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## THÈSE

Présentée pour obtenir

LE GRADE DE DOCTEUR EN SCIENCES DE  
L'UNIVERSITÉ PARIS-NORD XIII

Spécialité: Mathématiques

par

Mostapha BENHENDA

# **Estimées de la conjugaison à des rotations de difféomorphismes du cercle. Conjugaisons successives et réalisations différentiables**

Soutenue le 17 septembre 2012 devant la Commission d'examen:

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## Résumé

Cette thèse traite de quelques questions de dynamique différentiable. Elle se compose de deux parties relativement indépendantes, comprenant chacune deux chapitres. La première partie établit des estimées de la conjugaison à des rotations de difféomorphismes du cercle, et en obtient des applications. La seconde partie porte sur la méthode de construction par conjugaisons successives et le problème de réalisation différentiable.

Le premier chapitre part d'un théorème célèbre de Herman et Yoccoz, qui affirme que si un difféomorphisme  $C^\infty$  du cercle  $f$  a un nombre de rotation  $\alpha$  qui satisfait à une condition diophantienne, alors  $f$  est  $C^\infty$ -conjugué à une rotation. Nous établissons des relations explicites entre les normes  $C^k$  de cette conjugaison et la condition diophantienne sur  $\alpha$ . Pour obtenir ces estimées, nous modifions convenablement la preuve de Yoccoz.

Dans le deuxième chapitre, nous utilisons certaines de ces estimées pour montrer deux résultats. Le premier porte sur le problème de quasi-réductibilité : pour un ensemble Baire-dense de nombres  $\alpha$ , pour tout difféomorphisme  $f$  de nombre de rotation  $\alpha$ , il est possible d'accumuler  $R_\alpha$  avec une suite  $h_n f h_n^{-1}$ ,  $h_n$  étant un difféomorphisme. Le second résultat de ce chapitre est : pour un ensemble Baire-dense de nombres  $\alpha$ , étant donnés deux difféomorphismes  $f$  and  $g$  qui commutent, tels que  $f$  a  $\alpha$  pour nombre de rotation, il est possible d'approcher chacun d'eux par des difféomorphismes  $f_n$  et  $g_n$  qui commutent, et qui sont conjugués de manière différentiable à des rotations.

Le troisième chapitre traite du problème de réalisation lisse non-standard de translations du tore. Sur certaines variétés admettant une action du cercle, nous construisons des difféomorphismes préservant le volume, et métriquement isomorphes à des translations ergodiques du tore, tels qu'une coordonnée de la translation soit un nombre de Liouville arbitraire. Pour obtenir ce résultat, nous déterminons des conditions suffisantes sur des vecteurs de translation du tore qui permettent de construire explicitement la suite de conjugaisons successives dans la méthode d'Anosov-Katok, avec des estimées convenables de leur norme.

Dans le quatrième chapitre, sur les mêmes variétés que précédemment, et pour certains angles de Liouville  $\alpha$ , nous montrons que l'adhérence lisse de la classe de conjugaison lisse et préservant le volume de la rotation  $S_\alpha$  contient un difféomorphisme lisse et préservant le volume  $T$  qui est métriquement isomorphe à une rotation irrationnelle du cercle  $R_\beta$ , avec  $\alpha \neq \pm\beta$ , et avec  $\alpha$  et  $\beta$  choisis rationnellement dépendants ou rationnellement indépendants. En particulier, l'anneau fermé  $[0, 1] \times \mathbb{T}^1$  admet une pseudo-rotation lisse ergodique  $T$  d'angle  $\alpha$  qui est métriquement isomorphe à une rotation  $R_\beta$ .

**Mots-clefs** : systèmes dynamiques, théorie ergodique, difféomorphismes du cercle, nombre de rotation, méthode d'Anosov-Katok, problème de réalisation lisse.

ESTIMATES OF THE CONJUGACY TO ROTATIONS OF CIRCLE DIFFEOMORPHISMS. SUCCESSIVES CONJUGACIES  
AND SMOOTH REALIZATIONS

**Abstract**

This thesis deals with some questions on differentiable dynamical systems. It comprises two relatively independent parts, with two chapters each. The first part deals with estimates of the conjugacy to rotations of circle diffeomorphisms and their applications. The second part deals with successive conjugacies and the smooth realization problem.

The first chapter is based on a celebrated theorem by Herman and Yoccoz, which asserts that if the rotation number  $\alpha$  of a  $C^\infty$ -diffeomorphism of the circle  $f$  satisfies a Diophantine condition, then  $f$  is  $C^\infty$ -conjugated to a rotation. We establish explicit relationships between the  $C^k$  norms of this conjugacy and the Diophantine condition on  $\alpha$ . To obtain these estimates, we follow a suitably modified version of Yoccoz's proof.

In the second chapter, we use some of these estimates to show two related results. The first is on quasi-reducibility: for a Baire-dense set of  $\alpha$ , for any diffeomorphism  $f$  of rotation number  $\alpha$ , it is possible to accumulate  $R_\alpha$  with a sequence  $h_n f h_n^{-1}$ ,  $h_n$  being a diffeomorphism. The second result of this chapter is: for a Baire-dense set of  $\alpha$ , given two commuting diffeomorphisms  $f$  and  $g$ , such that  $f$  has  $\alpha$  for rotation number, it is possible to approach each of them by commuting diffeomorphisms  $f_n$  and  $g_n$  that are differentiably conjugated to rotations.

The third chapter deals with the problem of non-standard smooth realization of translations of the torus. On some manifolds admitting a circle action, we construct volume-preserving diffeomorphisms that are metrically isomorphic to ergodic translations on the torus, where one given coordinate of the translation is an arbitrary Liouville number. To obtain this result, we determine sufficient conditions on translation vectors of the torus that allow to explicitly construct the sequence of successive conjugacies in Anosov-Katok's method, with suitable estimates of their norm.

In the fourth chapter, on the same manifolds as previously, we show that the smooth closure of the smooth volume-preserving conjugation class of some Liouville rotations  $S_\alpha$  of angle  $\alpha$  contains a smooth volume-preserving diffeomorphism  $T$  that is metrically isomorphic to an irrational rotation  $R_\beta$  on the circle, with  $\alpha \neq \pm\beta$ , and with  $\alpha$  and  $\beta$  chosen either rationally dependent or rationally independent. In particular, the closed annulus  $[0, 1] \times \mathbb{T}^1$  admits a smooth ergodic pseudo-rotation  $T$  of angle  $\alpha$  that is metrically isomorphic to the rotation  $R_\beta$ .

**Keywords** : dynamical systems, ergodic theory, circle diffeomorphisms, rotation number, Anosov-Katok method, smooth realization problem.



## Remerciements

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## Introduction

Cette thèse se compose de deux parties relativement indépendantes. Une première partie (chapitres I et II) porte sur les difféomorphismes du cercle. Une seconde partie (chapitres III et IV) porte sur l'étude, par la méthode de construction par conjugaisons successives, de deux problèmes de réalisation lisse non-standard. Chaque chapitre correspond à un article autonome, et peut être lue de manière indépendante. De plus, dans la première partie, les rappels préliminaires du chapitre I sont repris dans ceux du chapitre II. Dans la seconde partie, certains résultats du chapitre III sont utilisés et rappelés dans le chapitre IV.

# 1 Estimées de la linéarisation de difféomorphismes du cercle et applications

## 1.1 Estimées de la linéarisation de difféomorphismes du cercle

Le cercle  $\mathbb{R}/\mathbb{Z}$  est noté  $\mathbb{T}^1$ . Nous travaillons souvent dans le revêtement universel  $D^r(\mathbb{T}^1)$  du groupe des difféomorphismes du cercle de classe  $C^r$  qui préservent l'orientation.  $D^r(\mathbb{T}^1)$  est le groupe des difféomorphismes  $f$  de classe  $C^r$  de la droite réelle tels que  $f - Id$  est  $\mathbb{Z}$ -périodique. Un exemple simple de difféomorphisme du cercle est la rotation  $R_\alpha$  d'angle  $\alpha \in \mathbb{T}^1$ . Si  $\alpha = p/q$ ,  $p$  et  $q$  premiers entre eux, alors toutes les orbites de  $R_\alpha$  sont périodiques d'ordre  $q$ . Si  $\alpha$  est irrationnel, toutes les orbites sont denses.

Le nombre de rotation  $\rho(f) \in \mathbb{T}^1$  d'un homéomorphisme du cercle  $f$  préservant l'orientation est donné par la classe modulo  $\mathbb{Z}$  de la limite  $\alpha$  de la suite  $(\tilde{f}^n(x) - x)/n$ , où  $\tilde{f}$  est un relevé quelconque de  $f$  dans  $D^0(\mathbb{T}^1)$  ( $\alpha$  est aussi appelé le nombre de rotation de  $\tilde{f}$ ). La limite de cette suite est indépendante du choix de  $x$ . Si  $\rho(f)$  est rationnel, alors  $f$  admet au moins une orbite périodique. Si  $\rho(f) = \alpha$  est irrationnel, il existe une unique application  $h$ , continue, surjective et préservant l'orientation, telle que  $h(0) = 0$  et  $h \circ f = R_\alpha \circ h$ . De plus, si  $h$  est un homéomorphisme, alors toutes les orbites de  $f$  sont denses. Sinon, aucune orbite de  $f$  n'est dense, et toutes les orbites s'accroissent sur un ensemble de Cantor invariant. Cependant, si  $f$  est de classe  $C^2$ , alors  $h$  est toujours un homéomorphisme : c'est le contenu du théorème de Denjoy.

Dans le cas où  $\rho(f) = \alpha$  est irrationnel, il est important de noter que la régularité de la conjugaison  $h$  ne dépend pas seulement de la régularité de  $f$ . Par exemple, considérons la famille d'Arnold :  $f_{a,b}(x) = x + a \sin(2\pi x) + b$  où  $0 < |a| < 1/2\pi$ ,  $b \in \mathbb{T}^1$ . Il existe des valeurs du paramètre  $b$  telles que la conjugaison  $h$  de  $f_{a,b}$  à une rotation n'est pas absolument continue, bien que  $f_{a,b}$  soit analytique [Arn65]. Pour obtenir des résultats de régularité sur  $h$ , il est nécessaire de supposer que  $\alpha$  est "mal approché" par les rationnels (i.e. diophantien).

Un nombre  $\alpha$  irrationnel est dit *diophantien* s'il existe  $\beta \geq 0$ ,  $C_d > 0$  tels que pour tous entiers  $p, q$  premiers entre eux :

$$|\alpha - p/q| > C_d/q^{2+\beta}$$

Si de plus  $\beta = 0$ , alors  $\alpha$  est dit de *type constant*. Si  $\alpha$  irrationnel n'est pas diophantien, il est dit *Liouville*.

Arnold, dans le cas analytique, et Moser, dans le cas lisse, ont montré que si  $\alpha$  est diophantien et si  $f$  est proche de  $R_\alpha$ , alors  $h$  est analytique si  $f$  l'est, et  $h$  est lisse si  $f$  l'est. Ils obtiennent ces résultats avec la méthode dite de KAM (Kolmogorov-Arnold-Moser).

Herman [Her79] permet de s'affranchir de l'hypothèse de proximité à une rotation, et obtient un résultat global : *il existe un ensemble de mesure de Lebesgue pleine tel que si le nombre de rotation d'un difféomorphisme lisse  $f$  appartient à cet ensemble, alors  $f$  est conjugué à une rotation par un difféomorphisme lisse.* En utilisant les méthodes d'Herman, Yoccoz compléta ce résultat pour inclure tous les nombres diophantiens, ce résultat étant optimal en classe  $C^\infty$ . Depuis, d'autres résultats ont enrichi ce domaine [KO89b, Yoc02, FK09a, KT09]. Pour notre part, en s'appuyant sur une version modifiée de la démonstration de Yoccoz, nous obtenons des estimées de la norme de la conjugaison  $h$  en fonction des normes de  $f$ , de son degré de différentiabilité, et des constantes diophantiennes de  $\alpha$ . Ce résultat permet de prolonger la théorie perturbative, où les théorèmes KAM fournissent habituellement une borne sur la norme de la conjugaison qui dépend de la norme de la perturbation et des constantes diophantiennes du nombre  $\alpha$  [Mos66, DIL99]. En effet, dans le chapitre I, nous établissons le résultat suivant :

**Theorem 1.1.** *Soit  $k \geq 3$  un entier et  $f \in D^k(\mathbb{T}^1)$ . Soit  $\alpha \in CD(C_d, \beta)$  le nombre de rotation de  $f$ . Si  $k > 2\beta + 1$ , alors pour tout  $\eta > 0$ , il existe un difféomorphisme  $h$  de classe  $C^{k-1-\beta-\eta}$  qui conjugue  $f$  à  $R_\alpha$ , et une fonction numérique  $C$ , qui vérifient l'estimation :*

$$\|h\|_{k-1-\beta-\eta} \leq C[k, \beta, \eta, C_d, \|Df\|_0, W(f), \|Sf\|_{k-3}]$$

où  $Sf$  désigne la dérivée schwartzienne de  $f$  et  $W(f)$  désigne la variation totale de  $\log Df$ .

Notre preuve s'appuie sur [Yoc84]. Dans une première étape, nous estimons la norme  $C^1$  de la conjugaison. Dans une seconde étape, nous estimons les normes de degré supérieur, en nous appuyant sur les estimées de la norme  $C^1$ .

La preuve de l'estimation de la norme  $C^1$  de la conjugaison s'appuie sur une estimation de  $M_n/m_n$ , où

$M_n = \max_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$  et  $m_n = \min_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$ . Pour cette estimation, nous procédons en deux étapes : d'abord, nous estimons  $M_{n+1}/M_n$  en fonction de  $M_n$ ,  $\alpha_{n+1}/\alpha_n$  et d'une constante  $C_1$  qui dépend de  $k$ ,  $\|Sf\|_{k-3}$  et de  $W(f)$ , et que nous calculons explicitement.

Dans la deuxième étape, l'idée principale est d'établir une alternative entre deux situations possibles pour les suites  $M_n$  et  $\alpha_n$  : la situation "favorable" ( $R_n$ ) et la situation "défavorable" ( $R'_n$ ). La situation "défavorable" se produit seulement un nombre fini de fois, en raison de la condition diophantienne sur  $\alpha$ . Ces situations sont décrites par la proposition suivante, qui est une version modifiée de [Yoc84, p. 346] :

**Proposition 1.2.** *Soit  $1 \geq \eta_n \geq 0$  une suite telle que  $\alpha_n = \alpha_{n+1}^{1-\eta_n}$ . Il existe un paramètre  $\theta > 0$ , et il existe un entier  $n_1$  tel que pour tout  $n \geq n_1$ , on peut définir des suites  $a_n$  et  $\rho_n < 1$  telles que  $M_n = a_n \alpha_n^{\rho_n}$ . La suite  $a_n$  est définie par :*

si

$$(R_n) \quad C_1 M_n^{(k+1)/2-\theta} \leq M_n \frac{\alpha_{n+1}}{\alpha_n} \quad \text{alors} \quad a_{n+1} = a_n \frac{1 + M_n^\theta}{1 - C_1 M_n^{1/2}}$$

et si

$$(R'_n) \quad C_1 M_n^{(k+1)/2-\theta} > M_n \frac{\alpha_{n+1}}{\alpha_n} \text{ alors } a_{n+1} = a_n$$

De plus, si  $(R_n)$  se produit, alors  $\rho_{n+1} \geq \rho_n + \eta_n(1 - \rho_n)$  ;

et si  $(R'_n)$  se produit, alors  $\rho_{n+1} \geq ((k+1)/2 - \theta)(1 - \eta_n)\rho_n$ . En particulier, la suite  $(\rho_n)_{n \geq n_1}$  est croissante.

Dans la situation "favorable"  $(R_n)$ , nous pouvons estimer  $M_{n+1}/\alpha_{n+1}$  en fonction de  $M_n/\alpha_n$  et de même, nous pouvons estimer  $\alpha_{n+1}/m_{n+1}$  en fonction de  $\alpha_n/m_n$ . Lorsque  $\rho_n$  franchit un certain seuil  $\rho$ , la situation "favorable"  $(R_n)$  se produit toujours. Par conséquent, au-delà d'un certain rang  $n_2$ , la situation "favorable"  $(R_n)$  se produit toujours, et nous pouvons estimer  $M_n/m_n$  en fonction de  $M_{n_2}/m_{n_2}$ . Nous relierons  $M_{n_2}/m_{n_2}$  à  $\alpha_{n_2}$ , et nous calculons un minorant de  $\alpha_{n_2}$ . La preuve de Yoccoz a besoin d'être modifiée parce que dans sa version originale, elle ne permet pas de calculer un minorant de  $\alpha_{n_2}$ .

En particulier, notre méthode montre que  $\|Dh\|_0$  est majoré par une fonction qui ne dépend que de  $k, \beta, C_d, \|Df\|_0, W(f), \|Sf\|_{k-3}$ . Au contraire, a priori, la preuve de Yoccoz (y compris si on y ajoute le calcul explicite des constantes) n'exclut pas le fait que l'ensemble :

$$E_A = \left\{ \|Dh\|_0 / \exists f \in \text{Diff}_+^k(\mathbb{T}^1), f = h^{-1}R_\alpha h, \alpha \in CD(\beta, C_d), \max(k, \beta, C_d, \|Df\|_0, W(f), \|Sf\|_{k-3}) \leq A \right\}$$

ne soit pas borné, pour tout  $A > 0$  fixé.

Nous pouvons préciser l'estimée dans un cas particulier. Lorsque  $k \geq 3\beta + 9/2$ , on obtient :

$$\|Dh\|_0 \leq e^{(3)} \wedge \left( C_2[\beta]C_3[C_d]C_4[\|Df\|_0, W(f), \|Sf\|_0]C_5[\|Sf\|_{[3\beta+3/2]}] \right) \quad (1)$$

Dans cette estimée, trois itérations de la fonction exponentielle apparaissent : une première exponentielle vient de l'estimation  $\|Df^{n_2}\|_0 \leq C\|Df\|_0^{2/\alpha_{n_2}}$ . Une seconde exponentielle vient de l'écriture  $\alpha_{n_2} = \alpha_0^{\prod_{n=0}^{n_2-1} (\frac{1}{1-\eta_n})}$ . Nous majorons chaque  $\frac{1}{1-\eta_n}$  grâce à la condition diophantienne, et une troisième exponentielle vient de l'estimation  $\prod_{n \in E_2} \left( \frac{1}{1-\eta_n} \right) \leq C^{n_3-n_1}$ , où  $E_2$  est l'ensemble et  $n_3 - n_1$  est le nombre de cas "défavorables".

Pour les estimées des normes d'ordre supérieur, la preuve se décompose en quatre étapes. Soient des nombres réels  $0 \leq \gamma_0 < \gamma_1 < g(\gamma_0)$  ( $g$  est une fonction vérifiant certaines conditions) et un entier  $N$ . Dans les trois premières étapes, nous calculons  $\|\log Df^N\|_{\gamma_1}$  en fonction de  $\sup_{p \geq 0} \|\log Df^p\|_{\gamma_0}$ . Dans la première étape, en utilisant des inégalités de convexité et une conséquence de la formule de Faa-d-Bruno, nous établissons une estimation de  $\|\log Df^{q_s}\|_{\gamma}$  pour  $0 \leq \gamma \leq k-1$ .

Dans la deuxième étape, nous obtenons une estimation de  $\|\log Df^{nq_s}\|_{\gamma}$ ,  $0 \leq n \leq q_{s+1}/q_s$ ,  $0 \leq \gamma \leq k-1$ . Nous y modifions très légèrement la preuve d'une proposition : en suivant strictement la preuve de Yoccoz, nous aurions obtenu une estimée qui diverge lorsque  $f$  devient proche d'une rotation.

Dans une troisième étape, nous écrivons  $N = \sum_{s=0}^S b_s q_s$ , avec  $b_s$  des entiers vérifiant  $0 \leq b_s \leq q_{s+1}/q_s$ , et nous obtenons ainsi une estimation de  $\|\log Df^N\|_{\gamma_1}$  en fonction de  $\sup_{p \geq 0} \|\log Df^p\|_{\gamma_0}$ .

Dans la quatrième étape, nous itérons ce raisonnement : nous disposons d'une estimée de  $h \in D^1(\mathbb{T}^1)$  tel que  $f = h \circ R_\alpha \circ h^{-1}$ . Nous supposons  $h \in D^{1+\gamma_0}(\mathbb{T}^1)$  pour un certain  $\gamma_0$ , et nous itérons l'estimée de  $\sup_{N \geq 0} \|\log Df^N\|_{\gamma_1}$  en fonction de  $\sup_{p \geq 0} \|\log Df^p\|_{\gamma_0}$  (le nombre  $\gamma_1$  de la  $n^{\text{ième}}$  étape devient le  $\gamma_0$  de la  $(n+1)^{\text{ième}}$  étape). Finalement, en utilisant nos estimées  $C^1$ , nous obtenons une estimation de  $\|h\|_{k-1-\beta-\eta}$ .

Dans cette partie sur les estimées d'ordre supérieur, il y a quelques nouveautés par rapport au travail initial de Yoccoz : le calcul explicite des constantes, et d'infimes modifications dans la preuve.

## 1.2 Application à la quasi-réductibilité et aux difféomorphismes qui commutent

Outre leur intérêt intrinsèque, ces estimées possèdent des applications à l'étude des difféomorphismes de nombre de rotation Liouville. Ces applications font l'objet du chapitre II.

En effet, ces estimées s'appliquent au problème de *quasi-réductibilité* : existe-t-il une suite de difféomorphismes  $(h_n)_{n \geq 0}$  telle que  $h_n^{-1} f h_n \rightarrow R_\alpha$  dans la norme  $C^k$  ? Cette question remonte à Herman [Her79, pp.93-99], qui a montré que pour tout  $f \in D^2(\mathbb{T}^1)$  de nombre de rotation irrationnel  $\alpha$ , il est possible d'accumuler  $R_\alpha$  dans la norme  $C^{1+vb}$  par une suite  $h_n^{-1} f h_n$ ,  $h_n$  étant un difféomorphisme  $C^2$ . Cette question est distincte de celle résolue par Yoccoz [Yoc95] : ce dernier a montré l'existence d'une suite de difféomorphismes  $(h_n)_{n \geq 0}$  telle que  $h_n R_\alpha h_n^{-1} \rightarrow f$ . Sur ce problème de quasi-réductibilité, nous obtenons le résultat suivant, qui vient compléter les résultats de Herman [Her79] et Yoccoz [Yoc84] dans le cas diophantien :

**Theorem 1.3.** *Il existe un ensemble dense au sens de Baire  $A \subset \mathbb{R}$  tel que si  $f \in D^\infty(\mathbb{T}^1)$  a pour nombre de rotation  $\alpha \in A$ , alors  $f$  est  $C^\infty$ -quasi-réductible : il existe une suite  $h_n \in D^\infty(\mathbb{T}^1)$  telle que  $h_n^{-1} f h_n \rightarrow R_\alpha$  pour la norme  $C^\infty$ .*

La quasi-réductibilité a des applications à l'étude de l'espace des actions différentiables de  $\mathbb{Z}^2$  dans  $\mathbb{T}^1$ . Cette espace est assez complexe, comme l'illustre l'étude de Yoccoz [Yoc95] des centralisateurs différentiables de difféomorphismes du cercle, étude où il fournit notamment un exemple de difféomorphisme lisse dont le centralisateur dans le groupe des difféomorphismes du cercle de classe  $C^2$  qui préservent l'orientation, est réduit à l'ensemble de ses itérés. La structure de cet espace d'actions différentiables de  $\mathbb{Z}^2$  dans  $\mathbb{T}^1$  demeure mal connue. Par exemple, on ne sait pas encore s'il est localement connexe par arcs : c'est une question ancienne posée par Rosenberg. Une question de Mather, qui lui est proche, est de savoir si, étant donnés deux difféomorphismes  $f, g$  qui commutent, ces derniers peuvent être accumulés par des suites de difféomorphismes réductibles (i.e. différentiablement conjugués à des rotations) et qui commutent. Sur cette question, nous obtenons le résultat suivant :

**Theorem 1.4.** *Il existe un ensemble dense au sens de Baire  $A \subset \mathbb{R}$  tel que si  $f \in D^\infty(\mathbb{T}^1)$  a pour nombre de rotation  $\alpha \in A$ , et si  $g \in D^\infty(\mathbb{T}^1)$  tel que  $fg = gf$ , alors il existe deux suites de difféomorphismes  $C^\infty$   $f_n$  et  $g_n$  qui sont  $C^\infty$ -conjugués à des rotations, tels que  $f_n g_n = g_n f_n$ , et tels que  $f_n$  et  $g_n$  convergent respectivement vers  $f$  et  $g$  dans la norme  $C^\infty$ .*

Donnons ici l'idée de la preuve du résultat sur la quasi-réductibilité. On note  $(q_n)_{n \geq 0}$  la suite des dénominateurs des réduites de  $\alpha$ , et  $(a_n)_{n \geq 1}$  la suite d'entiers telle que  $q_{n+1} = a_{n+1}q_n + q_{n-1}$ . On observe d'abord que pour toute suite  $\phi(n)$  tendant vers  $+\infty$ , l'ensemble des nombres  $\alpha$  tels que pour une infinité de  $n$ ,  $\sup_{k \leq n} a_k \leq \phi(n)$ , est Baire-dense.

La suite tronquée  $\alpha_n = [a_0, \dots, a_n, 1, \dots]$  est une suite de nombres de type constant, qui converge vers  $\alpha$  à une vitesse contrôlée :  $|\alpha - \alpha_n| \leq 4/2^n$ .

Suivant une idée de Herman [Her77], on perturbe  $f$  en un difféomorphisme  $R_{\lambda_n}f$  de nombre de rotation  $\alpha_n$ , qui est linéarisable par une conjugaison  $h_n$ . En écrivant

$$h_n^{-1}fh_n - R_\alpha = h_n^{-1}fh_n - h_n^{-1}R_{\lambda_n}fh_n + R_{\alpha_n} - R_\alpha$$

et en utilisant la formule de Faa-di-Bruno, on obtient un contrôle de la norme de  $h_n^{-1}fh_n - R_\alpha$  en fonction de celle de  $h_n$ , et en fonction de  $|\alpha - \alpha_n|$ . De plus, on dispose d'une estimée de la norme de  $h_n$  en fonction de  $\sup_{k \leq n} a_k$ .

Ainsi, si l'on choisit la vitesse de croissance de la suite  $\sup_{k \leq n} a_k$  suffisamment petite par rapport à la vitesse de convergence de  $\alpha_n$  vers  $\alpha$ , alors  $h_n^{-1}fh_n$  converge vers  $R_\alpha$ , et  $f$  est quasi-réductible.

Pour la preuve du résultat sur les difféomorphismes qui commutent, notons  $f'_n = h_n^{-1}fh_n$  et  $g'_n = h_n^{-1}gh_n$ , où  $h_n$  est la conjugaison construite dans la preuve du résultat sur la quasi-réductibilité.  $f'_n = h_n R_\alpha h_n^{-1}$  converge vers  $f$  et commute avec  $g_n = h_n R_{g'_n(0)} h_n^{-1}$ . Pour obtenir la convergence de  $g_n$  vers  $g$ , il suffit de montrer que la norme de  $g'_n - R_{g'_n(0)}$  tend vers 0 suffisamment vite.

Pour cela, l'idée de base est la suivante : on approche les points  $x$  du cercle par  $p(x)\alpha \pmod{1}$ , où  $p(x) \leq q_r$  est un entier, et où l'entier  $r$  sera fixé ultérieurement. On dispose d'un contrôle de  $|x - p(x)\alpha|$  en fonction de  $q_r$ . Ensuite, en utilisant l'hypothèse de commutation  $g'_n f_n'^p = f_n'^p g'_n$ , on peut écrire (on peut confondre  $x \in \mathbb{T}^1$  avec son relevé dans  $\mathbb{R}$ ) :

$$g'_n(x) - R_{g'_n(0)}(x) = g'_n(x) - g'_n(p\alpha) + g'_n(p\alpha) - g'_n f_n'^p(0) + f_n'^p g'_n(0) - R_{p\alpha}(g'_n(0)) + R_{g'_n(0)}(p\alpha) - R_{g'_n(0)}(x)$$

On utilise la proximité de  $f_n'^p$  à  $R_{p\alpha}$ , proximité qui dépend de  $q_r$  et de la norme de  $f'_n - R_\alpha$ . Cette dernière a été estimée dans la preuve du résultat de quasi-réductibilité. On utilise aussi des analogues  $C^k$ ,  $k \geq 2$ , de l'inégalité des accroissements finis, qui sont obtenus avec la formule de Faa-di-Bruno.

L'entier  $q_r$  doit être choisi suffisamment grand par rapport à la conjugaison  $h_n$ , afin que  $|x - p\alpha|$  soit suffisamment petit. Mais cet entier  $q_r$  ne doit pas être trop grand, afin de maintenir  $\|f_n'^p - R_{p\alpha}\|$  assez petit. Cet entier  $q_r$  est contrôlé à l'aide de  $\sup_{k \leq r} a_k$ , qui contrôle lui-même la norme de  $h_r$ . Ainsi, on choisit bien l'entier  $r$  en fonction de  $n$  afin d'obtenir la convergence de  $g_n$  vers  $g$ .

## 2 Conjugaisons successives et réalisations lisses

La théorie ergodique s'intéresse aux propriétés de transformations ou de flots qui préservent la mesure d'un espace mesuré standard (typiquement, un espace de Lebesgue). La



classification de ces actions ne serait pas envisageable [Kat77a] [Kat03, p.1], car les comportements les plus variés sont possibles. Ainsi, la théorie ergodique classique contient peu de résultats généraux, mais beaucoup d'exemples et de contre-exemples. C'est pourquoi les méthodes de construction d'exemples y occupent une place privilégiée. Parmi ces méthodes, se trouve les méthodes d'approximation périodique et de construction par conjugaison successives (méthode dite d'Anosov-Katok, [AK70], cf. aussi [KT97, GK00, Kat03, FK04, KT06, KL09]), qui jouent un rôle central dans les constructions différentiables. L'idée est de construire, par conjugaisons successives, une suite de difféomorphismes périodiques qui approche le difféomorphisme ayant la propriété souhaitée. Nous appliquons ces méthodes à des problèmes de réalisation lisse.

## 2.1 Le problème de réalisation lisse : translations du tore et couples non-standards d'angles

Un *isomorphisme de tribus*  $K : (\mathcal{B}, \mu) \rightarrow (\mathcal{B}', \mu')$  est une bijection préservant la mesure et d'inverse mesurable. Soient  $T : (\mathcal{B}, \mu) \rightarrow (\mathcal{B}, \mu)$  et  $T' : (\mathcal{B}', \mu') \rightarrow (\mathcal{B}', \mu')$  deux applications préservant la mesure. Les systèmes mesurés  $(X, \mathcal{B}, \mu, T)$  et  $(X', \mathcal{B}', \mu', T')$  sont *métriquement isomorphes* s'il existe un isomorphisme de tribus  $K : (\mathcal{B}, \mu) \rightarrow (\mathcal{B}', \mu')$  tel que  $KT^{-1} = T'^{-1}K$  (où  $T^{-1}$  et  $T'^{-1}$  sont vues comme des applications de tribus). de plus, si  $T$  et  $T'$  sont bijectives, alors cette dernière condition est équivalente à  $KT = T'K$ . Si de plus,  $X', \mu', T'$  sont lisses, alors  $(X, \mathcal{B}, \mu, T)$  admet une *réalisation lisse*. Si  $X$  et  $X'$  ne sont pas difféomorphes, alors cette réalisation lisse est *non-standard*.

Anosov et Katok [AK70] ont construit des exemples de réalisations lisses non-standards de translations ergodiques (par rapport à la tribu borélienne et la mesure de Lebesgue) du cercle et du tore de dimension  $h \geq 2$ . Cependant, ils ne disposent pas d'un contrôle sur les ensembles de translations qui peuvent être réalisés. Par ailleurs, Fayad, Saprikyina et Windsor [FSW07] ont obtenu une condition suffisante pour qu'une rotation du cercle admette une réalisation lisse non-standard. Ils ont montré qu'il suffisait que l'angle de la rotation soit un nombre de Liouville.

Dans le chapitre III, nous avons déterminé un ensemble de vecteurs du tore  $\mathbb{T}^h$ ,  $h \geq 2$ , qui admettent une réalisation lisse non-standard. Nous obtenons le résultat suivant :

**Theorem 2.1.** *Soit  $\beta \in \mathbb{T}$  un nombre de Liouville, et soit  $h \geq 2$  un entier. Soit  $M$  une variété compacte connexe lisse de dimension  $d \geq 2$ , admettant une action du cercle  $S_t$  lisse, effective, et préservant une mesure positive et lisse  $\mu$ . Alors il existe  $h - 1$  ensembles denses  $E_1(\beta, d), \dots, E_{h-1}(\beta, d) \subset \mathbb{T}$  de nombres de Liouville, tels que pour tout  $\beta_1 \in E_1(\beta, d), \dots, \beta_{h-1} \in E_{h-1}(\beta, d)$ , il existe un difféomorphisme lisse  $T \in \text{Diff}^\infty(M, \mu)$  préservant  $\mu$  et métriquement isomorphe à la translation ergodique de vecteur  $(\beta_1, \dots, \beta_{h-1}, \beta)$ .*

Le second problème de réalisation lisse non-standard qui nous intéresse est introduit de la façon suivante. Soit  $M$  une variété lisse compacte connexe de dimension  $d$ , sur laquelle existe une action  $S_t$  lisse et effective du cercle, préservant une mesure lisse positive  $\mu$ . Soit  $\mathcal{A}_\alpha$  la classe de conjugaison lisse de la rotation  $S_\alpha$ , et  $\bar{\mathcal{A}}_\alpha$  son adhérence dans la topologie lisse. Si  $M = \mathbb{T}^1$  et si  $\alpha$  est diophantien, alors  $\bar{\mathcal{A}}_\alpha = \mathcal{A}_\alpha$  par le théorème d'Herman-Yoccoz [Yoc84] (en effet, par continuité, le nombre de rotation d'un difféomorphisme  $T \in \bar{\mathcal{A}}_\alpha$  est  $\alpha$ ). D'autre

part, quand  $\alpha$  est Liouville,  $\tilde{\mathcal{A}}_\alpha \neq \mathcal{A}_\alpha$ . Supposons maintenant que  $M$  est de dimension  $d \geq 2$ . Si  $\tilde{\mathcal{A}}_\alpha$  contient une réalisation lisse non-standard de la rotation du cercle  $R_\beta$ , alors  $(\alpha, \beta)$  est appelé un *couple non-standard d'angles*.

Anosov and Katok [AK70] ont montré l'existence d'un angle  $\alpha$  tel que  $(\alpha, \alpha)$  est un couple non-standard. Fayad, Saprikyna et Windsor [FSW07] ont montré que pour tout  $\alpha$  Liouville,  $(\alpha, \alpha)$  est un couple non-standard d'angles. La question qui nous intéresse est la suivante : existe-t-il un couple non-standard d'angles  $(\alpha, \beta)$  avec  $\alpha \neq \pm\beta$  ?

En effet, il est utile de rappeler que deux rotations ergodiques du cercle  $R_\alpha$  et  $R_\beta$  sont métriquement isomorphes si et seulement si  $\beta = \pm\alpha$ . Si  $\beta = \alpha$ , l'isomorphisme est donné par l'identité, et si  $\beta = -\alpha$ , l'isomorphisme est donné par une symétrie d'axe quelconque passant par le centre du cercle. Par conséquent, en appliquant le résultat de Fayad, Saprikyna et Windsor [FSW07], il devient trivial de trouver un couple non-standard d'angles  $(\alpha, -\alpha)$ . Si, au lieu de considérer des automorphismes métriques du cercle, on considère des isomorphismes métriques entre  $M$  de dimension  $d \geq 2$  et le cercle, la situation devient plus riche : en effet, nous montrons l'existence de couples non-standards  $(\alpha, \beta)$ ,  $\alpha \neq \pm\beta$ , avec  $\alpha$  et  $\beta$  rationnellement dépendants ou rationnellement indépendants. Nous montrons l'énoncé suivant :

**Theorem 2.2.** *Soit  $M$  une variété lisse compacte connexe de dimension  $d \geq 2$ , sur laquelle existe une action lisse effective du cercle  $(S_t)_{t \in \mathbb{T}^1}$ , qui préserve une mesure positive lisse  $\mu$ . Pour tous  $u, v \in \mathbb{T}^1$ , pour tout  $\epsilon > 0$ , il existe  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  dans un  $\epsilon$ -voisinage de  $(u, v)$ , il existe  $T \in \text{Diff}^\infty(M, \mu)$ , tel que  $T \in \tilde{\mathcal{A}}_\alpha$  et tel que la rotation  $R_\beta$  d'angle  $\beta$  sur  $\mathbb{T}^1$  est métriquement isomorphe à  $T$ . De plus,  $\alpha$  et  $\beta$  peuvent être choisis rationnellement dépendants ou rationnellement indépendants.*

Lorsque  $M = \mathbb{A} = [0, 1] \times \mathbb{T}^1$  est l'anneau fermé, notre résultat contribue à l'étude des pseudo-rotations. Introduisons d'abord la notion de pseudo-rotation. Soit  $T$  un homéomorphisme de  $\mathbb{A}$  isotope à l'identité. L'ensemble de rotation de  $T$  mesure les vitesses de rotation asymptotiques des orbites de  $T$  autour de l'anneau. C'est une généralisation de la notion de nombre de rotation d'un homéomorphisme du cercle, introduite par Poincaré.  $T$  est une *pseudo-rotation irrationnelle* si son ensemble de rotation est réduit à un seul nombre irrationnel  $\alpha$ , appelé l'angle de  $T$ . Une question est posée par Béguin, Crovisier, Le Roux et Patou [BCLRP04] : dans quelle mesure la dynamique d'une pseudo-rotation irrationnelle  $T$  d'angle  $\alpha$  ressemble-t-elle à la dynamique de la rotation rigide  $S_\alpha$  d'angle  $\alpha$  ?

D'un point de vue topologique, une ressemblance entre  $S_\alpha$  et  $T$  a été montrée par Béguin et al. [BCLRP04] : la rotation  $S_\alpha$  est dans l'adhérence de la classe de conjugaison de  $T$ . Leur résultat est analogue à un théorème de Kwapisz [Kwa03] sur le tore  $\mathbb{T}^2$  (dans ce cas, l'angle de la pseudo-rotation est un élément de  $\mathbb{T}^2$ ). Jäger [Jäg09] et Wang [Wan11] ont aussi exploré cette question. Cependant, des différences entre  $S_\alpha$  et  $T$  sont également possibles. D'un point de vue métrique, Anosov et Katok [AK70] ont en effet construit une pseudo-rotation irrationnelle lisse de  $\mathbb{A}$  qui est métriquement isomorphe à une translation ergodique de  $\mathbb{T}^2$ . Béguin, Crovisier et Le Roux [BCLR07] ont construit sur  $\mathbb{T}^2$  une pseudo-rotation irrationnelle qui est minimale, uniquement ergodique, mais d'entropie positive. Pour notre part, notre résultat implique le corollaire suivant :

**Corollary 2.3.** *Soit  $M = [0, 1] \times \mathbb{T}^1$ ,  $\mu$  la mesure de Lebesgue. Pour  $t \in \mathbb{T}^1$ , soit  $S_t : M \rightarrow M$  défini par  $S_t(x, s) = (x, s + t)$ . Pour tous  $u, v \in \mathbb{T}^1$ , pour tout  $\epsilon > 0$ , il existe  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  dans un  $\epsilon$ -voisinage de  $(u, v)$ ,  $T \in \text{Diff}^\infty(M, \mu)$  une pseudo-rotation irrationnelle d'angle  $\alpha$ ,*

tels que la rotation  $R_\beta$  de  $\mathbb{T}^1$  d'angle  $\beta$  est métriquement isomorphe à  $T$ . De plus,  $\alpha$  et  $\beta$  peuvent être choisis rationnellement dépendants ou rationnellement indépendants.

Dans ce théorème,  $\alpha$  est Liouville. Notons qu'il est impossible d'obtenir un résultat analogue avec  $\alpha$  diophantien. En effet, un résultat de Herman (avec une preuve publiée par Fayad et Krikorian [FK09b]) implique que si une quasi-rotation lisse  $T$  de l'anneau fermé admet un angle diophantien, alors  $T$  ne peut pas être ergodique. A fortiori,  $T$  ne peut pas être métriquement isomorphe à une rotation ergodique. Cependant, la situation où  $\alpha$  est Liouville et  $\beta$  diophantien, qui n'est pas envisagée dans ce travail, n'est pas encore exclue. Or, l'existence d'une telle situation permettrait de répondre positivement à la question ouverte sur l'existence d'une réalisation lisse non-standard d'une rotation du cercle diophantienne [FK04].

## 2.2 Schémas des preuves : vecteurs limites et méthode de construction par conjugaisons successives

Les résultats d'Anosov-Katok, Fayad et al., ainsi que les nôtres, s'obtiennent par la méthode de construction par conjugaison successive. Ils s'appuient sur le lemme suivant [AK70] :

**Lemma 2.4** ([AK70]). *Soient  $M_1$  et  $M_2$  des espaces de Lebesgue, et soient  $\xi_n^{(i)}$  ( $i = 1, 2$ ) des suites de partitions mesurables finies, monotones et génératrices de  $M_i$ . Soient  $T_n^{(i)}$  des automorphismes de  $M_i$  tels que  $T_n^{(i)} \xi_n^{(i)} = \xi_n^{(i)}$  et  $T_n^{(i)} \rightarrow T^{(i)}$  dans la topologie faible. Supposons aussi qu'il existe des isomorphismes métriques  $L_n : M_1/\xi_n^{(1)} \rightarrow M_2/\xi_n^{(2)}$  tels que*

$$L_n T_n^{(1)} / \xi_n^{(1)} = T_n^{(2)} / \xi_n^{(2)} L_n$$

et

$$L_{n+1} \xi_n^{(1)} = \xi_n^{(2)}$$

alors  $(M_1, T_1)$  et  $(M_2, T_2)$  sont métriquement isomorphes.

Autrement dit, s'il existe des suites de partitions génératrices  $\xi_n^{(i)}$  ( $i = 1, 2$ ) et des suites d'automorphismes  $T_n^{(i)}$  convergeant faiblement vers  $T^{(i)}$ , et si, pour tout entier  $n$ , le diagramme suivant commute :

$$\begin{array}{ccc} T_n^{(1)} \left( \xi_n^{(1)} \right) & \xrightarrow{L_n} & \xi_n^{(2)} \left( T_n^{(2)} \right) \\ \downarrow & & \downarrow \\ \xi_{n+1}^{(1)} & \xrightarrow{L_{n+1}} & \xi_{n+1}^{(2)} \end{array}$$

alors  $(M_1, T_1)$  et  $(M_2, T_2)$  sont métriquement isomorphes.

Pour établir les résultats d'Anosov, Katok ainsi que de Fayad, Windsor et Saprikyina sur la réalisation lisse non-standard de rotations du cercle, ce lemme est appliqué de la façon suivante : on prend  $M_1 = \mathbb{T}^1$ ,  $M_2 = \mathbb{A}$ .  $\xi_n^{(1)} = \zeta_n = \{[i/q_n, (i+1)/q_n[, i = 0, \dots, q_n - 1\}$ ,  $\xi_n^{(2)} = \xi_n =$

$B_n^{-1}\eta_n$  où  $\eta_n = \{[0, 1] \times [i/q_n, (i+1)/q_n[, i = 0, \dots, q_n - 1\}$ , et où  $B_n$  est un difféomorphisme de  $\mathbb{A}$  construit tel que  $\xi_n = B_n^{-1}\eta_n$  soit génératrice. On prend aussi  $T_n^{(1)} = R_{\frac{p_n}{q_n}}$ ,  $T_n^{(2)} = T_n = B_n^{-1}S_{\frac{p_n}{q_n}}B_n$ , et  $L_n = B_n^{-1}K_n$ , où  $K_n : \mathcal{B}(\zeta_n) \rightarrow \mathcal{B}(\eta_n)$  est tel que  $K_n([i/q_n, (i+1)/q_n]) = [0, 1] \times [i/q_n, (i+1)/q_n[$  (figure 1).  $q_n$  est une suite d'entiers telle que  $q_n$  divise  $q_{n+1}$ , et qui croit suffisamment vite pour que  $T_n$  converge dans la topologie  $C^\infty$  vers un difféomorphisme lisse  $T$ .

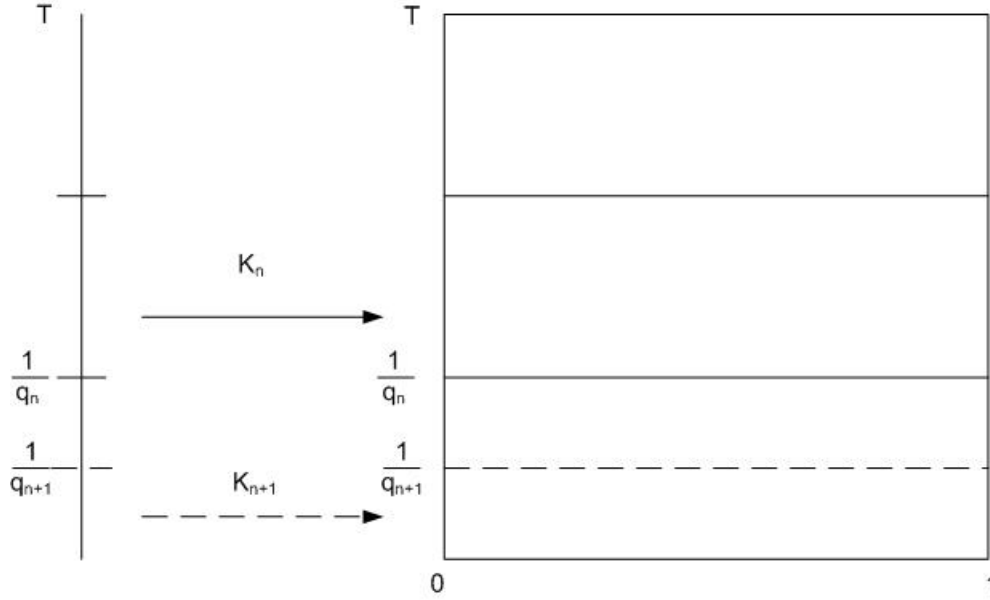


FIGURE 1 – Les isomorphismes métriques  $K_n$  (traits pleins) et  $K_{n+1}$  (pointillés) entre  $\mathbb{T}^1$  et  $\mathbb{A}$ . Ici,  $q_n = 3$  et  $q_{n+1} = 6$ .

La preuve du théorème 2.1 se décompose en trois parties. Dans la première partie, nous effectuons des modifications successives à partir d'une partition simple  $\zeta_n$  en parallélépipèdes, qui est stable par une translation périodique  $T_{\frac{p_n}{q_n}\gamma^{(n)}}$ , où  $(\gamma^{(n)} = (\gamma_1^{(n)}, \dots, \gamma_h^{(n)}))_{n \geq 0}$  est une suite de vecteurs à coordonnées entières vérifiant des hypothèses convenables. Ces modifications conduisent à une partition  $\zeta_n^\infty$ , qui est monotone et génératrice.

Dans la deuxième partie, nous déterminons des conditions suffisantes sur  $B_n \in \text{Diff}^\infty(M, \mu)$ , telles que si  $T_n = B_n^{-1}S_{\frac{p_n}{q_n}}B_n$  converge faiblement vers un automorphisme  $T$ , alors il existe un isomorphisme métrique entre  $(\mathbb{T}^h, R_\beta, \text{Leb})$  et  $(M, T, \mu)$ . Pour cela, nous appliquons le lemme 2.4 : nous construisons une suite monotone et génératrice de partitions  $\xi_n^\infty$  de  $M$ , et une suite d'isomorphismes  $\bar{K}_n^\infty : \mathcal{B}(\zeta_n^\infty) \rightarrow \mathcal{B}(\xi_n^\infty)$ , telles que  $\bar{K}_n^\infty T_{\frac{p_n}{q_n}\gamma^{(n)}} = T_n \bar{K}_n^\infty$  et  $\bar{K}_{n+1}^\infty|_{\mathcal{B}(\zeta_n^\infty)} = \bar{K}_n^\infty \cdot \bar{K}_n^\infty$  est construit par modifications successives d'un isomorphisme simple  $K_n$  entre  $\mathcal{B}(\zeta_n)$ , la tribu complétée engendrée par la partition simple  $\zeta_n$  sur  $\mathbb{T}^h$ , et  $\mathcal{B}(\eta_n)$ , la tribu complétée engendrée par une partition simple  $\eta_n$  sur  $M$ .

De plus, les éléments de  $\xi_n^\infty$  ne sont pas les plus élémentaires, car ils doivent être choisis de manière à assurer la monotonie de la suite  $\bar{K}_n^\infty$ . Cette condition de monotonie introduit des contraintes combinatoires sur les éléments de la partition  $\xi_n^\infty$ . Ce n'est pas le cas dans la construction de Fayad, Saprikyina et Windsor [FSW07], où il suffit de considérer une suite d'isomorphismes simples  $K_n$  entre des tribus complétées engendrées par des partitions simples du cercle et de  $M$ .

Dans la troisième partie, nous construisons des difféomorphismes  $T_n = B_n^{-1}S_{\frac{p_n}{q_n}}B_n$  sur  $M$ , tels que chaque  $T_n$  préserve la partition  $\xi_n^\infty$ . Nous construisons  $B_n$  successivement, i.e. nous

écrivons  $B_{n+1} = A_{n+1}B_n$ . A chaque étape,  $T_n$  est métriquement isomorphe à la translation périodique du tore  $T^{\frac{p_n}{q_n}\gamma^{(n)}}$ . Par passage à la limite, nous obtenons l'isomorphisme métrique  $\bar{K}^\infty$  et le difféomorphisme  $T$  recherchés.

Afin d'obtenir la convergence de la suite  $T_n$  dans la topologie  $C^\infty$ , nous montrons que cette suite est de Cauchy pour la norme  $C^k$ , norme dont la distance associée est notée  $d$ . En ajoutant l'hypothèse  $S_{\frac{p_n}{q_n}}A_{n+1} = A_{n+1}S_{\frac{p_n}{q_n}}$ , nous obtenons :

$$d(T_{n+1}, T_n) = d(B_{n+1}^{-1}S_{\frac{p_{n+1}}{q_{n+1}}}B_{n+1}, B_n^{-1}S_{\frac{p_n}{q_n}}B_n)$$

$$d(T_{n+1}, T_n) = d(B_{n+1}^{-1}S_{\frac{p_{n+1}}{q_{n+1}}}B_{n+1}, B_{n+1}^{-1}S_{\frac{p_n}{q_n}}B_{n+1}) \leq \|B_{n+1}\|_{k+1} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|$$

Si  $p_{n+1}/q_{n+1}$  est choisi suffisamment proche de  $p_n/q_n$ , alors la suite  $T_n$  est de Cauchy. C'est ainsi que procèdent Anosov et Katok [AK70].

Dans notre cas, nous effectuons une construction explicite de la suite de conjugaisons. C'est une différence importante avec la construction de Anosov-Katok. Notre construction combine les méthodes de Fayad et al. [FS05, FSW07] avec une version modifiée de la construction de Anosov-Katok [AK70]. Il est crucial de disposer d'un contrôle polynomial en  $q_n$  de la norme de  $B_{n+1}$ , afin d'obtenir la réalisation d'une translation du tore ayant un nombre de Liouville quelconque sur une coordonnée.

Une deuxième différence avec la construction de Anosov-Katok est que l'on ne suppose pas que les nombres  $p_n$  et  $q_n$  sont premiers entre eux. Cette hypothèse de primalité n'était pas nécessaire pour leur construction. Ainsi, contrairement à la construction d'Anosov-Katok, les éléments de la partition  $\zeta_n^\infty$  et  $\xi_n^\infty$  ne sont pas des domaines fondamentaux de  $T^{\frac{p_n}{q_n}\gamma^{(n)}}$  et  $T_n$ , respectivement. Les domaines fondamentaux de ces transformations sont plutôt obtenues comme réunions d'éléments de ces partitions. Cette généralisation est la même que dans [FSW07], sauf que ces auteurs ne la présentent pas de cette façon.

La preuve du théorème 2.2 suit partiellement le même schéma que les deux dernières parties de la preuve du théorème 2.1. Certaines parties des deux preuves pourraient même être formulées dans un cadre général. Cependant, nous n'avons pas fait ce choix. En effet, un cadre général a déjà été proposé par Anosov-Katok [AK70]. Le lecteur pourra bénéficier ici d'un autre point de vue, plus "concret", une construction abstraite n'étant pas toujours facile à comprendre, ni à ré-employer, ni à modifier.

Nous expliquons maintenant les principales différences entre les deux preuves. Nous construisons un isomorphisme  $\bar{K}_n^\infty : \mathcal{B}(\zeta_n) \rightarrow \mathcal{B}(\xi_n^\infty)$ , mais ici,  $\zeta_n$  est une partition simple sur le cercle. De plus,  $\bar{K}_n^\infty$  est construit par modifications successives d'un isomorphisme  $K_n$  entre  $(R_{\frac{p_n b_n}{q_n}}, \mathcal{B}(\zeta_n))$  et  $(S_{\frac{p_n}{q_n}}, \mathcal{B}(\eta_n))$ , où  $\eta_n$  est une partition simple sur  $M$ , et où  $b_n$  est un entier premier avec  $q_n$ , et vérifiant d'autres hypothèses.  $K_n$  découle de l'observation élémentaire que deux permutations cycliques de même ordre sont conjuguées.

Un autre point qui est absent de la preuve du théorème 2.1 est le suivant : afin de construire  $\bar{K}_n^\infty$ , nous utilisons  $K_{n+1|\mathcal{B}(\zeta_n)}$ . Or, a priori, chaque élément de  $K_{n+1}(\zeta_n)$  comportent  $q_{n+1}/q_n$  "tranches" de largeur  $1/q_{n+1}$ . Cependant, ce fait ne garantit pas la convergence de  $T_n$ , car il implique seulement l'estimation  $\|B_{n+1}\|_j \leq (q_{n+1})^{R(n)}$  pour une certaine suite fixée  $R(n)$ . Mais,

pour montrer la convergence de  $T_n$  dans la topologie  $C^\infty$  avec le raisonnement précédent, une meilleure estimée est nécessaire. Heureusement, le phénomène qui se produit est que les "tranches" de chaque élément de  $K_{n+1}(\zeta_n)$ , qui sont de largeur  $1/q_{n+1}$ , s'empilent les unes sur les autres. Cela donne  $b_{n+1}$  composantes connexes à chaque élément de  $K_{n+1}(\zeta_n)$ , chacune ayant une largeur de l'ordre de  $1/(q_n b_{n+1})$ . Cela permet une estimation de la forme  $\|B_{n+1}\|_j \leq (q_n b_{n+1})^{R(n)}$ . Si de plus,  $b_{n+1}$  est contrôlé indépendamment de  $q_{n+1}$ , cette estimation garantit la convergence de  $T_n$ .

Une dernière différence avec la preuve du théorème 2.1 est pour l'obtention de la génération de la suite de partitions  $\xi_n^\infty$ . En général, nous ne disposons pas d'une minoration de la croissance de la suite  $b_n$  (en revanche, dans la construction du théorème 2.1, la croissance de  $\gamma_h^{(n)}$  est minorée. Dans une certaine mesure,  $\gamma_h^{(n)}$  joue un rôle analogue à  $b_n$ ). En particulier, dans le cas  $(\alpha, \beta)$  rationnellement liés, la suite  $b_n$  est constante. De plus, en raison des contraintes arithmétiques qui lient  $b_n$  et  $q_n$ , nous devons procéder plus finement.

Mentionnons que dans le théorème 2.1, nous ne pouvons choisir un nombre de Liouville quelconque que sur une seule coordonnée. Une fois ce nombre choisi, la construction autorise peu de liberté dans le choix des autres coordonnées.

En effet, l'isomorphisme métrique entre l'application périodique  $T_n$  sur  $M$  et la translation périodique  $T_{\frac{p_n}{q_n}} \gamma^{(n)}$  sur le tore introduit des contraintes combinatoires entre les coefficients de  $\gamma^{(n)}$ , contraintes qui se propagent à la limite. Ces contraintes viennent du fait que sur  $M$ , il n'existe qu'un seul "degré de liberté" (le long de l'action du cercle sur  $M$ ), alors que sur le tore  $\mathbb{T}^h$ , il en existe a priori plusieurs (autant que la dimension  $h$ ).

De même, en raison de contraintes arithmétiques analogues, nous ne pouvons pas montrer le théorème 2.2 pour un  $\alpha$  Liouville quelconque. Ces contraintes portent ici sur les dénominateurs des convergents de  $\alpha$ .

Remarque sur les conventions de ponctuation dans les formules : conformément à un usage émergent [Hat], nous n'avons pas ponctué les formules en "display mode", car cela n'ajouterait ni à leur clarté, ni à leur esthétique.

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## Chapter I

# Estimates of the linearization of circle diffeomorphisms

## Summary

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A celebrated theorem by Herman and Yoccoz asserts that if the rotation number  $\alpha$  of a  $C^\infty$ -diffeomorphism of the circle  $f$  satisfies a Diophantine condition, then  $f$  is  $C^\infty$ -conjugated to a rotation. In this chapter, we establish explicit relationships between the  $C^k$  norms of this conjugacy and the Diophantine condition on  $\alpha$ . To obtain these estimates, we follow a suitably modified version of Yoccoz's proof.

### 3 Introduction

In his seminal work, M. Herman [Her79] shows the existence of a set  $A$  of Diophantine numbers of full Lebesgue measure such that for any rotation number  $\alpha \in A$  of a circle diffeomorphism  $f$  of class  $C^\omega$  (resp.  $C^\infty$ ), there is a  $C^\omega$ -diffeomorphism (resp.  $C^\infty$ -diffeomorphism)  $h$  such that  $hfh^{-1} = R_\alpha$ . In the  $C^\infty$  case, J. C. Yoccoz [Yoc84] extended this result to all Diophantine rotation numbers. Results in analytic class and in finite differentiability class subsequently enriched the global theory of circle diffeomorphisms [KS87, KO89a, KO89b, SK89, KH96, Yoc02, FK09a, KT09]. In the perturbative theory, KAM theorems usually provide a bound on the norm of the conjugacy that involves the norm of the perturbation and the Diophantine constants of the number  $\alpha$  (see [Her79, Mos66, DIL99] for example). We place ourselves in the global setting, we compute a bound on the norms of this conjugacy  $h$  in function of  $k$ ,  $|Df|_0$ ,  $W(f)$ ,  $|Sf|_{k-3}$ ,  $\beta$  and  $C_d$ .

To obtain these estimates, we follow a suitably modified version of Yoccoz's proof. Indeed, Yoccoz's proof needs to be modified because a priori, it does not exclude the fact that the set:

$$E_X = \left\{ |Dh|_0 / \exists f \in \text{Diff}_+^k(\mathbb{T}^1), f = h^{-1}R_\alpha h, \alpha \in DC(\beta, C_d), \max(k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}) \leq X \right\}$$

could be unbounded for any fixed  $X > 0$ .

These estimates have natural applications to the global study of circle diffeomorphisms with Liouville rotation number: in the next part, they allow to show the following results: 1) Given a diffeomorphism  $f$  of rotation number  $\alpha$ , for a Baire-dense set of  $\alpha$ , it is possible to accumulate  $R_\alpha$  with a sequence  $h_n f h_n^{-1}$ ,  $h_n$  being a diffeomorphism. 2) Given two commuting diffeomorphisms  $f$  and  $g$ , with the rotation number  $\alpha$  of  $f$  belonging to a specified Baire-dense set, it is possible to approach each of them by commuting diffeomorphisms  $f_n$  and  $g_n$  that are differentiably conjugated to rotations.

#### 3.1 Notations

We follow the notations of [Yoc84].

- The circle is noted  $\mathbb{T}^1$ . The group of orientation-preserving circle diffeomorphisms of class  $C^r$  is denoted  $\text{Diff}_+^r(\mathbb{T}^1)$ . The group of  $\mathbb{Z}$ -periodic diffeomorphisms of class  $C^r$  of the real line is noted  $C^r(\mathbb{T}^1)$ . We often work in the universal cover  $D^r(\mathbb{T}^1)$ , which is the group of diffeomorphisms  $f$  of class  $C^r$  of the real line such that  $f - Id \in C^r(\mathbb{T}^1)$ . Note that if  $f \in D^r(\mathbb{T}^1)$  and  $r \geq 1$ , then  $Df \in C^{r-1}(\mathbb{T}^1)$ .



- The derivative of  $f \in D^1(\mathbb{T}^1)$  is noted  $Df$ . The Schwartzian derivative  $Sf$  of  $f \in D^3(\mathbb{T}^1)$  is defined by:

$$Sf = D^2 \log Df - \frac{1}{2}(D \log Df)^2$$

- The total variation of the logarithm of the first derivative of  $f$  is:

$$W(f) = \sup_{a_0 \leq \dots \leq a_n} \sum_{i=0}^n |\log Df(a_{i+1}) - \log Df(a_i)|$$

- For any continuous and  $\mathbb{Z}$ -periodic function  $\phi$ , let:

$$\|\phi\|_0 = \|\phi\|_0 = \sup_{x \in \mathbb{R}} |\phi(x)|$$

- Let  $0 < \gamma' < 1$ .  $\phi \in C^0(\mathbb{T}^1)$  is Holder of order  $\gamma'$  if:

$$|\phi|_{\gamma'} = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^{\gamma'}} < +\infty$$

Let  $\gamma \geq 1$  be a real number. In all the paper, we write  $\gamma = r + \gamma'$  with  $r \in \mathbb{N}$  and  $0 \leq \gamma' < 1$ .

- A function  $\phi \in C^r(\mathbb{T}^1)$  is said to be of class  $C^\gamma$  if  $D^r \phi \in C^{\gamma'}(\mathbb{T}^1)$ . The space of these functions is noted  $C^\gamma(\mathbb{T}^1)$  and is given the norm:

$$\|\phi\|_\gamma = \max \left( \max_{0 \leq j \leq r} \|D^j \phi\|_0, |D^r \phi|_{\gamma'} \right)$$

If  $\gamma = 0$  or  $\gamma \geq 1$ , the  $C^\gamma$ -norm of  $\phi$  is indifferently denoted  $\|\phi\|_\gamma$  or  $|\phi|_\gamma$ . Thus, when possible, we favor the simpler notation  $|\phi|_\gamma$ .

- For  $\alpha \in \mathbb{R}$  (respectively,  $\alpha \in \mathbb{T}^1$ ), we denote  $R_\alpha \in D^\infty(\mathbb{T}^1)$  (respectively,  $R_\alpha \in \text{Diff}_+^\infty(\mathbb{T}^1)$ ), the map  $x \mapsto x + \alpha$ .
- An irrational number  $\alpha \in DC(C_d, \beta)$  satisfies a Diophantine condition of order  $\beta \geq 0$  and constant  $C_d > 0$  if for any rational number  $p/q$ , we have:

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{C_d}{q^{2+\beta}}$$

Moreover, if  $\beta = 0$ , then  $\alpha$  is of constant type  $C_d$ .

- Let  $\alpha_{-2} = \alpha$ ,  $\alpha_{-1} = 1$ . For  $n \geq 0$ , we define a real number  $\alpha_n$  (the *Gauss sequence* of  $\alpha$ ) and an integer  $a_n$  by the relations  $0 < \alpha_n < \alpha_{n-1}$  and

$$\alpha_{n-2} = a_n \alpha_{n-1} + \alpha_n$$

- In the following statements,  $C_i[a, b, \dots]$  denotes a positive numerical function of real variables  $a, b, \dots$ , with an explicit formula that we compute.  
 $C[a, b, \dots]$  denotes a numerical function of  $a, b, \dots$ , with an explicit formula that we do not compute.
- We use the notations  $a \wedge b = a^b$ ,  $e^{(n)} \wedge x$  the  $n^{\text{th}}$ -iterate of  $x \mapsto \exp x$ ,  $\lfloor x \rfloor$  for the largest integer such that  $\lfloor x \rfloor \leq x$ , and  $\lceil x \rceil$  for the smallest integer such that  $\lceil x \rceil \geq x$ .

We recall Yoccoz's theorem [Yoc84]:

**Theorem 3.1.** *Let  $k \geq 3$  an integer and  $f \in D^k(\mathbb{T}^1)$ . We suppose that the rotation number  $\alpha$  of  $f$  is Diophantine of order  $\beta$ . If  $k > 2\beta + 1$ , there exists a diffeomorphism  $h \in D^1(\mathbb{T}^1)$  conjugating  $f$  to  $R_\alpha$ . Moreover, for any  $\eta > 0$ ,  $h$  is of class  $C^{k-1-\beta-\eta}$ .*

## 3.2 Statement of the results

### 3.2.1 $C^1$ estimations

**Theorem 3.2.** *Let  $f \in D^3(\mathbb{T}^1)$  of rotation number  $\alpha$ , such that  $\alpha$  is of constant type  $C_d$ . There exists a diffeomorphism  $h \in D^1(\mathbb{T}^1)$  conjugating  $f$  to  $R_\alpha$ , which satisfies the estimation:*

$$|Dh|_0 \leq e \wedge \left( \frac{C_1[W(f), |Sf|_0]}{C_d} \right)$$

The expression of  $C_1$  is given page 33.

More generally, for a Diophantine rotation number  $\alpha \in DC(C_d, \beta)$ , we have:

**Theorem 3.3.** *Let  $k \geq 3$  be an integer and  $f \in D^k(\mathbb{T}^1)$ . Let  $\alpha \in DC(C_d, \beta)$  be the rotation number of  $f$ . If  $k > 2\beta + 1$ , there exists a diffeomorphism  $h \in D^1(\mathbb{T}^1)$  conjugating  $f$  to  $R_\alpha$ , which satisfies the estimation:*

$$|Dh|_0 \leq C_2[k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}] \quad (2)$$

The expression of  $C_2$  is given page 47.

Moreover, if  $k \geq 3\beta + 9/2$ , we have:

$$|Dh|_0 \leq e^{(3)} \wedge \left( C_3[\beta] C_4[C_d] C_5[|Df|_0, W(f), |Sf|_0] C_6[|Sf|_{\lceil 3\beta+3/2 \rceil}] \right) \quad (3)$$

The expressions of  $C_3, C_4, C_5, C_6$  are given page 53.

Let  $\delta = k - 2\beta - 1$ . When  $\delta \rightarrow 0$ , we have:

$$|Dh|_0 \leq e^{(3)} \wedge \left( \frac{1}{\delta^2} C_7[k, C_d, |Df|_0, W(f), |Sf|_0] + \frac{C[\delta]}{\delta^2} C[k, C_d, |Df|_0, W(f), |Sf|_0, |Sf|_{k-3}] \right) \quad (4)$$

where  $C[\delta] \rightarrow_{\delta \rightarrow 0} 0$ . The expression of  $C_7$  is given page 55.

*Remark 3.4.* Katznelson and Ornstein [KO89b] showed that the assumption  $k > 2\beta + 1$  in Yoccoz's theorem is not optimal (instead it is  $k > \beta + 2$ ). Therefore, the divergence of the bound given by estimation (4) is because we compute the bound of the conjugacy by following the Herman-Yoccoz method.

*Remark 3.5.* Let  $\alpha_n$  be the Gauss sequence associated with  $\alpha$ . Yoccoz's proof already gives the following result: if  $k \geq 3\beta + 9/2$  and if, for any  $n \geq 0$ ,

$$\frac{\alpha_{n+1}}{\alpha_n} \geq C_8[n, k, W(f), |Sf|_{k-3}] \quad (5)$$

then:

$$\|Dh\|_0 \leq \exp\left(C_9[k, W(f), |Sf|_{k-3}]^{C_{10}|\beta|}\right) \|Df\|_0^2$$

The expressions of  $C_8, C_9, C_{10}$  are given page 56.

### 3.2.2 $C^u$ estimations

**Theorem 3.6.** *Let  $k \geq 3$  an integer,  $\eta > 0$  and  $f \in D^k(\mathbb{T}^1)$ . Let  $\alpha \in DC(C_d, \beta)$  be the rotation number of  $f$ . If  $k > 2\beta + 1$ , there exists a diffeomorphism  $h \in D^{k-1-\beta-\eta}(\mathbb{T}^1)$  conjugating  $f$  to  $R_\alpha$ , which satisfies the estimation:*

$$\|Dh\|_{k-2-\beta-\eta} \leq e^{(\lceil \log((k-2-\beta)/\eta) / \log(1+1/(2\beta+3)) \rceil)} \wedge (C_{11}[\eta, k, \beta, C_d, \|Df\|_0, W(f), |Sf|_{k-3}]) \quad (6)$$

The expression of  $C_{11}$  is given page 77.

Moreover, if  $k \geq 3\beta + 9/2$ , we have:

$$\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq e \wedge \left( C_{12}[k] e^{(2)} \wedge (2 + C_3[\beta] C_4[C_d] C_5[\|Df\|_0, W(f), |Sf|_0] C_6[|Sf|_{k-3}]) \right) \quad (7)$$

The expression of  $C_{12}$  is given page 75.

If  $\alpha$  is of constant type, for any  $k > 3$ , we have:

$$\|Dh\|_{\frac{k}{4} - \frac{1}{2}} \leq e \wedge \left( C_{13}[k] \left[ C_{14}[W(f), |Sf|_{k-3}] + \frac{C_1[W(f), |Sf|_0]}{C_d} \right]^4 \right) \quad (8)$$

The expressions of  $C_{13}$  and  $C_{14}$  are given page 76.

## 4 Preliminaries

Let  $f \in D^0(\mathbb{T}^1)$  be a homeomorphism and  $x \in \mathbb{R}$ . When  $n$  tends towards infinity,  $(f^n(x) - x)/n$  admits a limit independent of  $x$ , noted  $\rho(f)$ . We call it the *translation number* of  $f$ . Two lifts of  $f \in \text{Diff}_+^0(\mathbb{T}^1)$  only differ by a constant integer, so this is also the case for their translation numbers. We call the class of  $\rho(f) \bmod \mathbb{Z}$  the *rotation number* of  $f$ . We still denote it  $\rho(f)$ . It is invariant by conjugacy. Let  $f \in D^2(\mathbb{T}^1)$ . When  $\alpha = \rho(f)$  is irrational, Denjoy showed that  $f$  is topologically conjugated to  $R_\alpha$ . However, this conjugacy is not always differentiable (see [Arn65, Her79, KH96, Yoc02]). The regularity of this conjugacy depends on the Diophantine properties of the rotation number  $\alpha$  (see Yoccoz's theorem 3.1).

Let  $\alpha$  be an irrational number. Let the distance of  $\alpha$  to the closest integer be:

$$\|\alpha\| = \inf_{p \in \mathbb{Z}} |\alpha - p|$$

For  $n \geq 1$ ,  $a_n \geq 1$ . Let  $\alpha = a_0 + 1/(a_1 + 1/(a_2 + \dots))$  be the development of  $\alpha$  in continued fraction. We denote it  $\alpha = [a_0, a_1, a_2, \dots]$ . Let  $p_{-2} = q_{-1} = 0$ ,  $p_{-1} = q_{-2} = 1$ . For  $n \geq 0$ , let  $p_n$  and  $q_n$  be:

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2}$$

We have  $q_0 = 1$ ,  $q_n \geq 1$  for  $n \geq 1$ . The rationals  $p_n/q_n$  are called the convergents of  $\alpha$ . They satisfy the following properties:

1.  $\alpha_n = (-1)^n (q_n \alpha - p_n)$
2.  $\alpha_n = \|q_n \alpha\|$ , for  $n \geq 1$
3.  $1/(q_{n+1} + q_n) < \alpha_n < 1/q_{n+1}$  for  $n \geq 0$ .
4.  $\alpha_{n+2} < \frac{1}{2} \alpha_n$ ,  $q_{n+2} \geq 2q_n$ , for  $n \geq -1$

We recall that  $DC(C_d, \beta)$  denotes the set of Diophantine numbers of constants  $\beta$  and  $C_d$ . One of the following relations characterizes  $DC(C_d, \beta)$ :

1.  $|\alpha - p_n/q_n| > C_d/q_n^{2+\beta}$  for any  $n \geq 0$
2.  $a_{n+1} < \frac{1}{C_d} q_n^\beta$  for any  $n \geq 0$
3.  $q_{n+1} < \frac{1}{C_d} q_n^{1+\beta}$  for any  $n \geq 0$
4.  $\alpha_{n+1} > C_d \alpha_n^{1+\beta}$  for any  $n \geq 0$

In all the paper, we denote  $C'_d = 1/C_d$ .

- Let  $m_n(x) = f^{q_n}(x) - x$ ,  $n \geq 1$ ,  $x \in \mathbb{T}^1$ , let  $M_n = \sup_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$  and  $m_n = \inf_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$ .
- For any  $\phi, \psi \in C^\gamma(\mathbb{T}^1)$ , we have:

$$|\phi\psi|_\gamma \leq \|\phi\|_0 |\psi|_\gamma + |\phi|_\gamma \|\psi\|_0 \quad (9)$$

$$\|\phi\psi\|_\gamma \leq \|\phi\|_0 |\psi|_\gamma + \|\phi\|_\gamma \|\psi\|_0 \quad (10)$$

- For any real numbers  $a$  and  $b$ ,  $a \vee b$  denotes  $\max(a, b)$ .

In the rest of the paper, for any integer  $i$ ,  $C_i^f$  denotes a constant depending only on  $W(f)$  and  $|Sf|_0$  (i.e.  $C_i^f$  is a numerical function of these variables).  $C_i^{f,k}$  denotes a constant depending only on  $k$ ,  $W(f)$ ,  $|Sf|_0$  and  $|Sf|_{k-3}$ .  $C_i$  denotes a constant that might depend on  $k$ ,  $W(f)$ ,  $|Sf|_0$ ,  $|Sf|_{k-3}$  and also  $\beta$  and  $C_d$ .

## 5 $C^1$ estimations: constant type

### 5.1 A 2-parameters family of homographies. A lower bound on the $C^1$ estimate

In this subsection, we show the existence of a lower bound on the norm of the conjugacy in function of  $C_d$  in the particular case of a 2-parameters family of homographies. We also establish an upper bound on the  $C^1$  norm of the conjugacy for this family. These bounds are similar to what is given by the local KAM theory. However, these bounds are very specific to this setting. Our general bounds given in theorems 3.2 and 3.3 are much larger. This study has been suggested by J.C. Yoccoz.

**Proposition 5.1.** *Let  $f : \{z \in \mathbb{C}/|z| = 1\} \rightarrow \{z \in \mathbb{C}/|z| = 1\}$  defined by  $f(z) = h^{-1}R_\theta h(z)$ , with  $R_\theta(z) = e^{i\theta}z$  and  $h$  is a homography defined by:*

$$h(z) = \frac{z - a}{az - 1}$$

*Let  $1 > a > 1/2$ , let  $C_d$  such that  $C_d^{-1} \geq 6$  is a positive integer; and  $0 < \theta = 2\pi C_d \leq \pi/3$ , (therefore,  $\theta/(2\pi) = [0, C_d^{-1}, 1]$  is of constant type  $C_d$ ). Let  $\tilde{f} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  the circle diffeomorphism induced by  $f$  and  $\tilde{h}$  the conjugacy induced by  $h$ . We have the following estimation:*

$$\frac{3}{16\pi} C_{15}(|D\tilde{f}(0)|, |D^2\tilde{f}(0)|)/C_d \leq |D\tilde{h}|_0 \leq C_{15}(|D\tilde{f}(0)|, |D^2\tilde{f}(0)|)/C_d$$

*Proof.* For any  $\phi \in \mathbb{R}/\mathbb{Z}$ , we can write  $h(e^{i\phi}) = e^{i\tilde{h}(\phi)}$ . By differentiating this expression, we have:

$$D\tilde{h}(\phi) = e^{i\phi} \frac{Dh(e^{i\phi})}{h(e^{i\phi})}$$

and

$$Dh(z) = \frac{(a-1)(a+1)}{(az-1)^2}$$

Therefore

$$D\tilde{h}(\phi) = e^{i\phi} \frac{a^2 - 1}{(ae^{i\phi} - 1)(e^{i\phi} - a)}$$

$|D\tilde{h}(\phi)|$  reaches its maximum for  $\phi = 0$ , and  $|D\tilde{h}|_0 = \frac{a+1}{1-a}$ .

Moreover, we have:

$$\frac{D^2\tilde{f}(\phi)}{D\tilde{f}(\phi)} = i + ie^{i\phi} \frac{D^2f(e^{i\phi})}{Df(e^{i\phi})} - ie^{i\phi} \frac{Df(e^{i\phi})}{f(e^{i\phi})}$$

Since  $\frac{D^2\tilde{f}(\phi)}{D\tilde{f}(\phi)} \in \mathbb{R}$  and  $D\tilde{f}(\phi) \in \mathbb{R}$ , we have:

$$\left| \frac{D^2 f(1)}{Df(1)} \right| = \left( \left( \frac{D^2 \tilde{f}(0)}{D\tilde{f}(0)} \right)^2 + (D\tilde{f}(0) - 1)^2 \right)^{1/2} = C_{15}(|D\tilde{f}(0)|, |D^2 \tilde{f}(0)|)$$

Therefore, in order to get the proposition, it suffices to show:

$$\frac{3}{16\pi C_d} \left| \frac{D^2 f(1)}{Df(1)} \right| \leq |Dh|_0 \leq \frac{1}{C_d} \left| \frac{D^2 f(1)}{Df(1)} \right|$$

Let us write

$$f(z) = \frac{(e^{i\theta} - a^2)z - a(e^{i\theta} - 1)}{a(e^{i\theta} - 1)z - (a^2 e^{i\theta} - 1)} = \frac{bz - c}{cz + d}$$

We have

$$Df(z) = \frac{db + c^2}{(cz + d)^2}$$

and

$$D^2 f(z) = -2 \frac{Df(z)}{z + d/c}$$

Moreover,

$$Df(1) = \frac{(1+a)^2 e^{i\theta}}{(ae^{i\theta} + 1)^2}$$

and

$$D^2 f(1) = -2Df(1) \frac{a(e^{i\theta} - 1)}{(1-a)(1+ae^{i\theta})}$$

We have:

$$\left| \frac{D^2 f(1)}{Df(1)} \right| = 2 \left| \frac{a}{1+ae^{i\theta}} \right| \frac{|e^{i\theta} - 1|}{1-a}$$

Since  $|e^{i\theta} - 1| \geq \sin \theta \geq \frac{2}{\pi} \theta$  (because  $0 \leq \theta \leq \pi/2$ ), then:

$$\left| \frac{D^2 f(1)}{Df(1)} \right| \geq \frac{4}{\pi} \frac{a}{1+a} \frac{\theta}{1-a}$$

Therefore,

$$\left| \frac{D^2 f(1)}{Df(1)} \right| \geq \frac{4}{\pi} \frac{a}{(1+a)^2} |Dh|_0 \theta$$

i.e.

$$\frac{2\pi}{\theta} \left| \frac{D^2 f(1)}{Df(1)} \right| \geq |Dh|_0$$

Hence the first part of the inequality.

On the other hand, since  $\theta \leq \pi/3$ , then  $|1 + ae^{i\theta}| \geq 1 + a \cos \theta \geq 1 + a/2 \geq \frac{1}{2}(a + 1)$ .

Furthermore,  $|e^{i\theta} - 1|^2 = 2 - 2 \cos \theta = 2 - 2(\cos^2 \theta/2 - \sin^2 \theta/2) = 4 \sin^2 \theta/2 \leq \theta^2$ . Therefore,

$$\left| \frac{D^2 f(1)}{Df(1)} \right| \leq \frac{\theta}{1-a} \frac{4a}{a+1} \leq \frac{4\theta}{1-a} = \frac{4\theta |Dh|_0}{a+1}$$

i.e.

$$\frac{3}{8} \left| \frac{D^2 f(1)}{\theta Df(1)} \right| \leq |Dh|_0$$

Hence the second part of the inequality. □

## 5.2 Proof of the C<sup>1</sup> estimate in the case of constant type

The proof of theorem 3.2 is divided in three steps. The first step is based on the improved Denjoy inequality, which estimates the C<sup>0</sup>-norm of  $\log Df^{q_i}$ . In the second step, we extend this estimation to  $\log Df^N$  for any integer  $N$ . To do this, following Denjoy and Herman, we write  $N = \sum_{s=0}^S b_s q_s$ , with  $b_s$  integers satisfying  $0 \leq b_s \leq q_{s+1}/q_s$  and we apply the chain rule. In the third step, we derive a C<sup>0</sup>-estimation of the derivative  $Dh$  of the conjugacy  $h$ .

The first step is based on the Denjoy inequality:

**Proposition 5.2.** *Let  $f \in \text{Diff}_+^3(\mathbb{T}^1)$  and  $x \in \mathbb{T}^1$ . We have:*

$$|\log Df^{q_i}(x)| \leq W(f)$$

Proposition 5.2 is used to obtain an improved version of Denjoy inequality [Yoc84, p.342]:

**Lemma 5.3.** *Let  $f \in \text{Diff}_+^3(\mathbb{T}^1)$ . We have:*

$$|\log Df^{q_i}|_0 \leq C_{16}^f M_l^{1/2}$$

$$|Df^{q_i} - 1|_0 \leq C_{17}^f M_l^{1/2}$$

Moreover, we can take:

$$C_{16}^f = 2\sqrt{2}(2e^{W(f)} + 1)e^{W(f)}(|Sf|_0)^{1/2}$$

and

$$C_{17}^f = 6\sqrt{2}e^{3W(f)}|Sf|_0^{1/2}$$

In the second step, we estimate  $D \log Df^N$  independently of  $N$ . This step is based on the following lemma:

**Lemma 5.4.** *Let  $f \in \text{Diff}_+^3(\mathbb{T}^1)$  and  $M_l = \sup_{x \in \mathbb{T}^1} |f^{q_l}(x) - x|$ . We have:*

$$\sum_{l \geq 0} \sqrt{M_l} \leq \frac{1}{\sqrt{C_{18}^f - C_{18}^f}}$$

with

$$C_{18}^f = \frac{1}{\sqrt{1 + e^{-C_{19}^f}}} \quad (11)$$

and:

$$C_{19}^f = 6\sqrt{2}e^{2W(f)}(|Sf|_0^{1/2} \vee 1) \quad (12)$$

*Proof.* To obtain this lemma, we need the claim:

**Claim 5.5.** *Let  $f \in \text{Diff}_+^2(\mathbb{T}^1)$  of rotation number  $\alpha$ , and let  $p_n/q_n$  be the convergents of  $\alpha$ . Then for all  $x \in \mathbb{T}^1$ , we have:*

$$[x, f^{2q_{l+2}}(x)] \subset [x, f^{q_l}(x)]$$

*Proof.* By topological conjugation, it suffices to examine the case of a rotation of angle  $\alpha$ . It is also sufficient to take  $x = 0$ .

By absurd, if the lemma was false, then we would have the following cyclic order on  $\mathbb{T}^1$ :  $-q_{l+2}\alpha \leq (q_{l+2} - q_l)\alpha \leq 0 \leq (q_l - q_{l+2})\alpha \leq q_{l+2}\alpha$ . In particular,  $(q_{l+2} - q_l)\alpha$  would be closer to 0 than  $q_{l+2}\alpha$ , which would contradict the fact that

$$\|q_{l+2}\alpha\| = \inf\{\|q\alpha\|/0 < q \leq q_{l+2}\}.$$

□

For any interval  $I$  of the circle, if  $|I|$  denotes the length of  $I$ , lemma 5.3 implies the estimation:

$$\frac{|f^{q_{l+2}}(I)|}{|I|} \geq e^{-C_{19}^f M_{l+2}^{1/2}}$$



Let  $x \in \mathbb{T}^1$  such that  $M_{l+2} = f^{q_{l+2}}(x) - x$  and let  $I = [x, f^{q_{l+2}}(x)]$ . The former estimation implies

$$|f^{2q_{l+2}}(x) - f^{q_{l+2}}(x)| \geq e^{-C_{19}^f M_{l+2}^{1/2}} M_{l+2} \quad (13)$$

By applying claim 5.5, and since  $M_n \leq 1$ , we obtain:

$$M_{n+2} + e^{-C_{19}^f} M_{n+2} \leq M_{n+2} + e^{-C_{19}^f M_{n+2}^{1/2}} M_{n+2} \leq M_n$$

Therefore, for any  $l \geq 0$ ,

$$M_l \leq (C_{18}^f)^{l-1} \quad (14)$$

with

$$C_{18}^f = \frac{1}{\sqrt{1 + e^{-C_{19}^f}}}$$

Estimation (14) above gives:

$$\sum_{l \geq 0} \sqrt{M_l} \leq \frac{1}{\sqrt{C_{18}^f}} \frac{1}{1 - \sqrt{C_{18}^f}} \leq \frac{1}{\sqrt{C_{18}^f} - C_{18}^f}$$

Hence lemma 5.4. □

*Remark 5.6.* To get estimation (13), we applied the improved Denjoy inequality. However, applying Denjoy's inequality would give a better estimate, because we need an estimate of the first terms. The improvement of the "improved" Denjoy inequality is only asymptotic. This will be done in the corrected version of the thesis.

Now, let  $N$  be an integer. Following Denjoy, since  $\alpha$  is of constant type, we can write  $N = \sum_{l=0}^s b_l q_l$ , with  $b_l$  integers satisfying  $0 \leq b_l \leq q_{l+1}/q_l \leq C_d^{-1}$ . By the chain rule and by lemma 5.3, since for all  $y \in \mathbb{T}^1$ ,  $Df^N(y) > 0$ , then :

$$\begin{aligned} |\log D(f^N)(y)| &= |\log D(f^{\sum_{l=0}^s b_l q_l})(y)| = |\sum_{l=0}^s \sum_{i=0}^{b_l} \log Df^{q_l} \circ f^{iq_l}(y)| \\ &\leq \sup_{0 \leq l \leq s} b_l \sum_{l=0}^s |\log |D(f^{q_l})||_0 \leq C_d^{-1} C_{19}^f \sum_{l \geq 0} M_l^{1/2} \end{aligned}$$

By taking the upper bound on  $y \in \mathbb{T}^1$  and  $N \geq 0$ , we obtain an estimation of  $\sup_{N \geq 0} |\log D(f^N)|$ .

We turn to the third step: we relate the norms of  $Dh$  and  $Df^N$ . By [Yoc84],  $h$  is  $C^1$  and conjugates  $f$  to a rotation. Therefore, we have:

$$\log Dh - \log Dh \circ f = \log Df$$

hence, for all  $n$  integer:

$$\log Dh - \log Dh \circ f^n = \log D(f^n)$$

Since there is a point  $z$  in the circle such that  $Dh(z) = 1$ , we then have:

$$|\log Dh \circ f^n(z)| = |\log D(f^n)(z)| \leq \sup_{i \geq 0} |\log D(f^i)|_0$$

Moreover, since  $(f^n(z))_{n \geq 0}$  is dense in the circle, and since  $Dh$  is continuous, then we obtain:

$$|\log Dh|_0 \leq \sup_{i \geq 0} |\log D(f^i)|_0$$

We conclude:

$$|Dh|_0 \leq \exp\left(C_d^{-1} C_{19}^f \sqrt{e^{C_{19}^f \max(M_0^{1/2}, M_1^{1/2})} + 1} (\sqrt{M_0} + \sqrt{M_1})\right) \quad (15)$$

Finally, since  $\max(M_0^{1/2}, M_1^{1/2}) \leq 1$ , we obtain:

$$|Dh|_0 \leq \exp(C_1^f / C_d)$$

where  $C_1^f = 2C_{19}^f \sqrt{e^{C_{19}^f} + 1}$ . We recall that:

$$C_{19}^f = 6 \sqrt{2} e^{2W(f)} (|Sf|_0^{1/2} \vee 1)$$

Hence the theorem.

**Corollary 5.7.** *Since  $\frac{1}{\min_{T^1} Dh} \leq \exp\left(\sup_{i \geq 0} |\log D(f^i)|_0\right)$ , the proof above also provides an estimation on  $\frac{1}{\min_{T^1} Dh}$ :*

$$\frac{1}{\min_{T^1} Dh} \leq \exp(C_1^f / C_d)$$

## 6 $C^1$ estimations: non-constant type

We have  $\max_{n \geq 0} |Df^n|_0 \leq \max_{n \geq 0} M_n / m_n$ , by [Yoc84, p.348]. Therefore, in order to prove theorem 3.3, we can estimate  $M_n / m_n$ . To that end, we proceed in two steps: first, we establish some preliminary results. The most important result is corollary 6.6, which gives an estimation of  $M_{n+1} / M_n$  in function of  $M_n$ ,  $\alpha_{n+1} / \alpha_n$  and a constant  $C_{25}^{f,k}$ . This estimation is already given in [Yoc84, p. 345], but we still recall the steps to reach it, because we need to estimate the constant  $C_{25}^{f,k}$  in function of  $k$ ,  $W(f)$ ,  $|Sf|_0$  and  $|Sf|_{k-3}$ .

In the second step, we establish an estimation of the  $C^1$ -conjugacy, based on a modification of the proof given in [Yoc84]. The main idea is to establish an alternative between two possible

situations for the sequences  $M_n$  and  $\alpha_n$ : the "favorable" situation ( $R_n$ ) and the "unfavorable" situation ( $R'_n$ ) (proposition 6.10). The "unfavorable" situation only occurs a finite number of times, due to the Diophantine condition on  $\alpha$  (propositions 6.12 and 6.14).

In the "favorable" situation ( $R_n$ ), we can estimate  $M_{n+1}/\alpha_{n+1}$  in function of  $M_n/\alpha_n$  (see estimation (28)) and likewise, we can estimate  $\alpha_{n+1}/m_{n+1}$  in function of  $\alpha_n/m_n$ . Therefore, we can estimate  $M_n/m_n$  in function of  $M_{n_4}/m_{n_4}$ , where  $n_4$  is the integer such that for any  $n \geq n_4$ , the favorable case occurs (see proposition 6.19). We relate  $M_{n_4}/m_{n_4}$  to  $|Df|_0^{\frac{2}{\alpha_{n_4}}}$  (proposition 6.17), and we compute a bound on  $\alpha_{n_4}$  (proposition 6.15). Yoccoz's proof needs to be modified because in its original version, it does not allow to compute a bound on  $\alpha_{n_4}$ .

## 6.1 Preliminary results

First, we recall the following lemmas, which are in [Yoc84] (lemmas 3,4 and 5):

**Lemma 6.1.** *For  $l \geq 1$  and  $x \in \mathbb{T}^1$ , we have:*

$$\sum_{i=0}^{q_{n+1}-1} (Df^i(x))^l \leq C_{20}^f \frac{M_n^{l-1}}{m_n(x)^l}$$

with  $C_{20}^f(l) = e^{lW(f)}$ .

*Remark 6.2.* This lemma is obtained by applying Denjoy inequality.

**Lemma 6.3.** *Let  $f \in \text{Diff}_+^k(\mathbb{T}^1)$ ,  $k \geq 3$ . For any  $x \in \mathbb{T}^1$ , any  $n \in \mathbb{N}$ , any  $0 \leq p \leq q_{n+1}$ , we have:*

$$|S f^p|_0 \leq C_{21}^f \frac{M_n}{m_n^2}$$

$$|S f^p(x)| \leq C_{21}^f \frac{M_n}{m_n(x)^2}$$

$$|D \log Df^p|_0 \leq C_{22}^f \frac{M_n^{1/2}}{m_n}$$

$$|D \log Df^p(x)| \leq C_{23}^f \frac{M_n^{1/2}}{m_n(x)}$$

with:

- $C_{21}^f = |S f|_0 e^{2W(f)}$
- $C_{22}^f = \sqrt{2} |S f|_0 e^{W(f)}$
- $C_{23}^f = 9 \sqrt{2} |S f|_0 e^{4W(f)}$

**Lemma 6.4.** *For  $1 \leq r \leq k-1$ ,  $n \geq 0$ ,  $0 \leq p \leq q_{n+1}$ ,  $x \in \mathbb{T}^1$ , we have:*

$$|D^r \log Df^p(x)| \leq C_{24}^f(r) \left[ \frac{M_n^{1/2}}{m_n(x)} \right]^r \quad (16)$$

with

$$C_{24}^f(1) = C_{23}^f, \quad C_{24}^f(2) = 82|Sf|_0 e^{8W(f)}$$

and, for  $r \geq 3$ :

$$C_{24}^f(r) = \left[ 82(2r)^{2r} (1 \vee |Sf|_{r-2})^2 e^{(r+8)W(f)} \right]^{r!}$$

In particular,

$$C_{24}^{f,k} := C_{24}^f(k-1) \leq \left[ 100(2k-2)^{2k-2} (1 \vee |Sf|_{k-3})^2 e^{(k+7)W(f)} \right]^{(k-1)!}$$

*Proof of lemma 6.4.* The proof follows the line of [Yoc84], lemma 5: see appendix 8.1. □

The important preliminary result, corollary 6.6, is obtained from the following proposition. It is obtained by computing the constants in proposition 2 of [Yoc84]:

**Proposition 6.5.** *Let*

$$C_{25}^{f,k} = (k+3)^{(k+3)!} e^{(k+2)W(f)} (\max(1, |Sf|_{k-3}))^{k!} \quad (17)$$

For any  $x \in \mathbb{T}^1$ , we have:

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq C_{25}^{f,k} \left[ M_n^{(k-1)/2} m_n(x) + M_n^{1/2} m_{n+1}(x) \right] \quad (18)$$

**Corollary 6.6.**

$$M_{n+1} \leq M_n \frac{\frac{\alpha_{n+1}}{\alpha_n} + C_{25}^{f,k} M_n^{(k-1)/2}}{1 - C_{25}^{f,k} M_n^{1/2}} \quad (19)$$

$$m_{n+1} \geq m_n \frac{\frac{\alpha_{n+1}}{\alpha_n} - C_{25}^{f,k} M_n^{(k-1)/2}}{1 + C_{25}^{f,k} M_n^{1/2}}$$

The proof of proposition 6.5 combines the following three lemmas [Yoc84, pp. 343-344] (lemmas 6, 7 and 8):

**Lemma 6.7.** *For any  $x \in \mathbb{T}^1$ , there exists  $y \in [x, f^{q_n}(x)]$ ,  $z \in [f^{q_{n+1}}(x), x]$  such that*

$$m_{n+1}(y) = \frac{\alpha_{n+1}}{\alpha_n} m_n(z)$$

**Lemma 6.8.** *Suppose that  $m_{n+1}$  is monotonous on an interval  $I_z = (z, f^q(z))$ ,  $z \in \mathbb{T}^1$ . Then, for any  $x \in \mathbb{T}^1$ , for any  $y \in I_x$  ( $I_x = (x, f^q(x))$ ), we have:*

$$\left| \frac{m_{n+1}(y)}{m_{n+1}(x)} - 1 \right| \leq C_{26}^{f,k} M_n^{1/2}$$

with

$$C_{26}^{f,k} = 2^9(k+2)e^{(11+k/2)W(f)}(C_{17}^f)^2 C_{23}^f$$

**Lemma 6.9.** *If  $m_{n+1}$  is not monotonous on any interval of the form  $I_z = (z, f^q(z))$ ,  $z \in \mathbb{T}^1$ , then for any  $x \in \mathbb{T}^1$ ,  $y \in I_x$ , we have:*

$$|m_{n+1}(y) - m_{n+1}(x)| \leq C_{27}^{f,k} M_n^{(k-1)/2} m_n(x)$$

with

$$C_{27}^{f,k} = (C_{24}^f(k-1))e^{W(f)} \left( e^{(k/2+2)W(f)} (1 + e^{W(f)})^2 \frac{e^{(k/2+2)W(f)} - 1}{e^{W(f)} - 1} \right)^{k-1}$$

*Proof of proposition 6.5.* Let us recall the proof of proposition 6.5 from these three lemmas. (see [Yoc84, p.344]). Let  $x \in \mathbb{T}^1$  and  $y \in I_x$ ,  $z \in [f^{q_{n+1}}(x), x]$  the points given by lemma 6.7. By combining lemmas 6.8 and 6.9, we obtain:

$$|m_{n+1}(y) - m_{n+1}(x)| \leq \left( \max(C_{26}^{f,k}, C_{27}^{f,k}) \right) \left( M_n^{1/2} m_{n+1}(x) + M_n^{(k-1)/2} m_n(x) \right)$$

Moreover, by lemma 5.3, we have:

$$|m_n(z) - m_n(x)| \leq C_{17}^f M_n^{1/2} |z - x| \leq C_{17}^f M_n^{1/2} m_{n+1}(x)$$

By applying lemma 6.7, and since  $\alpha_{n+1}/\alpha_n \leq 1$ , we get:

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq \left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(z) \right| + \frac{\alpha_{n+1}}{\alpha_n} |m_n(z) - m_n(x)|$$

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq |m_{n+1}(y) - m_{n+1}(x)| + |m_n(z) - m_n(x)|$$

Therefore, we have:

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq C_{28}^{f,k} \left( M_n^{1/2} m_{n+1}(x) + M_n^{(k-1)/2} m_n(x) \right)$$

with  $C_{28}^{f,k} = \max(C_{26}^{f,k}, C_{27}^{f,k}) + C_{17}^f$ .

Finally, let us estimate  $C_{28}^{f,k}$ . Since  $k \geq 3$ , then:

$[4(k/2 + 1)(200k)]^2 \leq (k+3)^{(k+3)(k+2)k}/2$  and therefore,

$$2^{2(k-1)}(k/2 + 1)^{k-1}(200k)^{2(k+1)(k-1)!} \leq (k+3)^{(k+3)!}/2$$

Therefore, we have:

$$C_{27}^{f,k} + C_{17}^f \leq (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!}$$

Since  $k \geq 3$ , we also have:

$$C_{26}^{f,k} + C_{17}^f \leq (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!}$$

Therefore,  $C_{28}^{f,k} \leq C_{25}^{f,k} = (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!}$ . Hence proposition 6.5.  $\square$

## 6.2 Estimation of the $C^1$ -conjugacy: general case

We choose an integer  $n_1$  such that for any  $n \geq n_1$ , we have:

$$C_{25}^{f,k} M_n^{1/2} \leq C_{25}^{f,k} (C_{18}^f)^{\frac{n-1}{2}} < 1/2 \quad (20)$$

We take:

$$n_1 = \left\lceil \frac{-\log\left(2C_{25}^{f,k}/(C_{18}^f)^{1/2}\right)}{\log\left((C_{18}^f)^{1/2}\right)} \right\rceil$$

We choose a parameter  $\theta$  such that  $(k+1)/2 - \theta > (1+\beta+\theta)(1+\theta)$  (for the interpretation of this parameter  $\theta$ , see the remark after proposition 6.10). We take:

$$\theta = \min\left(1/2, \left(\frac{3+\beta}{4}\right)\left(-1 + \left(1 + \frac{2(k-2\beta-1)^{1/2}}{(3+\beta)^2}\right)\right)\right) \quad (21)$$

(in the proof of estimation (3), we take  $\theta = 1/2$  instead).

We recall that for  $x \geq 0$ ,  $1+x \leq e^x$  and for  $0 \leq x \leq 1/2$ ,  $\log(1/(1-x)) \leq x/(1-x) \leq 2x$ . We apply estimation (20), we use the definition of  $n_1$  and the fact that  $\theta \leq 1/2$ . We get:

$$\begin{aligned} \prod_{n=n_1}^{+\infty} (1 + M_n^\theta) &\leq \exp\left(\sum_{n=n_1}^{+\infty} M_n^\theta\right) \leq \exp\left(\frac{1}{2C_{25}^{f,k}(1-(C_{18}^f)^\theta)}\right) \\ \prod_{n=n_1}^{+\infty} \left(\frac{1}{1-C_{25}^{f,k}M_n^{1/2}}\right) &\leq \exp\left(\sum_{n=n_1}^{+\infty} 2C_{25}^{f,k}M_n^{1/2}\right) \leq \exp\left(\frac{1}{1-(C_{18}^f)^{1/2}}\right) \end{aligned}$$

Therefore,

$$\prod_{n=n_1}^{+\infty} \left(\frac{1 + M_n^\theta}{1 - C_{25}^{f,k}M_n^{1/2}}\right) \leq \exp\left(\frac{2}{1 - (C_{18}^f)^\theta}\right) = C_{29} \quad (22)$$

Let:

$$C_{30} = \max\left(\left(4C_{25}^{f,k}\right)^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}, C_{29}\right) \quad (23)$$

For any

$$n \geq \frac{-\log\left(2(C_{30})^2\right)}{\log C_{18}^f} + 1 = C_{31} \quad (24)$$

we have:

$$M_n \leq (C_{18}^f)^{n-1} \leq \frac{1}{2C_{30}^2} \quad (25)$$

We use this estimation in the second step of the proof, to which we come now:

Let

$$\tilde{n}_2 = \max(n_1, \tilde{n}_2) \quad (26)$$

where  $n_2$  is the integer defined by

$$C_{31} + \frac{4}{\log 2} \log(1/C_d) + 1 \leq \tilde{n}_2 < C_{31} + \frac{4}{\log 2} \log(1/C_d) + 2 \quad (27)$$

Having defined the integer  $n_2$ , we can present the alternative between the "favorable" case  $(R_n)$  and the "unfavorable" case  $(R'_n)$ .

**Proposition 6.10.** *Let  $a_{n_2} = 1/((C_{30})^2)$ . Let  $1 \geq \eta_n \geq 0$  be a sequence such that  $\alpha_n = \alpha_{n+1}^{1-\eta_n}$ . For any  $n \geq n_2$ , we can define a sequence  $a_n$ ,  $1/((C_{30})^2) \leq a_n \leq 1/C_{30}$  and a sequence  $\rho_n < 1$  such that  $M_n = a_n \alpha_n^{\rho_n}$ . The sequence  $a_n$  is defined by:*

if

$$(R_n) \quad C_{25}^{f,k} M_n^{(k+1)/2-\theta} \leq M_n \frac{\alpha_{n+1}}{\alpha_n} \quad \text{then} \quad a_{n+1} = a_n \frac{1 + M_n^\theta}{1 - C_{25}^{f,k} M_n^{1/2}}$$

and if

$$(R'_n) \quad C_{25}^{f,k} M_n^{(k+1)/2-\theta} > M_n \frac{\alpha_{n+1}}{\alpha_n} \quad \text{then} \quad a_{n+1} = a_n$$

Moreover, if  $(R_n)$  holds, then  $\rho_{n+1} \geq \rho_n + \eta_n(1 - \rho_n)$ ;

and if  $(R'_n)$  holds, then  $\rho_{n+1} \geq ((k+1)/2 - \theta)(1 - \eta_n)\rho_n$ . In particular, the sequence  $(\rho_n)_{n \geq n_2}$  is increasing.

The threshold between the alternatives  $(R_n)$  and  $(R'_n)$  is controlled with a parameter  $\theta$ , which could be freely chosen such that  $\theta > 0$  and  $(k+1)/2 - \theta \geq (1 + \beta + \theta)(1 + \theta)$ . When  $\theta$  increases, the number  $n_3$  of occurrences of  $(R'_n)$  increases. When  $n_3$  increases, all other quantities being equal, the bound on the norm of the conjugacy increases. Moreover, if  $\theta$  gets too large, we can no longer show that  $n_3$  is finite (see proposition 6.14), and therefore, we can no longer estimate the norm of the conjugacy.

On the other hand, when  $\theta$  is smaller,  $C_{29}$  increases. It increases the number  $n_2$  above which we consider the alternatives  $(R_n)$  and  $(R'_n)$ .  $C_{32}$  increases too (see proposition 6.19). When  $C_{29}$  and  $C_{32}$  increase, all other quantities being equal, the bound on the norm of the conjugacy increases. Moreover, when  $\theta \rightarrow 0$ ,  $C_{29} \rightarrow +\infty$ , which makes this bound on the conjugacy diverge.

Thus, the variation of  $\theta$  has contradictory influences on the bound of the norm of the conjugacy, and there is a choice of  $\theta$  that optimizes this bound. However, in this paper, we do not seek this optimal  $\theta$ , since it would complicate further the expression of the final estimate. Instead, in estimation (3), we fix  $\theta = 1/2$ , which allows simplifying the expression of the estimate. In estimation (4), we take  $\theta \rightarrow 0$ , which also allows simplifying the estimate.

*Proof of proposition 6.10:* For any  $n \geq n_2$ , since  $n_2 \geq n_1$ ,

$$a_{n_2} = \frac{1}{C_{30}^2} \leq a_n \leq a_{n_2} \prod_{n=n_1}^{+\infty} \left( \frac{1 + M_n^\theta}{1 - C_{25}^{f,k} M_n^{1/2}} \right) \leq \frac{C_{29}}{C_{30}^2} \leq \frac{1}{C_{30}}$$

and since

$$\alpha_n^{\rho_n} > a_n \alpha_n^{\rho_n} = M_n \geq \alpha_n$$

then  $\rho_n < 1$ .

Second, if  $(R_n)$  holds, then by applying corollary 6.6, we have:

$$M_{n+1} \leq \frac{1 + M_n^\theta}{1 - C_{25}^{f,k} M_n^{1/2}} M_n \frac{\alpha_{n+1}}{\alpha_n} \quad (28)$$

Therefore,

$$M_{n+1} = a_{n+1} \alpha_{n+1}^{\rho_{n+1}} \leq a_{n+1} \alpha_{n+1} \alpha_n^{\rho_n - 1} = a_{n+1} \alpha_{n+1} \alpha_{n+1}^{(1-\eta_n)(\rho_n - 1)}$$

and then:

$$\rho_{n+1} - 1 \geq (1 - \eta_n)(\rho_n - 1)$$

hence the estimation:

$$\rho_{n+1} \geq \rho_n + \eta_n(1 - \rho_n)$$



If  $(R'_n)$  holds, since  $C_{25}^{f,k} M_n^{1/2} \leq 1/2$ , then by applying corollary 6.6, we obtain:

$$M_{n+1} \leq 4C_{25}^{f,k} M_n^{(k+1)/2-\theta}$$

Moreover, since  $a_n \leq 1/C_{30} < 1$ , then:

$$a_n^{(k+1)/2-\theta} \leq a_n^{(1+\beta+\theta)(1+\theta)} = a_n a_n^{(1+\beta+\theta)(1+\theta)-1} \leq \frac{a_n}{C_{30}^{(1+\beta+\theta)(1+\theta)-1}} \leq \frac{a_n}{4C_{25}^{f,k}}$$

Therefore, by combining these two estimations, we obtain:

$$a_{n+1} \alpha_{n+1}^{\rho_{n+1}} = M_{n+1} \leq 4C_{25}^{f,k} M_n^{(k+1)/2-\theta} \leq 4C_{25}^{f,k} a_n^{(k+1)/2-\theta} \alpha_n^{\rho_n((k+1)/2-\theta)} \leq a_n \alpha_n^{\rho_n((k+1)/2-\theta)}$$

Moreover, since  $a_{n+1} = a_n$ , then

$$1 \leq \alpha_{n+1}^{(\rho_n((k+1)/2-\theta))(1-\eta_n)-\rho_{n+1}}$$

hence the estimation:

$$\rho_{n+1} \geq (\rho_n((k+1)/2-\theta))(1-\eta_n)$$

□

The reader can notice that until now, we have not used the Diophantine condition on  $\alpha$  yet. Now, we introduce this condition in order to estimate  $\rho_{n_2}$  from below (proposition 6.11), and in order to determine a bound  $\rho$  above which  $(R_n)$  always occurs (proposition 6.12).

**Proposition 6.11.** *If  $\beta > 0$ , we have the estimation:*

$$\rho_{n_2} \geq \frac{\log 2}{((1+\beta)^{n_2+1} - 1) \log(1/C_d)/\beta}$$

*If  $\beta = 0$ , we have the estimation:*

$$\rho_{n_2} \geq \frac{\log 2}{(n_2 + 1) \log(1/C_d)}$$

*Proof.* Since  $\alpha$  is Diophantine, we have:  $\alpha_{n+1} \geq C_d \alpha_n^{1+\beta}$ . Therefore, for  $\beta > 0$ ,

$$\log\left(\frac{1}{\alpha_{n+1}}\right) + \frac{\log(1/C_d)}{\beta} \leq (1+\beta) \left( \log(1/\alpha_n) + \frac{\log(1/C_d)}{\beta} \right)$$

and since  $\alpha_{-1} = 1$ , then by iteration, for any  $n \geq 0$ ,

$$\log(1/\alpha_n) \leq \left( (1 + \beta)^{n+1} - 1 \right) \frac{\log(1/C_d)}{\beta}$$

If  $\beta = 0$ , we have:

$$\log(1/\alpha_n) \leq (n + 1) \log(1/C_d)$$

Moreover, since  $\rho_{n_2} = -\log(M_{n_2}/a_{n_2})/\log(1/\alpha_{n_2})$  and  $M_{n_2}/a_{n_2} \leq 1/2$ , then we get proposition 6.11. □

**Proposition 6.12.** Let  $\beta_1 = \beta + \frac{2\log(1/C_d)}{(n_2-1)\log 2}$ . If

$$\rho_n \geq \frac{\beta_1}{(k-1)/2 - \theta} = \rho \quad (29)$$

then  $(R_n)$  occurs.

*Remark 6.13.* Note that  $\rho < 1$ , because  $(k+1)/2 - \theta \geq (1 + \beta + \theta)(1 + \theta)$  and  $\beta_1 \leq \beta + 1/2$ .

*Proof.* Since  $\alpha_n \leq (1/2)^{\frac{n-1}{2}}$ , then

$$0 < \frac{\log C_d}{\log \alpha_n} \leq \frac{-\log C_d}{\frac{n-1}{2} \log 2} \quad (30)$$

Furthermore, since  $\alpha_{n+1} = \alpha_n^{\frac{1}{1-\eta_n}} \geq C_d \alpha_n^{1+\beta}$ , then

$$\frac{1}{1-\eta_n} \log \alpha_n \geq \log C_d + (1 + \beta) \log \alpha_n$$

and since  $\log \alpha_n < 1$  for  $n \geq 0$ , then by (30),

$$\frac{1}{1-\eta_n} - 1 \leq \beta + \frac{\log C_d}{\log \alpha_n} \leq \beta + \frac{\log(1/C_d)}{\frac{n-1}{2} \log 2}$$

Therefore, if estimation (29) holds, then

$$\left( \frac{k-1}{2} - \theta \right) \rho_n + 1 - \frac{1}{1-\eta_n} \geq 0$$

and therefore,

$$\left( \frac{1}{\alpha_n} \right)^{\left( \frac{k-1}{2} - \theta \right) \rho_n + 1 - \frac{1}{1-\eta_n}} \geq 1$$

Hence

$$M_n \frac{\alpha_{n+1}}{\alpha_n} = a_n \alpha_n^{\rho_n} \frac{\alpha_{n+1}}{\alpha_n} \geq a_n \alpha_n^{(\frac{k+1}{2}-\theta)\rho_n} = M_n^{\frac{k+1}{2}-\theta} a_n^{1-(\frac{k+1}{2}-\theta)} \geq M_n^{\frac{k+1}{2}-\theta} C_{30}^{\frac{k+1}{2}-\theta-1} \geq M_n^{\frac{k+1}{2}-\theta} C_{30}^{(1+\beta+\theta)(1+\theta)-1}$$

Therefore,

$$M_n \frac{\alpha_{n+1}}{\alpha_n} \geq C_{25}^{f,k} M_n^{\frac{k+1}{2}-\theta}$$

□

**Proposition 6.14.** *The alternative  $(R'_n)$  occurs less than  $n_3$  times, with*

$$n_3 - n_2 \leq \max \left( 0, \frac{\log(\rho/\rho_{n_2})}{\log \left( \frac{(k+1)/2-\theta}{1+\beta_1} \right)} \right) \quad (31)$$

*Proof.* If  $\rho_{n_2} \geq \rho$ , then  $(R'_n)$  does not occur for any  $n \geq n_2$ . We suppose  $\rho_{n_2} < \rho$ . For any  $n \geq n_2$ , since

$$((k+1)/2 - \theta)(1 - \eta_n) \geq \frac{(k+1)/2 - \theta}{1 + \beta_1}$$

then

$$\rho_n \geq \left( \frac{(k+1)/2 - \theta}{1 + \beta_1} \right)^{n-n_2} \rho_{n_2}$$

Moreover,

$$\left( \frac{(k+1)/2 - \theta}{1 + \beta_1} \right)^{n-n_2} \rho_{n_2} \geq \rho$$

when

$$n \geq n_2 + \frac{\log(\rho/\rho_{n_2})}{\log \left( \frac{(k+1)/2-\theta}{1+\beta_1} \right)}$$

□

The next proposition gives a lower bound on  $\alpha_{n_4}$ , which allows computing a bound on the  $C^1$ -conjugacy.

**Proposition 6.15.** *Let  $n_4 \geq 0$  be the smallest integer such that for any  $n \geq n_4$ ,  $(R_n)$  occurs. We have:*

$$\alpha_{n_4} \geq C_d^{\exp((n_3+1+\rho/(1-\rho))(1+\beta_1))}$$

*Proof.* First, we suppose  $n_4 \geq n_2 + 1$ . We need the lemma:

**Lemma 6.16.** *Let  $n_5 \geq n_2$  be the smallest integer such that*

$$\sum_{n=n_2}^{n_5} \eta_n \geq n_3 - n_2 + \rho/(1 - \rho)$$

*$n_5$  exists. Moreover, we have  $\rho_{n_5+1} \geq \rho$ . In particular, for this integer  $n_5$ , we have that for any  $n \geq n_5 + 1$ ,  $(R_n)$  occurs.*

*Proof.* First, let us show the existence of  $n_5$ . By absurd, suppose that

$$\sum_{n=n_2}^{+\infty} \eta_n < n_3 - n_2 + \rho/(1 - \rho)$$

For any  $1 > x \geq 0$ ,

$$\log\left(\frac{1}{1-x}\right) \leq \frac{x}{1-x}$$

Therefore, for any integer  $p \geq n_2 + 1$ ,

$$\prod_{n=n_2}^{p-1} \left(\frac{1}{1-\eta_n}\right) \leq \exp\left(\sum_{n=n_2}^{p-1} \frac{\eta_n}{1-\eta_n}\right)$$

Moreover,  $\frac{1}{1-\eta_n} \leq 1 + \beta_1$  for any  $n \geq 1$ . Therefore,

$$\sum_{n=n_2}^{p-1} \frac{\eta_n}{1-\eta_n} \leq (n_3 - n_2 + \rho/(1 - \rho))(1 + \beta_1)$$

Since  $\eta_n \leq 1$ , then  $\sum_{n=0}^{n_2-1} \eta_n \leq n_2$ . Therefore,

$$\sum_{n=0}^{p-1} \frac{\eta_n}{1-\eta_n} \leq (n_3 + \rho/(1 - \rho))(1 + \beta_1)$$

Moreover, since  $\alpha_0 = \alpha \geq C_d$  then for any  $p \geq n_2 + 1$ :

$$\alpha_p = \alpha_0^{\prod_{n=0}^{p-1} \left(\frac{1}{1-\eta_n}\right)} \geq C_d^{\exp((n_3 + \rho/(1 - \rho))(1 + \beta_1))}$$

However, since  $\alpha_p \geq 2\alpha_{p+2}$ , then  $\alpha_p \rightarrow 0$  when  $p \rightarrow +\infty$ . Hence the contradiction and the existence of  $n_5$ . Note that  $n_5 + 1 \geq n_4$ .

Second, let us show that  $\rho_{n_5+1} \geq \rho$ . If there is  $n_6 \leq n_5$  such that  $\rho_{n_6} \geq \rho$ , then  $\rho_{n_5+1} \geq \rho$  because the sequence  $\rho_n$  is increasing. Otherwise, for any  $n \leq n_5$ , we have:  $\rho_n \leq \rho$ .

Let  $E_1 = \{n_5 \geq n \geq n_2 / (R_n) \text{ occurs}\}$  and  $E_2 = \{n_5 \geq n \geq n_2 / (R'_n) \text{ occurs}\}$ .

We have:

$$n_3 - n_2 + \frac{\rho}{1 - \rho} \leq \sum_{n=n_2}^{n_5} \eta_n = \sum_{n \in E_1} \eta_n + \sum_{n \in E_2} \eta_n \leq \sum_{n \in E_1} \eta_n + n_3 - n_2$$

Therefore,

$$\sum_{n \in E_1} \eta_n \geq \rho / (1 - \rho)$$

Since  $\rho_n$  is increasing and  $\rho_n \leq \rho$ , we get:

$$\begin{aligned} \rho_{n_5+1} &= \rho_{n_2} + \sum_{n=n_2}^{n_5} \rho_{n+1} - \rho_n \\ \rho_{n_5+1} &\geq \rho_{n_2} + \sum_{n \in E_1} \rho_{n+1} - \rho_n \geq \rho_{n_2} + \sum_{n \in E_1} (1 - \rho_n) \eta_n \geq \rho_{n_2} + (1 - \rho) \sum_{n \in E_1} \eta_n \geq \rho \end{aligned}$$

□

Now, let us show proposition 6.15. Since  $\eta_n \leq 1$  for any  $n$ , then we have:

$$n_3 - n_2 + 1 + \frac{\rho}{1 - \rho} > \sum_{n=n_2}^{n_5} \eta_n \geq n_3 - n_2 + \frac{\rho}{1 - \rho}$$

Since

$$n_3 - n_2 + \frac{\rho}{1 - \rho} + 1 \geq \sum_{n=n_2}^{n_5} \eta_n \geq \sum_{n=n_2}^{n_4-1} \eta_n$$

then by proceeding in the same way as in the first part of the proof of lemma 6.16, we obtain:

$$\alpha_{n_4} \geq C_d^{\exp((n_3+1+\rho/(1-\rho))(1+\beta_1))} \quad (32)$$

Finally, if  $n_4 \leq n_2$ , then as in the proof of lemma 6.16,

$$\alpha_{n_2} = \alpha_0^{\prod_{n=0}^{n_2-1} \left(\frac{1}{1-\eta_n}\right)} \geq C_d^{\exp(n_2(1+\beta_1))}$$

Therefore, the estimation given in proposition 6.15 still holds.

□

Having bounded  $\alpha_{n_4}$  from below, we show how this bound is related to  $M_n/m_n$  (and therefore, how this is related to the conjugacy).

**Proposition 6.17.** *Let  $n \geq 1$ ,  $M_n = \max_{x \in \mathbb{T}^1} m_n(x)$ ,  $m_n = \min_{x \in \mathbb{T}^1} m_n(x)$ . For any  $j \leq n$ ,*

$$\frac{M_j}{m_j} \leq 3|Df|_0^{\frac{2}{\alpha_n}}$$

*Proof.* We need the following lemma, which is in [Yoc84, p. 339]:

**Lemma 6.18.** *For any  $x \in \mathbb{T}^1$ , let  $J_x = [f^{-q_n}(x), f^{q_n}(x)]$ . The intervals  $f^i(J_x)$ ,  $0 \leq i < q_{n+1}$  recover  $\mathbb{T}^1$ .*

First, note that since  $f(x+1) - f(x) = 1$  (in the universal cover), then  $|Df|_0 \geq 1$ . Let  $x, y \in \mathbb{T}^1$  such that  $M_n = m_n(x)$  and  $m_n = m_n(y)$ . Let  $0 \leq i < q_{n+1}$  such that  $x \in f^i(J_y)$ . Since we have the cyclic order  $f^{i-q_n}(y) \leq x \leq f^{i+q_n}(y)$  then we also have:  $f^i(y) \leq f^{q_n}(x) \leq f^{i+2q_n}(y)$ . Therefore,  $[x, f^{q_n}(x)] \subset [f^{i-q_n}(y), f^{i+2q_n}(y)]$ . This implies:

$$M_n \leq f^{i+2q_n}(y) - f^{i+q_n}(y) + f^{i+q_n}(y) - f^i(y) + f^i(y) - f^{i-q_n}(y)$$

$$M_n \leq (|Df^{i+q_n}|_0 + |Df^i|_0 + |Df^{i-q_n}|_0)(f^{q_n}(y) - y)$$

and therefore,

$$\frac{M_n}{m_n} \leq (|Df^{i+q_n}|_0 + |Df^i|_0 + |Df^{i-q_n}|_0)$$

Therefore,

$$\frac{M_n}{m_n} \leq 3|Df|_0^{q_n+q_{n+1}}$$

Likewise, for any  $j \leq n$ , we have:

$$\frac{M_j}{m_j} \leq 3|Df|_0^{q_n+q_{n+1}}$$

Since  $q_n + q_{n+1} \leq 2q_{n+1} \leq \frac{2}{\alpha_n}$ , we obtain proposition 6.17. □

**Proposition 6.19.** *For any  $n \geq 1$ ,*

$$\frac{M_n}{m_n} \leq C_{32} \frac{M_{n_4}}{m_{n_4}} \tag{33}$$

with:

$$C_{32} = \exp \left( \frac{2(2C_{30}^2)^\theta - 1}{(2C_{30}^2)^\theta - 1} \frac{(C_{18}^f)^{(n_2-1)\theta}}{1 - (C_{18}^f)^\theta} + 3C_{25}^{f,k} \frac{(C_{18}^f)^{(n_2-1)/2}}{1 - (C_{18}^f)^{1/2}} \right) \tag{34}$$

*Proof.* Since for any  $n \geq n_4$ ,  $(R_n)$  occurs, then by corollary 6.6, we have:

$$\frac{M_{n+1}}{M_n} \leq \frac{1 + M_n^\theta}{1 - C_{25}^{f,k} M_n^{1/2}} \frac{\alpha_{n+1}}{\alpha_n}$$

$$\frac{m_{n+1}}{m_n} \geq \frac{1 - M_n^\theta}{1 + C_{25}^{f,k} M_n^{1/2}} \frac{\alpha_{n+1}}{\alpha_n}$$

Therefore,

$$\frac{M_{n+1}/m_{n+1}}{M_n/m_n} \leq \frac{1 + M_n^\theta}{1 - M_n^\theta} \frac{1 + C_{25}^{f,k} M_n^{1/2}}{1 - C_{25}^{f,k} M_n^{1/2}} \quad (35)$$

Therefore, for any  $n \geq n_4$ ,

$$\frac{M_n}{m_n} \leq \frac{M_{n_4}}{m_{n_4}} \prod_{j=n_4}^{+\infty} \frac{1 + M_j^\theta}{1 - M_j^\theta} \frac{1 + C_{25}^{f,k} M_j^{1/2}}{1 - C_{25}^{f,k} M_j^{1/2}}$$

As in the computation page 37, and since  $n_4 \geq n_2$ , we have:

$$\prod_{j=n_4}^{+\infty} (1 + M_j^\theta) \leq \exp \left( \sum_{j=n_2}^{+\infty} M_j^\theta \right) \leq \exp \left( \frac{(C_{18}^f)^{(n_2-1)\theta}}{1 - (C_{18}^f)^\theta} \right)$$

$$\prod_{j=n_4}^{+\infty} (1 + C_{25}^{f,k} M_j^{1/2}) \leq \exp \left( C_{25}^{f,k} \sum_{j=n_2}^{+\infty} M_j^{1/2} \right) \leq \exp \left( C_{25}^{f,k} \frac{C_{18}^{f(n_2-1)/2}}{1 - (C_{18}^f)^{1/2}} \right)$$

$$\prod_{j=n_4}^{+\infty} \frac{1}{1 - C_{25}^{f,k} M_j^{1/2}} \leq \exp \left( C_{25}^{f,k} \sum_{j=n_2}^{+\infty} \frac{M_j^{1/2}}{1 - C_{25}^{f,k} M_j^{1/2}} \right) \leq \exp \left( 2C_{25}^{f,k} \frac{(C_{18}^f)^{(n_2-1)/2}}{1 - (C_{18}^f)^{1/2}} \right) \quad (36)$$

and since, for  $j \geq n_2$ ,  $M_j \leq 1/(2C_{30}^2)$ , we get:

$$\prod_{j=n_4}^{+\infty} \frac{1}{1 - M_j^\theta} \leq \exp \left( \sum_{j=n_2}^{+\infty} \frac{M_j^\theta}{1 - M_j^\theta} \right) \leq \exp \left( \frac{\frac{(C_{18}^f)^{(n_2-1)\theta}}{1 - (C_{18}^f)^\theta}}{1 - \frac{1}{(2C_{30}^2)^\theta}} \right)$$

Therefore,

$$\frac{M_n}{m_n} \leq \frac{M_{n_4}}{m_{n_4}} \exp \left( \frac{2(2C_{30}^2)^\theta - 1}{(2C_{30}^2)^\theta - 1} \frac{(C_{18}^f)^{(n_2-1)\theta}}{1 - (C_{18}^f)^\theta} + 3C_{25}^{f,k} \frac{(C_{18}^f)^{(n_2-1)/2}}{1 - (C_{18}^f)^{1/2}} \right) \quad (37)$$

Hence proposition 6.19.

□

*Proof of estimation (2).* By combining propositions 6.17 and 6.19, and since by [Yoc84, p. 348],

$|Dh|_0 \leq \sup_{n \geq 0} M_n/m_n$ , we get:

$$|Dh|_0 \leq C_{33} |Df|_0^{\frac{2}{\alpha_{n_4}}} \quad (38)$$

with:

$$C_{33} = 3C_{32}$$

We estimate  $C_{32}$ : since  $(2x-1)/(x-1) = 2 + 1/(x-1)$ , since

$(C_{18}^f)^{(n_2-1)\theta} \leq 1/(2(C_{30})^2)^\theta$  and since  $\theta \leq 1/2$ , then:

$$C_{32} \leq \exp \left( \left( 2 + \frac{1}{(2(C_{30})^2)^\theta - 1} + 3C_{25}^{f,k} \right) \frac{1}{(2(C_{30})^2)^\theta (1 - (C_{18}^f)^\theta)} \right)$$

Since  $C_{18}^f \geq 1$ , we get:

$$|Dh|_0 \leq C_{34} |Df|_0^{\frac{2}{\alpha_{n_4}}} \quad (39)$$

with:

$$C_{34} = 3e \wedge \left( \left( 2 + \frac{1}{(2(C_{30})^2)^\theta - 1} + 3C_{25}^{f,k} \right) \frac{1}{(2(C_{30})^2)^\theta (1 - (C_{18}^f)^\theta)} \right)$$

We estimate  $C_{34}$  using expressions of  $\theta$  (see (21)), of  $C_{18}^f$  (see lemma 5.4) and of  $C_{30}$  (see (6.2) and proposition 6.5).

We estimate  $\alpha_{n_4}$  using propositions 6.15, 6.14, 6.12, 6.11, and the expressions of  $n_2$  (see (26)) and estimates of  $\theta$ ,  $C_{18}^f$  and  $C_{30}$ . We get:

$$|Dh|_0 \leq C_2(k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3})$$

where  $C_2$  is the combination of the following functions:

1.  $C_{18}^f = \left( 1 + e \wedge \left( -6\sqrt{2}e^{2W(f)}(|Sf|_0^{1/2} \vee 1) \right) \right)^{-1/2}$   
(since  $|Sf|_0 \leq |Sf|_{k-3}$ , we can estimate  $C_{18}^f$  in function of  $W(f)$ ,  $|Sf|_{k-3}$ ).
2.  $C_{25}^{f,k} = (k+3)^{(k+3)!} e^{(k+3)W(f)} (\max(1, |Sf|_{k-3}))^{(k+1)!}$
3.  $\theta = \min \left( 1/2, \left( \frac{3+\beta}{4} \right) \left( -1 + \sqrt{1 + \frac{2(k-(2\beta+1))}{(3+\beta)^2}} \right) \right)$
4.  $C_{30} = \max \left( e^{\frac{2}{1-(C_{18}^f)^\theta}}, (4C_{25}^{f,k})^{\frac{1}{(1+\theta)(1+\theta-1)}} \right)$



5.  $n_2 = \lfloor \max \left( -\frac{\log(2C_{30}^2)}{\log C_{18}^f} + \frac{2\log(1/C_d)}{\theta \log 2} + 2, 2 + \frac{(2C_{25}^{f,k})}{\log((C_{18}^f)^{1/2})} \right) \rfloor$
6.  $\beta_1 = \beta + \frac{2\log(1/C_d)}{(n_2-1)\log 2}$
7.  $n_3 = \lceil \frac{1}{\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)} \left( n_2(1 + \log(1 + \beta)) + \log\left(\frac{(n_2+1)\log(1/C_d)}{\log 2}\right) \right) \rceil$
8.  $\rho = \frac{\beta_1}{\frac{k-1}{2}-\theta}$
9.  $\alpha'_{n_4} = C_d \wedge \left( e \wedge \left( (n_3 + 1 + \frac{\rho}{1-\rho})(1 + \beta_1) \right) \right)$
10.  $C_{34} = 3e \wedge \left( \left( 2 + \frac{1}{(2(C_{30})^2)^\theta - 1} + 3C_{25}^{f,k} \right) \frac{1}{(2(C_{30})^2)^\theta(1-(C_{18}^f)^\theta)} \right)$
11.  $|Dh|_0 \leq C_{34}|Df|_0^{\frac{2}{\alpha'_{n_4}}}$

Note that we have a bound  $\alpha'_{n_4} \leq \alpha_{n_4}$ , but we do not know the value of  $\alpha_{n_4}$ .

□

### 6.3 The case $k \geq 3\beta + 9/2$

In order to obtain relatively simple estimates, we can take the parameter  $\theta$  (defined in (21)) either vanishingly close to 0 (estimation (4)), or fixed independently of the other parameters (estimation (3)).

In the latter case, we need to assume that  $k - 2\beta - 1$  is sufficiently large, in order to keep  $(k + 1)/2 - \theta \geq (1 + \beta + \theta)(1 + \theta)$ . To illustrate this case, we take  $\theta = 1/2$ , which requires  $k \geq 3\beta + 9/2$  (for any fixed  $\theta$ , we cannot obtain an assumption of the form  $k \geq 2\beta + u$  for some number  $u$ : we necessarily have  $k \geq \lambda\beta + u$  with  $\lambda > 2$ ).

To simplify the function  $C_2$ , we successively estimate  $C_{32}$ ,  $\alpha'_{n_4}$  and  $n_2$ .

**Lemma 6.20.** *Let  $C_{32}$  and  $C_{30}$  defined in proposition 6.19 and page 38 respectively. We have:*

$$C_{32} \leq \exp\left(C_{30}^{\frac{3\beta+1}{2}}\right)$$

*Proof.* For any  $x \geq 2$ ,

$$(2x - 1)/(x - 1) = 2 + 1/(x - 1) \leq 3$$

Since  $C_{30} \geq C_{29} \geq e \geq 2$ , then

$$\begin{aligned} & \exp\left(\frac{2(2C_{30})^{1/2}-1}{(2C_{30})^{1/2}-1} (1/C_{18}^f)^{1/2} \frac{(C_{18}^f)^{n_2/2}}{1-(C_{18}^f)^{1/2}} + 3C_{25}^{f,k} (1/C_{18}^f)^{1/2} \frac{(C_{18}^f)^{n_2/2}}{1-(C_{18}^f)^{1/2}}\right) \\ & \leq \exp\left(3(1 + C_{25}^{f,k})(1/C_{18}^f)^{1/2} \frac{(C_{18}^f)^{n_2/2}}{1-(C_{18}^f)^{1/2}}\right) \end{aligned}$$

On the other hand, since  $n_2 \geq \frac{-\log(2(C_{30})^2/C_{18}^f)}{\log C_{18}^f}$  and  $C_{18}^f < 1$ , then:

$$e^{\frac{n_2}{2} \log C_{18}^f} \leq \left(2(C_{30})^2/C_{18}^f\right)^{-1/2}$$

Therefore,

$$\begin{aligned} (1/C_{18}^f)^{1/2} \frac{(C_{18}^f)^{n_2/2}}{1 - (C_{18}^f)^{1/2}} &= (1/C_{18}^f)^{1/2} \frac{e^{\frac{n_2}{2} \log C_{18}^f}}{1 - (C_{18}^f)^{1/2}} \\ &\leq (1/C_{18}^f)^{1/2} \frac{\left(2(C_{30})^2/C_{18}^f\right)^{-1/2}}{1 - (C_{18}^f)^{1/2}} = \frac{1}{\sqrt{2}C_{30}(1 - (C_{18}^f)^{1/2})} \end{aligned}$$

Moreover, since  $C_{25}^{f,k} \geq 1/(\sqrt{2} - 1)$  then  $(1 + C_{25}^{f,k})/\sqrt{2} \leq C_{25}^{f,k}$ . Therefore,

$$\exp\left(3(1 + C_{25}^{f,k})(1/C_{18}^f)^{1/2} \frac{(C_{18}^f)^{n_2/2}}{1 - (C_{18}^f)^{1/2}}\right) \leq \exp\left(\frac{3C_{25}^{f,k}}{C_{30}(1 - (C_{18}^f)^{1/2})}\right)$$

Since  $C_{30}^{\frac{3}{2}(3/2+\beta)-1} \geq 4C_{25}^{f,k}$ , then

$$\frac{C_{25}^{f,k}}{C_{30}} \leq \frac{1}{4}C_{30}^{\frac{1}{4}+\frac{3}{2}\beta}$$

We also have:

$$C_{30}^{\frac{1}{4}} \geq e^{\frac{1/2}{1-(C_{18}^f)^{1/2}}} \geq \frac{1}{4} \frac{1}{1 - (C_{18}^f)^{1/2}}$$

We obtain:

$$\exp\left(\frac{3C_{25}^{f,k}}{C_{30}(1 - (C_{18}^f)^{1/2})}\right) \leq \left(e^{\frac{1}{4}C_{30}^{\frac{1}{4}+\frac{3}{2}\beta} \frac{1}{1-(C_{18}^f)^{1/2}}}\right) \leq \exp\left(C_{30}^{\frac{3\beta+1}{2}}\right)$$

□

**Lemma 6.21.** Let  $\alpha'_{n_4}$  defined page 48 and  $n_2$  defined in (26). We have:

$$\frac{1}{\alpha'_{n_4}} \leq \left(\frac{1}{C_d}\right) \wedge e \wedge \left((\beta + 3/2) \left(2 + \frac{n_2}{\log(3/2)} (2 + \log(1 + \beta) + \log \log(1/C_d))\right)\right)$$

*Proof.* Since  $\beta_1 \leq \beta + 1/2$ , and  $\theta = 1/2$ , then

$$\log\left(\frac{(k+1)/2 - \theta}{1 + \beta_1}\right) \geq \log(3/2)$$

Therefore,

$$n_3 \leq \frac{n_2}{\log(3/2)} \left( 1 + \log(1 + \beta) + \frac{\log((n_2 + 1) \log(1/C_d))}{n_2 \log 2} \right)$$

We have  $\log(n_2 + 1)/n_2 \leq 1$  and  $n_2 \log 2 \geq 1$ .

Moreover,  $\rho \leq 2/3$ , and so  $\rho/(1 - \rho) \leq 2$ . Hence the lemma. □

**Lemma 6.22.** *Let  $n_2$  defined in (26). We have:*

$$n_2 \leq C_{35}(W(f), |Sf|_0)(k + 4)!(1 + \log(\max(1, |Sf|_{k-3}))(1 + \log(1/C_d))$$

with:

$$C_{35}(W(f), |Sf|_0) = e^{(2)} \wedge (3W(f) + 2 \log(\max(1, |Sf|_0)) + 4)$$

*Proof.* In order to estimate  $n_2$ , we need to estimate  $C_{30}$  (see page 47). We distinguish the cases  $C_{30} = (4C_{25}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}$  and  $C_{30} = C_{29}^{f,k}$ .

First, we suppose  $C_{30} = (4C_{25}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}$ . We have:  $C_{30} \leq 4C_{25}^{f,k}$ . Therefore,

$$n_2 \leq 2 + \frac{\log(2(4C_{25}^{f,k})^2)}{-\log(C_{18}^f)} + \frac{4}{\log 2} \log(1/C_d)$$

Moreover (see page 48),

$$\log(C_{25}^{f,k}) \leq (k + 3)! [\log(k + 3) + W(f) + \max(1, |Sf|_{k-3})]$$

And for any  $x > -1$ ,  $\log(1 + x) \geq x/(1 + x)$ . Therefore,

$$\frac{1}{-\log(C_{18}^f)} \leq \frac{1}{\frac{1}{2} \log(1 + e^{-C_{19}^f})} \leq 2 + 2e^{C_{19}^f}$$

Therefore,

$$n_2 \leq 2 + 2 \left( 1 + e^{C_{19}^f} \right) [5(\log 2) + 2(k + 3)! (W(f) + \log(k + 3) + \log(\max(1, |Sf|_{k-3})))] \\ + \frac{4}{\log 2} \log(1/C_d)$$

Moreover,

$$4e^{C_{19}^f} \leq \exp(\exp(2W(f) + \log(6\sqrt{2}(\max(1, |Sf|_0))^{1/2}) + 1)) = C_{36}^f$$

Moreover,  $(k + 4)!/3 \geq 2 + 5 \log 2$  and  $2(k + 4)/3 \geq 2 \log(k + 3)$ . Therefore,

$$(k + 4)! \geq 2 + 5 \log 2 + 2(k + 3)! \log(k + 3)$$

Moreover,  $2(1 + e^{C_{19}^f}) \leq 4e^{C_{19}^f}$ . We get:

$$n_2 \leq (k+4)! C_{36}^f (W(f) + 1 + \log(\max(1, |Sf|_{k-3})) + \frac{4}{\log 2} \log(1/C_d))$$

By using that  $\log(6\sqrt{2}) + 1 \leq 4$  and  $4/\log 2 \leq (k+4)!$ , we obtain the estimation of lemma 6.22.

If  $C_{30} = C_{29}$ , we need the following lemma:

**Lemma 6.23.** *Let  $C_{30}$  and  $C_{31}$  given page 38. If  $C_{30} = C_{29}$ , then*

$$C_{31} \leq 40e^{2C_{19}^f}$$

*Proof.* We have:

$$C_{31} = \frac{-\log(2(C_{30})^2/C_{18}^f)}{\log C_{18}^f} = 1 + \frac{\log 2}{\frac{1}{2} \log(1 + e^{-C_{19}^f})} + \frac{4}{(1 - (1 + e^{-C_{19}^f})^{-1/4}) \frac{1}{2} \log(1 + e^{-C_{19}^f})}$$

Since for any  $x > -1$ ,  $\log(1+x) \geq x/(1+x)$ , then:

$$\frac{1}{\log(1 + e^{-C_{19}^f})} \leq e^{C_{19}^f} (1 + e^{-C_{19}^f})$$

On the other hand,

$$1 - (1 + e^{-C_{19}^f})^{-1/4} = \int_0^{e^{-C_{19}^f}} \frac{1}{4(1+t)^{5/4}} dt \geq \frac{e^{-C_{19}^f}}{4(1 + e^{-C_{19}^f})^{5/4}}$$

Therefore,

$$C_{31} \leq 8(1 + e^{-C_{19}^f})^{9/4} e^{2C_{19}^f} \left( \frac{1}{8(1 + e^{-C_{19}^f})e^{2C_{19}^f}} + \frac{\log 2}{4e^{C_{19}^f}} + 4 \right)$$

Since  $C_{19}^f \geq 6\sqrt{2}$ , then:

$$e^{-C_{19}^f} \leq 2.07 \times 10^{-4} \tag{40}$$

Therefore,

$$\frac{1}{8(1 + e^{-C_{19}^f})e^{2C_{19}^f}} + \frac{\log 2}{4e^{C_{19}^f}} + 4 \leq 4.1$$

Therefore,

$$C_{31} \leq 40e^{2C_{19}^f}$$

□

Assuming that  $C_{30} = C_{29}$ , if  $n_2 = \lfloor 2 + \frac{(2C_{25}^{f,k})}{\log((C_{18}^f)^{1/2})} \rfloor$  then we can follow the first part of the proof and we still obtain the estimate of lemma 6.22.

If  $n_2 = \lfloor -\frac{\log(2C_{30}^2)}{\log C_{18}^f} + \frac{2\log(1/C_d)}{\theta \log 2} + 2 \rfloor$ , then:

$$\frac{-\log(2(C_{30})^2/C_{18}^f)}{\log C_{18}^f} = C_{31} \leq 20e^{2C_{19}^f} \leq 2(C_{36}^f)^2 \leq C_{35}^f$$

Therefore,

$$n_2 \leq -\frac{\log(2C_{30}^2)}{\log C_{18}^f} + \frac{2\log(1/C_d)}{\theta \log 2} + 2 \leq C_{35}^f + \frac{4\log(1/C_d)}{\log 2} + 1$$

The estimation of lemma 6.22 still holds. This completes the proof of lemma 6.22.

□

Now, we show estimation (3). We denote  $\log^{(2)}(x) = \log \log x$ . By applying estimation (38) and lemma 6.21, we have:

$$|Dh|_0 \leq e^{(3)} \wedge$$

$$\left( \log^{(2)}(C_{33}) + \log^{(2)}(|Df|_0) + \log 2 + \log^{(2)}(1/C_d) + (\beta + 3/2) \left( 2 + \frac{n_2}{\log(3/2)} (2 + \beta + \log^{(2)}(1/C_d)) \right) \right)$$

By lemma 6.20, we have:

$$\log^{(2)}(C_{33}) \leq \log^{(2)}(3C_{32}) \leq \log \left( \log 3 + C_{30}^{\frac{3\beta+1}{2}} \right)$$

Moreover, since  $C_{30} \geq e$ ,  $\frac{3\beta+1}{2} \geq 1/2$  and  $e(e^{1/2} - 1) \geq \log 3$ , then

$$\log 3 + C_{30}^{\frac{3\beta+1}{2}} \leq C_{30}^{\frac{3\beta+1}{2}+1}$$

Therefore,

$$\log^{(2)}(C_{33}) \leq \left( \frac{3\beta+1}{2} + 1 \right) C_{30}$$

Moreover, by estimation (40),  $e^{-C_{19}^f} \leq 2.07 \times 10^{-4}$ , and therefore,  $-\log(C_{18}^f) \leq 2/3$ . By applying the definition of  $n_2$  (page 38), we get:

$$\log^{(2)}(C_{33}) \leq 3 \frac{(\beta + 1)}{2} \log C_{30} \leq \frac{\beta + 1}{2} n_2$$

Moreover,

$$\frac{\beta + 1}{2} n_2 + \left( \frac{\beta + 3/2}{\log(3/2)} \right) (\beta + 2) n_2 + 2(\beta + 3/2) \leq \frac{\beta + 3/2}{\log(3/2)} (\beta + 3)(n_2 + 2)$$

Therefore,

$$|Dh|_0 \leq e^{(3)} \wedge \left( \frac{2}{\log(3/2)} (\beta + 3/2)(\beta + 3)(n_2 + 2)(1 + \log^{(2)}(|Df|_0))(1 + \log^{(2)}(1/C_d)) \right)$$

We have:  $n_2 + 2 \leq 2n_2$ . By lemma 6.22, and since  $4/\log(3/2) \leq 10$ , we get:

$$|Dh|_0 \leq e^{(3)} \wedge \left( (\beta + 3/2)(\beta + 3)(k + 4)! C_5 (1 + \log(1/C_d))^2 (1 + \log(\max(1, |Sf|_{k-3}))) \right) \quad (41)$$

with  $C_5 = 10(1 + \log^{(2)}(|Df|_0))e^{(2)} \wedge (3W(f) + 2\log(\max(1, |Sf|_0))) + 2$ .

This estimation of  $|Dh|_0$  is increasing with  $k$ . Therefore, to obtain a bound as low as we can, we take  $k = \lceil 3\beta + 9/2 \rceil$ . We obtain:

$$|Dh|_0 \leq e^{(3)} \wedge \left( C_3[\beta] C_4[C_d] C_5[|Df|_0, W(f), |Sf|_0] C_6[|Sf|_{\lceil 3\beta+3/2 \rceil}] \right)$$

with:

1.  $C_3[\beta] = (\lceil 3\beta + 21/2 \rceil)!$
2.  $C_4[C_d] = (1 + \log(1/C_d))^2$
3.  $C_5[|Df|_0, W(f), |Sf|_0] = 10(1 + \log^{(2)}(|Df|_0))e^{(2)} \wedge (3W(f) + 2\log(\max(1, |Sf|_0))) + 4$
4.  $C_6[|Sf|_{\lceil 3\beta+3/2 \rceil}] = 1 + \log(\max(1, |Sf|_{\lceil 3\beta+3/2 \rceil}))$

## 6.4 The case $k - 2\beta - 1 \rightarrow 0$

Let  $\delta = k - 2\beta - 1$  and  $\beta > 0$ . We make a Taylor expansion with  $\delta \rightarrow 0$  (since  $k \geq 3$ , this implies automatically  $\beta > 0$ ). To estimate  $|Dh|_0$ , we successively estimate  $n_2$ ,  $n_3$ ,  $\rho/(1 - \rho)$  and  $\alpha'_{n_4}$ .

We have:

$$\theta = \frac{\delta}{4(3 + \beta)} + o(\delta)$$

Since  $\beta > 0$ , then for  $\delta$  sufficiently small,  $C_{30} = e^{\frac{2}{1-(C_{18}^f)^\theta}}$ . This makes the dependence on  $k$  and  $|Sf|_{k-3}$  disappear. Moreover,

$$C_{30} = e \wedge \left( \frac{2}{\theta \log(1/C_{18}^f) + o(\theta)} \right)$$

$$n_2 = \left( \frac{4}{(\log C_{18}^f)^2} + \frac{2 \log(1/C_d)}{\log 2} \right) \frac{1}{\theta} + o\left(\frac{1}{\theta}\right)$$

We denote  $C_{37} = \frac{4}{(\log C_{18}^f)^2}$  and  $C_{38} = \frac{2 \log(1/C_d)}{\log 2}$ .

Since

$$\beta_1 = \beta + \frac{C_{38}}{n_2} + o(\theta)$$

then

$$\log \left( \frac{\frac{k+1}{2} - \theta}{1 + \beta_1} \right) = \frac{\frac{k+1}{2} - \theta}{1 + \beta_1} - 1 + o(\theta) = \frac{\delta/2 - \theta - C_{38}/n_2}{1 + \beta_1} + o(\theta)$$

Therefore,

$$n_3 = n_2(1 + \log(1 + \beta)) \left( \frac{1 + \beta_1}{\delta/2 - \theta - C_{38}/n_2} \right) + o\left(\frac{1}{\theta^2}\right)$$

$$n_3 \leq (1 + \beta)^2 \frac{n_2^2}{\delta n_2/2 - \theta n_2 - C_{38}} + o\left(\frac{1}{\theta^2}\right)$$

Moreover,  $\theta n_2 = C_{38} + C_{37} + o(1)$  and  $\delta n_2 = 4(3 + \beta)(C_{38} + C_{37}) + o(1)$ . Therefore,

$$n_3 \leq \frac{(1 + \beta)^2(C_{38} + C_{37})}{\theta^2(4 + 2\beta)} + o\left(\frac{1}{\theta^2}\right) \quad (42)$$

Moreover,

$$\rho = \frac{\beta_1}{\frac{k-1}{2} - \theta} = \frac{\beta + \frac{C_{38}}{C_{38} + C_{37}}\theta + o(\theta)}{\beta + \delta/2 - \theta} = 1 - \frac{\theta}{\beta} \left( 2(3 + \beta) - 1 - \frac{C_{38}}{C_{38} + C_{37}} \right) + o(\theta)$$

Therefore,  $\rho/(1 - \rho) = o(1/\delta^2)$  (we recall that  $1/\delta = o(1/\delta^2)$ ).

Let

$$C_7[k, C_d, |Df|_0, W(f), |Sf|_0] = \frac{(k+5)^2(k+1)^3}{2k \log 2} \left( \frac{2 \log 2}{(\log C_{18}^f)^2} + \log(1/C_d) \right)$$

Since  $k = 2\beta + 1 + \delta$ , and by applying estimation (42), we have:

$$\alpha'_{n_4} \geq C_d \wedge \left( e \wedge \left( \frac{C_7}{\delta^2} + o\left(\frac{1}{\delta^2}\right) \right) \right)$$

Therefore,

$$|Dh|_0 \leq C_{34}|Df|_0 \wedge \left( (1/C_d) \wedge \left( e \wedge \left( \frac{1}{\delta^2} C_7 + o(1/\delta^2) \right) \right) \right)$$

$$|Dh|_0 \leq C_{34}|Df|_0 \wedge \left( e \wedge \left( e \wedge \left( \log \log(1/C_d) + \frac{1}{\delta^2} C_7 + o(1/\delta^2) \right) \right) \right)$$

Since  $\log^{(2)}(1/C_d) = o(1/\delta^2)$ , then

$$|Dh|_0 \leq C_{34}|Df|_0 \wedge \left( e \wedge \left( e \wedge \left( \frac{1}{\delta^2} C_7 + o(1/\delta^2) \right) \right) \right)$$

Likewise, since  $|\log \log |Df|_0| \leq e^{o(1/\delta^2)}$  and  $|\log C_{34}| \leq e \wedge e \wedge (o(1/\delta^2))$ . We conclude:

$$|Dh|_0 \leq e^{(3)} \wedge \left( \frac{1}{\delta^2} C_7[k, C_d, W(f), |Sf|_0] + o(1/\delta^2) \right)$$

with:

$$C_7[k, C_d, W(f), |Sf|_0] = \frac{(k+5)^2(k+1)^3}{2k \log 2} \left( \frac{2 \log 2}{(\log C_{18}^f)^2} + \log(1/C_d) \right)$$

We recall that:

$$C_{18}^f = \left( 1 + e \wedge \left( -6 \sqrt{2} e^{2W(f)} (|Sf|_0^{1/2} \vee 1) \right) \right)^{-1/2}$$

In estimations (3) and (4), three iterations of the exponential appear. This calls for explanation. A first exponential comes from the estimation  $|Df^n|_0 \leq C|Df|_0^{2/\alpha_{n_4}}$ , where  $n_4$  is the rank above which the "favorable" case always occurs. A second exponential comes from writing  $\alpha_{n_4} = \alpha_0^{\prod_{n=0}^{n_4-1} \left(\frac{1}{1-\eta_n}\right)}$ . We bound each  $\frac{1}{1-\eta_n}$  using the Diophantine condition, and a third exponential comes from the estimation  $\prod_{n \in E_2} \left(\frac{1}{1-\eta_n}\right) \leq C^{n_3-n_2}$ , where  $E_2$  is the set and  $n_3 - n_2$  is the number of "unfavorable" cases.

This number is bounded logarithmically, by  $C \log C_{30}$ . However,  $C_{30}$  is bounded by an exponential of the parameters. Indeed, when  $\delta$  is small,  $C_{30} \sim e^{\frac{1}{\delta}}$ , which gives estimation (4). Otherwise,  $C_{30} \sim C_{25}^{f,k}$ . In this case,  $C_{25}^{f,k} \sim C^k$ . Indeed, in lemma 16, we need  $k - 1$  iterations to estimate  $|D^{k-1} \log Df^p(x)|_0$  ( $p \leq q_{n+1}$ ), an estimation that, in turn, gives an estimate of  $C_{25}^{f,k}$ . This gives estimation (3). Thus, we have explained the occurrence of three exponentials in the estimates.



Since the number of "unfavorable" cases drives the dominant term of these estimates, they can be substantially improved when the "favorable" case always occurs. In remark 3.5, we make this assumption, together with the assumption  $k \geq 3\beta + 9/2$ . Thus, we can take  $\theta = 1/2$ , and a sufficient condition for the occurrence of the "favorable" case is:

$$\frac{\alpha_{n+1}}{\alpha_n} \geq C_{25}^{f,k} (C_{18}^f)^{(n-1)\frac{k}{2}} = C_8(n, k, \beta, W(f), |Sf|_{k-3})$$

which decreases geometrically with  $n$ .

We recall that:

$$C_{18}^f = \left(1 + e \wedge \left(-6\sqrt{2}e^{2W(f)}(|Sf|_0^{1/2} \vee 1)\right)\right)^{-1/2}$$

$$C_{25}^{f,k} = (k+3)^{(k+3)!} e^{(k+3)W(f)} (\max(1, |Sf|_{k-3}))^{(k+1)!}$$

We obtain the following estimation:

$$|Dh|_0 \leq \exp\left(C_9[k, W(f), |Sf|_{k-3}]^{C_{10}(\beta)}\right) |Df|_0^2$$

with:

$$C_9[k, W(f), |Sf|_{k-3}] = \max\left(e^{\frac{2}{1-(C_{18}^f)^{1/2}}}, 4C_{25}^{f,k}\right)$$

$$C_{10}[\beta] = \frac{3\beta + 1}{2}$$

Finally, note that numbers of constant type do not always satisfy (5) for any  $n$  (they only satisfy it above some rank). Moreover, there are numbers satisfying (5) that are not of constant type.

## 7 $C^k$ estimations

In this section, we compute estimates of higher order derivatives of the conjugacy  $h$  in function of bounds on the first derivative of  $h$ . We compute the values of some of the constants appearing in Yoccoz's proof [Yoc84] (we do not compute the dependency in  $k$ ). However, in order to obtain our result, we need to slightly modify the proof of one proposition (proposition 7.10). If we strictly followed Yoccoz's proof, we would find an estimate that depends on the  $C^1$ -norm of  $h$ , and on  $k, \beta, C_d, W(f), |Sf|_{k-3}, |D^{k-1} \log Df|_0$ , but this estimate would diverge as  $f$  gets closer to a rotation. Moreover, we need to elaborate on the end of his proof.

The proof has four steps. We let real numbers  $0 \leq \gamma_0 < \gamma_1 < g(\gamma_0)$ , with

$g(\gamma_0) = ((1 + \beta)\gamma_0 + k - (2 + \beta)) / (2 + \beta)$ , and we let an integer  $N$ . In the first three steps, we compute  $\|\log Df^N\|_{\gamma_1}$  in function of  $\sup_{p \geq 0} \|\log Df^p\|_{\gamma_0}$  (estimation (43)). In the first step, using convexity estimations (proposition 7.7) and a consequence of the Faa-di-Bruno formula (lemma 7.6), we establish an estimation of  $\|\log Df^q\|_{\gamma}$  for  $0 \leq \gamma \leq k - 1$  (lemma 7.8).

In the second step, we obtain an estimation of  $\|\log Df^{nq_s}\|_\gamma$ ,  $0 \leq n \leq q_{s+1}/q_s$  for  $0 \leq \gamma \leq \gamma_1$  (estimation (56)).

In the third step, we write  $N = \sum_{s=0}^S b_s q_s$ , with  $b_s$  integers satisfying  $0 \leq b_s \leq q_{s+1}/q_s$ , in order to get an estimation of  $\|\log Df^N\|_{\gamma_1}$  in function of  $\sup_{p \geq 0} \|\log Df^p\|_{\gamma_0}$ . Thus, in these three steps, the aim is to establish the following proposition:

**Proposition 7.1.** *Let  $0 \leq \gamma_0 < \gamma_1 < g(\gamma_0) = \frac{(1+\beta)\gamma_0 + k - (2+\beta)}{2+\beta}$ . We have:*

$$\|\log Df^N\|_{\gamma_1} \leq e \wedge \left( C_{69}(k, \beta) \left( C_{70}^{f,k} + 4k^2 \log(C_d^{-1}) + 24k(k+1) \sup_{p \geq 0} \|\log Df^p\|_{\gamma_0} \right)^4 \right) \quad (43)$$

The expressions of  $C_{69}$  and  $C_{70}^{f,k}$  are given page 74.

In the fourth step, we iterate this reasoning: the inductive step is given by proposition 7.1: if we have an estimate of  $\sup_{N \geq 0} \|\log Df^N\|_{\gamma_i}$ , then we can get an estimate of  $\sup_{N \geq 0} \|\log Df^N\|_{\gamma_{i+1}}$  for  $\gamma_i < \gamma_{i+1} < g(\gamma_i)$ . We can initiate the induction with  $\gamma_0 = 0$ , because we have  $C^1$  estimates. We take  $\gamma_{i+1} = \frac{1}{2}(g(\gamma_i) + \gamma_i)$  and we have:

$\lim_{i \rightarrow +\infty} \gamma_i = k - 2 - \beta$ . Thus, we can obtain an estimation of  $\|Dh\|_{k-2-\beta-\eta}$ .

In all the rest of the paper, we denote:

$$M' = \exp \left( \sup_{i \geq 0} |\log D(f^i)|_0 \right)$$

$$M = \exp \left( \sup_{i \geq 0} \|\log D(f^i)\|_{\gamma_0} \right)$$

Note that  $M \geq M' \geq 1$ .

## 7.1 Estimation of $\|\log Df^{q_s}\|_\gamma$ , $0 \leq \gamma \leq k-1$

The following lemma is a converse of the implication used in [Yoc84, p. 348], according to which if  $M_n/m_n$  is bounded, then the conjugacy of  $f$  to a rotation is  $C^1$ :

**Lemma 7.2.** *Let  $M_n = \sup_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$ ,  $m_n = \inf_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$  and*

*$M' = \exp \left( \sup_{i \geq 0} |\log D(f^i)|_0 \right)$ . Then we have the following estimation:*

$$\frac{M_n}{m_n} \leq M'$$

*Proof.* Let  $\epsilon > 0$ ,  $x, y$  such that  $M_n = |f^{q_n}(x) - x|$  and  $m_n = |f^{q_n}(y) - y|$ .

Since  $f^p(y)_{p \geq 0}$  is dense in  $\mathbb{T}^1$ , then there is a positive integer  $l$  such

that  $|f^l(y) - x| \leq \min\left(\frac{\epsilon}{|Df^{q_n}|_0}, \epsilon\right)$ .

Then we obtain:

$$|f^{q_n}(x) - x| \leq |f^{q_n}(x) - f^{q_n}(f^l(y))| + |f^l(f^{q_n}(y)) - f^l(y)| + |f^l(y) - x| \leq |Df^l|_0 |f^{q_n}(y) - y| + 2\epsilon \leq M' m_n + 2\epsilon$$

for all  $\epsilon > 0$ . Hence the lemma. □

The  $C^\gamma$ -norms, when  $\gamma$  varies in  $\mathbb{R}^+$ , are related with each other by convexity inequalities (also called interpolation inequalities):

**Proposition 7.3.** *Let  $\gamma_2, \gamma_3 \in \mathbb{R}^+$  with  $0 \leq \gamma_2 \leq \gamma_3$  and  $\gamma_3 > 0$ . For any  $\phi \in C^{\gamma_3}(\mathbb{T}^1)$ , we have:*

$$\|\phi\|_{\gamma_2} \leq C_{39}(\gamma_3) \|\phi\|_0^{\frac{\gamma_3 - \gamma_2}{\gamma_3}} \|\phi\|_{\gamma_3}^{\frac{\gamma_2}{\gamma_3}}$$

with  $C_{39}(\gamma_3) = 2^{(\lfloor \gamma_3 \rfloor + 1)^2 + 1}$ .

*Proof.* See appendix. □

Using these convexity inequalities, we establish various relations, among which is the important corollary 7.5, which relate the norms of  $\log Df^n$  and those of  $Df^n - 1$ .

For  $0 \leq \gamma_1 \leq \gamma_2 \leq k - 1$ ,  $\gamma_2 \neq 0$ , and  $n \in \mathbb{Z}$ , proposition 7.3 gives:

$$\|\log Df^n\|_{\gamma_1} \leq C_{40} \|\log Df^n\|_{\gamma_2}^{\gamma_1/\gamma_2} \tag{44}$$

$$\|Df^n - 1\|_{\gamma_1} \leq C_{41} \|Df^n - 1\|_{\gamma_2}^{\gamma_1/\gamma_2} \tag{45}$$

with  $C_{40}(\gamma_1, \gamma_2) = C_{39}(\gamma_2)(\log M')^{1-\gamma_1/\gamma_2}$  and  $C_{41}(\gamma_1, \gamma_2) = C_{39}(\gamma_2)(M' + 1)^{1-\gamma_1/\gamma_2}$ .

For  $n \geq 0$ ,  $j \in \mathbb{Z}$ , we have:

$$\|(Df^n)^j\|_0 \leq M'^{|j|} \tag{46}$$

and, for  $0 < \gamma' < 1$ , since  $|Df^n - 1|_{\gamma'} = |Df^n|_{\gamma'}$ :

$$|(Df^n)^j|_{\gamma'} \leq |j| M'^{|j|-1} |Df^n - 1|_{\gamma'} \tag{47}$$

Therefore, for  $0 \leq \gamma' < 1$ ,  $\phi \in C^{\gamma'}(\mathbb{T}^1)$ , we get, by the relations (9) and (10):

$$\|(Df^n)^j \phi\|_{\gamma'} \leq C_{42} (\|\phi\|_{\gamma'} + \|Df^n - 1\|_{\gamma'} \|\phi\|_0) \tag{48}$$

with  $C_{42}(j) = (|j| + 1) M'^{|j|}$ .

Let  $\Delta = X_1^{j_1} \dots X_l^{j_l}$  be a monomial of  $l$  variables, such that  $l = \sum_{p=1}^l p j_p \geq 1$ . Let  $0 \leq \gamma' < 1$ ,  $n \in \mathbb{Z}$ . We estimate  $\|\Delta\|_{\gamma'}$  when  $X_i = D^i \log Df^n$  or when  $X_i = D^{i+1} f^n$ , supposing that  $l + \gamma' \leq k-1$ .

The relations (9) and (10) allow estimating  $\|\Delta\|_{\gamma'}$  by a sum of less than  $2^l$  terms of the form  $\|X_p\|_{\gamma'} \|\Delta/X_p\|_0$ ,  $1 \leq p \leq l$ ,  $j_p \neq 0$ . By relation (44), we have:

$$\begin{aligned} \|D^p \log Df^n\|_{\gamma'} &\leq C_{43} \|\log Df^n\|_{l+\gamma'}^{(p+\gamma')/(l+\gamma')} \\ \|D^{p+1} f^n\|_{\gamma'} &\leq C_{44} \|Df^n - 1\|_{l+\gamma'}^{(p+\gamma')/(l+\gamma')} \\ \left\| \frac{\Delta(D \log Df^n, \dots, D^l \log Df^n)}{D^p \log Df^n} \right\|_0 &\leq C_{45} \|\log Df^n\|_{l+\gamma'}^{(l-p)/(l+\gamma')} \\ \left\| \frac{\Delta(D^2 f^n, \dots, D^{l+1} f^n)}{D^{p+1} f^n} \right\|_0 &\leq C_{46} \|Df^n - 1\|_{l+\gamma'}^{(l-p)/(l+\gamma')} \end{aligned}$$

with  $C_{43}(p, l, \gamma') = C_{40}(p + \gamma', l + \gamma')$ ;  $C_{44}(p, l, \gamma') = C_{41}(p + \gamma', l + \gamma')$ ;

$C_{45}(p, l, \gamma') = (C_{40}(1, l + \gamma'))^{l-1}$ ;

$C_{46}(p, l, \gamma') = (C_{41}(1, l + \gamma'))^{l-1}$

*Scheme of the proof.* The first two estimates are straightforward. For the third estimate, we write

$$\frac{\Delta(D \log Df^n, \dots, D^l \log Df^n)}{D^p \log Df^n} = (D \log Df^n)^{j_1} \dots (D^p \log Df^n)^{j_{p-1}} \dots (D^l \log Df^n)^{j_l}$$

we apply estimation (44) to each  $D^i \log Df^n$  and we use that  $\sum j_k \leq l$ .

The proof of the fourth estimate is analogous, by noting that  $D^i(Df^n - 1) = D^{i+1} f^n$ , for  $i \geq 1$ .

□

Therefore, when  $X_i = D^i \log Df^n$ , we get:

$$\|\Delta\|_{\gamma'} \leq C_{47} \|\log Df^n\|_{l+\gamma'} \quad (49)$$

and when  $X_i = D^{i+1} f^n$ ,

$$\|\Delta\|_{\gamma'} \leq C_{48} \|Df^n - 1\|_{l+\gamma'} \quad (50)$$

with  $C_{47}(l) = 2^l \max_{1 \leq p \leq l} \sup_{0 \leq \gamma' < 1} C_{43}(p, l, \gamma') C_{45}(p, l, \gamma')$  and

$C_{48}(l) = 2^l \max_{1 \leq p \leq l} \sup_{0 \leq \gamma' < 1} C_{44}(p + \gamma, l + \gamma') C_{46}(p + \gamma', l + \gamma')$ .

Using appendix 8.4, this allows obtaining the following lemma:

**Lemma 7.4.** *Let  $P$  be one of the polynomials of appendix 8.4.  $P$  is a polynomial of  $l$  variables  $X_1, \dots, X_b$ , homogeneous of weight  $l$  if  $X_i$  has a weight of  $i$ . For all  $n \in \mathbb{Z}$ , all  $0 \leq \gamma' < 1$ , we have:*

$$\begin{aligned} \|P(D \log Df^n, \dots, D^l \log Df^n)\|_{\gamma'} &\leq C_{49} \|\log Df^n\|_{l+\gamma'} \\ \left\| P\left(\frac{D^2 f^n}{Df^n}, \dots, \frac{D^{l+1} f^n}{Df^n}\right) \right\|_{\gamma'} &\leq C_{50} \|Df^n - 1\|_{l+\gamma'} \end{aligned}$$

with  $C_{49}(l) = (4(l+1))^{4(l+1)} C_{47}(l)$  and

$$C_{50}(l) = (4(l+1))^{4(l+1)} C_{42}(-l) C_{48}(l) (1 + C_{41}(l, l + \gamma') C_{41}(\gamma', l + \gamma'))$$

*Scheme of the proof.* The first estimate comes from the preceding discussion. For the second estimate, we write a monomial of  $P$  as:

$$\left(\frac{D^2 f^n}{Df^n}\right)^{j_1} \dots \left(\frac{D^l f^n}{Df^n}\right)^{j_l} = (D^2 f^n)^{j_1} \dots (D^l f^n)^{j_l} (Df^n)^{-\sum j_k}$$

We apply estimate (48), (50), and estimate (45) twice.

□

**Corollary 7.5.** *For  $n \in \mathbb{Z}$ ,  $0 \leq \gamma \leq k-1$ ,  $\gamma = \lfloor \gamma \rfloor + \gamma'$ ,  $0 \leq \gamma' < 1$ . we have:*

$$C_{51}^{-1} \|Df^n - 1\|_{\gamma} \leq \|\log Df^n\|_{\gamma} \leq C_{51} \|Df^n - 1\|_{\gamma}$$

with  $C_{51}(\gamma) = M$  if  $0 \leq \gamma < 1$  and

$$C_{51}(\gamma) = [C_{49}(\lfloor \gamma \rfloor) C_{42}(1) (1 + M C_{40}(\lfloor \gamma \rfloor, \gamma) C_{40}(\gamma', \gamma))] \vee C_{50}(\lfloor \gamma \rfloor) \text{ if } \gamma \geq 1.$$

*Scheme of the proof.* For  $0 \leq \gamma < 1$ , we prove the estimates directly, using that  $\log x \leq x - 1$ .

When  $\gamma \geq 1$ , for the right-hand side of the estimation, we use formula (78) in appendix 8.4 and the second estimate of lemma 7.4.

For the left-hand side, we apply formula (77) in appendix 8.4, the first estimate of lemma 7.4, relation (48) with  $\phi = D^{\lfloor \gamma \rfloor + 1} f^n / Df^n$  and  $j = 1$ , the left-hand side of this estimate of corollary 7.5 with  $\gamma < 1$ , and relation (44) twice.

□

Using mainly the Faa-d-Bruno formula, we have the lemma [Yoc84, p. 350]:

**Lemma 7.6.** *Let  $\gamma_0 \geq \gamma \geq 0$ ,  $\psi \in D^{\max(1, \gamma)}(\mathbb{T}^1)$ ,  $\phi \in C^{\gamma}(\mathbb{T}^1)$ . We have:*

$$\|\phi \circ \psi\|_{\gamma} \leq C_{52} \|\phi\|_{\gamma}$$

with  $C_{52}(\gamma, \psi) = \|\psi\|_{\gamma}$  for  $0 \leq \gamma < 1$ , and  $C_{52}(\gamma, \psi) = ((2\lfloor \gamma \rfloor)!)^2 \|D\psi\|_{\gamma-1}^{\lfloor \gamma \rfloor + 1}$  for  $\gamma \geq 1$ .

When  $\psi = f^n$  for some integer  $n$ , or when  $\psi = h^{-1}$ , where  $h$  is the conjugacy of  $f$  to a rotation, we note:

$$C_{52}(\gamma, \psi) = C_{52}(\gamma) = M^\gamma \text{ if } 0 \leq \gamma < 1, \text{ and}$$

$$C_{52}(\gamma, \psi) = C_{52}(\gamma) = ((2\lfloor \gamma \rfloor)!)^2 M^{\lfloor \gamma \rfloor + 1} \text{ if } \gamma \geq 1.$$

*Proof.* See appendix 8.3. □

We have:  $f^n = hR_{n\alpha}h^{-1}$ . We apply lemma 7.6 with  $\psi = h^{-1}$  and  $\phi = hR_{n\alpha} - h - n\alpha$ . To estimate  $\|hR_{n\alpha} - h - n\alpha\|_\gamma$ , we use the  $C^\gamma$ -norm of  $Dh$ . We get:

**Proposition 7.7.** *For  $n \in \mathbb{Z}$ ,  $0 \leq \gamma \leq \gamma_0$ , we have:*

$$\|f^n - id - n\alpha\|_\gamma \leq C_{53}\|n\alpha\|$$

$$\text{with } C_{53}(\gamma) = 2MC_{52}(\gamma).$$

Let  $\alpha_n = (-1)^n(q_n\alpha - p_n)$  and let  $\Delta_s = \|D^{k-1} \log Df^{q_s}\|_0 + \alpha_s$  (the role of this additional  $\alpha_s$  is explained at the end of the proof of lemma 7.12). We could also have taken  $\Delta_s = \|D^{k-1} \log Df^{q_s}\|_0 \vee \alpha_s$ ). By applying lemma 6.4, and since  $M_{s-1} \leq M'\alpha_{s-1}$ ,  $1/m_{s-1} \leq M'/\alpha_{s-1}$ , and  $\alpha_{s-1} \leq 1/q_s$ , then we have:

$$\Delta_s \leq (C_{24}^{f,k} M'^{\frac{3}{2}(k-1)} + 1)q_s^{(k-1)/2} \quad (51)$$

Using corollary 7.5, convexity inequalities (proposition 7.3), proposition 7.7, and corollary 7.5 again, we obtain the following lemma:

**Lemma 7.8.** *Let  $\gamma \in [0, k-1]$  and  $s \geq 0$ . We have:*

$$\|\log Df^{q_s}\|_\gamma \leq C_{54}(\gamma, \gamma_0)q_{s+1}^{-1}(q_{s+1}\Delta_s)^{\max(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0})}$$

$$\text{with } C_{54}(\gamma, \gamma_0) = C_{39}(k)C_{51}(\gamma)^{1+\max(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0})}C_{53}(\gamma_0)^{1-\max(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0})}.$$

We make a remark on the method and notation: in this lemma 7.8, we estimate the  $C^\gamma$ -norm for  $0 \leq \gamma \leq k-1$ , instead of only estimating the  $C^{\gamma_1}$ -norm, because of two reasons: first, this lemma is used to obtain lemma 7.9, in which we need an estimation of all the norms of order  $\gamma \leq k-1$ . Second, in the proof of proposition 7.10, we need an estimate of  $\|\log Df^{q_s}\|_{k-1}$ .

## 7.2 Estimation of $\|\log Df^{nq_s}\|_\gamma$ , $0 \leq n \leq q_{s+1}/q_s$ , $0 \leq \gamma \leq k-1$

We use lemma 7.8 to estimate  $\|\log Df^{nq_s}\|_\gamma$ ,  $0 \leq \gamma \leq k-1$  (lemma 7.9) and second, we bootstrap this estimate (lemma 7.12). This bootstrapping allows getting a higher degree of differentiability  $\gamma_1$  at the end (see estimation (57)).

The Diophantine condition on  $\alpha$  implies  $q_{s+1} \leq C_d^{-1} q_s^{1+\beta}$ . Therefore, by applying estimation (51), we get:

$$(\Delta_s q_{s+1})^{1/k} q_s^{-1} \leq C_{55}(0) q_s^{-\epsilon} \quad (52)$$

With  $\epsilon = \frac{1}{2} - \frac{1+2\beta}{2k} > 0$  and  $C_{55}(0) = \left[ (C_{24}^{f,k} M^{\frac{3}{2}(k-1)} + 1) C_d^{-1} \right]^{1/k}$

The preceding estimates give the lemma:

**Lemma 7.9.** *Let  $\gamma \in [0, k-1]$ . For  $s \geq 0$ ,  $0 \leq n \leq q_{s+1}/q_s$ , we have:*

$$\| \log Df^{nq_s} \|_\gamma \leq C_{56}(\gamma, 0) q_s^{-1} (q_{s+1} \Delta_s)^{(\gamma+1)/k}$$

where, if  $0 \leq \gamma < 1$ ,  $C_{56}(\gamma, 0) = C_{54}(\gamma, 0)$ , and if  $\gamma \geq 1$ ,  $C_{56}(\gamma, 0)$  is defined inductively by:

$$C_{56}(\gamma, 0) = C_{56}(\gamma-1, 0) \lfloor \gamma \rfloor^2 C_{55}(0) C_{49}(\lfloor \gamma \rfloor - 1) C_{54}(\gamma, 0) M^\gamma [2 + C_{40}(\gamma', \gamma) C_{40}(\lfloor \gamma \rfloor, \gamma)]$$

*Scheme of the proof.* This lemma is shown by induction on  $r = \lfloor \gamma \rfloor$ . If  $r = 0$ , we write  $\log Df^{nq_s} = \sum_{i=0}^{n-1} \log Df^{q_s} \circ f^i$  and we apply lemma 7.8.

Suppose the lemma holds for  $r-1 + \gamma'$ , with  $0 \leq \gamma' < 1$ . We have, using the expression (80) in appendix 8.4, and using estimations (9) and (10):

$$\| D^r \log Df^{nq_s} \|_{\gamma'} \leq \sum_{l=0}^{r-1} \sum_{i=0}^{n-1} (A_{i,l} + B_{i,l} + C_{i,l}) \quad (53)$$

with:

$$A_{i,l} = \| D^{r-l} \log Df^{q_s} \circ f^{iq_s} \|_{\gamma'} \| (Df^{iq_s})^{r-l} \|_0 \| E_l^r \|_0$$

$$B_{i,l} = \| D^{r-l} \log Df^{q_s} \circ f^{iq_s} \|_0 \| (Df^{iq_s})^{r-l} \|_{\gamma'} \| E_l^r \|_0$$

$$C_{i,l} = \| D^{r-l} \log Df^{q_s} \circ f^{iq_s} \|_0 \| (Df^{iq_s})^{r-l} \|_0 \| E_l^r \|_{\gamma'}$$

$$E_l^r = E_l^r(D \log Df^{iq_s}, \dots, D^l \log Df^{iq_s})$$

We estimate  $E_l^r$  with lemma 7.4 (with the polynomial  $P = E_l^r$ ), with (44) (for  $B_{i,l}$ ) and with the induction assumption. We estimate  $\| D^{r-l} \log Df^{q_s} \circ f^{iq_s} \|_{\tilde{\gamma}}$ ,  $\tilde{\gamma} = 0$  or  $\gamma'$ , by applying lemma 7.6 with  $\phi = D^{r-l} \log Df^{q_s}$  and  $\psi = f^{iq_s}$ , and by applying lemma 7.8. We estimate  $\| (Df^{iq_s})^{r-l} \|_0$  with (46). For  $\| (Df^{iq_s})^{r-l} \|_{\gamma'}$ , we apply (47), corollary 7.5, (44) and the induction assumption. We get:

$$A_{i,l} \leq C_{56}(l, 0) C_{49}(l) C_{54}(\gamma' + r - l, 0) M^{\gamma' + r - l} q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{\frac{r-l+\gamma'+1}{k} + \frac{l+1}{k}}$$

$$B_{i,l} \leq C_{54}(r-l, 0)C_{49}(l)(r-l)M^{r-l}C_{40}(l, l+\gamma')C_{40}(\gamma', l+\gamma')C_{56}(l+\gamma', 0)q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\frac{r-l+1}{k} + \frac{l+\gamma'+1}{k}}$$

$$C_{i,l} \leq M^{r-l}C_{49}(l)C_{54}(r-l, 0)C_{56}(l+\gamma', 0)q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\frac{r-l+1}{k} + \frac{l+\gamma'+1}{k}}$$

Thus, we have:

$$A_{i,l} + B_{i,l} + C_{i,l} \leq C_{57}(l, \gamma', 0)q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{\frac{\gamma+1}{k} + 1/k}$$

with:

$$C_{57}(l, \gamma', 0) = C_{56}(l+\gamma', 0)C_{49}(l)C_{54}(r-l+\gamma', 0)M^{r-l} \left[ M^{\gamma'} + (r-l)C_{40}(\gamma', l+\gamma')C_{40}(l, l+\gamma') + 1 \right]$$

We conclude using estimation (52), and using the fact that the sum (53) has

$rn \leq \lfloor \gamma \rfloor q_{s+1}/q_s$  terms.

□

By applying this lemma 7.9, together with estimate (52), lemma 7.8 and lemma 7.4, we get the proposition [Yoc84, p.355]:

**Proposition 7.10.** *The sequence  $(\Delta_s/q_s)_{s \geq 0}$  is bounded by  $C_{58}$ .*

$C_{58}$  is defined by the following:

$$\begin{aligned} C_{59} &= (k-2)M^{k-1}C_{49}(k-2)C_{56}(k-2, 0)C_{55}(0)C_{54}(k-1, 0); \\ C_{58} &= 2C_{24}^{f,k}M^{k-1} \prod_{s=0}^{\infty} \left( 1 + \frac{5C_{59}}{q_s^\epsilon} \right) \end{aligned}$$

*Proof.* We slightly modify Yoccoz's proof. Let  $\Delta'_{-1} = 0$  and, for  $s \geq 0$ :

$$\Delta'_s = \sup\{|D^{k-1} \log Df^{q_t} \circ f^m (Df^m)^{k-1}|_0, 0 \leq t \leq s, m \geq 0\}$$

For  $s \geq 0$ , we have:  $\Delta_s \leq \Delta'_s + \alpha_s$  (This implies  $\Delta_s \leq C\Delta'_s$  when  $f$  is not a rotation, but contrary to Yoccoz's proof, we do not use this estimate, because the constant  $C$  is of the form  $C = 1 + \frac{M^{k-1}}{|D^{k-1} \log Df|_0}$ , which diverges as  $f$  gets closer to a rotation). We compute a bound on  $(\Delta'_s + \alpha_s)/q_s$ .

Let  $s \geq 0$  (this is another difference with Yoccoz's proof, which only considers  $s \geq 1$ ). We have:  $q_{s+1} = a_{s+1}q_s + q_{s-1}$  (we recall that  $q_{-1} = 0$ ). Using formula (79) in appendix 8.4 with  $g = f^{q_{s-1}}$  and  $h = f^{a_{s+1}q_s}$ , we can write:



$$(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1} = X' + Y' + Z'$$

with:

$$X' = (D^{k-1} \log Df^{q_{s-1}} \circ f^{a_{s+1}q_s+m})(Df^{a_{s+1}q_s} \circ f^m)^{k-1}(Df^m)^{k-1}$$

$$Y' = D^{k-1} \log Df^{a_{s+1}q_s} \circ f^m (Df^m)^{k-1}$$

$$Z' = \sum_{l=1}^{k-2} (D^{k-1-l} \log Df^{q_{s-1}} \circ f^{a_{s+1}q_s+m})(Df^{a_{s+1}q_s} \circ f^m)^{k-1-l}(Df^m)^{k-1} \\ G_l^{k-1}(D \log Df^{a_{s+1}q_s} \circ f^m, \dots, D^l \log Df^{a_{s+1}q_s} \circ f^m)$$

We have:

$$|X'|_0 \leq \Delta'_{s-1}$$

Using formula (80) in appendix 8.4 with  $g = f^{q_s}$ , we have:

$$Y' = \sum_{l=0}^{k-2} \sum_{n=0}^{a_{s+1}-1} (D^{k-1-l} \log Df^{q_s} \circ f^{nq_s+m})(Df^{nq_s+m})^{k-1-l} E_l^{k-1}(D \log Df^{nq_s}, \dots, D^l \log Df^{nq_s}) \circ f^m (Df^m)^l \\ = \sum_{l=0}^{k-2} Y'_l$$

(with the convention  $E_0^{k-1} = 1$ ). We have:  $|Y'_0|_0 \leq a_{s+1} \Delta'_s$ .

For  $l \geq 1$ , we estimate  $E_l^{k-1}(D \log Df^{nq_s}, \dots, D^l \log Df^{nq_s}) \circ f^m (Df^m)^l$  using lemma 7.6 (with  $\psi = f^m$  and  $\gamma = 0$ ), lemma 7.4 (with  $P = E_l^{k-1}$ ), lemma 7.9 (with  $\gamma = l$ ) and estimation (52). We get:

$$|E_l^{k-1}(D \log Df^{nq_s}, \dots, D^l \log Df^{nq_s}) \circ f^m (Df^m)^l|_0 \leq M^l C_{49}(l) C_{56}(l, 0) C_{55}(0) (\Delta_s q_{s+1})^{l/k} q_s^{-\epsilon}$$

By applying lemma 7.8 (with  $\gamma = k - 1 - l$  and  $\gamma_0 = 0$ ), and using that  $\Delta_s \leq \Delta'_s + \alpha_s$ , we get:

$$|Y'_l|_0 \leq a_{s+1} (\Delta'_s + \alpha_s) M^{k-1} C_{54}(k-1-l, 0) C_{49}(l) C_{56}(l, 0) C_{55}(0) q_s^{-\epsilon}$$

Therefore,

$$|Y'|_0 \leq a_{s+1} \Delta'_s + a_{s+1} (\Delta'_s + \alpha_s) C_{59} q_s^{-\epsilon}$$

Likewise, we can show that, for  $s \geq 1$ :

$$|Z'|_0 \leq C_{59} q_s^{-\epsilon} q_s^{-1} (q_s \Delta_{s-1})^{\frac{k-l}{k}} (q_{s+1} \Delta_s)^{l/k}$$

(Yoccoz concludes the estimation of  $|Z'|_0$  here, using the fact that  $q_s^{1-l/k} \leq q_{s+1}^{1-l/k}$  and using the fact that  $\Delta_t \leq C\Delta'_t$ ,  $t = s-1, s$ . We don't use these facts.)

Since  $\Delta_t \leq \Delta'_t + \alpha_t$ ,  $t = s-1, s$ , we get:

$$\begin{aligned} |Z'|_0 &\leq C_{59} q_s^{-\epsilon} \left(\frac{q_{s+1}}{q_s}\right)^{l/k} (\Delta'_{s-1} + \alpha_{s-1})^{1-l/k} (\Delta'_s + \alpha_s)^{l/k} \\ |Z'|_0 &\leq C_{59} q_s^{-\epsilon} \frac{q_{s+1}}{q_s} (\Delta'_s + \alpha_s) \left( \left(1 + \frac{\alpha_{s-1} - \alpha_s}{\Delta'_s + \alpha_s}\right) \left(\frac{q_s}{q_{s+1}}\right) \right)^{1-l/k} \end{aligned}$$

Since  $\Delta'_s \geq 0$ , and since  $a_{s+1} \leq q_{s+1}/q_s \leq 2a_{s+1}$  and  $\alpha_{s-1} \leq 2a_{s+1}\alpha_s$ , we get:

$$|Z'|_0 \leq 4C_{59} q_s^{-\epsilon} a_{s+1} (\Delta'_s + \alpha_s)$$

If  $s = 0$ ,  $Z' = 0$ . This estimate still holds.

Therefore, for  $s \geq 0$ ,

$$\alpha_{s+1} + |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \leq \alpha_{s+1} + \Delta'_{s-1} + a_{s+1}\Delta'_s + a_{s+1}(\Delta'_s + \alpha_s)5C_{59}q_s^{-\epsilon} \quad (54)$$

$$\alpha_{s+1} + |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \leq \alpha_{s+1} - a_{s+1}\alpha_s + \Delta'_{s-1} + a_{s+1}(\Delta'_s + \alpha_s)(1 + 5C_{59}q_s^{-\epsilon})$$

Moreover, we have:  $\alpha_{s-1} = a_{s+1}\alpha_s + \alpha_{s+1}$ . Therefore, for  $s \geq 1$ , since  $\alpha_{s+1} < \frac{1}{2}\alpha_{s-1}$ , then

$$\alpha_{s+1} - a_{s+1}\alpha_s = 2\alpha_{s+1} - \alpha_{s-1} < 0 \leq \alpha_{s-1}$$

Therefore,

$$\alpha_{s+1} + |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \leq \max_{t=s-1,s} \frac{\alpha_t + \Delta'_t}{q_t} (q_{s-1} + a_{s+1}q_s)(1 + 5C_{59}q_s^{-\epsilon})$$

Since  $q_{s-1} + a_{s+1}q_s = q_{s+1}$ , we get:

$$\frac{\alpha_{s+1} + |(D^{k-1} \log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0}{q_{s+1}} \leq \max_{t=s-1,s} \frac{\alpha_t + \Delta'_t}{q_t} (1 + 5C_{59}q_s^{-\epsilon})$$

If  $s = 0$ , we have:

$$\frac{\alpha_1 + \Delta'_1}{q_1} \leq \frac{\alpha_0 + \Delta'_0}{q_0} (1 + 5C_{59})$$

Let  $\theta_s = \max_{0 \leq t \leq s} \frac{\alpha_t + \Delta'_t}{q_t}$ . The preceding estimates give:

$$\theta_{s+1} \leq \theta_s (1 + 5C_{59}q_s^{-\epsilon})$$

Moreover,

$$\left( \frac{\Delta'_0 + \alpha_0}{q_0} \right) \leq 1 + M^{k-1} |D^{k-1} \log Df|_0$$

Therefore, for any  $s \geq 0$ ,

$$\frac{\Delta_s}{q_s} \leq \left( 1 + M^{k-1} |D^{k-1} \log Df|_0 \right) \prod_{s=0}^{+\infty} (1 + 5C_{59}q_s^{-\epsilon})$$

To conclude, we apply the claim:

**Claim 7.11.** *Let  $C_{24}^{f,k}$  defined in lemma 6.4. For any  $k \geq 3$ , we have:*

$$|D^{k-1} \log Df|_0 \leq \tilde{C}_{24}(k, |Sf|_{k-3}) \leq C_{24}^{f,k}$$

*Proof.* First, we recall the observation (see e.g. [Yoc84]) that if  $x_0$  is a point where  $(D \log Df)^2$  is maximal, then we have:

$$|Sf|_0 \geq |Sf(x_0)| = \left| D^2 \log Df(x_0) - \frac{1}{2} (D \log Df(x_0))^2 \right| = \left| \frac{1}{2} (D \log Df(x_0))^2 \right| = \left| \frac{1}{2} (D \log Df)^2 \right|_0$$

To prove the claim, we proceed by induction on  $k$ , using the fact that

$$|D \log Df|_0 \leq \sqrt{2|Sf|_0} \text{ and by applying formula (71) in appendix 8.1.}$$

If  $k = 3$ ,

$$|D^2 \log Df|_0 \leq |Sf|_0 + \frac{1}{2} |(D \log Df)^2|_0 \leq 2|Sf|_0$$

Suppose the estimate holds for all  $r \leq k$ . By formula (71), we have:

$$D^k \log Df = D^{k-2} Sf - G_k(D \log Df, \dots, D^{k-1} \log Df)$$

As in the proof of lemma 6.4 (see appendix 8.1), we have:

$$|G_k(D \log Df, \dots, D^{k-1} \log Df)| \leq \frac{(2(k-1))!}{2k} \left( \tilde{C}_{24}(k, |Sf|_{k-3}) \right)^k$$

We conclude as in the proof of lemma 6.4.

Let us make a remark: by using lemma 7.4, we can improve this estimate. However, lemma 7.4 cannot be used to improve the estimate of lemma 6.4, because it is a pointwise estimate: an estimate of  $|D^{k-1} \log Df^{q_s}(x)|$  in function of  $M_n^{1/2}/m_n(x)$ . If we only needed an estimate of  $|D^{k-1} \log Df^{q_s}|_0$  in function of  $M_n^{1/2}/m_n$ , this improvement would be possible.

□

□

With proposition 7.10, and by using the Diophantine condition  $q_{s+1} \leq C_d^{-1} q_s^{1+\beta}$ , we can generalize estimation (52) and lemma 7.9, for  $\gamma_0 > 0$ . The generalization of estimation (52) is:

$$(\Delta_s q_{s+1})^{1/(k-\gamma_0)} q_s^{-1} \leq C_{55}(\gamma_0) q_s^{\frac{\beta+2}{k-\gamma_0}-1} \quad (55)$$

$$\text{with } C_{55}(\gamma_0) = C_{58}^{\frac{1}{k-\gamma_0}} C_d^{\frac{-1}{k-\gamma_0}}.$$

The generalization of lemma 7.9 is:

**Lemma 7.12.** *Let  $\gamma \in [0, k-1]$ . For  $s \geq 0$ ,  $0 \leq n \leq q_{s+1}/q_s$ , we have:*

$$\|\log Df^{nq_s}\|_\gamma \leq C_{56}(\gamma, \gamma_0) q_s^{-1} (q_{s+1} \Delta_s)^{\frac{\gamma+1-\gamma_0}{k-\gamma_0} \vee 0}$$

where, if  $0 \leq \gamma < 1$ ,  $C_{56}(\gamma, \gamma_0) = \max(C_{51}(\gamma)C_{53}(\gamma+1), C_{54}(\gamma, \gamma_0))$ , and if  $\gamma \geq 1$ ,  $C_{56}(\gamma, 0)$  is defined inductively by:

$$C_{56}(\gamma, \gamma_0) = C_{56}(\gamma-1, \gamma_0) \lfloor \gamma \rfloor C_{55}(\gamma_0) C_{49}(\lfloor \gamma \rfloor - 1)$$

$$C_{56}(\gamma, \gamma_0) = 2^{1+0\vee \frac{\gamma+1-\gamma_0}{k-\gamma_0}} C_{56}(\gamma-1, \gamma_0) \lfloor \gamma \rfloor^2 C_{55}(\gamma_0) C_{49}(\lfloor \gamma \rfloor - 1) C_{54}(\gamma, \gamma_0) M^\gamma [2 + C_{40}(\gamma', \gamma) C_{40}(\lfloor \gamma \rfloor, \gamma)]$$

*Remark 7.13.* When  $\gamma \geq 1$ , the definitions of  $C_{56}(\gamma, \gamma_0)$  are analogous to those of  $C_{56}(\gamma, 0)$  given in lemma 7.9, by replacing 0 with  $\gamma_0$ , by a factor  $2^{1+0\vee \frac{\gamma+1-\gamma_0}{k-\gamma_0}}$ .

*Scheme of the proof.* We give the scheme of the proof in order to explain the additional  $\alpha_s$  in the definition of  $\Delta_s$  (this additional  $\alpha_s$  makes necessary our modification of Yoccoz's proof of proposition 7.10).

If  $\gamma_0 - 1 \leq \gamma < 1$ , we proceed as in lemma 7.9. If  $\gamma \leq \gamma_0 - 1$  and  $\gamma < 1$ , we apply corollary 7.5 and proposition 7.7. The induction step is analogous to the proof of lemma 7.9, except the end: indeed, by proceeding as in lemma 7.9, we have:

$$A_{i,l} \leq C_{56}(l, \gamma_0) C_{49}(l) C_{54}(\gamma' + r - l, \gamma_0) M^{\gamma' + r - l} q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{0\vee \frac{r-l+\gamma'+1-\gamma_0}{k-\gamma_0} + 0\vee \frac{l+1-\gamma_0}{k-\gamma_0}}$$

$$B_{i,l} \leq C_{54}(r-l, 0)C_{49}(l)(r-l)M^{r-l}C_{40}(l, l+\gamma')C_{40}(\gamma', l+\gamma')C_{56}(l+\gamma', 0)q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{0 \vee \frac{r-l+1-\gamma_0}{k-\gamma_0} + 0 \vee \frac{l+\gamma'+1-\gamma_0}{k-\gamma_0}}$$

$$C_{i,l} \leq M^{r-l}C_{49}(l)C_{54}(r-l, 0)C_{56}(l+\gamma', 0)q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{0 \vee \frac{r-l+1-\gamma_0}{k-\gamma_0} + 0 \vee \frac{l+\gamma'+1-\gamma_0}{k-\gamma_0}}$$

We have:

$$\begin{aligned} & \left( 0 \vee \frac{r-l+1-\gamma_0}{k-\gamma_0} + 0 \vee \frac{l+\gamma'+1-\gamma_0}{k-\gamma_0} \right) \vee \left( 0 \vee \frac{r-l+\gamma'+1-\gamma_0}{k-\gamma_0} + 0 \vee \frac{l+1-\gamma_0}{k-\gamma_0} \right) \\ & \leq 0 \vee \frac{\gamma+1-\gamma_0}{k-\gamma_0} + \frac{1}{k-\gamma_0} \end{aligned}$$

Moreover, since  $2q_{s+1}\Delta_s \geq 2q_{s+1}\alpha_s \geq 1$ , then

$$A_{i,l} + B_{i,l} + C_{i,l} \leq 2^{\frac{1}{k-\gamma_0} + 0 \vee \frac{\gamma+1-\gamma_0}{k-\gamma_0}} C_{57}(l, \gamma', \gamma_0)q_{s+1}^{-1}q_s^{-1}(\Delta_s q_{s+1})^{0 \vee \frac{\gamma+1-\gamma_0}{k-\gamma_0} + \frac{1}{k-\gamma_0}}$$

with:

$$C_{57}(l, \gamma', \gamma_0) = C_{56}(l+\gamma', \gamma_0)C_{49}(l)C_{54}(r-l+\gamma', \gamma_0)M^{r-l} \left[ M^{\gamma'} + (r-l)C_{40}(\gamma', l+\gamma')C_{40}(l, l+\gamma') + 1 \right]$$

(this is why we define  $\Delta_s = |D^{k-1} \log Df^{q_s}|_0 + \alpha_s$ . If we defined  $\Delta_s = |D^{k-1} \log Df^{q_s}|_0$  and if  $|D^{k-1} \log Df^{q_s}|_0$  was too small, we could not do this estimate).  $\square$

By using estimation (55) and lemma 7.12, we obtain, for  $0 \leq n \leq (q_{s+1})/q_s$ , and  $0 \leq \gamma \leq \gamma_1$  [Yoc84, p.357]:

$$\|\log Df^{nq_s}\|_\gamma \leq C_{60}(\gamma, \gamma_0)q_s^{\rho(\gamma, \gamma_0)} \quad (56)$$

with

$$\rho(\gamma, \gamma_0) = \frac{(2+\beta)(0 \vee (\gamma+1-\gamma_0))}{k-\gamma_0} - 1$$

and

$$C_{60}(\gamma, \gamma_0) = C_{56}(\gamma, \gamma_0) \left( C_d^{-1} C_{58} \right)^{\frac{\gamma+1-\gamma_0}{k-\gamma_0} \vee 0}$$

Notice that for any

$$\gamma_1 < g(\gamma_0) = \frac{(1+\beta)\gamma_0 + k - (2+\beta)}{2+\beta} \quad (57)$$

we have  $\rho(\gamma_1, \gamma_0) < 0$  (we will take  $\gamma_1 = \frac{1}{2}(g(\gamma_0) + \gamma_0)$ ).

This implies  $\sum_{s \geq 0} q_s^{\rho(\gamma_1, \gamma_0)} < +\infty$ , which will allow estimating  $\|\log Df^N\|_{\gamma_1}$ , as we will see in the next subsection.

A remark on the method and notation: we establish estimate (56) for any  $0 \leq \gamma \leq \gamma_1$  (and not just for  $\gamma_1$ ) because we need it for the estimate of the quantity  $Z$  defined below.

### 7.3 Estimation of $\|\log Df^N\|_{\gamma_1}$

**Proposition 7.14.** *Let  $N$  be an integer and let us write  $\gamma_1 = r + \gamma'_1$ , with  $0 \leq \gamma'_1 < 1$  and  $r$  integer. We have:*

$$\|\log Df^N\|_{\gamma_1} \leq C_{60}(\gamma_1, \gamma_0) \prod_{s=1}^{\infty} \left( 1 + \frac{C_{61}(\gamma_1, \gamma_0) + C_{63}(\gamma_1, \gamma_0)}{q_s^{-\rho(\gamma_1, \gamma_0)}} \right) = C_{64} \quad (58)$$

with:

$$C_{61}(\gamma_1, \gamma_0) = C_{42}(r)C_{52}(\gamma'_1)C_{60}(\gamma_1, \gamma_0)C_{51}(\gamma'_1)$$

$$C_{63}(\gamma_1, \gamma_0) = (r-1)C_{61}(\gamma_1, \gamma_0)C_{49}(r-1)(2 + C_{40}(\gamma'_1, \gamma_1)C_{40}(r, \gamma_1))$$

*Scheme of the proof.* We write  $N = \sum_{s=0}^S b_s q_s$  with  $0 \leq b_s \leq \frac{q_{s+1}}{q_s}$  and  $b_s$  integer. Let  $N_s = \sum_{t=0}^s b_t q_t$  for  $0 \leq s \leq S$ . Moreover, let us write  $\gamma_1 = r + \gamma'_1$ , with  $0 \leq \gamma'_1 < 1$  and  $r$  integer. By formula (79) in appendix 8.4, we can write  $D^r \log Df^{N_s} = X + Y + Z$  with:

$$\begin{aligned} X &= (D^r \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^r; \\ Y &= D^r \log Df^{N_{s-1}}; \\ Z &= \sum_{l=1}^{r-1} (D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l} G_l^r; \\ G_l^r &= G_l^r(D \log Df^{N_{s-1}}, \dots, D^l \log Df^{N_{s-1}}) \end{aligned}$$

We successively estimate  $X$  and  $Z$ . For  $X$ , we use estimate (48), corollary 7.5 and lemma 7.6 with  $\phi = D^r \log Df^{b_s q_s}$  and  $\psi = f^{N_{s-1}}$ . We also use estimate (56), and the fact that  $q_s^{\rho(r, \gamma_0)} \leq q_s^{\rho(\gamma_1, \gamma_0)}$ . We get:

$$\|X\|_{\gamma'_1} \leq C_{61} q_s^{\rho(\gamma_1, \gamma_0)} (1 + \|\log Df^{N_{s-1}}\|_{\gamma'_1})$$

with

$$C_{61}(\gamma_1, \gamma_0) = C_{42}(r)C_{52}(\gamma'_1)C_{60}(\gamma_1, \gamma_0)C_{51}(\gamma'_1)$$

We estimate  $Z$ . By applying estimation (10), we have:

$$\|Z\|_{\gamma'_1} \leq \sum_{l=1}^{r-1} \|(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_0|G_l^r|_{\gamma'_1} + \|(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_{\gamma'_1}|G_l^r|_0$$

As with  $X$ , we have:

$$\|(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_{\gamma'_1} \leq C_{62} q_s^{\rho(\gamma_1, \gamma_0)} (1 + \|\log Df^{N_{s-1}}\|_{\gamma'_1})$$

with:

$$C_{62}(\gamma_1, \gamma_0, l) = C_{42}(r-l)C_{52}(\gamma'_1)C_{60}(\gamma'_1 + r-l, \gamma_0)C_{51}(\gamma'_1) \leq C_{61}(\gamma_1, \gamma_0)$$

Moreover, by estimate (56), we also have:

$$\|(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_0 \leq M^{r-l} C_{60}(r-l, \gamma_0) q_s^{\rho(r-l, \gamma_0)} \leq C_{61} q_s^{\rho(\gamma_1, \gamma_0)}$$

For  $G_l^r$ , we use lemma 7.4 with the polynomial  $P = G_l^r$  (see appendix 8.4). We estimate  $\|Z\|_{\gamma'_1}$  by applying estimation (44) twice. We get:

$$\|Z\|_{\gamma'_1} \leq C_{63} q_s^{\rho(\gamma_1, \gamma_0)} \|\log Df^{N_{s-1}}\|_{\gamma_1}$$

with:

$$C_{63}(\gamma_1, \gamma_0) = (r-1)C_{61}(\gamma_1, \gamma_0)C_{49}(r-1)(2 + C_{40}(\gamma'_1, \gamma_1)C_{40}(r, \gamma_1))$$

Therefore, since  $\|Y\|_{\gamma'_1} = \|D^r \log Df^{N_{s-1}}\|_{\gamma'_1}$ , we get, for  $s \geq 1$ :

$$\|D^r \log Df^{N_s}\|_{\gamma'_1} \leq \|\log Df^{N_s}\|_{\gamma_1} \leq \left(1 + \frac{C_{61}(\gamma_1, \gamma_0) + C_{63}(\gamma_1, \gamma_0)}{q_s^{-\rho(\gamma_1, \gamma_0)}}\right) \|D^r \log Df^{N_{s-1}}\|_{\gamma'_1}$$

Moreover, by estimate (56), since  $N_0 = b_0$ , we also have:

$$\|D^r \log Df^{N_0}\|_{\gamma'_1} \leq C_{60}(\gamma_1, \gamma_0)$$

We conclude that:

$$\|\log Df^N\|_{\gamma_1} \leq C_{60}(\gamma_1, \gamma_0) \prod_{s=1}^{\infty} \left(1 + \frac{C_{61}(\gamma_1, \gamma_0) + C_{63}(\gamma_1, \gamma_0)}{q_s^{-\rho(\gamma_1, \gamma_0)}}\right) = C_{64}$$

□

### 7.3.1 Computation of the estimate of $\|\log Df^N\|_{\gamma_1}$ in function of $\sup_{p \geq 0} \|\log Df^p\|_{\gamma_0}$

The quantity  $C_{64}$  depends on  $\sup_{p \geq 0} \|\log Df^p\|_{\gamma_0}$ . We estimate  $C_{64}$ . First, we estimate  $C_{58}$ . Since  $5C_{59} \leq (C_{56}(k-1, 0))^2$ , we estimate  $C_{56}(\gamma, 0)$  for  $0 \leq \gamma \leq k-1$ . By combining the constants appearing in lemma 7.9, we get:

$$C_{56}(\gamma, 0) \leq C_{56}(\gamma', 0) \left[ (k-1)2C_{55}(0)C_{54}(k-1, 0)C_{49}(k-1)M^k(\log MC_{39}(k))^2 \right]^{k-1}$$

Moreover,

$$C_{49}(k-1) \leq (4k)^{4k} 2^{k-1} (C_{39}(k) \log M)^{k-1}$$

$$C_{50}(k-1) \leq (4k)^{4k} k M^{k-1} 2^{k-1} (C_{39}(k)(M+1))^{k-1} (1 + ((M+1)C_{39}(k))^2)$$

$$C_{50}(k-1) \leq (4k)^{4k} k 2^{2k+1} M^{2k} (C_{39}(k))^{k+1}$$

Note that for any  $l \geq 2$ ,  $C_{51}(l) = C_{50}(l)$ . Since

$$C_{54}(k-1, 0) \leq [C_{51}(k-1)]^2 2MC_{39}(k)$$

We get:

$$C_{54}(k-1, 0) \leq (4k)^{8k} k^2 2^{4k+3} (C_{39}(k))^{2k+3} M^{4k+1} = \tilde{C}_{54}(k-1, 0) \quad (59)$$

We have:

$$C_{56}(\gamma, 0) \leq C_{56}(\gamma', 0) \left[ C_{55}(0)(\tilde{C}_{54}(k-1, 0))^2 M^k \right]^{k-1}$$

Since  $C_{55}(0) \leq (2M^{\frac{3}{2}(k-1)} C_{24}^{f,k} C_d^{-1})^{1/k}$ , we get, for  $0 \leq \gamma \leq k-1$ :

$$C_{56}(\gamma, 0) \leq C_d^{-1} C_{24}^{f,k} 2M^{k^2 + \frac{3}{2}(k-1)} \left[ \tilde{C}_{54}(k-1, 0) \right]^{2k} = \tilde{C}_{56} \quad (60)$$

We get:

$$C_d^{-1} C_{58} \leq C_d^{-1} C_{24}^{f,k} M^{k-1} \prod_{s=0}^{\infty} \left( 1 + \frac{(\tilde{C}_{56})^2}{q_s^e} \right) \leq \prod_{s=0}^{\infty} \left( 1 + \frac{2C_d^{-1} C_{24}^{f,k} M^{k-1} (\tilde{C}_{56})^2}{q_s^e} \right) \quad (61)$$

We estimate  $C_{61}(\gamma_1, \gamma_0)$  and  $C_{63}(\gamma_1, \gamma_0)$ .

We have:  $C_{61}(\gamma_1, \gamma_0) \leq 2(r+1)M^{r+1}C_{60}(\gamma_1, \gamma_0)C_{52}(\gamma'_1)$ .

Therefore,



$$C_{61}(\gamma_1, \gamma_0) \leq \left( \max_{0 \leq \gamma \leq k-1} C_{56}(\gamma, \gamma_0) \right) (C_d^{-1} C_{58})^{\frac{k-1}{k-\gamma_0}} C_{52}(\gamma_1') 2(r+1) M^{r+1}$$

We estimate  $C_{63}(\gamma_1, \gamma_0)$ . We have:

$$C_{63}(\gamma_1, \gamma_0) \leq (r-1)(2 + ((\log M)C_{39}(k)))^2 C_{61}(\gamma_1, \gamma_0) C_{49}(r-1) \quad (62)$$

To complete the estimations of  $C_{61}(\gamma_1, \gamma_0)$  and  $C_{63}(\gamma_1, \gamma_0)$ , we need to estimate  $C_{56}(\gamma, \gamma_0)$ . By writing  $\gamma = \lfloor \gamma \rfloor + \gamma'$ , and by proceeding as for the estimation of  $C_{56}(\gamma, 0)$ , we have:

$$C_{56}(\gamma, \gamma_0) \leq C_{56}(\gamma', \gamma_0) (C_d^{-1} C_{58})^{\frac{k-1}{k-\gamma_0}} C_{54}(k-1, \gamma_0)^{2k} M^{k^2} 2^{2(k-1)}$$

Moreover,

$$C_{54}(k-1, \gamma_0) \leq C_{54}(k-1, 0) C_{52}(k-1)$$

We can also check that:

$$C_{56}(\gamma', \gamma_0) = 2C_{39}(k) M^{3\gamma'+1} \vee 2M^{\gamma'+1} C_{52}(\gamma'+1) \leq \tilde{C}_{54}(k-1, 0) C_{52}(k-1)$$

Therefore,

$$C_{61}(\gamma_1, \gamma_0) \leq (C_d^{-1} C_{58})^{\frac{2(k-1)}{k-\gamma_0}} \left( \tilde{C}_{54}(k-1, 0) \right)^{2k+1} (C_{52}(k-1))^{2k+2} 2k M^{k^2+k} \quad (63)$$

Now, let

$$\tau_1 = \frac{5(k-1)}{\beta + 2 + \eta} \quad (64)$$

let  $C_{65} = (k+2)^{400k^4} M^{24k(k+1)} (C_{24}^{f,k})^{4k} (C_d^{-1})^{4k^2}$ . Let also

$$\epsilon_1 = \min \left( \epsilon, \frac{\eta}{2(\beta + 2 + \eta)} \right) \quad (65)$$

We have:  $\epsilon_1 \leq \min(\epsilon, -\rho(\gamma_1, \gamma_0))$  and for any  $\gamma_0 \leq k - 2 - \beta - \eta$ , we have  $\tau_1 \geq \frac{2(k-1)}{k-\gamma_0}$ .

Note that  $C_{39}(k)$  only depends on  $k$ , and that  $\tilde{C}_{54}(k-1, 0)$  and  $C_{52}(k-1)$  only depend on  $k$  and  $M$ .

By combining estimations (59), (60), (61), (63) and (62), we can check that we have:

$$C_{64} \leq \prod_{s=0}^{\infty} \left( 1 + \frac{C_{65} \prod_{s=0}^{\infty} \left( 1 + \frac{C_{65}}{q_s^{\epsilon_1}} \right)^{\tau_1}}{q_s^{\epsilon_1}} \right)$$

Since  $q_s \geq (\sqrt{2})^{s-1}$ , we get:

$$C_{64} \leq \prod_{s \geq 0} \left( 1 + \frac{\sqrt{2}C_{65} \prod_{s \geq 0} \left( 1 + \frac{\sqrt{2}C_{65}}{2^{s \frac{\epsilon_1}{2}}} \right)^{\tau_1}}{2^{s \frac{\epsilon_1}{2}}} \right)$$

In order to obtain the final estimation, we need the claim:

**Claim 7.15.** *Let  $C \geq 10$ . For any  $2 \geq u > 1$ , we have:*

$$\prod_{n=0}^{\infty} \left( 1 + \frac{C}{u^n} \right) \leq e^{\frac{2^{2/3}}{\log u} (\log C)^2}$$

*Proof.*

$$\begin{aligned} \sum_{n \geq 0} \log(1 + C/u^n) &= \sum_{n \leq \frac{\log C}{\log u} - 1} \log(1 + C/u^n) + \sum_{n > \frac{\log C}{\log u} - 1} \log(1 + C/u^n) \\ &\leq \frac{\log C}{\log u} \log(1 + C) + \sum_{n > \frac{\log C}{\log u} - 1} C/u^n \\ &\leq \frac{\log C}{\log u} (\log(1 + C) + 1) \leq \frac{2^{2/3}}{\log u} (\log C)^2 \end{aligned}$$

for  $C \geq 10$ .

□

By applying this proposition twice, we get the claim:

**Claim 7.16.** *Let  $C \geq 10$ ,  $2 \geq u > 1$ ,  $\tau \geq 1$ . We have:*

$$\prod_{n \geq 0} \left( 1 + \frac{C \prod_{n=0}^{\infty} \left( 1 + \frac{C}{u^n} \right)^{\tau}}{u^n} \right) \leq e \wedge \left( \frac{18\tau^2}{(\log u)^3} (\log C)^4 \right)$$

Let  $C_{66} = \sqrt{2}C_{65}$ . We apply claim 7.16 with  $C = C_{66}$ ,  $u = 2^{\frac{\epsilon_1}{2}}$ ,  $\tau = \tau_1$ . We obtain:

$$C_{64} \leq e \wedge \left( \frac{18\tau_1^2}{\left(\frac{\epsilon_1}{2} \log 2\right)^3} (\log C_{66})^4 \right) \quad (66)$$

Moreover, let

$$C_{67} = \frac{18\tau_1^2}{\left(\frac{\epsilon_1}{2} \log 2\right)^3}$$

and let  $C_{68}^- = \frac{18 \times 5^2 \times 8}{(\log 2)^3}$  a numerical constant. We have:

$$C_{67} \leq C_{68}^- \frac{(k-1)^2}{\epsilon_1^2}$$

By using the definitions of  $\epsilon_1$  (see (65)) and  $\tau_1$  (see (64)), since  $\epsilon = \frac{k-(1+2\beta)}{2k}$  and since  $\eta \leq k-2-\beta$ , we have:

$$C_{67} \leq C_{68}^- \frac{k^2}{\left(\min\left(\frac{k-(2\beta+1)}{2k}, \frac{\eta}{2(\beta+2+\eta)}\right)\right)^3} \leq 8C_{68}^- \frac{k^5}{(\min(k-(2\beta+1), k-(\beta+2)))^3} = C_{69}(k, \beta)$$

Therefore, we get:

$$\|\log Df^N\|_{\gamma_1} \leq e \wedge \left( C_{69}(k, \beta) \left( C_{70}^{f,k} + 4k^2 \log(C_d^{-1}) + 24k(k+1) \sup_{p \geq 0} \|\log Df^p\|_{\gamma_0} \right)^4 \right) \quad (67)$$

with:

$$C_{70}^{f,k} = \log \left( \sqrt{2}(k+2)^{400k^4} (C_{24}^{f,k})^{4k} \right)$$

Hence proposition 7.1.

### 7.3.2 The case $k \geq 3\beta + 9/2$

By corollary 7.5, we have:

$$\|Df^N\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq C_{51}(k-1)(1 + \|\log Df^N\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}})$$

Moreover, we recall that:

$$C_{51}(k-1) \leq (4k)^{4k} k 2^{2k+1} M^{2k} (C_{39}(k))^{k+1}$$

We have:  $C_{51}(k-1) \leq C_{71}(k)M^{2k}$  with  $C_{71}(k) = (4k)^{4k} k 2^{(k+1)^3 + 3k+2}$

Moreover,  $\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq \sup_{N \geq 0} \|Df^N\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}}$ . We get:

$$\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq e \wedge \left( C_{69}(k, \beta) \left( C_{70}^{f,k} + 4k^2 \log C_d^{-1} + 24k(k+1) \log M' \right)^4 + \log(C_{71}(k)) + 2k \log M' \right)$$

Since  $2k \log M' \leq k(k+1) \log M'$ , we get:

$$\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq e \wedge \left( C_{69}(k, \beta) \left( C_{70}^{f,k} + 4k^2 \log C_d^{-1} + 25k(k+1) \log M' + \log(C_{71}(k)) \right)^4 \right) \quad (68)$$

We show estimation (7). We suppose  $k \geq 3\beta + 9/2$ . Let:

$$C_{72}(\beta, C_d, |Df|_0, W(f), |Sf|_0, |Sf|_{k-3}) = e^{(3)} \wedge (C_3(\beta)C_4(C_d)C_5(|Df|_0, W(f), |Sf|_0)C_6(|Sf|_{k-3}))$$

i.e. we consider the bound given by estimation (3), except that we replace  $|Sf|_{\lceil 3\beta+3/2 \rceil}$  with  $|Sf|_{k-3}$ .

$C_{24}^{f,k}$  depends on  $k$ ,  $|Sf|_{k-3}$  and  $W(f)$ . We have:

$$4k \log C_{24}(k, |Sf|_{k-3}, W(f)) \leq 4k(k+4)! \log C_{24}(3\beta + 9/2, |Sf|_{k-3}, W(f)) \leq 4(k+5)! \log C_{72}$$

Moreover, since  $M' \leq C_{72}$  and  $k \geq 5$ ,

$$4k^2 \log C_d^{-1} + 25k(k+1) \log M' + \log(C_{71}(k)) + 400k^4 \log(\sqrt{2}(k+2)) \leq C_{73}(k) \log C_{72}$$

with  $C_{73} = 4k^2 + 25k(k+1) + \log(C_{71}(k)) + 400k^4 \log(\sqrt{2}(k+2)) \leq (k+5)!$ . Therefore,

$$\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq e \wedge (C_{69}(k, \beta) (5(k+5)! \log C_{72})^4)$$

We also have:

$$C_{69}(k, \beta) \leq 8C_{68}^- \frac{k^5}{(\min(k - (2\beta + 1), k - (\beta + 2)))^3} \leq C_{68}^- k^5$$

Since  $C_{68}^- k^5 (5(k+5)!)^4 \leq C_{68}^- 5^4 ((k+7)!)^4$ , and since  $2 \geq 2 \log 2$ , we conclude:

$$\|Dh\|_{\frac{k}{2(\beta+2)} - \frac{1}{2}} \leq e \wedge (C_{12}(k) e^{(2)} \wedge (2 + C_3(\beta)C_4(C_d)C_5(|Df|_0, W(f), |Sf|_0)C_6(|Sf|_{k-3})))$$

with:

$$C_{12}(k) = \frac{18 \times 5^6 \times 8}{(\log 2)^3} ((k+7)!)^4$$

### 7.3.3 The case $\alpha$ of constant type

If  $\beta = 0$ , we can use the  $C^1$  estimate. We have:  $\log M' \leq C_1/C_d$  and therefore,

$$4k^2 \log(C_d^{-1}) + 25k(k+1) \log M' \leq (k+5)! C_1/C_d$$

Moreover,

$$C_{70}^{f,k} + \log(C_{71}(k)) \leq 4(k+5)! (\log(1 \vee |Sf|_{k-3}) + W(f) + 1)$$

then by using estimation (68), we obtain:

$$\|Dh\|_{\frac{k}{4}-\frac{1}{2}} \leq e \wedge \left( C_{13}(k) \left[ C_{14}[W(f), |Sf|_{k-3}] + \frac{C_1[W(f), |Sf|_0]}{C_d} \right]^4 \right)$$

with:

$$C_{13}(k) = C_{12}(k)$$

$$C_{14}[W(f), |Sf|_{k-3}] = \log(1 \vee |Sf|_{k-3}) + W(f) + 1$$

#### 7.4 Iteration of the reasoning: the general case $k > 2\beta + 1$

To obtain an estimation of the  $C^{k-1-\beta-\eta}$ -norm of the conjugacy, we iterate estimation (43). We take  $\gamma_0 = \gamma_i$  and  $\gamma_1 = \gamma_{i+1} = \frac{1}{2}(g(\gamma_i) + \gamma_i)$ . Thus,  $\gamma_{i+1} < g(\gamma_i)$  and

$\lim_{i \rightarrow +\infty} \gamma_i = k - 2 - \beta$ . We need to estimate the rank above which  $\gamma_i \geq k - 2 - \beta - \eta$ :

**Claim 7.17.** Let  $C_{74} = \frac{\beta+3/2}{\beta+2}$ ,  $C_{75} = \frac{k-2-\beta}{2(\beta+2)}$ . If

$$n \geq \log\left(\frac{C_{75}}{\eta(1-C_{74})}\right) / \log\left(\frac{1}{C_{74}}\right) = C_{76}$$

we have  $\gamma_n \geq k - 2 - \beta - \eta$ .

*Proof.* We have:  $\gamma_{n+1} = C_{74}\gamma_n + C_{75}$ . Therefore,

$$\gamma_n = C_{75} \frac{1-C_{74}^n}{1-C_{74}}. \text{ Therefore, } |\gamma_n - (k-2-\beta)| = \left| C_{75} \frac{C_{74}^n}{1-C_{74}} \right| \leq \eta \text{ if } n \geq C_{76}.$$

□

**Claim 7.18.** Let  $F(x) = e^{c(a+bx)^4}$ . For any  $x, c \geq 1, a, b \geq 4$ , and integer  $n \geq 1$ , we have:

$$F^n(x) \leq e^{(n)} \wedge \left( (3+n)c(a+bx)^4 \right)$$

*Proof.* We show this estimate by induction. If  $n = 1$ , this estimate holds. Suppose this estimate holds at rank  $n$ . We have:

$$F^{n+1}(x) = F^n\left(e^{c(a+bx)^4}\right) \leq e^{(n)} \wedge \left( (3+n)c(a+be^{c(a+bx)^4})^4 \right)$$

For any  $x \geq 4$ ,  $e^x \geq x^2$ . Since  $c(a+bx)^4 \geq 4$  and  $4n \geq 3+n$ , then:

$$e^{nc(a+bx)^4} \geq \left( nc(a+bx)^4 \right)^2 \geq (3+n)c(a+b)^4$$

$$e^{(4+n)c(a+bx)^4} \geq (3+n)c(a+b)^4 e^{4c(a+bx)^4} \geq (3+n)c(a+be^{c(a+bx)^4})^4$$

Hence the estimate at rank  $n + 1$ .

□

We apply proposition 7.1. In claim 7.18, we take  $x = \log M'$ ,  
 $a = C_{70}^{f,k} + 4k^2 \log(C_d^{-1})$ ,  $b = 24k(k+1)$ ,  $c = C_{69}(k, \beta)$ . Let

$$n_7 = \left\lceil \frac{\log((k-2-\beta)/\eta)}{\log(1+1/(2\beta+3))} \right\rceil$$

We have:  $C_{76} + 1 > n_7 \geq C_{76}$ . We get:

$$\|\log Df^N\|_{k-2-\beta-\eta} \leq e^{(n_7)} \wedge \left( (3+n_7)C_{69}(k, \beta)(C_{70}^{f,k} + 4k^2 \log(C_d^{-1}) + 24k(k+1) \log M')^4 \right)$$

Moreover, by corollary 7.5, we have:

$$\|Df^N\|_{k-2-\beta-\eta} \leq C_{51}(1 + \|\log Df^N\|_{k-2-\beta-\eta})$$

Since  $\|Dh\|_{k-2-\beta-\eta} \leq \|Df^N\|_{k-2-\beta-\eta}$ , we get:

$$\|Dh\|_{k-2-\beta-\eta} \leq e^{(n_7)} \wedge \left( (4+n_7)C_{69}(k, \beta) \left[ C_{70}^{f,k} + 4k^2 \log(C_d^{-1}) + 25k(k+1) \log M' \right]^4 \right)$$

Since  $M' \leq C_2$ , we let:

$$C_{11}[\eta, k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}] = (4+n_7)C_{69}(k, \beta) \left[ C_{70}^{f,k} + 4k^2 \log(C_d^{-1}) + 25k(k+1) \log C_2 \right]^4$$

We recall that:

$$n_7 = \left\lceil \frac{\log((k-2-\beta)/\eta)}{\log(1+1/(2\beta+3))} \right\rceil$$

$$C_{69}(k, \beta) = \frac{18 \times 5^2 \times 8^2}{(\log 2)^3} \frac{k^5}{(\min(k - (2\beta + 1), k - (\beta + 2)))^3}$$

$$C_{70}^{f,k} \leq (k+7)! (1 + W(f) + \log(1 \vee |Sf|_{k-3}))$$

We have:

$$\|Dh\|_{k-2-\beta-\eta} \leq e^{(\lceil \log((k-2-\beta)/\eta) / \log(1+1/(2\beta+3)) \rceil)} \wedge (C_{11}[\eta, k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}]) \quad (69)$$

## 8 Appendix: Omitted Proofs

### 8.1 Proof of lemma 6.4

We follow [Yoc84] but we give more details. Let  $p \leq q_{n+1}$ . The case  $r = 1$  stems from lemma 6.3. For the case  $r = 2$ , we also use lemma 6.3:

$$|D^2 \log Df^p(x)| \leq |Sf^p(x)| + \frac{1}{2}|D \log Df^p(x)|^2 \leq \left( C_{21}^f + \frac{1}{2}(C_{23}^f)^2 \right) \frac{M_n}{m_n(x)^2}$$

In particular, we can take

$$C_{24}^f(2) = 82|Sf|_0 e^{8W(f)}$$

For  $r > 2$ , we prove lemma 6.4 by induction. Suppose the lemma is proved up to  $r \geq 2$ . Since for any  $C^3$ -diffeomorphisms  $g$  and  $h$ ,

$$S(g \circ h) = (Sg \circ h)(Dh)^2 + Sh$$

then for  $p \geq 1$ ,

$$Sf^p = \sum_{i=0}^{p-1} (Sf \circ f^i)(Df^i)^2$$

and by differentiating this last equality, we get, for  $r \geq 0$ ,  $n \geq 1$ ,

$$D^r Sf^p = \sum_{l=0}^r \sum_{i=0}^{p-1} (D^{r-l} Sf \circ f^i)(Df^i)^{r-l+2} F_l^r(D \log Df^i, \dots, D^l \log Df^i) \quad (70)$$

where  $F_l^r$  is a polynomial in  $l$  variables  $X_1, \dots, X_l$ , homogenous of weight  $l$  if  $X_i$  is given the weight  $i$ . Moreover, since  $Sf = D^2 \log Df - \frac{1}{2}(D \log Df)^2$ , then for  $r \geq 2$ ,

$$D^{r-2} Sf = D^r \log Df + G_r(D \log Df, \dots, D^{r-1} \log Df) \quad (71)$$

where  $G_r$  is a polynomial in  $r - 1$  variables  $X_1, \dots, X_{r-1}$ , homogeneous of weight  $r$  if  $X_i$  is given the weight  $i$ . Therefore, in order to estimate  $|D^r \log Df|_0$ , it suffices to estimate  $F_l^r(D \log Df^i, \dots, D^l \log Df^i)$  and  $G_r(D \log Df, \dots, D^{r-1} \log Df)$ . These estimations are given by lemmas 8.1 and 8.2. They are used in [Yoc84] but we recall them here in order to compute the constants  $C_{77}^f(r)$  in lemma 8.1 and  $C_{78}^f(r)$  in lemma 8.2.

**Lemma 8.1.** *Under the induction assumption, for  $0 \leq l \leq r$  and  $0 \leq p \leq q_{n+1}$ , we have:*

$$|F_l^r(D \log Df^p(x), \dots, D^l \log Df^p(x))| \leq C_{77}^f(r) \left[ \frac{M_n^{1/2}}{m_n(x)} \right]^l$$

with:

$$C_{77}^f(r) = (r)! \frac{(2r)!}{2} \left( C_{24}^f(r) \right)^r$$

*Proof.* We follow [Yoc84]. By derivating equation (70), we get:

$$\begin{aligned} D^{r+1} S f^p &= \sum_{i=0}^{n-1} \sum_{l=0}^r (D^{r+1-l} S f \circ f^i) (Df^i)^{r+1-l+2} F_l^r(D \log Df^i, \dots, D^l \log Df^i) + \\ &\quad (D^{r-l} S f \circ f^i) (Df^i)^{r-l+2} (r-l+2) D \log Df^i \\ &F_l^r(D \log Df^i, \dots, D^l \log Df^i) + \sum_{j=1}^l \frac{\partial F_l^r}{\partial X_j} (D \log Df^i, \dots, D^l \log Df^i) D^{j+1} \log Df^i (Df^i)^{r-l+2} \end{aligned}$$

$$\begin{aligned} D^{r+1} S f^p &= \sum_{i=0}^{p-1} \sum_{l=0}^r (D^{r+1-l} S f \circ f^i) (Df^i)^{r+1-l+2} F_l^r(D \log Df^i, \dots, D^l \log Df^i) + \\ &\sum_{l=1}^{r+1} (D^{r+1-l} S f \circ f^i) (Df^i)^{r-l+3} (r-l+3) D \log Df^i F_{l-1}^r(D \log Df^i, \dots, D^{l-1} \log Df^i) + \\ &\sum_{l=1}^{r+1} \sum_{j=2}^l \frac{\partial F_{l-1}^r}{\partial X_{j-1}} (D \log Df^i, \dots, D^{l-1} \log Df^i) D^j \log Df^i (Df^i)^{r+1-l+2} \end{aligned}$$

Therefore, for  $1 \leq l \leq r$ ,

$$F_l^{r+1} = F_l^r + (r-l+3) X_1 F_{l-1}^r + \sum_{j=2}^l X_j \frac{\partial F_{l-1}^r}{\partial X_{j-1}} \quad (72)$$

for  $l = 0$ ,

$$F_l^{r+1} = F_l^r$$

and for  $l = r + 1$ ,

$$F_l^{r+1} = (r-l+3) X_1 F_{l-1}^r + \sum_{j=2}^l X_j \frac{\partial F_{l-1}^r}{\partial X_{j-1}}$$

Now, let us write

$$F_l^r = \sum_{i_1+2i_2+\dots+li_l=l} a_{l,r}(i_1, \dots, i_l) X_1^{i_1} \dots X_l^{i_l}$$

We have  $a_{l,r}(i_1, \dots, i_l) \geq 0$ . Let



$$a_{l,r} = \max_{i_1+2i_2+\dots+li_l=l} a_{l,r}(i_1, \dots, i_l)$$

and

$$\bar{a}_r = \max_{0 \leq j \leq r} a_{j,r}$$

Consider  $i_1, \dots, i_l$  such that  $a_{l,r}(i_1, \dots, i_l) = a_{l,r}$ . By applying equation (72), we have, for  $1 \leq l \leq r$ :

$$a_{l,r+1} \leq a_{l,r} + (r+3-l)a_{l-1,r} + (l-1)(\max i_j)a_{l-1,r} \leq (r+3-l+l^2-l)\bar{a}_{l,r} \leq (r+1)^2\bar{a}_{l,r}$$

For  $l=0$  or  $r+1$ , this estimate still holds. Therefore,  $\bar{a}_{r+1} \leq (r+1)^2\bar{a}_r$  and by iteration, we obtain:

$$\bar{a}_r \leq (r!)^2$$

Moreover, since

$$F_l^r(D \log Df^i, \dots, D^l \log Df^i) = \sum_{i_1+2i_2+\dots+li_l=l} a_{l,r}(i_1, \dots, i_l) (D \log Df^i)^{i_1} \dots (D^l \log Df^i)^{i_l}$$

and since  $\#\{(i_1, \dots, i_l) / i_1 + 2i_2 + \dots + li_l = l\} \leq \#\{(i_1, \dots, i_l) / i_1 + i_2 + \dots + i_l = l\} = \frac{(2l-1)!}{l!(l-1)!}$  (this classical equality can be shown by induction) then by applying the induction assumption,

$$|F_l^r(D \log Df^i(x), \dots, D^l \log Df^i(x))| \leq (r!)^2 \frac{(2l-1)!}{l!(l-1)!} \max_{i_1+2i_2+\dots+li_l=l} (C_{24}^f(1))^{i_1} \dots (C_{24}^f(l))^{i_l} \left[ \frac{M_n^{1/2}}{m_n(x)} \right]^l$$

and since the  $C_{24}^f(i)$  are increasing with  $i$ , we obtain:

$$|F_l^r(D \log Df^i(x), \dots, D^l \log Df^i(x))| \leq C_{77}^f(r) \left[ \frac{M_n^{1/2}}{m_n(x)} \right]^l$$

□

Likewise, the estimation of  $G_r(D \log Df^p, \dots, D^{l-1} \log Df^p)$  is given by the lemma:

**Lemma 8.2.** For any  $x \in \mathbb{T}^1$ ,  $0 \leq p \leq q_{n+1}$ ,  $r \geq 2$ ,

$$|G_r(D \log Df^p(x), \dots, D^{l-1} \log Df^p(x))| \leq C_{78}^f(r) \left[ \frac{M_n^{1/2}}{m_n(x)} \right]^r$$

with:

$$C_{78}^f(r+1) = \frac{(2r)!}{2^{(r+1)}} (C_{24}^f(r))^{r+1}$$

*Proof.* The polynomial  $G_r$  satisfies the following identity:

$$G_{r+1} = \sum_{j=2}^r X_j \frac{\partial G_r}{\partial X_{j-1}}$$

We denote

$$G_r = \sum_{i_1+2i_2+\dots+(r-1)i_{r-1}=r} b_r(i_1, \dots, i_{r-1}) X_1^{i_1} \dots X_{r-1}^{i_{r-1}}$$

(we have, for example,  $G_2 = -\frac{1}{2}X_1^2$ )

Let

$$b_r = \max_{i_1+2i_2+\dots+(r-1)i_{r-1}=r} |b_r(i_1, \dots, i_{r-1})|$$

For  $r \geq 2$ , we have  $b_{r+1} \leq r(\max_{1 \leq j \leq r-1} i_j) b_r \leq r^2 b_r$  and therefore,  $b_r \leq \frac{(r-1)!^2}{2}$

Therefore,

$$|G_{r+1}(D \log Df^p(x), \dots, D^r \log Df^p(x))| \leq \frac{r!}{2} \frac{(2r)!}{r!(r+1)!} \max_{i_1+2i_2+\dots+ri_r=r+1} (C_{24}^f(1))^{i_1} \dots (C_{24}^f(r))^{i_r} \left[ \frac{M_n^{1/2}}{m_n(x)} \right]^{r+1}$$

Since the constants  $C_{24}^f(r)$  are increasing with  $r$ , we can take:

$$C_{78}^f(r+1) = \frac{(2r)!}{2(r+1)} (C_{24}^f(r))^{r+1}$$

□

We can now show estimation (16). By applying equation (71), we have, for  $r \geq 2$ :

$$D^{r+1} \log Df^p = D^{r-1} S f^p - G_{r+1}(D \log Df^p, \dots, D^r \log Df^p)$$

Therefore, by equation (70) and lemma 6.1,

$$|D^{r+1} \log Df^p(x)| \leq (r C_{77}^f(r) |S f|_{r-1} e^{(r+1)W(f)} + C_{78}^f(r+1)) \left( \frac{M_n^{1/2}}{m_n(x)} \right)^{r+1}$$

$$|D^{r+1} \log Df^p(x)| \leq (C_{24}^f(r))^r \frac{(2r)!}{2} (|S f|_{r-1} e^{(r+1)W(f)} + C_{24}^f(r)) \left( \frac{M_n^{1/2}}{m_n(x)} \right)^{r+1}$$

We can show by induction on  $r$  that we can take, for  $r \geq 3$ ,

$$C_{24}^f(r) = \left[ C_{24}^f(2) (2r)^{2r} (\max(1, |S f|_{r-2})) e^{rW(f)} \right]^{r!}$$

## 8.2 Proof of proposition 7.3

This estimation is well-known (see e.g. the appendix in [Hor76]). We recall the proof in order to determine the constant. We write  $\gamma_2 = r_2 + \gamma'_2$ ,  $\gamma_3 = r_3 + \gamma'_3$  with  $0 < \gamma'_2, \gamma'_3 < 1$ , and we estimate  $|D^{\gamma_2}\phi|_{\gamma'_2}$  in function of  $|D^{\gamma_3}\phi|_{\gamma'_3}$  and  $|\phi|_0$ . We use [Her79, p. 110]:

**Proposition 8.3.** *Let  $0 < \beta < 1$  and  $\phi \in C^1(\mathbb{T}^1)$ . We have:*

$$|\phi|_\beta \leq 2|\phi|_0^{1-\beta}|D\phi|_0^\beta$$

**Proposition 8.4.** *Let  $0 < \beta < 1$  and  $\phi \in C^\beta(\mathbb{T}^1)$ . If  $\beta' < \beta$ , we have:*

$$|\phi|_{\beta'} \leq 2|\phi|_0^{1-\frac{\beta'}{\beta}}|\phi|_\beta^{\frac{\beta'}{\beta}}$$

**Proposition 8.5.** *Let  $0 < \beta \leq 1$  and  $\phi \in C^1(\mathbb{T}^1)$  such that  $D\phi \in C^\beta(\mathbb{T}^1)$ . We have:*

$$|D\phi|_0 \leq \frac{3}{2}2^{\frac{\beta}{1+\beta}}\left(|\phi|_0^\beta|D\phi|_\beta\right)^{\frac{1}{1+\beta}}$$

Moreover, for  $\phi \in C^2(\mathbb{T}^1)$ , we have:

$$|D\phi|_0 \leq \left(2|\phi|_0|D^2\phi|_0\right)^{1/2}$$

First, we suppose  $\gamma'_2 = \gamma'_3 = 0$ . We already know that for any  $j < l$ , there exists  $C_{79}(j, l)$  such that:

$$|D^j\phi|_0 \leq C_{79}(j, l)|\phi|_0^{1-\frac{j}{l}}|D^l\phi|_0^{\frac{j}{l}}$$

We want to estimate a possible  $C_{79}(j, l)$  in function of  $l$ . First, we show that  $C_{79}(1, l) = C_{79}(l-1, l) = (\sqrt{2})^{l-1}$ . We proceed inductively on  $l$ . For  $l = 2$ , by proposition 8.5,  $C_{79}(1, l) = C_{79}(l-1, l) = \sqrt{2}$ . For  $l \geq 3$ , we have:

$$|D\phi|_0 \leq C_{79}(1, l-1)|\phi|_0^{\frac{l-2}{l-1}}|D^{l-1}\phi|_0^{\frac{1}{l-1}} \quad (73)$$

$$|D^{l-2}\phi|_0 \leq C_{79}(l-2, l-1)|\phi|_0^{\frac{1}{l-1}}|D^{l-1}\phi|_0^{\frac{l-2}{l-1}} \quad (74)$$

By applying (74) to  $D\phi$ , we also have:

$$|D^{l-1}\phi|_0 \leq C_{79}(l-2, l-1)|D\phi|_0^{\frac{1}{l-1}}|D^l\phi|_0^{\frac{l-2}{l-1}}$$

By combining this estimate with (73), we get:

$$|D^{l-1}\phi|_0^{1-\frac{1}{(l-1)^2}} \leq C_{79}(l-2, l-1)(C_{79}(1, l-1))^{\frac{1}{l-1}}|\phi|_0^{\frac{l-2}{(l-1)^2}}|D^l\phi|_0^{\frac{l-2}{l-1}}$$

Therefore,

$$C_{79}(l-1, l) = \left( C_{79}(l-2, l-1) (C_{79}(1, l-1))^{\frac{1}{l-1}} \right)^{\frac{1}{1-\frac{1}{(l-1)^2}}}$$

Likewise, we can show that

$$C_{79}(1, l) = \left( C_{79}(1, l-1) (C_{79}(l-2, l-1))^{\frac{1}{l-1}} \right)^{\frac{1}{1-\frac{1}{(l-1)^2}}}$$

Let  $u_l = \log C_{79}(1, l)$ ,  $v_l = \log C_{79}(l-1, l)$ ,  $a_l = \frac{1}{1-\frac{1}{(l-1)^2}}$ . We have:

$$u_l = a_l(u_{l-1} + \frac{1}{l-1}v_{l-1})$$

$$v_l = a_l(v_{l-1} + \frac{1}{l-1}u_{l-1})$$

Let  $w_l = u_l + v_l$ . We have:  $w_l = \frac{l-1}{l-2}w_{l-1}$  and therefore,  $w_l = (l-1)w_2$ . Moreover, let  $x_l = u_l - v_l$ . We have:  $x_l = \frac{l}{l-1}x_{l-1}$ . Since  $x_2 = 0$ , then  $x_l = 0$  for any  $l \geq 3$ . Therefore,  $u_l = v_l = (l-1) \log \sqrt{2}$ . Therefore,  $C_{79}(1, l) = C_{79}(l-1, l) = (\sqrt{2})^{l-1}$ .

To show that  $C_{79}(j, l) \leq (\sqrt{2})^{l^2}$  for  $l-2 \geq j \geq 2$ , we proceed inductively on  $l$ . We know that  $C_{79}(1, 2) = \sqrt{2}$  and we suppose that  $C_{79}(j, l-1) \leq (\sqrt{2})^{(l-1)^2}$  for  $j = 2, \dots, l-2$ . We observe that:

$$|D^{j+1}\phi|_0 \leq C_{79}(j, l-1) |D\phi|_0^{1-\frac{j}{l-1}} |D^l\phi|_0^{\frac{j}{l-1}}$$

Therefore,

$$C_{79}(j+1, l) \leq C_{79}(j, l-1) (C_{79}(1, l))^{\frac{l-j-1}{l-1}} = C_{79}(j, l-1) (\sqrt{2})^{l-j-1}$$

By applying the induction assumption, we get:

$$C_{79}(j+1, l) \leq (\sqrt{2})^{(l-1)^2+l-1} \leq (\sqrt{2})^{l^2}$$

Hence the proposition for  $\gamma_2, \gamma_3$  integers.

Now, we suppose that  $\gamma_2$  and  $\gamma_3$  are not integers.

If  $r_2 < r_3$ , we have:

$$|D^{r_2}\phi|_{\gamma_2'} \leq 2 |D^{r_2}\phi|_0^{1-\gamma_2'} |D^{r_2+1}\phi|_0^{\gamma_2'}$$

Moreover, by the interpolation inequality in the integer case, we have:

$$|D^{r_2+1}\phi|_0 \leq C_{79}(r_3) |\phi|_0^{1-\frac{r_2+1}{r_3}} |D^{r_3}\phi|_0^{\frac{r_2+1}{r_3}}$$

$$|D^{r_2}\phi|_0 \leq C_{79}(r_3)|\phi|_0^{1-\frac{r_2}{r_3}} |D^{r_3}\phi|_0^{\frac{r_2}{r_3}}$$

Therefore, by proposition 8.3

$$|D^{r_2}\phi|_{\gamma'_2} \leq 2C_{79}(r_3)|\phi|_0^{(1-\gamma'_2)(1-\frac{r_2}{r_3})+\gamma'_2(1-\frac{r_2+1}{r_3})} |D^{r_3}\phi|_0^{(1-\gamma'_2)\frac{r_2}{r_3}+\gamma'_2\frac{r_2+1}{r_3}}$$

On the other hand, by proposition 8.5,

$$|D^{r_3}\phi|_0 \leq 3|D^{r_3-1}\phi|_0^{\frac{\gamma'_3}{1+\gamma'_3}} |D^{r_3}\phi|_{\gamma'_3}^{\frac{1}{1+\gamma'_3}}$$

and we also have:

$$|D^{r_3-1}\phi|_0 \leq C_{79}(r_3)|\phi|_0^{\frac{1}{r_3}} |D^{r_3}\phi|_0^{\frac{r_3-1}{r_3}}$$

Therefore,

$$|D^{r_3}\phi|_0^{1-(1-\frac{1}{r_3})\frac{\gamma'_3}{1+\gamma'_3}} \leq 3(C_{79}(r_3))^{\frac{\gamma'_3}{1+\gamma'_3}} |\phi|_0^{\left(\frac{\gamma'_3}{1+\gamma'_3}\right)\frac{1}{r_3}} |D^{r_3}\phi|_{\gamma'_3}^{\frac{1}{1+\gamma'_3}}$$

Therefore,

$$|D^{r_3}\phi|_0 \leq 3^{\frac{1+\gamma'_3}{1+\gamma'_3/r_3}} (C_{79}(r_3))^{\frac{\gamma'_3}{1+\gamma'_3/r_3}} |\phi|_0^{\left(\frac{\gamma'_3/r_3}{1+\gamma'_3/r_3}\right)\frac{1}{r_3}} |D^{r_3}\phi|_{\gamma'_3}^{\frac{1}{1+\gamma'_3/r_3}} \quad (75)$$

Therefore,

$$|D^{r_2}\phi|_{\gamma'_2} \leq C_{79}(\gamma_3)|\phi|_0^{1-\frac{\gamma_2}{\gamma_3}} |D^{r_3}\phi|_{\gamma'_3}^{\frac{\gamma_2}{\gamma_3}}$$

with  $C_{79}(\gamma_3) \leq 3^2 C_{79}(r_3) \leq 2^{(r_3+1)^2}$  if  $r_3 \geq 2$  and  $C_{79}(\gamma_3) \leq 2^{(r_3+1)^2+1}$  if  $r_3 = 1$ .

Finally, if  $r_2 = r_3 = r$  and  $\gamma'_2 < \gamma'_3$ , then

$$|D^r\phi|_{\gamma'_2} \leq 2|D^r\phi|_0^{1-\frac{\gamma'_2}{\gamma'_3}} |D^r\phi|_{\gamma'_3}^{\frac{\gamma'_2}{\gamma'_3}}$$

By applying estimation (75), we still have:

$$|D^{r_2}\phi|_{\gamma'_2} \leq C_{79}(\gamma_3)|\phi|_0^{1-\frac{\gamma_2}{\gamma_3}} |D^{r_3}\phi|_{\gamma'_3}^{\frac{\gamma_2}{\gamma_3}}$$

with  $C_{79}(\gamma_3) \leq 2^{(r_3+1)^2}$  if  $r_3 \geq 2$  and  $C_{79}(\gamma_3) \leq 2^{(r_3+1)^2+1}$  if  $r_3 = 1$ .

This completes the proof of proposition 7.3.

### 8.3 Proof of lemma 7.6.

If  $\gamma < 1$ , we prove the lemma directly. We prove lemma 7.6 for  $\gamma \geq 1$ . This is necessary to compute the constant. We write  $\gamma = r + \gamma'$  with  $r$  integer and  $0 \leq \gamma' < 1$ . We need the Faa-di-Bruno formula. We first recall this formula and a related property (lemma 8.6). After that, we prove the lemma.

The Faa-di-Bruno formula states that for any integer  $u \geq 1$  and functions  $f$  and  $g$  of class  $C^u$ ,

$$D^u [f(g(x))] = \sum_{j=0}^u D^j f(g(x)) B_{u,j} (Dg(x), D^2g(x), \dots, D^{(u-j+1)}g(x))$$

where  $B_{u,j}$  are the Bell polynomials, given by:

$$B_{u,j}(x_1, x_2, \dots, x_{u-j+1}) = \sum \frac{u!}{l_1! l_2! \dots l_{u-j+1}!} \left(\frac{x_1}{1!}\right)^{l_1} \left(\frac{x_2}{2!}\right)^{l_2} \dots \left(\frac{x_{u-j+1}}{(u-j+1)!}\right)^{l_{u-j+1}}$$

The sum extends over all sequences  $l_1, l_2, l_3, \dots, l_{u-j+1}$  of non-negative integers such that:  $l_1 + l_2 + \dots = j$  and  $l_1 + 2l_2 + 3l_3 + \dots = u$ . The cardinal of this set of sequences is less than  $\#\{(i_1, \dots, i_l) / i_1 + i_2 + \dots + i_l = l\} = \frac{(2l-1)!}{l!(l-1)!}$  (see page 80).

We obtain the estimation, for any  $x$ , and  $u \geq 1$ :

$$\left| B_{u,j} (Dg(x), D^2g(x), \dots, D^{(u-j+1)}g(x)) \right| \leq \frac{(2u-1)!}{(u-1)!} \|Dg\|_{u-1}^j \quad (76)$$

We also need the lemma:

**Lemma 8.6.** *Let  $u \geq 1$  and  $u \geq j \geq 0$  be integers. Let  $a_1, \dots, a_{u-j+1}, x_1, \dots, x_{u-j+1}$  be real numbers, let  $a \geq \max\{|a_k|; 1 \leq k \leq u-j+1\}$ ,  $x \geq \max\{|x_k| \vee 1; 1 \leq k \leq u-j+1\}$ . Suppose  $x+a \geq 1$ . Let  $B_{u,j}$  be a Bell polynomial. We have :*

$$|B_{u,j}(x_1 + a_1, \dots, x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, \dots, x_{u-j+1})| \leq u^2 \frac{(2u)!}{2} a(x+a)^u$$

*Proof.* Let  $u+1 \geq p \geq 1$  and  $l_1, \dots, l_p$  be integers. We have:

$$(x_1+a_1)^{l_1} \dots (x_p+a_p)^{l_p} - x_1^{l_1} \dots x_p^{l_p} = \sum_{i=1}^p x_1^{l_1} \dots x_{i-1}^{l_{i-1}} (x_i+a_i)^{l_i} \dots (x_p+a_p)^{l_p} - x_1^{l_1} \dots x_i^{l_i} (x_{i+1}+a_{i+1})^{l_{i+1}} \dots (x_p+a_p)^{l_p}$$

$$(x_1+a_1)^{l_1} \dots (x_p+a_p)^{l_p} - x_1^{l_1} \dots x_p^{l_p} = \sum_{i=1}^p x_1^{l_1} \dots x_{i-1}^{l_{i-1}} (x_{i+1}+a_{i+1})^{l_{i+1}} \dots (x_p+a_p)^{l_p} \left[ (x_i+a_i)^{l_i} - x_i^{l_i} \right]$$

(with the conventions  $x_1^{l_1} \dots x_0^{l_0} = 1$  and  $x_{p+1}^{l_{p+1}} \dots x_p^{l_p} = 1$ ).

Since  $|(x_i + a_i)^{l_i} - x_i^{l_i}| \leq l_i |a_i| (|x_i| + |a_i|)^{l_i-1} \leq l_i a (x + a)^{l_i-1}$ ,  $l_i \leq u$  and  $x + a \geq 1$ , we obtain:

$$|B_{u,j}(x_1 + a_1, \dots, x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, \dots, x_{u-j+1})| \leq a(u - j + 1)uB_{u,j}(x + a, \dots, x + a)$$

The formula giving the Bell polynomials implies:

$$B_{u,j}(x + a, \dots, x + a) \leq \frac{(2u)!}{2} (x + a)^u$$

□

For any  $1 \leq u \leq r$ ,

$$|D^u(\phi(\psi(x)))| \leq (u + 1) \|D\phi\|_{r-1} \frac{(2u)!}{2} \|D\psi\|_{u-1}^u$$

We estimate  $|D^r(\phi \circ \psi)|_{\gamma'}$ . The Faa-di-Bruno formula applied to  $\phi \circ \psi$  gives:

$$\begin{aligned} |D^r[\phi(\psi(x))] - D^r[\phi(\psi(y))]| &\leq \sum_{j=0}^r |D^j\phi(\psi(x))| \\ &|B_{r,j}(D\psi(x), \dots, D^{(r-j+1)}\psi(x)) - B_{r,j}(D\psi(y), \dots, D^{(r-j+1)}\psi(y))| + \\ &|D^j\phi(\psi(x)) - D^j\phi(\psi(y))| |B_{r,j}(D\psi(y), \dots, D^{(r-j+1)}\psi(y))| \end{aligned}$$

Moreover, for  $1 \leq j \leq r$ ,  $|D^j\phi(\psi(x))| \leq \|D\phi\|_{r-1}$ , and

$$|D^j\phi(\psi(x)) - D^j\phi(\psi(y))| \leq |D^j\phi|_{\gamma'} |\psi(x) - \psi(y)|^{\gamma'} \leq \|D\phi\|_{\gamma-1} |D\psi|_0^{\gamma'} |x - y|^{\gamma'}$$

Estimation (76) gives:

$$\left| B_{r,j}(D\psi(y), D^2\psi(y), \dots, D^{(r-j+1)}\psi(y)) \right| \leq \frac{(2r)!}{2} \|D\psi\|_{r-1}^r$$

We apply lemma 8.6: let  $x_i = D^i\psi(x)$  and  $a_i = D^i\psi(y) - D^i\psi(x)$ . Let  $x = \|D\psi\|_{r-1}$  and  $a = \max_{1 \leq i \leq r-j+1} |D^i\psi(x) - D^i\psi(y)|$ . We have:

$$\begin{aligned} &|B_{r,j}(D\psi(x), \dots, D^{(r-j+1)}\psi(x)) - B_{r,j}(D\psi(y), \dots, D^{(r-j+1)}\psi(y))| \\ &\leq r^2 \frac{(2r)!}{2} (2\|D\psi\|_{\gamma-1})^r \max_{1 \leq i \leq r-j+1} |D^i\psi(x) - D^i\psi(y)| \end{aligned}$$

$$|B_{r,j}(D\psi(x), \dots, D^{(r-j+1)}\psi(x)) - B_{r,j}(D\psi(y), \dots, D^{(r-j+1)}\psi(y))| \leq r^2 2^r \frac{(2r)!}{2} \|D\psi\|_{\gamma-1}^{r+1} |x - y|^{\gamma'}$$

Therefore, we get:

$$|D^r(\phi \circ \psi)|_{\gamma^r} \leq \|D\phi\|_{\gamma-1} \|D\psi\|_{\gamma-1}^{|L\gamma|+1} \frac{(2r)!}{2} (r+1)(r^2 2^r + 1) \leq \|D\phi\|_{\gamma-1} \|D\psi\|_{\gamma-1}^{|L\gamma|+1} ((2r)!)^2$$

## 8.4 Estimates on some polynomials

Lemma 7.4 is used for some specific polynomials. There exist  $A_l, B_l, G_l^r, E_l^r$ , polynomials of  $l$  variables  $X_1, \dots, X_l$  homogeneous of weight  $l$  if  $X_i$  has weight  $i$ , such that, for  $l \geq 1$ , and for any diffeomorphisms  $g$  and  $h$  sufficiently differentiable, we have [Yoc84, p. 337-338]:

$$D^{l+1}g = A_l(D \log Dg, \dots, D^l \log Dg) Dg \quad (77)$$

$$D^l \log Dg = B_l\left(\frac{D^2g}{Dg}, \dots, \frac{D^{l+1}g}{Dg}\right) \quad (78)$$

For  $r \geq 0$ ,

$$D^r \log D(g \circ h) = (D^r \log Dg \circ h)(Dh)^r + D^r \log Dh + \sum_{l=1}^{r-1} D^{r-l} \log Dg \circ h (Dh)^{r-l} G_l^r(D \log Dh, \dots, D^l \log Dh) \quad (79)$$

For  $r \geq 0$  and  $n \geq 1$ ,

$$D^r \log Dg^n = \sum_{l=0}^{r-1} \sum_{i=0}^{n-1} (D^{r-l} \log Dg \circ g^i)(Dg^i)^{r-l} E_l^r(D \log Dg^i, \dots, D^l \log Dg^i) \quad (80)$$

Lemma 7.4 uses the following estimate:

**Claim 8.7.** Let  $P = A_l, B_l, G_l^r$  or  $E_l^r$ . Write  $P = \sum_{\sum_{k=1}^l k j_k = l} a_{j_1, \dots, j_l} X_1^{j_1} \dots X_l^{j_l}$ . We have:

$$\sum_{\sum_{k=1}^l k j_k = l} |a_{j_1, \dots, j_l}| \leq (4(l+1))^{4(l+1)}$$

*Proof.* For example, if  $P = B_l$ , we have

$$B_{l+1}\left(\frac{D^2g}{Dg}, \dots, \frac{D^{l+2}g}{Dg}\right) = DB_l\left(\frac{D^2g}{Dg}, \dots, \frac{D^{l+1}g}{Dg}\right)$$

Since

$$D\left(\frac{D^u g}{Dg}\right) = \frac{D^{u+1}g}{Dg} - \frac{D^u g}{Dg} \frac{D^2g}{Dg}$$

then



$$B_{l+1} = \sum_{i=1}^l (X_{i+1} - X_i X_1) \frac{\partial B_l}{\partial X_i}$$

Let  $b_l = \max |a_{j_i, \dots, j_l}|$ . We have:  $b_{l+1} \leq 2l^2 b_l$ . Thus  $b_l \leq 2^l ((l-1)!)^2$ . Moreover, the sum  $\sum_{\sum_{k=1}^l k j_k = l} |a_{j_1, \dots, j_l}|$  has less than  $\frac{(2l-1)!}{l!(l-1)!}$  terms. Hence the claim for  $P = B_l$ . The computations are analogous for the other polynomials (and analogous to the proof of lemma 6.4 in appendix 8.1).

□

To obtain lemma 7.4, we apply estimations (49) and (50) to each monomial of  $P$ , and we apply this claim.

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## Chapter II

# Circle diffeomorphisms: quasi-reducibility and commuting diffeomorphisms

## Summary

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## Résumé

In this article, we show two related results on circle diffeomorphisms. The first result is on quasi-reducibility : for a Baire-dense set of  $\alpha$ , for any diffeomorphism  $f$  of rotation number  $\alpha$ , it is possible to accumulate  $R_\alpha$  with a sequence  $h_n f h_n^{-1}$ ,  $h_n$  being a diffeomorphism. The second result is : for a Baire-dense set of  $\alpha$ , given two commuting diffeomorphisms  $f$  and  $g$ , such that  $f$  has  $\alpha$  for rotation number, it is possible to approach each of them by commuting diffeomorphisms  $f_n$  and  $g_n$  that are differentiably conjugated to rotations.

In particular, it implies that if  $\alpha$  is in this Baire-dense set, and if  $\beta$  is an irrational number such that  $(\alpha, \beta)$  are not simultaneously Diophantine, then the set of commuting diffeomorphisms  $(f, g)$  with singular conjugacy, and with rotation numbers  $(\alpha, \beta)$  respectively, is  $C^\infty$ -dense in the set of commuting diffeomorphisms with rotation numbers  $(\alpha, \beta)$ .

**Mots-clefs** : systèmes dynamiques, théorie ergodique, difféomorphismes du cercle, nombre de rotation, méthode d'Anosov-Katok, problème de réalisation lisse.

## 9 Introduction

It is well-known that there are circle diffeomorphisms with Liouville rotation numbers (i.e. non-Diophantine) that are not smoothly conjugated to rotations [Arn65, Her79, KH96, Kat77b]. A natural question arises, namely, the problem of smooth quasi-reducibility: *given a smooth diffeomorphism  $f$  of rotation number  $\alpha$ , is it possible to accumulate  $R_\alpha$  in the  $C^\infty$ -norm, with a sequence  $h_n^{-1} f h_n$ ,  $h_n$  being a smooth diffeomorphism?* In this case, we say that  $f$  is smoothly quasi-reducible to  $R_\alpha$ . Quasi-reducibility is a question that has been studied by Herman [Her79, pp.93-99], who showed that for any  $C^2$ -diffeomorphism  $f$  of irrational rotation number  $\alpha$ , it is possible to accumulate  $R_\alpha$  in the  $C^{1+bv}$ -norm, with a sequence  $h_n^{-1} f h_n$ ,  $h_n$  being a  $C^2$ -diffeomorphism (i.e.  $h_n^{-1} f h_n \rightarrow R_\alpha$  in the  $C^1$ -norm, and the total variation of  $D(h_n^{-1} f h_n - R_\alpha)$  converges towards zero). Quasi-reducibility is also related to a problem solved by Yoccoz [Yoc95], who showed that it is possible to accumulate a smooth diffeomorphism  $f$  in the  $C^\infty$ -norm with a sequence  $h_n R_\alpha h_n^{-1}$ ,  $h_n$  being a smooth diffeomorphism. However, these two problems are not the same, and the method used by Yoccoz does not directly yield our result. In our case, we determine a Baire-dense set of rotation numbers  $\alpha$  such that for any smooth diffeomorphism  $f$  of rotation number  $\alpha$ ,  $f$  is smoothly quasi-reducible.

Connected to the problem of quasi-reducibility is the following question, raised by Mather: *given two commuting  $C^\infty$ -diffeomorphisms  $f$  and  $g$ , is it possible to approach each of them in the  $C^\infty$ -norm by commuting smooth diffeomorphisms that are smoothly conjugated to rotations?* In this paper, we determine a Baire-dense set of rotation numbers  $\alpha$  such that if  $f$  and  $g$  are commuting  $C^\infty$ -diffeomorphisms, with  $f$  of rotation number  $\alpha$ , then  $f$  and  $g$  are accumulated in the  $C^\infty$  norm by commuting  $C^\infty$ -diffeomorphisms that are  $C^\infty$ -conjugated to a rotation. This result is related to a theorem of Fayad and Khanin [FK09a]. They showed that if  $(\alpha, \alpha')$  are simultaneously Diophantine (i.e. there is  $C_d > 0, \beta \geq 0$  such that for any  $p, p' \in \mathbb{Z}$ , any  $q \geq 1$ ,  $\max(|\alpha - p/q|, |\alpha' - p'/q|) \geq C_d/q^{2+\beta}$ . This set includes some pairs  $(\alpha, \alpha')$  with  $\alpha$  and  $\alpha'$  Liouvillean), and if  $f$  and  $g$  are commuting  $C^\infty$ -diffeomorphisms, with  $f$  and  $g$  of rotation numbers  $\alpha$  and  $\alpha'$  respectively, then  $f$  and  $g$  are smoothly linearizable. Fayad and Khanin's result implies our result of quasi-reducibility in the particular case when the rotation numbers of  $f$  and  $g$  are simultaneously Diophantine. However, in general, our result is not implied by theirs. Indeed, our result holds for a set  $(\alpha, \alpha')$  that is Baire-dense in  $\mathbb{R}^2$  (because  $\alpha$  belongs to a Baire-dense set of  $\mathbb{R}$  and  $\alpha'$  is arbitrary), whereas the set of simultaneously

Diophantine numbers is not Baire-dense.<sup>1</sup>

Moreover, for Diophantine rotation numbers, which are of full Lebesgue measure, the question of quasi-reducibility and Mather's problem are trivial, because in this case, the diffeomorphism  $f$  is smoothly conjugated to a rotation. Therefore, these two questions remain open for a meagre set of rotation numbers of zero Lebesgue measure.

In order to derive our results, we use estimates of the conjugacy to rotations of diffeomorphisms having rotation numbers of Diophantine constant type. These estimates were obtained in the previous chapter.

The circle is denoted  $\mathbb{T}^1$ . For  $r \in \mathbb{R}_+ \cup \{+\infty\}$ , we work in the universal cover  $D^r(\mathbb{T}^1)$ , which is the group of diffeomorphisms  $f$  of class  $C^r$  of the real line such that  $f - Id$  is  $\mathbb{Z}$ -periodic. For  $\alpha \in \mathbb{R}$ , we denote  $R_\alpha \in D^\infty(\mathbb{T}^1)$  the map  $x \mapsto x + \alpha$ .

Let  $f \in D^0(\mathbb{T}^1)$  be a homeomorphism and  $x \in \mathbb{R}$ . The sequence  $((f^n(x) - x)/n)_{n \geq 1}$  admits a limit independent of  $x$ , denoted  $\rho(f)$ . This limit is called the *rotation number* of  $f$ . This is a real number invariant by conjugacy.

**Theorem 9.1.** *There is a Baire-dense set  $A_1 \subset \mathbb{R}$  such that for any  $f \in D^\infty(\mathbb{T}^1)$  of rotation number  $\alpha \in A_1$ , there is a sequence  $h_n \in D^\infty(\mathbb{T}^1)$  such that  $h_n^{-1} f h_n \rightarrow R_\alpha$  in the  $C^\infty$ -topology.*

**Theorem 9.2.** *There is a Baire-dense set  $A_2 \subset \mathbb{R}$  such that for any  $f \in D^\infty(\mathbb{T}^1)$  of rotation number  $\alpha \in A_2$  and any  $g$  of class  $C^\infty$  with  $fg = gf$ ,  $f$  and  $g$  are accumulated in the  $C^\infty$ -topology by commuting  $C^\infty$ -diffeomorphisms that are  $C^\infty$ -conjugated to rotations. Moreover, if  $\beta$  is the rotation number of  $g$ ,  $R_\alpha$  and  $R_\beta$  are accumulated in the  $C^\infty$ -topology by commuting  $C^\infty$ -diffeomorphisms that are  $C^\infty$ -conjugated to  $f$  and  $g$ .*

*Remark 9.3.* The proof of theorem 9.1 also gives that  $h_n R_\alpha h_n^{-1} \rightarrow f$  in the  $C^\infty$ -topology if  $\alpha \in A_1$ . However, a stronger result was already proved by Yoccoz [Yoc95].

*Remark 9.4.* The rotation number of  $g_n$  can be kept equal to  $\beta$ , by replacing  $g'_n(0)$  with

$g'_n(x_n) - x_n = \beta$ , for a suitable  $x_n$ , in the definition of  $g_n$ . On the other hand, the rotation number of  $f_n$  cannot be kept constant in general.

*Remark 9.5.* Combined with [FK09a, p. 965], theorem 9.2 implies that if  $\alpha \in A_2$ , and  $(\alpha, \beta)$  are not simultaneously Diophantine, then  $S_{\alpha, \beta}$ , the set of couples  $(f, g)$  of smooth commuting circle diffeomorphisms with singular conjugacies to  $R_\alpha$  and  $R_\beta$  respectively, is  $C^\infty$ -dense in  $F_{\alpha, \beta}$ , the set of couples  $(f, g)$  of smooth commuting circle diffeomorphisms with rotation numbers  $\alpha$  and  $\beta$  respectively.

---

1. The complementary in  $\mathbb{R}^2$  of simultaneously Diophantine numbers (noted  $SD^c$ ) is Baire-dense. Indeed, we have:

$$SD^c = \bigcap_{k \in \mathbb{N}^*} \bigcap_{n \in \mathbb{N}^*} \bigcup_{q \geq n} (A_{q,k} \times A_{q,k})$$

with:

$$A_{q,k} = \left\{ \alpha \in \mathbb{R} \mid \text{there is an integer } p \in \mathbb{Z}, \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^k} \right\}.$$

$A_{q,k}$  is open (and so is  $A_{q,k} \times A_{q,k}$ ), and for any integer  $n$ ,  $\bigcup_{q \geq n} (A_{q,k} \times A_{q,k})$  is dense, because it contains all pairs of rational numbers (if  $\alpha = p_1/q_1$  and  $\alpha' = p_2/q_2$ , then  $(\alpha, \alpha') \in (A_{jq_1q_2,k} \times A_{jq_1q_2,k})$  for any  $j, k \in \mathbb{N}^*$ ). Therefore,  $SD^c$  is Baire-dense.

Indeed, our result shows that  $O_{\alpha,\beta}$ , the set of couples  $(f, g)$  of smooth commuting circle diffeomorphisms with smooth conjugacies to  $R_\alpha$  and  $R_\beta$  respectively, is  $C^\infty$ -dense in  $F_{\alpha,\beta}$ . Moreover, in [FK09a, p.965], for  $(\alpha, \beta)$  not simultaneously Diophantine, Fayad and Khanin described the construction of a couple  $(f, g)$  of smooth commuting circle diffeomorphisms with singular conjugacies to  $R_\alpha$  and  $R_\beta$  respectively. This construction relies on the method of successive conjugacies, which can be made  $C^\infty$ -dense in  $O_{\alpha,\beta}$  [FK04].

Moreover, by slightly modifying [Her79, p.160, p.167], this implies that  $(O_{\alpha,\beta}^1)^c$ , the set of couples  $(f, g)$  of smooth commuting circle diffeomorphisms with non- $C^1$  conjugacies to rotations  $R_\alpha$  and  $R_\beta$ , is  $C^\infty$ -generic in  $F_{\alpha,\beta}$ . See appendix 13 for a short proof.

## 10 Preliminaries

### 10.1 Basic properties

When the rotation number  $\alpha$  of  $f$  is irrational, and if  $f$  is of class  $C^2$ , Denjoy showed that  $f$  is topologically conjugated to  $R_\alpha$ . However, this conjugacy is not always differentiable. It depends on the Diophantine properties of the rotation number  $\alpha$ .

Let  $\alpha = a_0 + 1/(a_1 + 1/(a_2 + \dots))$  be the development of  $\alpha \in \mathbb{R}$  in continued fraction (see [DK02]). It is denoted  $\alpha = [a_0, a_1, a_2, \dots]$ . Let  $p_{-2} = q_{-1} = 0$ ,  $p_{-1} = q_{-2} = 1$ . For  $n \geq 0$ , we define integers  $p_n$  and  $q_n$  by:

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2}.$$

We have  $q_0 = 1$ ,  $q_n \geq 1$  for  $n \geq 1$ . The rationals  $p_n/q_n$  are called the convergents of  $\alpha$ . Remember that  $q_{n+2} \geq 2q_n$ , for  $n \geq -1$ .

For any real number  $\beta \geq 0$ ,  $\alpha \in \mathbb{R} - \mathbb{Q}$  is Diophantine of order  $\beta$  and constant  $C_d$  (a set denoted  $DC(C_d, \beta)$ ) if there is a constant  $C_d > 0$  such that for any  $p/q \in \mathbb{Q}$ , we have:

$$\left| \alpha - \frac{p}{q} \right| > \frac{C_d}{q^{2+\beta}}.$$

Each of the following relations characterizes  $DC(C_d, \beta)$  (see e.g. [Yoc11, pp.50-51]):

1.  $|\alpha - p_n/q_n| > C_d/q_n^{2+\beta}$  for any  $n \geq 0$
2.  $a_{n+1} < \frac{1}{C_d} q_n^\beta$  for any  $n \geq 0$
3.  $q_{n+1} < \frac{1}{C_d} q_n^{1+\beta}$  for any  $n \geq 0$
4.  $\alpha_{n+1} > C_d \alpha_n^{1+\beta}$  for any  $n \geq 0$ .

$DC(C_d, 0)$  is the set of irrational numbers of constant type  $C_d$ . The first derivative of  $f \in D^1(\mathbb{T}^1)$  is denoted  $Df$ .

## 10.2 Some useful lemmas

For any  $n$  integer, let  $\alpha_n = [a_0, \dots, a_n, 1, \dots]$ .

Let  $V_\alpha : \mathbb{N} \rightarrow \mathbb{R}$  defined by:  $V_\alpha(n) = \max_{0 \leq i \leq n} a_i$ . Observe that  $\alpha_n \in DC(1/V_\alpha(n), 0)$ . We will need the lemma:

**Lemma 10.1.** *Let  $\alpha$  be an irrational number,  $q_n$  its convergents and  $\alpha_n = [a_0, \dots, a_n, 1, \dots]$ . We have:*

$$|\alpha_n - \alpha| \leq \frac{2}{q_n^2} \leq \frac{4}{2^n}.$$

*Proof.* Let  $\tilde{\alpha}_n = [a_0, \dots, a_n, 0, \dots]$ . By induction, we can show that  $\tilde{\alpha}_n = p_n/q_n$ . Moreover,  $\tilde{\alpha}_n$  is also the  $n^{\text{th}}$  convergent of  $\alpha_n$ . Therefore, by the best rational approximation theorem,  $|\alpha - p_n/q_n| \leq 1/q_n^2$  and  $|\alpha_n - p_n/q_n| \leq 1/q_n^2$ . Moreover, since  $q_{n+2} \geq q_n$ , then  $q_n \geq (\sqrt{2})^{n-1}$ .

□

We need the lemma:

**Lemma 10.2.** *Let  $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$  be such that  $\phi(n) \rightarrow_{n \rightarrow +\infty} +\infty$ . Let*

$$A = \{\alpha \in \mathbb{R} / V_\alpha(n) < \phi(n) \text{ for an infinity of } n\}.$$

*Then  $A$  is Baire-dense.*

*Proof.* First, we show that for any positive integers  $n$  and  $i$ ,

$A_{i,n} = \{\alpha \text{ such that } a_i < \phi(n)\}$  is open. Let  $u(x) = \lfloor x \rfloor$ ,  $v(x) = \frac{1}{x}$  and  $w(x) = v(x) - u(v(x))$ . We have:  $a_{k+1} = v(w^k(x)) - w^{k+1}(x)$ . Since  $v$  is continuous and  $u$  is upper semi-continuous and non-negative, then  $w$  is lower semi-continuous. Moreover,  $w$  is non-negative. Therefore,  $w^k$  and  $w^{k+1}$  are also lower semi-continuous and non-negative. Since  $v$  is decreasing, then  $v \circ w^k - w^{k+1}$  is upper semi-continuous. We conclude that  $A_{i,n}$  is open.

Moreover, for any  $p \geq 0$ ,

$$\bigcup_{n \geq p} \bigcap_{i \leq n} A_{i,n}$$

is dense. Indeed, since  $\phi(n) \rightarrow +\infty$ , then it contains all numbers of constant type, which are dense. This set is also open and therefore,

$$A = \bigcap_{p \geq 0} \bigcup_{n \geq p} \bigcap_{i \leq n} A_{i,n}$$

is Baire-dense.

□

## 10.3 Notations

- For any real numbers  $a$  and  $b$ ,  $a \vee b$  denotes  $\max(a, b)$ .
- For  $\phi$  a real  $\mathbb{Z}$ -periodic  $C^r$  function,  $0 \leq r < +\infty$ , we define:

$$\|\phi\|_r = \max_{0 \leq j \leq r} \max_{x \in \mathbb{R}} |D^j \phi(x)|.$$

Note that for  $f, g \in D^r(\mathbb{T}^1)$ ,  $f - g$  is  $\mathbb{Z}$ -periodic, and for  $1 \leq j \leq r$ ,  $D^j f$  is  $\mathbb{Z}$ -periodic. For  $f \in D^r(\mathbb{T}^1)$ , we also define:

$$\|f\|_r = \max\left(\|f - id\|_0, \max_{1 \leq j \leq r} \|D^j f\|_0\right).$$

Note that the notation  $\|f\|_r$  is not a norm when  $f \in D^r(\mathbb{T}^1)$ , since  $D^r(\mathbb{T}^1)$  is not a vector space.

- In all the paper,  $C$  denotes a constant depending on  $u$ .  $W(f)$  denotes the total variation of  $\log Df$ , and  $Sf$  denotes the Schwartzian derivative of  $f$ .

## 10.4 Estimates of the conjugacy

The following theorem gives an estimate of the linearization of a diffeomorphism having a rotation numbers of Diophantine constant type. This estimate, obtained in the previous chapter, is necessary to derive our results.

**Theorem 10.3.** *Let  $l \geq 3$  be an integer and  $\eta > 0$ . Let  $f \in D^l(\mathbb{T}^1)$  be of rotation number  $\alpha$ , such that  $\alpha$  is of constant type  $C_d$ . There exists a diffeomorphism  $h \in D^{l-1-\eta}(\mathbb{T}^1)$  conjugating  $f$  to  $R_\alpha$ , and a function  $B$  of  $C_d, l, \eta, W(f), \|Sf\|_{l-3}$ , which satisfy the estimate:*

$$\max\left(\frac{1}{\min Dh}, \|h\|_{l-1-\eta}\right) \leq B(C_d, l, \eta, W(f), \|Sf\|_{l-3}). \quad (81)$$

In particular, we remark that if  $f_n$  is a sequence of diffeomorphisms of rotation number  $\alpha_n$ , if the sequences  $W(f_n)$  and  $\|Sf_n\|_{l-3}$  are bounded (this will hold in our case, because we will take  $f_n = \lambda_n + f$  for a properly chosen  $\lambda_n \in \mathbb{R}$ ), if  $V_\alpha(n) \rightarrow +\infty$  and if  $h_n$  is the conjugacy to a rotation associated with  $f_n$ , then there is a real function  $E(V_\alpha(n))$  such that, for  $n$  sufficiently large, we have:

$$\max\left(\frac{1}{\min Dh_n}, \|h_n\|_{l-1-\eta}\right) \leq E(V_\alpha(n)).$$

## 11 Quasi-Reducibility

**Theorem 11.1.** *Let  $l \geq 3$  be an integer,  $f \in D^l(\mathbb{T}^1)$  be of rotation number  $\alpha \in \mathbb{T}^1$ . Let  $\eta > 0$  be a real number. There exists a numerical sequence  $F(n)$ , going to  $+\infty$  as  $n \rightarrow +\infty$ , such that, if*

$$\liminf \frac{V_\alpha(n)}{F(n)} = 0$$

*then there is a sequence  $h_n$  of class  $C^{l-1-\eta}$  such that  $h_n^{-1} f h_n \rightarrow R_\alpha$  in the  $C^{l-2-\eta}$ -topology.*

By applying lemma 10.2, we obtain the corollary:

**Corollary 11.2.** *There is a Baire-dense set  $A_1 \subset \mathbb{R}$  such that if  $l \geq 3$  is an integer,  $f \in D^l(\mathbb{T}^1)$  of rotation number  $\alpha \in A_1$  and  $\eta > 0$ , then  $f$  is  $C^{l-2-\eta}$ -quasi-reducible: there is a sequence  $h_n \in D^{l-1-\eta}(\mathbb{T}^1)$  such that  $h_n^{-1} f h_n \rightarrow R_\alpha$  in the  $C^{l-2-\eta}$ -topology.*

The idea of the proof of theorem 11.1 is the following. We observe that for any sequence  $\phi(n) \rightarrow +\infty$ , the set of numbers  $\alpha$  such that for an infinity of  $n$ ,

$$\sup_{k \leq n} a_k \leq \phi(n), \text{ is Baire-dense (lemma 10.2).}$$

The truncated sequence of constant type numbers  $\alpha_n = [a_0, \dots, a_n, 1, \dots]$  converges towards  $\alpha$  at a controlled speed:  $|\alpha - \alpha_n| \leq 4/2^n$  (lemma 10.1).

Following an idea of Herman [Her79], we perturbate  $f$  to  $R_{\lambda_n} f = f + \lambda_n$  of rotation number  $\alpha_n$ , which is linearizable by a conjugacy  $h_n$  (lemma 11.3). By writing:

$$h_n^{-1} f h_n - R_\alpha = h_n^{-1} f h_n - h_n^{-1} R_{\lambda_n} f h_n + R_{\alpha_n} - R_\alpha$$

and by applying the Faa-di-Bruno formula, we obtain a control of the norm of  $h_n^{-1} f h_n - R_\alpha$  in function of the norm of  $h_n$ , and in function of  $|\alpha - \alpha_n|$  (lemma 11.4). Moreover, we have an estimate of the norm of  $h_n$  in function of  $\sup_{k \leq n} a_k$ .

Thus, if we choose the speed of growth of the sequence  $\sup_{k \leq n} a_k$  sufficiently small with respect to the speed of convergence of  $\alpha_n$  towards  $\alpha$ , then  $h_n^{-1} f h_n$  converges towards  $R_\alpha$ , and  $f$  is quasi-reducible.

*Proof of theorem 9.1.* We let  $\eta = l/3$  in corollary 11.2. Since  $f$  is smooth, then there is a sequence  $(h_{n,l})_{n \geq 0} \in D^\infty(\mathbb{T}^1)$  such that, for any integer  $l \geq 3$  fixed,

$$\|h_{n,l}^{-1} f h_{n,l} - R_\alpha\|_{2(\frac{l}{3}-1)} \rightarrow_{n \rightarrow +\infty} 0.$$

In particular, there is  $n(l)$  such that:

$$\|h_{n(l),l}^{-1} f h_{n(l),l} - R_\alpha\|_{2(\frac{l}{3}-1)} \leq \frac{1}{l}.$$

Let  $h_l = h_{n(l),l}$ . Let  $\epsilon > 0$ , and let  $k > 0$  be an integer. There is  $l_0 \geq 0$  such that for any  $l \geq l_0$ , we have:  $\epsilon \geq 1/l$ ,  $k \leq 2(\frac{l}{3} - 1)$  and:

$$\|h_l^{-1} f h_l - R_\alpha\|_k \leq \|h_l^{-1} f h_l - R_\alpha\|_{2(\frac{l}{3}-1)} \leq \frac{1}{l} \leq \epsilon.$$

Therefore,  $h_l^{-1} f h_l \rightarrow_{l \rightarrow +\infty} R_\alpha$  in the  $C^k$ -topology, for any  $k$ , and therefore, this convergence holds in the  $C^\infty$ -topology.

□

## 11.1 The one-parameter family $R_\lambda f$

To prove theorem 11.1, we need to consider the one-parameter family  $R_\lambda f = f + \lambda$  (see [Her79, p.31]). We have the lemma:



**Lemma 11.3.** *Let  $l \geq 3$  be an integer,  $f \in D^l(\mathbb{T}^1)$ ,  $0 < \eta \leq l - 3$ ,  $\alpha = \rho(f)$ . Let  $\tilde{\alpha}$  be an irrational number of constant type. There exists  $\lambda_0 \in \mathbb{R}$  and a  $C^{l-1-\eta}$ -diffeomorphism  $h$  such that  $h^{-1}R_{\lambda_0}fh = R_{\tilde{\alpha}}$ . Moreover,*

$$\frac{|\lambda_0|}{\min Dh} \geq |\tilde{\alpha} - \alpha| \geq \frac{|\lambda_0|}{\|Dh\|_0}.$$

*Proof.* Let  $\mu(\lambda) = \rho(R_\lambda f)$ .  $\mu$  is continuous, non-decreasing and  $\mu(\mathbb{R}) = \mathbb{R}$  (see [Her79, p. 31]). Therefore, there exists  $\lambda_0 \in \mathbb{R}$  such that  $\tilde{\alpha} = \rho(R_{\lambda_0}f)$ . Since  $\tilde{\alpha}$  is of constant type, there exists a  $C^{l-1-\eta}$ -diffeomorphism  $h$  such that  $h^{-1}R_{\lambda_0}fh = R_{\tilde{\alpha}}$  and that satisfies estimate (81) of theorem 10.3. By the mean value theorem, for any  $x$ , there is  $c(x)$  such that:

$$\tilde{\alpha} + x - h^{-1}fh(x) = R_{\tilde{\alpha}}(x) - h^{-1}fh(x) = h^{-1}R_{\lambda_0}fh(x) - h^{-1}fh(x) = D(h^{-1})(c(x))\lambda_0.$$

By integrating this equation on an invariant measure of  $h^{-1}fh$ , we get lemma 11.3. Note that since  $h \in D^1(\mathbb{T}^1)$ , then  $Dh(x) > 0$  for any  $x$ , and  $\min Dh > 0$ . □

## 11.2 The speed of approximation of $R_\alpha$

The proof of theorem 11.1 is also based on the lemma:

**Lemma 11.4.** *Let  $l \geq 3$  be an integer,  $f \in D^l(\mathbb{T}^1)$ ,  $0 < \eta \leq l - 3$ ,  $\alpha = \rho(f)$ . Let  $\tilde{\alpha}$  be an irrational number of constant type, and let  $\lambda_0 \in \mathbb{R}$  and  $h$  the  $C^{l-1-\eta}$ -diffeomorphism be given by lemma 11.3. Recall that  $C$  denotes a constant that only depends on  $u$ ,  $0 \leq u \leq l - 2 - \eta$ . We have the estimate:*

$$\|h^{-1}fh - R_\alpha\|_u \leq C\|f\|_u^C\|h\|_{u+1}^C \frac{1}{(\min Dh)^C} |\tilde{\alpha} - \alpha|.$$

Before proving lemma 11.4, we show how theorem 11.1 is derived from it.

*Proof of theorem 11.1.* If  $\alpha$  is of constant type, then  $f$  is reducible and there is nothing to prove. Therefore, we can suppose that  $V_\alpha(n) \rightarrow_{n \rightarrow +\infty} +\infty$ . By applying theorem 10.3, there exists a real function  $\tilde{F}$  strictly increasing with  $V_\alpha(n)$ , such that for  $\alpha_n$ , and for its associated diffeomorphism  $h_n$  given by lemma 11.3, we have, for  $n$  sufficiently large:

$$\|h_n^{-1}fh_n - R_\alpha\|_{l-2-\eta} \leq \exp(\tilde{F}(V_\alpha(n))) |\alpha_n - \alpha|.$$

Let  $F(n) = \tilde{F}^{-1}(n^{1/2})$ . By extracting, we can suppose that  $\lim_{n \rightarrow +\infty} \frac{V_\alpha(n)}{F(n)} = 0$ . Therefore,  $V_\alpha(n) \leq F(n)$  for  $n$  sufficiently large and therefore,

$$\tilde{F}(V_\alpha(n)) \leq n^{1/2}.$$

We get, for  $n$  sufficiently large,

$$\|h_n^{-1} f h_n - R_\alpha\|_{l-2-\eta} \leq e^{-\frac{n \log 2}{4}} \rightarrow_{n \rightarrow +\infty} 0.$$

Hence theorem 11.1.

□

Now, we show lemma 11.4:

*Proof of lemma 11.4.* We need the Faa-di-Bruno formula (see e.g. [Cra05]):

**Lemma 11.5.** *For every integer  $u \geq 0$  and functions  $\phi$  and  $\psi$  of class  $C^u$ , we have:*

$$D^u [\phi(\psi(x))] = \sum_{j=0}^u D^j \phi(\psi(x)) B_{u,j} (D\psi(x), D^2\psi(x), \dots, D^{(u-j+1)}\psi(x)).$$

The  $B_{u,j}$  are the Bell polynomials, defined by  $B_{u,0} = 1$  and, for  $j \geq 1$ :

$$B_{u,j}(x_1, x_2, \dots, x_{u-j+1}) = \sum \frac{u!}{l_1! l_2! \dots l_{u-j+1}!} \left(\frac{x_1}{1!}\right)^{l_1} \left(\frac{x_2}{2!}\right)^{l_2} \dots \left(\frac{x_{u-j+1}}{(u-j+1)!}\right)^{l_{u-j+1}}.$$

The sum extends over all sequences  $l_1, l_2, l_3, \dots, l_{u-j+1}$  of non-negative integers such that:  $l_1 + l_2 + \dots = j$  and  $l_1 + 2l_2 + 3l_3 + \dots = u$ .

Therefore, for any  $x$ , we have the estimate:

$$\left| B_{u,j}(D\psi(x), D^2\psi(x), \dots, D^{(u-j+1)}\psi(x)) \right| \leq C (1 \vee \|\psi\|_u^j). \quad (82)$$

Combining this estimate with lemma 11.5, we obtain the corollary:

**Corollary 11.6.** *For every integer  $u \geq 0$  and functions  $\phi$  and  $\psi$  of class  $C^u$ , we have:*

$$\|\phi \circ \psi\|_u \leq C \max_{0 \leq j \leq u} \|D^j \phi \circ \psi\|_0 (1 \vee \|\psi\|_u^u).$$

We apply this corollary to estimate  $\|h^{-1}\|_u$ . We let  $\phi(x) = 1/x$  and  $\psi = Dh \circ h^{-1}$ . We observe that  $D(h^{-1}) = \frac{1}{Dh \circ h^{-1}} = \phi \circ \psi$ . Since there is  $x_0$  such that  $Dh(x_0) = 1$ , then  $\|Dh\|_0 \geq 1$  (and we also have  $1 \geq \min Dh > 0$ ). Therefore, we get:

$$\|D(h^{-1})\|_u \leq C \max_{0 \leq j \leq u} \frac{1}{\|(Dh \circ h^{-1})^{j+1}\|_0} \|Dh \circ h^{-1}\|_u^C.$$

By corollary 11.6, we also have:

$$\|Dh \circ h^{-1}\|_u \leq C \|Dh\|_u \|h^{-1}\|_u^C.$$

By combining these two estimates, we get:

$$\|D(h^{-1})\|_u \leq C \frac{1}{(\min Dh)^C} \|Dh\|_u^C \|h^{-1}\|_u^C.$$

We iterate this estimate to estimate  $\|h^{-1}\|_u$ , for  $u \geq 1$ . We get:

$$\|h^{-1}\|_{u+1} \leq C \frac{1}{(\min Dh)^C} \|h\|_{u+1}^C \|h^{-1}\|_1^C. \quad (83)$$

Now, we estimate the  $C^u$ -distance of  $h^{-1}fh$  to  $R_\alpha$ . Let  $\tilde{\alpha}, \lambda_0$  be as in lemma 11.3. We have:

$$h^{-1}fh - R_\alpha = h^{-1}fh - h^{-1}R_{\lambda_0}fh + R_{\tilde{\alpha}} - R_\alpha.$$

Therefore,

$$\|h^{-1}fh - R_\alpha\|_u \leq \|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u + |\tilde{\alpha} - \alpha|. \quad (84)$$

On the other hand, by the Faa-di-Bruno formula, we have:

$$D^u \left[ h^{-1}fh - h^{-1}R_{\lambda_0}fh \right] (x) = \sum_{j=0}^u B_{u,j} \left( D(fh)(x), \dots, D^{u-j+1}(fh)(x) \right) \left[ D^j(h^{-1})(fh(x)) - D^j(h^{-1})(fh(x) + \lambda_0) \right].$$

Since  $|D^j(h^{-1})(fh(x)) - D^j(h^{-1})(fh(x) + \lambda_0)| \leq \|D^{j+1}(h^{-1})\|_0 |\lambda_0|$ , then by applying estimate (82), we get:

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C \|f \circ h\|_u^C \|h^{-1}\|_{u+1} |\lambda_0|.$$

By applying corollary 11.6, we get:

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C \|f\|_u^C \|h\|_u^C \|h^{-1}\|_{u+1} |\lambda_0|.$$

By applying (83), we obtain:

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C \|f\|_u^C \|h\|_u^C \frac{1}{(\min Dh)^C} \|h\|_{u+1}^C \|h^{-1}\|_1^C |\tilde{\alpha} - \alpha| \|Dh\|_0$$

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C \|f\|_u^C \|h\|_{u+1}^C \frac{|\tilde{\alpha} - \alpha|}{(\min Dh)^C}.$$

By estimate (84), we obtain:

$$\|h^{-1}fh - R_\alpha\|_u \leq C \|f\|_u^C \|h\|_{u+1}^C \frac{1}{(\min Dh)^C} |\tilde{\alpha} - \alpha|. \quad (85)$$

Hence lemma 11.4.

□

## 12 Application to commuting diffeomorphisms

**Theorem 12.1.** *There exists a numerical sequence  $G(n)$ , going to  $+\infty$  as  $n \rightarrow +\infty$ , such that, for any  $l \geq 3$  an integer,  $f \in D^l(\mathbb{T}^1)$  of rotation number  $\alpha \in \mathbb{R}_1$ ,  $\eta > 0$  and  $g$  of class  $C^l$  such that  $fg = gf$ , if*

$$\liminf \frac{V_\alpha(n)}{G(n)} = 0$$

*then there exists two sequences of diffeomorphisms  $f_n$  and  $g_n$  that are  $C^{l-1-\eta}$ -conjugated to rotations, such that  $f_n g_n = g_n f_n$ , and with  $f_n$  and  $g_n$  converging respectively towards  $f$  and  $g$  in the  $C^{l-2-\eta}$ -norm. Moreover, if  $\beta$  is the rotation number of  $g$ ,  $R_\alpha$  and  $R_\beta$  are accumulated in the  $C^{l-2-\eta}$ -norm by commuting diffeomorphisms  $f'_n, g'_n$  that are  $C^{l-1-\eta}$ -conjugated to  $f$  and  $g$  respectively.*

**Corollary 12.2.** *There is a Baire-dense set  $A_2 \subset \mathbb{R}$  such that if  $l \geq 3$  is an integer,  $f \in D^l(\mathbb{T}^1)$  has a rotation number  $\alpha \in A_2$ ,  $g$  is of class  $C^l$  such that  $fg = gf$  and  $\eta \in \mathbb{R}_+$ , then there exists two sequences of diffeomorphisms  $f_n$  and  $g_n$  that are  $C^{l-1-\eta}$ -conjugated to rotations, such that  $f_n g_n = g_n f_n$  and with  $f_n$  and  $g_n$  converging respectively towards  $f$  and  $g$  in the  $C^{l-2-\eta}$ -norm. Moreover, if  $\beta$  is the rotation number of  $g$ ,  $R_\alpha$  and  $R_\beta$  are accumulated in the  $C^{l-2-\eta}$ -norm by commuting diffeomorphisms  $f'_n, g'_n$  that are  $C^{l-1-\eta}$ -conjugated to  $f$  and  $g$  respectively.*

We derive theorem 9.2 from corollary 12.2 by following the same argument as in the proof of theorem 9.1.

### 12.1 The speed of approximation of $g$ by a linearizable and commuting diffeomorphism

To prove theorem 12.1, we consider  $(h_n)_{n \geq 0}$ , the sequence of conjugating diffeomorphisms constructed in the proof of theorem 11.1,  $(\lambda_n)_{n \geq 0}$  the associated sequence of real numbers such that  $f_n = R_{\lambda_n} f = h_n R_{\alpha_n} h_n^{-1}$ . We also consider  $g'_n = h_n^{-1} g h_n$  and  $g_n = h_n R_{g'_n(0)} h_n^{-1}$ . The diffeomorphisms  $f_n$  and  $g_n$  commute, and  $f_n \rightarrow f$  in the  $C^{l-2-\eta}$ -norm. To prove theorem 12.1, it suffices to show that  $g_n \rightarrow g$  and  $g'_n \rightarrow R_\beta$  in the  $C^{l-2-\eta}$ -norm. These convergences are based on the lemma (there is an analogous lemma for showing  $g'_n \rightarrow R_\beta$ ):

**Lemma 12.3.** *Let  $l \geq 3$  be an integer,  $f \in D^l(\mathbb{T}^1)$  of rotation number  $\alpha \in \mathbb{R}$ ,  $\eta > 0$ ,  $0 \leq u \leq l - 2 - \eta$ , and  $g \in D^l(\mathbb{T}^1)$  be such that  $fg = gf$ . Let  $(q_i)_{i \geq 0}$  be the sequence of denominators of the convergents of  $\alpha$ , and let  $r \geq 0$  be an integer. Let  $\tilde{\alpha}$  be an irrational number of constant type,  $\lambda_0 \in \mathbb{R}$  the associated number and  $h$  the associated  $C^{l-1-\eta}$  diffeomorphism given by lemma 11.3. Let  $f' = h^{-1} f h$  and  $g' = h^{-1} g h$ . We have the estimate:*

$$\|g - h R_{g'(0)} h^{-1}\|_u \leq C \|h\|_{u+1}^C \|f\|_u^C \|g\|_{u+1}^C \left( \frac{1}{q_r} + |\tilde{\alpha} - \alpha| \left( \frac{(C \|h\|_{u+1} \|f\|_u)^{C q_r}}{(\min Dh)^C} \right) \right).$$

To show this lemma, the basic idea is the following: we approach modulo 1 points  $x \in \mathbb{R}$  by  $p(x)\alpha \bmod 1$ , where  $p(x) \leq q_r$  is an integer, and where the integer  $r$  will be fixed later.

We have a control of  $|x - p(x)\alpha| \bmod 1$  in function of  $q_r$ . Then, by using the assumption of commutation  $g'f'^p = f'^pg'$ , we can write:

$$g'(x) - R_{g'(0)}(x) = g'(x) - g'(p\alpha) + g'(p\alpha) - g'f'^p(0) + f'^pg'(0) - R_{p\alpha}(g'(0)) + R_{g'(0)}(p\alpha) - R_{g'(0)}(x).$$

We use the distance of  $f'^p$  to  $R_{p\alpha}$ , which depends on  $q_r$  and the norm of  $f' - R_\alpha$ . This distance has been estimated in the proof of the result of quasi-reducibility. We also use  $C^k$  analogues,  $k \geq 2$ , of the mean value theorem, obtained with the Faa-di-Bruno formula. This allows to estimate the norm of  $g - hR_{g'(0)}h^{-1}$  in function of the norm of  $g' - R_{g'(0)}$ .

To obtain theorem 12.1 from lemma 12.3, we take  $\tilde{\alpha} = \alpha_n$ , and we consider the associated sequences  $f_n, g_n, f'_n, g'_n, h_n$ . The integer  $q_r$  must be chosen sufficiently large with respect to the conjugacy  $h_n$ , so that  $|x - p\alpha| \bmod 1$  is sufficiently small. However, this integer  $q_r$  must not be too large, to keep the norm of  $f_n'^p - R_{p\alpha}$  sufficiently small. This integer  $q_r$  is controlled with  $\sup_{k \leq r} a_k$ , which itself controls the norm of  $h_r$ . Thus, it suffices to properly choose the integer  $r$  in function of  $n$ , in order to obtain the convergence of  $g_n$  towards  $g$ .

*Proof of theorem 12.1.* Assuming lemma 12.3, we show theorem 12.1.

Let  $\tilde{\alpha} = \alpha_n$  and  $h_n$  be the associated diffeomorphism given by lemma 11.3. Since  $V_\alpha(n) \rightarrow +\infty$ , by applying the estimate for the conjugacy  $h_n$ , there exists  $\tilde{G}(x)$  strictly increasing with  $x$  such that, for  $n$  sufficiently large:

$$\|g - h_n R_{g'_n(0)} h_n^{-1}\|_{l-2-\eta} \leq e^{C\tilde{G}(V_\alpha(n))} \left( \frac{1}{q_r} + \frac{e^{C\tilde{G}(V_\alpha(n))q_r}}{2^n} \right).$$

Moreover, since  $q_n = a_n q_{n-1} + q_{n-2}$ , and  $q_{n-2} \leq q_{n-1}$ , then

$$(\sqrt{2})^{n-1} \leq q_n \leq \prod_{k=1}^n (a_k + 1). \quad (86)$$

Therefore, we get:

$$\|g - h_n R_{g'_n(0)} h_n^{-1}\|_{l-2-\eta} \leq e^{C\tilde{G}(V_\alpha(n)) - \frac{1}{2}(r-1)\log 2} + e^{C\tilde{G}(V_\alpha(n)) + C\tilde{G}(V_\alpha(n))(V_\alpha(n)+1)^r - n\log 2}. \quad (87)$$

Let  $G(n) = \tilde{G}^{-1}((\log n)^{1/2})$ . By extracting in the sequence  $V_\alpha(n)/G(n)$ , we can suppose that:

$$\frac{V_\alpha(n)}{G(n)} \rightarrow 0.$$

Therefore, for  $n$  sufficiently large, we have:

$$\tilde{G}(V_\alpha(n)) \leq (\log n)^{1/2}.$$

Moreover, for  $n$  sufficiently large, we can take an integer  $r_n$  such that:

$$(\log n)^{3/4} \leq r_n \leq (\log n)^{7/8}.$$

We get:

$$(V_\alpha(r_n) + 1)^{r_n} = e^{r_n \log(V_\alpha(r_n)+1)} \leq e^{(\log n)^{15/16}}$$

The first term in estimate (87) tends towards 0. Moreover, since, for  $n$  sufficiently large,

$$(\log n)^{1/2} e^{(\log n)^{15/16}} \leq \frac{n}{2} \log 2$$

then the second term also tends towards 0. Hence the first part of theorem 12.1.

In order to obtain the second part of theorem 12.1, by estimation (94) it suffices to show that  $g'_n \rightarrow R_\beta$ . We already have that  $g'_n - R_{g'_n(0)} \rightarrow 0$ , so it suffices to show that  $R_\beta - R_{g'_n(0)} \rightarrow 0$ , i.e. that  $g'_n(0) \rightarrow \beta$ .

Let  $\epsilon > 0$ . Since  $g'_n - R_{g'_n(0)} \rightarrow 0$ , there exists  $n_0$  such that for any  $n \geq n_0$ , for any  $x \in \mathbb{R}$ ,

$$|g'_n(x) - x - g'_n(0)| \leq \epsilon$$

Let  $x_0 \in \mathbb{R}$ ,  $x_i = g_n^i(x_0)$  and  $p \in \mathbb{N}$ . We have:

$$\sum_{i=0}^{p-1} |g'_n(x_i) - x_i - g'_n(0)| \leq p\epsilon$$

$$\left| \frac{g_n^p(x_0) - x_0}{p} - g'_n(0) \right| \leq \epsilon$$

By taking  $p \rightarrow +\infty$ , and since the rotation number of  $g'_n$  is  $\beta$ , we obtain:

$$|\beta - g'_n(0)| \leq \epsilon$$

Therefore,  $g'_n(0) \rightarrow \beta$ .

□

## 12.2 Higher-order analogous of the mean value theorem

*Proof of lemma 12.3.* We need two higher-order analogous of the mean value theorem. The first one is a direct application of corollary 11.6:

**Lemma 12.4.** *Let  $u \geq 0$ ,  $s, t \in D^u(\mathbb{T}^1)$ . Let  $\delta \in \mathbb{R}$ . We have:*

$$\|st - R_\delta t\|_u \leq C \|s - R_\delta\|_u \|t\|_u^u.$$

*Proof.* We apply corollary 11.6 with  $\Phi = s - R_\delta$  and  $\psi = t$ . Moreover, since  $t \in D^u(\mathbb{T})$ , then  $t(x+1) = t(x) + 1$ , and there is  $x_t$  such that  $Dt(x_t) = 1$ , which implies  $\|t\|_u^u \geq 1$  (with the convention  $\|t\|_0^0 = 1$  if  $t(x) = x$  for any  $x$ ). We get lemma 12.4. □

The second higher-order analogous of the mean value theorem is:

**Lemma 12.5.** *Let  $u \geq 0$ ,  $s \in D^{u+1}(\mathbb{T}^1)$ ,  $t \in D^u(\mathbb{T}^1)$ ,  $\delta \in \mathbb{R}$ . We have:*

$$\|st - sR_\delta\|_u \leq C\|s\|_{u+1}\|t\|_u^u\|t - R_\delta\|_u.$$

Observe the presence of the term  $\|t\|_u$ , which is absent in the mean value formula. This presence is because of an estimate on the Bell polynomial, in the Faa-di-Bruno formula. However, this estimate is not optimal.

*Proof.* If  $u = 0$ , the estimate holds, by the mean value formula. We suppose  $u \geq 1$ . We use the following lemma:

**Lemma 12.6.** *Let  $u \geq 1$ ,  $j \leq u$  be integers and  $a_1, \dots, a_{u-j+1}, x_1, \dots, x_{u-j+1} \geq 0$ . Let  $x \geq \max\{|x_k| \vee 1; 1 \leq k \leq u - j + 1\}$  and let  $a \geq \max\{|a_k|; 1 \leq k \leq u - j + 1\}$ . Let  $B_{u,j}$  be a Bell polynomial. We have:*

$$|B_{u,j}(x_1 + a_1, \dots, x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, \dots, x_{u-j+1})| \leq Ca(x+a)^u.$$

*Proof.* Let  $p \geq 1$  and  $l_1, \dots, l_p$  be integers. Then we have:

$$(x_1+a_1)^{l_1} \dots (x_p+a_p)^{l_p} - x_1^{l_1} \dots x_p^{l_p} = \sum_{i=1}^p x_1^{l_1} \dots x_{i-1}^{l_{i-1}} (x_i+a_i)^{l_i} \dots (x_p+a_p)^{l_p} - x_1^{l_1} \dots x_i^{l_i} (x_{i+1}+a_{i+1})^{l_{i+1}} \dots (x_p+a_p)^{l_p}$$

$$(x_1 + a_1)^{l_1} \dots (x_p + a_p)^{l_p} - x_1^{l_1} \dots x_p^{l_p} = \sum_{i=1}^p x_1^{l_1} \dots x_{i-1}^{l_{i-1}} (x_{i+1} + a_{i+1})^{l_{i+1}} \dots (x_p + a_p)^{l_p} \left[ (x_i + a_i)^{l_i} - x_i^{l_i} \right]$$

(with the conventions  $x_1^{l_1} \dots x_0^{l_0} = 1$  and  $x_{p+1}^{l_{p+1}} \dots x_p^{l_p} = 1$ ).

Since  $x+a \geq 1$  (because  $x \geq 1$ ), and since  $(x_i + a_i)^{l_i} - x_i^{l_i} \leq l_i |a_i| (|x_i| + |a_i|)^{l_i-1} \leq l_i a (x+a)^{l_i}$ ,  $1 \leq l_i \leq u$ , we obtain, by taking  $p = u - j + 1$ :

$$|B_{u,j}(x_1 + a_1, \dots, x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, \dots, x_{u-j+1})| \leq a(u-j+1)uB_{u,j}(x+a, \dots, x+a).$$

By the formula giving the Bell polynomials, we have:

$$B_{u,j}(x+a, \dots, x+a) \leq C(x+a)^u.$$

□

To show lemma 12.5, For any  $0 \leq v \leq u$ , we write:

$$D^v(st)(x) - D^v(sR_\delta)(x) = \sum_{j=0}^v D^j s(t(x)) \left[ B_{v,j} \left( Dt(x), \dots, D^{v-j+1} t(x) \right) - B_{v,j} \left( DR_\delta(x), \dots, D^{v-j+1} R_\delta(x) \right) \right] + \\ \left[ D^j s(t(x)) - D^j s(R_\delta(x)) \right] B_{v,j} \left( DR_\delta(x), \dots, D^{v-j+1} R_\delta(x) \right).$$

We apply lemma 12.6 with  $a = \|t - R_\delta\|_u$  and  $x = \|R_\delta\|_u = 1$  (because  $DR_\delta = 1$ ).

$$\left| B_{v,j} \left( Dt(x), \dots, D^{v-j+1} t(x) \right) - B_{v,j} \left( DR_\delta(x), \dots, D^{v-j+1} R_\delta(x) \right) \right| \leq C \|t - R_\delta\|_u (1 + \|t - R_\delta\|_u)^u$$

Since  $t \in D^u(\mathbb{T}^1)$ , then  $t(x+1) - t(x) = 1$ , and there exists  $x_t$  such that  $Dt(x_t) = 1$ , and  $\|t\|_u^u \geq 1$  (with the convention  $\|t\|_0^0 = 1$  if  $t(x) = x$  for any  $x$ ). We get:

$$\left| B_{v,j} \left( Dt(x), \dots, D^{v-j+1} t(x) \right) - B_{v,j} \left( DR_\delta(x), \dots, D^{v-j+1} R_\delta(x) \right) \right| \leq C \|t - R_\delta\|_u (2 + \|t\|_u)^u \leq C \|t - R_\delta\|_u \|t\|_u^u.$$

□

## 12.3 Successive estimates

To prove lemma 12.3, we also need these successive estimates:

**Lemma 12.7.** *Let  $l \geq 3$  be an integer,  $f \in D^l(\mathbb{T}^1)$  of rotation number  $\alpha \in \mathbb{R}$ ,  $\eta > 0$ ,  $0 \leq u \leq l - 2 - \eta$ , and  $g \in D^l(\mathbb{T}^1)$  be such that  $fg = gf$ . Let  $(q_t)_{t \geq 0}$  be the sequence of denominators of the convergents of  $\alpha$ . Let  $\tilde{\alpha}$  be an irrational number of constant type,  $\lambda_0 \in \mathbb{R}$  the associated number and  $h$  the associated  $C^{l-1-\eta}$  diffeomorphism given by lemma 11.3. Let  $f' = h^{-1}fh$  and  $g' = h^{-1}gh$ . We have the estimates:*

$$A_{1,u} = \|h^{-1}\|_u \leq C \|h\|_u^C \frac{1}{(\min Dh)^C} \quad (88)$$

$$A_{2,u} = \|f'\|_u \leq CA_{1,u} \|f\|_u^C \|h\|_u^C \quad (89)$$

$$A_{3,u}(m) = \|f'^m\|_u \leq C^m A_{2,u}^{mC} \quad (90)$$

$$A_{4,u} = \|f' - R_\alpha\|_u \leq C \|h\|_{u+1}^C \|f\|_u^C \frac{1}{(\min Dh)^C} |\tilde{\alpha} - \alpha| \quad (91)$$

$$A_{5,u}(m) = \|f'^m - R_{m\alpha}\|_u \leq mCA_{4,u} A_{2,u}^C \max_{k \leq m-1} A_{3,u+1}(k) \quad (92)$$



$$A_{6,u} = \|g'\|_u \leq CA_{1,u} \|g\|_u^C \|h\|_u^C \quad (93)$$

and for any integer  $r \geq 0$ , we have:

$$A_{7,u} = \|g' - R_{g'(0)}\|_u \leq \frac{A_{6,u+1} + 1}{q_r} + \max_{m \leq 2q_r} (A_{6,u+1} A_{3,u}^C(m) A_{5,u}(m) + A_{6,u}^C A_{5,u}(m)) \quad (94)$$

$$A_{8,u} = \|g'h^{-1} - R_\alpha h^{-1}\|_u \leq CA_{7,u} A_{1,u}^C \quad (95)$$

$$A_{9,u} = \|hg'h^{-1} - hR_{g'(0)}h^{-1}\|_u \leq C \|g\|_u^C A_{8,u} A_{1,u}^C \|h\|_{u+1}. \quad (96)$$

The crucial estimate is (94), which is obtained by approaching modulo 1 each  $x \in \mathbb{R}$  by a  $m(x)\alpha$ , with  $m(x) \leq q_r$ . If  $q_r$  increases,  $x - m(x)\alpha$  is smaller modulo 1, but the bound on  $A_{3,u}(m(x))$  and  $A_{5,u}(m(x))$  increases. In the proof of theorem 12.1, we make a proper choice of  $r$  (and  $q_r$ ).

Estimate (91) corresponds to estimate (85) of the proof of the result of quasi-reducibility.

The other estimates, namely, estimates (88),(89),(90), (92),(93), (95) and (96) are derived from applications of the Faa-di-Bruno formula: either corollary 11.6, lemma 12.4 or lemma 12.5.

*Proof of lemma 12.7.* For  $A_{1,u}$ , by estimate (83), we have:

$$\|h^{-1}\|_u \leq C \|h\|_u^C \frac{1}{(\min Dh)^C}.$$

Hence estimate (88).

For  $A_{2,u}$ , by applying corollary 11.6 twice, we have,

$$\|f'\|_u \leq CA_{1,u} \|f\|_u^C \|h\|_u^C.$$

Hence estimate (89).

For  $A_{3,u}$ , by applying corollary 11.6 again, we have, for any  $m$ ,

$$\|f'^{m+1}\|_u \leq C \|f'^m\|_u \|f'\|_u^C$$

and therefore, by iteration, we get:

$$\|f'^m\|_u \leq C^m \|f'\|_u^{mC}.$$

Hence (90).

Estimate (91) is a direct application of estimate (85).

For estimate (92), we observe that for any  $0 \leq v \leq u$ :

$$D^v f'^m - D^v R_{m\alpha} = D^v \sum_{k=0}^{m-1} f'^{m-k} R_{k\alpha} - f'^{m-k-1} R_{(k+1)\alpha}$$

$$D^v f'^m - D^v R_{m\alpha} = \sum_{k=0}^{m-1} D^v (f'^{m-k-1} f') R_{k\alpha} - D^v (f'^{m-k-1} R_\alpha) R_{k\alpha}.$$

By applying lemma 12.5, and by noting that for any  $k$ ,  $\|f'^{m-k-1}\|_{u+1} \leq \max_{0 \leq k \leq m-1} \|f'^k\|_{u+1}$ , we get:

$$\|f'^m - R_{m\alpha}\|_u \leq mC \|f'\|_u^C \max_{0 \leq k \leq m-1} \|f'^k\|_{u+1} \|f' - R_\alpha\|_u.$$

Hence (92).

For  $A_{6,u}$ , estimate (93) is the same as (89):

$$\|g'\|_u \leq C \|h^{-1}\|_u \|g\|_u^C \|h\|_u^C.$$

Hence (93).

For  $A_{7,u}$ , let  $m \geq 0$  and  $u \geq v \geq 1$ . For any  $x$ ,  $D^v R_\alpha(x) = \int_0^1 D^v g'(y) dy$ . Therefore,

$$|D^v g'(x) - D^v R_\alpha(x)| = \left| D^v g'(x) - \int_0^1 D^v g'(y) dy \right| =$$

$$\left| \int_0^1 (D^v g'(x) - D^v g'(y)) dy \right| \leq \max_{x,y \in [0,1]} |D^v g'(x) - D^v g'(y)|.$$

On the other hand, we have:

$$D^v g'(x) - D^v g'(y) = D^v g'(x) - D^v g'(y + m\alpha) + D^v g'(R_{m\alpha}(y)) - D^v(g' f'^m(y)) +$$

$$D^v(f'^m g'(y)) - D^v g'(y).$$

Moreover, we have:

$$|D^v g'(x) - D^v g'(y + m\alpha)| \leq |D^{u+1} g'|_0 |x - y - m\alpha|.$$

By lemma 12.5, we also have:

$$|D^v g'(R_{m\alpha}(y)) - D^v(g' f'^m(y))| \leq C \|g'\|_{u+1} \|f'^m\|_u^C \|f'^m - R_{m\alpha}\|_u.$$

Finally, by lemma 12.4, we have:

$$|D^v(f'^m g'(y)) - D^v(R_{m\alpha} g'(y))| \leq C \|f'^m - R_{m\alpha}\|_u \|g'\|_u^C.$$

Since  $R_{m\alpha} g'(y) = g'(y) + m\alpha$ , and  $v \geq 1$ , then  $D^v(R_{m\alpha} g'(y)) = D^v g'(y)$ . Therefore, we also have:

$$|D^v(f'^m g'(y)) - D^v g'(y)| \leq C \|f'^m - R_{m\alpha}\|_u \|g'\|_u^C.$$

By combining these estimates, we obtain:

$$|D^v g'(x) - D^v g'(y)| \leq \|g'\|_{u+1} |x - y - m\alpha| + C \|g'\|_{u+1} \|f'^m\|_u^C \|f'^m - R_{m\alpha}\|_u + C \|f'^m - R_{m\alpha}\|_u \|g'\|_u^C.$$

Moreover, for any  $r \geq 0$ , any  $x, y \in \mathbb{R}$ , there is an integer  $m(x, y) \leq 2q_r$ , there are real numbers  $x', y'$  such that  $x' - x \in \mathbb{Z}$ ,  $y' - y \in \mathbb{Z}$  and such that  $|x' - y' - m(x, y)\alpha| \leq 1/q_r$ . Since  $v \geq 1$ , then  $|D^v g'(x) - D^v g'(y)| = |D^v g'(x') - D^v g'(y')|$ . We apply the former estimate with  $x'$  and  $y'$  and we get:

$$\max_{1 \leq v \leq u} \|D^v g' - D^v R_{g'(0)}\|_0 \leq \frac{A_{6,u+1} + 1}{q_r} + \max_{m \leq 2q_r} (A_{6,u+1} A_{3,u}^C(m) A_{5,u}(m) + A_{6,u}^C A_{5,u}(m)).$$

If  $v = 0$ , we note that for any  $r \geq 0$ , any  $x \in \mathbb{R}$ , there is an integer  $m(x) \leq q_r$  and a real number  $x' \in \mathbb{R}$  such that  $x' - x \in \mathbb{Z}$ , and such that  $|x' - m(x)\alpha| \leq 1/q_r$ . Moreover, we have:  $g'(x) - R_{g'(0)}(x) = g'(x') - R_{g'(0)}(x')$ , and

$$g'(x') - R_{g'(0)}(x') = g'(x') - g'(m\alpha) + g'(m\alpha) - g' f'^m(0) + f'^m g'(0) - R_{m\alpha}(g'(0)) + R_{g'(0)}(m\alpha) - R_{g'(0)}(x').$$

Hence estimate (94).

For  $A_{8,u}$ , estimate (95) follows immediately from lemma 12.4.

For  $A_{9,u}$ , let  $x \in \mathbb{R}$ . Let  $0 \leq v \leq u$ . By the Faa-di-Bruno formula:

$$\begin{aligned}
D^v(hg'h^{-1})(x) - D^v(hR_{g'(0)}h^{-1})(x) &= \\
&\sum_{j=0}^v D^j h(g'h^{-1}(x)) B_{v,j} \left( D(g'h^{-1})(x), \dots, D^{v-j+1}(g'h^{-1}(x)) \right) - \\
&D^j h(g'h^{-1}(x)) B_{v,j} \left( D(R_{g'(0)}h^{-1})(x), \dots, D^{v-j+1}(R_{g'(0)}h^{-1}(x)) \right) \\
&= \sum_{j=0}^v D^j h(g'h^{-1}(x)) \\
&\left[ B_{v,j} \left( D(g'h^{-1})(x), \dots, D^{v-j+1}(g'h^{-1}(x)) \right) - B_{v,j} \left( D(R_{g'(0)}h^{-1})(x), \dots, D^{v-j+1}(R_{g'(0)}h^{-1}(x)) \right) \right] - \\
&\left[ D^j h(R_{g'(0)}h^{-1}(x)) - D^j h(g'h^{-1}(x)) \right] B_{v,j} \left( D(R_{g'(0)}h^{-1})(x), \dots, D^{v-j+1}(R_{g'(0)}h^{-1}(x)) \right).
\end{aligned}$$

Since  $\|h^{-1}\|_u \geq 1$ , then lemma 12.6 gives,

$$\left| B_{v,j} \left( D(g'h^{-1})(x), \dots, D^{v-j+1}(g'h^{-1}(x)) \right) - B_{v,j} \left( D(R_{g'(0)}h^{-1})(x), \dots, D^{v-j+1}(R_{g'(0)}h^{-1}(x)) \right) \right| \leq C \|g'h^{-1}\|_u^C \|g'h^{-1} - R_{g'(0)}h^{-1}\|_u.$$

Since  $g'h^{-1} = h^{-1}g$  and  $\|h^{-1}g\|_u \leq C \|h^{-1}\|_u \|g\|_u^C$ , we get,

$$\begin{aligned}
\left| D^v(hg'h^{-1})(x) - D^v(hR_{g'(0)}h^{-1})(x) \right| &\leq \\
&C \|g\|_u^C \|h\|_u \|h^{-1}\|_u^C \|g'h^{-1} - R_{g'(0)}h^{-1}\|_u + C \|h\|_{u+1} \|g'h^{-1} - R_{g'(0)}h^{-1}\|_u \|h^{-1}\|_u^C.
\end{aligned}$$

Hence estimate (96). This completes the proof of lemma 12.7. □

By combining these estimates, we obtain:

$$\begin{aligned}
A_{9,u} &\leq CA_{1,u+1}^C \|h\|_{u+1}^C \|g\|_{u+1}^C \left( \frac{1}{q_r} + \max_{m \leq 2q_r} (A_{3,u}^C(m) A_{5,u}(m)) \right) \\
A_{9,u} &\leq C \|h\|_{u+1}^C \|f\|_u^C \|g\|_{u+1}^C \left( \frac{1}{q_r} + |\tilde{\alpha} - \alpha| \left( \frac{(C \|h\|_{u+1} \|f\|_u)^{Cq_r}}{(\min Dh)^C} \right) \right).
\end{aligned}$$

Hence lemma 12.3. Notice the loss of one derivative for  $h$ . □

### 13 Appendix: proof of the $C^\infty$ -genericity of $(O_{\alpha,\beta}^1)^c$ in $F_{\alpha,\beta}$

To show that  $(O_{\alpha,\beta}^1)^c$  is  $C^\infty$ -generic in  $F_{\alpha,\beta}$ , we slightly modify [Her79, p.160, p.167]. Let  $H : F_{\alpha,\beta} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  be defined by  $H(f, g) = \sup_{n \geq 1} (\|Df^n\|_0, \|Dg^n\|_0)$ .

The map  $H$  is lower semi-continuous, because it is an upper bound of a family of continuous maps. Therefore,  $\{(f, g) \in F_{\alpha,\beta} | H(f, g) > n\}$  is open, and

$H^{-1}(+\infty) = \bigcap_{n \geq 1} \{(f, g) \in F_{\alpha,\beta} | H(f, g) > n\}$  is a  $G_\delta$ -set (i.e. a countable intersection of open sets).

By [Her79, p.52],  $(O_{\alpha,\beta}^1)^c = H^{-1}(+\infty)$  ( $f$  and  $g$  are not  $C^1$ -conjugated to a rotation if and only if  $H(f, g) = +\infty$ ). By the first part of remark 9.5,  $S_{\alpha,\beta} \subset (O_{\alpha,\beta}^1)^c$  is  $C^\infty$ -dense. Since  $C^1$ -open sets are  $C^\infty$ -open (if  $\phi_n$  does not converge to  $\phi$  in the  $C^1$  norm, then  $\phi_n$  does not converge to  $\phi$  in the  $C^\infty$  norm), we conclude that  $(O_{\alpha,\beta}^1)^c$  is  $C^\infty$ -generic in  $F_{\alpha,\beta}$ .

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## Part III

# Non-standard smooth realization of translations on the torus

## Summary

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Let  $M$  be a smooth compact connected manifold of dimension greater than two, on which there exists an effective smooth circle action preserving a positive smooth volume. For example, in dimension 2,  $M$  can be the annulus  $[0, 1] \times \mathbb{T}$ , the torus  $\mathbb{T}^2$ , the sphere  $\mathbb{S}^2$ , the real projective plane  $\mathbb{R}P^2$  or the Klein bottle  $K^2$ . In this chapter, we construct volume-preserving diffeomorphisms on  $M$  that are metrically isomorphic to ergodic translations on the torus, where one given coordinate of the translation is an arbitrary Liouville number. To obtain this result, we determine sufficient conditions on translation vectors of the torus that allow to explicitly construct the sequence of successive conjugacies in Anosov-Katok's method, with suitable estimates of their norm. To visualize the construction, we include numerous graphics.

## 14 Introduction

In this paper, we construct non-standard smooth realizations of some ergodic translations on the torus, translations with one arbitrary Liouville coordinate. A *smooth realization* of an abstract system  $(X, T, \nu)$  is a triplet  $(M, f, \mu)$ , where  $M$  is a smooth compact manifold,  $\mu$  a smooth measure on  $M$  and  $f$  a smooth  $\mu$ -preserving diffeomorphism of  $M$ , such that  $(M, f, \mu)$  is metrically isomorphic to  $(X, T, \nu)$ . Moreover, a smooth realization is *non-standard* if  $M$  and  $X$  are not diffeomorphic.

In their 1969 seminal paper, Anosov and Katok [AK70] constructed ergodic rotations on the circle that admit non-standard smooth realizations. Fayad et al. [FSW07] showed that all Liouvillean rotations of the circle admit non-standard smooth realizations. [FS05] also enriched the original work of [AK70]. In their paper [AK70], Anosov and Katok also constructed ergodic translations on the torus  $\mathbb{T}^h$ ,  $h \geq 2$ , that admit non-standard smooth realizations. In this paper, we show the following result: for any Liouville number  $\beta$ , the ergodic translation on the torus  $\mathbb{T}^h$  of vector  $(\beta_1, \dots, \beta_{h-1}, \beta)$  admits a non-standard smooth realization, where the  $\beta_i, i = 1, \dots, h-1$  are chosen in a dense set of  $\mathbb{T}^{h-1}$ . More precisely, we show the following theorem:

**Theorem 14.1.** *Let  $\beta \in \mathbb{T}$  be a Liouville number,  $h \geq 2$  a positive integer. Let  $M$  be a smooth compact connected manifold of dimension  $d \geq 2$ , on which there exists an effective smooth circle action  $S_t$  preserving a positive smooth measure  $\mu$ . There exists a dense set  $E(\beta, d) \subset \mathbb{T}^{h-1}$  such that for any  $(\beta_1, \dots, \beta_{h-1}) \in E(\beta, d)$ , there is  $T \in \text{Diff}^\infty(M, \mu)$  metrically isomorphic to the ergodic translation of vector  $(\beta_1, \dots, \beta_{h-1}, \beta)$ .*

To obtain this result, we explicitly construct the sequence of successive conjugacies in Anosov-Katok's periodic approximation method, with suitable estimates of their norm. Moreover, to obtain this result, we need to suitably relax one of Anosov-Katok's original assumptions.

### 14.1 Definitions

An irrational number  $\beta$  is *Liouville* if, for any  $k > 0$ , there is a sequence of integers  $q_n \rightarrow +\infty$  such that  $q_n^k \inf_{p \in \mathbb{Z}} |q_n \beta - p| \rightarrow 0$ . Liouville numbers are the complementary of Diophantine numbers in the set of irrational numbers. Let  $\mathbb{T}^h = \mathbb{R}^h / \mathbb{Z}^h$  denote the  $h$ -dimensional torus and  $\mu_h$  the Haar measure on  $\mathbb{T}^h$ . Let  $\text{Diff}^\infty(M, \mu)$  be the class of smooth

diffeomorphisms of  $M$  preserving a smooth measure  $\mu$ . For  $B \in \text{Diff}^\infty(M, \mu)$  and  $j \in \mathbb{Z}^*$ , let  $D^j B$  be the  $j^{\text{th}}$  derivative of  $B$  if  $j > 0$ , and the  $-j^{\text{th}}$  derivative of  $B^{-1}$  if  $j < 0$ . For  $x \in M$ , let  $|D^j B(x)|$  be the norm of  $D^j B(x)$  at  $x$ . We denote  $\|B\|_k = \max_{0 < |j| \leq k} \max_{x \in M} |D^j B(x)|$ .

A *finite measurable partition*  $\bar{\xi}$  of a measured manifold  $(N, \nu)$  is the equivalence class of a finite set  $\xi$  of disjoint measurable subsets of  $N$  whose union is  $N$ , modulo sets of  $\nu$ -measure zero. In most of this paper, we do not distinguish a partition  $\xi$  with its equivalent class  $\bar{\xi}$  modulo sets of  $\nu$ -measure zero. In these cases, both are denoted  $\xi$ . Moreover, all partitions considered in this paper are representatives of a finite measurable partition.

Let  $\xi = \{c_1, \dots, c_{|\xi|}\}$  and  $\xi' = \{c'_1, \dots, c'_{|\xi'|}\}$  be finite measurable partitions of the same size. Their distance is defined by:

$$d(\xi, \xi') = \inf_{\sigma \in \text{Perm}(|\xi|)} \sum_{i=1}^{|\xi|} \nu(c_i \Delta c'_{\sigma(i)})$$

where  $\text{Perm}(|\xi|)$  denotes the group of permutations of  $\{1, \dots, |\xi|\}$ .

A partition  $\xi'$  is *subordinate* to a partition  $\xi$  if any element of  $\xi$  is a union of elements of  $\xi'$ , modulo sets of  $\nu$ -measure zero. In this case, if  $\mathcal{B}(\xi)$  denotes the completed algebra generated by  $\xi$ , then  $\mathcal{B}(\xi) \subset \mathcal{B}(\xi')$ . The inclusion map  $i : \mathcal{B}(\xi) \rightarrow \mathcal{B}(\xi')$  will be denoted  $\xi \hookrightarrow \xi'$ . This notation also means that  $\xi'$  is *subordinate* to  $\xi$ . A sequence of partitions  $\xi_n$  is *monotonic* if for any  $n$ ,  $\xi_n \hookrightarrow \xi_{n+1}$ . These definitions and properties are independent of the choice of the representatives  $\xi$  and  $\xi'$  of the equivalence classes  $\bar{\xi}$  and  $\bar{\xi}'$ .

A measure-preserving bijective bimeasurable map  $T : (M_1, \mu_1, \mathcal{B}_1) \rightarrow (M_2, \mu_2, \mathcal{B}_2)$  induces an *isomorphism* of measure algebras, still denoted  $T : (\mu_1, \mathcal{B}_1) \rightarrow (\mu_2, \mathcal{B}_2)$ . If  $\xi_1, \xi_2$  are partitions, and if  $\mathcal{B}_1 = \mathcal{B}(\xi_1)$  and  $\mathcal{B}_2 = \mathcal{B}(\xi_2)$ , we denote  $T : \xi_1 \rightarrow \xi_2$  this induced isomorphism of measure algebras. If  $M_1 = M_2$ ,  $\mu_1 = \mu_2$  and  $\mathcal{B}_1 = \mathcal{B}_2$ , then  $T$  is a *measure-preserving transformation*. Its induced isomorphism is an *automorphism* (see [Hal56, p.43] and [Wei72]).

A *metric isomorphism* of measure preserving transformations  $T_1 : (M_1, \mu_1, \mathcal{B}_1) \rightarrow (M_1, \mu_1, \mathcal{B}_1)$ ,  $T_2 : (M_2, \mu_2, \mathcal{B}_2) \rightarrow (M_2, \mu_2, \mathcal{B}_2)$  is a measure-preserving bijective bimeasurable map  $L : (M_1, \mu_1, \mathcal{B}_1) \rightarrow (M_2, \mu_2, \mathcal{B}_2)$  such that  $LT_1 = T_2L$  a.e. For convenience, when the measure is the Lebesgue measure and the algebra is the Borelian algebra, we omit to mention the measures and algebras, and we simply say that  $L : (M_1, T_1) \rightarrow (M_2, T_2)$  is a metric isomorphism.

Let  $\bar{\xi}$  be a measurable partition and  $\xi$  a representative of this equivalent class modulo sets of  $\mu$ -measure zero. For  $x \in M$ , we denote  $\xi(x)$  the element of the partition  $\xi$  such that  $x \in \xi(x)$ . A sequences of partitions  $\xi_n$  of measurable sets *generates* if there is a set of full measure  $F$  such that for any  $x \in F$ ,

$$\{x\} = F \bigcap_{n \geq 1} \xi_n(x)$$

This property of generation is independent of the choice of the representatives  $\xi_n$  of the equivalent class  $\bar{\xi}_n$  and therefore, we will say that the sequence of measurable partitions  $\bar{\xi}_n$  generates. Let  $M/\xi$  denote the equivalent class of the algebra generated by  $\xi$ , modulo sets of  $\mu$ -measure zero.  $M/\xi$  is independent of the choice of the representative  $\xi$  of the equivalent class



$\bar{\xi}$ . If  $T : M_1 \rightarrow M_2$  is a measure preserving map such that  $T(\xi_1) = \xi_2$   $\mu$ -almost everywhere, we can define a quotient map:  $T/\xi_1 : M/\xi_1 \rightarrow M/\xi_2$ .

An *effective action* of a group  $G$  on  $M$  is an action such that there is a set of full measure  $F \subset M$  such that for any  $x \in F$ , there is  $g \in G$  such that  $gx \neq x$ . When  $M = [0, 1]^{d-1} \times \mathbb{T}^1$ , we consider the periodic flow  $S_t$  defined by:

$$\begin{aligned} S_t : [0, 1]^{d-1} \times \mathbb{T}^1 &\rightarrow [0, 1]^{d-1} \times \mathbb{T}^1 \\ (x, s) &\mapsto (x, t + s \pmod{1}) \end{aligned}$$

For  $a, b \in \mathbb{T}^1$ , let  $[a, b[$  be the positively oriented circular sector between  $a$  and  $b$ , with  $a$  included and  $b$  excluded.

The *diameter*  $diam(\Gamma)$  of a domain  $\Gamma \subset M$  is defined by:  $diam(\Gamma) = \max_{x, y \in \Gamma} d(x, y)$ , where  $d(x, y)$  is the distance between  $x$  and  $y$ .

Let  $h \geq 2$  and  $\gamma = (\gamma_1, \dots, \gamma_h) \in \mathbb{Z}^h$ , with  $\gamma_i, i = 1, \dots, h$  relatively prime. Let  $\{T^{t\gamma}\}_{t \geq 0}$  a periodic flow on  $\mathbb{T}^h$ . This flow has a fundamental domain  $\Gamma \subset \mathbb{T}^{h-1} \times \{0\}$ . The boundary of  $\Gamma$  is of dimension  $h - 2$ .

For example, when  $h = 2$ , there is a fundamental domain of the flow  $\{T^{t\gamma}\}_{t \geq 0}$  that is a segment line of length  $1/\gamma_2$  (this can be seen using the Bezout identity for  $(\gamma_1, \gamma_2)$ ). Remark that the 0-volume of its boundary is equal to 2 (the 0-volume of a set of points is its cardinal).

Finally, a sequence  $T_n$  of  $\mu$ -preserving maps *weakly converges* to  $T$  if, for any measurable set  $E$ ,  $\mu(T_n E \Delta T E) \rightarrow 0$ , where  $A \Delta B = (A - B) \cup (B - A)$ .

## 14.2 Basic steps of the proof

The metric isomorphism of theorem 14.1 is obtained as the limit of isomorphisms of finite algebras: indeed, we use the lemma [AK70, p.18]:

**Lemma 14.2.** *Let  $M_1$  and  $M_2$  be Lebesgue spaces and let  $\xi_n^{(i)}$  ( $i = 1, 2$ ) be monotonic and generating sequences of finite measurable partitions of  $M_i$ . Let  $T_n^{(i)}$  be automorphisms of  $M_i$  such that  $T_n^{(i)} \xi_n^{(i)} = \xi_n^{(i)}$  and  $T_n^{(i)} \rightarrow T^{(i)}$  in the weak topology. Suppose there are metric isomorphisms  $L_n : M_1/\xi_n^{(1)} \rightarrow M_2/\xi_n^{(2)}$  such that*

$$L_n T_n^{(1)} / \xi_n^{(1)} = T_n^{(2)} / \xi_n^{(2)} L_n$$

and

$$L_{n+1} \xi_n^{(1)} = \xi_n^{(2)}$$

then  $(M_1, T_1)$  and  $(M_2, T_2)$  are metrically isomorphic.

Said otherwise, if we have generating sequences of partitions and sequences of automorphisms  $T_n^{(i)}$  weakly converging towards  $T^{(i)}$ , and if, for any integer  $n$ , the following diagram commutes:

$$\begin{array}{ccc}
T_n^{(1)} \curvearrowright \xi_n^{(1)} & \xrightarrow{L_n} & \xi_n^{(2)} \curvearrowright T_n^{(2)} \\
\downarrow & & \downarrow \\
\xi_{n+1}^{(1)} & \xrightarrow{L_{n+1}} & \xi_{n+1}^{(2)}
\end{array}$$

then  $(M_1, T_1)$  and  $(M_2, T_2)$  are metrically isomorphic.

The proof of theorem 14.1 is in two steps. In the first step (lemma 14.3), we determine sufficient conditions on a sequence  $(T_{q_n}^{\frac{p_n}{q_n} \gamma^{(n)}})_{n \geq 0}$  of periodic translations of  $\mathbb{T}^h$  such that there exists sequences of finite partitions and automorphisms satisfying the assumptions of lemma 14.2 with  $M_1 = \mathbb{T}^h$ ,  $M_2 = M$  and  $T_n^{(1)} = T_{q_n}^{\frac{p_n}{q_n} \gamma^{(n)}}$ ,  $T_n^{(2)} = T_n$ , where  $T_n$  is also a smooth diffeomorphism, and such that the limit  $T$  in the smooth topology of the sequence  $T_n$  is smooth.

In the second step (lemma 14.5), we construct a sequence of translations  $(T_{q_n}^{\frac{p_n}{q_n} \gamma^{(n)}})_{n \geq 0}$  satisfying those conditions, such that it converges towards the translation of vector  $(\beta_1, \dots, \beta_{h-1}, \beta)$ .

**Lemma 14.3.** *There exist explicit sequences of increasing integers  $R_2(n)$ ,  $R_3(n)$ , such that, if there exist sequences of integers  $p_n, q_n$  and  $\gamma^{(n)} = (\gamma_1^{(n)}, \dots, \gamma_h^{(n)}) \in \mathbb{Z}^h$  such that, for any integer  $n$ ,*

1. (primality)  $\gcd(\gamma_1^{(n)}, \dots, \gamma_h^{(n)}) = 1$
2. (vertical monotonicity, generation on  $M$ )  $(\gamma_h^{(n+1)} q_n)^{R_3(n)}$  divides  $q_{n+1}$
3. (horizontal monotonicity)  $\gamma_h^{(n)}$  divides  $\gamma_h^{(n+1)}$
4. (isomorphism, horizontal monotonicity for  $h \geq 3$ )  $q_n$  divides  $\gamma_1^{(n+1)} - \gamma_1^{(n)}, \dots, \gamma_h^{(n+1)} - \gamma_h^{(n)}$ .
5. (convergence of the diffeomorphism)

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \leq \frac{1}{(\gamma_h^{(n+1)} q_n)^{R_2(n)}}$$

6. (horizontal convergence of the partition, generation on  $\mathbb{T}^h$ ) There exists  $\Gamma^{(n)} \subset \mathbb{T}^{h-1} \times \{0\}$  a fundamental domain of the flow  $\{T^{t\gamma^{(n)}}\}$ , such that if  $d_n$  is the diameter of  $\Gamma^{(n)}$ , and  $\sigma_n$  is the  $(h-2)$ -dimensional volume of the boundary of  $\Gamma^{(n)}$ , then

$$d_{n+1} \leq \frac{1}{2^n \gamma_h^{(n)} \sigma_n}$$

7. (vertical convergence of the partition)

$$\sum_{n \geq 0} \frac{(\gamma_h^{(n)})^2 \sigma_n}{q_n} \left| \frac{\gamma_h^{(n+1)}}{\gamma_h^{(n+1)}} - \frac{\gamma_h^{(n)}}{\gamma_h^{(n)}} \right| < +\infty$$

then all these assumptions imply that there is  $\alpha \in \mathbb{T}^h$  such that

$$\frac{p_n}{q_n} \gamma^{(n)} \rightarrow \alpha$$

and there is a smooth ergodic measure preserving diffeomorphism  $T$  of  $M$  such that  $(\mathbb{T}^h, T^\alpha, \mu_h)$  is metrically isomorphic to  $(M, T, \mu)$ .

*Remark 14.4.* Assumption 3 is automatically derived from assumptions 2 and 4. However, the construction is easier to follow if assumption 3 is sorted out.

In assumption 2, we only use that  $\gamma_h^{(n)}$  and  $q_{n-1}$  divide  $q_n$ , except page 141, where the full assumption is used to get generation.

**Lemma 14.5.** *Let  $\beta$  Liouville. There exists a dense set  $E(\beta, d) \subset \mathbb{T}^{h-1}$ , such that for any  $(\beta_1, \dots, \beta_{h-1}) \in E(\beta, d)$ , there exist sequences of integers  $p_n, q_n$  and  $\gamma^{(n)} = (\gamma_1^{(n)}, \dots, \gamma_h^{(n)}) \in \mathbb{Z}^h$  satisfying the assumptions of lemma 14.3, such that*

$$\frac{p_n}{q_n} \gamma^{(n)} \xrightarrow{\text{mod } 1} (\beta_1, \dots, \beta_{h-1}, \beta)$$

We divide the proof of lemma 14.3 in three parts. In the first part of the proof, we construct a monotonic and generating partition on the  $h$ -dimensional torus  $\mathbb{T}$ , called  $\zeta_n^\infty$ , which stabilizes the translation  $T^{\frac{p_n}{q_n} \gamma^{(n)}}$ . To that end, we use assumptions 2, 3, 4, 6 and 7 (for  $h = 2$ , assumption 3 is not used and assumption 6 can be relaxed).

In the second part of the proof, we construct an isomorphism  $\bar{K}_n^\infty$  between  $\zeta_n^\infty$  on  $\mathbb{T}^h$  and a partition  $\xi_n^\infty$  on the manifold  $M$ . In the construction of this isomorphism, assumption 4 is important. Moreover, we will see that the elements of  $\xi_n^\infty$  are not the most elementary, because they must be chosen in a way that ensures the monotonicity of the sequence  $\bar{K}_n^\infty$ . This condition of monotonicity induces combinatorial constraints on the elements of the partition  $\xi_n^\infty$ .

In the third part of the proof, we construct diffeomorphisms  $T_n = B_n^{-1} S^{\frac{p_n}{q_n}} B_n$  on  $M$  stabilizing  $\xi_n^\infty$ , obtained by successive conjugations from the rotation  $S^{\frac{p_n}{q_n}}$ . The conjugacy  $B_n$  is constructed explicitly. This is a major difference with Anosov-Katok's original construction [AK70]. Ours rather elaborates upon the work of [FSW07]. Our construction provides suitable estimates of the norm of the conjugacy  $\|B_n\|_n$  (an estimate that is a polynomial function of  $q_{n-1} \gamma_h^{(n)}$ ). Combined with condition 5, these estimates ensure the convergence of the sequence of diffeomorphisms  $T_n$ .

A second substantive difference with Anosov-Katok's original work is condition 5: contrary to [AK70], we do not require the numbers  $p_n$  and  $q_n$  to be relatively prime. Thus, we obtain a larger limit set of numbers. This assumption of relative primality was not really needed in Anosov-Katok's original construction.<sup>2</sup> In order to obtain the convergence of the sequence of diffeomorphisms  $T_n$ , and the ergodicity of its limit  $T$ , it suffices to require that  $p_{n+1}/q_{n+1}$  and  $p_n/q_n$  are close enough. To obtain ergodicity, we slightly modify Anosov-Katok's argument (they kept  $p_n$  and  $q_n$  relatively prime to get ergodicity).

Thus, contrary to Anosov-Katok's construction, elements of the partitions  $\zeta_n^\infty$  and  $\xi_n^\infty$  are not fundamental domains of  $T^{\frac{p_n}{q_n} \gamma^{(n)}}$  and  $T_n$ , respectively. Instead, fundamental domains of these transformations are obtained as unions of elements of those partitions. In other words, we "de-maximize" partitions to the extent permitted by condition 5. This generalization is the same as in [FSW07], except that these authors do not present it in this way.

First, we prove lemma 14.5.

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2. On the other hand, though the assumption  $\gcd(\gamma_1^{(n)}, \dots, \gamma_h^{(n)}) = 1$  is not necessary, relaxing this assumption does not lead to something interesting. Indeed, condition 6 requires a control of the speed of growth of  $\gcd(\gamma_1^{(n)}, \dots, \gamma_h^{(n)})$ . Moreover, the explicit construction of the limit translation keeps constant this gcd. Therefore, "de-maximizing" in the  $\gamma_i^{(n)}$  does not seem to improve the result.

### 14.3 Constructions of the limit translations: proof of lemma 14.5

First, we show the existence of one translation with  $\beta$  at the  $h^{\text{th}}$  coordinate. Second, we show the density for the  $h - 1^{\text{th}}$  first coordinates. This will prove lemma 14.5. Moreover, to show the existence of one translation, we first tackle the case of  $h = 2$ , simpler to write and read. Then, we consider the general case  $h \geq 2$ .

#### 14.3.1 Proof of the existence of one translation for $h = 2$

We divide the proof of the case  $h = 2$  in three steps. Let  $\beta$  be a Liouville number. First, let  $P(n) = 5R_2(n) + 4R_2(n)R_3(n)$ . Since  $\beta$  is Liouville, we can find a sequence  $p'_n, q'_n$  such that  $p'_0 = 0, q'_0 = 1$ , such that for  $n \geq 1, q'_n \geq q'_{n-1}$  and such that:

$$\left| \beta - \frac{p'_n}{q'_n} \right| \leq \frac{1}{2q_n'^{P(n)}} \quad (97)$$

The second step is based on the following claim:

**Claim 14.6.** *There exists sequences  $\gamma_1^{(n)}, \gamma_2^{(n)}, q_n$  such that for any  $n$ :*

- (i)  $q'_n \geq \gamma_2^{(n)}$  and  $q'_n \geq q_{n-1}$
- (ii)  $q_n = q'_n(q_{n-1}\gamma_2^{(n)})^{R_3(n)}$
- (iii)  $\gamma_1^{(n)}$  and  $\gamma_2^{(n)}$  are relatively prime

*Proof.* We proceed by induction. Let  $q_0 = \gamma_1^{(0)} = \gamma_2^{(0)} = 1$  and suppose that  $\gamma_2^{(n)}, q_n$  are constructed. Let us define  $\gamma_2^{(n+1)}, q_{n+1}$ : by the Bezout theorem, since  $\gamma_1^{(n)}$  and  $\gamma_2^{(n)}$  are relatively prime, there are  $m_{1,n}, m_{2,n}$  such that  $m_{1,n}\gamma_2^{(n)} - m_{2,n}\gamma_1^{(n)} = 1$ . Moreover, we can take  $0 < m_{2,n} \leq (\gamma_2^{(n)})^2$  (for this upper bound, any  $(\gamma_2^{(n)})^p$ , with  $p \geq 1$  a fixed function of  $n$  would work).

Let  $\gamma_1^{(n+1)} = \gamma_1^{(n)} + m_{1,n}q_n$  and  $\gamma_2^{(n+1)} = \gamma_2^{(n)} + m_{2,n}q_n$ . Then  $\gamma_1^{(n+1)}$  and  $\gamma_2^{(n+1)}$  satisfy a Bezout equation and therefore, they are relatively prime. Moreover, since  $R_3(n-1) \leq R_3(n)$ ,

$$q'_{n+1} \geq q_n'^{1+2R_3(n)} \geq q_n$$

and

$$\gamma_2^{(n+1)} \leq \gamma_2^{(n)} + (\gamma_2^{(n)})^2 q_n \leq q'_n + q_n'^{3+2R_3(n)} \leq q_n'^{4+2R_3(n)} \leq q'_{n+1}$$

We have thus completed the construction at rank  $n + 1$ .

□

By claim 14.6, conditions 1, 2, 3 of lemma 14.3 hold. Moreover, by construction,  $\gamma_1^{(n+1)}$  and  $\gamma_2^{(n+1)}$  satisfy condition 4 of lemma 14.3. Moreover, conditions 6 and 7 of lemma 14.3 automatically hold: indeed, by property (ii), and since  $m_{2,n} > 0$ , then  $\gamma_2^{(n+1)} \geq 2^n \gamma_2^{(n)}$  and therefore,

$$\frac{1}{\gamma_2^{(n+1)}} \leq \frac{1}{2^n \gamma_2^{(n)}}$$

hence condition 6 (indeed, when  $\gcd(\gamma_1, \gamma_2) = 1$ , the diameter of the fundamental domain of the translation of vector  $(\gamma_1, \gamma_2)$  is  $1/\gamma_2$ ).

To obtain 7, note that since

$$\frac{\gamma_1^{(n+1)}}{\gamma_2^{(n+1)}} - \frac{\gamma_1^{(n)}}{\gamma_2^{(n)}} = \frac{q_n}{(\gamma_2^{(n)})^2(1 + m_{2,n}q_n/\gamma_2^{(n)})}$$

and since, by (ii),  $q_n/\gamma_2^{(n)} \geq 2^n$ , then

$$\left| \frac{\gamma_1^{(n+1)}}{\gamma_2^{(n+1)}} - \frac{\gamma_1^{(n)}}{\gamma_2^{(n)}} \right| \leq \frac{q_n}{(\gamma_2^{(n)})^2 2^n} \quad (98)$$

hence condition 7.

It remains to obtain condition 5. We define  $p_n$  recursively by:  $p_0 = 0$  and:

$$p_{n+1} = \frac{q_{n+1}}{\gamma_2^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right) + \frac{q_{n+1}}{q_n} p_n \quad (99)$$

Given the definitions of  $q_{n+1}$  and  $\gamma_2^{(n+1)}$ ,  $p_{n+1}$  is an integer. Let us check condition 5. By estimate (97), we have:

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \leq \frac{2}{q_n'^{P(n)} \gamma_2^{(n+1)}}$$

Moreover, by (ii) of claim 14.6,

$$(\gamma_2^{(n+1)} q_n)^{R_2(n)} \leq (q_n'^{5+4R_3(n)})^{R_2(n)} \leq q_n'^{P(n)}$$

Therefore,

$$q_n'^{P(n)} \gamma_2^{(n+1)} \geq (\gamma_2^{(n+1)} q_n)^{R_2(n)}$$

Hence condition 5.

Moreover, by condition 4 (a condition that holds, as shown above), we have, modulo 1:

$$\frac{p_{n+1}}{q_{n+1}} \gamma_2^{(n+1)} - \frac{p_n}{q_n} \gamma_2^{(n)} \equiv_{\text{mod } 1} \left( \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right) \gamma_2^{(n+1)} = \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n}$$

By summing this equality, since  $p_0 = p'_0 = 0$ , we obtain that

$$\frac{p_n}{q_n} \gamma_2^{(n)} \rightarrow_{\text{mod } 1} \beta$$

Likewise,

$$\frac{p_{n+1}}{q_{n+1}} \gamma_1^{(n+1)} - \frac{p_n}{q_n} \gamma_1^{(n)} =_{\text{mod } 1} \left( \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right) \gamma_1^{(n+1)} = \frac{\gamma_1^{(n+1)}}{\gamma_2^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right)$$

Moreover, by estimation (98),

$$\left| \frac{\gamma_1^{(n+1)}}{\gamma_2^{(n+1)}} \right| \leq \frac{q_n}{2^n} \leq \frac{q_n'^{1+2R_3(n)}}{2^n} \leq \frac{1}{2^n \left| \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right|}$$

Therefore,  $\sum_{n \geq 0} \frac{\gamma_1^{(n+1)}}{\gamma_2^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right)$  converges, and

$$\frac{p_n}{q_n} \gamma_1^{(n)} \rightarrow_{\text{mod } 1} \sum_{n \geq 0} \frac{\gamma_1^{(n+1)}}{\gamma_2^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right)$$

Hence the existence of one translation for  $h = 2$ .

*Remark 14.7.* The proof above also gives that:

$$\frac{p_n}{q_n} \rightarrow_{\text{mod } 1} \sum_{n \geq 0} \frac{1}{\gamma_2^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right)$$

### 14.3.2 Proof of the existence of one translation for $h \geq 2$

For  $h \geq 2$ , the proof of the theorem is a little more intricate, though the main scheme is the same: we still need estimation (97) (with another function  $T(n)$ ) and equation (99).

To the vector space  $\mathbb{R}^h$ , we give the norm  $\|(x_1, \dots, x_h)\| = \max_{1 \leq i \leq h} |x_i|$  and we consider its induced norm on  $\mathbb{T}^h$ .

We seek  $\gamma^{(n+1)}$  in the form:

$$\gamma^{(n+1)} = q_n v_1 + (c q_n + 1) \gamma^{(n)}$$

with  $v_1 \in \mathbb{Z}^h$  and  $c \in \mathbb{N}$ .

Thus,  $q_n$  always divides  $\gamma_i^{(n+1)} - \gamma_i^{(n)}$  for any  $i = 0, \dots, h-1$ , and condition 4 of lemma 14.3 always obtains. Likewise, since  $\gamma_h^{(n)}$  divides  $q_n$ , then  $\gamma_h^{(n)}$  divides  $\gamma_h^{(n+1)}$  and condition 3 also obtains.

To get condition 1 of relative primality, we add assumptions on  $v_1$ : we consider it of the form  $v_1 = A v_2$ , with  $v_{2,h} = 0$ , and with the  $v_{2,i}$ ,  $i = 0, \dots, h-1$  relatively prime ( $v_{2,i}$  is the  $i^{\text{th}}$  coordinate of  $v_2$ ). Moreover,  $A \in \text{SL}(h, \mathbb{Z})$  is such that its last column is  $\gamma^{(n)}$ .

Indeed, such matrix  $A$  exists by corollary 14.9 below, and these assumptions on  $v_2$  imply condition 1, because  $\mathrm{SL}(h, \mathbb{Z})$  stabilizes the set of integer vectors with relatively prime coordinates. Let  $\gamma$  be a vector. A quantity  $Q(\gamma)$  is *polynomially controlled* in  $\gamma$  if there is a polynomial function  $P$  such that  $|Q(\gamma)| \leq P(|\gamma|)$ . Corollary 14.9 is well-known, but we recall it to highlight the polynomial control in  $\gamma$ . To prove corollary 14.9, we need the following lemma:

**Lemma 14.8.** *Let  $\gamma_1, \dots, \gamma_h \in \mathbb{Z}$  such that  $\gcd(\gamma_1, \dots, \gamma_h) = d > 0$ . Then there exists a matrix  $T \in \mathrm{SL}(h, \mathbb{Z})$  polynomially controlled in the norm of  $\gamma = (\gamma_1, \dots, \gamma_h)$  such that  $T(\gamma_1, \dots, \gamma_h) = (d, 0, \dots, 0)$ . More precisely we have:*

$$\|T\| \leq h^{h-1} \|\gamma\|^{h-1}$$

*Proof.* We proceed by induction on  $h$ . For  $h = 1$ , the lemma is obvious. Suppose it true for  $h - 1$ . Let  $d' = \gcd(\gamma_1, \dots, \gamma_{h-1})$ . By the induction assumption, there is  $T' \in \mathrm{SL}(h - 1, \mathbb{Z})$  polynomially controlled in  $\gamma$  such that  $T'(\gamma_1, \dots, \gamma_{h-1}) = (d', 0, \dots, 0)$ . Let

$$\tilde{T} = \begin{pmatrix} T' & 0 \\ 0 & 1 \end{pmatrix}$$

Then  $\tilde{T}(\gamma_1, \dots, \gamma_{h-1}) = (d', 0, \dots, 0, \gamma_h)$

Let  $-\gamma_h^2 - d'^2 \leq m_1, m_2 \leq \gamma_h^2 + d'^2$  such that  $m_1 d' + m_2 \gamma_h = d$ . Let

$$T_h = \begin{pmatrix} m_1 & 0 & \cdots & 0 & m_2 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ \gamma_h/d & 0 & \cdots & 0 & -d'/d \end{pmatrix}$$

We have that  $T_h \in \mathrm{SL}(h, \mathbb{Z})$  and  $T_h(d', 0, \dots, 0, \gamma_h) = (d, 0, \dots, 0)$ . If we let  $T = T_h \tilde{T}$  then we obtain the property at rank  $h$ . □

**Corollary 14.9.** *If  $\gcd(\gamma_1, \dots, \gamma_h) = 1$ , then there exists  $A \in \mathrm{SL}(h, \mathbb{Z})$  polynomially controlled in  $\gamma$  such that  $Ae_h = (\gamma_1, \dots, \gamma_h)$ .*

*Proof.* Modulo a permutation of coordinates (which is an application of a permutation matrix, i.e. an isometry), it suffices to show that there is a matrix  $A \in \mathrm{SL}(h, \mathbb{Z})$  polynomially controlled in  $\gamma$  such that  $Ae_1 = (\gamma_1, \dots, \gamma_h)$ . By lemma 14.8, there is a matrix  $B \in \mathrm{SL}(h, \mathbb{Z})$  polynomially controlled in  $\gamma$  such that  $e_1 = B(\gamma_1, \dots, \gamma_h)$ . Let  $A = B^{-1}$ . Then  $A = \pm {}^t \mathrm{com}(B)$ , where  ${}^t \mathrm{com}(B)$  is the transposition of the comatrix of  $B$ . By the Cramer formula,  $\|A\|$  is polynomially controlled by  $\|B\|$ . Hence corollary 14.9. □

We obtain conditions 6 and 7 of lemma 14.3 with a suitable choice of  $v_2$  and  $c$ , and to obtain 5, it suffices to show that  $v_2$  and  $c$  are bounded by a fixed polynomial of  $q_n, \gamma_h^{(n)}$ . We first choose  $v_2$ , then we choose  $c$ .

Condition 7 is the most easy to obtain. Indeed, we have:

$$\frac{\gamma^{(n+1)}}{\gamma_h^{(n+1)}} - \frac{\gamma^{(n)}}{\gamma_h^{(n)}} = \frac{q_n v_1 + (c q_n + 1) \gamma^{(n)}}{q_n v_{1,h} + (c q_n + 1) \gamma_h^{(n)}} - \frac{\gamma^{(n)}}{\gamma_h^{(n)}} = \frac{q_n v_1 \gamma_h^{(n)} - q_n v_{1,h} \gamma^{(n)}}{\gamma_h^{(n)} (q_n v_{1,h} + (c q_n + 1) \gamma_h^{(n)})} \quad (100)$$

The parameter  $\sigma_n$  is a fixed polynomial in  $\gamma^{(n)}$ . Therefore, there exists an integer-valued function  $R_4(n)$  such that if  $c \geq (\gamma_h^{(n)} q_n)^{R_4(n)}$ , then condition 7 obtains.

It remains to obtain 5 and 6. Let  $E \subset F \subset \mathbb{T}^h$ . The set  $E$  is a  $\delta$ -net of  $F$  if there is  $\Gamma \subset \mathbb{T}^h$  such that  $\text{diam } \Gamma \leq \delta$  and

$$F \subset \bigcup_{e \in E} e + \Gamma$$

To obtain 5 and 6, it suffices to show that, for any integer  $p > 1$ , we can find a vector  $v_2 \in \mathbb{Z}^{h-1} \times 0$  and a parameter  $c \in \mathbb{N}$  such that  $\|v_2\|$  and  $c$  are polynomially controlled by  $p$  and, such that the orbit of 0 by the periodic flow  $T^{t\gamma^{(n+1)}}$  is a  $1/p$ -net of  $\mathbb{T}^h$ .

To show this fact, we first show the proposition:

**Proposition 14.10.** *We can choose  $v_2 \in \mathbb{Z}^{h-1}$  polynomially controlled by  $p$  such that  $T^{tv_2}$  intersects any square of sidelength  $1/p$ .*

*Proof.* Let  $(x_1, \dots, x_{h-1}) \in \mathbb{T}^{h-1}$ . There are integers  $l_1, \dots, l_{h-1}$  such that

$$C = \prod_{i=1}^{h-1} \left[ \frac{l_i}{2p}, \frac{l_i}{2p} + \frac{1}{2p} \right] \subset \prod_{i=1}^{h-1} \left[ x_i, x_i + \frac{1}{p} \right]$$

It suffices to show that we can find  $t \in \mathbb{R}$  and  $v_2$  polynomially controlled by  $p$  such that  $tv_2 \in C$ . Let  $s = 4(h-2)p$  and  $v_2 = (s^{h-2}, s^{h-3}, \dots, 1)$ .  $v_2$  is polynomially controlled by  $p$ . Let  $k_i = (h-2)(2l_i + 1)$ , and

$$t = \frac{1}{s^{h-2}} \left( \frac{k_1}{s} + k_2 + \dots + k_{h-1} s^{h-3} \right)$$

Let  $u = (k_1/s, \dots, k_{h-1}/s)$ . Then  $\|tv_2 - u\| \leq 1/(4p)$  and

$$u \in \prod_{i=1}^{h-1} \left[ \frac{2l_i + 1}{4p} - \frac{1}{4p}, \frac{2l_i + 1}{4p} + \frac{1}{4p} \right] = C$$

Therefore,  $tv_2 \in C$ . □

Let  $v_3 = q_n v_2 + (c q_n + 1) e_h$ . For a given choice of  $c$  sufficiently large, but still polynomially controlled by  $p$ ,  $(tv_3)_{t \geq 0}$  intersects  $\mathbb{T}^{h-1} \times \{0\}$  in a  $1/p$ -net (see figure 1).

Let  $b_1, \dots, b_{h-1}$  such that the hyperplan  $A^{-1}(\mathbb{R}^{h-1} \times \{0\})$  has an equation of the form  $x_h = \sum_{i=1}^{h-1} b_i x_i$ . Let  $b = \max(1, b_1, \dots, b_{h-1})$ .  $b$  is polynomially controlled by  $\|A^{-1}\|$  (which is itself



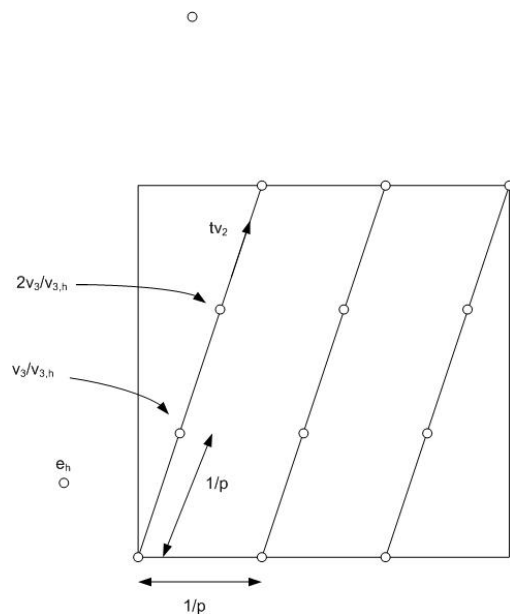


Figure 2: The small circles represent the intersections of  $tv_3$  with  $\mathbb{T}^{h-1} \times \{0\}$

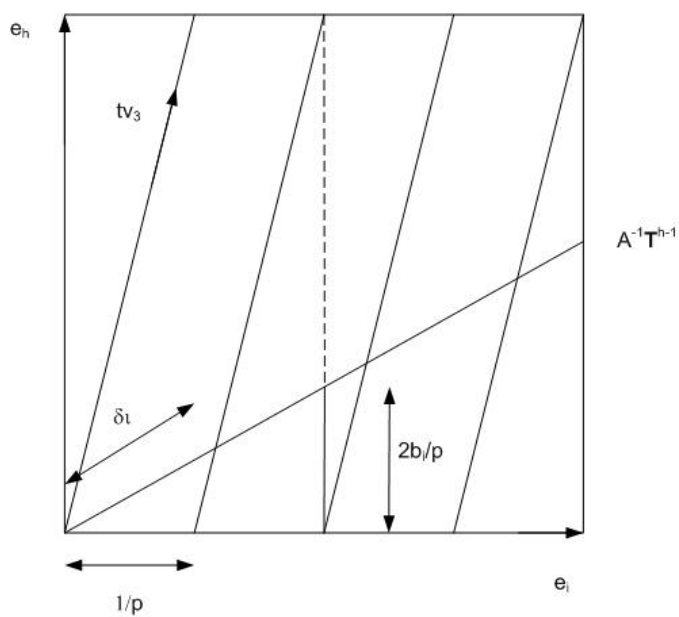


Figure 3: The flow  $tv_3$  in the plan  $(e_i, e_h)$ ,  $1 \leq i \leq h - 1$

polynomially controlled by  $\|\gamma^{(n)}\|$ , see corollary 14.9). Then the orbit of 0 by  $T^{lv_3}$  intersects  $A^{-1}(\mathbb{T}^{h-1} \times \{0\})$  in a  $\delta_A$ -net with  $\delta_A \leq 2b/p$  (see figure 3).

Finally, by applying the matrix  $A$ , we get that the orbit of 0 by  $T^{lAv_3} = T^{l\gamma^{(n+1)}}$  intersects  $\mathbb{T}^{h-1} \times \{0\}$  in a  $2b\|A\|/p$ -net. This completes the proof of the existence of one translation with  $\beta$  being the  $h^{\text{th}}$  coordinate of this translation.

The limit translations are of the form  $(\beta_1, \dots, \beta_{h-1}, \beta)$  with for any  $i = 0, \dots, h-1$ :

$$\beta_i = \sum_{n \geq 0} \frac{\gamma_i^{(n+1)}}{\gamma_h^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right)$$

Moreover, the rotation number  $\beta_0 \in E_0(\beta)$  of the diffeomorphism  $T$  on  $M$  is of the form:

$$\beta_0 = \sum_{n \geq 0} \frac{1}{\gamma_h^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right)$$

### 14.3.3 Proof of the density of the limit translations for $h \geq 2$

We directly tackle the case  $h \geq 2$ . We follow and slightly modify the previous proof. Let  $\epsilon > 0$  and  $\alpha_1, \dots, \alpha_{h-1} \in \mathbb{T}$ .

In relation (97), we can take a fixed function  $P(n)$  such that we will have, for  $n \geq 1$ ,

$$\left| \beta - \frac{p'_n}{q'_n} \right| \leq \frac{\epsilon}{q_n 2^n}$$

This fixes  $p'_1$  and  $q'_1$  (but it does not fix the other  $p'_n, q'_n, n \geq 1$ , because  $q_n$  depends on  $\gamma_h^{(n)}$ ). Moreover, we can take  $p'_1 > 0$ .

For  $i = 1, \dots, h-1$ , let  $\gamma_i^{(0)}$  and  $\gamma_h^{(0)}$  such that

$$\left| \frac{\gamma_i^{(0)}}{\gamma_h^{(0)}} - \frac{q'_1 \alpha_i}{p'_1} \right| \leq \frac{q'_1}{p'_1}$$

In relation (100), for  $n = 0$ , let  $c$  be sufficiently large such that

$$\left| \frac{\gamma_i^{(1)}}{\gamma_h^{(1)}} - \frac{\gamma_i^{(0)}}{\gamma_h^{(0)}} \right| \leq \frac{q'_1}{p'_1}$$

Therefore,

$$\left| \frac{p'_1 \gamma_i^{(1)}}{q'_1 \gamma_h^{(1)}} - \alpha_i \right| \leq 2\epsilon \tag{101}$$

For each  $n \geq 1$ , we also take  $c$  sufficiently large so that

$$\left| \frac{\gamma_i^{(n+1)}}{\gamma_h^{(n+1)}} \right| \leq q_n + \frac{\gamma_i^{(0)}}{\gamma_h^{(0)}} \leq q_n + \epsilon \frac{q'_1}{p'_1} + \alpha_i \frac{q'_1}{p'_1} \leq 4q_n$$

Therefore,

$$\sum_{n \geq 1} \frac{\gamma_i^{(n+1)}}{\gamma_h^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right) \leq \sum_{n \geq 1} \frac{\epsilon}{2^{n-2}} = 8\epsilon$$

By combining this estimation with (101), we finally obtain:

$$\left| \sum_{n \geq 0} \frac{\gamma_i^{(n+1)}}{\gamma_h^{(n+1)}} \left( \frac{p'_{n+1}}{q'_{n+1}} - \frac{p'_n}{q'_n} \right) - \alpha_i \right| \leq 10\epsilon$$

Hence the density of the limit translations.

Let us make two remarks. First, with our method, we cannot obtain substantially more translations, because the isomorphism condition 4 is constraining. Indeed, this condition introduces arithmetical constraints on the sequences  $\gamma_i^{(n)}$ , which limit the set of possible translations. We can interpret this limitation heuristically. On the manifold  $M$ , we only have one degree of freedom for the associated diffeomorphism  $T$ : the angle of the conjugating rotation. The isomorphism condition transfers this one-dimensional constraint to the multi-dimensional setting of the torus: although translations have  $h \geq 2$  dimensions, they only have one single "effective" degree of freedom.

Second, the set  $E_0(\beta)$  of possible rotation numbers  $\beta_0$  for the diffeomorphism  $T$  on  $M$  is infinite, and accumulates 0. We can note the following phenomenon: with our method, the closer we want to approximate a given irrational number  $\alpha_i$  on the  $i^{\text{th}}$  coordinate,  $i = 0, \dots, h-1$ , the smaller  $\beta_0$  becomes, because  $\gamma_h^{(1)}$  has to be taken large.

The rest of the paper is dedicated to the proof of lemma 14.3.

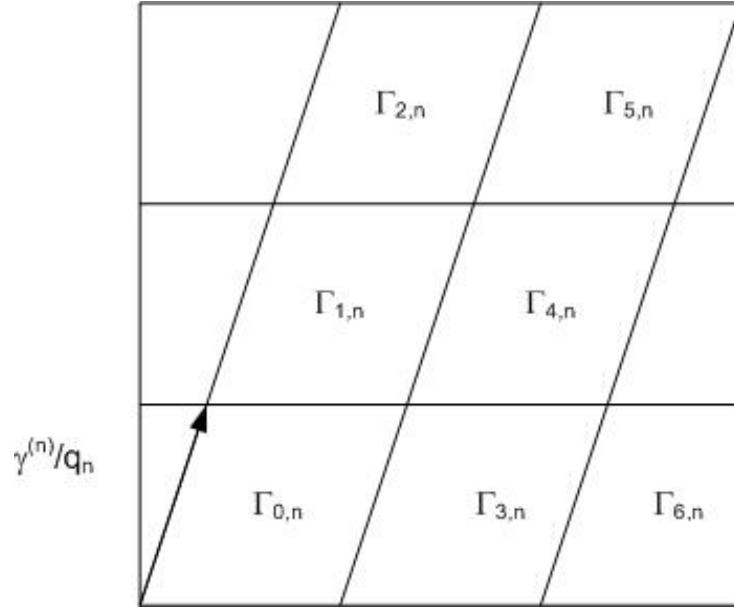
## 15 Partitions of the torus $\mathbb{T}^h$

The aim of this section is to show the following proposition:

**Proposition 15.1.** *If assumptions 1, 2, 3, 4, 6 and 7 of lemma 14.3 hold, there exists measurable partitions  $(\zeta_n^m)_{n \geq 0, n < m}$  of  $\mathbb{T}^h$ , such that  $\zeta_n^m$  is stable by the translation  $T^{\frac{p_n}{q_n} \gamma^{(n)}}$ , and such that at  $m$  fixed, for  $n < m$ ,  $\zeta_n^m \hookrightarrow \zeta_{n+1}^m$ .*

*Moreover, at  $n$  fixed,  $\zeta_n^m$  converges as  $m \rightarrow +\infty$  towards a partition  $\zeta_n^\infty$ , stable by the translation  $T^{\frac{\gamma^{(n)}}{q_n}}$ . Moreover, the sequence  $(\zeta_n^\infty)_{n \geq 0}$  is monotonous and generates.*

Proposition 15.1 is given by the combination of lemma 15.2 and 15.4. In all the paper, we denote  $r_n = q_n / \gamma_h^{(n)}$ ,  $k_n = \gamma_h^{(n+1)} / \gamma_h^{(n)}$ . They are integers by assumption.

Figure 4: The natural partition  $\zeta_n$ 

A natural partition stabilizing the translation  $T^{\frac{p_n}{q_n}\gamma^{(n)}}$  is given by (see figure 4):

$$\zeta_n = \{\Gamma_{k,n} = T^{\frac{k}{q_n}\gamma^{(n)}}\Gamma_{0,n}, 0 \leq k \leq q_n - 1\}$$

where

$$\Gamma_{0,n}^{(n)} = \bigcup_{0 \leq t \leq \frac{1}{q_n}} T^{t\gamma^{(n)}}\Gamma^{(n)}$$

and  $\Gamma^{(n)} \subset \mathbb{T}^{h-1}$  is a fundamental domain of the flow  $T^{t\gamma^{(n)}}$ .

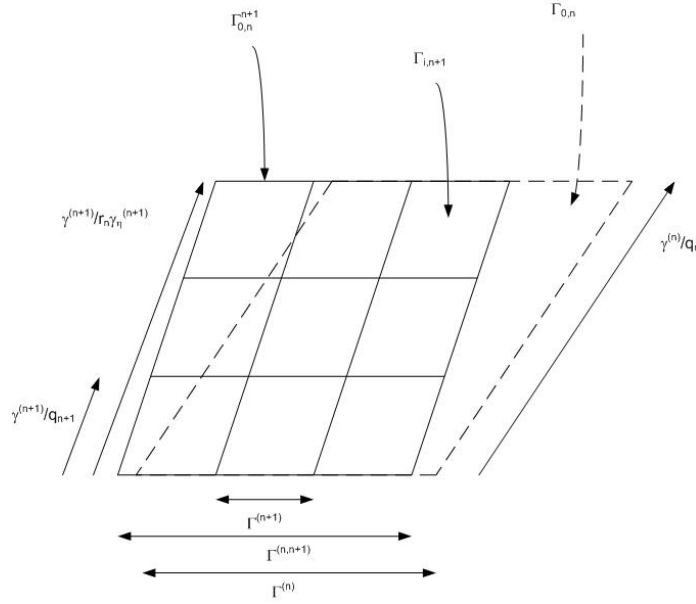
Remember that in order to apply lemma 14.2, we need a monotonous sequence of partitions. Moreover, in our theorem, we need the limit translation to be ergodic. However, note that if we take  $\gamma^{(n+1)}$  and  $\gamma^{(n)}$  not parallel, then the sequence of  $\zeta_n$  cannot be monotonous. On the contrary, if we take  $\gamma^{(n+1)}$  parallel to  $\gamma^{(n)}$ , then the limit translation cannot be ergodic: any band of the form

$$\bigcup_{t \geq 0} T^{t\gamma^{(n)}}B$$

where  $B \subset \mathbb{T}^{h-1}$ , and  $0 < \text{diam}B < \text{diam}\Gamma^{(n)}$ , is invariant by the flow  $T^{t\gamma^{(n)}}$ , and thus invariant by any translation included in this flow.

Therefore, we must take another partition. We obtain it by "monotonizing"  $\zeta_n$ . We begin with an informal description of the construction. This "monotonization" is performed in three steps: in the first step, we construct a partition  $\zeta_n^{n+1}$  stable by the translation  $T^{\frac{p_n}{q_n}\gamma^{(n)}}$  and such that  $\zeta_n^{n+1} \hookrightarrow \zeta_{n+1}$  (figure 5).

In the second step, we iterate this procedure, so as to obtain a partition  $\zeta_n^m$ , such that

Figure 5: The modified partition  $\zeta_n^{n+1}$ 

$\zeta_n^m \hookrightarrow \zeta_m$ . Thus, the element  $\Gamma_{k,n}^m$  of  $\zeta_n^m$  is a "pixelisation" of  $\Gamma_{k,n}$  with a resolution  $m$ . At  $m$  fixed, for  $n < m$ ,  $\zeta_n^m \hookrightarrow \zeta_{n+1}^m$ . These two steps are performed by lemma 15.2

In the third step, we take the limit  $m \rightarrow +\infty$ , which exists because of conditions 6 and 7. This gives a partition  $\zeta_n^\infty$  endowed with the required properties: monotonicity, generation and stability by the translation  $T_{q_n}^{\frac{p_n}{q_n} \gamma^{(n)}}$ .

**Lemma 15.2.** *If assumptions 2, 3 and 4 of lemma 14.3 hold, there exist measurable partitions  $(\zeta_n^m)_{n \geq 0, n < m}$  of  $\mathbb{T}^h$ , such that  $\zeta_n^m$  is stable by the translation  $T_{q_n}^{\frac{p_n}{q_n} \gamma^{(n)}}$ , and such that at  $m$  fixed, for  $n < m$ ,  $\zeta_n^m \hookrightarrow \zeta_{n+1}^m$ .*

*Proof.* We denote:

$$\zeta'_n = \left\{ \Gamma'_{k,n} = T_{\gamma_h}^{k \frac{\gamma^{(n)}}{\gamma_h}} \Gamma^{(n)}, k = 0, \dots, \gamma_h^{(n)} - 1 \right\}$$

Notice that  $T_{\gamma_h}^{\frac{\gamma^{(n)}}{\gamma_h}}$  can be seen as a translation of  $\mathbb{T}^{h-1}$ , and  $\zeta'_n$  as a partition of  $\mathbb{T}^{h-1}$ . Let

$$K = \left\{ 0 \leq k < \gamma_h^{(n+1)} \text{ such that } k \frac{\gamma^{(n+1)}}{\gamma_h^{(n+1)}} \in \Gamma^{(n)} \right\}$$

Note that  $|K| = k_n$ . We define (see figure 6):

$$\Gamma^{(n,n+1)} = \bigcup_{k \in K} \Gamma'_{k,n+1}$$

We have the lemma:

**Lemma 15.3.** *Under the assumptions 2, 3 and 4 of lemma 14.3,  $\Gamma^{(n,n+1)}$  is a fundamental domain of  $T^{\frac{\gamma^{(n)}}{\gamma_h^{(n)}}}$ , seen as a translation of  $\mathbb{T}^{h-1}$ .*

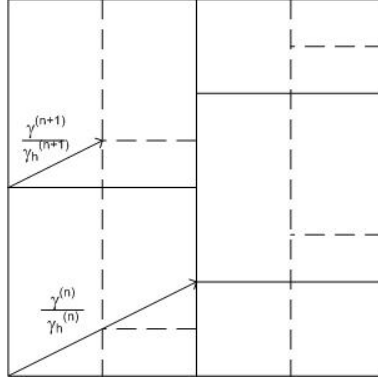


Figure 6: Partitions  $\zeta'_n$  (solid lines) and  $\zeta'_{n+1}$  (dashed lines) on  $\mathbb{T}^{h-1}$  for  $h = 3$ ,  $\gamma^{(n)} = (2, 1, 4)$  and  $\gamma^{(n+1)} = (2, 1, 8)$ . This example does not satisfy condition 6 of lemma 14.3, but it was chosen for illustrative purposes. To satisfy condition 6, we can take instead  $\gamma^{(n+1)} = (2, 9, 72)$  and  $n = 0$ . Indeed, in this case,  $d_1 = 1/8$  and  $\sigma_0 = 2$ .

*Proof.* Let  $V^{(n)} = T^{\frac{\gamma^{(n)}}{\gamma_h^{(n)}}}$ . By assumptions 2 and 4 of lemma 14.3,  $V^{(n)} = T^{\frac{\gamma^{(n+1)}}{\gamma_h^{(n+1)}}}$ . By assumptions 3 and 4,  $(V^{(n+1)})^{k_n} = V^{(n)}$ . For  $u = 0, \dots, \gamma_h^{(n)} - 1$ , let

$$K_u = \{l, 0 \leq l \leq \gamma_h^{(n+1)} - 1, (V^{(n+1)})^l(0) \in \Gamma'_{u,n}\}$$

Note that  $K_0 = K$  and that  $\{K_u, u = 0, \dots, \gamma_h^{(n)} - 1\}$  is a partition of  $\{0, \dots, \gamma_h^{(n+1)} - 1\}$ . Note also that  $l \in K_u$  if and only if  $l - k_n u \in K$ . We denote  $A \sqcup B$  the disjoint union of  $A$  and  $B$ . We have:

$$\mathbb{T}^{h-1} = \bigsqcup_{l=0}^{\gamma_h^{(n+1)}-1} (V^{(n+1)})^l \Gamma^{(n+1)} = \bigsqcup_{u=0}^{\gamma_h^{(n)}-1} \bigsqcup_{l \in K_u} (V^{(n+1)})^l \Gamma^{(n+1)} = \bigsqcup_{u=0}^{\gamma_h^{(n)}-1} \bigsqcup_{t \in K} (V^{(n+1)})^{t+k_n u} \Gamma^{(n+1)}$$

$$\mathbb{T}^{h-1} = \bigsqcup_{u=0}^{\gamma_h^{(n)}-1} (V^{(n+1)})^{k_n u} \Gamma^{(n,n+1)} = \bigsqcup_{u=0}^{\gamma_h^{(n)}-1} (V^{(n)})^u \Gamma^{(n,n+1)}$$

Therefore,  $\Gamma^{(n,n+1)}$  is a fundamental domain of  $T^{\frac{\gamma^{(n)}}{\gamma_h^{(n)}}}$ .

□

We define (see figure 5):

$$\Gamma_{q_n}^{(n,n+1)} = \bigcup_{0 \leq t < \frac{1}{r_n \gamma_h^{(n+1)}}} T^t \gamma^{(n+1)} \Gamma^{(n,n+1)}$$

Since  $r_n \gamma_h^{(n+1)} = k_n q_n$  divides  $q_{n+1}$ , then  $\Gamma_{q_n}^{(n,n+1)}$  is composed of elements of  $\zeta_{n+1}$ . Moreover,  $\Gamma_{q_n}^{(n,n+1)}$  has the same height as  $\Gamma_{k,n}$  by condition 2. Therefore, by lemma 15.3,  $\Gamma_{q_n}^{(n,n+1)}$  is also a fundamental domain of  $T^{\frac{\gamma^{(n)}}{q_n}}$ . Let

$$\zeta_n^{n+1} = \{\Gamma_{k,n}^{n+1} = T^{k \frac{\gamma^{(n)}}{q_n}} \Gamma_{q_n}^{(n,n+1)}, k = 0, \dots, q_n - 1\}$$

We have  $\zeta_n^{n+1} \hookrightarrow \zeta_{n+1}$ . Let

$$Q_n^{n+1} \Gamma_{k,n} = \Gamma_{k,n}^{n+1}$$

$Q_n^{n+1}$  is a measure-preserving bijection between partitions. For  $m > n$ , let

$$Q_n^m = Q_{m-1}^m \dots Q_n^{m-1}$$

The map  $Q_n^m$  is measure-preserving and injective as a composition of measure-preserving and injective maps. Let  $\zeta_n^m$  be the partition defined by:

$$\zeta_n^m = \{\Gamma_{k,n}^m = Q_n^m \Gamma_{k,n}, k = 0, \dots, q_n - 1\}$$

Notice that for  $n \leq m - 1$ ,  $\zeta_n^m \hookrightarrow \zeta_{n+1}^m$ . Moreover,  $\zeta_n^m$  is stable by the translation  $T^{\frac{p_n}{q_n} \gamma^{(n)}}$ .

□

In order to obtain a full sequence of monotonic partitions stable by this translation, we need to take  $m \rightarrow +\infty$ .

Indeed, by adding conditions 6 (horizontal convergence) and 7 (vertical convergence), we obtain the lemma:

**Lemma 15.4.** *If assumptions 2, 3, 4, 6 and 7 of lemma 14.3 hold, then for any fixed  $n$ , the partition  $\zeta_n^m$  converges to a partition  $\zeta_n^\infty$  when  $m \rightarrow +\infty$ . Moreover, the sequence  $(\zeta_n^\infty)_{n \geq 0}$  is monotonous, generating and stable by  $T^{\frac{\gamma^{(n)}}{q_n}}$ .*

*Proof.* We show that  $\zeta_n^m$  is a Cauchy sequence for the metric on measurable partitions. We recall that this metric is defined by:

$$d(\xi, \xi') = \inf \sum_{c \in \xi, c' \in \xi'} \mu(c \Delta c')$$

To show this fact, we need the lemma:

**Lemma 15.5.** *The numerical series  $\sum_{n \geq 0} \mu_h(\Gamma_{0,n} \Delta Q_n^{n+1} \Gamma_{0,n})$  is convergent.*

*Proof.* We first examine the horizontal difference between  $\Gamma_{0,n}$  and  $Q_n^{n+1} \Gamma_{0,n}$ . The element  $\Gamma_{q_n}^{(n,n+1)}$  is composed of all the  $\Gamma_{i,n+1}'$  that intersect  $\Gamma^{(n)}$ , except some, that still intersect the boundary of  $\Gamma^{(n)}$ . Therefore, the set  $\Gamma^{(n)} \setminus \Gamma_{q_n}^{(n,n+1)}$  is included in a neighborhood of width  $d_{n+1}$  of the boundary of  $\Gamma^{(n)}$  (figure 6).

Therefore,

$$\mu_{h-1} \left( \Gamma^{(n)} \setminus \Gamma^{(n,n+1)} \right) \leq \sigma_n d_{n+1}$$

Likewise, the elements  $\Gamma'_{i,n+1}$  of  $\Gamma^{(n,n+1)}$  that are not included in  $\Gamma^{(n)}$  always intersect the boundary of  $\Gamma^{(n)}$ . Therefore,

$$\mu_{h-1} \left( \Gamma^{(n,n+1)} \setminus \Gamma^{(n)} \right) \leq \sigma_n d_{n+1}$$

and therefore, by the condition 6 of horizontal convergence,

$$\sum_{n \geq 0} \gamma_h^{(n)} \mu_{h-1} \left( \Gamma^{(n,n+1)} \Delta \Gamma^{(n)} \right) < +\infty$$

Now, let

$$\hat{\Gamma}_{q_n}^{(n,n+1)} = \bigcup_{0 \leq t < \frac{1}{q_n}} T^{t\gamma^{(n)}} \Gamma^{(n,n+1)}$$

We have:

$$\mu_h \left( \Gamma_{0,n} \Delta \hat{\Gamma}_{q_n}^{(n,n+1)} \right) = \frac{1}{r_n} \mu_{h-1} \left( \Gamma^{(n)} \Delta \Gamma^{(n,n+1)} \right)$$

Therefore,

$$\sum_{n \geq 0} q_n \mu_h \left( \Gamma_{0,n} \Delta \hat{\Gamma}_{q_n}^{(n,n+1)} \right) < +\infty$$

In order to obtain the lemma, it suffices to show that

$$\sum_{n \geq 0} q_n \mu_h \left( \Gamma_{q_n}^{(n,n+1)} \Delta \hat{\Gamma}_{q_n}^{(n,n+1)} \right) < +\infty$$

i.e. that we have vertical convergence.

Let  $0 \leq t \leq 1$ . We have:

$$\mu_{h-1} \left( T^{t \frac{\gamma^{(n+1)}}{r_n \gamma_h^{(n+1)}}} \Gamma^{(n,n+1)} \Delta T^{t \frac{\gamma^{(n)}}{q_n}} \Gamma^{(n,n+1)} \right) \leq 2\sigma_n \left| \frac{\gamma^{(n+1)}}{r_n \gamma_h^{(n+1)}} - \frac{\gamma^{(n)}}{q_n} \right|$$

Therefore,

$$\mu_h \left( \Gamma_{q_n}^{(n,n+1)} \Delta \hat{\Gamma}_{q_n}^{(n,n+1)} \right) \leq 2 \frac{\sigma_n}{r_n} \left| \frac{\gamma^{(n+1)}}{r_n \gamma_h^{(n+1)}} - \frac{\gamma^{(n)}}{q_n} \right|$$

Moreover,



$$\frac{\sigma_n}{r_n} \left| \frac{\gamma^{(n+1)}}{r_n \gamma_h^{(n+1)}} - \frac{\gamma^{(n+1)}}{q_n} \right| = \frac{\sigma_n (\gamma_h^{(n)})^2}{q_n^2} \left| \frac{\gamma^{(n+1)}}{\gamma_h^{(n+1)}} - \frac{\gamma^{(n)}}{\gamma_h^{(n)}} \right|$$

Since  $\sum_{n \geq 0} \frac{\sigma_n (\gamma_h^{(n)})^2}{q_n} \left| \frac{\gamma^{(n+1)}}{\gamma_h^{(n+1)}} - \frac{\gamma^{(n)}}{\gamma_h^{(n)}} \right| < +\infty$  by condition 7,<sup>3</sup> then also

$$\sum_{n \geq 0} q_n \mu_h \left( \Gamma_{q_n}^{(n,n+1)} \Delta \hat{\Gamma}_{q_n}^{(n,n+1)} \right) < +\infty$$

Hence lemma 15.5. □

Now, we can prove the convergence of  $(\zeta_n^m)_{m > n}$ : for any fixed  $n$ ,  $n < m$ , we have:

$$d(\zeta_n^m, \zeta_n^{m+1}) = \sum_{\Gamma_{k,n}^m \in \zeta_n^m} \mu_h \left( \Gamma_{k,n}^m \Delta Q_m^{m+1} \Gamma_{k,n}^m \right) = \sum_{k=0}^{q_n-1} \mu_h \left( Q_n^m \Gamma_{k,n} \Delta Q_m^{m+1} Q_n^m \Gamma_{k,n} \right)$$

By volume conservation of the map  $Q_n^m$ ,  $Q_n^m \Gamma_{k,n}$  consists of  $\frac{q_m}{q_n}$  sets  $\Gamma_{l,m}$ , while  $Q_m^{m+1} Q_n^m \Gamma_{k,n}^m$  consists of the sets  $Q_m^{m+1} \Gamma_{l,m}^m$  with the same indices  $l$ .

Therefore, by lemma 15.5,

$$\sum_{m \geq 0} \sum_{\Gamma_{k,n}^m \in \zeta_n^m} \mu_h \left( \Gamma_{k,n}^m \Delta Q_m^{m+1} \Gamma_{k,n}^m \right) = \sum_{m \geq 0} \frac{q_m}{q_n} q_n \mu_h \left( \Gamma_{0,m} \Delta Q_m^{m+1} \Gamma_{0,m} \right) < +\infty$$

This completes the proof of the convergence of  $\zeta_n^m$  towards a partition  $\zeta_n^\infty$ . Now, we show that  $\zeta_n^\infty$  is monotonous. Let  $n \geq 0$  and  $\epsilon > 0$ . Let  $m > n$  such that  $d(\zeta_n^m, \zeta_n^\infty) < \epsilon/2$  and  $d(\zeta_{n+1}^m, \zeta_{n+1}^\infty) < \epsilon/2$ . Let  $\Gamma_{k,n}^\infty \in \zeta_n^\infty$ , and  $\Gamma_{k,n}^m \in \zeta_n^m$  such that  $\mu_h \left( \Gamma_{k,n}^\infty \Delta \Gamma_{k,n}^m \right) \leq \epsilon/2$ .

Since  $\zeta_n^m \hookrightarrow \zeta_{n+1}^m$ , we can write:

$$\Gamma_{k,n}^m = \bigcup_{l \in L} \Gamma_{l,n+1}^m$$

Since  $d(\zeta_{n+1}^m, \zeta_{n+1}^\infty) < \epsilon/2$ , we have:

$$\mu_h \left( \bigcup_{l \in L} \Gamma_{l,n+1}^m \Delta \bigcup_{l \in L} \Gamma_{l,n+1}^\infty \right) \leq \sum_{l \in L} \mu_h \left( \Gamma_{l,n+1}^m \Delta \Gamma_{l,n+1}^\infty \right) \leq \epsilon/2$$

Therefore,

$$\mu_h \left( \Gamma_{k,n}^\infty \Delta \bigcup_{l \in L} \Gamma_{l,n+1}^\infty \right) \leq \mu_h \left( \Gamma_{k,n}^\infty \Delta \Gamma_{k,n}^m \right) + \mu_h \left( \Gamma_{k,n}^m \Delta \bigcup_{l \in L} \Gamma_{l,n+1}^m \right) + \mu_h \left( \bigcup_{l \in L} \Gamma_{l,n+1}^m \Delta \bigcup_{l \in L} \Gamma_{l,n+1}^\infty \right)$$

3. We do not find the same condition as in Anosov-Katok [AK70, p.29]: there is a mistake in their paper.

$$\mu_h \left( \Gamma_{k,n}^\infty \Delta \bigcup_{l \in L} \Gamma_{l,n+1}^\infty \right) \leq \epsilon/2 + 0 + \epsilon/2 = \epsilon$$

Since this estimate holds for any  $\epsilon > 0$ , we conclude that:

$$\mu_h \left( \Gamma_{k,n}^\infty \Delta \bigcup_{l \in L} \Gamma_{l,n+1}^\infty \right) = 0$$

Therefore,  $\zeta_n^\infty \hookrightarrow \zeta_{n+1}^\infty$ . The proof that  $T^{\frac{2n}{q_n} \gamma^{(n)}}$  stabilizes  $\zeta_n^\infty$  is analogous.

Finally, let us show that  $(\zeta_n^\infty)_{n \geq 0}$  generates. Let  $G$  be a Lebesgue measurable set and let  $\epsilon > 0$ . Since  $\zeta_n$  generates, there exists  $n_0 \geq 0$  such that for any  $n \geq n_0$ , there is a  $\zeta_n$ -measurable set  $G_n$  such that  $\mu_h(G \Delta G_n) \leq \epsilon$ . Let  $I_n$  the (finite) set of indices such that

$$G_n = \bigcup_{i_n \in I_n} \Gamma_{i_n,n}$$

Let  $Q_n^\infty : \zeta_n \rightarrow \zeta_n^\infty$  be the limit isomorphism of the sequence  $Q_n^m$  (it exists by the first point of this lemma). Let

$$Q_n^\infty G_n = \bigcup_{i_n \in I_n} Q_n^\infty \Gamma_{i_n,n}$$

Since  $\zeta_n^m$  is a Cauchy sequence for the metric on partitions, we can fix an integer

$m_1 > n_0$  such that for any  $m \geq m_1$  and  $n \leq m$ :

$$\sum_{m' \geq m} \sum_{c \in \zeta_n} \mu_h(Q_n^{m'+1} c \Delta Q_n^{m'} c) \leq \epsilon$$

Let  $n \geq m_1$ . Since for any  $c \in \zeta_n$ ,  $Q_n^n c = c$ , then

$$\begin{aligned} \mu_h(Q_n^\infty G_n \Delta G_n) &= \mu_h(Q_n^\infty G_n \Delta Q_n^n G_n) \leq \sum_{m \geq n} \mu_h(Q_n^{m+1} G_n \Delta Q_n^m G_n) \\ &= \sum_{m \geq n} \sum_{i_n \in I_n} \mu_h(Q_n^{m+1} \Gamma_{i_n,n} \Delta Q_n^m \Gamma_{i_n,n}) \leq \sum_{m \geq n} \sum_{c \in \zeta_n} \mu_h(Q_n^{m+1} c \Delta Q_n^m c) \leq \epsilon \end{aligned}$$

Hence the generation of  $\zeta_n^\infty$ .

□

Let us make one remark about the assumptions of lemma 14.3. If  $h = 2$ , there is a natural choice of sequence of fundamental domains  $\Gamma^{(n)}$  of the flow  $(T^{\gamma^{(n)}})_{t \geq 0}$  such that  $\Gamma^{(n,n+1)} = \Gamma^{(n)}$ : we can choose  $\Gamma^{(n)} = [0, 1/(\gamma_2^{(n)})]$ . This choice allows to relax assumption 6, an assumption that is no longer useful for the convergence of  $\zeta_n^m$ . This relaxed assumption 6 remains useful to obtain the generation of the sequence of partitions  $(\zeta_n)_{n \geq 0}$ : this relaxed assumption is:

$d_n = 1/(\gamma_2^{(n)}) \rightarrow_{n \rightarrow +\infty} 0$ . However, this relaxation does not change the final statement of the theorem.

Moreover, if  $h = 2$ , assumption 4 is no longer used for the construction of  $\zeta_n^{n+1}$ . However, assumption 4 remains important for the construction of the metric isomorphism, see next section.

On the other hand, for  $h \geq 3$ , there is no natural choice of the sequence  $(\Gamma^{(n)})_{n \geq 0}$  that ensures that  $\Gamma^{(n, n+1)} = \Gamma^{(n)}$ . It is simpler to consider an arbitrary fundamental domain  $\Gamma^{(n)}$  of the flow  $(T^{t\gamma^{(n)}})_{t \geq 0}$ .

## 16 The metric isomorphism between $\mathbb{T}^h$ and $M = [0, 1]^{d-1} \times \mathbb{T}^1$

In this section, our aim is to elaborate sufficient conditions on  $B_n \in \text{Diff}^\infty(M, \mu)$  so that if  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$  weakly converges towards an automorphism  $T$ , then there exists a metric isomorphism between  $(\mathbb{T}^h, T^\alpha, \mu_h)$  and  $(M, T, \mu)$ , where  $M = [0, 1]^{d-1} \times \mathbb{T}^1$ .

To that end, we use lemma 14.2: with suitable assumptions, we construct a suitable sequence of partitions  $\xi_n^\infty$  of  $M$  and a sequence of isomorphisms  $\bar{K}_n^\infty : \mathbb{T}^h / \xi_n^\infty \rightarrow M / \xi_n^\infty$  so that  $\bar{K}_n^\infty T_{\frac{p_n}{q_n}}^{\gamma^{(n)}} = T_n \bar{K}_n^\infty$ . In this construction, condition 4 is critical. Let

$$\eta_n = \{\Delta_{i, q_n} = [0, 1]^{d-1} \times \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right], i = 0, \dots, q_n - 1\}$$

Let

$$\begin{aligned} K_n : \zeta_n &\rightarrow \eta_n \\ \Gamma_{i, n} &\mapsto \Delta_{i, q_n} \end{aligned}$$

We recall from section 15 that there exists an isomorphism  $Q_n^{n+1} : \zeta_n \rightarrow \zeta_n^{n+1}$ , with  $\zeta_n^{n+1} \hookrightarrow \zeta_{n+1}$  and with  $Q_n^{n+1} T_{\frac{\gamma^{(n)}}{q_n}} = T_{\frac{\gamma^{(n)}}{q_n}} Q_n^{n+1}$ . We have the lemma:

**Lemma 16.1.** *There exists a partition  $\eta_n^{n+1} \hookrightarrow \eta_{n+1}$  of  $M$  stable by  $S_{\frac{1}{q_n}}$ , and there exists a metric isomorphism  $K_n^{n+1} : \zeta_n \rightarrow \eta_n^{n+1}$  such that  $K_n^{n+1} = K_{n+1}|_{\zeta_n}$  and such that  $K_n^{n+1} T_{\frac{\gamma^{(n)}}{q_n}} = S_{\frac{1}{q_n}} K_n^{n+1}$ . There exists also a metric isomorphism  $C_n^{n+1} : \eta_n \rightarrow \eta_n^{n+1}$  such that  $C_n^{n+1} S_{\frac{a_n}{q_n}} = S_{\frac{a_n}{q_n}} C_n^{n+1}$  and  $K_n^{n+1} = C_n^{n+1} K_n$ . Said otherwise, we have the following commutative diagram:*

$$\begin{array}{ccc} \zeta_n & \xrightarrow{K_n} & \eta_n \\ \downarrow Q_n^{n+1} & & \downarrow C_n^{n+1} \\ T_{\frac{\gamma^{(n)}}{q_n}} \left( \zeta_n^{n+1} \right) & \xrightarrow{K_n^{n+1}} & \eta_n^{n+1} \left( S_{\frac{1}{q_n}} \right) \\ \downarrow & & \downarrow \\ \zeta_{n+1} & \xrightarrow{K_{n+1}} & \eta_{n+1} \end{array}$$

Moreover, there are integers  $a_n(i)$  such that:

$$R^{(n)} = \bigcup_{i=0}^{k_n-1} \Delta_{a_n(i)k_n+i, k_n q_n}$$

is a fundamental domain of  $S_{\frac{1}{q_n}}$ .

*Proof.* Let

$$\tilde{\Gamma}_{q_n}^{(n)} = \bigcup_{0 \leq t < \frac{1}{k_n q_n}} T^{t\gamma^{(n+1)}} \Gamma^{(n+1)}$$

and

$$\tilde{\zeta}_n = \{\tilde{\Gamma}_{i,n} = T^{\frac{i}{k_n q_n} \gamma^{(n+1)}} \tilde{\Gamma}_{q_n}^{(n)}, 0 \leq i \leq q_n - 1\}$$

Note that  $\zeta_n^{n+1} \hookrightarrow \tilde{\zeta}_n$ .

Let  $\tilde{\eta}_n = \{\Delta_{i, k_n q_n}, i = 0, \dots, k_n q_n - 1\}$  and  $\tilde{K}_n$  defined by:

$$\begin{aligned} \tilde{K}_n: \tilde{\zeta}_n &\rightarrow \tilde{\eta}_n \\ \tilde{\Gamma}_{i,n} &\mapsto \Delta_{i, k_n q_n} \end{aligned}$$

We have (figure 7):

$$\tilde{K}_n T^{\frac{\gamma^{(n)}}{q_n}} = S_{\frac{1}{q_n}} \tilde{K}_n$$

Indeed, on the one hand,

$$S_{\frac{1}{q_n}} \tilde{K}_n \tilde{\Gamma}_{i,n} = S_{\frac{1}{q_n}} \Delta_{i, k_n q_n} = \Delta_{i+k_n, k_n q_n}$$

On the other hand, by condition 4 of lemma 14.3,  $T^{\frac{\gamma^{(n)}}{q_n}} = T^{\frac{\gamma^{(n+1)}}{q_n}}$ . Therefore,

$$\tilde{K}_n T^{\frac{\gamma^{(n)}}{q_n}} \tilde{\Gamma}_{i,n} = \tilde{K}_n T^{\frac{\gamma^{(n+1)}}{q_n}} \tilde{\Gamma}_{i,n} = \tilde{K}_n T^{\frac{k_n \gamma^{(n+1)}}{k_n q_n}} \tilde{\Gamma}_{i,n} = \tilde{K}_n T^{\frac{k_n + i \gamma^{(n+1)}}{k_n q_n}} \tilde{\Gamma}_{q_n}^{(n)} = \tilde{K}_n \tilde{\Gamma}_{i+k_n, n} = \Delta_{i+k_n, k_n q_n}$$

Now, let  $K_n^{n+1} = \tilde{K}_n|_{\zeta_n^{n+1}}$ . It implies:  $K_n^{n+1} = K_{n+1}|_{\zeta_n^{n+1}}$ .

Since  $T^{\frac{\gamma^{(n)}}{q_n}}$  stabilizes  $\zeta_n^{n+1}$ , then we have:

$$K_n^{n+1} T^{\frac{\gamma^{(n)}}{q_n}} = S_{\frac{1}{q_n}} K_n^{n+1}$$

Let us denote  $R^{(n)} = K_n^{n+1} \Gamma_{0,n}^{n+1}$ . There are integers  $a_n(i)$  such that:

$$R^{(n)} = \bigcup_{i=0}^{k_n-1} \Delta_{a_n(i)k_n+i, k_n q_n}$$

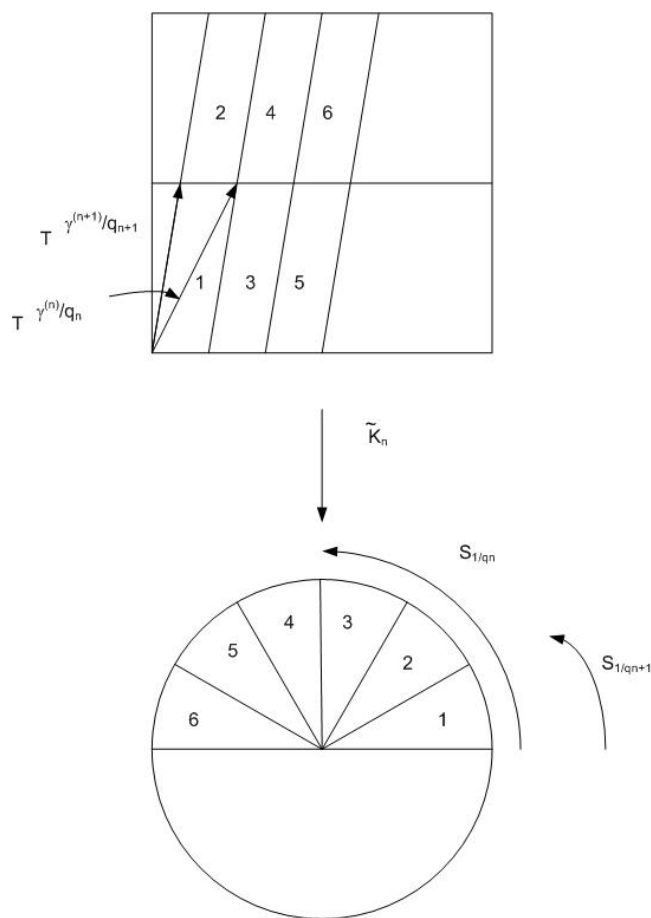


Figure 7: The isomorphism  $\tilde{K}_n$ . We took  $k_n = 3$ . The number  $i = 1, \dots, 6$  denotes  $\tilde{\Gamma}_{i,n}$  in the torus and  $\Delta_{i,k_n q_n}$  in the disk  $M$ . Moreover,  $R^{(n)} = \Delta_{1,k_n q_n} \cup \Delta_{3,k_n q_n} \cup \Delta_{5,k_n q_n}$

Denote also  $R_{k,n}^{n+1} = S_{\frac{k}{q_n}} R^{(n)}$ ,  $k = 0, \dots, q_n - 1$ .  $R^{(n)}$  is a fundamental domain of  $S_{\frac{1}{q_n}}$  and we have:

$$K_n^{n+1} \Gamma_{k,n}^{n+1} = S_{\frac{k}{q_n}} R^{(n)} = R_{k,n}^{n+1}$$

Let

$$\eta_n^{n+1} = C_n^{n+1} \eta_n = \{R_{i,n}^{n+1}, i = 0, \dots, q_n - 1\}$$

Let us also denote  $C_n^{n+1}$  the map defined by:

$$\begin{aligned} C_n^{n+1}: \eta_n &\rightarrow \eta_n^{n+1} \\ \Delta_{i,q_n} &\mapsto R_{i,n}^{n+1} \end{aligned}$$

$C_n^{n+1}$  is bijective.  $\eta_n^{n+1} = C_n^{n+1} \eta_n$  is a partition of  $M$  stable by  $S_{\frac{1}{q_n}}$ , and since  $k_n q_n$  divides  $q_{n+1}$ , then  $\eta_n^{n+1} \hookrightarrow \eta_{n+1}$ . □

By iterating lemma 16.1, we get a corollary that is important for the construction of the isomorphism:

**Corollary 16.2.** *For any  $m > n$ , there are partitions  $\eta_n^m \hookrightarrow \eta_{n+1}^m$  of  $M$  such that  $\eta_n^m$  is stable by  $S_{\frac{1}{q_n}}$  and there exists an isomorphism  $K_n^m: \zeta_n^m \rightarrow \eta_n^m$  such that  $K_n^m T_{\frac{1}{q_n}}^{\gamma(n)} = S_{\frac{1}{q_n}} K_n^m$  and  $K_n^m = K_{n+1}^m|_{\zeta_n^m}$ .*

Said otherwise, we have the following commutative diagram:

$$\begin{array}{ccc} T_{\frac{1}{q_n}}^{\gamma(n)} \left( \zeta_n^m \xrightarrow{K_n^m} \eta_n^m \right) S_{\frac{1}{q_n}} & & \\ \downarrow & & \downarrow \\ \zeta_{n+1}^m \xrightarrow{K_{n+1}^m} \eta_{n+1}^m & & \end{array}$$

*Proof.* The left hand side of the diagram comes from section 15. Since  $\eta_n^{n+1} \hookrightarrow \eta_{n+1}$ , we can define  $C_n^m = C_{m-1}^m \dots C_{n+1}^{n+2} C_n^{n+1}$ . Let  $\eta_n^m = C_n^m \eta_n$ . We have  $\eta_n^m \hookrightarrow \eta_{n+1}^m$ . Since  $C_n^m S_{\frac{1}{q_n}} = S_{\frac{1}{q_n}} C_n^m$ , then  $\eta_n^m$  is stable by  $S_{\frac{1}{q_n}}$ .

Moreover,  $Q_n^m: \zeta_n \rightarrow \zeta_n^m$  is bijective. Therefore, we can define

$$K_n^m = C_n^m K_n (Q_n^m)^{-1}. \text{ Since } Q_n^m T_{\frac{1}{q_n}}^{\gamma(n)} = T_{\frac{1}{q_n}}^{\gamma(n)} Q_n^m, \text{ and } C_n^m S_{\frac{1}{q_n}} = S_{\frac{1}{q_n}} C_n^m, \text{ we have: } K_n^m T_{\frac{1}{q_n}}^{\gamma(n)} = S_{\frac{1}{q_n}} K_n^m.$$

Let us check that  $K_n^m = K_{n+1}^m|_{\eta_n^m}$ . Let  $c \in \zeta_n$ . Since  $C_n^{n+1}$  is bijective,  $(C_n^{n+1})^{-1}$  is well-defined, and we have:

$$K_{n+1}^m (Q_n^m c) = C_{n+1}^m K_{n+1} (Q_{n+1}^m)^{-1} Q_n^m c = C_n^m (C_n^{n+1})^{-1} K_{n+1} Q_n^{n+1} c$$

Since  $Q_n^{n+1}c \in \zeta_n^{n+1}$ , then  $K_{n+1}(Q_n^{n+1}c) = K_n^{n+1}(Q_n^{n+1}c)$ .

Since  $(C_n^{n+1})^{-1}K_n^{n+1}Q_n^{n+1} = K_n$ , we get:

$$K_{n+1}^m(Q_n^m c) = C_{n+1}^m K_{n+1}(Q_{n+1}^m)^{-1} Q_n^m c = C_n^m K_n c$$

On the other hand,

$$K_n^m(Q_n^m c) = C_n^m K_n(Q_n^m)^{-1} Q_n^m c = C_n^m K_n c$$

Therefore,  $K_n^m = K_{n+1}^m|_{\eta_n^m}$ .

□

The sequence  $\eta_n^m$  must converge when  $m \rightarrow +\infty$ , in order to obtain a full sequence of monotonic partitions. Moreover, the possible limit sequence (i.e. a possible  $\eta_n^\infty$ ) must generate. Indeed, these assumptions are required to apply lemma 14.2. However, we can check that none of these assumptions are satisfied. Therefore, to obtain these assumptions, we pull back the partition  $\eta_n^m$  by a suitable smooth measure-preserving diffeomorphism  $B_m$ . The following lemma gives the conditions that  $B_m$  must satisfy:

**Lemma 16.3.** *Let  $B_m \in \text{Diff}^\infty(M, \mu)$ . Let  $A_{m+1} = B_{m+1}B_m^{-1}$ .*

1. *If  $A_{m+1}S_{\frac{1}{q_m}} = S_{\frac{1}{q_m}}A_{m+1}$  and if*

$$\sum_{m \geq 0} q_m \mu(\Delta_{0, q_m} \Delta A_{m+1}^{-1} R^{(m)}) < +\infty$$

*then for any fixed  $n$ , when  $m \rightarrow +\infty$ , the sequence of partitions  $\xi_n^m = B_m^{-1}\eta_n^m$  converges. We denote  $\xi_n^\infty$  the limit. The sequence  $\xi_n^\infty$  is monotonous and  $T_n = B_n^{-1}S_{\frac{p_n}{q_n}}B_n$  stabilizes each  $\xi_n^\infty$ .*

2. *If, moreover, the sequence  $\xi_n = B_n^{-1}\eta_n$  generates, then so does  $\xi_n^\infty$ .*

*Proof.* The proof of the first point share similarities with the proof of lemma 15.5. Let  $P_n^m = B_m^{-1}C_n^m B_n$ . By the Cauchy criterion for sequences of partitions, in order to show the convergence of  $B_m^{-1}\eta_n^m$ , it suffices to show that for any fixed  $n$ ,

$$\sum_{m \geq 0} \sum_{c \in \xi_n} \mu(P_n^m c \Delta P_n^{m+1} c) < +\infty$$

We have:

$$\sum_{c \in \xi_n} \mu(P_n^m c \Delta P_n^{m+1} c) = \sum_{c \in B_m^{-1}\eta_n^m} \mu(B_m^{-1}C_n^m B_n c \Delta B_{m+1}^{-1}C_n^{m+1}C_n^m B_n c) = \sum_{c' \in \eta_n^m} \mu(B_m^{-1}c' \Delta B_{m+1}^{-1}C_n^{m+1}c')$$

Since  $B_m$  is measure-preserving,

$$\sum_{c' \in \eta_n^m} \mu(B_m^{-1} c' \Delta B_{m+1}^{-1} C_m^{m+1} c') = \sum_{c' \in \eta_n^m} \mu(c' \Delta B_m B_{m+1}^{-1} C_m^{m+1} c')$$

The partition  $\eta_n^m$  consists of  $q_m/q_n$  sets  $c'$  of the form  $\Delta_{u,q_m}$ , and  $C_m^{m+1} c'$  is a set  $R_{u,m}^{m+1}$  with the same indices  $u$ . Since  $A_{m+1} S_{\frac{1}{q_m}} = S_{\frac{1}{q_m}} A_{m+1}$ , we get:

$$\sum_{m \geq 0} \sum_{c' \in \eta_n^m} \mu(c' \Delta B_m B_{m+1}^{-1} C_m^{m+1} c') \leq \sum_{m \geq 0} \frac{q_m}{q_n} \mu(\Delta_{0,q_m} \Delta A_{m+1}^{-1} R_{0,m}^{m+1}) < +\infty$$

by assumption. Hence the convergence of  $\xi_n^m$ . We let  $P_n^\infty : \xi_n \rightarrow \xi_n^\infty$  be the limit isomorphism of the sequence  $P_n^m$ . The proof that the sequence  $\xi_n^\infty$  is monotonous is the same as the proof that  $\zeta_n^\infty$  is monotonous. See section 15.

To show that  $T_n$  stabilizes  $\xi_n^\infty$ , let  $T_n^m = B_m^{-1} S_{\frac{p_n}{q_n}} B_m$ . Since  $S_{\frac{p_n}{q_n}}$  stabilizes  $\eta_n^m$ , then  $T_n^m$  stabilizes  $\xi_n^m$ . Since  $A_{n+1} S_{\frac{1}{q_n}} = S_{\frac{1}{q_n}} A_{n+1}$ , then  $T_n^m = T_n$ . As in the proof that  $T_{\frac{p_n}{q_n}}^{\gamma^{(n)}}$  stabilizes  $\zeta_n^\infty$ , we obtain that  $T_n$  stabilizes  $\xi_n^\infty$ .

Likewise, the proof of the second point, i.e. that the sequence  $\xi_n^\infty$  generates, is the same as the proof that  $\zeta_n^\infty$  generates. See section 15. □

By adding to lemma 16.3 the convergence of the sequence  $T_n$ , we obtain the required isomorphism:

**Corollary 16.4.** *If both conditions 1. and 2. of lemma 16.3 hold, and if  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$  weakly converges towards an automorphism  $T$ , then  $(\mathbb{T}^h, T^\alpha)$  and  $(M, T)$  are metrically isomorphic.*

*Proof.* By corollary 16.2,  $K_n^m T_{\frac{p_n}{q_n}}^{\gamma^{(n)}} = S_{\frac{1}{q_n}} K_n^m$ . By iteration,  $K_n^m T_{\frac{p_n}{q_n}}^{\gamma^{(n)} p_n} = S_{\frac{p_n}{q_n}} K_n^m$ . Therefore, the following diagram commutes:

$$\begin{array}{ccccc} & & S_{\frac{p_n}{q_n}} & & \\ & & \curvearrowright & & \\ & & \downarrow & & \\ \zeta_n^m & \xrightarrow{K_n^m} & \eta_n^m & \xrightarrow{B_m^{-1}} & \xi_n^m \\ \downarrow & & \downarrow & & \downarrow \\ \zeta_{n+1}^m & \xrightarrow{K_{n+1}^m} & \eta_{n+1}^m & \xrightarrow{B_m^{-1}} & \xi_{n+1}^m \end{array}$$

$T_{\frac{p_n}{q_n}}^{\gamma^{(n)}}$  (curved arrow on  $\zeta_n^m \rightarrow \zeta_{n+1}^m$ )       $T_n$  (curved arrow on  $\xi_n^m \rightarrow \xi_{n+1}^m$ )

We apply lemma 16.3, which gives a sequence of monotonous and generating partitions  $\xi_n^\infty$ , stable by  $T_n$ . Let  $\bar{K}_n^\infty : \zeta_n^\infty \rightarrow \xi_n^\infty$  defined by  $\bar{K}_n^\infty = P_n^\infty B_n^{-1} K_n(Q_n^\infty)^{-1}$ . We can show that  $\bar{K}_n^\infty T_{\frac{p_n}{q_n}}^{\gamma^{(n)}} = T_n \bar{K}_n^\infty$  and that  $\bar{K}_{n+1}^\infty|_{\zeta_n^\infty} = \bar{K}_n^\infty$  by using that  $\bar{K}_n^m T_{\frac{p_n}{q_n}}^{\gamma^{(n)}} = T_n \bar{K}_n^m$  and that  $\bar{K}_{n+1}^m|_{\zeta_n^m} = \bar{K}_n^m$ , where  $\bar{K}_n^m = P_n^m B_n^{-1} K_n(Q_n^m)^{-1}$ , i.e. we proceed as in the proof of the monotonicity of  $\zeta_n^\infty$ , in lemma 15.4 of section 15. This allows to apply lemma 14.2, which gives the required metric isomorphism. □

The next section is dedicated to the construction of the sequence of diffeomorphisms  $B_n$  satisfying the conditions of lemma 16.3.



## 17 The sequence of conjugacies

In this section, we construct a sequence of diffeomorphisms  $B_n$  on  $M$  satisfying the conditions of lemma 16.3 and such that  $\|B_n\|_n \leq (q_{n-1}k_{n-1})^{R_5(n-1)}$  for some  $R_5(n)$ .

**Proposition 17.1.** *There exists a sequence of diffeomorphisms  $B_n \in \text{Diff}^\infty(M, \mu)$  such that  $B_n$  and  $A_{n+1} = B_{n+1}B_n^{-1}$  satisfy the following conditions:*

1. (convergence of the partition  $\xi_n^m$  to  $\xi_n^\infty$ )

$$\sum_{m \geq 0} q_m \mu \left( \Delta_{0, q_m} \Delta A_{m+1}^{-1} R^{(m)} \right) < +\infty$$

2. (generation) There is a set  $E_{n+1} \subset M$  such that  $\sum_{n \geq 0} \mu(E_{n+1}^c) < +\infty$  and such that

$$\text{diam} \left( A_{n+1}^{-1} \left( \Delta_{0, q_{n+1}} \cap E_{n+1} \right) \right) \leq \frac{1}{2^n \|B_n\|_1}$$

3. (equivariance)

$$A_{n+1} S_{\frac{1}{q_n}} = S_{\frac{1}{q_n}} A_{n+1}$$

4. (polynomial estimation) There is a fixed sequence  $R_5(n) \in \mathbb{N}$  such that

$$\|A_{n+1}\|_{n+1} \leq (q_n k_n)^{R_5(n)}$$

5. (identity on a neighborhood of the boundary)  $B_n = \text{Id}$  on a neighborhood of  $\partial M$ .

*Remark 17.2.* Specification 2 above implies that  $\xi_n$  generates (and so  $\xi_n^\infty$ , by lemma 16.3), see proposition 17.9.

We construct  $B_n$  recursively. We suppose that  $B_n$  exists and satisfies these specifications, and we construct  $A_{n+1}$ .

The diffeomorphism  $A_{n+1}$  is constructed in two steps when  $d = 2$  (three steps when  $d \geq 3$ ), each step gives a smooth, measure-preserving, equivariant and polynomially controlled map. In the first step, lemma 17.3, we construct a smooth map  $A_{n+1}^1$  that "quasi-cuts"  $\Delta_{0, q_n}$  in  $k_n$  equal vertical slices, and then rotates each slice  $\Gamma_i$  by an angle  $a_n(i)/q_n$  along the periodic flow  $S_t$  (remember that the parameters  $0 \leq a_n(i) \leq q_n - 1$  are defined by  $R^{(n)} = \bigcup_{i=0}^{k_n-1} \Delta_{a_n(i)k_n+i, k_n q_n}$ ) (see figure 8).

In the second step, we construct a second map  $A_{n+1}^2$  that "quasi-sends" each vertical slice  $A_{n+1}^1(\Gamma_i)$  into the horizontal slice  $\Delta_{a_n(i)k_n+i, k_n q_n}$  (see figures 9 and 10). These two steps ensure that  $A_{n+1} = A_{n+1}^2 A_{n+1}^1$  "quasi-sends"  $\Delta_{0, q_n}$  to  $R^{(n)}$ . It ensures that  $\xi_n$  converges. Moreover, when  $d = 2$ , this also gives the generation of  $\xi_n$  (and so of  $\xi_n^\infty$ ).

For  $d \geq 3$  we need a third step. Inside each  $\Delta_{i, k_n q_n}$ , we quasi-rotate the slices  $\Delta_{j, (k_n q_n)^d}$  with  $A_{n+1}^3$ , so that the diameter of  $A_{n+1}^{-1}(\Delta_{i, q_{n+1}})$  is small, where  $A_{n+1} = A_{n+1}^3 A_{n+1}^2 A_{n+1}^1$  (see figures 11, 12, 13, 14). This guarantees that  $\xi_n$  generates (and so  $\xi_n^\infty$ ), which completes the construction.

First, we write the construction in the case  $M = [0, 1] \times \mathbb{T}$ , and then we extend it to  $M = [0, 1]^{d-1} \times \mathbb{T}$ .

## 17.1 Construction in dimension 2

The first step is based on the lemma (see figure 8):

**Lemma 17.3.** *Let  $\frac{1}{k_n} > \epsilon_1 > 0$  and for  $0 \leq i \leq k_n - 1$ , let  $\Gamma_i = [\frac{i}{k_n}, \frac{i+1}{k_n} - \epsilon_1] \times [0, \frac{1}{q_n}]$ . There is a smooth measure-preserving diffeomorphism  $A_{n+1}^1 : [0, 1] \times \mathbb{T} \rightarrow [0, 1] \times \mathbb{T}$  such that:*

1.

$$A_{n+1}^1 S_{\frac{1}{q_n}} = S_{\frac{1}{q_n}} A_{n+1}^1$$

2.

$$A_{n+1}^1(\Gamma_i) = S_{\frac{a_n(i)}{q_n}} \Gamma_i$$

3.

$$\|A_{n+1}^1\|_l \leq \frac{1}{\epsilon_1} \|\phi\|_l$$

where  $\phi$  is a fixed smooth diffeomorphism independent of  $n$  and  $\epsilon_1$ .

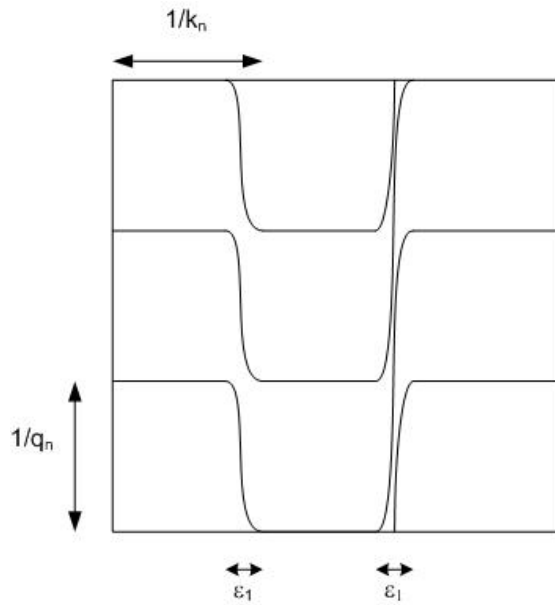


Figure 8: The image of the partition  $\eta_n$  after step 1.

*Proof.* Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth increasing function that equals 0 for  $x \leq -1$  and 1 for  $x \geq 0$ . Let  $\phi^{n,\epsilon_1} : [0, 1] \rightarrow [0, 1]$  defined by:

$$\phi^{n,\epsilon_1}(x) = \frac{a_n(0)}{q_n} + \frac{a_n(1) - a_n(0)}{q_n} \phi\left(\frac{x}{\epsilon_1} - \frac{1}{k_n \epsilon_1}\right) + \dots + \frac{a_n(k_n - 1) - a_n(k_n - 2)}{q_n} \phi\left(\frac{x}{\epsilon_1} - \frac{k_n - 1}{k_n \epsilon_1}\right)$$

For  $0 \leq i \leq k_n - 1$ , we have:  $\phi_{\left[\frac{i}{k_n}, \frac{i+1}{k_n} - \epsilon_1\right]}^{n,\epsilon_1} = \frac{a_n(i)}{q_n}$ , with the estimation:

$$\|\phi^{n,\epsilon_1}\|_l \leq \frac{1}{\epsilon_1} \|\phi\|_l$$

Finally, for  $x \in M$ , let  $A_{n+1}^1(x) = S_{\phi^{n,\epsilon_1}(x)}(x)$ . It satisfies all the specifications of the lemma.  $\square$

We take  $\epsilon_1 = \frac{1}{k_n 2^n}$  and we let

$$E_{n+1}^1 = \bigcup_{i=0}^{k_n-1} \left[ \frac{i}{k_n}, \frac{i+1}{k_n} - \epsilon_1 \right] \times \mathbb{T}$$

We have:

$$\mu(E_{n+1}^{1c}) = k_n \epsilon_1 = \frac{1}{2^n}$$

In the second step, we shrink  $\Gamma_i$  horizontally by a factor  $q_n$ , we expand it vertically by the same factor, and we rotate it by a  $\pi/2$  angle (except in a neighborhood of the border of  $\Delta_{0,q_n}$ ). Thus,  $\Gamma_i$  is quasi-sent to  $\Delta_{i,k_n q_n}$ . Note that with this operation, we automatically obtain the right combinatorics of  $R^{(n)}$ .

Our method is slightly different than in Anosov-Katok's original paper, but this difference is critical: in Anosov-Katok's original method, the fibered rotation in the first step does not depend on  $a_n(i)$ , and it does not make  $k_n$  cuts. This does not allow to get directly the right combinatorics of  $R^{(n)}$  by just using conjugacies of quasi-rotations (having an explicit construction is what allows us to get a quantitative control of the norm of  $B_n$ ).

We have the lemma (see figures 9 and 10):

**Lemma 17.4.** *Let  $\Gamma'_i = [\frac{i}{k_n}, \frac{i+1}{k_n}] \times [0, \frac{1}{q_n}]$ . There exists a map  $A_{n+1}^2$  equivariant by  $S_{\frac{1}{q_n}}$  and there exists a measurable set  $E_{n+1}^2$  such that  $\mu(E_{n+1}^2) \geq 1 - 4/2^n$ , such that  $E_{n+1}^2$  is globally invariant by  $S_{\frac{1}{q_n}}$  and  $A_{n+1}^2$ , and such that*

$$A_{n+1}^2(\Gamma'_i \cap E_{n+1}^2) = \Delta_{i,k_n q_n} \cap E_{n+1}^2$$

*In particular,*

$$\text{diam}\left((A_{n+1}^2)^{-1} \Delta_{i,k_n q_n} \cap E_{n+1}^2\right) \leq \max\left(\frac{1}{q_n}, \frac{1}{k_n}\right)$$

*Moreover, there is an explicit function  $R_2(j)$ , depending only on  $j$ , such that*

$$\|A_{n+1}^2\|_j \leq (q_n)^{R_2(j)} \|\phi_n\|_j$$

Combined with lemma 17.3, this lemma gives the convergence of the partition  $\xi_n$  to  $\xi_n^\infty$ :

**Corollary 17.5.** *We have the estimation:*

$$\mu\left(A_{n+1}^2 A_{n+1}^1(\Delta_{0,q_n}) \Delta R^{(n)}\right) \leq \frac{8}{2^n q_n}$$

*Proof of lemma 17.4.* We need to recall the definition of a "quasi-rotation" by  $\pi/2$  [FSW07]:

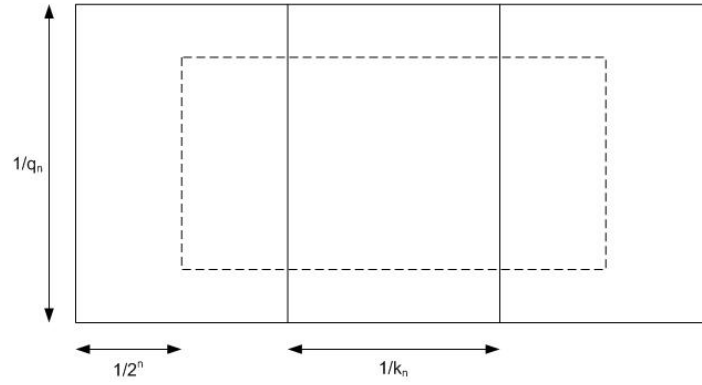


Figure 9: The partition  $A_{n+1}^1 \eta_n \cap [0, 1] \times \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right]$  before step 2 ( $\epsilon_1$  has been taken infinitesimally small in the illustration).

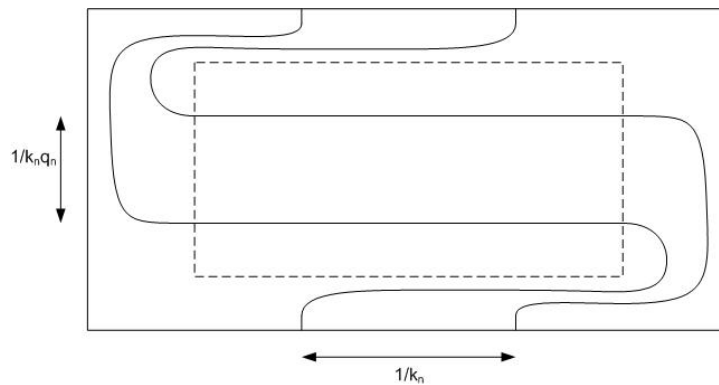


Figure 10: The partition  $A_{n+1}^1 \eta_n \cap [0, 1] \times \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right]$  after step 2 ( $\epsilon_1$  has been taken infinitesimally small in the illustration).

**Proposition 17.6.** For any  $n \geq 1$ , there is a smooth measure preserving map  $\phi_n : [0, 1]^2 \rightarrow [0, 1]^2$  (called "quasi-rotation") such that  $\phi_n = R_{\pi/2}$  on  $[\frac{1}{2^n}, 1 - \frac{1}{2^n}]^2$  and  $\phi_n = Id$  on  $[0, 1]^2 - [\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}]^2$ .

Let  $p \geq 2$  and

$$C_p : [0, 1] \times [0, \frac{1}{p}] \rightarrow [0, 1] \times [0, 1]$$

$$(x, y) \mapsto (x, py)$$

Let  $\phi_{n,p} = C_p^{-1} \phi_n C_p$ . The map  $\phi_{n,p}$  is measure preserving. By the Faa-di-Bruno formula, there exists a fixed function  $R_6(j)$  such that

$$\|\phi_{n,p}\|_j \leq p^{R_6(j)} \|\phi_n\|_j$$

Let  $A_{n+1}^2 = \phi_{n,q_n}$  on  $[0, 1] \times [0, \frac{1}{q_n}]$ , extended to  $[0, 1] \times \mathbb{T}$  by  $1/q_n$  periodicity. We have:

$$\|A_{n+1}^2\|_j \leq q_n^{R_6(j)} \|\phi_n\|_j$$

For  $r = 0, \dots, q_n - 1$ , let

$$E_{n+1,r}^2 = \left[ \frac{1}{2^n}, 1 - \frac{1}{2^n} \right] \times \left[ \frac{r}{q_n} + \frac{1}{q_n 2^n}, \frac{r+1}{q_n} - \frac{1}{q_n 2^n} \right]$$

and

$$E_{n+1}^2 = \bigcup_{r=0}^{q_n-1} E_{n+1,r}^2$$

The set  $E_{n+1}^2$  is invariant by  $S_{\frac{1}{q_n}}$  and  $A_{n+1}^2$ . Moreover,

$$A_{n+1}^2 \left( \Gamma'_i \bigcap E_{n+1}^2 \right) = A_{n+1}^2 \left( \Gamma'_i \bigcap E_{n+1,0}^2 \right) = \Delta_{i,k_n q_n} \bigcap E_{n+1}^2$$

□

*Proof of corollary 17.5.* By lemma 17.4 (using in particular that  $E_{n+1}^2$  is  $S_{\frac{1}{q_n}}$ -invariant, and that  $A_{n+1}^2$  is  $S_{\frac{1}{q_n}}$ -equivariant),

$$\begin{aligned} A_{n+1}^2 \left( \bigcup_{i=0}^{k_n-1} S_{\frac{a_n(i)}{q_n}} \Gamma'_i \bigcap E_{n+1}^2 \right) &= \bigcup_{i=0}^{k_n-1} A_{n+1}^2 \left( S_{\frac{a_n(i)}{q_n}} \Gamma'_i \bigcap E_{n+1}^2 \right) \\ &= \bigcup_{i=0}^{k_n-1} S_{\frac{a_n(i)}{q_n}} \Delta_{i,k_n q_n} \bigcap E_{n+1}^2 = R^{(n)} \bigcap E_{n+1}^2 \end{aligned}$$

On the other hand, by lemma 17.3,

$$A_{n+1}^1 \left( \Delta_{0,q_n} \bigcap E_{n+1}^1 \right) = \bigcup_{i=0}^{k_n-1} S_{\frac{a_n(i)}{q_n}} \Gamma'_i \bigcap E_{n+1}^1$$

Therefore, if we let  $A'_{n+1} = A_{n+1}^2 A_{n+1}^1$ , we obtain:

$$A'_{n+1} \left( \Delta_{0,q_n} \bigcap E_{n+1}^1 \bigcap (A_{n+1}^1)^{-1} (E_{n+1}^2) \right) = R^{(n)} \bigcap A_{n+1}^2 (E_{n+1}^1) \bigcap E_{n+1}^2$$

Thus,

$$\begin{aligned} \mu \left( A'_{n+1} \Delta_{0,q_n} \Delta R^{(n)} \right) &\leq \mu \left( A'_{n+1} \Delta_{0,q_n} \Delta A'_{n+1} \left( \Delta_{0,q_n} \bigcap E_{n+1}^1 \bigcap (A_{n+1}^1)^{-1} (E_{n+1}^2) \right) \right) + \\ &\quad \mu \left( A'_{n+1} \left( \Delta_{0,q_n} \bigcap E_{n+1}^1 \bigcap (A_{n+1}^1)^{-1} (E_{n+1}^2 \Delta R^{(n)}) \right) \right) \end{aligned}$$

But on the one hand, since  $A'_{n+1}$  is measure preserving,

$$\begin{aligned} \mu \left( A'_{n+1} \Delta_{0,q_n} \Delta A'_{n+1} \left( \Delta_{0,q_n} \bigcap E_{n+1}^1 \bigcap (A_{n+1}^1)^{-1} (E_{n+1}^2) \right) \right) &= \\ \mu \left( \Delta_{0,q_n} \Delta \left( \Delta_{0,q_n} \bigcap E_{n+1}^1 \bigcap (A_{n+1}^1)^{-1} (E_{n+1}^2) \right) \right) &= \mu \left( \Delta_{0,q_n} \bigcap \left( E_{n+1}^{1c} \cup (A_{n+1}^1)^{-1} (E_{n+1}^{2c}) \right) \right) \\ &\leq \mu \left( E_{n+1,0}^{1c} \right) + \mu \left( E_{n+1,0}^{2c} \right) \leq \frac{2}{2^n q_n} + \frac{2}{2^n q_n} = \frac{4}{2^n q_n} \end{aligned}$$

And likewise, by using  $S_{\frac{1}{q_n}}$ -invariance, since  $R^{(n)} = \bigcup_{i=0}^{k_n-1} S_{\frac{a_n(i)}{q_n}} \Delta_{i,k_n q_n}$ ,

$$\mu\left(R^{(n)} \cap A_{n+1}^2(E_{n+1}^1) \cap E_{n+1}^2\right) = \mu\left(\Delta_{0,q_n} \cap A_{n+1}^2(E_{n+1}^1) \cap E_{n+1}^2\right) \leq \frac{4}{2^n q_n}$$

□

Moreover, we automatically obtain generation of  $\xi_n$  (and so  $\xi_n^\infty$ ). Indeed, by applying the induction assumption,  $\|B_n\|_1 \leq (k_{n-1}q_{n-1})^{R_7(n-1)}$  for some  $R_7(n)$ . If, in assumption 2 of lemma 14.3, we choose  $R_3(n)$  such that  $q_n \geq 2^n(\gamma_h^{(n)}q_{n-1})^{R_3(n-1)}$  (remember that we took  $k_n = \gamma_h^{(n)}$ ), and since  $\gamma_h^{(n+1)} \geq q_n$ , then lemma 17.4 gives:

$$\text{diam}\left((A_{n+1}^2)^{-1}\Delta_{l,k_nq_n} \cap E_{n+1}^2\right) \leq \frac{1}{2^n\|B_n\|_1}$$

We obtain the conditions of proposition 17.9, which imply generation:

**Proposition 17.7.** *Let  $A_{n+1} = A_{n+1}^2A_{n+1}^1$  and  $E_{n+1} = E_{n+1}^2 \cap A_{n+1}^2(E_{n+1}^1)$ . We have  $\mu(E_{n+1}) \geq 1 - 8/2^n$ , and there exists an explicit sequence of integers  $R_3(n)$  depending only on  $n$ , such that, for any  $q_{n+1} = q'_{n+1}(\gamma_h^{(n+1)}q_n)^{R_3(n)}$ ,  $0 \leq l \leq q_{n+1} - 1$ , we have:*

$$\text{diam}\left((A_{n+1})^{-1}\left(\Delta_{l,q_{n+1}} \cap E_{n+1}\right)\right) \leq \frac{1}{2^n\|B_n\|_1}$$

## 17.2 Construction in higher dimensions

First, we consider the case  $M = [0, 1]^{d-1} \times \mathbb{T}$ . Then, in subsection 17.4, we extend the construction to more general manifolds. The first step is the same as in dimension 2: We let  $\Gamma_i = [\frac{i}{k_n}, \frac{i+1}{k_n} - \epsilon_1] \times [0, 1]^{d-2} \times [0, \frac{1}{q_n}]$  and for  $x = (x_1, \dots, x_d) \in M$ , we consider the map  $A_{n+1}^1(x) = S_{\phi_{n,\epsilon_1}(x_1)}(x) = (x_1, \dots, x_{d-1}, x_d + \phi_{n,\epsilon_1}(x_1))$ .

The second step is also the same as in dimension 2: we quasi-rotate in the  $(x_1, x_d)$  plan (the plan in which we sliced the  $\Gamma_i$ ). We write  $\phi_{n,q_n}(x, y) = (\phi_{n,q_n,1}(x, y), \phi_{n,q_n,2}(x, y))$ . We let  $A_{n+1}^2(x_1, \dots, x_d) = (\phi_{n,q_n,1}(x_1, x_d), x_2, \dots, x_{d-1}, \phi_{n,q_n,2}(x_1, x_d))$  on  $[0, 1] \times [0, \frac{1}{q_n}]$ , and we extend it to  $[0, 1] \times \mathbb{T}$  by  $1/q_n$  periodicity along the  $x_d$  coordinate.

The third step is different than in dimension two: to obtain generation, we have to combine all dimensions. We proceed as in [FSW07, pp.1808-1809]. We have the proposition:

**Proposition 17.8.** *There exists a smooth measure-preserving and  $S_{\frac{1}{k_nq_n}}$ -equivariant diffeomorphism  $A_{n+1}^3$ , and there is an explicit function  $R_8(j)$ , depending only on  $j$  and  $d$ , such that*

$$\|A_{n+1}^3\|_j \leq (k_nq_n)^{R_8(j)}\|\phi_n\|_j$$

and there is a set  $E_{n+1}$  satisfying the assumptions of proposition 17.9 such that if  $A_{n+1} = A_{n+1}^3A_{n+1}^2A_{n+1}^1$ , then for any choice of  $q_{n+1} = q'_{n+1}(k_nq_n)^{R_3(n)}$ , with  $R_3(n)$  multiple of  $d$ , we have:

$$\text{diam}\left((A_{n+1})^{-1}\left(\Delta_{l,q_{n+1}} \cap E_{n+1}\right)\right) \leq \frac{1}{2^n\|B_n\|_1}$$

*Proof.* For  $i = 0, \dots, d-1$ , let  $\phi_{n, k_n q_n}^i(x_1, \dots, x_d) = (x_1, \dots, x_{i-1}, \phi_{n, k_n q_n}(x_i, x_{i+1}), x_{i+2}, \dots, x_d)$ , extended by  $1/k_n q_n$ -periodicity along the  $x_d$  coordinate. We let (see figures 11, 12, 13, 14):

$$A_{n+1}^3(x_1, \dots, x_d) = \phi_{n, k_n q_n}^{d-1} \dots \phi_{n, k_n q_n}^1(x_1, \dots, x_d)$$

Let also  $A_{n+1} = A_{n+1}^3 A_{n+1}^2 A_{n+1}^1$ . For  $j = 1, \dots, d-1$ ,  $r = 0, \dots, k_n q_n - 1$ , let also

$$E_{j,r} = [0, 1]^{j-1} \times \left[ \frac{1}{2^n}, 1 - \frac{1}{2^n} \right] \times \left[ \frac{r}{k_n q_n} + \frac{1}{k_n q_n 2^n}, \frac{r+1}{k_n q_n} - \frac{1}{k_n q_n 2^n} \right] \times [0, 1]^{d-(j+1)}$$

(in the notations, we omit dependencies in  $n$ ) and

$$E_j = \bigcup_{r=0}^{k_n q_n - 1} E_{j,r}$$

Note that  $E_j$  is  $\phi_{n, k_n q_n}^j$ -invariant. We let

$$E_{n+1}^3 = E - \phi_{n, k_n q_n}^1(E_2) \bigcap \dots \bigcap \phi_{n, k_n q_n}^1 \dots \phi_{n, k_n q_n}^{d-2}(E_{d-1})$$

and

$$E_{n+1} = E_{n+1}^3 \cap A_{n+1}^3(E_{n+1}^2) \cap A_{n+1}^3 A_{n+1}^2(E_{n+1}^1)$$

We obtain, for  $l = 0, \dots, (k_n q_n)^d - 1$ :

$$\text{diam} \left( (A_{n+1}^3)^{-1} \Delta_{l, (k_n q_n)^d} \cap E_{n+1}^3 \right) \leq \frac{1}{k_n q_n}$$

Therefore, by lemma 17.4,

$$\text{diam} \left( (A_{n+1}^2)^{-1} \left( (A_{n+1}^3)^{-1} \left( \Delta_{l, (k_n q_n)^d} \cap E_{n+1}^3 \right) \right) \cap E_{n+1}^2 \right) \leq \frac{1}{k_n}$$

and therefore, we also have:

$$\text{diam} \left( (A_{n+1}^1)^{-1} \left( (A_{n+1}^2)^{-1} \left( (A_{n+1}^3)^{-1} \left( \Delta_{l, (k_n q_n)^d} \cap E_{n+1}^3 \right) \right) \cap E_{n+1}^2 \right) \cap E_{n+1}^1 \right) \leq \frac{1}{k_n} \leq \frac{1}{2^n \|B_n\|_1}$$

Finally, for any choice of  $q_{n+1} = q'_{n+1} (k_n q_n)^{R_3(n)}$ , with  $R_3(n)$  multiple of  $d$ , we obtain the wanted estimation:

$$\text{diam} \left( (A_{n+1})^{-1} \left( \Delta_{l, q_{n+1}} \cap E_{n+1} \right) \right) \leq \frac{1}{2^n \|B_n\|_1}$$

Thus, we get generation.

□

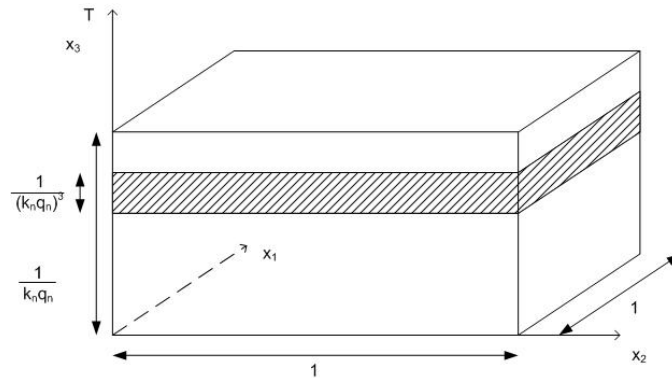


Figure 11: An element  $\Delta_{l,(k_n q_n)^d}$  for  $d = 3$ , before the application of  $(A_{n+1}^3)^{-1} = (\phi_{n,k_n q_n}^1)^{-1} (\phi_{n,k_n q_n}^2)^{-1}$ . Its size is  $1 \times 1 \times 1/(k_n q_n)^3$ .

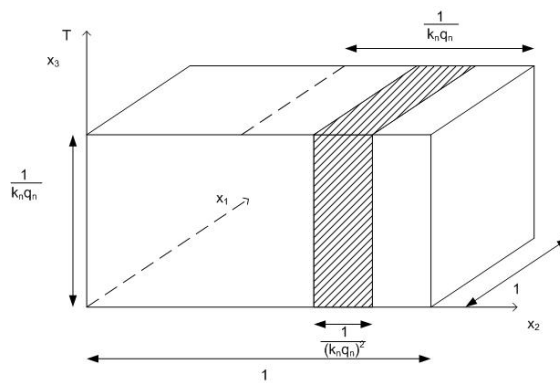


Figure 12: The element  $(\phi_{n,k_n q_n}^2)^{-1} (\Delta_{l,(k_n q_n)^d}) \cap E_{n+1}^3$ . Its size is less than  $1 \times 1/(k_n q_n)^2 \times 1/(k_n q_n)$ .

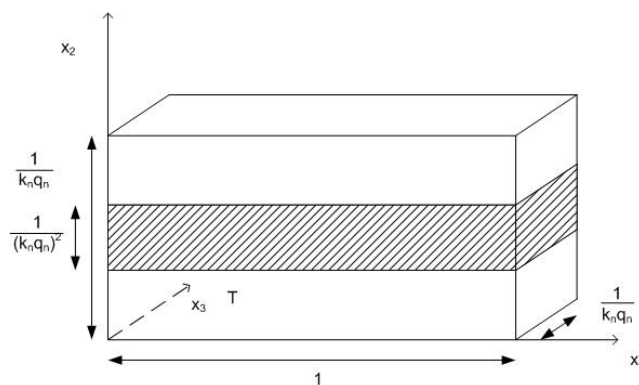


Figure 13: The element  $(\phi_{n,k_n q_n}^2)^{-1} (\Delta_{l,(k_n q_n)^d}) \cap E_{n+1}^3$ , in the plan  $(x_1, x_2)$ .



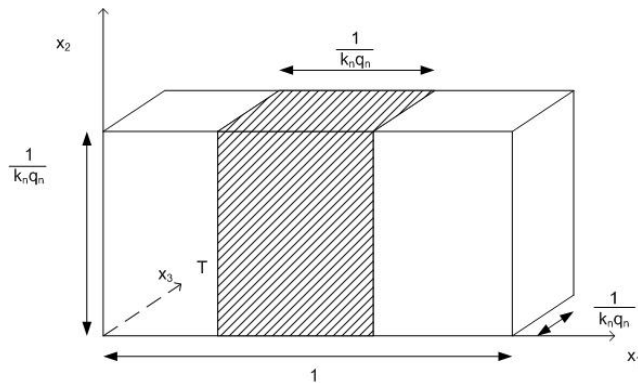


Figure 14: The element  $(\phi_{n,k_n q_n}^1)^{-1} (\phi_{n,k_n q_n}^2)^{-1} (\Delta_{l,(k_n q_n)^d}) \cap E_{n+1}^3$ , in the plan  $(x_1, x_2)$ . Its size is less than  $1/(k_n q_n) \times 1/(k_n q_n) \times 1/(k_n q_n)$ .

### 17.3 Generation of $\xi_n$ , convergence of the sequence of diffeomorphisms and ergodicity of the limit

By combining lemma 16.3, corollary 16.4, and proposition 17.1, in order to complete the proof of lemma 14.3, it remains to show that  $\xi_n$  generates, that  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$  converges in the smooth topology, and that the limit  $T$  of  $T_n$  is ergodic.

#### 17.3.1 Generation of $\xi_n$

**Proposition 17.9.** *If there is a set  $E_{n+1} \subset M$  such that  $\sum_{n \geq 0} \mu(E_{n+1}^c) < +\infty$  and such that for any  $0 \leq l \leq q_{n+1} - 1$ ,*

$$\text{diam} \left( A_{n+1}^{-1} \left( \Delta_{l,q_{n+1}} \cap E_{n+1} \right) \right) \leq \frac{1}{2^n \|B_n\|_1}$$

*then  $\xi_n$  generates.*

*Proof.* By the Borel-Cantelli lemma,

$$\mu(\liminf E_n) = 1$$

Let  $E = \liminf E_n$  and let  $x \in E$ . By extracting, we can suppose that  $x \in E_n$  for any  $n$ . Let  $c_n(x) \in \xi_n$  such that  $x \in c_n(x)$ . We have:  $\{x\} \subset \bigcap_{n \geq 0} E_n \cap c_n(x)$ . Moreover, since

$$\text{diam} \left( A_{n+1}^{-1} \left( \Delta_{l,q_{n+1}} \cap E_{n+1} \right) \right) \leq \frac{1}{2^n \|B_n\|_1}$$

then

$$\text{diam} \left( B_{n+1}^{-1} \left( \Delta_{l,q_{n+1}} \cap E_{n+1} \right) \right) \leq \frac{1}{2^n}$$

(note that this intersection may be empty for some  $l$ ), and therefore,

$\text{diam}(E_n \cap c_n(x)) \rightarrow_{n \rightarrow +\infty} 0$ . So  $\bigcap_{n \geq 0} E_n \cap c_n(x)$  is at most a singleton, and  $\xi_n$  generates.  $\square$

### 17.3.2 Convergence

To complete the proof of lemma 14.3 for  $M = [0, 1]^{d-1} \times \mathbb{T}$ , we need to show the convergence of  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$ . By the Cauchy criterion, it suffices to show that  $\sum_{n \geq 0} d_n(T_{n+1}, T_n)$  converges. We combine the fact that  $A_{n+1}$  commutes with  $S_{\frac{1}{q_n}}$ , the estimation of  $B_{n+1}$  and the assumption of closeness between  $p_{n+1}/q_{n+1}$  and  $p_n/q_n$  of lemma 14.3. We recall the lemma [FSW07, p.1812]:

**Lemma 17.10.** *Let  $k \in \mathbb{N}$ . There is a constant  $C(k, d)$  such that, for any  $h \in \text{Diff}(M)$ ,  $\alpha_1, \alpha_2 \in \mathbb{R}$ , we have:*

$$d_k(hS_{\alpha_1}h^{-1}, hS_{\alpha_2}h^{-1}) \leq C(k, d) \|h\|_{k+1}^{k+1} |\alpha_1 - \alpha_2|$$

Since  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n = B_{n+1}^{-1} S_{\frac{p_n}{q_n}} B_{n+1}$  (because  $A_{n+1}$  commutes with  $S_{\frac{1}{q_n}}$ ), and since, for  $n \geq 2$ ,  $\|\phi_n\|_{n+1} \leq q_n^{R_9(n)}$  for a sequence  $R_9(n)$  independent of  $q_n$  (because  $q_n \geq 2$  for  $n \geq 2$ ), we obtain, for a fixed sequence  $R_{10}(n)$  (that depends on  $n$  and on the dimension  $d$ ):

$$\begin{aligned} d_n(T_{n+1}, T_n) &= d_n(B_{n+1}^{-1} S_{\frac{p_{n+1}}{q_{n+1}}} B_{n+1}, B_{n+1}^{-1} S_{\frac{p_n}{q_n}} B_{n+1}) \\ &\leq C(k, d) \|B_{n+1}\|_{n+1}^{n+1} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \leq \left( \gamma_h^{(n+1)} q_n \right)^{R_{10}(n)} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \end{aligned}$$

For some choice of the sequence  $R_2(n)$  in lemma 14.3, this last estimate guarantees the convergence of  $T_n$  in the smooth topology.

### 17.3.3 Ergodicity

Ergodicity is a property preserved by metric isomorphism. Therefore, to show that  $T$  is ergodic, it is sufficient to show that the limit translation  $T^\alpha$  on the torus  $\mathbb{T}^h$  is ergodic. We apply a theorem by Katok and Stepin [KS67]:

**Theorem 17.11** (Katok-Stepin [KS67]). *Let  $U$  be an automorphism of a Lebesgue space  $(N, \nu)$ , let  $(U_n)_{n \geq 1}$  be a sequence of measure-preserving transformations, and let  $(\chi_n)_{n \geq 1}$  be a sequence of finite measurable partitions of  $N$ . Suppose that:*

- $U_n$  permutes the elements of  $\chi_n$  cyclically.
- $(\chi_n)_{n \geq 1}$  generates.
- $\sum_{c \in \chi_n} \nu(U(c)\Delta U_n(c)) = o(1/|\chi_n|)$  (where  $|\chi_n|$  is the cardinal of  $\chi_n$ ).

then  $U$  is ergodic.

Note that the sequence of partitions  $(\chi_n)_{n \geq 1}$  does not need to be monotonic. For our purpose, we cannot apply theorem 17.11 with  $\chi_n = \zeta_n$  or  $\zeta_n^\infty$ . Indeed, since  $p_n$  and  $q_n$  are not relatively prime in general (contrary to Anosov-Katok's original construction), then  $T_{\frac{p_n}{q_n}}^{\gamma^{(n)}}$  is not a cyclic permutation on  $\zeta_n$  nor  $\zeta_n^\infty$ , in general. Let  $g_n = \text{gcd}(p_n, q_n)$  and

$$\hat{\Gamma}_{0,n} = \bigcup_{j=0}^{g_n-1} T^{j \frac{\gamma^{(n)}}{q_n}} \Gamma_{0,n}$$

We have the lemma:

**Lemma 17.12.** *Let  $\hat{\zeta}_n$  be the partition defined by:*

$$\hat{\zeta}_n = \left\{ \hat{\Gamma}_{i,n} = T^{i \frac{g_n \gamma^{(n)}}{q_n}} \hat{\Gamma}_{0,n}, i = 0, \dots, \frac{q_n}{g_n} - 1 \right\}$$

$T^{\frac{p_n}{q_n} \gamma^{(n)}}$  is a cyclic permutation on  $\hat{\zeta}_n$ , and  $\hat{\zeta}_n$  generates.

*Proof.*  $T^{\frac{p_n}{q_n} \gamma^{(n)}}$  is a cyclic permutation on  $\hat{\zeta}_n$  because  $g_n = \gcd(p_n, q_n)$ . Since  $q_n$  divides  $q_{n+1}$ , there is an integer  $e_{n+1}$  such that:

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{e_{n+1}}{q_{n+1}}$$

Since

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \leq \frac{1}{(\gamma_h^{(n+1)} q_n)^{R_2(n)}}$$

Then:

$$|e_{n+1}| \leq \frac{q_{n+1}}{(\gamma_h^{(n+1)} q_n)^{R_2(n)}}$$

Moreover, since  $q_n^2$  divides  $q_{n+1}$ , then  $\gcd(p_{n+1}, q_{n+1}/q_n) = \gcd(p_{n+1}, q_{n+1}) = g_{n+1}$ . Moreover,

$$p_{n+1} - \frac{q_{n+1}}{q_n} p_n = e_{n+1}.$$

Since  $g_{n+1}$  divides  $p_{n+1}$  and  $q_{n+1}/q_n$ , then  $g_{n+1}$  divides  $e_{n+1}$ . In particular,  $g_{n+1} \leq |e_{n+1}|$ . By condition 7 of lemma 14.3, we get:

$$g_{n+1} \leq \frac{q_{n+1}}{(\gamma_h^{(n+1)} q_n)^{R_2(n)}} \leq \frac{q_{n+1}}{2^{n+1} \|\gamma^{(n+1)}\|}$$

Moreover, since:

$$\text{diam}(\Gamma_{0,n}) \leq \max \left( d_n, \frac{\|\gamma^{(n)}\|}{q_n} \right) \rightarrow_{n \rightarrow +\infty} 0$$

( $d_n = \text{diam}(\Gamma^{(n)})$ , where  $\Gamma^{(n)} \subset \mathbb{T}^{h-1} \times \{0\}$  is a fundamental domain of the flow  $\{T^{t\gamma^{(n)}}\}$ ) then:

$$\text{diam}(\hat{\Gamma}_{0,n}) \leq \max \left( d_n, \frac{g_n \|\gamma^{(n)}\|}{q_n} \right) \leq \max \left( d_n, \frac{1}{2^n} \right) \rightarrow_{n \rightarrow +\infty} 0$$

It shows that  $\hat{\zeta}_n$  generates. □

It remains to estimate  $\sum_{c \in \hat{\zeta}_n} \mu_h \left( T^\alpha c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right)$ . We have the lemma:

**Lemma 17.13.** *There is a choice of  $R_2(n)$  in condition 4 of lemma 14.3 such that:*

$$\sum_{c \in \hat{\zeta}_n} \mu_h \left( T^\alpha c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right) = o(g_n/q_n)$$

*Proof.* We have:

$$\sum_{c \in \hat{\zeta}_n} \mu_h \left( T^\alpha c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right) = \sum_{k \geq n} \sum_{c \in \hat{\zeta}_k} \mu_h \left( T^{\frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)}} c \Delta T^{\frac{p_k}{q_k} \gamma^{(k)}} c \right) = \sum_{k \geq n} \sum_{c \in \hat{\zeta}_k} \mu_h \left( T^{\frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)} - \frac{p_k}{q_k} \gamma^{(k)}} c \Delta c \right)$$

Let  $\tau_n$  be the  $(h-1)$ -volume of the border of an element of  $\hat{\zeta}_n$ . We have:

$$\mu_h \left( T^{\frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)} - \frac{p_k}{q_k} \gamma^{(k)}} c \Delta c \right) \leq \tau_k \left\| \frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)} - \frac{p_k}{q_k} \gamma^{(k)} \right\| = \tau_k \left\| \left( \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right) \gamma^{(k+1)} \right\|$$

Moreover, there is a fixed function  $f(h)$  that only depends on  $h$  such that:

$$\tau_n \leq g_n \left\| \frac{\gamma^{(n)}}{q_n} \right\| \text{Vol}(\Gamma^{(n)}) f(h) \leq \frac{q_n}{2^n \|\gamma^{(n)}\|} \frac{\|\gamma^{(n)}\|}{q_n} \text{Vol}(\Gamma^{(n)}) f(h)$$

Moreover,  $\text{Vol}(\Gamma^{(n)}) \leq 1$ . Therefore, there is a choice of  $R_2(n)$  such that:

$$\sum_{c \in \hat{\zeta}_n} \mu_h \left( T^\alpha c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right) = o(g_n/q_n)$$

□

By combining lemmas 17.12 and 17.13, and by applying theorem 17.11, we obtain that there is a choice of  $R_2(n)$  such that  $T^\alpha$  is ergodic.

## 17.4 Extension to more general manifolds

To extend the construction from  $M = [0, 1]^{d-1} \times \mathbb{T}$  to a general  $d$ -dimensional smooth compact connected manifold, admitting an effective volume-preserving circle action, we proceed as in [FSW07, p. 1805], but we give more details. Let  $\hat{S}$  be this effective circle action on  $M$ ,  $S$  be the circle action on  $[0, 1]^{d-1} \times \mathbb{T}$ . For  $q \geq 1$ , let  $F_q$  be the set of fixed points of  $\hat{S}_{1/q}$ . Let  $B = \partial M \cup_{q \geq 1} F_q$  be the set of exceptional points. We recall the proposition, stated in [FSW07], and which is a consequence of the proof of proposition 2.3 in [AK70, p.9]:

**Proposition 17.14** ([FSW07]). *Let  $M$  be a  $d$ -dimensional smooth compact connected manifold, with an effective circle action  $S$ , preserving a smooth volume  $\mu$ . Let  $S_1$  denote the circle action on  $[0, 1]^{d-1} \times \mathbb{T}$ . There exists a continuous surjective map  $\Gamma : [0, 1]^{d-1} \times \mathbb{T} \rightarrow M$  such that:*

1. *the restriction of  $\Gamma$  to  $]0, 1[^{d-1} \times \mathbb{T}$  is a smooth diffeomorphic embedding.*

2.  $\mu(\Gamma(\partial([0, 1]^{d-1} \times \mathbb{T}))) = 0$
3.  $B \subset \Gamma(\partial([0, 1]^{d-1} \times \mathbb{T}))$
4.  $\Gamma_*(Leb) = \mu$
5.  $\hat{S}\Gamma = \Gamma S$

*Proof of lemma 14.3 for a general  $M$ .* We apply this proposition at each step in lemma 14.2: we have that  $\Gamma : ]0, 1[^{d-1} \times \mathbb{T} \rightarrow E = \Gamma(]0, 1[^{d-1} \times \mathbb{T})$  is a diffeomorphism. Let  $\Gamma^{-1}$  its inverse. We need the claim:

**Claim 17.15.** *Let  $\hat{T}_n : M \rightarrow M$  defined by  $\hat{T}_n(x) = \Gamma B_n^{-1} S_{\frac{p_n}{q_n}} B_n \Gamma^{-1}(x)$  if  $x \in E$  and  $\hat{T}_n(x) = \hat{S}_{\frac{p_n}{q_n}}(x)$  otherwise.  $(\hat{T}_n)_{n \geq 0}$  is a sequence of smooth diffeomorphisms converging towards a smooth diffeomorphism  $\hat{T}$  in the smooth topology.*

*Proof.*  $\hat{T}_n$  is bijective,  $\hat{T}_n|_E$  and  $\hat{T}_n|_{E^c}$  are smooth diffeomorphisms. Moreover, let  $E_n = [1/2^{n+2}, 1 - 1/2^{n+2}]^{d-1} \times \mathbb{T}$  and  $G_n = ([0, 1]^{d-1} - [1/2^{n+2}, 1 - 1/2^{n+2}]^{d-1}) \times \mathbb{T}$ . of  $\partial([0, 1]^{d-1} \times \mathbb{T})$  inside  $[0, 1]^{d-1} \times \mathbb{T}$ , and stable by the flow  $S_t$ ,  $B_n = B_n^{-1} = Id$ . Therefore, for  $x \in \Gamma(G_n) \subset E$ ,

$$\hat{T}_n(x) = \Gamma B_n^{-1} S_{\frac{p_n}{q_n}} B_n \Gamma^{-1}(x) = \Gamma S_{\frac{p_n}{q_n}} \Gamma^{-1}(x) = \hat{S}_{\frac{p_n}{q_n}} \Gamma \Gamma^{-1}(x) = \hat{S}_{\frac{p_n}{q_n}}(x)$$

Therefore,  $\hat{T}_n$  is a smooth diffeomorphism. Finally, the sequence  $\hat{T}_n$  smoothly converges towards a smooth diffeomorphism  $\hat{T}$ . Indeed, we follow the same argument as in the proof of the convergence of  $T_n$  (for  $M = [0, 1]^{d-1} \times \mathbb{T}$ ), and we add that  $\sup_{E_n} \|\Gamma\|_{n+1} \leq q_n^{R_{11}(n)}$  for a sequence  $R_{11}(n)$  independent of  $q_n$  (because  $q_n \geq 2$  for  $n \geq 2$ ). We obtain, for a fixed sequence  $R_{12}(n)$  (that depends on  $n$  and on the dimension  $d$ ):

$$\begin{aligned} d_n(\hat{T}_{n+1}, \hat{T}_n) &= \max \left( d_n(\hat{T}_{n+1}|_{\Gamma(E_n)}, \hat{T}_n|_{\Gamma(E_n)}), d_n(\hat{T}_{n+1}|_{\Gamma(G_n)}, \hat{T}_n|_{\Gamma(G_n)}) \right) \\ &\leq C(k, d) \sup_{E_n} \|\Gamma\|_{n+1}^{n+1} \|B_{n+1}\|_{n+1}^{n+1} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \leq \left( \gamma_h^{(n+1)} q_n \right)^{R_{12}(n)} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \end{aligned}$$

□

For some choice of the sequence  $R_2(n)$  in lemma 14.3 (greater than in the case of  $[0, 1]^{d-1} \times \mathbb{T}$ ), this last estimate guarantees the convergence of  $\hat{T}_n$  in the smooth topology.

Let  $\hat{\xi}_n^\infty = \Gamma \xi_n^\infty$ . Since  $\Gamma$  is a diffeomorphism except on a set of zero Lebesgue measure, then the equivalent class of  $\hat{\xi}_n^\infty$  is a measurable partition. Moreover, since  $\Gamma$  is independent of  $n$ , then the sequence  $\hat{\xi}_n^\infty$  generates. Moreover,  $\hat{T}_n$  is  $\mu$ -measure preserving, stabilizes  $\hat{\xi}_n^\infty$  (because  $\hat{T}_n = \Gamma B_n^{-1} S_{\frac{p_n}{q_n}} B_n \Gamma^{-1}$   $\mu$ -almost everywhere) and by claim 17.15,  $\hat{T}_n$  weakly converges towards an automorphism of  $M$   $\hat{T}$ . Let  $\hat{K}_n^\infty = \Gamma \bar{K}_n^\infty$ . We have  $\bar{K}_n^\infty T_{\frac{p_n}{q_n}}^{\gamma^{(n)}} = \hat{T}_n \hat{K}_n^\infty$  and  $\hat{K}_{n+1}^\infty|_{\hat{\xi}_n^\infty} = \hat{K}_n^\infty$ . This allows to apply lemma 14.2, and combined with claim 17.15, we obtain lemma 14.3 for a general  $M$ .

□

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## Part IV

# Non-standard couples of angles of rotations

## Summary

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Let  $M$  be a smooth compact connected manifold, on which there exists an effective smooth circle action  $S_t$  preserving a positive smooth volume. In this chapter, we show that on  $M$ , the smooth closure of the smooth volume-preserving conjugation class of some Liouville rotations  $S_\alpha$  of angle  $\alpha$  contains a smooth volume-preserving diffeomorphism  $T$  that is metrically isomorphic to an irrational rotation  $R_\beta$  on the circle, with  $\alpha \neq \pm\beta$ , and with  $\alpha$  and  $\beta$  chosen either rationally dependent or rationally independent. In particular, if  $M$  is the closed annulus  $[0, 1] \times \mathbb{T}^1$ ,  $M$  admits a smooth ergodic pseudo-rotation  $T$  of angle  $\alpha$  that is metrically isomorphic to the rotation  $R_\beta$ . Moreover,  $T$  is smoothly tangent to  $S_\alpha$  on the boundary of  $M$ .

## 18 Introduction

Let  $\mathbb{A} = [0, 1] \times \mathbb{T}^1$  be the closed annulus and  $T$  be a homeomorphism isotopic to the identity. The *rotation set* of  $T$  measures the asymptotic speeds of rotation of the orbits of  $T$  around the annulus. It generalizes the notion of rotation number of a circle homeomorphism, introduced by Poincaré.  $T$  is an *irrational pseudo-rotation* if its rotation set is reduced to a single irrational number  $\alpha$ , called the *angle* of  $T$ . A broad question is raised by Béguin et al. [BCLRP04]: what are the similarities between the dynamics of the rigid rotation  $S_\alpha$  of angle  $\alpha$  and the dynamics of an irrational pseudo-rotation  $T$  of angle  $\alpha$ ?

From a topological viewpoint, a similarity between  $S_\alpha$  and  $T$  (but with  $T$  only  $C^0$ ) has been shown by Béguin et al. [BCLRP04]: the rotation  $S_\alpha$  is in the  $C^0$ -closure of the conjugacy class of  $T$ . Their result is analogous to a theorem by Kwapisz [Kwa03] on the torus  $\mathbb{T}^2$  (in this case, the angle of a pseudo-rotation is an element of  $\mathbb{T}^2$ ). Jäger [Jäg09] and Wang [Wan11] also investigated this broad question. However, there are also possible differences between  $S_\alpha$  and  $T$ . From a metric viewpoint, Anosov and Katok [AK70] constructed a smooth pseudo-rotation of  $\mathbb{A}$  that is metrically isomorphic to an ergodic translation of  $\mathbb{T}^2$ . Béguin et al. [BCLR07] constructed on  $\mathbb{T}^2$  a pseudo-rotation that is minimal, uniquely ergodic, but with positive entropy. In this paper, we construct a smooth pseudo-rotation of angle  $\alpha$  that is metrically isomorphic to an irrational rotation  $R_\beta$  with  $\alpha \neq \pm\beta$ . This is a construction of a non-standard smooth realization, based on the method of approximation by successive conjugations (see [FK04] for a presentation), a method that is often fruitful in smooth realization problems.

We recall that a *smooth realization* of an abstract system  $(X, f, \nu)$  is a triplet  $(M, T, \mu)$ , where  $M$  is a smooth compact manifold,  $\mu$  a smooth measure on  $M$  and  $T$  a smooth  $\mu$ -preserving diffeomorphism of  $M$ , such that  $(M, T, \mu)$  is metrically isomorphic to  $(X, f, \nu)$  (when  $(M, \mu)$  and  $(X, \nu)$  are implied, we just say that  $T$  is metrically isomorphic to  $f$ ). Moreover, a smooth realization is *non-standard* if  $M$  and  $X$  are not diffeomorphic.

Suppose there exists an ergodic pseudo-rotation  $T$  of angle  $\alpha$  that is a non-standard smooth realization of a rotation  $R_\beta$  on the circle. Then the couple  $(\alpha, \beta)$  is called a *non-standard couple of angles*. In this paper, we show that there exists non-standard couple of angles  $(\alpha, \beta)$ , such that  $\alpha \neq \pm\beta$ , with  $\alpha$  and  $\beta$  chosen either rationally dependent or rationally independent.

Anosov and Katok [AK70] showed the existence of an angle  $\alpha$  such that  $(\alpha, \alpha)$  is a non-standard couple of angles. Fayad et al. [FSW07] showed that for any  $\alpha$  Liouville,  $(\alpha, \alpha)$  is a non-standard couple of angles. The question arises about the existence of a non-standard couple of angles  $(\alpha, \beta)$  with  $\alpha \neq \beta$ .

It is worthy to recall that two ergodic rotations  $R_\alpha$  and  $R_\beta$  on the circle are metrically isomorphic if and only if  $\beta = \pm\alpha$ . If  $\beta = \alpha$ , the isomorphism is the identity, and if  $\beta = -\alpha$ , an isomorphism is given by a symmetry of axis going through the center of the circle. Therefore, by applying the result of Fayad et al. [FSW07], it becomes trivial to find a non-standard couple of angles  $(\alpha, -\alpha)$ . Our result shows that if, instead of considering metric automorphisms of the circle, we consider metric isomorphisms between the circle and the annulus, the situation becomes richer: we can have  $\alpha \neq \pm\beta$ , with  $\alpha$  and  $\beta$  either rationally dependent or rationally independent. However,  $\alpha$  needs to be Liouville. Indeed, a result by Herman (with a proof published by Fayad and Krikorian [FK09b]) implies that if a smooth quasi-rotation  $T$  of the closed annulus has Diophantine angle (i.e. non-Liouville), then  $T$  cannot be ergodic (and a fortiori,  $T$  cannot be metrically isomorphic to an ergodic rotation). However, the situation where  $\alpha$  is Liouville and  $\beta$  is Diophantine, though not addressed in this paper, is not excluded yet. The existence of this situation would reply positively to the open question about the existence of a non-standard smooth realization of a Diophantine circle rotation [FK04].

More generally, let  $M$  be a smooth compact connected manifold of dimension  $d$ , on which there exists an effective smooth circle action  $S_t$  preserving a positive smooth measure  $\mu$ . Let  $\mathcal{A}_\alpha$  be the smooth conjugation class of the rotation  $S_\alpha$ , and  $\bar{\mathcal{A}}_\alpha$  its closure in the smooth topology. If  $M = \mathbb{T}^1$  and if  $\alpha$  is Diophantine, then  $\bar{\mathcal{A}}_\alpha = \mathcal{A}_\alpha$  by Herman-Yoccoz theorem [Yoc84] (indeed, by continuity, the rotation number of a diffeomorphism  $T \in \bar{\mathcal{A}}_\alpha$  is  $\alpha$ ). On the other hand, when  $\alpha$  is Liouville,  $\bar{\mathcal{A}}_\alpha \neq \mathcal{A}_\alpha$ . In this paper, if  $M$  has a dimension  $d \geq 2$ , then for some Liouville  $\alpha$ , we show that  $\bar{\mathcal{A}}_\alpha$  contains non-standard smooth realizations of circle rotations  $R_\beta$ , with  $\alpha \neq \pm\beta$ , and with  $\alpha$  and  $\beta$  chosen either rationally dependent or rationally independent. In this case,  $(\alpha, \beta)$  is still called a *non-standard couple of angles*. More precisely, we show the following theorem:

**Theorem 18.1.** *Let  $M$  be a smooth compact connected manifold of dimension  $d \geq 2$ , on which there exists an effective smooth circle action  $(S_t)_{t \in \mathbb{T}^1}$  preserving a positive smooth measure  $\mu$ . For any  $u, v \in \mathbb{T}^1$ , for any  $\epsilon > 0$ , there exist  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  in a  $\epsilon$ -neighborhood of  $(u, v)$ ,  $T \in \text{Diff}^\infty(M, \mu)$ , such that  $T \in \bar{\mathcal{A}}_\alpha$  and such that the rotation  $R_\beta$  of angle  $\beta$  on  $\mathbb{T}^1$  is metrically isomorphic to  $T$ . Moreover,  $\beta$  can be chosen either rationally dependent or rationally independent of  $\alpha$ .*

Theorem 18.1 generalizes the particular case  $M = [0, 1]^{d-1} \times \mathbb{T}^1$ :

**Theorem 18.2.** *Let  $d \geq 2$ ,  $M = [0, 1]^{d-1} \times \mathbb{T}^1$ ,  $\mu$  the Lebesgue measure. For  $t \in \mathbb{T}^1$ , let  $S_t : M \rightarrow M$  defined by  $S_t(x, s) = (x, s + t)$ . For any  $u, v \in \mathbb{T}^1$ , for any  $\epsilon > 0$ , there exist  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  in a  $\epsilon$ -neighborhood of  $(u, v)$ ,  $T \in \text{Diff}^\infty(M, \mu)$ , such that for any  $j \in \mathbb{N}$ ,  $(D^j T)|_{\partial M} = (D^j S_\alpha)|_{\partial M}$  and such that the rotation  $R_\beta$  of angle  $\beta$  on  $\mathbb{T}^1$  is metrically isomorphic to  $T$ . Moreover,  $\beta$  can be chosen either rationally dependent or rationally independent of  $\alpha$ .*

In the case of the closed annulus  $M = [0, 1] \times \mathbb{T}^1$ , we obtain:

**Corollary 18.3.** *Let  $M = [0, 1] \times \mathbb{T}^1$ ,  $\mu$  the Lebesgue measure. For  $t \in \mathbb{T}^1$ , let  $S_t : M \rightarrow M$  defined by  $S_t(x, s) = (x, s + t)$ . For any  $u, v \in \mathbb{T}^1$ , for any  $\epsilon > 0$ , there exist  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  in a  $\epsilon$ -neighborhood of  $(u, v)$ ,  $T \in \text{Diff}^\infty(M, \mu)$  a pseudo-rotation of angle  $\alpha$ , such that the rotation  $R_\beta$  of angle  $\beta$  on  $\mathbb{T}^1$  is metrically isomorphic to  $T$ . Moreover,  $\beta$  can be chosen either rationally dependent or rationally independent of  $\alpha$ .*

To show these results, we suitably modify one of Anosov and Katok's constructions. In [AK70], they constructed ergodic translations on the torus  $\mathbb{T}^h$ ,  $h \geq 2$ , of coordinates



$(\beta_1, \dots, \beta_h)$ , translations that admit non-standard smooth realizations on  $[0, 1]^{d-1} \times \mathbb{T}^1$ ,  $d \geq 2$ , such that  $T|_{\partial M}$  is a rotation of angle  $\alpha$ . Moreover, in his construction,  $\alpha \neq \beta_i$ ,  $i = 1, \dots, h$ . In the previous chapter, we show that one  $\beta_i$  can be an arbitrarily chosen Liouville number. However, this construction does not apply directly to the one-dimensional case. This is why, to obtain our result, though we essentially follow the previous chapter, we still need some substantial modifications.

## 18.1 Definitions

Let  $\text{Diff}^\infty(M, \mu)$  be the class of smooth diffeomorphisms of  $M$  preserving the Lebesgue measure  $\mu$ . For  $B \in \text{Diff}^\infty(M, \mu)$  and  $j \in \mathbb{N}^*$ , let  $D^j B$  be the  $j^{\text{th}}$  derivative of  $B$  if  $j > 0$ , and the  $-j^{\text{th}}$  derivative of  $B^{-1}$  if  $j < 0$ . For  $x \in M$ , let  $|D^j B(x)|$  be the norm of  $D^j B(x)$  at  $x$ . We denote  $\|B\|_k = \max_{0 < |j| \leq k} \max_{x \in M} |D^j B(x)|$ .

A *finite measurable partition*  $\bar{\xi}$  of a measured manifold  $(N, \nu)$  is the equivalence class of a finite set  $\xi$  of disjoint measurable subsets of  $N$  whose union is  $N$ , modulo sets of  $\nu$ -measure zero. In most of this paper, we do not distinguish a partition  $\xi$  with its equivalent class  $\bar{\xi}$  modulo sets of  $\nu$ -measure zero. In these cases, both are denoted  $\xi$ . Moreover, all partitions considered in this paper are representatives of a finite measurable partition.

A partition  $\xi'$  is *subordinate* to a partition  $\xi$  if any element of  $\xi$  is a union of elements of  $\xi'$ , modulo sets of  $\nu$ -measure zero. In this case, if  $\mathcal{B}(\xi)$  denotes the completed algebra generated by  $\xi$ , then  $\mathcal{B}(\xi) \subset \mathcal{B}(\xi')$ . The inclusion map  $i : \mathcal{B}(\xi) \rightarrow \mathcal{B}(\xi')$  will be denoted  $\xi \hookrightarrow \xi'$ . This notation also means that  $\xi'$  is *subordinate* to  $\xi$ . A sequence of partitions  $\xi_n$  is *monotonic* if for any  $n$ ,  $\xi_n \hookrightarrow \xi_{n+1}$ . These definitions and properties are independent of the choice of the representatives  $\xi$  and  $\xi'$  of the equivalence classes  $\bar{\xi}$  and  $\bar{\xi}'$ .

A measure-preserving bijective bimeasurable map  $T : (M_1, \mu_1, \mathcal{B}_1) \rightarrow (M_2, \mu_2, \mathcal{B}_2)$  induces an *isomorphism* of measure algebras, still denoted  $T : (\mu_1, \mathcal{B}_1) \rightarrow (\mu_2, \mathcal{B}_2)$ . If  $\xi_1, \xi_2$  are partitions, and if  $\mathcal{B}_1 = \mathcal{B}(\xi_1)$  and  $\mathcal{B}_2 = \mathcal{B}(\xi_2)$ , we denote  $T : \xi_1 \rightarrow \xi_2$  this induced isomorphism of measure algebras. If  $M_1 = M_2$ ,  $\mu_1 = \mu_2$  and  $\mathcal{B}_1 = \mathcal{B}_2$ , then  $T$  is a *measure-preserving transformation*. Its induced isomorphism is an *automorphism* (see [Hal56, p.43] and [Wei72]).

A *metric isomorphism*  $L$  of measure-preserving transformations  $T_1 : (M_1, \mu_1, \mathcal{B}_1) \rightarrow (M_1, \mu_1, \mathcal{B}_1)$ ,  $T_2 : (M_2, \mu_2, \mathcal{B}_2) \rightarrow (M_2, \mu_2, \mathcal{B}_2)$  is a measure-preserving bijective bimeasurable map  $L : (M_1, \mu_1, \mathcal{B}_1) \rightarrow (M_2, \mu_2, \mathcal{B}_2)$  such that  $LT_1 = T_2L$  a.e. For convenience, when the measure is the Lebesgue measure and the algebra is the Borelian algebra, we omit to mention the measures and algebras, and we simply say that  $L : (M_1, T_1) \rightarrow (M_2, T_2)$  is a metric isomorphism.

Let  $\bar{\xi}$  be a measurable partition and  $\xi$  a representative of this equivalent class modulo sets of  $\mu$ -measure zero. For  $x \in M$ , we denote  $\xi(x)$  the element of the partition  $\xi$  such that  $x \in \xi(x)$ . A sequences of partitions  $\xi_n$  of measurable sets *generates* if there is a set of full measure  $F$  such that for any  $x \in F$ ,

$$\{x\} = F \bigcap_{n \geq 1} \xi_n(x)$$

This property of generation is independent of the choice of the representatives  $\xi_n$  of the

equivalent class  $\bar{\xi}_n$  and therefore, we will say that the sequence of measurable partitions  $\bar{\xi}_n$  generates. Let  $M/\xi$  denote the equivalent class of the algebra generated by  $\xi$ , modulo sets of  $\mu$ -measure zero.  $M/\xi$  is independent of the choice of the representative  $\xi$  of the equivalent class  $\bar{\xi}$ . If  $T : M_1 \rightarrow M_2$  is a measure-preserving map such that  $T(\xi_1) = \xi_2$   $\mu$ -almost everywhere, we can define a quotient map:  $T/\xi_1 : M/\xi_1 \rightarrow M/\xi_2$ .

An *effective action* of a group  $G$  on  $M$  is an action such that there is a set of full measure  $F \subset M$  such that for any  $x \in F$ , there is  $g \in G$  such that  $gx \neq x$ . A smooth effective circle action  $(S_t)_{t \in \mathbb{T}^1}$  on  $M$  can be seen as a 1-periodic smooth flow  $(S_t)_{t \geq 0}$ , we denote  $\mathcal{A}_\alpha = \{B^{-1}S_\alpha B, B \in \text{Diff}^\infty(M, \mu)\}$ . When  $M = [0, 1]^{d-1} \times \mathbb{T}^1$ , we consider the periodic flow  $S_t$  defined by:

$$\begin{aligned} S_t : [0, 1]^{d-1} \times \mathbb{T}^1 &\rightarrow [0, 1]^{d-1} \times \mathbb{T}^1 \\ (x, s) &\mapsto (x, t + s \pmod{1}) \end{aligned}$$

For  $a, b \in \mathbb{T}^1$ , let  $[a, b[$  be the positively oriented circular sector between  $a$  and  $b$ , with  $a$  included and  $b$  excluded.

A sequence  $T_n$  of  $\mu$ -preserving maps *weakly converges* to  $T$  if, for any measurable set  $E$ ,  $\mu(T_n E \Delta E) \rightarrow 0$ , where  $A \Delta B = (A - B) \cup (B - A)$ .

For  $\gamma \in \mathbb{R}$ , we denote:  $|\gamma|_{\pmod{1}} = \min_{k \in \mathbb{Z}} |k + \gamma|$

For  $t \in \mathbb{T}^1$  or  $\mathbb{R}$ , and  $A \subset I \times \mathbb{T}^1$ , we denote

$$t + A = \{(x, t + s \pmod{1}), (x, s) \in A\}$$

Suppose  $M = [0, 1] \times \mathbb{T}^1$  is the closed annulus. Let  $\tilde{M} = [0, 1] \times \mathbb{R}$  be the universal covering of  $M$ . Let  $T$  be a homeomorphism of  $M$  isotopic to the identity and  $\tilde{T}$  its lift to  $\tilde{M}$ . The *rotation set*  $\text{rot}(\tilde{T})$  of  $\tilde{T}$  is defined by:

$$\text{rot}(\tilde{T}) = \bigcap_{k \geq 0} \bigcup_{n \geq k} \left\{ \frac{\tilde{T}^n(\tilde{x}) - \tilde{x}}{n} / \tilde{x} \in \tilde{M} \right\}$$

We let the rotation set of  $T$ ,  $\text{rot}(T)$ , be the equivalent class modulo 1 of  $\text{rot}(\tilde{T})$ . If  $\text{rot}(T)$  is a singleton, and if  $T$  is isotopic to the identity, then  $T$  is a *pseudo-rotation*. Note that, if  $T|_{\partial M} = S_{\alpha|_{\partial M}}$ , then  $T$  is isotopic to the identity. Indeed,

$t \in [0, 1] \mapsto S_{t\alpha}$  is a continuous path between the identity map and  $S_\alpha$ , and by Alexander's trick, any homeomorphism equal to the identity on the boundary is isotopic to the identity. In this paper, all the diffeomorphisms that we construct are equal to a rotation on the boundary and therefore, they are all isotopic to the identity.

## 18.2 Basic steps of the proof

The metric isomorphism of theorem 18.2 is obtained as the limit of isomorphisms of finite algebras: indeed, we use the lemma [AK70, p.18]:

**Lemma 18.4.** *Let  $M_1$  and  $M_2$  be Lebesgue spaces and let  $\xi_n^{(i)}$  ( $i = 1, 2$ ) be monotonic and generating sequences of finite measurable partitions of  $M_i$ . Let  $T_n^{(i)}$  be automorphisms of  $M_i$  such that  $T_n^{(i)} \xi_n^{(i)} = \xi_n^{(i)}$  and  $T_n^{(i)} \rightarrow T^{(i)}$  in the weak topology. Suppose there are metric isomorphisms  $L_n : M_1/\xi_n^{(1)} \rightarrow M_2/\xi_n^{(2)}$  such that*

$$L_n T_n^{(1)} / \xi_n^{(1)} = T_n^{(2)} / \xi_n^{(2)} L_n$$

and

$$L_{n+1} \xi_n^{(1)} = \xi_n^{(2)}$$

then  $(M_1, T_1)$  and  $(M_2, T_2)$  are metrically isomorphic.

Said otherwise, if we have generating sequences of partitions and sequences of automorphisms  $T_n^{(i)}$  weakly converging towards  $T^{(i)}$ , and if, for any integer  $n$ , the following diagram commutes:

$$\begin{array}{ccc} T_n^{(1)} \left( \xi_n^{(1)} \right) & \xrightarrow{L_n} & \xi_n^{(2)} \left( T_n^{(2)} \right) \\ \downarrow & & \downarrow \\ \xi_{n+1}^{(1)} & \xrightarrow{L_{n+1}} & \xi_{n+1}^{(2)} \end{array}$$

then  $(M_1, T_1)$  and  $(M_2, T_2)$  are metrically isomorphic.

The proof of theorem 18.2 is in two steps. In the first step (lemma 18.5), we determine sufficient conditions on a sequence  $(R_{\frac{p_n}{q_n} b_n})_{n \geq 0}$  of periodic rotations of  $\mathbb{T}^1$  such that there exist sequences of finite partitions and automorphisms satisfying the assumptions of lemma 18.4 with  $M_1 = \mathbb{T}^1$ ,  $M_2 = M$ ,  $T_n^{(1)} = R_{\frac{p_n}{q_n} b_n}$ ,  $T_n^{(2)} = T_n$ , where  $T_n$  is also smooth diffeomorphism, and such that the limit  $T$  in the smooth topology of the sequence  $T_n$  is smooth, and  $T \in \bar{\mathcal{A}}_\alpha$  for  $\alpha = \lim p_n/q_n$ .

In the second step (lemma 18.6), we construct sequences of integers satisfying the conditions of the first step, such that  $p_n/q_n \rightarrow \alpha$ ,  $b_n p_n/q_n \rightarrow \beta$ , with  $(\alpha, \beta)$  that can be chosen arbitrarily close to any  $(u, v) \in \mathbb{T}^1 \times \mathbb{T}^1$ , and with  $(\alpha, \beta)$  either rationally dependent or rationally independent.

**Lemma 18.5.** *There exists an explicit sequence of integers  $R_{13}(n) \geq n$ , such that, if there exist increasing sequences of integers  $p_n, q_n, a_n, b_n \in \mathbb{N}^*$ , and a sequence  $s_n \in \mathbb{Z}^*$  such that, for any integer  $n$ ,*

1. (primality)  $a_n b_n - s_n q_n = 1$ .
2. (monotonicity)  $q_n$  divides  $q_{n+1}$  and  $q_n < q_{n+1}$ .
3. (isomorphism)  $q_n$  divides  $a_{n+1} - a_n$ .
4. (convergence of the diffeomorphism, generation)

$$0 < \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \leq \frac{1}{(b_{n+1} q_n)^{R_{13}(n)}}$$

then all these assumptions imply that there are  $\alpha, \beta \in \mathbb{T}^1$  such that

$$\frac{p_n}{q_n} \rightarrow_{\text{mod}1} \alpha, \quad \frac{p_n}{q_n} b_n \rightarrow_{\text{mod}1} \beta$$

and there is a smooth ergodic measure-preserving diffeomorphism  $T$  of  $M$  such that for any  $j \in \mathbb{N}$ ,  $(D^j T)_{|\partial M} = (D^j S_\alpha)_{|\partial M}$  and such that  $(\mathbb{T}^1, R_\beta, \text{Leb})$  is metrically isomorphic to  $(M, T, \mu)$ .

**Lemma 18.6.** For any  $u, v \in \mathbb{T}^1$ , for any  $\epsilon > 0$ , there exist  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  in a  $\epsilon$ -neighborhood of  $(u, v)$ , such that there exist sequences of integers  $p_n, q_n, a_n, b_n \in \mathbb{N}^*$ ,  $s_n \in \mathbb{Z}^*$  satisfying the assumptions of lemma 18.5, such that

$$\frac{p_n}{q_n} \rightarrow_{\text{mod}1} \alpha, \quad \frac{p_n}{q_n} b_n \rightarrow_{\text{mod}1} \beta$$

Moreover,  $\beta$  can be chosen either rationally dependent of  $\alpha$  or rationally independent of  $\alpha$ .

We divide the proof of lemma 18.5 in two main parts. In the first part of the proof, we elaborate sufficient conditions on  $B_n \in \text{Diff}^\infty(M, \mu)$ , where  $M = [0, 1]^{d-1} \times \mathbb{T}^1$ , so that if  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$  weakly converges towards an automorphism  $T$ , then there exists a metric isomorphism between  $(\mathbb{T}^1, R_\beta, \text{Leb})$  and  $(M, T, \mu)$ . To that end, we apply lemma 18.4: we construct a monotonous and generating sequence of partitions  $\xi_n^\infty$  of  $M$  and a sequence of isomorphisms  $\bar{K}_n^\infty : \mathbb{T}^1 / \zeta_n \rightarrow M / \xi_n^\infty$ , where  $\zeta_n = \{[i/q_n, (i+1)/q_n[, i = 0, \dots, q_n - 1\}$ , such that  $\bar{K}_n^\infty R_{\frac{p_n}{q_n}} = T_n \bar{K}_n^\infty$  and  $\bar{K}_{n+1}^\infty|_{\zeta_n} = \bar{K}_n^\infty$ . In the construction of this isomorphism, assumption 3 is important. Moreover, we will see that the elements of  $\xi_n^\infty$  are not the most elementary, because they must be chosen in a way that ensures the monotonicity of the sequence  $\bar{K}_n^\infty$ . This condition of monotonicity induces combinatorial constraints on the elements of the partition  $\xi_n^\infty$ . Though it follows a similar scheme, the construction of the sequence  $\bar{K}_n^\infty$  differs from the construction done in the previous chapter, especially because the assumption 1 is new.

In the second part of the proof, we construct diffeomorphisms  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$  on  $M$  stabilizing  $\xi_n^\infty$ , obtained by successive conjugations from the rotation  $S_{\frac{p_n}{q_n}}$ . The conjugacy  $B_n$  is constructed explicitly. In this second part, we essentially follow the previous chapter (which elaborated on [FSW07]), except for the obtention of the generation of the sequence of partitions  $(\xi_n^\infty)_{n \geq 1}$ , for which we need to slightly modify the construction.

Another change with respect to the previous chapter is in the construction of the limit angles  $\alpha$  and  $\beta$ , i.e. in the proof of lemma 18.6.

## 18.3 Construction of the limit angles $\alpha$ and $\beta$ : proof of lemma 18.6

### 18.3.1 The case $\beta = p\alpha$

We show:

**Lemma 18.7.** For any  $u, v \in \mathbb{T}^1$ , for any  $\epsilon > 0$ , there exist  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  in a  $\epsilon$ -neighborhood of  $(u, v)$ , such that there exist sequences of integers  $p_n, q_n, a_n, b_n \in \mathbb{N}^*$ ,  $s_n \in \mathbb{Z}^*$  satisfying the assumptions of lemma 18.5, such that

$$\frac{p_n}{q_n} \rightarrow_{\text{mod}1} \alpha, \quad \frac{p_n}{q_n} b_n \rightarrow_{\text{mod}1} \beta$$

Moreover,  $\beta$  can be chosen rationally dependent of  $\alpha$ .

*Proof.* Let  $u, v \in \mathbb{T}^1$  and  $\epsilon > 0$ . Let  $p_0, q_0, b_0$  be positive integers such that  $\gcd(b_0, q_0) = 1$ , and such that:

$$\left| \frac{p_0}{q_0} - u \right|_{\text{mod } 1} \leq \frac{\epsilon}{2}, \quad \left| \frac{p_0 b_0}{q_0} - v \right|_{\text{mod } 1} \leq \frac{\epsilon}{2}$$

By the Bezout theorem, there are integers  $a_0, s_0$ , with  $a_0 > 0$ , such that  $a_0 b_0 - s_0 q_0 = 1$ .

Suppose we have defined  $p_k, q_k, a_k, b_k, s_k$ , satisfying the assumptions of lemma 18.5, up to the rank  $k = n$ , and let us define  $p_{n+1}, q_{n+1}, a_{n+1}, b_{n+1}, s_{n+1}$ . (we will have  $s_n = 1$  for  $n \geq 1$ ). Let  $b_{n+1} = b_n$ .

Let  $c_n$  be an integer sufficiently large so that  $c_n \geq (b_{n+1} q_n)^{R_{13}(n)}$  and  $c_n \geq b_{n+1} 2^{n+1} / \epsilon$  ( $b_n = b_0$  is constant here, but this more general definition is used for the case  $(\alpha, \beta)$  rationally independent). Let

$$a_{n+1} = a_n + s_n c_n q_n$$

Therefore, assumption 3 holds. Let also

$$q_{n+1} = q_n s_n (1 + c_n b_n)$$

Therefore, assumption 2 holds. Moreover, we have:

$$a_{n+1} b_{n+1} - q_{n+1} = 1$$

Therefore, assumption 1 holds, with  $s_{n+1} = 1$ . Moreover, let  $p_{n+1} = p_n \frac{q_{n+1}}{q_n} + 1$ . Since  $q_{n+1} \geq (b_{n+1} q_n)^{R_{13}(n)}$ , we have:

$$0 < \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_{n+1}} \leq \frac{1}{(b_{n+1} q_n)^{R_{13}(n)}}$$

Therefore, assumption 4 holds. Moreover,

$$\frac{p_n}{q_n} b_n = \frac{p_0}{q_0} b_0 + \sum_{k=0}^{n-1} \left( \frac{p_{k+1}}{q_{k+1}} b_{k+1} - \frac{p_k}{q_k} b_k \right) = \text{mod } 1 \frac{p_0}{q_0} b_0 + \sum_{k=0}^{n-1} \left( \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right) b_{k+1} = \frac{p_0}{q_0} b_0 + \sum_{k=0}^{n-1} \frac{b_{k+1}}{q_{k+1}}$$

Since  $1/q_{n+1} \leq \epsilon / (2^{n+1} b_{n+1})$ , we get:

$$\left| \frac{p_n}{q_n} b_n - \frac{p_0}{q_0} b_0 \right| \leq \frac{\epsilon}{2}$$

Therefore,

$$\frac{p_n}{q_n} b_n \rightarrow \beta$$

with  $|\beta - u| \leq \epsilon$

Likewise,

$$\frac{p_n}{q_n} \rightarrow \alpha$$

with  $|\alpha - v| \leq \epsilon$ . Moreover, we have  $\beta = b_0\alpha$ .

□

### 18.3.2 The case $(\alpha, \beta)$ rationally independent

We show:

**Lemma 18.8.** *For any  $u, v \in \mathbb{T}^1$ , for any  $\epsilon > 0$ , there exist  $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$  in a  $\epsilon$ -neighborhood of  $(u, v)$ , such that there exist sequences of integers  $p_n, q_n, a_n, b_n \in \mathbb{N}^*$ ,  $s_n \in \mathbb{Z}^*$  satisfying the assumptions of lemma 18.5, such that*

$$\frac{p_n}{q_n} \xrightarrow{\text{mod } 1} \alpha, \quad \frac{p_n}{q_n} b_n \xrightarrow{\text{mod } 1} \beta$$

Moreover,  $\beta$  can be chosen rationally independent of  $\alpha$ .

*Proof.* The beginning of the construction is the same as in the case  $\beta = p\alpha$ , except that we take:

$$b_{n+1} = b_n + q_n$$

$$q_{n+1} = s_n q_n (1 + c_n b_n + c_n q_n + a_n)$$

This ensures that  $b_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ , and that

$$a_{n+1} b_{n+1} - q_{n+1} = 1$$

It only remains to show that the limit angles  $(\alpha, \beta)$  are rationally independent. To that aim, it suffices to show that the translation of vector  $(\alpha, \beta)$  on the torus  $\mathbb{T}^2$  is ergodic. We follow the proof of the ergodicity of the limit translation in the previous chapter, with a slight modification. We recall a theorem by Katok and Stepin [KS67]:

**Theorem 18.9** (Katok-Stepin [KS67]). *Let  $U$  be an automorphism of a Lebesgue space  $(N, \nu)$ , let  $(U_n)_{n \geq 1}$  be a sequence of measure-preserving transformations, and let  $(\chi_n)_{n \geq 1}$  be a sequence of finite partitions of  $N$  with measurable elements. Suppose that:*

- $U_n$  permutes the elements of  $\chi_n$  cyclically.
- $(\chi_n)_{n \geq 1}$  generates.
- $\sum_{c \in \chi_n} \nu(U(c)\Delta U_n(c)) = o(1/|\chi_n|)$  (where  $|\chi_n|$  is the cardinal of  $\chi_n$ ).

then  $U$  is ergodic.

Let  $\gamma^{(n)} = (1, b_n)$ ,  $g_n = \gcd(p_n, q_n)$ . Let  $\Gamma^{(n)} \subset \mathbb{T}^2$  a fundamental domain of the flow  $(T^{t\gamma^{(n)}})_{t \geq 0}$  on  $\mathbb{T}^2$ , where  $T^{t\gamma^{(n)}}$  is the translation of vector  $t\gamma^{(n)}$ . Note that the diameter of  $\Gamma^{(n)}$  is less than  $1/b_n$ . Let

$$\Gamma_{0,n} = \bigcup_{0 \leq t < \frac{g_n}{q_n}} T^{t\gamma^{(n)}} \Gamma^{(n)}$$

We have the lemma:

**Lemma 18.10.** *Let  $\hat{\zeta}_n$  be the partition defined by:*

$$\hat{\zeta}_n = \left\{ \Gamma_{i,n} = T^{i \frac{g_n \gamma^{(n)}}{q_n}} \Gamma_{0,n}, i = 0, \dots, \frac{q_n}{g_n} - 1 \right\}$$

$T^{\frac{p_n}{q_n} \gamma^{(n)}}$  is a cyclic permutation on  $\hat{\zeta}_n$ , and  $\hat{\zeta}_n$  generates.

*Proof.*  $T^{\frac{p_n}{q_n} \gamma^{(n)}}$  is a cyclic permutation on  $\hat{\zeta}_n$  because  $g_n = \gcd(p_n, q_n)$ .

To the vector space  $\mathbb{R}^2$ , we give the norm  $\|(x_1, x_2)\| = \max_{1 \leq i \leq 2} |x_i|$  and we consider its induced norm on  $\mathbb{T}^2$ .

Since

$$p_{n+1} - \frac{q_{n+1}}{q_n} p_n = 1$$

then  $p_{n+1}$  and  $\frac{q_{n+1}}{q_n}$  are relatively prime. Since  $g_{n+1}$  divides  $p_{n+1}$  and  $q_{n+1}$ , then  $g_{n+1}$  divides  $q_n$ . In particular,  $g_{n+1} \leq q_n$  (this is the slight difference with the proof in the previous chapter, in which we do not have:  $\gcd(p_{n+1}, \frac{q_{n+1}}{q_n}) = 1$ . But on the other hand, in the previous chapter, we have:  $\gcd(p_{n+1}, \frac{q_{n+1}}{q_n}) = \gcd(p_{n+1}, q_{n+1})$ . The important point is that in both cases,  $g_n$  is small enough).

Moreover, by assumption 4 of lemma 18.5,

$$\frac{b_{n+1} q_n}{q_{n+1}} \leq \frac{1}{(b_{n+1} q_n)^{R_{13}(n)-1}} \rightarrow_{n \rightarrow +\infty} 0$$

Therefore,

$$\text{diam}(\Gamma_{0,n}) \leq \max \left( \frac{1}{b_n}, \frac{g_n \|\gamma^{(n)}\|}{q_n} \right) \leq \max \left( \frac{1}{b_n}, \frac{q_{n-1} b_n}{q_n} \right) \rightarrow_{n \rightarrow +\infty} 0$$

It shows that  $\hat{\zeta}_n$  generates. □

It remains to estimate  $\sum_{c \in \hat{\zeta}_n} \mu_2 \left( T^\alpha c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right)$ , where  $\mu_2$  is the Lebesgue measure on  $\mathbb{T}^2$ . We have the lemma:

**Lemma 18.11.** *We have:*

$$\sum_{c \in \hat{\zeta}_n} \mu_2 \left( T^{(\alpha, \beta)} c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right) = o(g_n / q_n) = o(1 / |\hat{\zeta}_n|)$$

*Proof.* We have:

$$\sum_{c \in \hat{\zeta}_n} \mu_2 \left( T^{(\alpha, \beta)} c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right) = \sum_{k \geq n} \sum_{c \in \hat{\zeta}_k} \mu_2 \left( T^{\frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)}} c \Delta T^{\frac{p_k}{q_k} \gamma^{(k)}} c \right) = \sum_{k \geq n} \sum_{c \in \hat{\zeta}_k} \mu_2 \left( T^{\frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)} - \frac{p_k}{q_k} \gamma^{(k)}} c \Delta c \right)$$

Let  $\tau_k$  be the  $(h-1)$ -volume of the border of an element of  $\hat{\zeta}_k$ . We have:

$$\mu_2 \left( T^{\frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)} - \frac{p_k}{q_k} \gamma^{(k)}} c \Delta c \right) \leq \tau_k \left\| \frac{p_{k+1}}{q_{k+1}} \gamma^{(k+1)} - \frac{p_k}{q_k} \gamma^{(k)} \right\| = \tau_k \left\| \left( \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right) \gamma^{(k+1)} \right\| = \tau_k \frac{b_{k+1}}{q_{k+1}}$$

Moreover,

$$\tau_k \leq 2 \left( \frac{1}{b_k} + \frac{g_k b_k}{q_k} \right) \leq 2 \left( \frac{1}{b_k} + \frac{q_{k-1} b_k}{q_k} \right) \leq 4$$

Therefore, by applying assumption 4,

$$\sum_{c \in \hat{\zeta}_n} \mu_2 \left( T^{(\alpha, \beta)} c \Delta T^{\frac{p_n}{q_n} \gamma^{(n)}} c \right) = o(g_n/q_n)$$

□

By combining lemmas 18.10 and 18.11, and by applying theorem 18.9, we obtain that the translation of vector  $(\alpha, \beta)$  is ergodic with respect to the Lebesgue measure.

This completes the proof of lemma 18.6. □

### 18.3.3 Why not a construction for all $\alpha$ Liouville?

Let us make one remark. We were not able to show our theorem for any  $\alpha$  Liouville, because conditions 1-3 of lemma 18.5 introduce arithmetical constraints on the denominators of the convergents of  $\alpha$ . These conditions are analogous to those, in the previous chapter, which limit the set of possible translations of the  $h$ -dimensional torus,  $h \geq 2$ , that admit a non-standard smooth realization.

A sufficient condition for  $\alpha$  Liouville to belong to a non-standard couple of angles  $(\alpha, \beta)$  with  $\alpha \neq \pm\beta$ , is the following: if, for the sequence  $p'_n, q'_n$  of convergents of  $\alpha$ , there exist positive integers  $c_n, d_n$ , with  $d_n \leq q_n^{R(n)}$  for a fixed sequence  $R(n)$ , such that we can write:

$$q'_{n+1} = 1 + c_n b_n + d_n a_n + c_n d_n q_n$$

then there is  $\beta \neq \pm\alpha$  such that  $(\alpha, \beta)$  is a non-standard couple of angles. (in this construction, we take  $b_{n+1} = b_n + d_n q_n$ , with  $d_n \in \mathbb{N}$ )



## 18.4 Convergence modulo 1 of $\frac{p_n}{q_n}$ and $\frac{p_n}{q_n}b_n$ towards $\alpha$ and $\beta$ .

The rest of the paper is dedicated to the proof of lemma 18.5. Part of lemma 18.5 is straightforward, namely, the convergence modulo 1 of  $\frac{p_n}{q_n}$  and  $\frac{p_n}{q_n}b_n$  towards  $\alpha$  and  $\beta$  respectively:

*Partial proof of lemma 18.5.* By assumption 2, for  $n \geq 2$ ,  $q_n \geq 2$ . By assumption 3, and since  $R_{13}(n) \geq n$ ,  $p_n/q_n$  is Cauchy, and converges.

To show the convergence modulo 1 of  $\frac{p_n}{q_n}b_n$ , we note that assumptions 1 at ranks  $n$  and  $n+1$ , and assumption 3 at rank  $n$  imply that  $q_n$  divides  $b_{n+1} - b_n$ . Indeed, let us write  $b_{n+1} = b_n + k$ , with  $k$  integer, and let us show that  $q_n$  divides  $k$ . By the assumption 3 at rank  $n$ ,  $a_{n+1} = a_n + c_n q_n$ , with  $c_n$  integer. Therefore,

$$1 + s_{n+1}q_{n+1} = a_{n+1}b_{n+1} = (a_n + c_n q_n)(b_n + k) = a_n b_n + a_n k + q_n(c_n b_n + c_n k)$$

Therefore,

$$q_n \left( s_{n+1} \frac{q_{n+1}}{q_n} - s_n - c_n b_n - c_n k \right) = a_n k$$

Thus,  $q_n$  divides  $a_n k$ . Since  $q_n$  is relatively prime with  $a_n$ , then  $q_n$  divides  $k$ . Therefore,  $b_n/q_n = b_{n+1}/q_n \pmod{1}$ . Therefore,

$$\left| \frac{p_{n+1}b_{n+1}}{q_{n+1}} - \frac{p_n b_n}{q_n} \right| \equiv_{\text{mod } 1} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| |b_{n+1}| \leq \frac{1}{(b_{n+1}q_n)^{R_{13}(n)-1}}$$

Since for  $n \geq 1$ ,  $q_n \geq 2$  and  $R_{13}(n) - 1 \geq n - 1$ , then the sequence  $\left( \frac{p_n b_n}{q_n} \pmod{1} \right)_{n \geq 1}$  is Cauchy, and converges.

□

To show lemma 18.5, it remains to show that there is a smooth ergodic measure-preserving diffeomorphism  $T$  of  $M$  such that  $T \in \bar{\mathcal{A}}_\alpha$  and such that  $(\mathbb{T}^1, R_\beta, \text{Leb})$  is metrically isomorphic to  $(M, T, \mu)$ .

## 19 The metric isomorphism

In this section, our aim is to elaborate sufficient conditions on  $B_n \in \text{Diff}^\infty(M, \mu)$ , where  $M = [0, 1]^{d-1} \times \mathbb{T}^1$ , so that if  $T_n = B_n^{-1} S \frac{p_n}{q_n} B_n$  weakly converges towards an automorphism  $T$ , then there exists a metric isomorphism between  $(\mathbb{T}^1, R_\beta, \text{Leb})$  and  $(M, T, \mu)$ .

To that end, we use lemma 18.4: we construct a monotonous and generating sequence of partitions  $\xi_n^\infty$  of  $M$  and a sequence of isomorphisms  $\bar{K}_n^\infty : \mathbb{T}^1/\zeta_n \rightarrow M/\xi_n^\infty$ , where  $\zeta_n = \{[i/q_n, (i+1)/q_n[, i = 0, \dots, q_n - 1\}$ , such that  $\bar{K}_n^\infty R_{\frac{p_n}{q_n}} = T_n \bar{K}_n^\infty$  and  $\bar{K}_{n+1}^\infty|_{\zeta_n} = \bar{K}_n^\infty$ .

$\zeta_n$  is a partition of  $\mathbb{T}^1$  that is monotonic (because  $q_n$  divides  $q_{n+1}$ ) and that generates (because  $q_n \rightarrow +\infty$ ). Let  $\eta_n = \{I \times [j/q_n, (j+1)/q_n[, j = 0, \dots, q_n - 1\}$ .  $\eta_n$  is a monotonic partition of  $M$ .

The following lemma is straightforward, but important:

**Lemma 19.1.** *Let  $a_n$  and  $q_n$  two relatively prime integers, and let*

$$K_n : \begin{array}{ccc} \zeta_n & \rightarrow & \eta_n \\ \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right] & \mapsto & I \times \left[ \frac{ia_n}{q_n}, \frac{ia_n+1}{q_n} \right] \end{array}$$

$K_n$  is a metric isomorphism such that  $K_n R_{\frac{1}{q_n}} = S_{\frac{a_n}{q_n}} K_n$ . In other words, the following diagram commutes:

$$R_{\frac{1}{q_n}} \curvearrowright \zeta_n \xrightarrow{K_n} \eta_n \curvearrowright S_{\frac{a_n}{q_n}}$$

This lemma is related with two basic observations: the first is that both  $R_{\frac{1}{q_n}}$  and  $S_{\frac{a_n}{q_n}}$  are isomorphic to cyclic permutations of  $\{0, \dots, q_n - 1\}$  (this set is given the counting measure, i.e.  $\mu(A) = \#A$ ); the second observation is that two cyclic permutations of the same order are always conjugated.

The following lemma combines lemma 19.1 with the facts that  $\zeta_n \hookrightarrow \zeta_{n+1}$  and

$$\eta_n \hookrightarrow \eta_{n+1}:$$

**Lemma 19.2.** *Let  $a_n, a_{n+1}, q_n, q_{n+1} \in \mathbb{N}$  such that  $\gcd(a_n, q_n) = \gcd(a_{n+1}, q_{n+1}) = 1$ , such that  $q_n$  divides  $q_{n+1}$  and such that  $q_n$  divides  $a_{n+1} - a_n$ . There exists a partition  $\eta_n^{n+1} \hookrightarrow \eta_{n+1}$  of  $M$  stable by  $S_{\frac{a_n}{q_n}}$ , and there exists a metric isomorphism  $K_n^{n+1} : \zeta_n \rightarrow \eta_n^{n+1}$  such that  $K_n^{n+1} = K_{n+1}|_{\zeta_n}$  and such that  $K_n^{n+1} R_{\frac{1}{q_n}} = S_{\frac{a_n}{q_n}} K_n^{n+1}$ . There exists also a metric isomorphism  $C_n^{n+1} : \eta_n \rightarrow \eta_n^{n+1}$  such that  $C_n^{n+1} S_{\frac{a_n}{q_n}} = S_{\frac{a_n}{q_n}} C_n^{n+1}$  and  $K_n^{n+1} = C_n^{n+1} K_n$ . Said otherwise, we have the following commutative diagram:*

$$\begin{array}{ccc} R_{\frac{1}{q_n}} \curvearrowright \zeta_n & \xrightarrow{K_n} & \eta_n \curvearrowright S_{\frac{a_n}{q_n}} \\ \downarrow Id & & \downarrow C_n^{n+1} \\ R_{\frac{1}{q_n}} \curvearrowright \zeta_n & \xrightarrow{K_n^{n+1}} & \eta_n^{n+1} \curvearrowright S_{\frac{a_n}{q_n}} \\ \downarrow & & \downarrow \\ \zeta_{n+1} & \xrightarrow{K_{n+1}} & \eta_{n+1} \end{array}$$

*Proof.* Since  $\gcd(a_{n+1}, q_{n+1}) = 1$ , then by lemma 19.1,  $K_{n+1}$  is an isomorphism. Moreover, since  $q_n$  divides  $q_{n+1}$ , then  $\zeta_n \hookrightarrow \zeta_{n+1}$ . Therefore, we can define the isomorphism  $K_n^{n+1} = K_{n+1}|_{\zeta_n}$ . Let  $\eta_n^{n+1} = K_n^{n+1}(\zeta_n)$ . We have  $\eta_n^{n+1} \hookrightarrow \eta_{n+1}$ .

It remains to show that  $K_n^{n+1} R_{\frac{1}{q_n}} = S_{\frac{a_n}{q_n}} K_n^{n+1}$  (it automatically implies that  $\eta_n^{n+1}$  is stable by  $S_{\frac{a_n}{q_n}}$ , and that there is  $C_n^{n+1} : \eta_n \rightarrow \eta_n^{n+1}$  such that  $C_n^{n+1} S_{\frac{a_n}{q_n}} = S_{\frac{a_n}{q_n}} C_n^{n+1}$ ). Let  $0 \leq i \leq q_n - 1$ . We have:

$$\begin{aligned}
K_n^{n+1} R_{\frac{1}{q_n}} \left( \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right] \right) &= K_{n+1} R_{\frac{1}{q_n}} \left( \bigcup_{k=0}^{\frac{q_{n+1}}{q_n}-1} \left[ \frac{i}{q_n} + \frac{k}{q_{n+1}}, \frac{i}{q_n} + \frac{k+1}{q_{n+1}} \right] \right) \\
&= K_{n+1} \left( \bigcup_{k=0}^{\frac{q_{n+1}}{q_n}-1} \left[ \frac{i+1}{q_n} + \frac{k}{q_{n+1}}, \frac{i+1}{q_n} + \frac{k+1}{q_{n+1}} \right] \right) = \bigcup_{k=0}^{\frac{q_{n+1}}{q_n}-1} K_{n+1} \left( \left[ \frac{1+i}{q_n} + \frac{k}{q_{n+1}}, \frac{1+i}{q_n} + \frac{k+1}{q_{n+1}} \right] \right) \\
&= I \times \bigcup_{k=0}^{\frac{q_{n+1}}{q_n}-1} \left[ \frac{a_{n+1}}{q_n} + \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}}, \frac{a_{n+1}}{q_n} + \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}} + \frac{1}{q_{n+1}} \right]
\end{aligned}$$

Since  $a_{n+1}/q_n = a_n/q_n \pmod{1}$ , we get:

$$K_n^{n+1} R_{\frac{1}{q_n}} \left( \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right] \right) = I \times \bigcup_{k=0}^{\frac{q_{n+1}}{q_n}-1} \left[ \frac{a_n}{q_n} + \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}}, \frac{a_n}{q_n} + \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}} + \frac{1}{q_{n+1}} \right]$$

Therefore,

$$\begin{aligned}
K_n^{n+1} R_{\frac{1}{q_n}} \left( \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right] \right) &= \bigcup_{k=0}^{\frac{q_{n+1}}{q_n}-1} S_{\frac{a_n}{q_n}} \left( I \times \left[ \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}}, \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}} + \frac{1}{q_{n+1}} \right] \right) \\
&= S_{\frac{a_n}{q_n}} \left( I \times \bigcup_{k=0}^{\frac{q_{n+1}}{q_n}-1} \left[ \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}}, \frac{a_{n+1}i}{q_n} + \frac{a_{n+1}k}{q_{n+1}} + \frac{1}{q_{n+1}} \right] \right) = S_{\frac{a_n}{q_n}} K_n^{n+1} \left( \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right] \right)
\end{aligned}$$

□

Let us denote  $R^{(n)} = K_n^{n+1} \left( \left[ 0, \frac{1}{q_n} \right] \right)$ .

We also denote  $R_{i,n}^{n+1} = S_{\frac{ia_n}{q_n}} R^{(n)}$ ,  $i = 0, \dots, q_n - 1$ .  $R^{(n)}$  is a fundamental domain of  $S_{\frac{a_n}{q_n}}$ . Moreover, we have:

$$\begin{aligned}
C_n^{n+1} : \eta_n &\rightarrow \eta_n^{n+1} \\
\left[ \frac{ia_n}{q_n}, \frac{ia_n+1}{q_n} \right] &\mapsto R_{i,n}^{n+1}, \quad i = 0, \dots, q_{n+1} - 1
\end{aligned}$$

Note also that  $C_n^{n+1} S_{\frac{a_n}{q_n}} = S_{\frac{a_n}{q_n}} C_n^{n+1}$ . Moreover, by assumption 1 of lemma 18.5,  $a_n b_n / q_n = 1/q_n \pmod{1}$ . Therefore, we get:

$$C_n^{n+1} S_{\frac{1}{q_n}} = C_n^{n+1} S_{\frac{a_n b_n}{q_n}} = S_{\frac{a_n b_n}{q_n}} C_n^{n+1} = S_{\frac{1}{q_n}} C_n^{n+1}$$

By iterating lemma 19.2, we get a corollary that is important for the construction of the isomorphism:

**Corollary 19.3.** *For any  $m > n$ , there are partitions  $\eta_n^m \hookrightarrow \eta_{n+1}^m$  of  $M$  such that  $\eta_n^m$  is stable by  $S_{\frac{1}{q_n}}$  and there exists an isomorphism  $K_n^m : \zeta_n \rightarrow \eta_n^m$  such that  $K_n^m R_{\frac{1}{q_n}} = S_{\frac{a_n}{q_n}} K_n^m$  and  $K_n^m = K_{n+1}^m |_{\eta_n^m}$ .*

Said otherwise, the following diagram commutes:

$$\begin{array}{ccc} R_{\frac{1}{q_n}} \curvearrowright \zeta_n^m & \xrightarrow{K_n^m} & \eta_n^m \curvearrowright S_{\frac{a_n}{q_n}} \\ \downarrow & & \downarrow \\ \zeta_{n+1}^m & \xrightarrow{K_{n+1}^m} & \eta_{n+1}^m \end{array}$$

*Proof.* The proof is similar to the one found in the previous chapter.  $\square$

For any  $n$  fixed, the sequence of partitions  $(\eta_n^m)_{m \geq n}$  must converge when  $m \rightarrow +\infty$ , in order to obtain a full sequence of monotonic partitions. Moreover, the possible limit sequence (i.e. a possible  $\eta_n^\infty$ ) must generate. Indeed, these assumptions are required to apply lemma 18.4. However, we can check that none of these assumptions are satisfied, in general. Therefore, to obtain these assumptions, we pull back the partition  $\eta_n^m$  by a suitable smooth measure-preserving diffeomorphism  $B_m$ . The following lemma, already proved in the previous chapter, gives the conditions that  $B_m$  must satisfy:

**Lemma 19.4.** *Let  $B_m \in \text{Diff}^\infty(M, \mu)$ . Let  $A_{m+1} = B_{m+1} B_m^{-1}$ .*

1. *If  $A_{m+1} S_{\frac{1}{q_m}} = S_{\frac{1}{q_m}} A_{m+1}$  and if*

$$\sum_{m \geq 0} q_m \mu \left( (I \times [0, q_m[) \Delta A_{m+1}^{-1} R^{(m)} \right) < +\infty$$

*then for any fixed  $n$ , when  $m \rightarrow +\infty$ , the sequence of partitions  $\xi_n^m = B_m^{-1} \eta_n^m$  converges. We denote  $\xi_n^\infty$  the limit. The sequence  $\xi_n^\infty$  is monotonous and  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$  stabilizes each  $\xi_n^\infty$ .*

2. *If, moreover, the sequence  $\xi_n = B_n^{-1} \eta_n$  generates, then so does  $\xi_n^\infty$ .*

$C_m^{m+1}$  is not continuous in general, and  $A_{m+1}$  is its differentiable approximation. Lemma 19.4 is the reason why we need for  $M$  a manifold of dimension  $d \geq 2$ . Indeed, if we took  $M = \mathbb{T}^1$ , we could not find a diffeomorphism  $B_m$  satisfying the assumptions of this lemma, except for  $a_n = 1$  or  $a_n = q_n - 1$ . The choice  $a_n = 1$  gives that the rotation  $R_\alpha$  on the circle is isomorphic to itself. The choice  $a_n = q_n - 1$  gives that  $R_\alpha$  is isomorphic to  $R_{-\alpha}$ . The existence of these two isomorphisms are consistent with the fact, mentioned in the introduction, that  $R_\alpha$  and  $R_\beta$  are isomorphic, with  $\alpha$  irrational, if and only if  $\alpha = \pm\beta$ .

By adding to lemma 19.4 the convergence of the sequence  $T_n$ , we obtain the required isomorphism:

**Corollary 19.5.** *If both conditions 1. and 2. of lemma 19.4 hold, and if  $T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$  weakly converges towards an automorphism  $T$ , then  $(\mathbb{T}^1, R_\beta, \text{Leb})$  and  $(M, T, \mu)$  are metrically isomorphic.*

*Proof.* By corollary 19.3,  $K_n^m R_{\frac{1}{q_n}} = S_{\frac{a_n}{q_n}} K_n^m$ . By iteration,

$$K_n^m R_{\frac{b_n p_n}{q_n}} = S_{\frac{a_n b_n p_n}{q_n}} K_n^m$$

Since  $a_n b_n / q_n = 1 / q_n \pmod{1}$ , then

$$K_n^m R_{\frac{b_n p_n}{q_n}} = S_{\frac{p_n}{q_n}} K_n^m$$

Therefore, the following diagram commutes:

$$\begin{array}{ccccc}
 & & S_{\frac{p_n}{q_n}} & & \\
 & & \curvearrowright & & \\
 R_{\frac{b_n p_n}{q_n}} \curvearrowright & \zeta_n & \xrightarrow{K_n^m} & \eta_n^m & \xrightarrow{B_m^{-1}} & \xi_n^m \curvearrowright T_n \\
 \downarrow & \downarrow & & \downarrow & & \downarrow \\
 \zeta_{n+1} & \xrightarrow{K_{n+1}^m} & \eta_{n+1}^m & \xrightarrow{B_m^{-1}} & \xi_{n+1}^m
 \end{array}$$

Let  $\bar{K}_n^\infty : \zeta_n \rightarrow \xi_n^\infty$  be defined by  $\bar{K}_n^\infty = P_n^\infty B_n^{-1} K_n$ , where  $P_n^\infty : \xi_n \rightarrow \xi_n^\infty$  is the limit isomorphism of the sequence  $P_n^m = B_m^{-1} C_n^m B_n$ , defined in the same way as in the previous chapter.

As in the proof of corollary 3.4 in the previous chapter, we can show that  $\bar{K}_n^\infty R_{\frac{b_n p_n}{q_n}} = T_n \bar{K}_n^\infty$  and that  $\bar{K}_{n+1}^\infty|_{\zeta_n} = \bar{K}_n^\infty$ . This allows to apply lemma 18.4, which gives the required metric isomorphism. □

Let us make one remark. We consider the isomorphism between  $R_{\frac{b_n p_n}{q_n}}$  and

$T_n = B_n^{-1} S_{\frac{p_n}{q_n}} B_n$ , instead of the isomorphism between  $R_{\frac{p_n}{q_n}}$  and  $\check{T}_n = B_n^{-1} S_{\frac{a_n p_n}{q_n}} B_n$  (which seems to be a more "natural" choice), because in the latter case, we are not able to show the convergence of  $\check{T}_n$  towards a smooth diffeomorphism  $\check{T}$ . Indeed, we have:

$$d_k(\check{T}_{n+1}, \check{T}_n) \leq \|B_{n+1}\|_k \left| \frac{p_{n+1} a_{n+1}}{q_{n+1}} - \frac{p_n a_n}{q_n} \right|_{\text{mod } 1} = \|B_{n+1}\|_k a_{n+1} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \quad (102)$$

In the next section, we show that  $\|B_{n+1}\|_k \leq (b_{n+1} q_n)^{R_{13}(n)}$  for some fixed sequence  $R_{13}(n)$  (and we are not able to improve this estimate). Estimate (102) becomes:

$$d_k(\check{T}_{n+1}, \check{T}_n) \leq a_{n+1} (b_{n+1} q_n)^{R_{13}(n)} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|$$

Moreover, by assumption 1 of lemma 18.5,  $a_{n+1} b_{n+1} \geq q_{n+1}$ . Since  $\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \geq \frac{1}{q_{n+1}}$ , estimate (102) does not allow to show that  $\check{T}_n$  is Cauchy. On the other hand, applying this reasoning to show the convergence of  $T_n$  will be successful.

In order to construct the diffeomorphism  $A_{n+1}$  with suitable estimates of its norm, we need to control the width of the connected components of  $R^{(n)}$ . A priori,  $R^{(n)}$  consists of  $q_{n+1}/q_n$  "slices" of width  $1/q_{n+1}$ . However, this fact does not ensure the convergence of  $T_n$ , because it only implies that  $\|B_{n+1}\|_j \leq (q_{n+1})^{R(n)}$  for some fixed sequence of integers  $R(n)$ . In order to apply the reasoning above successfully, we need a better estimate. The following lemma shows that "slices" of  $R^{(n)}$  of width  $1/q_{n+1}$  stack on each other, which gives  $b_{n+1}$  connected components to  $R^{(n)}$ , each having a width of order  $1/(q_n b_{n+1})$ . This will allow an estimate of the form  $\|B_{n+1}\|_j \leq (q_n b_{n+1})^{R(n)}$ , which will ensure the convergence of  $T_n$ , by taking  $b_{n+1}$  small.

**Lemma 19.6.** *Let*

$$m_n = \frac{q_{n+1}}{q_n} - 1 - b_{n+1} \left\lfloor \frac{\frac{q_{n+1}}{q_n} - 1}{b_{n+1}} \right\rfloor$$

and for  $0 \leq l \leq b_{n+1} - 1$ , let

$$k_n(l) = \left\lfloor la_{n+1} \frac{q_n}{q_{n+1}} \right\rfloor$$

$$r_n(l) = la_{n+1} - \frac{q_{n+1}}{q_n} k_n(l)$$

We have:

$$R^{(n)} = \bigcup_{l=0}^{b_{n+1}-1} R^{(n),l}$$

with, if  $0 \leq l \leq m_n$ :

$$R^{(n),l} = I \times \left( \frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[ 0, \frac{\left\lfloor \frac{\frac{q_{n+1}}{q_n} - 1}{b_{n+1}} \right\rfloor + 1}{q_{n+1}} \right] \right)$$

and if  $m_n + 1 \leq l \leq b_{n+1} - 1$ :

$$R^{(n),l} = I \times \left( \frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[ 0, \frac{\left\lfloor \frac{\frac{q_{n+1}}{q_n} - 1}{b_{n+1}} \right\rfloor}{q_{n+1}} \right] \right)$$

*Proof.* We have:

$$R^{(n)} = I \times \bigcup_{i=0}^{q_{n+1}/q_n - 1} \left[ \frac{a_{n+1}i}{q_{n+1}}, \frac{a_{n+1}i}{q_{n+1}} + \frac{1}{q_{n+1}} \right]$$

For  $i = 0, \dots, q_{n+1}/q_n - 1$ , we make the Euclidean division of  $i$  by  $b_{n+1}$ . We get:

$$i = k_i b_{n+1} + r_i$$

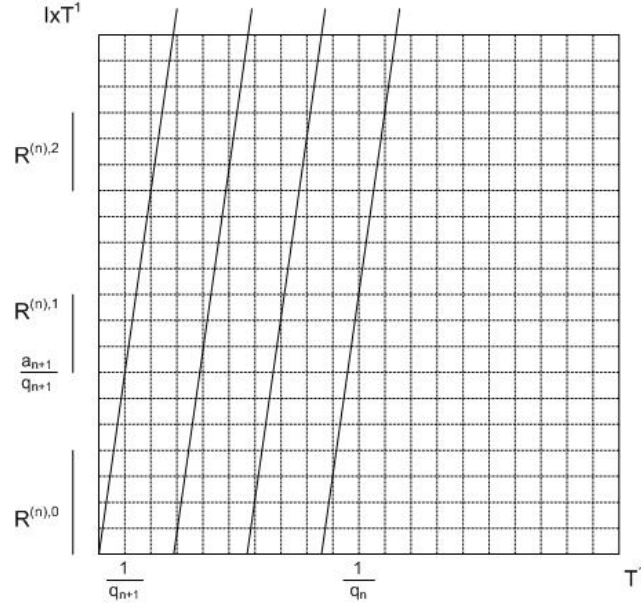


Figure 15: The set  $R^{(n)} = K_n^{n+1}(I \times [0, 1/q_n])$  for  $q_n = 2$ ,  $q_{n+1} = 20$ ,  $a_{n+1} = 7$ ,  $b_{n+1} = 3$ .  $R^{(n)} = R^{(n),0} \cup R^{(n),1} \cup R^{(n),2}$  has  $b_{n+1} = 3$  connected components. The oblique lines represent the graph of the map  $x \mapsto a_{n+1}x$  from  $\mathbb{T}^1$  to itself. In this illustration,  $k_n(0) = k_n(1) = 0$ ,  $k_n(2) = 1$ ,  $r_n(0) = 0$ ,  $r_n(1) = 7$ ,  $r_n(2) = 4$ .

with  $0 \leq r_i \leq b_{n+1} - 1$  and  $0 \leq k_i \leq \left\lfloor \frac{q_{n+1} - 1}{b_{n+1}} \right\rfloor$ . Since  $a_{n+1}b_{n+1}/q_{n+1} = 1/q_{n+1} \pmod{1}$ , we get:

$$R^{(n)} = I \times \bigcup_{i=0}^{q_{n+1}/q_n - 1} \left[ \frac{a_{n+1}r_i + k_i}{q_{n+1}}, \frac{a_{n+1}r_i + k_i}{q_{n+1}} + \frac{1}{q_{n+1}} \right]$$

Moreover, we have:

$$\begin{aligned} \{0, \dots, q_{n+1}/q_n - 1\} &= \{0, \dots, b_{n+1} - 1\} \bigcup (b_{n+1} + \{0, \dots, b_{n+1} - 1\}) \bigcup \dots \\ \dots \bigcup \left( b_{n+1} \left( \left\lfloor \frac{q_{n+1} - 1}{q_n} \right\rfloor / b_{n+1} \right) - 1 \right) &+ \{0, \dots, b_{n+1} - 1\} \bigcup \left( b_{n+1} \left( \left\lfloor \frac{q_{n+1} - 1}{q_n} \right\rfloor / b_{n+1} \right) + \{0, \dots, m_n\} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} R^{(n)} &= I \times \bigcup_{k_i=0}^{\lfloor (q_{n+1}-1)/b_{n+1} \rfloor - 1} \bigcup_{r_i=0}^{b_{n+1}-1} \left[ \frac{a_{n+1}r_i + k_i}{q_{n+1}}, \frac{a_{n+1}r_i + k_i}{q_{n+1}} + \frac{1}{q_{n+1}} \right] \bigcup \\ &\bigcup_{r_i=0}^{m_n} \left[ \frac{a_{n+1}r_i + \lfloor (q_{n+1} - 1)/b_{n+1} \rfloor}{q_{n+1}}, \frac{a_{n+1}r_i + \lfloor (q_{n+1} - 1)/b_{n+1} \rfloor}{q_{n+1}} + \frac{1}{q_{n+1}} \right] \\ R^{(n)} &= I \times \bigcup_{l=0}^{m_n} \left( \frac{a_{n+1}l}{q_{n+1}} + \bigcup_{k_i=0}^{\lfloor (q_{n+1}-1)/b_{n+1} \rfloor} \left[ \frac{k_i}{q_{n+1}}, \frac{k_i}{q_{n+1}} + \frac{1}{q_{n+1}} \right] \right) \bigcup \end{aligned}$$

$$\bigcup_{l=m_{n+1}}^{b_{n+1}-1} \left( \frac{a_{n+1}l}{q_{n+1}} + \bigcup_{k_i=0}^{\lfloor \frac{(q_{n+1}-1)/b_{n+1}}{q_n} \rfloor - 1} \left[ \frac{k_i}{q_{n+1}}, \frac{k_i}{q_{n+1}} + \frac{1}{q_{n+1}} \right] \right)$$

$$R^{(n)} = I \times \bigcup_{l=0}^{m_n} \left( \frac{a_{n+1}l}{q_{n+1}} + \left[ 0, \frac{\lfloor \frac{(q_{n+1}-1)/b_{n+1}}{q_n} \rfloor + 1}{q_{n+1}} \right] \right) \bigcup_{l=m_{n+1}}^{b_{n+1}-1} \left( \frac{a_{n+1}l}{q_{n+1}} + \left[ 0, \frac{\lfloor \frac{(q_{n+1}-1)/b_{n+1}}{q_n} \rfloor}{q_{n+1}} \right] \right)$$

Finally, the Euclidean division of  $la_{n+1}$  by  $q_{n+1}/q_n$  gives:

$$la_{n+1} = k_n(l)q_{n+1}/q_n + r_n(l)$$

with

$$k_n(l) = \left\lfloor la_{n+1} \frac{q_n}{q_{n+1}} \right\rfloor$$

$$r_n(l) = la_{n+1} - \frac{q_{n+1}}{q_n} \left\lfloor la_{n+1} \frac{q_n}{q_{n+1}} \right\rfloor$$

We get:

$$R^{(n)} = I \times \bigcup_{l=0}^{m_n} \frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[ 0, \frac{\lfloor \frac{q_{n+1}-1}{b_{n+1}} \rfloor + 1}{q_{n+1}} \right] \bigcup_{l=m_{n+1}}^{b_{n+1}-1} \frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[ 0, \frac{\lfloor \frac{q_{n+1}-1}{b_{n+1}} \rfloor}{q_{n+1}} \right]$$

□

The next section is dedicated to the construction of the sequence of diffeomorphisms  $B_n$  satisfying the conditions of lemma 19.4.

## 20 The sequence of conjugacies

In this section, we construct a sequence of diffeomorphisms  $B_n$  on  $M$  satisfying the conditions of lemma 19.4, such that  $\|B_n\|_n \leq (q_{n-1}b_n)^{R_1(n-1)}$  for some  $R_1(n)$ , and such that  $B_n = Id$  on a neighborhood of  $\partial M$ , in order to ensure that  $(D^j T)_{\partial M} = (D^j S_\alpha)_{\partial M}$  for any  $j \in \mathbb{N}$ .

**Proposition 20.1.** *There exists a sequence of diffeomorphisms  $B_n \in \text{Diff}^\infty(M, \mu)$  such that  $B_n$  and  $A_{n+1} = B_{n+1}B_n^{-1}$  satisfy the following conditions:*

1. (convergence of the partition  $\xi_n^m$  to  $\xi_n^\infty$ )

$$\sum_{m \geq 0} q_m \mu \left( \Delta_{0, q_m} \Delta_{m+1}^{-1} R^{(m)} \right) < +\infty$$



2. (generation) There is a set  $E_{n+1} \subset M$  such that  $\sum_{n \geq 0} \mu(E_{n+1}^c) < +\infty$  and such that

$$\text{diam}\left(A_{n+1}^{-1}\left(\Delta_{0,q_{n+1}} \cap E_{n+1}\right)\right) \leq \frac{1}{2^n \|B_n\|_1}$$

3. (equivariance)

$$A_{n+1} S_{\frac{1}{q_n}} = S_{\frac{1}{q_n}} A_{n+1}$$

4. (polynomial estimation) There is a fixed sequence  $R_1(n) \in \mathbb{N}$  such that

$$\|A_{n+1}\|_{n+1} \leq (q_n b_{n+1})^{R_1(n)}$$

5. (identity on a neighborhood of the boundary)  $B_n = \text{Id}$  on a neighborhood of  $\partial M$ .

*Remark 20.2.* Specification 2 above implies that  $\xi_n$  generates (and so  $\xi_n^\infty$ , by lemma 19.4), see the previous chapter and [FSW07].

We construct  $B_n$  recursively. We suppose that  $B_n$  exists and satisfies these specifications, and we construct  $A_{n+1}$ .

The diffeomorphism  $A_{n+1}$  is constructed in three steps, each step giving a smooth, measure-preserving, equivariant and polynomially controlled map. In the first step, lemma 20.3, we construct a smooth map  $A_{n+1}^1$  that "quasi-cuts"  $I \times [0, 1/q_n[$  into  $b_{n+1}$  vertical slices, and then rotates each slice  $\Gamma_l$  by an angle  $k_n(l)/q_n$  along the periodic flow  $S_t$ .

In the second step, we construct a second map  $A_{n+1}^2$  that "quasi-sends" each vertical slice  $A_{n+1}^1(\Gamma_l)$  into a suitable connected component of  $R^{(n)}$  (see lemma 19.6 for the decomposition of  $R^{(n)}$  into connected components). These two steps ensure that  $A_{n+1}$  "quasi-sends"  $I \times [0, 1/q_n[$  to  $R^{(n)}$ . It ensures that  $\xi_n$  converges.

In the third step, we obtain the generation of  $\xi_n$ . We use  $A_{n+1}^3$  to quasi-rotate slices inside each connected component of  $R^{(n)}$ . These slices are chosen sufficiently thin to ensure that the diameter of  $A_{n+1}^{-1}(I \times [l/q_{n+1}, (l+1)/q_{n+1}[$ ) is small, but these slices are not too thin to ensure that  $\|A_{n+1}\|_{n+1} \leq (b_{n+1} q_n)^{R_1(n)}$ , which enables the convergence of  $T_n$ . This last step completes the construction.

Let  $l_0, \dots, l_{b_{n+1}-1}$  integers such that  $0 = r_n(l_0) < \dots < r_n(l_{b_{n+1}-1}) \leq q_{n+1}/q_n - 1$ . Let  $l_{b_{n+1}} = b_{n+1}$  and  $r_n(l_{b_{n+1}}) = q_{n+1}/q_n$ .

## 20.1 Construction in dimension 2

We suppose  $M = [0, 1] \times \mathbb{T}^1$ . The first step is based on the following lemma, which analogous is found in the previous chapter:

**Lemma 20.3.** Let  $\frac{1}{b_{n+1}} > \epsilon_1 > 0$ , and  $\Gamma_i = \left[r_n(l_i) \frac{q_n}{q_{n+1}}, r_n(l_{i+1}) \frac{q_n}{q_{n+1}} - \epsilon_1\right] \times \left[0, \frac{1}{q_n}\right]$  for  $0 \leq i \leq b_{n+1} - 1$ . There is a smooth measure-preserving diffeomorphism  $A_{n+1}^1$  of  $[0, 1] \times \mathbb{T}^1$  such that:

1.

$$A_{n+1}^1 S_{\frac{1}{q_n}} = S_{\frac{1}{q_n}} A_{n+1}^1$$

2.

$$A_{n+1}^1(\Gamma_i) = S_{\frac{k_n(l_i)}{q_n}} \Gamma_i$$

3.

$$\|A_{n+1}^1\|_j \leq \frac{1}{\epsilon_1^j} \|\phi\|_j$$

where  $\phi$  is a fixed smooth diffeomorphism, independent of  $n$  and  $\epsilon_1$ .

We take  $\epsilon_1 = \frac{1}{2^n b_{n+1}}$  and we let:

$$E_{n+1}^1 = \bigcup_{i=0}^{b_{n+1}-1} \left[ r(l_i) \frac{q_n}{q_{n+1}}, r(l_{i+1}) \frac{q_n}{q_{n+1}} - \epsilon_1 \right] \times \mathbb{T}^1$$

We have:

$$\mu\left((E_{n+1}^1)^c\right) = b_{n+1} \epsilon_1 = \frac{1}{2^n}$$

In the second step, we shrink  $A_{n+1}^1(\Gamma_i)$  horizontally by a factor  $q_n$ , we expand it vertically by the same factor, and we rotate it by a  $\pi/2$  angle (except in a neighborhood of the border of  $I \times [0, 1/q_n[$ ). Thus,  $A_{n+1}^1(\Gamma_i)$  is quasi-sent to a connected component of  $R^{(n)}$ .

We have the lemma (see the previous chapter):

**Lemma 20.4.** For  $0 \leq i \leq b_{n+1} - 1$ , let  $\Gamma'_i = \left[ r_n(l_i) \frac{q_n}{q_{n+1}}, r_n(l_{i+1}) \frac{q_n}{q_{n+1}} \right] \times [0, \frac{1}{q_n}]$ . There exists a smooth measure-preserving diffeomorphism  $A_{n+1}^2$  of  $[0, 1] \times \mathbb{T}^1$ , equivariant by  $S_{\frac{1}{q_n}}$  and a measurable set  $E_{n+1}^2$  that is globally invariant by  $S_{\frac{1}{q_n}}$  and  $A_{n+1}^2$  such that:

$$A_{n+1}^2\left(\Gamma'_i \cap E_{n+1}^2\right) = I \times \left[ \frac{r_n(l_i)}{q_{n+1}}, \frac{r_n(l_{i+1})}{q_{n+1}} \right] \cap E_{n+1}^2$$

Moreover, there is an explicit function  $R_2(j)$ , depending only on  $j$ , such that:

$$\|A_{n+1}^2\|_j \leq (q_n)^{R_2(j)}$$

and such that if  $\Gamma \subset M$  with  $\text{diam}(\Gamma) \leq x$ , then:

$$\text{diam}\left((A_{n+1}^2)^{-1}\left(\Gamma \cap E_{n+1}^2\right)\right) \leq q_n x$$

Combined with lemma 20.3, lemma 20.4 gives the following corollary, which implies the convergence of the partition  $\xi_n$  to  $\xi_n^\infty$  (see the previous chapter):

**Corollary 20.5.** We have the estimation:

$$\mu\left(A_{n+1}^2 A_{n+1}^1\left(I \times [0, 1/q_n[ \right) \Delta R^{(n)}\right) \leq \frac{8}{2^n q_n}$$

In the third step, it remains to obtain the generation of the sequence of partitions  $\xi_n^\infty$ , without affecting the properties obtained in the first two steps. In particular, we cannot exactly proceed as in the previous chapter, because  $b_{n+1}$  can be a bounded sequence (while in the previous chapter, the sequence  $k_n$ , analogous to  $b_{n+1}$ , is larger than  $q_n$ ). We need to refine the approach of the previous chapter. This third step is based on the following lemma and its corollary:

**Lemma 20.6.** *For any integer  $w \geq 1$ , there is a smooth, measure-preserving, and  $S_{\frac{1}{q_n}}$ -equivariant diffeomorphism  $A_{n+1}^3$ , and an explicit sequence of integers  $R_2(n)$ , such that:*

$$\|A_{n+1}^3\|_{n+1} \leq \left(\frac{q_{n+1}}{w}\right)^{R_2(n)}$$

and there exists a  $S_{\frac{1}{q_n}}$ -invariant and  $A_{n+1}^3$ -invariant set  $E_{n+1}^3$  such that

$\mu(E_{n+1}^{3c}) \leq 4/2^n$ , and such that for any  $i = 0, \dots, q_{n+1} - 1$ , we have:

$$\text{diam}\left((A_{n+1}^3)^{-1}([0, 1] \times [i/q_{n+1}, (i+1)/q_{n+1}[) \cap E_{n+1}^3)\right) \leq \max\left(\frac{1}{w}, \frac{2w}{q_{n+1}}\right)$$

We obtain the corollary:

**Corollary 20.7.** *There exists an explicit sequence of integers  $R_3(n)$  depending only on  $n$ , there is a smooth, measure-preserving, and  $S_{\frac{1}{q_n}}$ -equivariant diffeomorphism  $A_{n+1}$ , such that:*

$$\|A_{n+1}\|_{n+1} \leq (b_{n+1}q_n)^{R_3(n)}$$

and there exists a  $S_{\frac{1}{q_n}}$ -invariant and  $A_{n+1}$ -invariant set  $E_{n+1}$  such that  $\mu(E_{n+1}^c) \leq 4/2^n$ , and such that for any  $i = 0, \dots, q_{n+1} - 1$ , we have:

$$\text{diam}\left((A_{n+1})^{-1}([0, 1] \times [i/q_{n+1}, (i+1)/q_{n+1}[) \cap E_{n+1})\right) \leq \frac{1}{2^n \|B_n\|_1}$$

*Proof of lemma 20.6.* We define  $A_{n+1}^3$  on  $R^{(n)}$ , and since  $R^{(n)}$  is a fundamental domain of  $S_{\frac{1}{q_n}}$ , we can extend it to all  $M$  by  $S_{\frac{1}{q_n}}$ -periodicity. To that aim, we define  $A_{n+1}^3$  on each connected component of  $R^{(n)}$  (see figure 16).

Let  $f_n(l) = \left\lfloor \frac{q_{n+1}-1}{b_{n+1}} \right\rfloor + 1$  if  $0 \leq l \leq m_n$  and  $f_n(l) = \left\lfloor \frac{q_{n+1}-1}{b_{n+1}} \right\rfloor$  if  $m_n + 1 \leq l \leq b_{n+1} - 1$  ( $f_n(l)/q_{n+1}$  is the width of a connected component of  $R^{(n)}$ , see lemma 19.6). We perform the Euclidean division of  $f_n(l)$  by  $w$ :

$$f_n(l) = h_n(l)w + t_n(l)$$

with  $0 \leq t_n(l) \leq w - 1$ .

We also need to recall the definition of a "quasi-rotation" by  $\pi/2$  [FSW07]:

**Proposition 20.8.** *For any  $n \geq 1$ , there is a smooth measure-preserving map*

$\phi_n : [0, 1]^2 \rightarrow [0, 1]^2$  (called "quasi-rotation") such that  $\phi_n = R_{\pi/2}$  on  $[\frac{1}{2^n}, 1 - \frac{1}{2^n}]^2$  and  $\phi_n = \text{Id}$  on a neighborhood of the boundary of  $[0, 1]^2$ .

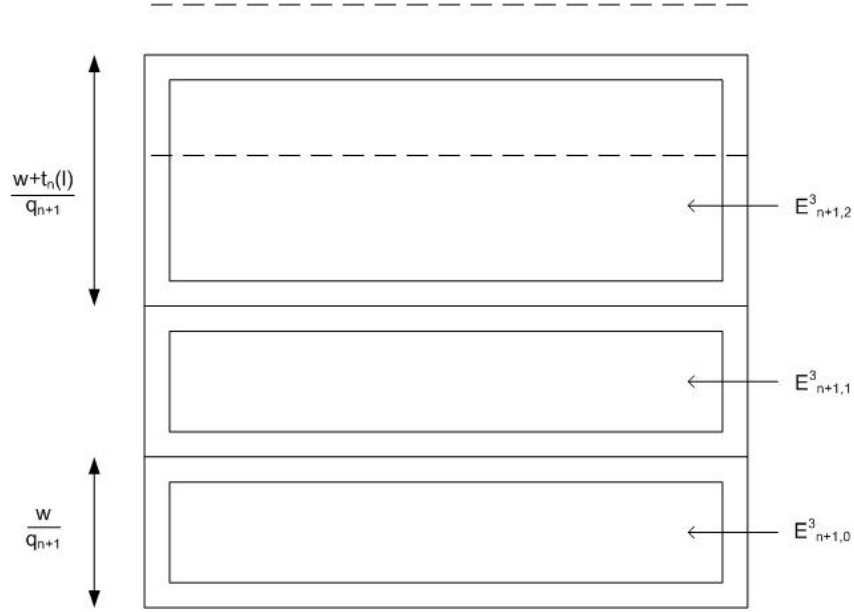


Figure 16: Illustration of the third step (generation): a connected component of  $R^{(n)}$  with  $h_n(l) = 3$ .

Let  $p \geq 2$  a real number and

$$C_p : [0, 1] \times \left[0, \frac{1}{p}\right] \rightarrow [0, 1] \times [0, 1]$$

$$(x, y) \mapsto (x, py)$$

Let  $\phi_{n,p} = C_p^{-1} \phi_n C_p$ . The map  $\phi_{n,p}$  is smooth and measure-preserving. By the Faa-di-Bruno formula, there exists a fixed function  $R_4(j)$  such that

$$\|\phi_{n,p}\|_j \leq p^{R_4(j)} \|\phi_n\|_j$$

Since  $\phi_n$  is fixed, by choosing a larger  $R_4(n)$ , we have:

$$\|\phi_{n,p}\|_{n+1} \leq p^{R_4(n)}$$

For  $0 \leq l \leq b_{n+1}$ , on  $[0, 1] \times \left(\frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[0, \frac{w}{q_{n+1}}\right]\right)$ , we let  $A_{n+1}^3 = \phi_{n, q_{n+1}/w}$  and

$$E_{n+1,0}^3 = \left[ \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}} \right] \times \left( \frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[ \frac{w}{2^{n+1}q_{n+1}}, \frac{w}{q_{n+1}} - \frac{w}{2^{n+1}q_{n+1}} \right] \right)$$

We extend  $A_{n+1}^3$  to  $[0, 1] \times \left(\frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[0, \frac{(h_n(l)-1)w}{q_{n+1}}\right]\right)$  by  $S_{\frac{w}{q_{n+1}}}$ -equivariance.

Likewise, for  $x = 1, \dots, h_n(l) - 2$ , we define:  $E_{n+1,x}^3 = \frac{xw}{q_{n+1}} + E_{n+1,0}^3$ .

On  $[0, 1] \times \left(\frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \left[\frac{(h_n(l)-1)w}{q_{n+1}}, \frac{h_n(l)w+t_n(l)}{q_{n+1}}\right]\right)$ , we let  $A_{n+1}^3 = \phi_{n, \frac{q_{n+1}}{w+t_n(l)}}$ . This completes the construction of  $A_{n+1}^3$  on  $R^{(n)}$ . By  $S_{\frac{1}{q_n}}$ -equivariance, we get the definition of  $A_{n+1}^3$  on the whole manifold  $M$ . Moreover, since  $0 \leq t_n(l) \leq w - 1$ , there exists  $R_2(n)$  such that:

$$\|A_{n+1}^3\|_{n+1} \leq \max\left(\|\phi_{n, \frac{q_{n+1}}{w}}\|_{n+1}, \max_{0 \leq l \leq b_{n+1}-1} \|\phi_{n, \frac{q_{n+1}}{w+t_n(l)}}\|_{n+1}\right) \leq \left(\frac{q_{n+1}}{w}\right)^{R_2(n)}$$

Let:

$$E_{n+1, h_n(l)-1}^3 = \left[ \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}} \right] \times \left( \frac{k_n(l)}{q_n} + \frac{r_n(l)}{q_{n+1}} + \frac{(h_n(l)-1)w}{q_{n+1}} + \left[ \frac{w+t_n(l)}{q_{n+1}2^{n+1}}, \frac{w+t_n(l)}{q_{n+1}} - \frac{w+t_n(l)}{q_{n+1}2^{n+1}} \right] \right)$$

We let  $E_{n+1}^{3,0} = \bigcup_{x=0}^{h_n(l)-1} E_{n+1,x}^3$ , and  $E_{n+1}^3 = \bigcup_{y=0}^{q_n-1} \frac{y}{q_n} + E_{n+1}^{3,0}$ . This completes the construction of  $E_{n+1}^3$ . Moreover, we have:  $\mu(E_{n+1}^{3c}) \leq 4/2^n$ .

Moreover, since  $t_n(l) \leq w$ , then for any  $i = 0, \dots, q_{n+1} - 1$ , we have:

$$\text{diam}\left((A_{n+1}^3)^{-1}([i/q_{n+1}, (i+1)/q_{n+1}] \times [0, 1]) \cap E_{n+1}^3\right) \leq \max\left(\frac{1}{w}, \max_{0 \leq l \leq b_{n+1}-1} \frac{w+t_n(l)}{q_{n+1}}\right) \leq \max\left(\frac{1}{w}, \frac{2w}{q_{n+1}}\right)$$

□

*Proof of corollary 20.7.* By the recurrence assumption on  $B_n$ , there exists  $R_5(n)$  such that  $\|B_n\|_1 \leq (b_n q_{n-1})^{R_5(n-1)}$ . Let

$$w = \left\lfloor \frac{q_{n+1}}{2^{n+1} q_n^2 (b_n q_{n-1})^{R_5(n-1)}} \right\rfloor$$

This choice of  $w$  determines  $A_{n+1}^3$  in lemma 20.6. Let  $A_{n+1} = A_{n+1}^3 A_{n+1}^2 A_{n+1}^1$  and  $E_{n+1} = E_{n+1}^3 \cap A_{n+1}^3 (E_{n+1}^2) \cap A_{n+1}^3 A_{n+1}^2 (E_{n+1}^1)$ . By lemma 20.6, we have:

$$\|A_{n+1}^3\|_{n+1} \leq (q_{n+1})^{R_2(n)} \left( \frac{2^{n+2} q_n^2 (b_n q_{n-1})^{R_5(n-1)}}{q_{n+1}} \right)^{R_2(n)} \leq (b_{n+1} q_n)^{R_6(n)}$$

for a fixed sequence  $R_7(n)$ . This ensures the existence of  $R_3(n)$  such that:

$$\|A_{n+1}\|_{n+1} \leq (b_{n+1} q_n)^{R_3(n)}$$

Moreover, we have:

$$w \leq \frac{q_{n+1}}{2^{n+1} q_n^2 \|B_n\|_1}$$

Since  $q_n$  divides  $q_{n+1}$ , and by the left-hand side of assumption 3 of lemma 18.5, we have:

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \geq \frac{1}{q_{n+1}}$$

Therefore, by the right-hand side of assumption 3,  $q_{n+1} \geq (b_{n+1} q_n)^{R_{13}(n)}$ .

We will choose an explicit sequence  $R_{13}(n)$  such that:

$$(b_{n+1}q_n)^{R_{13}(n)} \geq \left(2^{n+1}q_n^{3/2}(b_nq_{n-1})^{R_5(n-1)}\right)^2$$

This choice implies:

$$q_{n+1} \geq \left(2^{n+1}q_n^{3/2}(b_nq_{n-1})^{R_5(n-1)}\right)^2$$

Therefore,

$$w \geq \frac{q_{n+1}}{2^{n+1}q_n^2(b_nq_{n-1})^{R_5(n-1)}} - 1 \geq \frac{1}{2} \frac{\left(2^{n+1}q_n^{3/2}(b_nq_{n-1})^{R_5(n-1)}\right)^2}{2^{n+1}q_n^2(b_nq_{n-1})^{R_5(n-1)}} \geq 2^n q_n \|B_n\|_1$$

Therefore, by lemma 20.6, for any  $i = 0, \dots, q_{n+1} - 1$ , we have:

$$\text{diam}\left((A_{n+1}^3)^{-1}([0, 1] \times [i/q_{n+1}, (i+1)/q_{n+1}[) \cap E_{n+1}^3)\right) \leq \frac{1}{2^n q_n \|B_n\|_1}$$

Therefore, by lemma 20.4,

$$\text{diam}\left((A_{n+1}^2)^{-1}\left((A_{n+1}^3)^{-1}\left([0, 1] \times [i/q_{n+1}, (i+1)/q_{n+1}[) \cap E_{n+1}^3\right)\right) \cap E_{n+1}^2\right) \leq \frac{1}{2^n \|B_n\|_1}$$

and therefore, we also have:

$$\text{diam}\left((A_{n+1}^1)^{-1}\left((A_{n+1}^2)^{-1}\left((A_{n+1}^3)^{-1}\left([0, 1] \times [i/q_{n+1}, (i+1)/q_{n+1}[) \cap E_{n+1}^3\right)\right) \cap E_{n+1}^2\right) \cap E_{n+1}^1\right) \leq \frac{1}{2^n \|B_n\|_1}$$

□

## 20.2 Construction in higher dimensions

The construction in higher dimensions is slightly different of the previous chapter. The first two steps are the same as in dimension 2 (we make the construction in the plan  $(x_1, x_d)$ , see the previous chapter), and for the third step (generation), we combine all  $d-1$  dimensions. The following lemma generalizes lemma 20.6:

**Lemma 20.9.** *For any integers  $w_1, \dots, w_{d-1} \geq 1$  such that  $2 \prod_{i=1}^{d-1} w_i \leq q_{n+1}$ , there is a smooth, measure-preserving, and  $S_{\frac{1}{q_n}}$ -equivariant diffeomorphism  $A_{n+1}^3$ , and an explicit sequence of integers  $R_7(n)$ , such that:*

$$\|A_{n+1}^3\|_{n+1} \leq \left(\frac{q_{n+1}}{w_1}\right)^{R_7(n)}$$

and there exists a  $S_{\frac{1}{q_n}}$ -invariant and  $A_{n+1}^3$ -invariant set  $E_{n+1}^3$  such that

$\mu(E_{n+1}^{3c}) \leq 4/2^n$ , and such that for any  $i = 0, \dots, q_{n+1} - 1$ , we have:

$$\text{diam}\left((A_{n+1}^3)^{-1}\left([0, 1]^{d-1} \times [i/q_{n+1}, (i+1)/q_{n+1}[ \right) \cap E_{n+1}^3\right) \leq \max\left(\frac{1}{w_1}, \dots, \frac{1}{w_{d-1}}, \frac{2^{d-1}w_1 \dots w_{d-1}}{q_{n+1}}\right)$$

As in dimension 2, we let  $A_{n+1} = A_{n+1}^3 A_{n+1}^2 A_{n+1}^1$  and

$$E_{n+1} = E_{n+1}^3 \cap A_{n+1}^3(E_{n+1}^2) \cap A_{n+1}^3 A_{n+1}^2(E_{n+1}^1)$$

We obtain the corollary:

**Corollary 20.10.** *There exists an explicit sequence of integers  $R_8(n)$  depending only on  $n$ , there is a smooth, measure-preserving, and  $S_{\frac{1}{q_n}}$ -equivariant diffeomorphism  $A_{n+1}$ , such that:*

$$\|A_{n+1}\|_{n+1} \leq (b_{n+1}q_n)^{R_8(n)}$$

and there exists a  $S_{\frac{1}{q_n}}$ -invariant and  $A_{n+1}$ -invariant set  $E_{n+1}$  such that

$\mu(E_{n+1}^c) \leq 4/2^n$ , and such that for any  $i = 0, \dots, q_{n+1} - 1$ , we have:

$$\text{diam}\left((A_{n+1})^{-1}\left([0, 1]^{d-1} \times [i/q_{n+1}, (i+1)/q_{n+1}[ \right) \cap E_{n+1}\right) \leq \frac{1}{2^n \|B_n\|_1}$$

*Proof of lemma 20.9.* We denote:

$$\begin{aligned} \tilde{A}_{n+1,w}^3 : [0, 1] \times \mathbb{T}^1 &\rightarrow [0, 1] \times \mathbb{T}^1 \\ (x, y) &\mapsto (\tilde{A}_{n+1,1,w}^3(x, y), \tilde{A}_{n+1,2,w}^3(x, y)) \end{aligned}$$

the map  $A_{n+1}^3$  of the 2-dimensional case, given by lemma 20.6, associated with the integer  $w$ . For  $i = 1, \dots, d - 1$ , we denote:

$$A_{n+1,w}^{3,i}(x_1, \dots, x_d) = (x_1, \dots, x_{i-1}, \tilde{A}_{n+1,1,w}^3(x_i, x_d), x_{i+2}, \dots, \tilde{A}_{n+1,2,w}^3(x_i, x_d))$$

We let:  $A_{n+1}^3 = A_{n+1,w_1}^{3,1} A_{n+1,w_1 w_2}^{3,2} \dots A_{n+1,w_1 \dots w_{d-1}}^{3,d-1}$  (see figures 17, 18, 19). We define  $E_{n+1}^3$  by analogy with lemma 20.6. □

*Proof of corollary 20.10.* The proof is analogous to the proof of corollary 20.7. We let:

$$w_1 = \left\lfloor \frac{q_{n+1}}{(2^{n+1}q_n(b_n q_{n-1}))^{R_5(n-1)} d} \right\rfloor$$

and for  $i = 2, \dots, d - 1$ ,  $w_i = 2^n q_n (b_n q_{n-1})^{R_5(n-1)}$ .

As in lemma 20.6, there exists  $R_8(n)$  such that:

$$\|A_{n+1}\|_{n+1} \leq (b_{n+1}q_n)^{R_8(n)}$$

For  $i = 2, \dots, d - 1$ , we have:

$$\frac{1}{w_i} \leq \frac{1}{2^n q_n (b_n q_{n-1})^{R_5(n-1)}} \leq \frac{1}{2^n q_n \|B_n\|_1}$$

Moreover, we have:

$$\frac{2^{d-1} w_1 w_2 \dots w_{d-1}}{q_{n+1}} \leq \frac{1}{2^{n+1} q_n \|B_n\|_1}$$

We will choose an explicit sequence  $R_{13}(n)$  such that:

$$(b_{n+1}q_n)^{R_{13}(n)} \geq (2^n q_n (b_n q_{n-1})^{R_5(n-1)})^{d+1}$$

This choice implies:

$$q_{n+1} \geq (2^n q_n (b_n q_{n-1})^{R_5(n-1)})^{d+1}$$

Therefore,

$$\frac{1}{w_1} \leq \frac{(2^n q_n (b_n q_{n-1})^{R_5(n-1)})^{d-1}}{q_{n+1}} \leq \frac{1}{2^n q_n \|B_n\|_1}$$

By combining lemma 20.9 and lemma 20.4, we obtain the corollary.

□

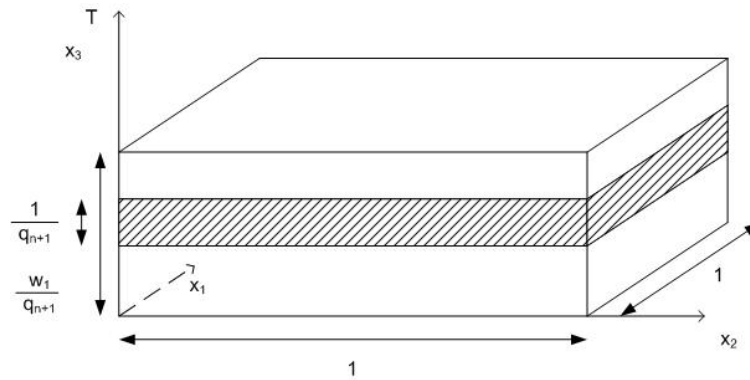


Figure 17: An element  $[0, 1]^2 \times [i/q_{n+1}, (i + 1)/q_{n+1}[$  (we take  $d = 3$ ), before the application of  $(A_{n+1}^3)^{-1} = (A_{n+1, w_1 w_2}^{3,2})^{-1} (A_{n+1, w_1}^{3,1})^{-1}$ . Its size is  $1 \times 1 \times 1/q_{n+1}$ .



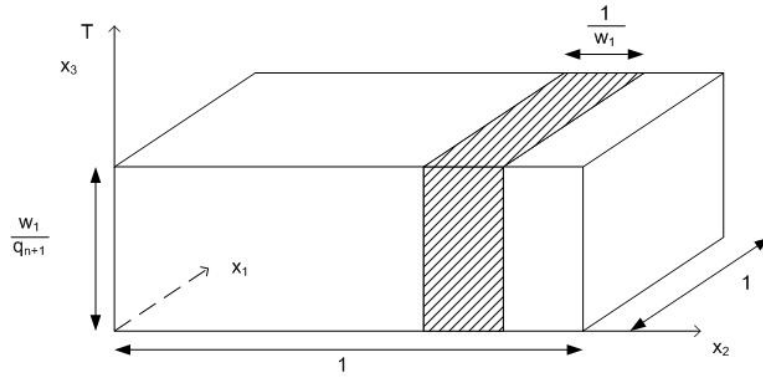


Figure 18: The element  $(A_{n+1,w_1}^{3,1})^{-1}([0, 1]^2 \times [i/q_{n+1}, (i + 1)/q_{n+1}[) \cap E_{n+1}^3$ . Its size is less than  $1 \times 1/w_1 \times w_1/q_{n+1}$ .

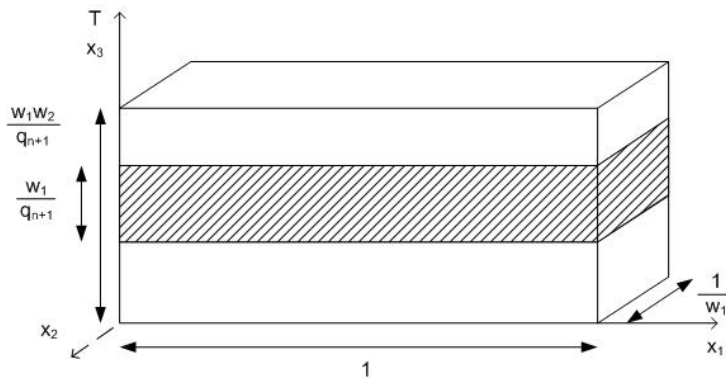


Figure 19:  $(A_{n+1,w_1}^{3,1})^{-1}([0, 1]^2 \times [i/q_{n+1}, (i + 1)/q_{n+1}[) \cap E_{n+1}^3$ , in the plan  $(x_1, x_3)$ .

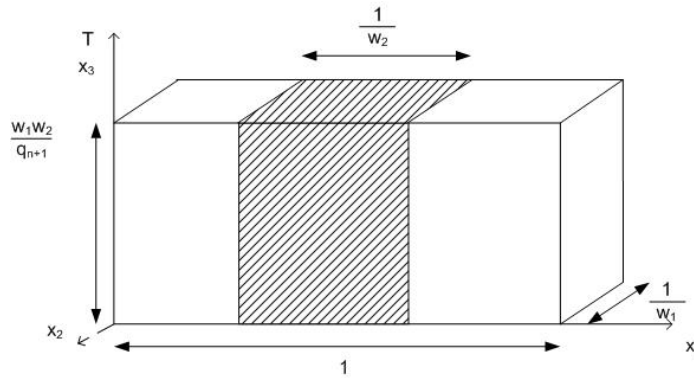


Figure 20:  $(A_{n+1,w_1w_2}^{3,2})^{-1}(A_{n+1,w_1}^{3,1})^{-1}([0, 1]^2 \times [i/q_{n+1}, (i + 1)/q_{n+1}[) \cap E_{n+1}^3$ , in the plan  $(x_1, x_3)$ . Its size is less than  $1/w_2 \times 1/w_1 \times w_1w_2/q_{n+1}$ .

### 20.3 Convergence of the sequence of diffeomorphisms and ergodicity of the limit $T$ . Proof that $T$ is a pseudo-rotation in dimension 2

By combining lemma 19.4, corollary 19.5, and proposition 20.1, and since  $\xi_n$  generates, then in order to complete the proof of lemma 18.5, it remains to show that  $T_n = B_n^{-1}S_{\frac{p_n}{q_n}}B_n$  converges in the smooth topology, and that the limit  $T$  of  $T_n$  is ergodic.

To show the convergence of  $T_n = B_n^{-1}S_{\frac{p_n}{q_n}}B_n$ , by the Cauchy criterion, it suffices to show

that  $\sum_{n \geq 0} d_n(T_{n+1}, T_n)$  converges. We combine the estimation of  $B_{n+1}$  and the assumption 3 of lemma 18.5 of closeness between  $p_{n+1}/q_{n+1}$  and  $p_n/q_n$ . We recall the lemma [FSW07, p.1812]:

**Lemma 20.11.** *Let  $k \in \mathbb{N}$ . There is a constant  $C(k, d)$  such that, for any  $h \in \text{Diff}(M)$ ,  $\alpha_1, \alpha_2 \in \mathbb{R}$ , we have:*

$$d_k(hS_{\alpha_1}h^{-1}, hS_{\alpha_2}h^{-1}) \leq C(k, d)\|h\|_{k+1}^{k+1}|\alpha_1 - \alpha_2|$$

Since  $T_n = B_n^{-1}S_{\frac{p_n}{q_n}}B_n = B_{n+1}^{-1}S_{\frac{p_n}{q_n}}B_{n+1}$ , we obtain, for a fixed sequence  $R_9(n)$  (that depends on  $n$  and on the dimension  $d$ ):

$$\begin{aligned} d_n(T_{n+1}, T_n) &= d_n(B_{n+1}^{-1}S_{\frac{p_{n+1}}{q_{n+1}}}B_{n+1}, B_{n+1}^{-1}S_{\frac{p_n}{q_n}}B_{n+1}) \leq C(k, d)\|B_{n+1}\|_{n+1}^{n+1} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \\ &\leq (b_{n+1}q_n)^{R_9(n)} \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| \end{aligned}$$

For some choice of the sequence  $R_{13}(n)$  in lemma 18.5, this last estimate guarantees the convergence of  $T_n$  in the smooth topology. Moreover, the limit  $T$  is ergodic, because it is metrically isomorphic to an irrational rotation of the circle, which is ergodic.

To show corollary 18.3, let us show that  $T$  is a pseudo-rotation when  $d = 2$ .

**Proposition 20.12.** *When  $d = 2$ , the limit  $T$  of  $T_n$  is a pseudo-rotation of angle  $\alpha$ .*

*Proof.* Since  $T|_{\partial M} = S_{\alpha}|_{\partial M}$ , then  $T$  is isotopic to the identity, and  $(0, \alpha) \in \text{rot}(T)$ , where  $\text{rot}(T)$  is the set of rotation vectors of  $T$ .

To show the proposition, it suffices to show that in any point of  $M$ , the rotation vector exists and is independent of the choice of the point. Let  $\tilde{T} : [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$  a lift of  $T$ ,  $\epsilon > 0$  and  $x \in [0, 1] \times \mathbb{R}$  and  $y \in \partial([0, 1] \times \mathbb{R})$ . For any integer  $n > 0$ , we have:

$$\frac{\tilde{T}^n(x) - x}{n} - \frac{\tilde{T}^n(y) - y}{n} = \frac{\tilde{T}^n(x) - \tilde{T}^n(y)}{n}$$

Let  $m$  such that:

$$\sum_{p \geq m} \|B_{p+1}\|_1 \left| \frac{p_{p+1}}{q_{p+1}} - \frac{p_p}{q_p} \right| \leq \epsilon$$

We have:

$$d_1(\tilde{T}^n, \tilde{T}^m) \leq \sum_{p \geq m} d_1(T_{p+1}^n, T_p^n) = d_1(B_{p+1}^{-1}S_{\frac{np_{p+1}}{q_{p+1}}}B_{p+1}, B_{p+1}^{-1}S_{\frac{np_p}{q_p}}B_{p+1}) \leq n\epsilon$$

Moreover,

$$\tilde{T}^n(x) - \tilde{T}^n(y) = \tilde{T}^n(x) - \tilde{T}_m^n(x) + \tilde{T}_m^n(x) - \tilde{T}_m^n(y) + \tilde{T}_m^n(y) - \tilde{T}^n(y)$$

$$\leq 2d_1(\tilde{T}^n, \tilde{T}_m^n) + |\tilde{T}_m^n(x) - \tilde{T}_m^n(y)| \leq 2n\epsilon + |\tilde{T}_m^n(x) - \tilde{T}_m^n(y)|$$

Moreover,

$$|\tilde{T}_m^n(x) - \tilde{T}_m^n(y)| \leq \|\tilde{T}_m^n\|_1 |x - y| \leq \|B_m\|_1^2 |x - y|$$

Therefore, for any  $n$  sufficiently large,

$$\left| \frac{\tilde{T}_m^n(x) - \tilde{T}_m^n(y)}{n} \right| \leq 2\epsilon + \frac{1}{n} \|B_m\|_1^2 |x - y| \leq 3\epsilon$$

Therefore, the translation vector of  $x$  exists and is equal to the translation vector of  $y$ , which is  $(0, \tilde{\alpha})$ , where  $\tilde{\alpha}$  is a lift of  $\alpha$ .

We conclude that  $\text{rot}(T) = \{(0, \alpha)\}$ .

□

## 20.4 Extension to more general manifolds

To extend the construction from  $[0, 1]^{d-1} \times \mathbb{T}$  to a general  $d$ -dimensional smooth compact connected manifold  $M$ , admitting an effective volume-preserving circle action  $\hat{S}_t$ , we proceed as in [FSW07, p. 1805] and the previous chapter. We keep denoting  $S_t$  the circle action on  $[0, 1]^{d-1} \times \mathbb{T}$ . For  $q \geq 1$ , let  $F_q$  be the set of fixed points of  $\hat{S}_{1/q}$ . Let  $B = \partial M \cup_{q \geq 1} F_q$  be the set of exceptional points. We recall the proposition:

**Proposition 20.13** ([FSW07]). *Let  $M$  be a  $d$ -dimensional smooth compact connected manifold, with an effective circle action  $\hat{S}_t$ , preserving a smooth volume  $\mu$ . Let  $S_t$  denote the circle action on  $[0, 1]^{d-1} \times \mathbb{T}$ . There exists a continuous surjective map  $\Gamma : [0, 1]^{d-1} \times \mathbb{T} \rightarrow M$  such that:*

1. *the restriction of  $\Gamma$  to  $]0, 1[^{d-1} \times \mathbb{T}$  is a smooth diffeomorphic embedding.*
2.  $\mu(\Gamma(\partial([0, 1]^{d-1} \times \mathbb{T}))) = 0$
3.  $B \subset \Gamma(\partial([0, 1]^{d-1} \times \mathbb{T}))$
4.  $\Gamma_*(\text{Leb}) = \mu$
5.  $\hat{S}_t \Gamma = \Gamma S_t$

We use this proposition at each step to apply lemma 18.4. We let  $\hat{T}_n : M \rightarrow M$  defined by  $\hat{T}_n(x) = \Gamma B_n^{-1} S_{\frac{pn}{qn}} B_n \Gamma^{-1}(x)$  if  $x \in \Gamma(]0, 1[^{d-1} \times \mathbb{T})$  and  $\hat{T}_n(x) = \hat{S}_{\frac{pn}{qn}}(x)$  otherwise. To show that  $\hat{T}_n$  is a smooth diffeomorphism (which implies that its limit is also smooth), we use the facts that  $\Gamma_{]0, 1[^{d-1} \times \mathbb{T}}$  is a smooth diffeomorphism, that  $B_n = \text{Id}$  on a neighborhood of  $\partial([0, 1]^{d-1} \times \mathbb{T})$  and that  $\hat{S}_t \Gamma = \Gamma S_t$ . To construct the metric isomorphism  $\hat{K}_n^\infty = \Gamma \hat{K}_n^\infty$ , we use the fact that the restriction of  $\Gamma$  to a set of full measure is a metric isomorphism. Details are in the previous chapter.

Finally, to show that  $\hat{T} \in \mathcal{A}_\alpha$ , where  $\hat{T}$  is the limit of  $\hat{T}_n$  in the smooth topology, we let  $\hat{H}_n : M \rightarrow M$  defined by  $\hat{H}_n(x) = \Gamma B_n \Gamma^{-1}(x)$  if  $x \in \Gamma(]0, 1[^{d-1} \times \mathbb{T})$  and  $\hat{H}_n(x) = x$  otherwise. We write

$$\hat{T} - \hat{H}_n^{-1} \hat{S}_\alpha \hat{H}_n = \hat{T} - \hat{T}_n + \hat{T}_n - \hat{H}_n^{-1} \hat{S}_\alpha \hat{H}_n$$

We know that  $\hat{T} - \hat{T}_n \rightarrow 0$  in the smooth topology. We show that  $\hat{T}_n - \hat{H}_n^{-1} \hat{S}_\alpha \hat{H}_n \rightarrow 0$  in the smooth topology by proceeding as in the proof of the convergence of  $T_n$  on  $[0, 1]^{d-1} \times \mathbb{T}$  in subsection 20.3.

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