Cinq essais sur les jeux d'oligopoles coopératifs

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Five Essays on Cooperative Oligopoly Games

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The theory of economics does not furnish a body of settled conclusions immediately applicable to policy. It is a method rather than a doctrine, an apparatus of the mind, a technique of thinking which helps its possessor to draw correct conclusions.

John Maynard Keynes

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Contents

Chapter 1

Introduction

1.1 The cooperative approach of oligopoly situations

A central question in oligopoly theory is the existence of collusive behaviors between firms. The main reason for which economists try to understand this phenomenon is that the formation of cartels impacts on both consumers' and producers' surplus, and so affects total welfare. In economic welfare analysis, it is a well-established and old idea that monopoly power can negatively affect social welfare. In his classic, *The Wealth of Nations*, Smith (1776) writes on both collusion between rival firms and on the exercise of monopoly power:

> People of the same trade seldom meet together, even for merriment or diversion, but the conversation ends in a conspiracy against the public, or in some contrivance to raise prices. The monopolists, by keeping the market constantly understocked, by never fully supplying the effectual demand, sell their commodities much above the natural price.

At the end of the 19th century, as a reaction to the formation of trusts in the United States, a consensus emerged on the necessity to maintain competition in industries. This led to the establishment of the first anti-trust law in the United States, the 1890 Sherman Act. While Section 1 of the act prohibited contracts, combinations and conspiracies "in restraint of trade", Section 2 of the act made illegal monopolization or any attempt to monopolize. This legislation permitted the United States government to sue and dismantle two famous trusts, the Standard Oil Company and American Tobacco. Afterwards, other anti-trust laws were established in the United States as the Clayton and Federal Trade Commission Acts in 1914.

In European Union, modern competition policy is principally established by the Treaty of the European Communities. It pays attention to both horizontal and vertical agreements (Article 81), abuse of dominant position (Article 82) and the regulation of mergers. A horizontal (vertical) agreement is an agreement among competitors on the same level (different levels) of production or distribution. Abuse of dominant position occurs when a firm uses a dominant position in the market to impede the maintenance of effective competition. Motta (2004) defines competition policy as: "the set of policies and laws which ensure that competition in the marketplace is not restricted in such a way as to reduce economic welfare." In this definition two elements should be underlined. The first is that firms may restrict competition in a way which is not necessarily detrimental. This is the case for some horizontal agreements where firms collaborate in research and development activities. By means of cross-licensing or patent pooling, firms can share research and development costs/results in order to reduce sales prices. This is in line with D'Aspremont and Jacquemin (1988) who show that cooperative behavior in research and development activities can positively affect economic welfare by generating spillover effects in industries having a few firms. This is also the case for many vertical restraints (vertical agreements) between a manufacturer and a retailer such as non-linear pricing (the price is composed of two parts: a lump-sum fee as well as a per-unit charge), quantity fixing (the retailer cannot buy less or more than a certain number of units fixed by the manufacturer) and exclusivity clauses (for instance, an exclusive territory clause specifies that there is only one retailer who sell a brand in a certain geographical area). The second is that economic welfare is the objective that competition policy pursues. Economic welfare is the standard concept which measures how well an industry performs. Economic welfare is given by the total surplus, i.e. the sum of consumer surplus and producer surplus. The surplus of a consumer is defined as the difference between the consumer's valuation for the good and the effective price for which he has to pay for it. Consumer surplus is the sum of the surplus of all consumers. The surplus of a producer is the profit it makes by selling the good. Producer surplus is the sum of profits of all producers in the industry.

Although cooperation on research and development activities may have beneficial welfare effects, both in the United States and European Union, many other horizontal agreements such as agreements on sales prices and division of the market shares are considered as negative for welfare. By keeping in mind these considerations, the study of cooperative oligopoly games is relevant insofar as it permits to establish the conditions under which a horizontal agreement on sales prices is likely to be stable, i.e. not contested by any firm, in an oligopolistic market, which is one of the main preoccupations of competition authorities. Precisely, this thesis tries to answer the question on the existence of stable horizontal agreements on sales prices from two angles: quantity competition (Part I) and price competition (Part II). While Part I deals with quantity competition where firms can indirectly control sales prices by engaging on the quantity produced, Part II analyzes price competition where firms can directly manipulate sales prices by agreeing on the prices charged. In both competition types, competitors form a cartel (coalition) in which they cooperate and agree on sales prices. Thus, the existence of horizontal agreements on sales prices is related to the problem of the formation of coalitions.

Aumann (1959) proposes to analyze the formation of coalitions, and so the horizontal agreements, by converting a strategic game into a cooperative game. An appropriate solution (for instance, the core) permits to deal with the stability of coalition structures resulting from the formation of coalitions (for instance, the whole set of players also called the grand coalition). We consider such a cooperative approach of oligopoly situations. The main difference with the non-cooperative approach is that firms are allowed to sign binding agreements in order to cooperate. This assumption allows to define a cooperative oligopoly game in which any cartel (coalition) can occur. It is commonly assumed that profit transfers between firms belonging to the same cartel are possible so that any cartel profit can be freely distributed among its members. The cooperative games consistent with this assumption are games with transferable utility or TU-games. Generally speaking, a TU-game is summarized by a set of players $N = \{1, 2, \ldots, n\}$ and a characteristic function $v : 2^N \longrightarrow \mathbb{R}$, with the convention that $v(\emptyset) = 0$, which assigns to every coalition $S \in 2^N \backslash \{\emptyset\}$, a worth $v(S) \in \mathbb{R}$. The number $v(S)$ is the total utility that is available for free division among the members of S. For example, consider an oligopolistic market in which there exist only three firms named 1, 2 and 3 respectively. The oligopoly TU-game is then summarized by the set of firms $N = \{1, 2, 3\}$, and the worths $v(\{1\})$, $v(\{2\})$ and $v(\{3\})$ for the one-member cartels, $v(\{1, 2\})$, $v(\{1, 3\})$ and $v(\{2, 3\})$ for the two-member cartels, and $v(\{1, 2, 3\})$ for the three-member cartel (the grand coalition).
A main characteristic of oligopolistic markets is that each firm's decision impacts on its rivals' profits, and so any cartel profit crucially depends on the strategies taken by the other firms competing on the market. Hence, the determination of the worth that a coalition can obtain requires to specify how firms outside the cartel, called outsiders, behave. In order to do that, we assume that outsiders facing a cartel behave according to some rules, called blocking rules, in the strategic oligopoly games. For instance, while some blocking rules specify that firms outside the cartel act so as to minimize the cartel profit, other blocking rules stipulate that each outsider maximizes its individual profit given the strategies taken by the cartel. By applying an appropriate solution concept to the strate-

gic oligopoly game allowing binding agreements among the cartel members and specifying outsiders' behavior, the resulting cooperating firms' profits determines the worth of the coalition, and so enables to define the induced oligopoly TU-game. In order to derive oligopoly TU-games from strategic oligopoly games, we follow three different approaches suggested by Aumann (1959) and Chander and Tulkens (1997).

Aumann (1959) proposes the first two approaches: according to the first, any cartel computes the total profit which it can guarantee itself regardless of what outsiders do; the second approach consists in computing the minimal profit for which outsiders can prevent the firms in the cartel from getting more. The characteristic functions obtained from these two assumptions are called the α and β-characteristic functions respectively. In the above example, assume that firms 1, 2 and 3 sell differentiated products, compete in price by choosing a price $p_i \in \mathbb{R}_+$, $i \in N$, operate at a constant and identical marginal cost equal to one and that the demand system is Shubik's (1980) so that the quantity demanded of firm i's brand, $i \in N$, is defined as:

$$D_i(p_1, p_2, p_3) = 5 - p_i - 2\left(p_i - \frac{1}{3}\sum_{j=1}^{3} p_j\right).$$

The quantity demanded of firm i's brand depends on its own price p_i and on the difference between p_i and the average price in the industry $\sum_{j=1}^{3} p_j/3$. This quantity is decreasing with respect to p_i and increasing with respect to any p_j such that $j \neq i$.

According to the α-approach, for any coalition $S \in 2^N \backslash \{\emptyset\}$, while the coalition members maximize the sum of their profits at a first period, the outsiders minimize the sum of coalition members' profits at a second period given coalition members' strategies. According to the β-approach, for any coalition $S \in 2^N \backslash \{\emptyset\}$, while outsiders minimize the sum of coalition members' profits at a first period, the coalition members maximize the sum of their profits at a second period given outsiders' strategies. In the above oligopolistic market, the α and β-characteristic functions are equal and given by: for any $i \in N$, $v_\alpha(\{i\}) = v_\beta(\{i\}) = 0.76$, for any $i \in N$ and any $j \in N$ such that $j \neq i$, $v_\alpha(\{i,j\}) = v_\beta(\{i,j\}) = 3.33$, and $v_\alpha(\{1,2,3\}) = v_\beta(\{1,2,3\}) = 12$. This equality between the α and β-characteristic functions holds in many oligopolistic markets.

However, the α and β-approaches can be questioned since outsiders probably cause substantial damages upon themselves by minimizing the profit of the cartel. A similar argument is developed by Rosenthal (1971). In the above example, it can be verified that outsiders charge prices equal to zero in order to minimize the cartel profit, and so obtain a negative profit.

This is why Chander and Tulkens (1997) introduce a third approach by considering a more credible blocking rule where firms outside the coalition choose their strategy individually as a best reply to the coalitional action. The characteristic function obtained from this assumption is called the γ-characteristic function. According to the γ-approach, for any coalition $S \in 2^N \backslash \{\emptyset\}$, while the coalition members maximize the sum of their profits, every outsider maximizes its individual profit. In other words, coalition S and every outsider play a Nash equilibrium. In the above oligopolistic market, the γ-characteristic function is given by: for any $i \in N$, $v_\gamma(\{i\}) = 3.36$, for any $i \in N$ and any $j \in N$ such that $j \neq i$, $v_\gamma(\{i,j\}) = 7.05$, and $v_\gamma(\{1,2,3\}) = 12$. Unsurprisingly, observe that while the worth of the grand coalition N is unchanged, the worth of any coalition $S \in 2^N \backslash \{\emptyset, N\}$ is greater under the γ-approach than under the α and β-approaches.
For more general oligopoly situations studied in the next chapters, we will have to make sure that the α, β and γ-characteristic functions are well-defined in strategic oligopoly games in order to define the associated oligopoly TU-games.

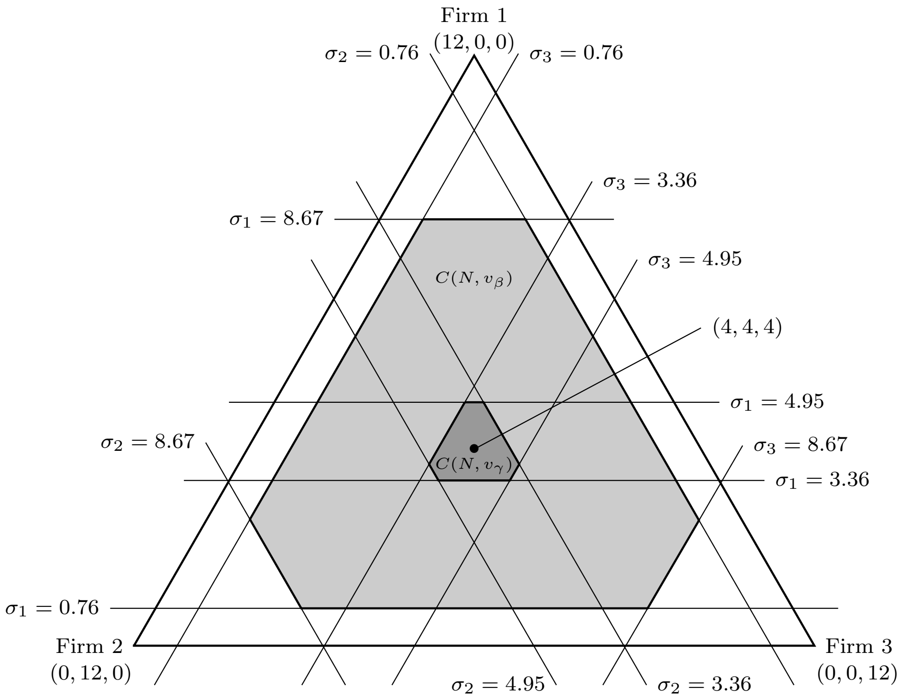
An appropriate set-valued solution for TU-games which deals with the stability of the grand coalition is the core. A payoff vector is in the core if no coalition can deviate from the grand coalition and obtain a better payoff for all its members. The stability of the grand coalition is then related to the non-emptiness of the core. For TU-games, the core is the set of all payoff vectors $\sigma \in \mathbb{R}^n$ such that $\sum_{i \in N} \sigma_i = v(N)$ and for any $S \in 2^N$, $\sum_{i \in S} \sigma_i \geq v(S)$. The first condition requires that the worth of the grand coalition is fully distributed among all the players. The second condition means that no subgroup of players can contest this sharing by breaking off from the grand coalition. In the above oligopolistic market, these two conditions imply that the core associated with the α and β-characteristic functions is:

$$C(N, v_\beta) = \left\{ \sigma \in \mathbb{R}^3 : \sum_{i=1}^{3} \sigma_i = 12 \text{ and } \forall i \in N,\ 0.76 \leq \sigma_i \leq 8.67 \right\},$$

while the core associated with the γ-characteristic function is:

$$C(N, v_\gamma) = \left\{ \sigma \in \mathbb{R}^3 : \sum_{i=1}^{3} \sigma_i = 12 \text{ and } \forall i \in N,\ 3.36 \leq \sigma_i \leq 4.95 \right\}.$$

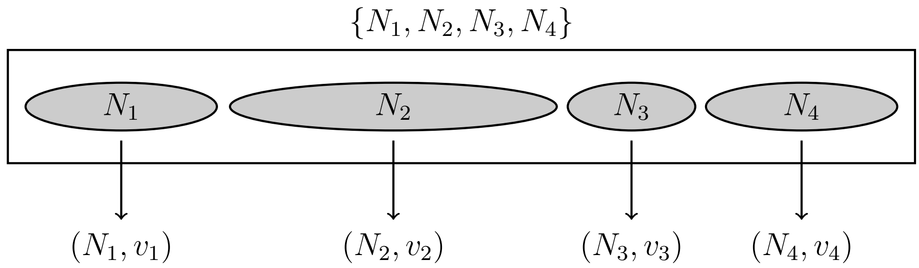
The 2-simplex below represents these two core configurations:



Note that the core shrinks when we switch from the α and β-approaches to the γ-approach. This is not surprising since we observed that while the worth of the grand coalition N is unchanged, the worth of any coalition $S \in 2^N \backslash \{\emptyset, N\}$ is greater under the γ-approach than under the α and β-approaches so that there are more incentives for coalitions to deviate from the grand coalition under the γ-approach. Hence, the core structure crucially depends on outsiders' behavior facing the deviating coalition. In the above oligopolistic market, the non-emptiness of the core implies that there exists some payoff distribution among all the firms (for instance, $\sigma = (4, 4, 4)$) which ensures the stability of the grand coalition. Thereafter, we will sustain that under basic assumptions, cooperative oligopoly games can be used to study the stability of any coalition structure and not only the stability of the grand coalition.

The study of cooperative oligopoly games is relevant to analyze the stability of coalition structures and explain cooperation mechanisms in many industries in which cooperation constitutes a strategic choice for firms such as the raw materials and telecommunications industries. Since the creation of GATT (General Agreement on Tariffs and Trade) in 1947 and WTO (World Trade Organization) in 1994, import and export barriers to trade and services have considerably reduced. The increasing of international flows of goods and services leads firms to make national and/or international strategic alliances. As a consequence, coop-

eration between firms in oligopolistic markets has become a strategic choice in order to reduce their costs, diversify their activities and increase their market shares.
The raw materials industries are characterized by important costs of extracting and transport. In such industries, exporting firms are likely to cooperate in order to reduce their transport cost. Massol and Tchung-Ming (2009) study the possibility of a profitable logistic coordination between twelve countries exporting liquefied natural gas. They provide the following example: in 2007, Trinidad and Tobago sent nearly 2.7 bcm (billion cubic metres) to Europe, while Algeria sent 2.1 bcm to the United States. In view of their geographical positions, these two exporting countries would have an opportunity for logistic cooperation in order to reduce their transport costs. By assuming that such agreements would not have any effect on market prices, Massol and Tchung-Ming define an oligopoly TU-game (calibrated on the year 2007) in which any coalition minimizes its members' transport costs. They conclude that the core of this game is empty even for a low amount of coordination costs, and so the credibility of a logistic cooperation scenario without any price manipulation is unlikely.
Another example are industries subject to frequent changes in technology. In such industries, firms often seek cooperation in order to extend their activities and develop their market shares. An industry in which technological changes and upturn and downturn are an inherent part is the telecommunications industry (Noam 2006). Since national operators are often restricted to their national boundaries, cooperation with other network operators is the only way to export their services as multinational companies. One example of cooperation is Unisource, a pan-European telecommunications company. We refer to Graack (1996) for an overview of the telecommunications industry in European Union. In such an industry, the question arises whether the market structure induced by the strategic alliances is stable or not. The study of cooperative oligopoly games provides an original approach in order to deal with the stability of the market structure. To this end, we assume that any coalition embedded in a coalition structure cannot communicate with the others so that only subsets of the existing coalitions can break up from the collusive agreements (Ray and Vohra (1997) make a similar assumption). For any cartel we can define an oligopoly TU-game in which the cartel in question is viewed as the grand coalition and the actions taken by the other cartels are considered as fixed, and so can be omitted. Hence, the stability of any cartel in the market structure is then related to the non-emptiness of the core of its associated oligopoly TU-game. For the market structure comprising four cartels $\{N_1, N_2, N_3, N_4\}$, this argument is illustrated in the figure below:



Many environmental problems can also be analyzed by means of cooperative oligopoly games also permits to deal with many environmental problems. For instance, the "tragedy of the commons" (Hardin 1968) is a famous dilemma in which a common pool resource is overused because of selfish behaviors. Notable examples include decreases of fish stocks and the deforestation in tropical countries. (Moulin 1997) shows that an oligopoly situation also describes a common pool situation. Consequently, the study of cooperative oligopoly games is relevant in order to deal with natural resource problems (Pham Do 2003). Funaki and Yamato (1999) study an economy with a common pool resource of fish by means of TU-games. They show that if any coalition has pessimistic expectations on the coalition formation of the outsiders, i.e. outsiders form singletons and act non-cooperatively, then the core is non-empty and so the tragedy of the commons can be avoided. Otherwise, if any coalition has optimistic expectations on the coalition formation of the outsiders, i.e. outsiders form the complementary coalition and play cooperatively, then the core is empty and so the tragedy of the commons cannot be avoided.

1.2 A review of game theory approaches on cooperation in oligopoly situations

Oligopoly theory deals with competition models which can be divided into two parts, i.e. the quantity competition (Cournot 1838) and the price competition (Bertrand 1883). For the quantity competition, Stackelberg (1934) incorporates the idea of commitment by proposing a leader-follower model. For each of these three oligopoly situations, some early works have already investigated cooperation between firms by means of both non-cooperative and cooperative oligopoly games.

As regards Cournot oligopoly situations, Salant et al. (1983) analyze the equilibrium distribution of outputs among the cartels and show that mergers may reduce cartel members' profits.

Norde et al. (2002) distinguish two different types of oligopoly situations, namely those with transferable technologies and those without transferable technologies. In the first type, a group of firms produces according to the cheapest technology used by the cartel members, whereas such a transfer of technologies is not possible for the second type. For Cournot oligopoly situations with or without transferable technologies, Zhao (1999a,b) shows that the α and β-characteristic functions are equal, and so the same set of Cournot oligopoly TU-games is associated with these two characteristic functions.
When technologies are transferable, Zhao (1999a) provides a necessary and sufficient condition in order to establish the convexity property in case the inverse demand and cost functions are linear. This property means that there are strong incentives to form the grand coalition insofar as the marginal contribution of a firm to some coalition increases if the coalition which it joins becomes larger. Although these games may fail to be convex in general, Norde et al. (2002) show they are nevertheless totally balanced which ensures the non-emptiness of the core.
When technologies are not transferable, Zhao (1999b) proves that the core is non-empty if any individual strategy set is compact and convex and any individual profit function is continuous and concave. More generally, by using a technical proof inspired from Scarf (1971), Zhao shows that the core is non-empty for general TU-games in which any individual strategy set is compact and convex, any individual utility function is continuous and concave, and the strong separability condition is satisfied. This latter condition requires that the utility function of a coalition and any of its members' individual utility functions have the same minimizers. Zhao proves that Cournot oligopoly TU-games satisfy this latter condition. Furthermore, Norde et al. (2002) show that these games are convex in case the inverse demand and cost functions are linear, and Driessen and Meinhardt (2005) provide economically meaningful sufficient conditions in order to guarantee the convexity property in a more general case.

Concerning Stackelberg oligopoly situations, in case there are a single leader and multiple followers in a quantity competition, Sherali et al. (1983) study strategic Stackelberg oligopoly games and prove the existence and uniqueness of the Nash equilibrium in case the inverse demand function is twice differentiable, strictly decreasing and satisfies for any output $X \in \mathbb{R}_+$, $p'(X) + Xp''(X) \leq 0$ (Sherali et al. provide an economic interpretation of this condition), and the individual cost functions are twice differentiable and convex. In particular, they show that the convexity of followers' reaction functions with respect to leader's output is crucial for the uniqueness of the Nash equilibrium.
For general TU-games, Marini and Currarini (2003) associate a two-stage structure with the γ-characteristic function. In this temporal sequence, any deviating

coalition possesses a first-mover advantage by acting as a leader while outsiders play their individual best reply strategies as followers. By assuming that any player's individual utility function is twice differentiable and strictly concave on its individual strategy set, they prove that if the players and externalities are symmetric (the players have identical individual utility functions and strategy sets, and externalities are either positive or negative) and the game has strategic complementarities, then the equal division solution belongs to the core. They apply their result to three economic models. First, they consider a quantity competition in which strategies are substitutes and show that the core associated with the two-stage game is non-empty. Then, they deal with a price competition and prove that for lower degrees of product differentiation, the core associated with the two-stage game becomes increasingly smaller than the core associated with the simultaneous game. Finally, they study an economy with two commodities, a public good and a private good, and show that the core associated with the two-stage game is empty.

Regarding Bertrand oligopoly situations, Kaneko (1978) considers a finite set of firms selling a homogeneous product to a continuum of consumers. He assumes that a subset of firms and consumers can cooperate by trading the good among themselves. The main result establishes that the core is empty when there are more than two firms. Deneckere and Davidson (1985) consider a Bertrand oligopoly situation with differentiated products in which the demand system is Shubik's (1980) and firms operate at a constant and identical marginal cost. They study the equilibrium distribution of prices and profits among the cartels and show that a merger of two cartels implies that all the firms charge higher prices, and so benefits all the industry. They also prove that these games have a superadditivity property in the sense that a merger of two disjoint cartels results in a joint after-merger profit for them which is greater than the sum of their pre-merger profits. For the same Bertrand oligopoly situation, Huang and Sjöström (2003) define a Bertrand oligopoly TU-game in which the worth of any coalition is defined by a recursive procedure applying the core solution to a reduced game in order to predict outsiders' behavior. They provide a necessary and sufficient condition for the non-emptiness of the core which requires that the substitutability parameter must be greater than or equal to some number that depends on the size of the industry. They conclude that the core is empty when there are more than ten firms.

1.3 Outline of the thesis

Up to now, we saw that few works have studied cooperative oligopoly games. As a counterpart to this lack of interest in the study of cooperative oligopoly games, this thesis studies cooperative oligopoly games in Cournot (Chapters 3 and 4), Stackelberg (Chapter 5) and Bertrand (Chapters 6 and 7) competitions. As regards Cournot oligopoly situations, we study Cournot oligopoly TU-games in γ-characteristic function form and Cournot oligopoly interval games in γ-set function form. We extend the previous analyses focusing on Cournot oligopoly TU-games in α and β-characteristic function forms (Zhao 1999a,b, Norde et al. 2002, Driessen and Meinhardt 2005) by providing sufficient condition on any individual profit function and any individual cost function under which the core is non-empty.
Concerning Stackelberg oligopoly situations, we deal with Stackelberg oligopoly TU-games in γ-characteristic function form. We relax Marini and Currarini's symmetric players assumption (2003) and extend their core allocation result by characterizing the core and providing a necessary and sufficient condition under which the core is non-empty.
Regarding Bertrand oligopoly situations, we study Bertrand oligopoly TU-games in α, β and γ-characteristic function forms. We show that the convexity property holds for Bertrand oligopoly TU-games in α and β-characteristic function forms. Moreover, we generalize the superadditivity result of Deneckere and Davidson (1985) by providing a sufficient condition under which Bertrand oligopoly TU-games in γ-characteristic function form are convex.
These contributions are detailed below.

Chapter 2 gives an overview of some notions in game theory which are frequently used in the next chapters. First, it briefly recalls some basic definitions of non-cooperative game theory such as strategic games, Nash equilibrium and partial agreement equilibrium. Then, it introduces some definitions of cooperative game theory such as TU-games, interval games and solutions such as the core, the Shapley value and the nucleolus. Finally, it proposes three approaches for converting a non-cooperative game into a cooperative game, i.e. the α and β-approaches (Aumann 1959) and the γ-approach (Chander and Tulkens 1997).

Chapter 3, which is based on Lardon (2009), focuses on Cournot oligopoly TU-games without transferable technologies. The main objective is to deal with the problem of the non-emptiness of the core for the set of Cournot oligopoly TU-games in γ-characteristic function form. We assume that the inverse demand function is differentiable, strictly decreasing and concave, and any individual cost function is continuous, strictly increasing and convex. We first show that

Cournot oligopoly TU-games in γ-characteristic function form are well-defined and study some of their properties concerning the equilibrium outputs. We go step further by adopting a more general approach in which we assume that any coalition structure can occur. In particular, the coalition structures in which any coalition faces outsiders acting individually can form. For any coalition structure, we construct an aggregated strategic Cournot oligopoly game for which a Nash equilibrium represents the aggregated equilibrium outputs of the embedded coalitions. Our first result proves that there exists a unique Nash equilibrium for any coalition structure. Therefore, it turns out that Cournot oligopoly TU-games in γ-characteristic function form are well-defined. Our second result shows that the equilibrium total output is decreasing with the coarseness of the coalition structure. This feature follows from two phenomena. Firstly, when some coalitions merge, the production of the new entity decreases, and secondly, the other coalitions respond by increasing their outputs. Thus, this equilibrium distribution of outputs differs from the equilibrium distribution of prices studied by Deneckere and Davidson (1985).

Then, we use these preliminaries results in order to study the non-emptiness of the core. We consider two approaches. The first shows that if the inverse demand function is differentiable and any individual profit function is continuous and concave on the set of strategy profiles, the corresponding Cournot oligopoly TU-game in γ-characteristic function form is balanced, and therefore has a non-empty core. This result extends Zhao's core non-emptiness result (1999b) to the set of Cournot oligopoly TU-games since the core associated with the γ-characteristic function is included in the core associated with the β-characteristic function. A drawback with this approach is that it does not point out any solution belonging to the core. The second approach provides a new single-valued solution in the core, called NP(Nash Pro rata) value, on the set of Cournot oligopoly TU-games in γ-characteristic function form with linear individual cost functions and asymmetric capacity constraints. The NP value distributes to every firm the worth of the grand coalition in proportion to his Nash individual output. This result generalizes Funaki and Yamato's core allocation result (1999) from no capacity constraint to asymmetric capacity constraints insofar as a Cournot oligopoly situation also describes a common pool situation (Moulin 1997). We characterize the NP value by means of four properties: efficiency, null firm, monotonicity and non-cooperative fairness. Efficiency requires that a solution distributes the worth of the grand coalition among the players. The null firm property stipulates that a firm with no production capacity obtains a zero payoff. Monotonicity specifies that if a firm has a production capacity greater than or equal to the production capacity of another firm, then former's payoff will be greater than or equal to latter's payoff. Non-cooperative fairness requires that a solution distributes to every player a payoff proportionally to his profit in the finest coalition structure. As far as we know, this is the first result that characterizes a solution belonging to the core for a set of Cournot oligopoly TU-games. Furthermore, we provide a Cournot oligopoly TU-game in γ-characteristic function form, where the inverse demand and individual cost functions are linear, which fails to be superadditive, and so convex. This proves that Norde et al.'s result (2002) cannot be extended to Cournot oligopoly TU-games in γ-characteristic function form.

In Chapter 4, which is based on Lardon (2010b), we relax the assumption on the differentiability of the inverse demand function in Chapter 3 which ensures that the γ-characteristic function is well-defined. Indeed, in many Cournot oligopoly situations the inverse demand function may not be differentiable. For instance, Katzner (1968) shows that demand functions derived from quite nice consumers' individual utility functions, even twice continuously differentiable, may not be differentiable everywhere. In order to guarantee that demand functions are at least continuously differentiable, many necessary and sufficient conditions are provided by Katzner (1968), Debreu (1972, 1976), Rader (1973, 1979) and Monteiro et al. (1996). This is why Chapter 4 focuses on Cournot oligopoly situations where the inverse demand function is continuous but not necessarily differentiable. As mentioned above, with such an assumption we cannot always define a Cournot oligopoly TU-game in γ-characteristic function form since the worth of every coalition is not necessarily unique. However, we show that we can always specify a Cournot oligopoly interval game in γ-set function form. An interval game assigns to every coalition a closed and bounded real interval that represents all its potential worths. Interval games are introduced by Branzei et al. (2003) to handle bankruptcy situations. We refer to Alparslan-Gok et al. (2009a) for an overview of recent developments in the theory of interval games. Regarding core solutions of these game types, we consider two extensions of the core: the interval core and the standard core. We use the term "standard core" instead of the term "core" in order to distinguish this core solution for interval games with the core for TU-games. The interval core is specified in a similar way to the core for TU-games by using the methods of interval arithmetic (Moore 1979). The standard core is defined as the union of the cores of all TU-games for which the worth of every coalition belongs to its worth interval. We consider the problem of the non-emptiness of the interval core and of the standard core for the set of Cournot interval games in γ-set function form. To this end, we use a decision theory criterion, the Hurwicz criterion (Hurwicz 1951), that consists in combining, for any coalition, the worst and the best worths that it can obtain in its worth interval. The first result states that the interval core is non-empty if and only if the Cournot oligopoly TU-game associated with the best worth of every coalition in its worth interval admits a non-empty core. However, we show that even for a very simple Cournot oligopoly situation, this condition fails to be satisfied. The second result states that the standard core is non-empty if and only if the Cournot oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty core. Moreover, we give some properties on individual profit functions and cost functions under which this condition always holds, which substantially extends the core non-emptiness results in Chapter 3.

For TU-games associated with a two-stage structure, Marini and Currarini's core allocation result (2003) raises two questions to which we answer in Chapter 5. The first concerns the core structure of such TU-games since they only provide a single-valued solution (the equal division solution) in the core. The second question turns on the role of the symmetric players assumption on the non-emptiness of the core. Chapter 5, which is based on Driessen, Hou, and Lardon (2011), answers both questions by considering the two-stage structure associated with the γ-characteristic function in a quantity competition. The set of cooperative oligopoly games associated with this temporal sequence is the set of Stackelberg oligopoly TU-games in γ-characteristic function form. Thus, contrary to Cournot oligopoly TU-games in γ-characteristic function form in which all the firms simultaneously choose their strategies, any deviating coalition produces an output at a first period and outsiders simultaneously and independently play a quantity at a second period. We assume that the inverse demand function is linear and firms operate at constant but possibly distinct marginal costs. Thus, contrary to Marini and Currarini (2003), the individual utility (profit) functions are not necessarily identical. First, we characterize the core by proving that it is equal to the set of imputations which answers the first question on the core structure of this game type. The reason is that the first-mover advantage gives too much power to singletons so that the worth of any deviating coalition is less than or equal to the sum of its members' individual worths except for the grand coalition. Then, we provide a necessary and sufficient condition under which the core is non-empty. Finally, we prove that this condition depends on the heterogeneity of firms' marginal costs, i.e. for a fixed number of firms the core is non-empty if and only if firms' marginal costs are not too heterogeneous. The more the number of firms is, the less the heterogeneity of firms' marginal costs must be in order to ensure the non-emptiness of the core which answers the second question on the role of the symmetric players assumption. Surprisingly, in case the inverse demand function is strictly concave, we provide an example in which the opposite result holds, i.e. when the heterogeneity of firms' marginal costs increases the core becomes larger.

Chapter 6, which is based on Lardon (2010a), studies cooperative oligopoly games in a price competition. We consider the same Bertrand oligopoly situation as Deneckere and Davidson (1985) and substantially extends their superadditivity result. In order to define Bertrand oligopoly TU-games, we consider successively the α, β and γ-characteristic functions. As for the set of Cournot oligopoly TU-games, we show that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions and we prove that the convexity property holds for this set of TU-games. Then, following Chander and Tulkens (1997) we consider the γ-characteristic function where firms react to a deviating coalition by choosing their individual best reply strategies. For this set of TU-games, we show that the equal division solution belongs to the core and we provide a sufficient condition under which such games are convex. This finding generalizes the superadditivity result of Deneckere and Davidson (1985) and contrasts sharply with the negative core non-emptiness results of Kaneko (1978) and Huang and Sjöström (2003). Note that these properties are also satisfied for Cournot oligopoly TU-games in γ-characteristic function form. In non-cooperative game theory, an important distinction between a strategic Cournot oligopoly game and a strategic Bertrand oligopoly game is that the former has strategic substitutabilities and the latter has strategic complementarities. Thus, although Cournot and Bertrand oligopoly games are basically different in their non-cooperative forms, it appears that their cooperative forms have the same core structure.

Chapter 7, which is based on Driessen, Hou, and Lardon (2010), goes further than Chapter 6 on the study of Bertrand oligopoly TU-games in α and β-characteristic function forms by assuming that the marginal costs are possibly distinct. First, we show that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions. On the one hand, we show that if the intercept of demand is sufficiently small then Bertrand oligopoly TU-games in β-characteristic function form have clear similarities with a well-known notion in statistics called variance with respect to the marginal costs. Although such games fail to be convex unless all the firms operate at an identical marginal cost, we prove that they are nevertheless totally balanced. On the other hand, we prove that if the intercept of demand is sufficiently large then Bertrand oligopoly TU-games in β-characteristic function form are convex, which extends the convexity result in Chapter 6. Finally, we give an appealing expression of the Shapley value for this second game type. We show that the Shapley value is determined by decomposing any Bertrand oligopoly TU-game in β-characteristic function form into the difference between two convex TU-games, besides two additive TU-games. Moreover, we provide an axiomatic characterization of the Shapley value by means of two properties: efficiency and individual monotonicity. Recall that efficiency requires that a solution distributes the worth of the grand coalition among the players. Individual monotonicity stipulates that the difference between the payoffs of two firms is equal to the difference between their individual worth weighted by some real number which depends on their average cost.

1.4 Alternative non-cooperative approaches

Many non-cooperative models have already dealt with the stability of coalition structures in oligopoly situations. It is known that standard strategic oligopoly games fail to explain the formation of cartels in oligopolistic markets. The main reason is that any cartel member has an individual interest in defecting from the collusive agreement. However, non-cooperative game theory has succeeded in providing the theoretical bases that justify the existence of collusive behaviors by means of repeated, strategic and extensive games.

As emphasized by Chamberlin (1929) and Stigler (1964), the maintenance of collusive behaviors, explicit or tacit, requires repeated interaction that allows firms to punish the deviants from the agreement. In the framework of repeated games, if the firms do not discount the future too much, each one do not have any interest in defecting from the collusive agreement because it rationally anticipates future punishments in the periods following its defection. On the basis of the paradigm called the "Folk theorem", Friedman (1971) shows that any profit vector that gives every firm strictly more than the static Nash equilibrium profits can be implemented by a strategy profile which is a subgame perfect Nash equilibrium of an infinitely repeated oligopoly game. This strategy profile specifies that, for any firm's defection, all the firms play the static Nash equilibrium at each of the next periods. For symmetric games, Abreu (1986, 1988) extends Friedman's result by showing that any profit vector which can be enforced by a subgame perfect Nash equilibrium can be implemented by a strategy profile comprising specific punishments with a two-phase structure which is also a subgame perfect Nash equilibrium of an infinitely repeated oligopoly game. Such punishment strategies have a simple two-phase stick-and-carrot structure and specify that, for any firm's defection, there is a one-period punishment (stick) corresponding to the worst possible symmetric subgame perfect Nash equilibrium, and for each of the next periods, firms revert to the best collusive sustainable output (carrot) corresponding to the best symmetric subgame perfect Nash equilibrium. In case another deviation from the best sustainable output (carrot) occurs, firms again revert to the one-period punishment (stick), and so on.

Although repeated oligopoly games provide a possible mechanism for the emergence of a tacit cooperation between firms, this approach does not deal with the stability of coalition structures in oligopoly situations.

D'Aspremont et al. (1983) are the first to analyze the stability of cartels embedded in coalition structures. They define the notions of internal and external stability. A cartel is internally stable if no member has an interest to leave it. A cartel is externally stable if there is no incentive for a new firm to join it. A cartel is stable if it is both internally and externally stable. They prove that there always exists a stable cartel in coalition structures with at most one cartel. In continuation of this work, Donsimoni et al. (1986) consider a model of "price leadership" with linear demand and individual cost functions. They show that there exists a unique stable cartel as long as firms are not too cost-efficient relative to market demand. Otherwise, there exist industry sizes for which two cartels are stable, among which the cartel comprising all the firms. Thoron (1998) shows that D'Aspremont et al.'s approach can be modeled as a strategic game in which firms' strategies have a binary form, i.e. cooperate or not. She proves the existence of a one-to-one correspondence between stable cartels and the Nash equilibria of this strategic game.
Hart and Kurz (1983) propose an original strategic approach incorporating cooperative concepts in order to deal with the stability of coalition structures. In their model, a player's strategy is his choice of the coalition to which he wants to belong. They consider two assumptions on the formation of coalitions: either a coalition forms if and only if all its members have chosen it, or a coalition forms if and only if all its members have chosen the same coalition. These two assumptions endogenize the formation of coalition structures which are the result of players' strategic choices. This induces a cooperative game in partition function form in which players' utilities are evaluated by using a "coalition structure" value, i.e. the Owen value (Owen 1977). Hart and Kurz define a strategic game in which player's strategy is his choice of the coalition to which he wants to belong, and player's utilities are given by the Owen value applied to the game in partition function form described above. They consider two notions of stability, i.e. δ and γ-stabilities, based on the strong Nash equilibrium, each one associated with the two above assumptions on the formation of coalitions. The notion of δ-stability corresponds to the strong Nash equilibrium related to the idea that for any deviation, the members of the coalitions concerned by this deviation remain together, and all the other coalitions remain unchanged. The notion of γ-stability corresponds to the strong Nash equilibrium based on the idea that for any deviation, the coalitions which are left by some members break up into singletons, while the other coalitions remain the same. Hart and Kurz show that there exist strategic games for which any coalition structure is neither

δ-stable nor γ-stable.
In this thesis, by contrast with Hart and Kurz's approach, we adopt a reverse approach in which we use non-cooperative concepts in order to define cooperative oligopoly games.

In the framework of extensive games, Bloch (1996) analyzes a sequential game of coalition formation where the rule of utility division is fixed and utilities depend on the whole coalition structure. In such a game, players are ranked according to an exogenous order. The first player starts the game by proposing the formation of a coalition. If all the potential members accept the proposal, the coalition is formed. Otherwise, the player who rejects the proposal has to propose the formation of a new coalition. Once a coalition is formed, the game is only played among the remaining players. Moreover, the coalitions which have been already formed cannot attract new members nor break apart. A coalition structure is core stable if there does not exist a coalition embedded into another coalition structure whose members receive strictly higher utilities. Bloch shows that any core stable coalition structure can be obtained as the outcome of a stationary perfect equilibrium, provided that the set of stationary perfect equilibria is non-empty. In the same spirit, Bloch (1995) studies the formation of cartels in an oligopoly with linear demand as a two-stage non-cooperative game. While in the first stage firms form cartels in order to decrease their costs, they compete on the market in the second stage. For such oligopoly games, Bloch shows that equilibrium coalition structures, i.e. coalition structures generated by a Markov-perfect equilibrium, are asymmetric and inefficient.
Ray and Vohra (1997) propose another approach in order to deal with the stability of coalition structures in which players are assumed to be farsighted, i.e. players compare their current situation to the ultimate situation induced by their actions and thus disregard the immediate profitability of their choices. As a consequence, when a coalition breaks away from a coalition structure, it takes into consideration that further deviations may occur after its own defection and that other deviating coalitions also apply a similar reasoning. A strategy profile is an equilibrium binding agreement for a coalition structure if no coalition can profitably deviate anticipating the ultimate consequence of its deviation. Hence, stable coalition structures are those supporting an equilibrium binding agreement. Ray and Vohra show that if any individual strategy set is non-empty, compact and convex, and any individual utility function is continuous and quasi-concave, then for any coalition structure there exists an equilibrium binding agreement. Diamantoudi (2003) extends Ray and Vohra's work by allowing a deviating coalition to be fearful of or hopeful about the ultimate consequences of its defection from a coalition structure. She introduces three solution concepts, each one associated with an optimistic, a pessimistic

and a strategic approach, and discusses the relation between them.
We refer to Hart and Mas-Colell (1997) and Ray (2007) for more details on the study of the formation of coalitions by means of non-cooperative games.

Chapter 2

Preliminaries

2.1 Introduction

Game theory studies problems of conflict among decision makers (players) in situations where a player's decision affects the other players. The basic assumption is that players are rational in the sense that they pursue well-defined objectives. It consists of a modeling part and a solution part. As regards the modeling part, the mathematical models can describe both strategic interaction and cooperation. As regards the solution part, the resulting utilities (payoffs) distributed to the players are determined according to certain solution concepts. Traditionally, game theoretical approaches are classified into two branches: non-cooperative and cooperative game theory.

Non-cooperative game theory models strategic interaction situations where players take into account only their own strategic objectives and thus binding agreements among the players are not possible. The emphasis is on players' strategies and the consequences of the strategic interaction on players' utilities. The main purpose of this approach is to make predictions on the "internal" stable outcome, i.e. a situation in which no player or no group of players has an incentive to deviate. The commonly used models are those of strategic and extensive games. The most well-known solution concept in non-cooperative game theory is the so-called Nash equilibrium (Nash 1950b).

While non-cooperative game theory describes the strategic interaction among the players, cooperative game theory ignores this strategic stage and assumes that players can sign binding agreements in order to cooperate. The emphasis is on the possibilities of cooperation among the players and/or the division of coalitional benefits in order to secure a sustainable agreement. In particular, this approach deals with the division of the worth of the grand coalition (all players together as a whole) among the players. The commonly used model in this

approach is that of games with transferable utility. The two most well-known solution are the core (Gillies 1953) and the Shapley value (Shapley 1953).
Although different in their mathematical modeling, both non-cooperative and cooperative approaches can be considered as two complementary ways of dealing with some conflict problems as cooperation (Hart and Mas-Colell 1997), and so there is a close relation between them. Nash (1951) proposes that cooperation among the players can be studied by means of bargaining models (Nash 1950a, Rubinstein 1982) in which the cooperative actions are the result of some bargaining process defined in terms of a similar solution concept to the Nash equilibrium. In this bargaining process, any player behaves according to some bargaining strategy that satisfies the same personal utility maximization criterion as in any other non-cooperative game. Therefore, non-cooperative game theory can be viewed as an instrument in order to obtain the cooperative results analyzed by cooperative game theory.
In the spirit of considering non-cooperative game theory prior to cooperative game theory in order to deal with cooperation, another approach consists in converting a strategic game into a game in characteristic function form (Aumann 1959, Chander and Tulkens 1997) and studying the cooperative game so induced. In order to do such a conversion, it is assumed that some players can sign binding agreements in order to cooperate in the strategic game. Then, by applying an appropriate solution concept allowing partial cooperation to the strategic game, the resulting cooperating players' utilities permit to define the induced cooperative game. The main purpose of this chapter is to introduce such a technical approach in order to apply it in the more specific framework of oligopoly theory.
In order to do that, the remainder of this chapter is structured as follows. In Section 2.2 we define strategic games and some of their solution concepts. Sections 2.3 and 2.4 introduce the theory of TU(Transferable Utility)-games and interval games respectively. Finally, Section 2.5 presents the technical approach which consists in associating cooperative games (TU-games and interval games) with strategic games.

2.2 Strategic games and solution concepts

2.2.1 Basic definitions

A situation in which every agent has to choose a strategy under complete information and obtains an utility depending on other agents' strategies can be described by a strategic game. A **strategic game** is a triplet $\Gamma = (N, (X_i, u_i)_{i \in N})$ defined as:

1. a finite **set of players** $N = \{1, 2, \ldots, n\}$ where 2^N is the **power set** of N;

2. for every $i \in N$, an **individual strategy set** X_i where x_i is a representative element of X_i;

3. for every $i \in N$, an **individual utility function** $u_i : \prod_{i \in N} X_i \longrightarrow \mathbb{R}$ where $\prod_{i \in N} X_i$ is the **set of strategy profiles** denoted by X_N.

We denote by $\mathcal{G}$ the **set of strategic games**.

Let $\mathcal{G}^* \subseteq \mathcal{G}$ be a subset of strategic games. A **solution concept** on $\mathcal{G}^*$ is a function φ which associates with every strategic game $\Gamma = (N, (X_i, u_i)_{i \in N}) \in \mathcal{G}^*$ the (possibly empty) subset of strategy profiles $\varphi(\Gamma) \subseteq X_N$.

2.2.2 Nash equilibrium and partial agreement equilibrium

Given a set of players N, we call a subset $S \in 2^N \backslash \{\emptyset\}$, a **coalition**. The **size** $s = |S|$ of coalition S is the number of players in S. For any coalition $S \in 2^N \backslash \{\emptyset\}$, we denote by $X_S = \prod_{i \in S} X_i$ the set of strategy profiles of players in S where $x_S = (x_i)_{i \in S}$ is a representative element of X_S.
The most well-known solution concept in non-cooperative game theory is the Nash equilibrium (Nash 1950a). Given a strategic game $\Gamma = (N, (X_i, u_i)_{i \in N}) \in \mathcal{G}$, a strategy profile $x^* \in X_N$ is a **Nash equilibrium** if:

$$\forall i \in N, \forall x_i \in X_i, \; u_i(x^*) \geq u_i(x_i, x^*_{N \backslash \{i\}}).$$

The function φ^N is the solution concept which associates with every strategic game $\Gamma = (N, (X_i, u_i)_{i \in N}) \in \mathcal{G}$ the (possibly empty) **set of Nash equilibria** $\varphi^N(\Gamma)$. The existence of a Nash equilibrium in strategic games is the object of a wide literature. We refer to Urai (2010) for an overview of recent developments in this area.

In the next chapters, it is assumed that players can sign binding agreements in order to form coalitions. An appropriate solution concept which permits some players to sign binding agreements in order to form a coalition while the other players pursue their own strategic objectives is the partial agreement equilibrium. The underlying assumption is that coalition members' utilities are transferable inside any coalition so that when a coalition forms, in the spirit of the Nash equilibrium it maximizes the sum of its members' individual utility functions while the other players choose their individual best reply strategies. For any coalition $S \in 2^N \backslash \{\emptyset\}$, the **coalition utility function** $u_S : X_S \times X_{N\backslash S} \longrightarrow \mathbb{R}$ is defined as:

$$u_S(x_S, x_{N\backslash S}) = \sum_{i \in S} u_i(x).$$

Given a coalition $S \in 2^N \backslash \{\emptyset\}$ and a strategic game $\Gamma = (N, (X_i, u_i)_{i \in N}) \in \mathcal{G}$, a strategy profile $(x^*_S, \tilde{x}_{N\backslash S}) \in X_S \times X_{N\backslash S}$ is a **partial agreement equilibrium** under S if:

$$\forall x_S \in X_S,\; u_S(x^*_S, \tilde{x}_{N\backslash S}) \geq u_S(x_S, \tilde{x}_{N\backslash S}),$$

and

$$\forall j \in N\backslash S, \forall x_j \in X_j,\; u_j(x^*_S, \tilde{x}_{N\backslash S}) \geq u_j(x^*_S, \tilde{x}_{N\backslash (S \cup \{j\})}, x_j).$$

We denote by $\varphi^{PA}(\Gamma, S)$ the set of partial agreement equilibria under S. The function φ^{PA} is the solution concept which associates with every strategic game $\Gamma = (N, (X_i, u_i)_{i \in N}) \in \mathcal{G}$ the (possibly empty) **set of partial agreement equilibria** $\varphi^{PA}(\Gamma) = \bigcup_{S \in 2^N \backslash \{\emptyset\}} \varphi^{PA}(\Gamma, S)$. Clearly, it holds that $\varphi^{PA}(\Gamma) \supseteq \varphi^N(\Gamma)$. The existence of a partial agreement equilibrium and its characterization in strategic games are studied in Béal et al. (2010).

2.3 TU-games and solutions

2.3.1 Basic definitions

A situation in which a group of agents obtains certain benefits by cooperation can be described by a cooperative game with transferable utility, or simply a TU-game. A **TU-game** is a pair (N, v) defined as:

1. a finite **set of players** $N = \{1, 2, \ldots, n\}$;
2. a **characteristic function** $v : 2^N \longrightarrow \mathbb{R}$ with the convention that $v(\emptyset) = 0$, which assigns a worth $v(S) \in \mathbb{R}$ to every coalition $S \in 2^N \backslash \{\emptyset\}$.

We denote by G the **set of TU-games**. A TU-game $(N, v) \in G$ is **non-negative** if:

$$\forall S \in 2^N,\; v(S) \geq 0.$$

A TU-game $(N, v) \in G$ is **monotone** if:

$$\forall S \in 2^N, \forall T \in 2^N : S \subseteq T,\; v(S) \leq v(T).$$

Clearly, a monotonic TU-game is non-negative. A TU-game $(N, v) \in G$ is **essential** if:

$$v(N) > \sum_{i \in N} v(\{i\}).$$

Otherwise, $(N, v) \in G$ is **non-essential**.
A TU-game $(N, v) \in G$ is **symmetric** if there exists a mapping $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that:

$$\forall S \in 2^N,\ v(S) = f(s).$$

Two TU-games $(N, v_1) \in G$ and $(N, v_2) \in G$ are **strategically equivalent** if there exist $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}^n$ such that:

$$\forall S \in 2^N,\ v_1(S) = av_2(S) + \sum_{i \in S} b_i.$$

A TU-game $(N, v) \in G$ is **additive** if:

$$\forall S \in 2^N,\ v(S) = \sum_{i \in S} v(\{i\}).$$

A TU-game $(N, v) \in G$ is **superadditive** if:

$$\forall S \in 2^N, \forall T \in 2^N : S \cap T = \emptyset,\ v(S) + v(T) \leq v(S \cup T).$$

A TU-game $(N, v) \in G$ is **convex** (or **supermodular**) if:

$$\forall S \in 2^N, \forall T \in 2^N,\ \ v(S) + v(T) \leq v(S \cup T) + v(S \cap T),$$

or equivalently (Shapley 1971, Topkis 1998):

$$\forall i \in N, \forall j \in N, \forall S \in 2^{N \setminus \{i,j\}},\ \ v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}).$$

A symmetric TU-game $(N, v) \in G$ is convex if:

$$\forall S \in 2^N : s \leq n - 2,\ f(s + 1) - f(s) \leq f(s + 2) - f(s + 1).$$

In a convex TU-game the marginal contribution of a player to some coalition increases if the coalition which he joins becomes larger. A characterization of convex TU-games can be found in Csóka, Herings, and Kóczy (2011). It is straightforward that a convex TU-game is superadditive.

A TU-game $(N, v) \in G$ is **average convex** if:

$$\forall S \in 2^N, \forall T \in 2^N : S \subseteq T,\ \sum_{i \in S} \big(v(S) - v(S \backslash \{i\})\big) \leq \sum_{i \in S} \big(v(T) - v(T \backslash \{i\})\big).$$

Given a TU-game $(N, v) \in G$, the **marginal contributions** to the grand coalition are defined as:

$$\forall i \in N,\ m_i(N, v) = v(N) - v(N \backslash \{i\}).$$

Given a TU-game $(N, v) \in G$, the **gap function** $g^{(N,v)} : 2^N \longrightarrow \mathbb{R}$, with the convention that $g^{(N,v)}(\emptyset) = 0$, is defined as:

$$\forall S \in 2^N \backslash \{\emptyset\},\ g^{(N,v)}(S) = \sum_{i \in S} m_i(N, v) - v(S).$$

A TU-game $(N, v) \in G$ is **1-concave** if:

$$\forall S \in 2^N \backslash \{\emptyset\},\ g^{(N,v)}(S) \leq g^{(N,v)}(N) \leq 0.$$

Given a TU-game $(N, v) \in G$, the **dual game** $(N, v^*) \in G$ of (N, v) is defined as:

$$\forall S \in 2^N,\ v^*(S) = v(N) - v(N \backslash S).$$

Given a TU-game $(N, v) \in G$ and a coalition $S \in 2^N \backslash \{\emptyset\}$, the **subgame** of (N, v) induced by S is a pair $(S, v^S) \in G$ such that:

$$\forall T \in 2^S,\ v^S(T) = v(T).$$

In a TU-game $(N, v) \in G$, every player $i \in N$ may receive a **payoff** $\sigma_i \in \mathbb{R}$. A vector $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$ is a **payoff vector**. A payoff vector $\sigma \in \mathbb{R}^n$ is **individually rational** if for any $i \in N$, $\sigma_i \geq v(\{i\})$, i.e. the payoff vector provides a payoff to every player that is at least as great as its individual worth. A payoff vector $\sigma \in \mathbb{R}^n$ is **acceptable** if for any coalition $S \in 2^N \backslash \{\emptyset\}$, $\sum_{i \in S} \sigma_i \geq v(S)$, i.e. the payoff vector provides a total payoff to the members of coalition S that is at least as great as its worth. A payoff vector $\sigma \in \mathbb{R}^n$ is **efficient** if $\sum_{i \in N} \sigma_i = v(N)$, i.e. the payoff vector provides a total payoff to all the players that is equal to the worth of the grand coalition. The **set of imputations** $I(N, v)$ of a TU-game $(N, v) \in G$ is the set of payoff vectors that are both individually rational and efficient:

$$I(N, v) = \Big\{ \sigma \in \mathbb{R}^n : \forall i \in N,\ \sigma_i \geq v(\{i\}) \text{ and } \sum_{i \in N} \sigma_i = v(N) \Big\}.$$

2.3.2 Basic properties of solutions

Let $G^* \subseteq G$ be a subset of TU-games. A **set-valued solution** on G^* is a function F which associates with every TU-game $(N, v) \in G^*$ the set of payoff vectors $F(N, v) \subseteq \mathbb{R}^n$. A solution F on G^* is **single-valued** if it assigns to every TU-game $(N, v) \in G^*$ a unique payoff vector.
Given a TU-game $(N, v) \in G$, two players $i \in N$ and $j \in N$ are **symmetric** in (N, v) if:

$$\forall S \in 2^{N \setminus \{i,j\}},\ v(S \cup \{i\}) = v(S \cup \{j\}).$$

Clearly, in a symmetric TU-game all the players are symmetric. A player $i \in N$ is a **null player** in (N, v) if:

$$\forall S \in 2^{N \setminus \{i\}},\ v(S \cup \{i\}) = v(S).$$

A player $i \in N$ is a **nullifying player** in (N, v) if:

$$\forall S \in 2^N : i \in S,\ v(S) = 0.$$

For any pair of TU-games $(N, v_1) \in G$, $(N, v_2) \in G$ and any $\alpha \in \mathbb{R}$, the TU-game $(N, \alpha v_1 + v_2) \in G$ is defined as:

$$\forall S \in 2^N,\ (\alpha v_1 + v_2)(S) = \alpha v_1(S) + v_2(S).$$

We recall some well-known properties of solutions. A set-valued solution F on $G^* \subseteq G$ satisfies:

- **non-emptiness** if for any $(N, v) \in G^*$, $F(N, v) \neq \emptyset$;
- **efficiency** if for any $(N, v) \in G^*$ and for any $\sigma \in F(N, v)$, $\sum_{i \in N} \sigma_i = v(N)$;
- **symmetry** if for any $(N, v) \in G^*$, for any $\sigma \in F(N, v)$ and any pair of symmetric players i and j in (N, v), $\sigma_i = \sigma_j$;
- the **null player property** if for any $(N, v) \in G^*$, for any $\sigma \in F(N, v)$ and any null player i in (N, v), $\sigma_i = 0$;
- the **nullifying player property** if for any $(N, v) \in G^*$, for any $\sigma \in F(N, v)$ and any nullifying player i in (N, v), $\sigma_i = 0$.

A single-valued solution F on $G^* \subseteq G$ satisfies:

- **additivity** if for any pair of TU-games $(N, v_1) \in G^*$ and $(N, v_2) \in G^*$ such that $(N, v_1 + v_2) \in G^*$, $F(N, v_1 + v_2) = F(N, v_1) + F(N, v_2)$;

- **linearity** if for any pair of TU-games $(N, v_1) \in G^*$ and $(N, v_2) \in G^*$ and any $\alpha \in \mathbb{R}$ such that $(N, \alpha v_1 + v_2) \in G^*$, $F(N, \alpha v_1 + v_2) = \alpha F(N, v_1) + F(N, v_2)$;

- **collusion neutrality** if for any TU-game $(N, v) \in G^*$, for any $i \in N$ and any $j \in N$, $F_i(N, v_{ij}) + F_j(N, v_{ij}) = F_i(N, v) + F_j(N, v)$ where for any coalition $S \in 2^N \backslash \{\emptyset\}$, it holds that:

$$v_{ij}(S) = \begin{cases} v(S \backslash \{j\}) & \text{if } i \in N \backslash S, \\ v(S \cup \{j\}) & \text{if } i \in S. \end{cases}$$

For a general introduction to the theory of TU-games and their solutions we refer to Peleg and Sudhölter (2003).

2.3.3 The core

The most well-known set-valued solution on G is the core (Gillies 1953). The **core** $C(N, v)$ of a TU-game $(N, v) \in G$ is the set of payoff vectors that are both acceptable and efficient:

$$C(N, v) = \left\{ \sigma \in \mathbb{R}^n : \forall S \in 2^N, \ \sum_{i \in S} \sigma_i \geq v(S) \text{ and } \sum_{i \in N} \sigma_i = v(N) \right\}.$$

A payoff vector in the core is stable in the sense that no coalition can do better and contest this sharing by breaking off from the grand coalition.

The balancedness property is a necessary and sufficient condition which guarantees the non-emptiness of the core (Bondareva 1963, Shapley 1967). This condition can be formulated in two equivalent ways (Owen 1995).
Firstly, given a TU-game $(N, v) \in G$, for any coalition $S \in 2^N \backslash \{\emptyset\}$, $e^S \in \mathbb{R}^n$ is the vector with coordinates equal to 1 in S and equal to 0 outside S. A map $\lambda : 2^N \backslash \{\emptyset\} \longrightarrow \mathbb{R}_+$ is a **balanced map** if $\sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S) e^S = e^N$. A TU-game $(N, v) \in G$ is **balanced** if for every balanced map λ it holds that:

$$\sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S) v(S) \leq v(N).$$

Secondly, given a TU-game $(N, v) \in G$, let $\mathcal{B} \subseteq 2^N \backslash \{\emptyset\}$ be a **family of coalitions** and denote by $\mathcal{B}_i = \{S \in \mathcal{B} : i \in S\}$ the subset of those coalitions of which player i is a member. Then $\mathcal{B}$ is a **balanced family** if for any $S \in \mathcal{B}$ there exists a balancing weight $\delta_S \in \mathbb{R}_+$ such that for any $i \in N$, $\sum_{S\in\mathcal{B}_i} \delta_S = 1$. A TU-game $(N, v) \in G$ is **balanced** if for every balanced family $\mathcal{B} \subseteq 2^N \backslash \{\emptyset\}$ it holds that:

$$\sum_{S\in\mathcal{B}} \delta_S v(S) \leq v(N).$$

Theorem 2.3.1 (Bondareva 1963, Shapley 1967) *A TU-game $(N, v) \in G$ has a non-empty core if and only if it is balanced.*

A TU-game $(N, v) \in G$ is **totally balanced** if for any coalition $S \in 2^N \backslash \{\emptyset\}$, the subgame (S, v^S) is balanced. It is well-known that any convex game is totally balanced, and so balanced (Shapley 1971, Ichiishi 1981).

2.3.4 The Shapley value

The most well-known single-valued solution on G is the Shapley value defined and characterized by Shapley (1953). In order to introduce the Shapley value, we need to define the following notions. Given the set of players N, a **permutation** of N is a bijective function $\tau : N \longrightarrow N$ such that for any $i \in N$, $\tau(i) \in N$ is the player indexed by number i in τ. A permutation of N is a listing of the n players in some specified order. The number of permutations of n players is equal to $n!$. We denote by Π_N the **set of permutations** of N. For any $\tau \in \Pi_N$ and any $i \in N$, the coalition $S^{\tau,i} \in 2^N \backslash \{\emptyset\}$ is the set of players with a smaller index than i in τ:

$$S^{\tau,i} = \big\{ j \in N : \tau^{-1}(j) \leq \tau^{-1}(i) \big\}.$$

Given a TU-game $(N, v) \in G$ and a permutation $\tau \in \Pi_N$, the **marginal vector** $m^\tau(N, v) \in \mathbb{R}^n$ is defined as:

$$\forall i \in N,\ m_i^\tau(N, v) = v(S^{\tau,i}) - v(S^{\tau,i} \backslash \{i\}).$$

Imagine that the players enter a room one by one in the ordering specified by permutation τ. The marginal vector $m^\tau(N, v)$ gives every player his marginal contribution to the coalition formed by his entrance. The **Shapley value** $Sh(N, v)$ of a TU-game $(N, v) \in G$ is the average of all marginal vectors:

$$\forall i \in N,\ Sh_i(N, v) = \frac{1}{n!} \sum_{\tau\in\Pi_N} m_i^\tau(N, v).$$

An alternative definition of the Shapley value is the following. The Shapley value assigns to any TU-game $(N, v) \in G$, the payoff vector $Sh(N, v)$ defined as:

$$\forall i \in N,\ Sh_i(N,v) = \sum_{S \in 2^N : i \in S} \frac{(n-s)!(s-1)!}{n!}\big(v(S) - v(S\backslash\{i\})\big).$$

The Shapley value can be interpreted as the expected marginal contribution of the players. Consider that the n players enter a room in some order and that all these $n!$ orderings are equally likely. Then, for any coalition $S \in 2^N\backslash\{\emptyset\}$ containing player i, the probability that player i enters the room to find precisely the players in $S\backslash\{i\}$ is equal to $((n-s)!(s-1)!)/n!$.

Theorem 2.3.2 (Shapley 1953, Shubik 1962) *The unique single-valued solution satisfying efficiency, additivity, symmetry and the null player property is the Shapley value.*

There are various other characterizations of the Shapley value that can be found in Young (1985), Hart and Mas-Colell (1988), Feltkamp (1995), van den Brink (2001), Hamiache (2001) and Casajus (2009). We refer to Ghintran (2011) for an overview of these characterizations.
The Shapley value is also known as the center of gravity of the Weber set. The **Weber set** is a set-valued solution on G which assigns to any TU-game $(N,v) \in G$ the convex hull of all marginal vectors:

$$W(N,v) = co\big\{m^\tau(N,v) : \tau \in \Pi_N\big\}.$$

It is known that the core is included in the Weber set. Shapley (1971) and Ichiishi (1981) show that a TU-game is convex if and only if the core is equal to the Weber set. Hence, for such games the Shapley value is the center of gravity of the core (Shapley 1971).
Another well-known single-valued solution on G is the equal division solution. The **equal division solution** $ED(N,v)$ of a TU-game $(N,v) \in G$ divides the worth of the grand coalition equally among the players:

$$\forall i \in N,\ ED_i(N,v) = \frac{v(N)}{n}.$$

For the set of symmetric TU-games, the Shapley value and the equal division solution coincide. van den Brink (2007) shows that by replacing the null player property with the nullifying player property in the characterization of the Shapley value yields the characterization of the equal division solution.

2.3.5 The nucleolus

Another single-valued solution on G is the nucleolus. In order to introduce the nucleolus, we need to define the following notions. Given a TU-game $(N, v) \in G$ and a payoff vector $\sigma \in \mathbb{R}^n$, the **excess** of any coalition $S \in 2^N \backslash \{\emptyset\}$ is defined as:

$$e(S, \sigma) = v(S) - \sum_{i \in S} \sigma_i.$$

We define the **excess function** $E : \mathbb{R}^n \longrightarrow \mathbb{R}^{(2^n-2)}$ which associates to any payoff vector $\sigma \in \mathbb{R}^n$ the $(2^n - 2)$-component vector $E(\sigma)$ composed of the excesses of all coalitions $S \in 2^N \backslash \{\emptyset, N\}$ in a non-increasing order, i.e. $E_1(\sigma) \geq E_2(\sigma) \geq \ldots \geq E_{(2^n-2)}(\sigma)$. Moreover, we denote by $\preceq_L$ the **lexicographic order** of vectors. The **nucleolus** $Nuc(N, v)$ of a TU-game $(N, v) \in G$ is the unique imputation which lexicographically minimizes the excess function E over the set of imputations, i.e. $Nuc(N, v) = \sigma$ such that:

$$\sigma \in I(N, v) \text{ and for any } \sigma' \in I(N, v),\ E(\sigma) \preceq_L E(\sigma').$$

The real number $e(S, \sigma)$ is a measure of the dissatisfaction of coalition S at the payoff vector σ. Hence, the vector $E(\sigma)$ orders the complaints of the coalitions according to their magnitude, i.e. the highest complaint first, the second-highest second, and so on. Thus, the nucleolus minimizes the dissatisfactions of the various coalitions according to the lexicographical order $\preceq_L$.
On the set of balanced TU-games, it is known that the nucleolus always belongs to the core. Moreover, Driessen et al. (2010) show that on the set of 1-concave TU-games, the nucleolus coincides with the center of gravity of the core.

2.4 Interval games

In some economic situations, the utilities obtained by a group of agents who cooperate are not known with certainty, and so we cannot use the theory of TU-games which assigns a unique worth to every coalition in order to study cooperative issues. This is why we consider the more general approach of interval games introduced by Branzei et al. (2003). This approach takes into account the uncertainty on the utilities of any group of players by associating with every coalition a closed and bounded real interval which represents all the potential utilities obtained by cooperation.

2.4.1 Basic definitions

A situation in which a group of agents knows with certainty only the lower and upper bounds of all the potential utilities obtained by cooperation can be described by an interval game. We denote by $I(\mathbb{R})$ the **set of all closed and bounded real intervals** $[a, b]$ such that $a \in \mathbb{R}$ and $b \in \mathbb{R}$. An **interval game** is a pair (N, w) defined as:

1. a finite **set of players** $N = \{1, 2, \ldots, n\}$;
2. a **set function** $w : 2^N \longrightarrow I(\mathbb{R})$, with the convention that $w(\emptyset) = [0, 0]$, which assigns a **worth interval** $w(S) \in I(\mathbb{R})$ to every coalition $S \in 2^N \backslash \{\emptyset\}$.

The worth interval $w(S)$ is denoted by $[\underline{w}(S), \overline{w}(S)]$ where $\underline{w}(S)$ and $\overline{w}(S)$ are the lower and the upper bounds of $w(S)$ respectively. We denote by IG the **set of interval games**. If any worth interval of an interval game $(N, w) \in IG$ is degenerate then (N, w) corresponds to the TU-game $(N, v) \in G$ where $v = \underline{w} = \overline{w}$. In this sense, the set of TU-games G is included in the set of interval games IG. The properties and solutions defined in Section 2.3 for the set of TU-games can be extended to the set of interval games by using the methods of interval arithmetic (Moore 1979). Let $J, K \in I(\mathbb{R})$ where $J = [\underline{j}, \overline{j}]$ and $K = [\underline{k}, \overline{k}]$. Then, we have $J + K = [\underline{j} + \underline{k}, \overline{j} + \overline{k}]$ and we say that J is **weakly better** than K, which we denote $J \succcurlyeq K$, if $\underline{j} \geq \underline{k}$ and $\overline{j} \geq \overline{k}$. For example, we have $[2, 3] \succcurlyeq [1, 2]$.

We denote by $I(\mathbb{R})^n$ the **set of n-dimensional interval vectors** where I is a representative element of $I(\mathbb{R})^n$. In an interval game $(N, w) \in IG$, every player $i \in N$ may receive a **payoff interval** $I_i \in I(\mathbb{R})$. An interval vector $I = (I_1, \ldots, I_n)$ is a **payoff interval vector**. An interval vector $I \in I(\mathbb{R})^n$ is **individually rational** if for any $i \in N$, $I_i \succcurlyeq w(\{i\})$, i.e. the payoff interval vector provides a payoff interval to every player that is weakly better than its individual worth interval. A payoff interval vector $I \in I(\mathbb{R})^n$ is **acceptable** if for any coalition $S \in 2^N \backslash \{\emptyset\}$, $\sum_{i \in S} I_i \succcurlyeq w(S)$, i.e. the payoff interval vector provides a total payoff interval to the members of coalition S that is weakly better than its worth interval. A payoff interval vector $I \in I(\mathbb{R})^n$ is **efficient** if $\sum_{i \in N} I_i = w(N)$, i.e. the payoff interval vector provides a total payoff interval to all the players that is equal to the worth interval of the grand coalition. The **set of imputations** $\mathcal{I}(N, w)$ of an interval game $(N, w) \in IG$ is the set of interval vectors that are both individually rational and efficient:

$$\mathcal{I}(N, w) = \left\{ I \in I(\mathbb{R})^n : \forall i \in N,\ I_i \succcurlyeq w(\{i\}) \text{ and } \sum_{i \in N} I_i = w(N) \right\}.$$

2.4.2 The cores

There are two main ways of generalizing the definition of a set-valued solution for the set of interval games. Let $IG^* \subseteq IG$ be a subset of interval games. A **set-valued solution** on IG^* is a function F which associates with every interval game $(N,w) \in IG^*$ either the set of payoff interval vectors $F(N,w) \subseteq I(\mathbb{R})^n$ or the set of payoff vectors $F(N,w) \subseteq \mathbb{R}^n$. We use each of these two generalizations in order to define the cores on the set of interval games.
The first extension of the core is due to Alparslan-Gok et al. (2008a). The **interval core** $\mathcal{C}(N,w)$ of an interval game $(N,w) \in IG$ is the set of all payoff interval vectors that are both acceptable and efficient:

$$\mathcal{C}(N,w) = \left\{ I \in I(\mathbb{R})^n : \forall S \in 2^N,\ \sum_{i \in S} I_i \succcurlyeq w(S) \text{ and } \sum_{i \in N} I_i = w(N) \right\}.$$

A payoff interval vector in the interval core is stable in the sense that no coalition can do better and contest this sharing by breaking off from the grand coalition.

While the strong-balancedness property is a sufficient condition in order to guarantee the non-emptiness of the interval core, the $\mathcal{I}$-balancedness property is a necessary and sufficient condition (Alparslan-Gok et al. 2008b). An interval game $(N,w) \in IG$ is **strongly balanced** if for every balanced map λ it holds that:

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\overline{w}(S) \leq \underline{w}(N).$$

An interval game $(N,w) \in IG$ is **$\mathcal{I}$-balanced** if for every balanced map λ it holds that:

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)w(S) \preccurlyeq w(N).$$

If all the worth intervals are degenerate both properties coincide with the balancedness property on the set of TU-games. Clearly, the strong-balancedness property is easier to verify than the $\mathcal{I}$-balancedness property.

Theorem 2.4.1 (Alparslan-Gok et al. 2008b)

(i) If an interval game $(N,w) \in IG$ is strongly balanced then it is $\mathcal{I}$-balanced;

(ii) An interval game $(N,w) \in IG$ has a non-empty interval core if and only if it is $\mathcal{I}$-balanced.

The second extension of the core is due to Alparslan-Gok et al. (2009b). Given an interval game $(N, w) \in IG$, a TU-game $(N, v) \in G$ is called a **selection** of (N, w) if for any coalition $S \in 2^N \backslash \{\emptyset\}$, $v(S) \in w(S)$. We denote by $Sel(N, w)$ the **set of all selections** of $(N, w) \in IG$. The **standard core** $C(N, w)$ of an interval game $(N, w) \in IG$ is defined as the union of the cores of all its selections $(N, v) \in G$:

$$C(N, w) = \bigcup_{(N,v)\in Sel(N,w)} C(N, v).$$

We use the term "standard core" instead of the term "core" in order to distinguish this core solution for interval games with the core solution for TU-games. The standard core $C(N, w)$ is non-empty if and only if there exists a balanced TU-game $(N, v) \in Sel(N, w)$.
We refer to Alparslan-Gok et al. (2009a) for an overview of recent developments in the theory of interval games.

2.5 Construction of cooperative games

When externalities are present such as in oligopoly situations, in order to calculate the worth of a coalition we have to specify the strategies taken by the non-members according to some rules, called blocking rules. These blocking rules are implemented by applying appropriate solution concepts to the strategic games allowing binding agreements among the coalition members and specifying outsiders' behavior. The resulting cooperating players' utilities will determine the worth of the coalition, and so permits to define the induced cooperative game. In this section, we associate two cooperative game types, i.e. TU-games and interval games, with strategic games by following three approaches. While the functions obtained from the first two approaches (Aumann 1959) are called the α and β-characteristic functions respectively, the functions obtained from the third approach (Chander and Tulkens 1997) are called the γ-characteristic and γ-set functions.

2.5.1 The α and β-characteristic functions

Aumann (1959) proposes two approaches of converting a non-cooperative game into a cooperative game. According to the first, called the α-approach, every coalition computes the total utility which it can guarantee itself regardless of what outsiders do. The second approach, called the β-approach, consists in computing the minimal utility for which outsiders can prevent the coalition members from getting more. Given a strategic game $\Gamma = (N, (X_i, u_i)_{i\in N}) \in$

$\mathcal{G}$, the associated **TU-games in α and β-characteristic function forms**, denoted by (N, v_α) and (N, v_β), are defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$v_\alpha(S) = \max_{x_S \in X_S} \min_{x_{N\backslash S} \in X_{N\backslash S}} u_S(x_S, x_{N\backslash S}),$$

and

$$v_\beta(S) = \min_{x_{N\backslash S} \in X_{N\backslash S}} \max_{x_S \in X_S} u_S(x_S, x_{N\backslash S}).$$

It is known that for any coalition $S \in 2^N \backslash \{\emptyset\}$, $v_\alpha(S) \leq v_\beta(S)$ so that $C(N, v_\beta) \subseteq C(N, v_\alpha)$. In Chapters 6 and 7, for any strategic Bertrand oligopoly game we assume that the demand and individual cost functions are linear in order to guarantee that the characteristic functions v_α and v_β are well-defined and lead to the same set of Bertrand oligopoly TU-games.

2.5.2 The γ-characteristic and γ-set functions

Chander and Tulkens (1997) question the resorting to the α and β-characteristic functions in order to derive cooperative games from non-cooperative games. For many economic situations, they sustain that outsiders may cause substantial damages upon themselves by minimizing the utility of a coalition. For instance, this is the case in the oligopolistic market with three firms described in the introduction. A similar argument is developed by Rosenthal (1971). They propose a more credible blocking rule, called the γ-approach, where players outside the coalition choose their strategy individually as a best reply to the coalitional action, which corresponds to the definition of a partial agreement equilibrium. Given a strategic game $\Gamma = (N, (X_i, u_i)_{i \in N}) \in \mathcal{G}$, the associated **TU-game in γ-characteristic function form**, denoted by (N, v_γ), is defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$v_\gamma(S) = u_S(x^*_S, \tilde{x}_{N\backslash S}),$$

where $(x^*_S, \tilde{x}_{N\backslash S}) \in \varphi^{PA}(\Gamma, S)$. It can be easily proved that for any coalition $S \in 2^N \backslash \{\emptyset\}$, $v_\beta(S) \leq v_\gamma(S)$ so that $C(N, v_\gamma) \subseteq C(N, v_\beta)$. In Chapter 3, for any strategic Cournot oligopoly game we assume that the inverse demand function is differentiable, strictly decreasing and concave, and any individual cost function is continuous, strictly increasing and convex, in order to guarantee that the characteristic function v_γ is well-defined. In Chapter 5, for any strategic Stackelberg oligopoly game we assume that the inverse demand and individual cost functions are linear and firms have no capacity constraint, in order to ensure that the characteristic function v_γ is well-defined.

In some economic situations, for some coalition the set of partial agreement equilibria does not support an unique worth, and so the characteristic function v_γ is not well-defined. For instance, for some Cournot oligopoly situations in which the inverse demand function is continuous but not necessarily differentiable, the worth of any coalition is not unique, and so we cannot define a Cournot oligopoly TU-game in γ-characteristic function form (Example 3.3.4). A more general approach consists in associating a closed and bounded real interval with the set of partial agreement equilibria. Given a strategic game $\Gamma = (N, (X_i, u_i)_{i \in N}) \in \mathcal{G}$, the associated **interval game in γ-set function form**, denoted by (N, w_γ), is defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$w_\gamma(S) = u_S(\varphi^{PA}(\Gamma, S)).$$

Recall that $w_\gamma(S) = [\underline{w}_\gamma(S), \overline{w}_\gamma(S)]$ where $\underline{w}_\gamma(S)$ and $\overline{w}_\gamma(S)$ are the minimal and the maximal utilities of coalition S enforced by $\varphi^{PA}(\Gamma, S)$ respectively. It is straightforward that for any TU-game $(N, v_\gamma) \in Sel(N, w_\gamma)$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$, $v_\beta(S) \leq v_\gamma(S)$. In Chapter 4, for any strategic Cournot oligopoly game we assume that the inverse demand function is continuous (but not necessarily differentiable), strictly decreasing and concave, and any individual cost function is continuous, strictly increasing and convex, in order to guarantee that the set function w_γ is well-defined.

Part I

Quantity competition

Chapter 3

The core in Cournot oligopoly TU-games with capacity constraints

3.1 Introduction

A Cournot oligopoly TU-game describes a quantity competition in which a group of firms obtains certain profits by cooperation. In this chapter, which is based on Lardon (2009), we study the core of Cournot oligopoly TU-games in γ-characteristic function form. We assume that the inverse demand function is differentiable, strictly decreasing and concave, and any individual cost function is continuous, strictly increasing and convex, in order to guarantee that the γ-characteristic function is well-defined and study some properties on the equilibrium outputs. To this end, we adopt a more general approach where any coalition structure can occur, in particular those supporting the partial agreement equilibria. Given a coalition structure, we construct an aggregated strategic Cournot oligopoly game in which a Nash equilibrium represents the aggregated equilibrium outputs of the embedded coalitions. We prove that such games admit a unique Nash equilibrium and conclude that the γ-characteristic function is well-defined. When some coalitions merge, we show that the total production of the new entities decreases while the other coalitions respond by increasing their outputs so that the equilibrium total production decreases.
We use these preliminaries results in order to obtain two core non-emptiness results. The first result shows that if the inverse demand function is differentiable and if any individual profit function is continuous and concave on the set of strategy profiles then the corresponding Cournot oligopoly TU-game in γ-characteristic function form is balanced. This result extends Zhao's core non-

emptiness result (1999b) since the core associated with the γ-characteristic function is included in the one associated with the β-characteristic function. However, a drawback with this approach is that it does not point out any payoff vector in the core. The second result provides a new single-valued solution in the core, called the NP(Nash pro rata) value, on the subset of Cournot oligopoly TU-games in γ-characteristic function form with linear individual cost functions and possibly distinct capacity constraints. The NP value distributes to every firm the worth of the grand coalition in proportion to its Nash individual output. Since the capacity constraints are possibly distinct, this result generalizes Funaki and Yamato's core non-emptiness result (1999) established with no capacity constraint insofar as a Cournot oligopoly situation also describes a common pool situation (Moulin 1997). We characterize the NP value by means of four properties: efficiency, null firm, monotonicity and non-cooperative fairness. Recall that efficiency requires that a solution distributes the worth of the grand coalition among the firms. The null firm property stipulates that a firm with no production capacity obtains a zero payoff. Monotonicity specifies that if a firm has a production capacity greater than or equal to the production capacity of another firm, then former's payoff will be greater than or equal to latter's payoff. Non-cooperative fairness requires that a single-valued solution distributes to every firm a payoff proportionally to its individual profit in the finest coalition structure. As far as we know, in oligopoly theory, this is the first result that characterizes a single-valued solution belonging to the core by means of appealing properties. Furthermore, we provide a linear Cournot oligopoly situation for which the corresponding Cournot oligopoly TU-game in γ-characteristic function form fails to be superadditive, and so convex. This proves that Norde et al.'s result (2002) establishing the convexity property for Cournot oligopoly TU-games in α and β-characteristic function forms cannot be extended to this set of games.

The remainder of this chapter is structured as follows. In section 3.2 we introduce the model and some notations. Section 3.3 presents some properties of the equilibrium outputs. Section 3.4 establishes our first core non-emptiness result by means of balancedness property. Section 3.5 shows that the NP value belongs to the core and provides an axiomatic characterization of this solution. Section 3.6 gives some concluding remarks.

3.2 The model

A **Cournot oligopoly situation** is a quadruplet $(N, (q_i, C_i)_{i \in N}, p)$ defined as:

1. a finite **set of firms** $N = \{1, 2, \ldots, n\}$;

2. for every $i \in N$, a **capacity constraint** $q_i \in \mathbb{R}_+$;
3. for every $i \in N$, an **individual cost function** $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$;
4. an **inverse demand function** $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which assigns to any aggregate quantity $X \in \mathbb{R}_+$ the unit price $p(X)$.

Throughout this chapter, we assume that:

(a) the inverse demand function p is differentiable, strictly decreasing and concave;

(b) every individual cost function C_i is continuous, strictly increasing and convex.

The **strategic Cournot oligopoly game** associated with the Cournot oligopoly situation $(N, (q_i, C_i)_{i \in N}, p)$ is a triplet $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N})$ defined as:

1. a finite **set of firms** $N = \{1, 2, \ldots, n\}$;
2. for every $i \in N$, an **individual strategy set** $X_i = [0, q_i] \subset \mathbb{R}_+$ where $x_i \in X_i$ represents the quantity produced by firm i;
3. for every $i \in N$, an **individual profit function** $\pi_i : X_N \longrightarrow \mathbb{R}_+$ defined as:

$$\pi_i(x) = p(X)x_i - C_i(x_i),$$

   where $X = \sum_{i \in N} x_i$ is the **total production**.

Note that firm i's profit depends on its individual output x_i and on the total output of its opponents $\sum_{j \in N \setminus \{i\}} x_j$. We denote by $\mathcal{G}_{co} \subseteq \mathcal{G}$ the **set of strategic Cournot oligopoly games**.

Now, we associate Cournot oligopoly TU-games in γ-characteristic function form with strategic Cournot oligopoly games. For any coalition $S \in 2^N \setminus \{\emptyset\}$, the **coalition profit function** $\pi_S : X_S \times X_{N \setminus S} \longrightarrow \mathbb{R}$ is defined as:

$$\pi_S(x_S, x_{N \setminus S}) = \sum_{i \in S} \pi_i(x).$$

As discussed in the introduction, the α and β-approaches (Aumann 1959) used to define Cournot oligopoly TU-games can be questioned insofar as the reaction of external firms to minimize the worth of a deviating coalition by increasing their output at full capacity probably implies substantial damages upon themselves. This is why, in this chapter, we consider the γ-approach (Chander and Tulkens 1997) where outsiders choose their strategy individually as a best reply facing the deviating coalition. Given a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$, the associated **Cournot oligopoly TU-game in γ-characteristic function form**, denoted by (N, v_γ), is defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$v_\gamma(S) = \pi_S(x_S^*, \tilde{x}_{N \backslash S}),$$

where $(x_S^*, \tilde{x}_{N \backslash S}) \in \varphi^{PA}(\Gamma_{co}, S)$. We denote by $G_{co}^\gamma \subseteq G$ the **set of Cournot oligopoly TU-games in γ-characteristic function form**. In the following section, we show that under assumptions (a) and (b), it is possible to define a Cournot oligopoly TU-game in γ-characteristic function form.

3.3 Properties of the equilibrium outputs

In this section, we study the existence of partial agreement equilibria in strategic Cournot oligopoly games. This will permit us to make sure that the γ-characteristic function is well-defined. Moreover, we analyze the variations of the equilibrium outputs enforced by the partial agreement equilibria. To this end, we adopt a more general approach in which any coalition structure can occur. Given a strategic Cournot oligopoly game and a coalition structure, we define an aggregated strategic Cournot oligopoly game in which a Nash equilibrium represents the aggregated equilibrium outputs of the embedded coalitions.

Given a set of firms $N = \{1, 2, \ldots, n\}$, a **coalition structure** $\mathcal{P}$ is a partition of N, i.e. $\mathcal{P} = \{S_1, \ldots, S_k\}$, $k \in \{1, \ldots, n\}$. An element of a coalition structure, $S \in \mathcal{P}$, is called an **admissible coalition** in $\mathcal{P}$. We denote by $\mathbf{P_N}$ the **set of coalition structures** with player set N.
Given a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$ and a coalition structure $\mathcal{P} \in \mathbf{P_N}$, we say that a strategy profile $(\hat{x}_S)_{S \in \mathcal{P}} \in \prod_{S \in \mathcal{P}} X_S$ is an **equilibrium under** $\mathcal{P}$ if:

$$\forall S \in \mathcal{P}, \forall x_S \in X_S,\ \pi_S(\hat{x}_S, \hat{x}_{N \backslash S}) \geq \pi_S(x_S, \hat{x}_{N \backslash S}).$$

We denote by $\varphi^E(\Gamma_{co}, \mathcal{P})$ the set of equilibria under $\mathcal{P}$. The function φ^E is the solution concept which associates with every strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$ the (possibly empty) **set of equilibria** $\varphi^E(\Gamma_{co}) = \bigcup_{\mathcal{P} \in \mathbf{P_N}} \varphi^E(\Gamma_{co}, \mathcal{P})$. For any coalition $S \in 2^N \backslash \{\emptyset\}$, we denote by

$\mathcal{P}^S = \{S\} \cup \{\{i\} : i \in N \backslash S\}$. Clearly, it holds that $\varphi^E(\Gamma_{co}, \mathcal{P}^S) = \varphi^{PA}(\Gamma_{co}, S)$.

Now, we deal with the problem of the existence of an equilibrium under any coalition structure. To this end, we define an aggregated strategic Cournot oligopoly game. The **aggregated strategic Cournot oligopoly game** associated with a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ and a coalition structure $\mathcal{P} \in \mathbf{P_N}$ is a triplet $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ defined as:

1. a **set of cartels** (or admissible coalitions) $\mathcal{P} = \{S_1, \ldots, S_k\}$;
2. for every $S \in \mathcal{P}$, an **aggregated coalition strategy set** $X^S = [0, \sum_{i\in S} q_i]$ where $x^S = \sum_{i\in S} x_i \in X^S$ represents the aggregated quantity produced by coalition S;
3. for every $S \in \mathcal{P}$, an **aggregated coalition cost function** $C_S : X^S \longrightarrow \mathbb{R}_+$ defined as:

$$C_S(x^S) = \min_{x_S \in I(x^S)} \sum_{i\in S} C_i(x_i),$$

where $I(x^S) = \{x_S \in X_S : \sum_{i\in S} x_i = x^S\}$ is the set of strategies of coalition S that permit it to produce the quantity x^S; for every $S \in \mathcal{P}$, an **aggregated coalition profit function** $\pi^S : \prod_{S\in\mathcal{P}} X^S \longrightarrow \mathbb{R}$ defined as:

$$\pi^S(x^{\mathcal{P}}) = p(X)x^S - C_S(x^S).$$

We denote by $X^{\mathcal{P}} = \prod_{S\in\mathcal{P}} X^S$ the **set of strategy profiles** and for any admissible coalition $S \in \mathcal{P}$, we denote by $X^{N\backslash S} = \prod_{T\in\mathcal{P}\backslash\{S\}} X^T$ the **set of outsiders' strategy profiles** where $x^{\mathcal{P}}$ and $x^{N\backslash S}$ are the representative elements of $X^{\mathcal{P}}$ and $X^{N\backslash S}$ respectively.

The following proposition establishes the existence of an equilibrium under any coalition structure.

Proposition 3.3.1 *Let* $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ *be a strategic Cournot oligopoly game. Then, for any coalition structure* $\mathcal{P} \in \mathbf{P}_N$*, there exists an equilibrium under* $\mathcal{P}$.

Proof: Let $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ be the aggregated strategic Cournot oligopoly game associated with the strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ and the coalition structure $\mathcal{P} \in \mathbf{P_N}$. We proceed in two

parts.
First, we show that there exists a strategy profile $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma^{\mathcal{P}}_{co})$ if and only if there exists a strategy profile $(\hat{x}_S)_{S\in\mathcal{P}} \in \varphi^E(\Gamma_{co},\mathcal{P})$ such that for any $S \in \mathcal{P}$, $\hat{x}_S \in I(\hat{x}^S)$.
[$\Longrightarrow$]: let $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma^{\mathcal{P}}_{co})$. By the definition of $\Gamma^{\mathcal{P}}_{co}$, for any $S \in \mathcal{P}$ there exists $\hat{x}_S \in X_S$ such that:

$$\sum_{i\in S} \hat{x}_i = \hat{x}^S \text{ and } \sum_{i\in S} C_i(\hat{x}_i) = C_S(\hat{x}^S) \tag{3.1}$$

For the sake of contradiction suppose that $(\hat{x}_S)_{S\in\mathcal{P}} \notin \varphi^E(\Gamma_{co},\mathcal{P})$. It follows that for some $S \in \mathcal{P}$, there exists $\check{x}_S \in X_S$ such that:

$$\pi_S(\hat{x}_S, \hat{x}_{N\setminus S}) < \pi_S(\check{x}_S, \hat{x}_{N\setminus S}) \tag{3.2}$$

We denote by $\check{x}^S \in X^S$ the corresponding strategy of coalition S such that $\check{x}^S = \sum_{i\in S} \check{x}_i$. By (3.1) and (3.2), it holds that:

$$\begin{aligned}
\pi^S(\hat{x}^{\mathcal{P}}) &= p(\hat{X})\hat{x}^S - C_S(\hat{x}^S) \\
&= p(\hat{X})\sum_{i\in S}\hat{x}_i - \sum_{i\in S} C_i(\hat{x}_i) \\
&= \pi_S(\hat{x}_S, \hat{x}_{N\setminus S}) \\
&< \pi_S(\check{x}_S, \hat{x}_{N\setminus S}) \\
&= p\left(\sum_{i\in S}\check{x}_i + \sum_{i\in N\setminus S}\hat{x}_i\right)\sum_{i\in S}\check{x}_i - \sum_{i\in S} C_i(\check{x}_i) \\
&\leq p(\check{x}^S + \hat{X} - \hat{x}^S)\check{x}^S - C_S(\check{x}^S) \\
&= \pi^S(\check{x}^S, \hat{x}^{N\setminus S}),
\end{aligned}$$

a contradiction with $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma^{\mathcal{P}}_{co})$.
[$\Longleftarrow$]: let $(\hat{x}_S)_{S\in\mathcal{P}} \in \varphi^E(\Gamma_{co},\mathcal{P})$. We denote by $\hat{x}^{\mathcal{P}} \in X^{\mathcal{P}}$ the strategy profile such that:

$$\forall S \in \mathcal{P},\ \hat{x}^S = \sum_{i\in S}\hat{x}_i \tag{3.3}$$

For any $S \in \mathcal{P}$, it follows from $\hat{x}_S \in I(\hat{x}^S)$ and $(\hat{x}_S)_{S\in\mathcal{P}} \in \varphi^E(\Gamma_{co},\mathcal{P})$ that $\sum_{i\in S} C_i(\hat{x}_i) = C_S(\hat{x}^S)$. For the sake of contradiction suppose that $\hat{x}^{\mathcal{P}} \notin \varphi^N(\Gamma^{\mathcal{P}}_{co})$. It follows that for some $S \in \mathcal{P}$, there exists $\check{x}^S \in X^S$ such that:

$$\pi^S(\hat{x}^S, \hat{x}^{N\setminus S}) < \pi^S(\check{x}^S, \hat{x}^{N\setminus S}) \tag{3.4}$$

By the definition of $\Gamma_{co}^{\mathcal{P}}$, there exists $\check{x}_S \in X_S$ such that:

$$\sum_{i \in S} \check{x}_i = \check{x}^S \text{ and } \sum_{i \in S} C_i(\check{x}_i) = C_S(\check{x}^S) \tag{3.5}$$

By (3.3), (3.4) and (3.5), it holds that:

$$\begin{aligned}
\pi_S(\hat{x}_S, \hat{x}_{N\setminus S}) &= p(\hat{X}) \sum_{i \in S} \hat{x}_i - \sum_{i \in S} C_i(\hat{x}_i) \\
&= p(\hat{X})\hat{x}^S - C_S(\hat{x}^S) \\
&= \pi^S(\hat{x}^S, \hat{x}^{N\setminus S}) \\
&< \pi^S(\check{x}^S, \hat{x}^{N\setminus S}) \\
&= p(\check{x}^S + \hat{X} - \hat{x}^S)\check{x}^S - C_S(\check{x}^S) \\
&= p\left(\sum_{i \in S} \check{x}_i + \sum_{i \in N\setminus S} \hat{x}_i\right) \sum_{i \in S} \check{x}_i - \sum_{i \in S} C_i(\check{x}_i) \\
&= \pi_S(\check{x}_S, \hat{x}_{N\setminus S}),
\end{aligned}$$

a contradiction with $(\hat{x}_S)_{S \in \mathcal{P}} \in \varphi^E(\Gamma_{co}, \mathcal{P})$.
Then, we show that the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P}}$ admits a unique Nash equilibrium. For any admissible coalition $S \in \mathcal{P}$, the aggregated coalition strategy set X^S is compact and convex and the aggregated coalition cost function C_S is continuous, strictly increasing and convex. The properties of the aggregated coalition cost function C_S follow from the continuity, the strict monotonicity and the convexity of any individual cost function C_i. Moreover, the inverse demand function p is differentiable, strictly decreasing and concave. It follows from Theorem 3.3.3 (page 30) in Okuguchi and Szidarovszky (1990) that the game $\Gamma_{co}^{\mathcal{P}} \in \mathcal{G}_{co}$ admits a unique Nash equilibrium. From the first part of the proof, we conclude that the game $\Gamma_{co} \in \mathcal{G}_{co}$ admits an equilibrium under $\mathcal{P}$.■

We deduce from Proposition 3.3.1 the following corollary.

Corollary 3.3.2 *Let $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$ be a strategic Cournot oligopoly game. Then, for any coalition $S \in 2^N \setminus \{\emptyset\}$ it holds that:*

(i) there exists a partial agreement equilibrium under S.

(ii) for any $(x_S^*, \tilde{x}_{N\setminus S}) \in \varphi^{PA}(\Gamma_{co}, S)$ and any $(y_S^*, \tilde{x}_{N\setminus S}) \in \varphi^{PA}(\Gamma_{co}, S)$, it holds that:

$$\sum_{i \in S} x_i^* = \sum_{i \in S} y_i^* \text{ and } \sum_{i \in S} C_i(x_i^*) = \sum_{i \in S} C_i(y_i^*).$$

Point *(i)* is a direct consequence of the equality $\varphi^{PA}(\Gamma_{co}, S) = \varphi^E(\Gamma_{co}, \mathcal{P}^S)$. Point *(ii)* follows from the uniqueness of the Nash equilibrium in the game $\Gamma_{co}^{\mathcal{P}^S} \in \mathcal{G}_{co}$. This stems from the fact that the members of coalition S can reallocate the total production among themselves. Moreover, point *(ii)* remains valid for any coalition structure $\mathcal{P} \in \mathbf{P_N}$, i.e. for any $(\hat{x}_S)_{S\in\mathcal{P}} \in \varphi^E(\Gamma_{co}, \mathcal{P})$ and any $(\check{x}_S)_{S\in\mathcal{P}} \in \varphi^E(\Gamma_{co}, \mathcal{P})$, it holds that:

$$\forall S \in \mathcal{P},\ \hat{x}^S = \check{x}^S \text{ and } \sum_{i\in S} C_i(\hat{x}_i) = \sum_{i\in S} C_i(\check{x}_i).$$

Hence, we deduce from Corollary 3.3.2 the following corollary.

Corollary 3.3.3 *Let $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ be a strategic Cournot oligopoly game. Then, for any coalition $S \in 2^N\backslash\{\emptyset\}$ the worth $v_\gamma(S)$ is unique so that the γ-characteristic function is well-defined.*

The following example shows that the continuity of the inverse demand function does not guarantee that the γ-characteristic function is well-defined. This explains why we have assumed the differentiability of the inverse demand function.

Example 3.3.4

Consider the Cournot oligopoly TU-game $(N, v_\gamma) \in G_{co}^\gamma$ associated with the Cournot oligopoly situation $(N, (q_i, C_i)_{i\in N}, p)$ taken from Norde et al. (2002) where $N = \{1, 2, 3\}$, $q_1 = 2$, $q_2 = 1$, $q_3 = 2$, $C_1(x_1) = 97x_1$, $C_2(x_2) = 98x_2$, $C_3(x_3) = 98x_3$, and the inverse demand function is defined as:

$$p(X) = \begin{cases} 103 - X & \text{if } 0 \le X \le 3, \\ 50(5 - X) & \text{if } 3 < X \le 5. \end{cases}$$

Clearly, p is continuous, piecewise linear and concave but it is not differentiable. Assume that coalition $\{2, 3\}$ forms. We show that any strategy profile $x \in X_N$ such that *(i)* $X = 3$ and *(ii)* $x_2+x_3 \in [1/25, 2]$ is a partial agreement equilibrium under $\{2, 3\}$. Let $x \in X_N$ satisfying *(i)* and *(ii)*. By *(i)* it holds that:

$$\pi_1(x) = 3x_1,$$

and

$$\pi_{\{2,3\}}(x) = 2(x_2 + x_3).$$

If firm 1 increases its output by $\epsilon \in]0, 2 - x_1]$, its new profit will be:

$$\pi_1(x_1+\epsilon, x_2, x_3) = (3-50\epsilon)(x_1+\epsilon).$$

Conversely, if it decides to decrease its output by $\delta \in]0, x_1]$, it will obtain:

$$\pi_1(x_1-\delta, x_2, x_3) = (3+\delta)(x_1-\delta).$$

Similarly, if coalition $\{2,3\}$ increases its output by $\epsilon+\epsilon' \in]0, 3-x_2-x_3]$ where $\epsilon \in [0, 1-x_2]$ and $\epsilon' \in [0, 2-x_3]$, its new coalition profit will be:

$$\pi_{\{2,3\}}(x_1, x_2+\epsilon, x_3+\epsilon') = \big(2-50(\epsilon+\epsilon')\big)(x_2+x_3+\epsilon+\epsilon').$$

On the contrary, if it decreases its output by $\delta+\delta' \in]0, x_2+x_3]$ where $\delta \in [0, x_2]$ and $\delta' \in [0, x_3]$, it will obtain:

$$\pi_{\{2,3\}}(x_1, x_2-\delta, x_3-\delta') = (2+\delta+\delta')(x_2+x_3-\delta-\delta').$$

In all cases, given *(ii)* neither firm 1 nor coalition $\{2,3\}$ can improve their profit. We conclude that any strategy profile $x \in X_N$ satisfying *(i)* and *(ii)* is a partial agreement equilibrium under $\{2,3\}$. It follows that the worth of coalition $\{2,3\}$ belongs to $[2/25, 4]$, and so the γ-characteristic function is not well-defined. □

Now, we study the variations of the equilibrium outputs of any coalition according to the coarseness of the coalition structure in which it is embedded. To this end, given a set of players N we introduce a binary relation $\leq^F$ on $\mathbf{P_N}$ defined as follows: we say that a coalition structure $\mathcal{P} \in \mathbf{P_N}$ is finer than a coalition structure $\mathcal{P}' \in \mathbf{P_N}$ (or $\mathcal{P}'$ is coarser than $\mathcal{P}$) which we write $\mathcal{P}' \leq^F \mathcal{P}$ if for any admissible coalition $S \in \mathcal{P}$ there exists an admissible coalition $T \in \mathcal{P}'$ such that $T \supseteq S$. Note that $(\mathbf{P_N}, \leq^F)$ is a complete lattice.
Moreover, we introduce the notions of forward and backward divided differences. Given a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and any positive number $\epsilon > 0$, the **forward and backward divided differences** $f^+ : \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ and $f^- : \mathbb{R}_+ \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ are defined as:

$$f^+(a,\epsilon) = \frac{1}{\epsilon}\big(f(a+\epsilon) - f(a)\big),$$

and

$$f^-(a,\epsilon) = \frac{1}{\epsilon}\big(f(a) - f(a-\epsilon)\big).$$

Given the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ and an admissible coalition S in $\mathcal{P}$, the functions $\phi_S^+ : X^S \times X^{N\setminus S} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ and $\phi_S^- : X^S \times X^{N\setminus S} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ are defined as:

$$\phi_S^+(x^S, x^{N\setminus S}, \epsilon) = p(X+\epsilon) + x^S p^+(X,\epsilon) - C_S^+(x^S,\epsilon),$$

and

$$\phi_S^-(x^S, x^{N\setminus S}, \epsilon) = p(X-\epsilon) + x^S p^-(X,\epsilon) - C_S^-(x^S,\epsilon).$$

We see that an admissible coalition $S \in \mathcal{P}$ cannot benefit from increasing its output x^S by ϵ if and only if $\phi_S^+(x^S, x^{N\setminus S}, \epsilon) \leq 0$; it cannot benefit from decreasing its output x^S by ϵ if and only if $\phi_S^-(x^S, x^{N\setminus S}, \epsilon) \geq 0$. Formally, the three following properties hold:

- $0 \in \arg\max_{x^S \in X^S} \pi^S(x^S, x^{N\setminus S}) \Longleftrightarrow \forall \epsilon > 0,\ \phi_S^+(0, x^{N\setminus S}, \epsilon) \leq 0$;
- $\sum_{i\in S} q_i \in \arg\max_{x^S \in X^S} \pi^S(x^S, x^{N\setminus S}) \Longleftrightarrow \forall \epsilon > 0,\ \phi_S^-(\sum_{i\in S} q_i, x^{N\setminus S}, \epsilon) \geq 0$;
- $\bar{x}^S \in \arg\max_{x^S \in X^S} \pi^S(x^S, x^{N\setminus S})$ such that $\bar{x}^S \in]0, \sum_{i\in S} q_i[\Longleftrightarrow \forall \epsilon, \epsilon' > 0$, $\phi_S^-(\bar{x}^S, x^{N\setminus S}, \epsilon) \geq 0 \geq \phi_S^+(\bar{x}^S, x^{N\setminus S}, \epsilon')$.

The following proposition compares equilibria under $\mathcal{P} \in \mathbf{P_N}$ and $\mathcal{P}' \in \mathbf{P_N}$ such that $\mathcal{P}' \leq^F \mathcal{P}$.

Proposition 3.3.5 *Let $\mathcal{P} \in \mathbf{P_N}$ and $\mathcal{P}' \in \mathbf{P_N}$ be two coalition structures such that $\mathcal{P}' \leq^F \mathcal{P}$. Let $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ and $\Gamma_{co}^{\mathcal{P}'} = (\mathcal{P}', (X^S, \pi^S)_{S\in\mathcal{P}'}) \in \mathcal{G}_{co}$ be the aggregated strategic Cournot oligopoly games such that $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}})$ and $\check{x}^{\mathcal{P}'} \in \varphi^N(\Gamma_{co}^{\mathcal{P}'})$ respectively. Then, it holds that:*

(i) $\sum_{S\in\mathcal{P}'} \check{x}^S = \check{X} \leq \hat{X} = \sum_{S\in\mathcal{P}} \hat{x}^S$;

(ii) $\forall (S,T) \in \mathcal{P} \times \mathcal{P}'$ *such that* $S \subseteq T$, $\hat{x}^S \leq \check{x}^T$;

(iii) $\sum_{T\in\mathcal{P}'\setminus\mathcal{P}} \check{x}^T \leq \sum_{S\in\mathcal{P}\setminus\mathcal{P}'} \hat{x}^S$.

When some coalitions merge, Proposition 3.3.5 states that the total production of the new entities decreases (point (iii)) while the other coalitions respond by increasing their outputs (point (ii)) so that the equilibrium total production decreases (point (i)). In order to establish the proof of Proposition 3.3.5, we need the following lemmas.

Lemma 3.3.6 *Let $\mathcal{P} \in \mathbf{P_N}$ and $\mathcal{P}' \in \mathbf{P_N}$ be two coalition structures for which there exist $T \in \mathcal{P}'$ and $S_l \in \mathcal{P}$, $l \in \{1,\ldots,p\}$, $p \in \{1,\ldots,n\}$, such that $T = \bigcup_{l=1}^p S_l$. Let $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ and $\Gamma_{co}^{\mathcal{P}'} = (\mathcal{P}', (X^S, \pi^S)_{S\in\mathcal{P}'}) \in \mathcal{G}_{co}$ be the aggregated strategic Cournot oligopoly games such that $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}})$ and $\check{x}^{\mathcal{P}'} \in \varphi^N(\Gamma_{co}^{\mathcal{P}'})$ respectively. If $\hat{X} \leq \check{X}$ then it holds that $\check{x}^T \leq \sum_{l=1}^p \hat{x}^{S_l}$.*

Proof: Let $\mathcal{P} \in \mathbf{P_N}$ and $\mathcal{P}' \in \mathbf{P_N}$ be two coalition structures satisfying the conditions of Lemma 3.3.6. For any $l \in \{1, \ldots, p\}$ denote by $\check{x}^{S_l}$ the output of subset S_l in coalition T so that $\sum_{l=1}^{p} \check{x}^{S_l} = \check{x}^T$. By the definition of a Nash equilibrium, for any $\epsilon > 0$, any $\epsilon' > 0$ and any $l \in \{1, \ldots, p\}$, it holds that:

$$\phi^+_{S_l}(\hat{x}^{S_l}, \hat{x}^{N\setminus S_l}, \epsilon) \leq 0,$$

and

$$\underbrace{p(\check{X} - \epsilon') + \check{x}^T p^-(\check{X}, \epsilon') - C^-_{S_l}(\check{x}^{S_l}, \epsilon')}_{A_l^{\epsilon'}} \geq 0,$$

where the second inequality means that under $\check{x}^{\mathcal{P}'} \in \varphi^N(\Gamma^{\mathcal{P}'}_{co})$, neither coalition T nor any of its subset S_l, $l \in \{1, \ldots, p\}$, can benefit from decreasing their output. For any $\epsilon > 0$, any $\epsilon' > 0$ and any $l \in \{1, \ldots, p\}$, define $Q^l_{\epsilon,\epsilon'} = \phi^+_{S_l}(\hat{x}^{S_l}, \hat{x}^{N\setminus S_l}, \epsilon) - A_l^{\epsilon'} \leq 0$. For the sake of contradiction suppose that there exists $l \in \{1, \ldots, p\}$ such that $\check{x}^{S_l} > \hat{x}^{S_l}$. It holds that:

$$\begin{aligned} Q^l_{\epsilon,\epsilon'} = {} & p(\hat{X} + \epsilon) - p(\check{X} - \epsilon') \\ & + \hat{x}^{S_l} p^+(\hat{X}, \epsilon) - \check{x}^T p^-(\check{X}, \epsilon') \\ & + C^-_{S_l}(\check{x}^{S_l}, \epsilon') - C^+_{S_l}(\hat{x}^{S_l}, \epsilon). \end{aligned}$$

In order to obtain a contradiction, it is sufficient to show that $Q^l_{\epsilon,\epsilon'}$ is positive for small enough ϵ and $\epsilon' = \epsilon$. First, take $\epsilon = \epsilon'$ and $\epsilon < \check{x}^{S_l} - \hat{x}^{S_l}$. By the convexity of individual cost functions and the definitions of $C^+_{S_l}$ and $C^-_{S_l}$, it holds that:

$$\begin{aligned} C^+_{S_l}(\hat{x}^{S_l}, \epsilon) & \leq C^+_{S_l}(\check{x}^{S_l} - \epsilon, \epsilon) \\ & = C^-_{S_l}(\check{x}^{S_l}, \epsilon'). \end{aligned}$$

By the above inequality and the differentiability of the inverse demand function p it follows that:

$$\begin{aligned} \lim_{\epsilon \longrightarrow 0} Q^l_{\epsilon,\epsilon} & \geq p(\hat{X}) - p(\check{X}) + \hat{x}^{S_l} \frac{dp}{dX}(\hat{X}) - \check{x}^T \frac{dp}{dX}(\check{X}) \\ & > 0, \end{aligned}$$

where the strict inequality follows from the fact that the inverse demand function p is strictly decreasing and concave and from the assumption $\hat{x}^{S_l} < \check{x}^{S_l} \leq \check{x}^T$. Hence, we obtain a contradiction with for any $\epsilon > 0$ and any $\epsilon' > 0$, $Q^l_{\epsilon,\epsilon'} = \phi^+_{S_l}(\hat{x}^{S_l}, \hat{x}^{N\setminus S_l}, \epsilon) - A_l^{\epsilon'} \leq 0$. So, for any $l \in \{1, \ldots, p\}$ we have $\check{x}^{S_l} \leq \hat{x}^{S_l}$ which implies $\check{x}^T = \sum_{l=1}^{p} \check{x}^{S_l} \leq \sum_{l=1}^{p} \hat{x}^{S_l}$. ■

Lemma 3.3.7 *Let $\mathcal{P} \in \mathbf{P_N}$ and $\mathcal{P}' \in \mathbf{P_N}$ be two coalition structures for which there exist $T \in \mathcal{P}'$ and $S_l \in \mathcal{P}$, $l \in \{1, \ldots, p\}$, $p \in \{1, \ldots, n\}$, such that $T = \bigcup_{l=1}^{p} S_l$. Let $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S \in \mathcal{P}}) \in \mathcal{G}_{co}$ and $\Gamma_{co}^{\mathcal{P}'} = (\mathcal{P}', (X^S, \pi^S)_{S \in \mathcal{P}'}) \in \mathcal{G}_{co}$ be the aggregated strategic Cournot oligopoly games such that $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}})$ and $\check{x}^{\mathcal{P}'} \in \varphi^N(\Gamma_{co}^{\mathcal{P}'})$ respectively. If $\check{X} \leq \hat{X}$ then it holds that for any $l \in \{1, \ldots, p\}$, $\check{x}^T \geq \hat{x}^{S_l}$.*

Proof: Let $\mathcal{P} \in \mathbf{P_N}$ and $\mathcal{P}' \in \mathbf{P_N}$ be two coalition structures satisfying the conditions of Lemma 3.3.7. By the definition of a Nash equilibrium, for any $\epsilon > 0$, any $\epsilon' > 0$ and any $l \in \{1, \ldots, p\}$, it holds that $\phi_T^+(\check{x}^T, \check{x}^{N \backslash T}, \epsilon) \leq 0$ and $\phi_{S_l}^-(\hat{x}^{S_l}, \hat{x}^{N \backslash S_l}, \epsilon') \geq 0$. For any $\epsilon > 0$, any $\epsilon' > 0$ and any $l \in \{1, \ldots, p\}$, define $Q_{\epsilon,\epsilon'}^l = \phi_T^+(\check{x}^T, \check{x}^{N \backslash T}, \epsilon) - \phi_{S_l}^-(\hat{x}^{S_l}, \hat{x}^{N \backslash S_l}, \epsilon') \leq 0$. For the sake of contradiction suppose that there exists $l \in \{1, \ldots, p\}$ such that $\hat{x}^{S_l} > \check{x}^T$. Then, it holds that:

$$\begin{aligned} Q_{\epsilon,\epsilon'}^l &= p(\check{X} + \epsilon) - p(\hat{X} - \epsilon') \\ &\quad + \check{x}^T p^+(\check{X}, \epsilon) - \hat{x}^{S_l} p^-(\hat{X}, \epsilon') \\ &\quad + C_{S_l}^-(\hat{x}^{S_l}, \epsilon') - C_T^+(\check{x}^T, \epsilon). \end{aligned}$$

As in the proof of Lemma 3.3.6, it is sufficient to show that $Q_{\epsilon,\epsilon'}^l$ is positive for small enough ϵ and $\epsilon' = \epsilon$. First, take $\epsilon = \epsilon'$ and $\epsilon < \hat{x}^{S_l} - \check{x}^T$. By the convexity of individual cost functions, the definitions of C_T^+, $C_{S_l}^+$ and $C_{S_l}^-$, and from the fact that it is less costly for coalition T to redistribute an extra cost than for any of its subsets S_l, $l \in \{1, \ldots, p\}$, it holds that:

$$\begin{aligned} C_T^+(\check{x}^T, \epsilon) &\leq C_T^+(\hat{x}^{S_l} - \epsilon, \epsilon) \\ &\leq C_{S_l}^+(\hat{x}^{S_l} - \epsilon, \epsilon) \\ &= C_{S_l}^-(\hat{x}^{S_l}, \epsilon'). \end{aligned}$$

By the above inequality and the differentiability of the inverse demand function p it follows that:

$$\begin{aligned} \lim_{\epsilon \longrightarrow 0} Q_{\epsilon,\epsilon}^l &\geq p(\check{X}) - p(\hat{X}) + \check{x}^T \frac{dp}{dX}(\check{X}) - \hat{x}^{S_l} \frac{dp}{dX}(\hat{X}) \\ &> 0, \end{aligned}$$

where the strict inequality follows from the fact that the inverse demand function p is strictly decreasing and concave and from the assumption $\hat{x}^{S_l} > \check{x}^T$. Hence, we obtain a contradiction with for any $\epsilon > 0$ and any $\epsilon' > 0$, $Q_{\epsilon,\epsilon'}^l =$

$\phi_T^+(\check{x}^T, \check{x}^{N\setminus T}, \epsilon) - \phi_{S_l}^-(\hat{x}^{S_l}, \hat{x}^{N\setminus S_l}, \epsilon') \leq 0$. So, for any $l \in \{1, \ldots, p\}$ we have $\check{x}^T \geq \hat{x}^{S_l}$. ■

Now, we are ready to establish the proof of Proposition 3.3.5.

Proof: First, in order to prove *(i)*, for the sake of contradiction suppose that $\hat{X} < \check{X}$. Since $\mathcal{P}' \leq^F \mathcal{P}$, for any $T \in \mathcal{P}'$ there exists $S_l \in \mathcal{P}$, $l \in \{1, \ldots, p\}$, $p \in \{1, \ldots, n\}$, such that $T = \bigcup_{l=1}^p S_l$. It follows from Lemma 3.3.6 that for any $T \in \mathcal{P}'$ and any $S_l \in \mathcal{P}$, $l \in \{1, \ldots, p\}$, we have $\check{x}^T \leq \sum_{l=1}^p \hat{x}^{S_l}$, and so $\check{X} \leq \hat{X}$, a contradiction.
Then, point *(ii)* follows directly from *(i)* and Lemma 3.3.7.
Finally, point *(iii)* is a consequence of points *(i)* and *(ii)*. Indeed, by *(i)* it holds that:

$$\begin{aligned}
\check{X} \leq \hat{X} \iff & \sum_{T\in\mathcal{P}'} \check{x}^T \leq \sum_{S\in\mathcal{P}} \hat{x}^S \\
\iff & \sum_{T\in\mathcal{P}'\setminus\mathcal{P}} \check{x}^T + \sum_{T\in\mathcal{P}'\cap\mathcal{P}} \check{x}^T \leq \sum_{S\in\mathcal{P}\setminus\mathcal{P}'} \hat{x}^S + \sum_{S\in\mathcal{P}\cap\mathcal{P}'} \hat{x}^S \\
\iff & \sum_{T\in\mathcal{P}'\cap\mathcal{P}} \check{x}^T - \sum_{S\in\mathcal{P}\cap\mathcal{P}'} \hat{x}^S \leq \sum_{S\in\mathcal{P}\setminus\mathcal{P}'} \hat{x}^S - \sum_{T\in\mathcal{P}'\setminus\mathcal{P}} \check{x}^T \qquad (3.6)
\end{aligned}$$

Moreover, by *(ii)* it follows that:

$$\sum_{T\in\mathcal{P}'\cap\mathcal{P}} \check{x}^T - \sum_{S\in\mathcal{P}\cap\mathcal{P}'} \hat{x}^S \geq 0 \qquad (3.7)$$

Using (3.6) and (3.7) we obtain $\sum_{T\in\mathcal{P}'\setminus\mathcal{P}} \check{x}^T \leq \sum_{S\in\mathcal{P}\setminus\mathcal{P}'} \hat{x}^S$. ■

Point *(iii)* of Proposition 3.3.5 cannot be improved in the sense that it does not always hold that for any $T \in \mathcal{P}'$ and any $S_l \in \mathcal{P}$, $l \in \{1, \ldots, p\}$, such that $T = \bigcup_{l=1}^p S_l$, we have $\check{x}^T \leq \sum_{l=1}^p \hat{x}^{S_l}$. This is illustrated in the following example.

Example 3.3.8

Consider the Cournot oligopoly situation $(N, (q_i, C_i)_{i\in N}, p)$ where $N = \{1, \ldots, 8\}$, for any $i \in N$, $q_i = 3/2$ and $C_i(x_i) = x_i$, and the inverse demand function is defined as $p(X) = 12 - X$. Let $\mathcal{P} = \{\{i\} : i \in N\} \in \mathbf{P_N}$ and $\mathcal{P}' = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}\} \in \mathbf{P_N}$ be two coalition structures such that $\mathcal{P}' \leq^F \mathcal{P}$. For the aggregated strategic Cournot oligopoly games $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ and $\Gamma_{co}^{\mathcal{P}'} = (\mathcal{P}', (X^S, \pi^S)_{S\in\mathcal{P}'}) \in \mathcal{G}_{co}$, the Nash equilibria of $\Gamma_{co}^{\mathcal{P}}$ and $\Gamma_{co}^{\mathcal{P}'}$ are given by, for any $i \in N$, $\hat{x}^{\{i\}} = 11/9$ and $\check{x}^{\mathcal{P}'} = (11/4, 11/4, 11/4)$ respectively. So, it holds that $\check{x}^{\{1,2\}} = 11/4 > 22/9 = \hat{x}^{\{1\}} + \hat{x}^{\{2\}}$. □

We saw that the Nash equilibrium outputs of the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S \in \mathcal{P}}) \in \mathcal{G}_{co}$ are equal to the aggregated equilibrium outputs under $\mathcal{P}$ of the strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$. For notational convenience, given a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ where $(x_S^*, \tilde{x}_{N \backslash S}) \in \varphi^{PA}(\Gamma_{co}, S)$, we denote by $X^{P,S}$ **the total production enforced by the partial agreement equilibrium** under S:

$$X^{P,S} = \sum_{i \in S} x_i^* + \sum_{i \in N \backslash S} \tilde{x}_i.$$

For any partial agreement equilibrium, we deduce from Proposition 3.3.5 the following corollary.

Corollary 3.3.9 *Let $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$ be a strategic Cournot oligopoly game. Let $S \in 2^N \backslash \{\emptyset\}$ and $T \in 2^N \backslash \{\emptyset\}$ such that $S \subseteq T$ where $(x_S^*, \tilde{x}_{N \backslash S}) \in \varphi^{PA}(\Gamma_{co}, S)$ and $(y_T^*, \tilde{y}_{N \backslash T}) \in \varphi^{PA}(\Gamma_{co}, T)$. Then it holds that:*

(i) $X^{P,S} \geq X^{P,T}$;

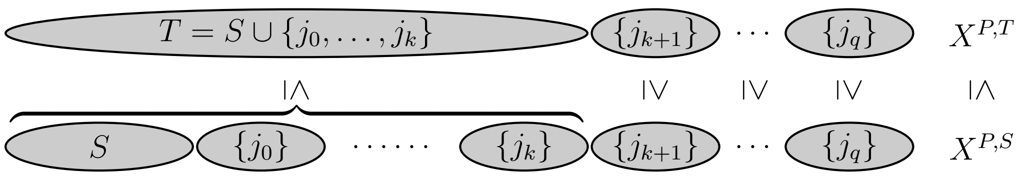
(ii) $\forall i \in N \backslash T$, $\tilde{x}_i \leq \tilde{y}_i$;

(iii) $\sum_{i \in S} x_i^* \leq \sum_{i \in T} y_i^*$;

(iv) $\sum_{i \in T} y_i^* \leq \sum_{i \in S} x_i^* + \sum_{i \in T \backslash S} \tilde{x}_i$.

Points *(ii)* and *(iii)* follow from *(ii)* of Proposition 3.3.5. Point *(i)* says that the total production decreases from the partial agreement equilibrium under S to the partial agreement equilibrium under T. Point *(ii)* stipulates that firms outside T increase their output from the partial agreement equilibrium under S to the partial agreement equilibrium under T. According to points *(iii)* and *(iv)*, the equilibrium output of coalition S is less than or equal to the one of coalition T but becomes greater if we add the equilibrium outputs of firms in T outside S. Corollary 3.3.9 is illustrated in the figure below:

$$N = S \cup \{j_0, j_1, \ldots, j_q\},\ q \in \{0, \ldots, n-2\},\ k \in \{0, \ldots, q-1\}$$



3.4 Concavity and balancedness property

In this section, we analyze the non-emptiness of the core of Cournot oligopoly TU-games in γ-characteristic function form. The main result states that if any individual profit function is continuous and concave on the set of strategy profiles, the corresponding Cournot oligopoly TU-game in γ-characteristic function form is balanced.

Theorem 3.4.1 *Let* $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ *be a strategic Cournot oligopoly game such that for any* $i \in N$*,* π_i *is concave on* X_N*. Then the corresponding Cournot oligopoly TU-game* $(N, v_\gamma) \in G^\gamma_{co}$ *is balanced.*

In order to establish the proof of Theorem 3.4.1, we need the following lemma. Given a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$, a balanced family of coalitions $\mathcal{B} \subseteq 2^N\backslash\{\emptyset\}$ where for any $S \in \mathcal{B}$, $(x^*_S, \tilde{x}_{N\backslash S}) \in \varphi^{PA}(\Gamma_{co}, S)$, we define the strategy profile $y \in X_N$ as:

$$\forall i \in N,\ y_i = \sum_{S\in\mathcal{B}_i} \delta_S x^*_i \tag{3.8}$$

Lemma 3.4.2 *Let* $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ *be a strategic Cournot oligopoly game such that* $z^* \in \varphi^N(\Gamma_{co})$*. Let* $y \in X_N$ *be a strategy profile as defined in* (3.8)*. Then it holds that:*

$$\forall j \in N,\ \sum_{S\in\mathcal{B}_j} \delta_S X^{P,S} \geq Y,$$

where $Y = \sum_{i\in N} y_i$*.*

Proof: Pick any $j \in N$. First, for any $S \in \mathcal{B}_j$ where $\mathcal{B}_j = \{S \in \mathcal{B} : j \in S\}$, we show that:

$$\sum_{S\in\mathcal{B}_j} \delta_S \sum_{i\in N\backslash S} \tilde{x}_i \geq \sum_{S\in\mathcal{B}\backslash\mathcal{B}_j} \delta_S \sum_{i\in S} x^*_i \tag{3.9}$$

By *(ii)* and *(iv)* of Corollary 3.3.9, it follows that:

$$\begin{aligned}
\sum_{S\in\mathcal{B}_j}\delta_S\sum_{i\in N\setminus S}\tilde{x}_i &\geq \sum_{S\in\mathcal{B}_j}\delta_S\sum_{i\in N\setminus S}z_i^* \\
&= \sum_{S\in\mathcal{B}_j}\delta_S\left(\sum_{i\in N}z_i^* - \sum_{i\in S}z_i^*\right) \\
&= \sum_{i\in N}z_i^* - \sum_{S\in\mathcal{B}_j}\delta_S\sum_{i\in S}z_i^* \\
&= \sum_{S\in\mathcal{B}}\delta_S\sum_{i\in S}z_i^* - \sum_{S\in\mathcal{B}_j}\delta_S\sum_{i\in S}z_i^* \\
&= \sum_{S\in\mathcal{B}\setminus\mathcal{B}_j}\delta_S\sum_{i\in S}z_i^* \\
&\geq \sum_{S\in\mathcal{B}\setminus\mathcal{B}_j}\delta_S\sum_{i\in S}x_i^*.
\end{aligned}$$

Thus, by (3.9) it holds that:

$$\begin{aligned}
\sum_{S\in\mathcal{B}_j}\delta_S X^{P,S} &= \sum_{S\in\mathcal{B}_j}\delta_S\sum_{i\in S}x_i^* + \sum_{S\in\mathcal{B}_j}\delta_S\sum_{i\in N\setminus S}\tilde{x}_i \\
&\geq \sum_{S\in\mathcal{B}_j}\delta_S\sum_{i\in S}x_i^* + \sum_{S\in\mathcal{B}\setminus\mathcal{B}_j}\delta_S\sum_{i\in S}x_i^* \\
&= \sum_{S\in\mathcal{B}}\delta_S\sum_{i\in S}x_i^* \\
&= \sum_{i\in N}\sum_{S\in\mathcal{B}_i}\delta_S x_i^* \\
&= Y,
\end{aligned}$$

which completes the proof. ∎

Helm (2001) obtains a similar result for pollution games. Now, we are ready to establish the proof of Theorem 3.4.1.

Proof (of Theorem 3.4.1): Let $\mathcal{B} \subseteq 2^N\setminus\{\emptyset\}$ be a balanced family of coalitions and $y \in X_N$ be a strategy profile as defined in (3.8). By the concavity of any individual profit function π_i on X_N, Lemma 3.4.2 and the strict monotonicity of the inverse demand function, and the Pareto efficiency of the worth of the grand coalition it holds that:

$$\begin{aligned}
\sum_{S\in\mathcal{B}} \delta_S v_\gamma(S) &= \sum_{S\in\mathcal{B}} \delta_S \pi_S(x_S^*, \tilde{x}_{N\setminus S}) \\
&= \sum_{S\in\mathcal{B}} \delta_S \sum_{i\in S} \pi_i(x_S^*, \tilde{x}_{N\setminus S}) \\
&= \sum_{i\in N} \sum_{S\in\mathcal{B}_i} \delta_S \pi_i(x_S^*, \tilde{x}_{N\setminus S}) \\
&\leq \sum_{i\in N} \pi_i\Big(\sum_{S\in\mathcal{B}_i} \delta_S (x_S^*, \tilde{x}_{N\setminus S})\Big) \\
&= \sum_{i\in N} \Big[p\Big(\sum_{S\in\mathcal{B}_i} \delta_S X^{P,S}\Big) y_i - C_i(y_i)\Big] \\
&\leq p(Y)Y - \sum_{i\in N} C_i(y_i) \\
&\leq v_\gamma(N),
\end{aligned}$$

which completes the proof. ■

The concave condition in Theorem 3.4.1 is a sufficient condition for the non-emptiness of the core but it is not a necessary one. This is illustrated in the following example.

Example 3.4.3

Consider the Cournot oligopoly TU-game $(N, v_\gamma) \in G_{co}^\gamma$ associated with the Cournot oligopoly situation $(N, (q_i, C_i)_{i\in N}, p)$ where $N = \{1, 2, 3\}$, $q_1 = 5$, $q_2 = 1$, $q_3 = 2$, $C_1(x_1) = x_1$, $C_2(x_2) = 2x_2$, $C_3(x_3) = 2x_3$ and the inverse demand function is defined as $p(X) = 10 - X$. Clearly, any individual profit function is not concave on X_N. The worth of any coalition $S \in 2^N\setminus\{\emptyset\}$ is given in the following table:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_\gamma(S)$	9	2	4	12.25	16	5.44	20.25

We can easily check that $\sigma = (13.25, 3, 4)$ is in the core. □

3.5 The Nash pro rata value

Although the concave condition seems to be a natural requirement in order to guarantee the non-emptiness of the core, many strategic Cournot oligopoly games fail to satisfy it. For instance, given a linear Cournot oligopoly situation where $p(X) = a - X$, $a \in \mathbb{R}_{++}$, and for any $i \in N$, $C_i(x_i) = c_i x_i$, $c_i \in \mathbb{R}_+$, any individual profit function is quadratic but it is not concave on X_N as in Example 3.4.3.

In this section, we adopt an alternative approach that consists in providing a single-valued solution, the Nash pro rata value, in the core without the concavity requirement. We succeed in doing that by assuming that individual cost functions are linear and that firms have identical marginal costs:

$$\exists c \in \mathbb{R}_+ : \forall i \in N,\ C_i(x_i) = cx_i \tag{3.10}$$

We do not impose any other condition on the capacity constraints and the inverse demand function. We denote by $G_{co}^{\gamma *} \subseteq G_{co}^{\gamma}$ the **set of Cournot oligopoly TU-games in γ-characteristic function form derived from Cournot oligopoly situations satisfying** (3.10).

For notational convenience, given the strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ and $z^* \in \varphi^N(\Gamma_{co})$ we denote by $X^{P,\emptyset} = \sum_{i\in N} z_i^*$ the total output under the Nash equilibrium. Given a Cournot oligopoly TU-game $(N, v_\gamma) \in G_{co}^{\gamma *}$, the **NP(Nash pro rata) value** is a single-valued solution defined as:

$$\forall i \in N, \mathrm{NP}_i(N, v_\gamma) = \begin{cases} \dfrac{z_i^*}{X^{P,\emptyset}} v_\gamma(N) & \text{if } X^{P,\emptyset} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The NP value distributes to every firm the worth of the grand coalition in proportion to its Nash individual output.

Theorem 3.5.1 *Let $(N, v_\gamma) \in G_{co}^{\gamma *}$ be a Cournot oligopoly TU-game associated with the strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$. Then, it holds that NP$(N, v_\gamma) \in C(N, v_\gamma)$.*

Proof: First, let $z^* \in \varphi^N(\Gamma_{co})$ and assume that $X^{P,\emptyset} = 0$. By *(i)* of Corollary 3.3.9, for any coalition $S \in 2^N \backslash \{\emptyset\}$ we have $X^{P,S} = 0$, and so $v_\gamma(S) = 0$. In this case, it is obvious that $\mathrm{NP}(N, v_\gamma) \in C(N, v_\gamma)$.

Then, let $z^* \in \varphi^N(\Gamma_{co})$ and assume that $X^{P,\emptyset} > 0$. For the sake of contradiction suppose that $\mathrm{NP}(N, v_\gamma) \notin C(N, v_\gamma)$, i.e. there exists a deviating coalition $S \in 2^N \backslash \{\emptyset\}$ such that $v_\gamma(S) > \sum_{i\in S} \mathrm{NP}_i(N, v_\gamma)$. It follows that $v_\gamma(S) > 0$ and so $X^{P,S} > 0$. Given $(x_S^*, \tilde{x}_{N\backslash S}) \in \varphi^{PA}(\Gamma_{co}, S)$, we define the payoff vector $\sigma_S \in \mathbb{R}^n$ as:

$$\forall i \in N,\ \sigma_{S,i} = \frac{v_\gamma(N)}{X^{P,S}} \alpha_i,$$

where

$$\alpha_i = \begin{cases} x_i^* \text{ if } i \in S, \\ \tilde{x}_i \text{ if } i \in N \backslash S. \end{cases}$$

From *(i)* and *(ii)* of Corollary 3.3.9, we know that $X^{P,S} \leq X^{P,\emptyset}$ and for any $i \in N \backslash S$, $z_i^* \leq \tilde{x}_i$ respectively. This implies that for any $i \in N \backslash S$, $\sigma_{S,i} \geq \mathrm{NP}_i(N, v_\gamma)$. Moreover, by the Pareto efficiency of the worth of the grand coalition and the contradicting assumption it holds that:

$$\begin{aligned} \sum_{i \in S} \sigma_{S,i} &= \frac{v_\gamma(N)}{X^{P,S}} \sum_{i \in S} x_i^* \\ &\geq \frac{1}{X^{P,S}} \left(p(X^{P,S}) X^{P,S} - c X^{P,S} \right) \sum_{i \in S} x_i^* \\ &= p(X^{P,S}) \sum_{i \in S} x_i^* - c \sum_{i \in S} x_i^* \\ &= v_\gamma(S) \\ &> \sum_{i \in S} \mathrm{NP}_i(N, v_\gamma). \end{aligned}$$

Thus, we obtain $\sum_{i \in N} \sigma_{S,i} > \sum_{i \in N} \mathrm{NP}_i(N, v_\gamma)$ and $\sum_{i \in N} \sigma_{S,i} = v_\gamma(N) = \sum_{i \in N} \mathrm{NP}_i(N, v_\gamma)$, a contradiction. ■

Note that for large capacity constraints the NP value is equal to the equal division solution that distributes $v_\gamma(N)$ equally among the firms since in this case for any $i \in N$ and any $j \in N$, $z_i^* = z_j^*$. Funaki and Yamato (1999) show that the equal division solution belongs to the core under pessimistic expectations of a common pool game without any capacity constraint. Since this game belongs to the set $G_{co}^{\gamma *}$, Theorem 3.5.1 generalizes their result to the case of asymmetric capacity constraints. Thus, the NP value is a single-valued solution that always belongs to the core and takes into account firms' capacity constraints.

From the regulator point of view it is interesting to know which properties are satisfied by the NP value. On the set $G_{co}^{\gamma *}$, the NP value can be characterized by means of four properties: efficiency, null firm, monotonicity and non-cooperative fairness. A single-valued solution F on $G_{co}^{\gamma *} \subseteq G$ satisfies:

- **efficiency**: if for any $(N, v_\gamma) \in G_{co}^{\gamma *}$, $\sum_{i \in N} F_i(N, v_\gamma) = v_\gamma(N)$; (EFF)
- **null firm**: if for any $(N, v_\gamma) \in G_{co}^{\gamma *}$, for any $i \in N$ such that $q_i = 0$, $F_i(N, v_\gamma) = 0$; (NF)

- **monotonicity**: if for any $(N, v_\gamma) \in G_{co}^{\gamma *}$, for any $i \in N$ and any $j \in N$ such that $q_i \geq q_j$, $F_i(N, v_\gamma) \geq F_j(N, v_\gamma)$; (M)

- **non-cooperative fairness**: if for any $(N, v_\gamma) \in G_{co}^{\gamma *}$, for any $i \in N$ and any $j \in N$, $v_\gamma(\{j\})F_i(N, v_\gamma) = v_\gamma(\{i\})F_j(N, v_\gamma)$. (NCF)

Theorem 3.5.2 *A single-valued solution F on $G_{co}^{\gamma *}$ satisfies (EFF), (NF), (M) and (NCF) if and only if $F = NP$.*

Proof: Pick any Cournot oligopoly TU-game $(N, v_\gamma) \in G_{co}^{\gamma *}$ associated with a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$ and let $z^* \in \varphi^N(\Gamma_{co})$.
Firstly, we show that the NP value satisfies (EFF). Assume that $X^{P,\emptyset} = 0$. It follows that for any $i \in N$, $\text{NP}_i(N, v_\gamma) = 0$. Moreover, by *(i)* of Corollary 3.3.9 we have $X^{P,N} = 0$, and so $\sum_{i \in N} \text{NP}_i(N, v_\gamma) = v_\gamma(N) = 0$. Then, assume that $X^{P,\emptyset} > 0$. By the definition of the NP value we see directly that $\sum_{i \in N} \text{NP}_i(N, v_\gamma) = v_\gamma(N)$.
Secondly, we show that the NP value satisfies (NF). Pick any $i \in N$ such that $q_i = 0$. Assume that $X^{P,\emptyset} = 0$. By the definition of the NP value we have $\text{NP}_i(N, v_\gamma) = 0$. Then, assume that $X^{P,\emptyset} > 0$. Since $q_i = 0$ it follows that $z_i^* = 0$, and so by the definition of the NP value we have $\text{NP}_i(N, v_\gamma) = 0$.
Thirdly, we show that the NP value satisfies (M). Pick any $i \in N$ and any $j \in N$ such that $q_j \geq q_i$. Assume that $X^{P,\emptyset} = 0$. By the definition of the NP value we have $\text{NP}_j(N, v_\gamma) = \text{NP}_i(N, v_\gamma) = 0$. Then, assume that $X^{P,\emptyset} > 0$. From $q_j \geq q_i$ and $(N, v_\gamma) \in G_{co}^{\gamma *}$ it follows that $z_j^* \geq z_i^*$, and so by the definition of the NP value we have $\text{NP}_j(N, v_\gamma) \geq \text{NP}_i(N, v_\gamma)$.
Fourthly, we show that the NP value satisfies (NCF). Assume that $X^{P,\emptyset} = 0$. It follows that for any $i \in N$, $\text{NP}_i(N, v_\gamma) = 0$, and so for any $i \in N$ and any $j \in N$, $v_\gamma(\{j\})\text{NP}_i(N, v_\gamma) = v_\gamma(\{i\})\text{NP}_j(N, v_\gamma) = 0$. Then, assume that $X^{P,\emptyset} > 0$. For any $i \in N$ and any $j \in N$ it holds that:

$$
\begin{aligned}
v_\gamma(\{j\})\text{NP}_i(N, v_\gamma) &= \big(p(X^{P,\emptyset}) - c\big) z_j^* \frac{z_i^*}{X^{P,\emptyset}} v_\gamma(N) \\
&= \big(p(X^{P,\emptyset}) - c\big) z_i^* \frac{z_j^*}{X^{P,\emptyset}} v_\gamma(N) \\
&= v_\gamma(\{i\})\text{NP}_j(N, v_\gamma).
\end{aligned}
$$

It remains to show that the NP value is the unique single-valued solution on the set $G_{co}^{\gamma *}$ that satisfies (EFF), (NF), (M) and (NCF). Pick any single-valued solution F on $G_{co}^{\gamma *}$ satisfying (EFF), (NF), (M) and (NCF) and prove that it is equal to the NP value. By (EFF), we know that $\sum_{i \in N} F_i(N, v_\gamma) = v_\gamma(N)$.

Moreover, (NF) and (M) ensures that for any $i \in N$, $F_i(N, v_\gamma) \geq 0$. Thus, there exists a mapping $\beta : G_{co}^{\gamma *} \longrightarrow \mathbb{R}_+^n$ such that:

$$\forall i \in N,\ F_i(N, v_\gamma) = \frac{\beta_i(N, v_\gamma)}{\sum_{j \in N} \beta_j(N, v_\gamma)} v_\gamma(N) \tag{3.11}$$

Assume that $X^{P,\emptyset} = 0$. By *(i)* of Corollary 3.3.9 we have $X^{P,N} = 0$, and so $v_\gamma(N) = 0$. It follows that for any $i \in N$, $F_i(N, v_\gamma) = \mathrm{NP}_i(N, v_\gamma) = 0$. Then, assume that $X^{P,\emptyset} > 0$. Without loss of generality, suppose that $\sum_{i \in N} \beta_i(N, v_\gamma) = X^{P,\emptyset}$. By (3.11) and (NCF) it holds that:

$$\forall i \in N, \forall j \in N,\ v_\gamma(\{j\})\beta_i(N, v_\gamma) = v_\gamma(\{i\})\beta_j(N, v_\gamma).$$

For any $i \in N$, by summing the equations above over all $j \in N$ it holds that:

$$\sum_{j \in N} v_\gamma(\{j\})\beta_i(N, v_\gamma) = v_\gamma(\{i\}) \sum_{j \in N} \beta_j(N, v_\gamma),$$

which is equivalent to

$$\big(p(X^{P,\emptyset}) - c\big) X^{P,\emptyset} \beta_i(N, v_\gamma) = \big(p(X^{P,\emptyset}) - c\big) z_i^* X^{P,\emptyset},$$

and so

$$\beta_i(N, v_\gamma) = z_i^*.$$

Thus, we conclude that for any $i \in N$, $F_i(N, v_\gamma) = \mathrm{NP}_i(N, v_\gamma)$. ■

One can ask whether Cournot oligopoly TU-games in γ-characteristic function form are superadditive or convex. Norde et al. (2002) establish that Cournot oligopoly TU-games in β-characteristic function form are convex in case the inverse demand function and individual cost functions are linear. The following example shows that this result cannot be extended on the set $G_{co}^{\gamma *} \subseteq G_{co}^{\gamma}$.

Example 3.5.3

Let $(N, v_\gamma) \in G_{co}^{\gamma *}$ be a Cournot oligopoly TU-game derived from the Cournot oligopoly situation $(N, (q_i, C_i)_{i \in N}, p)$ where $N = \{1, 2, 3\}$, $q_1 = 3/2$, $q_2 = 3$, $q_3 = 5/2$, for any $i \in N$, $C_i(x_i) = 2x_i$, and the inverse demand function is defined as $p(X) = 10 - X$. The worth of any coalition $S \in 2^N \backslash \{\emptyset\}$ is given in the following table:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_\gamma(S)$	3.25	4.69	4.69	7.56	7.56	10.56	16

Thus, $v_\gamma(\{1,2\}) < v_\gamma(\{1\}) + v_\gamma(\{2\})$. We conclude that $(N, v_\gamma) \in G_{co}^{\gamma *}$ is neither superadditive nor convex. $\square$

3.6 Concluding remarks

In this chapter, we have focused on the set of Cournot oligopoly TU-games in γ-characteristic function form. When a coalition forms, the underlying assumption is that external firms choose their action individually as a best reply to the coalitional action (Chander and Tulkens 1997). This assumption seems more appropriate than the maximin and minimax arguments suggested by Aumann (1959) which lead to the concepts of α and β-characteristic functions respectively.

In order to verify that the γ-characteristic function is well-defined, we have proved that an equilibrium under any coalition structure exists. We have studied the variations of equilibrium outputs of any coalition according to the coarseness of the coalition structure in which it is embedded. We have showed that total production equilibrium is decreasing with the coarseness of the coalition structure. This result is explained by the mergers between coalitions that occurred. Conversely, the other coalitions which do not merge increase their output.

Concerning the non-emptiness of the core, we have first established that Cournot oligopoly TU-games in γ-characteristic function form are balanced if the inverse demand function is differentiable and any individual profit function is continuous and concave on the set of strategy profiles. This result is a step forward beyond Zhao's theorem (1999b) for the set of Cournot oligopoly TU-games. However, many strategic Cournot oligopoly games fail to satisfy the concavity condition. In particular, this is the case when the inverse demand function and individual cost functions are linear. Thus, in case of the individual cost functions are linear and firms have identical marginal costs, we have introduced a new single-valued solution, the NP value, that distributes to every firm the worth of the grand coalition in proportion to its Nash individual output. We have showed that this solution belongs to the core. Insofar as our Cournot oligopoly situation also describes a common pool situation, this result generalizes Funaki and Yamato's core allocation result (1999) to the case of asymmetric capacity constraints. Moreover, we have provided an axiomatic characterization of the NP value by means of four appealing properties in oligopoly theory.

van den Brink (2008) proposes an axiomatic characterization of a set of single-valued proportional solutions with exogenous weights. We denote by $\Delta^n = \{\delta \in \mathbb{R}^n_+ : \sum_{i \in N} \delta_i = 1\}$ the $(n-1)$-dimensional unit simplex. Given a TU-game $(N, v) \in G$ and $\delta \in \Delta^n$, the single-valued proportional solution $F^\delta(N, v)$ is defined as:

$$\forall i \in N,\ F_i^{\delta}(N,v) = \frac{\delta_i}{\sum_{j\in N}\delta_j} v(N).$$

van den Brink (2008) characterizes this set of single-valued proportional solutions by means of three axioms: efficiency, collusion neutrality and linearity. We know that the NP value is efficient on $G_{co}^{\gamma *}$. However, it fails to satisfy collusion neutrality and linearity. We have characterized a single-valued proportional solution with endogenous weights (Nash individual outputs) on the set $G_{co}^{\gamma *}$. An axiomatic characterization of a single-valued proportional solution with endogenous weights on the set G would be of the greatest interest.
Other alternative blocking rules can be considered. For instance, firms in $N\backslash S$ can choose coalitional (rather than individual) best reply strategies. In this case, in order to determine the worth of any coalition $S \in 2^N\backslash\{\emptyset\}$, the equilibrium under the coalition structure $\mathcal{P} = \{S, N\backslash S\}$ must be considered. However, Funaki and Yamato (1999) show that the associated core under optimistic expectations of a common pool game is always empty for $n \geq 4$ in case of the individual cost functions are identical and linear. This result remains valid for our model.

Chapter 4

The core in Cournot oligopoly interval games

4.1 Introduction

In this chapter, which is based on Lardon (2010b), we relax the assumption on the differentiability of the inverse demand function in Chapter 3 which ensures that the γ-characteristic function is well-defined. Indeed, in many Cournot oligopoly situations the inverse demand function may not be differentiable. For instance, Katzner (1968) shows that demand functions derived from quite nice consumers' individual utility functions, even twice continuously differentiable, may not be differentiable everywhere.

So, we assume that the inverse demand function is continuous but not necessarily differentiable. With such an assumption, Example 3.3.4 shows that we cannot always define a Cournot oligopoly TU-game in γ-characteristic function form since the worth of any coalition is not necessarily unique. However, we prove that we can always specify a Cournot oligopoly interval game in γ-set function form, i.e. we can assign to any coalition a closed and bounded real interval that represents all its potential worths enforced by the set of partial agreement equilibria. We deal with the problem of the non-emptiness of two extensions of the core on the set of interval games: the interval core and the standard core. To this end, we use a decision theory criterion, the Hurwicz criterion (Hurwicz 1951), that consists in combining for any coalition the worst and the best worths that it can obtain in its worth interval. The first result states that the interval core is non-empty if and only if the Cournot oligopoly TU-game associated with the best worth of every coalition in its worth interval admits a non-empty core. However, we show that even for a very simple Cournot oligopoly situation, this condition fails to be satisfied. The second result states that the standard core is non-empty if and only if the Cournot oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty core. Moreover, we provide sufficient conditions on any individual profit function and any individual cost function under which this condition always holds. This result substantially extends the results in Theorems 3.4.1 and 3.5.1.

The remainder of this chapter is structured as follows. In Section 4.2, we introduce the model and prove that the γ-set function is well-defined. In Section 4.3, we introduce the Hurwicz criterion and provide a necessary and sufficient condition for the non-emptiness of each of the core solutions: the interval core and the standard core. Section 4.4 gives some concluding remarks.

4.2 The model

For the sake of clarity, we briefly recall some definitions established in Chapter 3. A **Cournot oligopoly situation** is a quadruplet $(N, (q_i, C_i)_{i \in N}, p)$ defined as:

1. a finite **set of firms** $N = \{1, 2, \dots, n\}$;
2. for every $i \in N$, a **capacity constraint** $q_i \in \mathbb{R}_+$;
3. for every $i \in N$, an **individual cost function** $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$;
4. an **inverse demand function** $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which assigns to any aggregate quantity $X \in \mathbb{R}_+$ the unit price $p(X)$.

Throughout this chapter, we assume that:

(c) the inverse demand function p is continuous, strictly decreasing and concave;

(d) every individual cost function C_i is continuous, strictly increasing and convex.

The **strategic Cournot oligopoly game** associated with the Cournot oligopoly situation $(N, (q_i, C_i)_{i \in N}, p)$ is a triplet $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N})$ defined as:

1. a finite **set of firms** $N = \{1, 2, \dots, n\}$;
2. for every $i \in N$, an **individual strategy set** $X_i = [0, q_i] \subset \mathbb{R}_+$ where $x_i \in X_i$ represents the quantity produced by firm i;

3. for every $i \in N$, an **individual profit function** $\pi_i : X_N \longrightarrow \mathbb{R}_+$ defined as:

$$\pi_i(x) = p(X)x_i - C_i(x_i),$$

where $X = \sum_{i \in N} x_i$ is the **total production**.

We denote by $\mathcal{G}_{co} \subseteq \mathcal{G}$ the **set of strategic Cournot oligopoly games**. For any coalition $S \in 2^N \backslash \{\emptyset\}$, the **coalition profit function** $\pi_S : X_S \times X_{N \backslash S} \longrightarrow \mathbb{R}$ is defined as:

$$\pi_S(x_S, x_{N\backslash S}) = \sum_{i \in S} \pi_i(x).$$

Given a set of firms $N = \{1, 2, \ldots, n\}$, a **coalition structure** $\mathcal{P}$ is a partition of N, i.e. $\mathcal{P} = \{S_1, \ldots, S_k\}$, $k \in \{1, \ldots, n\}$. An element of a coalition structure, $S \in \mathcal{P}$, is called an **admissible coalition** in $\mathcal{P}$. We denote by $\mathbf{P_N}$ the **set of coalition structures** with player set N.

The **aggregated strategic Cournot oligopoly game** associated with a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$ and a coalition structure $\mathcal{P} \in \mathbf{P_N}$ is a triplet $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S \in \mathcal{P}}) \in \mathcal{G}_{co}$ defined as:

1. a **set of cartels** (or admissible coalitions) $\mathcal{P} = \{S_1, \ldots, S_k\}$;

2. for every $S \in \mathcal{P}$, an **aggregated coalition strategy set** $X^S = [0, \sum_{i \in S} q_i]$ where $x^S = \sum_{i \in S} x_i \in X^S$ represents the aggregated quantity produced by coalition S;

3. for every $S \in \mathcal{P}$, an **aggregated coalition cost function** $C_S : X^S \longrightarrow \mathbb{R}_+$ defined as:

$$C_S(x^S) = \min_{x_S \in I(x^S)} \sum_{i \in S} C_i(x_i),$$

where $I(x^S) = \{x_S \in X_S : \sum_{i \in S} x_i = x^S\}$ is the set of strategies of coalition S that permit it to produce the quantity x^S; for every $S \in \mathcal{P}$, an **aggregated coalition profit function** $\pi^S : \prod_{S \in \mathcal{P}} X^S \longrightarrow \mathbb{R}$ defined as:

$$\pi^S(x^{\mathcal{P}}) = p(X)x^S - C_S(x^S).$$

We denote by $X^{\mathcal{P}} = \prod_{S \in \mathcal{P}} X^S$ the **set of strategy profiles** and for any admissible coalition $S \in \mathcal{P}$, we denote by $X^{N \setminus S} = \prod_{T \in \mathcal{P} \setminus \{S\}} X^T$ the **set of outsiders' strategy profiles** where $x^{\mathcal{P}}$ and $x^{N \setminus S}$ are the representative elements of $X^{\mathcal{P}}$ and $X^{N \setminus S}$ respectively.

Now, given a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$, we want to associate a cooperative Cournot oligopoly game following the blocking rule according to which outsiders choose their action individually as a best reply to the coalitional action (Chander and Tulkens 1997). As showed in Example 3.3.4, under assumptions (c) and (d) we cannot always define a Cournot oligopoly TU-game in γ-characteristic function form. Recall that given a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$, the associated **Cournot oligopoly TU-game in γ-characteristic function form**, denoted by (N, v_γ), is defined for any coalition $S \in 2^N \setminus \{\emptyset\}$ as:

$$v_\gamma(S) = \pi_S(x_S^*, \tilde{x}_{N \setminus S}),$$

where $(x_S^*, \tilde{x}_{N \setminus S}) \in \varphi^{PA}(\Gamma_{co}, S)$. We denote by $G_{co}^\gamma \subseteq G$ the **set of Cournot oligopoly TU-games in γ-characteristic function form**.

In this section, under assumptions (c) and (d) we show that it is possible to define a Cournot oligopoly interval game in γ-set function form. Given a strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{co}$, the associated **Cournot oligopoly interval game in γ-set function form**, denoted by (N, w_γ), is defined for any coalition $S \in 2^N \setminus \{\emptyset\}$ as:

$$w_\gamma(S) = \pi_S(\varphi^{PA}(\Gamma_{co}, S)).$$

We will show that the set function w_γ is well-defined, i.e. for any coalition $S \in 2^N \setminus \{\emptyset\}$, the worth interval $w_\gamma(S)$ is closed and bounded. The worth interval $w_\gamma(S)$ of any coalition $S \in 2^N \setminus \{\emptyset\}$ is denoted by $[\underline{w}_\gamma(S), \overline{w}_\gamma(S)]$ where $\underline{w}_\gamma(S)$ and $\overline{w}_\gamma(S)$ are the minimal and the maximal profits of S enforced by $\varphi^{PA}(\Gamma_{co}, S)$ respectively. Note that by the compacity of any individual strategy set and the continuity of any individual profit function, there exists a unique partial agreement equilibrium under N. Hence, the worth of the grand coalition N is unique, and so its worth interval $w_\gamma(N)$ is always degenerate, i.e. $\underline{w}_\gamma(N) = \overline{w}_\gamma(N)$. We denote by $IG_{co}^\gamma \subseteq IG$ the **set of Cournot oligopoly interval games in γ-set function form**.

In the remainder of this section we want to show that the γ-set function is well-defined. In order to do that, we adopt a more general approach in which any coalition structure can occur. First, we need to express the Nash equilibrium of any aggregated strategic Cournot oligopoly game as the fixed point of a one-dimensional correspondence. To this end, we define the following notions. Given the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$, for any admissible coalition $S \in \mathcal{P}$ the **aggregated coalition profit function*** $\psi_S : X^S \times X^S \times X^N \longrightarrow \mathbb{R}$ is defined as:

$$\forall x^S \leq X,\ \psi_S(y^S, x^S, X) = p(X - x^S + y^S)y^S - C_S(y^S),$$

and represents the profit of coalition S after changing its strategy from x^S to y^S when the total production was X. For any $S \in \mathcal{P}$, the **best reply correspondence** $R_S : X^N \twoheadrightarrow X^S$ is defined as:

$$R_S(X) = \left\{ x^S \in X^S : x^S \in \arg\max_{y^S \in X^S} \psi_S(y^S, x^S, X) \right\}.$$

The **one-dimensional correspondence** $R_{\mathcal{P}} : X^N \twoheadrightarrow X^N$ is defined as:

$$R_{\mathcal{P}}(X) = \left\{ Y \in X^N : Y = \sum_{S\in\mathcal{P}} x^S \text{ and } \forall S \in \mathcal{P},\ x^S \in R_S(X) \right\}.$$

Given an aggregated strategic Cournot oligopoly game, the following proposition expresses any Nash equilibrium as the fixed point of the one-dimensional correspondence defined above.

Proposition 4.2.1 *Let* $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ *be an aggregated strategic Cournot oligopoly game. Then, it holds that* $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}})$ *if and only if* $\hat{X} \in R_{\mathcal{P}}(\hat{X})$ *such that* $\hat{X} = \sum_{S\in\mathcal{P}} \hat{x}^S$.

Proof: [$\Longrightarrow$]: Let $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}})$ and $\hat{X} = \sum_{S\in\mathcal{P}} \hat{x}^S$. By the definition of the Nash equilibrium, for any $S \in \mathcal{P}$ it holds that:

$$\begin{aligned}
& \pi^S(\hat{x}^S, \hat{x}^{N\setminus S}) = \max_{x^S \in X^S} \pi^S(x^S, \hat{x}^{N\setminus S}) \\
\Longleftrightarrow\ & p(\hat{X} - \hat{x}^S + \hat{x}^S)\hat{x}^S - C_S(\hat{x}^S) = \max_{x^S \in X^S} p(\hat{X} - \hat{x}^S + x^S)x^S - C_S(x^S) \\
\Longleftrightarrow\ & \psi_S(\hat{x}^S, \hat{x}^S, \hat{X}) = \max_{x^S \in X^S} \psi_S(x^S, \hat{x}^S, \hat{X}) \\
\Longleftrightarrow\ & \hat{x}^S \in R_S(\hat{X}).
\end{aligned}$$

Hence, we conclude that $\hat{X} \in R_{\mathcal{P}}(\hat{X})$.
[$\Longleftarrow$]: Let $\hat{X} \in R_{\mathcal{P}}(\hat{X})$. By the definition of $R_{\mathcal{P}}$, it holds that $\hat{X} = \sum_{S\in\mathcal{P}} \hat{x}^S$ and for any $S \in \mathcal{P}$, $\hat{x}^S \in R_S(\hat{X})$. By a similar argument to the one in the first part of the proof, it follows that for any $S \in \mathcal{P}$, $\hat{x}^S \in \arg\max_{x^S \in X^S} \pi^S(x^S, \hat{x}^{N\setminus S})$, and

therefore $\hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}})$. ■

Given an aggregated strategic Cournot oligopoly game, the following proposition establishes some properties on the set of Nash equilibria.

Proposition 4.2.2 *Let $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$ be an aggregated strategic Cournot oligopoly game. Then, it holds that:*

(i) the set of Nash equilibria $\varphi^N(\Gamma_{co}^{\mathcal{P}})$ is a polyhedron;

(ii) the equilibrium total output is the same for any Nash equilibrium:

$$\exists \bar{X} \in X^N : \forall \hat{x}^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}}),\ \sum_{S\in\mathcal{P}} \hat{x}^S = \bar{X};$$

(iii) for any $S \in \mathcal{P}$, the set $\pi^S(\varphi^N(\Gamma_{co}^{\mathcal{P}}))$ is a compact (closed and bounded) real interval.

Proof: First, we show points *(i)* and *(ii)*. For any $S \in \mathcal{P}$, X^S is compact and convex and the aggregated coalition cost function C_S is continuous, strictly increasing and convex. The properties of the aggregated coalition cost function C_S follow from the continuity, the strict monotonicity and the convexity of any individual cost function C_i. Moreover, the inverse demand function p is continuous, strictly decreasing and concave. It follows from Theorem 3.3.3 (page 30) in Okuguchi and Szidarovszky (1990) that $\varphi^N(\Gamma_{co}^{\mathcal{P}})$ is a polyhedron and that the equilibrium total output $\bar{X}$ is the same for any Nash equilibrium which proves points *(i)* and *(ii)*.
Then, we prove point *(iii)*. From Lemma 3.3.1 (page 27) in Okuguchi and Szidarovszky (1990) we deduce for any $S \in \mathcal{P}$ and any $X \in X^N$ that $R_S(X)$ is a (possibly degenerate) closed interval which we denote by $[\alpha_S(X), \beta_S(X)]$. For any $S \in \mathcal{P}$, $R_S(X)$ is also bounded as the subset of the bounded aggregated coalition strategy set X^S. By point *(ii)*, we know that there exists a unique equilibrium total output $\bar{X}$. It follows that the polyhedron $\varphi^N(\Gamma_{co}^{\mathcal{P}})$ can be represented as the intersection of the orthotope (hyperrectangle) $\prod_{S\in\mathcal{P}} R_S(\bar{X}) = \prod_{S\in\mathcal{P}}[\alpha_S(\bar{X}), \beta_S(\bar{X})]$ and the hyperplane $\{x^{\mathcal{P}} \in X^{\mathcal{P}} : \sum_{S\in\mathcal{P}} x^S = \bar{X}\}$:

$$\varphi^N(\Gamma_{co}^{\mathcal{P}}) = \Big\{x^{\mathcal{P}} \in X^{\mathcal{P}} : \forall S \in \mathcal{P},\ x^S \in [\alpha_S(\bar{X}), \beta_S(\bar{X})] \text{ and } \sum_{S\in\mathcal{P}} x^S = \bar{X}\Big\}.$$

The polyhedron $\varphi^N(\Gamma_{co}^{\mathcal{P}})$ is compact and convex as the intersection of two compact and convex sets. Since a convex set is always connected, we deduce that the polyhedron $\varphi^N(\Gamma_{co}^{\mathcal{P}})$ is compact and connected. Moreover, the continuity of the inverse demand function p and any aggregated coalition cost function C_S implies that the aggregated coalition profit function π^S is continuous. It follows that the set $\pi^S(\varphi^N(\Gamma_{co}^{\mathcal{P}}))$ is compact and connected as the image of a compact and connected set by a continuous function. Since a subset of $\mathbb{R}$ is connected if and only if it is an interval, we conclude that $\pi^S(\varphi^N(\Gamma_{co}^{\mathcal{P}}))$ is a compact real interval, which proves point *(iii)*. ■

Point *(ii)* of Proposition 4.2.2 implies that $\bar{X}$ is the unique fixed point of the one-dimensional correspondence $R_{\mathcal{P}}$. Moreover, we deduce from *(iii)* of Proposition 4.2.2 the following corollary.

Corollary 4.2.3 *Let* $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ *be a strategic Cournot oligopoly game. Then for any coalition* $S \in 2^N\backslash\{\emptyset\}$*, the set of profits of* S *enforced by the set of partial agreement equilibria under* S*,* $\pi_S(\varphi^{PA}(\Gamma_{co}, S))$*, is a compact (closed and bounded) real interval.*

Proof: Take any coalition $S \in 2^N\backslash\{\emptyset\}$. Consider the coalition structure $\mathcal{P}^S = \{S\} \cup \{\{i\} : i \in N\backslash S\} \in \mathbf{P_N}$ and the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P}^S} = (\mathcal{P}^S, (X^T, \pi^T)_{T\in\mathcal{P}^S}) \in \mathcal{G}_{co}$. It follows from the first part of the proof of Proposition 3.3.1 that the set of profits of S enforced by $\varphi^{PA}(\Gamma_{co}, S)$ and the set of profits of S enforced by $\varphi^N(\Gamma_{co}^{\mathcal{P}^S})$ are equal:

$$\pi_S(\varphi^{PA}(\Gamma_{co}, S)) = \pi^S(\varphi^N(\Gamma_{co}^{\mathcal{P}^S})).$$

Hence, from point *(iii)* of Proposition 4.2.2 we conclude that $\pi_S(\varphi^{PA}(\Gamma_{co}, S))$ is a compact real interval. ■

Under assumptions (c) and (d), it follows from Corollary 4.2.3 that the γ-set function is well-defined, and so we can always specify a Cournot oligopoly interval game in γ-set function form.

4.3 The non-emptiness of the cores

In this section we deal with the problem of the non-emptiness of the interval core and the standard core on the set of Cournot oligopoly interval games in γ-set function form. First, we introduce a decision theory criterion, the Hurwicz criterion (Hurwicz 1951), which permits to choose, for every $(N, w_\gamma) \in IG_{co}^\gamma$, any of its selection $(N, v_\gamma) \in Sel(N, w_\gamma)$. Then, we provide a necessary and sufficient condition for the non-emptiness of each of the core solutions: the interval core and the standard core. The first result states that the interval core is non-empty if and only if the Cournot oligopoly TU-game associated with the best worth of every coalition in its worth interval admits a non-empty core. However, we show that even for a very simple Cournot oligopoly situation, this condition fails to be satisfied. The second result states that the standard core is non-empty if and only if the Cournot oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty core. Moreover, we give some properties on any individual profit function and any individual cost function under which this condition always holds. This result substantially extends the results in Theorems 3.4.1 and 3.5.1.

4.3.1 The Hurwicz criterion

A Cournot oligopoly interval game in γ-set function form $(N, w_\gamma) \in IG^\gamma_{co}$ fits all the situations where any coalition $S \in 2^N \backslash \{\emptyset\}$ knows with certainty only the lower and upper bounds $\underline{w}_\gamma(S)$ and $\overline{w}_\gamma(S)$ of all its potential worths. Consequently, the expectations of any coalition $S \in 2^N \backslash \{\emptyset\}$ on its potential worths are necessarily focused on its worth interval $w_\gamma(S)$. In order to define the expectations of any coalition $S \in 2^N \backslash \{\emptyset\}$, we use a decision theory criterion, the Hurwicz criterion (Hurwicz 1951), that consists in doing a convex combination of the lower and upper bounds of all its potential worths, i.e. $\mu_S \underline{w}_\gamma(S) + (1 - \mu_S)\overline{w}_\gamma(S)$ where $\mu_S \in [0, 1]$. The real number $\mu_S \in [0, 1]$ can be regarded as the **degree of pessimism** of coalition S. A vector $\mu = (\mu_S)_{S \in 2^N}$ is an **expectation vector**. Given an expectation vector $\mu \in [0, 1]^{2^N}$, the associated Cournot oligopoly TU-game, denoted by $(N, v^\mu_\gamma) \in G^\gamma_{co}$, is defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$v^\mu_\gamma(S) = \mu_S \underline{w}_\gamma(S) + (1 - \mu_S)\overline{w}_\gamma(S),$$

where $(N, v^\mu_\gamma) \in Sel(N, w_\gamma)$. Each of the two necessary and sufficient conditions provided below is derived from a particular selection of (N, w_γ), i.e. $(N, v^0_\gamma) = (N, \overline{w}_\gamma)$ and $(N, v^1_\gamma) = (N, \underline{w}_\gamma)$ respectively.

4.3.2 The non-emptiness of the interval core

Given a Cournot oligopoly interval game in γ-set function form, the following theorem states that the interval core is non-empty if and only if the Cournot oligopoly TU-game associated with the minimum degree of pessimism of every coalition admits a non-empty core.

Theorem 4.3.1 *The Cournot oligopoly interval game $(N, w_\gamma) \in IG^\gamma_{co}$ has a non-empty interval core if and only if the Cournot oligopoly TU-game $(N, v^0_\gamma) \in Sel(N, w_\gamma)$ has a non-empty core.*

Proof: [$\Longrightarrow$]: Assume that $\mathcal{C}(N, w_\gamma) \neq \emptyset$ and take any payoff interval vector $I \in \mathcal{C}(N, w_\gamma)$. Then, by the definition of the interval core it holds that $\sum_{i\in N} I_i = w_\gamma(N)$ implying that $\sum_{i\in N} \overline{I}_i = \overline{w}_\gamma(N)$, and for any $S \in 2^N$ it holds that $\sum_{i\in S} I_i \succcurlyeq w_\gamma(S)$ implying that $\sum_{i\in S} \overline{I}_i \geq \overline{w}_\gamma(S)$. Let $\sigma \in \mathbb{R}^n$ be a payoff vector such that for any $i \in N$, $\sigma_i = \overline{I}_i$. It follows from $\overline{w}_\gamma = v^0_\gamma$ that $\sum_{i\in N} \sigma_i = v^0_\gamma(N)$ and for any $S \in 2^N$, $\sum_{i\in S} \sigma_i \geq v^0_\gamma(S)$. Hence, we conclude that $\sigma \in C(N, v^0_\gamma)$.
[$\Longleftarrow$]: Assume that $C(N, v^0_\gamma) \neq \emptyset$. By the balancedness property, it holds for every balanced map λ that:

$$\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S) v^0_\gamma(S) \leq v^0_\gamma(N) \tag{4.1}$$

Since the worth interval of the grand coalition is always degenerate, we have $v^0_\gamma(N) = \overline{w}_\gamma(N) = \underline{w}_\gamma(N)$. Hence, from $v^0_\gamma = \overline{w}_\gamma$ and (4.1) we deduce that the Cournot oligopoly interval game $(N, w_\gamma) \in IG^\gamma_{co}$ is strongly balanced, i.e. for every balanced map λ it holds that:

$$\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S) \overline{w}_\gamma(S) \leq \underline{w}_\gamma(N).$$

By *(i)* and *(ii)* of Theorem 2.4.1, we conclude that $(N, w_\gamma) \in IG^\gamma_{co}$ is $\mathcal{I}$-balanced, and therefore has a non-empty interval core. ■

One can ask which properties on any individual profit function or any individual cost function guarantee the non-emptiness of the core $C(N, v^0_\gamma)$. The following example shows that even for a very simple Cournot oligopoly situation, this condition fails to be satisfied.

Example 4.3.2

Consider the Cournot oligopoly interval game $(N, w_\gamma) \in IG^\gamma_{co}$ associated with the strategic Cournot oligopoly game $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ derived from the Cournot oligopoly situation $(N, (q_i, C_i)_{i\in N}, p)$ where $N = \{1, 2, 3\}$, for any $i \in N$, $q_i = 5/3$ and $C_i(x_i) = 97x_i$, and the inverse demand function is defined as:

$$p(X) = \begin{cases} 103 - X & \text{if } 0 \leq X \leq 3; \\ 50(5 - X) & \text{if } 3 < X \leq 5. \end{cases}$$

Clearly, the inverse demand function p is continuous, piecewise linear and concave but it is not differentiable at point $\bar{X} = 3$. Assume that coalition $\{2, 3\}$ forms. We show that a strategy profile $x \in \varphi^{PA}(\Gamma_{co}, \{2, 3\})$ if and only if it

satisfies *(i)* $X = \bar{X}$ and *(ii)* $x_2 + x_3 \in [4/3, 147/50]$.
[$\Longleftarrow$]: Take any $x \in X_N$ satisfying *(i)* and *(ii)*. By *(i)* it holds that:

$$\pi_1(x) = 3x_1,$$

and

$$\pi_2(x) + \pi_3(x) = 3(x_2 + x_3).$$

If firm 1 increases its output by $\epsilon \in]0, 5/3 - x_1]$, its new profit will be:

$$\pi_1(x_1 + \epsilon, x_2, x_3) = (3 - 50\epsilon)(x_1 + \epsilon).$$

Conversely, if it decides to decrease its output by $\delta \in]0, x_1]$, it will obtain:

$$\pi_1(x_1 - \delta, x_2, x_3) = (3 + \delta)(x_1 - \delta).$$

Similarly, if coalition $\{2, 3\}$ increases its output by $\epsilon + \epsilon' \in]0, 10/3 - x_2 - x_3]$ where $\epsilon \in [0, 5/3 - x_2]$ and $\epsilon' \in [0, 5/3 - x_3]$, its new coalition profit will be:

$$\pi_{\{2,3\}}(x_1, x_2 + \epsilon, x_3 + \epsilon') = \big(3 - 50(\epsilon + \epsilon')\big)(x_2 + x_3 + \epsilon + \epsilon').$$

On the contrary, if it decreases its output by $\delta + \delta' \in]0, x_2 + x_3]$ where $\delta \in [0, x_2]$ and $\delta' \in [0, x_3]$, it will obtain:

$$\pi_{\{2,3\}}(x_1, x_2 - \delta, x_3 - \delta') = (3 + \delta + \delta')(x_2 + x_3 - \delta - \delta').$$

In all cases, given *(ii)*, neither firm 1 nor coalition $\{2, 3\}$ can improve their profits. We conclude that any strategy profile $x \in X_N$ satisfying *(i)* and *(ii)* is a partial agreement equilibrium under $\{2, 3\}$.
[$\Longrightarrow$]: Take any $x \in \varphi^{PA}(\Gamma_{co}, \{2, 3\})$. By the first part of the proof and by point *(ii)* of Proposition 4.2.2 it holds that $\bar{X} = 3$ is the unique equilibrium total output. It follows that $x \in \varphi^{PA}(\Gamma_{co}, \{2, 3\})$ is such that $X = \bar{X}$. Moreover, given *(i)* and by the above four equalities we deduce that any $x \in \varphi^{PA}(\Gamma_{co}, \{2, 3\})$ is such that $x_2 + x_3 \in [4/3, 147/50]$.
Hence, by *(i)* and *(ii)* we conclude that the worth interval of coalition $\{2, 3\}$ is $w_\gamma(\{2, 3\}) = [4, 8.82]$.
In a similar way, we can compute the worth intervals of the other coalitions $S \in 2^N \backslash \{\emptyset\}$ summarized in the following table:

S	$\{i\}$	$\{i, j\}$	N
$w_\gamma(S)$	$[0.18, 5]$	$[4, 8.82]$	$[9, 9]$

We can check that $\sum_{i\in N} v^0_\gamma(\{i\}) = 15 > 9 = v^0_\gamma(N)$, and so the Cournot oligopoly TU-game $(N, v^0_\gamma) \in Sel(N, w_\gamma)$ is non-essential which implies that $C(N, v^0_\gamma) = \emptyset$. It follows from Theorem 4.3.1 that the Cournot oligopoly interval game (N, w_γ) admits an empty interval core. This result is a consequence of the non-differentiability of the inverse demand function p at point $\bar{X} = 3$. Indeed, at this point it is possible for some deviating coalition to obtain a large coalition profit on a partial agreement equilibrium since there is no incentive for other firms to change their outputs on any neighborhood at point $\bar{X} = 3$. □

4.3.3 The non-emptiness of the standard core

Given a Cournot oligopoly interval game in γ-set function form, the following theorem states that the standard core is equal to the core of the Cournot oligopoly TU-game associated with the maximum degree of pessimism of every coalition.

Theorem 4.3.3 *Let $(N, w_\gamma) \in IG^\gamma_{co}$ be a Cournot oligopoly interval game such that $(N, v^1_\gamma) \in Sel(N, w_\gamma)$. Then $C(N, w_\gamma) = C(N, v^1_\gamma)$.*

Proof: First, it follows from $(N, v^1_\gamma) \in Sel(N, w_\gamma)$ that:

$$C(N, v^1_\gamma) \subseteq \bigcup_{(N,v^\mu_\gamma)\in Sel(N,w_\gamma)} C(N, v^\mu_\gamma) = C(N, w_\gamma).$$

Then, it remains to show that $C(N, w_\gamma) \subseteq C(N, v^1_\gamma)$. If $C(N, w_\gamma) = \emptyset$ we have obviously $C(N, w_\gamma) \subseteq C(N, v^1_\gamma)$. So, assume that $C(N, w_\gamma) \neq \emptyset$ and take any payoff vector $\sigma \in C(N, w_\gamma)$. Thus, there exists an expectation vector $\bar{\mu} \in [0, 1^{2^N}]$ such that $\sigma \in C(N, v^{\bar{\mu}}_\gamma)$:

$$\forall S \in 2^N,\ \sum_{i\in S} \sigma_i \geq v^{\bar{\mu}}_\gamma(S) \text{ and } \sum_{i\in N} \sigma_i = v^{\bar{\mu}}_\gamma(N) \tag{4.2}$$

Since the worth interval of the grand coalition N is degenerate we have $v^{\bar{\mu}}_\gamma(N) = v^1_\gamma(N)$, and therefore by (4.2), $\sum_{i\in N} \sigma_i = v^1_\gamma(N)$. Moreover, by the definition of the Hurwicz criterion it holds that $v^{\bar{\mu}}_\gamma \geq v^1_\gamma$ implying by (4.2) that for any $S \in 2^N$, $\sum_{i\in S} \sigma_i \geq v^1_\gamma(S)$. Hence, we conclude that $\sigma \in C(N, v^1_\gamma)$ which proves that $C(N, w_\gamma) \subseteq C(N, v^1_\gamma)$. ■

It follows from Theorem 4.3.3 that a Cournot oligopoly interval game $(N, w_\gamma) \in IG^\gamma_{co}$ has a non-empty standard core if and only if the Cournot oligopoly TU-game $(N, v^1_\gamma) \in Sel(N, w_\gamma)$ has a non-empty core. By defining the **standard core*** of an interval game $(N, w) \in IG$ as the intersection of the cores of all its selections $(N, v) \in G$:

$$C^*(N,w) = \bigcap_{(N,v)\in Sel(N,w)} C(N,v),$$

we obtain the opposite result to Theorem 4.3.3, i.e. $C^*(N,w_\gamma) = C(N,v_\gamma^0)$.

Once again, one can ask which properties on any individual profit function or any individual cost function guarantee the non-emptiness of the core $C(N,v_\gamma^1)$, and so by Theorem 4.3.3 the non-emptiness of the standard core $C(N,w_\gamma)$. In the remainder of this section, given a Cournot oligopoly interval game $(N,w_\gamma) \in IG_{co}^\gamma$, we show that under conditions in Theorems 3.4.1 and 3.5.1, the core of the Cournot oligopoly TU-game $(N,v_\gamma^1) \in Sel(N,w_\gamma)$ is non-empty which generalizes the results on the non-emptiness of the core in Theorems 3.4.1 and 3.5.1.
First, we denote by $\mathcal{X}$ the **denumerable set of points where the inverse demand function p is non-differentiable**. The concavity of the inverse demand function p ensures that $\mathcal{X}$ is at most denumerable. The Weierstrass approximation theorem states that any continuous function defined on a compact real interval can be uniformly approximated as closely as desired by a sequence of polynomial functions. Proposition 4.5.1 in the appendix states that there always exists a **sequence of differentiable, strictly decreasing and concave inverse demand functions**, denoted by $(p_\epsilon)_{\epsilon>0}$, that uniformly converges to the inverse demand function $p_0 = p$, i.e. for any $\zeta > 0$, there exists $\epsilon' > 0$ such that for any $\epsilon < \epsilon'$, it holds that:

$$\forall X \in X^N,\ |p_\epsilon(X) - p(X)| < \zeta.$$

Then, we generalize some above definitions. First, given a sequence $(p_\epsilon)_{\epsilon>0}$, for any $\epsilon > 0$, we denote by $\Gamma_{co}^\epsilon = (N,(X_i,\pi_i^\epsilon)_{i\in N}) \in \mathcal{G}_{co}$ the strategic Cournot oligopoly game where for any $i \in N$, the individual profit function $\pi_i^\epsilon : X_N \longrightarrow \mathbb{R}$ is defined as:

$$\pi_i^\epsilon(x) = p_\epsilon(X)x_i - C_i(x_i).$$

Given a strategic Cournot oligopoly game $\Gamma_{co}^\epsilon = (N,(X_i,\pi_i^\epsilon)_{i\in N}) \in \mathcal{G}_{co}$ the associated Cournot oligopoly TU-game in γ-characteristic function form, denoted by $(N,v_\gamma^\epsilon) \in G_{co}^\gamma$, is defined for any coalition $S \in 2^N\backslash\{\emptyset\}$ as:

$$v_\gamma^\epsilon(S) = \pi_S^\epsilon(x_S^*, \tilde{x}_{N\backslash S}),$$

where $(x_S^*, \tilde{x}_{N\backslash S}) \in \varphi^{PA}(\Gamma_{co}^\epsilon, S)$. Since p_ϵ is differentiable, Corollary 3.3.2 ensures that the γ-characteristic function v_γ^ϵ is well-defined.
Then, given a strategic Cournot oligopoly game $\Gamma_{co}^\epsilon = (N,(X_i,\pi_i^\epsilon)_{i\in N}) \in \mathcal{G}_{co}$, and a coalition structure $\mathcal{P} \in \mathbf{P_N}$, we denote by $\Gamma_{co}^{\mathcal{P},\epsilon} = (\mathcal{P},(X^S,\pi_\epsilon^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$

the aggregated strategic Cournot oligopoly game where for any $S \in \mathcal{P}$, the aggregated coalition profit function $\pi_\epsilon^S : X^{\mathcal{P}} \longrightarrow \mathbb{R}$ is defined as:

$$\pi_\epsilon^S(x^{\mathcal{P}}) = p_\epsilon(X)x^S - C_S(x^S).$$

Finally, given any aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P},\epsilon} = (\mathcal{P}, (X^S, \pi_\epsilon^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$, for any $S \in \mathcal{P}$ the aggregated coalition profit function* $\psi_S^\epsilon : X^S \times X^S \times X^N \longrightarrow \mathbb{R}$ is defined as:

$$\forall x^S \leq X,\ \psi_S^\epsilon(y^S, x^S, X) = p_\epsilon(X - x^S + y^S)y^S - C_S(y^S).$$

For any $S \in \mathcal{P}$, the best reply correspondence $R_S^\epsilon : X^N \twoheadrightarrow X^S$ is defined as:

$$R_S^\epsilon(X) = \left\{x^S \in X^S : x^S \in \arg\max_{y^S\in X^S} \psi_S^\epsilon(y^S, x^S, X)\right\}.$$

The one-dimensional correspondence $R_{\mathcal{P}}^\epsilon : X^N \twoheadrightarrow X^N$ is defined as:

$$R_{\mathcal{P}}^\epsilon(X) = \left\{Y \in X^N : Y = \sum_{S\in\mathcal{P}} x^S \text{ and } \forall S \in \mathcal{P},\ x^S \in R_S^\epsilon(X)\right\}.$$

Given a Cournot oligopoly interval game, the following result states that if any Cournot oligopoly TU-game $(N, v_\gamma^\epsilon) \in G_{co}^\gamma$ associated with the sequence $(p_\epsilon)_{\epsilon>0}$ admits a non-empty core, then the standard is non-empty.

Theorem 4.3.4 *Let $(N, w_\gamma) \in IG_{co}^\gamma$ be a Cournot oligopoly interval game and $(p_\epsilon)_{\epsilon>0}$ a sequence that uniformly converges to p. If for any $\epsilon > 0$, the Cournot oligopoly TU-game $(N, v_\gamma^\epsilon) \in G_{co}^\gamma$ admits a non-empty core, then by the definition of the Hurwicz criterion there exists an expectation vector $\bar{\mu} \in [0,1]^{2^N}$ such that $C(N, v_\gamma^{\bar{\mu}}) \neq \emptyset$, and so $C(N, w_\gamma) \neq \emptyset$.*

In order to establish the proof of Theorem 4.3.4, we first need the following two lemmas. In these lemmas, for any $\epsilon > 0$, we denote by $\hat{x}_\epsilon^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P},\epsilon})$ the unique Nash equilibrium of the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P},\epsilon} = (\mathcal{P}, (X^S, \pi_\epsilon^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$. This uniqueness result is established in the proof of Proposition 3.3.1. Moreover, from *(ii)* of Proposition 4.2.2 we denote by $\bar{X}$ the unique equilibrium total output of the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P}} = (\mathcal{P}, (X^S, \pi^S)_{S\in\mathcal{P}}) \in \mathcal{G}_{co}$.

Lemma 4.3.5 *Let $\mathcal{P} \in \mathbf{P_N}$ be a coalition structure, $(p_\epsilon)_{\epsilon>0}$ a sequence that uniformly converges to p and $(\hat{x}_\epsilon^{\mathcal{P}})_{\epsilon>0}$ the associated sequence of strategy profiles such that for any $\epsilon > 0$, $\hat{x}_\epsilon^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P},\epsilon})$. If the sequence $(\hat{x}_\epsilon^{\mathcal{P}})_{\epsilon>0}$ converges to a strategy profile $\hat{x}_0^{\mathcal{P}} \in X^{\mathcal{P}}$ then it holds that:*

(i) $\sum_{S\in\mathcal{P}} \hat{x}_0^S = \bar{X}$;

(ii) $\forall S \in \mathcal{P}$, $\hat{x}_0^S \in R_S(\bar{X})$;

(iii) $\hat{x}_0^{\mathcal{P}} \in \varphi^N(\Gamma_{co}^{\mathcal{P}})$.

Proof: From Proposition 4.2.1, for any $\epsilon > 0$ we have $\sum_{S\in\mathcal{P}} \hat{x}_\epsilon^S = \hat{X}_\epsilon \in R_{\mathcal{P}}^\epsilon(\hat{X}_\epsilon)$. By the definitions of R_S^ϵ and $R_{\mathcal{P}}^\epsilon$, for any $\epsilon > 0$ it holds that:

$$\forall S \in \mathcal{P},\ \psi_S^\epsilon(\hat{x}_\epsilon^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) = \max_{x^S\in X^S} \psi_S^\epsilon(x^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) \tag{4.3}$$

For any $S \in \mathcal{P}$, the uniform convergence of the sequence $(p_\epsilon)_{\epsilon>0}$ to p implies that the sequence $(\psi_S^\epsilon)_{\epsilon>0}$ uniformly converges to ψ_S. This result, the continuity of any aggregated coalition profit function* ψ_S^ϵ and (4.3) imply for any $S \in \mathcal{P}$ that:

$$\begin{aligned}
& \lim_{\epsilon\longrightarrow 0} \psi_S^\epsilon(\hat{x}_\epsilon^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) = \lim_{\epsilon\longrightarrow 0} \max_{x^S\in X^S} \psi_S^\epsilon(x^S, \hat{x}_\epsilon^S, \hat{X}_\epsilon) \\
\Longleftrightarrow\ & \lim_{\epsilon\longrightarrow 0} \psi_S^\epsilon\left(\hat{x}_\epsilon^S, \hat{x}_\epsilon^S, \sum_{T\in\mathcal{P}} \hat{x}_\epsilon^T\right) = \max_{x^S\in X^S} \lim_{\epsilon\longrightarrow 0} \psi_S^\epsilon\left(x^S, \hat{x}_\epsilon^S, \sum_{T\in\mathcal{P}} \hat{x}_\epsilon^T\right) \\
\Longleftrightarrow\ & \psi_S\left(\hat{x}_0^S, \hat{x}_0^S, \sum_{T\in\mathcal{P}} \hat{x}_0^T\right) = \max_{x^S\in X^S} \psi_S\left(x^S, \hat{x}_0^S, \sum_{T\in\mathcal{P}} \hat{x}_0^T\right) \\
\Longleftrightarrow\ & \hat{x}_0^S \in R_S\left(\sum_{T\in\mathcal{P}} \hat{x}_0^T\right)
\end{aligned} \tag{4.4}$$

It follows from (4.4) that $\sum_{S\in\mathcal{P}} \hat{x}_0^S \in R_{\mathcal{P}}(\sum_{S\in\mathcal{P}} \hat{x}_0^S)$. From *(ii)* of Proposition 4.2.2, $\bar{X}$ is the unique fixed point of $R_{\mathcal{P}}$. Hence, we deduce that $\sum_{S\in\mathcal{P}} \hat{x}_0^S = \bar{X}$, and therefore by (4.4) for any $S \in \mathcal{P}$, $\hat{x}_0^S \in R_S(\bar{X})$ which proves points *(i)* and *(ii)*.
Finally, point *(iii)* is a consequence of points *(i)* and *(ii)* by Proposition 4.2.1.■

Lemma 4.3.6 *Let $S \in 2^N\backslash\{\emptyset\}$ be a coalition, $(p_\epsilon)_{\epsilon>0}$ a sequence that uniformly converges to p and $(\hat{x}_\epsilon^{\mathcal{P}^S})_{\epsilon>0}$ the associated sequence of strategy profiles such that for any $\epsilon > 0$, $\hat{x}_\epsilon^{\mathcal{P}^S} \in \varphi^N(\Gamma_{co}^{\mathcal{P}^S,\epsilon})$. If the sequence $(\hat{x}_\epsilon^{\mathcal{P}^S})_{\epsilon>0}$ converges to a strategy profile $\hat{x}_0^{\mathcal{P}^S} \in X^{\mathcal{P}^S}$ then it holds that $\lim_{\epsilon\longrightarrow 0} v_\gamma^\epsilon(S) \in w_\gamma(S)$.*

Proof: Take any $\epsilon > 0$. By the first part of the proof of Proposition 3.3.1 we know that the set of profits of S enforced by $\varphi^{PA}(\Gamma_{co}^\epsilon, S)$ and the set of profits of S enforced by $\varphi^N(\Gamma_{co}^{\mathcal{P}^S,\epsilon})$ are equal, i.e. $\pi_S^\epsilon(\varphi^{PA}(\Gamma_{co}^\epsilon, S)) = \pi_\epsilon^S(\varphi^N(\Gamma_{co}^{\mathcal{P}^S,\epsilon}))$. Hence, for any $\epsilon > 0$ it holds that:

$$\begin{aligned} v_\gamma^\epsilon(S) &= \pi_S^\epsilon(x_S^*, \tilde{x}_{N\backslash S}) \\ &= \pi_\epsilon^S(\hat{x}_\epsilon^{\mathcal{P}^S}), \end{aligned}$$

where $\hat{x}_\epsilon^{\mathcal{P}^S} \in \varphi^N(\Gamma_{co}^{\mathcal{P}^S \epsilon})$ is the unique Nash equilibrium of the aggregated strategic Cournot oligopoly game $\Gamma_{co}^{\mathcal{P}^S,\epsilon}$. The uniform convergence of the sequence $(p_\epsilon)_{\epsilon>0}$ to p implies that the sequence $(\pi_\epsilon^S)_{\epsilon>0}$ uniformly converges to π^S. It follows from this result and the continuity of π^S that:

$$\begin{aligned} \lim_{\epsilon \longrightarrow 0} v_\gamma^\epsilon(S) &= \lim_{\epsilon \longrightarrow 0} \pi_\epsilon^S(\hat{x}_\epsilon^{\mathcal{P}^S}) \\ &= \pi^S(\hat{x}_0^{\mathcal{P}^S}). \end{aligned}$$

From *(iii)* of Lemma 4.3.5 it holds that $\hat{x}_0^{\mathcal{P}^S} \in \varphi^N(\Gamma_{co}^{\mathcal{P}^S})$. Hence, from the above equality we deduce that $\lim_{\epsilon \longrightarrow 0} v_\gamma^\epsilon(S) \in \pi^S(\varphi^N(\Gamma_{co}^{\mathcal{P}^S}))$. By the first part of the proof of Proposition 3.3.1, we know that the set of profits of S enforced by $\varphi^{PA}(\Gamma_{co}, S)$ and the set of profits of S enforced by $\varphi^N(\Gamma_{co}^{\mathcal{P}^S})$ are equal. Thus, by the definition of the γ-set function it holds that:

$$\begin{aligned} \pi^S(\varphi^N(\Gamma_{co}^{\mathcal{P}^S})) &= \pi_S(\varphi^{PA}(\Gamma_{co}, S)) \\ &= w_\gamma(S). \end{aligned}$$

Hence, we conclude that $\lim_{\epsilon \longrightarrow 0} v_\gamma^\epsilon(S) \in w_\gamma(S)$. ■

Now, we are ready to establish the proof of Theorem 4.3.4.

Proof (of Theorem 4.3.4): By the definition of the core, for any $\epsilon > 0$ there exists a payoff vector $\sigma^\epsilon \in \mathbb{R}^n$ such that:

$$\forall S \in 2^N, \ \sum_{i \in S} \sigma_i^\epsilon \geq v_\gamma^\epsilon(S) \text{ and } \sum_{i \in N} \sigma_i^\epsilon = v_\gamma^\epsilon(N) \tag{4.5}$$

By (4.5), the sequence $(\sigma^\epsilon)_{\epsilon>0}$ is bounded in $\mathbb{R}^n$. Thus, there exists a subsequence of $(\sigma^\epsilon)_{\epsilon>0}$ that converges to a point $\sigma^0 \in \mathbb{R}^n$. Without loss of generality we denote by $(\sigma^\epsilon)_{\epsilon>0}$ such a subsequence.

First, take any coalition $S \in 2^N \backslash \{\emptyset\}$ and consider the coalition structure $\mathcal{P}^S = \{S\} \cup \{\{i\} : i \notin S\}$. By the compacity of any aggregated coalition strategy set X^T, $T \in \mathcal{P}^S$, there exists a subsequence of $(\hat{x}_\epsilon^{\mathcal{P}^S})_{\epsilon>0}$, denoted by $(\hat{x}_{\epsilon_k}^{\mathcal{P}^S})_{\epsilon_k>0}$, $k \in \mathbb{N}$, that converges to a strategy profile $\hat{x}_0^{\mathcal{P}^S} \in \varphi^N(\Gamma_{co}^{\mathcal{P}^S})$ by point *(iii)* of Lemma 4.3.5. Thus, by (4.5) it holds that:

$$\lim_{\epsilon_k \longrightarrow 0} \sum_{i \in S} \sigma_i^{\epsilon_k} \geq \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(S) \Longleftrightarrow \sum_{i \in S} \sigma_i^0 \geq \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(S).$$

It follows from Lemma 4.3.6 that for any $S \in 2^N$, $\lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(S) \in w_\gamma(S)$. From this result, we deduce that there exists an expectation vector $\bar{\mu}$ such that:

$$\forall S \in 2^N, \ \sum_{i \in S} \sigma_i^0 \geq v_\gamma^{\bar{\mu}}(S) \tag{4.6}$$

Then, consider the grand coalition $N \in 2^N \backslash \{\emptyset\}$. By a similar argument to the one in the first part of the proof and (4.5) it holds that:

$$\lim_{\epsilon_k \longrightarrow 0} \sum_{i \in N} \sigma_i^{\epsilon_k} = \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(N) \Longleftrightarrow \sum_{i \in N} \sigma_i^0 = \lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(N).$$

It follows from Lemma 4.3.6 that $\lim_{\epsilon_k \longrightarrow 0} v_\gamma^{\epsilon_k}(N) \in w_\gamma(N)$. As the worth interval of the grand coalition is degenerate, it holds that:

$$\sum_{i \in N} \sigma_i^0 = v_\gamma^{\bar{\mu}}(N) \tag{4.7}$$

By (4.6) and (4.7) we conclude that $\sigma^0 \in C(N, v_\gamma^{\bar{\mu}}) \subseteq C(N, w_\gamma)$ since $(N, v_\gamma^{\bar{\mu}}) \in Sel(N, w_\gamma)$. ■

We deduce from Theorems 3.4.1, 3.5.1 and 4.3.4 the following theorem.

Theorem 4.3.7 *Let $(N, w_\gamma) \in IG_{co}^\gamma$ be a Cournot oligopoly interval game and $(p_\epsilon)_{\epsilon > 0}$ a sequence that uniformly converges to p where for any $\epsilon > 0$, the strategic Cournot oligopoly game $\Gamma_{co}^\epsilon = (N, (X_i, \pi_i^\epsilon)_{i \in N}) \in \mathcal{G}_{co}$ is such that: either any individual profit function π_i^ϵ is concave on X_N, or any individual cost function C_i satisfies:*

$$\exists c \in \mathbb{R}_+ : \forall i \in N, \ C_i(x_i) = cx_i.$$

Then, it holds that $C(N, w_\gamma) \neq \emptyset$.

Theorem 4.3.7 generalizes Theorems 3.4.1 and 3.5.1. Indeed, if the inverse demand function p is differentiable, all the worth intervals of $(N, w_\gamma) \in IG_{co}^\gamma$ are degenerate, i.e. $w_\gamma = \{v_\gamma\}$ where $(N, v_\gamma) \in G_{co}^\gamma$. Thus, the standard core of (N, w_γ) is equal to the core of (N, v_γ). It remains to take the constant sequence

$(\bar{p}_\epsilon)_{\epsilon>0}$ such that for any $\epsilon > 0$, $\bar{p}_\epsilon = p$ in order to obtain an equivalent formulation of Theorems 3.4.1 and 3.5.1.

The following theorem provides another equivalent generalization of Theorem 3.4.1.

Theorem 4.3.8 *Let $\Gamma_{co} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{co}$ be a strategic Cournot oligopoly game such that for any $i \in N$, π_i is concave on X_N. Then the corresponding Cournot oligopoly interval game $(N, w_\gamma) \in IG^\gamma_{co}$ admits a non-empty standard core.*

Proof: First, by a similar proof to the one in Proposition 4.5.1, for any concave individual profit function π_i on X_N (not necessarily differentiable), by using Bézier curves it is possible to construct a sequence of differentiable and concave individual profit functions $(\pi_i^\epsilon)_{\epsilon>0}$ which uniformly converges to $\pi_i^0 = \pi_i$. Hence, it follows from Theorem 3.4.1 that for any $\epsilon > 0$, the associated oligopoly TU-game $(N, v_\gamma^\epsilon) \in G^\gamma_{co}$ admits a non-empty core. Finally, by applying Theorem 4.3.4 we conclude from the definition of the Hurwicz criterion that there exists an expectation vector $\bar{\mu} \in [0,1]^{2^N}$ such that $C(N, v_\gamma^{\bar{\mu}}) \neq \emptyset$, and so $C(N, w_\gamma) \neq \emptyset$. ■

4.4 Concluding remarks

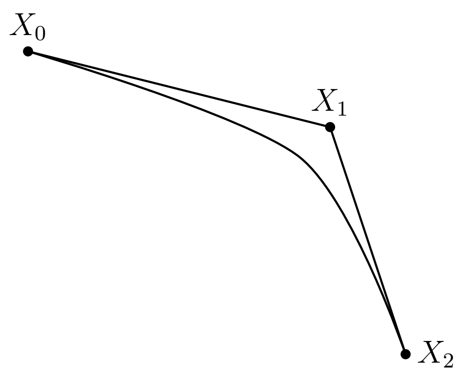
In this chapter, we have focused on Cournot oligopoly interval games in γ-set function form. Although Example 3.3.4 shows that the continuity of the inverse demand function is not sufficient in order to guarantee that the γ-characteristic function is well-defined, we have showed that we can always specify a Cournot oligopoly interval game. As far as we know, this is the first time that this set of games is modeled in oligopoly theory. We have dealt with two extensions of the core: the interval core and the standard core. We have provided a necessary and sufficient condition for the non-emptiness of each of these core solutions. The first result states that the interval core is non-empty if and only if the Cournot oligopoly TU-game associated with the best worth of every coalition in its worth interval admits a non-empty core. However, we have showed that even for a very simple Cournot oligopoly situation, this condition fails to be satisfied. The second result states that the standard core is non-empty if and only if the Cournot oligopoly TU-game associated with the worst worth of every coalition in its worth interval admits a non-empty core. Moreover, we have provided some properties on any individual profit function and any individual cost function under which this condition always holds which substantially extends the results in Theorems 3.4.1 and 3.5.1.

Many economic situations such that an economy with environmental externalities (Helm 2001) where any individual utility function is continuous but not necessarily differentiable can be described by means of interval games. In such models, we expect that similar conditions on players' individual utility functions will be sufficient in order to guarantee the non-emptiness of the interval core and the standard core.

4.5 Appendix

Given a continuous, strictly decreasing and concave inverse demand function p, we construct a sequence of differentiable, strictly decreasing and concave inverse demand functions denoted by $(p_\epsilon)_{\epsilon>0}$ that uniformly converges to p by using Bézier curves (Bézier 1976). Bézier curves are still the object of a wide literature in Mathematics and are used in industrial car design.

A **Bézier curve** is a parametric curve defined through specific points called **control points**. A particular set of Bézier curves are quadratic Bézier curves defined with three control points X_0, X_1 and X_2 as illustrated by the following figure:

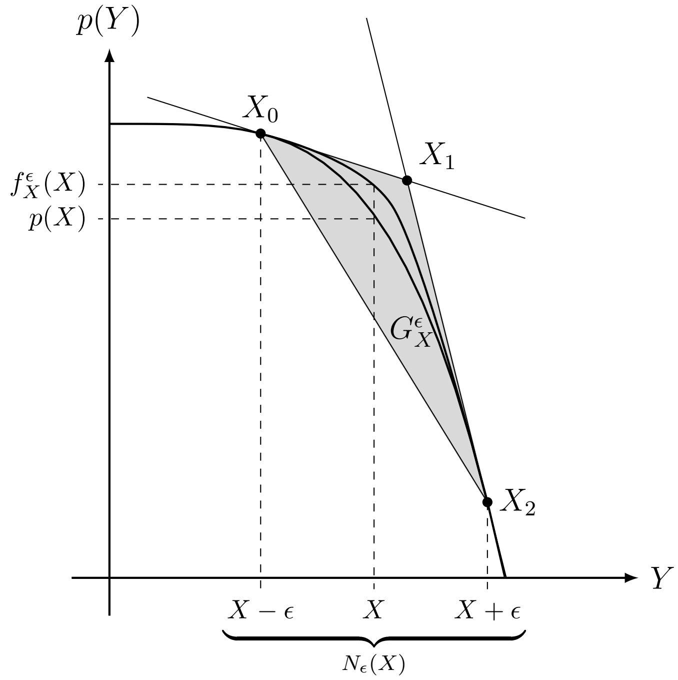


Formally, this quadratic Bézier curve is the path traced by the mapping $B : [0,1] \longrightarrow \mathbb{R}^2$ defined as:

$$B(t) = (1-t)^2 X_0 + 2(1-t)tX_1 + t^2 X_2 \tag{4.8}$$

Proposition 4.5.1 *Let p be a continuous, strictly decreasing and concave inverse demand function. Then, there exists a sequence of differentiable, strictly decreasing and concave inverse demand functions $(p_\epsilon)_{\epsilon>0}$ that uniformly converges to p.*

Proof: First, for any $X \in \mathcal{X}$ (the set of points where p is non-differentiable) and any $\epsilon > 0$, we define a quadratic Bézier curve. The steps of this construction are illustrated by the following figure:



For any $X \in \mathcal{X}$, the neighborhood of X with radius $\epsilon > 0$ is defined as:

$$N_\epsilon(X) = \{Y \in \mathbb{R}_+ : |Y - X| < \epsilon\}.$$

Since $\mathcal{X}$ is at most denumerable, there exists $\bar{\epsilon} > 0$ such that for any $\epsilon < \bar{\epsilon}$ it holds that:

$$\forall X \in \mathcal{X}, \forall X' \in \mathcal{X},\; N_\epsilon(X) \cap N_\epsilon(X') = \emptyset.$$

In the remainder of the proof, we assume everywhere that $\epsilon < \bar{\epsilon}$. Take any $X \in \mathcal{X}$. For any $\epsilon > 0$, in order to construct the quadratic Bézier curve, we consider three control points given by $X_0 = (\inf N_\epsilon(X), p(\inf N_\epsilon(X)))$, $X_2 = (\sup N_\epsilon(X), p(\sup N_\epsilon(X)))$ and X_1 defined as the intersection point between the tangent lines to the curve of p at points X_0 and X_2 respectively. Given these three control points, the quadratic Bézier curve is the path traced by the function $B^\epsilon_X : [0,1] \longrightarrow \mathbb{R}^2$ defined as in (4.8). It is well-known that the quadratic Bézier curve B^ϵ_X can be parametrized by a polynomial function which we denote by $f^\epsilon_X : \overline{N_\epsilon(X)} \longrightarrow \mathbb{R}_+$ where $\overline{N_\epsilon(X)}$ is the closure of $N_\epsilon(X)$.
Then, for any $\epsilon > 0$ the inverse demand function $p_\epsilon : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is defined as:

$$p_\epsilon(Y) = \begin{cases} f_X^\epsilon(Y) & \text{if for some } X \in \mathcal{X},\ Y \in N_\epsilon(X) \\ p(Y) & \text{otherwise} \end{cases} \tag{4.9}$$

By the construction of control points X_0, X_1 and X_2, it follows from the properties of the inverse demand function p and the quadratic Bézier curves defined above that p_ϵ as defined in (4.9) is differentiable, strictly decreasing and concave. It remains to show that the sequence $(p_\epsilon)_{\epsilon>0}$ uniformly converges to p. Take any $\zeta > 0$ and assume that $Y \notin \mathcal{X}$. It follows that there exists $\epsilon_1 > 0$ such that for any $\epsilon < \epsilon_1$ and any $X \in \mathcal{X}$, we have $Y \notin N_\epsilon(X)$. Hence, by (4.9) for any $\epsilon < \epsilon_1$ we have $p_\epsilon(Y) = p(Y)$, and so $|p_\epsilon(Y) - p(Y)| = 0 < \zeta$. Then, assume that $Y \in \mathcal{X}$. For any $\epsilon > 0$ we denote by G_Y^ϵ the convex hull of the set of control points $\{X_0, X_1, X_2\}$:

$$G_Y^\epsilon = co\{X_0, X_1, X_2\}.$$

By the construction of control points X_0, X_1 and X_2 it holds that:

$$\lim_{\epsilon \longrightarrow 0} G_Y^\epsilon = \{(Y, p(Y))\} \tag{4.10}$$

Moreover, recall that B_Y^ϵ is defined as a convex combination of control points X_0, X_1 and X_2. Hence, for any $\epsilon > 0$ we have $B_Y^\epsilon \subseteq G_Y^\epsilon$, and therefore $(Y, f_Y^\epsilon(Y)) \in G_Y^\epsilon$. By (4.10) we deduce that there exists $\epsilon_2 > 0$ such that for any $\epsilon < \epsilon_2$, it holds that:

$$|p_\epsilon(Y) - p(Y)| = |f_Y^\epsilon(Y) - p(Y)| < \zeta.$$

Finally, take $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$. For any $\epsilon < \epsilon_3$ it holds that:

$$\forall Y \in \mathbb{R}_+,\ |p_\epsilon(Y) - p(Y)| < \zeta,$$

which proves that the sequence $(p_\epsilon)_{\epsilon>0}$ uniformly converges to p. ■

Chapter 5

A necessary and sufficient condition for the non-emptiness of the core in Stackelberg oligopoly TU-games

5.1 Introduction

In Chapters 3 and 4, in order to define cooperative Cournot oligopoly games we assumed that all the firms simultaneously choose their outputs. However, as discussed in the introduction some firms may gain a strategic advantage by restricting in a credible way their choices. In order to endogenize this leadership role, a two-stage structure is introduced in strategic games. In oligopoly theory, such games are called strategic Stackelberg oligopoly games in which some firms called leaders produce an output at a first period while the other firms called followers play a quantity at a second period.
In this chapter, which is based on Driessen, Hou, and Lardon (2011), we associate such a two-stage structure with the γ-characteristic function in a quantity competition. The set of cooperative oligopoly games associated with this temporal sequence is the set of Stackelberg oligopoly TU-games in γ-characteristic function form. Thus, contrary to cooperative Cournot oligopoly games defined in Chapters 3 and 4 in which all the firms simultaneously play their quantities, any deviating coalition produces an output at a first period and outsiders simultaneously and independently play a quantity at a second period. We assume that the inverse demand function is linear and firms operate at constant but possibly distinct marginal costs. Thus, contrary to Marini and Currarini (2003), the individual utility (profit) functions are not necessarily identical. First, we provide an expression of the worth of any deviating coalition and prove that it is increasing with respect to outsiders' marginal costs and decreasing with re-

spect to the smallest marginal cost among its members. Then, we characterize the core by proving that it is equal to the set of imputations. The reason is that the first-mover advantage gives too much power to singletons so that the worth of any deviating coalition is less than or equal to the sum of its members' individual worths except for the grand coalition. This result goes further than Marini and Currarini (2003) who only provide a single-valued solution, the equal division solution, in the core when players are symmetric. Finally, we provide a necessary and sufficient condition under which the core is non-empty in case the players are symmetric. We prove that this condition depends on the heterogeneity of firms' marginal costs, i.e. for a fixed number of firms the core is non-empty if and only if firms' marginal costs are not too heterogeneous. The more the number of firms is, the less the heterogeneity of firms' marginal costs must be in order to ensure the non-emptiness of the core which turns on the role of the symmetric players assumption in Marini and Currarini (2003) for the non-emptiness of the core. Surprisingly, in case the inverse demand function is strictly concave, we provide an example in which the opposite result holds, i.e. when the heterogeneity of firms' marginal costs increases the core becomes larger.

The remainder of this chapter is structured as follows. In Section 5.2 we introduce the model and some notations and provide an expression of the worth of any deviating coalition. In Section 5.3, we show that the core is equal to the set of imputations and provide a necessary and sufficient condition under which the core is non-empty. Section 5.4 gives some concluding remarks.

5.2 The model

A **Stackelberg oligopoly situation** is a quintuplet $(L, F, (q_i, C_i)_{i\in N}, p)$ defined as:

1. the disjoint finite **sets of leaders and followers** L and F respectively where $L \cup F = \{1, 2, \ldots, n\}$ is the set of firms denoted by N;
2. for every $i \in N$, a **capacity constraint** $q_i \in \overline{\mathbb{R}}_+$;
3. for every $i \in N$, an **individual cost function** $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$;
4. an **inverse demand function** $p : \mathbb{R}_+ \longrightarrow \mathbb{R}$ which assigns to any aggregate quantity $X \in \mathbb{R}_+$ the unit price $p(X)$.

Throughout this chapter, we assume that:

(e) firms have no capacity constraint:

$$\forall i \in N,\, q_i = +\infty;$$

(f) firms operate at constant but possibly distinct marginal costs:

$$\forall i \in N,\, \exists c_i \in \mathbb{R}_+ : C_i(y_i) = c_i y_i,$$

where c_i is firm i's marginal cost, and $y_i \in \mathbb{R}_+$ is the quantity produced by firm i;

(g) firms face the linear inverse demand function:

$$p(X) = a - X,$$

where $X \in \mathbb{R}_+$ is the total production of the industry and $a \in \mathbb{R}_+$ is the prohibitive price (the intercept) of the inverse demand function p such that $a \geq 2n \times \max\{c_i : i \in N\}$.

Given assumptions (e), (f) and (g), a Stackelberg oligopoly situation is summarized by the 4-tuple $(L, F, (c_i)_{i\in N}, a)$. Without loss of generality we assume that the firms are ranked according to their marginal costs, i.e. $c_1 \leq \ldots \leq c_n$. For notational convenience, for any coalition $S \in 2^N \backslash \{\emptyset\}$ we denote the **minimal coalitional cost** by $\underline{c}_S = \min\{c_i : i \in S\}$ and by $i_S \in S$ the firm in S with the smallest index that operates at marginal cost $\underline{c}_S$.

The **strategic Stackelberg oligopoly game** associated with the Stackelberg oligopoly situation $(L, F, (c_i)_{i\in N}, a)$ is a quadruplet $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i\in N})$ defined as:

1. the disjoint finite **sets of leaders and followers** L and F respectively where $N = L \cup F$ is the **set of firms**;
2. for every $k \in N$, an **individual strategy set** X_k such that:
   - for every $i \in L$, $X_i = \mathbb{R}_+$ where $x_i \in X_i$ represents the quantity produced by leader i;
   - for every $j \in F$, X_j is the set of mappings $x_j : X_L \longrightarrow \mathbb{R}_+$ where $x_j(x_L)$ represents the quantity produced by follower j given leaders' strategy profile $x_L \in X_L$;

3. for every $k \in N$, an **individual profit function** $\pi_k : X_L \times X_F \longrightarrow \mathbb{R}_+$ such that:

   - for every $i \in L$, $\pi_i : X_L \times X_F \longrightarrow \mathbb{R}_+$ is defined as:

   $$\pi_i(x_L, x_F(x_L)) = p(X)x_i - c_i x_i;$$

   - for every $j \in F$, $\pi_j : X_L \times X_F \longrightarrow \mathbb{R}_+$ is defined as:

   $$\pi_j(x_L, x_F(x_L)) = p(X)x_j(x_L) - c_j x_j(x_L),$$

   where $X = \sum_{i \in L} x_i + \sum_{j \in F} x_j(x_L)$ is the **total production**.

Given a strategic Stackelberg oligopoly game $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i \in N})$, every leader $i \in L$ simultaneously and independently produces an output $x_i \in X_i$ at a first period while every follower $j \in F$ simultaneously and independently plays a quantity $x_j(x_L) \in X_j$ at a second period given leaders' strategy profile $x_L \in X_L$. We denote by $\mathcal{G}_{so} \subseteq \mathcal{G}$ the **set of strategic Stackelberg oligopoly games**. In case there is a single leader and multiple followers, Sherali et al. (1983) prove the existence and uniqueness of the Nash equilibrium in strategic Stackelberg oligopoly games under standard assumptions on the inverse demand function and the individual cost functions, i.e. the inverse demand function is twice differentiable, strictly decreasing and satisfies:

$$\forall X \in \mathbb{R}_+, \ \frac{dp}{dX}(X) + X\frac{d^2p}{dX^2}(X) \leq 0,$$

and the individual cost functions are twice differentiable and convex. In particular, they show that the convexity of followers' reaction functions with respect to leader's output is crucial for the uniqueness of the Nash equilibrium. Assumptions (e), (f) and (g) ensure that Sherali et al.'s result (1983) holds on $\mathcal{G}_{so}$ so that any strategic Stackelberg oligopoly game $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{so}$ such that $|L| = 1$ admits a unique Nash equilibrium.

Now, given a strategic Stackelberg oligopoly game $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{so}$, we want to associate a Stackelberg oligopoly TU-game in γ-characteristic function form (Chander and Tulkens 1997). In a dynamic oligopoly "à la Stackelberg" this assumption implies that the coalition members produce an output at a first period, thus anticipating outsiders' reaction who simultaneously and independently play a quantity at a second period. For any coalition $S \in 2^N \backslash \{\emptyset\}$ where $S = L$ and $N \backslash S = F$, the **coalition profit function** $\pi_S : X_S \times X_{N \backslash S} \longrightarrow \mathbb{R}$ is defined as:

$$\pi_S(x_S, x_{N\backslash S}(x_S)) = \sum_{i\in S} \pi_i(x_S, x_{N\backslash S}(x_S)).$$

Moreover, **followers' individual best reply strategies** $\tilde{x}_{N\backslash S} : X_S \longrightarrow X_{N\backslash S}$ are defined as:

$$\forall j \in N\backslash S, \forall x_S \in X_S,\ \tilde{x}_j(x_S) \in \arg \max_{x_j(x_S)\in X_j} \pi_j(x_S, \tilde{x}_{N\backslash(S\cup\{j\})}(x_S), x_j(x_S)).$$

For any coalition $S \in 2^N\backslash\{\emptyset\}$ and the induced strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$, the associated **Stackelberg oligopoly TU-game in γ-characteristic function form**, denoted by (N, v_γ), is defined for any coalition $S \in 2^N\backslash\{\emptyset\}$ as:

$$v_\gamma(S) = \pi_S(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)),$$

where $(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)) \in \varphi^{PA}(\Gamma_{so}, S)$. We denote by $G^\gamma_{so} \subseteq G$ the **set of Stackelberg oligopoly TU-games in γ-characteristic function form**.

Given a strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$, by assumptions (e), (f) and (g), and by the definition of the partial agreement equilibrium, any deviating coalition $S \in 2^N\backslash\{\emptyset\}$ can be represented by firm $i_S \in S$ acting as a single leader while the other firms in coalition S play a zero output. It follows from Sherali et al.'s result (1983) that the induced strategic Stackelberg oligopoly game $\Gamma_{so} = (\{i_S\}, N\backslash S, (X_i, \pi_i)_{i\in\{i_S\}\cup N\backslash S}) \in \mathcal{G}_{so}$ has a unique Nash equilibrium, and so the strategic Stackelberg oligopoly game Γ_{so} admits a partial agreement equilibrium under S. Indeed, in case there are at least two firms operate at the minimal marginal cost $\underline{c}_S$, the most efficient firms in coalition S can coordinate their output decision and reallocate the Nash equilibrium output of firm i_S among themselves. We conclude that there can exist several partial agreement equilibria under S which support the unique worth $v_\gamma(S)$. Hence, the γ-characteristic function is well-defined. The following proposition goes further by expressing the worth of any deviating coalition.

Proposition 5.2.1 *For any coalition $S \in 2^N\backslash\{\emptyset\}$ and the associated strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$, it holds that:*

$$v_\gamma(S) = \frac{1}{4(n-s+1)}\left(a + \sum_{j\in N\backslash S} c_j - \underline{c}_S(n-s+1)\right)^2.$$

Proof: Take any coalition $S \in 2^N \backslash \{\emptyset\}$ and consider the associated strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$. In order to compute the worth $v_\gamma(S)$ of coalition S, we have to successively solve the maximization problems derived from the definition of the partial agreement equilibrium. First, consider the profit maximization program of any follower $j \in N\backslash S$ at the second period:

$$\forall x_S \in X_S, \forall x_{N\backslash(S\cup\{j\})}(x_S) \in X_{N\backslash(S\cup\{j\})}, \max_{x_j(x_S)\in X_j} \pi_j(x_S, x_{N\backslash(S\cup\{j\})}(x_S), x_j(x_S)).$$

The first-order conditions for a maximum are:

$$\forall j \in N\backslash S, \forall x_{N\backslash\{j\}} \in X_{N\backslash\{j\}}, \frac{\partial \pi_j}{\partial x_j}(x_j, x_{N\backslash\{j\}}) = 0,$$

and imply that the unique maximizers $\tilde{x}_j(x_S)$, $j \in N\backslash S$, satisfy:

$$\forall j \in N\backslash S, \forall x_S \in X_S, \tilde{x}_j(x_S) = \frac{1}{2}\left(a - \sum_{i\in S} x_i - \sum_{k\in N\backslash(S\cup\{j\})} \tilde{x}_k(x_S) - c_j\right).$$

By solving the above system of equations, we deduce that followers' individual best reply strategies at the second period are given by:

$$\forall j \in N\backslash S, \forall x_S \in X_S, \tilde{x}_j(x_S) = \frac{1}{(n-s+1)}\left(a - \sum_{i\in S} x_i + \sum_{k\in N\backslash S} c_k\right) - c_j \quad (5.1)$$

Then, given $\tilde{x}_{N\backslash S}(x_S) \in X_{N\backslash S}$ consider the profit maximization program of coalition S at the first period:

$$\max_{x_S\in X_S} \pi_S(x_S, \tilde{x}_{N\backslash S}(x_S)).$$

Since the firms have no capacity constraint, it follows that the above profit maximization program of coalition S is equivalent to the profit maximization program of firm $i_S \in S$ given that the other members in S play a zero output:

$$\max_{x_{i_S}\in X_{i_S}} \pi_{i_S}(x_{i_S}, 0_{S\backslash\{i_S\}}, \tilde{x}_{N\backslash S}(x_{i_S}, 0_{S\backslash\{i_S\}})).$$

The first-order condition for a maximum is:

$$\frac{\partial \pi_{i_S}}{\partial x_{i_S}}(x_{i_S}, 0_{S\backslash\{i_S\}}, \tilde{x}_{N\backslash S}(x_{i_S}, 0_{S\backslash\{i_S\}})) = 0,$$

and implies that the unique maximizer $x^*_{i_S} \in X_{i_S}$ is given by:

$$x^*_{i_S} = \frac{1}{2}\left(a + \sum_{j \in N \backslash S} c_j - \underline{c}_S(n-s+1)\right) \tag{5.2}$$

By (5.1) and (5.2), for any $j \in N \backslash S$ it holds that:

$$\begin{aligned} \tilde{x}_j(x^*_S) &= \tilde{x}_j(x^*_{i_S}, 0_{S \backslash \{i_S\}}) \\ &= \frac{1}{2(n-s+1)}\left(a + \sum_{k \in N \backslash S} c_k + \underline{c}_S(n-s+1)\right) - c_j \end{aligned} \tag{5.3}$$

By (5.2) and (5.3), we deduce that:

$$\begin{aligned} v_\gamma(S) &= \pi_S(x^*_S, \tilde{x}_{N \backslash S}(x^*_S)) \\ &= \pi_{i_S}((x^*_{i_S}, 0_{S \backslash \{i_S\}}), \tilde{x}_{N \backslash S}(x^*_{i_S}, 0_{S \backslash \{i_S\}})) \\ &= \frac{1}{4(n-s+1)}\left(a + \sum_{j \in N \backslash S} c_j - \underline{c}_S(n-s+1)\right)^2, \end{aligned}$$

which completes the proof. ■

Thus, the worth of any deviating coalition is increasing with respect to outsiders' marginal costs and decreasing with respect to the smallest marginal cost among its members. Note that the condition $a \geq 2n \times \max\{c_i : i \in N\}$ (assumption (g)) ensures that the equilibrium outputs in (5.2) and (5.3) are positive.

5.3 The non-emptiness of the core

In this section, we study the core of Stackelberg oligopoly TU-games in γ-characteristic function form. First, we characterize the core by showing that it is equal to the set of imputations. Then, we provide a necessary and sufficient condition under which the core is non-empty. Finally, we prove that this condition depends on the heterogeneity of firms' marginal costs.

The following proposition provides a characterization of the core.

Proposition 5.3.1 *Let $(N, v_\gamma) \in G^\gamma_{so}$ be a Stackelberg oligopoly TU-game. Then, it holds that:*

$$C(N, v_\gamma) = I(N, v_\gamma).$$

In order to establish the proof of Proposition 5.3.1 we first need the following lemma. Given a family of marginal costs $\{c_i\}_{i \in N}$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ let $\alpha(S) = \sum_{j \in S \backslash \{i_S\}} (\underline{c}_S - c_j)^2$ and denote by:

$$A_1(S) = \frac{1}{2} \sum_{j \in S \backslash \{i_S\}} \sum_{k \in S \backslash \{i_S\}} (c_j - c_k)^2; \quad B_1(S) = (s-1)\big(\alpha(S) - A_1(S)\big);$$
$$C_1(S) = -(s-1)\big(s\alpha(S) + A_1(S)\big); \quad D_1(S) = -(s-1)\big(\alpha(S) + A_1(S)\big).$$

We define the functions $f_1 : \mathbb{N} \times 2^N \backslash \{\emptyset\} \longrightarrow \mathbb{R}$ and $f_2 : \mathbb{N} \times 2^N \backslash \{\emptyset\} \longrightarrow \mathbb{R}$ as:

$$f_1(n, S) = 3A_1(S)n^2 + \big(3A_1(S) + 2B_1(S)\big)n + A_1(S) + B_1(S) + C_1(S);$$

$$f_2(n, S) = A_1(S)n^3 + B_1(S)n^2 + C_1(S)n + D_1(S).$$

Lemma 5.3.2 *Let $\{c_i\}_{i \in N}$ be a family of marginal costs. Then, for any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$ it holds that (i) $f_1(n, S) \geq 0$, and (ii) $f_2(n, S) \geq 0$.*

Proof: First we show point (i). For any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = n - 1$ it holds that:

$$\begin{aligned} f_1(n, S) &= (n^2 - 4)\alpha(S) + (n^2 + 5n + 5)A_1(S) \\ &\geq 0 \end{aligned} \tag{5.4}$$

Then, we show that for any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_1(n, S) \geq 0$. We proceed by a double induction on the number of firms $n \geq 3$ and the size $s \in \{2, \ldots, n-1\}$ of coalition S respectively.
Initialisation: assume that $n = 3$ and take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = 2$. By (5.4) it holds that $f_1(3, S) \geq 0$.
Induction hypothesis: assume that for any $n \leq k$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_1(n, S) \geq 0$.
Induction step: we want to show that for $n = k + 1$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k\}$, $f_1(k + 1, S) \geq 0$. It follows from (5.4) that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = k$, $f_1(k + 1, S) \geq 0$. It remains to show that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$, $f_1(k + 1, S) \geq 0$. Take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$. Then it follows from the definition of f_1 and the induction hypothesis that:

$$\begin{aligned} f_1(k+1,S) &= f_1(k,S) + 6A_1(S)k + 6A_1(S) + 2B_1(S) \\ &= f_1(k,S) + A_1(S)(6k-2s+8) + 2(s-1)\alpha(S) \\ &\geq 0, \end{aligned}$$

which concludes the proof of point (i).
Then, we show point (ii). For any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = n-1$ it holds that:

$$\begin{aligned} f_2(n,S) &= (n^2-3n+2)\alpha(S) + (n^2+n+2)A_1(S) \\ &\geq 0 \end{aligned} \quad (5.5)$$

Then, we show that for any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_2(n,S) \geq 0$. We proceed by a double induction on the number of firms $n \geq 3$ and the size $s \in \{2, \ldots, n-1\}$ of coalition S respectively.
Initialisation: assume that $n = 3$ and take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = 2$. By (5.5) it holds that $f_2(3,S) \geq 0$.
Induction hypothesis: assume that for any $n \leq k$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_2(n,S) \geq 0$.
Induction step: we want to show that for $n = k+1$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k\}$, $f_2(k+1,S) \geq 0$. It follows from (5.5) that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = k$, $f_2(k+1,S) \geq 0$. It remains to show that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$, $f_2(k+1,S) \geq 0$. Take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$. Then it follows from the definitions of f_1 and f_2, the induction hypothesis and point (i) of Lemma 5.3.2 that:

$$\begin{aligned} f_2(k+1,S) &= f_2(k,S) + f_1(k,S) \\ &\geq 0, \end{aligned}$$

which concludes the proof of point (ii). ■

Now, we are ready to establish the proof of Proposition 5.3.1 which consists in showing that the first-mover advantage gives too much power to singletons so that the worth of any deviating coalition is less than or equal to the sum of its members' individual worths except for the grand coalition. Given a family of marginal costs $\{c_i\}_{i \in N}$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ we denote by:

$A_2(S) = \dfrac{(n-s)(s-1)}{4n(n-s+1)}$ (note that for $s \in \{2, \ldots, n-1\}$, $A_2(S) > 0$);

$$B_2(S) = \frac{1}{2n}\sum_{i\in S}\left(\sum_{j\in N\backslash\{i\}} c_j - nc_i\right) - \frac{1}{2(n-s+1)}\left(\sum_{j\in N\backslash S} c_j - \underline{c}_S(n-s+1)\right);$$

$$C_2(S) = \frac{1}{4n}\sum_{i\in S}\left(\sum_{j\in N\backslash\{i\}} c_j - nc_i\right)^2 - \frac{1}{4(n-s+1)}\left(\sum_{j\in N\backslash S} c_j - \underline{c}_S(n-s+1)\right)^2.$$

These quantities will be used in the following proof.

Proof (of Proposition 5.3.1): First, assume that $n = 2$. By the definitions of the core and the set of imputations it holds that $C(N, v_\gamma) = I(N, v_\gamma)$.
Then, assume that $n \geq 3$. The core is equal to the set of imputations if and only if:

$$\forall S \in 2^N\backslash\{\emptyset\} : s \in \{2,\ldots,n-1\},\ v_\gamma(S) \leq \sum_{i\in S} v_\gamma(\{i\}).$$

In order to prove the above condition, take any coalition $S \in 2^N\backslash\{\emptyset\}$ such that $s \in \{2,\ldots,n-1\}$. By Proposition 5.2.1 we deduce that:

$$\sum_{i\in S} v_\gamma(\{i\}) - v_\gamma(S) = A_2(S)a^2 + B_2(S)a + C_2(S).$$

Now, we define the mapping $P_S : \mathbb{R} \longrightarrow \mathbb{R}$ as:

$$P_S(y) = A_2(S)y^2 + B_2(S)y + C_2(S),$$

so that $P_S(a) = \sum_{i\in S} v_\gamma(\{i\}) - v_\gamma(S)$. We want to show that for any $y \in \mathbb{R}$, $P_S(y) \geq 0$. It follows from $A_2(S) > 0$ that the minimum of P_S is obtained at point $y^* \in \mathbb{R}$ such that:

$$y^* = -\frac{B_2(S)}{2A_2(S)}.$$

After some calculation steps, we obtain that the minimum of P_S is equal to:

$$P_S(y^*) = \frac{1}{4n(n-s)(s-1)} f_2(n, S),$$

where f_2 is defined as in Lemma 5.3.2. Hence, it follows from point (ii) of Lemma 5.3.2 that $P_S(y^*) \geq 0$, which implies that for any $y \in \mathbb{R}$, $P_S(y) \geq 0$. In particular, we conclude that $P_S(a) \geq 0$, and so $\sum_{i\in S} v_\gamma(\{i\}) - v_\gamma(S) \geq 0$. ■

Now, we provide a necessary and sufficient condition for the non-emptiness of the core of Stackelberg oligopoly TU-games in γ-characteristic function form as enunciated in the following proposition.

Proposition 5.3.3 *Let $(N, v_\gamma) \in G^\gamma_{so}$ be a Stackelberg oligopoly TU-game. Then, the core is non-empty if and only if:*

$$2a\left(\sum_{i\in N} c_i - n\underline{c}_N\right) \geq \sum_{i\in N}\left(\sum_{j\in N\setminus\{i\}} c_j - nc_i\right)^2 - n\underline{c}_N^2 \tag{5.6}$$

or equivalently

$$2a(\overline{c}_N - \underline{c}_N) \geq \frac{(n+1)^2}{n}\sum_{j\in N} c_j^2 - \frac{(n+2)}{n}\left(\sum_{j\in N} c_j\right)^2 - \underline{c}_N^2 \tag{5.7}$$

where $\overline{c}_N = \sum_{i\in N} c_i/n$ is the average cost of the grand coalition.

Proof: It follows from Proposition 5.3.1 that the core is non-empty if and only if $\sum_{i\in N} v_\gamma(\{i\}) \leq v_\gamma(N)$. By Proposition 5.2.1 it holds that:

$$\begin{aligned}
\sum_{i\in N} v_\gamma(\{i\}) &= \frac{1}{4n}\sum_{i\in N}\left(a + \sum_{j\in N\setminus\{i\}} c_j - nc_i\right)^2 \\
&= \frac{1}{4n}\sum_{i\in N}\left(a + \sum_{j\in N\setminus\{i\}} c_j - nc_i - \underline{c}_N + \underline{c}_N\right)^2 \\
&= \frac{1}{4n}\sum_{i\in N}\left[\left(a - \underline{c}_N\right)^2 + 2(a - \underline{c}_N)\left(\sum_{j\in N\setminus\{i\}} c_j - nc_i + \underline{c}_N\right)\right. \\
&\qquad\qquad\left. + \left(\sum_{j\in N\setminus\{i\}} c_j - nc_i + \underline{c}_N\right)^2\right] \\
&= v_\gamma(N) + \frac{1}{4n}\left[2a\left(n\underline{c}_N - \sum_{i\in N} c_i\right) + \sum_{i\in N}\left(\sum_{j\in N\setminus\{i\}} c_j - nc_i\right)^2 - n\underline{c}_N^2\right] \\
&= v_\gamma(N) + \frac{1}{4n}\left[2a\left(n\underline{c}_N - n\overline{c}_N\right) + (n+1)^2\sum_{j\in N} c_j^2\right. \\
&\qquad\qquad\left. - (n+2)\left(\sum_{j\in N} c_j\right)^2 - n\underline{c}_N^2\right],
\end{aligned}$$

which permits to conclude that $\sum_{i\in N} v_\gamma(\{i\}) \leq v_\gamma(N)$ if and only if inequalities (5.6) or (5.7) holds. ■

In case all the firms have the same marginal cost, the both sides of inequality (5.6) are equal to zero which implies that $\sum_{i\in N} v_\gamma(\{i\}) = v_\gamma(N)$, and so $(v_\gamma(\{i\}))_{i\in N} = (v_\gamma(N)/n)_{i\in N}$ is the unique core element which coincides with Marini and Currarini's core allocation result (2003).

Driessen, Hou, and Lardon (2011) show that for any TU-game, the core is non-empty and equal to the set of imputations if and only if its dual game is 1-concave. Moreover, Driessen et al. (2010) show that the nucleolus of any 1-concave TU-game coincides with the center of gravity of the core. Hence, for any Stackelberg oligopoly TU-game $(N, v_\gamma) \in G^\gamma_{so}$, the dual game (N, v^*_γ) is 1-concave if and only if inequality (5.6) holds, implying that the nucleolus $Nuc(N, v^*_\gamma)$ is the center of gravity of the core $C(N, v^*_\gamma)$.

The following theorem gives a more relevant expression of inequality (5.6). When the difference between any two successive marginal costs is constant, it provides an upper bound on the heterogeneity of firms' marginal costs below which inequality (5.6) holds.

Theorem 5.3.4 *Let $(N, v_\gamma) \in G^\gamma_{so}$ be a Stackelberg oligopoly TU-game such that:*

$$\exists \delta \in \mathbb{R}_+ : \forall i \in \{1, \ldots, n-1\},\ c_{i+1} = c_i + \delta \tag{5.8}$$

Then, inequality (5.6) holds if and only if:

$$\delta \leq \delta^*(n) = \frac{12(a - \underline{c}_N)(n-1)}{n^4 + 2n^3 + 3n^2 - 8n + 2} \tag{5.9}$$

Proof: It follows from (5.8) that inequality (5.6) can be expressed as:

$$\sum_{i\in N}\left(\delta\frac{(n^2 - 2ni + n)}{2} - c_i\right)^2 - n\underline{c}_N^2 \leq an(n-1)\delta.$$

By noting that:

$$\sum_{i\in N} c_i^2 - n\underline{c}_N^2 = \delta\underline{c}_N n(n-1) + \delta^2\frac{n(n-1)(2n-1)}{6},$$

we deduce that the above inequality is equivalent to:

$$\delta\left[\sum_{i\in N}\left(\frac{n^2-2ni+n}{2}\right)^2+\frac{n(n-1)(2n-1)}{6}\right]\leq(a-\underline{c}_N)n(n-1)$$
$$+\sum_{i\in N}(n^2-2ni+n)c_i \quad (5.10)$$

It remains to compute the two sums in inequality (5.10). First, it holds that:

$$\begin{aligned}\sum_{i\in N}\left(\frac{n^2-2ni+n}{2}\right)^2 &= \frac{n^3(n+1)^2}{4}-n^2(n+1)\sum_{i\in N}i+n^2\sum_{i\in N}i^2\\ &= \frac{1}{12}\big(2n^4(n+1)-n^3(n+1)^2\big) \quad (5.11)\end{aligned}$$

Then, it holds that:

$$\begin{aligned}\sum_{i\in N}c_i &= n\underline{c}_N+\delta\sum_{i=1}^{n-1}i\\ &= n\left(\underline{c}_N+\delta\frac{(n-1)}{2}\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in N}ic_i &= \underline{c}_N\sum_{i=1}^{n}i+\delta\sum_{i=1}^{n}i(i-1)\\ &= \underline{c}_N\frac{n(n+1)}{2}+\delta\frac{n(n+1)(2n-2)}{6}.\end{aligned}$$

Hence, we deduce that:

$$\begin{aligned}\sum_{i\in N}(n^2-2ni+n)c_i &= (n^2+n)\sum_{i\in N}c_i-2n\sum_{i\in N}ic_i\\ &= \frac{1}{6}\big(\delta n^2(n+1)(1-n)\big) \quad (5.12)\end{aligned}$$

By (5.11) and (5.12), we conclude that inequality (5.10) is equivalent to (5.9).■

By noting that:

$$\frac{d\delta^*}{dn}(n) = -\frac{36(a - \underline{c}_N)(n^2 + 2n + 2)}{(n^3 + 3n^2 + 6n - 2)^2} < 0,$$

and

$$\frac{d^2\delta^*}{dn^2}(n) = \frac{72(a - \underline{c}_N)(2n^4 + 8n^3 + 15n^2 + 20n + 14)}{(n^3 + 3n^2 + 6n - 2)^3} > 0,$$

we deduce that the bound $\delta^*(n)$ is strictly decreasing and strictly convex with respect to the number of firms n. Moreover, when n tends to infinity its limit is equal to 0. So, the more the number of firms is, the less the heterogeneity of firms' marginal costs must be in order to ensure the non-emptiness of the core. This result extends Marini and Currarini's core allocation result (2003) and shows that their result crucially depends on the symmetric players assumption.

We saw that when the heterogeneity of firms' marginal costs increases the core becomes smaller. Surprisingly, in case the inverse demand function is strictly concave, the following example shows that the opposite result may holds, i.e. when the heterogeneity of firms' marginal costs increases the core becomes larger.

Example 5.3.5

Consider the three Stackelberg oligopoly TU-games $(N, v_\gamma^1) \in G_{so}^\gamma$, $(N, v_\gamma^2) \in G_{so}^\gamma$ and $(N, v_\gamma^3) \in G_{so}^\gamma$ associated with the Stackelberg oligopoly situations $(L, F, (c_1, c_2^1), p)$, $(L, F, (c_1, c_2^2), p)$ and $(L, F, (c_1, c_2^3), p)$ respectively where $N = \{1, 2\}$, $c_1 = c_2^1 = 2$, $c_2^2 = 4$, $c_2^3 = 5$ and $p = 10 - X^2$. The worths of any coalition $S \in 2^N \backslash \{\emptyset\}$ are given in the following table:

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$v_\gamma^1(S)$	4.70	4.70	8.71
$v_\gamma^2(S)$	6.19	2.02	8.71
$v_\gamma^3(S)$	7.05	1.05	8.71

Thus, it holds that $\emptyset = C(N, v_\gamma^1) \subset C(N, v_\gamma^2) \subset C(N, v_\gamma^3)$, and so when the heterogeneity of firms' marginal costs increases the core becomes larger. □

5.4 Concluding remarks

In this chapter we have studied the set of Stackelberg oligopoly TU-games in γ-characteristic function form (Chander and Tulkens 1997) in which any deviating coalition produces an output at a first period and outsiders simultaneously and independently play a quantity at a second period. We have assumed that the inverse demand function is linear and that firms operate at constant but possibly distinct marginal costs. Thus, contrary to Marini and Currarini (2003), the individual utility (profit) functions are not necessarily identical. First, we have characterized the core by showing that it is equal to the set of imputations. Indeed, the first-mover advantage gives too much power to singletons so that the worth of any deviating coalition is less than or equal to the sum of its members' individual worths except for the grand coalition. Then, we have provided a necessary and sufficient condition under which the core is non-empty. Finally, we have proved that this condition depends on the heterogeneity of firms' marginal costs, i.e. the core is non-empty if and only if firms' marginal costs are not too heterogeneous. The more the number of firms is, the less the heterogeneity of firms' marginal costs must be in order to ensure the non-emptiness of the core. This last result extends Marini and Currarini's core allocation result (2003) and shows that their result crucially depends on the symmetric players assumption. Surprisingly, in case the inverse demand function is strictly concave, we have provided an example in which the opposite result holds, i.e. when the heterogeneity of firms' marginal costs increases the core becomes larger.

Instead of quantity competition, we can associate a two-stage structure with the γ-characteristic function in a price competition. Marini and Currarini's core allocation result (2003) applies to this framework and they provide examples in which the core of the sequential Bertrand oligopoly TU-games is included in the core of the static Bertrand oligopoly TU-games. A question concerns the effect of the heterogeneity of firms' marginal costs on the non-emptiness of the core. In the second part of this thesis, we study oligopoly TU-games in price competition.

Part II

Price competition

Chapter 6

Convexity in Bertrand oligopoly TU-games with differentiated products

6.1 Introduction

In the first part of the thesis, we have dealt with the non-emptiness of the core of cooperative oligopoly games in quantity competition. However, in many other oligopoly situations, firms compete in price rather than in quantity.
In this chapter, which is based on Lardon (2010a), we study the core of Bertrand oligopoly TU-games in α, β and γ-characteristic function forms when firms operate at a constant and identical marginal cost. The case with distinct marginal costs will be studied in Chapter 7. First, as for the set of Cournot oligopoly TU-games, we show that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions and we prove that the convexity property holds for this set of games, i.e. when any coalition has pessimistic expectations on its future coalition profits there exists a strong incentive to form the grand coalition. Then, we consider the set of Bertrand oligopoly TU-games in γ-characteristic function form. For this set of games, we show that the equal division solution belongs to the core and we provide a sufficient condition under which such games are convex. This finding generalizes the superadditivity result of Deneckere and Davidson (1985) and contrasts sharply with the negative core non-emptiness results of Kaneko (1978) and Huang and Sjöström (2003). In non-cooperative game theory, an important distinction between a strategic Cournot oligopoly game and a strategic Bertrand oligopoly game is that the former has strategic substitutabilities and the latter has strategic complementarities. Thus, although Cournot and Bertrand oligopoly games are basically different in their

non-cooperative forms, it appears that their cooperative forms have the same core structure.

The remainder of this chapter is structured as follows. In Section 6.2 we introduce the model and some notations. Section 6.3 establishes that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions and shows that the convexity property holds for this set of games. Section 6.4 proves that the equal division solution belongs to the core of Bertrand oligopoly TU-games in γ-characteristic function form and provides a sufficient condition under which such games are convex. Section 6.5 gives some concluding remarks.

6.2 The model

A **Bertrand oligopoly situation** is a triplet $(N, (D_i, C_i)_{i \in N})$ defined as:

1. a finite **set of firms** $N = \{1, 2, \dots, n\}$;
2. for every $i \in N$, a **demand function** $D_i : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ which assigns to any price vector $p \in \mathbb{R}^n_+$ the quantity demanded of firm i's brand $D_i(p)$;
3. for every $i \in N$, an **individual cost function** $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$.

Throughout this chapter, we assume that:

(h) the demand system is Shubik's (1980):

$$\forall i \in N,\ D_i(p_1, \dots, p_n) = V - p_i - r\left(p_i - \frac{1}{n}\sum_{j \in N} p_j\right),$$

where p_j is the price charged by firm j, $V \in \mathbb{R}_+$ is the intercept of demand and $r \in \mathbb{R}_{++}$ is the substitutability parameter. When r approaches zero, products become unrelated, and when r approaches infinity, products become perfect substitutes. The quantity demanded of firm i's brand depends on its own price p_i and on the difference between p_i and the average price in the industry $\sum_{j \in N} p_j/n$. This quantity is decreasing with respect to p_i and increasing with respect to any p_j such that $j \neq i$;

(i) firms operate at a constant and identical marginal cost:

$$\exists c \in \mathbb{R}_+ : \forall i \in N,\ C_i(x_i) = cx_i,$$

where $c \in \mathbb{R}_+$ is firm i's marginal cost, and $x_i = D_i(p_1, \dots, p_n) \in \mathbb{R}_+$ is the quantity demanded of firm i's brand.

Given assumptions (h) and (i), a Bertrand oligopoly situation is summarized by the 4-tuple (N, V, r, c).

The **strategic Bertrand oligopoly game** associated with the Bertrand oligopoly situation (N, V, r, c) is a triplet $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N})$ defined as:

1. a finite **set of firms** $N = \{1, 2, \dots, n\}$;
2. for every $i \in N$, an **individual strategy set** $X_i = \mathbb{R}_+$ where $p_i \in X_i$ represents the price charged by firm i;
3. for every $i \in N$, an **individual profit function** $\pi_i : X_N \longrightarrow \mathbb{R}$ defined as:

$$\pi_i(p) = D_i(p)(p_i - c).$$

Note that firm i's profit depends on its own price p_i and on the average price in the industry $\sum_{j \in N} p_j/n$. We denote by $\mathcal{G}_{bo} \subseteq \mathcal{G}$ the **set of strategic Bertrand oligopoly games**.

6.3 Pessimistic expectations

In this section, we want to associate Bertrand oligopoly TU-games in α and β-characteristic function forms with strategic Bertrand oligopoly games. First, we show that the α and β-characteristic functions are well-defined and we prove that the same set of Bertrand oligopoly TU-games is associated with these characteristic functions. This equality between the α and β-characteristic functions is a useful property, as it relieves us of the burden of choosing between the α and β-characteristic functions when describing coalition profits. Then, we prove that the convexity property holds for this set of games, i.e. when any coalition has pessimistic expectations on its future coalition profit there exists a strong incentive to form the grand coalition.

For any coalition $S \in 2^N \backslash \{\emptyset\}$, the **coalition profit function** $\pi_S : X_S \times X_{N \backslash S} \longrightarrow \mathbb{R}$ is defined as:

$$\pi_S(p_S, p_{N \backslash S}) = \sum_{i \in S} \pi_i(p).$$

Given a strategic Bertrand oligopoly game $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$, the associated **Bertrand oligopoly TU-games in** α **and** β**-characteristic function**

forms, denoted by (N, v_α) and (N, v_β), are defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$v_\alpha(S) = \max_{p_S \in X_S} \min_{p_{N\backslash S} \in X_{N\backslash S}} \pi_S(p_S, p_{N\backslash S}),$$

and

$$v_\beta(S) = \min_{p_{N\backslash S} \in X_{N\backslash S}} \max_{p_S \in X_S} \pi_S(p_S, p_{N\backslash S}).$$

We denote by $G^\alpha_{bo} \subseteq G$ and $G^\beta_{bo} \subseteq G$ the **set of Bertrand oligopoly TU-games in** α **and** β**-characteristic function forms** respectively.

The following proposition states that the β-characteristic function is well-defined.

Proposition 6.3.1 *Let* $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$ *be a strategic Bertrand oligopoly game and* $(N, v_\beta) \in G^\beta_{bo}$ *be the associated Bertrand oligopoly TU-game. Then, for any coalition* $S \in 2^N \backslash \{\emptyset\}$ *it holds that:*

$$v_\beta(S) = \pi_S(\overline{p}_S, \overline{p}_{N\backslash S}),$$

where $(\overline{p}_S, \overline{p}_{N\backslash S}) \in X_S \times X_{N\backslash S}$ *is given by:*

$$\forall i \in S,\ \overline{p}_i = \max\left\{c, \frac{V}{2\big(1 + r(n-s)/n\big)} + \frac{c}{2}\right\} \tag{6.1}$$

and

$$\sum_{j \in N\backslash S} \overline{p}_j = \max\left\{0, \frac{n}{r}\left(c\left(1 + r\frac{(n-s)}{n}\right) - V\right)\right\} \tag{6.2}$$

Proof: Take any coalition $S \in 2^N \backslash \{\emptyset\}$. In order to compute the worth $v_\beta(S)$ of coalition S, we have to successively solve the maximization and the minimization problems derived from the definition of the β-characteristic function. In order to do that, we define the **coalition best reply function** $b_S : X_{N\backslash S} \longrightarrow X_S$ as:

$$\forall p_{N\backslash S} \in X_{N\backslash S},\ \{b_S(p_{N\backslash S})\} = \arg\max_{p_S \in X_S} \pi_S(p_S, p_{N\backslash S}).$$

By the definition of b_S it holds that:

$$v_\beta(S) = \min_{p_{N\backslash S} \in X_{N\backslash S}} \pi_S(b_S(p_{N\backslash S}), p_{N\backslash S}).$$

In order to compute the worth $v_\beta(S)$, we have to verify that the function b_S is well-defined. Given $p_{N\backslash S} \in X_{N\backslash S}$ consider the profit maximization program of coalition S:

$$\forall p_{N\backslash S} \in X_{N\backslash S},\ \max_{p_S \in X_S} \pi_S(p_S, p_{N\backslash S}).$$

The first-order conditions for a maximum are:

$$\forall p_{N\backslash S} \in X_{N\backslash S}, \forall i \in S,\ \frac{\partial \pi_S}{\partial p_i}(p_S, p_{N\backslash S}) = 0,$$

and imply that the unique maximizer $b_S(p_{N\backslash S})$ is given by:

$$\forall p_{N\backslash S} \in X_{N\backslash S}, \forall i \in S,\ b_i(p_{N\backslash S}) = \frac{V + (r/n)\sum_{j\in N\backslash S} p_j}{2\big(1 + r(n-s)/n\big)} + \frac{c}{2} \tag{6.3}$$

so that b_S is well-defined.
Then, given $b_S(p_{N\backslash S}) \in X_S$ consider the profit minimization program of the complementary coalition $N\backslash S$:

$$\min_{p_{N\backslash S} \in X_{N\backslash S}} \pi_S(b_S(p_{N\backslash S}), p_{N\backslash S}).$$

The first-order conditions for a minimum are:

$$\forall j \in N\backslash S,\ \frac{\partial \pi_S}{\partial p_j}(b_S(p_{N\backslash S}), p_{N\backslash S}) = 0,$$

which are equivalent, for any $j \in N\backslash S$, to the following equality:

$$\sum_{j\in N\backslash S} p_j = \frac{n}{r}\left(c\left(1 + r\frac{(n-s)}{n}\right) - V\right).$$

Since for any $i \in N$, $X_i = \mathbb{R}_+$, it follows that any minimizer $\overline{p}_{N\backslash S} \in X_{N\backslash S}$ satisfies:

$$\sum_{j\in N\backslash S} \overline{p}_j = \max\left\{0, \frac{n}{r}\left(c\left(1 + r\frac{(n-s)}{n}\right) - V\right)\right\},$$

which proves (6.2). By substituting (6.2) into (6.3), we deduce that:

$$\forall i \in S,\ \overline{p}_i = b_i(\overline{p}_{N\backslash S}) = \max\left\{c, \frac{V}{2\big(1 + r(n-s)/n\big)} + \frac{c}{2}\right\},$$

which proves (6.1) and completes the proof. ■

Proposition 6.3.1 calls for some comments which will be useful for the sequel.

Remark 6.3.2

For any coalition $S \in 2^N \backslash \{\emptyset\}$, it holds that:

1. If $V \leq c\big(1 + r(n-s)/n\big)$, then by (6.1) any member $i \in S$ charges prices equal to their marginal cost, $\overline{p}_i = c$, and by (6.2) the outsiders charge a non-negative average price, $\sum_{j \in N \backslash S} \overline{p}_j/(n-s) \geq 0$. In this case, coalition S obtains a zero coalition profit, and so $v_\beta(S) = 0$.

2. If $V > c\big(1 + r(n-s)/n\big)$, then by (6.1) any member $i \in S$ charges prices strictly greater than their marginal cost, $\overline{p}_i > c$, and by (6.2) the outsiders charge a zero average price, $\sum_{j \in N \backslash S} \overline{p}_j/(n-s) = 0$. In this case, coalition S obtains a positive coalition profit, and so $v_\beta(S) > 0$.

3. The computation of the worth $v_\beta(S)$ is consistent with the fact that the quantity demanded of any firm i's brand, $i \in S$, is positive since for any $i \in S$, $D_i(\overline{p}) \geq 0$.

By solving successively the minimization and the maximization problems derived from the definition of the α-characteristic function, we can show that the α-characteristic function is well-defined. The proof is similar to the one for Proposition 6.3.1, and so it is not detailed. A useful property is that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions as enunciated in the following proposition.

Proposition 6.3.3 *Let $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$ be a strategic Bertrand oligopoly game, and $(N, v_\alpha) \in G^\alpha_{bo}$ and $(N, v_\beta) \in G^\beta_{bo}$ be the associated Bertrand oligopoly TU-games. Then, for any coalition $S \in 2^N \backslash \{\emptyset\}$ it holds that:*

$$v_\alpha(S) = v_\beta(S).$$

Proof: First, for any coalition $S \in 2^N \backslash \{\emptyset\}$ it holds that:

$$\begin{aligned} v_\alpha(S) &= \max_{p_S \in X_S} \min_{p_{N \backslash S} \in X_{N \backslash S}} \pi_S(p_S, p_{N \backslash S}) \\ &\leq \min_{p_{N \backslash S} \in X_{N \backslash S}} \max_{p_S \in X_S} \pi_S(p_S, p_{N \backslash S}) \\ &= v_\beta(S). \end{aligned}$$

Then, it remains to show that for any coalition $S \in 2^N \backslash \{\emptyset\}$, $v_\beta(S) \leq v_\alpha(S)$. We distinguish two cases:
- assume that $V \leq c\big(1+r(n-s)/n\big)$. It follows from point 1 of Remark 6.3.2 that for any $i \in S$, $\overline{p}_i = c$. Hence, for any $p_{N\backslash S} \in X_{N\backslash S}$ it holds that $\pi_S(\overline{p}_S, p_{N\backslash S}) = 0$.
- assume that $V > c\big(1 + r(n - s)/n\big)$. It follows from point 2 of Remark 6.3.2 that for any $i \in S$, $\overline{p}_i > c$, and $\overline{p}_{N\backslash S} = 0_{N\backslash S}$. Since for any $i \in S$, D_i is increasing on $X_{N\backslash S}$ we deduce that for any $p_{N\backslash S} \in X_{N\backslash S}$, $\pi_S(\overline{p}_S, p_{N\backslash S}) \geq \pi_S(\overline{p}_S, 0_{N\backslash S})$.
In both cases, it holds that:

$$\overline{p}_{N\backslash S} \in \arg \min_{p_{N\backslash S} \in X_{N\backslash S}} \pi_S(\overline{p}_S, p_{N\backslash S}).$$

Hence, we deduce for any coalition $S \in 2^N \backslash \{\emptyset\}$ that:

$$\begin{aligned} v_\beta(S) &= \pi_S(\overline{p}_S, \overline{p}_{N\backslash S}) \\ &= \min_{p_{N\backslash S} \in X_{N\backslash S}} \pi_S(\overline{p}_S, p_{N\backslash S}) \\ &\leq \max_{p_S \in X_S} \min_{p_{N\backslash S} \in X_{N\backslash S}} \pi_S(p_S, p_{N\backslash S}) \\ &= v_\alpha(S), \end{aligned}$$

which completes the proof. ■

Proposition 6.3.3 implies that outsiders' strategy profile $\overline{p}_{N\backslash S}$ that best punishes coalition S as a first mover (α-characteristic function) also best punishes S as a second mover (β-characteristic function). Zhao (1999b) obtains a similar result for general TU-games in which any individual strategy set is compact, any individual utility function is continuous, and the strong separability condition is satisfied. This latter condition requires that the utility function of a coalition and any of its members' individual utility functions have the same minimizers. We could have used Zhao's result (1999b) in order to prove Proposition 6.3.3. First, we compactify the individual strategy sets by assuming that for any $i \in N$, $X_i = [0, \mathbf{p}]$ where $\mathbf{p}$ is sufficiently large so that the maximization/minimization problems derived from the definitions of the α and β-characteristic functions have interior solutions. Then, it is clear that any individual profit function π_i is continuous. Finally, since the demand system is symmetric and firms operate at a constant and identical marginal cost, we can verify that the strong separability condition is satisfied. In order to be shorter and perfectly rigorous we prefer to give a constructive proof of Proposition 6.3.3 without using Zhao's result (1999b). We deduce from Proposition 6.3.3 the following corollary.

Corollary 6.3.4 *Let $(N, v_\alpha) \in G^\alpha_{bo}$ and $(N, v_\beta) \in G^\beta_{bo}$ be the Bertrand oligopoly*

TU-games associated with the same strategic Bertrand oligopoly game $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$*. Then, it holds that:*

$$C(N, v_\alpha) = C(N, v_\beta).$$

Now, we want to prove that the set of Bertrand oligopoly TU-games in α or β-characteristic function form satisfies the convexity property. Proposition 6.3.1 implies that Bertrand oligopoly TU-games in β-characteristic function form are symmetric. It follows from (6.1) that the members of any coalition $S \in 2^N \backslash \{\emptyset\}$ charge identical prices, i.e. there exists $\overline{p}^s \in \mathbb{R}_+$ such that for any $i \in S$, $\overline{p}_i = \overline{p}^s$. It follows from (6.2) that outsiders charge an average price $\overline{p}_{[n-s]}/(n-s)$ where $\overline{p}_{[n-s]} = \sum_{j \in N \backslash S} \overline{p}_j$. Hence, the worth $v_\beta(S)$ depends only on the size s of coalition S, i.e. there exists a function $f_\beta : \mathbb{N} \longrightarrow \mathbb{R}$ such that for any coalition $S \in 2^N \backslash \{\emptyset\}$, it holds that:

$$v_\beta(S) = f_\beta(s) = s(\overline{p}^s - c)\left(V - \overline{p}^s\left(1 + r\frac{(n-s)}{n}\right) + \frac{r}{n}\overline{p}_{[n-s]}\right).$$

The following theorem states that the convexity property holds for the set of Bertrand oligopoly TU-games in β-characteristic function form. This result implies that the core of such games is equal to the Weber set, i.e. the convex hull of all marginal vectors.

Theorem 6.3.5 *Any Bertrand oligopoly TU-game* $(N, v_\beta) \in G^\beta_{bo}$ *is convex.*

Proof: We want to prove the convexity property for the set of symmetric TU-games, i.e. for any $s \leq n-2$, $f_\beta(s+1) - f_\beta(s) \leq f_\beta(s+2) - f_\beta(s+1)$. Take any coalition $S \in 2^N \backslash \{\emptyset\}$ of size s such that $s \leq n-2$. First, we distinguish two cases:
- assume that $V \leq c\big(1 + r(n-s-1)/n\big)$. It follows from point 1 of Remark 6.3.2 that $\overline{p}^{s+1} = c$.
- assume that $V > c\big(1 + r(n-s-1)/n\big)$. This implies that $V > c\big(1 + r(n-s-2)/n\big)$, and it follows from point 2 of Remark 6.3.2 that $\overline{p}_{[n-s-1]} = \overline{p}_{[n-s-2]} = 0$.

In both cases, it holds that:

$$(\overline{p}^{s+1} - c)\overline{p}_{[n-s-2]} = (\overline{p}^{s+1} - c)\overline{p}_{[n-s-1]} \tag{6.4}$$

Since $\overline{p}^{s+2}$ is the unique maximizer for any coalition of size $s+2$ and from (6.4), it holds that:

$$\begin{aligned} f_\beta(s+2) &= (s+2)(\overline{p}^{s+2}-c)\left(V-\overline{p}^{s+2}\left(1+r\frac{(n-s-2)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-2]}\right) \\ &\geq (s+2)(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s-2)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-2]}\right) \\ &= (s+2)(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s-2)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \\ &= f_\beta(s+1)+(\overline{p}^{s+1}-c)\Bigg(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-3)}{n}\right) \\ &\qquad +\frac{r}{n}\overline{p}_{[n-s-1]}\Bigg) \qquad (6.5) \end{aligned}$$

Moreover, since $\overline{p}^s$ is the unique maximizer for any coalition of size s and $\overline{p}_{[n-s]} \geq \overline{p}_{[n-s-1]}$, we deduce that:

$$\begin{aligned} f_\beta(s) &= s(\overline{p}^{s}-c)\left(V-\overline{p}^{s}\left(1+r\frac{(n-s)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s]}\right) \\ &\geq s(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s]}\right) \\ &\geq s(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \qquad (6.6) \end{aligned}$$

It follows from the expression of $f_\beta(s+1)$ and (6.6) that:

$$f_\beta(s+1)-f_\beta(s) \leq (\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-1)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \qquad (6.7)$$

We conclude from (6.5) and (6.7) that:

$$\begin{aligned} f_\beta(s+1)-f_\beta(s) &\leq (\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-1)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \\ &\leq (\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-3)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \\ &\leq f_\beta(s+2)-f_\beta(s+1), \end{aligned}$$

which completes the proof. ■

The convexity property does not always hold in Bertrand oligopoly TU-games in β-characteristic function form when the firms operate at distinct marginal costs as illustrated in the following example.

Example 6.3.6

Consider the Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$ where $N = \{1, 2, 3\}$, $V = 2$, $r = 5$, $c_1 = 1$, $c_2 = 3$ and $c_3 = 5$. The Bertrand oligopoly TU-game $(N, v_\beta) \in G^\beta_{bo}$ is summarized in the following table:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_\beta(S)$	0	0	0	3	12	3	12

Note that $v_\beta(\{1, 2\}) + v_\beta(\{1, 3\}) = 15 > 12 = v_\beta(\{1, 2, 3\}) + v_\beta(\{1\})$, and so (N, v_β) is not convex. □

In Chapter 7, we provide a sufficient condition under which the convexity property holds for the set of Bertrand oligopoly TU-games in β-characteristic function form with distinct marginal costs.

6.4 Rational expectations

In this section, we associate Bertrand oligopoly TU-games in γ-characteristic function form with strategic Bertrand oligopoly games. First, we show that the γ-characteristic function is well-defined. Then, we prove that the core is non-empty by showing that the equal division solution belongs to the core. Finally, we provide a sufficient condition under which Bertrand oligopoly TU-games in γ-characteristic function form are convex, i.e. when any coalition has rational expectations on its future coalition profit there exists a strong incentive to form the grand coalition.

Given a strategic Bertrand oligopoly game $\Gamma_{bo} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{bo}$, the associated **Bertrand oligopoly TU-game in γ-characteristic function form**, denoted by (N, v_γ), is defined for any coalition $S \in 2^N\backslash\{\emptyset\}$ as:

$$v_\gamma(S) = \pi_S(p^*_S, \tilde{p}_{N\backslash S}),$$

where $(p^*_S, \tilde{p}_{N\backslash S}) \in \varphi^{PA}(\Gamma_{bo}, S)$. We denote by $G^\gamma_{bo} \subseteq G$ the **set of Bertrand oligopoly TU-games in γ-characteristic function form**.

Throughout this section, in addition to assumptions (h) and (i) we assume that:

(j) the intercept of demand $V \in \mathbb{R}_+$ is strictly greater than the marginal cost $c \in \mathbb{R}_+$.

For any partial agreement equilibrium, assumption (j) ensures that the quantity demanded of any firm's brand is non-negative as discussed below.

Deneckere and Davidson (1985) study strategic Bertrand oligopoly games with general coalition structures. Given a set of firms $N = \{1, 2, \ldots, n\}$, a **coalition structure** $\mathcal{P}$ is a partition of N, i.e. $\mathcal{P} = \{S_1, \ldots, S_k\}$, $k \in \{1, \ldots, n\}$. An element of a coalition structure, $S \in \mathcal{P}$, is called an **admissible coalition** in $\mathcal{P}$. The set of coalition structures with player set N is denoted by $\mathbf{P_N}$. The binary relation $\leq^F$ on $\mathbf{P_N}$ is defined as follows: we say that a coalition structure $\mathcal{P}' \in \mathbf{P_N}$ is finer than a coalition structure $\mathcal{P} \in \mathbf{P_N}$ (or $\mathcal{P}$ is coarser than $\mathcal{P}'$) which we write $\mathcal{P} \leq^F \mathcal{P}'$ if for any admissible coalition S in $\mathcal{P}'$ there exists an admissible coalition T in $\mathcal{P}$ such that $T \supseteq S$. Note that $(\mathbf{P_N}, \leq^F)$ is a complete lattice.

The **strategic Bertrand oligopoly game** associated with the strategic Bertrand oligopoly game $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$ and the coalition structure $\mathcal{P} \in \mathbf{P_N}$ is a triplet $\Gamma_{bo}^{\mathcal{P}} = (\mathcal{P}, (X_S, \pi_S)_{S \in \mathcal{P}}) \in \mathcal{G}_{bo}$ defined as:

1. a **set of cartels** (or admissible coalitions) $\mathcal{P} = \{S_1, \ldots, S_k\}$;
2. for every $S \in \mathcal{P}$, a **coalition strategy set** $X_S = \prod_{i \in S} X_i$;
3. for every $S \in \mathcal{P}$, a **coalition profit function** $\pi_S : \prod_{S \in \mathcal{P}} X_S \longrightarrow \mathbb{R}$ defined as:

$$\pi_S(p_S, p_{N \setminus S}) = \sum_{i \in S} \pi_i(p).$$

The following proposition is a compilation of different results in Deneckere and Davidson (1985).

Proposition 6.4.1 (Deneckere and Davidson 1985)

- *Let* $\Gamma_{bo}^{\mathcal{P}} = (\mathcal{P}, (X_S, \pi_S)_{S \in \mathcal{P}}) \in \mathcal{G}_{bo}$ *be a strategic Bertrand oligopoly game. Then:*

1. *there exists a unique Nash equilibrium* $p^* \in \varphi^N(\Gamma_{bo}^{\mathcal{P}})$ *such that:*

$$\forall S \in \mathcal{P}, \exists p^{*S} \in \mathbb{R}_+ : \forall i \in S,\ p_i^* = p^{*S}.$$

2. it holds that:

$$\forall S \in \mathcal{P}, \forall T \in \mathcal{P} : s \leq t,\ p^{*S} \leq p^{*T},$$

with strict inequality if $s < t$.

- *Let $\Gamma_{bo}^{\mathcal{P}} = (\mathcal{P}, (X_S, \pi_S)_{S \in \mathcal{P}}) \in \mathcal{G}_{bo}$ and $\Gamma_{bo}^{\mathcal{P}'} = (\mathcal{P}', (X_S, \pi_S)_{S \in \mathcal{P}'}) \in \mathcal{G}_{bo}$ be two strategic Bertrand oligopoly games such that $\mathcal{P} \leq^F \mathcal{P}'$ where $p^* \in \varphi^N(\Gamma_{bo}^{\mathcal{P}})$ and $p^{**} \in \varphi^N(\Gamma_{bo}^{\mathcal{P}'})$. Then:*

3. for any $i \in N$, $p_i^* \geq p_i^{**}$.

Point 1 of Proposition 6.4.1 establishes the existence of a unique Nash equilibrium for any strategic Bertrand oligopoly game $\Gamma_{bo}^{\mathcal{P}} = (\mathcal{P}, (X_S, \pi_S)_{S \in \mathcal{P}}) \in \mathcal{G}_{bo}$ and stipulates that the members of any admissible coalition $S \in \mathcal{P}$ charge identical prices. Point 2 of Proposition 6.4.1 characterizes the distribution of prices within a coalition structure and states that if the size t of an admissible coalition $T \in \mathcal{P}$ is greater than or equal to the size s of an admissible coalition $S \in \mathcal{P}$, then the firms in T charge higher prices than the firms in S. Point 3 of Proposition 6.4.1 analyses the variations of equilibrium prices according to the coarseness of the coalition structure and specifies that any firm charges higher prices when the coalition structure becomes coarser.

The following proposition states that the γ-characteristic function is well-defined.

Proposition 6.4.2 *Let $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$ be a strategic Bertrand oligopoly game. Then, for any coalition $S \in 2^N \backslash \{\emptyset\}$ there exists a unique partial agreement equilibrium under S.*

Proof: Take any coalition $S \in 2^N \backslash \{\emptyset\}$ and consider the coalition structure $\mathcal{P}^S = \{S\} \cup \{\{i\} : i \in N \backslash S\}$. It follows from the definition of the partial agreement equilibrium that a strategy profile $(p_S^*, \tilde{p}_{N \backslash S}) \in \varphi^{PA}(\Gamma_{bo}, S)$ if and only if $(p_S^*, \tilde{p}_{N \backslash S}) \in \varphi^N(\Gamma_{bo}^{\mathcal{P}^S})$. By point 1 of Proposition 6.4.1 we conclude that there exists a unique partial agreement equilibrium under S. ■

By solving the maximization problems derived from the definition of the partial agreement equilibrium, it follows that any $(p_S^*, \tilde{p}_{N \backslash S}) \in \varphi^{PA}(\Gamma_{bo}, S)$ is given by:

$$\forall i \in S,\ p_i^* = \frac{(V - c)\big(2n(1 + r) - r\big)n}{2\big(2n + r(n + s - 1)\big)\big(n + r(n - s)\big) - r^2 s(n - s)} + c \qquad (6.8)$$

and

$$\forall j \in N\backslash S,\, \tilde{p}_j = \frac{(V-c)\big(2n(1+r)-rs\big)n}{2\big(2n+r(n+s-1)\big)\big(n+r(n-s)\big)-r^2s(n-s)} + c \qquad (6.9)$$

When $c = 0$, Deneckere and Davidson (1985) provide equivalent expressions of the above equilibrium prices. We deduce from these equalities that Bertrand oligopoly TU-games in γ-characteristic function form are symmetric. Indeed, it follows from (6.8) that the members of any coalition $S \in 2^N\backslash\{\emptyset\}$ charge identical prices, i.e. there exists $p^{*s} \in \mathbb{R}_+$ such that for any $i \in S$, $p_i^* = p^{*s}$. It follows from (6.9) that outsiders charge identical prices, i.e. there exists $\tilde{p}^s \in \mathbb{R}_+$ such that for any $j \in N\backslash S$, $\tilde{p}_j = \tilde{p}^s$. Hence, the worth $v_\gamma(S)$ depends only on the size s of coalition S, i.e. there exists a mapping $f_\gamma : \mathbb{N} \longrightarrow \mathbb{R}$ such that for any coalition $S \in 2^N\backslash\{\emptyset\}$, it holds that:

$$v_\gamma(S) = f_\gamma(s) = s(p^{*s}-c)\bigg(V - p^{*s} + r\frac{(n-s)}{n}(\tilde{p}^s - p^{*s})\bigg).$$

With these notations in mind, Proposition 6.4.1 calls for some comments which will be useful for the sequel.

Remark 6.4.3

1. For any coalition $S \in 2^N\backslash\{\emptyset\}$, we deduce from point 2 of Proposition 6.4.1 that $p^{*s} \geq \tilde{p}^s$, i.e. the members of coalition S charge higher prices than the outsiders.

2. For any coalition $S \in 2^N\backslash\{\emptyset\}$ and any coalition $T \in 2^N\backslash\{\emptyset\}$ such that $S \subseteq T$, it follows from point 3 of Proposition 6.4.1 that $p^{*s} \leq p^{*t}$ and $\tilde{p}^s \leq \tilde{p}^t$.

3. For any coalition $S \in 2^N\backslash\{\emptyset\}$, let $(p_S^*, \tilde{p}_{N\backslash S}) \in \varphi^{PA}(\Gamma_{bo}, S)$. If $p^{*s} > c$ and $\tilde{p}^s > c$ then for any $i \in N$, $D_i(p_S^*, \tilde{p}_{N\backslash S}) \geq 0$. In order to prove this result, for the sake of contradiction, assume that there exists $i \in N$ such that $D_i(p_S^*, \tilde{p}_{N\backslash S}) < 0$, and $p^{*s} > c$ and $\tilde{p}^s > c$. We distinguish two cases:
   - if $i \in S$ then we deduce from point 1 of Proposition 6.4.1 that for any $j \in S$, $D_j(p_S^*, \tilde{p}_{N\backslash S}) = D_i(p_S^*, \tilde{p}_{N\backslash S}) < 0$. Hence, it follows from $p^{*s} > c$ that coalition S obtains a negative profit.
   - if $i \in N\backslash S$ then it follows from $\tilde{p}^s > c$ that outsider i obtains a negative profit.

In both cases, since coalition S or any outsider can guarantee a non-negative profit by charging $p^{*s} = c$ or $\tilde{p}^s = c$ respectively, we conclude that $(p^*_S, \tilde{p}_{N\setminus S}) \notin \varphi^{PA}(\Gamma_{bo}, S)$, a contradiction.
Note that $p^{*s} > c$ and $\tilde{p}^s > c$ is satisfied if and only if $V > c$, which corresponds to assumption (j). Thus, assumption (j) ensures that the quantity demanded of any firm's brand is non-negative.

As discussed in the introduction, the core associated with the γ-characteristic function is included in the core associated with the β-characteristic function as illustrated in the following example.

Example 6.4.4

Consider the Bertrand oligopoly situation (N, V, r, c) where $N = \{1, 2, 3\}$, $V = 5$, $r = 2$ and $c = 1$. The Bertrand oligopoly TU-games $(N, v_\beta) \in G^\beta_{bo}$ and $(N, v_\gamma) \in G^\gamma_{bo}$ are summarized in the following table:

s	1	2	3
$f_\beta(s)$	0.76	3.33	12
$f_\gamma(s)$	3.36	7.05	12

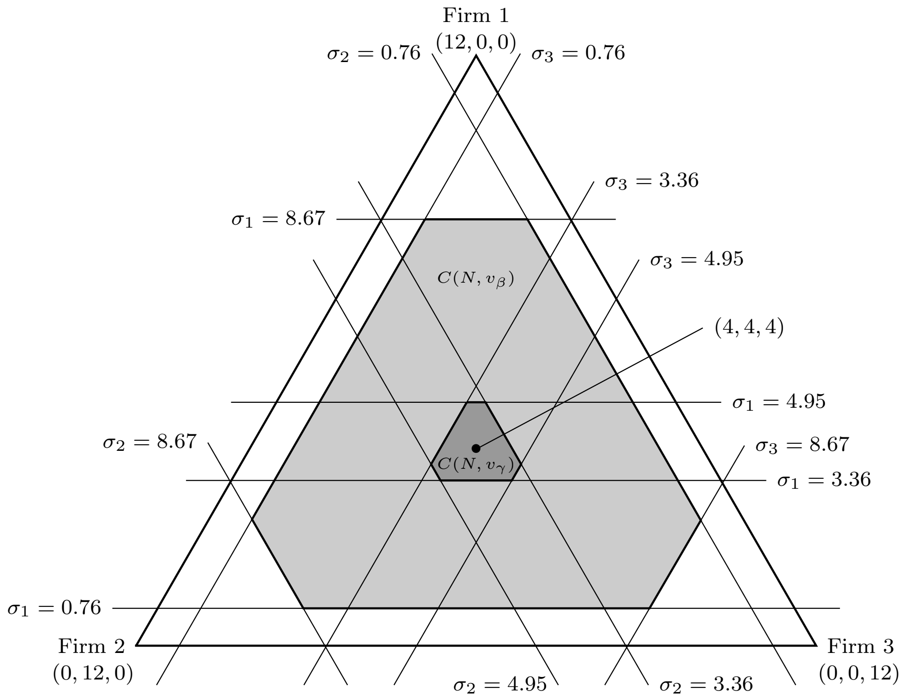
It follows that the cores $C(N, v_\beta)$ and $C(N, v_\gamma)$ are given by:

$$C(N, v_\beta) = \left\{\sigma \in \mathbb{R}^3 : \sum_{i \in N} \sigma_i = 12 \text{ and } \forall i \in \{1, 2, 3\},\ 0.76 \leq \sigma_i \leq 8.67\right\},$$

and

$$C(N, v_\gamma) = \left\{\sigma \in \mathbb{R}^3 : \sum_{i \in N} \sigma_i = 12 \text{ and } \forall i \in \{1, 2, 3\},\ 3.36 \leq \sigma_i \leq 4.95\right\}.$$

The 2-simplex below represents these two core configurations:



Thus, from the Bertrand oligopoly TU-game (N, v_β) to the Bertrand oligopoly TU-game (N, v_γ), we see that the core is substantially reduced. Two features must be noticed. The first is that the equal division solution $(4, 4, 4) \in \mathbb{R}^3$ is the center of gravity of both cores. The second is that the Bertrand oligopoly TU-game (N, v_γ) is convex. In the remainder of this section, we show that these properties still hold for some subset of Bertrand oligopoly TU-games in γ-characteristic function form.

Now, we show that the equal division solution belongs to the core of Bertrand oligopoly TU-games in γ-characteristic function form as enunciated in the following theorem.

Theorem 6.4.5 *For any Bertrand oligopoly TU-game $(N, v_\gamma) \in G^\gamma_{bo}$, it holds that $ED(N, v_\gamma) \in C(N, v_\gamma)$.*

Proof: In order to prove that $ED(N, v_\gamma) \in C(N, v_\gamma)$, we have to show that for any coalition $S \in 2^N \backslash \{\emptyset\}$, $v_\gamma(N)/n \geq v_\gamma(S)/s$. First, it follows from (6.8) that $p^{*n} = (V + c)/2$. Then, take any coalition $S \in 2^N \backslash \{\emptyset\}$. We deduce from points 1 and 2 of Remark 6.4.3 that:

$$\begin{aligned}
\frac{v_\gamma(N)}{n} - \frac{v_\gamma(S)}{s} &= (p^{*n} - c)(V - p^{*n}) - (p^{*s} - c)\left(V - p^{*s} + r\frac{(n-s)}{n}(\tilde{p}^s - p^{*s})\right) \\
&\geq (p^{*n} - c)(V - p^{*n}) - (p^{*s} - c)(V - p^{*s}) \\
&= (p^{*n} - p^{*s})(V + c - p^{*n} - p^{*s}) \\
&\geq (p^{*n} - p^{*s})(V + c - 2p^{*n}) \\
&= (p^{*n} - p^{*s})(V + c - V - c) \\
&= 0,
\end{aligned}$$

which completes the proof. ∎

Then, we provide a sufficient condition under which Bertrand oligopoly TU-games in γ-characteristic function form are convex. For any Bertrand oligopoly TU-game $(N, v_\gamma) \in G^\gamma_{bo}$, this condition is defined as:

$$\forall s \leq n-2, \ \frac{(s+2)(n-s-2)\tilde{p}^{s+2} + s(n-s)\tilde{p}^s + 2p^{*s+1}}{2(s+1)(n-s-1)} \geq \tilde{p}^{s+1} \qquad (6.10)$$

By noting that $(s+2)(n-s-2) + s(n-s) + 2 = 2(s+1)(n-s-1)$, condition (6.10) means that the convex combination of $\tilde{p}^{s+2}$, $\tilde{p}^s$ and p^{*s+1} must be greater than or equal to $\tilde{p}^{s+1}$. It follows from point 1 of Remark 6.4.3 that $p^{*s+1} \geq \tilde{p}^{s+1}$. It follows from point 2 of Remark 6.4.3 that $\tilde{p}^{s+2} \geq \tilde{p}^{s+1} \geq \tilde{p}^s$. Hence, if the difference between $\tilde{p}^{s+1}$ and $\tilde{p}^s$ is sufficiently small, then condition (6.10) holds. For instance, condition (6.10) is satisfied in Example 6.4.4.

Theorem 6.4.6 *Let $(N, v_\gamma) \in G^\gamma_{bo}$ be a Bertrand oligopoly TU-game such that condition* (6.10) *is satisfied. Then (N, v_γ) is convex.*

Proof: We want to prove the convexity property for the set of symmetric TU-games, i.e. for any $s \leq n-2$, $f_\gamma(s+2) + f_\gamma(s) \geq 2f_\gamma(s+1)$. Take any coalition $S \in 2^N \backslash \{\emptyset\}$ of size s such that $s \leq n-2$. Since p^{*s+2} is the unique maximizer for any coalition of size $s+2$, it holds that:

$$\begin{aligned}
f_\gamma(s+2) &= (s+2)(p^{*s+2} - c)\left(V - p^{*s+2} + r\frac{(n-s-2)}{n}(\tilde{p}^{s+2} - p^{*s+2})\right) \\
&\geq (s+2)(p^{*s+1} - c)\left(V - p^{*s+1} + r\frac{(n-s-2)}{n}(\tilde{p}^{s+2} - p^{*s+1})\right).
\end{aligned}$$

Similarly, since p^{*s} is the unique maximizer for any coalition of size s, it holds that:

$$
\begin{aligned}
f_\gamma(s) &= s(p^{*s}-c)\Big(V-p^{*s}+r\frac{(n-s)}{n}(\tilde{p}^s-p^{*s})\Big)\\
&\geq s(p^{*s+1}-c)\Big(V-p^{*s+1}+r\frac{(n-s)}{n}(\tilde{p}^s-p^{*s+1})\Big).
\end{aligned}
$$

For notational convenience, for any $s \leq n-2$, we denote by $A(s) = (s+2)(n-s-2)\tilde{p}^{s+2} + s(n-s)\tilde{p}^s + 2p^{*s+1}$ so that condition (6.10) becomes:

$$
\forall s \leq n-2,\ \ A(s) \geq 2(s+1)(n-s-1)\tilde{p}^{s+1} \tag{6.11}
$$

By the above two inequalities and (6.11) it holds that:

$$
\begin{aligned}
f_\gamma(s+2)+f_\gamma(s) &\geq (p^{*s+1}-c)\Big(2(s+1)(V-p^{*s+1})\\
&\qquad\qquad + \frac{r}{n}\big(A(s)-2(s+1)(n-s-1)p^{*s+1}\big)\Big)\\
&\geq (p^{*s+1}-c)\Big(2(s+1)(V-p^{*s+1})\\
&\qquad\qquad + \frac{r}{n}\big(2(s+1)(n-s-1)(\tilde{p}^{s+1}-p^{*s+1})\big)\Big)\\
&= 2(s+1)(p^{*s+1}-c)\Big(V-p^{*s+1}\\
&\qquad\qquad + r\frac{(n-s-1)}{n}(\tilde{p}^{s+1}-p^{*s+1})\Big)\\
&= 2f_\gamma(s+1),
\end{aligned}
$$

which completes the proof. ■

Note that condition (6.10) is not necessary for the convexity property as illustrated in the following example.

Example 6.4.7

Consider the Bertrand oligopoly situation (N, V, r, c) where $N = \{1, 2, 3, 4\}$, $V = 5$, $r = 6$ and $c = 0$. The Bertrand oligopoly TU-game $(N, v_\gamma) \in G^\gamma_{bo}$ is summarized in the following table:

s	1	2	3	4
$f_\gamma(s)$	3.25	6.96	12.58	25

Although (N, v_γ) is convex, condition (6.10) does not hold for $s = 2$ since $A(2) - 2(2+1)(n-2-1)\tilde{p}^{2+1} = -0.03$ where $A(s)$ is defined as in the proof of Theorem 6.4.6. □

6.5 Concluding remarks

In this chapter, we have considered Bertrand oligopoly TU-games in α, β and γ-characteristic function forms. We have assumed that the demand system is Shubik's (1980) and firms operate at a constant and identical marginal cost. First, we have showed that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions. Moreover, we have proved that the convexity property holds for this set of games. Then, for the set of Bertrand oligopoly TU-games in γ-characteristic function form, we have showed that the equal division solution belongs to the core and we have provided a sufficient condition under which such games are convex. This result substantially extends the superadditivity result of Deneckere and Davidson (1985) and contrasts sharply with the negative core non-emptiness results of Kaneko (1978) and Huang and Sjöström (2003). Thus, although Cournot and Bertrand oligopoly games are basically different in their non-cooperative forms, it appears that their cooperative forms have the same core structure. Hence, it follows from the convexity property that there exists a strong incentive for large scale cooperation in such games.

We have directly assumed that products are differentiated. Two other cases can be considered: when products are unrelated ($r = 0$) and when products are perfect substitutes ($r = +\infty$).

In the first case, the quantity demanded of any firm i's brand only depends on its own price. Hence, any coalition profit does not depend on outsiders' behavior, and so the α, β and γ-characteristic functions are equal. Moreover, for any coalition $S \in 2^N \backslash \{\emptyset\}$, the coalition profit function π_S is separable:

$$\forall x_S \in X_S,\, \pi_S(x_S) = \sum_{i \in S} \pi_i(x_i).$$

Thus, given the strategic Bertrand oligopoly game $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$, for any coalition $S \in 2^N \backslash \{\emptyset\}$ the unique Nash equilibrium $p^* \in \varphi^N(\Gamma_{bo})$ is also the unique partial agreement equilibrium under S, i.e. $\{p^*\} = \varphi^{PA}(\Gamma_{bo}, S)$. Hence, Bertrand oligopoly TU-games are additive, and so $(v(\{i\}))_{i \in N} \in \mathbb{R}^n$ is the unique core element.

In the second case, firms sell a homogeneous product. It follows that firm i's quantity demanded is positive if and only if it charges the smallest price. Since firms operate at a constant and identical marginal cost, for any coalition $S \in$

$2^N \backslash \{\emptyset, N\}$, outsiders charge prices equal to their marginal cost (this outsiders' behavior is consistent with the α, β and γ-approaches), and so coalition S obtains a zero profit. By charging the monopoly price, the grand coalition obtains a non-negative profit, and we conclude that the core is equal to the set of imputations.

Other alternative blocking rules can be considered. For instance, firms in $N\backslash S$ can choose coalitional rather than individual best reply strategies. In this case, the worth of coalition S is given by the unique Nash equilibrium of the strategic Bertrand duopoly game $\Gamma_{bo}^{\{S,N\backslash S\}} = (\{S, N\backslash S\}, (X_T, \pi_T)_{T\in\{S,N\backslash S\}}) \in \mathcal{G}_{bo}$. However, the following example shows that the non-emptiness of the core crucially depends on the substitutability parameter.

Example 6.5.1

We consider the two Bertrand oligopoly situations (N, V, r_1, c) and (N, V, r_2, c) where $N = \{1, 2, 3, 4\}$, $V = 1$, $r_1 = 1$, $r_2 = 3$ and $c = 0$. The two Bertrand oligopoly TU-games derived from the Bertrand oligopoly situations (N, V, r_1, c) and (N, V, r_2, c) are symmetric, and so the worths of any coalition $S \in 2^N\backslash\{\emptyset\}$ are summarized by the functions $f^{r_1} : \mathbb{N} \longrightarrow \mathbb{R}$ and $f^{r_2} : \mathbb{N} \longrightarrow \mathbb{R}$ respectively. The worths of any coalition $S \in 2^N\backslash\{\emptyset\}$ are given in the following table:

s	1	2	3	4
$f^{r_1}(s)$	0.252	0.480	0.719	1
$f^{r_2}(s)$	0.242	0.408	0.622	1

For the Bertrand oligopoly TU-game derived from the Bertrand oligopoly situation (N, V, r_1, c), we have $4f^{r_1}(1) = 1.008 > 1 = f^{r_1}(4)$, and so the core is empty. For the Bertrand oligopoly TU-game derived from the Bertrand oligopoly situation (N, V, r_2, c), the payoff vector $(0.25, 0.25, 0.25, 0.25) \in \mathbb{R}^4$ is a core element, and so the core is non-empty. □

According to Example 6.5.1, we observe that the core becomes non-empty when the substitutability parameter increases. A similar argument is used by Huang and Sjöström (2003) in order to guarantee the non-emptiness of the "recursive core". They prove that the "recursive core" is non-empty if and only if the substitutability parameter is greater than or equal to some number that depends on the number of firms. When firms in $N\backslash S$ choose coalitional best reply strategies, it is likely that a similar condition would ensure the non-emptiness of the core.

Chapter 7

Convexity and the Shapley value in Bertrand oligopoly TU-games with distinct marginal costs

7.1 Introduction

In Chapter 6, we have dealt with Bertrand oligopoly TU-games where all the firms operate at a constant and identical marginal cost and we have focused on the convexity property of such games.
In this chapter, which is based on Driessen, Hou, and Lardon (2010), we go further by studying Bertrand oligopoly TU-games in α and β-characteristic function forms in which firms' marginal costs are possibly distinct. First, we show that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions, which extends the result in Proposition 6.3.3. Then, on the one hand, we show that if the intercept of demand is sufficiently small, then Bertrand oligopoly TU-games in β-characteristic function form have clear similarities with a well-known notion in statistics called variance with respect to the marginal costs. Although such games fail to be convex unless all the firms operate at an identical marginal cost, we prove that they are nevertheless totally balanced. On the other hand, we prove that if the intercept of demand is sufficiently large, then Bertrand oligopoly TU-games in β-characteristic function form are convex which extends the result in Theorem 6.3.5. Finally, we give an appealing expression of the Shapley value for this second game type. We show that the Shapley value is fully determined by decomposing any Bertrand oligopoly TU-game in β-characteristic function form into the difference between two convex TU-games, besides two additive TU-games. Moreover, for this second game type we provide an axiomatic characterization of the Shapley value

by means of two properties: efficiency and individual monotonicity. Recall that efficiency requires that a solution distributes the worth of the grand coalition among the firms. Individual monotonicity stipulates that the difference between the payoffs of two firms is equal to the difference between their individual worth weighted by some real number which depends on their average cost.

The remainder of this chapter is structured as follows. In Section 7.2, we introduce the model and show that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions. In Section 7.3, we show that if the intercept of demand is sufficiently small then Bertrand oligopoly TU-games in β-characteristic function form fail to be convex but are nevertheless totally balanced. Conversely, if the intercept of demand is sufficiently large we prove that Bertrand oligopoly TU-games in β-characteristic function form are convex. In Section 7.4, for the second game type we give an appealing expression of the Shapley value and provide an axiomatic characterization. Section 5 gives some concluding remarks.

7.2 The model

A **Bertrand oligopoly situation** is a triplet $(N, (D_i, C_i)_{i \in N})$ defined as:

1. a finite **set of firms** $N = \{1, 2, \ldots, n\}$;
2. for every $i \in N$, a **demand function** $D_i : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ which assigns to any price vector $p \in \mathbb{R}^n_+$ the quantity demanded of firm i's brand $D_i(p)$;
3. for every $i \in N$, an **individual cost function** $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$.

Throughout this chapter, we assume that:

(k) the demand system is Shubik's (1980):

$$\forall i \in N,\ D_i(p_1, \ldots, p_n) = V - p_i - r\left(p_i - \frac{1}{n}\sum_{j \in N} p_j\right),$$

where p_j is the price charged by firm j, $V \in \mathbb{R}_+$ is the intercept of demand and $r \in \mathbb{R}_{++}$ is the substitutability parameter. When r approaches zero, products become unrelated, and when r approaches infinity, products become perfect substitutes. The quantity demanded of firm i's brand depends on its own price p_i and on the difference between p_i and the average price in the industry $\sum_{j \in N} p_j/n$. This quantity is decreasing with respect to p_i and increasing with respect to any p_j such that $j \neq i$;

(l) firms operate at possibly distinct marginal costs:

$$\forall i \in N,\ \exists c_i \in \mathbb{R}_+ : C_i(x_i) = c_i x_i,$$

where $c_i \in \mathbb{R}_+$ is firm i's marginal cost, and $x_i = D_i(p_1, \dots, p_n) \in \mathbb{R}_+$ is the quantity demanded of firm i's brand.

Given assumptions (k) and (l), a Bertrand oligopoly situation is summarized by the 4-tuple $(N, V, r, (c_i)_{i \in N})$. For notational convenience, for any coalition $S \in 2^N \backslash \{\emptyset\}$ we denote its **average coalitional cost** by $\bar{c}_S = (1/s) \sum_{i \in S} c_i$.

The **strategic Bertrand oligopoly game** associated with the Bertrand oligopoly situation $(N, V, r, (c_i)_{i \in N})$ is a triplet $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N})$ defined as:

1. a finite **set of firms** $N = \{1, 2, \dots, n\}$;
2. for every $i \in N$, an **individual strategy set** $X_i = \mathbb{R}_+$ where $p_i \in X_i$ represents the price charged by firm i;
3. for every $i \in N$, an **individual profit function** $\pi_i : X_N \longrightarrow \mathbb{R}$ defined as:

$$\pi_i(p) = D_i(p)(p_i - c_i).$$

Note that firm i's profit depends on its own price p_i and on the average price in the industry $\sum_{j \in N} p_j / n$. We denote by $\mathcal{G}_{bo} \subseteq \mathcal{G}$ the **set of strategic Bertrand oligopoly games**.

For any coalition $S \in 2^N \backslash \{\emptyset\}$, recall that the **coalition profit function** $\pi_S : X_S \times X_{N \backslash S} \longrightarrow \mathbb{R}$ is defined as:

$$\pi_S(p_S, p_{N \backslash S}) = \sum_{i \in S} \pi_i(p).$$

Given a strategic Bertrand oligopoly game $\Gamma_{bo} = (N, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{bo}$, the associated **Bertrand oligopoly TU-games in α and β-characteristic function forms**, denoted by (N, v_α) and (N, v_β), are defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$v_\alpha(S) = \max_{p_S \in X_S} \min_{p_{N \backslash S} \in X_{N \backslash S}} \pi_S(p_S, p_{N \backslash S}),$$

and

$$v_\beta(S) = \min_{p_{N\setminus S}\in X_{N\setminus S}} \max_{p_S\in X_S} \pi_S(p_S, p_{N\setminus S}).$$

We denote by $G^\alpha_{bo} \subseteq G$ and $G^\beta_{bo} \subseteq G$ the **set of Bertrand oligopoly TU-games in α and β-characteristic function forms** respectively.

The following proposition states that the β-characteristic function is well-defined, and so it generalizes Proposition 6.3.1.

Proposition 7.2.1 *Let $\Gamma_{bo} = (N, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{bo}$ be a strategic Bertrand oligopoly game and $(N, v_\beta) \in G^\beta_{bo}$ be the associated Bertrand oligopoly TU-game. Then, for any coalition $S \in 2^N\backslash\{\emptyset\}$ it holds that:*

$$v_\beta(S) = \pi_S(\overline{p}_S, \overline{p}_{N\setminus S}),$$

where $(\overline{p}_S, \overline{p}_{N\setminus S}) \in X_S \times X_{N\setminus S}$ is given by:

$$\forall i \in S,\ \overline{p}_i = \max\left\{\frac{\overline{c}_S + c_i}{2}, \frac{1}{2}\left(\frac{nV}{n + r(n-s)} + c_i\right)\right\} \tag{7.1}$$

and

$$\sum_{j\in N\setminus S} \overline{p}_j = \max\left\{0, \frac{n}{r}\left(\overline{c}_S\left(1 + r\frac{(n-s)}{n}\right) - V\right)\right\} \tag{7.2}$$

Proof: Take any coalition $S \in 2^N\backslash\{\emptyset\}$. In order to compute the worth $v_\beta(S)$ of coalition S, we have to successively solve the maximization and the minimization problems derived from the definition of the β-characteristic function. In order to do that, we define the **coalition best reply function** $b_S : X_{N\setminus S} \longrightarrow X_S$ as:

$$\forall p_{N\setminus S} \in X_{N\setminus S},\ \{b_S(p_{N\setminus S})\} = \arg\max_{p_S\in X_S} \pi_S(p_S, p_{N\setminus S}).$$

By the definition of b_S it holds that:

$$v_\beta(S) = \min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S(b_S(p_{N\setminus S}), p_{N\setminus S}).$$

In order to compute the worth $v_\beta(S)$, we have to verify that the function b_S is well-defined. Given $p_{N\setminus S} \in X_{N\setminus S}$ consider the profit maximization program of coalition S:

$$\forall p_{N\setminus S} \in X_{N\setminus S},\ \max_{p_S\in X_S} \pi_S(p_S, p_{N\setminus S}).$$

The first-order conditions for a maximum are:

$$\forall p_{N\backslash S} \in X_{N\backslash S}, \forall i \in S,\ \frac{\partial \pi_S}{\partial p_i}(p_S, p_{N\backslash S}) = 0,$$

and imply that the unique possible maximizer $b_S(p_{N\backslash S})$ is given by:

$$\begin{aligned}\forall p_{N\backslash S} \in X_{N\backslash S}, \forall i \in S,\ b_i(p_{N\backslash S}) \quad &= \quad \frac{1}{2(1+r)}\Bigg(V + \frac{r}{n}\Bigg(\sum_{j\in N\backslash S} p_j \\ &\quad + \sum_{j\in S}\big(2b_j(p_{N\backslash S}) - c_j\big)\Bigg)\Bigg) + \frac{c_i}{2}.\end{aligned}$$

We conclude that for any solution of the above maximization program there exists $\bar{b}_S \in \mathbb{R}_+$ such that for any $i \in S$, $b_i(p_{N\backslash S}) - c_i/2 = \bar{b}_S$. Through substitution in the above system of equations we deduce that:

$$2(1+r)\bar{b}_S = V + \frac{r}{n}\Bigg(\sum_{j\in N\backslash S} p_j + 2s\bar{b}_S\Bigg),$$

which is equivalent to:

$$\bar{b}_S = \frac{nV + r\sum_{j\in N\backslash S} p_j}{2\big(n + r(n-s)\big)}.$$

Hence, it holds that:

$$\forall p_{N\backslash S} \in X_{N\backslash S}, \forall i \in S,\ b_i(p_{N\backslash S}) = \frac{nV + r\sum_{j\in N\backslash S} p_j}{2\big(n + r(n-s)\big)} + \frac{c_i}{2} \tag{7.3}$$

so that b_S is well-defined.
Then, given $b_S(p_{N\backslash S}) \in X_S$ consider the profit minimization program of the complementary coalition $N\backslash S$:

$$\min_{p_{N\backslash S} \in X_{N\backslash S}} \pi_S(b_S(p_{N\backslash S}), p_{N\backslash S}).$$

The first-order conditions for a minimum are:

$$\forall j \in N\backslash S,\ \frac{\partial \pi_S}{\partial p_j}(b_S(p_{N\backslash S}), p_{N\backslash S}) = 0,$$

which are equivalent, for any $j \in N\backslash S$, to the following equality:

$$\sum_{j\in N\backslash S} p_j = \frac{n}{r}\left(\overline{c}_S\left(1+r\frac{(n-s)}{n}\right)-V\right).$$

Since for any $i \in N$, $X_i = \mathbb{R}_+$, it follows that any minimizer $\overline{p}_{N\backslash S} \in X_{N\backslash S}$ satisfies:

$$\sum_{j\in N\backslash S} \overline{p}_j = \max\left\{0, \frac{n}{r}\left(\overline{c}_S\left(1+r\frac{(n-s)}{n}\right)-V\right)\right\},$$

which proves (7.2). By substituting (7.2) into (7.3), we deduce that:

$$\forall i \in S, \overline{p}_i = b_i(\overline{p}_{N\backslash S}) = \max\left\{\frac{\overline{c}_S + c_i}{2}, \frac{1}{2}\left(\frac{nV}{n+r(n-s)}+c_i\right)\right\},$$

which proves (7.1) and completes the proof. ■

Proposition 7.2.1 generalizes the result in Proposition 6.3.1. Indeed, if there exists $c \in \mathbb{R}_+$ such that for any $i \in S$, $c_i = c$, then (7.1) and (7.2) coincide with (6.1) and (6.2) respectively. Moreover, Proposition 7.2.1 calls for some comments which will be useful for the sequel.

Remark 7.2.2

For any coalition $S \in 2^N\backslash\{\emptyset\}$, it holds that:

1. If $V \leq \overline{c}_S\big(1+r(n-s)/n\big)$, then by (7.1) any member $i \in S$ charges prices equal to the average between the average coalitional cost and its own marginal cost, $\overline{p}_i = (\overline{c}_S + c_i)/2$, and by (7.2) the outsiders charge a non-negative average price, $\sum_{j\in N\backslash S}\overline{p}_j/(n-s) \geq 0$. In this case, coalition S obtains:

$$v_\beta(S) = \frac{(1+r)}{4}\sum_{i\in S}(c_i - \overline{c}_S)^2 \qquad (7.4)$$

2. If $V > \overline{c}_S\big(1+r(n-s)/n\big)$, then by (7.1) any member $i \in S$ charges prices strictly greater than the average between the average coalitional cost and its own marginal cost, $\overline{p}_i > (\overline{c}_S + c_i)/2$, and by (7.2) the outsiders charge a zero average price, $\sum_{j\in N\backslash S}\overline{p}_j/(n-s) = 0$. In this case, coalition S obtains:

$$v_\beta(S) = \frac{s\big(nV - (n + r(n-s))\overline{c}_S\big)^2}{4n\big(n + r(n-s)\big)} + \frac{(1+r)}{4}\sum_{i \in S}(c_i - \overline{c}_S)^2 \tag{7.5}$$

or equivalently

$$v_\beta(S) = -\frac{V}{2}\sum_{i \in S} c_i - \frac{r}{4n}\bigg(\sum_{i \in S} c_i\bigg)^2 + \frac{(1+r)}{4}\sum_{i \in S} c_i^2 + \frac{s(nV)^2}{4n\big(n + r(n-s)\big)} \tag{7.6}$$

3. Contrary to point 3 of Remark 6.3.2, the computation of the worth $v_\beta(S)$ may not be consistent with the fact that the quantity demanded of firm i's brand, $i \in S$, is positive. Note that the condition $V > \big(1 + r(n-1)/n\big) \max_{i \in N} c_i$ guarantees that for any $i \in S$, $D_i(\overline{p}) \geq 0$.

By point 1 of Remark 7.2.2, Bertrand oligopoly TU-games in β-characteristic function form given by (7.4) have clear similiraties with the well-known notion in statistics called variance. Given a family of marginal costs denoted by $\mathcal{C} = \{c_i\}_{i \in N}$, the associated **variance TU-game** $(N, VAR_{\mathcal{C}}) \in G$ is defined for any coalition $S \in 2^N \backslash \{\emptyset\}$ as:

$$VAR_{\mathcal{C}}(S) = \sum_{i \in S}(c_j - \overline{c}_S)^2.$$

The notion of variance is well-known in the field of statistics and it refers to the sum of the squares of the differences between any coalition member's marginal cost and the average coalitional cost. By (7.4), any Bertrand oligopoly TU-game $(N, v_\beta) \in G_{bo}^\beta$ is strategically equivalent to the corresponding variance TU-game. For instance, for any family $\mathcal{C} = \{c_1, c_2, c_3\}$ for which there exists $c \in \mathbb{R}_+$ such that $c_1 = c_2 = c$ and $c_3 = c + 1$ the associated variance TU-game is given by $VAR_{\mathcal{C}}(S) = 0$ if $3 \in N \backslash S$ and $VAR_{\mathcal{C}}(S) = (s-1)/s$ if $3 \in S$. Note that firms 1 and 2 are substitutes in this variance TU-game.
Generally speaking, for any coalition $S \in 2^N \backslash \{\emptyset\}$ for which there exists $i \in S$ such that $c_i = \overline{c}_S$, it holds that $c_i = \overline{c}_{S \backslash \{i\}}$ and in turn, $VAR_{\mathcal{C}}(S \backslash \{i\}) = VAR_{\mathcal{C}}(S)$. Particularly, if $c_i = \overline{c}_N$ then we have $VAR_{\mathcal{C}}(N) - VAR_{\mathcal{C}}(N \backslash \{i\}) = 0$. Because the latter expression represents an upper bound for the core of the variance TU-game, any core allocation for player i degenerates to zero, provided that firm i's marginal cost coincides with the average coalitional cost of the grand coalition.
By point 2 of Remark 7.2.2, when all the firms operate at an identical marginal cost, the per-capita worth of any Bertrand oligopoly TU-game in β-characteristic

function form given by (7.5) is strategically equivalent to the square of a specific bankruptcy game. Indeed, given a Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$ for which there exists $c \in \mathbb{R}_{++}$ such that for any $i \in N$, $c_i = c$, and for any coalition $S \in 2^N\backslash\{\emptyset\}$, $V > c\big(1 + r(n-s)/n\big)$, the **proportional aggregate netto demand** E is defined as:

$$E = \frac{n(V-c)}{rc}.$$

By (7.5), for any coalition $S \in 2^N\backslash\{\emptyset\}$ it holds that:

$$\frac{v_\beta(S)}{s} = \frac{(rc)^2}{4n} \times \frac{\big(E-(n-s)\big)^2}{n+r(n-s)} > 0 \tag{7.7}$$

The non-zero coalitional worth in the symmetric Bertrand oligopoly TU-game in β-characteristic function form depends on the validity of the constraint $V > c\big(1 + r(n-s)/n\big)$ involving the intercept V or the equivalent constraint $E > (n-s)$ involving the proportional aggregate netto demand E. We interpret $n(V-c)$ as the aggregate netto demand when prices are equal to zero. Obviously, if a coalition S of size s meets the constraint $E \leq (n-s)$, yielding by point 1 of Remark 6.3.2, the zero worth $v_\beta(S) = 0$, then any coalition of the same size s or less inherits the same constraint yielding zero worth. Similarly, if a coalition T of size t meets the inverse constraint $E > (n-t)$, yielding by point 2 of Remark 6.3.2, the positive worth $v_\beta(T) > 0$, then any coalition of the same size or more inherits the same inverse constraint yielding a positive worth. By (7.7), the per-capita worth $v_\beta(S)/s$ is strategically equivalent to the quotient of the square of a bankruptcy game with estate E and unitary claims and a linearly decreasing symmetric game varying from levels $(1+r)n$ down to level n.

In summary, by points 1 and 2 of Remark 7.2.2, any Bertrand oligopoly TU-game in β-characteristic function form consists of two types of worths for any coalition $S \in 2^N\backslash\{\emptyset\}$ according to the relevant constraint $V > \overline{c}_S\big(1+r(n-s)/n\big)$. By (7.4), if the average coalitional cost is sufficiently large then the worth is fully determined by the multiple $(1+r)/4$ of the corresponding variance TU-game. Otherwise, by (7.5), if the average coalitional cost is sufficiently small then the worth counts, besides the corresponding variance TU-game, the positive worth in the symmetric Bertrand oligopoly TU-game, with the understanding that the constant marginal cost must be replaced by the average coalitional cost. The alternative decomposition (7.6) into four types of TU-games will be exploited in Sections 7.3 and 7.4.

By solving successively the minimization and the maximization problems derived from the definition of the α-characteristic function, we can show that the α-characteristic function is well-defined. The proof is similar to the one for Proposition 7.2.1, and so it is not detailed. A useful property is that the same set of Bertrand oligopoly TU-games is associated with the α and β-characteristic functions as enunciated in the following proposition.

Proposition 7.2.3 *Let $\Gamma_{bo} = (N,(X_i,\pi_i)_{i\in N}) \in \mathcal{G}_{bo}$ be a strategic Bertrand oligopoly game, and $(N,v_\alpha) \in G^\alpha_{bo}$ and $(N,v_\beta) \in G^\beta_{bo}$ be the associated Bertrand oligopoly TU-games. Then, for any coalition $S \in 2^N\backslash\{\emptyset\}$ it holds that:*

$$v_\alpha(S) = v_\beta(S).$$

Proof: First, for any coalition $S \in 2^N\backslash\{\emptyset\}$, it holds that:

$$\begin{aligned} v_\alpha(S) &= \max_{p_S\in X_S} \min_{p_{N\backslash S}\in X_{N\backslash S}} \pi_S(p_S,p_{N\backslash S}) \\ &\leq \min_{p_{N\backslash S}\in X_{N\backslash S}} \max_{p_S\in X_S} \pi_S(p_S,p_{N\backslash S}) \\ &= v_\beta(S). \end{aligned}$$

Then, it remains to show that for any coalition $S \in 2^N\backslash\{\emptyset\}$, $v_\beta(S) \leq v_\alpha(S)$. We distinguish two cases:
First, assume that $V \leq \bar{c}_S\big(1 + r(n-s)/n\big)$. It follows from point 1 of Remark 7.2.2 that for any $i \in S$, $\overline{p}_i = (\bar{c}_S + c_i)/2$. Hence, for any $p_{N\backslash S} \in X_{N\backslash S}$ it holds that:

$$\pi_S(\overline{p}_S,p_{N\backslash S}) = \sum_{j\in S}\left(\frac{\bar{c}_S - c_j}{2}\right)\left(V - (1+r)\frac{(\bar{c}_S + c_j)}{2} + \frac{rs}{n}\bar{c}_S + \frac{r}{n}\sum_{k\in N\backslash S} p_k\right).$$

For any of the partial derivatives of π_S with respect to p_k, $k \in N\backslash S$, it holds that:

$$\begin{aligned} \frac{\partial \pi_S}{\partial p_k}(\overline{p}_S,p_{N\backslash S}) &= \frac{r}{n}\sum_{j\in S}\left(\frac{\bar{c}_S - c_j}{2}\right) \\ &= \frac{r}{n}\left(\sum_{j\in S}\frac{\bar{c}_S}{2} - \sum_{j\in S}\frac{c_j}{2}\right) \\ &= \frac{r}{n}\left(\frac{s\bar{c}_S}{2} - \frac{s\bar{c}_S}{2}\right) \\ &= 0. \end{aligned}$$

Hence, we deduce that outsiders' prices don't affect the profit of coalition S. Indeed, after some calculations, for any $p_{N\setminus S} \in X_{N\setminus S}$ it holds that:

$$\pi_S(\overline{p}_S, p_{N\setminus S}) = \frac{(1+r)}{4}\sum_{j\in S}(c_j - \overline{c}_S)^2.$$

Then, assume that $V > \overline{c}_S\big(1 + r(n-s)/n\big)$. It follows from point 2 of Remark 7.2.2 that for any $i \in S$, $\overline{p}_i = (1/2)\big((nV)/(n + r(n-s)) + c_i\big)$ and $\overline{p}_{N\setminus S} = 0_{N\setminus S}$. Hence it holds that:

$$\begin{aligned}\pi_S(\overline{p}_S, p_{N\setminus S}) = \frac{1}{2}\sum_{j\in S}\left(\frac{nV}{n+r(n-s)} - c_j\right)\Bigg(V - \frac{(1+r)}{2}\left(\frac{nV}{n+r(n-s)} + c_j\right)\\ + \frac{r}{2n}\sum_{j\in S}\left(\frac{nV}{n+r(n-s)} + c_j\right) + \frac{r}{n}\sum_{k\in N\setminus S} p_k\Bigg).\end{aligned}$$

For any of the partial derivatives of π_S with respect to p_k, $k \in N\setminus S$, it holds that:

$$\begin{aligned}\frac{\partial \pi_S}{\partial p_k}(\overline{p}_S, p_{N\setminus S}) &= \frac{r}{2n}\sum_{j\in S}\left(\frac{nV}{n+r(n-s)} - c_j\right)\\ &= \frac{r}{2n}\left(\frac{snV}{n+r(n-s)} - \sum_{j\in S} c_j\right)\\ &= \frac{r}{2n}\left(\frac{snV}{n+r(n-s)} - s\overline{c}_S\right)\\ &= \frac{rs}{2n}\left(\frac{nV}{n+r(n-s)} - \overline{c}_S\right)\\ &> 0.\end{aligned}$$

We deduce that for any $p_{N\setminus S} \in X_{N\setminus S}$, $\pi_S(\overline{p}_S, p_{N\setminus S}) \geq \pi_S(\overline{p}_S, 0_{N\setminus S})$. Finally, in both cases it holds that:

$$\overline{p}_{N\setminus S} \in \arg\min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S(\overline{p}_S, p_{N\setminus S}) \tag{7.8}$$

For any coalition $S \in 2^N\setminus\{\emptyset\}$, it follows from (7.8) that:

$$\begin{aligned} v_\beta(S) &= \pi_S(\overline{p}_S, \overline{p}_{N\setminus S}) \\ &= \min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S(\overline{p}_S, p_{N\setminus S}) \\ &\leq \max_{p_S\in X_S} \min_{p_{N\setminus S}\in X_{N\setminus S}} \pi_S(p_S, p_{N\setminus S}) \\ &= v_\alpha(S), \end{aligned}$$

which completes the proof. ■

7.3 Totally balancedness and convexity properties

In this section, we distinguish Bertrand oligopoly TU-games in β-characteristic function form either given by (7.4) or given by (7.5) and (7.6). First, we show that any Bertrand oligopoly TU-game in β-characteristic function form of the first type fails to be convex unless all the firms operate at an identical marginal cost but is nevertheless totally balanced. Then, we prove that any Bertrand oligopoly TU-game in β-characteristic function form of the second type is convex.

The following proposition provides a subset of Bertrand oligopoly TU-games in β-characteristic function form given by (7.4) which fail to be convex unless all the firms operate at an identical marginal cost.

Proposition 7.3.1 *Let* $(N, v_\beta) \in G^\beta_{bo}$ *be a Bertrand oligopoly TU-game given by* (7.4) *for which there exists at least two firms that operate at an identical marginal cost* $c \in \mathbb{R}_+$. *Then* (N, v_β) *is not convex unless all the firms operate at the same marginal cost* c.

Proof: First, assume that all the firms operate at the same marginal cost $c \in \mathbb{R}_+$. It follows from point 1 of Remark 6.3.2 that for any coalition $S \in 2^N\backslash\{\emptyset\}$, $v_\beta(S) = 0$, and so (N, v_β) is trivially convex. This also follows from Theorem 6.3.5.
Then, assume that at least two firms but not all the firms have the same marginal cost $c \in \mathbb{R}_+$. Without loss of generality, we want to show that the associated variance TU-game $(N, VAR_\mathcal{C})$ is not convex. We denote by E the set of firms that operate at marginal cost c, i.e. $E = \{i \in N : c_i = c\}$. By assumption,

$2 \leq |E| < n$. Obviously, for any coalition $S \in 2^E \backslash \{\emptyset\}$, $\bar{c}_S = c$ and thus, $VAR_{\mathcal{C}}(S) = 0$. For any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $S \not\subseteq E$, it holds that:

$$\begin{aligned} \bar{c}_S &= \frac{1}{s} \sum_{i \in S} c_i \\ &= \frac{1}{s} \Big(|S \cap E| c + \sum_{i \in S \backslash E} c_i \Big) \\ &= c + \frac{1}{s} \sum_{i \in S \backslash E} (c_i - c) \end{aligned} \tag{7.9}$$

For notational convenience, we denote by $\theta_S = (1/s) \sum_{i \in S \backslash E} (c_i - c)$. By the definition of the variance TU-game $(N, VAR_{\mathcal{C}})$ and (7.9), it holds that:

$$\begin{aligned} VAR_{\mathcal{C}}(S) &= \sum_{i \in S \cap E} \big(c - \bar{c}_S\big)^2 + \sum_{i \in S \backslash E} \big(c_i - \bar{c}_S\big)^2 \\ &= \sum_{i \in S \cap E} (-\theta_S)^2 + \sum_{i \in S \backslash E} (c_i - c - \theta_S)^2 \\ &= \sum_{i \in S \cap E} \theta_S^2 + \sum_{i \in S \backslash E} (c_i - c)^2 - \sum_{i \in S \backslash E} 2\theta_S (c_i - c) + \sum_{i \in S \backslash E} \theta_S^2 \\ &= s\theta_S^2 + \sum_{i \in S \backslash E} (c_i - c)^2 - 2s\theta_S^2 \\ &= \sum_{i \in S \backslash E} (c_i - c)^2 - s\theta_S^2 \\ &= \sum_{i \in S \backslash E} (c_i - c)^2 - \frac{1}{s} \Big(\sum_{i \in S \backslash E} (c_i - c) \Big)^2. \end{aligned}$$

For any $i \in E$ and any coalition $S \in 2^{N \backslash \{i\}} \backslash \{\emptyset\}$ such that $S \not\subseteq E$, it holds that $S \cup \{i\} \not\subseteq E$ as well as $(S \cup \{i\}) \backslash E = S \backslash E$. Hence, by the latter equality we deduce on the one hand that:

$$VAR_{\mathcal{C}}(S \cup \{i\}) - VAR_{\mathcal{C}}(S) = \frac{1}{s(s+1)} \Big(\sum_{i \in S \backslash E} (c_i - c) \Big)^2.$$

Similarly, for any $i \in E$, any $j \in E$ and any coalition $S \in 2^{N \backslash \{i,j\}} \backslash \{\emptyset\}$ such that $S \not\subseteq E$, it holds that $S \cup \{i,j\} \not\subseteq E$ as well as $(S \cup \{i,j\}) \backslash E = (S \cup \{j\}) \backslash E = S \backslash E$. Hence, we deduce on the other hand that:

$$VAR_{\mathcal{C}}(S \cup \{i,j\}) - VAR_{\mathcal{C}}(S \cup \{j\}) = \frac{1}{(s+1)(s+2)} \Big(\sum_{i \in S \backslash E} (c_i - c) \Big)^2 .$$

Finally, take any $i \in E$ and any $j \in E$ such that $i \neq j$, and choose any $l \in N \backslash E$. Consider $S = \{l\}$ so that $S \not\subseteq E$. Then, it follows from the above two equalities that $VAR_{\mathcal{C}}(S \cup \{i,j\}) - VAR_{\mathcal{C}}(S \cup \{j\}) < VAR_{\mathcal{C}}(S \cup \{i\}) - VAR_{\mathcal{C}}(S)$ which completes the proof. ■

The following example illustrates that Bertrand oligopoly TU-games given by (7.4) may neither be convex nor average convex.

Example 7.3.2

Let $(N, V, r, (c_i)_{i \in N})$ be a Bertrand oligopoly situation where $N = \{1,2,3\}$, $r = 3$, $c_1 = c - 1$, $c_2 = c$ and $c_3 = c + 1$ where c and V are such that for any coalition $S \in 2^N \backslash \{\emptyset\}$ the constraint $V \leq \overline{c}_S\big(1+r(n-s)/n\big)$ is satisfied so that the Bertrand oligopoly TU-game $(N, v_\beta) \in G^\beta_{bo}$ is given by (7.4). Note that (N, v_β) is equal to the variance TU-game $(N, VAR_{\mathcal{C}}) \in G$ where $\mathcal{C} = \{c-1, c, c+1\}$. By (7.4), it holds that for any $i \in N$, $v_\beta(\{i\}) = 0$, $v(\{1,2\}) = v(\{2,3\}) = 1/2$ and $v(\{1,3\}) = v(\{1,2,3\}) = 2$. We conclude that this game is neither convex since $v(N) - v(\{1,3\}) = 0 < 1/2 = v(\{1,2\}) - v(\{1\})$, nor average convex since for $S = \{1,3\}$ and $T = N$ the inequality $\sum_{i \in S} \big(v(S) - v(S \backslash \{i\})\big) \leq \sum_{i \in S} \big(v(T) - v(T \backslash \{i\})\big)$ fails to hold. □

Although a large subset of Bertrand oligopoly TU-games in β-characteristic function form given by (7.4) fails to satisfy the convexity property, the following theorem states that they are nevertheless totally balanced.

Theorem 7.3.3 *Any Bertrand oligopoly TU-game $(N, v_\beta) \in G^\beta_{bo}$ given by* (7.4) *is totally balanced.*

Proof: Take any Bertrand oligopoly TU-game $(N, v_\beta) \in G^\beta_{bo}$ given by (7.4). By the strategic equivalence property between (N, v_β) and the corresponding variance TU-game $(N, VAR_{\mathcal{C}}) \in G$, it is sufficient to prove that $(N, VAR_{\mathcal{C}})$ is totally balanced. Let $\mathcal{B} \subseteq 2^N \backslash \{\emptyset\}$ be a balanced family of coalitions. By the definition of the variance TU-game it holds that:

$$\begin{aligned}
\sum_{S\in\mathcal{B}} \delta_S VAR_{\mathcal{C}}(S) &= \sum_{S\in\mathcal{B}} \delta_S \sum_{i\in S} (c_i - \overline{c}_S)^2 \\
&= \sum_{i\in N} \sum_{S\in\mathcal{B}_i} \delta_S (c_i - \overline{c}_S)^2 \\
&= \sum_{i\in N} \sum_{S\in\mathcal{B}_i} \delta_S \big((c_i - \overline{c}_N) + (\overline{c}_N - \overline{c}_S)\big)^2 \\
&= \sum_{i\in N} \sum_{S\in\mathcal{B}_i} \delta_S \big((c_i - \overline{c}_N)^2 + (\overline{c}_N - \overline{c}_S)^2 + 2(c_i - \overline{c}_N)(\overline{c}_N - \overline{c}_S)\big) \\
&= \sum_{i\in N} (c_i - \overline{c}_N)^2 + \sum_{i\in N} \sum_{S\in\mathcal{B}_i} \delta_S (\overline{c}_N - \overline{c}_S)\big((\overline{c}_N - \overline{c}_S) + 2(c_i - \overline{c}_N)\big) \\
&= \sum_{i\in N} (c_i - \overline{c}_N)^2 + \sum_{S\in\mathcal{B}} \delta_S \sum_{i\in S} (\overline{c}_N - \overline{c}_S)(2c_i - \overline{c}_N - \overline{c}_S) \\
&= VAR_{\mathcal{C}}(N) - \sum_{S\in\mathcal{B}} \delta_S s (\overline{c}_N - \overline{c}_S)^2 \\
&\le VAR_{\mathcal{C}}(N).
\end{aligned}$$

Hence, we conclude that $(N, VAR_{\mathcal{C}})$ is balanced. Since for any coalition $T \in 2^N\backslash\{\emptyset\}$, any subgame $(T, VAR_{\mathcal{C}}^T) \in G$ of $(N, VAR_{\mathcal{C}})$ is also a variance TU-game (associated with the same data restricted to the subset T), a similar argument permits to conclude that $(T, VAR_{\mathcal{C}}^T)$ is balanced, and so we conclude that $(N, VAR_{\mathcal{C}})$ is totally balanced. ■

Now, we want to prove that Bertrand oligopoly TU-games in β-characteristic function form given by (7.6) are convex. For that purpose, we assume that $V > \big(1 + r(n-1)/n\big)\max_{i\in N} c_i$ in order to guarantee that for any coalition $S \in 2^N\backslash\{\emptyset\}$ the constraint $V > \overline{c}_S\big(1 + r(n-s)/n\big)$ is satisfied. By (7.6), any Bertrand oligopoly TU-game $(N, v_\beta) \in G_{bo}^{\beta}$ can be decomposed into four TU-games denoted by $(N, v_1) \in G$, $(N, v_2) \in G$, $(N, v_3) \in G$ and $(N, v_4) \in G$ defined for any coalition $S \in 2^N\backslash\{\emptyset\}$ as:

$$v_1(S) = \sum_{i\in S} c_i; \qquad v_2(S) = v_1(S)^2;$$

$$v_3(S) = \sum_{i\in S} c_i^2; \qquad v_4(S) = \frac{sn(1+r)}{n + r(n-s)}.$$

Hence, it holds that:

$$v_\beta(S) = \frac{-V}{2} v_1(S) - \frac{r}{4n} v_2(S) + \frac{(1+r)}{4} v_3(S) + \frac{V^2}{4(1+r)} v_4(S).$$

The above decomposition calls for some comments.
First, both TU-games $(N, v_k) \in G$, $k \in \{1, 3\}$, arising from the distinct marginal costs and their squares respectively, are additive. It is known that additivity property is redundant for the convexity property.
Then, concerning the TU-game $(N, v_2) \in G$, the square of an additive TU-game is convex too because the marginal contribution of a fixed player i with respect to any coalition $S \in 2^{N\setminus\{i\}}\setminus\{\emptyset\}$ are non-decreasing with respect to the set inclusion:

$$\begin{aligned} v_2(S \cup \{i\}) - v_2(S) &= \big(v_1(S \cup \{i\})\big)^2 - \big(v_1(S)\big)^2 \\ &= \big(v_1(S) + v_1(\{i\})\big)^2 - \big(v_1(S)\big)^2 \\ &= 2v_1(\{i\})v_1(S) + v_1(\{i\})^2. \end{aligned}$$

Hence, it holds that:

$$\begin{aligned} \big(v_2(S \cup \{i, j\}) - v_2(S \cup \{j\})\big) - \big(v_2(S \cup \{i\}) - v_2(S)\big) &= 2v_1(\{i\})v_1(\{j\}) \\ &\geq 0 \end{aligned} \tag{7.10}$$

Finally, it remains to study the TU-game $(N, v_4) \in G$. In order to do that, for any $r > 0$ and $r_n = r/\big(n(1 + r)\big)$, we define the function $f : [0, 1/r_n) \longrightarrow \mathbb{R}$ as:

$$f(x) = \frac{x}{(1 - r_n x)}.$$

Take any coalition $S \in 2^N\setminus\{\emptyset\}$. By the definition of f it holds that:

$$\begin{aligned} f(s) &= \frac{s}{(1 - r_n s)} \\ &= \frac{sn(1 + r)}{\big(n(1 + r) - rs\big)} \\ &= \frac{sn(1 + r)}{\big(n + r(n - s)\big)} \\ &= v_4(S) \end{aligned} \tag{7.11}$$

Moreover, by noting that:

$$\begin{aligned} \frac{V}{1 + r} f(s) - s\bar{c}_S &= \frac{snV}{n + r(n - s)} - s\bar{c}_S \\ &= s\bigg(\frac{nV}{n + r(n - s)} - \bar{c}_S\bigg), \end{aligned}$$

we deduce from the definition of the variance TU-game and (7.5) that:

$$v_\beta(S) = (1+r)\frac{\left(\frac{V}{1+r}f(s) - s\bar{c}_S\right)^2}{4f(s)} + \frac{(1+r)}{4}VAR_{\mathcal{C}}(S) \tag{7.12}$$

The first and second derivatives of f are given by:

$$\frac{df}{dx}(x) = \frac{1}{(1-r_n x)^2} > 0,$$

and

$$\frac{d^2 f}{dx^2}(x) = \frac{2r_n}{(1-r_n x)^3} > 0,$$

respectively. Hence, we deduce that the function f is strictly increasing and strictly convex, and so by (7.11) the TU-game $(N, v_4) \in G$ is strictly convex. Moreover, note that the marginal returns of function f satisfy:

$$f(s+1) - f(s) = \frac{n^2(1+r)^2}{\big(n + r(n-s)\big)\big(n + r(n-s-1)\big)} \tag{7.13}$$

and

$$f(s+2) - 2f(s+1) + f(s) = \frac{2rn^2(1+r)^2}{\big(n + r(n-s)\big)\big(n + r(n-s-1)\big)\big(n + r(n-s-2)\big)} \tag{7.14}$$

In summary, all four TU-games $(N, v_k) \in G$, $k \in \{1, 2, 3, 4\}$, are convex. Since any Bertrand oligopoly TU-game in β-characteristic function form given by (7.6) is the difference of two convex games, it is not straightforward that it is convex too. However, the following theorem states that the convexity property still holds for this game type.

Theorem 7.3.4 *Any Bertrand oligopoly TU-game* $(N, v_\beta) \in G^\beta_{bo}$ *given by* (7.6) *is convex.*

Proof: Take any Bertrand oligopoly TU-game $(N, v_\beta) \in G^\beta_{bo}$ given by (7.6). It follows from the additivity property of the TU-games $(N, v_k) \in G$, $k \in \{1, 3\}$, that (N, v_β) is convex if and only if the TU-game $(N, nV^2 v_4 - r(1+r)v_2) \in G$ is convex. By (7.10) and (7.11) the convexity property of the TU-game $(N, nV^2 v_4 - r(1+r)v_2) \in G$ holds if for any $i \in N$, any $j \in N$ such that $i \neq j$, and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \leq n-2$ it holds that:

$$\big(f(s+2)-f(s+1)\big)-\big(f(s+1)-f(s)\big)\geq \frac{2r(1+r)}{nV^2}c_ic_j.$$

By assumption, it holds that for any $k\in\{i,j\}$, $c_k\leq (nV)/\big(n+r(n-1)\big)$. Hence, the above inequality holds if:

$$\big(f(s+2)-f(s+1)\big)-\big(f(s+1)-f(s)\big)\geq \frac{2nr(1+r)}{(n+r(n-1))^2} \tag{7.15}$$

By noting that the right hand of (7.14) is non-decreasing with respect to the coalition size s and attains its minimum at $s=0$, we deduce that:

$$\begin{aligned} f(s+2)-2f(s+1)+f(s) &\geq \frac{2rn^2(1+r)^2}{(n+rn)(n+r(n-1))(n+r(n-2))} \\ &= \frac{2rn(1+r)}{(n+r(n-1))(n+r(n-2))} \\ &\geq \frac{2nr(1+r)}{(n+r(n-1))^2}, \end{aligned}$$

which completes the proof. ■

7.4 The Shapley value

In this section, we give an appealing expression of the Shapley value for any Bertrand oligopoly TU-game in β-characteristic function form given by (7.6). Then, we provide an axiomatic characterization of the Shapley value. The decomposition (7.6) of any Bertrand oligopoly TU-game $(N,v_\beta)\in G^\beta_{bo}$ into four types of TU-games determines its Shapley value by using linearity, efficiency and symmetry. For notational convenience, for any coalition $S\in 2^N\backslash\{\emptyset\}$ of size $s\leq n-1$ we denote by:

$$p_n(s)=\frac{1}{n\binom{n-1}{s}}.$$

Theorem 7.4.1 *Let $(N,v_\beta)\in G^\beta_{bo}$ be a Bertrand oligopoly TU-game given by (7.6). Then, it holds that:*

$$\forall i\in N,\; Sh_i(N,v_\beta)=\frac{(V-c_i)^2}{4}+\frac{r}{4}c_i(c_i-\bar{c}_N).$$

Proof: By (7.6) and the linearity property, it is sufficient to compute the Shapley value of the four TU-games $(N, v_k) \in G$, $k \in \{1, 2, 3, 4\}$. First, due to its probabilistic interpretation, the Shapley value of any additive game equals the vector of the individual worths. Hence, it holds that:

$$\forall i \in N,\ Sh_i(N, v_1) = c_i \text{ and } Sh_i(N, v_3) = c_i^2 \tag{7.16}$$

Then, by symmetry and efficiency, the Shapley value of any symmetric TU-game coincides with the equal division solution. Hence, it holds that:

$$\forall i \in N,\ Sh_i(N, v_4) = \frac{v_4(N)}{n} = 1 + r \tag{7.17}$$

Finally, it remains to compute the Shapley value of the TU-game $(N, v_2) \in G$. Due to the probabilistic interpretation of the Shapley value, for any $i \in N$ it holds that:

$$\begin{aligned}
Sh_i(N, v_2) &= \sum_{S \in 2^{N \setminus \{i\}}} p_n(s) \Big(c_i^2 + 2c_i \sum_{j \in S} c_j \Big) \\
&= c_i^2 + 2c_i \sum_{S \in 2^{N \setminus \{i\}}} p_n(s) \sum_{j \in S} c_j \\
&= c_i^2 + 2c_i \sum_{j \in N \setminus \{i\}} c_j \sum_{\substack{S \in 2^{N \setminus \{i\}}: \\ j \in S}} p_n(s) \\
&= c_i^2 + c_i \sum_{j \in N \setminus \{i\}} c_j \\
&= c_i n \bar{c}_N
\end{aligned} \tag{7.18}$$

By (7.16), (7.17) and (7.18), and the linearity of the Shapley value, for any $i \in N$ we deduce that:

$$\begin{aligned}
Sh_i(N, v_\beta) &= \frac{-V}{2} c_i - \frac{r}{4n} n c_i \bar{c}_N + \frac{1 + r}{4} c_i^2 + \frac{V^2}{4} \\
&= \frac{(V - c_i)^2}{4} + \frac{r}{4} c_i (c_i - \bar{c}_N),
\end{aligned}$$

which completes the proof. ■

In words, the Shapley value of a Bertrand oligopoly TU-game in β-characteristic function form given by (7.6) is divided into two parts. Precisely, it involves two types of payoffs for any firm $i \in N$, namely the square of the netto demand intercept $V - c_i$, as well as a proportional part c_i of firm's deviation from the average coalitional cost of the grand coalition $c_i - \overline{c}_N$. We refer to Driessen, Hou, and Lardon (2010) for another expression of the Shapley value in Bertrand oligopoly TU-games in β-characteristic function form given by (7.4).

Now, we provide an axiomatic characterization of the Shapley value by means of two properties: efficiency and individual monotonicity. Recall that efficiency requires that a solution distributes the worth of the grand coalition among the firms. Individual monotonicity stipulates that the difference between the payoffs of two firms is equal to the difference between their individual worth weighted by some real number which depends on their average cost. A single-valued solution F on G^{β}_{bo} satisfies:

- **efficiency**: if for any $(N, v_\beta) \in G^{\beta}_{bo}$, $\sum_{i \in N} F_i(N, v_\gamma) = v_\beta(N)$; (EFF)
- **individual monotonicity**: if for any $(N, v_\beta) \in G^{\beta}_{bo}$, for any $i \in N$ and any $j \in N$, $F_i(N, v_\beta) - F_j(N, v_\beta) = \alpha_{ij}\big(v_\beta(\{i\}) - v_\beta(\{j\})\big)$ where $\alpha_{ij} = (\beta_{ij} + r\overline{c}_N)/(\beta_{ij} + 2(r/n)\overline{c}_{\{ij\}})$ and $\beta_{ij} = 2\big(V - (1+r)\overline{c}_{\{ij\}}\big)$. (IM)

Note that $\alpha_{ij} > 0$. Indeed, by assumption $V > (1 + r(n-1)/n)\max_{i \in N} c_i$, it holds that:

$$\begin{aligned}\beta_{ij} + 2(r/n)\overline{c}_{\{ij\}} &> 2\bigg((1 + r(n-1)/n)\max_{k \in N} c_k - (1+r)\overline{c}_{\{ij\}}\bigg) + 2(r/n)\overline{c}_{\{ij\}} \\ &\geq 2r\big((n-1)/n\big)\overline{c}_{\{ij\}} - 2r\overline{c}_{\{ij\}} + 2(r/n)\overline{c}_{\{ij\}} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\beta_{ij} + r\overline{c}_N &> 2\bigg(\big(1 + r(n-1)/n\big)\max_{k \in N} c_k - (1+r)\overline{c}_{\{ij\}}\bigg) + r\overline{c}_N \\ &\geq 2r\overline{c}_{\{ij\}}\big(((n-1)/n) - 1\big) + r\overline{c}_N \\ &= (r/n)\sum_{k \in N} c_k - 2(r/n)\overline{c}_{\{ij\}} \\ &= (r/n)\bigg(\sum_{k \in N} c_k - c_i - c_j\bigg) \\ &\geq 0.\end{aligned}$$

Individual monotonicity requires that the difference between the payoffs of two firms i and j is equal to the difference between their individual worth weighted by some real number α_{ij} which depends on their average cost $\bar{c}_{\{ij\}}$. Since $\alpha_{ij} > 0$, it holds that if $v_\beta(\{i\}) \geq v_\beta(\{j\})$, then by individual monotonicity, $F_i(N, v_\beta) \geq F_j(N, v_\beta)$. In words, if firm i's individual worth is greater than or equal to firm j's individual worth, then former's payoff must be greater than or equal to latter's payoff.

Theorem 7.4.2 *A single-valued solution F on G^β_{bo} satisfies (EFF) and (IM) if and only if $F = Sh$.*

Proof: It is known that the Shapley value satisfies (EFF). In order to prove that the Shapley value satisfies (IM), note that:

$$v_\beta(\{i\}) - v_\beta(\{j\}) = \left(\frac{c_j - c_i}{4}\right)\big(2V - (1 + r(n-1)/n)(c_i + c_j)\big),$$

and

$$Sh_i(N, v_\beta) - Sh_j(N, v_\beta) = \left(\frac{c_j - c_i}{4}\right)\big(2V - (1 + r)(c_i + c_j) + r\bar{c}_N\big).$$

Hence, it holds that:

$$Sh_i(N, v_\beta) - Sh_j(N, v_\beta) = \alpha_{ij}\big(v_\beta(\{i\}) - v_\beta(\{j\})\big).$$

It remains to show that the Shapley value is the unique single-valued solution that satisfies (EFF) and (IM). Pick any single-valued solution F on G^β_{bo}. By (IM) it holds that:

$$\forall i \in N, \forall j \in N,\ F_i(N, v_\beta) - F_j(N, v_\beta) = \alpha_{ij}\big(v_\beta(\{i\}) - v_\beta(\{j\})\big).$$

By summing up the above equalities over all $i \in N$ and by (EFF), for any $j \in N$ it holds that:

$$\begin{aligned}
&\sum_{i \in N} F_i(N, v_\beta) - nF_j(N, v_\beta) = \sum_{i \in N} \alpha_{ij}\big(v_\beta(\{i\}) - v_\beta(\{j\})\big) \\
\iff &F_j(N, v_\beta) = \left(v_\beta(N) - \sum_{i \in N} \alpha_{ij}\big(v_\beta(\{i\}) - v_\beta(\{j\})\big)\right)n^{-1}.
\end{aligned}$$

By a similar argument, for any $j \in N$ it holds that:

$$Sh_j(N, v_\beta) = \Big(v_\beta(N) - \sum_{i \in N} \alpha_{ij}\big(v_\beta(\{i\}) - v_\beta(\{j\})\big)\Big)n^{-1},$$

and so,

$$\forall j \in N,\ F_j(N, v_\beta) = Sh_j(N, v_\beta),$$

which concludes the proof. ■

7.5 Concluding Remarks

In Chapter 6, we have dealt with the set of Bertrand oligopoly TU-games in β-characteristic function form but only with reference to identical marginal costs. In this chapter, we have studied the more general situation with possibly distinct marginal costs. Surprisingly, if the intercept of demand is sufficiently small the Bertrand oligopoly TU-games in β-characteristic function form agrees with the fundamental notion in statistics called variance with respect to the marginal costs. We have proved that such games may fail to be convex and average convex but are nevertheless totally balanced. If the intercept of demand is sufficiently large, the complexity of the description of the Bertrand oligopoly TU-games in β-characteristic function form is compensated by its decomposition into four TU-games, namely two additive TU-games, one symmetric TU-game, and the square of one of these two additive TU-games. Although it concerns the difference of two convex TU-games, we have proved that such games are convex too which extends the result in Theorem 6.3.5. Its current proof technique by decomposition differs from the proof of convexity for the Bertrand oligopoly TU-game in β-characteristic function form with identical marginal costs in Chapter 6. Finally, for this game type we give an appealing expression of the Shapley value and provide an axiomatic characterization by means of two properties: efficiency and individual monotonicity.
When firms operate at possibly distinct marginal costs, the study of Bertrand oligopoly TU-games in γ-characteristic function form would be of the greatest interest. However, the expression of the worth of any coalition appears very complicated so that it seems difficult to analyze the properties of such games.

General conclusion and future research

In this thesis we have developed five essays on cooperative oligopoly games. The contributions are summarized in the following table:

	$\alpha = \beta$	γ
Cournot	**With transferable technologies:** - totally balanced (Norde et al. 2002) - convex (Zhao 1999a) **Without transferable technologies:** - balanced (Zhao 1999b) - convex (Norde et al. 2002, Driessen and Meinhardt 2005)	**TU-game:** - balanced (Theorem 3.4.1) - NP value (Theorems 3.5.1, 3.5.2) **Interval game:** - $\mathcal{I}$-balanced (Theorem 4.3.1) - strongly-balanced (Theorems 4.3.3, 4.3.4, 4.3.7, 4.3.8)
Stackelberg	**With capacity constraints:** = Cournot	**Without capacity constraints:** - $C(N, v) = I(N, v)$ (Proposition 5.3.1) - balanced (Theorems 5.3.3, 5.3.4)
Bertrand	**With identical costs:** - convex (Theorem 6.3.5) **With distinct costs:** - non-convex (Proposition 7.3.1) - totally balanced (Theorem 7.3.3) - convex (Theorem 7.3.4) - Shapley value (Theorems 7.4.1, 7.4.2)	**With identical costs:** - $ED(N, v) \in C(N, v)$ (Theorem 6.4.5) - convex (Theorem 6.4.6)

In summary, we have showed that the core is non-empty for Cournot oligopoly situations. As regards Stackelberg oligopoly situations, we have proved that the core is non-empty if and only if firms' marginal costs are not too heterogeneous. Concerning Bertrand oligopoly situations, we have established the convexity property which means that there are strong incentives to form the grand coalition. Thus, we conclude that for most of oligopoly situations a horizontal agreement is likely to be stable except if deviating coalitions have a first-mover advantage and firms' marginal cost are too heterogeneous. As discussed in the introduction, it is considered that horizontal agreements on sales prices negatively affect welfare. Thus, the results in this thesis suggest and justify an interventionist policy of competition authorities in most of oligopolistic markets.

The contributions of this thesis establish the bases for further research on cooperative oligopoly games.
In Example 3.5.3, the Cournot oligopoly TU-game in γ-characteristic function form fails to be superadditivity or convex. It would be interesting as future work to study which conditions on any individual profit function would ensure the convexity in this set of games.
The proof of Theorem 4.3.4 ensures the existence of an expectation vector for which the standard core is non-empty. It would be of great interest to provide an expression of such an expectation vector in order to obtain more information on the degree of pessimism from which each Cournot oligopoly TU-game associated with a lower degree of pessimism has a non-empty core.
In order to define Stackelberg oligopoly TU-games in γ-characteristic function form, we have assumed that firms do not have any capacity constraint. Proposition 5.3.1 and Theorem 5.3.3 depend on such an assumption. Thus, future research trajectory may consist in introducing capacity constraints in Stackelberg oligopoly situations. Moreover, Theorem 5.3.3 states that the core is empty in case that firms' marginal costs are too heterogeneous. We conjecture this result continues to be true for any Stackelberg oligopoly TU-game in γ-characteristic function form where the inverse demand function is strictly concave and firms operate at a constant and identical marginal cost.
Theorem 6.4.6 provides a sufficient condition which ensures the convexity of Bertrand oligopoly TU-games in γ-characteristic function form. Up to now, we have not succeeded to provide a Bertrand oligopoly TU-game in γ-characteristic function form which fails to be convex. It would be interesting to establish either the proof of the convexity property for any Bertrand oligopoly TU-game in γ-characteristic function form or the construction of an example in which the convexity property fails to hold.
When firms have distinct marginal costs, we have dealt with Bertrand oligopoly TU-games in α and β-characteristic function forms by assuming that technologies are not transferable. Future research of this work would consist in studying Bertrand oligopoly TU-games in α, β and γ-characteristic function forms in which technologies are transferable.

Résumé

Une question inhérente à la théorie de l'oligopole concerne l'existence de comportements collusifs dans les marchés oligopolistiques. La raison principale pour laquelle les économistes s'intéressent à ce phénomène est que la formation de cartels affecte à la fois le surplus des consommateurs et le surplus des producteurs et donc le surplus total, i.e. le bien-être économique. En économie du bien-être, l'idée communément admise est que le pouvoir de monopole impacte négativement le surplus des agents économiques, et plus particulièrement celui des consommateurs.

À la fin du 19ème siècle, suite à la formation de cartels aux États-Unis, un consensus a émergé sur la nécessité de maintenir une concurrence effective dans les industries. Cette prise de conscience a conduit à la mise en place en 1890 de la première loi anti-trusts aux États-Unis, le Sherman Act composé de deux sections. La première section prohibe les ententes illicites qui restreignent les échanges et le commerce. La deuxième section sanctionne les monopoles et les tentatives de monopoliser plus connues sous l'expression d'«abus de position dominante». Cette législation a permis au gouvernement américain de l'époque de poursuivre en justice plusieurs cartels célèbres tels que l'American Tobacco et la Standard Oil Company qui furent démantelés en 1911. Par la suite, d'autres lois anti-trusts ont été mises en place comme le Clayton Act et le Federal Trade Commission Act en 1914.

Dans l'Union Européenne, la politique de concurrence moderne est principalement établie par les traités de Rome (1957) et d'Amsterdam (1997). Le droit européen de la concurrence distingue parmi les firmes les accords dits «horizontaux» des accords dits «verticaux» (Article 81), et prohibe les abus de position dominante. Un accord horizontal est un accord ou une pratique concertée entre entreprises opérant au même niveau de la chaîne de production ou de distribution. Un accord vertical est un accord ou une pratique concertée entre entreprises opérant à différents niveaux de la chaîne de production ou de distribution, et réglant les conditions sous lesquelles les parties peuvent acheter, vendre ou revendre certains biens ou services. L'abus de position dominante concerne une entreprise en situation de domination grâce à son pouvoir de marché, qui

profite de cette position pour s'émanciper des contraintes que devrait lui imposer la libre concurrence. Motta (2004) définit la politique de concurrence comme «l'ensemble des politiques et des lois assurant que la concurrence sur le marché n'est pas restreinte de manière à réduire le bien-être économique». Dans cette définition, deux éléments doivent être soulignés. Le premier élément est que les entreprises peuvent restreindre la concurrence d'une manière qui n'est pas forcément préjudiciable pour le bien-être économique. C'est principalement le cas des accords horizontaux qui concernent les activités de recherche et développement des firmes (D'Aspremont et Jacquemin 1988). En utilisant les échanges de licence ou encore la mise en commun de brevets, les entreprises partagent les coûts et les bénéfices des découvertes et peuvent ainsi réduire leur prix de vente. C'est aussi le cas de nombreuses restrictions verticales (accords verticaux) entre un producteur et un ou plusieurs détaillants telles que les prix non-linéaires (le détaillant paie le coût marginal de production plus une somme fixe au producteur), l'imposition d'un quota (le détaillant doit acheter au moins ou au plus une certaine quantité de biens fixée par le producteur) et les clauses d'exclusivité (un accord d'exclusivité territorial spécifie qu'un détaillant est le seul à pouvoir vendre un certain bien sur une zone géographique délimitée). Le second élément est que le bien-être économique est l'objectif poursuivi par les autorités de la concurrence. Le bien-être économique est défini comme la somme des surplus des consommateurs et des producteurs. Le surplus d'un consommateur est la différence entre le prix qu'il est prêt à payer pour acquérir un bien et le montant qu'il paye effectivement lors de l'achat du bien. Le surplus d'un producteur est le profit qu'il réalise en vendant le bien en question.

Bien que les accords horizontaux qui concernent dans les activités de recherche et développement puissent être socialement désirables, à la fois aux États-unis et en Union européenne, de nombreux autres accords horizontaux concernant le prix de vente ou la division des parts de marché sont considérés comme préjudiciables pour le bien-être économique.

En gardant à l'esprit ces considérations, l'étude des jeux d'oligopole coopératifs est pertinente dans la mesure où elle permet d'établir les conditions inhérentes au marché sous lesquelles un accord horizontal sur les prix de vente est susceptible d'apparaître. Comme nous l'avons démontré, la mise en place de tels accords constitue une des principales préoccupations des autorités de la concurrence. Cette thèse est composée de deux parties. La première partie considère un cadre de concurrence en quantité dans lequel les entreprises peuvent indirectement contrôler le prix de vente en passant des accords horizontaux sur leur quantités produites. La seconde partie analyse un cadre de concurrence en prix dans lequel les entreprises peuvent directement manipuler le prix de vente en passant des accords horizontaux sur leur prix fixés. D'une manière générale, la coopération entre entreprises passe par formation d'un cartel (coalition) dans lequel les entreprises passent des accords horizontaux.

Aumann (1959) propose d'analyser la formation des coalitions, et donc les accords horizontaux sur le prix de vente, en convertissant un jeu non-coopératif en un jeu coopératif. Un concept solution approprié (par exemple, le cœur) permet alors de traiter de la stabilité des structures de coalitions (par exemple, l'ensemble de tous les joueurs, aussi appelé la «grande coalition»). Dans cette thèse, nous considérons cette approche coopérative dans les situations d'oligopole. La principale différence avec l'approche non-coopérative est que les entreprises sont autorisées à signer des accords contraignants dans le but de coopérer. Cette hypothèse permet de définir un jeu d'oligopole coopératif dans lequel tout cartel (coalition) est susceptible de se former. Il est communément admis que les transferts de profits entre les firmes appartenant à un même cartel sont possibles de telle sorte que le profit retiré par un cartel peut être distribué librement entre ses membres. Les jeux coopératifs associés à cette hypothèse de transferts de profits sont les jeux à utilité transférable ou encore jeux sous forme caractéristique. D'une manière générale, un jeu sous forme caractéristique est la donnée d'un ensemble de joueurs $N = \{1, 2, \ldots, n\}$ et d'une fonction caractéristique $v : 2^N \longrightarrow \mathbb{R}$ qui assigne à chaque coalition $S \in 2^N \backslash \{\emptyset\}$ une capacité $v(S)$. Par convention, on suppose que $v(\emptyset) = 0$. Le nombre $v(S)$ est l'utilité totale dont disposent les membres de la coalition S. Par exemple, on considère un marché oligopolistique comprenant trois entreprises, dénommées 1, 2 et 3 respectivement. Le jeu d'oligopole sous forme caractéristique associé est alors la donnée d'un ensemble de firmes $N = \{1, 2, 3\}$, et des capacités $v(\{1\})$, $v(\{2\})$ et $v(\{3\})$ pour les coalitions à une seule firme, $v(\{1, 2\})$, $v(\{1, 3\})$ et $v(\{2, 3\})$ pour les coalitions à deux firmes, et $v(\{1, 2, 3\})$ pour la coalition à trois firmes (la grande coalition).
Une des principales caractéristiques des marchés oligopolistiques est que le profit de chaque entreprise dépend des quantités produites ou des prix fixés par les autres firmes. Le profit d'un cartel dépend donc également des stratégies choisies par les firmes à l'extérieur du cartel. Par conséquent, pour déterminer la capacité d'une coalition, nous devons spécifier la manière dont les firmes n'appartenant pas au cartel agissent. Pour ce faire, nous supposons que les entreprises à l'extérieur du cartel se comportent selon certaines règles, appelées «règles de blocage», dans un jeu non-coopératif et plus précisément dans un jeu stratégique. Par exemple, certaines règles de blocage spécifient que les firmes n'appartenant pas au cartel sélectionnent les stratégies qui minimisent le profit du cartel. D'autres règles de blocage stipulent que toute firme à l'extérieur du cartel vise à maximiser son profit individuel étant donné les stratégies choisies par les autres entreprises. Un concept solution pour le jeu d'oligopole stratégique qui autorisent les firmes membres d'un même cartel à signer des accords contraignants

et qui spécifient les stratégies choisies par les firmes à l'extérieur du cartel. La capacité d'une coalition est alors égale au profit qu'elle obtient dans le jeu d'oligopole stratégique, et donc le jeu d'oligopole coopératif associé est entièrement spécifié. En particulier, afin de définir un jeu d'oligopole coopératif à partir d'un jeu d'oligopole stratégique, nous suivons trois approches suggérées par Aumann (1959), et Chander et Tulkens (1997).

Aumann (1959) propose les deux premières approches. La première approche stipule que chaque cartel calcule le profit qu'il peut obtenir indépendamment du choix des firmes à l'extérieur. La seconde approche consiste à calculer le profit minimum pour lequel les firmes à l'extérieur du cartel peuvent empêcher le cartel d'obtenir un meilleur profit. Ces deux approches sont associées à deux fonctions caractéristiques, appelées fonctions caractéristiques α et β respectivement. Dans l'exemple d'oligopole ci-dessus, supposons que les entreprises vendent des biens différenciés, se fassent une concurrence en prix en choisissant un prix $p_i \in \mathbb{R}^+$, $i \in N$, produisent à un coût marginal constant et identique égal à un, et que le système de demande soit de Shubik (1980), i.e. la quantité demandée à chaque firme i, $i \in N$, sera définie par:

$$D_i(p_1, p_2, p_3) = 5 - p_i - 2\Bigg(p_i - \frac{1}{3}\sum_{j=1}^{3} p_j\Bigg).$$

La quantité demandée à la firme i dépend donc de son propre prix p_i et de la différence entre son prix p_i et du prix moyen pratiqué dans l'industrie $\sum_{i=1}^{3} p_j/3$. Cette quantité demandée est décroissante en p_i et croissante en p_j avec $j \neq i$.

En suivant l'approche α, pour toute coalition $S \in 2^N\backslash\{\emptyset\}$, tandis que le cartel maximise la somme des profits de ses membres dans un premier temps, les firmes n'appartenant pas au cartel minimisent le profit du cartel dans un second temps étant donné les stratégies préalablement choisies par les membres du cartel.

En suivant l'approche β, pour toute coalition $S \in 2^N\backslash\{\emptyset\}$, tandis que les firmes à l'extérieur du cartel minimisent le profit du cartel dans un premier temps, le cartel maximise la somme des profits de ses membres dans un second temps. Dans l'exemple d'oligopole ci-dessus, les fonctions caractéristiques α et β sont égales et données par: pour tout $i \in N$, $v_\alpha(\{i\}) = v_\beta(\{i\}) = 0.76$, pour tout $i \in N$ et tout $j \in N$ tel que $j \neq i$, $v_\alpha(\{i,j\}) = v_\beta(\{i,j\}) = 3.33$, et $v_\alpha(\{1,2,3\}) = v_\beta(\{1,2,3\}) = 12$.

Cependant, les approches α et β semblent peu appropriées pour étudier la formation des coalitions dans les marchés oligopolistiques dans la mesure où en minimisant le profit d'un cartel, les firmes à l'extérieur du cartel subissent elles-mêmes des pertes. Un argument similaire est développé par Rosenthal (1971). Dans l'exemple d'oligopole ci-dessus, on peut vérifier que les firmes à l'extérieur d'un cartel fixent des prix nuls afin de minimiser le profit du cartel, et donc

obtiennent des profits négatifs.
C'est la raison pour laquelle Chander et Tulkens (1997) proposent une autre règle de blocage plus crédible selon laquelle les firmes n'appartenant pas au cartel choisissent leur meilleure réponse individuelle face aux stratégies choisies par les membres du cartel. Cette approche est associée à une fonction caractéristique, appelée fonction caractéristique γ. En suivant l'approche γ, pour toute coalition $S \in 2^N \backslash \{\emptyset\}$, tandis que le cartel maximise la somme des profits de ses membres, chaque firme à l'extérieur du cartel maximise simultanément son profit individuel. Autrement dit, la coalition S et chaque firme à l'extérieur du cartel jouent un équilibre de Nash. Dans le marché oligopolistique ci-dessus, la fonction caractéristique γ est donnée par: pour tout $i \in N$, $v_\gamma(\{i\}) = 3.36$, pour tout $i \in N$ et tout $j \in N$ tel que $j \neq i$, $v_\gamma(\{i,j\}) = 7.05$, et $v_\gamma(\{1,2,3\}) = 12$. On observe que la capacité de la grande coalition N est égale pour les trois approches α, β et γ, et que la capacité de toute autre coalition $S \in 2^N \backslash \{\emptyset, N\}$ est plus grande sous l'approche γ que sous les approches α et β.

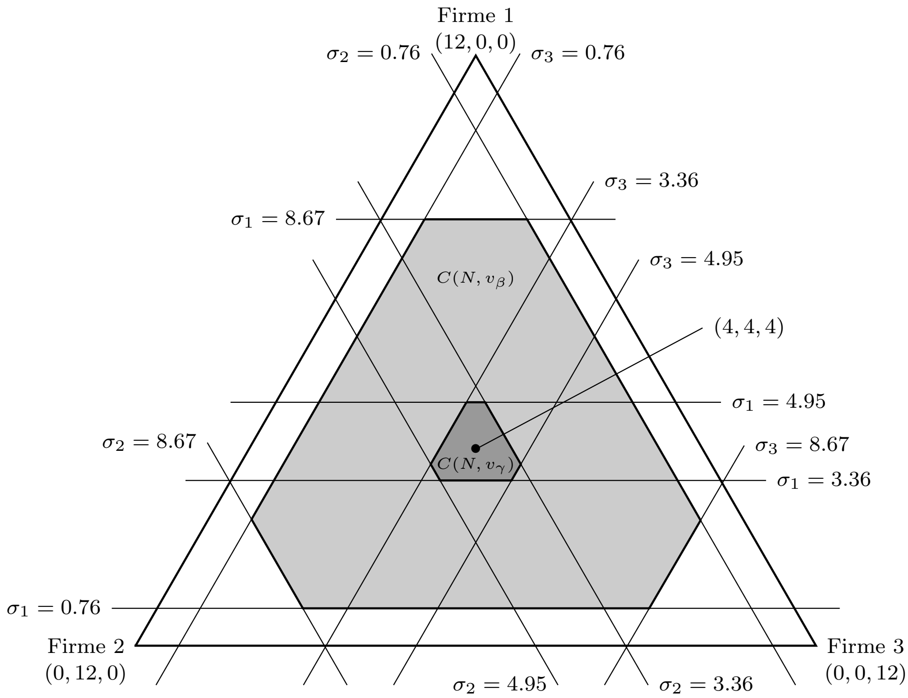
Un concept solution approprié pour les jeux sous forme caractéristique qui permet de traiter de la stabilité de la grande coalition est le cœur. Un vecteur de paiements appartient au cœur s'il n'existe aucune coalition qui peut faire scission de la grande coalition et distribuer un paiement strictement supérieur à l'ensemble de ses membres. La stabilité de la grande coalition est alors associée à la non-vacuité du cœur. Pour les jeux sous forme caractéristique, le cœur est l'ensemble des vecteurs de paiements $\sigma \in \mathbb{R}^n$ tels que $\sum_{i \in N} \sigma_i = v(N)$ et pour toute coalition $S \in 2^N \backslash \{\emptyset\}$, $\sum_{i \in S} \sigma_i \geq v(S)$. La première condition stipule que la capacité de la grande coalition est entièrement distribuée à l'ensemble des joueurs. La seconde condition signifie qu'il n'existe aucun sous-groupe de joueurs qui conteste ce partage en faisant scission de la grande coalition. Dans le marché oligopolistique ci-dessus, ces deux conditions impliquent que le cœur associé aux fonctions caractéristiques α et β est donné par:

$$C(N, v_\beta) = \left\{ \sigma \in \mathbb{R}^3 : \sum_{i=1}^{3} \sigma_i = 12 \text{ et } \forall i \in N,\, 0.76 \leq \sigma_i \leq 8.67 \right\},$$

tandis que le cœur associé à la fonction caractéristique γ est donné par:

$$C(N, v_\gamma) = \left\{ \sigma \in \mathbb{R}^3 : \sum_{i=1}^{3} \sigma_i = 12 \text{ et } \forall i \in N,\, 3.36 \leq \sigma_i \leq 4.95 \right\}.$$

Le 2-simplexe ci-dessous représente ces deux structures géométriques du cœur:

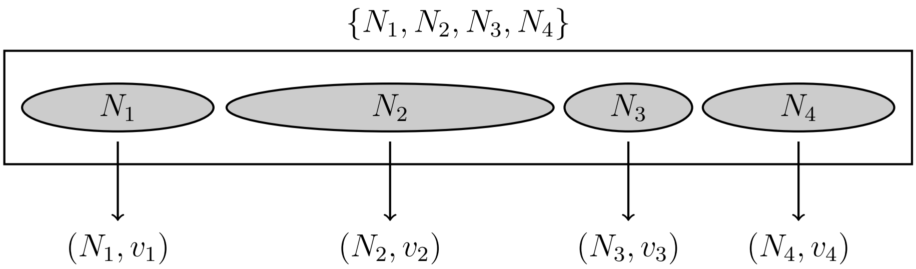


Notons que le cœur se réduit très fortement lors du passage des approches α et β à l'approche γ. Ceci n'est pas surprenant dans la mesure où nous avons observé que la capacité de la grande coalition N est égale pour les trois approches α, β et γ, et que la capacité de toute autre coalition $S \in 2^N \backslash \{\emptyset, N\}$ est plus grande sous l'approche γ que sous les approches α et β de telle sorte qu'il existe moins d'incitations pour les coalitions à faire scission de la grande coalition sous l'approche γ.
Par conséquent, la structure géométrique du cœur dépend très fortement de la règle de blocage utilisée pour définir un jeu d'oligopole coopératif. Dans le marché oligopolistique ci-dessus, la non-vacuité du cœur signifie qu'il existe un vecteur de paiements (par exemple, $\sigma = (4, 4, 4)$) qui permet à la grande coalition de rester stable. Par la suite, nous montrerons que sous des hypothèses standard, les jeux d'oligopole coopératifs permettent d'étudier la stabilité de toute structure de coalitions et pas uniquement de la grande coalition.

L'étude des jeux d'oligopole coopératifs apparaît pertinente pour révéler les mécanismes de coopération dans de nombreuses industries au sein desquelles les accords entre entreprises constituent un choix stratégique telles que les industries de matières premières et de télécommunications. Depuis la création du GATT (Accord général sur les tarifs douaniers et le commerce) en 1947 et de l'OMC (Organisation Mondiale du Commerce) en 1994, les barrières à l'import et à l'export des biens et des services ont considérablement diminué. L'augmentation des flux internationaux de biens et services ont poussé les entreprises à faire des alliances stratégiques nationales et internationales. Dès lors, la coopération entre firmes est devenue un choix stratégique dans le but de réduire leurs coûts, de diversifier leurs activités et d'accroître leurs parts de marché.

Les industries de matières premières sont caractérisées par d'importants coûts d'extraction et de transports. Dans ce type d'industries, les entreprises exportatrices sont susceptibles de coopérer afin de réduire leurs coûts de transports. Massol et Tchung-Ming (2009) étudient la possibilité d'une coopération logistique profitable entre douze pays exportateurs de gaz naturel liquéfié. Ils fournissent l'exemple suivant: en 2007, Trinidad et Tobago ont expédié près de 2.7 Gm^3 (giga mètre cube) en Europe, pendant que l'Algérie a exporté 2.1 Gm^3 aux États-Unis. Au regard de leur position géographique respective, ces deux pays exportateurs auraient un intérêt commun à coopérer sur leur logistique afin de réduire leur coût de transports. En supposant que de tels accords de coopération n'auraient aucun effet sur les prix du marché, Massol et Tchung-Ming définissent un jeu d'oligopole sous forme caractéristique (calibré sur l'année 2007) dans lequel chaque coalition minimise les coûts de transports de ses membres. Ils montrent que le cœur de ce jeu est vide même pour un faible coût de coordination entre les douze pays exportateurs. Par conséquent, ils concluent que la crédibilité d'une coopération logistique sans effet sur les prix de marché est faible.

Un autre exemple concerne les industries dans lesquelles l'innovation occupe une place prépondérante pour soutenir la compétitivité des entreprises. Dans de telles industries, les entreprises coopèrent afin d'étendre leurs activités et d'augmenter leurs parts de marché. C'est dans les industries de télécommunications que des changements technologiques se produisent fréquemment (Noam 2006). Dans la mesure où les opérateurs sont restreints à leurs territoires nationaux respectifs, la coopération avec d'autres opérateurs constitue l'unique moyen d'étendre leur services en tant qu'entreprise multinationale. Un exemple de coopération est Unisource, une entreprise pan-européenne de télécommunications. Nous renvoyons le lecteur à Graak (1996) pour une étude sur les industries de télécommunications en Union Européenne. Dans ce type d'industries, on peut se poser la question de la stabilité de la structure de marché induite par les alliances stratégiques. L'étude des jeux d'oligopole coopératifs constitue une approche originale afin de traiter de la stabilité des structures de marché. Pour ce faire, nous supposons que chaque coalition appartenant à une structure de coalitions ne peut pas communiquer avec les autres coalitions de telle sorte que les seules coalitions susceptibles de faire scission de la structure de marché sont nécessairement des sous-groupes de firmes des coalitions déjà existantes (Ray et Vohra (2007) formulent une hypothèse similaire). Pour chaque cartel, on définit alors un jeu d'oligopole sous forme caractéristique dans lequel le cartel en question est considéré comme la grande coalition du jeu et où les choix des autres cartels sont considérés comme fixés et donc, dans une certaine mesure, peuvent être omis. La stabilité d'un cartel appartenant à une structure de marché est alors associée à la non-vacuité du cœur du jeu d'oligopole sous forme caractéristique associé. Pour une structure de marché comprenant quatre cartels $\{N_1, N_2, N_3, N_4\}$, cet argument est illustré par la figure ci-dessous:



Ainsi, l'étude des jeux d'oligopole coopératifs permet de traiter de la stabilité des structures de coalitions et constitue une approche alternative aux modèles non-coopératifs de la formation des coalitions (D'Aspremont et al. 1983, Hart and Kurz 1983, Bloch 1996, Ray et Vohra 1997).

L'étude des jeux d'oligopole coopératifs permet également de traiter de certains problèmes environnementaux. La «tragédie des communs» est un célèbre dilemme dans lequel une ressource commune est surexploitée (Hardin 1968), comme par exemple la diminution des stocks de poissons ou encore la déforestation des forêts tropicales. Funaki et Yamato (1999) étudient une économie disposant d'une ressource commune de poissons en utilisant les jeux sous forme caractéristique. Ils montrent que si chacune des coalitions a des anticipations pessimistes sur la formation des coalitions des pêcheurs extérieurs à la coalition, i.e. si ces pêcheurs restent isolés et jouent de manière non-coopérative, alors le cœur est non-vide et la tragédie des communs peut être évitée. Autrement, si chaque coalition a des anticipations optimistes sur la formation des coalitions des pêcheurs extérieurs à la coalition, i.e. si ces pêcheurs s'associent et jouent de manière coopérative face au cartel, alors le cœur est vide et la tragédie des communs ne peut pas être évitée. Étant donné qu'une situation d'oligopole décrit aussi une situation de ressource commune (Moulin 1997), l'étude des jeux d'oligopole coopératifs est pertinente pour traiter des problèmes de partage d'une ressource naturelle (Pham Do 2003).

La théorie de l'oligopole traite de modèles de concurrence qui peuvent schématiquement être divisés en deux parties, i.e. les modèles de concurrence en quantité (Cournot 1838) et les modèles de concurrence en prix (Bertrand 1883). Pour un cadre de concurrence en quantité, Stackelberg (1934) incorpore l'idée d'engagement en proposant le modèle de «meneur-suiveur». Pour chacune de ces trois situations d'oligopole, des travaux précurseurs ont déjà étudié la coopération dans les marchés oligopolistiques en modélisant des jeux d'oligopole non-coopératifs et coopératifs.

En ce qui concerne les situations d'oligopole de Cournot, Salant et al. (1983) analysent les quantités d'équilibres produites par les cartels et montrent que certaines fusions entre firmes peuvent réduire leur profits.
Norde et al. (2002) distinguent deux situations d'oligopole différentes, i.e. celles ayant des technologies transférables et celles ayant des technologies non-transférables. Dans le premier cas, un groupe de firmes produit en utilisant la technologie la plus efficace parmi les technologies utilisées par les membres du cartel. Dans le second cas, un tel transfert de technologies n'est pas possible. Pour les situations d'oligopole de Cournot avec ou sans technologies transférables, Zhao (1999a, b) montrent que les fonctions caractéristiques α et β sont égales, et que, par conséquent, le même ensemble de jeux d'oligopole de Cournot sous forme caractéristique est associé à ces deux fonctions caractéristiques.
Lorsque les technologies sont transférables, Zhao (1999a) donne une condition nécessaire et suffisante qui garantit la propriété de convexité en supposant que la fonction de demande inverse et les fonctions de coûts sont linéaires. Cette propriété signifie qu'il existe de très fortes incitations à former la grande coalition. Bien que dans un cadre plus général de tels jeux ne satisfassent pas toujours à la propriété de convexité, Norde et al. (2002) montrent que ces jeux satisfont à la propriété de balancement total qui assure la non-vacuité du cœur.
Lorsque les technologies ne sont pas transférables, Zhao (1999b) prouve que le cœur est non-vide si les ensembles de stratégies individuelles sont compactes et convexes et que les fonctions de profit individuel sont continues et concaves sur l'ensemble des profils de stratégies. En utilisant une technique similaire à celle de Scarf (1971), Zhao montre que le cœur est non-vide pour les jeux sous forme caractéristique dans lesquels les ensembles de stratégies individuelles sont compacts et convexes, les fonctions de profit individuel sont continues et concaves sur l'ensemble des profils de stratégies, et la propriété de séparabilité forte est vérifiée. Cette dernière condition stipule que la fonction d'utilité d'une coalition et les fonctions d'utilité individuelle de chacun de ses membres ont les mêmes arguments qui les minimisent. Zhao montre que les jeux d'oligopole de Cournot sous forme caractéristique satisfont à cette dernière condition. En outre, Norde et al. (2002) montrent que ces jeux satisfont à la propriété de convexité dans le cas où la fonction de demande inverse et les fonctions de coût individuel sont linéaires. Enfin, Driessen et Meinhardt (2005) donnent des conditions suffisantes

qui garantissent la propriété de convexité dans un cadre plus général.

Concernant les situations d'oligopole de Stackelberg, dans lesquels il y a un seul meneur et plusieurs suiveurs qui se font concurrence en quantité, Sherali et al. (1983) prouvent l'existence d'un unique équilibre de Nash dans les jeux d'oligopole de Stackelberg stratégique où la fonction de demande inverse est deux fois dérivable, strictement décroissante et satisfait pour toute quantité $X \in \mathbb{R}_+$, $(dp/dX)(X) + X(d^2p/dX^2)(X) \leq 0$, et où les fonctions de coût individuel sont deux fois dérivables et convexes. En particulier, ils montrent que la convexité des fonctions de réaction des suiveurs joue un rôle prépondérant pour obtenir l'unicité de l'équilibre de Nash.
Pour les jeux sous forme caractéristique, Marini et Currarini (2003) associent une séquence à deux étapes à la fonction caractéristique γ. Dans cette séquence, une coalition faisant scission de la grande coalition joue à la première période en tant que meneur tandis que les autres firmes jouent leurs meilleures réponses individuelles en tant que suiveurs. En supposant que la fonction d'utilité individuelle d'un joueur est deux fois dérivable et strictement concave sur son ensemble de stratégies, Marini et Currarini montrent que si les joueurs et les externalités sont symétriques (les joueurs ont des fonctions d'utilité individuelle et des ensembles de stratégies individuelles identiques, les externalités sont positives ou bien négatives) et que le jeu posséde des complémentarités stratégiques, alors la valeur de partage égalitaire appartient au cœur. Par la suite, Marini et Currarini appliquent leur résultat à trois modèles économiques. Tout d'abord, ils considèrent une concurrence en quantité dans laquelle les stratégies sont substituts et montrent que le cœur des jeux sous forme caractéristique γ associés à une séquence à deux étapes est non-vide. Ensuite, ils traitent d'une concurrence en prix et prouvent que pour un faible degré de différentiation des produits, le cœur associé au jeu à deux étapes se réduit de plus en plus relativement au cœur associé au jeu simultané. Finalement, ils étudient une économie avec deux commodités, un bien public et un bien privé, et montrent que le cœur associé au jeu à deux étapes est toujours vide.

En ce qui concerne les situations d'oligopole de Bertrand, Kaneko (1978) considère un ensemble fini d'entreprises vendant un bien homogène à un continuum de consommateurs. Il suppose qu'un sous-ensemble de firmes et de consommateurs peuvent coopérer en négociant le bien entre eux. Le résultat principal de Kaneko établit que le cœur est toujours vide lorsqu'il y a plus de deux firmes. Deneckere et Davidson (1985) considèrent une situation d'oligopole de Bertrand avec biens différenciés associée à un système de demande de Shubik (1980) où les firmes produisent à un coût marginal constant et identique. Deneckere et Davidson étudient les prix d'équilibre pratiqués par les cartels et montrent qu'une fusion entre deux cartels implique que toutes les firmes dans l'industrie fixent des prix plus élevés, ce qui accroît donc le profit de toutes les firmes dans l'industrie. Ils prouvent que ce type de jeu satisfait à la propriété de superadditivité au sens où la fusion de deux cartels permet d'obtenir un profit plus élevé que la somme des profits de ces cartels avant la fusion. Pour la même situation d'oligopole de Bertrand, Huang et Sjöström (2003) définissent un jeu d'oligopole sous forme caractéristique dans lequel la capacité d'une coalition est définie par une procédure récursive qui consiste à appliquer le concept solution du cœur à un jeu réduit afin de prédire le comportement des firmes à l'extérieur du cartel. Ils donnent une condition nécessaire et suffisante pour la non-vacuité du cœur qui stipule que le paramètre de substituabilité entre les biens doit être supérieur ou égal à un certain nombre qui dépend de la taille de l'industrie. Huang et Sjöström concluent que le cœur est vide lorsqu'il y a plus de dix firmes.

Nous avons constaté qu'un nombre limité de travaux ont étudié les jeux d'oligopole coopératifs. En contrepartie de ce manque d'intérêt pour ce type de jeux, cette thèse traite des jeux d'oligopole coopératifs dans lesquels les entreprises se livrent à une concurrence «à la Cournot» (Chapitres 3 et 4), «à la Stackelberg» (Chapitre 5) et «à la Bertrand» (Chapitres 6 et 7).
Pour les situations d'oligopole de Cournot, nous étudions les jeux d'oligopole de Cournot sous forme caractéristique et sous forme d'intervalle γ. Nous étendons les cadres d'analyse précédents qui traitent des jeux d'oligopole de Cournot sous les formes caractéristiques α et β (Zhao 1999a,b, Norde et al. 2002, Driessen et Meinhardt 2005) en donnant des conditions suffisantes sur les fonctions de profit individuel et les fonctions de coût individuel qui assurent la non-vacuité du cœur.
Pour les situations d'oligopole de Stackelberg, nous étudions les jeux d'oligopole de Stackelberg sous forme caractéristique γ. Nous relâchons l'hypothèse de joueurs symétriques de Marini et Currarini (2003) et nous généralisons leur résultat en donnant une caractérisation du cœur et en formulant une condition nécessaire et suffisante qui assure sa non-vacuité.
Pour les situations d'oligopole de Bertrand, nous étudions les jeux d'oligopole de Bertrand sous forme caractéristiques α, β et γ. Nous montrons que les jeux sous les deux premières formes caractéristiques satisfont à la propriété de convexité. De plus, nous généralisons le résultat de superadditivité de Deneckere et Davidson (1985) en formulant une condition suffisante pour que les jeux sous forme caractéristique γ satisfassent à la propriété de convexité.
Les contributions de cette thèse sont détaillées ci-dessous.

Le Chapitre 2 introduit les concepts de théorie des jeux que nous utilisons dans cette thèse. Tout d'abord, nous rappelons quelques définitions de la théorie des jeux non-coopératifs tels que les jeux stratégiques, l'équilibre de Nash et l'équilibre d'accord partiel. Ensuite, nous introduisons quelques concepts de la théorie des jeux coopératifs tels que les jeux sous forme caractéristique, les jeux sous forme d'intervalle et les solutions comme le cœur, la valeur de Shapley et le nucléole. Enfin, nous décrivons trois approches qui permettent de convertir un jeu non-coopératif en un jeu coopératif, i.e. les approches α et β (Aumann 1959), ainsi que l'approche γ (Chander et Tulkens 1997).

Le Chapitre 3, basé sur l'article de Lardon (2009), traite des jeux d'oligopole de Cournot sous forme caractéristique γ lorsque les technologies ne sont pas transférables. Nous traitons du problème de la non-vacuité du cœur pour les jeux d'oligopole de Cournot sous forme caractéristique γ. Nous supposons que la fonction de demande inverse est dérivable, strictement décroissante et concave, et que les fonctions de coût individuel sont continues, strictement décroissantes et convexes. Tout d'abord, nous montrons que les jeux d'oligopole de Cournot sous forme caractéristique γ sont bien définis et nous étudions les propriétés des quantités d'équilibre. Pour ce faire, nous considérons une approche plus générale dans laquelle nous supposons que toute structure de coalitions est susceptible de se former. En particulier, peuvent se former les structures de coalitions dans lesquelles une coalition fait face à des firmes isolées. Pour toute structure de coalitions, nous construisons un jeu d'oligopole de Cournot stratégique agrégé dans lequel un équilibre de Nash représente les quantités d'équilibre agrégées des coalitions appartenant à la structure de coalitions. Nous montrons qu'il existe un unique équilibre de Nash ce qui permet de conclure que les jeux d'oligopole de Cournot sous forme caractéristique γ sont bien définis. Nous démontrons que la quantité totale d'équilibre décroît lorsque la structure de coalitions en question devient moins concurrentielle. Ce phénomène s'explique d'une part par le fait que lorsque deux coalitions fusionnent, la production de la nouvelle entité décroît, et d'autre part par le fait que les autres firmes augmentent leur quantité. Cette distribution des quantités d'équilibre diffère très fortement de la distribution des prix d'équilibres étudiée par Deneckere et Davidson (1985).
Ensuite, en utilisant ces résultats préliminaires, nous étudions la non-vacuité du cœur. Pour cela, nous considérons deux approches. La première approche montre que si la fonction de demande inverse est dérivable et les fonctions de profit individuel sont continues et concave sur l'ensemble des profils de stratégies, alors le jeu d'oligopole de Cournot sous forme caractéristique γ satisfait à la propriété de balancement et donc admet un cœur non-vide. Ce résultat prolonge le résultat de Zhao (1999b) établissant que les jeux d'oligopole de Cournot sous forme caractéristique β satisfont à la propriété de balancement. L'inconvénient de cette approche est qu'elle ne fournit aucune règle d'allocation appartenant au cœur. C'est la raison pour laquelle la seconde approche définit une nou-

velle règle d'allocation appartenant au cœur, appelée «valeur au prorata de Nash», sur l'ensemble des jeux d'oligopole de Cournot sous forme caractéristique γ pour lesquels les fonctions de coût individuel sont linéaires. La valeur au prorata de Nash donne à chaque firme la capacité de la grande coalition au prorata de sa quantité produite à l'équilibre de Nash dans le jeu d'oligopole de Cournot stratégique. Ce résultat généralise le résultat de Funaki et Yamato (1999) d'un cadre sans contraintes de capacité à un cadre avec contraintes de capacité possiblement asymétriques. De plus, nous caractérisons la valeur au prorata de Nash par quatre propriétés: efficience, firme nulle, monotonie et équité non-coopérative. La propriété d'efficience stipule qu'une solution distribue entièrement la capacité de la grande coalition aux firmes. Le propriété de firme nulle spécifie qu'une firme n'ayant aucune capacité de production obtient un paiement nul. La propriété de monotonie stipule que si une firme a une capacité de production plus élevée qu'une autre firme alors la première firme obtiendra un meilleur paiement que la seconde firme. La propriété d'équité non-coopérative spécifie qu'une solution donne à chaque firme un paiement proportionnel à son profit individuel obtenu à l'équilibre de Nash dans le jeu d'oligopole de Cournot stratégique. Ce résultat est le premier qui caractérise une solution appartenant au cœur sur un ensemble de jeux d'oligopole de Cournot coopératifs. De plus, nous donnons l'exemple d'un jeu d'oligopole de Cournot sous forme caractéristique γ dans lequel les fonctions de coût individuel sont linéaires, qui ne satisfait pas à la propriété de superadditivité, et donc de convexité. Ceci montre que le résultat de convexité de Norde et al. (2002) ne tient pas sur l'ensemble de jeux d'oligopole de Cournot sous forme caractéristique γ.

Dans le Chapitre 3, nous avons supposé que la fonction de demande inverse est dérivable afin que la fonction caractéristique γ soit bien définie. Cependant, dans de nombreuses situations d'oligopole de Cournot, la fonction de demande inverse n'est pas dérivable. En effet, Katzner (1968) montre que certaines fonctions de demande construites à partir de fonctions d'utilité individuelle de consommateurs deux fois continument dérivable peuvent ne pas être elle-même dérivable. Afin de garantir que les fonctions de demande sont au moins une fois continument dérivable, de nombreuses conditions nécessaires et suffisantes ont été données par Katzner (1968), Debreu (1972, 1976), Rader (1973, 1979) et Monteiro et al. (1996).
C'est la raison pour laquelle le Chapitre 4, basé sur l'article de Lardon (2010b), se focalise sur les situations d'oligopole de Cournot où la fonction de demande inverse est continue mais pas nécessairement dérivable. Dans un tel cadre, il n'est pas toujours possible de définir un jeu d'oligopole de Cournot sous forme caractéristique γ puisque la capacité d'une coalition n'est pas nécessairement unique. Cependant, nous montrons que l'on peut toujours définir un jeu d'oligopole

de Cournot sous forme d'intervalle γ. Un jeu sous forme d'intervalle associe à chaque coalition un intervalle de capacités fermé et borné. Ces jeux ont été introduits par Branzei et al. (2003) pour traiter des situations de banqueroutes. Nous renvoyons le lecteur à l'article de Alparslan-Gok et al. (2009a) pour une présentation générale des récents développements sur les jeux sous forme d'intervalle. Nous considérons deux extensions du cœur pour ce type de jeux: le cœur intervalle et le cœur standard. Nous utilisons le terme «cœur standard» plutôt que le terme «cœur» pour dissocier le cœur défini sur l'ensemble des jeux sous forme d'intervalle du cœur défini sur l'ensemble des jeux sous forme caractéristique. Le cœur intervalle est défini de manière similaire au cœur pour les jeux sous forme caractéristique en utilisant l'arithmétique propre aux intervalles (Moore 1979). Le cœur standard est défini comme l'union de tous les cœurs des jeux sous forme caractéristique pour lesquels la capacité de chaque coalition appartient à son intervalle de capacités dans le jeu sous forme d'intervalle. Nous traitons alors du problème de la non-vacuité du cœur intervalle et du cœur standard sur l'ensemble des jeux d'oligopole de Cournot d'intervalle γ. Dans ce but, nous utilisons un critère de la théorie de la décision, le critère d'Hurwicz (Hurwicz 1951), qui consiste à combiner pour chaque coalition la plus faible et la meilleure capacité qu'elle peut obtenir dans son intervalle de capacités. Notre premier résultat montre que le cœur intervalle est non-vide si et seulement si le jeu d'oligopole de Cournot sous forme caractéristique γ associé à la meilleure capacité qu'obtient chaque coalition dans son intervalle de capacités admet un cœur non-vide. Cependant, nous montrons que même pour une situation d'oligopole de Cournot linéaire, cette condition n'est pas satisfaite. Notre second résultat établit que le cœur standard est non-vide si et seulement si le jeu d'oligopole de Cournot sous forme caractéristique γ associé à la plus faible capacité qu'obtient chaque coalition dans son intervalle de capacités admet un cœur non-vide. De plus, nous formulons des conditions suffisantes sur les fonctions d'utilité individuelle et les fonctions de coût individuel qui garantissent que cette condition est satisfaite, ce qui généralise les résultats du Chapitre 3.

Pour les jeux sous forme caractéristique associés à une séquence à deux étapes, le résultat de non-vacuité du cœur de Marini et Currarini (2003) soulève deux questions. La première concerne la structure géométrique du cœur dans ce type de jeux puisque ils donnent seulement une règle d'allocation (la valeur de partage égalitaire) appartenant au cœur. La seconde question concerne le rôle de l'hypothèse de joueurs symétriques sur la non-vacuité du cœur. Le Chapitre 5, basé sur l'article de Driessen, Hou et Lardon (2011), répond à ces deux questions en considérant une séquence à deux étapes associée à la fonction caractéristique γ dans une concurrence en quantité. L'ensemble des jeux sous forme caractéristique associé à cette séquence temporelle est l'ensemble des jeux d'oligopole de Stackelberg sous forme caractéristique γ. Ainsi, contrairement aux jeux d'oligopole de Cournot sous forme caractéristique γ dans lesquels toutes les firmes jouent simultanément, chaque coalition faisant scission de la grande coalition produit une quantité à la première période, et les autres firmes choisissent simultanément et indépendamment une stratégie à la seconde période. Nous supposons que la fonction de demande inverse est linéaire et que les firmes produisent à des coûts marginaux constants mais possiblement distincts. Ainsi, contrairement à Marini et Currarini (2003), les fonctions de profit individuel ne sont pas nécessairement identiques. Premièrement, nous caractérisons le cœur en montrant qu'il est égal à l'ensemble des imputations ce qui répond à la première question concernant la caractérisation du cœur. En effet, l'avantage de meneur donné aux coalitions permet à chaque singleton d'avoir une capacité suffisamment élevée de telle sorte que la capacité de chaque coalition, hormis la grande coalition, est inférieure ou égal à la valeur de la somme des capacités individuelles de ses membres. Ensuite, nous donnons une condition nécessaire et suffisante qui assure la non-vacuité du cœur. Finalement, nous prouvons que cette condition dépend de l'hétérogénéité des coûts marginaux des firmes, i.e. pour un nombre donné de firmes, le cœur est non-vide si et seulement si les coûts marginaux des firmes ne sont pas trop hétérogènes. Plus le nombre de firmes est important, moins les coûts marginaux des firmes doivent être hétérogènes afin de garantir la non-vacuité du cœur ce qui répond à la seconde question concernant le rôle de l'hypothèse de joueurs symétriques. Cependant, dans le cas où la fonction de demande inverse est strictement concave, nous donnons un exemple dans lequel nos obtenons le résultat opposé, i.e. lorsque l'hétérogénéité des coûts marginaux des firmes augmente, alors le cœur devient plus large.

Le Chapitre 6, basé sur l'article de Lardon (2010a), étudie les jeux d'oligopole coopératifs dans un cadre de concurrence en prix. Nous considérons la même situation d'oligopole de Bertrand que Deneckere et Davidson (1985) et prolongeons leur résultat. Afin de définir des jeux d'oligopole de Bertrand coopératifs, nous considérons successivement les fonctions caractéristiques α, β et γ. Tout d'abord, comme pour les jeux d'oligopole de Cournot coopératifs, nous montrons que les fonctions caractéristiques α et β sont égales. Le résultat principal montre que les jeux d'oligopole de Bertrand sous les formes caractéristiques α ou β satisfont à la propriété de convexité. Ensuite, en suivant l'approche suggérée par Chander et Tulkens (1997), nous considérons la fonction caractéristique γ pour laquelle les firmes font face à une coalition déviante en choisissant leur meilleure réponse individuelle. Pour cet ensemble de jeux, nous montrons que la valeur de partage égalitaire appartient au cœur et nous donnons une condition suffisante qui assure que ces jeux satisfont à la propriété de convexité. Ce résultat généralise le résultat de superadditivité de Deneckere et Davidson

(1985) et contraste avec le résultat de vacuité du cœur de Kaneko (1978) et de Huang et Sjöström (2003). Notons que ces propriétés sont aussi satisfaites pour les jeux d'oligopole de Cournot sous forme caractéristique γ. En théorie des jeux non-coopératifs, une distinction importante entre un jeu d'oligopole de Cournot stratégique et un jeu d'oligopole de Bertrand stratégique est que le premier possède des complémentarités stratégiques tandis que le deuxième présente des substituabilités stratégiques. Ainsi, bien que ces deux types de jeux présentent des différences fondamentales dans leur forme non-coopérative, il apparaît que le cœur de leur forme coopérative respective présente la même structure géométrique.

Le Chapitre 7, basé sur l'article de Driessen, Hou et Lardon (2010), prolonge l'analyse du Chapitre 6 en étudiant l'ensemble des jeux d'oligopole de Bertrand sous les formes caractéristiques α et β dans lequel les firmes produisent à des coûts marginaux possiblement distincts. Tout d'abord, nous prouvons que les fonctions caractéristiques α et β sont égales. D'une part, nous montrons que si la constante de la demande est suffisamment petite, alors le jeu d'oligopole de Bertrand sous forme caractéristique β a une structure similaire à une notion bien connue en statistiques, i.e. la variance des coûts marginaux. Bien que de tels jeux ne satisfont pas à la propriété de convexité hormis si les firmes produisent à des coûts marginaux identiques, nous prouvons qu'ils satisfont à la propriété de balancement total. D'autre part, nous montrons que si la constante de la demande est suffisamment grande, alors les jeux d'oligopole de Bertrand sous forme caractéristique β satisfont à la propriété de convexité ce qui généralise le résultat de convexité du Chapitre 6. Finalement, nous formulons une expression de la valeur de Shapley pour ce second type de jeux. Plus précisément, nous montrons que la valeur de Shapley est déterminée en décomposant le jeu d'oligopole de Bertrand sous forme caractéristique β comme la différence de deux jeux sous forme caractéristique convexes, en plus de la somme de deux jeux sous forme caractéristique additifs. De plus, nous donnons une caractérisation de la valeur de Shapley en utilisant deux propriétés: efficience et monotonie individuelle. La propriété d'efficience stipule qu'une solution distribue entièrement la capacité de la grande coalition aux firmes. La propriété de monotonie individuelle spécifie que la différence entre les paiements de deux firmes est égale à la différence de leur capacité individuelle pondérée par un certain nombre qui dépend de leur coût moyen.

Dans cette thèse nous avons développé cinq modèles de jeux d'oligopole coopératifs. Les contributions sont résumées dans le tableau ci-dessous:

	$\alpha = \beta$	γ
Cournot	**Avec technologies transférables :** - totalement balancé (Norde et al. 2002) - convexe (Zhao 1999a) **Sans technologies transférables :** - balancé (Zhao 1999b) - convexe (Norde et al. 2002, Driessen et Meinhardt 2005)	**Jeux sous forme caractéristique :** - balancé (Théorème 3.4.1) - NP value (Théorèmes 3.5.1, 3.5.2) **Jeux sous forme d'intervalle :** - $\mathcal{I}$-balancé (Théorème 4.3.1) - fortement balancé (Théorèmes 4.3.3, 4.3.4, 4.3.7, 4.3.8)
Stackelberg	**Avec contraintes de capacité :** = Cournot	**Sans contraintes de capacité :** - $C(N, v) = I(N, v)$ (Proposition 5.3.1) - balancé (Théorèmes 5.3.3, 5.3.4)
Bertrand	**Avec coûts identiques :** - convexe (Théorème 6.3.5) **Avec coûts distincts :** - non-convexe (Proposition 7.3.1) - totalement balancé (Théorème 7.3.3) - convexe (Théorème 7.3.4) - valeur de Shapley (Théorèmes 7.4.1, 7.4.2)	**Avec coûts identiques :** - $ED(N, v) \in C(N, v)$ (Théorème 6.4.5) - convexe (Théorème 6.4.6)

En résumé, nous avons montré que le cœur est non-vide pour les situations d'oligopole de Cournot. En ce qui concerne les situations d'oligopole de Stackelberg, nous avons prouvé que le cœur est non-vide si et seulement si les coûts marginaux des firmes ne sont pas trop hétérogènes. Concernant les situations d'oligopole de Bertrand, nous avons établi que de tels jeux satisfont à la propriété de convexité ce qui signifie qu'il existe de fortes incitations à former la grande coalition. Ainsi, pour la plupart des situations d'oligopole, un accord horizontal sur le prix de vente est susceptible d'apparaître hormis si les coalitions faisant scission de la grande coalition possèdent un avantage de meneurs et les firmes ont des coûts marginaux suffisamment hétérogènes. Puisque les accords horizontaux sur le prix de vente diminuent le bien-être économique, les résultats établis dans cette thèse suggèrent et justifient une politique interventionniste de la part des autorités de la concurrence dans la plupart des marchés oligopolistiques.

Les contributions de cette thèse établissent les bases pour de futurs travaux de recherche sur les jeux d'oligopole coopératifs.
Dans l'exemple 3.5.3, le jeu d'oligopole de Cournot sous forme caractéristique γ ne satisfait pas à la propriété de superadditivité et de convexité. Par la suite, il serait intéressant d'étudier les conditions sur les fonctions de profit individuel qui assurent que de tels jeux satisfont à la propriété de convexité.

La preuve du Théorème 4.3.4 assure l'existence d'un vecteur d'anticipations «seuil» pour lequel le cœur standard est non-vide. Un futur travail de recherche consisterait à fournir l'expression de ce vecteur d'anticipations afin d'obtenir plus d'informations sur le degré de pessimisme à partir duquel le cœur standard est non-vide.

Pour définir un jeu d'oligopole de Stackelberg sous forme caractéristique γ, nous avons supposé que les firmes n'avaient aucune contrainte de capacité. Les résultats établis dans la Proposition 5.3.1 et le Théorème 5.3.3 dépendent fortement de cette hypothèse. Une future trajectoire de recherche consisterait à introduire des contraintes de capacité dans les situations d'oligopole de Stackelberg. De plus, le Théorème 5.3.3 établit la vacuité du cœur si les coûts marginaux des firmes sont suffisamment hétérogènes. Nous conjecturons que ce résultat reste vrai pour toute situation d'oligopole de Satckelberg dans laquelle la fonction de demande inverse est strictement concave et les firmes produisent à des coûts marginaux identiques.

Le résultat du Théorème 6.4.6 donne une condition suffisante qui assure que les jeux d'oligopole de Bertrand sous forme caractéristique γ satisfont à la propriété de convexité. Jusqu'à maintenant, nous n'avons pas réussi à donner un exemple de jeu d'oligopole de Bertrand sous forme caractéristique γ qui ne satisfait pas à la propriété de convexité. Il serait donc intéressant soit d'établir la preuve que la propriété de convexité est satisfaite sur cet ensemble de jeux, soit de donner un contre-exemple.

Lorsque les coûts marginaux des firmes sont possiblement distincts, nous avons traité des jeux d'oligopole de Bertrand sous les formes caractéristiques α et β en supposant que les technologies n'étaient pas transférables. Un prolongement de ce travail consisterait à étudier les jeux d'oligopole de Bertrand sous les formes caractéristiques α, β et γ, dans le cas où les technologies seraient transférables.

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Index

Abstract: In the first essay, we study Cournot oligopoly TU-games in γ-characteristic function form. First, we prove that if any individual profit function is concave, such games are balanced. Then, when the individual cost functions are linear, we provide a solution in the core, called NP(Nash Pro rata) value.
The second essay considers Cournot oligopoly interval game in γ-set function form. The first (second) result states that the interval (standard) core is non-empty if and only if the Cournot oligopoly TU-game associated with the best (worst) worth of every coalition in its worth interval admits a non-empty core.
In the third essay, we focus on Stackelberg oligopoly TU-games in γ-characteristic function form. First, we prove that the core is equal to the set of imputations. Then, we provide a necessary and sufficient condition, depending on the heterogeneity of firms' marginal costs, under which the core is non-empty.
In the fourth essay, we show that Bertrand oligopoly TU-games in α and β-characteristic function forms are convex. Then, we prove that the equal division solution is in the core of Bertrand oligopoly TU-games in γ-characteristic function form and we give a sufficient condition under which such games are convex.
The fifth essay studies the case where the marginal costs are distinct. If the intercept of demand is sufficiently small then games in β-characteristic function form are totally balanced. Otherwise, these games are convex.

Résumé : Tout d'abord, nous traitons des jeux d'oligopole de Cournot sous forme caractéristique γ. Nous montrons que ces jeux sont balancés lorsque les fonctions de profit individuel sont concave. Ensuite, lorsque les fonctions de coût individuel sont linéaires, la «valeur au prorata de Nash» appartient au cœur.
Par la suite, nous étudions les jeux d'oligopole de Cournot sous forme d'intervalle γ. Nous prouvons que le cœur intervalle (standard) est non-vide si et seulement si le jeu d'oligopole de Cournot sous forme caractéristique γ associé à la meilleure (plus faible) capacité qu'obtient chaque coalition admet un cœur non-vide.
Ensuite, nous analysons les jeux d'oligopole de Stackelberg sous forme caractéristique γ. Nous montrons que le cœur est égal à l'ensemble des imputations. Ensuite, nous donnons une condition nécessaire et suffisante, qui dépend de l'hétérogénéité des coûts marginaux, assurant la non-vacuité du cœur.
Enfin, nous considérons les jeux d'oligopole de Bertrand. Nous prouvons que les jeux sous les formes caractéristiques α ou β satisfont à la propriété de convexité. Ensuite, nous prouvons que la valeur de partage égalitaire appartient au cœur des jeux sous forme caractéristique γ et nous donnons une condition suffisante qui assure que ces jeux satisfont à la propriété de convexité.
Nous prolongeons cette analyse en supposant que les coûts marginaux sont distincts. Si la constante de la demande est suffisamment petite, alors les jeux sous forme caractéristique β satisfont à la propriété de balancement total. Autrement, ces jeux satisfont à la propriété de convexité.