hamiltonian cycle. This result may be viewed as a counterpart to Dirac's well known result for general graphs [30] , insofar as the conditions involved deal only with degree conditions and with nothing else.

Theorem 3.1.23. *[3] Let G^c be a c-edge-colored multigraph of order n with minimum colored degree greater than or equal to $\lceil \frac{n+1}{2} \rceil$.*
I)If $c = 2$, then G^c has a properly edge-colored Hamiltonian cycle when n is even, and a properly edge-colored cycle of length $n-1$, when n is odd.
II) If $c \geq 3$, then G^c has a properly edge-colored Hamiltonian cycle.

Based on this theorem they obtained a series of corollaries for properly edge-colored Hamiltonian paths, namely :

Corollary 3.1.24. *[3] Let G^c be a c-edge colored multigraph, $c \geq 3$. Assume that for each color $i \in \{1, 2, ..., c\}$ and for each vertex x of G^c, $d^i(x) \geq \lceil \frac{n}{2} \rceil$. Then G^c has a properly edge-colored Hamiltonian path.*

In next corollary they give interesting results for properly edge-colored Hamiltonian paths with fixed end-points.

Corollary 3.1.25. *[3] Let x, y be two fixed vertices in G^c, $c \geq 2$. Assume that $\forall v \in V(G^c)$, $d^i(v) \geq \lceil \frac{n+3}{2} \rceil$ for each color $i \in \{1, 2, ..., c\}$. Then G^c has a properly edge-colored Hamiltonian path with endpoints x, y.*

In next theorem, they go further by showing that under the conditions of Theorem 3.1.23, G^c has cycles of many lengths. An edge-colored multigraph G^c of order n is called pancyclic if it contains properly edge-colored cycles of all possible lengths $2, 3, 4, 5, \ldots, n$. Similarly, G^c is even-pancyclic if it contains properly edge-colored cycles of all possible even lengths $2, 4, 6, 8, \ldots, 2\lfloor \frac{n}{2} \rfloor$.

Theorem 3.1.26. *[3] Let G^c be a c-edge colored multigraph, $c \geq 2$. Assume that $\forall x \in V(G^c)$, $d^i(x) \geq \lceil \frac{n+1}{2} \rceil$ for each colori $\in \{1, 2, ..., c\}$.*
i)If $c = 2$, then G^c is even-pancyclic.
ii)If $c \geq 3$, then G^c is pancyclic.

Now let us see another interesting result on $edge-colored$ multigraphs. Clearly, it is interesting to find a maximum 1-path-cycle subgraph or maximum cycle subgraph of a c-edge-colored graph multigraph. Based on the polynomial algorithm for finding a maximum weight perfect matching in a weighted graph in $O(n^3)$ time ([58] Ch. 11), Bang-Jensen and Gutin [10] give the following theorem.

Theorem 3.1.27. *[10] One can construct a maximum cycle subgraph and a maximum 1-path-cycle subgraph, respectively in a c-edge-colored multigraph G on n vertices in time $O(n^3)$.*

In order to explain some of the following results, we introduce the notion of color-connectivity, proposed by Saad [62].

Definition 3.1.28. *A pair of vertices x, y of a c-edge-colored multigraph G is called* color-connected *if there exist alternating (x, y)-paths $P = xx', ..., y'y$ and $P' = xu', ..., v'y$, such that $c(xx') \neq c(xu')$ and $c(y'y') \neq c(v'y)$. We define a vertex x to be color-connected to itself. We say that G is color-connected if every pair of vertices of G is color-connected.*

A very useful observation is that every Hamiltonian c-edge-colored graph multigraph is color-connected. The next logical question is how to check if the given two vertices are or not color-connected. Bang-Jensen and Gutin [11] show that it is possible to check the color-connectivity between two given vertices in $O(|E|)$ time.

Lemma 3.1.29. *[11] Let $G = (V, E)$ be a connected 2-edge-colored multigraph and let x and y be distinct vertices of G. For each choice of $i, j \in \{1, 2\}$ we can find an alternating path $P = x_1x_2...x_k$ with $x_1 = x$, $x_k = y$, $c(x_1x_2) = i$ and $c(x_{k-1}x_k) = j$ in time $O(|E|)$ (if one exists).*

3.2 Bipartite multigraphs

Let us see now some results on properly edge-colored bipartite multigraphs. Hilton [47] proved the following theorem :

Theorem 3.2.1. *[47] Let G be a 2-edge-colored regular bipartite graph such that each of the partite sets of G has m vertices and let G" be the blue subgraph of G. If the degree $d(G)$ of G is at least $\frac{m}{2} + 1$ and G" is regular of degree at least $\frac{m}{2}$ and at most $d(G) - 1$, then G has a Hamiltonian alternating cycle.*

Theorem 3.2.2. *[11] A 2-edge-colored complete bipartite multigraph is Hamiltonian iff it is color-connected and has a spanning cycle subgraph. There is an algorithm for constructing a Hamiltonian alternating cycle in a color-connected 2-edge-colored complete bipartite multigraph on n vertices in time $O(n^{2,5})$ (if one exists).*

Another condition for a 2-edge-colored complete multigraph to contain a Hamiltonian alternating cycle was obtained by Chetwynd and Hilton [27] :

Theorem 3.2.3. *[27] A 2-edge-colored complete bipartite graph B with partite sets U and W ($|U| = |W| = n$) is Hamiltonian iff B has a spanning cycle subgraph and, for every $k = 2, ..., n-1$ and every pair of k-sets X and Y such that $X \subset U$, $Y \subset W$, $d_1(X) + d_2(Y) > k^2$ and $d_2(X) + d_1(Y) > k^2$.*

The longest properly edge-colored cycle problem for $2 - edge - colored$ complete bipartite multigraphs is polynomially solvable.

Theorem 3.2.4. *[11] The length of the longest alternating cycle in a color-connected 2-edge-colored complete bipartite multigraph G is equal to the number of vertices in maximum cycle subgraph of G. There is an algorithm for finding a longest alternating cycle in a color-connected 2-edge-colored complete bipartite multigraph on n vertices in time $O(n^3)$.*

Let r be a positive integer. The 2-edge-colored complete bipartite graph $B(r)$ has partite sets $\{x_1, ..., x_2\}$ and $\{y_1, ..., y_2\}$. The set of red edges of $B(r)$ is$\{x_i y_j : 1 \leq i \leq r, 1 \leq j \leq r\} \cup \{x_i y_j : r + 1 \leq i \leq 2r, r + 1 \leq j \leq 2r\}$.

Theorem 3.2.5. *[11] A 2-edge-colored complete bipartite multigraph is vertex evenpancyclic iff it is Hamiltonian and not isomorphic to one of the graphs $B(r)$ $(r = 2, 3, ...)$.*

Corollary 3.2.6. *[11] A 2-edge-colored complete bipartite multigraph is evenpancyclic if and only if it is Hamiltonian and not isomorphic to one of the graphs $B(r)$ $(r = 2, 3, ...)$.*

3.3 Heterochromatic subgraphs

In this section, we present some important results on heterochromatic subgraphs in edge-colored graphs. More details can be found in a survey by Kano and Li [50].
As mentioned before, heterochromatic subgraph is also called multicolored, rainbow colored, colorful or polychromatic subgraph. As one can easily understand from its name, a heterochromatic subgraph is a subgraph where each edge is of a distinct color, i.e. there is as many colors as edges in the subgraph.
One of the problems on heterochromatic subgraphs is the **Partition problem**. There are two natural ways to make a partition, one is vertex partition and the second one is edge partition.
We present some existing results on special types of heterochromatic subgraphs such that heterochromatic trees, cycles and paths. Clearly the heterochromatic spanning subgraphs are important as well car they are properly edge-colored and we can use classical algorithms i order to find properly edge-colored spanning trees for the initial graph.

3.3.1 Vertex Partitions

The vertex partition problem is the problem of partition by heterochromatic subgraphs, the initial c-edge-colored graph. In fact the goal is to find the minimum number of vertex-disjoint heterochromatic subgraphs. If the minimum is equal to one, then the heterochromatic trees, cycles and paths becomes spanning heterochromatic subgraphs.

If the c-edge-colored graph is monochromatic, i.e. all it's edges are colored in same color, then

the minimum heterochromatic tree partition problem is equivalent to finding a maximum matching.
Li and Zhang [53] studied the complexity of the heterochromatic tree, cycle and path partition problems.

Theorem 3.3.1. *[53] The minimum heterochromatic tree (cycle) partition problem is NP-Complete, and there does not exist a constant factor approximation algorithm for it. Actually, the minimum heterochromatic tree partition problem is NP-Complete for bipartite graphs.*

Theorem 3.3.2. *[53] The minimum heterochromatic path partition problem is NP-Complete for 2-edge-colored graphs, and therefore, it is NP-Complete for general graphs.*

Another result for complete bipartite graphs is due to Chen et al. [26]. They gave an explicit formula for the heterochromatic tree partition number ($hetetree_r$) of an r-edge-colored complete bipartite graph $K_{m,n}$.

Theorem 3.3.3. *[26] Let n, m and r be integers such that $2 \leq m \leq n$, $1 \leq r \leq mn$. Then the heterochromatic tree partition number of an r -edge-colored complete bipartite graph $K_{m,n}$ is*

$$hetetree_r(K_{m,n}) = \begin{cases} n & \text{if } 1 \leq r \leq n; \\ 1 & \text{if } m(n-1)+1 \leq r \leq mn; \\ 2 & \text{if } m = n \text{ and } r = n^2 - 2n + 2; \\ n - \lfloor \frac{n}{m} \rfloor & \text{if } m+1 \leq r \leq m(n-1), \text{and } r \equiv 0, 1 \ (mod\ m); \\ n - \lfloor \frac{n}{m} \rfloor - 1 & \text{otherwise.} \end{cases}$$

Jin and Li [49] give the heterochromatic tree partition number of an r-edge-colored complete graph.

Theorem 3.3.4. *[49] Let $3 \leq n$, $2 \leq r \leq \binom{n}{2}$ and $\binom{t}{2} + 2 \leq r \leq \binom{t+1}{2} + 1$. Then $hetetree_r\ (K_n) = \lceil \frac{(n-t)}{2} \rceil$.*

3.3.2 Edge Partitions

The next results concern a heterochromatic tree partition of the edge set of a properly edge colored complete graph. Constantine [29] gave a result on the existence of a proper edge coloring of complete graphs, such that their edges can be partitioned into heterochromatic spanning trees.

Theorem 3.3.5. *[29] For $n \neq 1, 3$, K_{2n} can be properly edge-colored with $2n-1$ colors in such a way that the edges can be partitioned into edge-disjoint heterochromatic isomorphic spanning trees.*

In graph theory, a factor of a graph G is a spanning subgraph, i.e., a subgraph that has the same vertex set as G. A k-factor of a graph is a spanning k-regular subgraph, and a $k-factorization$ partitions the edges of the graph into disjoint k-factors. A graph G is said to be $k-factorable$ if it admits a k-factorization. In particular, a $1-factor$ is a $perfect\ matching$, and a $1-factorization$ of a k-regular graph is an edge coloring with k colors.

The degree condition for a properly edge-colored graph to have a heterochromatic H-factor, was studied by Yuster [71]. He found the following result.

Theorem 3.3.6. *[71] Let H be a graph and $n \geq 2$ be an integer such that $|H|$ divides n. Then there exists an integer $k = k(H)$ such that every properly edge-colored graph of order n and with minimum degree at least $(1 - \frac{1}{\chi(H)})n + k$ has a heterochromatic $H-factor$.*

Suppose that the edges of the complete K_{2n} are colored with $2n-1$ colors in such a way that the edges of any single color form a perfect matching. Such a coloring is called a *coloring of $factorization$*. It is easy to see that every complete graph K_{2n} with a coloring of factorization contains a heterochromatic spanning tree : namely, the star $K_{1,2n-1}$ at any vertex.

A complete graph on $2n$ vertices K_{2n} can be partitioned into n spanning trees. So Brualdi and Hollingsworth [21] formulated the following conjecture.

Conjecture 3.3.7. *[21] If the complete graph K_{2n} has a coloring of factorization, then the edges of K_{2n} can be partitioned into n heterochromatic spanning trees.*

And they proved the following.

Theorem 3.3.8. *[21] If the complete graph K_{2n} $(n \geq 3)$ has a coloring of factorization, then there exist two edge-disjoint heterochromatic spanning trees.*

Kaneko, Kano and Suzuki [1] extended the theorem 3.3.8 to properly edge-colored complete graphs and gave the following theorem and conjecture.

Theorem 3.3.9. *[1] Every properly edge-colored complete graph K_n $(n \geq 6)$ has three edge-disjoint heterochromatic spanning trees.*

Conjecture 3.3.10. *[1] Every properly edge-colored complete graph K_n $(n \geq 6)$ has $\lfloor \frac{n}{2} \rfloor$ edge-disjoint heterochromatic spanning trees. In particular, if n is even then the edges of K_n can be partitioned into $\frac{n}{2}$ heterochromatic spanning trees.*

Akbari and Alipourn [5] generalized the theorem 3.3.8 for a graph colored by r colors with $colorsdistribution$ $(a_1, a_2, \cdots, a_r)$ there a_i is the number of edges colored in color i , $1 \leq i \leq r$, as follows.

Theorem 3.3.11. *[5] If the r -edge-colored complete graph K_n has a color distribution $(a_1, \cdots, a_r)$ with $1 \leq a_1 \leq \cdots \leq a_r \leq (n+3)/2$ and $r \geq n-1$, then K_n has a heterochromatic spanning tree.*

Theorem 3.3.12. *[5] Suppose that the r-edge-colored complete graph K_n has a color distribution $(a_1, \cdots, a_r)$ with $2 \leq a_1 \leq \cdots \leq a_r \leq (n+1)/2$. If T is a non-star heterochromatic spanning tree of K_n, then $K_n - T$ has a heterochromatic spanning tree, where T is regarded as an edge subset.*

Theorem 3.3.13. *[5] If the complete graph K_n is r -edge-colored so that its color distribution $(a_1, \cdots, a_r)$ satisfies $1 \leq a_1 \leq \cdots \leq a_r \leq (n)/2$ and $r \geq n-1$, then K_n has two edge-disjoint heterochromatic spanning trees.*

Theorem 3.3.14. *[5] If the complete graph K_n, $n \geq 3$, is r-edge-colored and $r \geq \binom{n-2}{2} + 2$, then K_n has a heterochromatic spanning tree.*

Theorem 3.3.15. *[5] If the complete graph K_n, $n \geq 6$, is r-edge-colored and $r \geq \binom{n-2}{2} + 3$, then K_n has two edge-disjoint heterochromatic spanning trees.*

Brualdi and Hollingsworth [22] obtained a necessary and sufficient condition using color distribution for an r-edge-colored complete bipartite graph to have a heterochromatic spanning tree.

Theorem 3.3.16. *[22] Every $(2n-1)$-edge-colored complete bipartite graph $K_{n,n}$ with color distribution $(a_1, \cdots, a_{2n-1})$ such that $1 \leq a_1 \leq \cdots \leq a_{2n-1}$ has a heterochromatic spanning tree if and only if for every integer k with $k \leq 2n-1$, it follows $\sum_{i=1}^{k}(a_i) > \frac{k^2}{4}$.*

3.3.3 Heterochromatic Hamiltonian Cycle

A $k-$*bounded edge*$-$*colored* graph is a graph there the edges are colored using no color more than k times. The problem to find a large bound $k = k(n)$ such that every $k-$*bounded edge*$-$*colored complete* graph K_n contains a heterochromatic Hamiltonian cycle was mentioned by Erdos, Nesetril and Rodl [33]. They addressed it as an Erdos-Stein problem and showed that k can be any constant.
Hahn and Thomassen [45] obtained that k could grow as fast as $n^{\frac{1}{3}}$ to guarantee that a k-bounded edge-colored K_n contains a heterochromatic Hamiltonian cycle. They conjectured that the growth rate of k could in fact be linear.

Theorem 3.3.17. *[45] There exists a constant number c such that if $n \geq ck^3$ then every k-bounded edge-colored complete graph K_n has a heterochromatic Hamiltonian cycle.*

Frieze and Reed [38] made further progress.

Theorem 3.3.18. *[38] There exists a constant number c such that if n is sufficiently large and $k \leq \frac{n}{c\ ln(n)}$, then every k-bounded edge-colored complete graph K_n contains a heterochromatic Hamiltonian cycle.*

In [6], Albert, Frieze and Reed proved the conjecture of [45].

Theorem 3.3.19. *[6] Let $c < \frac{1}{32}$. If n is sufficiently large and $k \leq \lceil cn \rceil$, then every k-bounded edge-colored complete graph K_n contains a heterochromatic Hamiltonian cycle.*

Albert, Frieze and Reed also obtained a similar result in the case of directed graphs.

Theorem 3.3.20. *[6] Let $c < \frac{1}{64}$. If n is sufficiently large and $k \leq \lceil cn \rceil$, then every k-bounded edge-colored complete digraph $\overrightarrow{K_n}$ contains a heterochromatic Hamiltonian directed cycle.*

Hahn and Thomassen [45] considered the heterochromatic Hamiltonian path in infinite complete graphs K_w. Since we are not dealing with infinite graphs in this thesis, we are not giving further details about these results.

Corollary 3.3.21. *[45] If K_w is edge-colored so that no color is used more than k times for a fixed k, then it contains a heterochromatic Hamiltonian path.*

3.3.4 Heterochromatic Spanning Tree

In the previous section we gave some results on the heterochromatic spanning trees. We continue with different results.

An edge coloring of a graph G is called an *edge coloring with complete bipartite decomposition* if each color class form a complete bipartite subgraph of G.

In unpublished work, Caen conjectured that if a complete graph K_n is edge-colored with complete bipartite decomposition using $n-1$ colors, then K_n has a heterochromatic spanning tree.
Alon, Brualdi and Shader[7] proved the following stronger result.

Theorem 3.3.22. *[7] Every complete graph K_n having an edge coloring with complete bipartite decomposition contains a heterochromatic spanning tree.*

Suzuki [66] studied necessary and sufficient conditions for the existence of a heterochromatic spanning tree in an edge-colored graph.

Theorem 3.3.23. *[66] An edge-colored connected graph G of order n has a heterochromatic spanning tree, if and only if, for any k colors ($1 \leq k \leq n-2$), the removal of all the edges colored with these k colors from G results in a graph having at most $k+1$ components.*

Using this Theorem 3.3.23 Suzuki proved the following.

Theorem 3.3.24. *[66] The $(\frac{n}{2})$-bounded edge-colored complete graph K_n has a heterochromatic spanning tree.*

Jin and Li [48] generalized Theorem 3.3.23 to the following theorem.

Theorem 3.3.25. *[48] Let G^c be an edge-colored connected graph and $1 \leq k \leq |G|-1$ be an integer. Then G has a spanning tree with at least k colors if and only if for any color subset $S \subseteq c(e) : e \in E(G)$, it holds that*

$$w(G - E_S) \leq n - k + |S|,$$

where E_S is the set of edges with colors in S, and $w(G - E_S)$ denotes the number of components in $G - E_S$.

Akbari and Alipour [5] gave another necessary and sufficient condition for the existence of a heterochromatic spanning tree in an edge-colored connected graph.

Theorem 3.3.26. *[5] An edge-colored connected graph G has a heterochromatic spanning tree if and only if for every partition of $V(G)$ into t parts, where $1 \leq t \leq |V(G)|$, there exist at least $t - 1$ edges with distinct colors that join different partition sets.*

Broersma and Li[20] obtained some results on the complexity problem of finding a spanning tree with as many different colors as possible, and of finding one with as few different colors as possible .

Theorem 3.3.27. *[20] Finding a spanning tree with as many different colors as possible is equivalent to finding a common independent set of maximum cardinality in two matroids, in particular, there is a polynomial time algorithm.*

Theorem 3.3.28. *[20] Finding a spanning tree with as few different colors as possible is NP-hard.*

Later, Chang and Leu [24] showed that finding a spanning tree with as few different colors as possible is NP-Complete even for the complete graphs by using set cover problem, which implies that there is no constant factor approximation algorithm unless P = NP.

3.3.5 Proper trees and weak proper trees in edge colored graphs

Abouelaoualim, Manoussakis et al. [2] defined for the first time a *proper spanning tree* (PST) and the PST problem. They gave some results on that using the *m-tree-cycle-proper-edge-colored subgraph.*

Definition 3.3.29. *Let G^c be a $c-edge-colored$ graph, a* **m-tree-cycle-proper-edge-colored subgraph** *of C^c is a subgraph composed by m proper edge colored trees and a collection of proper edge colored cycles, all vertex disjoint.*

Using this definition they prove the following lemma :

Lemma 3.3.30. *Let K_n^c be an c-edge-colored complete graph, T a proper-colored tree and C a proper colored cycle, such that T and C a vertex disjoint and $V(k_n^c) = V(T) \cup V(C)$. If at least one of two following conditions holds, then K_n^c has a PST.*

(i) It exist $x \in V(C)$ and $t, t' \in V(T)$ such that $tt' \in E(T)$ is an edge of T and ($c(tx) \neq c(tt')$, where t is a leaf of T) or ($c(t'x) \neq c(tt')$, t' is a leaf of T).

(ii) It exist $x \in V(C)$ and $t \in V(T)$ such that $c(tx) \neq c(tt'), \forall t' \in N_T(t)$.

(iii) It exist $t, t' \in V(T)$ such that $tt' \in E(T)$ is an edge of T and $c(tC) = c(tt') = c(t'C)$.

They also gave an algorithm that uses the fact of existence of a tree and of a cycle that spans the graph, and proved the following lemma.

Lemma 3.3.31. *Let K_n^c be an c-edge-colored complete graph, T a proper-edge-colored tree and C a proper edge colored cycle, such that C and T are vertex disjoint and $V(k_n^c) = V(T) \cup V(C)$. Then K_n^c has a PST, and its can be found in $O(|V(T)||V(C)|)$ time.*

The following theorem is a generalization of the lemma.

Theorem 3.3.32. *Let K_n^c be a c-edge-colored complete graph, T a proper-edge-colored tree and $C_1, C_2 \cdots, C_k$ proper edge colored cycles , such that $C_1, C_2 \cdots, C_k$ and T are pairwise vertex disjoint and $V(K_n^c) = V(T) \cup V(C_1) \cup V(C_2) \cdots \cup V(C_k)$. Then K_n^c has a PST and it can be found in $O((k-1)^2 s^2)$ where $s = max\{V(T), V(C_i)\ i : 1, \cdots, k\}$.*

Theorem 3.3.33. *Let p be an even integer, and G^c a c-edge-colored graph, where $c = \frac{n}{p}$. If the monochromatic subgraph G_i in color $\forall i \in \{1, 2, \cdots, c\}$ is $\frac{p}{2} - edge - connected$, then G^c has a PST.*

3.4 Conclusion

As we have seen in this chapter, there is a lot of results concerning Hamiltonian paths, Hamiltonian cycles, heterochromatic subgraphs and trees. The authors give necessary and sufficient conditions for the existence of proper Hamiltonian paths, cycles, heterochromatic spanning trees in an edge-colored connected graph. The proper spanning tree and the weak spanning tree problems have first been defined by A. Abouelaoualim, Y. Manoussakis et al. [2]. They provided some results by using the m-tree-cycle-proper-edge-colored subgraphs. However, both problems are relatively new and have not been much investigated. Therefore, in this thesis our aim is to contribute to the study of these problems, and we begin with the maximum proper and weak proper trees in edge-colored graphs, in the next chapter.

Chapitre 4

Algorithmics aspects of PST and WST problems

Contents

In this chapter we introduce the optimization version of the two problems PST and WST [1]. Also in this chapter are studied the maximum proper colored tree (MPT) and the maximum weak-proper tree (MWT) problems, in a general edge-colored graph.

4.1 NP-completeness and non-approximability results

Definition 4.1.1. *Let G^c be a $c-edge-colored$ graph. T is a proper spanning tree of G^c if and only if $V(T) = V(G^c)$ and any two adjacent edges in $E(T)$ have different colors.*

Definition 4.1.2. *Let G^c be a $c-edge-colored$ graph. T is a weak proper spanning tree of G^c if and only if T is rooted, $V(T) = V(G^c)$ and all root-to-leaf paths are proper colored.*

1. This is the joint work done with Wenceslas Fernandez de La Vega, Yannis Manoussakis, Carlos Martinhon, Rahul Muthu, and Rachid Saad.

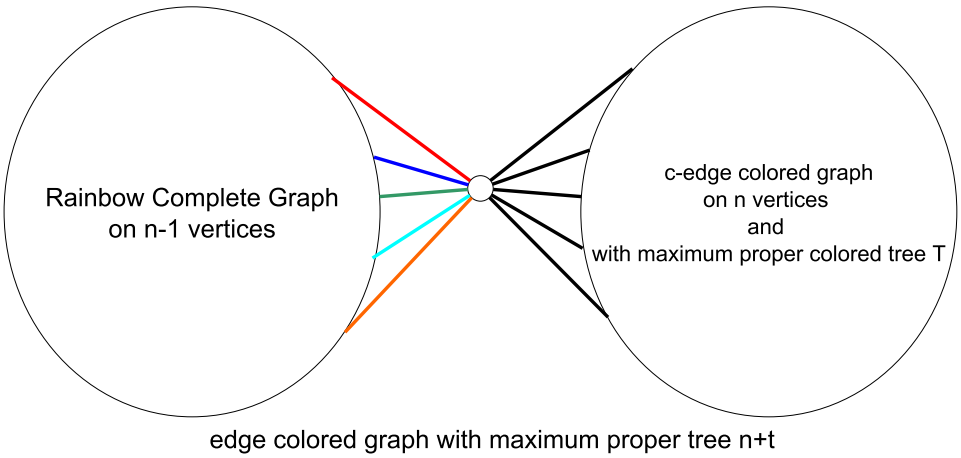

FIGURE 4.1 – Graph with $\Omega(n^2)$ colors

Let formally define the optimization version of the problems.

Problem :	maximum proper colored tree problem (MPT)
Input :	Edge-colored graph and a constant $k \in \mathbb{N}$.
Question :	Is there a proper tree of order k ?

Problem :	maximum weak-proper tree problem (MWT)
Input :	Edge-colored graph and a constant $k \in \mathbb{N}$.
Question :	Is there a weak-proper tree of order k ?

Consider initially a 2-edge colored graph G^c on n vertices. It is evident that a PST in this case corresponds to Hamiltonian path. But the maximum proper tree (MPT) corresponds to the Longest path in the graph. The PST and MPT problems obviously belong to NP. As you can see for 2-edge colored graph both problems are NP-Complete.

Now consider a c-edge colored graph G^c on n vertices. If the number of colors c is constant, we proved that the MPT problem is NP-Complete.

The next result shows that the proper tree problem remains NP-hard even for graphs with $c = \Omega(n^2)$ colors. The proof is based in the example in Figure 4.1.

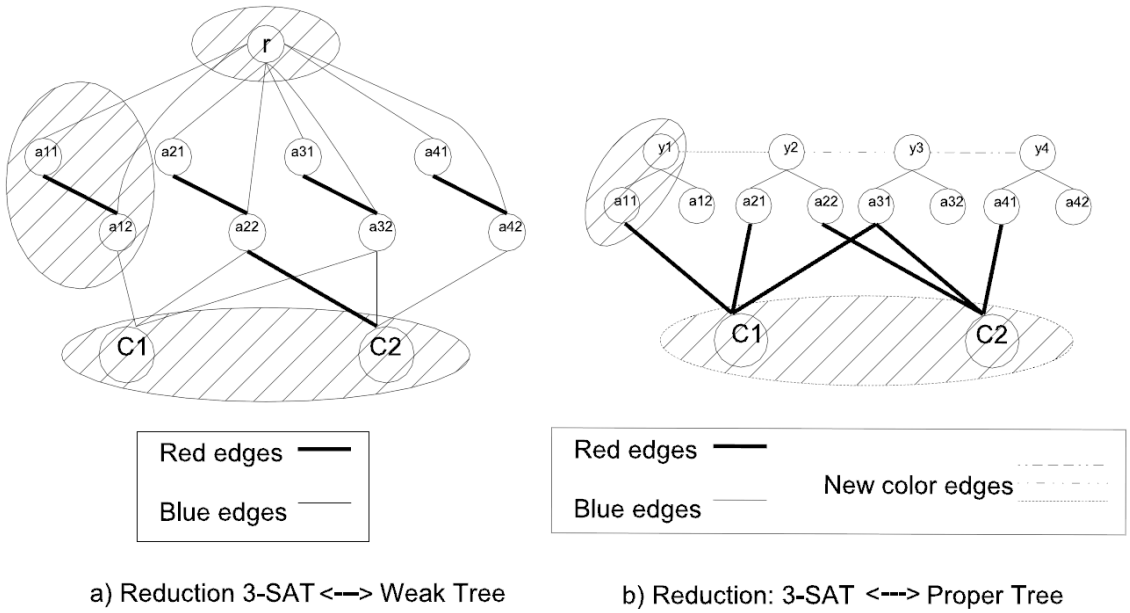

FIGURE 4.2 – Redution 3-SAT formula $B = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x_2} \vee x_3 \vee x_4)$ to WST (fig.(a)) and PST (fig.(b)).

Theorem 4.1.3. *The maximum proper tree problem on G^c is NP-hard even for $c = \Omega(n^2)$.*

Démonstration. Let G^c be an instance of the MPT problem with n nodes and c colors. Construct a complete graph $K_n^{c'}$ with n nodes and color each of its edges with a different color. Add new edges between some fixed vertex of $K_n^{c'}$ and all vertices of G^c and give them all the same color different from those used on G^c and $K_n^{c'}$. Clearly, this new graph has $c = \Omega(n^2)$ colors and contains a proper tree on $n + t$ nodes if and only if G^c has one on t nodes. □

Before giving some nonapproximability results on MPT Problem we will present the proof of the NP-Completness of the MWT problem.

Theorem 4.1.4. *Given a vertex r in G^c, $c \geq 2$, the* WST *problem rooted at r is NP-complete.*

Démonstration. The WST problem obviously belongs to NP. To show that WST is NP-hard we construct a reduction from the 3-SAT problem as follows. Consider a boolean expression B in conjunctive normal form (CNF) with variables $x_1, \ldots, x_s$ and clauses $c_1, \ldots, c_t$. In addition, suppose that B constains exactly 3 literals per clause (actually, we may also consider clauses of arbitrary size). We show how to construct a 2-edge-colored graph G^c such that, B is true if and only if G contains a WST with root r.

The vertex set $V(G^c)$ consists of $2s + t + 1$ nodes and is defined as :

$V = \{r, a_{11}, a_{12}, a_{21}, a_{22}, \ldots, a_{s1}, a_{s2}, c'_1, \ldots, c'_t\}$.

The vertex r is the root, vertices a_{i1}, a_{i2} for $i = 1, \ldots, s$ are associated respectively with variables x_i of B, and all vertices c'_j (for $j = 1, \ldots, t$) are associated with the clauses c_j of B.

The edge set $E(G^c)$ is constructed in the following way. All edges between the root r and the vertices a_{i1}, a_{i2}, for each $i \in \{1, \ldots, s\}$, are added with color blue. Each pair a_{i1}, a_{i2}, for each $i \in \{1, \ldots, s\}$, is connected by a red edge. For each occurrence of x_i in the positive form in the clause c_j we add a blue edge $a_{i2}c'_j$. Analogously, for each ocurrence of x_i in the negative form in the clause c_j we add a red edge $a_{i2}c'_j$. See the example of Figure 4.2 (a).

Therefore given a truth assignment for B, we obtain a Weak Spanning Tree T in G^c as follows. For each variable x_i which is true, we select edges ra_{i1}, $a_{i1}a_{i2}$ and all blue incident to a_{i2}. Similarly, for each variable x_i which is false, we select edges ra_{i2} and all red edges incident to a_{i2}.

Conversely, if T is a Weak Spanning Tree, an assigment for all variables of B is obtained as follows. If the last edge on the path from the root r to some c'_j, namely $a_{i2}c'_j$ is colored blue, then the corresponding variable x_i is set to true, otherwise it is set to false. □

In the rest of this section, we present nonapproximability results for the MWT and MPT problems. Recall that in the MWT, the objective is to maximize the number of nodes covered by a tree T with root r. Initially, consider the following auxiliary result relating the maximum number of covered vertices $Opt(G)$ and the maximum number of satisfied clauses $Opt(B)$.

Lemma 4.1.5. *$Opt(G) = Opt(B) + 2s + 1$.*

Démonstration. Consider the construction described in the proof of the previous theorem, correlating a 3-CNF-formula and a corresponding graph. As in that proof, for every assignment of values to the variables of the formula, we have a weak tree rooted at r, covering all the $a_{11}, a_{12}, \ldots, a_{s1}, a_{s2}$ and all c'_j associated with the set of satisfied clauses of B. Conversely, given any weak tree rooted at r, we obtain an assignment of values to the variables in B, such that we satisfy at least as many clauses as the number of paths from r to c'_j. It follows that $Opt(G) = Opt(B) + 2s + 1$. □

Now lets look to the following nonapproximabilty results for MWT problem. To prove the nonapproximability bounds we are using a classical negative result for the Max-3-Sat problem. As proved in Hastad [46], the Max-3-Sat cannot be approximated within $7/8 + \epsilon$, unless P=NP.

Theorem 4.1.6. *The MWT problem is nonapproximable within $63/64 + \epsilon$, for $\epsilon > 0$, unless $P = NP$.*

Démonstration. Again, consider a boolean expression B with s variables and t clauses. In addition, suppose that B constains exactly 3 literals per clause. Then, by the gap reduction technique we prove that :
1) if $Opt(B) \geq t$ then $Opt(G) \geq f(s,t)$, where $f(s,t) = 2s + t + 1$ and,

2) if $Opt(B) < (7/8+\epsilon)t$ then $Opt(G) < (63/64+\epsilon)f(s,t)$, for $\epsilon > 0$.

The first condition follows directly since $Opt(G) = Opt(B) + 2s + 1$ by Lemma 4.1.5. Thus, consider $Opt(B) < (7/8+\epsilon)t$. From this inequality, and Lemma 4.1.5, it follows that $Opt(G) < (7/8+\epsilon)t + 2s + 1 = (7/8+\epsilon)t + f(s,t) - t$. Therefore $Opt(G) < (\epsilon - 1/8)t + f(s,t)$.

Now, from the definition of the 3-SAT problem it follows that $3 \leq s \leq 3t$. Therefore, $f(s,t) \leq 2s + t + s/3 = 7s/3 + t \leq 8t$. Thus, for $0 < \epsilon < 1/8$, it follows that $Opt(G) < (\epsilon - 1/8)f(s,t)/8 + f(s,t) = (\epsilon/8 - 1/64 + 1)f(s,t)$. Finally, $Opt(G) < (63/64 + \epsilon')f(s,t)$ where $\epsilon' = \epsilon/8$. □

Now, we deal with the MST problem.

Lemma 4.1.7. $Opt(G) = Opt(B) + 2s$

Démonstration. We construct a tree associated with any formula in 3-CNF as follows. We have three vertices y_i, a_{i1}, a_{i2}, for each variable $x_i, 1 \leq i \leq s$, in the formula. We also have vertices $c'_1, \ldots, c'_t$ corresponding to each of the t clauses in the formula. We have a red edge between a_{i1} and c'_i if variable x_j occurs in positive form in the clause C_j. In case the variable x_i occurs in the negative form in clause C_j, we put a red edge between a_{i2} and c'_j. We have blue edges between y_i and a_{i1} and y_i and a_{i2}, for each $i \in \{1, \ldots, s\}$. The vertex pairs y_i and y_{i+1} are connected each by different colors other than red and blue for each $i \in \{1, \ldots, s-1\}$.

It can be easily deduced by looking at Figure 4.2 (b), that for any $i \in \{1, \ldots, s\}$, at most two of y_i, a_{i1}, a_{i2} can be covered by a proper tree. The vertices corresponding to the satisfiied clauses can always be covered. Thus, clearly $Opt(G) \geq Opt(B) + 2s$.

Now suppose that we are able to also cover a vertex corresponding to an unsatisfied clause. This implies that the vertex, say c'_j is connected to some a_{i1} or a_{i2}, only one of which is present in the tree. Also, every satisfied clause is covered, and this is independent of the assignment made to the variable x_i. This means we can satisfy an extra clause contradicting the optimality of the assignment. Thus $Opt(G) \leq Opt(B) + 2s$. □

Theorem 4.1.8. *The* MST *problem is non approximable within* $55/56 + \epsilon$*, for* $\epsilon > 0$*, unless* $P = NP$.

Démonstration. Consider a boolean expression B with s variables, t clauses and exactly 3 literals per clause. We want to show that :
1) if $Opt(B) \geq t$, then $Opt(G) \geq f(s,t)$, where $f(s,t) = 2s + t$ and,
2) if $Opt(B) < (7/8+\epsilon)t$, then $Opt(G) < (53/54+\epsilon)f(s,t)$, for $\epsilon > 0$.

Case (1) follows immediately from Lemma 4.1.7.

For case (2), we again apply the gap reduction technique using the Max-3-Sat problem. Clearly, $Opt(G) = Opt(B) + 2s < (7/8+\epsilon)t + 2s = (7/8+\epsilon)t + (f(s,t) - t) = (\epsilon - 1/8)t + f(s,t)$. Now, from the definition of the 3-SAT problem it follows that $3 \leq s \leq 3t$. Therefore, $f(s,t) \leq$

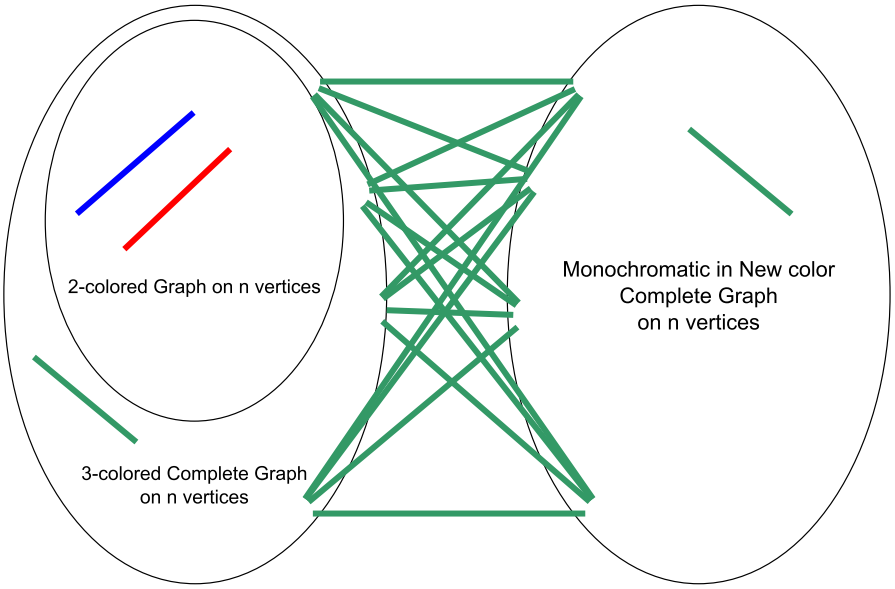

FIGURE 4.3 – A Complete graph on 3 colors

$2s+t \leq 6t+t = 7t$. Thus, for $0 < \epsilon < 1/8$, it follows that $Opt(G) \leq (\epsilon - 1/8)f(s,t)/7 + f(s,t) = (\epsilon/7 - 1/56)f(s,t) + f(s,t)$. Finally, $Opt(G) < (55/56 + \epsilon')f(s,t)$ where $\epsilon' = \epsilon/7$. □

It is known that it exist a polynomial algorithm to find Hamiltonian path in 2 edge-colored complete graphs [15]. Looking for PST Problem for complete edge-colored graphs with at least 3 colors, we have the following result.

Theorem 4.1.9. *The* PST *is NP-complete for complete graphs* K_n^c*, colored with* $c \geq 3$ *colors.*

Démonstration. Let $G^c = (V, E)$ be an instance of the PST problem with n nodes and $c \geq 2$ colors. Construct a new colored complete graph H^{c+1} on $2n$ vertices and $(c+1)$-colors, as follows. Add all edges in the complement graph $\overline{G^c}$ using a new color, and retain the edges of G^c with their original color. Also, use the extra color to form a complete graph on a new set of n vertices as well as a complete bipartite graph between the old vertices and new vertices.

Observe, that all the new vertices, being monochromatic, are necessarily leaves in any PST. Additionally, no two of them may be adjacent to the same vertex, since all edges incident to the entire set of new vertices are of the same color. Thus, it is necessary for the original graph G^c to have a PST. It is also a sufficient condition, since a PST of G^c does not use any edges of the new color, and hence the set of new vertices can be attached as leaves to such a tree to get a PST of H^{c+1}.

Since the PST problem is NP-complete for arbitrary graphs colored with two or more colors, it is also NP-complete for complete graphs colored with three or more colors. □

4.1.1 Conclusion

In this section we has proved that the problem of determining whether an edge-colored graph has a PST or a WST is NP-Complete for general edge-colored graphs. We proved that the proper tree problem remains NP-hard even for graphs with $c = \Omega(n^2)$ colors. We proved that the proper tree problem remains NP-complete for complete edge-colored graphs with at least 3 colors. It is evident that in the case of properly edge-colored graphs both problems are polinomialy solvable, since we can use Prim's or Kruskal's algorithm to solve this problems. In the rest of this thesis we study same problems in the case of special families of edge-colored graphs.

4.2 Maximum colored trees in acyclic edge-colored graphs

As mentioned and proved in an earlier section, the problem of determining whether an edge-colored graph has a PST is NP-Complete. This is also the case with the WST problem. In this section, we prove that these problems can be solved efficiently when we restrict our attention to the class of colored acyclic graphs. When there is no spanning tree, our algorithms can be directly adapted to find a tree of maximum cardinality.

While some ideas are common, the proofs and conditions for the PST and WST problems on acyclic edge-colored graphs, differ significantly. Consequently, we have divided the presentation into two seperate subsections, for the sake of clarity.

4.2.1 PST

We begin with a very simple theorem which shows, when a given acyclic graph has a PST. The algorithm which results from this proof has complexity $O(cn^{2.5})$. Subsequently, we present an alternative, more complicated, proof *but* with a much better algorithmic complexity of $O(n^{2.5})$. Since that proof is more involved, we divide it into two parts. We prove the result first for acyclic complete graphs and then generalize it to all acyclic graphs. In addition to the superior running time, in case the graph does not contain a PST this latter algorithm can be modified to produce a MPT. Finally, it is possible that the ideas used in the latter results can be adapted to develop approximation algorithms for the MPT problem on general graphs.

Theorem 4.2.1. *An acyclic graph G^c has a* PST *iff the union of maximum matchings of each of the colors in χ_c contains exactly $(n-1)$ edges.*

Démonstration. We use, here, $\mathcal{M}_i$ to denote a maximum matching in color i and $\mathcal{T}_i$ to denote the edge set of color i in a PST. Any subgraph induced by $\bigcup_{i=1}^{c} \mathcal{M}_i$ is clearly a proper one. Thus if $|\bigcup_{i=1}^{c} \mathcal{M}_i| \geq n$, then the subgraph contains a proper cycle, contradicting the acyclicity of G^c. Now suppose $|\bigcup_{i=1}^{c} \mathcal{M}_i| < (n-1)$. Clearly, if G^c has a PST then for some color i, $|\mathcal{M}_i| < |\mathcal{T}_i|$, since $|\bigcup_{i=1}^{c} \mathcal{M}_i| < (n-1) = |\bigcup_{i=1}^{c} \mathcal{T}_i|$. This is a contradiction of the maximality of $\mathcal{M}_i$. □

We now provide alternative results specifying conditions under which an acyclic graph has a PST. We also develop an efficient algorithm to construct one if it exists. If the given graph does not have an PST, then our algorithm can be adapted in a straightforward manner to produce a *proper spanning forest* (PSF) with the smallest possible number of components (trees). This adaptation is just a greedy procedure, which finds a tree of largest size, and then repeats the procedure on the subgraph induced by the residual vertices. We first prove all these results for acyclic edge-colored *complete* graphs and then extend them for *general* acyclic edge-colored graphs.

Let $G_1, \ldots, G_p$, $p \geq 1$ be the components of $G^c \setminus \{u\}$. By Theorem 3.1.9 if G^c is acyclic, for some u, the edges between u and G_i are monochromatic, for all $i \in \{1, \ldots, p\}$. We call such a vertex u, a *yeo-vertex*. We refer to a yeo-vertex which uses different colors for the edges to

different components as a *rainbow-yeo-vertex*. We call any other yeo-vertex *non-rainbow-yeo*, if the need arises to distinguish them.

We first prove an elementary result on colored acyclic graphs, which we use in Section 4.3.

Lemma 4.2.2. *For any colored acyclic graph G^c, $c \leq n-1$.*

Démonstration. Assume, G^c is an acyclic colored graph with $c \geq n$. Consider a subgraph induced by a set of n edges of distinct colors. Since, this subgraph has at most n vertices, the presence of n edges implies a cycle. It is a properly colored cycle as the set of selected edges are all of distinct colors. This contradicts the assumption that G^c is acyclic. □

Based on Theorem 3.1.9, we state below an easy lemma.

Lemma 4.2.3. *If an acyclic edge-colored graph contains a non-rainbow-yeo vertex, then it has no* PST*.*

Démonstration. This is because there are at least two components in the graph obtained by deleting this vertex which must be connected to each other in a potential PST by a path of length two through this vertex. These edges are necessarily of the same color and hence the resulting spanning tree is not a proper one. □

In Theorem 4.2.5 we characterize acyclic edge-colored complete graphs having proper spanning trees. With a view to proving that result, we first give a structural characterization of acyclic edge colored complete graphs.

Proposition 4.2.4. *An acyclic edge-colored complete graph consists of a unique sequence of induced cliques (called blocks) $\mathcal{B}_1, \ldots, \mathcal{B}_k$ such that :*
i) each $\mathcal{B}_i$ is a maximal monochromatic clique in the subgraph induced by $\bigcup_{j=i}^{k} \mathcal{B}_j$,
ii) all edges between each $\mathcal{B}_i$ and $\bigcup_{j=i+1}^{k} \mathcal{B}_j$ are monochromatic in the same color as the edges of $\mathcal{B}_i$.

Démonstration. For a complete graph, the induced subgraph obtained by deleting any vertex has exactly one component, because it is again a complete graph with one fewer vertex. Thus for any such graph which is acyclic, a yeo-vertex is necessarily a rainbow-yeo-vertex. In other words it is a monochromatic vertex (all edges incident to it are of the same color). Thus if there is more than one yeo-vertex in the given graph, then they are monochromatic in the same color, since they also have a direct edge between themselves (as the graph is complete).

Thus, the entire set of yeo-vertices of an acyclic edge-colored complete graph induce a monochromatic clique, and additionally, they are all connected to every other vertex in the graph by edges of this same color.

Note that if we remove this entire clique of vertices, the resultant smaller graph is also an acyclic edge-colored complete graph on fewer vertices.

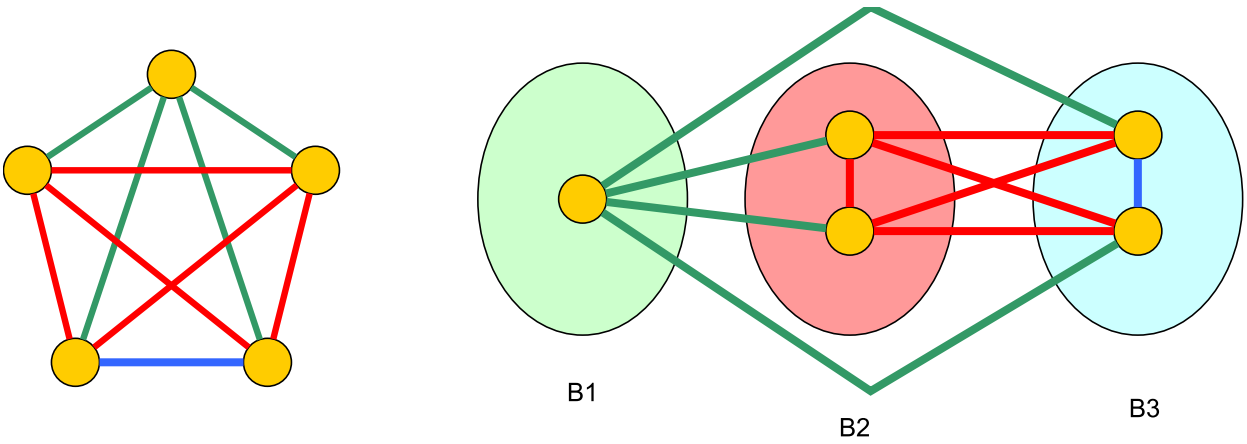

FIGURE 4.4 – Acyclic Complete graph decomposition

Thus, it follows, that there exists a similar clique (of a color distinct from the one just removed) in the remainder of the graph. We delete this clique and place the constituent vertices in a second group. We repeat this procedure of collecting vertices in this type of groups and obtaining a smaller graph until all the vertices have been removed. The partial order we use in finding a PST, is precisely the order determined by these groupings.

□

We now state a theorem, which given an acyclic edge-colored complete graph, determines whether or not it contains a PST. The decision is made on the basis of the cardinalities of the blocks $\mathcal{B}_1, \ldots, \mathcal{B}_k$ in the ordering described in Proposition 4.2.4. To facilitate the description of our results we let the color of the ith block be denoted c_i. For any fixed i, $1 \leq i \leq k$, we let t_i denote the total number of vertices in the set of blocks from $\mathcal{B}_i$ to $\mathcal{B}_k$, i.e., $t_i = |V(\mathcal{B}_i)| + \ldots + |V(\mathcal{B}_k)|$. We define $\mathcal{S}_i = \{\mathcal{B}_j | i \leq j \leq k; c_j = c_i\}$. Define $\mathcal{S}_i^{'} = \bigcup x \in \mathcal{S}_i$. Now, we define $t_i^{c_i} = \left|\mathcal{S}_i^{'}\right|$

With the terminology above, we may state the following.

Theorem 4.2.5. *An acyclic edge-colored complete graph has a* PST *if and only if :*
i) The last block $\mathcal{B}_k$ has two vertices, and
ii) for each $i < k$, if block $\mathcal{B}_i$ has the same color as the last one $\mathcal{B}_k$, then $t_i^{c_i} \leq \frac{t_i}{2} + 2$. Else $t_i^{c_i} \leq \frac{t_i}{2}$.

Démonstration. First assume that the given graph has a PST, and let $\mathcal{T}$ represent any such possible PST. Notice that in the partial ordering of the vertices according to Proposition 4.2.4, the group of vertices in block $\mathcal{B}_i$ are necessarily leaves in the subtree induced by the vertices of $\bigcup_{j=i}^{k} \mathcal{B}_j$. Moreover, each of these vertices are attached to distinct internal vertices of the subtree (since all edges incident to these vertices in the subgraph induced by $\bigcup_{j=i}^{k} \mathcal{B}_j$ are of the same color). A necessary condition is, thus, that there is a proper tree $\mathcal{T}_i$ spanning $\bigcup_{j=i}^{k} \mathcal{B}_j$ for each $i \in \{1, \ldots, k\}$.

Thus, in order to add the vertices of $\mathcal{B}_{i+1}$ as leaves to the subtree $\mathcal{T}_i$, to construct $\mathcal{T}_{i+1}$, there must be at least $|\mathcal{B}_{i+1}|$ vertices not having any edge of color c_i incident to them in $\mathcal{T}_i$. However, all vertices, in blocks $i, \ldots, k$ whose color is c_{i+1} necessarily use an edge of this color in the tree. Additionally, each of them (except if they are in the last block $\mathcal{B}_k$), are also appended to some other vertex in the tree $\mathcal{T}_i$, with color c_i. The vertices of $\mathcal{B}_{i+1}$ must therefore be attached as leaves to $|\mathcal{B}_{i+1}|$ distinct vertices different from the vertices accounted for above. This proves the second condition of the theorem.

The first condition states that the last block $\mathcal{B}_k$ must be of size exactly two. From the earlier arguments, these vertices must contain a proper tree spanning them. Since they induce a monochromatic clique, it is possible only if there are exactly two vertices.

Conversely, if the conditions are on the cardinalities are satisfied, we show that a PST exists. We construct a partial tree consisting of the edge between the two vertices of the last block $\mathcal{B}_k$. Subsequently, we consider in order the vertices of the blocks $\mathcal{B}_{k-1}, \ldots, \mathcal{B}_1$ and at each stage, we pick a set of vertices of size $|\mathcal{B}_i|$ in the partial tree $\mathcal{T}_{i-1}$, which are free of the color c_i and attach the vertices of $\mathcal{B}_i$ as leaves to distinct vertices in this set, to augment the tree to $\mathcal{T}_i$. □

Now based on Theorem 4.2.5 we describe, an algorithm which computes a PST for any acyclic edge-colored complete graph, if one exists. If one does not exist then the algorithm can easily be adapted to produce a PSF with the minimum possible number of components.

We now describe our algorithm to construct the PST.

The running time of the above algorithm is $O(n^2)$. This is the cost using Breadth-First-Search (BFS), to compute the order in Step 1. The rest of the algorithm consists in finding for each new block of vertices, a certain type of matching saturating them, which enables them to be attached as leaves of the partial tree. If there is no PST, our algorithm finds a maximum size proper tree, and then repeats the procedure on the subgraph induced by the residual vertices, resulting in an SSF with the fewest possible number of trees.

We now show how Theorem 4.2.5 as well as the algorithm can be extended to find a PST in a general acyclic graph, if one exists. We define a canonical auxiliary tree associated with any acyclic edge-colored graph. It is similar to the linear order for acyclic edge-colored complete graphs. We assume that the graph has no non-rainbow-yeo-vertex. The root of the auxiliary tree is any yeo-vertex of the graph. The number of children of the root is the number of components resulting from the deletion of this vertex from the graph. The root of the auxiliary tree is connected to each subtree using an edge of the same color as the edge between the original

Algorithm 1 PST for K_n^c-acyclic

compute the order described above
if last block $\mathcal{B}_k$ has more than two vertices **then return** "No PST"
if last block $\mathcal{B}_k$ has two vertices **then** connect the two vertices of $\mathcal{B}_k$ to get an initial Proper Tree
for $i = k-1$ to 1 **do**
  if condition 2 of Lemma 4.2.5 is true **then**
    join the vertices of $\mathcal{B}_i$ as leaves, to distinct vertices already incorporated in the tree which have not used an edge of color c_i in the partial proper tree obtained in the previous iteration.
  else
    return "NO PST"
  end if
end for
return the PST

yeo-vertex and the corresponding component. The root of the subtrees are likewise computed recursively. Thus we get an auxiliary rooted edge-colored tree $\mathcal{T}$.

First we need to define some associated auxiliary graphs, which we use to characterize acyclic edge-colored graphs that have a PST.

Definition 4.2.6 (auxiliary tree)**.** *Given an acyclic edge-colored graph G^c, we define an associated auxiliary tree $\mathcal{T}$ as follows.*
i) $n(\mathcal{T}) = n(G^c)$.
ii) The root $r(\mathcal{T})$ is associated with a yeo-vertex v_0 of G^c.
iii) There is one subtree corresponding to each component $G^c \setminus \{v_0\}$.
iv) The roots of these subtrees are computed recursively and are attached as children of v_0.
v) The color of the edge between two nodes of $\mathcal{T}$ is the same as the color of the edges between the corresponding component and yeo-vertex in G^c.
vi) Note that the presence of an edge between two nodes in $\mathcal{T}$ does not imply the presence of an edge between the corresponding vertices in G^c.

We now define a set of bipartite graphs one for each color in $\{1, \ldots, c\}$.

Definition 4.2.7. *Associated with each color $l \in \{1, \ldots, c\}$ we have a bipartite graph as follows.*
i)If a node in the auxiliary tree (of Definition 4.2.6) corresponding to a vertex has a child of color l then it is placed in the left part of the bipartite graph corresponding to color l.
ii) Any other vertex with an edge of color l incident to it in the graph G^c is placed in the right part.
iii) The edges are all those in the original graph of the color l, crossing this partition.

Theorem 4.2.8. *An acyclic edge-colored graph has a* PST *iff for each color in $\{1, \ldots, c\}$ the corresponding bipartite graph has a matching saturating the vertices of the left partite set.*

Démonstration. By Lemma 4.2.3, if there exists a non-rainbow-yeo-vertex, then we immediately conclude that the given graph has no PST. It follows that we need only consider the case where all the yeo-vertices are of the rainbow type. Assume that v_0 is one such vertex. Let the components of $G^c \setminus \{v_0\}$ be $C_1, \ldots, C_t$. It is straightforward to verify that G^c has a PST iff the subgraph of G^c induced by the vertex set $V(C_i) \cup \{v_0\}$ has a PST for each $i \in \{1, \ldots, t\}$.

We are in effect able to use a rainbow-yeo-vertex to divide the problem into smaller and independent subproblems. This immediately suggests a recursive appproach to solving the problem. We use the recursion tree of Definition 4.2.6 and the corresponding bipartite graphs of Definition 4.2.7 to divide the problem.

In fact, if any C_i does not have a PST then G^c also does not have a PST. We conclude that G^c has a PST iff each C_i has a PST which can be extended to include the vertex v_0. The connection of component C_i to v_0 must be by an edge of color i. The feasibility of this is checked using cardinalities, like in the case of complete acyclic graphs. We do not have a simple linear structure here, unlike in that case, and hence check the condidions using matching in the auxiliary bipartite graph instead. The existence and computation of the matching can be done using any of the standard algorithms. The subtrees rooted at the children of v_0 are computed recursively in the same way.

Thus, using the recursion tree obtained, we construct the bipartite graphs described above and then solve the PST problem by transforming it to a series of matching problems. □

Suppose the graph G^c does not have a PST. This either means that there is no rainbow yeo-vertex, or the cardinality conditions fail.

In the former case, let us denote the components of $G^c \setminus v_0$, by $C_1, \ldots, C_t$. We find the maximum proper trees for each C_i. We also compute the maximum proper trees for each $G_i \cup v_0$, which use touch vertex v_0. From the latter set of trees, we pick the ones of largest cardinality corresponding to distinct colors of the edge incident to v_0 and take their union. A maximum proper tree is the largest among the first set of trees and the tree obtained by combining the second set as described above.

The second case is merely a special case of the previous one, wherein, when we compute the tree combining the second sequence, we consider all the components.

The next theorem follows easily from the preceeding analysis and results.

Theorem 4.2.9. *Given an acyclic edge-colored graph G^c, a* PST*, if any, can be found in G^c in time $O(n^{2.5})$.*

Démonstration. This cost is dominated by the time to compute matchings in graphs bipartite graphs, which we use in the algorithm for SPT. □

4.2.2 WST

We now show how to construct a WST, if one exists, in an acyclic edge-colored graph G^c, with a given root vertex r. Like we did for the case of the PST, we define here as well, an associated

auxiliary tree.

Definition 4.2.10 (auxiliary tree)**.** *Given an acyclic edge-colored graph G^c, and a specified vertex r, we define an associated auxiliary tree $\mathcal{T}'$ as follows.*
i) The tree $\mathcal{T}$ is computed according to the Definition 4.2.6.
ii) The tree $\mathcal{T}'$ is then obtained by re-rooting $\mathcal{T}$ at the node R, corresponding to the vertex r in G^c.

Theorem 4.2.11. *An acyclic edge-colored graph G^c, has a* WST *iff in the auxiliary tree $\mathcal{T}'$ every path from the root R to any leaf is properly colored.*

Démonstration. The above auxiliary tree $\mathcal{T}'$, obtained by re-rooting at the vertex corresponding to the specified root of the WST provides a direct way to compute a WST if one exists. A WST rooted at r is computed for each subgraph induced by $\{r\} \cup C_i$, $i \in \{1, \ldots, t\}$. Here, $C_1, \ldots, C_t$ are the components of $G^c \setminus \{r\}$. These trees are then merely fused together at their only common vertex r, to get a WST. Such a tree is a weak one rooted at r, because in the auxiliary tree any path from R to any leaf is a properly colored one. □

Theorem 4.2.12. *Given an acyclic edge-colored graph G^c, a* WST*, if any, can be found in G^c in time $O(n^{2.5})$.*

Démonstration. It is almost identical to the proof of Theorem 4.2.9. □

4.3 Proper spanning forests in acyclic graphs

In this section, we show how to find in polynomial time, a maximum proper spanning forest (MPSF) in G^c, the maximality being in terms of the number of edges. If the graph has a PST, then naturally, one is produced by our algorithm for maximum forest. This section differs from the previous one, for graphs which do not contain a PST. In the previous section, the algorithm we describe produces a largest possible tree, whereas, here the algorithm produces a forest with, possibly, many trees, in such a way that the total number of edges is maximized. We conclude the section by showing how to decide, in polynomial time, whether G^c contains a proper spanning forest (PSF), satisfying given degree constraints on each vertex. We also show how to construct the forest if one exists in this latter case.

Basically, for the MPSF problem, the idea is to construct a new colored multigraph $G^{c'}$, with color set $\chi_{c'} = \chi_c \cup \{0\}$ (where 0 is a new color) and an associated non-colored weighted graph G, which always contains a perfect matching. The multigraph $G^{c'}$, has multiplicity at most two. Moreover, the color of two edges between the same pair of vertices always differ. If there are two edges between a pair of vertices, then one of them is always colored 0. We prove that a maximum weight perfect matching in G, whose construction is described below, is associated with a set of properly edge-colored closed trails in $G^{c'}$, such that the number of edges with a color from χ_c is

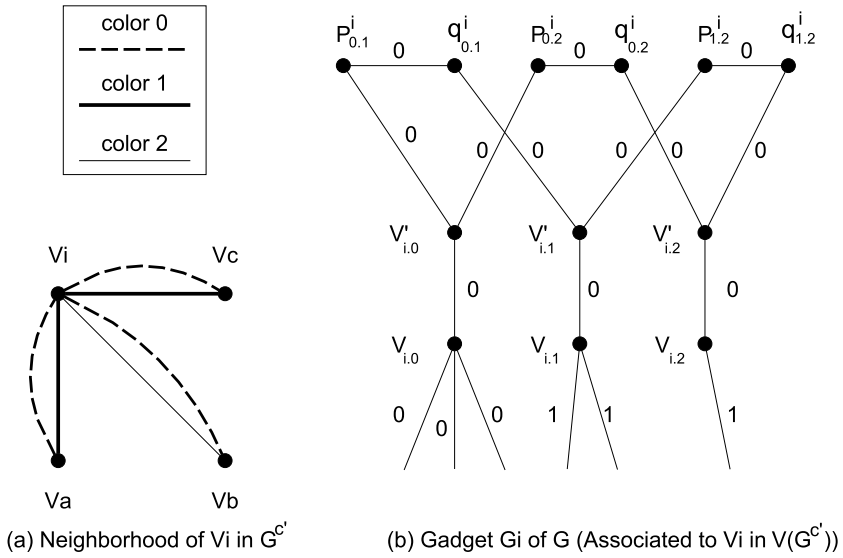

FIGURE 4.5 – Construction of a Maximum Proper Spanning Forest

maximized. As a result, deleting all edges colored 0 in this set, we directly produces a maximum proper spanning forest in G^c.

$G^{c'}$ is constructed by taking the union of G^c with a complete monochromatic graph (in color 0) on the same vertex set. Let K_n^0 denote the resulting monochromatic complete subgraph of $G^{c'}$ with the edges of color 0. Thus, we have $V(G^{c'}) = V(G^c) = V(K_n^0)$ and $E(G^{c'}) = E(G^c) \cup E(K_n^0)$.

Before we describe the construction of G, we first define gadgets G_i associated with each vertex v_i of $G^{c'}$, as depicted in the sequel (see Figure 4.5). We use these gadgets in our construction of G. Formally :

- $V(G_i) = (\bigcup_{\gamma \in \chi_{c'}} \{v_{i,\gamma}, v'_{i,\gamma} : N^{\gamma}_{G^{c'}}(v_i) \neq \varnothing\}) \cup (\bigcup_{\alpha,\beta \in \chi_{c'}} \{p^i_{\alpha,\beta}, q^i_{\alpha,\beta} : \alpha < \beta,\ N^{\alpha}_{G^{c'}}(v_i) \neq \varnothing$ and $N^{\beta}_{G^{c'}}(v_i) \neq \varnothing\})$
- $E(G_i) = (\bigcup_{\gamma \in \chi_{c'}} \{v_{i,\gamma} v'_{i,\gamma} : N^{\gamma}_{G^{c'}}(v_i) \neq \varnothing\}) \cup (\bigcup_{\alpha,\beta \in \chi_{c'}} \{v'_{i,\alpha} p^i_{\alpha,\beta}, p^i_{\alpha,\beta} q^i_{\alpha,\beta},$ $q^i_{\alpha,\beta} v'_{i,\beta} : \alpha < \beta\})$

Now, the weighted non-colored graph $G = (V', E')$ with $w : E' \to \{0, 1\}$ is constructed as follows :

- $V' = \bigcup_{v_i \in V(G^{c'})} V(G_i)$, and
- $E' = (\bigcup_{v_i \in V(G^{c'})} E(G_i)) \cup (\bigcup_{\gamma \in \chi_{c'}} \{v_{i,\gamma} v_{j,\gamma} : v_i v_j \in E^{\gamma}(G^{c'})\})$
- $w(v_{i,\gamma} v_{j,\gamma}) = 1$, for every $v_{i,\gamma} v_{j,\gamma} \in E'$ with $\gamma \in \chi_c$. The remaining edge weights of E' will be settled to 0, *i.e.,* $w(v_{i,0} v_{j,0}) = 0$ for every $v_{i,0} v_{j,0} \in E'$ and $w(xy) = 0$, for every $xy \in E(G_i)$.

After constructing $G^{c'}$ and G as above, we solve the maximum perfect matching problem

over G (see [34]). Observe that graph G always contains a perfect matching. To see that, it suffices to exhibit a perfect matching (with cost 0) by only choosing edges of $E(G_i)$. Thus, if M_i denotes a perfect matching in G_i (which is unique in this case), the subset $M = \bigcup_{v_i \in V(G^{c'})} M_i$ obviously defines a perfect matching in G. Therefore, we can establish the following result :

Theorem 4.3.1. *Let G^c be an acyclic c-edge-colored graph. Then, the maximum proper spanning forest problem can be solved in polynomial time in the order of G^c.*

Démonstration. Basically, the idea is to prove that maximum proper spanning forests in G^c are associated with maximum perfect matchings in G, and vice-versa. Initially, suppose we have a proper spanning forest T^* in G^c with the maximum number of edges. Let $T_1^*, T_2^*, \ldots, T_k^*$ (for $k \geq 1$) be the subtrees of T^*. Note that, for every subtree T_i^* of T^*, the number of vertices of T_i^* with odd degree is even. As a consequence of that, the total number of vertices with odd degrees in T^* is also even. Let P_{T^*} be this subset of vertices. Now, consider the multigraph $G^{c'}$ as above, obtained after the addition of edges with color 0 to G^c. Further, consider H^0 (with all edges colored 0) the complete subgraph of $G^{c'}$ induced by the vertices of P_{T^*}. Let $M^0 \subseteq E(H^0)$ be an arbitrary perfect matching of H^0 and $H^{c'} = (\overline{V}, \overline{E})$ with $\overline{V} = V(G^{c'})$ and $\overline{E} = M^0 \cup E(T^*)$, the associated subgraph. Note that subgraph $H^{c'}$ is not necessarily connected and all edges incident to $v \in \overline{V}$ have a different color. Further, notice that all vertices of $H^{c'}$ have an even degree and each connected component, say CT_i (for $1 \leq i \leq k'$ and $k' \leq k$), contains a properly edge-colored closed trail, *i.e.*, each CT_i defines an eulerian trail.

Now, given $\overline{E}$, we construct a perfect matching M^* in $G = (V', E')$ as described in the sequel. Initially, we set $M^* = \emptyset$ and add to M^* all edges $v_{i,0}v_{j,0}$ of E' with $v_iv_j \in M^0$. Now, we increase M^* by choosing all edges $v_{i,\gamma}v_{j,\gamma}$ of E' with $v_iv_j \in E^\gamma(T^*)$ and $\gamma \in \chi_c$. The remaining edges in the gadgets G_i (with cost 0) are now directly obtained. Note that $c(M^*) = |E(T^*)|$. Finally, we prove that M^* is a maximum perfect matching in G. Suppose, by contradiction, we have some new perfect matching M' with cost $c(M') > c(M^*)$ and an associated proper edge-colored subgraph H' of G^c with $c(H') > |E(T^*)|$. In this case, there are two possibilities : a) If H' defines a new proper spanning forest, this contradicts the fact that T^* is a maximum proper forest in G^c ; b) If H' does not define a proper spanning forest, we would have some properly edge-colored cycle in H', contradicting the fact that G^c is acyclic.

Conversely, consider the weighted graph $G = (V', E')$ associated to G^c as above, and M^* a maximum perfect matching with cost $c(M^*)$ in G. Let $\overline{M} = (E' \setminus E(G_i)) \cap M^*$ be a subset of M^*. Now, to obtain $G^{c'}$ from G it suffices to color edges $v_{i,\gamma}v_{j,\gamma}$ of G with $\gamma \in \chi_{c'}$ and contract all gadgets G_i to vertex v_i. Note that all edges of $G^{c'}$ associated to $\overline{M}$ define a subset of proper spanning closed trails in G^c (denoted by $CT_1, \ldots, CT_{k'}$) with the maximum number of edges with colors in χ_c. Hence, since G^c is acyclic, all properly edge-colored cycles in CT_i (for $i \in \{1, \ldots, k'\}$) contain at least one edge-colored 0. After deleting all these edges, one directly obtains a proper spanning forest $T^* = \bigcup_{\ell=1}^{k} T_\ell$ in G^c with cost $c(T^*) = c(M^*)$ and $k \geq k'$. Finally, note that T^* contains a maximum number of edges since no edges with unitary costs in G were eliminated in the process.

Now, we show that the algorithm above has a polynomial time complexity in the order of G^c. Initially, observe that each gadget G_i of G (with $c \geq 2$) contains at most $c(c+1)$ vertices. From Lemma 4.2.2, we know that $c = \emptyset(n)$. Thus, the non-colored graph G contains $\Theta(n^3)$ vertices, in the worst-case. However, note that a maximum perfect matching in any graph G can be obtained in time $O(|V(G)|^{2.5})$ (see [34] for details). Therefore, the algorithm for the proper spanning forest has total complexity equal to $O(n^{7.5})$. □

Now, we conclude with the following result regarding proper spanning forests with given degrees.

Theorem 4.3.2. *Let G^c be a c-edge-colored acyclic graph and $d : V(G^c) \to \{0, 1, \ldots, n-1\}$, an integer function. In addition, consider $0 \leq d(v_i) \leq |\{\gamma \in \chi_c : N^{\gamma}_{G^c}(v_i) \neq \emptyset\}|$. Then we can find in polynomial time, provided that one exists, a proper spanning forest in G^c satisfying d.*

Démonstration. Initially, given an acyclic edge-colored graph G^c, we construct a non-colored graph G, as described in the sequel. For each $v_i \in V(G^c)$ with $d(v_i) > 0$ we define gadgets G_i in the following manner :

- $V(G_i) = (\bigcup_{\gamma \in \chi_c} \{v_{i,\gamma}, v'_{i,\gamma} : N^{\gamma}_{G^c}(v_i) \neq \varnothing\}) \cup \{v^i_1, \ldots, v^i_{d(v_i)}\}$
- $E(G_i) = (\bigcup_{\gamma \in \chi_c} \{v_{i,\gamma} v'_{i,\gamma} : N^{\gamma}_{G^c}(v_i) \neq \varnothing\}) \cup (\bigcup_{\gamma \in \chi_c} \{v'_{i,\gamma} v^i_j : j = 1, \ldots, d(v_i)\}$

Above, if $d(v_i) = 0$ for some $v_i \in V(G^c)$, we just set $V(G_i) = (\bigcup_{\gamma \in \chi_c} \{v_{i,\gamma}, v'_{i,\gamma} : N^{\gamma}_{G^c}(v_i) \neq \varnothing\})$ and $E(G_i) = (\bigcup_{\gamma \in \chi_c} \{v_{i,\gamma} v'_{i,\gamma} : N^{\gamma}_{G^c}(v_i) \neq \varnothing\})$.

Now, a non-colored graph $G = (V', E')$ is constructed as follows :

- $V' = \bigcup_{v_i \in V(G^c)} V(G_i)$, and
- $E' = (\bigcup_{v_i \in V(G^c)} E(G_i)) \cup (\bigcup_{\gamma \in \chi_c} \{v_{i,\gamma} v_{j,\gamma} : v_i v_j \in E^{\gamma}(G^c))$

We show that G^c contains a proper spanning forest T satisfying the degree constraint d, if and only if, $G = (V', E')$ contains a perfect matching. Hence, our result follows since the perfect matching problem can be solved polynomial time.

Initially, consider M^*, a perfect matching in G, if any. In this case, we can obtain a proper spanning forest $H^{c'}$ in G^c (for $c' \leq c$) in the following manner. Initially, let $\overline{M} = (E' \backslash E(G_i)) \cap M^*$ be a subset of M^*. Now, we color all edges $v_{i,\gamma} v_{j,\gamma} \in E'$ with color $\gamma \in \chi_c$ and contract all gadgets G_i to vertex v_i. Finally, we construct $H^{c'}$ by choosing all edges of G^c associated to $\overline{M}$. Notice by the construction of G_i, that we have exactly $d(v_i) > 0$ edges incident to v_i in the resulting edge-colored subgraph $H^{c'}$. Further, all edges incident to v_i have a different color and v_i is isolated if $d(v_i) = 0$. Therefore, since G^c is an acyclic edge-colored graph, the subgraph $H^{c'}$ contains no properly colored cycles and defines a proper spanning forest in G^c satisfying d.

Conversely, consider $H^{c'}$ a proper spanning forest in G^c, and the graph G as above. Initially, set $M^* = \emptyset$ in G. Now, we obtain the associated matching M^* in two steps :

(1) For every edge $v_i v_j \in E^{\gamma}(H^{c'})$ with $\gamma \in \chi_c$ we add edges $v_{i,\gamma} v_{j,\gamma}$ to M^* ;

(2) The remaining edges of M^* present in the gadgets G_i are now directly obtained.

It is easy to see that M^* constructed as above defines a perfect matching in G. □

4.4 Random graphs

Let $c \geq 3$ be a fixed integer and let $G(n,p)$ be the random graph on n vertices where each edge is present with independent probability p. We define $G^{(c)}(n,p)$ as the edge union of c independent copies of $G(n, \frac{p}{c})$ where for each $j \in \{1, \ldots, c\}$, the edges of the j^{th} copy are colored by j.

Theorem 4.4.1. *If $p = \frac{\lambda \log n}{n}$, where λ is a sufficiently large constant, then with probability tending to 1 as n tends to infinity, $G^{(c)}(n,p)$ contains a proper spanning tree.*

Démonstration. Let $G = G^{(c)}(n,p)$ and let c_0 be a fixed color. We describe an algorithm to prospectively construct a proper spanning tree in our graph and prove that it works with probability close to 1. The algorithm proceeds in two stages.

(1) Construct a sequence of proper trees $T_0, T_1, T_2, ...T_t$. Here, T_0 is a star with an arbitrary root and with ν edges with colors pairwise distinct and different from c_0. We define t and ν later. For each $i \in \{1, \ldots, t\}$, T_i is obtained from T_{i-1} by expanding a pendant vertex with edges of all the colours distinct from c_0, and also distinct from the color of the edge incident to this vertex in T_{i-1}.

(2) From T_t obtain a proper spanning tree by adding a pairing of the internal vertices of T_t other than the root with the remaining vertices of G using edges of color c_0.

This concludes the description of the algorithm.

Note that T_t has a total of $t(c-2)+\nu+1$ vertices among which exactly t are internal vertices, distinct from the root. We want to match these t vertices with the remaining vertices of G, that is, we want

$$\begin{aligned} t &= n - t(c-2) - \nu - 1 \\ \text{or, } t &= \frac{n-\nu-1}{c-1}, \end{aligned}$$

and this will be an integer for precisely one value of ν with $1 \leq \nu \leq c-1$. We fix this value for ν.

To prove the correctness of the first stage of the algorithm it suffices to prove that at each point in the construction we can find with probability $1 - o(1/n)$ an edge with any particular fixed colour linking a point v, say, already in the current tree and the external set which has size at least t. Now, when we look for such an edge of colour j, say, we have not looked previously at the edges of colour j linking v to the remaining vertices. Hence, the conditional distribution of these edges given the previous steps of the algorithm is the same as their unconditional distribution, namely, each is present with probability $\frac{\lambda \log n}{cn}$ and they are independent. Therefore the probability that at least one is present is at least

$$\begin{aligned} 1 - (1 - \frac{\lambda \log n}{cn})^t &\geq 1 - \exp(-\frac{\lambda \log n}{c(c-1)}) \\ &= 1 - o(1/n) \end{aligned}$$

if $\lambda \geq c^2$.

We turn now to the second stage. Since we have not looked yet at the edges of colour c_0, we just have to check that with high probability there exist in a random graph with edge probability p a pairing between to given disjoint sets with sizes t. (We have not looked before at the edges with color c.) This amounts to asking for a perfect matching in the random bipartite graph $B(t,p)$. Now $p = \frac{\lambda \log n}{cn} \geq \frac{\theta \log n}{t}$ with $\theta = \frac{2\lambda}{3c(c-1)}$. This graph is known to have, almost surely, a perfect matching for any fixed θ greater than $1/2$ and thus for $\lambda \geq c(c-1)/2$ (see Corollary 13, page 159 in [17]). Putting together this estimate with the bound already found, we infer that the theorem holds for $\lambda \geq c^2$. On the other direction, the theorem does *not* hold for $\lambda \leq c$ by connectivity considerations. □

4.5 Conclusions

In this chapter we recapitulate the notion of various types of colored trees in edge-colored graphs. We obtain results reflecting the computational difficulties involved in their solution and provide efficient algorithms for the specific family of acyclic graphs. We give a mathematical characterization of complete graphs which contain a PST. We study the trends of random graphs with reference to the problem of PST.

We list here some possible future directions for research in this area.

(1) Algorithms to solve the PST and WST problems on other special classes of graphs like planar graphs or hypercubes.

(2) Approximation algorithms with performance guarantee of at least logarithm of the optimal solution.

(3) The minimum number of edges whose colors need to be changed in order to render the graph acyclic. This is closely related to the feedback arc set problem.

Chapitre 5

Sufficient conditions for PST

Contents

In this chapter, we consider sufficient conditions that guarantee the existence of proper spanning trees in graphs whose edges are colored (not necessarily properly) with any number of colors. The conditions we impose are relationships between various parameters of colored graphs, such as total number of colors, number of vertices, connectivity and number of differently colored edges incident to a vertex. These special conditions are of interest because there are no good characterizations for the existence of such colored trees in general edge-colored graphs. We state some general results about rainbow spanning subgraphs using all colors in the edge-coloredgraph. After that, we give sufficient conditions on the total number of colors of the graph which guarantee the existence of proper spanning trees in G^c. The number of colors depends on the connectivity. In next section, we obtain conditions on the number of colors and the rainbow degree of the graph for the existence of proper spanning trees. All the above results hold for graphs with a lower bound on the number of vertices, which are specified in the theorems'statements.

5.1 Some general results

Before explaining the main results of this section we present some general results for edge-colored connected graphs [1]. We consider here a rainbow spanning subgraph using all colors in G^c which gives us a tool to construct proper spanning trees. In the following lemmas we consider the largest component of such a rainbow spanning subgraph and study its order and connectivity.

1. This is a joint work with Raquel Agueda, Yannis Manoussakis, Gervais Mendy and Rahul Muthu.

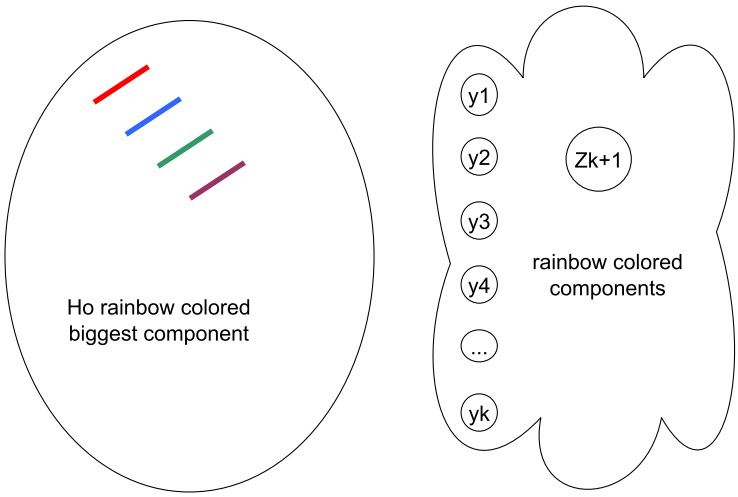

FIGURE 5.1 – Rainbow partition

Lemma 5.1.1. *Let G^c be a c-edge-colored connected graph such that $|V(G^c)| = n$. If $n \geq \frac{k^2+3k+4}{2}$, for some $k \in \mathbb{N}$, and $c \geq \frac{(n-k-1)(n-k-2)}{2} + k + 1$, then the size of the largest component in any rainbow spanning subgraph using all colors is at least $n-k-1$.*

Démonstration. Consider a rainbow spanning subgraph $\mathcal{H}_{rs}$ in G^c using all the colors. We shall use $\mathcal{H}_0$ to denote its largest component and $\mathcal{H}_1$ to denote the subgraph induced on the rest of the vertices.

Let $p \geq 0$ represent $|V(\mathcal{H}_1)|$. Thus, $|V(\mathcal{H}_0)| = n-p$. As the subgraph $\mathcal{H}_{rs}$ is a rainbow using all the colors which are present in the graph, the number of edges in such a subgraph is equal to the number of colors in the original graph, whose lower bound is $\frac{(n-k-1)(n-k-2)}{2} + k + 1$. Further, the number of edges in the subgraph must be at most $\frac{p(p-1)}{2} + \frac{(n-p)(n-p-1)}{2}$, assuming all possible edges within the vertices of each part of the partition are present.

Combining these two facts, we get

$$\frac{p(p-1)}{2} + \frac{(n-p)(n-p-1)}{2} \geq \frac{(n-k-1)(n-k-2)}{2} + k + 1,$$

which reduces to the following inequality

$$2p^2 - 2np + 2n + 2nk - k^2 - 5k - 4 \geq 0$$

Solving this inequality on p, we get that

$$p \in \{0, \ldots, k+1\} \cup \{n-k-1, \ldots, n\}.$$

Moreover, since $|V(\mathcal{H}_1)| = p$, we have $p \leq k+1$. Otherwise, let us suppose that $p \geq n-k-1$, then the size of the largest component $\mathcal{H}_0$ - and hence the size of each component in $\mathcal{H}_{rs}$ - is at most $k+1$. In this case the total number of edges in the subgraph is at most $\frac{n}{k+1}\frac{(k+1)\,k}{2} = \frac{nk}{2}$, which is strictly less than $\frac{(n-k-1)\,(n-k-2)}{2} + k + 1$, a lower bound on the number of colors for $n \geq \frac{k^2+3k+4}{2}$. This proves the lemma. □

Lemma 5.1.2. *Let G^c be a c-edge-colored connected graph such that $|V(G^c)| = n$ with $n > \frac{k^2+3k+4}{2}$ and $c \geq \frac{(n-k-1)\,(n-k-2)}{2} + k + 1$. Let us denote by $\mathcal{H}_{rs}$ a rainbow spanning subgraph using all colors in G^c and by $\mathcal{H}_0$ its largest component, where $|\mathcal{H}_0| = n-p \geq n-k-1$. Then the number of bridges in $\mathcal{H}_0$ is less or equal to $k-p+1$.*

Démonstration. Recall that the number of edges in $\mathcal{H}_{rs}$ is exactly c, which is at least $\frac{(n-k-1)(n-k-2)}{2} + k+1$. As in the previous lemma, $\mathcal{H}_1$ denotes the subgraph induced on the vertices $V(G^c) - V(\mathcal{H}_0)$, so $|V(\mathcal{H}_1)| = p$ and the number of edges in $\mathcal{H}_1$ is at most $\frac{p(p-1)}{2}$. Suppose that the number of bridges in $E(\mathcal{H}_0)$ is $k-p+2$ or greater, thus the number of edges in $\mathcal{H}_{rs}$ is less or equal to $\frac{(n-k-2)(n-k-3)}{2} + k - p + 2 + \frac{p(p-1)}{2}$, corresponding to the case where the deletion of the $k-p+2$ bridges decomposes $\mathcal{H}_0$ into a "big" component and $k-p+2$ singletons.

The following inequality holds :

$$\frac{(n-k-1)(n-k-2)}{2} + k + 1 \leq \frac{(n-k-2)(n-k-3)}{2} + k - p + 2 + \frac{p(p-1)}{2}$$

This inequality reduces to

$$n - k - 2 \leq \frac{(p-1)\,(p-2)}{2},$$

and since $p \leq k+1$, implies that

$$n - k - 2 \leq \frac{k(k-1)}{2}$$

which contradicts the hypothesis $n > \frac{k^2+3k+4}{2}$. □

Lemma 5.1.3. *Let G^c be a c-edge-colored connected graph such that $|V(G^c)| = n$ with $n > \frac{k^2+3k+4}{2}$ and $c \geq \frac{(n-k-1)\,(n-k-2)}{2} + k + 1$. Let us denote by $\mathcal{H}_{rs}$ a rainbow spanning subgraph using all colors in G^c and by $\mathcal{H}_0$ its largest component. If the deletion of the bridges in $\mathcal{H}_0$ disconnects $\mathcal{H}_0$ into a component $\widetilde{\mathcal{H}}_0$ and m singletons, then $\widetilde{\mathcal{H}}_0$ is at least $(k-m+2)$-edge-connected.*

Démonstration. Recall that the number of edges in $\mathcal{H}_{rs}$ is exactly c, which is at least $\frac{(n-k-1)(n-k-2)}{2} + k+1$. As in the previous results $\mathcal{H}_1$ denotes the subgraph induced on the vertices $V(G^c) - V(\mathcal{H}_0)$.

The number of edges in $\mathcal{H}_1$ is at most $\frac{(k+1)k}{2}$ since $|\mathcal{H}_1| \leq k+1$ by Lemma 5.1.1. We have m bridges in $E(\mathcal{H}_0)$ connecting m singletons to the "big" component $\widetilde{\mathcal{H}}_0$, so that the number of edges in $\widetilde{\mathcal{H}}_0$ is at least $\frac{(n-k-1)(n-k-2)}{2} + k + 1 - \frac{(k+1)k}{2} - m$. Therefore at least $\frac{(n-k-1)(n-k-2)}{2} - \frac{k(k+1)}{2}$ edges remain after the removal of any $k-m+1$ edges from that component, *i.e.,* since $\frac{(n-k-1)(n-k-2)}{2}$ precisely corresponds to the number of edges in a complete graph on $n-k-1$ vertices, the former deletion results in a deficit of at most $\frac{(k+1)k}{2}$ edges. Since $n > \frac{k^2+3k+4}{2}$, we have $\frac{k(k+1)}{2} = \frac{k^2+3k+4}{2} - (k+2) < n-k-2$, the edge-connectivity of a complete graph on $n-k-1$ vertices. □

5.2 Minimum number of colors in terms of the connectivity

In this section we consider G^c to be a c-edge-colored simple k-connected graph. Our main theorem of this section, Theorem 5.2.3, is stated at the end. It follows as a direct consequence of two propositions which consider different cases depending on the number of bridges that can appear in the largest component of a rainbow spanning subgraph using all colors in G^c.

Proposition 5.2.1. *Let G^c be a c-edge-colored simple k-connected graph such that $|V(G^c)| = n$ with $n \geq \frac{k^2+3k+4}{2}$ and $c \geq \frac{(n-k-1)(n-k-2)}{2} + k + 1$. Let $\mathcal{H}_{rs}$ be a rainbow spanning subgraph using all colors in G^c and $\mathcal{H}_0$ its largest component, where $|\mathcal{H}_0| = n-p$ and the number of bridges in $\mathcal{H}_0$ is $m \leq k-p$. Then G^c has a properly edge-colored spanning tree.*

Démonstration. By Lemma (5.1.1) $\mathcal{H}_0$ has at least $n-k-1$ vertices. In this case we set the parameter k to be the connectivity of the graph and, since the subgraph is rainbow, every spanning tree of its components is proper.

Let $\mathcal{H}_1$ denote, again, the subgraph induced on the vertices $V(G^c) - V(\mathcal{H}_0)$. Since $|V(\mathcal{H}_1)| = p$ and G^c is k-connected, then $|N_{\mathcal{H}_0}(x)| \geq k-p+1$ for all $x \in \mathcal{H}_1$. Let $\widetilde{B}_0$ be the set of vertices of $\mathcal{H}_0$ that have neighbors both in $\mathcal{H}_0$ and in $\mathcal{H}_1$ and let $\widetilde{B}$ be the set of all vertices in $\widetilde{B}_0$ is the following set :

$$\left\{x \in \widetilde{B}_0, \text{ such that } E(x,\mathcal{H}_0) = B(x,\mathcal{H}_0) \text{ and } c(E(x,\mathcal{H}_0)) = c(E(x,\mathcal{H}_1))\right\}.$$

Let us now consider the bipartite graph

$$\widetilde{G} := \left(V(G^c),\ E(G^c) - \left(E(\mathcal{H}_0) \cup E(\mathcal{H}_1) \cup E\left(\widetilde{B}, \mathcal{H}_1\right)\right)\right).$$

Let us apply Hall's theorem (see [19]) to the subgraph $\widetilde{G}$. This subgraph contains a matching of size k crossing the cut $(\mathcal{H}_0, \mathcal{H}_1)$ if and only if $|S| \leq \left|N^{\widetilde{G}}_{\mathcal{H}_0}(S)\right|$ for all subsets S of $\mathcal{H}_1$where $N^{\widetilde{G}}_{\mathcal{H}_0}(S)$ denotes the neighbors of S in $\mathcal{H}_0$ joined by edges in $E\left(\widetilde{G}\right)$, i.e.,

$$N^{\widetilde{G}}_{\mathcal{H}_0}(S) = \left\{x \in \mathcal{H}_0 \text{ such that } xy \in E\left(\widetilde{G}\right) \text{ for some } y \in S\right\}.$$

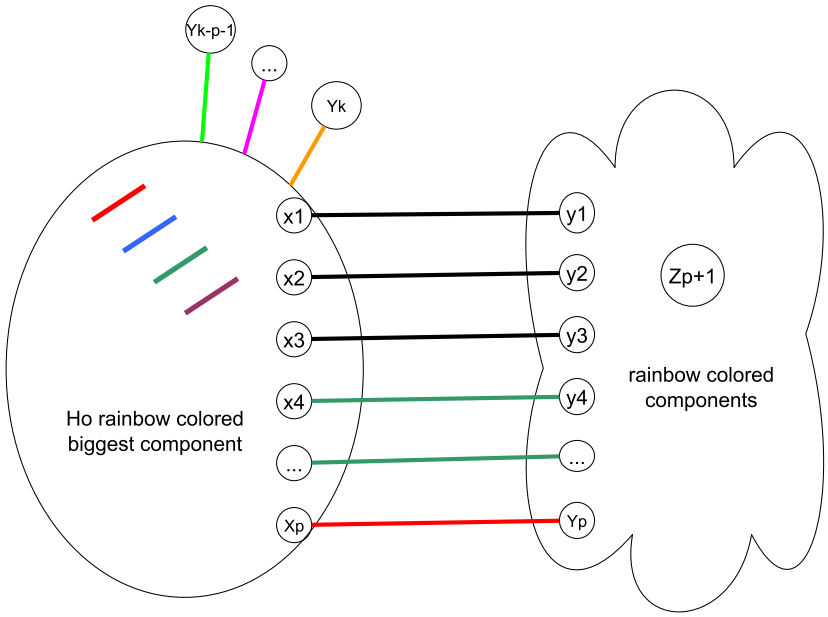

FIGURE 5.2 – K-connected case

To see this, let us assume that such a matching does not exist. Then there is some subset S of $\mathcal{H}_1$ such that $\left|N^{\widetilde{G}}_{\mathcal{H}_0}(S)\right| < |S| = s$. Under this assumption, let us study the order of $N_G(S)$. This number is given by :

$$|N_G(S)| = |N_{\mathcal{H}_0}(S)| + |N_{\mathcal{H}_1}(S)| = \left|N^{\widetilde{G}}_{\mathcal{H}_0}(S)\right| + \left|N^{G-\widetilde{G}}_{\mathcal{H}_0}(S)\right| + |N_{\mathcal{H}_1}(S)| < s + k - p + p - s = k,$$

since $\left|N^{G-\widetilde{G}}_{\mathcal{H}_0}(S)\right|$ is bounded by the number of bridges in $\mathcal{H}_0$.

This contradicts the fact that G is k-connected, since there is a subset S of G that can be disconnected from the rest of the graph by removing a number of vertices strictly smaller than k. Therefore we have proved that $\widetilde{G}$ contains a matching of $\mathcal{H}_0$ into $\mathcal{H}_1$ and therefore G contains a matching of size k crossing the cut $(\mathcal{H}_0, \mathcal{H}_1)$ where all the problems to attach the vertices in $\mathcal{H}_1$ to $\mathcal{H}_0$ have been avoided and the result holds. □

Proposition 5.2.2. *Let G^c be a c-edge-colored simple k-connected graph such that $|V(G^c)| = n$ with $n \geq \dfrac{k^2+3k+4}{2}$ and $c \geq \dfrac{(n-k-1)(n-k-2)}{2} + k + 1$. Let $\mathcal{H}_{rs}$ be a rainbow spanning subgraph using all colors in G^c and $\mathcal{H}_0$ its largest component, where $|\mathcal{H}_0| = n-p$ and the number of bridges in $\mathcal{H}_0$ is $m = k - p + 1$. Then G^c has a properly edge-colored spanning tree.*

Démonstration. We shall denote $Y_0 = \{y_1, \ldots, y_m\}$ the set of vertices in $\mathcal{H}_0$, such that for every $i \in \{1, \ldots, m\}$ there exists a unique $x_i \in N_{\mathcal{H}_0}(y_i)$ and $b_i = x_i y_i$ is a bridge. The deletion of $\{b_1, \ldots, b_m\}$ divides $\mathcal{H}_0$ in a total of $m+1$ components where m of them are singletons.

Let us denote $\widetilde{\mathcal{H}}_0 := (V(\mathcal{H}_0) - Y_0, E(\mathcal{H}_0) - \{b_1, \ldots, b_m\})$. By Lemma (5.1.3) $\widetilde{\mathcal{H}}_0$ is at least $(k-m+2)$-edge-connected.

Since the graph G^c is k-connected, there is a matching of size k crossing the cut $\left(\widetilde{\mathcal{H}}_0, Y_0 \cup \mathcal{H}_1\right)$. Denote the edges of this matching by $E_{\mathcal{M}} = \{b_1, \ldots, b_m, e_{m+1}, \ldots, e_k\}$. Let the endpoints of these edges in $\widetilde{\mathcal{H}}_0$ be respectively $X = \{x_1, \ldots, x_k\}$ and the corresponding ones in $\widetilde{\mathcal{H}}_1$ be $Y_1 = \{y_{m+1}, \ldots, y_k\}$. Let Y be $Y_0 \cup Y_1$ and z be the unmatched vertex in $\mathcal{H}_1$, then $\widetilde{\mathcal{H}}_1 := (V(\mathcal{H}_1) \cup Y_0, E(\mathcal{H}_1))$ where $V(\mathcal{H}_1) \cup Y_0 = Y \cup \{z\}$, and we shall distinguish two cases according to $N(z)$.

Case 1 : $N(z) \nsubseteq X \cup Y$.

Let $z' \in N_{\widetilde{\mathcal{H}}_0}(z) - X$. In this case, we add zz' and $b_1, \ldots, b_m, e_{m+1}, \ldots, e_k$ to $\mathcal{H}_{rs}$ deleting any edge in $\mathcal{H}_{rs}$ incident at z' with color $c(zz')$, and also any edge incident to e_i with color $c(e_i)$ for all $i \in \{m+1, \ldots, k\}$. Any such deletion on account of e_i happens at exactly one of x_i or y_i, but not both. Observe that there is no edge in $\widetilde{\mathcal{H}}_0$ incident to b_i for $i \in \{1, \ldots, m\}$ with color $c(b_i)$ since b_i was the unique edge in $\mathcal{H}_{rs}$ colored in $c(b_i)$. So this entire process leads to the deletion of at most $k-m+1$ edges from $\widetilde{\mathcal{H}}_0$, which thus remains connected.

All the vertices of $\widetilde{\mathcal{H}}_1$ have been connected to $\widetilde{\mathcal{H}}_0$ to get a connected spanning properly colored subgraph. Hence this subgraph, and also G^c, has a proper spanning tree.

Case 2 : $\mathbf{N(z) \subseteq X \cup Y}$

If there is a vertex $x_{i_0} \in N_X(z)$ such that $c(z, x_{i_0}) \neq c(b_{i_0})$ if $i_0 \in \{1, \ldots, m\}$ (respectively, $c(z, x_{i_0}) \neq c(e_{i_0})$ if $i_0 \in \{m+1, \ldots, k\}$), then we simply add all the edges in $E_{\mathcal{M}}$ and (z, x_{i_0}), deleting any edge incident to x_i of the same color for $i \in \{m+1, \ldots, k\}$ (recall again that there is no edge in $\widetilde{\mathcal{H}}_0$ incident to b_{i_0} for $i \in \{1, \ldots, m\}$ with color $c(b_i)$ since b_{i_0} was the unique edge in $\mathcal{H}_{rs}$ colored in $c(b_i)$). Similarly, if there is an edge of the form (z, y_{i_0}) with $c(z, y_{i_0}) \neq c(e_{i_0})$ if $i_0 \in \{1, \ldots, m\}$ (respectively, $c(z, x_{i_0}) \neq c(b_{i_0})$ if $i_0 \in \{m+1, \ldots, k\}$). In either case, this connects all the vertices of $\widetilde{\mathcal{H}}_1$ to the rest of the graph, while preserving the connectedness of $\widetilde{\mathcal{H}}_0$. This yields a spanning connected properly edge colored subgraph implying the existence of a proper spanning tree for this subgraph and G^c.

We now consider the scenario where for every edge (z, x_i) and (z, y_j) we have $c(z, x_i) = c(e_i)$ and $c(z, y_j) = c(e_j)$ for all $i \in \{1, \ldots, k\}$ such that $(z, x_i),\ (z, y_j) \in E(G^c)$. Since the number of edges in $\mathcal{H}_{rs}$ is exactly c, which is at least $\dfrac{(n-k-1)(n-k-2)}{2} + k + 1$ and $\left|V\left(\widetilde{\mathcal{H}}_0\right)\right| \leq \dfrac{(n-k-1)(n-k-2)}{2}$, there are at least $k+1$ differently colored edges in $\widetilde{\mathcal{H}}_1$. Thus one of these edges must be of the form (y_i, y_j) in $E\left(\widetilde{\mathcal{H}}_1\right)$ where $c(y_i, y_j) \notin c(E(z, Y_1))$. We may assume, by relabeling the edges in $E_{\mathcal{M}}$, if necessary, along with their endpoints in X and Y_2, that this edge is indeed (y_{m+1}, y_{m+2}).

If there exists a vertex $z' \in N(z) \cap \{x_{m+1}, x_{m+2}, y_{m+1}, y_{m+2}\}$, we connect z using $e = zz'$ and add up (y_{m+1}, y_{m+2}) and all edges in $E_{\mathcal{M}}$ except for the one which is incident to e in order to connect all the points in Y_1. We again must delete all the edges in $\widetilde{\mathcal{H}}_0$ which are incident to any e_i colored with $c(e_i)$ for all $i \in \{m+1, \ldots, k\}$. This preserves the connectedness of $\widetilde{\mathcal{H}}_0$, the

properness and the resultant subgraph is spanning and connected.

If $N(z) \cap \{x_{m+1}, x_{m+2}, y_{m+1}, y_{m+2}\}$ is the empty set, then $N_G(\mathcal{H}_1) \subseteq \{x_3, \ldots, x_k\}$. Then we disconnect $\mathcal{H}_1$ by deleting $k-2$ vertices from original graph G^c, which contradicts the k-connectivity of G^c. □

Theorem 5.2.3. *Let G^c be a c-edge-colored simple k-connected graph such that $|V(G^c)| = n$. If $n \geq \frac{k^2+3k+4}{2}$ and $c \geq \frac{(n-k-1)(n-k-2)}{2} + k + 1$, then G^c has a properly edge-colored spanning tree.*

Démonstration. By Lemma 5.1.2 the number of bridges on $E(\mathcal{H}_0)$ is less or equal to $k-p+1$. The result then holds directly from Proposition 5.2.1 and Proposition 5.2.2. □

This bound is the best possible, in the sense that there exist edge-colored simple k-connected graphs colored with $\frac{(n-k-1)(n-k-2)}{2} + k$ colors and without a proper spanning tree. More precisely, consider a rainbow complete edge-colored simple graph on $n-k-1$ vertices. Add $k+1$ new vertices $x_1, \ldots, x_{k+1}$. Consider, also, k new colors, say $c_1, \ldots, c_k$, not already used in the rainbow complete graph. Pick now k vertices $y_1, \ldots, y_k$ in the complete graph and then add all monochromatic edges on color c_i, between each x_i and all y_j, $1 \leq j \leq k$, $i = 1, \ldots, k$. Finally add edges $x_{k+1}x_i$ each on color c_i, for $i = 1, \ldots, k$. The resulting graph, although colored with $\frac{(n-k-1)(n-k-2)}{2} + k$ colors, has no proper spanning tree as vertex x_{k+1} cannot be appended to each component of any proper spanning forest of the rest of the graph.

5.3 Minimum number of colors in terms of the rainbow degree

In this section we consider G^c to be a connected graph with rainbow degree $rd(G^c) = k$. As in the previous section, the main result, Theorem 5.3.3, is presented as a corollary of two propositions which consider different cases depending on the order of the largest component of a rainbow spanning subgraph using all colors in G^c.

Proposition 5.3.1. *Let G^c be a c-edge-colored connected graph with rainbow degree $rd(G^c) = k$ such that $|V(G^c)| = n$ with $n \geq \frac{k^2+3k+4}{2}$ and $c \geq \frac{(n-k-1)(n-k-2)}{2} + k + 1$. Let $\mathcal{H}_{rs}$ be a rainbow spanning subgraph using all colors in G^c such that the order of its largest component is $n-k-1$, then G^c has a properly edge-colored spanning tree.*

Démonstration. Let $\mathcal{H}_0$ the largest component of $\mathcal{H}_{rs}$ and $\mathcal{H}_1$ the subgraph induced on the rest of the vertices. Since $|V(\mathcal{H}_0)| = n-k-1$, $|V(\mathcal{H}_1)| = k+1$. Let X be the set $\{x_1, x_2, \ldots, x_k\}$ and $V(\mathcal{H}_1) = \{x\} \cup X$.

We shall consider two different cases depending on $rd_{\mathcal{H}_1}(\mathcal{H}_1)$.

Case 1 : $rd_{\mathcal{H}_1}(\mathcal{H}_1) = k$.

The graph G^c is connected, then $N_{\mathcal{H}_0}(\mathcal{H}_1) \neq \varnothing$. Suppose without loss of generality that there exists $N_{\mathcal{H}_0}(x) \neq \varnothing$. Since $rd_{\mathcal{H}_1}(x) = k$, then $xx_i \in E(G^c)$ and $c(xx_i) \neq c(xx_j)$ for all $i,\ j \in \{1, \ldots,\ k\}$ such that $i \neq j$.

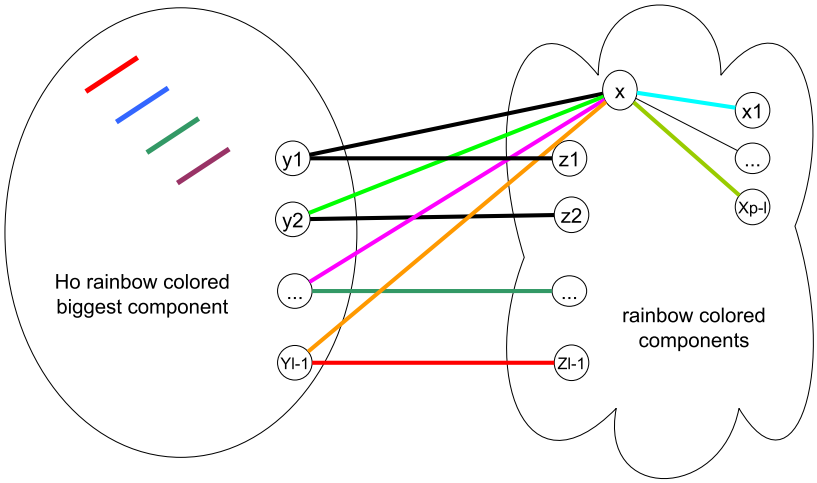

FIGURE 5.3 – Rainbow degree case

Let $y \in N_{\mathcal{H}_0}(x)$ and $e = xy$. In this case we can consider $\mathcal{H}_0 \cup e \cup E(x,\ X)$ unless there is an edge $b \in E(\mathcal{H}_{rs})$, adjacent to e, such that $c(b) = c(e)$.

1.A. : If $b \in E(\mathcal{H}_0)$, consider $\mathcal{H}_0 \cup e \cup E(x,\ X) - b$. The subgraph $\mathcal{H}_0$ remains connected after the deletion of the edge b since it is $(k+2)$-edge-connected (see Lemma 5.1.3). We then have a connected, spanning and properly colored subgraph and we can extract a properly edge-colored spanning tree.

1.B. : If $b \in E(\mathcal{H}_1)$, *i.e.*, b is of the form xx_i for some $i \in \{1, 2, \ldots, k\}$, say $i = 1$. We consider $N_{\mathcal{H}_1}(x_2)$, the neighborhood of x_2 in $\mathcal{H}_1$. Since $N_{\mathcal{H}_1}(x_2) = V(\mathcal{H}_1)$, necessarily $x_1 \in N_{\mathcal{H}_1}(x_2)$. We also know that $c(x_1x_2) \neq c(xx_2)$ since $rd_{\mathcal{H}_1}(x_2) = k$, then we join x_1 to $\mathcal{H}_0$ via the path yxx_2x_1.

Case 2 : $rd_{\mathcal{H}_1}(\mathcal{H}_1) < k$.

We shall consider two cases :

2.A. : There is a $(k+1)$-matching $\mathcal{M}$ between $\mathcal{H}_0$ and $\mathcal{H}_1$.

Let us denote $E_{\mathcal{M}} = \{e, e_1, \ldots, e_k\}$ the set of edges running between $\mathcal{H}_0$ and $\mathcal{H}_1$. We shall use this matching in order to join every vertex in $V(\mathcal{H}_1)$ to $\mathcal{H}_0$ and delete any edge $b, b_i \in E(\mathcal{H}_0)$ adjacent to e and e_i, respectively, such that $c(e) = c(b)$ and $c(e_i) = c(b_i)$. There are at most $k+1$ such edges, therefore the deletion of all of them preserves the connectivity of $\mathcal{H}_0$ since $\mathcal{H}_0$ is $(k+2)$-edge-connected.

2.B. : There is no $(k+1)$-matching $\mathcal{M}$ between $\mathcal{H}_0$ and $\mathcal{H}_1$.

Let $l \in \mathbb{N}$ such that $k - l$ is the maximum rainbow degree in $\mathcal{H}_1$ for all vertices in $V(\mathcal{H}_1)$. Suppose again without loss of generality that $k - l = rd_{\mathcal{H}_1}(x)$ and $\{x_1, \ldots,\ x_{k-l}\} = N_{\mathcal{H}_1}(x)$.

Since $rd(x) \geq k$, there also exist at least l vertices, $y_1, \ldots, y_l$, in $N_{\mathcal{H}_0}(x)$ such that all edges xx_i and xy_j are pairwise differently colored. Let Y be $\{y_1, \ldots, y_l\}$. Our aim is to connect the vertices of the set $V(\mathcal{H}_1) - (N_{\mathcal{H}_1}(x) \cup \{x\}) = \{x_{k-l+1}, \ldots, x_k\}$ to $\mathcal{H}_0$. For the sake of readability, let us rename this set as $Z = \{z_1, z_2, \ldots, z_l\}$. Since $rd_{\mathcal{H}_1}(z_i) \leq k-l$ and $rd(z_i) \geq k$ for all $i \in \{1, \ldots, l\}$, thus all these vertices must have at least l neighbors in $\mathcal{H}_0$ and therefore we can always find a l-matching $(Z, N_{\mathcal{H}_0}(Z))$. If for some $i \in \{1, \ldots, l\}$, there exists $y_{l+1} \in N_{\mathcal{H}_0}(z_i) - Y$, then there is a $(l+1)$-matching $(Z \cup x, N_{\mathcal{H}_0}(Z \cup x))$ and by deleting, if necessary, all edges adjacent to the extreme vertices of the matching in $\mathcal{H}_0$ colored the same way, we get the desired result.

Otherwise $Y = N_{\mathcal{H}_0}(z_i) = N_{\mathcal{H}_0}(x)$ for all $i \in \{1, \ldots, l\}$, and all but one vertex in Z, say z_1, are connected via the edges $y_i z_i$ for all $i \in \{2, \ldots, l\}$ and $y_1 x$. We have to find a new way to connect z_1 to $\mathcal{H}_0$.

Since the number of colors in $c(E(\mathcal{H}_1))$ is greater or equal to $k+1$, there is at least one extra color apart from the ones in $E(x, X)$. We shall study different cases depending on where this new colored edge lies, i.e., the new colored edge is of the form :

$z_i z_j$ or $z_i x_j$. Then we can add the edge $z_1 y_i$, delete the edge of the matching $y_i z_i$ and use the path $y_j z_j z_i$ or $y_1 x x_j z_i$, respectively to connect back z_i.

$x_i x_j$, for some $i, j \in \{1, \ldots, k-l\}$. In this case, as $|N_{\mathcal{H}_1}(z_1)| = k-l$, we simply change the roles that x and z_1 play and we will be back to the previous case.

□

Proposition 5.3.2. *Let G^c be a c-edge-colored connected graph with rainbow degree $rd(G^c) = k$ such that $|V(G^c)| = n$ with $n \geq \frac{k^2+3k+4}{2}$ and $c \geq \frac{(n-k-1)(n-k-2)}{2} + k + 1$. Let $\mathcal{H}_{rs}$ be a rainbow spanning subgraph using all colors in G^c such that the order of its largest component is greater than $n-k-1$, then G^c has a properly edge-colored spanning tree.*

Démonstration. Let $\mathcal{H}_0$ the largest component of $\mathcal{H}_{rs}$ and $\mathcal{H}_1$ the subgraph induced on the rest of the vertices. Let $p \in \mathbb{N}$ represent $|V(\mathcal{H}_1)|$, so that $|V(\mathcal{H}_0)| = n-p$ and let $l \in \mathbb{N}$ such that $p-l$ is the maximum rainbow degree in $\mathcal{H}_1$ for all vertices in $V(\mathcal{H}_1)$. Let $x \in V(\mathcal{H}_1)$ be a vertex such that $rd_{\mathcal{H}_1}(x) = p-l$, then $|rd_{\mathcal{H}_0}(x)| \geq k-p+l$. This number is greater or equal to $k-p+1$, which is the maximum number of bridges that can appear in $E(\mathcal{H}_0)$ as it was proved in Lemma (5.1.2).

We shall study two different cases depending on the value of l.

Case 1 : $l = 1$.

Since $rd_{\mathcal{H}_1}(x) = p-1$, x is adjacent to every point in $V(\mathcal{H}_1)$ and the edges of type xx_i are pairwise differently colored. Additionally, $|rd_{\mathcal{H}_0}(x)| \geq k-p+1 \geq 1$ since $p \leq k$. Therefore, there exists $y \in N_{\mathcal{H}_0}(x)$ such that $c(xy) \notin c(x, \mathcal{H}_1)$. The subgraph $\mathcal{H}_0$ remains connected after removing a possible edge in $E(\mathcal{H}_0)$ adjacent to xy and colored in $c(xy)$, so we get the desired result.

Case 2 : $l > 1$.

Let us set $N_{\mathcal{H}_1}(x) = \{x_1, x_2, \ldots, x_{p-l}\}$, since $rd_{\mathcal{H}_0}(x) \geq k-p+l$ and the number of bridges in $\mathcal{H}_0$ is less or equal than $k-p-1$, there are at least $l-1$ vertices, $y_1,\ y_2, \ldots, y_{l-1}$, in $N_{\mathcal{H}_0}(x)$ such that all edges xy_i are pairwise differently colored and such that there is no bridge in $E(\mathcal{H}_0)$ incident to them. Let Y be $\{y_1, \ldots, y_{l-1}\}$ ($|Y| = l-1$). Consider $Z = V(\mathcal{H}_1) - (N_{\mathcal{H}_1}(x) \cup \{x\}) = \{\ z_1,\ z_2, \ldots,\ z_{l-1}\}$. Our aim is again to connect the vertices of the set Z to $\mathcal{H}_0$. Since $rd_{\mathcal{H}_0}(z_i)$ are greater or equal to $k-p+l$ for all $i \in \{1, 2, \ldots, l-1\}$, there is a $(l-1)$-matching between $N_{\mathcal{H}_0}(\{x\} \cup Z)$ and $\mathcal{H}_0$ so that none of the adjacent edges in $E(\mathcal{H}_0)$ is a bridge. Then only one vertex, say z_1, is left to be joined. Since $rd(z_1) \geq k$ and $rd_{\mathcal{H}_1}(z_1) \leq p-l$, then $rd_{\mathcal{H}_0}(z_1) \geq k-p+l$ which is greater or equal to l since $k \geq p$. Thus there exists a vertex $y \in V(\mathcal{H}_0) - Y$ such that $yz_1 \in E(G^c)$ and $c(yz_1) \notin c(z_1, V(\mathcal{H}_1) \cup Y)$ and we obtain a l-matching between $N_{\mathcal{H}_0}(\{x\} \cup Z)$ and $\mathcal{H}_0$. Hence the proof is finished.

□

Theorem 5.3.3. *Let G^c be a c-edge-colored connected graph with rainbow degree $rd(G^c) = k$ and $n \geq \dfrac{k^2+3k+4}{2}$. If $c \geq \dfrac{(n-k-1)(n-k-2)}{2} + k + 1$, then G^c has a properly edge-colored spanning tree.*

Démonstration. Let us consider $\mathcal{H}_{rs}$, a rainbow spanning subgraph in G^c using all colors and denote $\mathcal{H}_0$ its largest component and $\mathcal{H}_1$ the subgraph induced on the rest of the vertices. By Lemma (5.1.1), we know that $|\mathcal{H}_0| \geq n-k-1$, so the proof of the Theorem follows directly from Proposition 5.3.1 and Proposition 5.3.2. □

5.4 Conclusion

In this chapter, we considered sufficient conditions that guarantee the existence of proper spanning trees in edge-colored graphs. The conditions that we imposed are relationships between various parameters of colored graphs, such as total number of colors, number of vertices, connectivity and number of differently colored edges incident to a vertex.

In the next chapter we will consider edge-colored multigraps. Since proper Hamiltonian paths are also proper spanning trees, we give sufficient conditions in relationships between total number of edges, number of colors and number of vertices, guarantee the existence of proper Hamiltonian paths in edge-colored multigraphs.

Chapitre 6

Colored trees in Multigraphs

Contents

6.1 Introduction

In this chapter we focus on edge-colored multigraphs [1]. Since we try to find proper and weak proper trees, there is no interest to suppose that the parallel edges are colored with the same color. We give sufficient conditions involving various parameters, such as the number of edges, rainbow degree, etc., in order to guarantee the existence of properly edge-colored hamiltonian paths in edge-colored multigraphs. Results involving only degree conditions can be found in [3].

In this chapter all multigraphs are assumed to be connected.
We consider two families of multigraphs $H_{k,k+3}$ and $H_{k,k+2}$ without proper hamiltonian paths :

(1) $H_{k,k+3}$ denotes a 2-edge-colored multigraph on $2k+3$ vertices, $k \geq 1$, defined as follows : consider a complete red graph on k vertices joined with red edges to an independent set of $k+3$ vertices. Finally, superpose it with a complete blue graph on $2k+3$ vertices.

(2) $H_{k,k+2}$ denotes a c-edge-colored multigraph on $2k+2$ vertices, $k \geq 1$ and $c \geq 3$. Consider a rainbow complete graph on k vertices joined with edges of all possible colors to an independent set of $k+2$ vertices.

Before presenting the main results of this chapter, we introduce some of our preliminary results for connected graphs, that can be of independent interest.

1. This is a joint work with Raquel Agueda Mate, Yannis Manoussakis, Gervais Mendy and Leandro Montero.

6.2 Preliminary results

Lemma 6.2.1. *Let G be a simple non-colored graph on $n \geq 14$ vertices. If $m \geq \frac{(n-3)(n-4)}{2} + 4$, then G has a matching M of size $|M| \geq \lceil \frac{n-2}{2} \rceil$.*

Démonstration. We can assume that the graph is connected, otherwise it is easy to see that the only possible case is when G has two components, the first one having two adjacent vertices and the second one having $n-2$ vertices and at least $\frac{(n-3)(n-4)}{2} + 3$ edges. Then, the result immediately follows from the well known Theorem in [23].
Therefore, let G be a graph of order n. Let M be a maximum matching in G and let U be the set of unmatched vertices. We shall prove that $|U| \leq 2$ by contradiction.

Assume that $|U| \geq 3$. In fact, it is enough to prove it for $|U| = 3$. Set $U = \{u_1, u_2, u_3\}$. Since M is a maximum matching, neither we can add any new edge to M nor we can replace a set of edges in M, in order to obtain a new matching in G which is larger than M. Necessarily, U is an independent set, otherwise we could add a new edge to M.

Let us denote the complement of G by $\overline{G}$. We will count the number of edges in $E(\overline{G})$, i.e. those edges which cannot be present in $E(G)$ because otherwise the matching M would not be maximum.

Since $m = |E(G)| \geq \frac{(n-3)(n-4)}{2} + 4$, then $\left|E(\overline{G})\right| \leq 3n - 10$. In the sequel we will prove this lemma for n odd, the case even being very similar.

Since the graph is connected, we shall prove the result depending on the size of a maximum matching $\widetilde{M}$ between U and $G - U$.

We distinguish three cases, depending on the cardinality of $|\widetilde{M}|$.

Case 1 : $\left|\widetilde{M}\right| = 1$. There is a unique vertex in $N(U)$ and therefore there are at least three vertices of degree 1 in $V(G)$. This leads to a contradiction, since the number of edges in $E(G)$ would be at most $\frac{(n-3)(n-4)}{2} + 3$ (the edges which are present in a complete graph on $n-3$ vertices, and three more, which connect the vertices of degree one to the complete graph), which contradicts the hypothesis of the lemma.

Case 2 : $\left|\widetilde{M}\right| = 2$. In this case, there are two vertices $v_1, v_2 \in G - U$, $v_1 \neq v_2$, such that $u_1v_1, u_2v_2, u_3v_2 \in E(G)$ and $N(\{u_2, u_3\}) \subseteq \{v_1, v_2\}$, otherwise we would have a larger matching between the sets U and $G - U$. Therefore, all edges of the type u_2v and u_3v, for any $v \in G - U - \{v_1, v_2\}$, are in $E(\overline{G})$.
Also, $\{u_1, u_2, u_3\}$ is an independent set, there are at least three more edges in $E(\overline{G})$.
Since M is a perfect matching in $G - U$, the vertices v_1 and v_2 are the extremities of some edge in M. If $v_1v_2 \in M$, then we can replace v_1v_2 by u_1v_1 and u_2v_2, contradicting the fact that M is maximum. Thus, there exist $w_1, w_2 \in G - U - \{v_1, v_2\}$, $w_1 \neq w_2$, such that $v_1w_1, v_2w_2 \in M$.
Necessarily, $w_1w_2 \in E(\overline{G})$. Otherwise we can replace $\{v_1w_1, v_2w_2\}$ by $\{u_1v_1, u_2v_2, w_1w_2\}$ and

obtain a larger matching. The edge u_1w_2 is also in $E(\overline{G})$, otherwise we can replace v_2w_2 in M by u_1w_2 and u_2v_2 in order to get a larger matching.
So far we have at least $2(n-5)+3+1+1 = 2n-5$ edges in $E(\overline{G})$.

There are $\frac{n-7}{2}$ edges, $e_1, e_2, ..., e_{\frac{n-7}{2}}$, in $M - \{v_1w_1, v_2w_2\}$. We shall study the conditions given by the possible connections between the edges e_i, for $i = 1, ..., \frac{n-7}{2}$, and the set $\{w_1, w_2\}$. Necessarily, $|N_{e_i}(\{w_1, w_2\})| \leq 2$, $\forall\ i = 1, 2, ..., \frac{n-7}{2}$. Otherwise we can find a matching of size two, say $\{f_{i1},\ f_{i2}\}$, connecting e_i and $\{w_1, w_2\}$ and we can replace $\{e_i, v_1w_1, v_2w_2\}$ by $\{u_1v_1,\ u_2v_2,\ f_{i1}, f_{i2}\}$. We can assure then, that there are at least $n-7$ new edges in $E(\overline{G})$. By adding these edges to the $2n-5$ edges obtained before, we conclude that there are at least $3n-12$ edges in $E(\overline{G})$.

We shall study two different subcases.

(1) $N(\{v_1\}) \cap \{u_2, u_3, w_2\} = \varnothing$. Then 3 new edges belong to $E(\overline{G})$. By adding these new edges to those we have counted previously, we have $3n-12+3 = 3n-9$ edges in $E(\overline{G})$. This number is larger than $3n-10$, the result follows.

(2) $N(v_1) \cap \{u_2, u_3, w_2\} \neq \varnothing$. Then $u_1w_1 \in E(\overline{G})$. In fact, if for example, $u_2v_1 \in E(G)$, then we can replace v_1w_1 in M by $\{u_1w_1, u_2v_1\}$, so one more edge is in $E(\overline{G})$. Now, if u_2v_1 or u_3v_1 are in $E(G)$ then all edges of the type u_1v, for any $v \in G - U - \{v_1, v_2\}$, are in $E(\overline{G})$ since $\left|\widetilde{M}\right| = 2$. Therefore both u_2v_1 and u_3v_1 are missing, or $n-5-2 = n-7$ edges are missing (since we have already counted u_1w_1 and u_1w_2). In the first case we arrive to $3n-9$, which is a contradiction. In the second case we have $3n-11+n-7 = 4n-18 > 3n-10$, which is again a contradiction.

Case 3 : $\left|\widetilde{M}\right| = 3$. In this case there are three distinct vertices $v_1, v_2, v_3 \in G - U$, such that $u_iv_i \in E(G)$, for $i = 1, 2, 3$. Since there is a perfect matching in $G - U$, the vertices v_1, v_2 and v_3 are the extremities of some edge in M. If $v_iv_j \in M$, then we can replace v_iv_j by u_iv_i and u_jv_j contradicting that M is maximum. Thus, there exist three distinct vertices $w_1, w_2, w_3 \in G - U - \{v_1, v_2, v_3\}$, such that $v_1w_1, v_2w_2, v_3w_3 \in M$.
Necessarily, $w_iw_j \in E(\overline{G})$ for all $i, j = 1, 2, 3$. Otherwise we can replace v_iw_i and v_jw_j by u_iv_i, u_jv_j and w_iw_j and obtain a larger matching. In the same way, $u_iw_j \in E(\overline{G})$ for all $i, j = 1, 2, 3$, $i \neq j$, otherwise we can replace v_jw_j by $\{u_jv_j, u_jv_j\}$. Since U is an independent set, there are at least $3+3+6 = 12$ edges in $E(\overline{G})$.
If $u_iw_i \in E(G)$ then $N(v_i) \cap \{u_j, w_j\} = \varnothing$ for $j \neq i$. Otherwise, if for example $u_jv_i \in E(G)$, we can replace v_iw_i in M by u_iw_i and u_jv_i. Thus, there are 3 more edges in $E(\overline{G})$.
Now, there are $\frac{n-9}{2}$ edges, $e_1, e_2, ..., e_{\frac{n-9}{2}}$, in $M - \{v_1w_1, v_2w_2, v_3w_3\}$. We shall study the conditions given by the possible connections between the edges e_i, for $i = 1, ..., \frac{n-9}{2}$, and the sets

$U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3\}$.
Necessarily, $|N_{e_i}(W)| \leq 3$, for all $i = 1, 2, ..., \frac{n-9}{2}$. Otherwise, we can find a matching of size two, say $\{f_{ij},\ f_{ik}\}$, $j, k \in \{1, 2, 3\}$, connecting e_i and $\{w_j, w_k\}$, and we can replace $\{e_i, v_j w_j,\ v_k w_k\}$ by $\{u_j v_j,\ u_k v_k,\ f_{ij},\ f_{ik}\}$.
Also $|N_{e_i}(U)| \leq 3$, for all $i = 1, 2, ..., \frac{n-9}{2}$. Otherwise, we can find a matching of size two, say $\{g_{ij},\ g_{ik}\}$, $j, k \in \{1, 2, 3\}$, connecting e_i and $\{u_j, u_k\}$, and we can replace e_i by $\{\ g_{ij},\ g_{ik}\}$. We can assure then, that there are at least $2(3\frac{n-9}{2}) = 3n - 27$ new edges in $E(\overline{G})$.
By adding up these edges to the 15 edges considered before, we obtain $3n - 12$ edges in $E(\overline{G})$.
Recall that we have denoted the extremities of the edge e_i by p_i and q_i. Without loss of generality, let us assume that if $|N_{e_i}(u_j)| = 1$, for all $i = 1, 2, ..., \frac{n-9}{2}$ and $j = 1, 2, 3$, then $N_{e_i}(u_j) = p_i$. Thus, either $|N_{e_i}(U)| \leq 1$ or $N_{q_i}(U) = \varnothing$, $i = 1, 2, 3$. Otherwise we can find a matching of size 2 between U and e_i and again, we can define a larger matching.
Now, for $n \geq 15$, there are at least 3 edges in $M - \{v_1 w_1, v_2 w_2, v_3 w_3\}$. If $N_{e_i}(U) = \varnothing$, then 3 more edges are in $E(\overline{G})$ and, by adding these new edges to the $3n - 12$ that we have already found, the result follows.
If $|N_{e_i}(U)| = 1$, then either there exists $j \neq i$ such that $N_{e_i}(U) \neq N_{e_j}(U)$ or $N_{e_i}(U) = N_{e_j}(U)$ for all $i \neq j$. In the first case $q_i q_j \in E(\overline{G})$ and either $p_1 u_1$ or $p_1 q_2$ is in $E(\overline{G})$. We again have 3 new edges in $E(\overline{G})$ and the proof is finished. In the latter case all edges connecting the edges in $M - \{v_1 w_1, v_2 w_2, v_3 w_3\}$ and U have a common vertex, say u_1. Again, at least 3 more edges are in $E(\overline{G})$.

Only one case is left : if $N_{q_i}(U) = \varnothing$, then $q_i q_j \in E(\overline{G})$ for all $i = 1, 2, 3$ and we also get 3 new edges in $E(\overline{G})$.
This completes the proof. □

Lemma 6.2.2. *Suppose that for every vertex x in G^c, $c = 2$, $rd(x) = 2$ and $n \geq 14$. If $m \geq (n-3)(n-4) + 3n - 2$, then G^c has two matchings M^r and M^b in colors, say red and blue, such that $|M^r| = \lfloor \frac{n}{2} \rfloor$ and $|M^b| \geq \lceil \frac{n-2}{2} \rceil$.*

Démonstration. Let us denote by $E^r(G^c)$ and $E^b(G^c)$ the set of edges colored in red and blue, of sizes $|E^r(G^c)| = m^r$ and $|E^b(G^c)| = m^b$, respectively. Observe that, since for every vertex x in G^c, $rd(x) = 2$, we have $d^i(x) \geq 1$ for $i \in \{r, b\}$. Observe also that $m^i \geq \frac{(n-3)(n-4)}{2} + 4$ for $i \in \{r, b\}$, since this threshold is tight when the multigraph is complete on one of the colors.

Let us see the case where n is odd. By Lemma 6.2.1, there exist two matchings M^r and M^b each of the size $\frac{n-1}{2}$, so the result is straightforward.

Let us see now the case where n is even. Then, again by Lemma 6.2.1, there exist two matchings, M^r and M^b, each of the size at least $\frac{n-2}{2}$. We shall prove the result by contradiction.

Let us consider monochromatic subgraphs in colors r and b respectively, and suppose that $|M^r| = |M^b| = \dfrac{n-2}{2}$.
Let $U = \{u_1, u_2\}$ denote the independent set of unmatched vertices in M^r. The vertices u_1 and u_2 are connected to the extremities of the edges in M^r.
We claim that there exist two distinct vertices v_1, v_2 in $V(G) - \{u_1, u_2\}$, such that $u_1v_1, u_2v_2 \in E^r(G^c)$. Otherwise, if $N^r(\{u_1, u_2\}) = v_1$, this vertex is the extremity of some edge v_1w_1 in M^r and then we distinguish two cases.
First, if $N^r(w_1) = v_1$, then we have three distinct vertices of degree one which leads us to a contradiction with the total number of edges.
Second, there exist $w_2 \in N^r(w_1) - \{v_1\}$ and v_2, such that $v_2w_2 \in M^r$. Then we can replace $\{v_1w_1, v_2w_2\}$ by $\{u_1v_1, w_1w_2\}$. After this permutation, we have a new matching of same size but u_1 is replaced by v_2.
So, as we claimed, there always exist two distinct vertices v_1, v_2 in $V(G) - \{u_1, u_2\}$ such that $u_1v_1,\ u_2v_2 \in E^r(G^c)$ (u_2 and v_2 are those vertices in the case we have just seen).
Now, the edge v_1v_2 is not in M^r, otherwise we can replace it by $\{u_1v_1, u_2v_2\}$ and get a larger matching in color r, contradicting that M^r is maximum.
So, there are two vertices w_1 and w_2 in $V(G) - \{u_1, u_2, v_1, v_2\}$ such that $v_1w_1,\ v_2w_2 \in M^r$. Observe that there can be at most two edges connecting the endpoints of any edge in M^r to the set $\{u_1, u_2\}$, i.e. there are at least two missing edges for each edge in M^r, in total $2(\frac{n-2}{2})$. Similarly, for w_1 and w_2, we have the same constraints and therefore, this also means two edges are missing for each edge in $M^r - \{v_1w_1, v_2w_2\}$, which gives the total of $2(\frac{n-6}{2})$. Otherwise, suppose that there is an edge v_3w_3 with three edges between its endpoints to the vertices w_1 and w_2. So we can replace v_1w_1 and v_2w_2 by say w_1v_3, w_2w_3, v_1u_1 and v_2u_2 to obtain a perfect matching.
Finally, as u_1 and u_2 are independent, the edge u_1u_2 is missing. If we sum up these numbers, we conclude that there are at least $(n-2) + (n-6) + 1 = 2n - 7$ edges missing in color red.
The same reasoning can be done with the matching M^b to obtain $2n-7$ blue missing edges. So, the total number of missing edges in colors red and blue is $4n - 14$. Since the complement of G^c has the edge set of the size less than or equal to $3n - 10$, for $n \geq 6$ we have a contradiction. Therefore, the result holds. □

Lemma 6.2.3. *Let G^c, $c \geq 2$ be a connected c-edge-colored multigraph. Suppose that G^c contains a proper path $P = x_1y_1x_2y_2 \dots x_py_p$, $p \geq 2$, such that each edge x_iy_i is red. If G^c does not contain a proper cycle C with vertex set $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$, then there are at least $(c-1)(2p-2)$ missing edges in G^c.*

Démonstration. Let $P = x_1y_1x_2y_2 \dots x_py_p$ be a proper path, $p \geq 2$, such that each edge x_iy_i is red. Let blue be the second color. The blue edge x_1y_p cannot be in G^c, otherwise $C = x_1y_p \dots x_1$ is a proper cycle.
Suppose that the blue edges x_1x_i are present in G^c, for $i = 2, \dots, p$. Then, the blue edges $y_{i-1}y_p$

can not be in G^c, otherwise we have the proper cycle $C = x_1x_i \dots y_py_{i-1} \dots x_1$, that contradicts our hypothesis. Therefore, for each edge $y_{i-1}x_i$ in the path, either the blue edge x_1x_i or the blue edge $y_{i-1}y_p$ is missing. So there are $\frac{2p-2}{2}$ missing blue edges.
Now, suppose that the blue edges x_1y_i are present in G^c, for $i = 3, \dots, p-2$. Then, the blue edges $x_{i+1}y_p$ cannot be in G^c at the same time as x_iy_{i+1}, $y_{i-1}x_{i+2}$ or $y_{i-1}y_{i+1}$, x_ix_{i+2}, otherwise we have the proper cycles $C = x_1y_ix_iy_{i+1}x_{i+1}y_p \dots\ x_{i+2}y_{i-1} \dots x_1$ or $C = x_1y_ix_ix_{i+2} \dots y_px_{i+1}y_{i+1}y_{i-1} \dots x_1$. The minimum in this case corresponds to one missing edge $x_{i+1}y_p$ for each edge $y_{i-1}x_i$ in the path, for $i = 2, \dots, p-1$. Therefore, there are $\frac{2p-6}{2}$ missing blue edges.

For the moment we have $2p-3$ missing blue edges. To obtain the last missing edge, suppose that the blue edge x_2y_p is present in G^c. Then, it cannot be at same time with x_1y_2, y_1x_3 or x_1x_3, y_1y_2, otherwise we obtain the proper cycles $C = x_1y_2 \dots x_2y_p \dots x_3y_1x_1$ or $C = x_1x_3 \dots y_px_2 \dots y_2y_1x_1$. The minimum in this case corresponds to one missing edge x_2y_p. We remark that the blue edges x_2y_p, y_1y_2 and y_1x_3 were not counted before. The edges x_1x_3 and x_1y_2 were supposed to exist, otherwise in order to obtain the last missing edge we consider the symmetric case, i.e. using the blue edge $y_{p-1}x_1$ (if exists).

In total, there are $\frac{2p-2}{2} + \frac{2p-6}{2} + 2 = (2p-2)$ missing blue edges in G^c. As we have $c-1$ colors different from red, that gives us $(c-1)(2p-2)$ missing edges, as desired.
Note that this number of missing edges is the same as in the simplest case, this is, if all edges different from red x_1x_i and x_1y_i, for $i = 2, \dots, p$ are not present in G^c. □

Lemma 6.2.4. *Let G^c be a connected c-edge-colored multigraph, $c \geq 2$. Let M be a matching of G^c in one color, say red, of size $|M| \geq \lceil \frac{n-2}{2} \rceil$. Let $P = x_1y_1x_2y_2 \dots x_py_p$, $p \geq 2$, be a maximum proper path compatible with M. Then, the following cases are possible :*

- ***(1)*** *n is even, $|M| = \frac{n}{2}$ and $2p < n$*
  - ***(1a)*** *If P is not a proper cycle, then there are at least $(n-2+pn-2p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-4)(c-1)$ for $p = \frac{n-2}{2}$.*
  - ***(1b)*** *If P is a proper cycle, then there are at least $(2pn-4p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-4)(c-1)$ for $p = \frac{n-2}{2}$.*
- ***(2)*** *n is odd, $|M| = \frac{n-1}{2}$ and $2p < n-1$*
  - ***(2a)*** *If P is not a proper cycle, then there are at least $(n-3-p+pn-2p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-6)(c-1)$ for $p = \frac{n-3}{2}$.*
  - ***(2b)*** *If P is a proper cycle, then there are at least $(2pn-2p-4p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-6)(c-1)$ for $p = \frac{n-3}{2}$.*
- ***(3)*** *n is even, $|M| = \frac{n-2}{2}$ and $2p < n-2$*

- ***(3a)*** *If P is not a proper cycle, then there are at least $(n-4-2p+pn-2p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-8)(c-1)$ for $p=\frac{n-4}{2}$.*
- ***(3b)*** *If P is a proper cycle, then there are at least $(2pn-4p-4p^2)(c-1)$ missing edges in G^c different from red and the minimum value of this function is $(2n-8)(c-1)$ for $p=\frac{n-4}{2}$.*

Démonstration. Before starting the proof, we remark that the edges x_1y_1 and x_py_p are red. Otherwise we can easily extend the path by adding an edge of the matching to P.

First, suppose that n is even and $2p<n$. Since M is of the size $\frac{n}{2}$, there are $\frac{n-2p}{2}$ red edges outside P. Let us denote these edges by e_i for $i=1,\ldots,\frac{n-2p}{2}$. Suppose that P is not a proper cycle. Let blue be the second color. By Lemma 6.2.3 there are $(2p-2)$ missing blue edges. As the path is maximum, we cannot extend P having an edge e_i neither at the beginning nor at the end of it, thus there are no blue edges between the vertices x_1, y_p and the edges e_i. Therefore, there are $4\frac{n-2p}{2}$ blue missing edges. Finally, as we cannot add any edge e_i in between the path, then there at most 2 blue edges between the edges e_i and the edges y_ix_{i+1}, $i=1,\ldots,p-1$. So, there are $2\frac{n-2p}{2}\frac{2p-2}{2}$ missing blue edges different from red. By summarizing and simplifying all these numbers and having $c-1$ colors different from red, we obtain that there are $(n-2+pn-2p^2)(c-1)$ missing edges in G^c different from red. If we calculate the minimum value of this function, we obtain $(2n-4)(c-1)$ for $p=\frac{n-2}{2}$ and case **(1a)** holds.
Now, if P is also a proper cycle, then there cannot exist any edge, different from red between all vertices of P and the edges e_i, and, therefore there are $2\frac{n-2p}{2}2p(c-1)=(2pn-4p^2)(c-1)$ missing edges different from red. Again, minimizing the function we obtain the same result as above and the case **(1b)** holds.

Suppose now that n is odd, $M=\frac{n-1}{2}$ and $2p<n-1$, or n is even, $M=\frac{n-2}{2}$ and $2p<n-2$. In both cases, same arguments as before apply, just by replacing n with $n-1$ or $n-2$ respectively, in the number of missing edges. This, because we have $n-1$ matched vertices and one non-matched vertex for the first case and $n-2$ matched vertices and two non-matched vertices for the second case. □

Lemma 6.2.5. *Let G be a connected non-colored simple graph on n vertices, $n\geq 9$. If $m\geq \frac{(n-2)(n-3)}{2}+3$, then G has a matching M of the size $|M|=\lfloor\frac{n}{2}\rfloor$.*

Démonstration. By a theorem of [23], a 2-connected graph on $n\geq 10$ vertices and $m\geq \frac{(n-2)(n-3)}{2}+5$ edges, has a hamiltonian cycle. So, if we add a new vertex v to G, joined to all its vertices, we obtain that $G+\{v\}$ has $m\geq \frac{(n-2)(n-3)}{2}+3+n=\frac{(n-1)(n-2)}{2}+5$ edges. Therefore, $G+\{v\}$ has a hamiltonian cycle, i.e. G has a hamiltonian path and this implies that there exists a matching M in G of the size $|M|=\lfloor\frac{n}{2}\rfloor$. □

Definition 6.2.6. *A path P is compatible with a matching M if its edges belong alternatively to M and not to M.*

Definition 6.2.7. *A vertex is called monochromatic if all its incident edges are colored by the same color.*

6.3 Main results

Theorem 6.3.1. *Let G^c be a c-edge-colored multigraph, $c = 2$. Suppose that for every vertex x in G^c, $rd(x) = 2$ and $n \geq 14$. If $m \geq (n-3)(n-4) + 3n - 2$, then G^c has a proper hamiltonian path.*

Démonstration. Let us suppose that G^c has not a proper hamiltonian path. We will show that $E(\overline{G^c})$ has more than $3n - 10$ edges, i.e., G^c has less than $(n-3)(n-4) + 3n - 2$ edges, a contradiction to the hypothesis of the theorem.

- n even : By Lemma 6.2.2, G^c has two matchings M^r, M^b, such that $|M^r| = \frac{n}{2}$ and $|M^b| \geq \frac{n-2}{2}$. Take the longest proper path $P = x_1y_1x_2y_2 \dots x_py_p$ compatible with the matching M^r and the longest proper path $P' = x'_1y'_1x'_2y'_2 \dots x'_{p'}y'_{p'}$ compatible with the matching M^b. Suppose $2p < n$ and $2p' < n - 2$. Since $|M^r| = \frac{n}{2}$, $c(x_1y_1) = c(x_py_p) = r$. Otherwise, we can easily extend the path by adding an edge of the matching to P. Suppose also that $c(x'_1y'_1) = c(x'_{p'}y'_{p'}) = b$. So, as P (resp. P') is properly colored, the edges x_iy_i (resp. $x'_iy'_i$) are red (resp. blue) for $i = 1, \dots, p$ (resp. p') and the edges y_ix_{i+1} (resp. $y'_ix'_{i+1}$) are blue (resp. red) for $i = 1, \dots, p-1$ (resp. $p'-1$). We have the following possibilites :
  - Neither P nor P' are proper cycles.
  - P' is a proper cycle but P is not.
  - P is a proper cycle but P' is not.
  - Both P and P' are proper cycles.

  First, observe that in any case, by Lemma 6.2.4, there are at least $2n - 4$ blue missing edges and $2n - 8$ red ones. We obtain $4n - 12 > 3n - 10$ missing edges. A contradiction. Therefore we conclude that $2p = 2p' = n - 2$.

  Now suppose that P is not a proper cycle, then again by Lemma 6.2.4, there are at least $2n - 4$ blue missing edges. Now, let v_1, w_1 be the vertices in $G^c - P'$. It is clear that if there exists a blue edge joining v_1 and w_1, we are under the same conditions of the red matching, and therefore in contradiction with the number of edges. Otherwise we count red missing edges if we cannot extend P' to a proper hamiltonian path. First, if we have 3 or more red edges between v_1, w_1 and x'_1,$y'_{p'}$, we have a hamiltonian path and we are done. So, there are at least 2 red missing edges. Second, if the path P' is not a proper cycle again by Lemma 6.2.3 there are $2p' - 2 = n - 4$ red missing edges. Now we combine and sum up all these number and we arrive to $2n - 4 + n - 4 + 2 = 3n - 6 > 3n - 10$ missing edges. Again a contradiction. Finally, if the path P' is a proper cycle we have that, if there are 3 or more red edges between v_1, w_1 and the endpoints of the red edges

of the cycle, we can add v_1 and w_1 to the path P' as following. Suppose without losing generality that v_1 is adjacent to y'_i and w_1 to x'_{i+1}, both with red edges and the red edge $y'_i x'_{i+1}$ is present in the proper cycle. So we form the proper hamiltonian path starting with the red edge $v_1 y'_i$ then following the proper cycle in the direction of x'_i until we arrive to x'_{i+1} and finally we take the red edge $x'_{i+1} w_1$. Therefore if there are not these 3 or more red edges, they are missing $2\frac{2p'-2}{2} = n-4$ red edges. If we sum up again we arrive to $2n-4+n-4+2 = 3n-6 > 3n-10$ missing edges. Again a contradiction.

Suppose now, that P is a proper cycle. Let e be the red edge of the matching outside the path P. If there existe a blue edge between any endpoint of e to the cycle, we easily obtain a proper hamiltonian path, taking the edge e, any blue edge going to the cycle and then following it in the appropriate direction. Otherwise, as the graph is connected, all edges between the endpoints of e to the path are blue. Now, as $rd(G^c) = 2$, we have that there must exist a blue edge parallel to e and therefore we obtain the proper hamiltonian path just as before but starting with the blue edge parallel to e. So, we covered all the cases for n even.

– n odd : Again by Lemma 6.2.2, G^c has two matchings M^r, M^b, such that $|M^r| = |M^b| = \frac{n-1}{2}$. We take the longest proper paths P and P' compatibles with the matching just as before. Suppose now that $2pn-1$ and $2p' < n-1$. As the even case, we have exactly the same four possibilities.

Observe that in any case, by Lemma 6.2.4, there are at least $2n-6$ blue missing edges and $2n-6$ red ones. We obtain $4n-12 > 3n-10$ missing edges. A contradiction. Therefore we have that $2p = 2p' = n-1$. We will count blue missing edges if we cannot extend P to a proper hamiltonian path. Same argument holds for red edges since we are under the same conditions. Now, let v be vertex outside the path P. It is clear that is P is also a proper cycle we trivially obtain a proper hamiltonian path since the graph is connected and $rd(v) = 2$. So, as P is not a proper cycle we have by Lemma 6.2.3, that there are $2p-2 = n-3$ blue missing edges. If we have blue edges between v and x_1 or y_p, we have a hamiltonian path and we are done. So, there are 2 blue missing edges. Finally, if there exists a blue edge between x_1 and x_i $(i = 2, \ldots, p)$, then it cannot exist the blue edge vy_{i-1} $(i = 2, \ldots, p)$, otherwise we form the proper hamiltonian path $vy_{i-1} \ldots x_1 x_i \ldots y_p$. Similar, if we have the blue edges y_p and y_i, with vx_{i+1} $(i = 1, \ldots, p-1)$. Note that since P is not also a proper cycle we cannot have at the same time the blue edges $x_1 x_i$ and $y_p y_{i-1}$ $(i = 2, \ldots, p)$. Therefore, there are $\frac{2p-2}{2} = \frac{n-3}{2}$ blue missing edges. Concluding, if we make the sum and multiply it by two, since the same number of red missing edges is obtained with P' we obtain that there are $2(n-3+2+\frac{n-3}{2}) = 3n-5$ missing edges. A contradiction.

Since we covered all the cases, the theorem is proved. □

Theorem 6.3.1 is the best possible for n odd, $n \geq 14$, consider a complete blue graph, say A, on $n-3$ vertices. Add 3 new vertices v_1, v_2, v_3 and join them to a vertex v in A with blue

edges. Finally, superpose the obtained graph with a complete red graph on n vertices. Although the resulting 2-edge-colored multigraph has $(n-3)(n-4)+3n-3$ edges, it has no proper hamiltonian path since one of the vertices v_1, v_2, v_3 cannot belong to such a path.

The two lemmas below will be useful in view of Theorem 6.3.4.

Lemma 6.3.2. *Let G^c be a 2-edge-colored multigraph. Assume that G^c, contains a proper cycle C, of length at most $n-2$ and that there exists a red edge xy in $G^c - C$. If $d^b_C(x) + d^b_C(y) > |C|$, then G^c has a proper cycle of length $|C|+2$ containing xy.*

Démonstration. Set $C = x_1y_1x_2y_2 \dots x_sy_sx_1$, where x_iy_i are the red edges of C, $i = 1, 2, \dots, s$. Then $d^b_{\{x_i,y_i\}}(x) + d^b_{\{x_i,y_i\}}(y) \leq 2$, otherwise if $d^b_{\{x_i,y_i\}}(x) + d^b_{\{x_i,y_i\}}(y) \geq 3$, then the cycle $x_1y_1x_2y_2 \dots x_ixyy_i \dots x_sy_sx_1$ is the desired one. It follows that $\sum_i d^b_{\{x_i,y_i\}}(x) + d^b_{\{x_i,y_i\}}(y) \leq 2\frac{|C|}{2} = |C|$, a contradiction to the hypothesis of the lemma. This completes the proof. □

Lemma below is established in view of Theorem 6.3.4

Lemma 6.3.3. *If $m \geq f_0(n) = (n-1)(n-2)+n$, then G^c has a proper hamiltonian cycle if n is even, and a proper cycle of length $n-1$ otherwise.*

Démonstration. Let red and blue be the two colors of G^c. The proof is by induction on n. The theorem is true for small values of n, say $n = 2, 3, 4$. Let us suppose that $n \geq 5$ and that the theorem is true until $n-1$. We will prove it for n. By a Theorem of [3], if for any vertex $d^r(x) \geq \lceil \frac{n+1}{2} \rceil$ and $d^b(x) \geq \lceil \frac{n+1}{2} \rceil$, then G^c has a proper hamiltonian cycle for n even and a proper cycle of length $n-1$ for n odd. Let us suppose therefore that for some vertex, say x, and for some color, say red, $d^r(x) \leq \lceil \frac{n+1}{2} \rceil - 1$. Notice now that $d^r(x) > 0$ and $d^b(x) > 0$, otherwise, for example if $d^r(x) = 0$, then $m \leq n(n-1) - (n-1) = (n-1)^2 < (n-1)(n-2) + n$, a contradiction. Similarly $d^r(x) + d^b(x) \geq 3$, otherwise, if $d^r(x) + d^b(x) \leq 2$, then $m \leq n(n-1) - 2n + 4 = n^2 - 3n + 4 < (n-1)(n-2) + n$, again a contradiction. Thus we may conclude that there are two distinct neighbors, say y and z, of x such that $c(xy) = r$ and $c(xz) = b$ in G^c. Replace now the vertices x, y, z by a new vertex s such that $N^b(s) = N^b_{G^c - \{x,y,z\}}(y)$ and $N^r(s) = N^r_{G^c - \{x,y,z\}}(z)$. The obtained graph, say G', has $n-2$ vertices and at least $(n-1)(n-2)+n-[(n-1)+\lceil \frac{n+1}{2} \rceil - 1 + 2(n-2)] = n^2 - \frac{11n}{2} + 8 > f_0(n-2) = n^2 - 6n + 8$ edges. By the induction hypothesis, G' has a proper hamiltonian cycle for $n-2$ even and a proper cycle of length $n-3$ otherwise. If G' has a proper hamiltonian cycle then coming back to G^c we may easily find a proper hamiltonian cycle in G^c. Assume now that $n-2$ (and thus n) is odd. Let C be a proper cycle of length $n-3$ in G'. If s belongs to C, then as previously we may easily find a proper cycle of length $n-1$ in G^c. Assume therefore that s does not belong to C. By Lemma 6.3.2, if $d^b_C(x) + d^b_C(y) > |C|$ or $d^r_C(x) + d^r_C(z) > |C|$) then we may integrate the edge xy (respectively xz) in C in order to obtain a cycle of length $n-1$. Assume therefore that $d^b_C b(x) + d^b_C(y) \leq |C|$ and $d^r_C(x) + d^r_C(z) \leq |C|$. But then the number of edges of G^c is at most $n(n-1) - (n-3) - (n-3) = n^2 - 3n + 6 < f_0(n) = n^2 - 2n + 2$, again a contradiction. This completes the argument and the proof. □

Theorem 6.3.4. *Let G^c be a 2-edge-colored multigraph. For $n \geq 14$, if $m \geq f_1(n) = n^2-3n+4$, then G^c has a proper hamiltonian path.*

Démonstration. By Theorem 6.3.1, as $n^2 - 3n + 4 \geq (n-3)(n-4) + 3n - 2$, it suffices to prove that for every vertex x of G^c, $rd(x) = 2$, i.e., $d^r(x) \neq 0$ and $d^b(x) \neq 0$. Assume therefore that either $d^r(x) = 0$ or $d^b(x) = 0$ but not both, as $d^r(x) = 0$ and $d^b(x) = 0$ can be easily checked as not possible. In that case $G^c - x$ has at least $n^2 - 3n + 4 - (n-1) = n^2 - 4n + 5 = [(n-1)-1][(n-1)-2]+(n-1)$ edges. By Lemma 6.3.3, $G^c - x$ has either a proper hamiltonian cycle or a cycle of length $n-2$. If $G^c - x$ has a proper hamiltonian cycle then in a trivial maner we join x and the cycle in order to obtain a proper hamiltonian path. Assume therefore that $n-1$ is odd and that $G^c - x$ has a proper cycle, say C, of length $n-2$. Let y be the vertex outside C in $G^c - x$. By using arguments similar to those of Lemma 6.3.2 we may show that $d^r(x) + d^r(y) \leq |C|$ and $d^b(x) + d^b(y) \leq |C|$. It follows that the number of edges of G^c is at most $n(n-1) - 2(n-2) + 2 = n^2 - 3n + 4$ which again is a contradiction unless all inequalities become strict equalities. In particular there is a red and a blue edge between x and y. In that case find a proper path starting at, say x, containing all the vertices of C. Then join y to x by using one edge xy with the appropriate color. The obtained proper path is a hamiltonian one. □

Theorem 6.3.4 is the best possible for $n \geq 14$. In fact, consider a rainbow complete 2-edge-colored multigraph on $n-2$ vertices for n odd. Add two new vertices x_1 and x_2. Then add the red edge x_1x_2 and all red edges between $\{x_1, x_2\}$ and the complete graph. Although the resulting graph has $n^2 - 3n + 3$ edges, it has no proper hamiltonian path, since at least one of the vertices x_1 or x_2 cannot be attached to any such path. Indeed, for n odd, the two extremal edges of any proper hamiltonian path must differ in colors. If $n = 5, 7$, Theorem 6.3.4 does not hold for the graphs $H_{k,k+3}$, $k = 1, 2$.

In the rest of this chapter we will deal with c-edge-colored multigraphs, for $c \geq 3$. Now, we will present a result that allows us to consider just the case $c = 3$.

Lemma 6.3.5. *Let G^c be a c-edge-colored connected multigraph on n vertices, $c \geq 3$ and $m \geq cf(n) + 1$ edges. There exists one color c_j such that if we color its edges with another used color and we delete parallel edges with same color, then the resulting $(c-1)$-edge-colored multigraph is connected and has $m' \geq (c-1)f(n) + 1$ edges, such that if G^{c-1} has a proper hamiltonian path then G^c has one too. Moreover, if $rd(G^c) = k$, $1 \leq k \leq c$, then $rd(G^{c-1}) = k - 1$.*

Démonstration. Let c_i denote the color i, for $i = 1, \ldots, c$, in G^c, and denote by $|c_i|$ the number of edges with color i. Let c_j be the color with less number of edges. Color the edges on color c_j with another used color, say c_l, and delete (if necessary) parallel edges with that color. Call this graph G^{c-1}. By this, we delete at most $|c_j|$ edges. It is clear that this graph is connected since we delete just parallel edges. Also, if is G^{c-1} has a proper hamiltonian path, then, this path is also proper hamiltonian in G^c but maybe with some edges on color c_j (in the case that they have been recolored with c_l). Observe also that if $rd(G^c) = k$ then $rd = (G^{c-1}) = k - 1$

since only the color c_j disappeared. We will show now that $m' \geq (c-1)f(n)+1$. We have two cases. First, if $|c_j| > f(n)$, then clearly $m' \geq (c-1)f(n)+1$ edges since for all i, $|c_i| > f(n)$. For the second case, we have that $|c_j| \leq f(n)$. Now, $m = \sum_{i=1}^{c} |c_i| \geq cf(n)+1$ and therefore $\sum_{i=1, i\neq j}^{c} |c_i| \geq cf(n) - |c_j| + 1 = (c-1)f(n) + f(n) - |c_j| + 1$. This last expression is greater or equal than $(c-1)f(n)+1$ since $f(n)-|c_j| \geq 0$. Finally, we have that G^{c-1} has $m' \geq (c-1)f(n)+1$ edges as desired. □

Theorem 6.3.6. *Let G^c be a c-edge-colored multigraph on n vertices, $n \geq 3$ and $c \geq 3$. If $m \geq \frac{c(n-1)(n-2)}{2} + 1$, then G^c has a proper hamiltonian path.*

Démonstration. First of all by Lemma 6.3.5 we can assume that $c = 3$ and $m \geq \frac{3(n-1)(n-2)}{2} + 1$. Since there exists one color, say red, such that the number of edges are at least $\frac{(n-1)(n-2)}{2} + 1$ then by a theorem in [23], there is a hamiltonian red path and therefore a perfect or almost perfect matching. Also, as $m \geq \frac{3(n-1)(n-2)}{2} + 1$ we have that $|E(\overline{G^c})|$ is at most $3n-4$. Now, we will split our proof in the even and odd case.

- n even. Let M^r be the perfect matching in color red, i.e. $|M^r| = \frac{n}{2}$. Take the longest proper path $P = x_1y_1x_2y_2 \dots x_py_p$ compatible with the matching M^r. Set $2p = |P|$. Since $|M^r| = \frac{n}{2}$, $c(x_1y_1) = c(x_py_p) = r$. Otherwise, we can easily extend the path by adding an edge of the matching to P. So, as P is properly colored, the edges x_iy_i are red ($i = 1, \dots, p$) and the edges y_ix_{i+1} are in colors different from red ($i = 1, \dots, p-1$). Now, by Lemma 6.2.4, if $2p < n$ (i.e. P not hamiltonian) then there are at least $2(n-2+pn-2p^2)$ missing edges in G^c different from red if P is not also a proper cycle and $2(2pn-4p^2)$ otherwise. We start with the first case.
  - P is not a proper cycle. Since there are at least $2(n-2+pn-2p^2)$ missing edges different from red the folllowing inequality must be satisfied : $2(n-2+pn-2p^2) \leq 3n-4$. So, if we solve it, we can observe that it is only satisfied for $n = 4$ and $p = \frac{n-2}{2}$. Therefore, for all other values we have a contradiction with the number of edges of G^c. The case $n = 4$ and $p = \frac{n-2}{2}$ is easy to check. So, P must be a proper hamiltonian path.
  - P is a proper cycle. As before, $2(2pn-4p^2) \leq 3n-4$ must be satisfied, and this only happens, again, if $n = 4$ and $p = \frac{n-2}{2}$. Finally P must be a proper hamiltonian path.
- n odd. We will prove the result by induction. Cases $n = 3, 5$ are easy to check. Now, let M^r be the almost perfect matching in color red, i.e. $|M^r| = \frac{n-1}{2}$. Take the longest proper path $P = x_1y_1x_2y_2 \dots x_py_p$ compatible with the matching M^r. As in the even case, $c(x_1y_1) = c(x_py_p) = r$. By Lemma 6.2.4, if $2p < n-1$ then there are at least $2(n-3-p+pn-2p^2)$ missing edges in G^c different from red if P is not also a proper cycle and $2(2pn-2p-4p^2)$ otherwise. We start with the first case.
  - P is not a proper cycle. Since there are at least $2(n-3-p+pn-2p^2)$ missing edges different from red the folllowing inequality must be satisfied : $2(n-2+pn-2p^2) \leq 3n-4$. If we solve it, we have that is true for $n = 5, 7$ and $p = \frac{n-3}{2}$. We will show first the case where $2p = n-1$, this is, we have just one vertex, say v, not in the proper path P. We

have the following cases depending on the degree and neighbors of v.

– v has at least two neighbors, say x and y, in two different colors, say c_1 and c_2 respectively. If we cannot attach v to the path P we have that there are 4 missing edges in colors different from red between v and the vertices x_1, y_p. Also, we have at most 2 edges different from red between v and the edges y_i, x_{i+1} of the path. Therefore there are $2\frac{n-3}{2}$ missing edges. Adding up all this, we conclude that the degree of v in colors different from red is at most $n-3$. So, if we contract the three vertices v, x, y to a new one, say v', we will delete $n-3$ edges different from red and $n-1$ red edges from v. At most $n-3$ edges of color c_1 from x, $n-3$ of color c_2 from y, 3 parallel edges between x and y. And finally all the edges in color c_3 from x but from y, we just delete the edges of color c_3 that are not incident to a common neighbor of x with color c_3. All this is at most $n-3$. Adding up all these numbers we arrive that, after the contraction, we deleted $5n-10$ edges. Now, it is easy to the new 3-edge-colored multigraph on $n-2$ vertices, say G'^c, has at least $\frac{(n-1)(n-2)}{2}+1-(5n-10) \geq \frac{(n-3)(n-4)}{2}+1$, for $n \geq 5$, therefore, by inductive hyphotesis, we have a proper hamiltonian path P' in G'^c. To build the proper hamiltonian path for G^c, we have the following situations. If v' is an endpoint of the path P', this is $v'P''$, and the edge in the path incident to v' has color c_1 (c_2), this means that, because of the edges that we have deleted, this edge must be incident to y (x) in the graph G^c. Therefore, we obtain the proper hamiltonian path in G^c, $xvyP''$ ($yvxP''$) with $c(xv) = c_1$, $c(vy) = c_2$ and $c(yP'') = c_1$ (with $c(yv) = c_2$, $c(vx) = c_1$ and $c(xP'') = c_2$). If the edge in the path P' incident to v' has color c_3, we have the proper hamiltonian path $xvyP''$ or $yvxP''$ in G^c, since the endpoint was a common neighbor with color c_3 to x and y. Now, it the path P' is of the form $P_1v'P_2$ and let v_1, v_2 be the vertices in P_1, P_2 respectively adjacent to v' we have the following situations. If $c(v_1v') = c_1$ and $c(v'v_2) = c_2$, or $c(v_1v') = c_1$ and $c(v'v_2) = c_3$ we have, by the way we deleted the edges for the contraction, the proper hamiltonian path in G^c, P_1yvxP_2 with $c(v_1y) = c_1$, $c(yv) = c_2$, $c(vx) = c_1$ and $c(xv_2) = c_2$, or P_1yvxP_2 with $c(v_1y) = c_1$, $c(yv) = c_2$, $c(vx) = c_1$ and $c(xv_2) = c_3$. The other cases are just symmetric or analog.

– v is monochromatic in red. We have these situations. First, $d(v) \leq n-2$, consider the $G^c - \{v\}$ and delete from a neighbor of v, say w, all edges in two colors in order to have w monochromatic not in red. Call this graph G'^c. Observe that we can always do this since, it is impossible to have 2 monochromatic vertices. So, by this, we delete at most $n-2+n-2+n-2 = 3n-6$ edges. It is easy to see that the graph G'^c on $n-1$ vertices has at least $\frac{(n-1)(n-2)}{2}+1-(3n-6) = \frac{(n-2)(n-3)}{2}+1$, then, by even case, we have a proper hamiltonian path P' in G'^c. Since w is monochromatic not in red, w is either in the beggining or in the end of P' and therefore it is trivial to add v to P' in order to find a proper hamiltonian path in G^c. If this does not hold, we have that $d(v) = n-1$. Therefore the graph $G^c - \{v\}$ on $n-1$ vertices has at least

$\frac{(n-1)(n-2)}{2} + 1 - (n-1) \geq \frac{(n-2)(n-3)}{2} + 1$, then, again, we have a proper hamiltonian path P' in G'^c. Now, if the path either starts or ends with a color different from red, we trivially add v to the path. If not, both of them finish with red. Now, if we can take any parallel edge of these without losing the property of being properly colored we have again that it is easy to add v to the path. Otherwise, we have, without losing generality, the degree in some color, say c_1 of the first vertex of the path, say w, is at most $n-3$. So, we are in the same case as the first one, since we take the graph $G^c - \{v\}$ and we delete from w the edges in that color c_1 and in another color in order to have w monochromatic not in red. By this, we delete $n-1+n-2+n-3 = 3n-6$ edges and finally, the result follows exactly as in the first situation.

- v has one neighbor, say w, in at least two colors. This case is analog to the last one just taking the graph $G^c - \{v\}$ and deleting from w edges in the appropriate two colors in order to have it monochromatic. By this we delete $3 + 2(n-2) = 2n+1$ edges and we have that $\frac{(n-1)(n-2)}{2} + 1 - (2n+1) \geq \frac{(n-2)(n-3)}{2} + 1$ for $n \geq 7$. As $n = 5$ was proved the case is finished.

To finish the case that P is not a proper cycle, we have just to check when $n = 7$ and $p = \frac{n-3}{2}$. So, let $P = x_1y_1x_2y_2$ be the path compatible with the matching, let uv be the edge of the matching not in the path and let w be the non-matched vertex. Since P is maximum with the compatibility property, we cannot add the edge uv to it and since P is not a proper cycle we can check that either the degree of x_1 is at most 10 or the degree of y_2 is at most 10. So, if we contract without losing generality x_1, y_1 and one neighbor of x_1 or y_1 we will delete as most 25 edges. We can check that the new graph satisfy the hyphotesis of the theorem for $n = 5$ and the result will hold with the same argument for the contraction as before.

- P is a proper cycle. As before, $2(2pn - 2p - 4p^2) \leq 3n - 4$ must be satisfied, and this only happens, again, if $n = 5, 7$ and $p = \frac{n-3}{2}$. The case where $2p = n - 1$ it is trivial, since as the graph is connected, the only vertex outside the path, must be connected to it, and therefore, as the path is also a cycle, we can obtain a proper hamiltonian path just taking this vertex connected to the path and the following the cycle in the appropriate direction. For the case $n = 7$ and $p = \frac{n-3}{2}$, let $P = x_1y_1x_2y_2$ be the path compatible with the matching (that is also a proper cycle), let uv be the edge of the matching not in the path and let w be the non-matched vertex. Now, as P is maximum, we cannot add the edge uv to it, so we have just red edges between the vertices u, v and the vertices in the path, meaning that the degree of u and v is at most 10. So, we will contract the vertices u, v and one neighbor of u or v, as we did before. By this, we will delete at most 25 edges. The result will hold with the same argument as last case.

□

Theorem 6.3.6 is the best possible. Indeed, consider a rainbow complete graph on $n-1$ vertices with c colors and add a new isolated vertex x. The resulting graph, although it has

$c\frac{(n-1)(n-2)}{2}$ edges, contains no proper hamiltonian path, as x cannot belong to such a path.

Notice that in the above theorem there is no condition guaranting the connectivity of the underling graph.

Theorem 6.3.7. *Let G^c be a connected 3-edge-colored multigraph on 9 vertices. If $m \geq 72$, then G^c has a proper hamiltonian path.*

Démonstration. Before starting the proof, we will show that if there is no almost perfect matching in some color, then there are at least 15 missing edges in that color. We will use this result through the proof.
CLAIM : If there exist a maximum matching M^i such that $|M^i| \leq 3$ then there are at least 15 missing edges.
Proof of the CLAIM : So, suppose that there is a maximum matching M in some color of size less than or equal to $\frac{n-3}{2} = 3$ and just consider the subgraph of G^c in that color. We distinguish two cases.

- There exist 2 isolated vertices. Then, clearly, there are at least $n-1+n-2 = 2n-3 = 15$ missing edges.
- There are no 2 isolated vertices. We consider that M is maximum. Let q be the number of edges of M, then $M = \{e_1, \cdots, e_q\}$ with $1 \leq q \leq 3$. So, there are $n-2q$ vertices without incident edges between them. Let I be the set of those vertices. Since any arbitrary couple of vertices in I can not have more than two incident edges with e_j $1 \leq j \leq q$, then the number of missing edges between M and I is at least $\frac{2q\binom{n-2q}{2}}{n-2q-1} = q(n-2q)$. This number is justified by the number of combinations of 2 among $|I| = n-2q$ and each vertex of I in this number is counted $n-2q-1$ times. Inside of I, the number of missing edges of color r and b is $(n-2q)(n-2q-1)$. Since I is an independent set, the number of missing edges is $\frac{(n-2q)(n-2q-1)}{2}$. Adding up both numbers, we have at least $\frac{(n-1)(n-2q)}{2} = \frac{8(9-2q)}{2}$ missing edges. For $q = 1, 2$, we have $\frac{8(9-2q)}{2} \geq 20$ missing edges and for $q = 3$, just 12. Therefore, we will study the case $q = 3$. Observe the subgraph induced by the vertices of M. Let us assume first that there are two edges of M such that the underlying subgraph is a clique. Otherwise we have 3 missing edges to add to $\bar{G}$. We distinguish now the following cases :

  - One vertex of I has 2 neighbors in the clique.
    That implies that there is no edge between the others 2 vertices of I and the clique, so the number of missing edges grows up to 4. In same time, since at least one vertex from 2 vertices of I are connected to the edge not in the clique then the second extremity is not connected to the clique and results in 4 more missing edges. The total number in this case is $\frac{8(9-2q)}{2} + 4 + 4 = 12 + 8 = 20$ missing edges.

  - No vertex has more than one neighbor in the clique.
    Then, for any vertex of I, we have 1 more missing edge and that give us 3 more edges.

In same time if the 3 vertices in I are connected to the edge not in the clique then the second extremity is not connected to the clique and results in 4 more missing edges, otherwise if not, then 3 more edges are missing. That give us $\frac{8(9-2q)}{2}+3+4 = 12+7 = 19$ or $\frac{8(9-2q)}{2} + 3 + 3 = 12 + 6 = 18$ missing edges. So we can conclude that the minimum number of missing edges is when M has no clique of size 4, and therefore there are at least $\frac{8(9-2q)}{2} + 3 = 15$ missing edges if $M = 3$.

Since there are no more cases to observe we conclude that if there is no almost perfect matching, there are always 15 missing edges.
This completes the proof of the CLAIM.

Through the proof, if $M^{i,j}$ is a matching in colors i and j, we will denote the path compatible with it as $P_{i,j} = x_1y_1, \ldots, x_py_p$ Also, suppose now that $|M^{i,j}| = \frac{n-1}{2}$ and v is the unmatched vertex. Let v_kw_k be the edges in $M^{i,j}$ for $k = 1, \ldots, \frac{n-1}{2}$. If we have an edge vv_k in color i (or j) from v to some vertex v_k in the matching and the color of the edge v_kw_k is j (i) we will use the set $M'_{i,j} = M^{i,j} \cup \{vv_k\}$.

Since $m \geq 72$ we have that $E(\bar{G^c}) \leq 36$, so we will count the number of missing edges if G^c has no proper hamiltonian path or it if has no matchings in different sizes. If we arrive to 37 missing edges we will have a contradiction.

We have then, the following cases :

– There exist two colors, say i and j, such that $\nexists M'_{i,j} = \frac{n+1}{2}$.
Take the matching $M^{i,j}$ such that $\nexists M'_{i,j} = \frac{n+1}{2}$. Let v be the unmatched vertex. Since it is not possible to create $M'_{i,j}$ there are no edges with color i from v to the edges with color j of the matching, and vice-versa. Another situation is when v has no incident edges at all with colors i and j. Otherwise observe that if it exist an edge in color j from v to an edge in color j (or both in color i) of the matching then, we can replace that edges of the matching by the edge incident to v and restart the operation. Observe that we can continue to find new vertices with bad connections until we return to the same vertices. So if we can not create $M'_{i,j}$, is because it exists a set of vertices S, such that there are no edges in color i between S and the edges in color j of the matching, and vice-versa.
Observe that for each edge in $M^{i,j}$, if v has an edge in color j connecting v with an edge in color j also, of $M^{i,j}$ in one extremity say v', then the second one, say v'', can take his place. Let us suppose vv' is in color j. So at this step $|S| = 2$ and there are 2 edges going to same vertex v'. We have three possibilities :
  – There is one more edge from v going to v'' in color j in this case $S = \{v, v', v''\}$ and suppose there are no more edges in color j and i from vertices in S to the rest of edges in $M^{i,j}$. In this case there are $2(3(n-3)) = 36$ edges in color i and j missing between S and $V(G^c) \setminus S$, and 3 more in color i inside S. That makes 39 in total, a contradiction with the number of edges in G^c.

– There is one more edge from v or v'' going to another edge of the matching with extremities w and w'. In this case $S = \{v, v'', w\}$. Then there are $2(3(n-5)) = 24$ missing edges between S and $V(G^c) \setminus (S \cup \{v', w\})$, 6 missing edges inside S and 6 missing edges parallel to those that can exist between S and $\{v', w\}$. That gives us the total of 36 missing edges. Since, we can suppose that the rest $V(G^c) \setminus S$ are complete in color i and j, there is a cycle in two colors i and j of order $n-3$. Now, using edges in color k and j we can insert say v in this cycle, after that using color k we can add the rest of vertices w and v'' to some edge in color i or j not incident to the edge of color k used to insert v. Unless there is at least one more missing edge that will give us a contradiction.
– There are no more edges at v and v''. Then in this case there are $2(2(n-3)) = 24$ edges in color i and j missing and 2 more edges in color i parallel to vv' and $v'v''$ plus 2 between them, that give us 28 missing edges. Therefore the maximum matching in color i has size at most $\frac{n-3}{2}$. Suppose now that $M^j = \frac{n-1}{2}$. Take the longest proper path $P_j = x_1y_1, \ldots, x_py_p$ compatible with M^j using just the edges in colors j and k. The worst case is when $|P_j| = n-1$. Let x be the vertex outside the path. If this path is also a proper cycle then we can find a proper hamiltonian path since G^c is connected. Otherwise, there are, by Lemma 6.2.3, $2p-2 = n-3 = 6$ missing edges in color k, and since we can not add x to P_j there are 2 more missing edges in color k. Now, if we have the edge xy_1 with color k we can not have the edge x_1x_2 in color k, otherwise we would have a proper hamiltonian path. Therefore there is one more missing edges and with it, we arrive to 37, a contradiction. The case when $M^i \leq \frac{n-3}{2}$ and $M^j \leq \frac{n-3}{2}$ as it is the same when the set S has size 1, will be analyzed later.

As we have just showed, it remains to study just the case $|S| = 1$. Suppose that colors i and j are colors red and blue (r and b respectively) and suppose that given a matching $M^{r,b}$, we have that $\nexists M'_{r,b} = \frac{n+1}{2}$. Let v be the vertex outside $M^{r,b}$. It is clear that there are 16 missing edges in colors r and b from v. Now we have the following situations depending on the size of maximum matchings in colors r and b.

– $|M^r| = |M^b| = \frac{n-1}{2}$. Let P_r and P_b the longest proper paths compatibles with matchings M^r and M^b. Clearly v is outside both paths.

  Now, if $|P_r| < n-1$ (analog if $|P_b| < n-1$) then by Lemma 6.2.4 there are at least $2(2n-6)$ missing edges in colors b and g. Therefore, adding this number with the 16 missing edges from v we have $24 + 16 = 40$ missing edges. A contradiction.

  If $|P_r| = |P_b| = n-1$ we have two situations. If any of them is also a proper cycle, it is clear that G^3 has a proper hamiltonian path, since it is connected therefore there exists at least one edge from v to the cycle. Otherwise, if neither P_r nor P_b are proper cycles, by Lemma 6.2.3, there are at least $3(n-3) = 18$ missing edges in colors r, b and g. Also there are no edges in color g from v to the extremities of the paths. This adds two more missing edges (because the extremities of both paths can be the same vertices). Adding up all these numbers with the 16 missing edges in colors r and b from v, we arrive to

36 missing edges. Finally, for finding the last missing edge to get a contradiction, take for example the path P_r. If we have the green edge vy_1 then we cannot have the green edge x_1x_2 otherwise we obtain the proper hamiltonian path $vy_1x_1x_2, \ldots, y_p$. Also, if we have the green edge vx_2 then we cannot have the green edge y_py_1 otherwise we obtain the proper hamiltonian path $vx_2, \ldots, y_py_1x_1$. Now, since in Lemma 6.2.3, just one of the green edges x_1x_2, y_py_1 were counted, we have at least one more missing edge, giving us 37 missing edges in total. Again a contradiction.

- $|M^r| = \frac{n-1}{2}$ and $|M^b| \leq \frac{n-3}{2}$ (the symmetric case is analog). For this case we will observe the size of the maximum green matching M^g. If $|M^g| \leq \frac{n-3}{2}$ we have that there area at least 15 green missing edges and the same for M^r. Also since we have 8 red missing edges from v (the blue ones might be already counted) we arrive to 38 missing edges. A contradiction. Now, consider when $|M^g| = \frac{n-1}{2}$. Let P_r and P_g the longest proper paths compatibles with matchings M^r and M^g. Clearly v is outside P_r. If $|P_r| < n-1$ we have by Lemma 6.2.4 that there are at least 24 missing edges in colors b and g. Adding these edges with the 16 missing edges from v we arrive to 40 missing edges. A contradiction. Now if $|P_r| = n-1$ we have, as before, that clearly P_r is not also a proper cycle. We obtain by Lemma 6.2.3 that there are 12 missing edges in colors b and g. Moreover, there are two green missing edges between v and the extremities of the path P_r. Now, if we have the green edge vy_1 then we cannot have the green edge x_1x_2 otherwise we obtain the proper hamiltonian path $vy_1x_1x_2, \ldots, y_p$. Also, if we have the green edge vx_2 then we cannot have the green edge y_py_1 otherwise we obtain the proper hamiltonian path $vx_2, \ldots, y_py_1x_1$. Now, since in Lemma 6.2.3, just one of the green edges x_1x_2, y_py_1 were counted, we have at least one more missing edge. As same argument applies for green edges vy_2 or vx_3, and vy_3 or vx_4, we have at least 3 more missing edges. If $|P_g| < n-1$, we have, by Lemma 6.2.4, that there are at least 12 edges in color red. Finally, we have 8 more missing edges from v in color b (red ones might be already counted). Adding up all these numbers we arrive to 37 missing edges, therefore, a contradiction. The last case to observe is when $|P_g| = n-1$. We have already counted from P_r, 17 missing edges in colors b and g. Now, same arguments apply to P_g counting red missing edges. That give us 6 missing edges if P_g is not a proper cycle, 2 more missing edges between the non-matched vertex and the extremities of P_g and 3 more between the non-matched vertex and the interior vertices of P_g. So we have $17 + 11$ missing edges from the paths and 8 more edges in blue color from v (again red ones might be already counted). This gives us 37 missing edges. For the last one, observe that, since $|M^b| \leq \frac{n-3}{2}$ we have that at least one blue edge is not parallel to the edges in the matching M^r, otherwise we have that $|M^b| = \frac{n-1}{2}$. Finally, we have 37 missing edges. A contradiction.
- $|M^r| \leq \frac{n-3}{2}$ and $|M^b| \leq \frac{n-3}{2}$. Since both matchings are of size lower or equal than $\frac{n-3}{2}$ we have that there are at least 15 red missing edges and 15 blue ones. Suppose now that there exists a matching in two colors r and b of size $\frac{n-1}{2}$. Take the longest proper

path $P_{r,b}$ compatible with this matching. If $|P_{r,b}| < n-1$ we have by Lemma 6.2.4 that there are 12 green missing edges therefore 42, a contradiction. If $|P_{r,b}| = n-1$, it can not be a proper cycle, so by Lemma 6.2.3 there are 6 green missing edges and 2 more between v and the extremities of the path, therefore 38. Again a contradiction. Now, it does not exist a matching in two colors r and b of size $\frac{n-1}{2}$. As colors r and b have at least 15 missing edges there are at most 21 edges in color r and 21 edges in color b. Therefore, there are at least 30 edges in color g where the maximum number is 36, so the green subgraph is almost complete. Let S_r be the set of unmatched vertices by the red matching and let S_b be the set of unmatched vertices by the blue matching. Clearly, S_r and S_b are independent sets in colors r and b respectively. So, if there are not blue edges between S_r and M^r, and no edges between S_b and M^b we have 42 missing edges. A contradiction. Then, there must be at least 6 edges between any of them. Suppose that we have a blue edge between S_r and M^r. Since the green color is almost complete, it is easy to find a proper hamiltonian path using those 6 edges alternating with the matching edges and the green ones.

– For all colors r, b, g we have that $M'_{r,b} = M'_{r,g} = M'_{b,g} = \frac{n+1}{2}$. Let e_i, f_j the adjacent edges that complete the sets $M'_{i,j}$ for $i \neq j$, $i, j \in \{r, b, g\}$. Take the longest proper paths $P_{i,j}$, compatibles with these matchings, where the edges e_i, f_j are taken as a single one. Clearly, for all colors i, j, $|P_{i,j}| < n-1$. Now, by Lemma 6.2.4 we have that if each path $P_{i,j}$ is not a proper cycle, then there 12 are missing edges in color k, where $i \neq j \neq k$. If at least one of them is a proper cycle, it is trivial to find a proper hamiltonian path. That gives us 36 missing edges.

  Suposse that we have two colors say i, j such that we can find a proper path compatible with $M'_{i,j}$ of size 5, without using the edges e_i, f_j. Denote the endpoints of the edge e_i by x and z, and the second endpoint of the edge f_j by w. Then we will find one more missing edge in color k by the following observation. If $e_i f_j$ (vzw) has an edge in color k from x_1 to z and an edge in color k from v to w, we can find the proper hamiltonian path $P = wvzx_1y_1 \dots x_py_p$. Otherwise, there is one more missing edge and therefore a contradiction.

  Otherwise suppose that for all colors we can not find such a paths. Then suppose $|P_{i,j}| = 3$ and there the double edge vzw and the edge e_3 outside the path. We will count again the missing edges in color k. As we have so far 24 missing edges in colors i and j, we need at least 13 in color k to arrive to a contradiction. We distinguish two situations :

  – $P_{i,j}$ is a cycle. Then there are no edges in color k between $P_{i,j}$ and e_3. This give us 8 missing edges. Suppose the case that there are no edges in color k between the extremities of e_3 and v, and between the extremities of e_3 and w. In this case there are 4 missing edges more in color k. Observe that $P_{i,j}$ and vzw form a cycle, otherwise there is at least one more missing edge. If there is no cycle $P_{i,j}$ with vzw, then we have 13 missing edges in color k, a contradiction. If there is a cycle $P_{i,j}$ with vzw then we can use the

edges in color k from the extremities of e_3 and z to create a hamiltonian path, unless they are missing. In this case we arrive to 14 missing edges and again a contradiction.

– $P_{i,j}$ is not a cycle. In this case, by Lemma 6.2.4, there are 8 missing edges in color k in interior of $P_{i,j}$ and from e_3 to the extremities of $P_{i,j}$. There are at most 2 edges in color k from e_3 to the edge y_1x_2. We have now the possible situations :

(1) Suppose that both extremities of e_3 are connected with edges in color k to y_1. And suppose that there are 2 edges connecting e_3 to v. In this case there are 2 missing edges from e_3 to w. Since we can not add e_3 and vzw directly to $P_{i,j}$ there are 3 missing edges more from w to x_1, y_2 and x_2. That give us 13 missing edges in color k. A contradiction.

(2) Suppose that both extremities of e_3 are connected with edges in color k to y_1. And suppose that there are 4 edges connecting e_3 to v, and w. In this case there are 3 missing edges from w to x_1, y_2 and x_2, We have the same situation at v. That give us 14 missing edges in color k. A contradiction.

(3) Suppose that both extremities of e_3 are connected with edges in color k to y_1. And suppose that there are no edges in color k connecting e_3 to v, and w. In this case there are 4 missing edges more. Denote the endpoints of e_3 by x_3 and y_3. Suppose that there is and edge from any of them in color k to z. Suppose without losing generality that we have the edge y_3z. Also suppose that there are the edges vx_2 and y_2w, in color k. Then we have the hamiltonian path $P = x_1y_1x_3y_3zvx_2y_2w$, otherwise there is at least one more missing edge. A contradiction.

(4) suppose e_3 are not connected to $P_{i,j}$. Then there are 2 more missing edges that give us a total of 10 missing edges in color k. If we suppose that there is no edges from e_3 to vzw then we have contradiction since there are 4 more missing edges. Suppose there is edges from extremities of e_3 to v. Since there is no to w there are 2 missing edges more in color k. There are 2 more missing edges from w to x_1 and y_2. Suppose now that there are edges from e_3 to v and w then there can not be edges from v and w to x_1 and y_2, that give us 4 more missing edges in color k. By this a contradiction.

(5) Suppose that one extremity of e_3, say x_3, is connected to y_1 and x_2 with edges in color k. We restart to count the missing edges in color k. There 3 missing edges in $P_{i,j}$, 4 more missing from y_3 to $P_{i,j}$, and 2 more from x_3 to x_1 and y_2. This give us in total 9 missing edges in color k. If we suppose that there are no edges from e_3 to vzw then we have contradiction, since 4 missing edges more are added. Otherwise, if there are edges only to v, then there are no edges wx_1 and wy_2 in color k, or if there are edges from e_3 to w and v, then there can not be edge from wx_1, wy_2, vx_1 and vy_2. Therefore a contradiction.

Suppose now that $|P_{i,j}| = 1$ for all $i \neq j$, $i, j \in \{r, b, g\}$. This can arrive when the matchings

are parallel. Therefore there are no edges between the edges of the matchings in any color. Now, we can see that these edges are fully connected to the vertices v, z and w. Then we can take alternatively the edges in colors i, j and k to make a hamiltonian path.

Since we covered all cases, the proof is completed. □

Theorem 6.3.8. *Let G^c be a connected 3-edge-colored multigraph on 10 vertices. If $m \geq 94$, then G^c has a proper hamiltonian path.*

Proof. Before beginning the proof we state the following.
CLAIM : For any two colors, say r and b, $|M^{r,b}| \geq 4$

Let us assume $|M^{r,b}| < 4$. We consider that $M^{r,b}$ is maximum. Let q be the number of edges of $M^{r,b}$, then $M^{r,b} = \{e_1, \cdots, e_q\}$ with $1 \leq q \leq 3$. So there are $10 - 2q$ vertices without edge of color r and b between them. Let I be the set of those vertices. Now we are going to count the number of missing edges of G^c. Since any arbitrary couple of vertices in I can not have more than two incident edges with color r and b with e_i, then the number of missing edges of color r or b between $M^{r,b}$ and I is at least $\frac{4q\binom{10-2q}{2}}{10-2q-1} = 2q(10-2q)$. This number is justified by the number of combinations of 2 among $|I| = 10 - 2q$ and each vertex of I in this number is counted $10 - 2q - 1$ times. Inside of I, the number of missing edges of color r and b is $(10-2q)(10-2q-1)$. By addition, we have at least $9(10-2q)$ missing edges. Getting a contradiction about the number of edges, means to have the number of missing edges greater than 41 which the maximum number of edges of $\bar{G}^c$. For $q = 1, 2$, we have $9(10-2q) > 41$. Then we have to study the particular case of $q = 3$. We shall focus our attention on the induced subgraph of $M^{r,b}$. Let assume there are two edges of $M^{r,b}$ such that the underlying subgraph is a clique. Otherwise we add at least 6 more missing edges and by this we arrive to contradiction. We distinguish now the following cases :

- Case 1 : One vertex of I has 2 neighbors in the clique
  That implies there no edge of color r or b between the others 3 vertices of I and the clique, so the number of missing edges is 24. The number of missing edges of color r and b inside of I is 12. We know also that the minimum number of missing edges between I and an edge of $M^{r,b}$ (we consider the no used edge of the clique) is 8. Then at least the total number of missing edges is 44.

- Case 2 : No vertex of I has more than one neighbor in the clique
  Then, for any vertex of I, the number of missing edges with the clique is at least 6. Then between the clique and I, the number of missing edges is at least 24. By adding the missing edges inside of I and since any arbitrary couple of vertices in I can not have more than two incident edges with color r and b with the edge of $M^{r,b}$ not considered in the clique, we obtain at least 44 missing edges.
  In both cases, the number of missing edges is greater than 41.

In conclusion, we say that $|M^{i,j}| \geq 4$, $\forall i, j \in \{r, b, g\}, j \neq i$. Therefore the claim is proved.

Now let prove by contradiction that G^c has no proper hamiltonian path.

- Let us assume that there is a perfect monochromatic matching in some color
  We will prove that G^c has a proper hamiltonian path. Without loss of generality, let set

$|M^r| = \frac{n}{2}$ and $|M^{b,g}| \geq \frac{n-2}{2}$ and let us consider the longest compatible paths P_r and $P_{b,g}$ with M^r and $M^{b,g}$. Let us set $P_r = x_1y_1x_2y_2\cdots x_py_p$ and $P_{b,g} = s_1t_1s_2t_2\cdots s_{p'}t_{p'}$ such that $c(x_iy_i) = r$, $c(s_jt_j) = b$ *or* g. Let us assume that there is no proper hamiltonian path in G^c. We are going to count the missing edges into the two following cases :

Let us first suppose that $|M^r| = \frac{n}{2}$ and $|M^{b,g}| = \frac{n}{2}$.

(1) No one of the paths P_r or $P_{b,g}$ is a cycle
Since P_r is not a cycle, by lemma 6.2.4, there are at least $2(n-2+pn-2p^2)$ missing edges in G^c in color b and g. There are also at least $n-2+p'n-2p'^2$ missing edges in color r because $P_{b,g}$ is a not cycle. By addition the missing edges are at least

$$f(p,p') = -4p^2 + 2np - 2p'^2 + np' + 3n - 6$$

Now we compare $f(p,p')$ with $5n-9$ witch is the maximum number of $\bar{G^c}$. It suffices to have the minimum value of $f(p,p')$ greater than $5n-9$ for to conclude that there is a proper hamiltonian path. We also know that $1 \leq p,p' \leq \frac{n-2}{2}$
By studying the optimum of the function $f(p,p')$, we find the minimum value for $(1,1)$ and $(\frac{n-2}{2},\frac{n-2}{2})$.

$$f(1,1) = f(\frac{n-2}{2},\frac{n-2}{2}) = 6n-12$$

Since $6n-12 \geq 5n-9$ for $n = 10$, then we say G^c has a proper hamiltonian path in this case.

(2) At least one of the paths P_r or $P_{b,g}$ is a cycle
we have to solve the three following combinations :
– P_r is a cycle and $P_{b,g}$ is not a cycle
– P_r is a not a cycle and $P_{b,g}$ is a cycle
– P_r is a cycle and $P_{b,g}$ is a cycle
We apply the lemma 6.2.4 to obtain, the functions $g_1(p,p')$, $g_2(p,p')$ and $g_3(p,p')$, $1 \leq p,p' \leq \frac{n-2}{2}$, for the different combinations.

$$g_1(p,p') = 2(2pn-4p^2) + n + np' - 2p'^2 - 2$$

$$g_2(p,p') = 2n + 2np - 4p^2 - 4 + 2p'n - 4p'^2$$

$$g_3(p,p') = 2(2pn-4p^2) + 2p'n - 4p'^2$$

we have $g_1(p,p') - f(p,p') = 2(g_2(p,p') - f(p,p')) = 2np - 2n - 4p^2 + 4 \geq 0$ if $p \in \{1,\cdots,\frac{n}{2}-1\}$. we have also $g_3(p,p') - f(p,p') = (g_1(p,p') - f(p,p')) + (g_2(p,p') - f(p,p')) \geq 0$.
We conclude that in this case G^c has a hamiltonian path.

Let us now suppose that $|M^r| = \frac{n}{2}$ and $|M^{b,g}| = \frac{n-2}{2}$.

(1) So, in the case that P_r and $P_{b,g}$ are not cycles, we use again lemma 6.2.4 to get missing edges

$$f(p,p') = -4p^2+2np+2n-4-2p'^2+np'-2p'+n-4 = -4p^2+2np-2p'^2+np'-2p'+3n-8$$

for $1 \leq p \leq \frac{n-2}{2}$, $1 \leq p' \leq \frac{n-4}{2}$. The minimum value of this function is $6n-16$ and is greater than $5n-9$ for all $n \geq 8$. Since G^c has no hamiltonian path, then $p' = \frac{n-2}{2}$. Since P_r is a not a cycle, there is at least $2(n-2+pn-2p^2)$ missing edges in G^c in color b and g. By lemma 6.2.3 there are also at least $n-4$ missing edges of color r. By considering the matching M^r, we know there are at least 2 missing edges of color b and g, parallel to the edges of the matching . Finally by considering the matching $M^{b,g}$, we add 2 missing edges of color r at the extremities of $P_{b,g}$, otherwise it will be a proper hamiltonian path. By addition, we obtain a function with one variable

$$f(p) = -4p^2 + 2np + 3n - 4$$

For all $p = 1, \cdots, \frac{n}{2}$, the latter function is greater than $5n-9$ because its minimum value is $5n-8$.

(2) P_r is a cycle and $P_{b,g}$ is a not a cycle. By lemma 6.2.4, we have $g_1(p,p') = 2(2pn - 4p^2)+(n-4+np'-2p'-2p'^2)$, for $1 \leq p \leq \frac{n-2}{2}$, $1 \leq p' \leq \frac{n-4}{2}$ and $g_1(p,p')-f(p,p') = 2np-2n-4p^2+4 \geq 0$. For $p' = \frac{n-2}{2}$, we add $n-4$ ($n-4$ from the lemma 6.2.3) to $2(2pn-4p^2)$ There are also 2 missing parallel edges of color r and b for the extremities. In total we $n-2$ for the color r and b for the extremities to $2(2pn-4p^2)$ and the minimum value is $5n-8$.

(3) P_r is a not a cycle and $P_{b,g}$ is a cycle. Then $g_2(p,p') = 2(n-2+pn-2p^2)+(2p'n-4p'-4p'^2)$, for $1 \leq p \leq \frac{n-2}{2}$, $1 \leq p' \leq \frac{n-4}{2}$ and $g_1(p,p')-f(p,p') \geq 0$. For $p' = \frac{n-2}{2}$, when we consider the cycle $P_{b,g}$, for each edge of color r of $P_{b,g}$, there are 2 missing edges of color r between this edge and the 2 vertices out of $P_{b,g}$. In total there are $n-2$ missing edges in color r in which we add 2 missing parallel edges of color b and g from M^r. The resultant function is $2(n-2+pn-2p^2)+n$ and its minimum value is $5n-8$.

(4) P_r is a cycle and $P_{b,g}$ is a cycle. $g_3(p,p') = 2(2pn-4p^2)+(2p'n-4p'-4p'^2)$ for $1 \leq p \leq \frac{n-2}{2}$, $1 \leq p' \leq \frac{n-4}{2}$ and we have $g_3(p,p')-f(p,p') = (g_1(p,p')-f(p,p'))+(g_2(p,p')-f(p,p')) \geq 0$. For $p' = \frac{n-2}{2}$,as previous, we add n to $2(2pn-4p^2)$ and the total has $5n-8$ as minimum value. The contradiction about the number of edges means that G^c has a proper hamiltonian path in this case.

– Now we deal with the case that there is no monochromatic perfect matching
Let say that the color r is the dominating color in G^c. It verifies the number of edges of the conditions of lemma 6.2.5. It means that the underlying subgraph on color r is not connected. It will be easy to verify this subgraph contains a big component of size $n-1$ and an isolated vertex z. Since G^c is connected, it is trivial that there is an edge in some color on z, such that this edge and the component have by lemma 6.2.5, a perfect matching. In this matching, all edges are in color r except one. We cannot have at the same time $|M^b| \leq 3$ and $|M^g| \leq 3$, otherwise we have a contradiction about the number of missing edges. To see this, we have to analyze and calculate like the first step. Let us assume that $|M^{r,b}| = \frac{n}{2}$. If $|M^{r,g}| = \frac{n}{2}$, then the missing edges in color g are $2n-4$ obtained from the lemma 6.2.4, and one edge more not parallel to the edges of $M^{r,b}$ since M^g is not perfect. By $|M^{r,g}| = \frac{n}{2}$, the missing edges in color b are $2n-3$ and the missing edges in color r are $n-1$ since $d^r(z) = 0$. Adding all these numbers lead to $5n-7$ missing edges.
Let us now suppose that $|M^{r,g}| < \frac{n}{2}$. Then it implies that $d^g(z) = 0$.
Now we are going to distinguish to cases : first we consider that $|M^g| = \frac{n-2}{2}$ and second $|M^g| < \frac{n-2}{2}$

Case 1 : $|M^g| = \frac{n-2}{2}$.

Let us consider the longest compatible path $P_g = x_1y_1x_2y_2\cdots x_py_p$ using edges from M^g and edge of color r. From $|M^{r,b}| = \frac{n}{2}$, the missing edges in color g are $2n-4$ in which we add one parallel edge to $M^{r,b}$. Since M^b is not perfect, there are $n-1$ missing edges in color b. There are $n-1$ missing edges in color r, since $d^r(z) = 0$. Let assume that the order of P_g is smaller than $n-2$. It means that there is at least one edge of color g out of the path P_g and two vertices among which z. The missing edges in color r from P_g are at least $2n-8$ since we cannot add the last edge. So the total number of these missing edges is $6n-13$ which 47 for $n = 10$. Now let suppose that the order of P_g is exactly $n-2$. So, the vertex z and an other vertex, say z' are out of the path P_g. Clearly, there is no edge in color b between z and z' , otherwise $|M^{b,g}| = \frac{n}{2}$ which is not possible.

Now let us suppose that P_g is a cycle

Let assume that z' cannot be inserted in P_g so as to obtain a new proper cycle containing all the vertices of G^c except z. Otherwise we get a proper hamiltonian path because G^c is connected and we easily link z and choose an appropriate direction. Since we cannot insert z' to P_g , then z' is incident to some edges y_ix_{i+1} (if any) of P_g in only color r or only b , or z' is adjacent on 2 colors (parallel edges) to only one extremity of some edges y_ix_{i+1}.

Let's denote the set of edges in color r that has monochromatic edges to z' by Z and the rest by $(P_b \setminus M^b) \setminus Z$. There is no edge from z to Z otherwise we get proper hamiltonian path with z and z' as extremities and there is one edge from z to each edge of $(P_b \setminus M^b) \setminus Z$ to same extremity. So in total there are 4 edges missing for

each edge in Z (2 since z' can not be inserted and 2 since there is no edges from z to this edges, otherwise we can brake the cycle) and 3 for each in $(P_b \setminus M^b) \setminus Z$ (2 since z' can not be inserted and 1 since there is no edges from z to second extremity of the edges, otherwise we can brake the cycle). The minimum correspond to at least $3\frac{n-2}{2} = 12$ missing edges, where $|Z| = 0$.

Let us now consider the two following subcases :

- There is edge zx_i in color b and edges $z'x_{i-1}$ in color b and r
  Then we can make a new path almost alternating in color g in one edge $P = z'x_{i-1}y_{i-2}\dots x_iz$ such that y_{i-1} is outside the path. If now we can insert the vertex y_{i-1} in place of an edges in color r from the path, where are $|(P_b \setminus M^b) \setminus \{y_{i-1}x_i\}| = \frac{n-2}{2} - 1 = \frac{n-4}{2}$ edges in color r to consider, then we have a hamiltonian path. Since is not the case there are at least 2 edges missing in color r and b at y_{i-1} for each edge in $(P_b \setminus M^b) \setminus \{y_{i-1}x_i\}$, that give us $n-4$ edges more. So $2n-3$ in color g plus $n-1$ in color r at z, plus 12 plus $n-4$, give us $4n+4$ that is a contradiction with number of edges in G^c.
- There is edge zx_i in color b and edges $z'x_{i-1}$ in color b or r
  Suppose that we cannot make a new path almost alternating in color k in one edge $P = z'x_{i-1}y_{i-2}\dots x_iz$ such that y_{i-1} is outside the path. So there are $\frac{n-2}{2}$ possibilities we miss one edge per each possibility. Then we have $2n-3$ in color k plus $n-1$ in color r at z, plus $3\frac{n-2}{2} + \frac{n-2}{2}$ give us $5n-8$ missing edges, result contradiction.
  Let us suppose that P_g is a not cycle.
  In this case by Lemma 6.2.3 there is $(2p-2) = n-4$ missing edges in color r, between the vertices of P_g.
  Let consider the longest compatible path P'_g using edges from M^g and edges of color b . Suppose that P'_g is not a cycle of order $n-2$, otherwise we return to previous case.
  By the same lemma, there are $n-4$ edges missing in color b between the vertices of P'_g, otherwise we return to the previous case. There are also 2 missing edges between z and z' and the two endpoints of P'_g because we can not have more than 2 edges in color b between them, otherwise we will get a proper hamiltonian path. Then we distinguish two cases :
  - The missing edges in color b with the extremities of P'_g are only at z
    Then observe that z' has parallel edges in color r and b connected to both extremities that allow to make a proper cycle of order $n-1$. Hence it will be easy to connect z so as to get a proper hamiltonian path. Then there are 2 more missing edges at z' for not to have a such path.
  - The missing edges in color b with the extremities of P'_g are on z and z'
    Then there are two edges in color b going in one of the extremities of P'_g. So it

is not allowed to have an edge of color r between the other extremity and the vertex z' Otherwise, we will have a proper hamiltonian path.

In total, there are $2n-3$ edges in color g , $n-1$ at z in color r, $n-4$ between vertices of P_g in color r, $n-4$ in color b between vertices of P_g and at least 3 between z, z' and the extremities of P_g, plus the edge between z and z' in color b , and we arrive to $5n-8$ missing edges that contradicts the number of edges in G^c.

Case 2 : $|M^g| < \frac{n-2}{2}$

Since the matchings M^r and M^b are not perfect, there are at least $(n-1) = 9$ edges missing in color r and $(n-1) = 9$ in color b . This correspond to 18 edges in $\bar{G^c}$. For $|M^g| = \frac{n-4}{2}$ there are several situations but only one doesn't give direct contradiction. It is the case that there is no clique in color g using 2 edges from M^g. Following the same reasoning presented above there are only one situation that don't give a contradiction, is in the case then we miss 21 edges in color g, but this case suppose that all vertices has at least 3 incident edges in color g . Since in our case now it exist a vertex v without color g incident to it we conclude that there are 3 edges more missed in color g , that give us 24 missing edges in color g . In total, there are 42 missing edges that contradicts the number of edges in G^c.

□

Theorem 6.3.9. *Let G^c be a connected c-edge-colored multigraph on n vertices, $n \geq 9$ and $c \geq 3$. If $m \geq \frac{c(n-2)(n-3)}{2} + n$, then G^c has a proper hamiltonian path.*

Démonstration. By Lemma 6.3.5 it is enough to prove the theorem for $c = 3$.

The proof is by induction on n. For $n = 9$, 10 the theorem is true, by Theorems 6.3.7 and 6.3.8, respectively. We distinguish 2 cases, depending if the multigraph contains or not a monochromatic vertex.

(1) There exists a monochromatic vertex in G^c. Let us denote this vertex by x. Clearly, there exists a vertex $y \in N(x)$, such that y is not monochromatic, otherwise there is no enough edges in G^c. Suppose that the edge xy is of color b. Replace the vertices x and y by a new vertex, say z, such that $N^b(z) = N^r(z) = \emptyset$ and $N^g(z) = N^g(y)$ (or $N^b(z) = N^g(z) = \emptyset$ and $N^r(z) = N^g(y)$). This process is called the contraction.
Observe that, if the resulting multigraph on $n-1$ vertices is connected (as we will show later) it has a proper hamiltonian path, then, since s is monochromatic, it can only be the endpoint of the path. We can replace back s with x and y using the edge xy in color b and find the proper hamiltonian path for the initial multigraph.

We now count the edges that are deleted by contraction process. We check the maximum number of edges that are deleted by the contraction process. We delete all the edges

incident to x, at most $n-1$, and the edges in color b and say g (or r) incident to y, at most $2(n-2)$. This gives us a total of $3n-5$ edges. We shall choose x, y for the contraction process, such that we can guarantee that we delete at most $f(n)-f(n-1)$ $= 3\frac{(n-2)(n-3)}{2}+n-3\frac{(n-3)(n-4)}{2}-n+1 = 3n-8$ edges needed for the induction hypothesis to hold.

Let us study the different cases, depending on the degree in color b at x.

- $d^b(x) = n-1$. Consider the subgraph $G^c - \{x\}$. By a Theorem of [3], if for every vertex $z \in V(G^c - \{x\})$, $d^i(z) \geq \lceil \frac{n}{2} \rceil$, where $i \in \{r, b, g\}$, then $G^c - \{x\}$ has a proper hamiltonian cycle. We can conclude that in $G^c - \{x\}$ exists a vertex such that $d^i(z) < \lceil \frac{n}{2} \rceil$. Without loss of generality let us suppose that say $d^r(z) < \lceil \frac{n}{2} \rceil$. Then in the contraction process we delete at most $n-1$ edges in color b at x and $n-2$ in color b at z and $\frac{n}{2}$ at z in color r. In total we have $\frac{(5n-6)}{2}$ deleted edges that is less than $3n-8$ for $n > 10$.
- $d^b(x) = n-2$. Using the same arguments as in the previous case we can conclude that in $G^c - \{x\}$ there exists a vertex such that $d^i(w) < \lceil \frac{n}{2} \rceil$, $w \in V(G^c - \{x\})$ and $i \in \{r, b, g\}$. If we suppose that there exists the edge xw in color b then we return to the previous case, otherwise suppose there is no edge xw in color b. Then there exists a vertex $z \in N^b(x)$, such that there is no edge zw in color say r. Consider the subgraph $V(G^c) - \{x, w\}$ vertices with the edge set in color r and b. By a Theorem of [3], if for any vertex $z \in G^c - \{x, w\}$, $d^i(z) \geq \lceil \frac{n-1}{2} \rceil$, where $i \in \{r, b\}$, then $G^c - \{x, w\}$ has a proper hamiltonian cycle, alternating in color b. Let us see now the two possible situations :

  - There is an alternating cycle in color b of order $n-2$. Since x is connected to all vertices of the cycle we can use edges in color g and b at w to break the cycle at one end point of some edge in color b and add x and w to the extremities of this edge. Since G^c has no hamiltonian path we conclude that there is no edges in color b and g at w. Then we can use any vertex from the cycle for the contraction process as vertex x, such that we delete at most $n-2$ edges in color b at x and at most $2(n-3)$ in color b and g at the vertex in the cycle. Therefore there are $3n-8$ deleted edges and the contraction process can be done.
  - There is no alternating cycle in color b of order $n-2$. If such is the case, then there exists a vertex z' such that the degree in color say r : $d^r(z') < \lceil \frac{n-1}{2} \rceil$. If we suppose that there are the edges in color r, $z'w$ then $d^r(z) < \lceil \frac{n+1}{2} \rceil$. Then in the contraction process we delete $n-2$ edges in color b at x and $n-2$ in color b at z' and $\frac{n+1}{2}$ at z in color r. In total we have $\frac{(5n-7)}{2}$ deleted edges that is less than $3n-8$ for $n > 9$.
- $d^b(x) \leq n-3$ Let us suppose that $d^b(x) = n-3$. Otherwise we can use any vertex $z \in N(x)$ for the contraction process, since we delete at most $n-4$ edges in color b at x and $2n-4$ in color b and say r at z, that is enough to respect the induction hypothesis. Using the same arguments as in the previous case, we can conclude that in $G^c - \{x\}$ exists a vertex such that $d^i(w) < \lceil \frac{n}{2} \rceil$, $w \in V(G^c - \{x\})$ and $i \in \{r, b, g\}$. If we suppose

that there exists the edge xw in color b then we return to the previous case. Otherwise suppose that there is no edge xw. Then there exists a vertex $z \in N^b(x)$, such that there is no edge zw in color say r. Then we delete $n-3$ edges in color b of x and $n-2$ in color b of z plus $n-3$ in color r of z, which gives us the total of $3n-8$ that is enough for the contraction process.

(2) There is no monochromatic vertex in G^c. First suppose that there exists a vertex x such that $N(x) = 1$. Take then x and y, where $y \in N(x)$, for the contraction process. We delete at most 3 edges at x and at most $2n-4$ in color say r and b at y. So in total $2n-1$ edges will be deleted, which guarantees the induction hypothesis for $n \geq 8$.

In what follows, we suppose that $N(x) \geq 2$ for all $x \in G^c$. We describe now a new contraction process, between 3 vertices.

Let us take a vertex x, y and z, where $y, z \in N(x)$. Suppose that the edge xy is in color b and the edge xz in color r. Replace x, y and z by a new vertex say s, such that $N^r(s) = N^r(y)$, $N^b(s) = N^b(z)$ and $N^g(s) = (N^g(y) \cap N^g(z))$.

Suppose now that the contraction of 3 vertices has been done and the resulting multigraph is connected (we will show this later). So, as the multigraph on $n-2$ vertices has a proper hamiltonian path, then s can be replaced back with x, y and z using the initial edges as follows. Suppose that s is neither at the beggining nor at the end of the path and suppose that usv is part of the path with $c(us) = r$ and $c(sv) = b$. To obtain the proper hamiltonian path for the initial graph we replace usv in the path with $uyxzv$ such that $c(uy) = r$, $c(yx) = b$, $c(xz) = r$ and $c(zv) = b$. Now, if we have usv with $c(us) = g$ and $c(sv) = r$ (the case that $c(sv) = b$ is similar) then we replace it with $uzxyv$ such that $c(uz) = g$, $c(zx) = r$, $c(xy) = b$ and $c(yv) = r$. The case that s is either at the beggining or at the end of the path is easy to observe.

Let us count now the maximum number of edges that are deleted in the contraction process. In general, we would delete at most $3(n-1)$ edges at x and $(n-3)$ at y in color b, $(n-3)$ at z in color r and $(n-3)$ in color g at z and y, plus 3 edges between z and y. This gives a total of $6n-9$ deleted edges at x, z and y.

First of all we shall find a triplet of vertices, x, z and y. We shall furthermore choose x, z and y so that we guarantee that we delete at most $3\frac{(n-2)(n-3)}{2} + n - (3\frac{(n-4)(n-5)}{2} + n - 2)$ $= 6n - 19$ edges, necessary for the induction hypothesis.

So we need to find some particular vertices, x, z and y, so that we can guarantee that the number of edges that have been deleted is at most $6n-19$. We should prove that there exist a vertex in G^c, such that its total color degree is less than or equal to $6n-19-(3n-6) = 3n-13$ and use this vertex as x, or that exist x y and z such that the total number of deleted edges in the contraction process is less than or equal to $6n-19$.

The proof is split in two cases, depending on the parity of n. In what follows we show how to find this triplet.

Let E^b be the set of edges in color b and let's suppose that the color b is the dominating

color. The monochromatic subgraph in color b has at least $m^b \geq \frac{(n-2)(n-3)}{2} + \lceil \frac{n}{3} \rceil$. We distinguish two possibilities.
If this subgraph is connected, then by Lemma 6.2.5 there is a perfect matching in color b of the size $\frac{n}{2}$ for n even and $\frac{n-1}{2}$ if n is odd. Otherwise, if the subgraph is disconnected, then there is a matching of size $\frac{n-2}{2}$ for n even and $\frac{n-1}{2}$ if n is odd.

Let M^b denote the maximum matching in E^b. We deal with the longest proper compatible path P with M^b.

– n is even, and $M^b \geq \frac{n-2}{2}$.
Let $P = x_1y_1x_2y_2 \dots x_py_p$ be the longest proper compatible path with M^b. It is easy to check that $|P| \geq 4$, otherwise there is no enough edges in C^c.
Suppose now that P is also a proper cycle. Then we distinguish 2 cases :

(a) $|P| \leq n-8$. Let us suppose that $M^b = \frac{n}{2}$. Observe that between the vertices in the cycle and the vertices in $G^c \setminus P$ there are no edges in color r and g, otherwise a longest proper path can be found. Take the triplet x, y and z from the vertices of the cycle (consecutive). We count the edges incident to x. There is at most $n-1$ edges in color b. Observe that the maximum number of edges in color r and color g, is when the vertices of the cycle form a complete multigraph in color r and g. In the worst case we delete at x at most $2(n-9)$ edges in color r and g and $n-1$ in color b. We can see that there is at most $2n-18+n-1 = 3n-19$ edges incident at x in this case. Using this triplet we can do the contraction.
Let us suppose that $M^b = \frac{(n-2)}{2}$. In this case as in the previous one, between the vertices in the cycle and the vertices in $M^b \setminus (P \cap M^b)$, there is no edges in color r and g. As before, we take x, y and z from the vertices of the cycle. Observe that the maximum number of edges in color r and color g is when the vertices of the cycle form a complete multigraph in color r and g and then there are edges in both colors to the two unmatched vertices in $G^c \setminus M^b$. Then we delete at x as previous at most $n-1$ edges in color b plus $2(n-9)+4 = 2n-14$ edges in color r and g. Using this triplet we can do the contraction.

(b) $n-8 < |P| < n-2$. Let us suppose that $M^b = \frac{n}{2}$. As previously, there is no edges in color r and g between the vertices in cycle and the vertices in $G^c \setminus P$. Let us take the triplet x, y and z from the vertices of $M^b \setminus (P \cap M^b)$. We count the edges incident to x. There is at most $n-1$ edges in color b. Observe that the maximum number of edges in color r and color g, is when the vertices of $M^b \setminus (P \cap M^b)$ form a complete multigraph in color r and g. Since $|P| \geq n-6$, there are 6 vertices in $G^c \setminus P$, therefore in the worst case we delete at x at most $2(5)$ edges in color r and g and $n-1$ in color b. We can see that there is at most $n-9$ edges incident to x in this case. Using this triplet we can do the contraction.

Let us suppose that $M^b = \frac{(n-2)}{2}$. In this case, between the vertices in the cycle and the vertices in $M^b \setminus (P \cap M^b)$, there is no edges in color r and g. Suppose that there is at least 1 edge in $M^b \setminus (P \cap M^b)$. Take one vertex of the edges as x. Observe that the maximum number of edges in color r and color g is when the vertices $(M^b \setminus (P \cap M^b)) \cup (G^c \setminus M^b)$ form a complete multigraph in color r and g. Then we delete at x at most $n-1$ edges in color b plus 2(5) edges in color r and g. So using this triplet we can do the contraction.

(c) $|P| = n-2$ Let us suppose that $M^b = \frac{n}{2}$. By the same observations as above we can use one vertex in $M^b \setminus (P \cap M^b)$ as x for the contraction process. Let us suppose that $M^b = \frac{(n-2)}{2}$. Since there is a cycle of order $n-2$ we can use the edges in color r and b at vertices in $(G^c \setminus M^b)$ to add them to the cycle using the extremities of some edge $y_{i-1}x_i$. Observe that one vertex is enough to add to the cycle, since G^c is connected, we can then construct a hamiltonian path. Therefore, there are at least $\frac{(n-2)}{2}$ missing edges in color r and g. Using this vertex as x we respect the induction hypothesis.

Let us suppose the case when P is not a proper cycle.

(a) Let us suppose that $M^b = \frac{n}{2}$. Since P is not hamiltonian there is at least one edge in $M^b \setminus (P \cap M^b)$. Clearly, there are no edges in color r and g at the extremities of P to the edges in $M^b \setminus (P \cap M^b)$. By Lemma 6.2.4 we can check that $|P| \geq n-4$ for $n \geq 11$.
Let us suppose that $|P| = n-4$. Since we can not add one more edge from $M^b \setminus (P \cap M^b)$ to P there are at most 4 missing edges in color r and g from x_{p+1} and y_{p+1}, to each edge $y_{i-1}x_i$ in P. There are $\frac{(n-6)}{2}$ edges $y_{i-1}x_i$, this implies that there are $4\frac{(n-6)}{2}$ edges from x_{p+1} and y_{p+1} to the vertices in P.
Suppose that x_{p+1} is the vertex with minimum degree in r and g. Then, there can be at most $(n-6)+4 = n-2$ edges incident in color r and g at x_{p+1}. We can use this vertex as x for the contraction process for $n \geq 9$.

Suppose now that $|P| = n-2$. In this case doing the same observations as below we conclude that there is a vertex with minimum degree in color r and g that has at most n edges.

(b) Let us suppose that $M^b = \frac{(n-2)}{2}$ For $|P| = n-4$ the proof is analog. Then, suppose that $|P| = n-2$. There are two vertices, w and s in $G^c \setminus M^b$. We insert both vertices into the path at vertices y_{i-1} and x_i if it is possible. We distinguish two situations :
– Suppose we add s to P. But we cannot add w to $P \cup \{s\}$. Since, there are now $\frac{(n-6)}{2}$ edges $y_{i-1}x_i$ (since one edge was used to insert s) we can conclude that at w there are at most $(n-6)$ edges in color r and g plus 6 possible edges to s and 2

of its neighbors in P. This implies that the degree in color r and g is n, therefore we can use this vertex as x for the contraction process if $n \geq 11$.

- If we cannot add any vertex to P, then there are at most $2\frac{(n-4)}{2}$ edges in color r and g from each vertex to the internal vertices in P and at most 2 to the extremities of P. Observe that there can not be more than 2 edges from one vertex to the extremities of P without having a cycle of order $n-1$, that will contradict the hypothesis of the nonexistence of a proper hamiltonian path in G^c. By this observations we have that at, for example, w there are at most $n-2$ edges in color r and g and so we can use this vertex as x for our contraction process.

- Let us suppose that n is odd, and $M^b \geq \frac{n-1}{2}$.
  Clearly there are no edges in color r and g at the extremities of P to the edges in $M^b \setminus (P \cap M^b)$. By Lemma 6.2.4 we can check that $|P| \geq n-7$ for $n \geq 11$. Then let us distinguish two cases :
  - P is also a proper cycle. In this case, as for the case of n even, we can observe that if $|P| = n-7$ we can use a vertex from P as x for our contraction process. If $n-5 \leq |P| \leq n-3$ there are at most 5 vertices in $G^c \setminus P$, therefore there can be a vertex w in $M^b \setminus (P \cap M^b)$ with maximum degree in color r and g at most 2(4). This vertex can be used for the contraction process if $n \geq 10$.
  - P is not a cycle. Since P is not cycle we will analyze the degree in color r and g at x_1 and y_p. There since there are no crossing edges x_1x_i and $y_{i-1}y_p$ we can suppose that there are $2p-2$ missing edges in both colors at x_i and y_p. There is 2 more missing edges between x_1 and y_p. We can say that there exists a vertex with minimum degree in color r and g between x_1 and y_p. This vertex has the missing degree in color r and g at most $\frac{(2p-2)}{2}+2$. Then we distinguish two cases depending on size of $M^b \setminus (P \cap M^b)$
    (a) $(M^b \setminus (P \cap M^b)) \neq \emptyset$. Then, since $p \leq \frac{(n-3)}{2}$, there can be 2 more edges in color r and g to the vertex in $G^c \setminus M^b$. Let suppose without loosing generality that x_1 is the vertex with the minimum degree. Then the total degree at x_1 is $2(2p-1)+2-\frac{(2p-2)}{2}-2 = \frac{(6p-2)}{2}$. For the maximum value of $p \leq \frac{(n-5)}{2}$, $\frac{(6p-2)}{2}$ is less than or equal to $2n-12$ and it is for $n \geq 11$.
    Suppose now that $p = \frac{(n-3)}{2}$ then we use the fact that where can be at most 4 edges to each edge $y_{i-1}x_i$ from $x_{p+1}y_{p+1}$ we can affirm that there are at most $n-5$ edges in color r and g at the vertex with minimum degree between x_{p+1} and y_{p+1} to the vertices in P and 2(2) to the rest of vertices in $G^c - P$. Then since $n-1 \leq 2n-12$ for $n \geq 11$ we can use this vertex for the contraction process.
    (b) $(M^b \setminus (P \cap M^b)) = \emptyset$. In this case $p = \frac{(n-1)}{2}$. If we cannot add the last vertex to P, it is because its degree in color r and g is at most $n-3$, so we can use this vertex for the contraction process.

  We now check the connectivity of the resulting multigraph after the contraction of 2 and of 3 vertices.

Let us see first the case when we contract 2 vertices, v and his neighbor, say y. In this case let us suppose that after the contraction of v and y a set of vertices S is disconnected. Taking in consideration the total number of edges that has the resulting multigraph $|S|$ is equal to one. Denote the vertex in S by q. Then there can be at most two parallel edges incidents to q. Take then q and contract with y, there is another vertex disconnected q'. This implies that G^c has at most $3\frac{(n-2)(n-3)}{2}+4$ edges. A contradiction.

Let us suppose that after the contraction of v, y and z, where v is connected to y and to z by edges in different colors, there is set of $|S|$ vertices disconnected from the rest of the multigraph. Taking in consideration the total number of edges that has the resulting multigraph $|S|$ is equal to one.

Denote by w the vertex in S. Clearly, there can be at most 3 edges between w and vertices y and z, there can be at most 3 edges between w and v. Since $1 < d^{r,g,b}(w) \leq 6$ and $1 < N^{r,g,b}(w) \leq 3$, we can take the triplet w, y' and z' (where y' and z' in $\{v, y, z\}$) and contract it. Suppose there is one disconnected vertex w'. Then if $w' \neq z \neq y \neq v$ then there are at most 3 incident edges to w'. Otherwise there can be at most 6 incident edges to w' but there are 3 of them connected to w. In conclusion the multigraph G^c can have at most $3\frac{(n-2)(n-3)}{2}+6+3$ edges. A contradiction.
Now as the connectivity is proved the theorem holds.

□

Theorem 6.3.10. *Let G^c be a c-edge-colored multigraph on n vertices, $n \geq 11$ and $c \geq 3$. Assume that for every vertex x of G^c, $rd(x) = c$. If $m \geq f(n) = \frac{c(n-2)(n-3)}{2} + 2c + 1$, then G^c has a proper hamiltonian path.*

Proof. By Lemma 6.3.5 it is enough to prove the theorem for $c = 3$. As $m \geq f(n) = \frac{3(n-2)(n-3)}{2} + 7$ then $E(\bar{G^c}) \leq 6n-16$. The proof will be done by construction of a proper hamiltonian path or if it is not possible, by reduction to Theorem 6.3.9, i.e., a connected c-edge-colored multigraph on $n' \geq 9$ and $m' \geq g(n') = \frac{3(n'-2)(n'-3)}{2} + n'$ has a proper hamiltonian path. We will do this reduction by contracting 2 or 3 vertices depending if there exists a vertex x in G^c, such that $|N^{r,g,b}(x)| = 1$ or not.

- There exist $x \in G^c$ such that $N^{r,g,b}(x) = 1$. Let denote the neighbor of x by y. We will replace x and y by a new vertex s, such that $N^r(s) = N^r(y)$ and $N^b(s) = N^g(s) = \emptyset$. Clearly, we will delete 3 edges between x and y and at most $2(n-2)$ edges in color b and g incident to y. This gives us a total of $2n-1$ deleted edges. For the reduction hypothesis we can delete at most $f(n) - g(n-1) = \frac{3(n-2)(n-3)}{2} + 7 - \frac{3(n-3)(n-4)}{2} - n + 1 = 2n - 1$ edges, where $n' = n - 1$. Since we delete exactly $2n - 1$ edges the reduction hypothesis holds.

Observe that, if the resulting multigraph on $n-1$ vertices is connected (as we will show later) it has a proper hamiltonian path by Theorem 6.3.9, then, since s is monochromatic, it can only be the endpoint of the path. We can replace back s with x and y using the edge xy in color b and find the proper hamiltonian path for the initial multigraph.

– There is no vertex x in G^c, such that $N^{r,g,b}(x) = 1$.
Let us suppose that there are 3 vertices x, y and z, such that xy is in color, say b, and xz in color, say r. Then we will replace x, y and z by a new vertex s, such that $N^r(s) = N^r(y)$, $N^b(s) = N^b(z)$ and $N^g(s) = N^g(y) \cap N^g(z)$. Clearly, we will delete at most $3(n-1)$ edges incident to x, $n-3$ edges in b incident to y, $n-3$ edges in r incident to z, and $n-3$ in color g incident to y and z, plus 3 edges between y and z. In total we will delete at most $6n-9$ edges. But only $f(n) - g(n-2) = \frac{3(n-2)(n-3)}{2} + 7 - \frac{3(n-4)(n-5)}{2} - n + 2 = 5n - 12$ edges can be deleted to make the reduction. Therefore, we need to find a vertex such that its total degree is less than or equal to $2n-6$.
Suppose now that the contraction of 3 vertices has been done and the resulting multigraph is connected (we will show this later). So, as the multigraph on $n-2$ vertices has a proper hamiltonian path, then s can be replaced back with x, y and z using the initial edges as follows. Suppose that s is neither at the beggining nor at the end of the path and suppose that usv is part of the path with $c(us) = r$ and $c(sv) = b$. To obtain the proper hamiltonian path for the initial graph we replace usv in the path with $uyxzv$ such that $c(uy) = r$, $c(yx) = b$, $c(xz) = r$ and $c(zv) = b$. Now, if we have usv with $c(us) = g$ and $c(sv) = r$ (the case that $c(sv) = b$ is similar) then we replace it with $uzxyv$ such that $c(uz) = g$, $c(zx) = r$, $c(xy) = b$ and $c(yv) = r$. The case that s is either at the beggining or at the end of the path is easy to observe.
Suppose without losing generality that $|E^b| \geq |E^r| \geq |E^g|$, then $|E^b| \geq \frac{3(n-2)(n-3)}{2} + 3$, otherwise we have contradiction with the number of edges in G^c. Since the subgraph in color b is connected because of the rainbow degree of the vertices, we have by Lemma 6.2.5 that there is a matching $|M^b| = \frac{n}{2}$ for n even and $|M^b| = \frac{(n-1)}{2}$ for n odd.
The proof is divided in two cases depending on the parity of n. We will distinguish two sub-cases depending if the longest proper path $P = x_1y_1x_2y_2 \dots x_iy_i \dots x_py_p$ compatible with M^b is also a proper cycle or not. Let us suppose that we take M^b such that the resulting P is the longest possible. Suppose that G^c has no proper hamiltonian path. And suppose that the vertex x_{p+1} is the vertex with minimum degree (between x_{p+1} and y_{p+1}), where x_{p+1} and y_{p+1} extremities of an edge in $M^b \setminus (M^b \cap P)$.

(1) n is even.
   – P is a proper cycle.
   By Lemma 6.2.4 we can check that $|P| \geq n-6$. Clearly that there can not be edges in color r and g between $M^b \setminus (M^b \cap P)$ and P. Since there are at most 3 edges in $M^b \setminus (M^b \cap P)$ the degree in color r and g of any vertex is at most $2(n-2p-1)$.

Suppose now that there is one edge in color say r (resp. g) between $x_{p+1}y_{p+1}$, then using edges $x_{p+1}x_i$, $y_{p+1}y_i$ or $y_{p+1}x_i$, $x_{p+1}y_i$ in color b, we can insert the edge $x_{p+1}y_{p+1}$ in color r into the cycle. Since this will contradict our hypothesis we can conclude that there are 2 missing edges in color b from $x_{p+1}y_{p+1}$ to each edge x_iy_i in P. In the worst case there will be one edge missing in color b for x_{p+1} and one for y_{p+1}, for each edge x_iy_i. Since there are $\frac{(2p)}{2}$ edges x_iy_i we conclude that vertex x_{p+1} has $d^{r,g,b}(x_{p+1}) \leq 3(n-2p-1) + \frac{(2p)}{2}$. As $p \geq \frac{n-6}{2}$, then for $n \geq 12$, $3(n-2p-1) + \frac{(2p)}{2}$ is less than or equal to $2n-6$.

Suppose that there are no edges in color r and g between $x_{p+1}y_{p+1}$. Clearly $p \leq \frac{(n-4)}{2}$, otherwise for $p = \frac{(n-2)}{2}$ by rainbow degree condition we have both edges and we are in the previous case.

Then in the worst case $d^{r,g,b}(x_{p+1}) \leq 2(n-2p-2)+n-1$. If $d^b(x_{p+1}) < n-1$, as $p \geq \frac{(n-6)}{2}$ then $d^{r,g,b}(x_{p+1}) \leq 2(n-2p-2)+n-2$ is less than or equal to $2n-6$ for $n \geq 12$. If $d^b(x_{p+1}) = n-1$ we can choose a different matching replacing $x_{p+1}y_{p+1}$ and $x_{p+2}y_{p+2}$ by $x_{p+1}x_{p+2}$ and $y_{p+1}y_{p+2}$. Since there are no edges in color r and g between $x_{p+1}x_{p+2}$ (otherwise previous case applies) we conclude that the degree in all the colors of $d^{r,g,b}(x_{p+1}) \leq 2(n-2p-3)+n-1$ and this is less than or equal to $2n-6$ for $n \geq 11$.

– P is not a proper cycle

By Lemma 6.2.4 we can check that $|P| \geq n-6$. Clearly, there can not be edges in color r and g between $M^b \setminus (M^b \cap P)$ and the extremities of P. Also, there can be at most 4 edges in color r and g from each edge in $M^b \setminus (M^b \cap P)$ and edges $y_{i-1}x_i$ in P. Then we can conclude that in the worst case there are at most $2\frac{(2p-2)}{2}$ edges in color r and g, between x_{p+1} and vertices in P.

Suppose now that there is one edge in color say r (resp. g) between x_{p+1} and y_{p+1}, then using edges $x_{p+1}x_i$, $y_{p+1}y_i$ or $y_{p+1}x_i$, $x_{p+1}y_i$ in color b, we can insert the edge $x_{p+1}y_{p+1}$ in color r into P. Since this will contradict our hypothesis we can conclude that there are 2 missing edges in color b from $x_{p+1}y_{p+1}$ to each edge x_iy_i in P. There will be one edge missing in color b between each vertex x_{p+1}, y_{p+1} and each edge x_iy_i. For $|P| = n-6$ and $|P| = n-4$ we already have a contradiction with the number of edges. Now, since there are $\frac{(2p)}{2}$ edges x_iy_i we conclude that the vertex x_{p+1} has $d^b(x_{p+1}) \leq \frac{(2p)}{2} + (n-2p-1)$. In total we have that $d^{r,g,b}(x_{p+1}) \leq 3(n-2p-1) + \frac{(2p)}{2} + (2p-2) = 3n-3p-5$, and for $p = \frac{(n-2)}{2}$ this is less than or equal to $2n-6$.

Suppose there are no edges in color r and g between x_{p+1} and y_{p+1}. Then the degree in color r and g at x_{p+1} is $d^{r,g}(x_{p+1}) = 2(n-2p-2) + (2p-2)$. For $|P| = n-6$ we have already contradiction as before.

If $|P| = n-4$ there are $6n-20$ missing edges in color r and g. Since there are no edges in color r and g between x_{p+1} and y_{p+1}, x_{p+2} and y_{p+2} we conclude 4 missing

edges more in color r and g. Observe that if we can replace the edges $x_{p+1}y_{p+1}$ and $x_{p+2}y_{p+2}$ in M^b by $x_{p+1}x_{p+2}$ and $y_{p+1}y_{p+2}$, then there are missing the 4 parallel edges in color r and g. A contradiction. Otherwise if not, there are 2 more missing edges and again a contradiction.

If $|P| = n-2$, since there are no parallel edges $x_{p+1}y_{p+1}$ we have that $d^{r,g,b}(x_{p+1}) \leq 2n-5$ and $d^{r,g,b}(y_{p+1}) \leq 2n-5$. We can use one of these vertices unless both inequalities become equalities. In this case we will try to replace the edges x_iy_i in P by $x_ix_{p+1}y_i$ and $x_jy_{p+1}y_j$, $i \neq j$. Suppose that we have the edges $x_{p+1}x_i$ in color r and g, $x_{p+1}y_i$ in color b, $y_{p+1}x_j$ in color r and g, and $y_{p+1}y_j$ in color b. Then we have a proper hamiltonian path $P' = x_1y_1 \dots x_ix_{p+1}y_ix_{i+1} \dots x_jy_{p+1}y_j \dots x_py_p$. Otherwise at least one of those edges is missing and therefore either $d^{r,g,b}(x_{p+1}) \leq 2n-6$ or $d^{r,g,b}(y_{p+1}) \leq 2n-6$.

(2) n is odd.

– P is a proper cycle.

By Lemma 6.2.4 we can check that $|P| \geq n-5$. The case $|P| = n-1$ is trivial. Clearly, there are no edges in colors r and g between $M^b \setminus (M^b \cap P)$ and P. As before we can see that $d^{r,g}(x_{p+1}) \leq 2(n-2p-1)$.

If there is an edge $x_{p+1}y_{p+1}$ in color r or g, then we can use the edges in color b between $x_{p+1}y_{p+1}$ and x_iy_i to extend P. Since it is not possible there are at most $\frac{2p}{2} + (n-2p-1)$ edges in color b incident at x_{p+1}. Then the degree in all colors of x_{p+1} is less than or equal $\frac{2p}{2} + 3(n-2p-1) = 3n-5p-3$. As $p \geq \frac{(n-5)}{2}$, $3n - 5\frac{(n-5)}{2} - 3 \leq 2n-6$ for $n \geq 11$.

– P is not a proper cycle.

By Lemma 6.2.4 we can check that $|P| \geq n-7$.

Suppose $|P| < n-1$. There can not be edges in colors r and g to the extremities of P and there can be only 4 edges in colors r and g between $M^b \setminus (M^b \cap P)$ and $y_{i-1}x_i$ in P. Suppose now that there exists an edge $x_{p+1}y_{p+1}$ in color r or g, then we can use edges in color b between $x_{p+1}y_{p+1}$ and x_iy_i to extend P. Since it is not possible we conclude that there are only $\frac{2p}{2} + (n-2p-1)$ edges in color b at x_{p+1}. For $|P| = n-7$ and $|P| = n-5$ we already have a contradiction. Now, as $p = \frac{n-3}{2}$ then $d^{r,g,b}(x_{p+1}) \leq \frac{2p}{2} + (2p-2) + 3(n-2p-1) = 3n-3p-5 \leq 2n-6$.

Suppose there are no parallel edges in color r and g to the edges in $M^b \setminus (M^b \cap P)$. If $|P| = n-7$ then there are 6 missing edges. Now, if we can replace the three edges in $M^b \setminus (M^b \cap P)$ we are either in the previous case or there are 6 more parallel missing edges and by Lemma 6.2.4 there are $8n - 48 - 6 - 6 = 8n - 36$ missing edges. A contradiction. Otherwise there are at least 5 missing edges in color b and again by Lemma 6.2.4 there are $8n - 48 - 6 - 5 = 8n - 37$ missing edges. Again a contradiction.

If $|P| = n-5$ there are 4 parallel missing edges to the edges in $M^b \setminus (M^b \cap P)$ and

by Lemma 6.2.4 there are $6n-26$ more missing edges in color r and g.
Now, since there are no 2 independent edges to those in $M^b \setminus (M^b \cap P)$ there are at least 2 missing edges in color b, otherwise we will miss more parallel edges. Also there are 4 edges in color b between the extremities of the edges in $M^b \setminus (M^b \cap P)$ and the unmatched vertex, since otherwise we can construct 4 different matchings and therefore 8 more missing edges. In conclusion we have at least 6 missing edges in color b and $6n-22$ in color r and g, that give us a total of $6n-16$ missing edges.
Now suppose that there are parallel edges x_1y_1 and x_py_p in color r and g, then we can make a proper cycle of order $n-1$, $C = x_1y_1 \dots x_py_py_{p+2}y_{p+1}x_{p+1}x_{p+2}x_1$, if we have the edges x_py_p and x_1y_1 in color r or g, y_py_{p+2} and $x_{p+2}x_1$ in color b, $x_{p+1}x_{p+2}$ and $y_{p+1}y_{p+2}$ in color r or g and $x_{p+1}y_{p+1}$ in color b. Now as it is trivial to attach the unmatched vertex to the proper cycle there is at least one of those edges missing and therefore a contradiction.
If $|P| = n-3$, we have as before that $d_P^{r,g}(x_{p+1}) \leq n-5$. As always there are no parallel edges in color r and g to the edge $x_{p+1}y_{p+1}$. Now, if there are no edges between $x_{p+1}y_{p+1}$ and the unmatched vertex z in color r and g, then $d^{r,g,b}(x_{p+1}) \leq 2n-6$. Otherwise if there are edges in color r and g to the vertex z then there can not be the edges in color b parallel to them since we could replace $x_{p+1}y_{p+1}$ by one of these and therefore, the previous case. By rainbow degree there is an edge between z and some edge x_iy_i in color b. Suppose that the edge is zx_i, then there can not be the edge $x_{p+1}y_i$ in color b, otherwise a longest proper path exists. We can conclude then that the degree in all colors of x_{p+1} are less than or equal to $n-5+2+n-3 = 2n-6$.
Let us suppose that $|P| = n-1$. Then for the same reasons as before $d^{r,g}(z) \leq n-3$. If we suppose that there are edges in color r and g between z and y_i, then there can not be the edges x_iz in color b, otherwise a hamiltonian path can be found. Suppose there is no such edges then either x_1 or y_p have degree in colors r and g at most $n-3$ and all parallel. If there are at least one of the edges zy_1 or zx_p in color b, then we can replace z by either x_1 or y_p. In both cases we arrive that $d^{r,g,b}(x_1) \leq 2n-6$ or $d^{r,g,b}(y_p) \leq 2n-6$, otherwise $d^{r,g,b}(z) \leq 2n-6$.

Since all cases were covered, there always exists a vertex x such that $d^{r,g,b}(x) \leq 2n-6$.

We will check now the connectivity of the resulting multigraph after the contraction of 2 or 3 vertices.

We start with the case when v has only one neighbor, say y. Let us suppose that after the deletion of two colors at y a vertex w is disconnected. This means that the rainbow degree of w is at most 2, contradicting the hypothesis of the theorem.

We continue with the case of 3 vertices.

Let us suppose that after the contraction of v, y and z, with $d^{r,g,b}(v) \leq 2n-6$, $c(vy) = r$

and $c(vz) = b$ there is set X of vertices disconnected from the rest of the multigraph. Since the resulting multigraph on $n' = n - 2$ vertices has $m' \geq 3\frac{(n'-2)(n'-3)}{2} + n'$ edges, we can verify $|X| = 1$.

Denote by w the vertex in X. Clearly, there can be at most 3 edges between w and vertices y and z, there can be at most 3 edges between w and v. We will distinguish 3 cases :

(1) Let us suppose that there exist 2 new triplets v, y', z' and v, y'', z'', where $y \neq y' \neq y''$ and $z \neq z' \neq z''$. Then suppose that after each contraction there is a vertex w' and w'' disconnected respectively. Clearly $w \neq w' \neq w''$, otherwise we contradict the fact that the resulting multigraph after the contraction was disconnected. Since there can be at most 6 edges incident at each vertex w, w' and w'', we have a contradiction with the number of edges in G^c.

(2) Let us suppose that there exists only 1 new triplet v, y', z'. This case can appear when say $N^r(v) \leq n - 1$ and $N^{b,g}(v) \leq 2$ or when $N^{r,g,b}(v) \leq 5$ that implies that there are at most $n-1+4 = n+3$ or 15 incident edges to v. For each triplet there exist w and w' and there can be at most 6 edges between w, w' and y, z and y', z'. Take now the triplet v, y and z'', such that $z'' \neq z \neq z'$, this is always possible if $6 \leq N^r(v) \leq n - 1$. Then there exist a third vertex w'' with at most 6 incident edges to it. We conclude that G^c has at most $3\frac{(n-3)(n-4)}{2} + 6 + 4 + 6$ or $3\frac{(n-4)(n-5)}{2} + n + 14$ edges. A contradiction.

(3) Suppose there are no new triplets. Then $N^r(v) \leq n-1$ and $N^{g,b} = 1$ or $N^{r,g,b}(v) \leq 3$, that implies that there are at most $n-1+2 = n+1$ or 9 edges incident to v. Suppose $N^r(v) \geq 4$. Then after the contraction of v, y and z there is vertex w that is disconnected and there can be at most 4 edges between w and the vertices of the triplet. Take then the triplet v, y and z', where $z \neq z'$. Suppose there is another vertex $w' \neq w$ what is disconnected with at most 4 incident edges at it. This implies that G^c has $3\frac{(n-3)(n-4)}{2} + n + 9$ edges. A contradiction.

Suppose that $N^{r,g,b}(v) = 3$. Then take the triplet v, y and $z' \neq z$ and make the contraction. There is one more vertex w' disconnected. If $w, w' \in N^{r,g,b}(v)$ then there are at most 6 edges incident to each vertex, otherwise there are at most 3 edges incident to each of them if $w, w' \notin N^{r,g,b}(v)$. This implies that G^c can have at most $3\frac{(n-3)(n-4)}{2} + 9 + 3 + 3$ edges. A contradiction.

There is one more case to study when $N^{r,g,b}(v) \leq 2$. Then $w \notin N^{r,g,b}(v)$ and since $d^{r,g,b}(w) = 3$ we can use the triplet w, z and y for the contraction. If there is one vertex more disconnected, then there can be 2 possibilities first $w' = v$ and second $w' \neq v$. If $w' = v$ then there are at most 3 edges incident to v and this implies that G^c can have at most $3\frac{(n-2)(n-3)}{2} + 6$ edges. A contradiction. If $w' \neq v$ then G^c can have at most $3\frac{(n-3)(n-4)}{2} + 6 + 3 + 3$ edges. A contradiction.

Now as the connectivity is proved the theorem holds.

□

The Theorem 6.3.10 is the best possible for $n \geq 9$. In fact, consider a rainbow complete multigraph, say A, on $n-2$ vertices. Add 2 new vertices v_1, v_2 and then join them to a vertex v of A with all possible colors. The resulting c-edge-colored multigraph has $\frac{c(n-2)(n-3)}{2} + 2c$ edges and clearly has no proper hamiltonian path. If $n = 6, 8$, the graphs $H_{k,k+2}$, $k = 2, 3$, are exceptions for Theorem 6.3.10.

6.4 Summary and remarks

In this chapter, we studied the existence of proper hamiltonian paths in edge colored multigraphs with bounded number of edges and fixed rainbow degree.

Here, the notable fact is that the proofs were sometimes long and tedious, despite the lower bounds for the edges in the considered multigraphs were really high.

It should be also interesting to study similar conditions for other patterns such as cycles, trees, etc.

Chapitre 7

Proper connection of graphs

Contents

Clearly, if a graph is k-proper connected, then it is also k-connected. Conversely, any k-connected graph has an edge coloring that makes it k-proper connected ; the number of colors is easily bounded by the edge chromatic number which is well known to be at most $\Delta(G)$ or $\Delta(G)+1$ by Vizing's Theorem [68] (where $\Delta(G)$, or simply Δ, is the maximum degree of a vertex in G over all its vertices). Thus $pc_k(G) \leq \Delta + 1$ for any k-connected graph G.

An edge-colored graph is connected if the underlying non-colored graph is connected. We denote the *connection number* of a graph by $\kappa(G)$. Throughout this chapter, all edge-colored graphs are considered to be connected unless otherwise specified. Given a colored path $P = v_1v_2 \dots v_{s-1}v_s$ between any two vertices v_1, v_s, we denote by $start(P)$ the color of the first edge in the path, i.e. $c(v_1, v_2)$, and by $end(P)$ the last color, i.e. $c(v_{s-1}, v_s)$. If P is just the edge v_1v_s then $start(P) = end(P) = c(v_1, v_s)$.

This chapter is organized as follows : In Section 7.1 we study $pc_k(G)$ for bipartite graphs. We state a conjecture, prove several small results and finally we prove the conjecture for $k = 1$, that is, for $pc(G)$. In Section 7.2, we study $pc(G)$ for general graphs and prove non-trivial bounds, improving Vizing's trivial bound of $\Delta + 1$. Then, motivated by both of these sections, we state a conjecture regarding $pc_k(G)$ for general graphs. In Section 7.3 we prove a bound concerning the minimum degree of G. Finally we present the conclusions of our work [1] and some open problems.

1. This is a joint work with Shinya Fujita, Aydin Gerek, Colton Magnant, Yannis Manoussakis, Leandro Montero, Zsolt Tuza.

7.1 Bipartite graphs

In this section, we study proper connection numbers in bipartite graphs. We state a general conjecture for $pc_k(G)$ where G is a bipartite graph with some specific connectivity that depends on k. Following that, we show that this conjecture is best possible in the sense of connectivity. Later, we prove some results for specific classes of graphs such as complete bipartite graphs with lower connectivity assumptions than that which is required for the conjecture. Then, we prove that the conjecture is true for complete bipartite graphs. Finally, we study the case $k = 1$ and obtain results for trees and other graphs depending on their connectivity. We end the section by obtaining, as main result, the proof of the conjecture for the special case $k = 1$ and some corollaries stemming from it.

Conjecture 7.1.1. *If G is a $2k$-connected bipartite graph with $k \geq 1$, then $pc_k(G) = 2$.*

If true, Conjecture 7.1.1 is the best possible in the sense of connectivity. In the following we present a family of bipartite graphs which are $(2k-1)$-connected with the property that $pc_k(G) > 2$. It is also clear that we cannot exchange the vertex connectivity for edge connectivity since it is easy to find graphs with connectivity 1 which have edge connectivity $2k$.

Consider the complete bipartite graph $G = K_{p,q}$ with $p = 2k-1$ $(k \geq 1)$ and $q > 2^p$ where $G = V \cup W$, $V = \{v_1, v_2, \ldots, v_p\}$ and $W = \{w_1, w_2, \ldots, w_q\}$. Clearly, G is $(2k-1)$-connected. We will show that $pc_k(G) > 2$.

Proposition 7.1.2. *Let $p = 2k-1$ $(k \geq 1)$ and $q > 2^p$. Then $pc_k(K_{p,q}) > 2$.*

Démonstration. Suppose that $pc_k(G) = 2$ and consider a k-proper connected coloring of G with 2 colors. For each vertex $w_i \in W$, there exists a p-tuple $C_i = (c_1, c_2, \ldots, c_p)$ so that $c(v_j, w_i) = c_j$ for $1 \leq j \leq p$. Therefore, each vertex $w_i \in W$ has 2^p different ways of coloring its incident edges using 2 colors. Since $q > 2^p$, there exist at least two vertices $w_i, w_j \in W$ such that $C_i = C_j$. As $pc_k(G) = 2$, there exist k internally disjoint proper paths in G between w_i, w_j. Using this, we will arrive to a contradiction. First, observe that one of these paths between w_i, w_j (say P) must have only one intermediate vertex $v_l \in V$ since otherwise, if all the paths have at least two intermediate vertices in V, we would have $|V| \geq 2k$, which is a contradiction. Hence, as $C_i = C_j$ we have $c(v_l, w_i) = c(v_l, w_j)$ and therefore the path P is not properly colored, leading to a contradiction. □

Based on the previous result we prove the following.

Theorem 7.1.3. *Let $G = K_{n,3}$ then*

$$pc_2(G) = \begin{cases} 2 & \text{if } 3 \leq n \leq 6 \\ 3 & \text{if } 7 \leq n \leq 8 \\ \lceil \sqrt[3]{n} \rceil & \text{if } n \geq 9 \end{cases}$$

Démonstration. It is easy to check that $pc_2(G) = 2$ for $3 \leq n \leq 6$ and $pc_2(G) = 3$ for $7 \leq n \leq 8$. Now let $n \geq 9$. We will give a 2-proper coloring of G using $c = \lceil \sqrt[3]{n} \rceil$ colors and we will also

show that this is the best possible. Consider the bipartition of $G = V \cup W$ such that $|V| = n$ and $|W| = 3$. Let $V = \{v_1, ..., v_n\}$ and $W = \{w_1, w_2, w_3\}$. For each vertex $v_i \in V$, we consider a 3-tuple $C_i = (c_1, c_2, c_3)$ so that $c(v_i, w_j) = c_j$ for $1 \leq j \leq 3$. Therefore, each vertex $v_i \in V$ has c^3 different ways of coloring its incident edges using c colors. We then color the edges of G as follows. If $c \geq 4$ then we color the edges of $(c-1)^3$ vertices of V with all the different triples of $c-1$ colors and, for the remaining vertices, we choose different triples but this time using the c^{th} color. If $c = 3$, we just choose different triples of colors but starting with the $c!$ colorings with all three colors different. By this coloring we have that for each pair of vertices $v_i, v_j \in V$ we have that $C_i \neq C_j$ for all $1 \leq i \neq j \leq n$.

Before proving that this coloring is 2-proper, it is easy to see that G cannot be colored to make it 2-proper connected using fewer than c colors by following the same argument as in Proposition 7.1.2. That is, if we use fewer than c colors, there must exist at least two vertices $v_i, v_j \in V$ such that $C_i = C_j$, a contradiction.

Now consider two vertices $v_i, v_j \in V$ and we would like show the existence of 2-proper paths between them. Since $C_i \neq C_j$, we know that at least one of the three colors is different. If two or three are different, then we have 2-proper paths of the form v_i, w_k, v_j and v_i, w_l, v_j such that $c(v_i, w_k) \neq c(v_j, w_k)$ and $c(v_i, w_l) \neq c(v_j, w_l)$. Suppose now that exactly one of the three colors is different, say c_1 without losing generality, then v_i, w_1, v_j is a proper path. For the second path, there exists a vertex $v_k \in V$ such that, by construction of the coloring, $c(v_i, w_2) \neq c(v_k, w_2)$, $c(v_j, w_3) \neq c(v_k, w_3)$ and $c(v_k, w_2) \neq c(v_k, w_3)$. Therefore v_i, w_2, v_k, w_3, v_j is a proper path between v_i and v_j.

Next consider $w_i, w_j \in W$, it is clear that there exist two vertices $v_k, v_l \in V$ such that C_k and C_l have both colors different to w_i, w_j. Therefore w_i, v_k, w_j and w_i, v_l, w_j are proper paths. Finally, we consider the case where $v_i \in V$ and $w_j \in W$. The edge $v_i w_j$ provides a trivial proper path. For the second path, simply choose other appropriate vertices $v_k \in V$ and $w_l \in W$ such that v_i, w_l, v_k, w_k results in a proper path. These vertices exist by the constructed coloring of G. As no cases are left, the theorem holds. □

Now we prove the conjecture for complete bipartite graphs.

Theorem 7.1.4. *Let $G = K_{n,m}$, $m \geq n \geq 2k$ for $k \geq 1$. Then $pc_k(G) = 2$*

Démonstration. Take the bipartition of $G = A \cup B$. Then split each set A and B into the sets A_1, A_2, B_1, B_2 such that $|A_i|, |B_i| \geq k$ for $i = 1, 2$. This is clearly possible since $|A|, |B| \geq 2k$. Now color the graph in the following way. Put $c(v, w) = 1$ for all $v \in A_1$ and $w \in B_1$, and for all $v \in A_2$ and $w \in B_2$. Finally put color 2 to the rest of the edges, that is, $c(v, w) = 2$ for all $v \in A_1$ and $w \in B_2$, and for all $v \in A_2$ and $w \in B_1$ (see Figure 7.1). Now we prove that this coloring produces k proper paths between each pair of vertices of G. First, consider two vertices $v, w \in A_1$ (an identical argument holds for pairs in other sets). Since the cardinality of each set is at least k, we form k proper paths v, b_1, a_2, b_2, w choosing $b_1 \in B_1, a_2 \in A_2$ and $b_2 \in B_2$. If $v \in A_1$ and $w \in A_2$ (similarly for $v \in B_1$ and $w \in B_2$) we have at least $2k$ proper paths formed

as v, b, w for each choice of $b \in B$. The final case is when $v \in A_1$ and $w \in B_1$ (that is, v and w are adjacent). Here we have at least $k+1$ proper paths, as follows. One path is simply the edge vw while the k that remain are of the form v, b_2, a_2, w for each choice of $b_2 \in B_2$ and $a_2 \in A_2$. This completes the proof. □

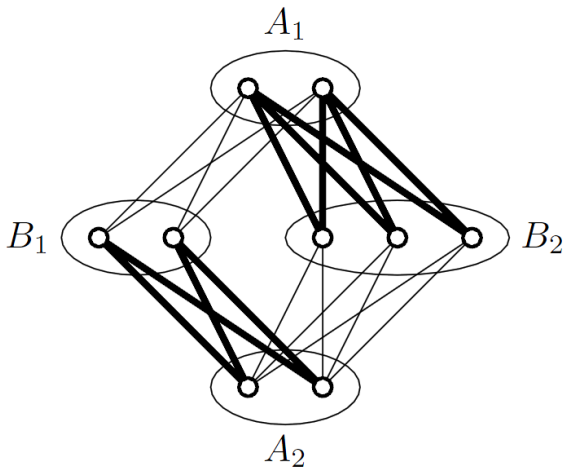

FIGURE 7.1 – Coloring of $K_{4,5}$. Normal edges represent color 1 and bold edges color 2.

Now we will study the case $k = 1$, that is $pc(G)$. By König's Bipartite Theorem [51] we have that the edge chromatic number is Δ for bipartite graphs and therefore Δ is a trivial upper bound for $pc(G)$ for any bipartite graph G. Then, we obtain this trivial corollary.

Corollary 7.1.5. *If G is a tree then $pc(G) = \Delta$.*

We present now the following proposition.

Proposition 7.1.6. *If $pc(G) = 2$ then $pc(G \cup v) = 2$ as long as $d(v) \geq 2$.*

Démonstration. Let u, w be two neighbors of v in G. Since we have assumed there is a 2-coloring of G so that G is properly connected, there is a properly colored path P from u to w in G. Color the edge uv so that $c(u, v) \neq start(P)$ and color vw so that $c(v, w) \neq end(P)$. Since every vertex of G has a properly colored path to a vertex of P, every vertex has a properly colored path to v through either u or w, thereby completing the proof. □

The following theorem is the main result of the section. It improves upon the upper bound of Δ by König to the best possible whenever the graph is bipartite and 2-edge-connected.

Theorem 7.1.7. *Let G be a graph. If G is bipartite and 2-connected then $pc(G) = 2$ and there exists a 2-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exists two paths P_1, P_2 between them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

An *ear* of a graph G is a maximal path whose internal vertices have degree 2 in G. An *ear decomposition* of G is a decomposition $P_0, P_1, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \ldots \cup P_k$. A graph is 2-connected if and only if it has an ear decomposition. An *close ear* in a graph G is a cycle C such that all vertices of C expect one have degree 2 in G. An $close - ear\ decomposition$ of G is a decomposition $P_0, P_1, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is either an (open) ear or a closed ear in $P_0 \cup \ldots \cup P_k$. A graph is 2-edge-connected if and only if it has an closed-ear decomposition.

Given a 2-connected graph G, let G_1 be an instance of the graph $G \setminus P$ where P is the set of internal vertices of the last ear of an ear decomposition of a G. Similarly, if the graph is 2-edge-connected, there is a (closed) ear decomposition in which an ear may attach to the previous structure at a single vertex. Therefore, using the same argument, one could easily show the result also holds for a 2-edge-connected graph G.

Démonstration. Suppose G is 2-connected and bipartite and consider a spanning minimally 2-connected subgraph (meaning that the removal of any edge would leave G 1-connected). For the sake of simplicity, we call this subgraph G. This proof is by induction on the number of ears in an ear decomposition of G. The base case of this induction is when G is simply an even cycle and we alternate colors on the edges.

Let P be the last ear added where the ends u and v of P are in G_1 and all internal vertices of P are in $G \setminus G_1$. Since G is minimally 2-edge-connected, we know that the length of P is at least 2. By induction on the number of ears, we obtain a 2-coloring of G_1 so that G_1 has the strong property. Color P with alternating colors.

Finally we show that this coloring of G is proper connected with the strong property. Every pair of vertices in C has the strong property since C is an alternating even cycle. Also, by induction, every pair of vertices in G_1 has the strong property. Let $x \in G \setminus C$ and let $y \in P$. The pair xu has the strong property so there exists a path Q_u from x to u so that xQ_uuPy forms a proper path Q'_u. Similarly the pair xv has the strong property so there exists a path Q_v from x to v so that xQ_vvPy is a proper path Q'_v. Since C is a proper cycle, Q'_u and Q'_v must have different colors on the edges incident to y. Note also that, since G is bipartite, the parity of the length of Q'_u is the same as the parity of the length of Q'_v. Hence, Q'_u and Q'_v must also have different colors on the edges incident to x. This shows that x and y have the strong property, thereby completing the proof. □

As a result of Theorem 7.1.7 we obtain the following corollaries.

Corollary 7.1.8. *Let G be a graph. If G is 3-connected, then $pc(G) = 2$ and there exists a 2-edge-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exist two paths P_1, P_2 between them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

Démonstration. By [59], any 3-connected graph has a spanning 2-connected bipartite subgraph. Then the result holds by Theorem 7.1.7 □

7.2 General graphs

We begin this section by studying $pc(G)$ for a general graph G. We show some easy results for specific classes such as complete graphs and cycles. Following this, we prove a result analogous to that obtained in the previous section for 2-connected graphs but using 3 colors instead of 2. We also show that this bound is sharp by presenting a 2-connected graph for which 2 colors are not enough to make it proper connected. As a main result of the section, we state an upper bound for $pc(G)$ for general graphs that can be possibly reached as we saw in the previous section. Based on the results of 2-connected graphs we extend Conjecture 7.1.1 to general graphs and finally we prove this for complete graphs.

By Vizing's Theorem [68], we have that the edge chromatic number of any graph is at most $\Delta + 1$ and therefore $\Delta + 1$ is a trivial upper bound for $pc(G)$ for any graph G. First we present some easy results.

Fact 7.2.0.1. *A graph G has $pc(G) = 1$ if and only if G is complete.*

By using alternating colors, it is easy to see that any path of length at least 2 and any cycle of length at least 4 has proper connection number 2. Also it is clear that the addition of an edge to G cannot increase $pc_k(G)$.

Fact 7.2.0.2. *For $n \geq 3$, $pc(P_n) = 2$ and if $n \geq 4$, $pc(C_n) = 2$. Furthermore, pc_k is monotone decreasing with respect to edge addition.*

The following theorem improves the Vizing's $\Delta + 1$ upper bound whenever the graph is 2-connected. This result is a natural extension of Theorem 7.1.7.

Theorem 7.2.1. *Let G be a graph. If G is 2-connected, then $pc(G) \leq 3$ and there exists a 3-edge-coloring of G that makes it proper connected with the following strong property. For any pair of vertices v, w there exist two paths P_1, P_2 between them (not necessarily disjoint) such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$.*

As in Theorem 7.1.7, we note that an edge-connected version of this result is immediate from the proof.

Démonstration. Suppose G is a 2-connected graph and consider a spanning minimally 2-connected subgraph (meaning that the removal of any edge would leave G 1-connected). For the sake of simplicity, we call this subgraph G. This proof is by induction on the number of ears in an ear decomposition of G. The base case of this induction is when G is simply a cycle and we properly color the edges with at most 3 colors.

Let P be the last ear added in an ear decomposition of G and let G_1 be the graph after removal of the internal vertices of P. Since G is assumed to be minimally 2-connected, we know that P has at least one internal vertex. Let u and v be the vertices of $P \cap G_1$ so $P = uu_1u_2 \dots u_pv$. By induction, there is a 3-coloring of G_1 which is proper connected with the strong property. Color the edges of G_1 as such.

Within this coloring, there exist two paths P_1 and P_2 from u to v such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$. If possible, properly color P so that $c(u, u_1) \notin \{start(P_1), start(P_2)\}$ and $c(u_p, v) \notin \{end(P_1), end(P_2)\}$. Note that this is always possible if either P has at least 2 internal vertices or $\{start(P_1), start(P_2)\} \cup \{end(P_1), end(P_2)\} = \{1, 2, 3\}$. It will become clear that this is the easier case so will assume this is not the case, namely that P has only one internal vertex x and $\{start(P_1), start(P_2)\} \cup \{end(P_1), end(P_2)\} = \{1, 2\}$.

Color the edge xu with color 3 and xv with color 2 (supposing that $end(P_2) = 2$). We will show that this coloring of G is proper connected with the strong property. For any pair of vertices in G_1, there is a pair of proper paths connecting them with the strong property by induction. Since $P \cup P_1$ forms a proper cycle, any pair of vertices in this cycle also have the desired paths. Let $y \in G_1 \setminus P_1$ and note that our goal is to find two proper paths from x to y with the strong property.

Since y and u are both in G_1, there exist a pair of paths P_{u_1} and P_{u_2} starting at y and ending at u with the strong property. Similarly, there exist two paths P_{v_1} and P_{v_2} starting at y and ending at v with the strong property. Since these paths have the strong property, we know that $Q_1 = xuP_{u_i}y$ (note that the implied orientation on P_{u_i} is reversed when traversing the path from u to y) is a proper path for some $i \in \{1, 2\}$ (suppose $i = 1$) and similarly $Q_2 = xvP_{v_j}y$ is a proper path for some $j \in \{1, 2\}$ (suppose $j = 1$). These paths form the desired pair if $end(Q_1) \neq end(Q_2)$ so suppose $start(P_{v_1}) = start(P_{u_1})$.

Next consider walk $R_1 = xuP_1vP_{v_2}y$ and the path $R_2 = Q_2$. If R_1 is a path, then R_1 and R_2 are the desired pair of paths since $end(P_1) \neq c(x, v) = end(P_{v_2})$, meaning that R_1 is a proper walk. Hence, suppose R_1 is not a path and let z be the vertex closest to y on P_{v_2} which is in $P_1 \cap P_{v_2}$. Now if the path $R_1' = xuP_1zP_{v_2}y$ is a proper path, then R_1' and R_2 are the desired pair of paths so we may assume that $end(uP_1z) = start(zP_{v_2}y)$.

Finally we show that the paths $S_1 = xvP_1zP_{v_2}y$ and $S_2 = Q_1 = xuP_{u_1}y$ are proper paths from x to y with the strong property. Certainly, as noted above, S_2 is a proper path. Also, S_1 is a proper path since P_1 is proper so $end(vP_1z) \neq end(uP_1z) = start(zP_{v_2}y)$. Finally since $end(zP_{v_2}y) = start(P_{v_2}) \neq start(P_{v_1}) = start(P_{u_1})$, we see that S_1 and S_2 have the strong property. □

It is important to mention that there exist 2-connected graphs with $pc(G) = 3$ and therefore the bound obtained by Theorem 7.2.1 is reached. Now we give an example (see Fig. 7.2) of such a graph and prove why two colors are not enough.

Proposition 7.2.2. *Any graph G consisting of an even cycle with the addition of three ears creating disjoint odd cycles such that each uninterupted segment has at least 4 edges has $pc(G) = 3$.*

The assumption that each uninterupted segment has length at least 4 is mostly for convenience. Note that the graph G (in Figure 7.2) does not satisfy this condition but it can still be shown that $pc(G) = 3$ by a similar argument.

Démonstration. By Theorem 7.2.1, we know that $pc(G) \leq 3$ so it suffices to show that $pc(G) \neq 2$.

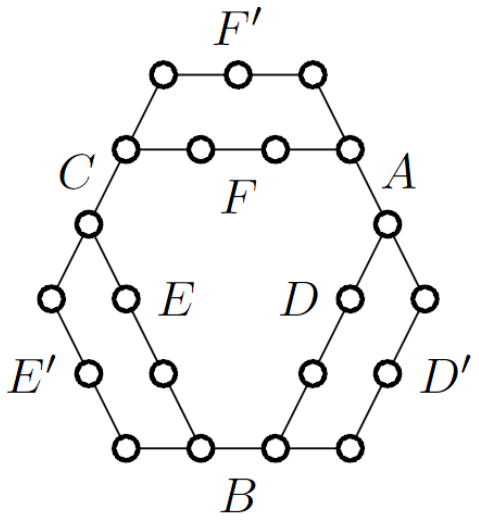

FIGURE 7.2 – Smallest 2-connected graph with $pc(G) = 3$

Suppose we have a 2-coloring of G which is properly connected. Label the segments of G as in Figure 7.2. Note that we may assume there are no three edges in a row of the same color within an uninterupted segment since we could switch the color of the middle edge (making that subsegment alternating) without disturbing the proper connectivity.

We would first like to show the segments A, B and C are all alternating. If two of these segments are not alternating, suppose A and B, then any vertex in D cannot be properly connected to any vertex of C so this is clearly not the case. This means that at most one segment, suppose A, is non-alternating. Suppose the edges uv and vw have the same color for some $u, v, w \in A$ (see Figure 7.3). There must exist a proper path from u to w so suppose there is such a path using the segments $FCEBD$. Since the following argument does not rely on the parity of this path, this assumption, as opposed to using any of D', E' or F', does not lose any generality.

Let x be a vertex in the interior of B. We already know there is a proper path from x to v using D. Since $D \cup D'$ forms an odd cycle, there can be no proper path from x to v through D'. Let $y \in E'$. In order for y to have a proper path to w, it must use the segments BD (as opposed to BD') and similarly to reach u, it must use CF (as opposed to CF'). Since $E \cup E'$ forms an odd cycle, and yet y can reach both u and w, we know that the edges on either side of y must have the same color. This holds for all $y \in E'$, clearly a contradiction. Therefore we know that A, B and C are all alternating segments.

Next we would like to show that at least one of D or D' must be alternating (and similarly

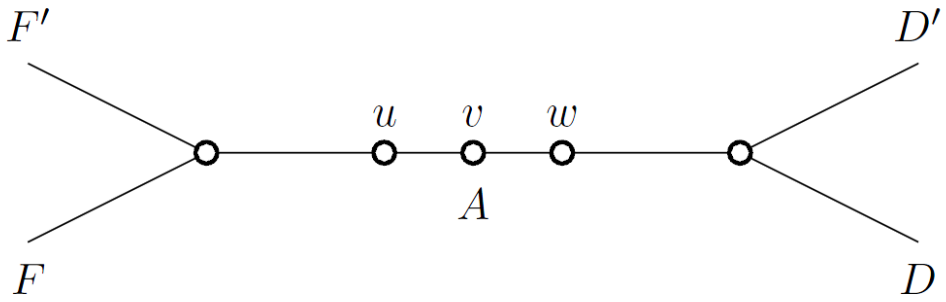

FIGURE 7.3 – Placement of vertices.

at least one of E or E' and one of F or F'). Suppose D and D' are both non-alternating. Let v be an interior vertex in D which has two edges of the same color and let y be a vertex of D' with two edges of the same color. Let u and w be the neighbors of v and let x and z be the neighbors of y (see Figure 7.4). Clearly there can be at most one pair (in this case D and D') in which neither segment is alternating since there must be an alternating path from u to w and it must pass through the other segments. Also, there can be no other pairs of adjacent monochromatic edges within D and D' since u, v and w (likewise x, y and z) must have alternating paths out of the segment and we have assumed that there are no three edges of the same color in a row. Note that, in the figure, possibly $x = a$, $u = a$, $z = b$ or $w = b$.

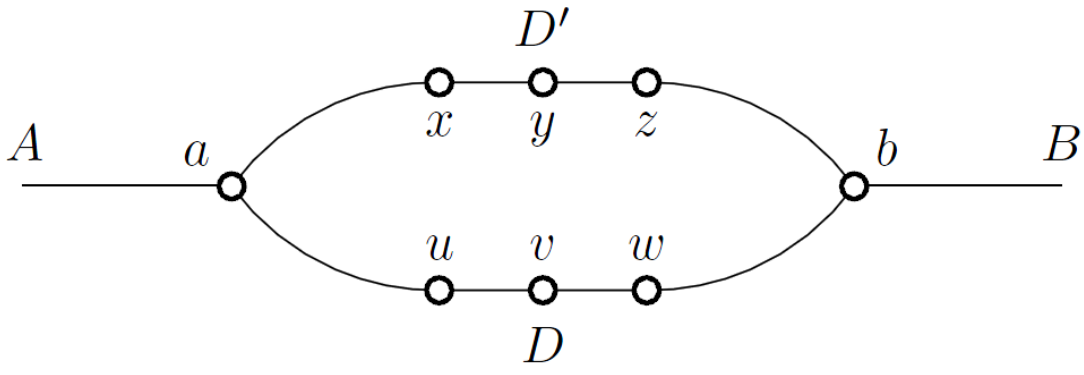

FIGURE 7.4 – Placement of vertices.

Let $Q = D \cup D'$ and let a and b be the vertices in $D \cap D' \cap A$ and $D \cap D' \cap B$ respectively. If we let $c \in C$, then each of u, w, x and z must have an alternating path to c. Suppose the edge of A incident to a has color 1. Then both edges incident to a in Q must have color 2. This

means that both edges of Q which are incident to a must be the same color (and similarly both edges of Q incident to b must have the same color). Therefore, there are exactly 4 vertices in Q for which both edges of Q have the same color. Unless $x = a$ (or possibly $z = b$, $u = a$ or $w = b$), this means that Q is even, a contradiction. Suppose $x = a$ so, in order for $z \neq b$ to have a proper path to w, we must also have $w = b$, meaning that $u \neq a$ and z so again Q is even for a contradiction. Hence, we know that at least one of D or D' must be alternating (and similarly for the other odd ears). Without loss of generality, suppose D, E and F are all alternating.

Our next goal is to show that $Q = A \cup B \cup C \cup D \cup E \cup F$ forms an alternating cycle (with the possible replacement of D with D', E with E' or F with F'). As we have shown, the only places where we can have a problem is at the intersections so let a and b be (as before) the end-vertices of D (the same argument may be applied for E or F) and suppose a is between two edges of the same colors (suppose color 1) on Q. Let u, v, w be the neighbors of a with $u \in A$, $v \in D'$ and $w \in D$ so we have assumed the edges au and aw both have color 1 (see Figure 7.5 where the darker edges represent edges that must have color 1). In order for an alternating path to get from u to w, we must either use $D' \cup D$ or Q (with the possible replacements noted above). If the path uses D', then $D \cup D'$ forms an alternating (and hence even) cycle, a contradiction. Hence, we may assume there is an alternating path from u to w through $BECFA$ (recall again that E may be replaced with E' or F with F' in this argument).

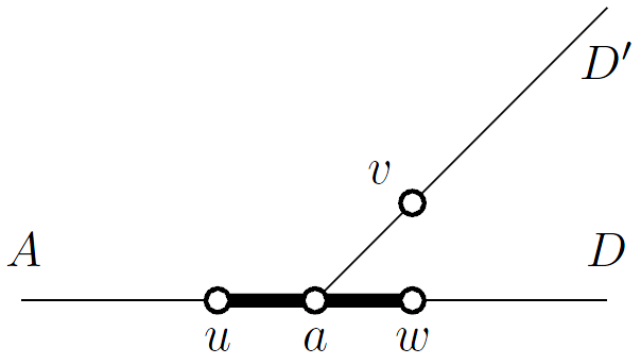

FIGURE 7.5 – Placement of vertices.

Let $x \in E'$. There is an alternating path from u to x and from w to x. Since $E \cup E'$ forms an odd cycle but x has an alternating path through B (to get to w) and through C and A (to get to u), we know that x must have two edges of the same color within E'. Since x was chosen arbitrarily, this is clearly a contradiction. This means that Q is an alternating (and hence even) cycle.

Now we simply consider one vertex in each of D', E' and F'. Since these ears form odd cycles, there exists a vertex in each segment from which (and to which) an alternating path can only go one direction on Q. By the pigeon hole principle, at least two of them must go the same direction, meaning there is no alternating path between them. This completes the proof

of Proposition 7.2.2. □

If the diameter is small, then the proper connection number is also small. More formally, we get the following result.

Theorem 7.2.3. *If $diam(G) = 2$ and G is 2-connected, then $pc(G) = 2$.*

Démonstration. If G is 3-connected, Corollary 7.1.8 implies that $pc(G) = 2$ so we may assume $\kappa(G) = 2$. Let $C = \{c_1, c_2\}$ be a (minimum) 2-cut of G and let $H_1, \ldots, H_t$ be the components of $G \setminus C$. Order components so that there is an integer $0 \leq s \leq t$ such that every vertex of H_i is adjacent to both c_1 and c_2 for $i > s$. Note that if $s = 0$, we have all edges from C to $G \setminus C$ so G contains a spanning 2-connected bipartite graph and by Theorem 7.1.7, $pc(G) = 2$. For each component H_i with $i \leq s$, define subsets $H_{i,1} = N(c_1) \cap H_i$ and $H_{i,2} = N(c_2) \cap H_i$. Since each component is connected and C is a minimum cut, there must be an edge from $H_{i,1}$ to $H_{i,2}$. Let $e_i = v_{i,1}v_{i,2}$ be one such edge in each component H_i. Define the graph $G_0 = G[C \cup (\bigcup_{i=1}^{s}\{v_{i,1}, v_{i,2}\})]$. This graph is 2-connected and bipartite so $pc(G_0) = 2$ and notice that $|G_0| = 2 + 2s$. Let G_1 be a subgraph of G obtained by adding a vertex to G_0 which has at least 2 edges into G_0. Furthermore, let G_i be a subgraph of G obtained by adding a vertex to G_{i-1} which has at least 2 edges into G_{i-1}. By Proposition 7.1.6, $pc(G_i) = 2$ for all i. We claim that there exists such a sequence of subgraphs of G such that $G_{n-(2+2s)}$ is a spanning subgraph of G. In order to prove this, suppose that G_i is the largest such subgraph of G and suppose there exists a vertex $v \in G \setminus G_i$. Certainly every vertex which is adjacent to both c_1 and c_2 is in G_i. This means $v \in H_j$ for some $1 \leq j \leq s$. Since H_j is connected, there exists a path from $v_{i,1}$ to v within H_j. Let w be the first vertex on this path which is not in G_i. Since $diam(G) = 2$, we know that w must be adjacent to at least one vertex of C. This means that $d_{G_i}(w) \geq 2$ so we may set $G_{i+1} = G_i \cup w$ for a contradiction. This completes the proof. □

Finally we prove an upper bound for $pc(G)$ for general graphs which is best possible as we saw before.

Theorem 7.2.4. *Let G be a connected graph. Consider $\widetilde{\Delta}(G)$ as the maximum degree of a vertex which is an endpoint of a bridge in G. Then $pc(G) \leq \widetilde{\Delta}(G)$ if $\widetilde{\Delta}(G) \geq 3$ and $pc(G) \leq 3$ otherwise.*

Démonstration. Let $B_1, B_2, \ldots B_s$ be the blocks of G with at least 3 vertices. For each block of B_i we have the following cases.

- B_i is bipartite or 3-connected : Then by Theorem 7.1.7 and Corollary 7.1.8, B_i can be colored with 2 colors having the strong property. We color B_i in such a way.
- $\kappa(B_i) = 2$: Then by Theorem 7.2.1, B_i can be colored with 3 colors having the strong property. We color B_i in such a way.

It is easy to see that G is proper connected if there are no more uncolored edges in G since each B_i has the strong property. Thus, suppose that there remain uncolored edges in G. It is clear that these edges induce a forest F in G. We color them as follows. Take one of the blocks,

say B_1, which contains a vertex $v \in B_1$ which is incident with some uncolored edges. Clearly, v is an endpoint of a bridge in G. We color these uncolored edges incident to v with different colors starting with color $rd_{B_1}(v)+1$. Then, we have that $rd_G(v) \leq \widetilde{\Delta}(G)$. We do the same for the rest of the vertices incident to bridges in B_1. Then, we extend our coloring for each tree going out from B_1 in a Breadth First Search (BFS) way, coloring its edges with different colors (observe from Corollary 7.1.5 that $rd_G(w) \leq \widetilde{\Delta}(G) \leq \Delta$ for each vertex w in the interior of a tree) until we reach the rest of the blocks. And finally, for each of these blocks (in this order), we repeat the previous step. Before proving that this coloring makes G proper connected, it is important to mention that, if we reach a block B_i with some color $c \geq rd_{B_i}(w)+1$, and the corresponding vertex, say w, of B_i has more than $c - rd_{B_i}(w)$ uncolored incident edges, then, when we color these edges, we do not repeat color c. Also, it is important to remark that, by coloring F in this way, we have that in any path that traverses some block from one tree in F to another, at least one of the colors before or after traversing the block is not used in the block.

We now prove that G is proper connected. Let v, w be vertices of G. It is clear that if both belong to the same block B_i, then there exists a proper path between them and the same happens if they belong to the same tree outside the blocks. If $v \in B_i$, $w \in B_j$ and $B_i \cap B_j = \{u\}$, then there exist two paths P_1, P_2 between v and u in B_i, and two paths P_3, P_4 between u and w in B_j with the strong property. Suppose without losing generality that $end(P_1) \neq start(P_3)$ and $end(P_2) \neq start(P_4)$, then we obtain the paths P_1P_3 and P_2P_4 between v and w. It is clear that $start(P_1P_3) \neq start(P_2P_4)$ and $end(P_1P_3) \neq end(P_2P_4)$ since $start(P_1P_3) = start(P_1) \neq start(P_2) = start(P_2P_4)$ and $end(P_1P_3) = end(P_3) \neq end(P_4) = end(P_2P_4)$. Therefore, these paths are proper. Now, if $B_i \cap B_j = \emptyset$ and there is a tree T in F such that $B_i \cap T = \{u_1\}$ and $B_j \cap T = \{u_2\}$, we form a proper path between v and w as follows. Let P_1 be the unique (proper) path in the tree T between u_1 and u_2. Let P_2 be the proper path in B_i between v and u_1 such that $end(P_2) \neq start(P_1)$. This path exists since we have the strong property in each block. Analogously, let P_3 be the proper path in B_j between u_2 and w such that $end(P_1) \neq start(P_3)$. Finally the path $P = P_2P_1P_3$ is proper between v and w. The same idea applies if v is in a block B_i and w is in a tree T in F such that $B_i \cap T = \{u\}$. The idea also applies in the case that v is in a tree T_i in F, w is in a tree T_j in F and there is a block B such that $T_i \cap B = \{u_1\}$ and $T_j \cap B = \{u_2\}$. Finally, the result holds by induction on the number of trees and blocks between vertices v and w using the remark stated before to guarantee the paths always traverse the blocks. Therefore, $pc(G) \leq \widetilde{\Delta}(G)$ if $\widetilde{\Delta}(G) \geq 3$ and $pc(G) \leq 3$ otherwise. □

To end the section, based on the Theorem 7.2.1 and the previous section, we extend the Conjecture 7.1.1 to general graphs.

Conjecture 7.2.5. *If G is a $2k$-connected graph with $k \geq 1$, then $pc_k(G) \leq 3$.*

This conjecture is proved for $k = 1$ in Theorem 7.2.1. Now we prove a stronger result for complete graphs.

Theorem 7.2.6. *Let $G = K_n$, $n \geq 4$, and $k > 1$. If $n \geq 2k$ then $pc_k(G) = 2$*

Démonstration. – Case $n = 2p$ for $p \geq 2$:

Take a hamiltonian cycle $C = v_1, v_2, \ldots, v_{2p}$ of G and alternate colors on the edges using colors 1 and 2 starting with color 1. Color the rest of the edges using color 1. It is clear that there are $p \geq k$ edges with color 2. We will prove that this coloring gives us k proper paths between each pair of vertices of G. Take two vertices v, w such that $c(v, w) = 2$. This edge colored with color 2 is one proper path between v and w. Now, since there are at least other $p-1 \geq k-1$ edges colored with color 2 and the rest of the edges are colored with color 1, we have at least $k-1$ proper paths between v and w using these edges. That is, for each vertices v', w' such that $c(v', w') = 2$ we form the proper path v, v', w', w. The case where $c(v, w) = 1$ is similar.

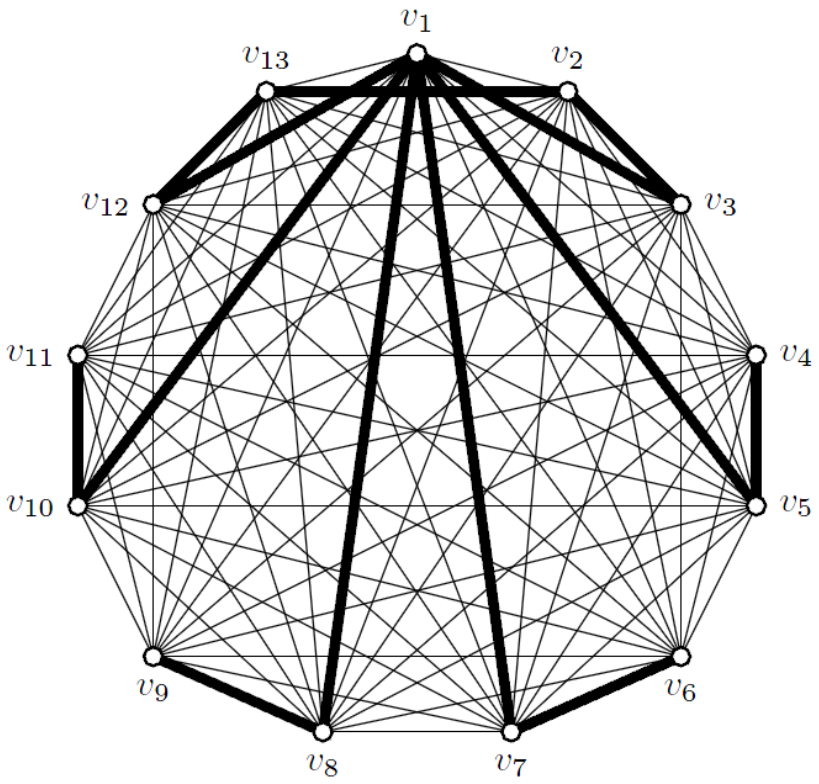

FIGURE 7.6 – Coloring of K_{13}. Normal edges represent color 1 and bold edges color 2.

– Case $n = 2p - 1$ for $p \geq 2$:

Take a hamiltonian cycle $C = v_1, v_2, \ldots, v_{2p-1}$ of G and alternate colors on the edges using colors 1 and 2 starting with color 1. We have p edges with color 1 and $p-1$ edges with color 2 so far since $c(v_1, v_2) = 1$ and $c(v_1, v_{2p-1}) = 1$. Now, put $c(v_2, v_{2p-1}) = 2$, $c(v_1, v_3) = 2$, $c(v_1, v_{2p-2}) = 2$ and for each edge with color 2, different from v_2, v_3 and v_{2p-2}, v_{2p-1}, choose one of the endpoints, say v', and put $c(v_1, v') = 2$ (see Fig. 7.6).

Finally, color the rest of the edges with color 1. We now show that this coloring gives k proper paths between each pair of vertices v and w of G. First, take $v = v_1$ and $w = v_2$ (or similarly taking $w = v_{2p-1}$). We have the edge v_1v_2 and the path v_1, v_{2p-1}, v_2. Now since $n = 2p - 1 \geq 2k$ we have at least $(p-1) - 2 \geq k - 2$ edges in the cycle C with color 2 different from v_2, v_3 and v_{2p-2}, v_{2p-1} and therefore we form the following $k-2$ proper paths between v_1 and v_2 of the form v_1, v', v_2 where v' is an endpoint of each of these edges such that $c(v_1, v') = 2$. Now take $v = v_1$ and $w = v_3$ (analog taking $w = v_{2p-2}$). This case is similar to the previous except changing the second formed path to v_1, v_2, v_3. Suppose now that $v = v_1$ and $w = w'$ with $w' \notin \{v_2, v_3, v_{2p-2}, v_{2p-1}\}$. We take the edge v_1w' and now, since there are at least $i(p-1) - 1 \geq k - 1$ edges in the cycle C with color 2 with endpoints different from v', we form the following $k-1$ proper paths between v_1 and w' of the form v_1, v', w' where v' is an endpoint of each of these edges such that $c(v_1, v') = 2$. The rest of the cases are similar to those described before in the case $n = 2p$ forming most of the proper paths with length 3.

□

7.3 Minimum degree

In this section, we prove the following result concerning minimum degrees.

Theorem 7.3.1. *If G is a connected non-complete graph with $n \geq 68$ vertices and $\delta(G) \geq n/4$, then $pc(G) = 2$.*

The minimum degree condition is best possible. To see this, we construct the following graph. Let G_i be a complete graph with $n/4$ vertices for $i = 1, 2, 3, 4$, and take a vertex $v_i \in G_i$ for each $1 \leq i \leq 4$. Let G be a graph obtained from $G_1 \cup G_2 \cup G_3 \cup G_4$ by joining v_1 and v_j with an edge for each $2 \leq j \leq 4$. Then the resulting graph G is connected and it has $\delta(G) = n/4 - 1$ and $pc(G) = 3$. To prove Theorem 7.3.1, we will make use of the following theorems.

Theorem 7.3.2 ([30])**.** *Let G be a graph with n vertices. If $\delta(G) \geq \frac{n-1}{2}$, then G has a hamiltonian path. Moreover, if $\delta(G) \geq n/2$, then G has a hamiltonian cycle. Also, if $\delta(G) \geq \frac{n+1}{2}$, then G is hamilton-connected.*

Theorem 7.3.3 ([69])**.** *Let G be a graph with n vertices. If $\delta(G) \geq \frac{n+2}{2}$ then G is panconnected meaning that, between any pair of vertices in G, there is a path of every length from 2 up to $n-1$.*

Theorem 7.3.4 ([57])**.** *Let G be a 3-connected graph with n vertices and $\delta(G) \geq n/4+2$. Then, for any longest cycle C in G, every component of $G - C$ has at most two vertices.*

Theorem 7.3.5 ([32])**.** *Let G be a connected graph with n vertices and $\delta(G) \geq n/3$. Then one of the following holds :*

(i) G contains a hamiltonian path.

(ii) For any longest cycle C of G, $G - C$ has no edge.

Also we use the following easy fact as a matter of course.

Fact 7.3.5.1. *Every 2-connected graph G with $\delta(G) \geq 2$ is either hamiltonian or contains a cycle C with at least $2\delta(G)$ vertices.*

For this statement, we use the following notation. For a path $P = v_1 v_2 \cdots v_\ell$, we let $endpoints(P) = \{v_1, v_\ell\}$.

Lemma 7.3.6. *The following graphs H_i, for $(i = 1, 2, \ldots, 6)$, have $pc(H_i) = 2$.*

(1) The graph H_1 obtained from a path P with $|P| \geq 2$ and $m \geq 0$ isolated vertices $v_1, \ldots, v_m$ by joining each v_i for $(i \leq m)$ within P with at least two edges.

(2) The graph H_2 obtained from a path P with $|P| \geq 1$ and even cycle C by identifying exactly one vertex (i.e., $|P \cap C| = 1$).

(3) The graph H_3 obtained from H_2 and $m \geq 0$ isolated vertices $v_1, \ldots, v_m$ by joining each v_i for $(i \leq m)$ with at least two edges to either $P - C$ or $C - P$ in H_2.

(4) The graph H_4 obtained from an even cycle C and two paths P_1 and P_2 by identifying an end of each path to a vertex of C. As in H_3, we may also join vertices each with at least 2 edges to either a path P_i or C.

(5) The graph H_5 obtained from the union of two disjoint cycles which are connected by two disjoint paths to form a 2-connected graph. Furthermore, we may also add vertices each with at least 2 edges to this structure.

(6) The graph H_6 obtained from H_5 by removing an edge from one of the cycles. Again we may add vertices each with at least 2 edges to this structure.

Démonstration. One can easily get a 2-coloring of H_i which forces $pc(H_i) = 2$ for $i = 1, 2, \ldots, 6$. For example, as for H_1, by Fact 7.2.0.2 and Proposition 7.1.6, there is a 2-coloring of H_1 that is properly connected. □

Proof of Theorem 7.3.1: If $\kappa(G) \geq 3$, then by Corollary 7.1.8, we have $pc(G) = 2$. So we may assume that $\kappa(G) = 1$ or 2. We divide the proof into two cases according to the value of $\kappa(G)$.

Case 1 : $\kappa(G) = 1$.

Let v be a cutvertex of G and let $C_1, \ldots, C_\ell$ be the components of $G \backslash v$ such that $|C_1| \leq \ldots \leq |C_\ell|$. By the minimum degree condition, we see that $\ell = 2$ or 3 and $|C_1| \geq n/4$. We further divide the proof into two subcases :

Subcase 1.1 : $\ell = 2$.

In this case note that $|C_1| \leq (n-1)/2$ and, by the minimum degree condition, $|C_2| \leq 3n/4 - 1$. Utilizing Theorem 7.3.2 and the minimum degree condition, it is easy to check that $\langle \{v\} \cup C_1 \rangle$ contains a hamiltonian path P_1 such that $v \in endpoints(P_1)$. If $\kappa(C_2) \geq 3$, then let C be a longest cycle of C_2. Since G is connected, there is a path P' from v to C. Now $H = P_1 \cup P' \cup C$

satisfies the conditions of H_2 in Lemma 7.3.6. This means that $pc(H) = 2$. By Theorem 7.3.4, every component of $C_2 \setminus C$ has at most 2 vertices. By the minimum degree condition and since we assume $n \geq 12$, for each $x \in C_2 \setminus H$, we have $|E(x, H)| \geq \frac{n}{4} - 1 \geq 2$. Hence, G contains a spanning subgraph which satisfies the properties of H_3 in Lemma 7.3.6 so $pc(G) = 2$. Thus we may assume that $\kappa(C_2) = 1$ or 2. Let S be a cutset in C_2 with $1 \leq |S| \leq 2$. By the minimum degree condition, it is easy to check that there are exactly two components C_{21}, C_{22} with $|C_{21}| \leq |C_{22}|$ in $C_2 - S$. Note that $n/4 - |S| \leq |C_{21}| \leq |C_{22}| \leq (3n/4 - 1) - |S| - (n/4 - |S|) = n/2 - 1$ because $\delta(G) \geq n/4$ and $|C_{21}| \leq (3n/4 - 1 - |S|)/2 = 3n/8 - (|S| + 1)/2 \leq 3n/8 - 1$. Hence by Theorem 7.3.2, C_{21} contains a hamiltonian cycle C'_{21}. Since $\delta(C_{22}) \geq n/4 - 3$, C_{22} is either hamiltonian or contains a cycle C'_{22} with $|C'_{22}| \geq n/2 - 6$. Now take a path P_2 with $v \in endpoints(P_2)$ so that

(1) P_2 contains a longer segment of C'_{2j} for each $j = 1, 2$, and subject to condition (1),

(2) $|P_2|$ is as large as possible.

By the choice of P_2, note that $P_2 \cap S \neq \emptyset$. Let P be a path joining P_1 and P_2 at the common vertex v. Then, utilizing P and the assumption $\delta(G) \geq n/4$, we will find a spanning subgraph which has a property of H_1 in Lemma 7.3.6. In order to show this, we need only show that each vertex in $G \setminus P$ has at least 2 edges to P. As previously discussed, we know that all vertices in C_1 have at least 2 edges to P_1 so we need only check vertices $x \in C_2 \setminus P_2$. If $x \in C_{21}$ then since $|P \cap C_{21}| \geq |C_{21}|/2$ and $|C_{21}| \leq 3n/8 - 1$, by the minimum degree condition, x has at least $n/4 - 3n/16 \geq 2$ edges to P since $n \geq 32$. For $x \in C_{22}$, we know $|C_{22}| \leq n/2 - 2$ and either C_{22} is hamiltonian or contains a cycle of length at least $n/2 - 6$. In either case, the same arguments easily show that x has at least 2 edges to P, meaning that $pc(G) = 2$.

Subcase 1.2 : $\ell = 3$.

In this case, by the minimum degree condition, we see that $n/4 \leq |C_1| \leq (n-1)/3 \leq |C_3| \leq n/2 - 1$, and $|C_2| \leq 3n/8 - 1/2$. Hence by Theorem 7.3.2, each C_i with $i = 1, 2$ is hamilton-connected. Also, by the minimum degree condition and since $n \geq 36$, we see that $\delta(C_i) \geq (|C_i| + 2)/2$ for $i = 1, 2$ so for any vertex $z \in C_i$, $C_i - z$ is hamilton-connected. By Theorem 7.3.2, C_3 is hamiltonian so it contains a spanning path P with $v \in endpoints(P)$. If $|E(v, C_i)| \geq 2$ holds for $i = 1$ or 2 (suppose $i = 1$), then we can find an even cycle C in $C_1 \cup v$ such that $v \in C$ and $|C_1| \leq |C| \leq |C_1| + 1$. Using a hamiltonian path of C_2 ending at v, together with the path P and the even cycle C, we can easily find a spanning subgraph which satisfies the property of H_3 in Lemma 7.3.6, and hence $pc(G) = 2$. Thus we may assume that $|E(v, C_1)| = |E(v, C_2)| = 1$. This implies $|C_1| \geq n/4 + 1$, because there is a vertex of C_1 which is not adjacent to v. Then we get $|C_3| \leq n/2 - 3$ so $\delta(C_3) \geq n/4 - 1 \geq (|C_3| + 1)/2$. If $|C_3|$ is odd, then by Theorem 7.3.2, C_3 is hamiltonian connected. Hence, we can find an even cycle using all of C_3 and v and a single path through v using all of C_1 and $C2$. This provides a spanning subgraph satisfying the properties of H_3 in Lemma 7.3.6. If $|C_3|$ is even, then $\delta(C_3) \geq \lceil \frac{|C_3|+1}{2} \rceil = \frac{|C_3|+2}{2}$ so, by Theorem 7.3.3, C_3 is panconnected. Thus we can find an even cycle through $v \cup C_3$ which avoids exactly 1 vertex of C_3 again easily providing a subgraph satisfying the conditions of H_3 in Lemma 7.3.6. This

shows that $pc(G) = 2$ and completes the proof of this case.

Case 2 : $\kappa(G) = 2$**.**

Let u and v be a minimum cutset of G. Again we let $C_1, C_2, \ldots, C_\ell$ be the components of $G \setminus \{u, v\}$ with $|C_i| \leq |C_j|$ for $i \leq j$ and break the rest of the argument into cases based on the value of ℓ. Note that, since $\delta(G) \geq n/4$, we have $2 \leq \ell \leq 4$.

Subcase 2.1 : $\ell = 4$

Since $\delta(G) \geq n/4$, we know that $n/4 - 1 \leq |C_1| \leq (n-2)/4 \leq |C_4| \leq n/4 + 1$. This means that $\delta(C_i) \geq |C_i| - 2$ for all i. The graph G is 2-connected so there are two independent edges from $\{u, v\}$ to each component C_i. With $n \geq 26$, we see that $|C_i| \geq 6$ so the minimum degree condition $\delta(C_i) \geq |C_i| - 2$ implies, by Theorem 7.3.3, that each component C_i is panconnected. This means that, if $|C_3 \cup C_4|$ is even, we may find a cycle through $\{u, v\} \cup C_3 \cup C_4$ using all the vertices, and if $|C_3 \cup C_4|$ is odd, we may find a similar cycle which misses exactly one vertex $w \in C_4$. This cycle, along with a spanning path of $u \cup C_1 \cup C_2$ and possibly w provides a spanning subgraph of G satisfying the properties of H_3 from Lemma 7.3.6, meaning that $pc(G) = 2$.

Subcase 2.2 : $\ell = 3$

Since $\delta(G) \geq n/4$, we have $n/4 - 1 \leq |C_1| \leq |C_2| \leq (5n-4)/12$ and $\delta(C_i) \geq n/4 - 2$ so $\delta(C_i) \geq \frac{3|C_i|+1}{5} - 2$ for $i = 1, 2$. Since $n \geq 23$, this implies that $\delta(C_i) \geq \frac{|C_i|+1}{2}$ for $i = 1, 2$ so C_1 and C_2 are both hamiltonian-connected by Theorem 7.3.2. This means we may create a single cycle D_{12} using all of $C_1 \cup C_2$. If $\kappa(C_3) \geq 2$, then let D_3 be a longest cycle in C_3. Since $\delta(C_3) \geq \frac{n}{4} - 2$, we know $|D_3| \geq \min\{|C_3|, \frac{n}{2} - 4\}$. In either case every vertex of C_3 has at least 2 edges to H_3. Now since G is 2-connected, there exist two disjoint paths from D_{12} to D_3 meaning there is a spanning subgraph of G satisfying the conditions of the graph H_5. By Lemma 7.3.6, we have $pc(G) = 2$. If $\kappa(C_3) = 1$, then by Theorem 7.3.5, there is a spanning path P of C_3. The vertices u and v must each have at least one edge to P so $P \cup D_{12}$ forms a spanning subgraph of G satisfying the conditions of the graph H_6 in Lemma 7.3.6. Hence, $pc(G) = 2$.

Subcase 2.3 : $\ell = 2$

If C_1 and C_2 are both 3-connected, then by Corollary 7.1.8, there is a 2-coloring of each with that strong property. Along with these colorings, we also color all edges between $\{u, v\}$ and C_i with color i. This coloring clearly shows that $PC(G) = 2$ so we may assume that at least one component C_i has $1 \leq \kappa(C_i) \leq 2$. Next we will suppose that $1 \leq \kappa(C_i) \leq 2$ for both $i = 1, 2$. In this case, by the minimum degree condition and the fact that G is 2-connected, we may easily show that each component is hamiltonian connected (since n is large) so G is hamiltonian. This means $pc(G) = 2$.

Finally, if we suppose C_1 is 3-connected while $1 \leq \kappa(C_2) \leq 2$, each possible case contains a large (almost spanning) subgraph with the properties of H_4 from Lemma7.3.6, meaning that $pc(G) = 2$. This completes the proof of Theorem 7.3.1. □

7.4 Conclusion

From Theorem 7.3.1, it is clear that if G is 2-connected and $\delta(G) \geq \frac{n}{4}$, then $pc(G) = 2$. We believe that this degree condition can be greatly improved in the 2-connected case. In particular, we propose the following conjecture.

Conjecture 7.4.1. *If $\kappa(G) = 2$ and $\delta(G) \geq 3$, then $pc(G) = 2$.*

By the proof of Theorem 7.2.1 and the standard ear decomposition of a 2-connected graph, it is easy to produce a linear-time algorithm to 3-color any 2-connected graph to be proper connected with the strong property. Also since there is an $O(n + m)$ algorithm for finding a block decomposition of a graph G with $\kappa(G) = 1$ on n vertices with m edges, we can find an $O(n + m)$ algorithm to produce a proper connected coloring of such graphs. Therefore, in practice, these colorings are not difficult to find.

Chapitre 8

Conclusion and Perspectives

Contents

In the present thesis, we studied proper spanning trees and weak-proper spanning trees in edge-colored graphs and multigraphs. Our main objective was to focus on particular families of edge-colored graphs, and to contribute in the following aspects :

(1) Demonstrate the NP-completeness and give some nonapproximability bounds for both PST and WST problems,

(2) Focus on the edge-colored complete graphs and show that the problem remains NP-complete,

(3) Provide methods to solve the PST and WST problems, and devise algorithms for a more general problem of finding Maximum Proper Tree and Maximum Weak Proper tree in the case of edge-colored acyclic graphs.

(4) Give sufficient conditions for edge-colored graphs and multigraphs to have PST.

We believe that the results presented in the thesis contribute significantly to the understanding of the PST and WST problem in edge-colored graphs. In what follows, we recall briefly the results presented in this thesis and conclude with a few remarks on the future work and perspectives.

8.1 Contribution summary

As we have seen in Chapter 4, both problems PST and WST as well as their optimization versions are NP-Complete. We gave some nonapproximability bounds for both problems. For this reason we have studied these problems for some particular families of edge-colored graphs. We have seen that in the case of the edge-colored complete graphs the PST Problem (PSTP) remains NP-Complete. However, in the case of WST Problem (WSTP), it becomes trivial. We have also seen that in the case of the edge-colored graphs with many colors of order $\Omega(n^2)$, the

PSTP remains NP-Complete.
Next, we focused on the PSTP and WSTP in the acyclic edge-colored graphs. We gave three different ways to solve the problem if any, PST or WST, exists, and a polynomial time algorithm that finds a Maximum Proper Tree (MPT) and Maximum Weak Proper Tree (MWT) in a given acyclic edge-colored graph.
The first approach consists in finding the maximum matchings for each monochromatic subgraph, and then in taking the union of edges in all the matchings and as a result we find a PST, if any. It is obvious that a PST is also a WST. However, if no PST exists in a given edge-colored graph, we can not determine whether a WST exists. As we have shown, such an easy and intuitive approach doesn't apply in the case of WST.
The second approach uses an algorithm based on the Breadth-First-Search. This algorithm finds a PST in a complete acyclic edge-colored graph. In the case of general acyclic edge-colored graphs, we presented similar algorithms based on the maximum matching algorithm for edge-colored graphs. These algorithms return the MPT and MWT in case where neither PST nor WST exist in the graph.
The third approach is an algorithm based again on the maximum matching algorithm, though it returns the maximum proper forest (maximum in terms of edges). This can be interesting for designing an approximation algorithm for MSTP.

We also studied sufficient conditions for edge-colored graphs to have PST. We gave some sufficient conditions related to the number of vertices, number of colors and the connectivity. We also gave a similar relation between the minimum rainbow degree of a graph, the number of colors and the order of the graph.
Next, we studied the same problem in the case of the edge-colored multigraphs. Since for 2-edge-colored multigraphs PST are equivalent to Hamiltonian proper paths, we give sufficient conditions based on the number of edges and of vertices for a 2-edge-colored multigraph to have a Hamiltonian path. We have done the same work for c-edge-colored multigraphs.
Finally, other parallel results were presented on the coloring of simple graphs in order to guarantee some proper connectivity.

8.2 Perspectives for future work

In order to complete the study of PST and WST problems, we would like to pursue our work in the following directions.
First, we would like to develop algorithms for solving the PST and the WST problems in other special classes of graphs, such as planar graphs.
Second, concerning the existence of PST and WST, we would like to propose sufficient conditions in the case of edge-colored multigraphs, and to give more sufficient conditions for the WST problem in edge-colored graphs. Also, we would like to find such conditions for Proper Hamiltonian cycles

in edge-colored multigraphs, as well as for cycles partition and 1-tree-cycles partition in edge-colored graphs and multigraphs.
Finally, the following are conjectures we formulated and we believe we can prove, which would allow to provide necessary conditions for the WST problem.

Conjecture 8.2.1. *Let G^c be be a k-connected edge-colored graph such that $n \geq \frac{(k+j)^2+3(k+j)-2}{2}$ and*
$c \geq \frac{(n-k-j)(n-k-j-1)}{2} + 2$, with $j = \lceil \frac{1+\sqrt{1+4k}}{2} \rceil$. Then G^c has a weak spanning tree.

Conjecture 8.2.1, if true, is the best possible. Indeed, consider a rainbow complete graph on $n-k-j$ with suppose $k=1$ and $j=2$, and $c \geq \frac{(n-k-j)(n-k-j-1)}{2} + 1$. Since $k=1$ the three vertices say v, w and z such that v are connected to the complete graph on $n-k-j$ vertices and to w and z only to w by edges in one color. Then the resulting graph has no WST since if you choose as root a vertex in the complete graph you can not connect w and z, if you will choose as root v you can not connect z and if you will choose w as root you can not connect the complete graph.

Conjecture 8.2.2. *Let G^c be a c-edge-colored connected graph with rainbow degree $rd\,(G^c) = k$ and $n \geq 4k+2$. If $c \geq \frac{(n-3k-1)(n-3k-2)}{2} + k(k-1) + \frac{k(k+1)}{2} + 2$, then G^c has a weakly proper edge-colored spanning tree.*

Conjecture 8.2.2, if true, is the best possible. Indeed, consider a rainbow complete graphs on $n-3k-1 \geq k+1$ vertices, one on $k+1$ and 2 on k. Choose a vertex x and y in the graph on $n-3k-1$ and $k+1$ vertices. Connect now the x with all vertices of one graph in k vertices, y with vertices of the second one and x and y between them by edges in new color. The resulting graph has $c \geq \frac{(n-3k-1)(n-3k-2)}{2} + k(k-1) + \frac{k(k+1)}{2} + 1$, and the rainbow degree equal k. This graph has no WST since no matter in what of the 4 graphs we will choose the root, one of the 4 graphs can not be added to the WT.

In addition, the following Conjectures 8.2.3, 8.2.4, 8.2.5 have already been mentioned previously, concern the coloring of the graph in order to guarantee the proper connectivity.

Conjecture 8.2.3. *If $\kappa(G) = 2$ and $\delta(G) \geq 3$, then $pc(G) = 2$.*

Conjecture 8.2.4. *If G is a $2k$-connected bipartite graph with $k \geq 1$, then $pc_k(G) = 2$.*

Conjecture 8.2.5. *If G is a $2k$-connected graph with $k \geq 1$, then $pc_k(G) \leq 3$.*

In conclusion, our future work and long term objective is to provide a complete study of the PST and WST problems in different kinds of graphs and multigraphs, including sufficient and necessary conditions for PST and WST to exist, as well as approximation algorithms with performance guarantee of at least logarithm of the optimal solution.

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