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Anne-Claire Egloffé

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Étude de quelques problèmes inverses  
pour le système de Stokes.  
Application aux poumons.

**THÈSE DE DOCTORAT**

Présentée par

**Anne-Claire EGLOFFE**

pour obtenir le grade de

**Docteur de  
l'Université Pierre et Marie Curie - Paris VI**

**Spécialité : Mathématiques Appliquées**

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*A mes parents*  
*A mes grands-parents*  
*Au petit Anatole : bienvenu !*





## Résumé

Dans cette thèse, nous nous intéressons à la résolution de problèmes inverses provenant d'une modélisation multi-échelle de l'écoulement de l'air dans les poumons. Dans un premier temps, nous considérons une version simplifiée du modèle de l'écoulement de l'air dans les poumons : l'écoulement est modélisé par les équations de Stokes incompressibles avec des conditions aux limites de type Robin sur une partie du bord. Nous cherchons à identifier le coefficient de Robin défini sur une partie non accessible du bord à partir de mesures de la vitesse et de la pression disponibles sur une autre partie du bord.

Après avoir quantifié des résultats de continuation unique pour le système de Stokes, nous établissons deux inégalités de stabilité logarithmiques, l'une valable en dimension 2 et l'autre valable en toute dimension. Toutes deux sont basées sur des inégalités de Carleman, globale dans le premier cas et locales dans le second. Les inégalités de stabilité sont d'abord montrées sur le problème stationnaire puis la théorie des semi-groupes permet de passer au problème non stationnaire. De plus, sous l'hypothèse *a priori* que le coefficient de Robin est constant par morceaux, nous prouvons une inégalité de stabilité Lipschitzienne pour le problème stationnaire. Nous concluons cette thèse en revenant au problème initial pour lequel nous imposons des conditions au bord non-standard faisant intervenir le flux. En particulier, nous obtenons des premiers résultats numériques encourageants concernant l'identification de certains paramètres du modèle (résistances à l'écoulement de l'air, élasticité des tissus).

**Mots-clés:** Problème inverse, Inégalité de stabilité, Inégalité de Carleman, Système de Stokes, Conditions aux limites mixtes, Modélisation de l'appareil respiratoire.



## Abstract

In this work, we are interested in the resolution of some inverse problems arising from a multi-scale modeling of the airflow in the lungs. As a first step, we focus on a simplified model of the airflow in the lungs: we consider the incompressible Stokes equations with Robin boundary conditions on a part of the boundary. We want to identify the Robin coefficient defined on this non accessible part of the boundary from measurements of the velocity and the pressure available on another part of the boundary.

We first prove quantification results for the unique continuation property for the Stokes system, then we establish two logarithmic stability inequalities, one valid in dimension 2 and the other one valid in any dimension. Both are based on Carleman estimates, global in the first case and local in the second one. Our stability estimates are first established for the stationary problem and the semigroup theory allows to deduce from the stationary case stability estimates for the non-stationary problem. Moreover, under the *a priori* assumption that the Robin coefficient is piecewise constant, we provide a Lipschitz stability estimate for the stationary problem. We conclude by coming back to the initial model which involves non-standard boundary conditions with the flux. In particular, we obtain encouraging first numerical results concerning the identification of some parameters of the model (resistances to air flow, tissue elasticity).

**Keywords:** Inverse problem, Stability estimate, Carleman inequality, Stokes system, Mixed boundary conditions, Modeling of the respiratory tract.



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## Chapitre 1

# Introduction générale



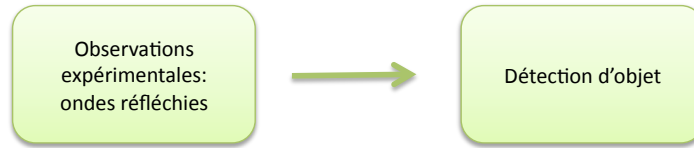


FIGURE 1.1 – Exemple de problème inverse : le radar. Un émetteur envoie des ondes radios qui sont réfléchies par la cible et détectées par un récepteur. La position de la cible est estimée grâce au temps de retour du signal.

## 1.1 Introduction

Un problème inverse consiste à retrouver les causes d'un phénomène sachant que l'on en connaît les effets. On retrouve des problèmes inverses dans de nombreux domaines scientifiques : l'imagerie médicale (échographie, scanner, rayon X, ...), ingénierie pétrolière, le radar (voir la Figure 1.1), ...

D'un point de vue scientifique, la résolution de problèmes inverses passe par une étape de modélisation du problème direct. Ensuite, à partir de mesures expérimentales, la démarche consiste à déterminer les paramètres qui permettent de restituer ces mesures, ce qui se traduit mathématiquement par l'optimisation d'une fonctionnelle.

On peut voir le diagnostic médical comme un problème inverse qui n'est pas basé sur une modélisation mais sur l'expérience. Le patient consulte son médecin car il présente des symptômes. L'examen médical permet alors au médecin de déterminer les signes cliniques physiques de la maladie (manifestations objectives de la maladie). Grâce à ses connaissances et à sa pratique de la médecine, le médecin est alors capable de déterminer de quelle pathologie souffre le patient et de lui proposer une thérapie adaptée.

Le travail de cette thèse concerne l'étude de problèmes inverses provenant d'une modélisation multi-échelle de l'écoulement de l'air dans les poumons et s'insère dans un cadre plus général concernant la modélisation de l'appareil respiratoire. On cherche à identifier certains paramètres qui interviennent dans le modèle à partir de mesures effectuées au niveau de la bouche, et plus généralement on cherche à obtenir des inégalités de stabilité de ces paramètres par rapport aux mesures : par exemple, si deux mesures de volumes effectuées au niveau de la bouche sont « proches » alors on cherche à montrer que les paramètres sont également « proches ». Parmi ces paramètres, on peut distinguer la résistance à l'écoulement de l'air dans les bronches ou encore l'élastance pulmonaire qui est la variation de pression nécessaire pour obtenir une variation de volume des poumons. L'élastance pulmonaire apporte des renseignements sur l'élasticité des tissus composant l'appareil pulmonaire. Ces paramètres peuvent varier significativement lors de pathologies. C'est pourquoi parvenir à les identifier présente un réel intérêt.

Concernant la structure de ce chapitre introductif, nous allons commencer par présenter la modélisation de l'appareil respiratoire qui constitue le point de départ de cette thèse dans la Section 1.2.2. Puis nous passerons en revue, dans la Section 1.3, les différents résultats existant dans la littérature concernant la résolution théorique du problème inverse qui nous intéresse. Nous présenterons ensuite, dans la Section 1.4, les apports de cette

thèse. Enfin, dans la Section 1.5, nous présenterons le plan de la thèse chapitre après chapitre.

## 1.2 Modèle de l'écoulement de l'air dans les poumons

### 1.2.1 Introduction

Les tests spirométriques sont un élément important dans le diagnostic et le suivi d'un certain nombre de maladies pulmonaires, notamment des pathologies pulmonaires impliquant des anomalies du comportement mécanique des poumons. Ces tests consistent à mesurer le débit et le volume d'air mobilisé lors d'un cycle respiratoire, au repos et à l'exercice. En particulier, une courbe volume-débit, correspondant à la courbe paramétrée  $t \rightarrow (\text{volume}(t), \text{débit}(t))$ , est obtenue. On renvoie à la Figure 1.2a pour une illustration de courbe volume-débit normale. Pour parvenir à un diagnostic relativement objectif, une étude quantitative est effectuée : la courbe est comparée à des profils « normaux » provenant d'un large nombre de patients présentant les mêmes caractéristiques (âge, sexe, taille,...). De plus, en complément de l'analyse quantitative, le praticien effectue également une analyse qualitative de l'allure de la courbe. En effet, les pathologies modifiant le comportement mécanique du poumon sont usuellement divisées en deux catégories (obstructive et restrictive) avec des courbes volume-débit présentant des profils différents (voir les Figures 1.2b et 1.2c). Les courbes volume-débit contiennent des informations importantes concernant l'état de santé des poumons d'un patient.

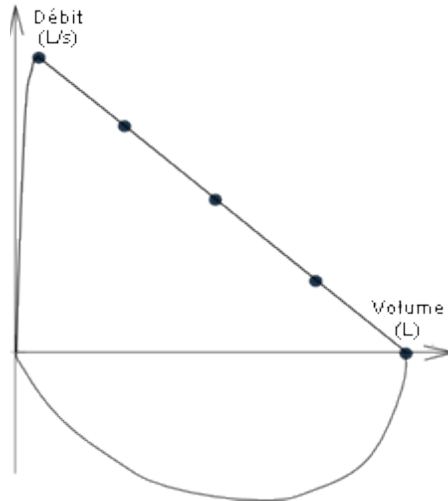
Du point de vue de la modélisation, il est intéressant de voir si l'on est capable de reproduire ces courbes volume-débit. Il existe dans la littérature un certain nombre de modèles, allant du plus simple modèle 0d ( voir [62], [9], [14], ...) au plus élaboré modèle 3d (voir [7], [63], [42], ...). Le fait d'enrichir le modèle permet de se rapprocher au plus près de la réalité. Les modèles 3d permettent notamment de prendre en compte les effets inertiels dans la partie supérieure de l'arbre bronchique ou encore de mettre en évidence l'influence de la géométrie. En contre-partie, l'analyse mathématique du problème est plus délicate et les simulations numériques sont plus coûteuses.

Nous allons présenter dans la sous-section suivante le modèle multi-échelle de l'écoulement de l'air dans les poumons, proposé par L. Baffico, C. Grandmont et B. Maury dans [7], qui est le point de départ de ce travail de recherche.

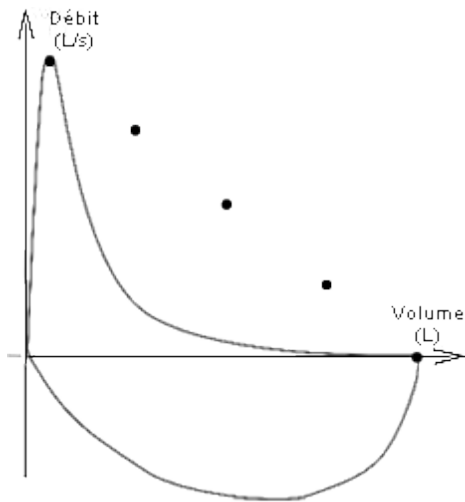
### 1.2.2 Présentation du modèle

Les voies aériennes de l'appareil respiratoire ont principalement deux fonctions : une partie permet la conduction de l'air et l'autre est le lieu des échanges gazeux entre l'air extérieur et le sang. Le modèle que l'on va présenter, introduit dans [7], se concentre sur l'écoulement de l'air dans les voies aériennes.

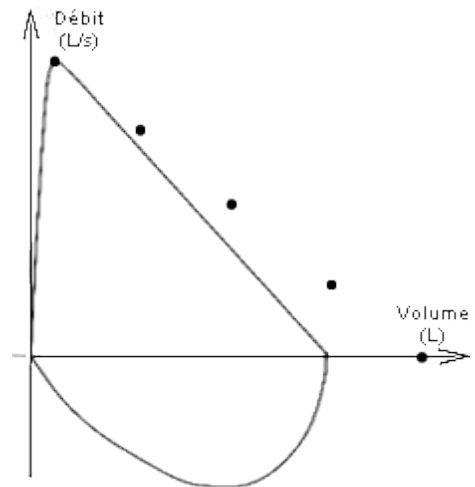
La géométrie « fractale » de l'arbre bronchique est particulièrement complexe : la trachée se divise en deux bronches, une pour chaque poumon, chacune des bronches se divise en deux et ainsi de suite (voir la Figure 1.3a). Ce processus dichotomique se produit pendant environ vingt-trois générations. Par conséquent, effectuer des simulations numériques sur l'arbre tout entier n'est pas envisageable. C'est pourquoi le modèle considère l'écoulement de l'air dans un domaine simplifié correspondant aux premières générations de l'arbre bronchique (voir la Figure 1.3b). Le fait de travailler sur un domaine tronqué



(a) Courbe volume-débit normale.



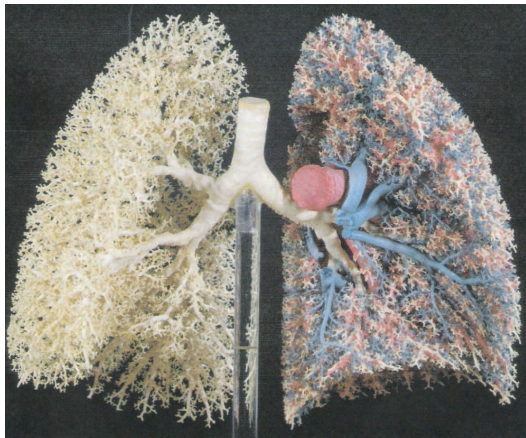
(b) Courbe volume-débit correspondant à une pathologie obstructive.



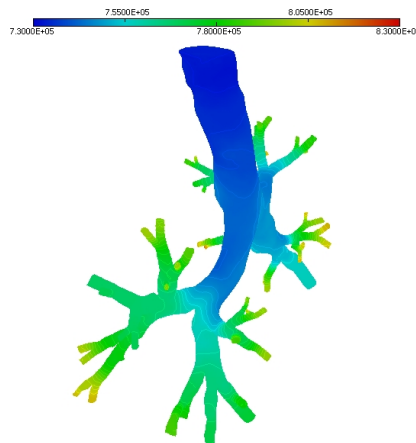
(c) Courbe volume-débit correspondant à une pathologie restrictive.

FIGURE 1.2 – Courbes volume-débit. Les images proviennent de [76].





(a) Modélisation des poumons humain réalisée par E. R. Weibel.



(b) Arbre bronchique reconstruit numériquement (l'image provient de [7]).

FIGURE 1.3

fait apparaître des frontières artificielles. Comme aucune mesure *in vivo* de vitesse ni de pression n'est disponible, il faut trouver un modèle permettant de décrire la physique intervenant dans la partie distale de l'arbre respiratoire. On peut, par exemple, imposer des conditions aux limites prenant en compte la dissipation de l'énergie dans les bronches inférieures. L'utilisation de frontières artificielles est une méthode classique pour simplifier un mécanisme complexe afin de le modéliser. On la retrouve notamment dans des modélisations du système cardiovasculaire (on renvoie aux références [69] et [77]).

La ventilation pulmonaire ou respiration est le renouvellement de l'air contenu dans les poumons par l'action des muscles respiratoires dont le principal est le diaphragme. Pour décrire la ventilation, l'appareil respiratoire peut être divisé en trois parties où des modèles mécaniques différents vont être utilisés :

- La partie proximale (les dix premières générations environ), où les équations de Navier-Stokes incompressibles décrivent l'écoulement de l'air.
- La partie distale (de la onzième à la dix-septième générations environ), qui correspond à la partie géométriquement complexe, située en aval de la partie proximale et destinée à être condensée.
- Les acini, où la diffusion d'oxygène a lieu. A ce niveau, le poumon est représenté par une boîte remplie d'un matériau amorphe et incompressible qui décrit le parenchyme qui entoure les alvéoles. Une paroi de la boîte est un piston mobile représentant l'action des muscles de la paroi thoracique et du diaphragme.

La Figure 1.4 schématise cette décomposition.

Soit  $d = 2, 3$  et  $\Omega \subset \mathbb{R}^d$  un domaine simplifié correspondant aux premières générations de l'arbre bronchique. On note  $\mathbf{u}$  la vitesse,  $p$  la pression de l'air dans l'arbre et  $\mathbf{n}$  la normale extérieure au bord du domaine  $\Omega$ . La frontière de  $\Omega$  est donnée par

$$\partial\Omega = \Gamma_0 \cup \Gamma_l \left( \bigcup_{i=1}^N \Gamma_i \right),$$

où les  $\Gamma_i$  correspondent aux frontières artificielles (voir la Figure 1.4).

Dans la partie supérieure, l'écoulement est décrit par les équations de Navier-

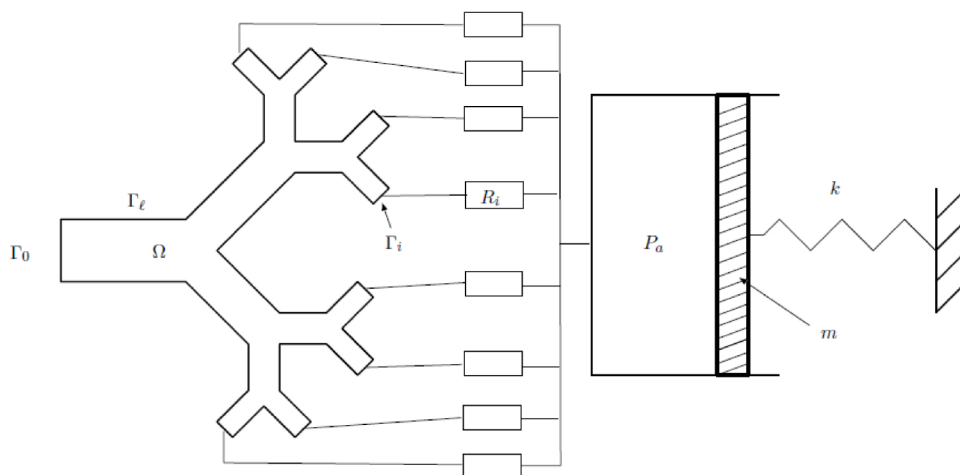


FIGURE 1.4 – Schéma de la modélisation de l'appareil respiratoire : on divise l'appareil respiratoire en trois parties.

Stokes incompressibles :

$$\left\{ \begin{array}{ll} \rho u_t + \rho(u \cdot \nabla)u - \mu \Delta u + \nabla p & = 0, \quad \text{dans } (0, T) \times \Omega, \\ \operatorname{div} u & = 0, \quad \text{dans } (0, T) \times \Omega, \\ u & = 0, \quad \text{sur } (0, T) \times \Gamma_l, \\ \mu \frac{\partial u}{\partial n} - pn & = -P_0 n, \quad \text{sur } (0, T) \times \Gamma_0, \\ \mu \frac{\partial u}{\partial n} - pn & = -\pi_i n, \quad \text{sur } (0, T) \times \Gamma_i, \quad \text{pour } i = 1, \dots, N. \end{array} \right.$$

La densité  $\rho$  est constante et donnée, tout comme la viscosité  $\mu$ . La pression  $P_0$ , qui correspond à la pression atmosphérique, est connue alors que les pressions  $\pi_i$  sur les frontières artificielles sont inconnues. Dans la partie distale, on peut supposer que l'écoulement est laminaire. Chaque branche locale du sous-arbre commençant en  $\Gamma_i$  est alors caractérisée par une résistance. Par analogie avec ce qui se passe en électricité, on condense ce sous-arbre en un conduit équivalent caractérisé par une unique résistance  $R_i$  qui prend en compte toutes les résistances des branches locales sous  $\Gamma_i$ . On renvoie à [65] pour plus de détails concernant les résistances équivalentes et à la Figure 1.5 pour une illustration.

Le saut de pression  $P_i - \pi_i$  (où  $P_i$  est la pression dans les acini associée à la  $i^{\text{ème}}$  branche de l'arbre) est alors proportionnelle au flux et le coefficient de proportionnalité est la résistance équivalente  $R_i$ . Il s'agit de la loi de Poiseuille :

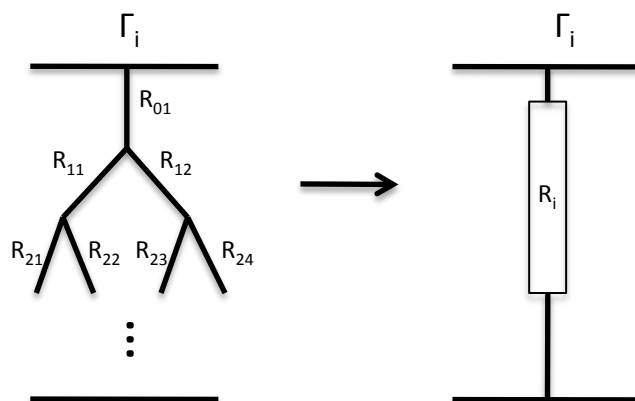
$$\pi_i - P_i = R_i \left( \int_{\Gamma_i} u \cdot n \right).$$

Grâce à cette relation, les conditions aux limites sur  $\Gamma_i$  s'écrivent

$$\mu \frac{\partial u}{\partial n} - pn = -P_i n - R_i \left( \int_{\Gamma_i} u \cdot n \right) n, \quad (1.1)$$

pour  $i = 1, \dots, N$ . Il s'agit de conditions aux limites non standards que nous appellerons dans la suite **conditions aux limites dissipatives**.

Nous allons ensuite coupler ce système avec une équation différentielle ordinaire qui modélise le déplacement du diaphragme. Soit  $t > 0$ . On note  $x(t)$  la position du piston


 FIGURE 1.5 – On condense le sous-arbre commençant en  $\Gamma_i$  en un conduit équivalent.

à l'instant  $t$ . La somme des forces qui s'appliquent sur la paroi mobile de la boîte est composée de la force musculaire, notée  $f_{ext}$ , de la force de rappel élastique  $-kx$ , où  $k$  est la constante de raideur qui caractérise le diaphragme, et de la force de pression  $f_P$ . D'après le principe fondamental de la dynamique, on a :

$$m\ddot{x} = -kx + f_{ext} + f_P,$$

où  $m$  est la masse totale des poumons. On suppose que la pression est uniforme dans la boîte qui contient les alvéoles. Autrement dit,  $P_i = P_a$  pour tout  $i = 1, \dots, N$ . Par conséquent, la force de pression  $f_P$  est égale à

$$f_P = SP_a,$$

où  $S$  est la surface de la paroi de la boîte qui se déplace. Les conduits de l'arbre sont supposés rigides et le parenchyme est supposé incompressible. Par conséquent, toute variation de volume dans les poumons est due à une variation de volume des alvéoles, et toute variation de volume au niveau de la trachée résulte de la même variation de volume dans la boîte, ce qui se traduit par :

$$S\dot{x} = \sum_{i=1}^N \int_{\Gamma_i} u \cdot n = - \int_{\Gamma_0} u \cdot n.$$

Pour résumer, le système prenant en compte les trois niveaux de l'arbre respiratoire s'écrit :

$$\left\{ \begin{array}{ll} \rho u_t + \rho(u \cdot \nabla)u - \mu \Delta u + \nabla p & = 0, & \text{dans } (0, T) \times \Omega, \\ \operatorname{div} u & = 0, & \text{dans } (0, T) \times \Omega, \\ u & = 0 & \text{sur } (0, T) \times \Gamma_l, \\ \mu \frac{\partial u}{\partial n} - pn & = -P_0 n, & \text{sur } (0, T) \times \Gamma_0, \\ \mu \frac{\partial u}{\partial n} - pn & = -P_a n - R_i \left( \int_{\Gamma_i} u \cdot n \right) n, & \text{sur } (0, T) \times \Gamma_i, \\ & & \text{pour } i = 1, \dots, N, \\ m\ddot{x} + kx & = f_{ext} + SP_a, & \text{sur } (0, T) \\ S\dot{x} & = \sum_{i=1}^N \int_{\Gamma_i} u \cdot n = - \int_{\Gamma_0} u \cdot n. & \text{sur } (0, T). \end{array} \right. \quad (1.2)$$

Il s'agit d'un modèle multi-échelle où le mouvement du diaphragme et les résistances des petites voies aériennes sont prises en compte de manière simplifiée.

**Remarque 1.1.** Dans le système (1.2), les équations de Navier-Stokes et l'équation différentielle sont couplées à travers la pression alvéolaire  $P_a$  qui intervient à la fois dans les conditions aux limites sur  $\Gamma_i$  et dans l'équation différentielle.

Le système (1.2) doit être complété avec un choix de conditions initiales appropriées :

$$\begin{aligned} (u, x, \dot{x})_{t=0} &= (u_0, x_0, x_1), \text{ avec } \operatorname{div} u_0 = 0, \\ u_0 &= 0 \text{ sur } \Gamma_l, Sx_1 = - \int_{\Gamma_0} u_0 \cdot n. \end{aligned} \quad (1.3)$$

### 1.2.3 Problème direct et problème inverse

Nous allons voir dans le Théorème 1.3 que le problème direct, qui consiste à déduire des paramètres  $R_i, m, k, f_{ext}$  et  $S$  et des données  $P_0, x_0, x_1$  et  $u_0$  des informations sur le comportement de  $(u, p)$  solution du système (1.2) associé, est bien posé : ce système admet une unique solution régulière.

Plus précisément, supposons que :

- (A) le bord du domaine  $\partial\Omega = \Gamma_l \cup \left(\bigcup_{i=0}^N \Gamma_i\right)$  est tel que  $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset$ , pour  $i \neq j$ ,  $(i, j) \in \{0, \dots, N\}$  et  $\Gamma_i$  rencontre le bord latéral  $\Gamma_l$  avec un angle égal à  $\frac{\pi}{2}$  pour  $i = 0, \dots, N$ ,
- (B) le bord du domaine  $\Omega$  est localement régulier :  $\Gamma_l$  et  $\Gamma_i$  sont réguliers, pour  $i = 0, \dots, N$ ,
- (C) soit  $T > 0$ ,  $P_0 \in L^2(0, T)$ ,  $f_{ext} \in L^2(0, T)$ ,  $u_0 \in \{u \in H^1(\Omega) / u|_{\Gamma_l} = 0 \text{ et } \operatorname{div} u = 0\}$ .

L'existence de solution pour le système (1.2) peut être obtenue en utilisant la méthode de Galerkin avec une base de Galerkin convenablement choisie : dans [7], L. Baffico, C. Grandmont et B. Maury introduisent un opérateur de Stokes  $A_0$  (voir la Définition 1.2) associé aux conditions aux limites dissipatives (1.1) sur  $\Gamma_i$  dont les vecteurs propres constitueront la base de Galerkin utilisée pour construire la suite d'approximation.

Afin d'être plus précis, nous allons introduire des espaces fonctionnels :

$$\mathbf{V}_{\Gamma_l} = \{v \in H^1(\Omega) / v|_{\Gamma_l} = 0 \text{ et } \operatorname{div} v = 0\},$$

$$\mathbf{H}_{\Gamma_l} = \overline{\mathbf{V}}^{L^2(\Omega)}.$$

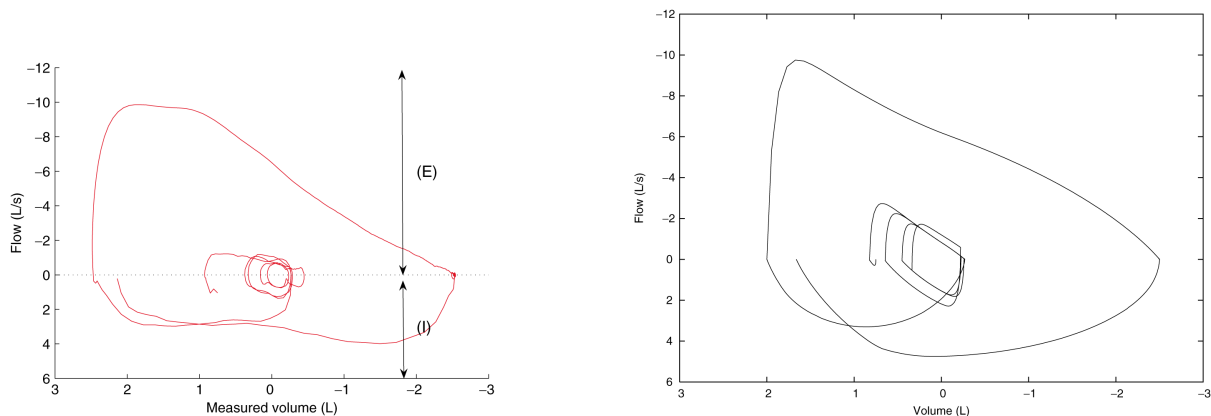
De plus, nous introduisons les notations suivantes :

$$(v, w)_0 = \rho \int_{\Gamma} v \cdot w + \frac{m}{S^2} \left( \int_{\Gamma_0} v \cdot n \right) \left( \int_{\Gamma_0} w \cdot n \right),$$

on note  $\|\cdot\|_0$  la norme associée et

$$a_{R, \mu}(v, w) = \mu \int_{\Omega} \nabla v : \nabla w + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} v \cdot n \right) \left( \int_{\Gamma_i} w \cdot n \right).$$

On introduit maintenant la définition de l'opérateur  $A_0$ .



(a) Exemple de courbe volume-débit provenant de données réelles : (E) expiration, (I) inspiration.

(b) Exemple de courbe volume-débit obtenue numériquement.

FIGURE 1.6 – Courbes volume-débit. Les images proviennent de [7].

**Définition 1.2.** *L'opérateur  $A_0$  est défini comme suit :*

$$\mathcal{D}(A_0) = \{v \in V_{\Gamma_1} / |a_{R,\mu}(v, w)| \leq C\|w\|_0, \forall w \in V_{\Gamma_1}\},$$

$$(A_0 v, w)_0 = a_{R,\mu}(v, w), \forall (v, w) \in \mathcal{D}(A_0) \times V_{\Gamma_1}.$$

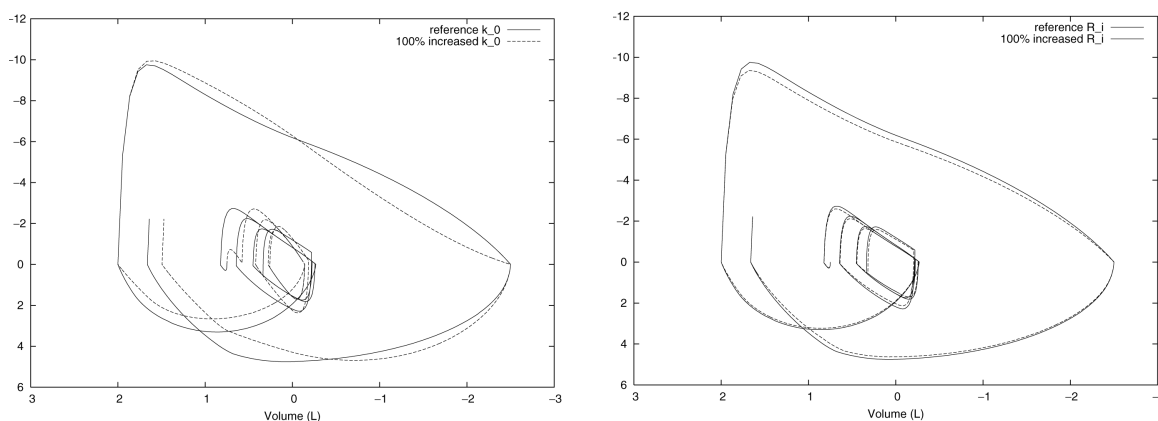
Nous pouvons énoncer le résultat d'existence et de régularité suivant pour le problème direct, prouvé dans [7] :

**Théorème 1.3.** *Sous les hypothèses (A), (B) et (C) précédentes, il existe un intervalle de temps  $[0, t^*)$  sur lequel le système (1.2) complété avec les conditions initiales (1.3) admet une unique solution  $u \in L^2(0, t; \mathcal{D}(A_0)) \cap L^\infty(0, t; V_{\Gamma_1}) \cap H^1(0, t; H_{\Gamma_1})$ , avec  $t < t^*$ . De plus, dans le cas particulier où  $k = 0$  et pour des données petites, la solution est définie globalement en temps et il existe  $C > 0$  tel que pour tout  $t > 0$ ,*

$$\|\nabla u\|_{L^2(\Omega)}^2 + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right)^2 \leq C.$$

D'un point de vue numérique, il est intéressant de constater que le modèle est capable de reproduire des courbes volume-débit (voir la Figure 1.6) dans une géométrie réaliste telle que la Figure (1.3b), avec des données provenant de la littérature. De plus, en modifiant les paramètres  $R$  et  $k$ , on obtient des courbes volume-débit différentes (voir la Figure 1.7), ce qui est positif du point de vue de la modélisation puisque ces paramètres peuvent varier significativement lors de pathologies. Par exemple, les résistances augmentent en cas d'asthme.

Dans le précédent modèle, un certain nombre de paramètres sont à caler, en particulier, les résistances  $R_i$  et la constante de raideur du diaphragme  $k$ . Parvenir à les identifier présente un double intérêt : d'une part, s'il n'y a pas stabilité, on peut se poser des questions quant à la pertinence du modèle et d'autre part, identifier ces paramètres à partir de mesures faites par des professionnels de la santé donne une information sur l'état de santé du patient. C'est le problème inverse qui a motivé ce travail de thèse : on cherche à récupérer la constante de raideur  $k$  qui caractérise l'élasticité des tissus et les



(a) Courbes volume-débit obtenues pour des valeurs différentes de  $k$ .

(b) Courbes volume-débit obtenues pour des valeurs différentes de la résistance distale.

FIGURE 1.7 – Courbes volume-débit : on fait varier les paramètres. Les images proviennent de [7].

résistances  $R_i$  à partir de mesures effectuées au niveau de la bouche. Les questions que l'on se pose sont classiques dans le domaine des problèmes inverses : on s'intéresse aux questions d'**unicité** et de **stabilité** des paramètres par rapport aux mesures.

Plus précisément, soit  $\Gamma \subseteq \Gamma_0$  et  $T > 0$  et  $(u_l, p_l)$  la solution de (1.2) associée à  $R_i = R_i^l$  pour  $i = 1, \dots, N$  et  $k = k_l$  pour  $l = 1, 2$ . On définit  $\mathcal{M}_{(0,T) \times \Gamma}$  l'opérateur de mesures :  $\mathcal{M}_{(0,T) \times \Gamma}(u, p)$  correspond aux mesures faites sur  $(0, T) \times \Gamma$  pour  $(u, p)$  solution de (1.2). Les questions que l'on se pose sont :

- (i) Est-ce que  $\mathcal{M}_{(0,T) \times \Gamma}(u_1, p_1) = \mathcal{M}_{(0,T) \times \Gamma}(u_2, p_2)$  implique que  $k_1 = k_2$  et  $R_i^1 = R_i^2$  pour tout  $i \in \{1, \dots, N\}$  ?
- (ii) Est-il possible d'obtenir une inégalité de stabilité du type :

$$|k_1 - k_2| + \sum_{i=1}^N |R_i^1 - R_i^2| \leq f(\|\mathcal{M}_{(0,T) \times \Gamma}(u_1, p_1) - \mathcal{M}_{(0,T) \times \Gamma}(u_2, p_2)\|), \quad (1.4)$$

où  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  est une fonction croissante qui satisfait  $\lim_{x \rightarrow 0} f(x) = 0$  ?

Dans un souci de clarté, nous introduisons du vocabulaire caractérisant les inégalités de stabilité.

**Définition 1.4.** • Quand la fonction  $f$  intervenant dans l'inégalité (1.4) est du type :

$$f(x) = \frac{C_0}{\log\left(\frac{C_1}{x}\right)},$$

avec  $C_0 > 0$  et  $C_1 > 0$ , nous dirons que l'inégalité de stabilité est **logarithmique**.

- Quand la fonction  $f$  intervenant dans l'inégalité (1.4) est du type :

$$f(x) = C_0 x,$$

avec  $C_0 > 0$ , nous dirons que l'inégalité de stabilité est **Lipschitzienne**.

### 1.3 État de l'art

Le résolution du problème inverse introduit précédemment est, à notre connaissance, un problème largement ouvert.

Concernant l'unicité, C. Fabre et G. Lebeau ont prouvé dans [39] un résultat de continuation unique pour les équations de Stokes avec un potentiel borné à partir duquel on peut apporter une réponse à la question (i) pour le problème de Stokes. Pour ce faire, ils ont établi une variante de l'inégalité de Carleman locale classique pour le Laplacien.

Différents types de problèmes inverses concernant le système de Stokes ont été étudiés. Nous avons choisi de parler de certains d'entre eux. Il s'agit bien sûr d'une liste non exhaustive, mais qui permet de rendre compte des différents résultats qui y sont obtenus ainsi que des outils en jeu.

O.Y. Imanuvilov et M. Yamamoto se sont intéressés dans [53] à l'estimation d'un terme source dans le système de Stokes avec des conditions de Dirichlet homogène, à partir de mesures internes dans un ouvert  $\omega \subset \Omega$  et de mesures en un temps donné sur tout le domaine. Ils obtiennent une inégalité de stabilité Lipschitzienne. Pour ce faire, ils utilisent une variante de la méthode de A.L. Bukhgeim et M.V. Klibanov [24]. A.L. Bukhgeim et M.V. Klibanov ont introduit une méthode efficace pour résoudre des problèmes inverses lorsque le paramètre que l'on cherche à récupérer est présent dans l'équation aux dérivées partielles en question. Cette méthode consiste à considérer la dérivée en temps du problème initial. Ce faisant, on retrouve le paramètre que l'on cherche à estimer dans la condition initiale. Puis, utilisant bilan d'énergie et inégalité de Carleman, il est alors possible d'obtenir des inégalités de stabilité. On renvoie à [18] pour une mise en oeuvre de cette méthode dans le cas de l'équation de la chaleur avec un second membre non linéaire et des conditions de Neumann homogène et à [34] pour le cas d'un système visco-élastique. Cette méthode n'est pas adaptée à notre cas car le paramètre que nous cherchons à récupérer intervient sur une partie du bord : en dérivant le problème en temps, on ne retrouve pas le paramètre dans la condition initiale. Dans [8], A. Ballerini considère un autre type de problème inverse. Elle s'est intéressée à la détection d'un corps immergé dans un conteneur rempli d'un fluide qui obéit aux équations de Stokes, à partir de mesure de la vitesse et de la force de pression qui s'exerce sur le bord du domaine. Elle précise dans ce papier que la question de la stabilité pour ce genre de problème reste une question largement ouverte. La question de l'unicité a quant à elle été démontrée dans [6] en utilisant des techniques de continuation unique. A. Ballerini a obtenu une estimation de la stabilité du type log-log, ce qui signifie que la fonction  $f$  impliquée dans l'estimation de la stabilité est  $\frac{C_0}{\log(|\log(x)|)}$ ,  $C_0 > 0$ . Le principal outil utilisé est l'inégalité des trois boules pour la puissance  $l^{\text{ème}}$  du Laplacien. Notez que la preuve utilise le fait que, puisque  $-\Delta u + \nabla p = 0$  et  $u$  étant à divergence nulle, on a  $\Delta p = \text{div}(\Delta u + \nabla p) = 0$ . On renvoie également à [6] et [29] pour d'autres résultats concernant la détection de la forme et de la position d'un corps rigide en mouvement dans un fluide visqueux incompressible à partir de mesures frontières. Pour terminer ce paragraphe, notons que dans [58], C.L. Lin, G. Uhlmann et J.N. Wang prouvent une inégalité des trois boules pour  $u$  solution d'une équation de Stokes et ils en déduisent ensuite un résultat quantitatif de la propriété forte de continuation unique pour les équations de Stokes (quand le potentiel est non borné, contrairement au résultat de continuation unique de C. Fabre et G. Lebeau) prouvé par R. Regbaoui dans [71]. Pour cela, ils se ramènent à des équations elliptiques en utilisant la divergence nulle et le rotationnel.

La particularité du problème inverse qui nous intéresse réside dans le fait que l'on cherche à récupérer des paramètres définis sur une partie du bord à partir de mesures effectuées sur une autre partie du bord. Au vu des résultats cités ci-dessus, on est tenté de regarder le système de Stokes comme un système de  $d$  équations de la chaleur. C'est pourquoi on s'est intéressé dans un premier temps à un problème inverse similaire pour le Laplacien et qui consiste à récupérer un paramètre défini au bord du domaine à partir de mesures accessibles sur une autre partie du bord.

*Estimer un coefficient de Robin défini sur une partie du bord du domaine à partir de mesures disponibles sur une autre partie du bord du domaine pour  $u$  solution de :*

$$\begin{cases} -\Delta u & = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} & = g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} + qu & = 0, & \text{on } \Gamma_{out}, \\ u(0) & = u_0, & \text{in } \Omega. \end{cases}$$

avec  $\Gamma_{out} \cup \Gamma_0 \subseteq \partial\Omega$ .

La littérature concernant la résolution de ce problème inverse est riche. Il s'agit en général d'un problème provenant de la détection de corrosion et qui consiste à déterminer un coefficient de Robin sur une partie inaccessible de la frontière à partir de mesures électrostatiques effectuées sur une partie accessible de la frontière.

La plupart du temps, une inégalité de stabilité logarithmique est obtenue (voir [4], [10], [26] et [28]). S. Chaabane and M. Jaoua ont démontré dans [27] une inégalité de stabilité lipschitzienne locale, selon la terminologie introduite par H. Bellout et A. Friedman dans [12] et qui a été utilisée ensuite par de nombreux auteurs, ainsi qu'une inégalité de stabilité lipschitzienne globale monotone, sous l'hypothèse que le flux  $g$  est positif. Affaiblissant cette contrainte en imposant à la place un contrôle sur le caractère oscillant du flux  $g$ , G. Alessandrini, L. Del Piero et L. Rondi ont obtenu dans [4] une inégalité de stabilité logarithmique. Plus récemment, E. Sincich a prouvé dans [74], une inégalité de stabilité lipschitzienne, en supposant que le coefficient de Robin  $q$  est constant par morceaux. Elle démontre notamment que la constante impliquée dans l'inégalité de stabilité dépend exponentiellement du nombre de morceaux considérés.

Dans ces papiers, différentes approches sont développées. Une première approche consiste à utiliser les propriétés des fonctions analytiques complexes. C'est le cas par exemple dans [4] et [26]. Une caractéristique de cette méthode est qu'elle n'est valable qu'en dimension 2. Une autre approche est basée sur les inégalités de Carleman qui s'avèrent être un outil très efficace pour obtenir des résultats de continuation unique et des inégalités de stabilité. Par exemple, dans [68], K.D. Phung a quantifié un résultat de continuation unique pour le Laplacien dans un domaine borné de classe  $C^\infty$  quand les conditions au bord du domaine sont *a priori* inconnues : il a établi une dépendance de type logarithmique. Puis, dans [10], M. Bellassoued, J. Cheng et M. Choulli ont utilisé cette inégalité pour obtenir une inégalité de stabilité logarithmique. Il est intéressant de noter que l'inégalité de K.D. Phung a été étendue au cas d'ouvert de classe  $C^{1,1}$  par L. Bourgeois dans [19] puis au cas d'ouvert lipschitzien par L. Bourgeois et J. Dardé [20]. Ajoutons que J. Cheng, M. Choulli, et J. Lin dans [28] ont également utilisé une inégalité de Carleman démontrée par A.L. Bukhgeim dans [23] avec des fonctions poids astucieusement choisies pour établir leur inégalité de stabilité.

En ce qui concerne la résolution du problème inverse pour le système instationnaire, peu de résultats existent, à notre connaissance. Dans [10], M. Bellassoued, J. Cheng



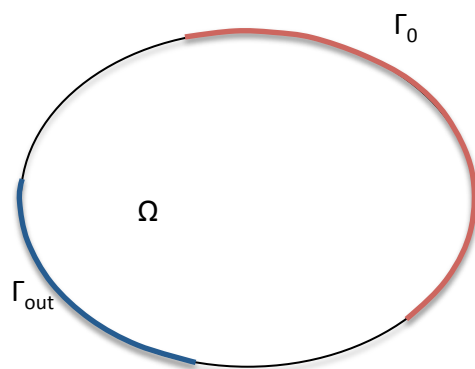


FIGURE 1.8 – On cherche à estimer des paramètres définis sur  $\Gamma_{out}$  à partir de mesures disponibles sur  $\Gamma_0$ .

et M. Choulli ont déduit de l'inégalité de stabilité obtenue pour le cas stationnaire, une inégalité de stabilité logarithmique pour le problème instationnaire. Ils ont pour cela fait appel à la théorie des semi-groupes. Cela conduit à effectuer des mesures en temps infini.

## 1.4 Apport de la thèse

Nous nous intéressons à une version simplifiée du modèle de l'écoulement de l'air dans les poumons et qui consiste à supprimer le terme convectif non linéaire : on a considéré les équations de Stokes incompressibles. De plus, on a remplacé les conditions aux limites dissipatives (1.1), sur une partie du bord noté  $\Gamma_{out}$ , par des conditions aux limites de type Robin :

$$\frac{\partial u}{\partial n} - pn + qu = 0, \text{ sur } \Gamma_{out}. \quad (1.5)$$

Autrement dit, nous considérons le système suivant :

$$\begin{cases} u_t - \Delta u + \nabla p = 0, & \text{dans } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{dans } (0, T) \times \Omega, \\ \frac{\partial u}{\partial n} - pn = g, & \text{sur } (0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu = 0, & \text{sur } (0, T) \times \Gamma_{out}, \\ u(0) = u_0, & \text{dans } \Omega, \end{cases} \quad (1.6)$$

avec  $\Gamma_0 \cup \Gamma_{out} \subseteq \partial\Omega$ .

On cherche à récupérer le coefficient de Robin  $q$  défini sur  $\Gamma_{out}$  à partir de mesures disponibles sur  $\Gamma \subseteq \Gamma_0$  (voir la Figure 1.8).

Soit  $\Gamma \subseteq \Gamma_0$ ,  $T > 0$  et  $(u_i, p_i)$  la solution de (1.6) associée à  $q = q_i$  pour  $i = 1, 2$ . On note toujours par  $\mathcal{M}_{(0,T) \times \Gamma}$  l'opérateur de mesures. Les questions qui nous intéressent sont :

- (iii) Est-ce que  $\mathcal{M}_{(0,T) \times \Gamma}(u_1, p_1) = \mathcal{M}_{(0,T) \times \Gamma}(u_2, p_2)$  implique que  $q_1 = q_2$  sur  $\Gamma_{out}$  ?
- (iv) Est-il possible d'obtenir des inégalités de stabilité du type :

$$\|(q_1 - q_2)|_{\Gamma_{out}}\| \leq f(\|\mathcal{M}_{(0,T) \times \Gamma}(u_1, p_1) - \mathcal{M}_{(0,T) \times \Gamma}(u_2, p_2)\|),$$

où  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  est une fonction croissante qui satisfait  $\lim_{x \rightarrow 0} f(x) = 0$  ?

Nous considérerons les deux cas suivants :

- (a)  $\Gamma_0 \cup \Gamma_{out} = \partial\Omega$  et  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ ,
- (b)  $\partial\Omega \setminus (\Gamma_0 \cup \Gamma_{out}) = \Gamma_l \neq \emptyset$  et  $u = 0$  sur  $\Gamma_l$ .

Le cas (a) nous assure de la régularité globale sur la solution du système (1.6) sans aucune condition de compatibilité des conditions aux bords. En revanche, dans le second cas, nous n'avons pas *a priori* de régularité globale sur la solution car il s'agit d'un problème aux limites mixtes de type Dirichlet-Neumann.

Dans cette thèse, nous avons établi dans le cas (a) deux inégalités de stabilité logarithmiques, l'une valable en dimension 2 et l'autre valable en toute dimension. Nous détaillons ces résultats dans la Sous-section 1.4.2.

Dans le cas (b), nous avons établi une inégalité de stabilité Lipschitzienne pour le problème stationnaire sous l'hypothèse *a priori* que le coefficient de Robin est constant par morceaux. Notons que si l'on s'affranchit de cette condition *a priori* sur le coefficient de Robin, il est possible de prouver une inégalité de stabilité logarithmique locale.

Nous résumons ci-dessous les différents cas en considération.

**Cas (a) :** soit  $\Omega$  un ouvert tel que  $\partial\Omega = \Gamma_0 \cup \Gamma_{out}$  et  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ . On considère le système de Stokes suivant :

$$\left\{ \begin{array}{ll} u_t - \Delta u + \nabla p & = 0, \quad \text{dans } (0, T) \times \Omega, \\ \operatorname{div} u & = 0, \quad \text{dans } (0, T) \times \Omega, \\ \frac{\partial u}{\partial n} - pn & = g, \quad \text{sur } (0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = 0, \quad \text{sur } (0, T) \times \Gamma_{out}, \\ u(0) & = u_0, \quad \text{dans } \Omega. \end{array} \right.$$

$\rightsquigarrow$  Récupérer le coefficient de Robin  $q$  à partir de mesures disponibles sur  $\Gamma_0$ .

**Cas (b) :** soit  $\Omega$  un ouvert tel que  $\Omega = \Gamma_l \cup \Gamma_0 \cup \Gamma_{out}$  avec  $\Gamma_{out} = \bigcup_{i=1}^N \Gamma_i$ . On considère le système suivant :

$$\left\{ \begin{array}{ll} -\Delta u + \nabla p & = 0, \quad \text{dans } \Omega, \\ \operatorname{div} u & = 0, \quad \text{dans } \Omega, \\ u & = 0, \quad \text{dans } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = g, \quad \text{sur } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = 0, \quad \text{sur } \Gamma_{out}, \end{array} \right.$$

avec  $q$  **constant par morceaux** sur  $\Gamma_{out}$  :

$$q_{\Gamma_i} = q_i \text{ sur } \Gamma_i \text{ pour } 1 \leq i \leq N,$$

avec  $(q_i)_{1 \leq i \leq N} \in \mathbb{R}_+^N$ .

$\rightsquigarrow$  Récupérer le coefficient de Robin  $q$  à partir de mesures disponibles sur  $\Gamma_0$ .

Les **résultats d'identifiabilité** sont obtenus comme un corollaire du résultat de continuation unique de C. Fabre et G. Lebeau [39]. Leur résultat est résumé dans le théorème suivant.

**Théorème 1.5.** *On note  $Q = (0, T) \times \Omega$  et  $O$  un ouvert inclus dans  $Q$ . La composante horizontale de  $O$  est*

$$C(O) = \{(t, x) \in Q, \exists x_0 \in \Omega, (t, x_0) \in O\}.$$

Soit  $(u, p) \in L^2(0, T; H^1_{loc}(\Omega)) \times L^2_{loc}(Q)$  solution faible de

$$\begin{cases} u_t - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \quad (1.7)$$

dans  $Q$  satisfaisant  $u = 0$  dans  $O$ , alors  $u = 0$  et  $p$  est constant dans  $C(O)$ .

Nous en déduisons facilement le résultat suivant duquel on obtient les résultats d'identifiabilité :

**Corollaire 1.6.** *Soit  $\Gamma \subseteq \partial\Omega$ ,  $\delta > 0$ ,  $x \in \Gamma$ ,  $r > 0$  et  $\gamma = ]t_0 - \delta, t_0 + \delta[ \times (\mathcal{B}(x, r) \cap \Gamma)$  un ensemble ouvert inclus dans  $(0, T) \times \Gamma$ . Soit  $(u, p) \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \times L^2(0, T; H^1(\Omega))$  solution de (1.7) dans  $(0, T) \times \Omega$  satisfaisant  $u = 0$  et  $\frac{\partial u}{\partial n} - pn = 0$  sur  $\gamma$ . Alors  $u = 0$  et  $p = 0$  dans  $]t_0 - \delta, t_0 + \delta[ \times \Omega$ .*

Grâce à ce corollaire, il est possible de montrer, sous l'hypothèse que le flux  $g$  est non identiquement nul sur  $\Gamma_0$ , l'unicité du coefficient de Robin sur  $\Gamma_{out}$  par rapport à des mesures faites sur  $(0, T) \times \Gamma \subseteq (0, T) \times \Gamma_0$ . Cela permet d'apporter une réponse positive à la question (iii).

Concernant les inégalités de stabilité, elles reposent, dans cette thèse, sur des inégalités de Carleman. Les inégalités de Carleman sont couramment utilisées pour prouver des résultats de continuation unique ([39], [58], [68], ...), des résultats de contrôlabilité ([25], [40], ...) et des inégalités de stabilité ([10], [28], ...). Il s'agit dans notre cas d'inégalités permettant de majorer la norme  $H^1$  avec poids de  $u$  en fonction du Laplacien de  $u$  plus éventuellement des termes de bord. Elles sont de deux types : globales (c'est-à-dire l'intégrale porte sur tout le domaine  $\Omega$ ) et locales. Nous utiliserons notamment des inégalités de Carleman locales pour quantifier des résultats de continuation unique pour le système de Stokes dont nous détaillons le contenu dans la Sous-section 1.4.1.

Nous tenons à souligner que dans les résultats présentés dans les sous-sections suivantes, certains énoncés sont **volontairement simplifiés** afin de mettre en valeur les résultats obtenus. En particulier, la dépendance des constantes par rapport aux données intervenant dans les inégalités de stabilité est volontairement non explicitée.

### 1.4.1 Quantification de résultats de continuation unique pour le système de Stokes

On se propose de quantifier des résultats de continuation unique pour le système de Stokes :

$$\begin{cases} -\Delta u + \nabla p = 0, & \text{dans } \Omega, \\ \operatorname{div} u = 0, & \text{dans } \Omega. \end{cases} \quad (1.8)$$

Nous prouvons deux types de résultats : une estimation locale qui est de type Hölder et deux estimations globales qui sont logarithmiques.

Le résultat de stabilité locale, énoncé dans le Théorème 1.7, permet d'estimer la vitesse en norme  $H^1$  et la pression en norme  $L^2$  sur un compact  $K \subset \Omega$ .

**Théorème 1.7.** *Soit  $\omega$  un ouvert non vide et  $K$  un compact, tout deux inclus dans  $\Omega$ .*

*Alors, il existe  $c > 0$  et  $0 < \beta < 1$  tel que pour tout  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  solution de (1.8), on a :*

$$\|u\|_{H^1(K)} + \|p\|_{L^2(K)} \leq c (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)})^\beta (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)})^{1-\beta}. \quad (1.9)$$

Concernant les estimations globales, nous estimons dans la première  $(u, p)$  solution de (1.8) en norme  $H^1$  dans tout le domaine par rapport à la norme  $L^2$  de  $(u|_\Gamma, p|_\Gamma)$  et de  $(\frac{\partial u}{\partial n}|_\Gamma, \frac{\partial p}{\partial n}|_\Gamma)$ , où  $\Gamma$  est une partie du bord du domaine  $\Omega$ . Dans la seconde, nous obtenons une estimation de  $(u, p)$  solution de (1.8) en norme  $H^1$  dans tout le domaine par rapport à la norme  $H^1$  de  $u$  et  $p$  dans un ensemble ouvert  $\omega \subset \Omega$ . Pour être plus précis, nous prouvons le théorème suivant :

**Théorème 1.8.** *Supposons que  $\Omega$  est de classe  $C^\infty$ . Soit  $0 < \nu \leq \frac{1}{2}$ ,  $\Gamma$  une partie ouverte non vide du bord du domaine  $\Omega$  et  $\omega$  un ouvert non vide inclus dans  $\Omega$ .*

*Alors, il existe  $d_0 > 0$  tel que pour tout  $\beta \in ]0, \frac{1}{2} + \nu[$ , pour tout  $d > d_0$ , il existe  $c > 0$ , tel que l'on ait*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left( \ln \left( d \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \|\frac{\partial u}{\partial n}\|_{L^2(\Gamma)} + \|\frac{\partial p}{\partial n}\|_{L^2(\Gamma)}} \right) \right)^\beta}, \quad (1.10)$$

et

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left( \ln \left( d \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}} \right) \right)^\beta}, \quad (1.11)$$

pour tout couple  $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$  solution de (1.8).

Notons que dans le Théorème 1.8, nous avons cherché à exprimer la régularité minimale nécessaire sur  $(u, p)$  pour que l'estimation soit vraie : on a besoin de  $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ .

Nous tenons à souligner que nos estimations de stabilité ne sont peut-être pas optimales au vue des résultats de continuation unique du Théorème 1.5 et du Corollaire 1.6 (on renvoie à [58] pour une inégalité des trois boules optimale pour le système de Stokes). On peut notamment remarquer que nos estimations de stabilité nécessitent davantage de mesures. En effet, dans l'inégalité (1.9), nous avons besoin d'informations sur  $u$  et  $p$  dans  $\omega$  alors que le Théorème 1.5 ne nécessite des informations que sur  $u$ . De même, la contrainte  $\frac{\partial u}{\partial n} - pn$  qui apparaît dans le Corollaire 1.6 est divisée en deux parties dans l'inégalité (1.10) :  $\frac{\partial u}{\partial n}$  d'un côté et  $pn$  de l'autre. Il y a également un terme supplémentaire, la dérivée normale de  $p$ . Notons cependant que pour la résolution du problème inverse qui nous intéresse, nous aurons besoin d'une estimation sur la pression. Par exemple, une variante de l'estimation de stabilité (1.9) (voir l'inégalité des trois boules énoncée dans le Lemme 1.27) nous sera utile pour prouver une inégalité de stabilité Lipschitzienne dans le cas (b) (voir la Sous-section 1.4.3) et l'inégalité (1.10) nous sera utile dans la Sous-section 1.4.2 pour prouver une inégalité de stabilité de type Logarithmique.

Notons que les Théorèmes 1.7 et 1.8 sont respectivement équivalents aux Théorèmes 1.9 et 1.10 suivants. On passe des uns aux autres par un argument d'optimisation.

**Théorème 1.9.** *Soit  $\omega$  un ouvert non vide et  $K$  un ensemble compact, tout deux inclus dans  $\Omega$ . Alors, il existe  $c > 0$  et  $s > 0$  tel que pour tout  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  solution de (1.8), et pour tout  $\epsilon > 0$  on a :*

$$\|u\|_{H^1(K)} + \|p\|_{L^2(K)} \leq \frac{c}{\epsilon} (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}). \quad (1.12)$$

**Théorème 1.10.** *Supposons que  $\Omega$  est de classe  $C^\infty$ . Soit  $0 < \nu \leq \frac{1}{2}$ ,  $\Gamma$  une partie ouverte non vide du bord du domaine  $\Omega$  et  $\omega$  un ouvert non vide inclus dans  $\Omega$ . Alors, pour tout  $\beta \in ]0, \frac{1}{2} + \nu[$ , il existe  $c > 0$ , tel que pour tout  $\epsilon > 0$ , on a :*

$$\begin{aligned} \|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{\epsilon}{c}} \left( \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) \\ + \epsilon^\beta (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}), \end{aligned} \quad (1.13)$$

et

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{\epsilon}{c}} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}), \quad (1.14)$$

pour tout couple  $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$  solution de (1.8).

Notons que dans (1.9) seule la norme de  $p$  dans  $L^2$  apparaît. Cela repose sur l'inégalité de Cacciopoli qui nous permet d'estimer le gradient de  $p$  dans une boule en fonction de la norme  $L^2$  de  $p$  dans une boule de rayon strictement plus grand. Plus précisément,

**Lemme 1.11** (Inégalité de Caccioppoli). *Soit  $v$  solution faible de  $\Delta v = 0$  dans  $\Omega \subset \mathbb{R}^d$ . Alors, il existe  $C > 0$  tel que pour tout  $x_0 \in \Omega$  et  $0 < \rho < R < d(x_0, \partial\Omega)$ , on a*

$$\int_{B(x_0, \rho)} |\nabla v|^2 \leq \frac{C}{(R - \rho)^2} \int_{B(x_0, R)} |v|^2.$$

Nous allons maintenant donner une idée des preuves de ces différents résultats. La preuve du Théorème 1.8 est divisée en trois résultats intermédiaires qui illustrent la façon dont se transmet l'information d'une partie du bord du domaine à une autre. La preuve du Théorème 1.7 est quant à elle une conséquence directe de l'un de ces résultats intermédiaires.

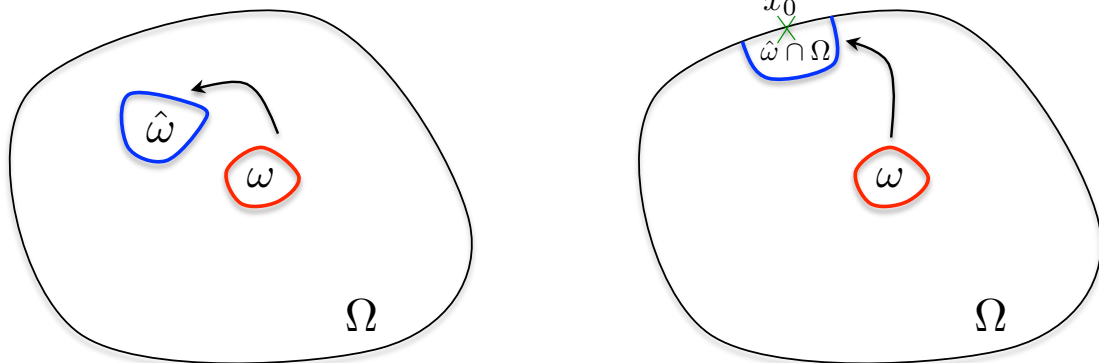
La première proposition permet de transmettre l'information d'un ouvert à un ensemble relativement compact inclus dans  $\Omega$ . On renvoie à la Figure 1.9a pour une illustration.

**Proposition 1.12.** *Soit  $\omega$  un ensemble non vide inclus dans  $\Omega$  et soit  $\hat{\omega}$  un ouvert relativement compact dans  $\Omega$ . Alors, nous avons :*

$$\begin{cases} \exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution de (1.8),} \\ \|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}), \end{cases} \quad (1.15)$$

et

$$\begin{cases} \exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times L^2(\Omega) \text{ solution de (1.8),} \\ \|u\|_{H^1(\hat{\omega})} + \|p\|_{L^2(\hat{\omega})} \leq \frac{c}{\epsilon} (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}). \end{cases} \quad (1.16)$$



(a) Figure illustrant comment se transmet l'information dans la Proposition 1.12 : d'un ouvert  $\omega$  à un ouvert relativement compact  $\hat{\omega}$ .

(b) Figure illustrant comment se transmet l'information dans la Proposition 1.13 : d'un ouvert  $\omega$  à un voisinage du bord  $\hat{\omega} \cap \Omega$ .

FIGURE 1.9

La seconde proposition permet de transmettre l'information d'un ouvert relativement compact inclus dans  $\Omega$  à un voisinage du bord du domaine. On renvoie à la Figure 1.9b pour une illustration.

**Proposition 1.13.** *Supposons que  $\Omega$  est de classe  $C^\infty$ . Soit  $0 < \nu \leq \frac{1}{2}$ ,  $x_0 \in \partial\Omega$  et  $\omega$  un ensemble ouvert inclus dans  $\Omega$ . Il existe un voisinage  $\hat{\omega}$  de  $x_0$  tel que :*

$$\forall \beta \in \left] 0, \frac{1}{2} + \nu \right[ , \exists c > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.8),}$$

$$\|u\|_{H^1(\hat{\omega} \cap \Omega)} + \|p\|_{H^1(\hat{\omega} \cap \Omega)} \leq e^{\frac{\epsilon}{c}} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}). \quad (1.17)$$

Finalement, la troisième proposition permet de transmettre l'information d'une partie du bord du domaine  $\Omega$  à un ensemble relativement compact inclus dans  $\Omega$ . On renvoie à la Figure 1.10 pour une illustration.

**Proposition 1.14.** *Supposons que  $\Omega$  est de classe  $C^\infty$ . Soit  $0 < \nu \leq \frac{1}{2}$ ,  $\Gamma$  une partie ouverte non vide du bord du domaine  $\Omega$  et  $\hat{\omega}$  un ensemble relativement compact inclus dans  $\Omega$ . Alors, on a :*

$$\exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (1.8),}$$

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon} \left( \|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

La nature logarithmique des inégalités (1.10) et (1.11) vient de la Proposition 1.13 où une exponentielle apparaît devant le premier terme du membre de droite de l'inégalité alors que les estimations des Propositions 1.12 et 1.14 conduisent à des estimations de type Hölder.

Pour prouver ces résultats, nous utilisons les deux inégalités de Carleman locales suivantes. On renvoie à [52] pour une preuve de la première et à [56] pour la seconde.

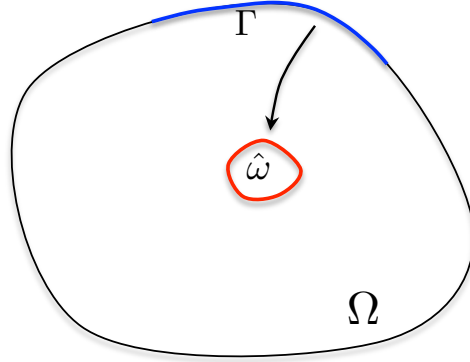


FIGURE 1.10 – Figure illustrant comment se transmet l’information dans la Proposition 3.22 : d’une partie du bord  $\Gamma$  à un ouvert relativement compact  $\hat{\omega}$ .

**Proposition 1.15.** *Soit  $U$  un ouvert inclus dans  $\mathbb{R}^n$  et  $K$  un compact inclus dans  $U$ . Soit  $\phi \in C^\infty(\mathbb{R}^n)$  et  $P = \Delta$ . On suppose que la fonction  $\phi$  satisfait*

$$|\nabla\phi| > 0 \text{ dans } U,$$

*et la propriété d’hypoellipticité de Hörmander sur  $U$ .*

*Alors, il existe  $c > 0$  et  $h_1 > 0$  tel que pour tout  $h \in ]0, h_1[$  et pour toute fonction  $y \in C_0^\infty(K)$ , on a*

$$\int_K |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_K |\nabla y(x)|^2 e^{2\phi(x)/h} dx \leq ch^3 \int_K |\Delta y(x)|^2 e^{2\phi(x)/h} dx.$$

Localement au voisinage du bord, on peut se ramener au cas du demi-espace par un changement de coordonnées. On a l’inégalité de Carleman suivante.

**Proposition 1.16.** *Soit  $K = \{x \in \mathbb{R}_+^n / |x| \leq R_0\}$  et  $\Sigma = \{x \in \partial K / x_n = 0\}$ . On note  $C_{0, \partial K \setminus \Sigma}^\infty(\bar{K})$  la restriction à  $K$  de fonctions appartenant à  $C_0^\infty(\bar{B}(0, R_0))$ .*

*Soit  $P$  un opérateur différentiel du second ordre dont les coefficients sont  $C^\infty$  dans un voisinage de  $K$ , défini par  $P(x, \partial_x) = -\partial_{x_n}^2 + R(x, \frac{1}{i}\partial_{x'})$ , et  $\phi = \phi(x) \in C^\infty$  une fonction définie dans un voisinage de  $K$ . On note par  $r(x, \xi')$  le symbole principal de  $R$  et on suppose que  $r(x, \xi') \in \mathbb{R}$  et qu’il existe une constante  $c > 0$  telle que  $(x, \xi') \in K \times \mathbb{R}^{n-1}$ , on a  $r(x, \xi') \geq c|\xi'|^2$ .*

*On suppose que la fonction  $\phi$  satisfait la propriété d’hypoellipticité de Hörmander sur  $U$  et*

$$\partial_{x_n}\phi(x) \neq 0, \forall x \in K.$$

*Alors, il existe  $c > 0$  et  $h_1 > 0$  tel que pour tout  $h \in ]0, h_1[$  et pour toute fonction  $y \in C_{0, \partial K \setminus \Sigma}^\infty(\bar{K})$ , on a :*

$$\begin{aligned} \int_{\mathbb{R}_+^n} |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{\mathbb{R}_+^n} |\nabla y(x)|^2 e^{2\phi(x)/h} dx &\leq ch^3 \int_K |P(x, \partial_x)y(x)|^2 e^{2\phi(x)/h} dx \\ &+ c \int_{\mathbb{R}^{n-1}} (|y(x', 0)|^2 + |h\partial_{x'}y(x', 0)|^2 + |h\partial_{x_n}y(x', 0)|^2) e^{2\phi(x', 0)/h} dx'. \end{aligned}$$

Dans chaque cas, cela consiste à appliquer simultanément l'inégalité de Carleman à  $u$  and  $p$ , en utilisant le fait que  $\Delta u = \nabla p$  et  $\Delta p = \operatorname{div}(\Delta u) = 0$ .

Les résultats présentés dans cette sous-partie vont nous servir pour l'obtention de certaines inégalités de stabilité détaillées ci-dessous.

### 1.4.2 Système de Stokes avec des conditions aux limites de type Robin

Soit  $\Omega \subset \mathbb{R}^d$  un ouvert tel que  $\partial\Omega = \Gamma_0 \cup \Gamma_{out}$  et  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ . On suppose que le coefficient de Robin  $q$  ne dépend que de la variable d'espace et on considère le système de Stokes (1.6).

Notons que grâce aux hypothèses sur le bord du domaine, on obtient sans problème de la régularité sur  $(u, p)$  solution de (1.6) à partir de résultat de régularité pour le système de Stokes avec des conditions aux limites de type Neumann (on renvoie à [21] pour des résultats de régularité concernant le système de Stokes).

Pour ce problème, nous prouvons le résultat d'unicité suivant :

**Proposition 1.17.** *Soit  $x_0 \in \Gamma_0$  et  $r > 0$ . On suppose que  $g$  est non identiquement nul sur  $\Gamma_0$ . Soit  $(u_k, p_k) \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \times L^2(0, T; L^2(\Omega))$  solution du système (1.6) avec  $q = q_k$  pour  $k = 1, 2$ .*

*Si  $u_1 = u_2$  sur  $(0, T) \times (\mathcal{B}(x_0, r) \cap \Gamma_0)$  alors  $q_1 = q_2$  sur  $\Gamma_{out}$ .*

Cette proposition est une conséquence directe du Corollaire 1.6.

En ce qui concerne la stabilité, nous établissons tout d'abord des inégalités de stabilité pour le problème stationnaire puis nous en déduisons ensuite des inégalités de stabilité pour le problème non stationnaire en utilisant le théorème des semi-groupes analytiques.

#### Le problème stationnaire

Introduisons le système stationnaire :

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{dans } \Omega, \\ \operatorname{div} u &= 0, & \text{dans } \Omega, \\ \frac{\partial u}{\partial n} - pn &= g, & \text{sur } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu &= 0, & \text{sur } \Gamma_{out}. \end{cases} \quad (1.18)$$

Nous prouvons deux inégalités de stabilité pour le problème stationnaire : l'une n'est valable qu'en dimension 2 et l'autre est valable en toute dimension. Toutes deux reposent sur des inégalités de Carleman.

Avant d'en dire davantage sur ces deux inégalités, notons que l'inégalité suivante est satisfaite sur  $\Gamma_{out}$  :

$$|u_1| |q_1 - q_2| \leq \left| \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right| + |p_1 - p_2| + |q_2| |u_1 - u_2|,$$

où  $(u_k, p_k)$  est solution du système (1.18) avec  $q = q_k$  pour  $k = 1, 2$ .

Nous voyons donc que pour estimer la différence entre les deux coefficients de Robin sur  $\tilde{\Gamma} \subseteq \Gamma_{out}$ , il est nécessaire que  $u_1$  soit non nul sur  $\tilde{\Gamma}$ . Si l'on intègre sur un



compact  $K \subset \{x \in \Gamma_{out}/u_1 \neq 0\}$ , on obtient, en supposant que  $q_2$  est borné :

$$\|q_1 - q_2\|_{L^2(K)} \leq C \left( \left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right\|_{L^2(K)} + \|p_1 - p_2\|_{L^2(K)} + \|u_1 - u_2\|_{L^2(K)} \right). \quad (1.19)$$

Reste donc à estimer les termes de bords sur  $K$  par rapport à des termes de bords sur  $\Gamma \subseteq \Gamma_0$ .

Chronologiquement, la première inégalité de stabilité que nous avons obtenue est résumée dans le théorème suivant.

**Théorème 1.18.** *Soit  $\Omega \subset \mathbb{R}^2$  un ouvert de classe  $\mathcal{C}^{3,1}$  et  $\Gamma \subseteq \Gamma_0$ . On suppose que  $(u_k, p_k) \in H^4(\Omega) \times H^3(\Omega)$  est solution du système (1.18) avec  $q = q_k$  pour  $k = 1, 2$ . Soit  $K$  un compact inclus dans  $\{x \in \Gamma_{out}/u_1 \neq 0\}$ .*

*Alors, il existe  $C > 0$  et  $C_1 > 0$  tel que :*

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C}{\left( \ln \left( \frac{C_1}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^{\frac{1}{2}}}.$$

Cette inégalité est basée sur l'inégalité de Carleman globale suivante, prouvée par A. L. Bukhgeim dans [23], avec un choix astucieux de la fonction poids  $\Psi$ , dépendant d'un paramètre  $s > 0$ .

**Lemme 1.19.** *Soit  $\Psi \in \mathcal{C}^2(\bar{\Omega})$ . Nous avons :*

$$\int_{\Omega} (\Delta \Psi |u|^2 + (\Delta \Psi - 1) |\nabla u|^2) e^{\Psi} \leq \int_{\Omega} |\Delta u|^2 e^{\Psi} + \int_{\partial \Omega} \frac{\partial \Psi}{\partial n} \left( |u|^2 + |\nabla u|^2 + 2 \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) e^{\Psi}, \quad (1.20)$$

*pour tout  $u \in \mathcal{C}^2(\bar{\Omega})$ .*

Notons que l'inégalité du Lemme 1.19 n'est valable qu'en dimension 2. C'est pourquoi l'inégalité de stabilité qui en résulte n'est valable qu'en dimension 2. L'idée consiste à appliquer l'inégalité de Carleman du Lemme 1.19 deux fois : une fois sur la vitesse  $u$  puis une autre fois sur la pression  $p$ . Lorsque l'on applique une première fois l'inégalité à  $u$ , il apparaît un terme en gradient de  $p$  à droite de l'inégalité et c'est pour s'en débarrasser que l'on applique une nouvelle fois l'inégalité de Carleman à  $p$ . En effet, en sommant les deux inégalités ainsi obtenues, des termes en gradient de  $p$  apparaissent de chaque côté de l'inégalité et grâce au choix de la fonction poids, le terme de droite est absorbé par le terme de gauche. Puis l'on sépare les termes de bord sur  $\Gamma_{out}$  et  $\Gamma$  et on explicite la dépendance par rapport au paramètre  $s > 0$ . On se retrouve avec l'inégalité suivante :

$$\int_{\Gamma_{out}} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq C \left( e^{ks} \left( \int_{\Gamma} |u|^2 + |p|^2 + \left| \frac{\partial u}{\partial n} \right|^2 + \left| \frac{\partial p}{\partial n} \right|^2 \right)^{\frac{1}{4}} + \frac{1}{s} \right), \quad \forall s > 0.$$

On conclut en optimisant l'inégalité précédente par rapport à  $s > 0$ .

La seconde inégalité que l'on obtient est résumée dans le théorème suivant.

**Théorème 1.20.** Soit  $\Omega \subset \mathbb{R}^d$ ,  $\Gamma$  un sous-ensemble ouvert non vide de  $\Gamma_0$  et  $k \in \mathbb{N}^*$  tel que  $k+2 > \frac{d}{2}$ . On suppose que  $\Gamma$  et  $\Gamma_{out}$  sont de classes  $C^\infty$  et que  $(u_i, p_i) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$  est solution du système (1.18) avec  $q = q_i$  pour  $i = 1, 2$ . Soit  $K$  un compact inclus dans  $\{x \in \Gamma_{out}/u_1 \neq 0\}$ .

Alors, pour tout  $\beta \in ]0, 1[$ , il existe  $C > 0$  et  $C_1 > 0$  tel que

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C}{\left( \ln \left( \frac{C_1}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^{\frac{3}{4}\beta}}. \quad (1.21)$$

Cette inégalité est une conséquence du Théorème 1.8. Donnons une esquisse de la preuve du Théorème 1.20. Soit  $0 < \epsilon < \frac{1}{2}$ . Repartant de (1.19), puis utilisant la continuité de l'application trace et une inégalité d'interpolation, on a :

$$\|q_1 - q_2\|_{L^2(K)} \leq C \left( \|u\|_{H^1(\omega)}^\theta + \|p\|_{H^1(\omega)}^\theta \right),$$

où  $\theta = \frac{3}{4}(1 - \frac{2\epsilon}{3})$  et  $\omega \subset \Omega$  un ouvert de classe  $C^\infty$  tel que  $K \subset \partial\omega$  et  $\Gamma \subset \partial\omega$ . Il suffit ensuite d'appliquer l'inégalité (1.10) du Théorème 1.8 avec  $\beta$  convenablement choisi. Soit  $\beta' \in ]0, 1[$  fixe. On choisit  $\epsilon$  suffisamment petit pour que  $\beta_0 = \frac{\beta'}{1 - \frac{2\epsilon}{3}}$  appartient à  $]0, 1[$ . On note  $A = \|u\|_{H^2(\omega)} + \|p\|_{H^2(\omega)}$  et  $B = \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}$ .

Appliquant l'inégalité (1.10) du Théorème 1.8 avec  $\nu = \frac{1}{2}$  et avec  $\beta = \beta_0$ , on obtient l'existence de  $d_0 > 0$  tel que pour tout  $d > d_0$ , il existe  $C > 0$  tel que

$$\|q_1 - q_2\|_{L^2(K)} \leq C \frac{A^\theta}{\left( \ln \left( \frac{A}{B} \right) \right)^{\beta_0 \theta}}. \quad (1.22)$$

On conclut la preuve du Théorème 1.20 en étudiant les variations de la fonction  $f_y(x) = \frac{x}{(\ln(\frac{x}{y}))^{\beta_0}}$  sur  $(y, +\infty)$ , pour  $y = \frac{B}{d}$ .

## Le problème non stationnaire

Expliquons maintenant comment nous étendons les estimations de stabilité des Théorèmes 1.18 et 1.20 au cas instationnaire. Nous distinguons deux cas :

- le flux  $g$  ne dépend pas du temps,
- le flux  $g$  dépend du temps.

L'idée est la même dans les deux cas, elle consiste à utiliser l'inégalité de stabilité prouvée pour le cas stationnaire. Notons simplement que dans le second cas des hypothèses supplémentaires sur le comportement asymptotique de  $g$  quand  $t$  tend vers l'infini sont nécessaires. Nous ne parlerons dans cette introduction générale que du cas où  $g$  ne dépend pas du temps.

Le Théorème 1.21 est l'équivalent du Théorème 1.18 dans le cas non stationnaire.

**Théorème 1.21.** Soit  $\Omega \subset \mathbb{R}^2$  un ouvert de classe  $C^{3,1}$  et  $\Gamma \subseteq \Gamma_0$ . On note  $(u_i, p_i) \in L^\infty(0, +\infty; H^3(\Omega)) \times L^\infty(0, +\infty; H^2(\Omega))$  la solution du problème non stationnaire (1.6) avec  $q = q_i$  et  $(v_i, \xi_i) \in H^4(\Omega) \times H^3(\Omega)$  la solution du problème stationnaire (1.18) associée, pour  $i = 1, 2$ . Soit  $K$  un compact inclus dans  $\{x \in \Gamma_{out}/v_1 \neq 0\}$ .

Alors, il existe  $C > 0$  et  $C_1 > 0$  tel que

$$\begin{aligned} & \|q_1 - q_2\|_{L^2(K)} \\ & \leq \frac{C}{\left( \ln \left( \frac{C_1}{\|u_1 - u_2\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \|p_1 - p_2\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0,+\infty;L^2(\Gamma))}} \right) \right)^{\frac{1}{2}}}. \end{aligned}$$

Le Théorème 1.22 est l'équivalent du Théorème 1.20 dans le cas non stationnaire.

**Théorème 1.22.** *Soit  $\Omega \subset \mathbb{R}^d$ ,  $\Gamma$  un sous-ensemble ouvert non vide de  $\Gamma_0$  et  $k \in \mathbb{N}^*$  tel que  $k + 2 > \frac{d}{2}$ . On suppose que  $\Gamma$  et  $\Gamma_{out}$  sont de classe  $C^\infty$ . On note  $(u_i, p_i) \in L^\infty(0, +\infty; H^3(\Omega)) \times L^\infty(0, +\infty; H^2(\Omega))$  la solution du problème non stationnaire (1.6) avec  $q = q_i$  et  $(v_i, \xi_i) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$  la solution du problème stationnaire (1.18) associée, pour  $i = 1, 2$ . Soit  $K$  un compact inclus dans  $\{x \in \Gamma_{out} / v_1 \neq 0\}$ .*

Alors, pour tout  $\beta \in ]0, 1[$ , il existe  $C > 0$  et  $C_1 > 0$  tel que

$$\begin{aligned} & \|q_1 - q_2\|_{L^2(K)} \\ & \leq \frac{C}{\left( \ln \left( \frac{C_1}{\|u_1 - u_2\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \|p_1 - p_2\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0,+\infty;L^2(\Gamma))}} \right) \right)^{\frac{3\beta}{4}}}. \end{aligned}$$

La preuve des deux théorèmes précédents repose sur le lemme suivant, permettant de majorer des termes de bords sur  $\Gamma$  pour  $(v, \xi)$  solution du problème stationnaire par rapport à des termes de bords sur  $\Gamma$  pour  $(u, p)$  solution du problème non stationnaire :

**Lemme 1.23.** *On suppose que  $\Omega$  est un ouvert de classe  $C^{2,1}$ . Soit  $\Gamma \subseteq \Gamma_0$ . On note  $(u, p) \in L^\infty(0, +\infty; H^3(\Omega)) \times L^\infty(0, +\infty; H^2(\Omega))$  la solution du problème non stationnaire (1.6) et  $(v, \xi) \in H^3(\Omega) \times H^2(\Omega)$  la solution du problème stationnaire (1.18). Alors, on a :*

$$\|v\|_{L^2(\Gamma)} + \|\xi\|_{L^2(\Gamma)} + \left\| \frac{\partial \xi}{\partial n} \right\|_{L^2(\Gamma)} \leq \|u\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \|p\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \left\| \frac{\partial p}{\partial n} \right\|_{L^\infty(0,+\infty;L^2(\Gamma))}. \quad (1.23)$$

Pour prouver le Lemme 1.23, on introduit l'opérateur de Stokes associé au système (1.18), noté  $A_q$ , qui est tel que  $-A_q$  engendre un semi-groupe analytique sur  $L^2(\Omega)$ . De ce fait, on peut utiliser l'inégalité suivante : il existe  $C > 0$  et  $\mu > 0$  tel que

$$\|A_q^\eta e^{-tA_q}\| \leq C \frac{e^{-\mu t}}{t^\eta}, \quad t > 0, \eta > 0,$$

où  $\| \cdot \|$  est la norme d'opérateur. On renvoie à [67] et [72] pour plus de détails concernant les semi-groupes.

Introduisons  $(w, \pi) = (u - v, p - \xi)$ . Grâce à l'inégalité précédente, on montre que

$$\|w(t, \cdot)\|_{H^3(\Omega)} + \|\pi(t, \cdot)\|_{H^2(\Omega)} \leq G(t),$$

où  $G$  est une fonction qui tend vers 0 quand  $t$  tend vers l'infini.

Nous pouvons maintenant présenter rapidement la preuve du Théorème 1.21. Grâce au Théorème 1.18, on est en mesure d'estimer  $\|q_1 - q_2\|_{L^2(K)}$  par rapport à une fonction croissante qui dépend de  $(v_1 - v_2)|_\Gamma$ ,  $(\xi_1 - \xi_2)|_\Gamma$  et  $\left(\frac{\partial \xi_1}{\partial n} - \frac{\partial \xi_2}{\partial n}\right)|_\Gamma$ . Grâce au Lemme 1.23, on peut comparer le comportement asymptotique quand  $t$  tend vers l'infini de  $u_1 - u_2$  et  $p_1 - p_2$  sur  $\Gamma$  par rapport aux solutions du système stationnaire  $v_1 - v_2$  et  $\xi_1 - \xi_2$  sur  $\Gamma$  :

$$\begin{aligned} & \|v_1 - v_2\|_{L^2(\Gamma)} + \|\xi_1 - \xi_2\|_{L^2(\Gamma)} + \left\| \frac{\partial \xi_1}{\partial n} - \frac{\partial \xi_2}{\partial n} \right\|_{L^2(\Gamma)} \\ & \leq \|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \|p_1 - p_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma))}. \end{aligned} \quad (1.24)$$

Appliquant le Théorème 1.21 à  $(v_j, \xi_j)$  pour  $j = 1, 2$ , nous obtenons l'existence de deux constantes positives  $C$  and  $C_1$  telles que

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{C}{\left( \ln \left( \frac{C_1}{\|v_1 - v_2\|_{L^2(\Gamma)}^2 + \|\xi_1 - \xi_2\|_{L^2(\Gamma)} + \left\| \frac{\partial \xi_1}{\partial n} - \frac{\partial \xi_2}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^{\frac{1}{2}}}.$$

On conclut en utilisant l'inégalité (1.24) et le fait que la fonction  $x \rightarrow \frac{1}{\ln(\frac{1}{x})}$  croît sur  $\mathbb{R}_+^*$ .

La preuve du Théorème 1.22 est très similaire (on utilise le Théorème 1.20 à la place du Théorème 1.18).

### 1.4.3 Système de Stokes avec des conditions aux limites mixtes de type Dirichlet/Robin

Soit  $\Omega \subset \mathbb{R}^d$ , avec  $d = 2, 3$ , un ouvert tel que  $\Omega = \Gamma_l \cup \Gamma_0 \cup \Gamma_{out}$  où  $\Gamma_{out} = \bigcup_{i=1}^N \Gamma_i$ . On considère le système suivant :

$$\begin{cases} -\Delta u + \nabla p & = 0, & \text{dans } \Omega, \\ \operatorname{div} u & = 0, & \text{dans } \Omega, \\ u & = 0, & \text{dans } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = g, & \text{sur } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu & = 0, & \text{sur } \Gamma_{out}. \end{cases} \quad (1.25)$$

L'une des différences principales par rapport au problème inverse traité dans la section précédente est que l'on a ajouté des conditions de Dirichlet homogène sur une partie du bord. De plus, on relaxe l'hypothèse  $\bar{\Gamma}_{out} \cap \bar{\Gamma}_0 = \emptyset$ . Une conséquence directe est une perte de régularité globale sur  $(u, p)$  solution du système (1.25).

Dans cette sous-section, on suppose que le coefficient de Robin  $q$  est constant par morceaux sur  $\Gamma_{out}$  :

$$q|_{\Gamma_i} = q_i, \text{ sur } \Gamma_i \text{ pour } i = 1, \dots, N,$$

avec  $(q_i)_{1 \leq i \leq N} \in \mathbb{R}_+^N$ .

Nous avons le résultat d'unicité suivant :

**Proposition 1.24.** Soit  $x_0 \in \Gamma_0$  et  $r > 0$ . On suppose que  $g$  est non identiquement nul sur  $\Gamma_0$ . Soit  $(u_k, p_k) \in H^1(\Omega) \times L^2(\Omega)$  solution du système (1.25) avec  $q = q^k$  pour  $k = 1, 2$ .

Si  $u_1 = u_2$  sur  $\mathcal{B}(x_0, r) \cap \Gamma_0$  alors  $q^1 = q^2$  sur  $\Gamma_{out}$ .

Concernant la stabilité, nous prouvons une inégalité de stabilité Lipschitz :

**Théorème 1.25.** On suppose que  $\Gamma_0$  est de classe  $\mathcal{C}^\infty$  et  $\Gamma_i$  est de classe  $\mathcal{C}^{2,1}$  pour  $i = 1, \dots, N$ . Soit  $(q_i^k)_{1 \leq i \leq N} \in \mathbb{R}_+^N$  pour  $k = 1, 2$  et  $\Gamma \subseteq \Gamma_0$  un ensemble ouvert non vide du bord de  $\Omega$  tel que  $(\bar{\Gamma} \cap \bar{\Gamma}_l) \cup (\bar{\Gamma} \cap \bar{\Gamma}_{out}) = \emptyset$ . On suppose que  $g$  est non identiquement nul sur  $\Gamma_0$ . Soit  $(u_k, p_k) \in H^1(\Omega) \times L^2(\Omega)$  la solution du système (1.25) avec  $q = q^k = \sum_{i=1}^N q_i^k \mathbb{1}_{\Gamma_i}$ , pour  $k = 1, 2$ .

Alors, il existe  $C > 0$  tel que :

$$\|q^1 - q^2\|_{L^\infty(\Gamma_{out})} \leq C \left( \|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right). \quad (1.26)$$

Donnons une idée de la preuve du Théorème 1.25. Considérons :

$$(w, \pi) = \left( \frac{u_1 - u_2}{\sum_{j=1}^N |q_j^1 - q_j^2|}, \frac{p_1 - p_2}{\sum_{j=1}^N |q_j^1 - q_j^2|} \right). \quad (1.27)$$

Puisque pour  $k = 1, 2$ ,  $q^k$  est constant par morceaux,  $(w, \pi)$  est solution de :

$$\left\{ \begin{array}{ll} -\Delta w + \nabla \pi & = 0, & \text{dans } \Omega, \\ \operatorname{div} w & = 0, & \text{dans } \Omega, \\ w & = 0, & \text{sur } \Gamma_l, \\ \frac{\partial w}{\partial n} - \pi n & = 0, & \text{sur } \Gamma_0, \\ \frac{\partial w}{\partial n} - \pi n + q_1 w & = \frac{(q^2 - q^1)}{\sum_{j=1}^N |q_j^1 - q_j^2|} u_2, & \text{sur } \Gamma_{out}. \end{array} \right. \quad (1.28)$$

Notons que ce n'est pas le cas en général quand  $q^k$  n'est pas constant par morceaux : il apparaît alors des termes en fonction de la dérivée de  $q_k$  dans le système (1.28).

Nous allons montrer qu'il existe  $C > 0$  tel que :

$$\|w\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \geq C,$$

ce qui, utilisant la Définition (1.27) de  $(w, \pi)$  impliquera directement l'inégalité (1.26). Pour cela, nous allons utiliser les estimations de continuation unique de la Sous-section 1.4.1.

Soit  $1 \leq i \leq N$  et  $x_i \in \left\{ x \in \Gamma_i / d(x, \overline{\partial\Omega \setminus \Gamma_i}) > 0 \right\}$ . Donnons une idée sur la façon dont l'information se transmet de  $\Gamma$  à un voisinage  $V_{x_i}$  de  $x_i$ . D'après la Proposition 1.26 suivante, on peut transmettre l'information de  $\Gamma$  à un ouvert relativement compact  $B_0$  inclus dans  $\Omega$ . Cette proposition s'obtient comme un corollaire de la Proposition 1.14.

**Proposition 1.26.** Soit  $\eta > 0$ . Supposons que  $\Gamma_0$  est de classe  $\mathcal{C}^\infty$ . Soit  $\Gamma \subset \Gamma_0$  un sous-ensemble ouvert non vide du bord de  $\Omega$  tel que  $(\bar{\Gamma} \cap \bar{\Gamma}_l) \cup (\bar{\Gamma} \cap \bar{\Gamma}_{out}) = \emptyset$  et  $\hat{\omega} \subset \Omega$  un ouvert relativement compact. Soit  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  la solution du système (1.25).

Alors, il existe  $C > 0$ ,  $0 < \delta < 1$  tel que l'inégalité suivante est satisfaite :

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq C \left( \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right)^\delta. \quad (1.29)$$

Puis nous utilisons une suite de boules  $(B(\zeta_{i,k}, \rho_k))_{k \in \mathbb{N}}$  avec des rayons décroissants pour approcher le bord du domaine en  $x_i$ . Nous appliquons une inégalité des trois boules, résumé dans le Lemme 1.27 à cette suite de boules.

**Lemme 1.27** (Inégalité des trois boules). *Soit  $\rho > 0$  et  $q \in \mathbb{R}^d$ . Il existe  $C > 0$ ,  $\alpha > 0$  tel que pour toute fonction  $(u, p) \in H^1(B(q, 8\rho)) \times H^1(B(q, 8\rho))$  solution dans  $B(q, 8\rho)$  de*

$$\begin{cases} -\Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \quad (1.30)$$

l'inégalité suivante est satisfaite :

$$\begin{aligned} & \|u\|_{H^1(B(q, 3\rho))} + \|p\|_{L^2(B(q, 3\rho))} \\ & \leq C \left( \|u\|_{H^1(B(q, \rho))} + \|p\|_{L^2(B(q, \rho))} \right)^\alpha \left( \|u\|_{H^1(B(q, 8\rho))} + \|p\|_{L^2(B(q, 8\rho))} \right)^{1-\alpha}, \end{aligned} \quad (1.31)$$

avec  $\alpha = \frac{g(3\rho) - g(\frac{7}{2}\rho)}{g(\frac{6}{3}\rho) - g(\frac{7}{2}\rho)}$  et avec  $g(r) = e^{-\lambda r^2}$  avec  $\lambda$  suffisamment large.

Soit  $\epsilon > 0$ . On s'approche du bord jusqu'à ce que  $B(\zeta_{i,k}, \rho_k) \subset B(x_i, \epsilon)$ . Notons  $k_0$  le premier entier tel que cette inclusion soit satisfaite. Utilisant ensuite les conditions aux limites sur  $\Gamma_i$  et le fait que  $(u, p) \in \mathcal{C}^{1,\beta}(B(x_i, \epsilon)) \times \mathcal{C}^{0,\beta}(B(x_i, \epsilon))$ , on obtient :

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} \leq C \left( |\nabla w(y_i)| + |\pi(y_i)| + |w(y_i)| + C\epsilon^\beta \right).$$

On intègre en norme  $L^2$  sur  $B(\zeta_{i,k_0}, \rho_{k_0})$  :

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} \leq \frac{C}{\rho_{k_0}^{\frac{d}{2}}} \left( \|w\|_{H^1(B(\zeta_{i,k_0}, \rho_{k_0}))} + \|\pi\|_{L^2(B(\zeta_{i,k_0}, \rho_{k_0}))} \right) + C\epsilon^\beta.$$

Sommant l'inégalité précédente par rapport à  $i$  pour  $i = 1, \dots, N$  puis combinant le Lemme 1.27 et la Proposition 1.26, on conclut par un argument d'optimisation. La Figure 1.11 illustre cette transmission d'information.

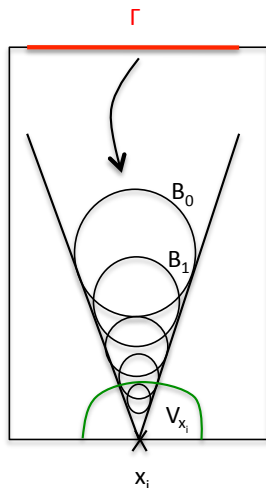


FIGURE 1.11 – Schéma représentant la transmission de l'information dans la preuve du Théorème 1.25.

## 1.5 Plan de le thèse

Dans cette section, nous résumons les différents chapitres constituant cette thèse. On rappelle que les cas (a) et (b) sont présentés au début de la section 1.4.

**Chapitre 2** : On s'intéresse au problème direct non couplé avec le ressort : on considère le système de Navier-Stokes avec des conditions aux limites dissipatives. Nous commençons par passer en revue quelques résultats d'existence concernant ce système. Puis nous montrons que si l'on contraint la vitesse à être normale au bord du domaine au niveau des sorties  $\Gamma_i$  et de l'entrée  $\Gamma_0$ , il est possible d'obtenir davantage de régularité sur la solution. Pour ce faire, on se ramène à prouver de la régularité sur le système suivant :

$$\left\{ \begin{array}{ll} -\mu\Delta u + \nabla p & = 0, \quad \text{dans } \Omega, \\ \operatorname{div} u & = 0, \quad \text{dans } \Omega, \\ u & = 0 \quad \text{sur } \Gamma_l, \\ \left( \mu \frac{\partial u}{\partial n} - pn \right) \cdot n & = 0, \quad \text{sur } \Gamma_i, \text{ pour } i = 0, \dots, N, \\ u \cdot \tau_k & = 0, \quad \text{sur } \Gamma_i, \text{ pour } i = 0, \dots, N, \text{ et } k = 1, \dots, d-1, \end{array} \right.$$

$\tau = (\tau_1, \dots, \tau_{d-1})$  étant  $d-1$  vecteurs tel que  $(n, \tau)$  est une base orthogonale dans  $\mathbb{R}^d$ . L'idée consiste à prolonger  $(u, p)$  en une solution d'un système de Stokes avec des conditions aux limites homogène et ce par symétrie par rapport à l'hyperplan  $\Gamma_i$  pour  $i = 0, \dots, N$ .

**Chapitre 3** : On quantifie des résultats de continuation unique pour le système de Stokes : on développe les résultats de la Sous-section 1.4.1.

**Chapitre 4** : On s'intéresse au cas (a) stationnaire. On commence par mentionner des résultats de régularité globale concernant le problème direct, on prouve un résultat

d'identifiabilité, puis l'on prouve deux inégalités de stabilité logarithmiques : l'une valable en dimension 2 et l'autre valable en toute dimension.

**Chapitre 5** : On s'intéresse au cas (a) non stationnaire. On commence par mentionner des résultats de régularité globale concernant le problème direct ainsi qu'un résultat d'identifiabilité. On introduit ensuite l'opérateur de Stokes associé au système et on montre qu'il engendre un semi-groupe analytique sur  $L^2$ . Puis nous prouvons deux inégalités de stabilité pour le problème non stationnaire, l'une valable en dimension 2 et l'autre valable en toute dimension. Notons que l'on distingue les cas où  $g$  dépend du temps et où  $g$  ne dépend pas du temps. Lorsque  $g$  dépend du temps, une hypothèse sur le comportement asymptotique de  $g$  quand  $t$  tend vers l'infini est nécessaire pour obtenir les inégalités de stabilité.

**Chapitre 6** : On s'intéresse au cas (b). On développe les résultats de la Sous-section 1.4.3 : on commence par mentionner des résultats de régularité locales, on énonce un résultat d'identifiabilité puis on prouve une inégalité de stabilité Lipschitzienne.

**Chapitre 7** : Dans ce chapitre, on cherche à revenir au modèle présenté dans la Sous-section 1.2.2, c'est-à-dire que l'on tente d'apporter une réponse aux questions (i) et (ii) énoncées à la fin de la Sous-section 1.2.3. Ce chapitre est divisé en deux parties : dans la première, nous nous intéressons à la résolution théorique du problème inverse concernant le système (1.6) lorsque l'on remplace les conditions aux limites de Robin (1.5) par des conditions aux limites dissipatives (1.1) sur  $\Gamma_{out}$  et dans l'autre nous nous intéressons à la résolution numérique du problème inverse concernant le système initial (1.2). Il s'agit d'un travail préliminaire, qui ouvre de nouvelles perspectives de recherche.

### Choix de la langue

La suite de cette thèse est rédigée en anglais. Une partie des résultats des chapitres 4 et 5 ont donné lieu à un article [16] actuellement accepté pour publication dans *Mathematical control and related field*. Le chapitre 3 et une partie des résultats présentés dans le chapitre 4 sont un projet de papier [17]. De même, le Chapitre 6 est également un projet de papier en cours de préparation [37]. Afin de garder une certaine cohérence dans la structure de cette thèse, nous avons choisi de rédiger les autres chapitres en anglais.





Part I

Mathematical analysis of the direct  
problem



## Chapter 2

# Mathematical analysis of the direct problem

In this chapter, we are interested in existence and regularity results concerning incompressible Navier–Stokes system with mixed Dirichlet and Neumann boundary conditions or dissipative boundary conditions that can be encountered in blood flow modeling or air flow modeling. To be more precise, the aim of this chapter is two-fold: on the one hand, we review some of the results that have been obtained concerning existence and regularity of solutions for the Navier–Stokes system with this type of boundary conditions, and on the other hand, we prove more regularity for the solution of this system in a special case, when we enforce the velocity to be normal to the outlets.

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## 2.1 Introduction

As we have seen in the introduction, the air flow in the proximal part of the bronchial tree can be modeled thanks to the Navier–Stokes equations coupled with  $0D$  models that represent the downstream part of the bronchial tree. In this section, we will focus on the simplest  $0D$  model, namely we will consider the Navier–Stokes equations with dissipative boundary conditions, and we will review some existence results as well as regularity properties verified by the solution.

Let  $N \in \mathbb{N}^*$ . Throughout this chapter will assume that  $\Omega \subset \mathbb{R}^d$  with  $d = 2, 3$  is a Lipschitz, bounded and connected open set such that

$$\partial\Omega = \Gamma_l \cup \left( \bigcup_{i=0}^N \Gamma_i \right) \text{ and } \bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset \text{ for } i \neq j, (i, j) \in \{0, \dots, N\}. \quad (2.1)$$

We denote by  $\mathbf{n}$  the exterior unit normal to  $\Omega$ .

Let  $T > 0$ . We consider the following Navier–Stokes system with dissipative boundary conditions:

$$\left\{ \begin{array}{ll} \rho u_t + \rho(u \cdot \nabla)u - \mu \Delta u + \nabla p = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0, T) \times \Omega, \\ u = 0, & \text{on } (0, T) \times \Gamma_l, \\ \mu \frac{\partial u}{\partial n} - pn = -P_0 n, & \text{on } (0, T) \times \Gamma_0, \\ \mu \frac{\partial u}{\partial n} - pn = -P_i n - R_i \left( \int_{\Gamma_i} u \cdot n \right) n, & \text{on } (0, T) \times \Gamma_i, \text{ for } i = 1, \dots, N, \\ u(0) = u_0, & \text{in } \Omega, \end{array} \right. \quad (2.2)$$

where  $R_i \geq 0$ , for  $i = 1, \dots, N$  is the resistance at the outlet  $\Gamma_i$ ,  $P_0, P_i$  are given pressures and  $\rho$  and  $\mu$  denote respectively the fluid density and its viscosity.

We are interested in existence and regularity results concerning system (2.2). We need to introduce functional spaces:

$$V_{\Gamma_l} = \{v \in H^1(\Omega) / v|_{\Gamma_l} = \mathbf{0} \text{ and } \operatorname{div} v = 0\}, \quad (2.3)$$

and

$$H_{\Gamma_l} = \overline{V}_{\Gamma_l}^{L^2(\Omega)}. \quad (2.4)$$

Note that we will not distinguish vector valued functions and scalar valued functions. Let us consider the variational formulation of system (2.2). We will say that  $u$  is a **weak solution** of system (2.2) on  $(0, T)$  if  $u \in L^2(0, T; V_{\Gamma_l})$  is such that for all  $v \in V_{\Gamma_l}$ , we have in the distributive sense on  $(0, T)$ :

$$\begin{aligned} \rho \frac{d}{dt} \int_{\Omega} u \cdot v + \rho \int_{\Omega} (u \cdot \nabla)u \cdot v + \mu \int_{\Omega} \nabla u : \nabla v \\ + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} v \cdot n \right) = - \sum_{i=0}^N P_i \left( \int_{\Gamma_i} v \cdot n \right), \end{aligned} \quad (2.5)$$

and for all  $v \in V_{\Gamma_l}$ ,

$$\int_{\Omega} u(0) \cdot v = \int_{\Omega} u_0 \cdot v,$$

where we denote by  $\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i=1}^d \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i$ .

To prove that the problem has at least one weak solution a first step is to derive energy estimates. In order to emphasize the difficulties which occur when dealing with the Navier–Stokes equations with Neumann boundary conditions, we focus on them.

Let us assume that  $u$  is a solution of (2.2) that is regular enough to perform all the following computations. We take  $v = u$  in the variational formulation (2.5). After integrations by parts, using the incompressibility of the fluid, it leads to:

$$\rho \int_{\Omega} \partial_t u \cdot u + \rho \int_{\Omega} (u \cdot \nabla) u \cdot u + \mu \int_{\Omega} |\nabla u|^2 + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right)^2 + \sum_{i=0}^N P_i \left( \int_{\Gamma_i} u \cdot n \right) = 0.$$

The first term can be rewritten in order to reveal the derivative of the kinetic energy of the fluid:

$$\rho \int_{\Omega} \partial_t u \cdot u = \frac{\rho}{2} \frac{d}{dt} \int_{\Omega} |u|^2 = \frac{d}{dt} E_{kf}.$$

In addition, for all  $0 \leq i \leq N$ :

$$P_i \left( \int_{\Gamma_i} u \cdot n \right) = P_i Q_i = -\tau_{P_i},$$

where  $\tau_{P_i}$  is the power developed by the force associated to the pressure  $P_i$  on  $\Gamma_i$ . By integrating by parts the nonlinear term, since  $u$  is divergence free, we obtain:

$$\rho \int_{\Omega} (u \cdot \nabla) u \cdot u = \frac{\rho}{2} \sum_{i=0}^N \int_{\Gamma_i} |u|^2 u \cdot n = \mathcal{F}_{kf}.$$

where  $\mathcal{F}_{kf}$  denotes the kinetic energy flux. Finally, we obtain:

$$\begin{aligned} & \underbrace{\frac{d}{dt} E_{kf}}_{\text{Time derivative of the kinetic energy}} = - \underbrace{\mu \int_{\Omega} |\nabla u|^2}_{\text{Energy dissipated by the fluid}} \\ & - \underbrace{\sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right)^2}_{\text{Energy dissipated by the resistive outlets}} + \underbrace{\sum_{i=0}^N \tau_{P_i}}_{\text{Pressure load}} - \underbrace{\mathcal{F}_{kf}}_{\text{Fluxes of kinetic energy}}. \end{aligned} \quad (2.6)$$

We know the sign of the two first terms in the right hand-side of (2.6) and we can easily estimate the powers  $\tau_{P_i}$  by using Lemma 2.4 below. The problem lies in the kinetic energy fluxes  $\mathcal{F}_{kf}$  whose sign is not known and which has to be estimated. This is a typical difficulty that arises when considering Navier–Stokes system with Neumann boundary conditions. Note that when considering Dirichlet boundary conditions, we have enough information to estimate it (furthermore, if you consider homogeneous Dirichlet boundary conditions, this term disappears). Note however that in dimension 2, it is possible to estimate  $\mathcal{F}_{kf}$  using Sobolev injection and interpolation inequalities, as we will see in Subsection 2.3.1. Nevertheless, this argument does not allow us to conclude in dimension 3. Thus, in the next Subsection 2.3.2, we will present three results which developed different strategies to estimate the nonlinear terms and prove existence results of weak or strong solutions.

## 2.2 Preliminaries: useful tools

Before reviewing existence results for system (2.2) we first introduce below some common tools that will be useful to prove these existence and regularity results. All these results will be used in various places in this thesis.

In the sequel, when it is not specified otherwise,  $C > 0$  is a generic constant whose value may change and we will use the following notation.

Let us denote by  $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$  the image of  $H^1(\Omega)$  by the trace operator and  $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$  the dual space of  $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ .

**Notation 2.1.** Let  $g \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$  and  $v \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ . We denote by  $\langle g, v \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega}$  the image of  $v$  by the linear form  $g$ .

**Proposition 2.2** (Stokes formula). *Let us introduce:*

$$E(\Omega) = \{u \in L^2(\Omega) / \operatorname{div} u \in L^2(\Omega)\}.$$

We have the Stokes formula:

$$\forall u \in E(\Omega), \forall w \in H^1(\Omega), \int_{\Omega} u \cdot \nabla w dx + \int_{\Omega} w \operatorname{div} u dx = \langle u \cdot n, w \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega}.$$

We refer to [21] for more details about this result.

Young Lemma will be very useful to obtain estimates necessary to prove existence of a solution in Galerkin method.

**Lemma 2.3** (Young inequality). *Let  $a, b \geq 0$  and  $(p, q)$  be two positive real numbers such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then, the following inequality holds:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

*Proof of Lemma 2.3.* This is a direct consequence of the convexity of the exponential function.  $\square$

Lemma 2.4 will be useful to estimate the flux of the fluid velocity at the outlet  $\Gamma_i$ , for  $i = 1, \dots, N$ .

**Lemma 2.4.** *Assume that  $\Omega$  satisfies (2.1) and let  $u \in H_{\Gamma_l}$ . Then,  $\int_{\Gamma_i} u \cdot n$  has a weak sense for all  $i = 0, \dots, N$ . Moreover, there exists  $c_0 > 0$  such that for all  $i = 0, \dots, N$ ,*

$$\left| \int_{\Gamma_i} u \cdot n \right| \leq c_0 \|u\|_{L^2(\Omega)}.$$

*Proof of Lemma 2.4.* Thanks to Stokes formula, the quantity  $\int_{\Gamma_i} u \cdot n$  has a weak sense by means of the standard duality as:

$$\langle u \cdot n, g_i \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega} = \int_{\Omega} u \cdot \nabla g_i, \quad (2.7)$$

where  $g_i \in H^1(\Omega)$  is any function such that  $g_i = 1$  on  $\Gamma_i$  and which vanishes on  $\Gamma_j$  for  $j \neq i$ ,  $j \in \{0, \dots, N\}$ . Such a function exists because  $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset$ , for  $i \neq j$ .



Note that if  $u \in V_{\Gamma_l}$ , then (2.7) coincides with  $\int_{\Gamma_i} u \cdot n$  because on the lateral boundary  $\Gamma_l$ ,  $u \cdot n$  vanishes. □

Let us recall Aubin Theorem which is very useful to pass to the limit in the nonlinear terms in the variational formulation.

**Theorem 2.5** (Aubin Theorem). *Let  $B_0 \subset B_1 \subset B_2$  be three Banach spaces. We assume that the first injection is continuous and compact and the second one is continuous. Let  $p, r$  be such that  $1 \leq p, r \leq +\infty$ . For  $T > 0$ , we denote by*

$$E_{p,r} = \left\{ v \in L^p(0, T; B_0) / \frac{dv}{dt} \in L^r(0, T; B_2) \right\}.$$

Then,

- (i) *If  $p < +\infty$ , the injection  $E_{p,r} \hookrightarrow L^p(0, T; B_1)$  is compact.*
- (ii) *If  $p = +\infty$  and if  $r > 1$ , the injection  $E_{p,r} \hookrightarrow C^0(0, T; B_1)$  is compact.*

We refer to [21] for a proof of Theorem 2.5.

A way to prove existence results for the Stokes problem is to look at it as a constraint problem and to use saddle point formulation. The following proposition gives the existence of solution for saddle point formulation in an abstract framework.

**Theorem 2.6.** *Let  $(X, \| \cdot \|_X)$  and  $(M, \| \cdot \|_M)$  be two Hilbert spaces. We denote by  $\langle \cdot, \cdot \rangle$  the duality pairing between the spaces  $X$  and  $X'$  or  $M$  and  $M'$ . Let us introduce two bilinear continuous forms:*

$$a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}, \quad b(\cdot, \cdot) : X \times M \rightarrow \mathbb{R}.$$

We associate with the bilinear form  $b$ , the linear operator  $B \in \mathcal{L}(X, M')$  defined by:

$$\langle Bv, \mu \rangle = b(v, \mu), \forall v \in X, \forall \mu \in M.$$

Let us consider the following problem: given  $l \in X'$  and  $\xi \in M'$ , find a pair  $(u, \lambda) \in (X, M)$  such that

$$\begin{aligned} a(u, v) + b(v, \lambda) &= \langle l, v \rangle, \forall v \in X, \\ b(u, \mu) &= \langle \xi, \mu \rangle, \forall \mu \in M. \end{aligned} \tag{2.8}$$

We set  $V = \ker(B)$ . Assume that there exists  $\alpha > 0$  such that  $a(v, v) \geq \alpha \|v\|_X^2$ , for all  $v \in V$  and that the operator  $B$  is surjective. Then, there exists a unique  $(u, \lambda) \in X \times M$  solution of (2.8).

We refer to [46] for a proof of Theorem 2.6.

Another way to prove existence results for Stokes system consists in using Lax-Milgram and De Rham Theorem.

**Theorem 2.7** (Lax Milgram Theorem). *Let  $(X, \| \cdot \|_X)$  be a Hilbert space and let us introduce a bilinear continuous form:*

$$a(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}.$$

Assume that  $a$  is coercive on  $X$ . Then, for all  $\phi \in X'$ , there exists a unique  $u \in X$  such that:

$$a(u, v) = \langle \phi, v \rangle, \forall v \in X.$$

We refer to [22] for a proof of Theorem 2.7 and to [21] for a proof of Theorem 2.8.

**Theorem 2.8** (De Rham Theorem). *Let  $f \in H^{-1}(\Omega)$  such that for all function  $\phi \in C_c^\infty(\Omega)$  satisfying  $\operatorname{div} \phi = 0$ , we have  $\langle f, \phi \rangle_{H^{-1}, H_0^1} = 0$ , where  $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$  denotes the duality pairing between the space  $H_0^1(\Omega)$  and its dual space  $H^{-1}(\Omega)$ . Then, there exists a unique function  $p \in L_0^2(\Omega) = \{p \in L^2(\Omega) / \int_\Omega p = 0\}$  such that  $f = \nabla p$ .*

Moreover the following Lemma will be useful to obtain an estimate on the pressure.

**Lemma 2.9.** *There exists a linear continuous operator  $\Pi$  from  $L_0^2(\Omega)$  onto  $H_0^1(\Omega)$  such that for all  $h \in L_0^2(\Omega)$ , the function  $u = \Pi(h)$  satisfies  $\operatorname{div} u = h$ .*

We refer to [21] for the proofs of Theorem 2.8 and Lemma 2.9.

Coercivity is a crucial property to obtain the existence of a solution. The following inequality provides this property.

**Proposition 2.10** (Generalized Poincaré inequality). *Let  $T : H^1(\Omega) \rightarrow \mathbb{R}$  be a linear form such that the image by  $T$  of a non zero constant function is non zero. Then, there exists a constant  $C_2 > 0$  such that:*

$$\|u\|_{L^2(\Omega)} \leq C_2 (|Tu| + \|\nabla u\|_{L^2(\Omega)}), \text{ for all } u \in H^1(\Omega).$$

*Proof of Proposition 2.10.* This proposition can be proved by contradiction (see [64]).  $\square$

Now let us recall some regularity results for the Stokes system with Dirichlet boundary conditions or Neumann boundary conditions.

**Proposition 2.11.** *Let  $k \in \mathbb{N}$ . Assume that  $\Omega$  is of class  $C^{k+1,1}$ . Let  $(u, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$  a be weak solution of:*

$$\begin{cases} -\Delta u + \nabla p &= f, & \text{in } \Omega, \\ \operatorname{div} u &= g & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

*Assume that:*

$$f \in H^k(\Omega) \text{ and } g \in H^{k+1}(\Omega).$$

*Then,  $(u, p)$  belongs to  $H^{k+2}(\Omega) \times H^{k+1}(\Omega)$  and there exists a constant  $C > 0$  such that:*

$$\|u\|_{H^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \leq C \left( \|g\|_{H^{k+1}(\Omega)} + \|f\|_{H^k(\Omega)} \right).$$

**Proposition 2.12.** *Let  $k \in \mathbb{N}$ . Assume that  $\Omega$  of class  $C^{k+1,1}$ . Let  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  be a weak solution of:*

$$\begin{cases} -\Delta u + \nabla p &= f & \text{in } \Omega, \\ \operatorname{div} u &= g, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn &= h, & \text{on } \partial\Omega, \end{cases}$$

*If we assume that:*

$$(f, g, h) \in H^k(\Omega) \times H^{k+1}(\Omega) \times H^{k+\frac{1}{2}}(\partial\Omega).$$

*Then,  $(u, p)$  belongs to  $H^{k+2}(\Omega) \times H^{k+1}(\Omega)$  and there exists a constant  $C > 0$  such that:*

$$\|u\|_{H^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \leq C \left( \|h\|_{H^{\frac{1}{2}+k}(\partial\Omega)} + \|g\|_{H^{k+1}(\Omega)} + \|f\|_{H^k(\Omega)} \right).$$

We refer to [21] for more details about the two previous results. For regularity results in the case of mixed boundary conditions in domain with corners and edges we refer to [66].

## 2.3 State of the art for the existence results

### 2.3.1 A result valid in dimension 2

In dimension 2, one can prove existence result globally in time under the assumption that the data are small enough. Let us explain the outline of the proof. The proof can be separated in several standard steps:

- Build  $(u_n)$  an approximated sequence of solutions;
- Obtain *a priori* bounds on  $(u_n)$  independent of  $n$ ;
- Pass to the limit in the sequence of approximated problems.

Since, in the case of Neumann boundary conditions, the second point is not straightforward we will focus mostly on the derivation of energy estimates. Moreover we will detail all these steps for the special case developed at Section 2.4.

Let  $\{a^k\}_{k \geq 0}$  be a family of functions which is complete and orthogonal both in  $H_{\Gamma_l}$  and  $V_{\Gamma_l}$ . Let  $n \in \mathbb{N}$ . We approximate  $V_{\Gamma_l}$  by the subspace  $\mathcal{V}_n = \text{Span}(a^i)_{1 \leq i \leq n}$ . Let us introduce the approximated variational formulation: find  $u_n(t) \in \mathcal{V}_n$  such that for all  $v \in \mathcal{V}_n$ ,

$$\begin{aligned} \rho \frac{d}{dt} \int_{\Omega} u_n \cdot v + \rho \int_{\Omega} (u_n \cdot \nabla) u_n \cdot v + \mu \int_{\Omega} \nabla u_n : \nabla v \\ + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u_n \cdot n \right) \left( \int_{\Gamma_i} v \cdot n \right) = - \sum_{i=0}^N P_i \left( \int_{\Gamma_i} v \cdot n \right), \end{aligned} \quad (2.10)$$

and for all  $v \in \mathcal{V}_n$ ,

$$\int_{\Omega} u(0) \cdot v = \int_{\Omega} u_0 \cdot v. \quad (2.11)$$

The nonlinear differential system (2.10) completed with the initial condition (2.11) has a unique maximal solution  $u_n = \sum_{i=1}^n \alpha_i a^i$  defined on some interval  $(0, t_n)$ . To get an estimate on  $u_n$  we take successively  $v = a^i$ , for  $i = 1, \dots, n$ , in the variational formulation (2.10) and multiply the corresponding equation by the coefficient  $\alpha_i$ . We have to estimate two kinds of terms: the approximated flux of kinetic energy and the powers developed by the force associated to  $P_i$ . It is easy to estimate the second one thanks to Lemma 2.4. We can control the flux of kinetic energy using the following lemma.

**Lemma 2.13.** *Let  $u \in V_{\Gamma_l}$ . Then, there exists  $C > 0$  such that:*

$$\left| \int_{\Omega} (u \cdot \nabla) u \cdot u \right| \leq \begin{cases} D_1 \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 & \text{if } d=2, \\ D_2 \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)}^{\frac{5}{2}} & \text{if } d=3. \end{cases}$$

*Proof of Lemma 2.13.* One can prove this lemma using Sobolev injection and interpolation inequality. We refer to [75] for a proof of this lemma.  $\square$

Using the first inequality of Lemma 2.13 (i.e. for  $d = 2$ ), we obtain:

$$\begin{aligned} \rho \frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + \mu \|\nabla u_n\|_{L^2(\Omega)}^2 + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u_n \cdot n \right)^2 \\ \leq \rho D_1 \|u_n\|_{L^2(\Omega)} \|\nabla u_n\|_{L^2(\Omega)}^2 + C \left( \sum_{i=0}^N |P_i| \right) \|u_n\|_{L^2(\Omega)}, \end{aligned} \quad (2.12)$$

By using Poincaré inequality and Young inequality for the second term in the right hand-side of inequality (2.12), we obtain:

$$\rho \frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|\nabla u_n\|_{L^2(\Omega)}^2 + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u_n \cdot n \right)^2 \leq \rho D_1 \|u_n\|_{L^2(\Omega)} \|\nabla u_n\|_{L^2(\Omega)}^2 + C f^2,$$

where

$$f(t) = \sum_{i=0}^N |P_i(t)|. \quad (2.13)$$

Assuming that the initial conditions is small enough and as long as  $\frac{\mu}{2} - \rho D_1 \|u_n\|_{L^2(\Omega)} > 0$ , the first term in the right hand-side is absorbed by the second one in the left hand-side. Consequently, thanks to Gronwall Lemma, the solution satisfies an energy estimate locally in time and for small enough initial data. Thus  $t_n$  can be taken independently of  $n$  and there exists  $t^*$  such that  $u_n$  is bounded uniformly with respect to  $n$  in  $L^2(0, t^*; V_{\Gamma_l}) \cap L^\infty(0, t^*; H_{\Gamma_l})$ . Standard arguments enable then to prove compactness in time of the sequence  $u_n$  in order to pass to the limit in the weak formulation and prove the existence of a weak solution.

Furthermore, one can obtain existence globally in time but with additional smallness assumptions of the initial data as well as on the applied forces. Indeed, if  $\|u_n\|_{L^2(\Omega)}$  is small enough (i. e. on every time interval such that  $\frac{\mu}{2} - \rho D_1 \|u_n\|_{L^2(\Omega)} \geq \frac{\mu}{4}$  for instance), using Poincaré inequality yields:

$$\rho \frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + C \frac{\mu}{4} \|u_n\|_{L^2(\Omega)}^2 \leq C f^2,$$

This inequality implies that, if the data are small enough, then the solution will verify  $\frac{\mu}{2} - \rho D_1 \|u_n\|_{L^2(\Omega)} \geq \frac{\mu}{4}$  at any time.

Note that Lemma 2.13 does not allow us to conclude in dimension 3 using the same reasoning. A different approach has to be developed to estimate the nonlinear terms in dimension 3 as we will see in the next subsection.

### 2.3.2 Result valid in dimension 3

Concerning the general case, we first refer to the work of J. G. Heywood, R. Rannacher and S. Turek [51]. In this paper, they study the Navier–Stokes system with the *so called* defective boundary conditions and in particular study the existence of solutions for the Navier–Stokes system with Neumann boundary conditions. Here we choose to describe three other results:

1. A. Quarteroni and A. Veneziani, 2003 ([69]),
2. C. Grandmont, B. Maury and A. Soualah, 2008 ([47]),
3. L. Baffico, C. Grandmont and B. Maury, 2010 ([7]).

The first one concerns the existence of strong solutions for the Navier–Stokes system with Robin like boundary conditions on the artificial boundaries and followed the same lines as in [51]. It relies in particular on an *a priori* assumption summarized in Lemma 2.15 and is valid for small enough resistances  $R_i$ . The second one deals with the existence of weak solutions for the special case where the fluid velocity at the outlets is proportional to a

given velocity. The additional constrain enables to estimate the flux of kinetic energy at the outlets. The third one concerns the full problem and the existence of strong solutions. It does not rely on an *a priori* assumption and is valid for any value of the resistances  $R_i$ .

We are going to present these three results chronologically.

### 1. A. Quarteroni and A. Veneziani, 2003.

In this paper the authors consider slightly different boundary conditions that write

$$\mu \frac{\partial u}{\partial n} - pn + R_i (u \cdot n) n = -P_i n,$$

on  $\Gamma_i$  for  $i = 1, \dots, N$ . Note that the variational formulation is also slightly different from (2.5):  $\sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} v \cdot n \right)$  is replaced by  $\sum_{i=1}^N R_i \int_{\Gamma_i} (u \cdot n)(v \cdot n)$ . These boundary conditions are used in the modeling of blood flow. In [69], local existence of strong solutions is proved under some condition on the size of the  $R_i$ 's coefficients. Moreover, the authors obtain global existence under additional assumptions of smallness of the data. To do this, they extend similar result holding for the mean pressure problem proved in [51] by J. G. Heywood, R. Rannacher and S. Turek.

More precisely, let us explain the outline of the proof and in particular the derivation of *a priori* estimates. First, we define the Stokes operator  $\tilde{\Delta}$  associated with mixed Neumann–Dirichlet homogeneous boundary conditions as introduced in [51]:

**Definition 2.14.** We define the set  $\mathcal{D}(\tilde{\Delta})$  as follows:

$$\mathcal{D}(\tilde{\Delta}) = \{u \in V_{\Gamma_l} / \exists C > 0, \forall v \in V_{\Gamma_l}, |(\nabla u, \nabla v)_{L^2(\Omega)}| \leq C \|v\|_{L^2(\Omega)}\},$$

and the operator  $\tilde{\Delta} : \mathcal{D}(\tilde{\Delta}) \subset H_{\Gamma_l} \rightarrow H_{\Gamma_l}$  by:

$$\forall u \in \mathcal{D}(\tilde{\Delta}), (\nabla u, \nabla v)_{L^2(\Omega)} = (\tilde{\Delta} u, v)_{L^2(\Omega)}, \forall v \in V_{\Gamma_l}. \quad (2.14)$$

One can prove that  $\tilde{\Delta}$  is well-defined and that:

- $\tilde{\Delta} \in \mathcal{L}(\mathcal{D}(\tilde{\Delta}), H_{\Gamma_l})$  is invertible and its inverse is compact on  $H_{\Gamma_l}$ .
- $\tilde{\Delta}$  is selfadjoint.

As a consequence  $\tilde{\Delta}$  admits a family of eigenvalues  $\{\tilde{a}^k\}_{k \geq 0}$  which is complete and orthogonal both in  $H_{\Gamma_l}$  and  $V_{\Gamma_l}$ . The  $\{\tilde{a}^k\}_{k \geq 0}$  will be chosen as a special basis for the Galerkin approximation of the problem.

The key argument to prove the existence of a strong solution is then the following inequality:

**Lemma 2.15.** There exists  $c_i > 0$ ,  $i = 1, 2$ , such that, for  $w \in \mathcal{D}(\tilde{\Delta})$ ,

$$\|w\|_{L^\infty(\Omega)} \leq c_1 \|\nabla w\|_{L^2(\Omega)}^{\frac{1}{2}} \|\tilde{\Delta} w\|_{L^2(\Omega)}^{\frac{1}{2}} \quad (2.15)$$

and

$$\|\nabla w\|_{L^2(\Omega)} \leq c_2 \|\tilde{\Delta} w\|_{L^2(\Omega)} \quad (2.16)$$

It is asserted in [51] that the first estimate may be valid for bounded two or three-dimensional domains such that  $\Gamma_l$  and  $\Gamma_i$  intersect with a right angle in analogy with a result for the Laplacian proved by W. Xie in [78]. Unfortunately, estimate (2.15) holds if  $\mathcal{D}(\tilde{\Delta}) \subset H^2(\Omega)$  and V. Maz'ya and J. Rossmann proved in a very recent paper [66] that this is not the case in dimension 2 or 3. This is the reason why L. Baffico, C. Grandmont and B. Maury prove a slightly different estimate in [7] based on the fact that  $\mathcal{D}(\tilde{\Delta}) \subset H^{\frac{3}{2}+\epsilon}(\Omega)$ , for some  $\epsilon > 0$  (see Lemma 2.21). Note that the argument developed in [7] could apply here, leading to the existence of strong solutions. Moreover we will see that in the special case studied in Section 2.4 the result of Lemma 2.15 is valid.

Let  $n \in \mathbb{N}$ . We approximate  $V_{\Gamma_l}$  by the subspace  $\mathcal{V}_n = \text{Span}(\tilde{a}^i)_{1 \leq i \leq n}$ . Let us introduce the approximated variational formulation: find  $u_n(t) \in \mathcal{V}_n$  such that for all  $v \in \mathcal{V}_n$ ,

$$\begin{aligned} \rho \int_{\Omega} \partial_t u_n \cdot v + \rho \int_{\Omega} (u_n \cdot \nabla) u_n \cdot v + \mu \int_{\Omega} \nabla u_n : \nabla v \\ + \sum_{i=1}^N R_i \int_{\Gamma_i} (u_n \cdot n)(v \cdot n) = - \sum_{i=0}^N P_i \left( \int_{\Gamma_i} v \cdot n \right), \end{aligned} \quad (2.17)$$

and for all  $v \in \mathcal{V}_n$ ,

$$\int_{\Omega} u(0) \cdot v = \int_{\Omega} u_0 \cdot v. \quad (2.18)$$

We take  $v = \tilde{\Delta} u_n(t)$  in the approximated variational formulation (2.17), which is admissible thanks to the choice of the Galerkin basis. Indeed,  $\tilde{\Delta} u_n(t) \in \mathcal{V}_n$ . Note that, by construction of the operator  $\tilde{\Delta}$ ,  $\int_{\Omega} \nabla u_n : \nabla(\tilde{\Delta} u_n) = \|\tilde{\Delta} u_n\|_{L^2(\Omega)}^2$  and  $\int_{\Omega} \partial_t u_n \cdot \tilde{\Delta} u_n = \frac{1}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2$ .

Let us denote by  $R = \max_{1 \leq i \leq N} R_i$ . We estimate  $\int_{\Gamma_i} (u_n \cdot n)(\tilde{\Delta} u_n \cdot n)$  by using the following lemma.

**Lemma 2.16.** *There exists a positive constant  $c_3$  such that for all  $w \in \mathcal{D}(\tilde{\Delta})$ ,*

$$\left| \sum_{i=1}^N R_i \int_{\Gamma_i} (w \cdot n)(\tilde{\Delta} w \cdot n) \right| \leq c_3 R \|\nabla w\|_{L^2(\Omega)} \|\tilde{\Delta} w\|_{L^2(\Omega)}.$$

*Proof of Lemma 2.16.* Let  $w \in \mathcal{D}(\tilde{\Delta})$ . Thanks to Stokes formula, we have for all  $v \in H^1(\Omega)$ :

$$\sum_{i=0}^N \int_{\Gamma_i} (\tilde{\Delta} w \cdot n) v = \int_{\Omega} v \operatorname{div}(\tilde{\Delta} w) + \int_{\Omega} \nabla v \cdot \tilde{\Delta} w = \int_{\Omega} \nabla v \cdot \tilde{\Delta} w, \quad (2.19)$$

since  $\tilde{\Delta} w$  is a solenoidal function. Then, let us consider:

$$\begin{cases} \Delta \phi = 0, & \text{in } \Omega, \\ \phi = R_i w \cdot n, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \\ \phi = 0, & \text{on } \Gamma_l \cup \Gamma_0. \end{cases}$$

Note that  $\phi|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$  because  $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset$  for  $i \neq j$ . Thus, we deduce that  $\phi \in H^1(\Omega)$  and there exists  $C > 0$  such that  $\|\nabla \phi\|_{L^2(\Omega)} \leq CR \|w \cdot n\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq CR \|\nabla w\|_{L^2(\Omega)}$ , the latter inequality being a consequence of Poincaré and trace inequalities.

To conclude the proof, it is sufficient to apply equality (2.19) with  $v = \phi$  and to apply Cauchy-Schwarz inequality.  $\square$

By using Lemmas 2.15 and 2.16, we obtain:

$$\left| \sum_{i=1}^N R_i \int_{\Gamma_i} (u_n \cdot n)(\tilde{\Delta}u_n \cdot n) \right| \leq c_2 c_3 R \|\tilde{\Delta}u_n\|_{L^2(\Omega)}^2. \quad (2.20)$$

We estimate the other boundary terms without difficulty using Lemma 2.4 and Young inequality:

$$\sum_{i=0}^N |P_i| \left| \int_{\Gamma_i} \tilde{\Delta}u_n \cdot n \right| \leq \frac{\mu}{4} \|\tilde{\Delta}u_n\|_{L^2(\Omega)}^2 + \frac{C}{\mu} f^2, \quad (2.21)$$

where  $f$  is defined in (2.13). Finally, let us emphasize how the nonlinear terms are bounded. Thanks to Cauchy-Schwarz inequality, we have:

$$\left| \int_{\Omega} (u_n \cdot \nabla)u_n \cdot \tilde{\Delta}u_n \right| \leq \|u_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^2(\Omega)} \|\tilde{\Delta}u_n\|_{L^2(\Omega)}.$$

Using inequality (2.15), we obtain:

$$\left| \int_{\Omega} (u_n \cdot \nabla)u_n \cdot \tilde{\Delta}u_n \right| \leq c_1 \|\nabla u_n\|_{L^2(\Omega)}^{\frac{3}{2}} \|\tilde{\Delta}u_n\|_{L^2(\Omega)}^{\frac{3}{2}}. \quad (2.22)$$

Using moreover Young inequality, we obtain:

$$\left| \rho \int_{\Omega} (u_n \cdot \nabla)u_n \cdot \tilde{\Delta}u_n \right| \leq \frac{C\rho^2}{\mu^3} \|\nabla u_n\|_{L^2(\Omega)}^6 + \frac{\mu}{4} \|\tilde{\Delta}u_n\|_{L^2(\Omega)}^2. \quad (2.23)$$

Let us denote by  $\kappa = \mu - 2c_2 c_3 R$  and assume that  $\kappa > 0$  (which means that the resistances  $R_i$  are small enough for all  $i = 1, \dots, N$ ). By combining (2.20), (2.21) and (2.23), we obtain:

$$\rho \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \kappa \|\tilde{\Delta}u_n\|_{L^2(\Omega)}^2 \leq \frac{C}{\mu} f^2 + \frac{C\rho^2}{\mu^3} \|\nabla u_n\|_{L^2(\Omega)}^6.$$

Then, using nonlinear Gronwall lemma, it leads to estimate locally in time: there exists  $t^* > 0$  such that the solution  $u_n$  is bounded uniformly with respect to  $n$  in  $L^2(0, t^*; \mathcal{D}(\tilde{\Delta})) \cap L^\infty(0, t^*; V_{\Gamma_l})$ . Next by taking  $\partial_t u_n$  as a test function in the variational formulation, we obtain that it is bounded uniformly in  $L^2(0, t^*; H_{\Gamma_l})$ . These estimates allow to obtain (by applying Aubin Theorem) compactness to pass to the limit in the variational formulation and prove existence of a strong solution defined over  $(0, t^*)$ .

One can also obtain an existence result globally in time. Indeed, notice that by combining inequalities (2.16) and (2.22), we obtain:

$$\left| \int_{\Omega} (u_n \cdot \nabla)u_n \cdot \tilde{\Delta}u_n \right| \leq c_1 \sqrt{c_2} \|\nabla u_n\|_{L^2(\Omega)} \|\tilde{\Delta}u_n\|_{L^2(\Omega)}^2. \quad (2.24)$$

Thus, by combining (2.20), (2.21) and (2.24), we obtain:

$$\rho \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + (\kappa - 2c_1 \sqrt{c_2} \|\nabla u_n\|_{L^2(\Omega)}) \|\tilde{\Delta}u_n\|_{L^2(\Omega)}^2 \leq \frac{C}{\mu} f^2$$

If we assume that at initial time  $(\kappa - 2c_1\sqrt{c_2}\rho\|\nabla u_n(0)\|_{L^2(\Omega)}) \geq \frac{\kappa}{2}$  then, as long as  $(\kappa - 2c_1\sqrt{c_2}\rho\|\nabla u_n\|_{L^2(\Omega)}) \geq \frac{\kappa}{2}$  we have

$$\rho \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \frac{\kappa}{2} \|\tilde{\Delta} u_n\|_{L^2(\Omega)}^2 \leq \frac{C}{\mu} f^2$$

Consequently, (2.16) implies

$$\rho \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \frac{\kappa}{2c_2} \|\nabla u_n\|_{L^2(\Omega)}^2 \leq \frac{C}{\mu} f^2$$

It proves, under well chosen smallness assumption on the data (initial conditions and exterior pressures) and using Gronwall lemma, that the solution will satisfy  $(\kappa - 2c_1\sqrt{c_2}\rho\|\nabla u_n\|_{L^2(\Omega)}) \geq \frac{\kappa}{2}$  at any time. The global existence in time follows.

### C. Grandmont, B. Maury and A. Soualah, 2008.

This result concerns the existence of weak solutions. In order to estimate the nonlinear terms in [47], C. Grandmont, B. Maury and A. Soualah assume that, at each outlet  $\Gamma_i$ , the trace of the fluid velocity is proportional to a given velocity profile  $U_i$  for  $i = 0, \dots, N$ . The proportionality coefficient  $\lambda_i$  measures the velocity flux at the interface and is an unknown of the problem. To be more precise, the boundary conditions on  $\Gamma_i$  are replaced by:

$$\begin{cases} u & = \lambda_i U_i \\ \int_{\Gamma_i} \mu(\nabla u \cdot n - pn) \cdot U_i & = -P_i \left( \int_{\Gamma_i} U_i \cdot n \right) - R_i \left( \int_{\Gamma_i} U_i \cdot n \right), \end{cases}$$

for  $i = 0, \dots, N$  and with  $R_0 = 0$ .

For such a system, the authors proved that there exist a time  $t^* > 0$  and a weak solution  $(u, (\lambda_i)_{0 \leq i \leq N}) \in L^2(0, T^*; V_{\Gamma_i}) \cap L^\infty(0, t^*; H_{\Gamma_i}) \times L^\infty(0, t^*)$ .

Let us give an outline of the proof and in particular focus on the derivation of *a priori* estimates. To do so, we introduce the space

$$W = \{(u, (\lambda_i)_{0 \leq i \leq N}) \in V \times \mathbb{R}^{N+1} / u = \lambda_i U_i \text{ on } \Gamma_i \text{ for } i = 0, \dots, N\}.$$

The key lemma is the following:

**Lemma 2.17.** *For all  $j \in \{0, \dots, N\}$  such that  $U_j \cdot n$  is not identically equal to zero over  $\Gamma_j$ , there exists a constant  $C_1 > 0$  such that for all  $(u, (\lambda_i)_{0 \leq i \leq N}) \in W$ , we have:*

$$|\lambda_j| \leq C_1 \|u\|_{L^2(\Omega)}.$$

Let us consider  $(w_i)_{i \in \mathbb{N}}$  a Galerkin basis of the space  $W$ . Let  $n \in \mathbb{N}$  and  $\mathcal{V}_n = \text{Span}(w_i)_{0 \leq i \leq n} \subseteq W$ . The approximated variational formulation admits a solution  $(u_n, \Lambda_n)$ . By taking  $v = u_n(t)$  as a test function and estimating the nonlinear terms using Lemma 2.17:

$$\left| \sum_{i=0}^n \int_{\Gamma_i} |u_n|^2 u_n \cdot n \right| \leq \sum_{i=0}^n |\lambda_i|^3 \int_{\Gamma_i} |U_i|^2 |U_i \cdot n| \leq K_1 C_1^3 \|u_n\|_{L^2(\Omega)}^3, \quad (2.25)$$



with  $K_1 = \sum_{i=0}^N \int_{\Gamma_i} |U_i|^2 |U_i \cdot n|$ , leads to:

$$\frac{\rho}{2} \frac{d}{dt} \|u_n\|_{L^2(\Omega)}^2 + \mu \|\nabla u_n\|_{L^2(\Omega)}^2 + \sum_{i=1}^N R_i (\lambda_i)^2 \left( \int_{\Gamma_i} U_i \cdot n \right)^2 \leq C \left( \|u_n\|_{L^2(\Omega)}^3 + \|u_n\|_{L^2(\Omega)}^2 + \sum_{i=1}^N |P_i|^2 \right).$$

Thanks to nonlinear Gronwall lemma, *a priori* bounds locally in time follows. Next additional bounds can be derived in a standard way and compactness enable to pass to the limit in the nonlinear terms. The existence of a weak solution defined locally in time on  $(0, t^*)$  follows.

Furthermore, once again, one can prove that the solution is global provided that the data are small enough.

Note that the key point here is that one has a finite number of degree of freedom that define the fluid velocity at the outlets. It enables to control the kinetic fluid fluxes at the interface and thus the nonlinear term.

### L. Baffico, C. Grandmont and B. Maury, 2010.

In this paper, an existence result for the Navier–Stokes equations (2.2) coupled with a differential equation representing the motion of the thoracic cage, is proved. This coupled system has been introduced in Chapter 1 as system (1.2) and we have already enunciated the regularity result in Theorem 1.3: the authors obtain the existence of at least one strong solution defined locally in time for any data and, in the special case where the spring stiffness is equal to zero, defined globally in time, provided that the data are small enough. Let us present briefly the main steps of the proof of the local existence in the special case  $m = 0$ .

The idea is to introduce a Stokes operator associated to the special boundary conditions on  $\Gamma_i$ ,  $i = 0, \dots, N$  and whose eigenvectors will constitute the Galerkin basis used to build the sequence of approximate solutions. By doing this, no smallness assumption on the resistances  $R_i$  will be needed to derive *a priori* bounds.

More precisely, we denote by

$$a_{R,\mu}(u, v) = \mu \int_{\Omega} \nabla u : \nabla v + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} v \cdot n \right). \quad (2.26)$$

**Definition 2.18.** *The operator  $A$  is defined on  $H_{\Gamma_l}$  as follows:*

$$\mathcal{D}(A) = \{v \in V_{\Gamma_l} / \exists C > 0, |a_{R,\mu}(v, w)| \leq C \|w\|_{L^2(\Omega)}, \forall w \in V_{\Gamma_l}\},$$

$$(Av, w)_{L^2(\Omega)} = a_{R,\mu}(v, w), \forall (v, w) \in \mathcal{D}(A) \times V_{\Gamma_l}.$$

Let us make the link between the Stokes operator  $A$  and a solution of Stokes system. We can notice that  $Au = f$  if and only if there exists a pressure field  $p$  over  $\Omega$  such that  $(u, p)$  is weak solution to the following Stokes system:

$$\begin{cases} -\mu \Delta u + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_l, \\ \mu \frac{\partial u}{\partial n} - pn = 0, & \text{on } \Gamma_0, \\ \mu \frac{\partial u}{\partial n} - pn = -R_i \left( \int_{\Gamma_i} u \cdot n \right) n, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N. \end{cases} \quad (2.27)$$

For the Stokes system (2.27) a regularity result can be proved under some geometrical assumption on the open set  $\Omega$ . This is summarized in the following proposition.

**Proposition 2.19.** *We assume that the  $\Gamma_i$ 's meet with lateral boundary  $\Gamma_l$  at angle  $\frac{\pi}{2}$ . Let  $A$  be as in Definition 2.18. There exist  $\epsilon > 0$  and a constant  $C > 0$  such that, for any  $f \in H_{\Gamma_l}$ ,  $Au = f$ , it holds:*

$$\|u\|_{H^{\frac{3}{2}+\epsilon}(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \quad (2.28)$$

**Remark 2.20.** *The idea is to reduce the system (2.27) to the case of a Stokes system with mixed Neumann and Dirichlet homogeneous boundary conditions, thanks to a change of variables in the pressure. Then, since  $\Gamma_i$ 's meet the lateral boundary  $\Gamma_l$  with the angle  $\frac{\pi}{2}$ , a regularity result for the Stokes system with mixed Neumann and Dirichlet homogeneous boundary condition [66] implies Proposition 2.19.*

Next, one can prove that  $A$  satisfies the following properties:

1.  $A \in \mathcal{L}(\mathcal{D}(A), H_{\Gamma_l})$  is invertible and its inverse is compact on  $H_{\Gamma_l}$ .
2.  $A$  is selfadjoint.

As a consequence  $A$  admits a family of eigenvalues  $\{\phi_j\}_{j \geq 0}$  which is complete and orthogonal both in  $H_{\Gamma_l}$  and  $V_{\Gamma_l}$ . Then, the key argument to estimate the nonlinear terms is the following inequality:

**Lemma 2.21.** *Under the geometrical assumption of Proposition 2.19, there exists  $C > 0$  such that the following inequality holds:*

$$\|v\|_{L^\infty(\Omega)} \leq C\|Av\|_{L^2(\Omega)}, \quad v \in \mathcal{D}(A),$$

and there exists  $\theta \in (0, 1)$  such that:

$$\|v\|_{L^\infty(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}^\theta \|Av\|_{L^2(\Omega)}^{1-\theta}.$$

*Proof of Lemma 2.21.* The first estimate is a direct consequence of the injection  $H^{\frac{3}{2}+\epsilon}(\Omega) \hookrightarrow L^\infty(\Omega)$  and (2.28). Let  $0 < \epsilon' < \epsilon$ . We have

$$\|v\|_{H^{\frac{3}{2}+\epsilon'}(\Omega)} \leq \|v\|_{H^1(\Omega)}^\theta \|v\|_{H^{\frac{3}{2}+\epsilon}(\Omega)}^{1-\theta},$$

with  $\frac{3}{2} + \epsilon' = \theta + (1 - \theta)(\frac{3}{2} + \epsilon)$ . Then, the desired inequality follows from (2.28) and Poincaré inequality.  $\square$

Let  $n \in \mathbb{N}$  and  $\mathcal{V}_n = \text{Span}(\phi_i)_{0 \leq i \leq n}$ . We take  $v = Au_n(t)$  in the approximated variational formulation. By construction of  $A$ ,  $a_{R,\mu}(u_n, Au_n) = \|Au_n\|_{L^2(\Omega)}^2$ . Let us remark that, thanks to the Definition 2.18 of  $A$  where appears the resistances  $R_i$ , no assumption is needed on the size of the resistances to obtain the existence result. Let us detail the way the nonlinear terms are bounded:

$$\left| \int_{\Omega} (u_n \cdot \nabla) u_n \cdot Au_n \right| \leq \|u_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^2(\Omega)} \|Au_n\|_{L^2(\Omega)}.$$

Then, thanks to Lemma 2.21 and Young inequality, we obtain:

$$\begin{aligned} \left| \int_{\Omega} (u_n \cdot \nabla) u_n \cdot Au_n \right| &\leq C\|\nabla u_n\|_{L^2(\Omega)}^{1+\theta} \|Au_n\|_{L^2(\Omega)}^{2-\theta}, \quad \text{with } \theta \in (0, 1), \\ &\leq C\|\nabla u_n\|_{L^2(\Omega)}^{2(1+\theta)/\theta} + \frac{1}{4}\|Au_n\|_{L^2(\Omega)}^2. \end{aligned}$$

The term  $\frac{1}{4}\|Au_n\|_{L^2(\Omega)}^2$  will be absorbed in the left hand-side by  $\|Au_n\|_{L^2(\Omega)}^2$ . Thanks to nonlinear Gronwall lemma, there exists  $t^* > 0$  such that  $u_n$  is uniformly bounded in  $L^2(0, t^*; \mathcal{D}(A)) \cap L^\infty(0, t^*; V_{\Gamma_l})$ . Moreover by taking  $\partial_t u_n$  as a test function (and under suitable assumptions on the smoothness of the initial data), we obtain that  $\partial_t u_n$  is bounded in  $L^2(0, t^*; H_{\Gamma_l})$ . This leads to the existence of solutions locally in time: there exists  $t^* > 0$  such that  $u \in L^2(0, t^*; \mathcal{D}(A)) \cap L^\infty(0, t^*; V_{\Gamma_l}) \cap H^1(0, t^*; H_{\Gamma_l})$ . Note that  $\mathcal{D}(A)$  is only included in  $H^{\frac{3}{2}+\epsilon}(\Omega)$ .

Once again, one can also obtain existence globally in time for small data. The same approach is developed in the proof of Proposition 2.29, where we prove more regularity in space in a special case.

## 2.4 An existence and regularity result for specific boundary conditions

In this subsection, we will study a special case: we modify the boundary conditions at the outlets. This change will imply that the domain of the associated Stokes operator is embedded at least in  $H^2(\Omega)$ . Consequently an analogue of Lemma 2.15 will be satisfied and the solution  $u$  will belong to  $L^2(0, t; H^2(\Omega))$ .

We assume that  $\Gamma_i$  is contained in an hyperplane of  $\mathbb{R}^d$  and meets  $\Gamma_l$  at angle  $\frac{\pi}{2}$ , for  $i = 0, \dots, N$ .

Let  $\tau = (\tau_1, \dots, \tau_{d-1})$  be  $d - 1$  vectors such that  $(n, \tau)$  is an orthogonal basis of  $\mathbb{R}^d$ .

We modify the Neumann boundary: we replace the boundary condition

$$\frac{\partial u}{\partial n} - pn = -P_i n - R_i \left( \int_{\Gamma_i} u \cdot n \right) n, \quad \text{on } \Gamma_i, \quad (2.29)$$

by

$$\begin{cases} \left( \frac{\partial u}{\partial n} - pn \right) \cdot n &= -P_i - R_i \left( \int_{\Gamma_i} u \cdot n \right), & \text{on } \Gamma_i, \\ u \cdot \tau_k &= 0, & \text{on } \Gamma_i, \text{ for } k = 1, \dots, d-1, \end{cases} \quad (2.30)$$

for  $i = 1, \dots, N$  and with  $R_0 = 0$ .

Let us remark that by assuming  $u \cdot \tau_k = 0$  on  $\Gamma_i$  for  $k = 1, \dots, d-1$ , we constrain the velocity to be normal to  $\Gamma_i$ , for  $i = 1, \dots, N$ .

As explained in Remark 2.20, the fact that  $u \in H^{\frac{3}{2}+\epsilon}(\Omega)$ , for  $u$  solution of system (2.2), comes from a regularity result for the Stokes system with mixed homogeneous Neumann and Dirichlet boundary conditions. Here we will focus on the following Stokes system:

$$\begin{cases} -\mu \Delta u + \nabla p &= f, & \text{in } \Omega, \\ \operatorname{div} u &= 0, & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_l, \\ \left( \mu \frac{\partial u}{\partial n} - pn \right) \cdot n &= 0, & \text{on } \Gamma_i, \text{ for } i = 0, \dots, N, \\ u \cdot \tau_k &= 0, & \text{on } \Gamma_i, \text{ for } i = 0, \dots, N \text{ and for } k = 1, \dots, d-1. \end{cases} \quad (2.31)$$

We are going to prove more regularity in space than  $H^{\frac{3}{2}+\epsilon}(\Omega)$  for  $u$  solution of (2.31). Then, we will deduce from it more regularity in space on the solution of the following system

$$\left\{ \begin{array}{ll} \rho u_t + \rho(u \cdot \nabla)u - \mu \Delta u + \nabla p = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0, T) \times \Omega, \\ u = 0, & \text{on } (0, T) \times \Gamma_l, \\ \left( \mu \frac{\partial u}{\partial n} - pn \right) \cdot n = -P_0, & \text{on } (0, T) \times \Gamma_0, \\ \left( \mu \frac{\partial u}{\partial n} - pn \right) \cdot n = -P_i - R_i \left( \int_{\Gamma_i} u \cdot n \right), & \text{on } (0, T) \times \Gamma_i, \text{ for } i = 1, \dots, N, \\ u \cdot \tau_k = 0, & \text{on } (0, T) \times \Gamma_i, \text{ for } i = 0, \dots, N \\ & \text{and for } k = 1, \dots, d-1, \\ u(0) = u_0, & \text{in } \Omega. \end{array} \right. \quad (2.32)$$

### 2.4.1 Statement of the regularity results

We recall that  $V_{\Gamma_l}$  is defined in (2.3). Let us begin this section by introducing useful functional spaces:

$$\tilde{V} = \{u \in V_{\Gamma_l} / u \cdot \tau_k = 0 \text{ on } \Gamma_i, \text{ for } i = 0, \dots, N \text{ and } k = 1, \dots, d-1\}, \quad (2.33)$$

and

$$\tilde{H} = \overline{\tilde{V}}^{L^2(\Omega)}. \quad (2.34)$$

Concerning the homogeneous mixed Neumann–Dirichlet Stokes problem we can state the following proposition:

**Proposition 2.22.** *Let  $f \in L^2(\Omega)$ . There exists a unique  $(u, p) \in (H^2(\Omega) \cap \tilde{V}) \times H^1(\Omega)$  solution of Stokes system (2.31) and there exists  $C > 0$  such that:*

$$\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Moreover, if  $f$  belongs to  $H^1(\Omega)$  and is such that  $f \cdot \tau_k = 0$  on  $\Gamma_i$  for  $i = 0, \dots, N$  and  $k = 1, \dots, d-1$ , then  $(u, p) \in (H^3(\Omega) \cap \tilde{V}) \times H^2(\Omega)$  and there exists  $C > 0$  such that

$$\|u\|_{H^3(\Omega)} + \|p\|_{H^2(\Omega)} \leq C \|f\|_{H^1(\Omega)}.$$

The proof of Proposition 2.22 is developed in Subsection 2.4.3. We are now going to see how this regularity properties is transmitted to the solution of the Navier–Stokes system (2.32). Let us introduce the Stokes operator associated to the dissipative boundary conditions on  $\Gamma_i$  for  $i = 1, \dots, N$ .

**Definition 2.23.** *The operator  $\tilde{A}$  is defined on  $\tilde{H}$  as follows:*

$$\begin{aligned} \mathcal{D}(\tilde{A}) &= \{v \in \tilde{V} / \exists C > 0, |a_{R,\mu}(v, w)| \leq C \|w\|_{L^2(\Omega)}, \forall w \in \tilde{V}\}, \\ (\tilde{A}v, w)_{L^2(\Omega)} &= a_{R,\mu}(v, w), \forall (v, w) \in \mathcal{D}(\tilde{A}) \times \tilde{V}, \end{aligned} \quad (2.35)$$

where  $a_{R,\mu}$  is defined by (2.26).

**Remark 2.24.** Let us have a word about the definition of the operator  $\tilde{A}$ . For all  $u \in \mathcal{D}(\tilde{A})$ , we can extend the application  $\phi : v \in \tilde{V} \rightarrow a_{R,\mu}(u, v)$  in a continuous linear form on  $\tilde{H}$ . Indeed,  $\phi$  is continuous on  $\tilde{V}$  for the topology on  $\tilde{H}$  and  $\tilde{V}$  is dense in  $\tilde{H}$ . Thanks to Riesz-Fréchet Theorem, there exists a unique  $\tilde{A}u \in \tilde{H}$  such that  $(\tilde{A}u, w)_{L^2(\Omega)} = a_{R,\mu}(u, w)$  for all  $w \in \tilde{V}$ . This proves the well-posedness of the operator  $\tilde{A}$ .

Let us explain the link between the Stokes operator  $\tilde{A}$  and a solution of Stokes system. Let us notice that  $\tilde{A}u = f$  if and only if there exists a pressure field  $p$  such that  $(u, p)$  is a weak solution to the following Stokes system:

$$\left\{ \begin{array}{ll} -\mu\Delta u + \nabla p & = f, & \text{in } \Omega, \\ \operatorname{div} u & = 0, & \text{in } \Omega, \\ u & = 0 & \text{on } \Gamma_l, \\ \left( \mu \frac{\partial u}{\partial n} - pn \right) \cdot n & = 0, & \text{on } \Gamma_0, \\ \left( \mu \frac{\partial u}{\partial n} - pn \right) \cdot n & = -R_i \left( \int_{\Gamma_i} u \cdot n \right), & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N \\ u \cdot \tau_k & = 0, & \text{on } \Gamma_i, \text{ for } i = 0, \dots, N \text{ and for } k = 1, \dots, d-1. \end{array} \right. \quad (2.36)$$

System (2.36) is well-posed and that the following proposition is verified:

**Proposition 2.25.** Let  $R_M > 0$ ,  $f \in L^2(\Omega)$  and  $0 \leq R_i \leq R_M$ , for  $i = 1 \dots N$ . There exists a unique  $(u, p) \in \tilde{V} \times L^2(\Omega)$  solution of Stokes system (2.36). Moreover, there exists  $C > 0$  such that:

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(R_M) \|f\|_{L^2(\Omega)}. \quad (2.37)$$

We refer to Section 2.4.3 for a proof of this proposition. Coming back to the operator  $\tilde{A}$ , we have

**Proposition 2.26.** The operator  $\tilde{A}$  has the following properties:

1.  $\tilde{A} \in \mathcal{L}(\mathcal{D}(\tilde{A}), \tilde{H})$  is invertible and its inverse is compact on  $\tilde{H}$ .
2.  $\tilde{A}$  is selfadjoint.

As a consequence,  $\tilde{A}$  admits a family of eigenvectors  $\{\tilde{\phi}_j\}_{j \geq 0}$

$$\tilde{A}\tilde{\phi}_l = \lambda_l \tilde{\phi}_l \text{ with } 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \text{ and } \lim_{j \rightarrow \infty} \lambda_j = +\infty,$$

which is complete and orthogonal both in  $\tilde{H}$  and  $\tilde{V}$ . Moreover,  $\{\tilde{\phi}_j\}_{j \geq 0}$  is orthonormal in  $\tilde{H}$ .

*Proof of Proposition 2.26.* We are going to prove that  $\tilde{A}$  is maximal monotone, which will imply, since  $a_{R,\mu}$  is symmetric, that  $\tilde{A}$  is selfadjoint.

Firstly, for  $v \in \mathcal{D}(\tilde{A})$ , we have  $(\tilde{A}v, v)_{L^2(\Omega)} = a_{R,\mu}(v, v) \geq 0$  which proves that  $\tilde{A}$  is monotone. Then, let us prove that for all  $f \in \tilde{H}$ , there exists  $u \in \mathcal{D}(\tilde{A})$  such that for all  $v \in \tilde{V}$ :

$$\int_{\Omega} u \cdot v + \mu \int_{\Omega} \nabla u : \nabla v + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} v \cdot n \right) = \int_{\Omega} f \cdot v. \quad (2.38)$$

The bilinear form  $b : (u, v) \mapsto \int_{\Omega} u \cdot v + \mu \int_{\Omega} \nabla u : \nabla v + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} v \cdot n \right)$  is continuous and coercive on  $\tilde{V}$  and the linear form  $v \rightarrow \int_{\Omega} f \cdot v$  is continuous on  $\tilde{V}$ . Thanks

to Lax Milgram theorem, there exist a unique  $u \in \tilde{V}$  solution of (2.38) and  $C > 0$  such that:

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \quad (2.39)$$

From  $a_{R,\mu}(u, v) = \int_{\Omega}(f - u) \cdot v$  and (2.39), we deduce that  $u \in \mathcal{D}(\tilde{A})$ .

Let  $f \in \tilde{H}$ , we want to find  $u \in \tilde{V}$  such that  $a_{R,\mu}(u, v) = (f, v)_{L^2(\Omega)}$  for all  $v \in \tilde{V}$  (note that if such a  $u$  exists, it belongs to  $\mathcal{D}(\tilde{A})$ ). One can easily verified that the assumptions of the Lax Milgram Theorem are satisfied, which assures us the existence and uniqueness of such  $u$ . Therefore,  $\tilde{A} \in \mathcal{L}(\mathcal{D}(\tilde{A}), \tilde{H})$  is bijective, then it is invertible.

Moreover, the injection  $\mathcal{D}(\tilde{A}) \hookrightarrow \tilde{H}$  is compact. Thus  $\tilde{A}^{-1} : \tilde{H} \rightarrow \tilde{H}$  is compact too. Finally, the existence of a family of eigenvalues like in Proposition 2.26 relies on classical argument concerning compact operator. We refer to [22] or [70] for such result. To conclude, note that  $\lambda_1 > 0$  comes from  $\lambda_1 = (\tilde{A}\tilde{\phi}_1, \tilde{\phi}_1)_{L^2(\Omega)} = a_{R,\mu}(\tilde{\phi}_1, \tilde{\phi}_1) \geq \mu\|\tilde{\phi}_1\|_{L^2(\Omega)}^2 = \mu$ , where  $\mu$  is the coercivity constant associated with the bilinear form  $a_{R,\mu}$ .  $\square$

**Proposition 2.27.** *Let  $R_M > 0$ ,  $f \in \tilde{H}$  and assume that  $0 \leq R_i \leq R_M$  for  $i = 1, \dots, N$ . There exists a constant  $C(R_M) > 0$  such that, for any  $f \in \tilde{H}$  such that  $\tilde{A}u = f$ , it holds:*

$$\|u\|_{H^2(\Omega)} \leq C(R_M)\|f\|_{L^2(\Omega)}. \quad (2.40)$$

Moreover, for any  $f \in \tilde{V}$  such that  $\tilde{A}u = f$ , we have

$$\|u\|_{H^3(\Omega)} \leq C(R_M)\|f\|_{H^1(\Omega)}. \quad (2.41)$$

*Proof of Proposition 2.27.* This proposition will be obtained thanks to Proposition 2.22 by coming back to homogeneous mixed Dirichlet and Neumann boundary conditions. More precisely, we consider the auxiliary problem:

$$\begin{cases} \Delta \check{p} = 0, & \text{in } \Omega, \\ \frac{\partial \check{p}}{\partial n} = 0, & \text{on } \Gamma_l, \\ \check{p} = 0, & \text{on } \Gamma_0, \\ \check{p} = R_i \left( \int_{\Gamma_i} u \cdot n \right), & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N. \end{cases} \quad (2.42)$$

One can prove that there exists a unique  $\check{p} \in H^2(\Omega)$  solution of (2.42). We refer to [48] for the proofs of more general results concerning regularity of the solution of Laplacian problems with mixed Neumann and Dirichlet boundary conditions. Using moreover Lemma 2.4 and (2.37), we have that there exists  $C(R_M) > 0$  such that:

$$\|\check{p}\|_{H^2(\Omega)} \leq C(R_M)\|u\|_{L^2(\Omega)} \leq C(R_M)\|f\|_{L^2(\Omega)}.$$

Therefore,  $(u, \bar{p}) = (u, p - \check{p})$  satisfies:

$$\begin{cases} -\mu\Delta u + \nabla \bar{p} = f - \nabla \check{p}, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_l, \\ \left( \mu \frac{\partial u}{\partial n} - \bar{p}n \right) \cdot n = 0, & \text{on } \Gamma_0, \\ \left( \mu \frac{\partial u}{\partial n} - \bar{p}n \right) \cdot n = 0, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \\ u \cdot \tau_k = 0, & \text{on } \Gamma_i, \text{ for } i = 0, \dots, N \text{ and } k = 1, \dots, d-1. \end{cases}$$

Note that  $\nabla\check{p} \in \tilde{V}$ . Indeed  $\operatorname{div}(\nabla\check{p}) = \Delta\check{p} = 0$  and, since  $\check{p}$  is constant on  $\Gamma_i$ , then  $\nabla\check{p} \cdot \tau_k = \partial_k\check{p} = 0$  on  $\Gamma_i$  for  $i = 0, \dots, N$  and  $k = 1, \dots, d-1$ . We conclude by using Proposition 2.22.  $\square$

Then, we are going to use the Galerkin method to obtain existence of  $(u, p)$  strong solution of (2.32). The key argument to estimate the nonlinear terms is the following lemma, which is the analogue of Lemma 2.15:

**Lemma 2.28.** *There exists  $C > 0$  such that the following inequalities hold for all  $v \in \mathcal{D}(\tilde{A})$ :*

$$\|v\|_{L^\infty(\Omega)} \leq C\|\tilde{A}v\|_{L^2(\Omega)},$$

and

$$\|v\|_{L^\infty(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}^{\frac{1}{2}}\|\tilde{A}v\|_{L^2(\Omega)}^{\frac{1}{2}}. \quad (2.43)$$

*Proof.* The proof relies on the fact that  $\mathcal{D}(\tilde{A}) \subset H^2(\Omega)$  (which is embedded in  $L^\infty(\Omega)$ ) and on interpolations and Gagliardo inequalities and can be found in [30] in the case of homogeneous Dirichlet boundary conditions.  $\square$

We are now able to state existence of regular solution for the Navier–Stokes system (2.32).

**Proposition 2.29.** *Let  $T > 0$ ,  $P_0 \in L^2(0, T)$ ,  $P_i \in L^2(0, T)$ ,  $R_i \geq 0$ , for  $i=1, \dots, N$  and  $u_0 \in \tilde{V}$ .*

*Then, there exists  $t^* \in (0, T)$  such that  $u \in L^2(0, t^*; H^2(\Omega)) \cap L^\infty(0, t^*; \tilde{V}) \cap H^1(0, t^*; \tilde{H})$  is solution on  $(0, t^*)$ , for all  $v \in \tilde{V}$ , to:*

$$\begin{aligned} \rho \frac{d}{dt}(u, v)_{L^2(\Omega)} + a_{R, \mu}(u, v) + b(u, u, v) &= l_1(v), \\ (u(0), v)_{L^2(\Omega)} &= (u_0, v)_{L^2(\Omega)}, \end{aligned} \quad (2.44)$$

where  $b(u, v, w) = \rho \int_{\Omega} (u \cdot \nabla) v \cdot w$  and  $l_1(v) = -\sum_{i=0}^N P_i \left( \int_{\Gamma_i} v \cdot n \right)$ .

Moreover, if  $\|u_0\|_{H^1(\Omega)}$  and  $\sum_{i=0}^N |P_i(t)|$  are small enough, then the solution can be defined globally in time.

**Remark 2.30.** *We can rewrite the first line of (2.32) as:  $\tilde{A}u = -\rho(u \cdot \nabla)u - \rho u_t$ . To prove that  $u \in L^2(0, t^*; H^3(\Omega))$ , according to Proposition 2.27, we need that  $(u \cdot \nabla)u$  and  $u_t$  belong to  $L^2(0, t^*; \tilde{V})$ . By differentiating system (2.32), one could prove that  $u_t \in L^2(0, t^*; \tilde{V})$ , but it is not clear that  $f = (u \cdot \nabla)u$  satisfies  $f \cdot \tau_k = 0$  on  $\Gamma_i$  for  $i = 0, \dots, N$  and  $k = 1, \dots, d-1$ .*

*Proof of Proposition 2.29.* We follow the same steps as in the proof of the existence of a regular for the coupled Navier–Stokes system (1.2) proved in [7]. Note that we use Lemma 2.28 instead of Lemma 2.21 to estimate the nonlinear term. We focus on local and global regularity and we refer to [7] for the proof of uniqueness.

**Galerkin approximation** We recall that  $(\tilde{\phi}_i)_{i \in \mathbb{N}}$  was introduced in Proposition 2.26.

Let  $n \in \mathbb{N}$ . The approximated variational formulation is: find  $u_n(t) \in \mathcal{V}_n = \text{Span}(\tilde{\phi}_i)_{0 \leq i \leq n}$  such that

$$\rho \left( \frac{\partial u_n}{\partial t}, v \right)_{L^2(\Omega)} + a_{R,\mu}(u_n, v) + b(u_n, u_n, v) = l_1(v), \text{ for all } v \in \mathcal{V}_n, \quad (2.45)$$

$$(u_n(0) - u_0, v)_{L^2(\Omega)}, \text{ for all } v \in \mathcal{V}_n. \quad (2.46)$$

The nonlinear differential system (2.45) with the initial condition (2.46) has a maximal solution  $u_n$  defined on some interval  $(0, t_n)$ . We are going to prove that this system has a unique solution defined on  $(0, t^*)$  where  $t^*$  does not depend on  $n$ . We choose  $v = \tilde{A}u_n(t)$  in (2.45), which is admissible thanks to the choice of the Galerkin basis. Indeed,  $\tilde{A}u_n(t) \in \mathcal{V}_n$ . By construction of  $\tilde{A}$ , we have:

$$\left( \frac{\partial u_n}{\partial t}, \tilde{A}u_n \right)_{L^2(\Omega)} = \mu \frac{1}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{i=1}^N R_i \frac{d}{dt} \left( \int_{\Gamma_i} u_n \cdot n \right)^2,$$

and

$$a_{R,\mu}(u_n, \tilde{A}u_n) = \|\tilde{A}u_n\|_{L^2(\Omega)}^2.$$

Thus, by taking into account the two previous equalities, we obtain:

$$\frac{\mu}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{i=1}^N R_i \frac{d}{dt} \left( \int_{\Gamma_i} u_n \cdot n \right)^2 + \|\tilde{A}u_n\|_{L^2(\Omega)}^2 = -b(u_n, u_n, \tilde{A}u_n) + l_1(\tilde{A}u_n). \quad (2.47)$$

Let us estimate the right hand-side of equality (2.47). Using Lemma 2.28 and Young inequality, we have:

$$\begin{aligned} \left| b(u_n, u_n, \tilde{A}u_n) \right| &\leq \|u_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^2(\Omega)} \|\tilde{A}u_n\|_{L^2(\Omega)}, \\ &\leq C \|\nabla u_n\|_{L^2(\Omega)}^{\frac{3}{2}} \|\tilde{A}u_n\|_{L^2(\Omega)}^{\frac{3}{2}}, \\ &\leq C \|\nabla u_n\|_{L^2(\Omega)}^6 + \frac{1}{4} \|\tilde{A}u_n\|_{L^2(\Omega)}^2. \end{aligned}$$

Moreover, using Lemma 2.4 and Young inequality, we obtain:

$$\begin{aligned} |l_1(\tilde{A}u_n)| &\leq C \|\tilde{A}u_n\|_{L^2(\Omega)} \left( \sum_{i=0}^N |P_i| \right), \\ &\leq C \left( \sum_{i=0}^N |P_i| \right)^2 + \frac{1}{4} \|\tilde{A}u_n\|_{L^2(\Omega)}^2. \end{aligned}$$

It leads to:

$$\frac{d}{dt} \phi_n(t) + \psi_n(t) \leq C \phi_n^3(t) + f(t) \quad (2.48)$$

where  $f$  is defined in (2.13) and where we denote by

$$\phi_n(t) = \mu \|\nabla u_n(t)\|_{L^2(\Omega)}^2 + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u_n(t) \cdot n \right)^2, \quad (2.49)$$

and  $\psi_n(t) = \frac{1}{2} \|\tilde{A}u_n(t)\|_{L^2(\Omega)}^2$ . By choosing  $v = \tilde{A}_{R,\mu}u_n(0)$ , in (2.46), we obtain:

$$(u_n(0), \tilde{A}_{R,\mu}u_n(0))_{L^2(\Omega)} = (u_0, \tilde{A}_{R,\mu}u_n(0))_{L^2(\Omega)},$$

which leads, by construction of  $\tilde{A}$ , to:

$$\phi_n(0) = a_{R,\mu}(u_n(0), u_n(0)) = a_{R,\mu}(u_0, u_n(0)) \leq C \|u_0\|_{L^2(\Omega)}. \quad (2.50)$$



Let us denote by  $t^*$  a positive time such that the solution  $F$  to:

$$F'(t) = CF^3(t) + f(t) \text{ and } F(0) = C\|u_0\|_{L^2(\Omega)}.$$

is well-defined on  $(0, t^*)$ . Thanks to (2.48), the function  $\phi_n$  is a sub-solution of  $F$ . Then on  $(0, t_n)$ , we have:

$$\phi_n(t) = \mu\|\nabla u_n\|_{L^2(\Omega)}^2 + \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u_n \cdot n \right)^2 \leq F(t). \quad (2.51)$$

We deduce from non explosion theorem for differential equations that  $t^* \leq t_n$  (otherwise, we could extend  $\phi_n$ , which is in contradiction with  $t_n$  maximal). By integrating (2.48) over  $(0, t^*)$  we obtain, using (2.51):

$$\int_0^{t^*} \|\tilde{A}u_n\|_{L^2(\Omega)}^2 \leq C \left( \|F\|_{L^\infty(0, t^*)}^3 + \sum_{i=0}^N |P_i(t)|^2 + \|u_0\|_{H^1(\Omega)}^2 \right). \quad (2.52)$$

Thus, combining (2.51) and (2.52), we deduce that  $u_n$  is uniformly bounded in  $L^\infty(0, t^*; \tilde{V}) \cap L^2(0, t^*; \mathcal{D}(\tilde{A}))$ .

Let us now prove that  $\frac{\partial u_n}{\partial t}$  is uniformly bounded in  $L^2(0, t^*; L^2(\Omega))$ . Indeed, by choosing  $v = \frac{\partial u_n}{\partial t}$  in (2.45), we obtain:

$$\rho \left\| \frac{\partial u_n}{\partial t} \right\|_{L^2(\Omega)}^2 = -\rho \int_{\Omega} (u_n \cdot \nabla) u_n \cdot \frac{\partial u_n}{\partial t} - \left( \tilde{A}u_n, \frac{\partial u_n}{\partial t} \right)_{L^2(\Omega)} + l_1 \left( \frac{\partial u_n}{\partial t} \right),$$

and it follows

$$\left\| \frac{\partial u_n}{\partial t} \right\|_{L^2(\Omega)}^2 \leq C \left( \|u_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^2(\Omega)} + \|\tilde{A}u_n\|_{L^2(\Omega)} + f(t) \right) \left\| \frac{\partial u_n}{\partial t} \right\|_{L^2(\Omega)}.$$

We deduce, using Lemma 2.28, that:

$$\left\| \frac{\partial u_n}{\partial t} \right\|_{L^2(\Omega)} \leq C \left( \|\tilde{A}u_n\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla u_n\|_{L^2(\Omega)}^{\frac{3}{2}} + \|\tilde{A}u_n\|_{L^2(\Omega)} + f(t) \right).$$

Since  $u_n$  is uniformly bounded in  $L^\infty(0, t^*; \tilde{V}) \cap L^2(0, t^*; \mathcal{D}(\tilde{A}))$ , we obtain the desired result.

**Passage to the limit** Consequently, there exists a subsequence of  $(u_n)_{n \in \mathbb{N}}$ , also denoted  $(u_n)_{n \in \mathbb{N}}$ , such that:

$$\begin{cases} u_n \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(0, t^*; \tilde{V}), \\ \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{weakly in } L^2(0, t^*; L^2(\Omega)). \end{cases}$$

Thanks to Aubin compactness Theorem (see Theorem 2.5), we have, up to a subsequence:

$$u_n \rightarrow u \text{ in } L^2(0, t^*; L^2(\Omega)).$$

It enables us to pass to the limit in the approximated variational formulation and to obtain existence locally in time, of at least one solution  $u$  of (2.44).

**Global existence** We come back to equation (2.47). Using Lemma 2.4, Young inequality

and Lemma 2.28, we obtain:

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{i=1}^N R_i \frac{d}{dt} \left( \int_{\Gamma_i} u_n \cdot n \right)^2 + \|\tilde{A}u_n\|_{L^2(\Omega)}^2 \\ & \leq \|u_n\|_{L^\infty(\Omega)} \|\nabla u_n\|_{L^2(\Omega)} \|\tilde{A}u_n\|_{L^2(\Omega)} + \|\tilde{A}u_n\|_{L^2(\Omega)} \left( \sum_{i=0}^N |P_i| \right) \\ & \leq C \|\nabla u_n\|_{L^2(\Omega)} \|\tilde{A}u_n\|_{L^2(\Omega)}^2 + C \left( \sum_{i=0}^N |P_i| \right)^2 + \frac{1}{2} \|\tilde{A}u_n\|_{L^2(\Omega)}^2, \end{aligned}$$

that can be rewritten as:

$$\mu \frac{d}{dt} \|\nabla u_n\|_{L^2(\Omega)}^2 + \sum_{i=1}^N R_i \frac{d}{dt} \left( \int_{\Gamma_i} u_n \cdot n \right)^2 + (1 - C \|\nabla u_n\|_{L^2(\Omega)}) \|\tilde{A}u_n\|_{L^2(\Omega)}^2 \leq C \left( \sum_{i=0}^N |P_i| \right)^2$$

Since  $\|\tilde{A}u_n\|_{L^2(\Omega)}^2 \geq C \|\nabla u_n\|_{L^2(\Omega)}^2$  (thanks to Proposition 2.22) and  $\left| \int_{\Gamma_i} u_n \cdot n \right| \leq C \|\nabla u_n\|_{L^2(\Omega)}$ , there exist constants  $B_1 > 0$ ,  $B_2 > 0$  and  $C > 0$  such that:

$$\frac{d}{dt} \phi_n + B_1 \left( 1 - B_2 \sqrt{\phi_n} \right) \phi_n \leq C f,$$

where  $\phi_n$  is defined in (2.49) and  $f$  in (2.13). Let  $\delta > 0$  and  $\tilde{t}_n \leq t^*$  be the maximal positive time such that on  $(0, \tilde{t}_n)$ ,

$$1 - B_2 \sqrt{\phi_n(t)} \geq \delta \Leftrightarrow \phi_n(t) \leq \left( \frac{1 - \delta}{B_2} \right)^2.$$

Note that under smallness assumption on the data  $u_0$ , we know that such a  $\tilde{t}_n$  exists. Assume that there exists  $K_1 > 0$  such that

$$C f(t) \leq K_1, \text{ for all } t > 0.$$

Then, we have on  $(0, \tilde{t}_n)$ ,

$$\frac{d}{dt} \phi_n(t) + B_1 \delta \phi_n(t) \leq K_1,$$

which implies, using Gronwall Lemma and inequality (2.50):

$$\phi_n(t) \leq C \|u_0\|_{L^2(\Omega)} e^{-B_1 \delta t} + \frac{K_1}{B_1 \delta} \left( 1 - e^{-B_1 \delta t} \right).$$

Then, if  $C \|u_0\|_{L^2(\Omega)} < \frac{1}{2} \left( \frac{1 - \delta}{B_2} \right)^2$  and  $K_1 < \frac{B_1 \delta}{2} \left( \frac{1 - \delta}{B_2} \right)^2$ , it leads, for  $t \in (0, \tilde{t}_n)$ , to:

$$\phi_n(t) < \left( \frac{1 - \delta}{B_2} \right)^2.$$

We deduce that, if the data are small enough, the solution is defined globally in time.  $\square$

## 2.4.2 Background and notations

In this section, we introduce some notations and properties which will be used in the next Subsection 2.4.3, which is dedicated to the proof of Proposition 2.22.

**Notation 2.31.** Let  $u = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}$  be a vectorial function. We will denote by  $\check{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_{d-1} \end{pmatrix}$ . As a consequence,  $u = \begin{pmatrix} \check{u} \\ u_d \end{pmatrix}$ .

**Notation 2.32.** Let  $x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^d$ . We will denote by  $x' = \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} \in \mathbb{R}^{d-1}$ . As a consequence,  $x = \begin{pmatrix} x' \\ x_d \end{pmatrix}$ .

**Notation 2.33.** Let us denote by  $\nabla_{x'} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_{d-1}} \end{pmatrix}$ .

**Notation 2.34.** Let  $0 \leq i \leq N$ . We denote by  $\gamma_i$  the reflexion through the hyperplane which contains  $\Gamma_i$ .

To prove Proposition 2.22, we extend  $(u, p)$  solution of (2.31) outside of  $\Omega$  using the reflexion  $\gamma_i$  through the hyperplane which contains  $\Gamma_i$  and we prove that the extension  $(U, P)$  is the weak solution of a Stokes system. To do so, Lemmas 2.35 and 2.36 will be very useful.

Let  $R > 0$ . Let us denote by

$$K_R = \{y = (y', y_d) \mid \frac{1}{2} \in \mathbb{R}^{d-1} \times \mathbb{R}^+ / \|y\| < R\}, \quad (2.53)$$

and

$$\tilde{K}_R = \{y = (y', y_d) \mid \frac{1}{2} \in \mathbb{R}^{d-1} \times \mathbb{R}^- / \|y\| < R\}.$$

**Lemma 2.35.** Let  $w \in H^1(K_R)$  and  $W$  defined as follows:

$$W(x', x_d) = \begin{cases} w(x', x_d), & \text{when } x \in K_R, \\ \tilde{w}(x', x_d) = w(x', -x_d), & \text{when } x \in \tilde{K}_R. \end{cases}$$

Then,  $w \in H^1(K_R \cup \tilde{K}_R)$ .

We refer to [22] for a proof of Lemma 2.35.

**Lemma 2.36.** Let  $w \in H^1(K_R)$  and  $W$  be defined as follows:

$$W(x', x_d) = \begin{cases} w(x', x_d), & \text{when } x \in K_R, \\ \tilde{w}(x', x_d) = -w(x', -x_d), & \text{when } x \in \tilde{K}_R. \end{cases}$$

Assume that

$$w = 0 \text{ on } \{x \in K_R / x_d = 0\}. \quad (2.54)$$

Then,  $w \in H^1(K_R \cup \tilde{K}_R)$ .

The proof of Lemma 2.36 is classical. Nevertheless, for the sake of completeness, we refer to Annex C for a proof of this lemma.

### 2.4.3 Proof of Proposition 2.22

Note that the existence of a solution  $(u, p) \in \tilde{V} \times L^2(\Omega)$  of system (2.31) is a special case of Proposition 2.25 with  $R_i = 0$  for  $i = 1, \dots, N$ . For the sake of completeness, we will give the proof of Proposition 2.25 below.

Then, we decompose the proof of Proposition 2.22 into two intermediate results, namely Lemma 2.38 and Lemma 2.39.

*Proof of Proposition 2.25.* The variational formulation of the problem is: find  $u \in \tilde{V}$  such that for all  $v \in \tilde{V}$ ,

$$a_{R,\mu}(u, v) = L(v), \quad (2.55)$$

where  $a_{R,\mu}$  is defined in (2.26) and  $L(v) = \int_{\Omega} f \cdot v$ . According to Poincaré inequality, the bilinear form  $a_{R,\mu}$  is coercive on  $\tilde{V}$ . On the other hand,  $L$  is a continuous linear form on  $\tilde{V}$ . Then, Lax-Milgram theorem implies that there exists a unique  $u \in \tilde{V}$  solution of (2.55). Moreover, there exists  $C > 0$  such that

$$\|u\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \quad (2.56)$$

We are now going to prove the existence and the uniqueness of  $\acute{p}$  in  $L_0^2(\Omega)$  using De Rham theorem. Then, we will prove that there exists  $C_0 \in \mathbb{R}$  such that  $(u, p) = (u, \acute{p} - C_0)$  satisfies the boundary conditions on  $\partial\Omega$ . It will imply the uniqueness of  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  weak solution of Stokes system (2.36).

Let us introduce the following linear form  $L_2$ :

$$\begin{aligned} L_2 : H_0^1(\Omega) &\rightarrow \mathbb{R}, \\ v &\rightarrow -\mu \int_{\Omega} \nabla u : \nabla v + \int_{\Omega} f \cdot v. \end{aligned}$$

It is continuous and it vanishes on  $\{\phi \in \mathcal{C}_c^\infty(\Omega) / \text{div } \phi = 0\}$ . Let  $v \in \mathcal{C}_c^\infty(\Omega)$ . According to De Rham theorem, there exists a unique pressure  $\acute{p}$  in  $L_0^2(\Omega)$  such that:

$$L_2(v) = \int_{\Omega} \acute{p} \text{div } v.$$

We denote by  $\sigma = -\mu \nabla u + \acute{p} Id$ . We infer from:

$$\left| \int_{\Omega} \sigma \cdot \nabla v \right| = \left| \int_{\Omega} f \cdot v \right| \leq C \|v\|_{L^2(\Omega)}, \quad \forall v \in \mathcal{C}_c^\infty(\Omega),$$

that  $\text{div } \sigma$  belongs to  $L^2(\Omega)$  and satisfies:

$$\int_{\Omega} \text{div } \sigma \cdot v = \int_{\Omega} f \cdot v.$$

The previous equality is valid for all  $v \in \mathcal{C}_c^\infty(\Omega)$ , so we have almost everywhere in  $\Omega$ :

$$\text{div}(-\mu \nabla u + \acute{p} Id) = f. \quad (2.57)$$

Let  $w \in \tilde{V}$ . By virtue of Stokes formula, we have:

$$\int_{\Omega} w \cdot \operatorname{div}(-\mu \nabla u + \dot{p} Id) = \left\langle \frac{1}{2} - \mu \frac{\partial u}{\partial n} + \dot{p} n, w \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} + \mu \int_{\Omega} \nabla w : \nabla u.$$

Then, we deduce from (2.57):

$$\int_{\Omega} f \cdot w = \left\langle \frac{1}{2} - \mu \frac{\partial u}{\partial n} + \dot{p} n, w \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} + \mu \int_{\Omega} \nabla w : \nabla u.$$

Substituting this identity in the variational formulation (2.55), we obtain  $\forall w \in \tilde{V}$ :

$$\left\langle \frac{1}{2} - \mu \frac{\partial u}{\partial n} + \dot{p} n, w \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} - \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} w \cdot n \right) = 0. \quad (2.58)$$

Let us denote by

$$\underline{V} = \{v \in H^{\frac{1}{2}}(\partial \Omega) / w|_{\Gamma_i} = 0 \text{ and } v \cdot \tau_k = 0 \text{ on } \Gamma_i \text{ for } i = 1, \dots, N \text{ and } k = 1, \dots, d-1\},$$

and by  $L_3$  the following linear form:

$$\begin{aligned} L_3 : \underline{V} &\rightarrow \mathbb{R}, \\ v &\rightarrow \left\langle \frac{1}{2} - \mu \frac{\partial u}{\partial n} + \dot{p} n, v \right\rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} - \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} v \cdot n \right). \end{aligned}$$

From (2.58), we have that  $L_3$  is zero against trace of functions in  $\tilde{V}$ .

Let  $\Psi \in \underline{V}$  be a function such that  $\int_{\partial \Omega} \Psi \cdot n = 0$ . There exists a unique couple  $(\phi, \pi) \in \tilde{V} \times L_0^2(\Omega)$  solution of the following Stokes system (we refer to [21] for a proof of this result):

$$\begin{cases} -\Delta \phi + \nabla \pi &= 0, & \text{dans } \Omega, \\ \operatorname{div} \phi &= 0, & \text{dans } \Omega, \\ \phi &= \Psi, & \text{sur } \partial \Omega. \end{cases}$$

Thus,  $\Psi$  is the trace of  $\phi$  which belongs to  $\tilde{V}$ . Therefore, we have for all  $\Psi \in \underline{V}$  such that  $\int_{\partial \Omega} \Psi \cdot n = 0$  that:

$$\langle L_3, \Psi \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} = 0.$$

Let us denote by  $\tilde{\Gamma}_{out} = \bigcup_{i=0}^N \Gamma_i$ ,  $\delta > 0$  and let us consider  $\Phi_0 = \chi n \in \underline{V}$ , where  $\chi \in \mathcal{C}_c^\infty(\tilde{\Gamma}_{out})$  is such that  $\chi \geq 0$  on  $\tilde{\Gamma}_{out}$  and  $\chi \geq \delta$  on  $\Gamma$ , where  $\Gamma$  is a non empty set included in  $\tilde{\Gamma}_{out}$ . We have  $\int_{\tilde{\Gamma}_{out}} \Phi_0 \cdot n = \int_{\tilde{\Gamma}_{out}} \chi \geq \delta |\Gamma|$ , where  $|\Gamma|$  is the measure of  $\Gamma$  in  $\mathbb{R}^{d-1}$ . Let  $\Phi \in \underline{V}$  and

$$\Phi_1 = \Phi - \frac{1}{\left| \int_{\tilde{\Gamma}_{out}} \Phi_0 \cdot n \right|} \left( \int_{\tilde{\Gamma}_{out}} \Phi \cdot n \right) \Phi_0. \quad (2.59)$$

By construction,  $\Phi_1 \in \underline{V}$  satisfies  $\int_{\partial \Omega} \Phi_1 \cdot n = 0$ . As a consequence,

$$\langle L_3, \Phi_1 \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} = 0,$$

we can rewrite it, using (2.59), as follows:

$$\langle L_3, \Phi \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} = \frac{1}{\left| \int_{\tilde{\Gamma}_{out}} \Phi_0 \cdot n \right|} \left( \int_{\tilde{\Gamma}_{out}} \Phi \cdot n \right) \langle L_3, \Phi_0 \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega}.$$

Let us denote by:

$$C_0 = \frac{1}{\left| \int_{\tilde{\Gamma}_{out}} \Phi_0 \cdot n \right|} \langle L_3, \Phi_0 \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega}. \quad (2.60)$$

For all function  $\Phi \in \mathbb{V}$ , we have:

$$\langle L_3, \Phi \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega} = \int_{\partial\Omega} \Phi \cdot (C_0 n).$$

It proves that  $L_3 = C_0 n$  in  $\mathbb{V}'$ . We see that if we set  $p = \acute{p} - C_0$ , we obtain:

$$\langle \frac{1}{2} - \mu \frac{\partial u}{\partial n} + pn, \Phi \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega} - \sum_{i=1}^N R_i \left( \int_{\Gamma_i} u \cdot n \right) \left( \int_{\Gamma_i} \phi \cdot n \right) = 0, \quad (2.61)$$

for all  $\Phi \in \mathbb{V}$ .

To conclude, let us prove the estimate on  $p$ . Thanks to Lemma 2.9, there exists  $w \in H_0^1(\Omega)$  such that  $\operatorname{div} w = \acute{p}$  and  $C > 0$  such that  $\|w\|_{H^1(\Omega)} \leq C \|\acute{p}\|_{L_0^2(\Omega)}$ . By virtue of Stokes formula, we have:

$$\int_{\Omega} w \cdot \operatorname{div}(-\mu \nabla u + \acute{p} Id) = \int_{\Omega} \nabla w : (\mu \nabla u - \acute{p} Id) = \mu \int_{\Omega} \nabla w : \nabla u - \int_{\Omega} \acute{p} \operatorname{div} w.$$

Then, using (2.57) and with the choice of  $w$ , we obtain:

$$\int_{\Omega} |\acute{p}|^2 = \mu \int_{\Omega} \nabla w : \nabla u - \int_{\Omega} w \cdot f.$$

We deduce from Cauchy-Schwarz inequality and (2.56) that

$$\|\acute{p}\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

It remains to estimate the constant  $C_0$ . To do so, reminding the definition (2.60) of  $C_0$ , we use the previous estimate on  $\acute{p}$ , Stokes formula, Lemma 2.4 and estimate (2.56). We deduce that there exists  $C(R_M) > 0$  such that

$$\|p\|_{L^2(\Omega)} \leq C(R_M) \|f\|_{L^2(\Omega)}.$$

□

**Remark 2.37.** Let  $0 \leq i \leq N$ . Modulo a change of basis, we can assume that  $n(x) = \begin{pmatrix} 0_{\mathbb{R}^{d-1}} \\ -1 \end{pmatrix}$ , for  $x \in \Gamma_i$ . Consequently, if  $u$  satisfies  $u \cdot \tau_k = 0$  on  $\Gamma_i$  for  $k = 1, \dots, d-1$ , it implies that  $\ddot{u} = 0$  on  $\Gamma_i$ , then  $\partial_j u_j = 0$  on  $\Gamma_i$  for  $j \in \{1, \dots, d-1\}$ . Since  $u$  is divergence free, we deduce that  $\partial_d u_d = 0$  on  $\Gamma_i$ . Thus, if  $(u, p)$  is solution of system (2.31),  $p = 0$  on  $\Gamma_i$ .

**Lemma 2.38.** Let  $0 \leq i \leq N$ ,  $x \in \bar{\Gamma}_i \cap \bar{\Gamma}_l$  and  $\omega$  be a Lipschitz open set such that  $\omega$  is the restriction to  $\Omega$  of a neighborhood of  $x$ . Let  $R > 0$  be small enough such that  $B(x, R) \cap \Omega \subset \omega$ . Let us denote by  $\tilde{\omega} = \gamma_i(\omega)$  and  $\Upsilon = \omega \cup \tilde{\omega}$ , where  $\gamma_i$  is defined in Notation 2.34.

Let  $(u, p)$  be solution of system (2.31). We extend  $u$ ,  $p$  and  $f$  in  $\tilde{\omega}$  as follows:

$$U(x', x_d) = \begin{cases} u(x', x_d) = \begin{pmatrix} \ddot{u}(x', x_d) \\ u_d(x', x_d) \end{pmatrix}, & \text{when } (x', x_d) \in \omega, \\ \tilde{u}(x', x_d) = \begin{pmatrix} -\ddot{u}(x', -x_d) \\ u_d(x', -x_d) \end{pmatrix}, & \text{when } (x', x_d) \in \tilde{\omega}, \end{cases} \quad (2.62)$$

$$P(x', x_d) = \begin{cases} p(x', x_d) & \text{when } (x', x_d) \in \omega, \\ \tilde{p}(x', x_d) = -p(x', -x_d), & \text{when } (x', x_d) \in \tilde{\omega}, \end{cases} \quad (2.63)$$

$$F(x', x_d) = \begin{cases} f(x', x_d) = \begin{pmatrix} \check{f}(x', x_d) \\ f_d(x', x_d) \end{pmatrix}, & \text{when } (x', x_d) \in \omega, \\ \tilde{f}(x', x_d) = \begin{pmatrix} -\check{f}(x', -x_d) \\ f_d(x', -x_d) \end{pmatrix}, & \text{when } (x', x_d) \in \tilde{\omega}. \end{cases} \quad (2.64)$$

Then, for all  $W \in \mathcal{C}_c^\infty(\Upsilon)$ , we have:

$$\mu \int_{\Upsilon} \nabla U : \nabla W - \int_{\Upsilon} P \operatorname{div} W = \int_{\Upsilon} F \cdot W. \quad (2.65)$$

Moreover,  $\operatorname{div} U = 0$ .

*Proof of Lemma 2.38.* We deduce from Lemmas (2.35) and (2.36) that  $U \in H^1(\Upsilon)$ . The fact that  $\operatorname{div} U = 0$  is an immediate consequence of the construction (2.62) of  $U$ . Then, we decompose the proof of (2.65) into two steps.

Step 1: We prove that  $(\tilde{u}, \tilde{p})$  satisfies for all  $\tilde{w} \in \mathcal{C}_c^\infty(\tilde{\omega})$ ,

$$\mu \int_{\tilde{\omega}} \nabla \tilde{u} : \nabla \tilde{w} - \int_{\tilde{\omega}} \tilde{p} \operatorname{div} \tilde{w} = \int_{\tilde{\omega}} \tilde{f} \cdot \tilde{w}. \quad (2.66)$$

Using the definition of  $\tilde{u}$  given by (2.62), we obtain:

$$\begin{aligned} & \int_{\tilde{\omega}} \nabla \tilde{u}(x', x_d) : \nabla \tilde{w}(x', x_d) dx' dx_d \\ &= \int_{\tilde{\omega}} -\nabla_{x'} \check{u}(x', -x_d) : \nabla_{x'} \check{w}(x', x_d) + \partial_d \check{u}(x', -x_d) \cdot \partial_d \check{w}(x', x_d) dx' dx_d \\ &+ \int_{\tilde{\omega}} \nabla_{x'} u_d(x', -x_d) \cdot \nabla_{x'} \tilde{w}_d(x', x_d) - \partial_d u_d(x', -x_d) \partial_d \tilde{w}_d(x', x_d) dx' dx_d. \end{aligned} \quad (2.67)$$

Let  $(x', x_d) \in \omega$ . We denote by  $\check{w}(x', x_d) = -\check{w}(x', -x_d)$  and  $w_d(x', x_d) = \tilde{w}_d(x', -x_d)$ . By performing the change of variables  $x_d = -x_d$  in (2.67), we get:

$$\int_{\tilde{\omega}} \nabla \tilde{u} : \nabla \tilde{w} = \int_{\omega} \nabla u : \nabla w.$$

We use now the fact that  $(u, p)$  is weak solution of (2.31):

$$\mu \int_{\tilde{\omega}} \nabla \tilde{u} : \nabla \tilde{w} = \int_{\omega} f \cdot w + \int_{\omega} p \operatorname{div} w.$$

Finally, performing another change of variables in the right hand-side of the previous equality, we directly obtain (2.66).

Let us remark that (2.66) implies that we have, almost everywhere in  $\tilde{\omega}$ :

$$\operatorname{div}(-\mu \nabla \tilde{u} + \tilde{p} Id) = \tilde{f}. \quad (2.68)$$

Indeed, let us denote by  $\tilde{\sigma} = -\mu \nabla \tilde{u} + \tilde{p} Id$ . We infer from:

$$\left| \int_{\tilde{\omega}} \tilde{\sigma} \cdot \nabla \tilde{w} \right| = \left| \int_{\tilde{\omega}} \tilde{f} \cdot \tilde{w} \right| \leq C \|\tilde{w}\|_{L^2(\tilde{\omega})}, \quad \forall \tilde{w} \in \mathcal{C}_c^\infty(\tilde{\omega}),$$

that  $\operatorname{div} \tilde{\sigma}$  belongs to  $L^2(\tilde{\omega})$  and satisfies:

$$\int_{\tilde{\omega}} \operatorname{div} \tilde{\sigma} \cdot \tilde{w} = \int_{\tilde{\omega}} \tilde{f} \cdot \tilde{w}.$$

The previous inequality is valid for all  $\tilde{w} \in C_c^\infty(\tilde{\omega})$ , so it implies (2.68).

Step 2: Let us now prove (2.65). Let  $W \in C_c^\infty(\Upsilon)$ . We have:

$$\mu \int_{\Upsilon} \nabla U : \nabla W - \int_{\Upsilon} P \operatorname{div} W = \mu \int_{\omega} \nabla u : \nabla W - \int_{\omega} p \operatorname{div} W + \mu \int_{\tilde{\omega}} \nabla \tilde{u} : \nabla W - \int_{\tilde{\omega}} \tilde{p} \operatorname{div} W.$$

Thanks to Stokes formula and since  $W \in C_c^\infty(\Upsilon)$ , we get:

$$\mu \int_{\omega} \nabla u : \nabla W - \int_{\omega} p \operatorname{div} W = \int_{\Omega} \operatorname{div}(-\mu \nabla u + p \operatorname{Id}) \cdot W + \langle \mu \frac{\partial u}{\partial n} - pn, W \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega}.$$

Moreover, thanks to a change of variable, we obtain:

$$\begin{aligned} \mu \int_{\tilde{\omega}} \nabla \tilde{u} : \nabla W - \int_{\tilde{\omega}} \tilde{p} \operatorname{div} W &= \mu \int_{\omega} \nabla u : \nabla \tilde{W} - \int_{\omega} p \operatorname{div} \tilde{W} \\ &= \int_{\Omega} \operatorname{div}(-\mu \nabla u + p \operatorname{Id}) \cdot \tilde{W} + \langle \mu \frac{\partial u}{\partial n} - pn, \tilde{W} \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega}, \end{aligned}$$

where  $\check{W}(x', x_d) = -\tilde{W}(x', -x_d)$  and  $\check{W}_d(x', x_d) = W_d(x', -x_d)$ . Combining the three previous equality, we obtain

$$\begin{aligned} \mu \int_{\Upsilon} \nabla U : \nabla W - \int_{\Upsilon} P \operatorname{div} W &= \int_{\omega} f \cdot W + \langle \mu \frac{\partial u}{\partial n} - pn, W \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} \\ &\quad + \int_{\omega} f \cdot \tilde{W} + \langle \mu \frac{\partial u}{\partial n} - pn, \tilde{W} \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega}. \end{aligned} \quad (2.69)$$

The functions  $(W \cdot n)n$  and  $(\tilde{W} \cdot n)n$  belong to  $\underline{V}$ , thus, thanks to (2.61) (with  $R_i = 0$  for  $i = 1, \dots, N$ ), we deduce that:

$$\langle \mu \frac{\partial u}{\partial n} - pn, (W \cdot n)n \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} = \langle \mu \frac{\partial u}{\partial n} - pn, (\tilde{W} \cdot n)n \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} = 0.$$

Moreover, by definition of  $\tilde{W}$ , we have for all  $k = 1, \dots, d-1$ ,

$$\langle \mu \frac{\partial u}{\partial n} - pn, (W \cdot \tau_k)\tau_k \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega} = - \langle \mu \frac{\partial u}{\partial n} - pn, (\tilde{W} \cdot \tau_k)\tau_k \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial \Omega}.$$

Then, taking into account the two previous equalities, the desired inequality (2.65) follows from (2.69), by performing another change of variables.  $\square$

**Lemma 2.39.** *Let  $0 \leq i \leq N$ ,  $x \in \bar{\Gamma}_i \cap \bar{\Gamma}_l$ . Let us denote by  $\Upsilon_R = (B(x, R) \cap \Omega) \cup (\gamma_i(B(x, R) \cap \Omega))$ . We have, for all  $R > 0$  small enough, that  $(U, P)$  defined by (2.62) and (2.63) is solution of:*

$$\begin{cases} -\mu \Delta U + \nabla P &= F, \\ \operatorname{div} U &= 0, \end{cases} \quad (2.70)$$

on  $\Upsilon_R$  and where  $F$  is defined in (2.64). Thus,  $(U, P)$  belongs to  $H^2(\Upsilon_R) \times H^1(\Upsilon_R)$  and there exists  $C_x > 0$  such that

$$\|U\|_{H^2(\Upsilon_R)} + \|P\|_{H^1(\Upsilon_R)} \leq C_x \|f\|_{L^2(\Omega)}.$$



Moreover, if  $f \in H^1(\Omega)$  is such that  $f \cdot \tau_k = 0$  on  $\Gamma_i$  for  $k = 1, \dots, d-1$ , then  $(U, P) \in H^3(\Upsilon_R) \times H^2(\Upsilon_R)$  and there exists  $C_x > 0$  such that

$$\|U\|_{H^3(\Upsilon_R)} + \|P\|_{H^2(\Upsilon_R)} \leq C_x \|f\|_{H^1(\Omega)}.$$

*Proof of Lemma 2.39.* We keep notations of Lemma 2.38. Let  $\tilde{\Upsilon}$  be an open set of class  $\mathcal{C}^{1,1}$  such that  $\Upsilon_R \subset \tilde{\Upsilon} \subset \Upsilon$  and  $\Xi \in \mathcal{C}_c^\infty(\tilde{\Upsilon})$  such that  $\Xi = 1$  on  $\Upsilon_R$ .

By performing integration by parts, one can prove that  $(V, Q) = (\Xi U, \Xi P)$  is solution of:

$$\begin{cases} -\mu \Delta V + \nabla Q &= g, & \text{in } \tilde{\Upsilon}, \\ \operatorname{div} V &= h, & \text{in } \tilde{\Upsilon}, \\ V &= 0 & \text{on } \partial \tilde{\Upsilon}, \end{cases} \quad (2.71)$$

where  $g = -\Delta \Xi U - 2\nabla U \nabla \Xi - P \nabla \Xi + \Xi F$  and  $h = \nabla \Xi \cdot U$ . Let us remark that, thanks to Lemmas 2.35 and 2.36,  $U$  belongs to  $H^1(\tilde{\Upsilon})$ . We conclude by using regularity result for the Stokes system with homogeneous Dirichlet boundary condition summarized in Proposition 2.11: we have  $(g, h) \in L^2(\tilde{\Upsilon}) \times H^1(\tilde{\Upsilon})$ , then  $(V, Q) \in H^2(\tilde{\Upsilon}) \times H^1(\tilde{\Upsilon})$ . Since  $\Xi = 1$ , on  $\Upsilon_R$ , we deduce that  $(U, P) \in H^2(\Upsilon_R) \times H^1(\Upsilon_R)$  and there exists  $C_x > 0$  such that

$$\|U\|_{H^2(\Upsilon_R)} + \|P\|_{H^1(\Upsilon_R)} \leq C_x \|f\|_{L^2(\Omega)}.$$

To prove that  $(U, P) \in H^3 \times H^2$  near  $x$ , we perform the same reasoning as previously, except the fact that we consider an open set  $\tilde{\Upsilon}$  of class  $\mathcal{C}^{2,1}$  such that  $\Upsilon_R \subset \tilde{\Upsilon} \subset \Upsilon$ . Then, we obtain that  $(U, P) \in H^3(\Upsilon_R) \times H^2(\Upsilon_R)$  and that there exists  $C_x > 0$  such that

$$\|U\|_{H^3(\Upsilon_R)} + \|P\|_{H^2(\Upsilon_R)} \leq C_x \|f\|_{H^1(\Omega)}.$$

□

We are now able to prove Proposition 2.22.

*Proof of Proposition 2.22.* In dimension 2,  $\bar{\Gamma}_i \cap \bar{\Gamma}_l$  consists of two corners, whereas in dimension 3,  $\bar{\Gamma}_i \cap \bar{\Gamma}_l$  is a closed curve. According to Lemma 2.39, there exists a neighborhood  $\omega_i$  of  $\bar{\Gamma}_i \cap \bar{\Gamma}_l$  such that  $(u, p) \in H^2(\omega_i \cap \Omega) \times H^1(\omega_i \cap \Omega)$ . Regularity is not an issue away from  $\bar{\Gamma}_i \cap \bar{\Gamma}_l$ . Thus, we obtain that  $(u, p) \in H^2(\Omega) \times H^1(\Omega)$ . To obtain the estimate, we work with a finite sub-cover of  $\Omega$ . The same reasoning works to prove that  $(u, p) \in H^3(\Omega) \times H^2(\Omega)$ . □

## 2.5 Conclusion

By constraining the velocity to be normal to the outlets, we obtained more regularity on the solution of the Stokes system with mixed homogeneous Dirichlet and Neumann boundary conditions. Moreover, we deduced from this result more regularity on  $(u, p)$  solution of the Navier–Stokes system (2.32) with mixed Dirichlet and dissipative boundary conditions. Note that regularity results are important to derive stability estimates for inverse problems.

In [38], the authors are concerned with blood flow simulations and consider different methods to prevent backflow divergence in finite element Navier–Stokes solvers. One of them consists in constraining the velocity to be normal to the outlet. Thus, it

could be interesting to know if this method also prevents backflow divergence in numerical simulation of 3D multi-scale model of the respiratory tract in realistic domain.



Part II

# Unique continuation estimates



## Chapter 3

# Stability estimates for the unique continuation properties of the Stokes system

In this chapter, we consider the incompressible Stokes system in a smooth and bounded open set. We propose a quantification of continuation results for the Stokes system. Following the vocabulary introduced by F. John [54] and borrowing techniques from K. D. Phung [68], we obtain two different types of stability estimates: a local Hölder stability estimate and two global logarithmic stability estimates. In both cases, boundary conditions are unknown. These results are established employing local Carleman estimates: an interior Carleman estimate and a Carleman estimate near the boundary.

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Results of this chapter have been obtained in collaboration with Muriel Boulakia and Céline Grandmont.



### 3.1 Introduction

Let  $n \in \mathbb{N}^*$  and  $\Omega \subset \mathbb{R}^n$  be a bounded and connected open set,  $\Omega$  being locally on one side of its boundary. We consider the Stokes system:

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{in } \Omega, \\ \operatorname{div} u &= 0, & \text{in } \Omega. \end{cases} \quad (3.1)$$

C. Fabre and G. Lebeau proved in [39] a unique continuation result for the unsteady Stokes equations with a non-smooth potential. In the particular case of the steady problem (3.1), their result is the following:

**Theorem 3.1.** *Let  $\omega$  be an nonempty open set in  $\Omega$  and  $(u, p) \in H_{loc}^1(\Omega) \times L_{loc}^2(\Omega)$  be a weak solution of system (3.1) satisfying  $u = 0$  in  $\omega$  then  $u = 0$  and  $p$  is constant in  $\Omega$ .*

We easily deduce from the previous theorem the following result.

**Corollary 3.2.** *Let  $x_0 \in \partial\Omega$ ,  $r > 0$ ,  $\gamma = \mathcal{B}(x_0, r) \cap \partial\Omega$  and  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  be a weak solution of system (3.1) satisfying  $u = 0$  and  $\frac{\partial u}{\partial n} - pn = 0$  on  $\gamma$ . Then  $u = 0$  and  $p = 0$  in  $\Omega$ .*

*Proof of Corollary 3.2.* We extend  $u$  and  $p$  by 0 on  $(\mathcal{B}(x_0, r) \cap \Omega^c)$ :

$$\tilde{u} \text{ (resp } \tilde{p}) = \begin{cases} u \text{ (resp } p), & \text{in } \Omega, \\ 0, & \text{in } (\mathcal{B}(x_0, r) \cap \Omega^c), \end{cases}$$

and we denote  $\tilde{\Omega} = \Omega \cup \mathcal{B}(x_0, r)$ . Let us verify that  $(\tilde{u}, \tilde{p}) \in H^1(\tilde{\Omega}) \times L^2(\tilde{\Omega})$  is still a solution of the Stokes equations in  $\tilde{\Omega}$ . Let  $v \in \mathcal{D}(\tilde{\Omega})$ . We check by integration by parts that:

$$\int_{\tilde{\Omega}} \nabla \tilde{u} \cdot \nabla v - \int_{\tilde{\Omega}} \tilde{p} \operatorname{div} v = 0.$$

Moreover  $\operatorname{div} \tilde{u} = 0$  in  $\tilde{\Omega}$ . Therefore, we can apply Theorem 3.1 to  $(\tilde{u}, \tilde{p})$ :  $(\tilde{u}, \tilde{p}) = (0, 0)$  in  $\tilde{\Omega}$  which implies that  $u = 0$  and  $p$  is constant in  $\Omega$ . At last, the fact that  $\frac{\partial u}{\partial n} - pn = 0$  on  $\gamma$  implies that  $p = 0$  in  $\Omega$ .  $\square$

**Remark 3.3.** *Since  $-\nabla u + pId \in L^2(\Omega)$  and  $\operatorname{div}(-\nabla u + pId) \in L^2(\Omega)$ ,  $\frac{\partial u}{\partial n} - pn = 0$  on  $\gamma$  has a sense in  $\left(H_{00}^{\frac{1}{2}}(\gamma)\right)'$ , where  $H_{00}^{\frac{1}{2}}(\gamma)$  denotes the subspace of  $H^{\frac{1}{2}}(\gamma)$  corresponding to the functions whose extension by zero on  $\partial\Omega \setminus \gamma$  belong to  $H^{\frac{1}{2}}(\partial\Omega)$ .*

The purpose of this Chapter is to obtain stability estimates for the unique continuation of the Stokes system (3.1). We want to emphasize that these estimates may not be optimal from the point of view of the unique continuation results stated previously (see for instance [58] for a paper dealing with optimal three balls inequality for the Stokes system) but have the advantage of providing an estimate both on  $u$  and  $p$ , which will be useful in Chapters 4 and 6 to solve the inverse problem of identifying a Robin coefficient defined on some part of the boundary from measurements available on another part of the boundary.

Firstly, we state a local stability result which allows to estimate the velocity and the pressure on a compact set. This inequality is of hölderian type:



**Theorem 3.4.** *Let  $\omega$  be a nonempty open set and  $K$  be a compact set, both included in  $\Omega$ . Then, there exists  $c > 0$  and  $0 < \beta < 1$  such that for all  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  solution of (3.1), we have:*

$$\|u\|_{H^1(K)} + \|p\|_{L^2(K)} \leq c \left( \|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)} \right)^\beta \left( \|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \right)^{1-\beta}. \quad (3.2)$$

Then, we are going to prove two global logarithmic estimates. In the first one, we estimate  $(u, p)$  solution of (3.1) in the  $H^1$ -norm on the whole domain with respect to the  $L^2$ -norm of  $(u|_\Gamma, p|_\Gamma)$  and  $(\frac{\partial u}{\partial n}|_\Gamma, \frac{\partial p}{\partial n}|_\Gamma)$ , where  $\Gamma$  is a part of the boundary of  $\Omega$ . In the second one, we obtain an estimate of  $(u, p)$  solution of (3.1) in the  $H^1$ -norm on the whole domain with respect to the  $H^1$ -norm of  $u$  and  $p$  in an open set  $\omega \subset \Omega$ . To be more specific, we prove the following theorem:

**Theorem 3.5.** *Assume that  $\Omega$  is of class  $C^\infty$ . Let  $0 < \nu \leq \frac{1}{2}$ . Let  $\Gamma$  be a nonempty open subset of the boundary of  $\Omega$  and  $\omega$  be a nonempty open set included in  $\Omega$ .*

*Then, there exists  $d_0 > 0$  such that for all  $\beta \in (0, \frac{1}{2} + \nu)$ , for all  $d > d_0$ , there exists  $c > 0$ , such that we have*

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left( \ln \left( d \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \|\frac{\partial u}{\partial n}\|_{L^2(\Gamma)} + \|\frac{\partial p}{\partial n}\|_{L^2(\Gamma)}} \right) \right)^\beta}, \quad (3.3)$$

and

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\left( \ln \left( d \frac{\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}}{\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}} \right) \right)^\beta}, \quad (3.4)$$

for all couple  $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$  solution of (3.1).

We can notice that our stability estimates require more measurements than the Fabre-Lebeau unique continuation result. Indeed, in inequality (3.2), we need information on  $u$  and  $p$  on  $\omega$  whereas the result in Theorem 3.1 only requires information on  $u$ . Moreover, note that, the constraint  $\frac{\partial u}{\partial n} - pn$  which appears in Corollary 3.2 is divided into two terms in inequality (3.3):  $\frac{\partial u}{\partial n}$  in one hand and  $pn$  in the other hand. There is also an additional term, the normal derivative of  $p$ .

In [68], K. D. Phung gives a quantification of Holmgren uniqueness theorem for the Laplace equation. The proof is based on local Carleman inequalities (inside the domain and near the boundary) coming from pseudo-differential calculus. He obtained a stability estimate of Hölder type in a subdomain which does not include a neighborhood of the boundary where limit conditions are unknown, and a stability estimate of logarithmic type in the whole domain.

We can make the connection between logarithmic stability estimate and the fact that the Cauchy problem is ill-posed in the sense of Hadamard (see [50]): the dependence of the solution on the measurements is of logarithm type. In particular, this leads to difficulties in the numerical resolution of the problem because a small error on the measurements may induce a large error on the solution and requires regularization (see for instance [20] for the Laplace equation and references therein and [13] for the Stokes system).

Let us describe the content of the chapter. The second section is dedicated to the statement of Carleman inequalities. Adapting the method of [68], we will use these Carleman inequalities to prove Theorems 3.4 and 3.5 in the third section. The proof of Theorem 3.5 is divided into three intermediate results which illustrate how the information spreads from a part of the boundary to another, whereas the proof of Theorem 3.4 is a direct consequence of one of the intermediate results. The fact that, in the right-hand side of inequality (3.2), we only need the  $L^2$ -norm of  $p$  is due to Caccioppoli inequality. As in [68], we use two kinds of local Carleman inequalities, one near the boundary and one in the interior of the open set  $\Omega$ . In each case, it consists in applying simultaneously the Carleman estimate to  $u$  and  $p$ , by using the fact that  $\Delta u = \nabla p$  and  $\Delta p = \operatorname{div}(\Delta u) = 0$ , in order to free ourselves from terms in the right-hand side of the inequalities. It is interesting to note that if we directly apply estimate coming from [68] to  $(u, p)$  solution of Stokes equations, and if we perform the same reasoning as explained above, we obtain  $\nabla p$  in  $L^2$  norm over all  $\Omega$  in the right-hand side of the inequality which we can not discard. Consequently, we can not prove Theorems 3.4 and 3.5 without going deeply in the heart of the proof. Note that using the fact that, since  $u$  is divergence free,  $p$  is a solution elliptic equation, is a classic idea. For instance, this idea is used by E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov and J.-P. Puel in [41] to prove a local exact controllability of the Navier-Stokes system.

If not specified otherwise,  $c$  is a generic constant, whose value may change and which only depends on the geometry of the open set  $\Omega$ .

For  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we denote by  $x' \in \mathbb{R}^{n-1}$  the  $n - 1$  first coordinates of  $x$ . We will also use the following notation:  $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n / x_n \geq 0\}$ .

## 3.2 Local Carleman inequalities

In this section, we recall local Carleman inequalities, firstly inside  $\Omega$ , then on the boundary of  $\Omega$ . These inequalities are based on Gårding inequality, which is itself a consequence of pseudo-differential calculus.

**Definition 3.6.** Let  $h > 0$ ,  $P$  be an operator and  $\phi \in C^\infty(\mathbb{R}^n)$ . Let us define the conjugate operator  $P_\phi = h^2 e^{\phi/2} \circ P \circ e^{-\phi/2}$  and  $p_\phi$  its principal symbol. We recall that the Poisson bracket between  $\operatorname{Re} p_\phi$  and  $\operatorname{Im} p_\phi$  is defined by:

$$\{\operatorname{Re} p_\phi, \operatorname{Im} p_\phi\} = \nabla_\xi \operatorname{Re} p_\phi \nabla_x \operatorname{Im} p_\phi - \nabla_x \operatorname{Re} p_\phi \nabla_\xi \operatorname{Im} p_\phi.$$

We say that  $\phi$  satisfies the Hörmander hypoellipticity property on  $K$  if:

$$\exists c_1 > 0, \forall (x, \xi) \in K \times \mathbb{R}^n, p_\phi(x, \xi) = 0 \Rightarrow \{\operatorname{Re} p_\phi, \operatorname{Im} p_\phi\}(x, \xi) \geq c_1. \quad (3.5)$$

**Proposition 3.7.** Let  $U$  be an open set in  $\mathbb{R}^n$  and  $K$  be a compact set included in  $U$ . Let us consider  $\phi \in C^\infty(\mathbb{R}^n)$  and  $P = \Delta$ . We assume that the function  $\phi$  satisfies

$$|\nabla \phi| > 0 \text{ in } U,$$

and the Hörmander hypoellipticity property on  $U$  given by (3.5).

Then, there exists  $c > 0$  and  $h_1 > 0$  such that for all  $h \in (0, h_1)$  and for all function  $y \in C_0^\infty(K)$ , we have

$$\int_K |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_K |\nabla y(x)|^2 e^{2\phi(x)/h} dx \leq ch^3 \int_K |\Delta y(x)|^2 e^{2\phi(x)/h} dx.$$

*Proof.* We refer to [52] for a proof of this inequality.  $\square$

**Remark 3.8.** We can extend the previous inequality to functions which belong to  $\{y \in H_0^1(K)/\Delta y \in L^2(K)\}$  by a density argument.

Locally near the boundary, we can go back to the half-space by a change of coordinates. We have the following Carleman inequality:

**Proposition 3.9.** Let  $K = \{x \in \mathbb{R}_+^n/|x| \leq R_0\}$  and  $\Sigma = \{x \in \partial K/x_n = 0\}$ . Let us denote by  $\mathcal{C}_{0,\partial K \setminus \Sigma}^\infty(\overline{K})$  the restriction to  $K$  of  $\mathcal{C}_0^\infty(\overline{B(0, R_0)})$  functions.

Let  $P$  be a second-order differential operator whose coefficients are  $\mathcal{C}^\infty$  in a neighborhood of  $K$ , defined by  $P(x, \partial_x) = -\partial_{x_n}^2 + R(x, \frac{1}{i}\partial_{x'})$  and  $\phi = \phi(x) \in \mathcal{C}^\infty$  be a function defined in a neighborhood of  $K$ . Let us denote by  $r(x, \xi')$  the principal symbol of  $R$  and assume that  $r(x, \xi') \in \mathbb{R}$  and that there exists a constant  $c > 0$  such that  $(x, \xi') \in K \times \mathbb{R}^{n-1}$ , we have  $r(x, \xi') \geq c|\xi'|^2$ .

We assume that the function  $\phi$  satisfies (3.5) and

$$\partial_{x_n}\phi(x) \neq 0, \forall x \in K.$$

Then, there exists  $c > 0$  and  $h_1 > 0$  such that for all  $h \in (0, h_1)$  and for all function  $y \in \mathcal{C}_{0,\partial K \setminus \Sigma}^\infty(\overline{K})$ , we have:

$$\begin{aligned} \int_{\mathbb{R}_+^n} |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{\mathbb{R}_+^n} |\nabla y(x)|^2 e^{2\phi(x)/h} dx &\leq ch^3 \int_K |P(x, \partial_x)y(x)|^2 e^{2\phi(x)/h} dx \\ &+ c \int_{\mathbb{R}^{n-1}} (|y(x', 0)|^2 + |h\partial_{x'}y(x', 0)|^2 + |h\partial_{x_n}y(x', 0)|^2) e^{2\phi(x', 0)/h} dx'. \end{aligned}$$

*Proof.* We refer to [56] for a proof of this inequality.  $\square$

**Remark 3.10.** Let us denote by  $H_{0,\partial K \setminus \Sigma}^{\frac{3}{2}+\nu}(K)$  the restriction to the set  $K$  of functions in  $H_0^{\frac{3}{2}+\nu}(B(0, R_0))$ . We can extend the previous inequality to functions which belong to  $H_{0,\partial K \setminus \Sigma}^{\frac{3}{2}+\nu}(K)$  by a density argument.

The key point to apply the previous Propositions consists in the construction of a function  $\phi$  which satisfies the Hörmander hypoellipticity property (3.5). The two following lemmas are proved in [68]. The first one gives an example of function which satisfies the hypoellipticity assumption inside an open set.

**Lemma 3.11.** Let  $0 < \delta < M$ ,  $\lambda > 0$  and  $q \in \mathbb{R}^n$ . The function  $\phi(x) = e^{-\lambda|x-q|^2}$  satisfies the Hörmander hypoellipticity property on the set  $\{(x, \xi) \in \mathbb{R}^n/\delta < |x-q| < M\}$  as soon as  $\lambda$  is large enough.

The following lemma gives us functions which satisfies the Hörmander hypoellipticity property near the boundary.

**Lemma 3.12.** Let  $\lambda > 0$  and  $R_0 > 0$ . We denote by  $K = \{x \in \mathbb{R}_+^n/|x| \leq R_0\}$ .

Then,  $\phi(x) = e^{\lambda x_n}$  satisfies the Hörmander hypoellipticity condition on  $K$  as soon as  $\lambda$  is large enough.

Moreover, the functions

- $\phi(x) = e^{-\lambda x_n}$ ,
- $\phi(x) = e^{-\lambda(x_n + |x|^2)}$ ,

also satisfy the Hörmander hypoellipticity condition on  $K$  as soon as  $\lambda$  is large enough and  $R_0$  is small enough.

We end this section by a lemma which will be very useful in the following.

**Lemma 3.13.** *Let  $P$  be a second-order differential operator defined in an open set  $M$  and  $\chi \in C_0^\infty(M)$  such that  $\chi = 1$  in a subdomain  $\Pi$  of  $M$ . Then,  $P(\chi y) = \chi P y + [P, \chi]y$  with  $[P, \chi]$  a first-order operator with support in  $M \setminus \Pi$ . Moreover, we have the following inequality: there exists  $c > 0$ , for all  $y \in H^1(M)$ ,*

$$\|[P, \chi]y\|_{L^2(M)} \leq c \|y\|_{H^1(M \setminus \Pi)}.$$

### 3.3 Proof of the stability estimates

In this section, we give a proof of Theorems 3.4 and 3.5.

#### 3.3.1 Intermediate results

In this first subsection, we state some intermediate results. We first state two theorems, Theorem 3.14 and Theorem 3.15, and we prove that Theorem 3.4 and Theorem 3.5 are respectively equivalent to Theorem 3.14 and Theorem 3.15. Next, we state Propositions 3.19, 3.21 and 3.22 which will allow to prove Theorems 3.14 and 3.15.

**Theorem 3.14.** *Let  $\omega$  be a nonempty open set and  $K$  be a compact set, both included in  $\Omega$ . Then, there exists  $c > 0$  and  $s > 0$  such that for all  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  solution of (3.1), and for all  $\epsilon > 0$  we have:*

$$\|u\|_{H^1(K)} + \|p\|_{L^2(K)} \leq \frac{c}{\epsilon} (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}). \quad (3.6)$$

**Theorem 3.15.** *Assume that  $\Omega$  is of class  $C^\infty$ . Let  $0 < \nu \leq \frac{1}{2}$ ,  $\Gamma$  be a nonempty open subset of the boundary of  $\Omega$  and  $\omega$  a nonempty open set included in  $\Omega$ . Then, for all  $\beta \in (0, \frac{1}{2} + \nu)$ , there exists  $c > 0$ , such that for all  $\epsilon > 0$ , we have*

$$\begin{aligned} \|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}} \left( \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) \\ + \epsilon^\beta (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}), \end{aligned} \quad (3.7)$$

and

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}), \quad (3.8)$$

for all couple  $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$  solution of (3.1).

**Proposition 3.16.** *Theorem 3.14 and Theorem 3.4 are equivalent.*

*Proof of Proposition 3.16.* The fact that Theorem 3.14 implies Theorem 3.4 is a direct consequence of Lemma 3.17 below with

$$A = c (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}), \quad B = \|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}, \quad C_1 = 1, \quad C_2 = \beta \quad \text{and} \quad D = \|u\|_{H^1(K)} + \|p\|_{L^2(K)}.$$

Moreover, the fact that Theorem 3.4 implies Theorem 3.14 is a consequence of Young inequality by writing:

$$\begin{aligned} c (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)})^\beta (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)})^{1-\beta} \\ = \left(\frac{c}{\epsilon} (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)})\right)^\beta \left(\epsilon^{\frac{\beta}{1-\beta}} (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)})\right)^{1-\beta}. \end{aligned}$$

□

**Lemma 3.17.** *Let  $A > 0$ ,  $B > 0$ ,  $C_1 > 0$ ,  $C_2 > 0$  and  $D > 0$ . We assume that there exists  $\gamma_1 > 0$  such that*

$$D \leq Ae^{C_1\gamma} + Be^{-C_2\gamma}, \quad (3.9)$$

for all  $\gamma \geq \gamma_1$  and  $c_0 > 0$  such that  $D \leq c_0B$ . Then, there exists  $C > 0$  such that:

$$D \leq CA^{\frac{C_2}{C_1+C_2}} B^{\frac{C_1}{C_1+C_2}}.$$

*Proof of Lemma 3.17.* Let  $\gamma_0 = \frac{1}{C_1+C_2} \ln\left(\frac{B}{A}\right)$ . Two cases arise:

- if  $\gamma_0 \geq \gamma_1$ , we directly obtain the desired result by applying inequality (3.9) with  $\gamma = \gamma_0$ ,
- if  $\gamma_0 < \gamma_1$  then  $B < e^{(C_1+C_2)\gamma_1} A$ , which implies the desired inequality since by assumption  $D \leq c_0B$ .

□

**Proposition 3.18.** *Theorem 3.15 and Theorem 3.5 are equivalent.*

*Proof of Proposition 3.18.* Let us prove the equivalence between inequality (3.3) of Theorem 3.5 and inequality (3.7) of Theorem 3.15, the equivalence between inequalities (3.4) and (3.8) can be proved in the same way.

We denote by

$$A = \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \quad \text{and} \quad B = \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}.$$

Assume that inequality (3.7) is true. By continuity of the trace mapping, we know that there exists a constant  $d_0 > 0$  such that  $B \leq Ad_0$ . Thus, for all  $d > d_0$ , we have  $\frac{dA}{B} > 1$ . By choosing  $\epsilon = \frac{c}{1-\beta} \left(\ln\left(\frac{dA}{B}\right)\right)^{-1}$ , then replacing in (3.7), we obtain the existence of  $C > 0$  such that:

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq CA \left( \left(\frac{B}{dA}\right)^\beta + \frac{1}{\left(\ln\left(\frac{dA}{B}\right)\right)^\beta} \right).$$

Then we use the fact that for all  $x > 1$ ,  $\frac{1}{x} \leq \frac{1}{\ln(x)}$  to conclude.

Reciprocally, assume that inequality (3.3) holds. Thus, for all  $\beta \in (0, \frac{1}{2} + \nu)$ , for all  $d > d_0$ , there exists  $c > 0$ , for all couple  $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$ , solution of (3.1)

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq c \frac{A}{(\ln(d \frac{A}{B}))^\beta}. \quad (3.10)$$

This implies that:

$$\frac{dA}{B} \leq \exp \left( \left( \frac{cA}{\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}} \right)^{\frac{1}{\beta}} \right). \quad (3.11)$$

Let  $\epsilon > 0$ . We will consider the two following cases

$$\left( \frac{cA}{\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}} \right)^{\frac{1}{\beta}} \leq \frac{1}{\epsilon} \quad (3.12)$$

and

$$\left( \frac{cA}{\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}} \right)^{\frac{1}{\beta}} > \frac{1}{\epsilon} \quad (3.13)$$

separately to obtain inequality (3.7). If inequality (3.12) is satisfied, then (3.11) implies that  $\frac{dA}{B} \leq e^{\frac{1}{\epsilon}}$  and we conclude by using the fact that  $H^{\frac{3}{2}+\nu}(\Omega) \hookrightarrow H^1(\Omega)$ . If inequality (3.13) is satisfied, we obtain directly:

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq cAe^\beta, \quad (3.14)$$

which allows us to conclude.  $\square$

Let us now state three propositions. Theorem 3.15 will be proved using these three intermediate results whereas Theorem 3.14 will be a consequence of inequality (3.16) of Proposition 3.19.

The first proposition allows to transmit information from an open set to any relatively compact open set in  $\Omega$ . We refer to Figure 3.1a for an illustration.

**Proposition 3.19.** *Let  $\omega$  be a nonempty open set included in  $\Omega$  and let  $\hat{\omega}$  be a relatively compact open set in  $\Omega$ . Then, we have the following estimates:*

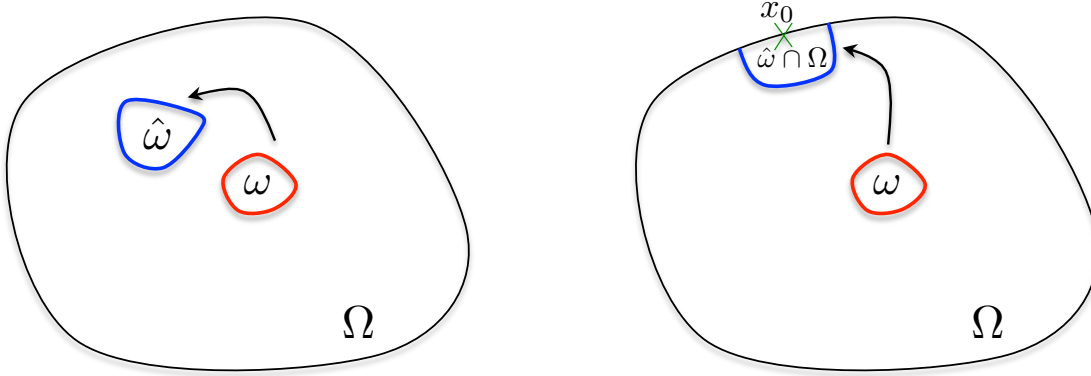
$$\begin{cases} \exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (3.1),} \\ \|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}), \end{cases} \quad (3.15)$$

and

$$\begin{cases} \exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times L^2(\Omega) \text{ solution of (3.1),} \\ \|u\|_{H^1(\hat{\omega})} + \|p\|_{L^2(\hat{\omega})} \leq \frac{c}{\epsilon} (\|u\|_{H^1(\omega)} + \|p\|_{L^2(\omega)}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}). \end{cases} \quad (3.16)$$

**Remark 3.20.** *Note that the difference between inequalities (3.15) and (3.16) lies in the fact that in (3.16) we only have the  $L^2$ -norm of  $p$  instead of the  $H^1$ -norm in each hand-side.*

The second proposition allows to transmit information from a relatively compact open set in  $\Omega$  to a neighborhood of the boundary. We refer to Figure 3.1b for an illustration.



(a) Figure illustrating how informations spread in Proposition 3.19: from an open set  $\omega$  to a relatively compact open set  $\hat{\omega}$ .

(b) Figure illustrating how informations spread in Proposition 3.21: from an open set  $\omega$  to a neighborhood of the boundary  $\hat{\omega} \cap \Omega$ .

Figure 3.1

**Proposition 3.21.** *Assume that  $\Omega$  is of class  $C^\infty$ . Let  $0 < \nu \leq \frac{1}{2}$ . Let  $x_0 \in \partial\Omega$  and let  $\omega$  be an open set in  $\Omega$ . There exists a neighborhood  $\hat{\omega}$  of  $x_0$  such that:*

$$\forall \beta \in \left(0, \frac{1}{2} + \nu\right), \exists c > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\|u\|_{H^1(\hat{\omega} \cap \Omega)} + \|p\|_{H^1(\hat{\omega} \cap \Omega)} \leq e^{\frac{c}{\epsilon}} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}). \quad (3.17)$$

Finally, the third proposition allows to transmit information from a part of the boundary of  $\Omega$  to a relatively compact open set in  $\Omega$ . We refer to Figure 3.2 for an illustration.

**Proposition 3.22.** *Assume that  $\Omega$  is of class  $C^\infty$ . Let  $0 < \nu \leq \frac{1}{2}$ . Let  $\Gamma$  be a non empty open subset of the boundary of  $\Omega$ . Let  $\hat{\omega}$  be a relatively compact open set in  $\Omega$ . Then, we have the following assertion:*

$$\exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{c}{\epsilon} \left( \|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

**Remark 3.23.** *The logarithmic nature of inequalities (3.7) and (3.8) comes from Proposition 3.21 where an exponential appears in front of the first term of the right-hand side whereas the estimates in Propositions 3.19 and 3.22 lead to Hölderian estimates, as a consequence of Lemma 3.17.*

**Remark 3.24.** *In Proposition 3.22, the regularity of  $u$  and  $p$  in  $H^{\frac{3}{2}+\nu}$  is necessary to give a sense to the normal derivatives.*

The next subsection is dedicated to the proof of Proposition 3.19. In the third subsection, we prove Propositions 3.21 and 3.22. Finally, in the last subsection, we conclude with the proof of the main theorems.

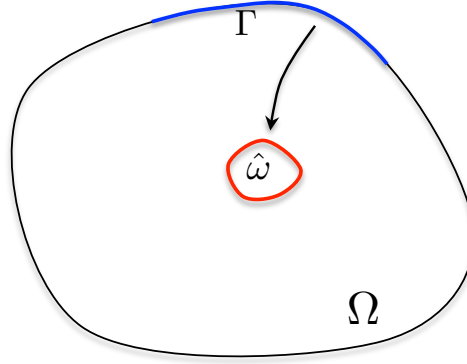


Figure 3.2: Figure illustrating how informations spread in Proposition 3.22: from a part of the boundary  $\Gamma$  to a relatively compact open set  $\hat{\omega}$ .

### 3.3.2 Estimates on relatively compact open sets: proof of Proposition 3.19

Let us begin by a lemma which will be useful to prove Proposition 3.19.

**Notation 3.25.** Let  $q \in \mathbb{R}^n$  and  $0 < r < r'$ . Let us denote by  $\mathbf{A}_q(\mathbf{r}, \mathbf{r}')$  the annulus delimited by the area between two concentric circles of centre  $q$  and of respective radius  $r$  and  $r'$ :  $A_q(r, r') = \{x \in \mathbb{R}^n / r < |x - q| < r'\}$ .

**Lemma 3.26.** Let  $q \in \mathbb{R}^n$  and  $0 < r_1 < r_2 < r_3 < r_4 < r_5$ . Then, there exists  $c > 0$ ,  $h_1 > 0$ ,  $c_1 > 0$  and  $c_2 > 0$  such that for all  $0 < h < h_1$  and for all function  $(u, p) \in H^1(B(q, r_5)) \times H^1(B(q, r_5))$  solution of

$$\begin{cases} -\Delta u + \nabla p = 0, & \text{in } B(q, r_5), \\ \operatorname{div} u = 0, & \text{in } B(q, r_5), \end{cases} \quad (3.18)$$

the following inequality is satisfied :

$$\begin{aligned} & \|u\|_{H^1(A_q(r_2, r_3))} + \|p\|_{H^1(A_q(r_2, r_3))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q, r_2))} + \|p\|_{H^1(B(q, r_2))}) + e^{-c_2/h} (\|u\|_{H^1(B(q, r_5))} + \|p\|_{H^1(B(q, r_5))}) \right), \end{aligned} \quad (3.19)$$

with  $c_1 = g(r_1) - g(r_3) > 0$  and  $c_2 = g(r_3) - g(r_4) > 0$ , where  $g(x) = e^{-\lambda x^2}$  and  $\lambda$  large enough.

*Proof of Lemma 3.26.* Let  $r_0$  and  $r_6$  be such that  $0 < r_0 < r_1$  and  $r_5 < r_6$ . We are going to apply Proposition 3.7 with

$$U_0 = A_q(r_0, r_6), \quad K_0 = \overline{A_q(r_1, r_5)},$$

and  $\phi(x) = e^{-\lambda|x-q|^2}$  for  $\lambda$  large enough, so that assumptions of Proposition 3.7 are verified on  $U_0$  according to Lemma 3.11.



Let  $\chi \in \mathcal{C}_c^\infty(B(q, r_6))$  such that  $0 \leq \chi \leq 1$ ,  $\chi = 1$  on  $\Pi = A_q(r_2, r_4)$  and  $\chi = 0$  in the exterior of  $K_0$ . Note that:

$$\Pi \subset K_0 \subset U_0.$$

Thanks to Remark 3.8, we can apply Proposition 3.7 successively to  $\chi u$  and  $\chi p$  where  $(u, p)$  is solution of (3.18): there exists  $c > 0$  and  $h_1 > 0$  such that for all  $h \in (0, h_1)$  and for all function  $(u, p) \in H^1(B(q, r_5)) \times H^1(B(q, r_5))$  solution of (3.18), we have, thanks to Lemma 3.13:

$$\begin{aligned} & \int_{\Pi} |u(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{\Pi} |\nabla u(x)|^2 e^{2\phi(x)/h} dx \\ & \leq ch^3 \int_{K_0} |\chi \nabla p(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K_0 \setminus \Pi} |[\Delta, \chi]u(x)|^2 e^{2\phi(x)/h} dx, \end{aligned} \quad (3.20)$$

and since  $\Delta p = \operatorname{div}(\Delta u) = 0$ :

$$\int_{\Pi} |p(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{K_0} |\nabla(\chi p)(x)|^2 e^{2\phi(x)/h} dx \leq ch^3 \int_{K_0 \setminus \Pi} |[\Delta, \chi]p(x)|^2 e^{2\phi(x)/h} dx.$$

Note that:

$$|\nabla(\chi p)(x)|^2 = |\chi(x)\nabla p(x) + p(x)\nabla\chi(x)|^2 = |\chi(x)\nabla p(x)|^2 + |p(x)\nabla\chi(x)|^2 + 2\chi(x)\nabla p(x) \cdot p(x)\nabla\chi(x).$$

Using Cauchy-Schwarz inequality and Young inequality, we have:

$$2\chi(x)\nabla p(x) \cdot p(x)\nabla\chi(x) \geq -2|\chi(x)\nabla p(x)||p(x)\nabla\chi(x)| \geq -\frac{|\chi(x)\nabla p(x)|^2}{2} - 8|p(x)\nabla\chi(x)|^2.$$

Thus, it follows:

$$\begin{aligned} & \int_{\Pi} |p(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{K_0} |\chi \nabla p(x)|^2 e^{2\phi(x)/h} dx \\ & \leq ch^3 \int_{K_0 \setminus \Pi} |[\Delta, \chi]p(x)|^2 e^{2\phi(x)/h} dx + ch^2 \int_{K_0 \setminus \Pi} |p(x)|^2 e^{2\phi(x)/h} dx. \end{aligned} \quad (3.21)$$

We add up inequalities (3.20) and (3.21): there exists  $h_1 > 0$  such that for all  $h \in (0, h_1)$ ,

$$\begin{aligned} & e^{g(r_3)/h} \int_{A_q(r_2, r_3)} (|u(x)|^2 + |p(x)|^2 + h^2(|\nabla u(x)|^2 + |\nabla p(x)|^2)) dx \\ & \leq ch^2 e^{g(r_1)/h} \int_{A_q(r_1, r_2)} (|[\Delta, \chi]u(x)|^2 + |[\Delta, \chi]p(x)|^2 + |p(x)|^2) dx \\ & \quad + ch^2 e^{g(r_4)/h} \int_{A_q(r_4, r_5)} (|[\Delta, \chi]u(x)|^2 + |[\Delta, \chi]p(x)|^2 + |p(x)|^2) dx. \end{aligned}$$

We divide the previous inequality by  $h^2$ . Using again Lemma 3.13, we obtain that there exists  $c > 0$ ,  $h_1 > 0$ ,  $c_1 = g(r_1) - g(r_3) > 0$  and  $c_2 = g(r_3) - g(r_4) > 0$  such that for all  $h \in (0, h_1)$  and for all function  $(u, p) \in H^1(B(q, r_5)) \times H^1(B(q, r_5))$  solution of (3.18), we have

$$\begin{aligned} & \|u\|_{H^1(A_q(r_2, r_3))} + \|p\|_{H^1(A_q(r_2, r_3))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q, r_2))} + \|p\|_{H^1(B(q, r_2))}) + e^{-c_2/h} (\|u\|_{H^1(B(q, r_5))} + \|p\|_{H^1(B(q, r_5))}) \right). \end{aligned}$$

□

Let us introduce the notion of  $\delta$ -sequence of balls between two points.

**Definition 3.27.** Let  $\delta > 0$  and  $(x_0, x)$  be two points in  $\Omega$ . We say that a sequence of balls  $(B(q_j, \delta))_{j=0, \dots, N}$  is a  $\delta$ -sequence of balls between  $x_0$  and  $x$  if

$$\begin{cases} q_0 = x_0, \\ x \in \overline{B(q_N, \delta)}, \\ B(q_{j+1}, \delta) \subset B(q_j, 2\delta), \text{ for } j = 0, \dots, N-1, \\ B(q_j, 3\delta) \subset \Omega. \end{cases}$$

**Lemma 3.28.** Let  $x_0$  and  $x$  be given in  $\Omega$ . There exists  $\delta_0 > 0$  such that for all  $0 < \delta < \delta_0$ , there exists a  $\delta$ -sequence of balls between  $x_0$  and  $x$ .

*Proof of Lemma 3.28.* We refer to [73] for a proof of this lemma. Let us just mention that in [73], it is asserted that  $x \in B(q_N, 2\delta)$ , but looking carefully at the proof, we see that  $x \in \overline{B(q_N, \delta)}$ .  $\square$

We are now able to prove Proposition 3.19.

*Proof of Proposition 3.19.* Let us begin by proving inequality (3.15).

Let  $x_0 \in \omega$  and  $r_0 > 0$  such that  $B(x_0, r_0) \subset \omega$ . For all  $x \in \overline{\omega}$ , there exists, thanks to Lemma 3.28, a  $\delta_x$ -sequence of balls  $(B(q_j^x, \delta_x))_{j=0, \dots, N_x}$  between  $x_0$  and  $x$ . Remark that we can assume that  $\delta_x < r_0$ , for all  $x \in \overline{\omega}$ . The compact  $\overline{\omega}$  is included in  $\bigcup_{x \in \overline{\omega}} B(q_{N_x}^x, \delta_x)$ , thus we can extract a finite sub-covering: there exists  $\kappa \in \mathbb{N}^*$  and  $(x_j)_{j=1, \dots, \kappa} \in \overline{\omega}$  such that

$$\overline{\omega} \subset \bigcup_{j=1, \dots, \kappa} B(q_{N_j}^j, \delta_j) \subset \bigcup_{j=1, \dots, \kappa} B(q_{N_j}^j, \delta), \quad (3.22)$$

where we have denoted for  $j = 1, \dots, \kappa$ ,  $N_j = N_{x_j}$ ,  $\delta_j = \delta_{x_j}$ ,  $q_i^j = q_i^{x_j}$  for  $i = 0, \dots, N_j$  and where  $\delta = \max_{j=1, \dots, \kappa} \delta_j$ . Remark that we can assume that  $N_j = N$  for all  $j = 1, \dots, \kappa$  (if necessary, we consider several times the same ball).

To prove (3.15), it is sufficient to show that

$$\begin{cases} \exists c, s > 0, \forall j = 1, \dots, \kappa, \forall i = 0, \dots, N-1, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (3.1),} \\ \|u\|_{H^1(B(q_{i+1}^j, \delta))} + \|p\|_{H^1(B(q_{i+1}^j, \delta))} \leq \frac{c}{\epsilon} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{H^1(B(q_i^j, \delta))}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}). \end{cases} \quad (3.23)$$

Indeed, if this is the case, there exists  $c, s > 0$  such that for all  $j \in \{1, \dots, \kappa\}$ , for all  $\bar{\epsilon} > 0$  and  $\tilde{\epsilon} > 0$  we have:

$$\begin{aligned} & \|u\|_{H^1(B(q_N^j, \delta))} + \|p\|_{H^1(B(q_N^j, \delta))} \\ & \leq \frac{c}{\bar{\epsilon}} (\|u\|_{H^1(B(q_{N-1}^j, \delta))} + \|p\|_{H^1(B(q_{N-1}^j, \delta))}) + \bar{\epsilon}^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \\ & \leq \frac{c}{\bar{\epsilon}\tilde{\epsilon}} (\|u\|_{H^1(B(q_{N-2}^j, \delta))} + \|p\|_{H^1(B(q_{N-2}^j, \delta))}) + (\bar{\epsilon}^s + c\frac{\tilde{\epsilon}^s}{\bar{\epsilon}}) (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}). \end{aligned}$$

Let  $\epsilon > 0$ . Choosing successively  $\tilde{\epsilon} = \frac{\epsilon^{s+1}}{\bar{\epsilon}^s}$  then  $\bar{\epsilon} = \epsilon^{\frac{s}{2s+1}}$ , we obtain that there exists  $s > 0$  such that:

$$\|u\|_{H^1(B(q_N^j, \delta))} + \|p\|_{H^1(B(q_N^j, \delta))} \leq \frac{c}{\epsilon} (\|u\|_{H^1(B(q_{N-2}^j, \delta))} + \|p\|_{H^1(B(q_{N-2}^j, \delta))}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

By iterating the process, we obtain the existence of  $c, s > 0$  such that for all  $j = 1, \dots, \kappa$  and  $(u, p) \in H^1(\Omega) \times H^1(\Omega)$  solution of (3.1):

$$\|u\|_{H^1(B(q_N^j, \delta))} + \|p\|_{H^1(B(q_N^j, \delta))} \leq \frac{c}{\epsilon} (\|u\|_{H^1(B(q_0^j, \delta))} + \|p\|_{H^1(B(q_0^j, \delta))}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

Note that  $\delta < r_0$  and that, for all  $j = 1, \dots, \kappa$ ,  $q_0^j = x_0$ . By summing up the previous inequality for  $j = 1, \dots, \kappa$  and using (3.22), we obtain (3.15).

To prove (3.23), it suffices, thanks to the definition of a  $\delta$ -sequence of balls, to prove the following inequality:

$$\begin{cases} \exists c, s > 0, \forall j = 1, \dots, \kappa, \forall i = 0, \dots, N-1, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (3.1),} \\ \|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{H^1(B(q_i^j, 2\delta))} \leq \frac{c}{\epsilon} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{H^1(B(q_i^j, \delta))}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}). \end{cases} \quad (3.24)$$

Let us emphasize that, thanks to Lemma 3.28, we can choose  $\delta > 0$  in (3.22) small enough such that  $B(q_i^j, 5\delta) \subset \Omega$  for all  $j = 1, \dots, \kappa$  and  $i = 0, \dots, N-1$  (it suffices to take  $\delta \leq \frac{3\delta_0}{5}$ ).

Let  $j \in \{1, \dots, \kappa\}$  and  $i \in \{0, \dots, N\}$ . We are going to apply Lemma 3.26 with  $q = q_i^j$ ,  $r_1 = \frac{\delta}{4}$ ,  $r_2 = \frac{\delta}{2}$ ,  $r_3 = 2\delta$ ,  $r_4 = \frac{9\delta}{4}$ ,  $r_5 = \frac{5\delta}{2}$ .

We obtain that there exists  $c > 0$ ,  $h_1 > 0$ ,  $c_1 > 0$  and  $c_2 > 0$  such that for all  $h \in (0, h_1)$  and for all function  $(u, p) \in H^1(B(q_i^j, \frac{5\delta}{2})) \times H^1(B(q_i^j, \frac{5\delta}{2}))$  solution of (3.18), we have

$$\begin{aligned} & \|u\|_{H^1(A_{q_i^j}(\frac{\delta}{2}, 2\delta))} + \|p\|_{H^1(A_{q_i^j}(\frac{\delta}{2}, 2\delta))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q_i^j, \frac{\delta}{2}))} + \|p\|_{H^1(B(q_i^j, \frac{\delta}{2}))}) + e^{-c_2/h} (\|u\|_{H^1(B(q_i^j, \frac{5\delta}{2}))} + \|p\|_{H^1(B(q_i^j, \frac{5\delta}{2}))}) \right). \end{aligned} \quad (3.25)$$

Moreover, for all  $h$  small enough and for all functions  $(u, p) \in H^1(\Omega) \times H^1(\Omega)$  solution of (3.1), the following inequality is obviously true:

$$\begin{aligned} & \|u\|_{H^1(B(q_i^j, \frac{\delta}{2}))} + \|p\|_{H^1(B(q_i^j, \frac{\delta}{2}))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q_i^j, \frac{\delta}{2}))} + \|p\|_{H^1(B(q_i^j, \frac{\delta}{2}))}) + e^{-c_2/h} (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \right). \end{aligned}$$

Since  $B(q_i^j, \frac{5\delta}{2}) \subset \Omega$  and  $B(q_i^j, \delta) \hookrightarrow B(q_i^j, \frac{\delta}{2})$ , we get, summing up the two previous inequalities:

$$\begin{aligned} & \|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{H^1(B(q_i^j, 2\delta))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{H^1(B(q_i^j, \delta))}) + e^{-c_2/h} (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \right). \end{aligned} \quad (3.26)$$

Let us consider  $\epsilon = e^{-c_1/h}$ . We obtain that there exists  $c > 0$ ,  $s = \frac{c_2}{c_1} > 0$ , such that for all  $0 < \epsilon < \epsilon_1 = e^{-c_1/h_1}$ , for all  $(u, p) \in H^1(\Omega) \times H^1(\Omega)$  solution of (3.1), we have

$$\|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{H^1(B(q_i^j, 2\delta))} \leq c \left( \frac{1}{\epsilon} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{H^1(B(q_i^j, \delta))}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \right).$$

Observe that this inequality is still valid for  $\epsilon \geq \epsilon_1$ :

$$\begin{aligned} & \|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{H^1(B(q_i^j, 2\delta))} \leq \frac{\epsilon^s}{\epsilon^s} (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \leq \frac{\epsilon^s}{\epsilon^s} (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \\ & \leq c \left( \frac{1}{\epsilon} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{H^1(B(q_i^j, \delta))}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \right). \end{aligned}$$

This ends the proof of inequality (3.15).

**Remark 3.29.** Let  $\epsilon > 0$ . Note that, from inequality (3.26), we could conclude the proof of Proposition 3.19 by using Lemma 3.17: there exists  $C > 0$  such that,

$$\begin{aligned} & \|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{H^1(B(q_i^j, 2\delta))} \\ & \leq C \left( \|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{H^1(B(q_i^j, \delta))} \right)^{\frac{c_2}{c_1+c_2}} \left( \|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \right)^{\frac{c_1}{c_1+c_2}} \\ & \leq C \left( \epsilon (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{H^1(B(q_i^j, \delta))}) \right)^{\frac{c_2}{c_1+c_2}} \left( \epsilon^{\frac{c_2}{c_1}} (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}) \right)^{\frac{c_1}{c_1+c_2}}. \end{aligned}$$

Then, we conclude by using Young inequality.

Let us now prove inequality (3.16). As before, it is sufficient to prove the following inequality:

$$\begin{cases} \forall j = 1, \dots, \kappa, \forall i = 0, \dots, N-1, \exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times L^2(\Omega) \text{ solution of (3.1),} \\ \|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{L^2(B(q_i^j, 2\delta))} \leq \frac{c}{\epsilon} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{L^2(B(q_i^j, \delta))}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}). \end{cases}$$

Note that, since  $(u, p)$  is solution of (3.1),  $(u, p)$  belongs to  $H_{loc}^2(\Omega) \times H_{loc}^1(\Omega)$  (we refer to [21] for a proof of this result). Considering again inequality (3.25), we will use Caccioppoli inequality to get rid of the  $L^2$ -norm of  $\nabla p$ .

**Lemma 3.30** (Caccioppoli inequality). *Let  $v$  be a weak solution of  $\Delta v = 0$  in  $\Omega \subset \mathbb{R}^n$ . Then, there exists  $C > 0$  such that for all  $x_0 \in \Omega$  and  $0 < \rho < R < d(x_0, \partial\Omega)$ , we have*

$$\int_{B(x_0, \rho)} |\nabla v|^2 \leq \frac{C}{(R - \rho)^2} \int_{B(x_0, R)} |v|^2.$$

We refer to [44] for more details about Caccioppoli inequality. Thus, thanks to Caccioppoli inequality, there exists  $c > 0$  such that:

$$\|\nabla p\|_{L^2(B(q_i^j, \frac{\delta}{2}))} \leq c \|p\|_{L^2(B(q_i^j, \delta))} \quad \text{and} \quad \|\nabla p\|_{L^2(B(q_i^j, \frac{5\delta}{2}))} \leq c \|p\|_{L^2(B(q_i^j, 5\delta))}.$$

By coupling this with inequality (3.25), we obtain that:

$$\begin{aligned} & \|u\|_{H^1(A_{q_i^j}(\frac{\delta}{2}, 2\delta))} + \|p\|_{L^2(A_{q_i^j}(\frac{\delta}{2}, 2\delta))} \leq \|u\|_{H^1(A_{q_i^j}(\frac{\delta}{2}, 2\delta))} + \|p\|_{H^1(A_{q_i^j}(\frac{\delta}{2}, 2\delta))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{L^2(B(q_i^j, \delta))}) + e^{-c_2/h} (\|u\|_{H^1(B(q_i^j, 5\delta))} + \|p\|_{L^2(B(q_i^j, 5\delta))}) \right). \end{aligned} \tag{3.27}$$

Moreover, for all  $h$  small enough and for all functions  $(u, p) \in H^1(B(q_i^j, 5\delta)) \times L^2(B(q_i^j, 5\delta))$ , the following inequality is obviously true:

$$\begin{aligned} & \|u\|_{H^1(B(q_i^j, \frac{\delta}{2}))} + \|p\|_{L^2(B(q_i^j, \frac{\delta}{2}))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{L^2(B(q_i^j, \delta))}) + e^{-c_2/h} (\|u\|_{H^1(B(q_i^j, 5\delta))} + \|p\|_{L^2(B(q_i^j, 5\delta))}) \right). \end{aligned} \tag{3.28}$$

By summing up inequalities (3.27) and (3.28), we get:

$$\begin{aligned} & \|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{L^2(B(q_i^j, 2\delta))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{L^2(B(q_i^j, \delta))}) + e^{-c_2/h} (\|u\|_{H^1(B(q_i^j, 5\delta))} + \|p\|_{L^2(B(q_i^j, 5\delta))}) \right). \end{aligned} \tag{3.29}$$

Since  $B(q_i^j, 5\delta) \subset \Omega$ , this leads to:

$$\begin{aligned} & \|u\|_{H^1(B(q_i^j, 2\delta))} + \|p\|_{L^2(B(q_i^j, 2\delta))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q_i^j, \delta))} + \|p\|_{L^2(B(q_i^j, \delta))}) + e^{-c_2/h} (\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}) \right). \end{aligned}$$

In the same way that we concluded the proof of inequality (3.15), we can finish the proof of inequality (3.16) by considering  $\epsilon = e^{-c_1/h}$  or by using Lemma 3.17 (see remark 3.29).  $\square$

**Remark 3.31.** *Let us notice that inequality (3.15) of Proposition 3.19 implies that:*

$$\begin{cases} \forall \beta > 0, \exists c > 0, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (3.1),} \\ \|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq e^{\frac{\epsilon}{\epsilon}} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}). \end{cases}$$

Indeed, we have for all  $\beta > 0$ :

$$\begin{cases} \exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (3.1),} \\ \|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \left(\frac{\epsilon}{\epsilon}\right)^{\frac{\beta}{s}} (\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}) + \epsilon^\beta (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}), \end{cases}$$

and  $\left(\frac{\epsilon}{\epsilon}\right)^{\frac{\beta}{s}} = e^{\frac{\beta}{s} \ln(\frac{\epsilon}{\epsilon})} \leq e^{\frac{\epsilon}{\epsilon}}$ , since for all  $x > 0$ ,  $\ln(x) \leq x$ .

Let  $0 < \nu \leq \frac{1}{2}$ . Note that we will use this remark for  $\beta \in (0, \frac{1}{2} + \nu)$  in the proof of Theorem 3.15.

**Remark 3.32.** *The proof of Proposition 3.19 contains all the tools needed to prove an interesting result which is, in the case of the Stokes system, a three balls inequality involving the velocity in  $H^1$  norm and the pressure in  $L^2$  norm. For reasons of clarity, we do not set out this result here and we refer to Lemma 6.8 for a statement of it. This lemma will be used in Chapter 6 to prove a Lipschitz stability estimate for the Stokes system with Robin boundary condition, under the a priori assumption that  $q$  is piecewise constant.*

**Remark 3.33.** *An alternative to get rid of the  $L^2$  norm of  $\nabla p$  in the proof of Proposition 3.19 is to use, instead of the Carleman inequality of Proposition 3.7 another Carleman inequality proved by C. Fabre and G. Lebeau in [39]. Let us state it with the same notations and hypothesis as in Proposition 3.7.*

For  $s \in \mathbb{N}$  and  $f = (f_1, \dots, f_s)$ , we define the operator  $L_1$  as:

$$L_1(f) = \sum_{(j,k) \in \{1, \dots, s\} \times \{1, \dots, n\}} a_{jk} \frac{\partial f_j}{\partial x_k},$$

where  $a_{jk} \in C^\infty(\mathbb{R}^n)$ . There exists  $c > 0$  and  $h_1 > 0$  such that for all  $(y, f) \in H_0^1(U) \times L^2(U)$  satisfying  $\Delta y - L_1 f \in L^2(U)$  and  $\text{supp}(y) \cup \text{supp}(\Delta y - L_1 f) \cup \text{supp}(f) \subset K$ , and for all  $0 < h < h_1$ , we have:

$$\begin{aligned} & \int_K |y(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_K |\nabla y(x)|^2 e^{2\phi(x)/h} dx \\ & \leq ch \int_K |f(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_K |\Delta y(x) - L_1 f(x)|^2 e^{2\phi(x)/h} dx. \quad (3.30) \end{aligned}$$

We now give a sketch of the proof of an alternative way to get rid of  $\nabla p$  in the proof of Proposition 3.19. We use the notations of the proof of Lemma 3.26. We recall that  $(v, q) = (\chi u, \chi p)$ . Note that:

$$\Delta v = \Delta \chi u + 2\nabla u \nabla \chi + \nabla q - \nabla \chi p,$$

and

$$\Delta q = \chi \Delta p + \operatorname{div}(2\nabla \chi p) - \Delta \chi p.$$

We apply inequality (3.30) to  $y = v_k$ ,  $L_1 = \partial_k$  and  $f = q$ . We obtain for all  $0 < h < h_1$ ,

$$\begin{aligned} & \int_{\Pi} |u(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{\Pi} |\nabla u(x)|^2 e^{2\phi(x)/h} dx \\ & \leq ch \int_{K_0} |q(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K_0} |\Delta \chi u + 2\nabla u \nabla \chi - \nabla \chi p|^2 e^{2\phi(x)/h} dx. \end{aligned} \quad (3.31)$$

Then, we apply inequality (3.30) to  $y = q$ ,  $L_1 = \operatorname{div}$  and  $f = 2\nabla \chi p$ . We obtain for all  $0 < h < h_1$ ,

$$\begin{aligned} & \int_{K_0} |q(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{K_0} |\nabla q(x)|^2 e^{2\phi(x)/h} dx \\ & \leq ch \int_{K_0} |\nabla \chi p(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K_0} |\chi \Delta p - \Delta \chi p|^2 e^{2\phi(x)/h} dx. \end{aligned} \quad (3.32)$$

Note that  $\Delta p = 0$ . Then, we multiply inequality (3.32) by  $h$  and we sum up the resulting inequality with (3.31). Since  $\chi = 1$  in  $\Pi$ , we obtain for all  $0 < h < h_1$ ,

$$\begin{aligned} & \int_{\Pi} |u(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_{\Pi} |\nabla u(x)|^2 e^{2\phi(x)/h} dx + h \int_{\Pi} |p(x)|^2 e^{2\phi(x)/h} dx \\ & \leq ch^2 \int_{K_0 \setminus \Pi} (|u|^2 + |\nabla u|^2 + |p|^2) e^{2\phi(x)/h} dx. \end{aligned} \quad (3.33)$$

Remark that we only have terms of order zero in  $p$  in the right hand-side of the previous inequality. By dividing inequality (3.33) by  $h^2$ , this leads to

$$\begin{aligned} & \|u\|_{H^1(A_q(r_2, r_3))} + \|p\|_{L^2(A_q(r_2, r_3))} \\ & \leq c \left( e^{c_1/h} (\|u\|_{H^1(B(q, r_2))} + \|p\|_{L^2(B(q, r_2))}) + e^{-c_2/h} (\|u\|_{H^1(B(q, r_5))} + \|p\|_{L^2(B(q, r_5))}) \right). \end{aligned}$$

Finally, one can conclude by performing the same reasoning as in Proposition 3.19.

### 3.3.3 Estimates near the boundary: proof of Propositions 3.21 and 3.22

To prove Propositions 3.21 and 3.22, we are going to apply the local Carleman inequality near the boundary given by Proposition 3.9. To do this, we must locally go back to the half-plane: we use the system of geodesic normal coordinates. It is well known that in the system of geodesic normal coordinates, the Laplace operator is transported to an operator satisfying the assumptions of Proposition 3.9 (see [52]). We refer to Annex A for an illustration of this point in dimension 3.

We begin with a lemma which will be used to prove Proposition 3.21.

**Lemma 3.34.** *Let  $0 < \nu \leq \frac{1}{2}$ ,  $0 < r_0 \leq R_0$ ,  $K = \{x \in \mathbb{R}_+^n / |x| \leq R_0\}$ ,  $(f, g) \in L^2(K) \times L^2(K)$ ,  $B \in GL_n(\mathcal{C}^\infty(K))$  and  $P$  be a second-order differential operator whose coefficients are  $\mathcal{C}^\infty$  in a neighborhood of  $K$ , defined by  $P(x, \partial_x) = -\partial_{x_n}^2 + R(x, \frac{1}{i}\partial_{x'})$ . Let us denote by  $r(x, \xi')$  the principal symbol of  $R$ . We assume that  $r(x, \xi') \in \mathbb{R}$  and that there exists a constant  $c > 0$  such that  $(x, \xi') \in K \times \mathbb{R}^{n-1}$ , we have  $r(x, \xi') \geq c|\xi'|^2$ .*

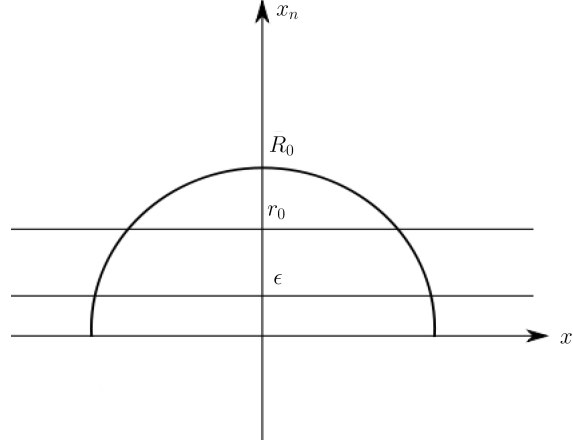


Figure 3.3: Considered sets in the proof of Proposition 3.21 (in dimension 2).

We denote by  $K(r, r') = \{x \in K / r < x_n < r'\}$ , for  $0 < r < r' < R_0$ . Then, for all  $\beta \in (0, \frac{1}{2} + \nu)$ , there exists  $c > 0$  such that for all  $\epsilon > 0$ , the following inequality holds

$$\begin{aligned} & \|v\|_{H^1(K(0, r_0))} + \|q\|_{H^1(K(0, r_0))} \\ & \leq e^{\frac{\epsilon}{c}} \left( \|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right) + \epsilon^\beta \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right). \end{aligned}$$

for all  $(v, q) \in H_{0, \partial K \setminus \Sigma}^{\frac{3}{2}+\nu}(K) \times H_{0, \partial K \setminus \Sigma}^{\frac{3}{2}+\nu}(K)$  solution of

$$\begin{cases} -Pv + B\nabla q = f, & \text{in } K, \\ Pq = g, & \text{in } K. \end{cases} \quad (3.34)$$

*Proof of Lemma 3.34.* Let  $0 < \epsilon < \epsilon_0 < r_0 < R_0$ , we denote by  $U = K(0, r_0)$  and  $U_\epsilon = K(\epsilon, r_0)$ . We are going to apply Proposition 3.9 with  $\phi(x) = e^{\lambda x_n}$ , for  $\lambda$  large enough. Let  $\chi \in C^\infty(K)$  be a function equal to zero in  $K^c$ , such that  $\chi = 1$  in  $U$ ,  $0 \leq \chi \leq 1$  in  $K \setminus U$ . Notice that:

$$U_\epsilon \subset U \subset K.$$

We apply successively Carleman inequality stated in Proposition 3.9 to  $\chi v$  and  $\chi q$ , taking into account Lemma 3.13:  $\exists c > 0, h_1 > 0, \forall 0 < h < h_1, \forall (v, q) \in H_{0, \partial K \setminus \Sigma}^{\frac{3}{2}+\nu}(K) \times H_{0, \partial K \setminus \Sigma}^{\frac{3}{2}+\nu}(K)$  solution of (3.34):

$$\begin{aligned} & \int_U |v(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_U |\nabla v(x)|^2 e^{2\phi(x)/h} dx \\ & \leq ch^3 \int_K |\chi P v(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K \setminus U} |[P, \chi]v(x)|^2 e^{2\phi(x)/h} dx \\ & + c \int_{\mathbb{R}^{n-1}} (|\chi v(x', 0)|^2 + |h \partial_{x'}(\chi v)(x', 0)|^2 + |h \partial_{x_n}(\chi v)(x', 0)|^2) e^{2\phi(x', 0)/h} dx', \quad (3.35) \end{aligned}$$

and

$$\begin{aligned}
 & \int_U |q(x)|^2 e^{2\phi(x)/h} dx + h^2 \int_K |\chi \nabla q(x)|^2 e^{2\phi(x)/h} dx \\
 & \leq ch^3 \int_K |\chi Pq(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K \setminus U} |[P, \chi]q(x)|^2 e^{2\phi(x)/h} dx + ch^2 \int_{K \setminus U} |q(x)|^2 e^{2\phi(x)/h} dx \\
 & \quad + c \int_{\mathbb{R}^{n-1}} (|\chi q(x', 0)|^2 + |h \partial_{x'}(\chi q)(x', 0)|^2 + |h \partial_{x_n}(\chi q)(x', 0)|^2) e^{2\phi(x', 0)/h} dx'. \quad (3.36)
 \end{aligned}$$

Since  $(v, q)$  satisfies (3.34), we can respectively replace  $\chi P v$  and  $\chi P q$  in the two previous inequality by  $-\chi B \nabla q + \chi f$  and  $\chi g$ . Note that there exists a constant  $c > 0$  such that  $|B \nabla q|^2 \leq c |\nabla q|^2$ . Then, by summing up inequalities (3.35) and (3.36), the term  $ch^3 \int_K |\chi \nabla q(x)|^2 e^{2\phi(x)/h} dx$  which appears in the right hand side of the resulting inequality will be absorb in the left hand side of the resulting inequality by  $h^2 \int_K |\chi \nabla q(x)|^2 e^{2\phi(x)/h} dx$ , for  $h$  small enough. By dividing by  $h^2$ , we obtain, for  $h$  small enough:

$$\begin{aligned}
 & \int_{U_\epsilon} (|v(x)|^2 + |q(x)|^2) e^{2\phi(x)/h} dx + \int_{U_\epsilon} (|\nabla v(x)|^2 + |\nabla q(x)|^2) e^{2\phi(x)/h} dx \\
 & \leq c \int_K |f(x)|^2 e^{2\phi(x)/h} dx + c \int_K |g(x)|^2 e^{2\phi(x)/h} dx + c \int_{K \setminus U} |q(x)|^2 e^{2\phi(x)/h} dx \\
 & + c \int_{K \setminus U} (|[P, \chi]v(x)|^2 + |[P, \chi]q(x)|^2) e^{2\phi(x)/h} dx + \frac{c}{h^2} \int_{\mathbb{R}^{n-1}} (|\chi v(x', 0)|^2 + |\chi q(x', 0)|^2) e^{2\phi(x', 0)/h} dx' \\
 & + \frac{c}{h^2} \int_{\mathbb{R}^{n-1}} (|\partial_{x'}(\chi v)(x', 0)|^2 + |\partial_{x'}(\chi q)(x', 0)|^2 + |\partial_{x_n}(\chi v)(x', 0)|^2 + |\partial_{x_n}(\chi q)(x', 0)|^2) e^{2\phi(x', 0)/h} dx'.
 \end{aligned}$$

By replacing  $\phi(x)$  by  $e^{\lambda x_n}$ , using Lemma 3.13 and thanks to the trace inequality, the previous inequality becomes

$$\begin{aligned}
 & e^{\frac{\lambda \epsilon}{h}} (\|v\|_{H^1(K(\epsilon, r_0))} + \|q\|_{H^1(K(\epsilon, r_0))}) \\
 & \leq ce^{\frac{\lambda R_0}{h}} (\|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}) \\
 & \quad + \frac{c}{h} e^{\frac{1}{h}} (\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}).
 \end{aligned}$$

Remark that for all  $\epsilon \geq 0$ ,  $-e^{\lambda \epsilon} + 1 \leq -\epsilon$  as long as  $\lambda$  is large enough. Thus:

$$\begin{aligned}
 & \|v\|_{H^1(K(\epsilon, r_0))} + \|q\|_{H^1(K(\epsilon, r_0))} \\
 & \leq ce^{\frac{\epsilon}{h}} (\|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}) \\
 & \quad + \frac{c}{h} e^{-\frac{\epsilon}{h}} (\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}).
 \end{aligned}$$

Moreover, for all  $\epsilon \geq 0$ ,  $\frac{1}{h} \leq \frac{2}{\epsilon} e^{\frac{\epsilon}{2h}}$ , which implies:

$$\begin{aligned}
 & \|v\|_{H^1(K(\epsilon, r_0))} + \|q\|_{H^1(K(\epsilon, r_0))} \\
 & \leq ce^{\frac{\epsilon}{h}} (\|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)}) \\
 & \quad + \frac{c}{\epsilon} e^{-\frac{\epsilon}{2h}} (\|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)}).
 \end{aligned}$$



According to Lemma 3.17, we obtain:

$$\begin{aligned} & \|v\|_{H^1(K(\epsilon, r_0))} + \|q\|_{H^1(K(\epsilon, r_0))} \\ & \leq c \left( \|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right)^{\frac{\epsilon}{\epsilon+c}} \left( \frac{1}{\epsilon} \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right) \right)^{1-\frac{\epsilon}{\epsilon+c}}. \end{aligned}$$

Let  $s > 0$  and  $\mu > 1$ . The previous estimate can be rewritten as:

$$\begin{aligned} & \|v\|_{H^1(K(\epsilon, r_0))} + \|q\|_{H^1(K(\epsilon, r_0))} \\ & \leq c \left( \epsilon^{-\frac{\epsilon}{\epsilon+c}(s+1)} \left( \|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right) \right)^{\frac{\epsilon}{\epsilon+c}} \times \\ & \quad \left( \epsilon^s \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right) \right)^{1-\frac{\epsilon}{\epsilon+c}} \\ & \leq c \epsilon^{-\frac{c(s+1)}{\epsilon}} \left( \|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right) \\ & \quad + \epsilon^s \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right). \end{aligned}$$

But  $\epsilon^{-\frac{c(s+1)}{\epsilon}} = \exp\left(\frac{c}{\epsilon}(s+1)\ln\left(\frac{1}{\epsilon}\right)\right) \leq \exp\left(\frac{c(s+1)}{(\mu-1)\epsilon^\mu}\right)$  since  $(\mu-1)\ln\left(\frac{1}{\epsilon}\right) \leq \frac{1}{\epsilon^{\mu-1}} \leq \frac{1}{\epsilon^\mu}$  for  $\epsilon$  small enough. Finally,  $\forall s > 0, \forall \mu > 1, \exists c > 0, \forall 0 < \epsilon < \epsilon_0$ ,

$$\begin{aligned} & \|v\|_{H^1(K(\epsilon, r_0))} + \|q\|_{H^1(K(\epsilon, r_0))} \\ & \leq c \epsilon^{\frac{c}{\epsilon^\mu}} \left( \|v\|_{H^1(K(r_0, R_0))} + \|q\|_{H^1(K(r_0, R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right) \\ & \quad + \epsilon^s \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right), \quad (3.37) \end{aligned}$$

for all  $(v, q)$  solution of (3.34).

It remains to estimate  $\|v\|_{H^1(K(0, \epsilon))} + \|q\|_{H^1(K(0, \epsilon))}$  uniformly in  $\epsilon$ . This is a consequence of Hardy inequality:

**Lemma 3.35** (Hardy inequality). *Let  $0 < \tau < \frac{1}{2}$ . There exists  $c > 0$  such that for all  $h \in H^\tau(\mathbb{R}_+^n)$ , we have*

$$\left\| \frac{h}{x_n^\tau} \right\|_{L^2(\mathbb{R}_+^n)} \leq c \|h\|_{H^\tau(\mathbb{R}_+^n)}.$$

We refer to [33] for a proof of Lemma 3.35.

We extend  $v$  and  $q$  by zero in  $\mathbb{R}_+^n \setminus K$ . Note that these extensions, denoted respectively by  $\tilde{v}$  and  $\tilde{q}$ , belong to  $H^{\frac{3}{2}+\nu}(\mathbb{R}_+^n)$  (see [59]). Let  $\tilde{\chi}$  be a function which belongs to  $C_c^\infty(\{(x', x_n) \in \mathbb{R}_+^n / x_n < r_0\})$  such that  $\tilde{\chi} = 1$  on  $K(0, \epsilon)$  and  $0 \leq \tilde{\chi} \leq 1$  elsewhere. The functions  $\tilde{\chi}\tilde{v}$  and  $\tilde{\chi}\tilde{q}$  belong to  $H^{\frac{3}{2}+\nu}(\mathbb{R}_+^n)$ , therefore as a result of Hardy inequality we have that for all  $0 < \tau < \frac{1}{2}$ , that there exists  $c > 0$ , such that

$$\left\| \frac{v}{x_n^\tau} \right\|_{L^2(K(0, \epsilon))} \leq \left\| \frac{\tilde{\chi}\tilde{v}}{x_n^\tau} \right\|_{L^2(\mathbb{R}_+^n)} \leq c \|\tilde{\chi}\tilde{v}\|_{H^\tau(\mathbb{R}_+^n)}.$$

Since  $\tilde{\chi}\tilde{v} = 0$  in  $(\mathbb{R}_+^n \setminus K) \cup K(r_0, R_0)$ , we obtain

$$\left\| \frac{v}{x_n^\tau} \right\|_{L^2(K(0, \epsilon))} \leq c \|v\|_{H^\tau(K(0, r_0))} \leq c \|v\|_{H^{\frac{1}{2}}(K(0, r_0))}.$$

Consequently, for all  $\tau \in (0, \frac{1}{2})$ , there exists  $c > 0$ , such that for all  $\alpha > 0$ ,

$$\begin{aligned} \|v\|_{L^2(K(0,\epsilon))} &\leq c\epsilon^\tau \|v\|_{H^{\frac{1}{2}}(K(0,r_0))} \leq c\epsilon^\tau \|v\|_{H^{\frac{1}{2}}(K)} \|v\|_{L^2(K(0,r_0))}^{\frac{1}{2}} \\ &\leq c \left( \frac{\epsilon^{2\tau}}{\alpha} \|v\|_{H^1(K)} + \alpha \|v\|_{L^2(K(0,r_0))} \right), \end{aligned}$$

where we used an interpolation inequality and Young inequality. In the same way, we have for  $\nabla v$ :

$$\begin{aligned} \|\nabla v\|_{L^2(K(0,\epsilon))} &\leq c\epsilon^\tau \|\nabla v\|_{H^{\frac{1}{2}}(K(0,r_0))} \leq c\epsilon^\tau \|\nabla v\|_{H^{\frac{1}{2}+\nu}(K)}^{\frac{1}{1+2\nu}} \|\nabla v\|_{L^2(K(0,r_0))}^{\frac{2\nu}{1+2\nu}} \\ &\leq c \left( \epsilon^{\tau(1+2\nu)} \frac{1}{\alpha^{2\nu}} \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \alpha \|v\|_{H^1(K(0,r_0))} \right). \end{aligned}$$

To summarize, for all  $\beta \in (0, \frac{1}{2} + \nu)$ , there exists  $c > 0$ , such that for all  $0 < \alpha < 1$ ,

$$\|v\|_{H^1(K(0,\epsilon))} \leq c \left( \frac{\epsilon^\beta}{\alpha} \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \alpha \|v\|_{H^1(K(0,r_0))} \right).$$

The same inequality also holds for  $q$ . Thus, for all  $\beta \in (0, \frac{1}{2} + \nu)$ , there exists  $c > 0$ , such that for all  $0 < \alpha < 1$ ,

$$\begin{aligned} &\|v\|_{H^1(K(0,\epsilon))} + \|q\|_{H^1(K(0,\epsilon))} \\ &\leq c \left( \frac{\epsilon^\beta}{\alpha} \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right) + \alpha \left( \|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))} \right) \right). \end{aligned} \quad (3.38)$$

We can choose  $\alpha$  small enough such that by combining (3.37) and (3.38) we have the following assertion:  $\forall \beta \in (0, \frac{1}{2} + \nu)$ ,  $\forall \mu > 1$ ,  $\exists c > 0$ ,  $\forall 0 < \epsilon < \epsilon_0$ ,

$$\begin{aligned} &\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))} \\ &\leq c\epsilon^{\frac{\epsilon}{\epsilon^\mu}} \left( \|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right) \\ &\quad + c\epsilon^\beta \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right). \end{aligned}$$

By a change of variables, we obtain:  $\forall \beta \in (0, \frac{1}{2} + \nu)$ ,  $\exists c > 0$ ,  $\forall 0 < \epsilon < \epsilon_0^\mu$ ,

$$\begin{aligned} &\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))} \\ &\leq c\epsilon^{\frac{\epsilon}{\epsilon^\mu}} \left( \|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right) \\ &\quad + c\epsilon^\beta \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right). \end{aligned} \quad (3.39)$$

This last inequality remains true for  $\epsilon \geq \epsilon_0^\mu$ , since  $H^{\frac{3}{2}+\nu}(K) \hookrightarrow H^1(K(0,r_0))$ :

$$\begin{aligned} &\|v\|_{H^1(K(0,r_0))} + \|q\|_{H^1(K(0,r_0))} \\ &\leq c \frac{\epsilon^\beta}{\epsilon^\beta} \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right) \leq c \frac{\epsilon^\beta}{(\epsilon_0^\mu)^\beta} \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right) \\ &\leq c\epsilon^{\frac{\epsilon}{\epsilon^\mu}} \left( \|v\|_{H^1(K(r_0,R_0))} + \|q\|_{H^1(K(r_0,R_0))} + \|f\|_{L^2(K)} + \|g\|_{L^2(K)} \right) \\ &\quad + c\epsilon^\beta \left( \|v\|_{H^{\frac{3}{2}+\nu}(K)} + \|q\|_{H^{\frac{3}{2}+\nu}(K)} \right), \end{aligned}$$

which ends the proof of Lemma 3.34.  $\square$

Let us now prove Proposition 3.21.

*Proof of Proposition 3.21.* We are first going to prove that there exists an open neighborhood  $\hat{\omega}$  of  $x_0$  and two relatively compact open sets  $\tilde{\omega}_1 \subset \Omega$  and  $\tilde{\omega}_2 \subset \Omega$  such that:

$$\forall \beta \in \left(0, \frac{1}{2} + \nu\right), \exists c > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\begin{aligned} \|u\|_{H^1(\hat{\omega} \cap \Omega)} + \|p\|_{H^1(\hat{\omega} \cap \Omega)} &\leq e^{\frac{\epsilon}{\bar{\epsilon}}} \left( \|u\|_{H^1(\tilde{\omega}_1)} + \|p\|_{H^1(\tilde{\omega}_1)} + \|u\|_{H^1(\tilde{\omega}_2)} + \|p\|_{H^1(\tilde{\omega}_2)} \right) \\ &\quad + \epsilon^\beta \left( \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right). \end{aligned} \quad (3.40)$$

Then, to pass from (3.40) to (3.17) to obtain the estimate for any  $\omega$ , it suffices to apply Proposition 3.19. Indeed, let  $\epsilon' > 0$  and  $\bar{\epsilon} > 0$ , we obtain:

$$\begin{aligned} \|u\|_{H^1(\hat{\omega} \cap \Omega)} + \|p\|_{H^1(\hat{\omega} \cap \Omega)} &\leq e^{\frac{\epsilon}{\bar{\epsilon}}} \frac{c}{\epsilon'} \left( \|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)} \right) + \epsilon'^s e^{\frac{\epsilon}{\bar{\epsilon}}} \left( \|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \right) \\ &\quad + \bar{\epsilon}^\beta \left( \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right), \end{aligned}$$

choosing successively  $\epsilon' = e^{-\frac{c}{\bar{\epsilon}\beta} \frac{\beta}{s}}$  and  $\bar{\epsilon} = \frac{\epsilon}{2}$  and using the Sobolev's injection  $H^{\frac{3}{2}+\nu}(\Omega) \hookrightarrow H^1(\Omega)$ , we obtain the desired result.

By passing in normal geodesic coordinates, it is possible to straighten locally in a neighborhood  $\mathcal{V}$  of  $x_0$  simultaneously the Laplace operator and the boundary. We assume that  $\Omega \cap \mathcal{V} = \{(x', x_n) \in \mathcal{V} / x_n > \sigma(x')\}$  with  $\sigma \in \mathcal{C}^\infty$ . Restricting if necessary the open set  $\mathcal{V}$ , we can assume that there exists a neighborhood  $\tilde{\mathcal{V}} \subset \mathcal{V}$  of  $x_0$ , a surface  $S$  such that  $S \cap \tilde{\mathcal{V}} = \partial\Omega \cap \tilde{\mathcal{V}}$  and  $S$  is deformed inwardly the open set  $\Omega$  in  $\mathcal{V} \setminus \tilde{\mathcal{V}}$  (there exists  $s \in \mathcal{C}^\infty$  such that  $S = \{(x', x_n) \in \mathcal{V} / x_n = s(x')\}$  with  $s = \sigma$  in  $\tilde{\mathcal{V}}$  and  $s > \sigma$  in  $\mathcal{V} \setminus \tilde{\mathcal{V}}$ ) and a diffeomorphism, denoted  $\psi$ , which straightens both  $S$  and the Laplace operator. Let us denote by  $\tilde{\Omega} = \{(x', x_n) \in \mathcal{V} / x_n > s(x')\}$ . In other words, we have

1.  $\psi(x_0) = 0$ ,
2. there exists  $R_0 > 0$  such that  $\psi(\tilde{\Omega} \cap \mathcal{V}) = \{x \in \mathbb{R}_+^n / |x| < R_0\}$ ,
3.  $\psi(S) = \{(x', x_n) \in \mathbb{R}^n / x_n = 0 \text{ and } |x| < R_0\}$ ,
4. the transported operator  $P$  satisfies the assumptions of Proposition 3.9 in  $K = \{x \in \mathbb{R}_+^n / |x| \leq R_0\}$ .

Note that, by construction, there exists  $0 < r_3 < R_0$ , such that  $\psi^{-1}(\{x \in K / r_3 < |x|\})$  is a relatively compact open set of  $\Omega$ . Let  $\xi \in \mathcal{C}_c^\infty(\bar{K})$  be such that  $\xi = 1$  in  $\{x \in \mathbb{R}_+^n / |x| \leq r_3\}$  and  $0 \leq \xi \leq 1$  elsewhere. Let us denote by  $\varrho = \xi \circ \psi$ . Note that since  $(v, q) = (\varrho u, \varrho p)$  is solution in  $\tilde{\Omega} \cap \mathcal{V}$  of

$$\begin{cases} -\Delta v + \nabla q &= f, \\ \Delta q &= g, \end{cases}$$

with  $f = -u\Delta\varrho - 2\nabla u \nabla\varrho + \nabla\varrho p$  and  $g = \Delta\varrho p + 2\nabla\varrho \cdot \nabla p$ , then  $(w, \pi) = ((\varrho u) \circ \psi^{-1}, (\varrho p) \circ \psi^{-1})$  is solution in  $K$  of

$$\begin{cases} -Pw + (\nabla\psi)^T \nabla\pi &= f \circ \psi^{-1}, \\ P\pi &= g \circ \psi^{-1}. \end{cases} \quad (3.41)$$

We apply Lemma 3.34 to  $(w, \pi)$ . We obtain that for all  $\beta \in (0, \frac{1}{2} + \nu)$ , there exists  $c > 0$  such that for all  $\epsilon > 0$ ,

$$\begin{aligned} & \|w\|_{H^1(K(0,r_0) \cap B(0,r_3))} + \|\pi\|_{H^1(K(0,r_0) \cap B(0,r_3))} \\ & \leq e^{\frac{\epsilon}{c}} \left( \|w\|_{H^1(K(r_0,R_0))} + \|\pi\|_{H^1(K(r_0,R_0))} + \|f \circ \psi^{-1}\|_{L^2(K)} + \|g \circ \psi^{-1}\|_{L^2(K)} \right) \\ & \quad + \epsilon^\beta \left( \|w\|_{H^{\frac{3}{2}+\nu}(K)} + \|\pi\|_{H^{\frac{3}{2}+\nu}(K)} \right). \end{aligned}$$

In other words, there exists an open neighborhood  $\tilde{\omega}$  of  $x_0$  and a relatively compact open set  $\tilde{\omega}_1 \subset \Omega$  such that:

$$\forall \beta \in \left(0, \frac{1}{2} + \nu\right), \exists c > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\begin{aligned} \|u\|_{H^1(\tilde{\omega} \cap \Omega)} + \|p\|_{H^1(\tilde{\omega} \cap \Omega)} & \leq e^{\frac{\epsilon}{c}} \left( \|u\|_{H^1(\tilde{\omega}_1)} + \|p\|_{H^1(\tilde{\omega}_1)} + \|f\|_{L^2(\tilde{\Omega} \cap \mathcal{V})} + \|g\|_{L^2(\tilde{\Omega} \cap \mathcal{V})} \right) \\ & \quad + \epsilon^\beta \left( \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right). \end{aligned}$$

To conclude, let us remark that since  $\xi = 1$  in  $\{x \in \mathbb{R}_+^n / |x| \leq r_3\}$ ,  $\text{supp}(\nabla \xi) \subset \{x \in K/r_3 < |x|\}$  and then  $\text{supp}(\nabla \varrho) \subset \psi^{-1}(\{x \in K/r_3 < |x|\})$  which is a relatively compact open set of  $\Omega$ . Then, remembering the definition of  $f$  and  $g$ , we obtain that there exists a relatively compact open set  $\tilde{\omega}_2 \subset \Omega$  such that:

$$\forall \beta \in \left(0, \frac{1}{2} + \nu\right), \exists c > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\begin{aligned} \|u\|_{H^1(\tilde{\omega} \cap \Omega)} + \|p\|_{H^1(\tilde{\omega} \cap \Omega)} & \leq e^{\frac{\epsilon}{c}} \left( \|u\|_{H^1(\tilde{\omega}_1)} + \|p\|_{H^1(\tilde{\omega}_1)} + \|u\|_{H^1(\tilde{\omega}_2)} + \|p\|_{H^1(\tilde{\omega}_2)} \right) \\ & \quad + \epsilon^\beta \left( \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right). \end{aligned}$$

We obtain (3.40) which ends the proof.  $\square$

Let us end this subsection with the proof of Proposition 3.22.

*Proof of Proposition 3.22.* Let  $x_0 \in \Gamma$ . We are going to prove that there exists a neighborhood  $\omega$  of  $x_0$  such that:

$$\exists c, s > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\begin{aligned} \|u\|_{H^1(\omega \cap \Omega)} + \|p\|_{H^1(\omega \cap \Omega)} & \leq \frac{c}{\epsilon} \left( \|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) \\ & \quad + \epsilon^s \left( \|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \right). \end{aligned} \quad (3.42)$$

This inequality implies Proposition 3.22 thanks to inequality (3.15) of Proposition 3.19,  $\epsilon$  being suitably chosen (as in the beginning of proof of Proposition 3.21).

Near the boundary, in a neighborhood of  $x_0$ , we go back to the half-plane by passing in geodesic normal coordinates: let  $\psi$  and  $\mathcal{V}$  be such that

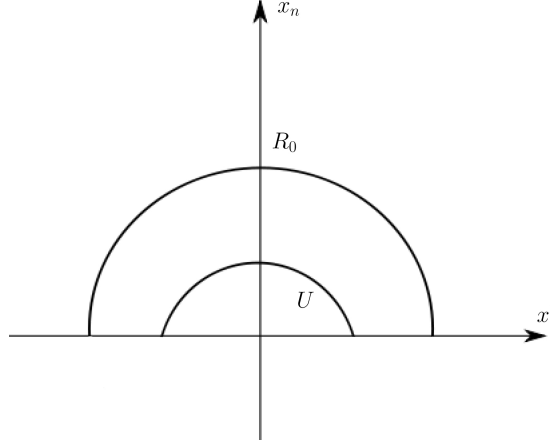


Figure 3.4: Considered sets in the proof of Proposition 3.22 (in dimension 2).

1.  $\psi(x_0) = 0$ ,
2. there exists  $R_0 > 0$  such that  $\psi(\Omega \cap \mathcal{V}) = \{x \in \mathbb{R}_+^n / |x| < R_0\}$ ,
3.  $\psi(\partial\Omega \cap \mathcal{V}) = \{(x', x_n) \in \mathbb{R}^n / x_n = 0 \text{ and } |x| < R_0\}$ ,
4. the transported operator  $P$  satisfies the assumptions of Proposition 3.9 in  $K = \{x \in \mathbb{R}_+^n / |x| \leq R_0\}$ .

We can always assume that  $\mathcal{V}$  is small enough in order to have  $\partial\Omega \cap \mathcal{V} \subset \Gamma$ . In the sequel, we denote by  $\Sigma = \psi(\partial\Omega \cap \mathcal{V}) \subset \mathbb{R}^{n-1}$ . Let us denote by  $(v, q) = (u \circ \psi^{-1}, p \circ \psi^{-1})$ . Note that  $(v, q)$  is solution in  $K$  of

$$\begin{cases} -Pv + (\nabla\psi)^T \nabla q & = 0, \\ Pq & = 0. \end{cases} \quad (3.43)$$

We will show that there exists  $\theta$  a neighborhood of 0 such that:

$\exists c, s > 0, \forall \epsilon > 0, \forall (v, q) \in H^{\frac{3}{2}+\nu}(K) \times H^{\frac{3}{2}+\nu}(K)$  solution of (3.43),

$$\begin{aligned} \|v\|_{H^1(K \cap \theta)} + \|q\|_{H^1(K \cap \theta)} &\leq \frac{c}{\epsilon} (\|v\|_{H^1(\Sigma)} + \|q\|_{H^1(\Sigma)} + \|\partial_{x_n} v\|_{L^2(\Sigma)} + \|\partial_{x_n} q\|_{L^2(\Sigma)}) \\ &\quad + \epsilon^s (\|v\|_{H^1(K)} + \|q\|_{H^1(K)}). \end{aligned}$$

To obtain this inequality, we apply Proposition 3.9 with  $\phi(x) = e^{-\lambda(x_n + |x|^2)}$  and  $\lambda$  large enough. Indeed, let  $U = \{x \in K / x_n + |x|^2 \leq r_0\}$  with  $r_0$  small enough (see Figure 3.4) and  $\chi \in C_c^\infty(\bar{K})$  be such that  $\chi = 1$  on  $U$ ,  $0 \leq \chi \leq 1$  in  $K \setminus U$ . By successive applications of Proposition 3.9 to  $\chi v$  and to  $\chi q$ , we obtain (in the same way as in the proof of Proposition

3.21):

$$\begin{aligned}
 & \exists c > 0, h_1 > 0, \forall 0 < h < h_1, \forall (v, q) \in H^{\frac{3}{2}+\nu}(K) \times H^{\frac{3}{2}+\nu}(K) \text{ satisfying (3.43)} \\
 & \int_U (|v(x)|^2 + |q(x)|^2) e^{2\phi(x)/h} dx + h^2 \int_U (|\nabla v(x)|^2 + |\nabla q(x)|^2) e^{2\phi(x)/h} dx \leq ch^3 \int_{K \setminus U} |\nabla q(x)|^2 e^{2\phi(x)/h} dx \\
 & \quad + ch^2 \int_{K \setminus U} |q(x)|^2 e^{2\phi(x)/h} dx + ch^3 \int_{K \setminus U} (|[P, \chi]v(x)|^2 + |[P, \chi]q(x)|^2) e^{2\phi(x)/h} dx \\
 & + c \int_{\mathbb{R}^{n-1}} (|h\partial_{x'}(\chi v)(x', 0)|^2 + |h\partial_{x'}(\chi q)(x', 0)|^2 + |h\partial_{x_n}(\chi v)(x', 0)|^2 + |h\partial_{x_n}(\chi q)(x', 0)|^2) e^{2\phi(x', 0)/h} dx' \\
 & \quad + c \int_{\mathbb{R}^{n-1}} (|\chi v(x', 0)|^2 + |\chi q(x', 0)|^2) e^{2\phi(x', 0)/h} dx'.
 \end{aligned}$$

We denote by  $R(r, r') = \{x \in K / r < x_n + |x|^2 < r'\}$ . The previous inequality becomes, with  $0 < z_1 < r_0 < z_2 < R_0$ :

$$\begin{aligned}
 e^{\frac{\epsilon^{-\lambda z_1}}{h}} (\|v\|_{H^1(R(0, z_1))} + \|q\|_{H^1(R(0, z_1))}) & \leq ce^{\frac{\epsilon^{-\lambda z_2}}{h}} (\|v\|_{H^1(R(z_2, R_0))} + \|q\|_{H^1(R(z_2, R_0))}) \\
 & \quad + ce^{\frac{1}{h}} (\|v\|_{H^1(\Sigma)} + \|\partial_{x_n} v\|_{L^2(\Sigma)} + \|q\|_{H^1(\Sigma)} + \|\partial_{x_n} q\|_{L^2(\Sigma)}).
 \end{aligned}$$

Accordingly:

$$\begin{aligned}
 & \exists c, h_1 > 0, \forall 0 < h < h_1, \forall (v, q) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.43),} \\
 & \|v\|_{H^1(R(0, z_1))} + \|q\|_{H^1(R(0, z_1))} \leq ce^{-\frac{1}{h}} (\|v\|_{H^1(K)} + \|q\|_{H^1(K)}) \\
 & \quad + ce^{\frac{\epsilon}{h}} (\|v\|_{H^1(\Sigma)} + \|\partial_{x_n} v\|_{L^2(\Sigma)} + \|q\|_{H^1(\Sigma)} + \|\partial_{x_n} q\|_{L^2(\Sigma)}).
 \end{aligned}$$

We can conclude the proof in the same way that we concluded the proof of inequality (3.15), by considering  $\epsilon = e^{-1/h}$  or by using Lemma 3.17 (see remark 3.29): we obtain inequality (3.42) with  $\omega \cap \Omega = \psi^{-1}(R(0, z_1))$ .  $\square$

### 3.3.4 Global estimate: proof of Theorems 3.14 and 3.15

In this subsection, we conclude the proofs of Theorems 3.14 and 3.15.

Let us first prove Theorem 3.15. Let  $\hat{\omega}$  be a relatively compact open set in  $\Omega$ . For each  $x \in \partial\Omega$ , we deduce from Proposition 3.21, that there exist a neighborhood  $\omega_x$  of  $x$ , such that for all  $\beta \in (0, \frac{1}{2} + \nu)$ , there exists  $c > 0$ , for all  $\epsilon > 0$ , for all  $(u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$  solution of (3.1), inequality (3.17) is satisfied. We point out that  $\partial\Omega \subseteq \bigcup_{x \in \partial\Omega} \omega_x$  and that  $\partial\Omega$  is compact. Thus, we can extract a finite subcover: there exists  $N \in \mathbb{N}$  and  $x_i \in \partial\Omega$ ,  $i = 1, \dots, N$ , such that  $\partial\Omega \subset \bigcup_{i=1}^N \omega_{x_i}$ . Let us denote  $\omega_i = \omega_{x_i}$  for  $i = 1, \dots, N$ . As a result, we obtain:

$$\forall \beta \in \left(0, \frac{1}{2} + \nu\right), \exists c > 0, \forall i \in \{1, \dots, N\}, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\|u\|_{H^1(\omega_i \cap \Omega)} + \|p\|_{H^1(\omega_i \cap \Omega)} \leq e^{\frac{\epsilon}{h}} (\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})}) + \epsilon^\beta \left( \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right).$$

We denote by  $\Upsilon = \bigcup_{i=1}^N (\omega_i \cap \Omega)$ . Let  $r > 0$ . Let us consider a finite subcover of  $\Omega \setminus \bar{\Upsilon}$ : there exist  $\tilde{N} \in \mathbb{N}$  and  $y_i \in \Omega$ ,  $i = 1, \dots, \tilde{N}$  such that  $\Omega \setminus \bar{\Upsilon} \subset \bigcup_{i=1}^{\tilde{N}} B(y_i, r)$ . For all

$i = 1, \dots, \tilde{N}$ , up to a decreasing  $r$ ,  $B(y_i, r)$  is a relatively compact open set in  $\Omega$  where we can apply inequality (3.15) of Proposition 3.19:

$$\exists c, s > 0, \forall i \in \{1, \dots, \tilde{N}\}, \forall \epsilon > 0, \forall (u, p) \in H^1(\Omega) \times H^1(\Omega) \text{ solution of (3.1),}$$

$$\|u\|_{H^1(\mathcal{B}(y_i, r))} + \|p\|_{H^1(\mathcal{B}(y_i, r))} \leq \frac{c}{\epsilon} (\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})}) + \epsilon^s (\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

Thus, by summing up the previous inequalities, taking into account Remark 3.31, we obtain:

$$\forall \beta \in \left(0, \frac{1}{2} + \nu\right), \exists c > 0, \forall \epsilon > 0, \forall (u, p) \in H^{\frac{3}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega) \text{ solution of (3.1),}$$

$$\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}} (\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})}) + \epsilon^\beta \left( \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right). \quad (3.44)$$

It remains to pass from a relatively compact open set  $\hat{\omega}$  to an open set  $\omega$  (not necessarily relatively compact): we use inequality (3.16) of Proposition 3.19, with  $\epsilon$  suitably chosen as in the beginning of the proof of Proposition 3.21, in order to bound  $\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})}$  in inequality (3.44) by  $\|u\|_{H^1(\omega)} + \|p\|_{H^1(\omega)}$ . It directly gives us inequality (3.8) of Theorem 3.15.

Now, if we apply Proposition 3.22, we obtain,  $\epsilon$  being suitably chosen:

$$\begin{aligned} \|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} \leq e^{\frac{c}{\epsilon}} \left( \|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) \\ + \epsilon^\beta (\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)}). \end{aligned} \quad (3.45)$$

Let  $\theta = \frac{1}{1+\nu} \in (0, 1)$ . Using an interpolation inequality, we obtain that there exists  $c > 0$  such that:

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left( \|u\|_{L^2(\Gamma)}^{1-\theta} \|u\|_{H^{1+\nu}(\Gamma)}^\theta + \|p\|_{L^2(\Gamma)}^{1-\theta} \|p\|_{H^{1+\nu}(\Gamma)}^\theta \right).$$

If we write that

$$\|u\|_{L^2(\Gamma)}^{1-\theta} \|u\|_{H^{1+\nu}(\Gamma)}^\theta = e^{\frac{2c\theta}{\epsilon}} \|u\|_{L^2(\Gamma)}^{1-\theta} e^{-\frac{2c\theta}{\epsilon}} \|u\|_{H^{1+\nu}(\Gamma)}^\theta,$$

and

$$\|p\|_{L^2(\Gamma)}^{1-\theta} \|p\|_{H^{1+\nu}(\Gamma)}^\theta = e^{\frac{2c\theta}{\epsilon}} \|p\|_{L^2(\Gamma)}^{1-\theta} e^{-\frac{2c\theta}{\epsilon}} \|p\|_{H^{1+\nu}(\Gamma)}^\theta,$$

according to Young inequality and to the continuity of the trace operator from  $H^{\frac{3}{2}+\nu}(\Omega)$  onto  $H^{1+\nu}(\Gamma)$ , we obtain:

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left( e^{\frac{-2c}{\epsilon}} \left( \|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \|p\|_{H^{\frac{3}{2}+\nu}(\Omega)} \right) + e^{\frac{2c}{\epsilon\nu}} (\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)}) \right).$$

Using the fact that  $e^{\frac{-c}{\epsilon}} \leq C\epsilon^\beta$  for all  $\epsilon > 0$ , it allows us to replace  $\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)}$  in the right hand-side of inequality (3.45) by  $\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)}$ . This proves inequality (3.7) of Theorem 3.15.

In the same way, using a partition of  $K$  by relatively compact open sets and according to inequality (3.16) of Proposition 3.21, we immediately deduce Theorem 3.14.

### 3.4 Conclusion

By borrowing the approach developed by K. D. Phung in [68], we prove some stability estimates that quantify the unique continuation result of C. Fabre and G. Lebeau in [39]. The Carleman estimate we use near the boundary is a consequence of pseudo-differential calculus. To apply this technique, the domain has to be very regular. In [19], L. Bourgeois proved that the stability estimates proved by K. D. Phung in [68] for  $C^\infty$  domains still holds for domains of class  $C^{1,1}$ . To do so, he used another technique to derive the same estimates near the boundary: he used a global Carleman estimate near the boundary on the initial geometry, by following method of [43]. Moreover, in [20], L. Bourgeois and J. Dardé complete the results obtained in [19]: they proved a conditional stability estimate related to the ill-posed Cauchy problem for Laplace equation in domain with Lipschitz boundary. For such non smooth domains, difficulties occur when one wants to estimate the function in a neighborhood of  $\partial\Omega$ : the authors use an interior Carleman estimate and a technique based on a sequence of balls which approaches the boundary, which is inspired by [3]. Let us emphasize the fact that the inequality obtained by this way is valid for a regular solution  $u$  ( $u$  belongs to  $C^{1,\alpha}(\Omega)$  and is such that  $\Delta u \in L^2(\Omega)$ ) and that boundary conditions are known on a part of the boundary. These two results suggest that it could be possible to extend results of this chapter to less regular open sets. Another improvement could be to reduce the number of observations needed on the boundary in the stability estimate of Proposition 3.22. A possibility could be to use the curl operator instead of the divergence operator as they did in [58] to end up with elliptic equations where the pressure is not involved. Finally, let us remark that we have not used the boundary conditions on the boundary of the domain in the proof of the different stability estimates. A perspective of research could be to use them in order to improve the results, as in [31], maybe to improve the logarithm character of the stability estimate.

Note that these kinds of stability estimates can be used to different purposes. For example, K. D. Phung uses his stability estimate for the Laplace equation to establish an estimate on the cost of an approximate control function for an elliptic model equation. L. Bourgeois and J. Dardé use their stability estimates to study the convergence rate for the method of quasi-reversibility introduced in [55] to solve the Cauchy problems. As far as we are concerned, we will use them in Chapters 4 and 6 to solve inverse problems.





## Part III

# Resolution of some inverse problems



## Chapter 4

# Logarithmic stability estimates for the stationary Stokes system

In this chapter, we study the inverse problem of identifying a Robin coefficient on some non accessible part of the boundary from measurements available on the other part of the boundary, for  $(u, p)$  solution of the stationary Stokes system with mixed Neumann and Robin boundary conditions. We first investigate the identifiability. Then, we prove two stability estimates, one valid in dimension two and the other one valid in any dimension, under the assumption that the velocity of a given reference solution stays far from 0 on a part of the boundary where Robin conditions are prescribed. We prove the first one using a global Carleman estimate proved by A. L. Bukhgeim in [23] and we prove the second one by using stability estimates for the unique continuation properties of the Stokes system from Chapter 3.

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Results of this chapter have been obtained in collaboration with Muriel Boulakia and Céline Grandmont.



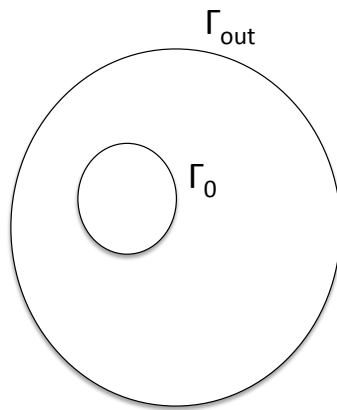


Figure 4.1: Example of an open set  $\Omega \subset \mathbb{R}^2$  such that  $\partial\Omega = \Gamma_0 \cup \Gamma_{out}$  and  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ .

## 4.1 Introduction

Let  $d \in \mathbb{N}^*$ . When we are not more specific,  $\Omega \subset \mathbb{R}^d$  is an open connected bounded Lipschitz set such that  $\partial\Omega = \Gamma_0 \cup \Gamma_{out}$  and  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ . We refer to Figure 4.1 for an example in dimension 2.

We consider the following Stokes system:

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{in } \Omega, \\ \operatorname{div} u &= 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn &= g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu &= 0, & \text{on } \Gamma_{out}. \end{cases} \quad (4.1)$$

We are concerned in this chapter with **uniqueness** and **stability properties** for the Robin coefficient  $q$  from measurements available on  $\Gamma \subseteq \Gamma_0$  for  $(u, p)$  solution of system (4.1).

In this chapter, we present two logarithmic stability estimates under the assumption that the velocity of a given reference solution stays far from 0 on a part of the boundary where Robin conditions are prescribed. The first one, stated in Section 4.4 is valid only in dimension 2. The second one, stated in Section 4.5 is valid in any space dimension but more regularity is needed on the open set  $\Omega$ , at least locally. We refer to Figure 4.2 for a summary of the main characteristics of the two stability estimates proved in this chapter.

Let us explain the structure of this chapter. We start with regularity results for system (4.1) in Section 4.2. Then, we state an identifiability result in Section 4.3 and finally we state and prove two logarithmic stability estimates, respectively in Section 4.4 and in Section 4.5.

## 4.2 Regularity results

**Notation 4.1.** *Let us denote by:*

$$V = \{v \in H^1(\Omega) / \operatorname{div} v = 0 \text{ in } \Omega\},$$

	Regularity needed on the open set	Regularity needed on $(u, p)$	Valid in dimension	Kind of inequality
Theorem 4.8	$\mathcal{C}^{3,1}$	$(u, p) \in H^4(\Omega) \times H^3(\Omega)$	2	Logarithmic
Theorem 4.16	locally $\mathcal{C}^\infty$	$(u, p) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ for $k \in \mathbb{N}^*$ such that $k + 2 > \frac{d}{2}$	in any dimension	Logarithmic

Figure 4.2: We point out the main differences between the stability results obtained in this chapter.

and

$$\mathbf{H} = \overline{\mathbf{V}}L^2(\Omega).$$

**Proposition 4.2.** *Let  $\alpha > 0$ ,  $M > 0$ ,  $f \in L^2(\Omega)$ ,  $g \in H^{-\frac{1}{2}}(\Gamma_0)$  and  $q \in L^\infty(\Gamma_{out})$  such that  $\alpha \leq q$  almost everywhere on  $\Gamma_{out}$ . System*

$$\begin{cases} -\Delta u + \nabla p &= f, & \text{in } \Omega, \\ \operatorname{div} u &= 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn &= g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu &= 0, & \text{on } \Gamma_{out}, \end{cases} \quad (4.2)$$

admits a unique solution  $(u, p) \in V \times L^2(\Omega)$ . Moreover, there exists a constant  $C(\alpha, M) > 0$  such that for every  $q \in L^\infty(\Gamma_{out})$  such that  $\|q\|_{L^\infty(\Gamma_{out})} \leq M$ ,

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(\alpha, M)(\|g\|_{H^{-\frac{1}{2}}(\Gamma_0)} + \|f\|_{L^2(\Omega)}). \quad (4.3)$$

*Proof of Proposition 4.2.* The saddle point formulation of the problem is: find  $(u, p) \in H^1(\Omega) \times L^2(\Omega)$  such that for all  $(v, q) \in H^1(\Omega) \times L^2(\Omega)$ :

$$\begin{cases} \int_{\Omega} \nabla u : \nabla v + \int_{\Gamma_{out}} qu \cdot v + \int_{\Omega} p \operatorname{div} v &= \langle g, v|_{\Gamma_0} \rangle_{-\frac{1}{2}, \frac{1}{2}, \Gamma_0} + \int_{\Omega} f \cdot v, \\ \int_{\Omega} q \operatorname{div} u &= 0. \end{cases} \quad (4.4)$$

We denote by,  $\forall (u, v) \in V$ ,

$$a_q(u, v) = \int_{\Omega} \nabla u : \nabla v + \int_{\Gamma_{out}} qu \cdot v, \quad (4.5)$$

and  $\forall v \in V$ ,

$$L_1(v) = \langle g, v|_{\Gamma_0} \rangle_{-\frac{1}{2}, \frac{1}{2}, \Gamma_0} + \int_{\Omega} f \cdot v.$$

We easily verify that  $a_q$  is a continuous symmetric bilinear form. Since  $q \geq \alpha > 0$ , according to the generalized Poincaré inequality, the bilinear form  $a_q$  is coercive. On the other hand,  $L_1$  is a continuous linear form on  $V$  and the divergence operator:

$$\begin{cases} B : H^1(\Omega) &\rightarrow L^2(\Omega), \\ u &\rightarrow \operatorname{div} u, \end{cases} \quad (4.6)$$

is surjective. Indeed, let  $p \in L^2(\Omega)$  and let us consider  $\phi$

$$\begin{cases} \Delta\phi = p, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.7)$$

By construction,  $\phi \in H^2(\Omega)$  and there exists  $C > 0$  such that:

$$\|\phi\|_{H^2(\Omega)} \leq C\|p\|_{L^2(\Omega)}.$$

Remark that  $\nabla\phi \in H^1(\Omega)$  satisfies  $\operatorname{div}(\nabla\phi) = \Delta\phi = p$ . Then, Theorem 2.6 implies that there exists a unique  $(u, p)$  solution of (4.4). By taking successively  $v = u$  and  $v = \nabla\phi$  in the first line of the saddle point formulation (4.4), we obtain (4.3).  $\square$

In order to study the Stokes system with Robin boundary conditions, one needs to specify to which space the Robin coefficient  $q$  belongs. As stated in Proposition 4.4, we will assume that  $q$  belongs to some Sobolev space  $H^s(\Gamma_{out})$  where  $s$  is large enough so that  $qu|_{\Gamma_{out}}$  belongs to  $H^r(\Gamma_{out})$  if  $u|_{\Gamma_{out}}$  belongs to  $H^r(\Gamma_{out})$ . This stability in the Sobolev spaces will allow to apply regularity result for the Stokes system with Neumann boundary condition (see Proposition 2.12). Before stating the regularity result, let us state the following lemma:

**Lemma 4.3.** *Let  $r, s \in \mathbb{R}$ , with  $s > \frac{d-1}{2}$  and  $0 \leq r \leq s$ . Let  $q \in H^s(\Gamma_{out})$ . The linear operator*

$$\begin{aligned} T : H^r(\Gamma_{out}) &\rightarrow H^r(\Gamma_{out}) \\ u &\mapsto qu \end{aligned}$$

*is continuous. Furthermore, the following estimate holds true*

$$\|qu\|_{H^r(\Gamma_{out})} \leq C\|q\|_{H^s(\Gamma_{out})}\|u\|_{H^r(\Gamma_{out})}.$$

*Proof of Lemma 4.3.* Since  $s > \frac{d-1}{2}$ ,  $H^s(\Gamma_{out})$  is a Banach algebra (see [2]) and thus  $T \in \mathcal{L}(H^s(\Gamma_{out}), H^s(\Gamma_{out}))$  and  $\|T\|_s = \sup_{u \in H^s(\Gamma_{out}), u \neq 0} \frac{\|Tu\|_{H^s(\Gamma_{out})}}{\|u\|_{H^s(\Gamma_{out})}} \leq \|q\|_{H^s(\Gamma_{out})}$ . Moreover, since  $H^s(\Gamma_{out}) \hookrightarrow L^\infty(\Gamma_{out})$ ,  $T \in \mathcal{L}(L^2(\Gamma_{out}), L^2(\Gamma_{out}))$  and  $\|T\|_0 = \sup_{u \in L^2(\Gamma_{out}), u \neq 0} \frac{\|Tu\|_{L^2(\Gamma_{out})}}{\|u\|_{L^2(\Gamma_{out})}} \leq \|q\|_{L^\infty(\Gamma_{out})} \leq C\|q\|_{H^s(\Gamma_{out})}$ . Thus, the result follows by interpolation (see [15] or [61]).  $\square$

From regularity result for the Stokes system with Neumann boundary condition (see Proposition 2.12) and Lemma 4.3, we deduce the following result:

**Proposition 4.4.** *Assume that  $\Omega$  is of class  $C^{k+1,1}$ . Let  $k \in \mathbb{N}$ ,  $s > \frac{d-1}{2}$  and  $s \geq \frac{1}{2} + k$ ,  $\alpha > 0$ ,  $M > 0$ ,  $f \in H^k(\Omega)$ ,  $g \in H^{\frac{1}{2}+k}(\Gamma_0)$  and  $q \in H^s(\Gamma_{out})$  be such that  $\alpha \leq q$  on  $\Gamma_{out}$ . Then, the solution  $(u, p)$  of system (4.2) belongs to  $H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ . Moreover, there exists a constant  $C(\alpha, M) > 0$  such that for every  $q \in H^s(\Gamma_{out})$  satisfying  $\|q\|_{H^s(\Gamma_{out})} \leq M$ ,*

$$\|u\|_{H^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \leq C(\alpha, M)(\|g\|_{H^{k+\frac{1}{2}}(\Gamma_0)} + \|f\|_{H^k(\Omega)}).$$

*Proof of Proposition 4.4.* Let us prove the result for  $k = 0$ . Let  $h = -qu|_{\Gamma_{out}} + g$ . According to Proposition 4.2,  $u$  belongs to  $H^1(\Omega)$ . We obtain from Lemma 4.3 for  $r = 1/2$  that  $qu|_{\Gamma_{out}} \in H^{\frac{1}{2}}(\Gamma_{out})$ , which implies, since  $g \in H^{\frac{1}{2}}(\Gamma_0)$  and  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ , that  $h \in H^{\frac{1}{2}}(\partial\Omega)$ . Using Proposition 2.12 with  $k = 0$  we obtain that  $(u, p) \in H^2(\Omega) \times H^1(\Omega)$  and:

$$\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C(\|h\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|f\|_{L^2(\Omega)}).$$



But, since by assumption,  $\|q\|_{H^s(\Gamma_{out})} \leq M$ , we have from Lemma 4.3 with  $r = 1/2$ , that:

$$\|h\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(M)(\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_0)}),$$

We obtain:

$$\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C(M)(\|g\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}).$$

Thus, we obtain the result for  $k = 0$  using the inequality of Proposition 4.2. We then proceed by recurrence to prove the result for any  $k \in \mathbb{N}$ .  $\square$

**Remark 4.5.** *Note that the space to which the Robin coefficient  $q$  belongs is not optimal. One could surely obtain similar regularity result for a less regular Robin coefficient. In fact, the key argument to proceed by induction in the proof of Proposition 4.4 is that  $qu|_{\Gamma_{out}} \in H^{k+\frac{1}{2}}(\Gamma_{out})$ , for  $u \in H^{k+\frac{1}{2}}(\Gamma_{out})$  (this property allows to apply the regularity result given by Proposition 2.12).*

### 4.3 An identifiability result

Proposition 4.6 is a consequence of the unique continuation result obtained by C. Fabre and G. Lebeau, we have recalled in Theorem 3.1. More precisely, we use Corollary 3.2 to prove it.

**Proposition 4.6.** *Let  $\alpha > 0$ ,  $x_0 \in \Gamma_0$ ,  $r > 0$ ,  $s > \frac{d-1}{2}$ ,  $g \in H^{-\frac{1}{2}}(\Gamma_0)$  be non identically zero and  $q_j \in H^s(\Gamma_{out})$  such that  $\alpha \leq q_j$  almost everywhere on  $\Gamma_{out}$  for  $j = 1, 2$ . Let  $(u_j, p_j)$  be the weak solutions of (4.1) with  $q = q_j$  for  $j = 1, 2$ .*

*We assume that  $u_1 = u_2$  on  $\mathcal{B}(x_0, r) \cap \Gamma_0$ . Then  $q_1 = q_2$ .*

**Remark 4.7.** *Note that Proposition 4.6 still holds if we do not assume anymore that  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$  because we do not need global regularity on the solution of the Stokes system (4.1).*

*Proof of Proposition 4.6.* We are going to prove Proposition 4.6 by contradiction: we assume that  $q_1$  is not identically equal to  $q_2$ .

We define  $u = u_1 - u_2$  and  $p = p_1 - p_2$ . Let us notice that  $(u, p)$  is the solution of the following problem:

$$\left\{ \begin{array}{ll} -\Delta u + \nabla p & = 0, \quad \text{in } \Omega, \\ \operatorname{div} u & = 0, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn & = 0, \quad \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + q_1 u_1 - q_2 u_2 & = 0, \quad \text{on } \Gamma_{out}. \end{array} \right.$$

By assumption,  $u = 0$  and  $\frac{\partial u}{\partial n} - pn = 0$  on  $\mathcal{B}(x_0, r) \cap \Gamma_0$ . Thus, according to Corollary 3.2,  $u_1 = u_2$  and  $p_1 = p_2$  in  $\Omega$ . Consequently, we deduce from

$$\begin{aligned} \frac{\partial u_1}{\partial n} - p_1 n + q_1 u_1 &= 0, \quad \text{on } \Gamma_{out}, \\ \frac{\partial u_1}{\partial n} - p_1 n + q_2 u_1 &= 0, \quad \text{on } \Gamma_{out}, \end{aligned}$$

that

$$u_1(q_1 - q_2) = 0 \text{ on } \Gamma_{out}. \quad (4.8)$$

By assumption,  $q_1$  is not identically equal to  $q_2$ . Since  $s > \frac{d-1}{2}$ ,  $q_1$  and  $q_2$  are continuous on  $\Gamma_{out}$ . Then, we can find an open set  $\kappa \subset \Gamma_{out}$  with a positive measure such that:

$$(q_1 - q_2)(x) \neq 0, \forall x \in \kappa.$$

Equation (4.8) implies that  $u_1 \equiv 0$  on  $\kappa$  and then  $u_1$  is the solution of

$$\begin{cases} -\Delta u_1 + \nabla p_1 = 0, & \text{in } \Omega, \\ \operatorname{div} u_1 = 0, & \text{in } \Omega, \\ u_1 = 0, & \text{on } \kappa, \\ \frac{\partial u_1}{\partial n} - p_1 n = 0, & \text{on } \kappa. \end{cases}$$

Applying again Corollary 3.2, we obtain that  $u_1 = 0$  and  $p_1 = 0$  in  $\Omega$ . This is in contradiction with the assumption that  $g$  is non identically zero.  $\square$

## 4.4 A stability estimate valid in dimension 2

We assume in this subsection that  $d = 2$ . The main result of this section is given in Theorem 4.8.

**Theorem 4.8.** *Let  $\Gamma \subseteq \Gamma_0$  be a non empty set. We assume that  $\Omega \subset \mathbb{R}^2$  is of class  $\mathcal{C}^{3,1}$ . Let  $\alpha > 0$ ,  $M_1 > 0$ ,  $M_2 > 0$ ,  $(g, q_j) \in H^{\frac{5}{2}}(\Gamma_0) \times H^{\frac{5}{2}}(\Gamma_{out})$  for  $j = 1, 2$  such that  $g$  is not identically zero,  $\|g\|_{H^{\frac{5}{2}}(\Gamma_0)} \leq M_1$ ,  $\alpha \leq q_j$  on  $\Gamma_{out}$  and  $\|q_j\|_{H^{\frac{5}{2}}(\Gamma_{out})} \leq M_2$ . We denote by  $(u_j, p_j)$  the solution of (4.1) associated to  $q = q_j$  for  $j = 1, 2$ . Let  $K$  be a compact subset of  $\{x \in \Gamma_{out}/u_1(x) \neq 0\}$  and let  $m > 0$  be a constant such that  $|u_1| \geq m$  on  $K$ .*

*Then, there exists positive constants  $C(M_1, M_2, \alpha)$  and  $C_1(M_1, M_2, \alpha)$  such that*

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(M_1, M_2, \alpha)}{\left( \ln \left( \frac{C_1(M_1, M_2, \alpha)}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right)} \right)^{\frac{1}{2}}. \quad (4.9)$$

**Remark 4.9.** *Since  $g$  is not identically zero, Corollary 3.2 ensures that  $\{x \in \Gamma_{out}/u_1(x) \neq 0\}$  is not empty. Moreover, according to Proposition 4.4,  $u_1$  is continuous, thus we obtain the existence of a compact  $K$  and a constant  $m$  as in Theorem 4.8. We notice however that the constants involved in the estimate (4.9) and the set  $K$  depend on  $u_1$ . Finding a uniform lower bound for a solution  $u$  of system (4.1) remains an open question. We refer to [27], [4] and [5] for the case of the scalar Laplace equation.*

**Remark 4.10.** *Comparing this result with Proposition 4.6, we can notice that we need additional measurements on the solution. In Proposition 4.6, we only need that  $u_1 = u_2$  and  $\frac{\partial u_1}{\partial n} - p_1 n = \frac{\partial u_2}{\partial n} - p_2 n$  on  $\Gamma \subseteq \Gamma_0$  in order to get identifiability of Robin coefficient  $q$ . Here, besides a measurement on  $u_1 - u_2$ , we need measurements on  $\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}$  (or  $p_1 - p_2$ ) and  $\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}$ .*

**Remark 4.11.** *In [28], the same kind of inequality is proved for the Laplacian problem with Robin boundary conditions under the hypothesis that the measurements on all  $\Gamma_0$  are small enough. Here, we free ourselves from this smallness assumption on the measurements and we only need measurements on some part  $\Gamma$  of the boundary included in  $\Gamma_0$ .*

The logarithmic stability estimate obtained here is based on a *so called* Carleman inequality proved by Bukhgeim and valid only in dimension  $d = 2$ :

**Lemma 4.12.** *Let  $\Psi \in C^2(\overline{\Omega})$ . We have:*

$$\int_{\Omega} (\Delta \Psi |u|^2 + (\Delta \Psi - 1) |\nabla u|^2) e^{\Psi} \leq \int_{\Omega} |\Delta u|^2 e^{\Psi} + \int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left( |u|^2 + |\nabla u|^2 + 2 \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) e^{\Psi} \quad (4.10)$$

for all  $u \in C^2(\overline{\Omega})$ .

*Proof of Lemma 4.12.* This inequality is only valid in two dimensions: it uses computational properties of function defined on  $\mathbb{C}$  (in particular, the fact that  $4\partial_{\bar{z}}\partial_z = \Delta$ ). We refer to [23] for more details.  $\square$

**Remark 4.13.** *This lemma is still true for  $u \in H^3(\Omega)$ . Indeed, let us consider  $u \in H^3(\Omega)$ . There exists  $(u_n)_{n \in \mathbb{N}} \in C^2(\overline{\Omega})^{\mathbb{N}}$  such that*

$$u_n \rightarrow u \text{ in } H^3(\Omega). \quad (4.11)$$

We can apply Lemma 4.12 to  $u_n$ , for all  $n \in \mathbb{N}$ . Let us prove that:

$$\lim_{n \rightarrow \infty} \int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left| \frac{\partial |\nabla u_n|^2}{\partial \tau} \right| e^{\Psi} = \int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| e^{\Psi}. \quad (4.12)$$

Note first that  $\int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| e^{\Psi}$  has a meaning for  $u \in H^3(\Omega)$ :

$$\int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| e^{\Psi} \leq 2 \|\Psi\|_{C^1(\overline{\Omega})} \|e^{\Psi}\|_{C^0(\overline{\Omega})} \left( \sum_{i=1}^2 \int_{\partial\Omega} |\nabla u| \cdot |\nabla \partial_i u| \right) < \infty.$$

We have:

$$\begin{aligned} & \int_{\partial\Omega} \left| \frac{\partial \Psi}{\partial n} \right| \left| \left| \frac{\partial |\nabla u_n|^2}{\partial \tau} \right| - \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right| e^{\Psi} \\ & \leq C \|\Psi\|_{C^1(\overline{\Omega})} \|e^{\Psi}\|_{C^0(\overline{\Omega})} \left[ \sum_{i,j=1}^2 \left( \int_{\partial\Omega} |\partial_j u|^2 \right)^{\frac{1}{2}} \left( \int_{\partial\Omega} |\partial_{ij} u_n - \partial_{ij} u|^2 \right)^{\frac{1}{2}} \right. \\ & \quad \left. + \left( \int_{\partial\Omega} |\partial_{ij} u_n|^2 \right)^{\frac{1}{2}} \left( \int_{\partial\Omega} |\partial_j u_n - \partial_j u|^2 \right)^{\frac{1}{2}} \right]. \end{aligned}$$

According to (4.11), the sequence  $(\partial_{ij} u_n)_{n \in \mathbb{N}}$  converges in  $L^2(\partial\Omega)$  towards  $\partial_{ij} u$  and  $\|\partial_{ij} u_n\|_{L^2(\partial\Omega)}$  is bounded by a constant independent of  $n$ . Then, equality (4.12) follows by passing in the limit in the last inequality.

Let us give an idea of the proof of Theorem 4.8. Firstly, we apply inequality (4.10) to  $u$ . Doing so, there is a term in  $\nabla p$  which appears in the right hand-side. So, in order to get rid of it, we apply inequality (4.10) to  $p$ , keeping in mind that  $\Delta p = \text{div}(\nabla u) = 0$  and we add up the two inequalities. Now, terms in  $\nabla p$  appear in each side of the inequality. A suitable choice of the weight function  $\Psi$  allows us to eliminate the term in  $\nabla p$  in the right hand-side (by passing it in the left hand-side, it is absorbed by the term in  $\nabla p$  in the left hand-side). Consequently, it remains in the right hand-side boundary terms which are bigger than positive terms. We separate the boundary terms on  $\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)$  and on

$\Gamma$ . Thanks to a suitable choice of  $\Psi$ , boundary terms on  $\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)$  are negative. We can pass them in the left hand-side: we obtain an estimation of boundary term on  $\Gamma_{out}$  by boundary terms on  $\Gamma$ . Then, using an optimization argument, we obtain a logarithmic stability estimate on boundary term on  $\Gamma_{out}$  with respect to boundary term on  $\Gamma$  (see Lemma 4.14). Finally, the use of the boundary condition on  $\Gamma_{out}$  for  $(u, p)$  solution of system (4.1) gives us the stability estimate of Theorem 4.8.

Let us begin by proving the following intermediate result which gives us a logarithmic estimate of the traces of  $u$ ,  $\nabla u$ ,  $p$ ,  $\nabla p$  over  $\Gamma_{out}$  with respect to the ones over  $\Gamma \subseteq \Gamma_0$ .

**Lemma 4.14.** *Let  $\Gamma \subseteq \Gamma_0$  be a non empty set. Let  $(u, p) \in H^3(\Omega) \times H^3(\Omega)$  be the solution in  $\Omega$  of*

$$\begin{cases} -\Delta u + \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{cases}$$

Then, there exist  $C > 0$ ,  $C_1 > 0$  and  $d_0 > 0$  such that for all  $\tilde{d} > d_0$ :

$$\begin{aligned} & \|u\|_{L^2(\Gamma_{out})} + \|\nabla u\|_{L^2(\Gamma_{out})} + \|p\|_{L^2(\Gamma_{out})} + \|\nabla p\|_{L^2(\Gamma_{out})} \\ & \leq \tilde{d}C \frac{\|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)}}{\left( \ln \left( \tilde{d}^2 C_1 \frac{\|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)}}{\|u\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right)} \right)^{\frac{1}{2}}. \end{aligned}$$

*Proof of Lemma 4.14.* The proof is based on the Carleman inequality of Lemma 4.12 for an appropriate choice of  $\Psi$ . Note that we will use (4.10) twice: one time for the velocity  $u$  and one time for the pressure  $p$ . The weight function  $\Psi$  is chosen in order to estimate the traces on  $\Gamma_{out}$  with respect to the ones on  $\Gamma$ .

Step 1: choice of  $\Psi$ .

We slightly modify the weight function  $\Psi$  constructed in [28] in order to take measurements only on  $\Gamma$  included in  $\Gamma_0$ .

There exists  $\Psi_0 \in \mathcal{C}^2(\overline{\Omega})$  non identically zero such that:

$$\Delta \Psi_0 = 0 \text{ in } \Omega, \quad \Psi_0 = 0 \text{ on } \Gamma_{out} \cup (\Gamma_0 \setminus \Gamma), \quad \Psi_0 \geq 0 \text{ on } \Gamma, \quad \frac{\partial \Psi_0}{\partial n} < 0 \text{ on } \Gamma_{out} \cup (\Gamma_0 \setminus \Gamma).$$

Indeed, let  $\chi \in \mathcal{C}^2(\partial\Omega)$  such that

$$\chi = 0 \text{ on } \Gamma_{out} \cup (\Gamma_0 \setminus \Gamma), \quad \chi \geq 0 \text{ on } \Gamma,$$

and  $\chi$  non identically zero on  $\Gamma$ . The boundary value problem:

$$\begin{cases} \Delta \Psi_0 &= 0, & \text{in } \Omega, \\ \Psi_0 &= \chi, & \text{on } \partial\Omega, \end{cases}$$

has a unique solution  $\Psi_0 \in \mathcal{C}^2(\overline{\Omega})$ . Note that  $\Psi_0$  is not constant because  $\chi$  is non identically zero. So, from the strong maximum principle,  $\Psi_0 > 0$  in  $\Omega$ . According to Hopf Lemma, we have  $\frac{\partial \Psi_0}{\partial n} < 0$  on  $\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)$ .

We refer to Annex B for a statement of the strong maximum principle and of Hopf Lemma.

Let  $\lambda > 0$ . Let us denote by  $\Psi_1 \in C^2(\bar{\Omega})$  the unique solution of the boundary value problem:

$$\begin{cases} \Delta \Psi_1 = \lambda, & \text{in } \Omega, \\ \Psi_1 = 0, & \text{on } \partial\Omega. \end{cases}$$

From the comparison principle and the strong maximum principle, we have  $\Psi_1 < 0$  in  $\Omega$ . Moreover, according to the Hopf Lemma, we have  $\frac{\partial \Psi_1}{\partial n} > 0$  on  $\partial\Omega$ .

Let us consider  $\Psi = \Psi_1 + s\Psi_0$  for  $s > 0$ . To summarize, the function  $\Psi$  has the following properties:  $\Delta \Psi = \lambda$  in  $\Omega$ ,  $\Psi = 0$  on  $\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)$ ,  $\Psi \geq 0$  on  $\Gamma$ , and  $s \frac{\partial \Psi_0}{\partial n} \leq \frac{\partial \Psi}{\partial n} \leq \frac{\partial \Psi_1}{\partial n}$  on  $\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)$ .

Step 2: We apply Lemma 4.12 to  $u$ . Using the fact that  $\Delta u = \nabla p$ , we have:

$$\int_{\Omega} (\Delta \Psi |u|^2 + (\Delta \Psi - 1) |\nabla u|^2) e^{\Psi} \leq \int_{\Omega} |\nabla p|^2 e^{\Psi} + \int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left( |u|^2 + |\nabla u|^2 + 2 \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) e^{\Psi}. \quad (4.13)$$

We then apply once again Lemma 4.12 to  $p$ , it yields to:

$$\int_{\Omega} (\Delta \Psi |p|^2 + (\Delta \Psi - 1) |\nabla p|^2) e^{\Psi} \leq \int_{\Omega} |\Delta p|^2 e^{\Psi} + \int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left( |p|^2 + |\nabla p|^2 + 2 \left| \frac{\partial |\nabla p|^2}{\partial \tau} \right| \right) e^{\Psi}. \quad (4.14)$$

We have  $\Delta p = \operatorname{div}(\Delta u) = 0$ , hence  $\int_{\Omega} |\Delta p|^2 e^{\Psi} = 0$ . We now choose  $\lambda \geq 2$ . By summing up inequalities (4.13) and (4.14) and eliminating the integrals over  $\Omega$  in the left hand-side which are positive terms, we obtain:

$$\int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left( |u|^2 + |\nabla u|^2 + 2 \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) e^{\Psi} + \int_{\partial\Omega} \frac{\partial \Psi}{\partial n} \left( |p|^2 + |\nabla p|^2 + 2 \left| \frac{\partial |\nabla p|^2}{\partial \tau} \right| \right) e^{\Psi} \geq 0.$$

We now specify the dependence with respect to  $s$ . We denote  $\theta = \min_{\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)} \left| \frac{\partial \Psi_0}{\partial n} \right|$ .

We note that on  $\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)$ ,  $e^{\Psi} = 1$ . Consequently:

$$\begin{aligned} & -s\theta \int_{\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) + \int_{\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)} \frac{\partial \Psi_1}{\partial n} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \\ & + 2 \int_{\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)} \frac{\partial \Psi}{\partial n} \left( \left| \frac{\partial |\nabla p|^2}{\partial \tau} \right| + \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) + 2 \int_{\Gamma} \frac{\partial \Psi}{\partial n} \left( \left| \frac{\partial |\nabla p|^2}{\partial \tau} \right| + \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) e^{\Psi} \\ & + \int_{\Gamma} \frac{\partial \Psi}{\partial n} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) e^{\Psi} \geq 0. \quad (4.15) \end{aligned}$$

Let us study each of the terms. We have:

$$\int_{\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)} \frac{\partial \Psi_1}{\partial n} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq C(\|u\|_{H^3(\Omega)}^2 + \|p\|_{H^3(\Omega)}^2).$$

Moreover, since  $\frac{\partial \Psi}{\partial n} \leq \frac{\partial \Psi_1}{\partial n}$  on  $\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)$ :

$$2 \int_{\Gamma_{out} \cup (\Gamma_0 \setminus \Gamma)} \frac{\partial \Psi}{\partial n} \left( \left| \frac{\partial |\nabla p|^2}{\partial \tau} \right| + \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) \leq C(\|u\|_{H^3(\Omega)}^2 + \|p\|_{H^3(\Omega)}^2).$$

Since, on  $\Gamma$ ,  $\left| \frac{\partial \Psi}{\partial n} \right| \leq sC$  for  $s \geq 1$ , we have

$$\int_{\Gamma} \frac{\partial \Psi}{\partial n} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) e^{\Psi} \leq Cs \int_{\Gamma} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) e^{\Psi}.$$

Using moreover Cauchy-Schwarz inequality, we obtain:

$$2 \int_{\Gamma} \frac{\partial \Psi}{\partial n} \left( \left| \frac{\partial |\nabla p|^2}{\partial \tau} \right| + \left| \frac{\partial |\nabla u|^2}{\partial \tau} \right| \right) e^{\Psi} \leq sC (\|p\|_{H^3(\Omega)} + \|u\|_{H^3(\Omega)}) \left( \int_{\Gamma} (|\nabla p|^2 + |\nabla u|^2) e^{2\Psi} \right)^{\frac{1}{2}}.$$

Note that  $e^{\Psi}$  depends on  $s$  over  $\Gamma$ . Hence, reassembling these inequalities, inequality (4.15) becomes:

$$\theta \int_{\Gamma_{out}} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq C \left( K_s + \frac{1}{s} (\|u\|_{H^3(\Omega)}^2 + \|p\|_{H^3(\Omega)}^2) \right), \quad (4.16)$$

where

$$K_s = \int_{\Gamma} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) e^{\Psi} + (\|p\|_{H^3(\Omega)} + \|u\|_{H^3(\Omega)}) \left( \int_{\Gamma} (|\nabla p|^2 + |\nabla u|^2) e^{2\Psi} \right)^{\frac{1}{2}}.$$

In order to precise the dependence with respect to  $s$  of  $K_s$ , we denote by:

$$\delta = \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}. \quad (4.17)$$

Remark that, thanks to classical interpolation inequalities (see [1]), there exists  $C > 0$  such that for all  $f \in H^2(\Gamma)$ :

$$\left\| \frac{\partial f}{\partial \tau} \right\|_{L^2(\Gamma)} \leq \|f\|_{H^1(\Gamma)} \leq C \|f\|_{L^2(\Gamma)}^{\frac{1}{2}} \|f\|_{H^2(\Gamma)}^{\frac{1}{2}}.$$

Applying the previous inequality, there exists  $C > 0$  such that:

$$\int_{\Gamma} \left| \frac{\partial u}{\partial \tau} \right|^2 \leq C \|u\|_{H^3(\Omega)} \|u\|_{L^2(\Gamma)}, \quad \text{and} \quad \int_{\Gamma} \left| \frac{\partial p}{\partial \tau} \right|^2 \leq C \|p\|_{H^3(\Omega)} \|p\|_{L^2(\Gamma)}. \quad (4.18)$$

We obtain, using the fact that  $\nabla v = \frac{\partial v}{\partial n} n + \frac{\partial v}{\partial \tau} \tau$  on  $\partial\Omega$  for any scalar function  $v \in H^2(\Omega)$ , and inequality (4.18), that there exists  $C > 0$  such that:

$$\begin{aligned} & \left( \int_{\Gamma} (|\nabla p|^2 + |\nabla u|^2) e^{2\Psi} \right)^{\frac{1}{2}} \\ & \leq e^{ks} \left( \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}^2 + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)}^2 + C \|u\|_{H^3(\Omega)} \|u\|_{L^2(\Gamma)} + C \|p\|_{H^3(\Omega)} \|p\|_{L^2(\Gamma)} \right)^{\frac{1}{2}} \\ & \leq C e^{ks} (\|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)})^{\frac{1}{2}} \left( \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} \right)^{\frac{1}{2}} \\ & \leq C e^{ks} (\|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)})^{\frac{1}{2}} \delta^{\frac{1}{2}} \end{aligned}$$

where  $k = \max_{\Gamma} \Psi_0$ . Similarly, we prove that

$$\int_{\Gamma} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) e^{\Psi} \leq C e^{ks} (\|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)}) \delta \leq C e^{ks} (\|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)})^{\frac{3}{2}} \delta^{\frac{1}{2}}.$$

Thus, combining the two previous inequalities and remembering the definition of  $K_s$ , we obtain,

$$K_s \leq C e^{ks} (\|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)})^{\frac{3}{2}} \delta^{\frac{1}{2}}.$$

Let us denote by  $A = \|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)}$ . Hence we get from (4.16):

$$\int_{\Gamma_{out}} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq C \left( A^{\frac{3}{2}} e^{ks} \delta^{\frac{1}{2}} + \frac{A^2}{s} \right),$$

for all  $s \geq 1$ . Remark that this inequality is trivially verified for  $0 < s \leq 1$  by continuity of the trace mapping. Let  $\tilde{d} \geq 1$ . To summarize, we have proved that:

$$\int_{\Gamma_{out}} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq C A^{\frac{3}{2}} \left( e^{ks} \delta^{\frac{1}{2}} + \frac{\tilde{d} A^{\frac{1}{2}}}{s} \right), \forall s > 0.$$

We now optimize the upper bound with respect to  $s$ . We denote by  $f(s) = e^{ks} \delta^{\frac{1}{2}} + \frac{\tilde{d} A^{\frac{1}{2}}}{s}$ . Let us study the function  $f$  in  $\mathbb{R}_+^*$ . We have:

$$\begin{cases} \lim_{s \rightarrow 0} f(s) = +\infty, \\ \lim_{s \rightarrow \infty} f(s) = +\infty. \end{cases}$$

So since  $f$  is continuous on  $\mathbb{R}_+^*$ ,  $f$  reaches its minimum at a point  $s_0 > 0$ . In this point,

$$f'(s_0) = 0 \Leftrightarrow \delta^{\frac{1}{2}} = \frac{e^{-ks_0} \tilde{d} A^{\frac{1}{2}}}{k s_0^2}, \text{ thus } f(s_0) = \frac{\tilde{d} A^{\frac{1}{2}}}{k s_0^2} + \frac{\tilde{d} A^{\frac{1}{2}}}{s_0}.$$

Hence:

$$\int_{\Gamma_{out}} (|u|^2 + |\nabla u|^2 + |p|^2 + |\nabla p|^2) \leq \frac{C \tilde{d} A^2}{s_0^\beta} \left( \frac{1}{k} + 1 \right),$$

where  $\beta = 1$  if  $s_0 \geq 1$  and  $\beta = 2$  otherwise. But:

$$\frac{1}{\delta^{\frac{1}{2}}} = \frac{k s_0^2 e^{ks_0}}{\tilde{d} A^{\frac{1}{2}}} \leq \frac{k e^{(k+2)s_0}}{\tilde{d} A^{\frac{1}{2}}},$$

that is to say:

$$\frac{1}{s_0} \leq \frac{k+2}{\ln \left( \frac{\tilde{d} A^{\frac{1}{2}}}{k \delta^{\frac{1}{2}}} \right)},$$

for all  $\tilde{d}$  large enough. In the same way, when  $s_0 < 1$ , we obtain:

$$\frac{1}{s_0^2} \leq \frac{k+1}{\ln \left( \frac{\tilde{d} A^{\frac{1}{2}}}{k e^k \delta^{\frac{1}{2}}} \right)},$$

for all  $d$  large enough. Using the fact that  $\ln \left( x^{\frac{1}{2}} \right) = \frac{1}{2} \ln(x)$  for all  $x > 0$  and remembering the definition (4.17) of  $\delta$ , the desired result follows.  $\square$

Let us now prove Theorem 4.8.

*Proof of Theorem 4.8.* Let us emphasize the fact that, thanks to Proposition 4.4, there exists  $C(\alpha, M_1, M_2) > 0$  such that:

$$\|u_j\|_{H^4(\Omega)} + \|p_j\|_{H^3(\Omega)} \leq C(\alpha, M_1, M_2), \text{ for } j = 1, 2. \quad (4.19)$$

In the following, we denote by  $u = u_1 - u_2$  and  $p = p_1 - p_2$ . We have:

$$(q_2 - q_1)u_1 = q_2u + \frac{\partial u}{\partial n} - pn, \text{ on } \Gamma_{out}. \quad (4.20)$$

Consequently, since  $|u_1| \geq m > 0$  on  $K$ :

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m}C(M_2) \left( \|u\|_{L^2(K)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(K)} + \|p\|_{L^2(K)} \right). \quad (4.21)$$

Let us denote by  $A = \|u\|_{H^3(\Omega)} + \|p\|_{H^3(\Omega)}$  and  $B = \|u\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}$ . By applying Lemma 4.14, we obtain that there exists  $C(M_2) > 0$ ,  $C > 0$  and  $d_0 > 0$  such that for all  $\tilde{d} > d_0$ :

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \tilde{d} C(M_2) \frac{A}{\left( \ln \left( \tilde{d}^2 C_1 \frac{A}{B} \right) \right)^{\frac{1}{2}}}. \quad (4.22)$$

We are going to conclude the proof by studying the variation of the function defined by  $f_y(x) = \frac{x}{(\ln(\frac{x}{y}))^{\frac{1}{2}}}$  on  $(y, +\infty)$ , for  $y = \frac{B}{C_1 \tilde{d}^2}$ . We have  $f'_y(x) = \frac{\ln(\frac{x}{y}) - \frac{1}{2}}{\ln(\frac{x}{y})^{\frac{3}{2}}}$ . Let us denote by  $x_0 = ye^{\frac{1}{2}}$ . The function  $f_y$  is decreasing on  $(y, x_0]$  and is increasing on  $[x_0, +\infty)$ . For  $\tilde{d}$  large enough, we have  $A \geq x_0$ . Using (4.19) and since  $f$  is increasing on  $[x_0, +\infty)$ , we directly deduce that  $f_{\frac{B}{\tilde{d}}} (A) \leq f_{\frac{B}{\tilde{d}}} (C(\alpha, M_1, M_2))$ .

Using this result in (4.22), we get that there exist  $C(\alpha, M_1, M_2) > 0$  and  $C_1(\alpha, M_1, M_2) > 0$  such that:

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left( \ln \left( \frac{C_1(\alpha, M_1, M_2)}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^{\frac{1}{2}}}.$$

Since  $\frac{\partial u}{\partial n} = pn$  on  $\Gamma$ , we obtain the desired inequality.  $\square$

**Remark 4.15.** Note that the assumption that  $|u_1| \geq m > 0$  on  $K$  is essential to pass from (4.20) to (4.21). Outside the set  $K$ , an estimate of  $q_1 - q_2$  may be undetermined or highly unstable. In particular, an estimate of the Robin coefficients on the whole set  $\Gamma_{out}$  might be worst than of logarithmic type (see [11]).

Note however that for a simplified problem, it is in fact possible to obtain a logarithmic stability estimate on the whole set  $\Gamma_{out}$  which does not depend on a given reference solution. Assume that  $g = g_e n$  and  $q \in \mathbb{R}$  are such that

- (A)  $g_e \in \mathbb{R}$  satisfies  $\beta \leq g_e \leq M_1$ ,
- (B)  $\alpha \leq q \leq M_2$ ,



for some  $\alpha > 0$ ,  $\beta > 0$ ,  $M_1 > 0$  and  $M_2 > 0$ .

We denote by  $(u_{g_e, q}, p_{g_e, q})$  the solution of system (4.1) associated to  $q$  and  $g = g_e n$ . Thanks to the weak formulation of the problem,  $\int_{\Gamma_0} u_{g, q} \cdot n > 0$ . Moreover, one can prove by contradiction and thanks to the continuity of the solution with respect to the data that there exists  $m_1 > 0$  which depends on  $M_1$ ,  $M_2$ ,  $\alpha$  and  $\beta$  such that for all  $(g_e, q) \in \mathbb{R}^2$  which satisfies (A) and (B),

$$\int_{\Gamma_0} u_{g_e, q} \cdot n \geq m_1.$$

For  $i = 1, 2$ , let  $q_i \in \mathbb{R}$  satisfy the assumption (B) above. We define  $(u_i, p_i) = (u_{g_e, q_i}, p_{g_e, q_i})$  the solution of system (4.1) associated with  $g = g_e n$  and  $q = q_i$  for  $i=1, 2$ . If we multiply (4.20) by the unit normal  $n$  and we integrate on  $\Gamma_{out}$ , we obtain:

$$(q_2 - q_1) \int_{\Gamma_{out}} u_1 \cdot n = q_2 \int_{\Gamma_{out}} (u_1 - u_2) \cdot n + \int_{\Gamma_{out}} \left( \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right) \cdot n - \int_{\Gamma_{out}} (p_1 - p_2).$$

Since  $u_1$  is divergence free,  $\left| \int_{\Gamma_{out}} u_1 \cdot n \right| = \left| \int_{\Gamma_0} u_1 \cdot n \right| \geq m_1$ . Thus, we get

$$|q_1 - q_2| \leq C(M_1, M_2, \alpha, \beta) \left( \|u_1 - u_2\|_{L^2(\Gamma_{out})} + \left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right\|_{L^2(\Gamma_{out})} + \|p_1 - p_2\|_{L^2(\Gamma_{out})} \right).$$

We conclude as in the proof of Theorem 4.8 and obtain that positive constants  $C(M_1, M_2, \alpha)$  and  $C_1(M_1, M_2, \alpha, \beta)$  such that

$$|q_1 - q_2| \leq \frac{C(M_1, M_2, \alpha, \beta)}{\left( \ln \left( \frac{C_1(M_1, M_2, \alpha)}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^{\frac{1}{2}}}.$$

## 4.5 A stability estimate valid in any space dimension.

In [10], M. Bellassoued, J. Cheng and M. Choulli used a unique continuation estimate for the Laplacian proved by K. D. Phung in [68] to obtain a logarithmic stability estimate for the Laplacian. In the same spirit, we apply unique continuation estimates for the Stokes system proved in Chapter 3 to obtain a logarithmic stability estimate. The main result of this subsection is the following theorem.

**Theorem 4.16.** *Let  $\Gamma \subseteq \Gamma_0$  be a nonempty open set of the boundary of  $\Omega$ . We assume that  $\Gamma$  and  $\Gamma_{out}$  are of class  $\mathcal{C}^\infty$ . Let  $k \in \mathbb{N}^*$  such that  $k + 2 > \frac{d}{2}$ ,  $\alpha > 0$ ,  $M_1 > 0$ ,  $M_2 > 0$  and  $s \in \mathbb{R}$  be such that  $s > \frac{d-1}{2}$  and  $s \geq \frac{1}{2} + k$ . We assume that  $(g, q_j) \in H^{\frac{1}{2}+k}(\Gamma_0) \times H^s(\Gamma_{out})$ , for  $j = 1, 2$ , are such that  $g$  is non identically zero,  $\|g\|_{H^{\frac{1}{2}+k}(\Gamma_0)} \leq M_1$ ,  $q_j \geq \alpha$  on  $\Gamma_{out}$  and  $\|q_j\|_{H^s(\Gamma_{out})} \leq M_2$ . We denote by  $(u_j, p_j)$  the solution of system (4.1) associated to  $q = q_j$ . Let  $K$  be a compact set of  $\{x \in \Gamma_{out}, u_1 \neq 0\}$  and  $m > 0$  be such that  $|u_1| \geq m$  on  $K$ .*

*Then, there exists  $C(\alpha, M_1, M_2) > 0$  and  $C_1(\alpha, M_1, M_2) > 0$  such that*

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left( \ln \left( \frac{C_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^{\frac{3}{4}\beta}}. \quad (4.23)$$

**Remark 4.17.** Note that Remark 4.9 is still relevant here.

*Proof of Theorem 4.16.* Let us emphasize the fact that, thanks to Proposition 4.4, there exists  $C(\alpha, M_1, M_2) > 0$  such that:

$$\|u_j\|_{H^{k+2}(\Omega)} + \|p_j\|_{H^{k+1}(\Omega)} \leq C(\alpha, M_1, M_2), \text{ for } j = 1, 2. \quad (4.24)$$

In the following, we denote by  $u = u_1 - u_2$  and  $p = p_1 - p_2$ . We have:

$$(q_2 - q_1)u_1 = q_2u + \frac{\partial u}{\partial n} - pn, \text{ on } \Gamma_{out}.$$

Consequently, since  $|u_1| \geq m > 0$  on  $K$ :

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m}C(M_2) \left( \|u\|_{L^2(K)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(K)} + \|p\|_{L^2(K)} \right).$$

Since  $K$  and  $\Gamma$  are in  $\mathcal{C}^\infty$ , we can construct an open set  $\omega \subset \Omega$  of class  $\mathcal{C}^\infty$  such that  $K \subset \partial\omega$  and  $\Gamma \subset \partial\omega$ . Then, for all  $0 < \epsilon < \frac{3}{2}$ , using the trace continuity and an interpolation inequality, we have

$$\begin{aligned} \|q_1 - q_2\|_{L^2(K)} &\leq \frac{1}{m}C(M_2) \left( \|u\|_{H^{3/2+\epsilon}(\omega)} + \|p\|_{H^1(\omega)} \right) \\ &\leq \frac{1}{m}C(M_2) \left( \|u\|_{H^1(\omega)}^\theta \|u\|_{H^3(\omega)}^{1-\theta} + \|p\|_{H^1(\omega)} \right) \end{aligned} \quad (4.25)$$

where  $\theta = \frac{3}{4} \left( 1 - \frac{2\epsilon}{3} \right)$ . According to inequality (4.24), we then deduce:

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m}C(\alpha, M_1, M_2) \left( \|u\|_{H^1(\omega)}^\theta + \|p\|_{H^1(\omega)}^\theta \right).$$

Let  $\beta \in (0, 1)$  be fixed. We choose  $0 < \epsilon < \frac{3}{2}$  small enough such that  $\beta' = \frac{\beta}{1 - \frac{2\epsilon}{3}}$  belongs to  $(0, 1)$ . We denote by  $A = \|u\|_{H^2(\omega)} + \|p\|_{H^2(\omega)}$  and  $B = \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}$ .

Applying inequality (3.3) of Theorem 3.5 with  $\nu = \frac{1}{2}$  and with  $\beta'$ , we get that there exists  $d_0 > 0$  such that for all  $\tilde{d} > d_0$ , there exists  $C(\alpha, M_1, M_2) > 0$ , :

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m}\tilde{d}C(\alpha, M_1, M_2) \frac{A^\theta}{\left( \ln \left( \frac{\tilde{d}A}{B} \right) \right)^{\beta'\theta}}. \quad (4.26)$$

We conclude by studying the variation of the function defined by  $f_y(x) = \frac{x}{(\ln(\frac{x}{y}))^{\beta'}}$  on  $(y, +\infty)$ , for  $y = \frac{B}{\tilde{d}}$ . We have  $f'_y(x) = \frac{\ln(\frac{x}{y}) - \beta'}{\ln(\frac{x}{y})^{\beta'+1}}$ . Let us denote by  $x_0 = ye^{\beta'}$ . The function  $f_y$  is decreasing on  $(y, x_0]$  and is increasing on  $[x_0, +\infty)$ . For  $\tilde{d}$  large enough,  $A \geq x_0$ . Thanks to (4.24) and since  $f$  is increasing on  $[x_0, +\infty)$ , we directly deduce that  $f_{\frac{B}{\tilde{d}}}(A) \leq f_{\frac{B}{\tilde{d}}}(C(\alpha, M_1, M_2))$ . Using this result in (4.26), we get that there exists  $C(\alpha, M_1, M_2) > 0$  and  $C_1(\alpha, M_1, M_2) > 0$  such that:

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left( \ln \left( \frac{C_1(\alpha, M_1, M_2)}{\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)}} \right) \right)^{\beta'\theta}},$$

and since  $\beta'\theta = \frac{3}{4}\beta$  and  $\frac{\partial u}{\partial n} = pn$  on  $\Gamma$ , this concludes the proof of the theorem.  $\square$

**Remark 4.18.** Remark 4.15 is still relevant here.

**Remark 4.19.** In inequality (4.23), the power  $\frac{3}{4}\beta$  is directly linked to the regularity of the solution  $(u, p)$ . If we are more precise in our estimates, we can notice that this power may be improved by a power which depends on  $k$ . Indeed, coming back to the inequalities (4.25) and using that  $(u, p) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ , we get that

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} C(M_2) \left( \|u\|_{H^1(\omega)}^{\tilde{\theta}} \|u\|_{H^{k+2}(\omega)}^{1-\tilde{\theta}} + \|p\|_{H^1(\omega)} \right)$$

where  $\tilde{\theta} = \frac{1/2+k}{1+k} - \frac{\epsilon}{1+k}$ . This estimate allows to obtain the power  $\frac{1/2+k}{1+k}\beta$  instead of  $\frac{3}{4}\beta$  in inequality (4.23) (when  $k = 1$ , these powers coincide).

**Remark 4.20.** In Theorem 4.16, the regularity assumptions on  $g$  and  $q_j$  allow to apply Proposition 4.4 with  $k \in \mathbb{N}^*$ . By this way, the solution  $(u_j, p_j)$ ,  $1 \leq j \leq 2$ , belongs to  $H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ . In fact, we can still obtain inequality (4.23) by enforcing less regularity on the solutions  $(u_j, p_j)$ ,  $1 \leq j \leq 2$ . In particular, if we consider the case when  $n \leq 5$ , it is sufficient to assume that for  $j = 1, 2$ ,  $(u_j, p_j)$  belongs to  $H^{\frac{5}{2}+\nu}(\Omega) \times H^{\frac{3}{2}+\nu}(\Omega)$  and  $q_j$  belongs to  $L^\infty(\Gamma_{out})$  and that

$$\|u_j\|_{H^{\frac{5}{2}+\nu}(\Omega)} + \|p_j\|_{H^{\frac{3}{2}+\nu}(\Omega)} \leq M_1, \quad \|q_j\|_{L^\infty(\Gamma_{out})} \leq M_2, \quad \text{for } j = 1, 2.$$

In this case, the velocity  $u_1$  is still continuous and with the same reasons as in Remark 4.17, there exist a compact  $K$  and a constant  $m > 0$  like in Theorem 4.16. Next, instead of the inequalities (4.25), we use that:

$$\begin{aligned} \|q_1 - q_2\|_{L^2(K)} &\leq \frac{1}{m} C(M_2) \left( \|u\|_{H^{3/2+\nu/3}(\omega)} + \|p\|_{H^1(\omega)} \right) \\ &\leq \frac{1}{m} C(M_2) \left( \|u\|_{H^1(\omega)}^{2/3} \|u\|_{H^{5/2+\nu}(\omega)}^{1/3} + \|p\|_{H^1(\omega)} \right) \end{aligned}$$

By performing the same reasoning as above, we get that there exists  $d_0 > 0$  such that for all  $\beta \in (0, 1)$ , there exists  $C(M_1, M_2) > 0$  and  $C_1(M_1, M_2) > 0$  such that

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(M_1, M_2)}{\left( \ln \left( \frac{C_1(M_1, M_2)}{\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right)} \right)^{\frac{2}{3}\beta}.$$

Let us notice that, due to the fact that the solution is less regular, the power in this inequality is weaker than in inequality (4.23) ( $\frac{2}{3}\beta$  instead of  $\frac{3}{4}\beta$  for  $\beta \in (0, 1)$ ).

## 4.6 Conclusion

We have proved two stability estimates for the inverse problem of identifying a Robin coefficient on some non accessible part of the boundary from measurements available on the other part of the boundary, for  $(u, p)$  solution of the stationary Stokes system with mixed Neumann and Robin boundary conditions. We will see in Chapter 5 to what extent we can extend these stability estimates to the case of the non stationary problem.

One may wonder if the stability estimates presented in this chapter remain true when we replace the Robin boundary condition on  $\Gamma_{out}$ ,  $\frac{\partial u}{\partial n} - pn + qu = 0$  by the boundary

conditions used by A. Quarteroni and A. Veneziani in [69], namely  $\frac{\partial u}{\partial n} - pn + R(u \cdot n)n = 0$ . Let us introduce the following Stokes system:

$$\left\{ \begin{array}{ll} -\Delta u + \nabla p & = 0, \quad \text{in } \Omega, \\ \operatorname{div} u & = 0, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn & = g, \quad \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + R(u \cdot n)n & = 0, \quad \text{on } \Gamma_{out}, \end{array} \right. \quad (4.27)$$

with  $R \in \mathbb{R}_+$ . Let  $(u_k, p_k)$  be solution of system (4.27) with  $R = R_k$  for  $k = 1, 2$ . In the following, we denote by  $u = u_1 - u_2$  and  $p = p_1 - p_2$ . Using the boundary condition on  $\Gamma_{out}$ , we obtain:

$$(R_2 - R_1)(u_1 \cdot n) = R_2(u \cdot n) + \frac{\partial u}{\partial n} \cdot n - p, \text{ on } \Gamma_{out}.$$

We are then able to perform the same reasoning as in the proof of Theorems 4.8 and 4.16 above and obtain logarithmic stability estimates on a compact  $K \subseteq \Gamma_i$  on which  $u_1 \cdot n \neq 0$ . Note that we can not conclude to the existence of such a compact set by using C. Fabre and G. Lebeau unique continuation result (see Theorem 3.1). The case of dissipative boundary conditions will be discussed in Chapter 7.

Our result which could be improved in different ways. In the stability estimates obtained in this chapter, the Robin coefficients are estimated on a compact subset  $K \subset \Gamma_{out}$  which is not a fixed inner portion of  $\Gamma_{out}$  but is unknown and depends on a given reference solution. Up to our knowledge, to obtain an estimate of Robin coefficients on the whole set  $\Gamma_{out}$  or on any compact subset  $K \subset \Gamma_{out}$  is still an open question. Moreover, we have seen that we need more measurements in our stability estimates than in the identifiability result. An improvement could be to obtain stability estimates in which less measurements is needed. Another natural issue concerns the optimality of the stability estimates: is it possible to obtain better than logarithmic stability estimate? We will see in Chapter 6 that under the *a priori* assumption that the Robin coefficient is piecewise constant, it is possible to prove a Lipschitz stability estimate.



# Logarithmic stability estimates for the nonstationary Stokes system

In this chapter, we study the inverse problem of identifying a Robin coefficient on some non accessible part of the boundary from measurements available on the other part of the boundary, for  $(u, p)$  solution of the non stationary Stokes system with mixed Neumann and Robin boundary conditions. We first investigate the identifiability. Then, we prove two stability estimates, one valid in dimension two and the other one valid in any dimension, under the assumption that the velocity of a given reference solution of the stationary problem stays far from 0 on a part of the boundary where Robin conditions are prescribed. To do so, we use the stability estimates proved in Chapter 4 for the stationary problem and the analytic semigroup theory. We distinguish two cases: we treat separately the case when the flux does not depend on time and when the flux depends on time.

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Results of this chapter have been obtained in collaboration with Muriel Boulakia and Céline Grandmont.



## 5.1 Introduction

Let  $d \in \mathbb{N}^*$  and  $\Omega \subset \mathbb{R}^d$  be an open connected set. We assume that  $\partial\Omega$  is composed of two sets  $\Gamma_0$  and  $\Gamma_{out}$  such that  $\Gamma_0 \cup \Gamma_{out} = \partial\Omega$  and  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ .

We deal in this chapter with the non stationary problem. We consider the following boundary problem:

$$\left\{ \begin{array}{lll} u_t(t, x) - \Delta u(t, x) + \nabla p(t, x) & = & 0, \quad \text{in } \Omega, \text{ for } t > 0, \\ \operatorname{div} u(t, x) & = & 0, \quad \text{in } \Omega, \text{ for } t > 0, \\ \frac{\partial u}{\partial n}(t, x) - p(t, x)n(x) & = & g(t, x), \quad \text{on } \Gamma_0, \text{ for } t > 0, \\ \frac{\partial u}{\partial n}(t, x) - p(t, x)n(x) + q(x)u(t, x) & = & 0, \quad \text{on } \Gamma_{out}, \text{ for } t > 0, \\ u(0, x) & = & u_0(x), \quad \text{in } \Omega. \end{array} \right. \quad (5.1)$$

Note that  $q$  depends only on the space variable.

As in the previous Chapter, we are concerned with **uniqueness result** and **stability properties** for the Robin coefficient  $q$  from measurements of  $(u, p)$  solution of system (5.1) available on  $\Gamma \subseteq \Gamma_0$ , for  $t > 0$ .

We obtain two stability estimates, one valid in dimension 2 and the other one valid in any dimension, under the assumption that the velocity of a given reference solution of the stationary problem stays far from 0 on a part of the boundary where Robin conditions are prescribed. The idea to obtain stability estimates in the non stationary case is to use the stability properties proved in Chapter 4 for the stationary problem. To do so, we use an inequality coming from analytic semigroup properties to estimate the difference between  $(u, p)$  solution of (5.1) and the solution of the associated stationary problem. Note that M. Bellassoued, J. Cheng and M. Choulli follow the same lines in [10] to extend their logarithmic stability estimate obtained for the Laplacian to the heat equation. Let us emphasize that, due to the method employed, we have to take measurements during an infinite time.

Let us explain the outline of this chapter. In Section 5.2, we start with regularity results concerning system (5.1). Then, we state an identifiability result which is the counterpart of Proposition 4.6 in the non stationary case. In order to use semigroup properties, we introduce in Section 5.4, the Stokes operator associated with the Robin boundary conditions on  $\Gamma_{out}$ . Finally, in Section 5.5, we assume that the flux  $g$  does not depend on time, whereas in Section 5.6 we assume that the flux  $g$  depends on time. In each case ( $g$  depending on time or not), we obtain two logarithmic stability estimates for the non stationary problem (5.1). One of the logarithmic estimates is valid only in dimension 2 and the other one is valid in any dimension. Note that when the flux depends on time, we have to do some asymptotic assumptions on the flux  $g$  to obtain the stability estimate.

## 5.2 Regularity results

**Theorem 5.1.** *Assume that  $\Omega$  is a bounded and connected open set in  $\mathbb{R}^d$  of class  $C^{1,1}$ . Let  $T > 0$ ,  $\alpha > 0$ ,  $s > \frac{d-1}{2}$  and  $u_0 \in V$ . We assume that  $g \in H^1(0, T; H^{\frac{1}{2}}(\Gamma_0))$ ,  $q \in H^s(\Gamma_{out})$  is such that  $\alpha \leq q$  on  $\Gamma_{out}$ .*

*Then, problem (5.1) admits a unique solution  $(u, p)$  which belongs to  $(L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V)) \times L^2(0, T; H^1(\Omega))$ .*



We are going to prove Theorem 5.1. First of all, as a preliminary step, we prove the following existence result:

**Proposition 5.2.** *Let  $T > 0$ ,  $\alpha > 0$  and  $u_0 \in H$ . We assume that  $g \in L^2(0, T; L^2(\Gamma_0))$  and that  $q \in L^\infty(\Gamma_{out})$  such that almost everywhere on  $\Gamma_{out}$ ,  $\alpha \leq q$ .*

*There exists  $u \in L^2(0, T; V)$  such in the distribution sense on  $(0, T)$ :*

$$\frac{d}{dt} \int_{\Omega} u \cdot v + \int_{\Omega} \nabla u : \nabla v + \int_{\Gamma_{out}} qu \cdot v = \int_{\Gamma_0} g \cdot v, \text{ for all } v \in V, \quad (5.2)$$

and

$$\int_{\Omega} u(0) \cdot v = \int_{\Omega} u_0 \cdot v, \text{ for all } v \in V. \quad (5.3)$$

*Proof of Proposition 5.2.* We begin to prove, using a Galerkin method, that there exists  $u \in L^2(0, T; V)$  such that

$$\begin{aligned} & \forall v \in V, \forall \psi \in \mathcal{C}^1(0, T) \text{ such that } \psi(T) = 0 \\ & - \int_0^T \int_{\Omega} u(t, x) \cdot v(x) \psi'(t) dx dt + \int_0^T \int_{\Omega} \nabla u(t, x) : \nabla v(x) \psi(t) dx dt \\ & + \int_0^T \int_{\Gamma_{out}} q(x) u(t, x) \cdot v(x) \psi(t) dx dt - \psi(0) \int_{\Omega} u_0(x) \cdot v(x) dx = \int_0^T \int_{\Gamma_0} g(t, x) \cdot v(x) \psi(t) dx dt. \end{aligned} \quad (5.4)$$

Let  $(w_i)_{i \in \mathbb{N}}$  be a Hilbert basis of  $V$  which is also an orthogonal basis of  $H$ . For each  $n \in \mathbb{N}$ , we define an approximate solution as follows: we search  $u_n \in V_n = \text{Vect}\{w_i\}_{1 \leq i \leq n}$  that satisfies

$$\begin{cases} \int_{\Omega} u_{n,t} \cdot w_j + \int_{\Omega} \nabla u_n \cdot \nabla w_j + \int_{\Gamma_{out}} q u_n \cdot w_j = \int_{\Gamma_0} g \cdot w_j, \forall j \in \{1, \dots, n\}, \\ u_n(0) = \sum_{k=1}^n (u_0, w_k)_{L^2(\Omega)} w_k. \end{cases} \quad (5.5)$$

Note that  $u_{n,t}$  denotes  $\partial_t u_n$ . Let  $t \in (0, T)$ . We decompose  $u_n(t, \cdot)$  in the Hilbert basis:

$$u_n(t, \cdot) = \sum_{i=1}^n \xi_i(t) w_i.$$

We denote by

$$\begin{aligned} A &= \left[ \int_{\Omega} w_i(x) \cdot w_j(x) dx \right]_{1 \leq i, j \leq n} \\ B &= \left[ \int_{\Omega} \nabla w_i(x) : \nabla w_j(x) + \int_{\Gamma_{out}} q(x) w_i(x) \cdot w_j(x) dx \right]_{1 \leq i, j \leq n} \\ L(t) &= \left( \int_{\Gamma_0} g(t, x) \cdot w_i(x) dx \right)_{1 \leq i \leq n} \end{aligned}$$

and

$$\xi(t) = (\xi_i(t))_{1 \leq i \leq n}.$$

We can rewrite the system (5.5) in the form:

$$\begin{cases} A\xi'(t) + B\xi(t) = L(t), \\ \xi(0) = [(u_0, w_i)_{L^2(\Omega)}]_{1 \leq i \leq n}. \end{cases}$$

Since the matrix  $A$  is invertible, the linear system has a unique global solution  $\xi \in H^1(0, T)$ . We are now going to prove that there exists a constant  $C > 0$  independent of  $n \in \mathbb{N}$  such that:

$$\sup_{t \in (0, T)} \int_{\Omega} |u_n|^2 + \int_0^T \int_{\Omega} |\nabla u_n|^2 + \int_0^T \int_{\Omega} |u_n|^2 \leq C. \quad (5.6)$$

Let  $t \in (0, T)$ . Multiplying the first equation of (5.5) by  $\xi_j$  and summing over  $j$  for  $j = 1, \dots, n$  we obtain:

$$\int_0^t \int_{\Omega} u_{n,t} \cdot u_n + \int_0^t \int_{\Omega} |\nabla u_n|^2 + \int_0^t \int_{\Gamma_{out}} q |u_n|^2 = \int_0^t \int_{\Gamma_0} g \cdot u_n \quad (5.7)$$

Let  $\epsilon > 0$ . Thus:

$$\begin{aligned} \int_0^t \int_{\Gamma_0} g \cdot u_n &\leq C \int_0^T \int_{\Gamma_0} |g|^2 + \epsilon \int_0^t \int_{\Gamma_0} |u_n|^2, \\ &\leq C \int_0^T \int_{\Gamma_0} |g|^2 + \epsilon \int_0^t \|u_n\|_{H^1(\Omega)}^2. \end{aligned}$$

Choosing a small enough  $\epsilon$  and using the fact that  $q \geq \alpha$  almost everywhere on  $\Gamma_{out}$ , we obtain:

$$\sup_{t \in (0, T)} \int_{\Omega} |u_n|^2 + \int_0^T \int_{\Omega} |\nabla u_n|^2 + \int_0^T \int_{\Omega} |u_n|^2 \leq C \left( \int_0^T \int_{\Gamma_0} |g|^2 + \int_{\Omega} |u_0|^2 \right). \quad (5.8)$$

This gives (5.6). According to inequality (5.6), there exists  $u \in L^2(0, T; V)$  such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } L^2(0, T; V).$$

Let  $j \in \mathbb{N}$ . Multiplying the first equation of (5.5) by  $\psi \in \mathcal{C}^1(0, T)$  such that  $\psi(T) = 0$  then integrating over  $(0, T)$ , we get,  $\forall n \geq j$ :

$$\begin{aligned} &\int_0^T \int_{\Omega} u_{n,t}(t, x) \cdot w_j(x) \psi(t) dx dt + \int_0^T \int_{\Gamma_{out}} q(x) u_n(t, x) \cdot w_j(x) \psi(t) dx dt \\ &+ \int_0^T \int_{\Omega} \nabla u_n(t, x) : \nabla w_j(x) \psi(t) dx dt = \int_0^T \int_{\Gamma_0} g(t, x) \cdot w_j(x) \psi(t) dx dt. \end{aligned} \quad (5.9)$$

Taking into account that:

$$\begin{aligned} &\int_0^T \int_{\Omega} u_{n,t}(t, x) \cdot w_j(x) \psi(t) dx dt \\ &= - \int_0^T \int_{\Omega} u_n(t, x) \cdot w_j(x) \psi'(t) dx dt - \int_{\Omega} u_n(0, x) \cdot w_j(x) \psi(0) dx, \end{aligned}$$

we easily pass to the limit when  $n$  goes to infinity in (5.9). Remark that this inequality is still valid if we replace  $w_j$  by any  $v \in V$  by continuity. This ends the proof of the existence of  $u \in L^2(0, T; V)$  which satisfies (5.2) in the distribution sense on  $(0, T)$ .

Let us finish the proof of Proposition 5.2 by verifying that the initial condition (5.3) is satisfied. We deduce from equality (5.4) that  $\frac{d}{dt}(u, v)_{L^2(\Omega)} \in L^2(0, T)$ . Consequently, the function  $t \rightarrow (u(t), v)_{L^2(\Omega)}$  is continuous. This gives a sense to  $(u(0), v)_{L^2(\Omega)}$ . Let  $\psi \in \mathcal{C}^1(0, T)$  such that  $\psi(T) = 0$ . Multiplying (5.2) by  $\psi$ , we obtain:

$$-\int_0^T (u, v)_{L^2(\Omega)^d} \psi'(t) dt + \int_0^T a_q(u, v) \psi(t) dt = (u(0, \cdot), v)_{L^2(\Omega)^d} \psi(0) + \int_0^T l(v) \psi(t) dt.$$

Comparing with equality (5.4), we obtain  $\psi(0)(u(0, \cdot) - u_0, v)_{L^2(\Omega)} = 0$ . Let  $\psi$  be such that  $\psi(0) \neq 0$ , we have  $(u(0) - u_0, v)_{L^2(\Omega)} = 0, \forall v \in V$ .  $\square$

We are now able to prove Theorem 5.1.

*Proof of Theorem 5.1.* We will begin by proving that  $u_t \in L^2(0, T; H)$ , then we will conclude by using the regularity result for the stationary problem from Proposition 4.4. We use the same notations as in the proof of Proposition 5.2.

Let  $t \in (0, T)$ . Multiplying the first equation of (5.5) by  $\xi_j^t$  and summing over  $j$  for  $j = 1, \dots, n$  we obtain:

$$\int_0^t \int_{\Omega} |u_{n,t}|^2 + \int_0^t \int_{\Gamma_{out}} q u_n u_{n,t} + \int_0^t \int_{\Omega} \nabla u_n : \nabla u_{n,t} = \int_0^t \int_{\Gamma_0} g \cdot u_{n,t}.$$

We have:

$$\int_0^t \int_{\Gamma_0} g \cdot u_{n,t} = - \int_0^t \int_{\Gamma_0} g_t u_n - \int_{\Gamma_0} g(0) u_n(0) + \int_{\Gamma_0} g(t) u_n(t). \quad (5.10)$$

Let  $\epsilon > 0$ . Thanks to Cauchy-Schwarz inequality and estimate (5.8), there exists  $C > 0$ :

$$\left| \int_0^t \int_{\Gamma_0} g_t u_n \right| \leq C \left( \int_0^T \int_{\Gamma_0} |g_t|^2 + \sup_{t \in (0, T)} \int_{\Gamma_0} |g|^2 + \|u_0\|_{H^1(\Omega)} + \epsilon \int_{\Omega} |\nabla u_n(t)|^2 \right).$$

Using successively integration by parts over  $(0, T)$  we finally obtain, choosing  $\epsilon$  small enough:

$$\sup_{t \in (0, T)} \int_{\Omega} |\nabla u_n|^2 + \int_0^T \int_{\Omega} |u_{n,t}|^2 \leq C \left( \|u_0\|_{H^1(\Omega)}^2 + \int_0^T \int_{\Gamma_0} |g_t|^2 + \sup_{t \in (0, T)} \int_{\Gamma_0} |g|^2 \right). \quad (5.11)$$

We deduce that  $(u_n)_{n \in \mathbb{N}}$  is bounded in  $H^1(0, T; H) \cap L^\infty(0, T; V)$  and therefore  $u \in H^1(0, T; H) \cap L^\infty(0, T; V)$ .

Then, we use the regularity result for the stationary problem. For all  $t \in (0, T)$ , we have  $u_t(t) \in L^2(\Omega)$ , so by applying Proposition 4.2, there exists  $p(t)$  such that  $(u(t), p(t)) \in V \times L^2(\Omega)$  is weak solution of (5.1). Moreover, since the map

$$\begin{aligned} L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma_0) &\rightarrow H^2(\Omega) \times H^1(\Omega) \\ (u_t(t), g(t)) &\mapsto (u(t), p(t)) \end{aligned}$$

is linear and continuous,  $(u(t), p(t))$  belongs to  $H^2(\Omega) \times H^1(\Omega)$ , thanks to Proposition 4.4. Since  $(u_t, g) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; H^{\frac{1}{2}}(\Gamma_0))$ , we deduce that  $(u, p) \in L^2(0, T; H^2(\Omega)) \times L^2(0, T; H^1(\Omega))$ .

Let us now prove the uniqueness of solution for problem (5.1). Assume that  $u_1$  and  $u_2$  are two solutions of (5.1) and let  $w = u_1 - u_2$ . Then  $w \in H^1(0, T; H) \cap L^2(0, T; V)$  and we have for all  $v \in V$ :

$$\int_{\Omega} w_t(t) \cdot v + \int_{\Omega} \nabla w(t) : \nabla v + \int_{\Gamma_{out}} qw(t) \cdot v = 0, \quad w(0) = 0. \quad (5.12)$$

Taking  $v = w(t)$  in (5.12), we find:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |w(t)|^2 + \int_{\Omega} |\nabla w(t)|^2 + \int_{\Gamma_{out}} q|w(t)|^2 = 0,$$

that is to say

$$\int_{\Omega} |w(t)|^2 \leq \int_{\Omega} |w(0)|^2 = 0, \quad \text{for all } t \in (0, T).$$

So  $u_1 = u_2$  on  $(0, T) \times \Omega$ . To conclude, thanks to system (5.1), we obtain  $p_1 = p_2$ .  $\square$

**Remark 5.3.** Let  $\alpha > 0$ . To obtain  $u_t \in L^2(0, T; L^2(\Omega))$  it is sufficient to assume, thanks to inequality (5.11), that  $u_0 \in V$ ,  $g \in H^1(0, T; L^2(\Gamma_0))$  and  $q \in L^\infty(\Gamma_{out})$  is such that  $q \geq \alpha$  on  $\Gamma_{out}$ .

The following corollary will be useful when we will prove logarithmic stability estimate for the evolution problem (5.1).

**Corollary 5.4.** Assume that  $\Omega$  is a bounded and connected open set in  $\mathbb{R}^d$  of class  $C^{2,1}$ . Let  $T > 0$ ,  $\alpha > 0$ ,  $u_0 \in H^3(\Omega) \cap H$  and  $s \in \mathbb{R}$  be such that  $s > \frac{d-1}{2}$  and  $s \geq \frac{3}{2}$ . We assume that  $g \in H^2(0, T; H^{\frac{3}{2}}(\Gamma_0))$  and  $q \in H^s(\Gamma_{out})$  is such that  $\alpha \leq q$  on  $\Gamma_{out}$ .

Then, problem (5.1) admits a unique solution  $(u, p)$  which belongs to

$$(L^\infty(0, T; H^3(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap H^2(0, T; L^2(\Omega))) \times (L^\infty(0, T; H^2(\Omega)) \cap H^1(0, T; H^1(\Omega))).$$

The proof of the previous corollary consists in applying Theorem 5.1 to  $u_t$ . Let us give the sketch of the proof.

*Proof of Corollary 5.4.* Let  $(u, p)$  be the solution of (5.1). Let us consider the following system:

$$\begin{cases} v_t - \Delta v + \nabla \tau & = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} v & = 0, & \text{in } (0, T) \times \Omega, \\ \frac{\partial v}{\partial n} - \tau n & = g_t, & \text{on } (0, T) \times \Gamma_0, \\ \frac{\partial v}{\partial n} - \tau n + qv & = 0, & \text{on } (0, T) \times \Gamma_{out}, \\ v(0) & = \Delta u_0 - \nabla p_0, & \text{in } \Omega, \end{cases} \quad (5.13)$$

where  $p_0 \in H^2(\Omega)$  is defined as the solution of the following elliptic boundary problem:

$$\begin{cases} \Delta p_0 & = 0, & \text{in } \Omega, \\ p_0 & = \frac{\partial u_0}{\partial n} \cdot n - g|_{t=0} \cdot n, & \text{on } \Gamma_0, \\ p_0 & = \frac{\partial u_0}{\partial n} \cdot n + qu_0 \cdot n, & \text{on } \Gamma_{out}. \end{cases}$$

According to Theorem 5.1, we obtain that  $(v, \tau)$  exists and belongs to  $(L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))) \cap L^\infty(0, T; V) \times L^2(0, T; H^1(\Omega))$ . Remark that  $(u_t, p_t)$  is solution of equations (5.13) in the distribution sense on  $(0, T)$ . Thus,  $(v, \tau) = (u_t, p_t)$ . Then, since  $q \in H^s(\Gamma_0)$ , we deduce from Proposition 4.4 that the linear map:

$$\begin{aligned} H^1(\Omega) \times H^{\frac{3}{2}}(\Gamma_0) &\rightarrow H^3(\Omega) \times H^2(\Omega) \\ (u_t(t), g(t)) &\mapsto (u(t), p(t)) \end{aligned}$$

is continuous. Since  $(u_t, g) \in L^\infty(0, T; V) \times L^\infty(0, T; H^{\frac{3}{2}}(\Gamma_0))$ , we deduce that  $(u, p) \in L^\infty(0, T; H^3(\Omega)) \times L^\infty(0, T; H^2(\Omega))$ .  $\square$

## 5.3 An identifiability result

### 5.3.1 Unique continuation

**Theorem 5.5.** *We note  $Q = (0, T) \times \Omega$  and let  $O$  be an open set in  $Q$ . The horizontal component of  $O$  is*

$$C(O) = \{(t, x) \in Q, \exists x_0 \in \Omega, (t, x_0) \in O\}.$$

Let  $(u, p) \in L^2(0, T; H^1_{loc}(\Omega)) \times L^2_{loc}(Q)$  be a weak solution of

$$\begin{cases} u_t - \Delta u + \nabla p = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0, T) \times \Omega, \end{cases}$$

satisfying  $u = 0$  in  $O$  then  $u = 0$  and  $p$  is constant in  $C(O)$ .

From this theorem, we easily deduce the following result which will be very useful in the next subsection.

**Corollary 5.6.** *Let  $\Gamma \subseteq \partial\Omega$ ,  $\delta > 0$ ,  $x \in \Gamma$  and  $r > 0$  such that  $\gamma = (t_0 - \delta, t_0 + \delta) \times (\mathcal{B}(x, r) \cap \Gamma)$  is an open set in  $(0, T) \times \Gamma$ . Let  $(u, p) \in (L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))) \times L^2(0, T; L^2(\Omega))$  be solution of:*

$$\begin{cases} u_t - \Delta u + \nabla p = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0, T) \times \Omega, \end{cases}$$

satisfying  $u = 0$  and  $\frac{\partial u}{\partial n} - pn = 0$  on  $\gamma$ . Then  $u = 0$  and  $p = 0$  in  $(t_0 - \delta, t_0 + \delta) \times \Omega$ .

*Proof of Corollary 5.6.* We extend  $u$  and  $p$  by 0 on  $(t_0 - \delta, t_0 + \delta) \times (\mathcal{B}(x, r) \cap \Omega^c)$ :

$$\tilde{u} \text{ (resp. } \tilde{p}) = \begin{cases} u \text{ (resp. } p) & \text{in } (t_0 - \delta, t_0 + \delta) \times \Omega \\ 0 & \text{in } (t_0 - \delta, t_0 + \delta) \times (\mathcal{B}(x, r) \cap \Omega^c) \end{cases}$$

and we denote  $\tilde{\Omega} = \Omega \cup \mathcal{B}(x, r)$ . Let us verify that  $(\tilde{u}, \tilde{p}) \in L^2(0, T; H^1(\Omega)) \times L^2(0, T; L^2(\Omega))$  is still a solution of the Stokes equations in  $\tilde{\Omega}$ . Let  $v \in \mathcal{D}(\tilde{\Omega})$ . We check by integration by parts in space that almost everywhere in  $t \in (t_0 - \delta, t_0 + \delta)$ :

$$\int_{\tilde{\Omega}} \tilde{u}_t \cdot v + \int_{\tilde{\Omega}} \nabla \tilde{u} \cdot \nabla v - \int_{\tilde{\Omega}} \tilde{p} \operatorname{div} v = 0.$$

Moreover  $\operatorname{div} \tilde{u} = 0$  in  $(t_0 - \delta, t_0 + \delta) \times \tilde{\Omega}$ . Therefore, we can apply Theorem 5.5 to  $(\tilde{u}, \tilde{p})$ :  $(\tilde{u}, \tilde{p}) = (0, 0)$  in  $(t_0 - \delta, t_0 + \delta) \times \tilde{\Omega}$  which implies that  $u = 0$  and  $p$  is constant in  $(t_0 - \delta, t_0 + \delta) \times \Omega$ . At last, the fact that  $\frac{\partial u}{\partial n} - pn = 0$  on  $\gamma$  implies that  $p = 0$  in  $(t_0 - \delta, t_0 + \delta) \times \Omega$ .  $\square$

**Remark 5.7.** Since  $-\nabla u + pId \in L^2(0, T; L^2(\Omega))$  and  $\operatorname{div}(-\nabla u + pId) = u_t \in L^2(0, T; L^2(\Omega))$ ,  $\frac{\partial u}{\partial n} - pn = 0$  on  $(0, T) \times \omega$  has a sense in  $L^2\left(0, T; \left(H_{00}^{\frac{1}{2}}(\omega)\right)'\right)$ .

### 5.3.2 Application

**Proposition 5.8.** Let  $T > 0$ ,  $\alpha > 0$ ,  $x_0 \in \Gamma_0$ ,  $r > 0$ ,  $s > \frac{d-1}{2}$ ,  $g \in H^1(0, T; L^2(\Gamma_0))$  be non identically zero,  $u_0 \in V$  and  $q_j \in H^s(\Gamma_{out})$  be such that  $q_j \geq \alpha$  on  $\Gamma_{out}$  for  $j = 1, 2$ . Let  $(u_j, p_j)$  be the weak solutions to (5.1) for  $t \in (0, T)$  with  $q = q_j$  for  $j = 1, 2$ .

We assume that  $u_1 = u_2$  on  $(0, T) \times (\mathcal{B}(x_0, r) \cap \Gamma_0)$ . Then  $q_1 = q_2$  on  $\Gamma_{out}$ .

*Proof of Proposition 5.8.* We are going to prove Proposition 5.8 by contradiction: we assume that  $q_1$  is not identically equal to  $q_2$  on  $\Gamma_{out}$ .

Thanks to Remark 5.3, we have  $(u_j, p_j) \in (L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))) \times L^2(0, T; L^2(\Omega))$  for  $j = 1, 2$  thanks to Theorem 5.1. We define  $u = u_1 - u_2$  and  $p = p_1 - p_2$ . Let us notice that  $(u, p)$  is the solution of the following problem:

$$\begin{cases} u_t - \Delta u + \nabla p & = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u & = 0, & \text{in } (0, T) \times \Omega, \\ \frac{\partial u}{\partial n} - pn & = 0, & \text{on } (0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + q_1 u_1 - q_2 u_2 & = 0, & \text{on } (0, T) \times \Gamma_{out}. \end{cases}$$

By assumption,  $u = 0$  and  $\frac{\partial u}{\partial n} - pn = 0$  on  $(0, T) \times (\mathcal{B}(x_0, r) \cap \Gamma_0)$ . Thus, according to Corollary 5.6,  $u_1 = u_2$  and  $p_1 = p_2$  in  $(0, T) \times \Omega$ . Consequently, we deduce from

$$\begin{aligned} \frac{\partial u_1}{\partial n} - p_1 n + q_1 u_1 &= 0, & \text{on } (0, T) \times \Gamma_{out}, \\ \frac{\partial u_1}{\partial n} - p_1 n + q_2 u_1 &= 0, & \text{on } (0, T) \times \Gamma_{out}, \end{aligned}$$

that

$$u_1(q_1 - q_2) = 0 \text{ on } (0, T) \times \Gamma_{out}. \quad (5.14)$$

By hypothesis,  $q_1$  is not identically equal to  $q_2$ . Since  $s > \frac{d-1}{2}$ ,  $q_1$  and  $q_2$  are continuous on  $\Gamma_{out}$ . Then, we can find an open set  $\kappa \subset \Gamma_{out}$  with a positive measure such that:

$$(q_1 - q_2)(x) \neq 0, \forall x \in \kappa.$$

Equation (5.14) implies that  $u_1 \equiv 0$  on  $(0, T) \times \kappa$  and then  $u_1$  verifies

$$\begin{cases} u_{1t} - \Delta u_1 + \nabla p_1 & = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u_1 & = 0, & \text{in } (0, T) \times \Omega, \\ u_1 & = 0, & \text{on } (0, T) \times \kappa, \\ \frac{\partial u_1}{\partial n} - p_1 n & = 0, & \text{on } (0, T) \times \kappa. \end{cases}$$

Applying again Corollary 5.6, we obtain that  $u_1 = 0$  and  $p_1 = 0$  in  $(0, T) \times \Omega$ . This is in contradiction with the assumption that  $g$  is non identically zero.  $\square$

## 5.4 Properties of the Stokes operator

Let us recall that the functional spaces  $\mathbf{V}$  and  $\mathbf{H}$  are introduced in Chapter 4 in Notation 4.1 and the bilinear form  $a_q$  is also defined in Chapter 4 by (4.5).

**Definition 5.9.** We define the set  $\mathcal{D}(A_q)$  as follows:

$$\mathcal{D}(A_q) = \{u \in V / \exists C > 0, \forall v \in V, |a_q(u, v)| \leq C \|v\|_{L^2(\Omega)}\},$$

and the operator  $A_q : \mathcal{D}(A_q) \subset H \rightarrow H$  by:

$$\forall u \in \mathcal{D}(A_q), a_q(u, v) = (A_q u, v)_{L^2(\Omega)}, \forall v \in V. \quad (5.15)$$

**Proposition 5.10.** Let  $\alpha > 0$  and  $q \in L^\infty(\Gamma_{out})$  such that  $q \geq \alpha$  almost everywhere on  $\Gamma_{out}$ . The operator  $A_q$  has the following properties:

1.  $A_q \in \mathcal{L}(\mathcal{D}(A_q), H)$  is invertible and its inverse is compact on  $H$ .
2.  $A_q$  is selfadjoint.

As a consequence,  $A_q$  admits a family of eigenvectors  $\phi_q^l$

$$A_q \phi_q^l = \lambda_q^l \phi_q^l \text{ with } 0 < \lambda_q^1 \leq \lambda_q^2 \leq \dots \leq \lambda_q^j \text{ and } \lim_{j \rightarrow \infty} \lambda_q^j = +\infty,$$

which is complete and orthogonal both in  $H$  and  $V$ . Moreover, the family  $(\phi_q^l)_{l \in \mathbb{N}}$  is orthonormal in  $H$ .

The proof of Proposition 5.10 is similar to the proof of Proposition 2.26.

**Remark 5.11.** Let  $\alpha > 0$ . There exists a constants  $\mu > 0$  such that for all  $q \in L^\infty(\Gamma_{out})$  such that  $q \geq \alpha$ , for  $l \in \mathbb{N}$ :

$$\lambda_q^l \geq \mu. \quad (5.16)$$

Indeed,  $\lambda_q^l \geq \lambda_q^1 = (A_q \phi_q^1, \phi_q^1)_{L^2(\Omega)} = a_q(\phi_q^1, \phi_q^1) \geq a_\alpha(\phi_q^1, \phi_q^1) \geq \mu \|\phi_q^1\|_{L^2(\Omega)}^2 = \mu$ , where  $\mu$  is the coercivity constant associated with the bilinear form  $a_\alpha$ .

In the following, we will use powers of the operator  $A_q$ . Let us introduce them.

**Definition 5.12.** Let  $s > 0$ . Let us denote by:

$$\mathcal{D}(A_q^s) = \left\{ u \in H / \sum_{k=1}^{+\infty} (\lambda_q^k)^{2s} (u, \phi_q^k)_{L^2(\Omega)}^2 < +\infty \right\}.$$

For all  $u \in \mathcal{D}(A_q^s)$ , we set:

$$A_q^s u = \sum_{k=1}^{+\infty} (\lambda_q^k)^s (u, \phi_q^k)_{L^2(\Omega)} \phi_q^k.$$

Finally,  $\mathcal{D}(A_q^s)$  is equipped with the natural scalar product:

$$(u, v)_{\mathcal{D}(A_q^s)} = \sum_{k=1}^{+\infty} \left( 1 + (\lambda_q^k)^{2s} \right) (u, \phi_q^k)_{L^2(\Omega)} (v, \phi_q^k)_{L^2(\Omega)}.$$

One can prove the following properties:

**Proposition 5.13.** 1. For all  $s > 0$ ,  $\mathcal{D}(A_q)$  is a Hilbert space,  $A_q^s$  is a positive self-adjoint operator and is an isomorphism from  $\mathcal{D}(A_q^s)$  onto  $H$ . Moreover, the family  $(\phi_q^k)_{k \in \mathbb{N}}$  is complete in  $\mathcal{D}(A_q^s)$ .

2. For all  $0 \leq s < s'$ , we have,  $\mathcal{D}(A_q^{s'}) \hookrightarrow \mathcal{D}(A_q^s)$ , the inclusion is strict and the injection compact.

*Proof.* We refer to [21] for a proof of this result.  $\square$

**Corollary 5.14.** *The operator  $A_q^{\frac{1}{2}}$  is an isometry from  $(V, a_q(\cdot, \cdot)^{\frac{1}{2}})$  onto  $(H, \|\cdot\|_{L^2(\Omega)})$ .*

*Proof of Corollary 5.14.* Let  $k \in \mathbb{N}^*$ . We have, using the fact that  $A_q^{\frac{1}{2}}$  is self-adjoint:

$$\|A_q^{\frac{1}{2}} \phi_q^k\|_{L^2(\Omega)}^2 = (A_q^{\frac{1}{2}} \phi_q^k, A_q^{\frac{1}{2}} \phi_q^k)_{L^2(\Omega)} = (A_q \phi_q^k, \phi_q^k)_{L^2(\Omega)}.$$

Combining this with the definition (5.15) of  $A_q$ , we obtain  $\|A_q^{\frac{1}{2}} \phi_q^k\|_{L^2(\Omega)}^2 = a_q(\phi_q^k, \phi_q^k)$ .  $\square$

Let us make the link between the Stokes operator  $A_q$  and the solutions of Stokes system.

**Proposition 5.15.** *Let  $\alpha > 0$ ,  $k \in \mathbb{N}$ ,  $s > \frac{d-1}{2}$  and  $s \geq \frac{1}{2} + k$ . We assume that  $\Omega$  is of class  $\mathcal{C}^{k+1,1}$  and  $q \in H^s(\Gamma_{out})$  is such that  $\alpha \leq q$  on  $\Gamma_{out}$ .*

*Then for each  $f \in H \cap H^k(\Omega)$ , there exists  $u \in H^{k+2}(\Omega)$  solution of  $A_q u = f$  if and only if there exists  $p \in H^{k+1}(\Omega)$  such that  $(u, p)$  is solution of the following problem:*

$$\begin{cases} -\Delta u + \nabla p &= f, & \text{in } \Omega, \\ \operatorname{div} u &= 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} - pn &= 0, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu &= 0, & \text{on } \Gamma_{out}. \end{cases}$$

*Moreover, there exists a constant  $C(\alpha, M) > 0$  such that for every  $q \in H^s(\Gamma_{out})$  satisfying  $\|q\|_{H^s(\Gamma_{out})} \leq M$ , we have  $\|u\|_{H^{2+k}(\Omega)} \leq C(\alpha, M) \|f\|_{H^k(\Omega)}$ .*

*Proof of Proposition 5.15.* It follows from the construction of the operator  $A_q$  and from Proposition 4.4.  $\square$

**Corollary 5.16.** *Let  $\alpha > 0$ ,  $k \in \mathbb{N}^*$ ,  $s > \frac{d-1}{2}$  and  $s \geq \frac{1}{2} + 2(k-1)$ . We assume that  $\Omega$  is of class  $\mathcal{C}^{2k-1,1}$  and  $q \in H^s(\Gamma_{out})$  is such that  $q \geq \alpha$  on  $\Gamma_{out}$ . Then  $\mathcal{D}(A_q^k) \hookrightarrow H^{2k}(\Omega) \cap H$ .*

*Proof of Corollary 5.16.* For  $k = 1$ , it is clear. Take now  $k = 2$ . Let  $u \in \mathcal{D}(A_q^2)$ . We have

$$A_q^2 u = f \Leftrightarrow \begin{cases} A_q u = v \\ A_q v = f \end{cases}$$

But  $v \in \mathcal{D}(A_q) \subset H^2(\Omega) \cap H$  by assumption, so  $u \in H^4(\Omega) \cap H$  thanks to Proposition 5.15. We conclude by induction on  $k$ .  $\square$

**Remark 5.17.** *Let us remark that  $\mathcal{D}(A_q^k)$  is not equal to  $H^{2k}(\Omega) \cap H$ : it comes from the boundary conditions.*

Finally, we end this section with the fact that  $-A_q$  generates an analytic semi-group on  $H$ . This will be very useful to prove the stability estimate for the unsteady Stokes system, in particular we will use resulting properties.



**Proposition 5.18.** *Let  $\alpha > 0$  and  $q \in L^\infty(\Gamma_{out})$  such that  $q \geq \alpha$  almost everywhere on  $\Gamma_{out}$ . The operator  $-A_q$  generates an analytic semigroup on  $H$ . This analytic semigroup is explicitly given by:*

$$e^{-tA_q} f = \sum_{l \geq 1} e^{-t\lambda_q^l} (\phi_q^l, f)_{L^2(\Omega)} \phi_q^l,$$

for all  $f \in H$ .

*Proof of Proposition 5.18.* It follows from the construction of the operator  $A_q$ . We refer to [67] and [32] for more details.  $\square$

## 5.5 Stability estimates when the flux $g$ does not depend on $t$

Throughout this section, we assume that  $g$  does not depend on time.

### 5.5.1 Preliminary result

The following lemma gives an estimate of a solution of the non stationary system (5.1) with respect to the solution of the stationary problem (4.1).

**Lemma 5.19.** *We assume that  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . Let  $\Gamma \subseteq \Gamma_0$ ,  $M_1 > 0$ ,  $M_2 > 0$ ,  $\alpha > 0$  and  $s \in \mathbb{R}$  be such that  $s > \frac{d-1}{2}$  and  $s \geq \frac{3}{2}$ . Let  $u_0 \in H \cap H^3(\Omega)$ . Let  $g \in H^{\frac{3}{2}}(\Gamma)$  be non identically zero and such that  $\|g\|_{H^{\frac{3}{2}}(\Gamma)} \leq M_1$ . Let  $q \in H^s(\Gamma_{out})$  be such that  $\|q\|_{H^s(\Gamma_{out})} \leq M_2$  and  $\alpha \leq q$  on  $\Gamma_{out}$ . We denote by  $(u, p)$  the solution of (5.1) associated with  $g$  and by  $(v, \zeta)$  the solution of the stationary system (4.1) associated with  $g$ .*

*Then, the following inequality holds:*

$$\|v\|_{L^2(\Gamma)} + \|\zeta\|_{L^2(\Gamma)} + \left\| \frac{\partial \zeta}{\partial n} \right\|_{L^2(\Gamma)} \leq \|u\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \|p\|_{L^\infty(0,+\infty;L^2(\Gamma))} + \left\| \frac{\partial p}{\partial n} \right\|_{L^\infty(0,+\infty;L^2(\Gamma))}. \quad (5.17)$$

*Proof of Lemma 5.19.* According to Proposition 4.4,  $(v, \zeta)$  belongs to  $H^3(\Omega) \times H^2(\Omega)$  and moreover there exists a constant  $C(\alpha, M_1, M_2) > 0$  such that

$$\|v\|_{H^3(\Omega)} + \|\zeta\|_{H^2(\Omega)} \leq C(\alpha, M_1, M_2). \quad (5.18)$$

We denote  $(w, \pi) = (u - v, p - \zeta)$ . We are going to prove that:

$$\|w(t, \cdot)\|_{H^3(\Omega)} + \|\pi(t, \cdot)\|_{H^2(\Omega)} \leq G(t),$$

where  $G$  is a function which tends to 0 when  $t$  goes to  $+\infty$ .

We have that  $(w, \pi)$  is the solution of the following problem: for  $t > 0$ ,

$$\begin{cases} \partial_t w - \Delta w + \nabla \pi & = 0, & \text{in } \Omega, \\ \operatorname{div} w & = 0, & \text{in } \Omega, \\ \frac{\partial w}{\partial n} - \pi n & = 0, & \text{on } \Gamma_0, \\ \frac{\partial w}{\partial n} - \pi n + qw & = 0, & \text{on } \Gamma_{out}, \\ w(0) & = u_0 - v, & \text{in } \Omega. \end{cases} \quad (5.19)$$

Let  $t > 0$ . We have from the theory of analytic semigroup that:

$$w(t, \cdot) = e^{-tA_q} w(0, \cdot). \quad (5.20)$$

Let  $\eta > 0$ . There exists a constant  $C > 0$  independent of  $q$  such that:

$$\|A_q^\eta e^{-tA_q}\| \leq C \frac{e^{-\mu t}}{t^\eta}, \quad t > 0, \eta > 0, \quad (5.21)$$

where  $\mu$  is given by (5.16) and where  $\| \cdot \|$  is the operator norm (see [67] or [60]). Using the regularity result for the stationary case given in Proposition 4.4, we have that:

$$\|w(t, \cdot)\|_{H^3(\Omega)} + \|\pi(t, \cdot)\|_{H^2(\Omega)} \leq \|\partial_t w(t, \cdot)\|_{H^1(\Omega)}. \quad (5.22)$$

Note that, thanks to Proposition 5.15 we have:

$$\|\partial_t w(t, \cdot)\|_{H^1(\Omega)} = \|A_q w(t, \cdot)\|_{H^1(\Omega)}.$$

Then, since  $w(t, \cdot)$  is given by (5.20), and using Corollary 5.14 together with estimate (5.21) with  $\eta = \frac{3}{2}$  and (5.18), it follows:

$$\begin{aligned} \|w(t, \cdot)\|_{H^3(\Omega)} + \|\pi(t, \cdot)\|_{H^2(\Omega)} &\leq C(\alpha, M_2) \|A_q^{\frac{3}{2}} e^{-tA_q} w(0, \cdot)\|_{L^2(\Omega)} \\ &\leq C(\alpha, M_2) \frac{e^{-\mu t}}{t^{\frac{3}{2}}} (\|u_0\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}) \\ &\leq C(\alpha, u_0, M_1, M_2) \frac{e^{-\mu t}}{t^{\frac{3}{2}}}. \end{aligned} \quad (5.23)$$

We have from (5.23):

$$\|v\|_{L^2(\Gamma)} \leq C(\alpha, u_0, M_1, M_2) \frac{e^{-\mu t}}{t^{\frac{3}{2}}} + \|u\|_{L^\infty(0, +\infty; L^2(\Gamma))}.$$

Then, passing to the limit when  $t$  goes to infinity, we get:

$$\|v\|_{L^2(\Gamma)} \leq \|u\|_{L^\infty(0, +\infty; L^2(\Gamma))}.$$

In the same way, we obtain:

$$\|\zeta\|_{L^2(\Gamma)} \leq \|p\|_{L^\infty(0, +\infty; L^2(\Gamma))},$$

and

$$\left\| \frac{\partial \zeta}{\partial n} \right\|_{L^2(\Gamma)} \leq \left\| \frac{\partial p}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma))}.$$

These three inequalities directly imply (5.17).  $\square$

**Remark 5.20.** Remark that the right hand side of inequality (5.17) is finite:  $(u, p) \in L^\infty(0, +\infty; H^3(\Omega)) \times L^\infty(0, +\infty; H^2(\Omega))$ . Indeed, let  $\nu > 0$ . Thanks to equation (5.23), we obtain that  $(w, \pi) \in L^\infty(\nu, +\infty; H^3(\Omega)) \times L^\infty(\nu, +\infty; H^2(\Omega))$  and since  $u = w + v$  and  $p = \pi + \zeta$ , we deduce that  $(u, p) \in L^\infty(\nu, +\infty; H^3(\Omega)) \times L^\infty(\nu, +\infty; H^2(\Omega))$ . Moreover, thanks to Corollary 5.4, we have  $(u, p) \in L^\infty(0, \nu; H^3(\Omega)) \times L^\infty(0, \nu; H^2(\Omega))$ .

### 5.5.2 A stability estimate valid in dimension 2

We assume in this subsection that  $d = 2$ . As a direct consequence of Theorem 4.8 and Lemma 5.19, we deduce a stability estimate for the non stationary problem valid only in dimension 2. This is summarized in the following theorem.

**Theorem 5.21.** *Let  $\Gamma \subseteq \Gamma_0$ . We assume that  $\Omega$  is of class  $\mathcal{C}^{3,1}$ . Let  $M_1 > 0$ ,  $M_2 > 0$  and  $\alpha > 0$ . Let  $u_0 \in H \cap H^3(\Omega)$ . Let  $g \in H^{\frac{5}{2}}(\Gamma_0)$  be non identically zero and such that  $\|g\|_{H^{\frac{5}{2}}(\Gamma_0)} \leq M_1$ . Let  $q \in H^{\frac{5}{2}}(\Gamma_{out})$  be such that  $\|q\|_{H^{\frac{5}{2}}(\Gamma_{out})} \leq M_2$  and  $\alpha \leq q$  on  $\Gamma_{out}$ . We denote by  $(u_j, p_j)$  the solution of (5.1) associated with  $q = q_j$  for  $j = 1, 2$ . Let  $K$  be a compact subset of  $\{x \in \Gamma_{out} \mid v_1 \neq 0\}$  where  $(v_1, \zeta_1)$  is the solution of the stationary system (4.1) with  $q = q_1$  and let  $m > 0$  be a constant such that  $|v_1| \geq m$  on  $K$ .*

*Then, there exist  $C(\alpha, M_1, M_2) > 0$  and  $C_1(\alpha, M_1, M_2) > 0$  such that*

$$\begin{aligned} & \|q_1 - q_2\|_{L^2(K)} \\ & \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left( \ln \left( \frac{C_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \|p_1 - p_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma))}} \right)} \right)^{\frac{1}{2}}. \end{aligned}$$

**Remark 5.22.** *Remark that, due to the method employed, we have to take the measurements during an infinite time.*

**Remark 5.23.** *Note that, thanks to Remark 5.20, we know that the measurements made on the boundary terms in infinite time are finite.*

*Proof of Theorem 5.21.* Let  $j = 1, 2$  and  $(v_j, \zeta_j)$  be the solution of the stationary problem (4.1) with  $q = q_j$ . We denote by  $(w_j, \pi_j)$  the difference  $(u_j - v_j, p_j - \zeta_j)$ . Thanks to Theorem 4.8, we are able to estimate  $\|q_1 - q_2\|_{L^2(K)}$  with respect to an increasing function of  $(v_1 - v_2)|_\Gamma$ ,  $(\zeta_1 - \zeta_2)|_\Gamma$  and  $\left( \frac{\partial \zeta_1}{\partial n} - \frac{\partial \zeta_2}{\partial n} \right)_\Gamma$ . We are interested in comparing the asymptotic behavior of  $(u_1 - u_2)|_\Gamma$  and  $(p_1 - p_2)|_\Gamma$  to the solution of the stationary problem  $(v_1 - v_2)|_\Gamma$  and  $(\zeta_1 - \zeta_2)|_\Gamma$ . This is done in Lemma 5.19: we have

$$\begin{aligned} & \|v_1 - v_2\|_{L^2(\Gamma)} + \|\zeta_1 - \zeta_2\|_{L^2(\Gamma)} + \left\| \frac{\partial \zeta_1}{\partial n} - \frac{\partial \zeta_2}{\partial n} \right\|_{L^2(\Gamma)} \\ & \leq \|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \|p_1 - p_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma))}. \end{aligned} \tag{5.24}$$

Applying Theorem 4.8 to  $(v_j, \zeta_j)$  for  $j = 1, 2$ , we obtain the existence of positive constants  $C(M_1, M_2, \alpha)$  and  $C_1(M_1, M_2, \alpha)$  such that

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(M_1, M_2, \alpha)}{\left( \ln \left( \frac{C_1(M_1, M_2, \alpha)}{\|v_1 - v_2\|_{L^2(\Gamma)} + \|\zeta_1 - \zeta_2\|_{L^2(\Gamma)} + \left\| \frac{\partial \zeta_1}{\partial n} - \frac{\partial \zeta_2}{\partial n} \right\|_{L^2(\Gamma)}} \right)} \right)^{\frac{1}{2}}.$$

We conclude by using inequality (5.24) and the fact that the function  $x \rightarrow \frac{1}{\ln(\frac{1}{x})}$  increases on  $\mathbb{R}_+^*$ .  $\square$

### 5.5.3 A stability estimate valid in any dimension $d$

Theorem 4.16 and Lemma 5.19 allow to get a stability estimate for the non stationary problem valid only in any dimension. This is stated in the following theorem.

**Theorem 5.24.** *Let  $\Gamma \subseteq \Gamma_0$  be a nonempty open subset of the boundary of  $\Omega$ . We assume that  $\Gamma$  and  $\Gamma_{out}$  are of class  $C^\infty$ . Let  $k \in \mathbb{N}^*$  be such that  $k + 2 > \frac{d}{2}$ ,  $M_1 > 0$ ,  $M_2 > 0$  and  $s \in \mathbb{R}$  be such that  $s > \frac{d-1}{2}$  and  $s \geq \frac{1}{2} + k$ . We assume that  $(g, q_j) \in H^{\frac{1}{2}+k}(\Gamma_0) \times H^s(\Gamma_{out})$ , for  $j = 1, 2$ , are such that  $g$  is non identically zero,  $\|g\|_{H^{\frac{1}{2}+k}(\Gamma_0)} \leq M_1$ ,  $\alpha \leq q_j$  on  $\Gamma_{out}$  and  $\|q_j\|_{H^s(\Gamma_{out})} \leq M_2$ . We denote by  $(u_j, p_j)$  the solution of (5.1) associated with  $q = q_j$  for  $j = 1, 2$ . Let  $K$  be a compact set of  $\{x \in \Gamma_{out} \mid v_1 \neq 0\}$  where  $(v_1, \zeta_1)$  is the solution of the stationary system (4.1) with  $q = q_1$  and  $m > 0$  be such that  $|v_1| \geq m$  on  $K$ .*

*Then, for all  $\beta \in (0, 1)$ , there exists  $C(\alpha, M_1, M_2) > 0$  and  $C_1(\alpha, M_1, M_2) > 0$  such that*

$$\begin{aligned} & \|q_1 - q_2\|_{L^2(K)} \\ & \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left( \ln \left( \frac{C_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \|p_1 - p_2\|_{L^\infty(0, +\infty; L^2(\Gamma))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma))}} \right)^{\frac{3\beta}{4}}}. \end{aligned}$$

*Proof of Theorem 5.24.* The proof follows exactly the same lines as previously.  $\square$

**Remark 5.25.** *Remarks 5.22 and 5.23 are still relevant here.*

## 5.6 Stability estimates when the flux $g$ depends on $t$

We restrict ourselves to the case where  $g$  is colinear to the outgoing normal  $n$ :  $g = \kappa n$ .

In the following, we will use the following assumption:

$$\kappa \in H_{loc}^2(0, +\infty; H^{\frac{3}{2}}(\Gamma_0)), \quad (5.25)$$

Moreover, we assume that there exists  $h \in H^{\frac{3}{2}}(\Gamma_0)$ , such that

$$\lim_{t \rightarrow \infty} \left( \|\kappa(t, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)} + \|\partial_t \kappa(t, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)} + \left( \int_0^t e^{-\mu(t-s)} \|\partial_t \kappa(s, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)}^2 ds \right)^{\frac{1}{2}} \right) = 0, \quad (5.26)$$

where  $\mu$  is given by inequality (5.16).

### 5.6.1 Preliminary result

The idea is the same as in the previous section, it consists in comparing the asymptotic behavior as  $t$  goes to infinity of the solution of the non stationary problem, to the solution of the stationary system. The following lemma gives an estimate of a solution of the non stationary system (5.1) associated with  $q$  and  $g = \kappa n$  with respect to a solution of the stationary problem (4.1) associated with  $q$  and  $hn$ .

**Lemma 5.26.** *Assume that  $\Omega$  is of class  $\mathcal{C}^{2,1}$ . Let  $M_1 > 0$ ,  $M_2 > 0$ ,  $\alpha > 0$  and  $s \in \mathbb{R}$  be such that  $s > \frac{d-1}{2}$  and  $s \geq \frac{3}{2}$ . Let  $u_0 \in H \cap H^3(\Omega)$ . Let  $h \in H^{\frac{3}{2}}(\Gamma_0)$  be non identically zero and such that  $\|h\|_{H^{\frac{3}{2}}(\Gamma_{out})} \leq M_1$ . We assume that  $\kappa$  satisfies (5.25) and that  $(h, \kappa)$  satisfies (5.26). Let  $q \in H^s(\Gamma_{out})$  be such that  $\|q\|_{H^s(\Gamma_{out})} \leq M_2$  and  $\alpha \leq q$  on  $\Gamma_{out}$ . We denote by  $(u, p)$  the solution of (5.1) associated with  $q$  and by  $(v, \zeta)$  the solution of the stationary system:*

$$\begin{cases} -\Delta v + \nabla \zeta & = 0, & \text{in } \Omega, \\ \operatorname{div} v & = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} - \zeta n & = hn, & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial n} - \zeta n + qv & = 0, & \text{on } \Gamma_{out}. \end{cases}$$

Then, the following inequality holds:

$$\|v\|_{L^2(\Gamma_0)} + \|\zeta\|_{L^2(\Gamma_0)} + \left\| \frac{\partial \zeta}{\partial n} \right\|_{L^2(\Gamma_0)} \leq \|u\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \|p\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \left\| \frac{\partial p}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}.$$

**Remark 5.27.** *Let  $l \in H_{loc}^2(0, +\infty; H^{\frac{3}{2}}(\Gamma_0))$  and  $h \in H^{\frac{3}{2}}(\Gamma_0)$ . Assume that there exists  $\theta > \mu$  such that:*

$$\sup_{t \geq 0} e^{t\theta} \left( \|l(t, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)} + \|\partial_t l(t, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)} \right) < +\infty,$$

Then  $\kappa = h + l$  satisfies (5.26). We note that a particular case of function satisfying (5.26) is given by  $l(t, x) = \omega(t)\rho(x)$  where  $\omega \in H_{loc}^2(0, +\infty)$ ,  $\rho \in H^{\frac{3}{2}}(\Gamma_0)$  and  $\lim_{t \rightarrow \infty} e^{t\theta}\omega(t) = \lim_{t \rightarrow \infty} e^{t\theta}\omega'(t) = 0$ .

*Proof of Lemma 5.26.* We decompose  $u$  into  $u = v + w$  where  $(v, \zeta) \in H^3(\Omega) \times H^2(\Omega)$  is the solution of the stationary problem:

$$\begin{cases} -\Delta v + \nabla \zeta & = 0, & \text{in } \Omega, \\ \operatorname{div} v & = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} - \zeta n & = hn, & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial n} - \zeta n + qv & = 0, & \text{on } \Gamma_{out}. \end{cases}$$

and  $(w, \pi)$  is solution of the following problem:

$$\begin{cases} \partial_t w - \Delta w + \nabla \pi & = 0, & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} w & = 0, & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial w}{\partial n} - \pi n & = (\kappa - h)n, & \text{on } (0, +\infty) \times \Gamma_0, \\ \frac{\partial w}{\partial n} - \pi n + qw & = 0, & \text{on } (0, +\infty) \times \Gamma_{out}, \\ w(0, x) & = u_0(x) - v(x), & \text{in } \Omega. \end{cases}$$

We are going to prove that:

$$\|w(t, \cdot)\|_{H^3(\Omega)} + \|\pi(t, \cdot)\|_{H^2(\Omega)} \leq G(t),$$

where  $G$  is a function which tends to 0 when  $t$  goes to  $+\infty$ . We would like to perform the same reasoning as in Lemma 5.19. Since the function  $\kappa$  depends on  $t$ , there will be one more step than in Lemma 5.19 and that is why we assume (5.26).

We split  $(w, \pi)$  into two terms:  $w = u^0 + \tilde{w}$  and  $\pi = p^0 + \tilde{\pi}$ , where  $(u^0, p^0)$  is solution of

$$\begin{cases} \partial_t u^0 - \Delta u^0 + \nabla p^0 = 0, & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} u^0 = 0, & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial u^0}{\partial n} - p^0 n = (\kappa - h)n, & \text{on } (0, +\infty) \times \Gamma_0, \\ \frac{\partial u^0}{\partial n} - p^0 n + q u^0 = 0, & \text{on } (0, +\infty) \times \Gamma_{out}, \\ u^0(0, x) = 0, & \text{in } \Omega, \end{cases}$$

and  $(\tilde{w}, \tilde{\pi})$  is solution of

$$\begin{cases} \partial_t \tilde{w} - \Delta \tilde{w} + \nabla \tilde{\pi} = 0, & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} \tilde{w} = 0, & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial \tilde{w}}{\partial n} - \tilde{\pi} n = 0, & \text{on } (0, +\infty) \times \Gamma_0, \\ \frac{\partial \tilde{w}}{\partial n} - \tilde{\pi} n + q \tilde{w} = 0, & \text{on } (0, +\infty) \times \Gamma_{out}, \\ \tilde{w}(0, x) = u_0(x) - v(x), & \text{in } \Omega. \end{cases}$$

Using the same arguments as in the previous section, we prove that:

$$\|\tilde{w}(t, \cdot)\|_{H^3(\Omega)} + \|\tilde{\pi}(t, \cdot)\|_{H^2(\Omega)} \leq C(\alpha, u_0, M_1, M_2) \frac{e^{-\mu t}}{t^{\frac{3}{2}}}.$$

It remains for us to bound  $\|u^0(t, \cdot)\|_{H^3(\Omega)}$  and  $\|p^0(t, \cdot)\|_{H^2(\Omega)}$ . Let  $t > 0$ . We are going to prove that there exists a constant  $C(\alpha, M_2) > 0$  such that:

$$\begin{aligned} \|u^0(t, \cdot)\|_{H^3(\Omega)} + \|p^0(t, \cdot)\|_{H^2(\Omega)} &\leq C(\alpha, M_2) \left( \|\kappa(t, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)} + \|\partial_t \kappa(t, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)} \right) \\ &+ C(\alpha, M_2) \left( e^{-\mu t} \|\kappa(0, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)} + \left( \int_0^t e^{-\mu(t-s)} \|\partial_t \kappa(s, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)}^2 ds \right)^{\frac{1}{2}} \right). \end{aligned} \quad (5.27)$$

Note that the right hand side tends to zero as  $t$  goes to infinity thanks to assumption (5.26). Let  $T > 0$ . We introduce  $(y, \rho)$  the solution of

$$\begin{cases} -\Delta y + \nabla \rho = 0, & \text{in } \Omega, \\ \operatorname{div} y = 0, & \text{in } \Omega, \\ \frac{\partial y}{\partial n} - \rho n = (\kappa - h)n, & \text{on } \Gamma_0, \\ \frac{\partial y}{\partial n} - \rho n + qy = 0, & \text{on } \Gamma_{out}, \end{cases}$$

for all  $t > 0$ . We know that  $(y, \rho) \in H^3(\Omega) \times H^2(\Omega)$  and satisfies,

$$\|y(t, \cdot)\|_{H^3(\Omega)} + \|\rho(t, \cdot)\|_{H^2(\Omega)} \leq C(\alpha, M_2) \|\kappa(t, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)}. \quad (5.28)$$

Remark that  $y(t, \cdot)$  belongs to  $\mathcal{D}(A_q^{\frac{3}{2}})$ . Indeed, there exists a unique  $\tilde{p}(t, \cdot) \in H^3(\Omega)$  solution of:

$$\begin{cases} \Delta \tilde{p} = 0, & \text{in } \Omega, \\ \tilde{p} = \kappa - h, & \text{on } \Gamma_0, \\ \tilde{p} = 0, & \text{on } \Gamma_{out}, \end{cases} \quad (5.29)$$

for all  $t > 0$  and there exists a constant  $C > 0$  such that

$$\|\tilde{p}(t, \cdot)\|_{H^3(\Omega)} \leq C \|\kappa(t, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)}. \quad (5.30)$$

Then  $(y, \rho + \tilde{p})$  is solution of:

$$\begin{cases} -\Delta y + \nabla(\rho + \tilde{p}) &= \nabla \tilde{p}, & \text{in } \Omega, \\ \operatorname{div} y &= 0, & \text{in } \Omega, \\ \frac{\partial y}{\partial n} - (\rho + \tilde{p})n &= 0, & \text{on } \Gamma_0, \\ \frac{\partial y}{\partial n} - (\rho + \tilde{p})n + qy &= 0, & \text{on } \Gamma_{out}, \end{cases}$$

for all  $t > 0$ . Remark that, since  $\nabla \tilde{p} \in H$ , we have that  $y(t) \in \mathcal{D}(A_q)$  by definition of  $\mathcal{D}(A_q)$ . Notice that the fact that  $g$  is colinear to  $n$  is important here and enable the introduction of a new pressure. We deduce from  $A_q y(t) = \nabla \tilde{p}(t) \in V = \mathcal{D}(A_q^{\frac{1}{2}})$  that  $y(t) \in \mathcal{D}(A_q^{\frac{3}{2}})$ . Moreover, using Corollary 5.14 and inequality (5.30), there exists a constant  $C(M_2) > 0$  such that:

$$\|A_q^{\frac{3}{2}} y(t, \cdot)\|_{L^2(\Omega)} \leq C(M_2) \|A_q y(t, \cdot)\|_{H^1(\Omega)} = C(M_2) \|\nabla \tilde{p}(t)\|_{H^1(\Omega)} \leq C(M_2) \|\kappa(t, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)}, \quad (5.31)$$

that is to say:

$$\|y(t, \cdot)\|_{\mathcal{D}(A_q^{\frac{3}{2}})} \leq C(\alpha, M_2) \|\kappa(t, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)}. \quad (5.32)$$

We can use the same argument, replacing  $\kappa - h$  by  $\partial_t \kappa$ , to prove that  $\partial_t y(t, \cdot) \in L^2(0, +\infty; \mathcal{D}(A_q^{\frac{3}{2}}))$  together with the estimate

$$\|\partial_t y(t, \cdot)\|_{\mathcal{D}(A_q^{\frac{3}{2}})} \leq C(\alpha, M_2) \|\partial_t \kappa(t, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)}. \quad (5.33)$$

Let us consider  $\bar{w} = u^0 - y$  and  $\bar{p} = p^0 - \rho$ . The couple  $(\bar{w}, \bar{p})$  is solution of

$$\begin{cases} \partial_t \bar{w} - \Delta \bar{w} + \nabla \bar{p} &= -\partial_t y, & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} \bar{w} &= 0, & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial \bar{w}}{\partial n} - \bar{p}n &= 0, & \text{on } (0, +\infty) \times \Gamma_0, \\ \frac{\partial \bar{w}}{\partial n} - \bar{p}n + q\bar{w} &= 0, & \text{on } (0, +\infty) \times \Gamma_{out}, \\ \bar{w}(0, x) &= -y(0, x), & \text{in } \Omega. \end{cases} \quad (5.34)$$

We know that  $\bar{w}$  is given by:

$$\bar{w}(t, \cdot) = -e^{-tA_q} y(0, \cdot) - \int_0^t e^{-(t-s)A_q} \partial_t y(s, \cdot) ds.$$

Using the family  $(\phi_q^l)_{l \geq 1}$  defined by Proposition 5.10, we have:  $\bar{w}(t, \cdot) = \sum_{l \geq 1} C_l(t) \phi_q^l$ , with

$$C_l(t) = -e^{-t\lambda_q^l} (y(0, \cdot), \phi_q^l)_{L^2(\Omega)} - \int_0^t e^{-(t-s)\lambda_q^l} (\partial_t y(s, \cdot), \phi_q^l)_{L^2(\Omega)} ds.$$

Thus, recalling that  $(\lambda_q^l)_{l \geq 1}$  satisfies (5.16) and by using Cauchy-Schwarz inequality, there exists  $C > 0$  such that:

$$C_l(t)^2 \leq 2e^{-2t\mu} (y(0, \cdot), \phi_q^l)_{L^2(\Omega)}^2 + C \int_0^t e^{-(t-s)\mu} (\partial_t y(s, \cdot), \phi_q^l)_{L^2(\Omega)}^2 ds.$$

We obtain from estimates (5.32) and (5.33):

$$\|\bar{w}(t, \cdot)\|_{\mathcal{D}(A_q^{\frac{3}{2}})} \leq C(\alpha, M_2) \left( e^{-\mu t} \|\kappa(0, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)} + \left( \int_0^t e^{-\mu(t-s)} \|\partial_t \kappa(s, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)}^2 ds \right)^{\frac{1}{2}} \right). \quad (5.35)$$

Remark that, thanks to Proposition 5.15 and Corollary 5.14, we have:

$$\|\bar{w}(t, \cdot)\|_{H^3(\Omega)} \leq C(\alpha, M_2) \|A_q \bar{w}(t, \cdot)\|_{H^1(\Omega)} \leq C(\alpha, M_2) \|A_q^{\frac{3}{2}} \bar{w}(t, \cdot)\|_{L^2(\Omega)} \leq C(\alpha, M_2) \|\bar{w}(t, \cdot)\|_{\mathcal{D}(A_q^{\frac{3}{2}})}. \quad (5.36)$$

To summarize, using (5.36) and (5.35), we obtain the desired estimate:

$$\|\bar{w}(t, \cdot)\|_{H^3(\Omega)} \leq C(\alpha, M_2) \left( e^{-\mu t} \|\kappa(0, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)} + \left( \int_0^t e^{-\mu(t-s)} \|\partial_t \kappa(s, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)}^2 ds \right)^{\frac{1}{2}} \right). \quad (5.37)$$

Using now the regularity result for the stationary problem given in Proposition 4.4, we have:

$$\|\bar{p}(t, \cdot)\|_{H^2(\Omega)} \leq C(\alpha, M_2) (\|\partial_t y(t, \cdot)\|_{H^1(\Omega)} + \|\partial_t \bar{w}(t, \cdot)\|_{H^1(\Omega)}).$$

Since  $A_q \bar{w} = -\partial_t y - \partial_t \bar{w}$ , we obtain:

$$\|\bar{p}(t, \cdot)\|_{H^2(\Omega)} \leq C(\alpha, M_2) (\|\partial_t y(t, \cdot)\|_{H^1(\Omega)} + \|A_q \bar{w}(t, \cdot)\|_{H^1(\Omega)}).$$

Thanks to Corollary 5.14, we know that  $\|A_q \bar{w}(t, \cdot)\|_{H^1(\Omega)} \leq C(\alpha) \|A_q^{\frac{3}{2}} \bar{w}(t, \cdot)\|_{L^2(\Omega)}$ . Therefore, using (5.33) and (5.35), we obtain:

$$\begin{aligned} & \|\bar{p}(t, \cdot)\|_{H^2(\Omega)} \\ & \leq C(\alpha, M_2) \left( e^{-\mu t} \|\kappa(0, \cdot) - h\|_{H^{\frac{3}{2}}(\Gamma_0)} + \|\partial_t \kappa(t, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)} + \left( \int_0^t e^{-\mu(t-s)} \|\partial_t \kappa(s, \cdot)\|_{H^{\frac{3}{2}}(\Gamma_0)}^2 ds \right)^{\frac{1}{2}} \right). \end{aligned} \quad (5.38)$$

The estimate (5.27) follows from the identities  $u^0 = \bar{w} + y$ ,  $p^0 = \bar{w} + \rho$  by combining inequalities (5.28), (5.37) and (5.38).  $\square$

## 5.6.2 A stability estimate valid in dimension 2

We assume in this subsection that  $d = 2$ . As a direct consequence of Theorem 4.8 and Lemma 5.26, we deduce a logarithmic stability estimate for the non stationary problems valid only in dimension 2. This is summarized in Theorem 5.28.

**Theorem 5.28.** *Assume that  $\Omega$  is of class  $\mathcal{C}^{3,1}$ . Let  $M_1 > 0$ ,  $M_2 > 0$  and  $\alpha > 0$ . Let  $u_0 \in H \cap H^3(\Omega)$ . Let  $h \in H^{\frac{5}{2}}(\Gamma_0)$  be non identically zero and such that  $\|h\|_{H^{\frac{5}{2}}(\Gamma_{out})} \leq M_1$ .*

*We assume that  $\kappa$  satisfies (5.25) and that  $(h, \kappa)$  satisfies (5.26). Let  $q \in H^{\frac{5}{2}}(\Gamma_{out})$  be such that  $\|q\|_{H^{\frac{5}{2}}(\Gamma_{out})} \leq M_2$  and  $\alpha \leq q$  on  $\Gamma_{out}$ . We denote by  $(u_j, p_j)$  the solution of (5.1) associated with  $q = q_j$  and  $g = \kappa n$ , for  $j = 1, 2$ . Let  $K$  be a compact subset of  $\{x \in \Gamma_{out} \mid v_1 \neq 0\}$  where  $(v_1, \zeta_1)$  is the solution of*

$$\begin{cases} -\Delta v + \nabla \zeta & = 0, & \text{in } \Omega, \\ \operatorname{div} v & = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} - \zeta n & = hn, & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial n} - \zeta n + q_1 v & = 0, & \text{on } \Gamma_{out}, \end{cases}$$

*and let  $m > 0$  be a constant such that  $|v_1| > m$  on  $K$ . We assume that (5.26) is verified.*



Then, there exist  $C(\alpha, M_1, M_2) > 0$  and  $C_1(\alpha, M_1, M_2) > 0$  such that

$$\begin{aligned} & \|q_1 - q_2\|_{L^2(K)} \\ & \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left( \ln \left( \frac{C_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \|p_1 - p_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}} \right) \right)^{\frac{1}{2}}}. \end{aligned}$$

*Proof of Theorem 5.28.* For  $j = 1, 2$ , we decompose  $u_j$  into  $u_j = v_j + w_j$  where  $(v_j, \zeta_j) \in H^4(\Omega) \times H^3(\Omega)$  is the solution of the stationary problem:

$$\begin{cases} -\Delta v + \nabla \zeta & = 0, & \text{in } \Omega, \\ \operatorname{div} v & = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} - \zeta n & = hn, & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial n} - \zeta n + q_j v & = 0, & \text{on } \Gamma_{out}. \end{cases}$$

and  $(w_j, \pi_j)$  is solution of the following problem:

$$\begin{cases} \partial_t w - \Delta w + \nabla \pi & = 0, & \text{in } (0, +\infty) \times \Omega, \\ \operatorname{div} w & = 0, & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial w}{\partial n} - \pi n & = (\kappa - h)n, & \text{on } (0, +\infty) \times \Gamma_0, \\ \frac{\partial w}{\partial n} - \pi n + q_j w & = 0, & \text{on } (0, +\infty) \times \Gamma_{out}, \\ w(0, x) & = u_0(x) - v_j(x), & \text{in } \Omega. \end{cases}$$

Thanks to Lemma 5.26, we have:

$$\begin{aligned} & \|v_1 - v_2\|_{L^2(\Gamma_0)} + \|\zeta_1 - \zeta_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial \zeta_1}{\partial n} - \frac{\partial \zeta_2}{\partial n} \right\|_{L^2(\Gamma_0)} \\ & \leq \|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \|p_1 - p_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}. \end{aligned} \quad (5.39)$$

Applying Theorem 4.8 to  $(v_j, \zeta_j)$  for  $j = 1, 2$ , we obtain the existence of positive constants  $C(M_1, M_2, \alpha)$  and  $C_1(M_1, M_2, \alpha)$  such that

$$\|q_1 - q_2\|_{L^2(K)} \leq \frac{1}{m} \frac{C(m, M_1, M_2, \alpha)}{\left( \ln \left( \frac{C_1(M_1, M_2, \alpha)}{\|v_1 - v_2\|_{L^2(\Gamma_0)} + \|\zeta_1 - \zeta_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial \zeta_1}{\partial n} - \frac{\partial \zeta_2}{\partial n} \right\|_{L^2(\Gamma_0)}} \right) \right)^{\frac{1}{2}}}.$$

We conclude by using inequality (5.39) and the fact that the function  $x \mapsto \frac{1}{\ln(\frac{1}{x})}$  increases on  $\mathbb{R}_+^*$ .  $\square$

### 5.6.3 A stability estimate valid in any dimension $d$

Theorem 4.16 and Lemma 5.26 imply a logarithmic stability estimate for the non stationary problem valid only in any dimension. This is stated in Theorem 5.29.

**Theorem 5.29.** *We assume that  $\Gamma_{out}$  is of class  $C^\infty$ . Let  $u_0 \in H^3(\Omega) \cap H$ ,  $k \in \mathbb{N}^*$  be such that  $k + 2 > \frac{d}{2}$ ,  $M_1 > 0$ ,  $M_2 > 0$ ,  $\alpha > 0$  and  $s \in \mathbb{R}$  be such that  $s > \frac{d-1}{2}$  and*

$s \geq \frac{1}{2} + k$ . Let  $h \in H^{\frac{1}{2}+k}(\Gamma_0)$  be non identically zero and such that  $\|h\|_{H^{\frac{1}{2}+k}(\Gamma_0)} \leq M_1$ . We assume that  $\kappa$  satisfies (5.25) and that  $(h, \kappa)$  satisfies (5.26). Let  $q_j \in H^s(\Gamma_{out})$  be such that  $\alpha \leq q_j$  on  $\Gamma_{out}$  and  $\|q_j\|_{H^s(\Gamma_{out})} \leq M_2$  for  $j = 1, 2$ . We denote by  $(u_j, p_j)$  the solution of (5.1) associated with  $q = q_j$  and  $g = \kappa n$  for  $j = 1, 2$ . Let  $K$  be a compact subset of  $\{x \in \Gamma_{out} \mid v_1 \neq 0\}$  where  $(v_1, \zeta_1)$  is the solution of

$$\begin{cases} -\Delta v + \nabla \zeta & = 0, & \text{in } \Omega, \\ \operatorname{div} v & = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} - \zeta n & = h n, & \text{on } \Gamma_0, \\ \frac{\partial v}{\partial n} - \zeta n + q_1 v & = 0, & \text{on } \Gamma_{out}. \end{cases}$$

We assume that (5.26) is verified.

Then, there exists  $d_0 > 0$  such that for all  $\beta \in (0, 1)$ , there exists  $C(\alpha, M_1, M_2) > 0$  and  $C_1(\alpha, M_1, M_2) > 0$  such that

$$\begin{aligned} & \|q_1 - q_2\|_{L^2(K)} \\ & \leq \frac{1}{m} \frac{C(\alpha, M_1, M_2)}{\left( \ln \left( \frac{C_1(\alpha, M_1, M_2)}{\|u_1 - u_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \|p_1 - p_2\|_{L^\infty(0, +\infty; L^2(\Gamma_0))} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^\infty(0, +\infty; L^2(\Gamma_0))}} \right) \right)^{\frac{3\beta}{4}}. \end{aligned}$$

*Proof of Theorem 5.29.* It is exactly the same approach as in the proof of Theorem 5.28.  $\square$

## 5.7 Conclusion

We have proved two stability estimates for the inverse problem of identifying a Robin coefficient on some non accessible part of the boundary from measurements available on the other part of the boundary, for  $(u, p)$  solution of the non stationary Stokes system with mixed Neumann and Robin boundary conditions. To do so, we have used the stability estimates proved for the stationary problem in Chapter 4 and the theory of analytic semigroup. Note that global regularity is needed on  $(u, p)$ .

Due to the method employed, we have to take measurements during infinite time. Up to our knowledge, questions concerning stability estimates in finite time, even for the inverse problem of identifying a Robin coefficient defined on some part of the boundary from measurements available on the other part of the boundary for the Laplace equation, remain largely open.



# A Lipschitz stability estimate for the stationary Stokes system

In this chapter, we study the inverse problem of identifying a Robin coefficient on some non accessible part of the boundary from measurements available on the other part of the boundary, for  $(u, p)$  solution of the stationary Stokes system with mixed Dirichlet, Neumann and Robin boundary conditions. We prove a Lipschitz stability estimate, under the *a priori* assumption that the Robin coefficient is piecewise constant and under the assumption that the velocity of a given reference solution stays far from 0 on a part of the boundary where Robin boundary conditions are prescribed. To do so, we follow the approach used in [74] where E. Sincich solves a similar problem for the Laplacian equation.

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## 6.1 Introduction

Let  $\Omega \subset \mathbb{R}^d$ , with  $d = 2, 3$ , be a Lipschitz bounded connected open set such that  $\partial\Omega = \Gamma_l \cup \Gamma_0 \cup \Gamma_{out}$ .

We consider the following Stokes system:

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{in } \Omega, \\ \operatorname{div} u &= 0, & \text{in } \Omega, \\ u &= 0, & \text{in } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn &= g, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + qu &= 0, & \text{on } \Gamma_{out}, \end{cases} \quad (6.1)$$

where  $\Gamma_{out} = \bigcup_{i=1}^N \Gamma_i$  and  $\mathbf{n}$  the exterior normal to  $\Omega$ .

Note that the difference between system (6.1) and system (4.1) is we introduce in (4.1) homogeneous boundary condition on one part of the boundary. Moreover, we do not assume anymore that  $\bar{\Gamma}_0 \cap \bar{\Gamma}_{out} = \emptyset$ . It directly results a loss of global regularity on  $(u, p)$  solution of system (6.1). Nevertheless, we still have local regularity inside the domain and in a neighborhood of the boundaries, as long as we stay away from the junction between two different boundary conditions.

Due to the local regularity, it could be possible to obtain logarithmic stability estimates similar to those stated in Theorems 4.8 and 4.16 for solutions of system (6.1), except that these estimates would be local, which means that we would estimate the Robin coefficient only on a compact set  $\mathcal{K} \subseteq K$ , where  $K$  is defined in Theorems 4.8 and 4.16, such that  $\bar{\mathcal{K}} \cap \bar{\Gamma}_l = \emptyset$  and  $\bar{\mathcal{K}} \cap \bar{\Gamma}_0 = \emptyset$ .

We also want to highlight that the identifiability result stated in Proposition 4.6 still stands for the solutions of system (6.1) under the same assumption on the data but, due to the homogeneous boundary conditions, we do not need anymore that  $q \geq \alpha$ , it is sufficient to assume  $q \geq 0$ .

In this chapter, we provide a **Lipschitz stability estimate**, under the *a priori* assumption that the **Robin coefficient is piecewise constant** on  $\Gamma_{out}$ :

$$q|_{\Gamma_i} = q_i, \text{ on } \Gamma_i, \text{ for } 1 \leq i \leq N, \quad (6.2)$$

with  $(q_i)_{1 \leq i \leq N} \in \mathbb{R}_+^N$ .

Let us explain the structure of this chapter. In Section 6.2, we begin with existence and regularity results on  $(u, p)$  solution of system (6.1): we state global and local regularity results which will be useful to prove the Lipschitz stability estimate. Then, we give unique continuation estimates in Section 6.3. Finally, Section 6.4 is dedicated to the main result: we provide a Lipschitz stability estimate.

Throughout this chapter, we will use the following notations:

- For  $x \in \mathbb{R}^d$  and  $r > 0$ , we denote  $\mathbf{B}_r(\mathbf{x})$  the ball of center  $x$  and of radius  $r$ .
- Let  $\Gamma \subset \partial\Omega$  be a non empty part of the boundary. We denote by

$$\Gamma^{in} = \{x \in \Gamma / d(x, \overline{\partial\Omega \setminus \Gamma}) > 0\}.$$

Since the open set  $\Omega$  is Lipschitz, it satisfies the cone property:

**Definition 6.1** (Cone property). *We say that  $\Omega$  satisfies the cone property if there exists  $\theta \in (0, \frac{\pi}{2})$  and  $R_0 > 0$  such that for all  $x_0 \in \partial\Omega$ , there exists  $\xi \in \mathbb{R}^d$ ,  $|\xi| = 1$  such that the finite cone*

$$\mathcal{C} = \{x \in \mathbb{R}^d / (x - x_0) \cdot \xi > |x - x_0| \cos \theta, |x - x_0| \leq R_0\}$$

*is included in  $\Omega$ .*

## 6.2 Regularity results

We focus in this section on global and local regularity results for system (6.1). Even if global  $H^2 \times H^1$  regularity is not expected in general due to the mixed Dirichlet and Neumann boundary conditions, we obtain local regularity.

### 6.2.1 Global regularity

We recall the definition of  $\mathbf{V}_{\Gamma_l}$  which is given by (2.3).

**Proposition 6.2.** *Let  $\mathbb{1}_{\Gamma_0}g \in H^{-\frac{1}{2}}(\partial\Omega)$  and assume that  $q$  satisfies (6.2).*

*Then, system (6.1) admits a unique solution  $(u, p) \in V_{\Gamma_l} \times L^2(\Omega)$ . Moreover, there exists a constant  $C(R_M) > 0$  such that:*

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(R_M) \|\mathbb{1}_{\Gamma_0}g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \quad (6.3)$$

*Proof of Proposition 6.2.* The variational formulation of the problem is: find  $u \in V_{\Gamma_l}$  such that for every  $v \in V_{\Gamma_l}$ ,

$$\int_{\Omega} \nabla u : \nabla v + \int_{\Gamma_{out}} qu \cdot v = \langle \mathbb{1}_{\Gamma_0}g, v \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega}. \quad (6.4)$$

We denote by

$$a_q(u, v) = \int_{\Omega} \nabla u : \nabla v + \int_{\Gamma_{out}} qu \cdot v, \quad (6.5)$$

and

$$L(v) = \langle \mathbb{1}_{\Gamma_0}g, v \rangle_{-\frac{1}{2}, \frac{1}{2}, \partial\Omega}.$$

We easily verify that  $a_q$  is a continuous symmetric bilinear form on  $V_{\Gamma_l}$ . According to Poincaré inequality, the bilinear form  $a_q$  is coercive on  $V_{\Gamma_l}$ . On the other hand,  $L$  is a continuous linear form on  $V_{\Gamma_l}$ . Thus we get the existence and uniqueness of  $u \in V_{\Gamma_l}$  solution of equations (6.1) using the Lax-Milgram Theorem. We prove the existence and uniqueness of  $p \in L^2(\Omega)$  in a classical way, by using De Rham Theorem and the boundary conditions, as we did in the proof of Proposition 2.25.  $\square$

### 6.2.2 Local regularity

Inside the domain  $\Omega$ , we have local regularity: this is resumed in Proposition 6.3. Moreover, locally near the boundary, as long as we stay away from the junction between two different boundary conditions, we can also obtain local regularity. We refer to Propositions 6.4 and 6.6 for a statement of these regularity results.

**Proposition 6.3.** *Let  $m \in \mathbb{N}^*$  and  $\hat{\omega} \subset \Omega$  be a relatively compact open set. Let  $\mathbb{1}_{\Gamma_0}g \in H^{-\frac{1}{2}}(\partial\Omega)$  and assume that  $q$  satisfies (6.2).*

*Then, the solution  $(u, p)$  of system (6.1) belongs to  $H^{m+1}(\hat{\omega}) \times H^m(\hat{\omega})$ . Moreover, there exists a constant  $C(R_M) > 0$  such that:*

$$\|u\|_{H^{m+1}(\hat{\omega})} + \|p\|_{H^m(\hat{\omega})} \leq C(R_M) \|\mathbb{1}_{\Gamma_0}g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \quad (6.6)$$

Although the proof of this result is classical, we give here a sketch of the proof for the sake of completeness.

*Proof of Proposition 6.3.* We prove this proposition by induction on  $m$ . For  $m = 0$ , the result is given by Proposition 6.2.

We assume that Proposition 6.3 holds for some fixed  $m$ . Let us prove that the proposition also holds for  $m + 1$ . Let  $\omega$  be an open set of class  $\mathcal{C}^{m+1,1}$  such that  $\hat{\omega} \subset \omega \subset \Omega$ . We localize in the neighborhood of  $\hat{\omega}$ . Let  $\chi \in \mathcal{C}_c^\infty(\omega)$  be such that  $\chi = 1$  in  $\hat{\omega}$  and  $0 \leq \chi \leq 1$  everywhere else. We denote by  $(v, \pi) = (\chi u, \chi p)$ . Note that  $(v, \pi)$  is solution if the following problem:

$$\begin{cases} -\Delta v + \nabla \pi &= -\Delta \chi u - 2\nabla u \nabla \chi + \nabla \chi p, & \text{in } \omega, \\ \operatorname{div} v &= \nabla \chi \cdot u, & \text{in } \omega, \\ \frac{\partial v}{\partial n} - \pi n &= 0, & \text{on } \partial\omega. \end{cases}$$

Let us denote by  $f = -\Delta \chi u - 2\nabla u \nabla \chi + \nabla \chi p$  and  $h = \nabla \chi \cdot u$ . By the induction assumption, we deduce that the  $(f, h)$  belongs to  $H^m(\omega) \times H^{m+1}(\omega)$ . By application of Proposition 2.12 with  $k = m$ , we obtain that  $(v, \pi)$  belongs to  $H^{m+2}(\omega) \times H^{m+1}(\omega)$ . Since  $\chi = 1$  on  $\hat{\omega}$ , we obtain the desired result.  $\square$

We now study the regularity near the boundary of the domain. Proposition 6.4 states regularity result in the restriction to  $\Omega$  of a neighborhood of any point  $x_j \in \Gamma_j^{in}$ , for  $i = j, \dots, N$ .

**Proposition 6.4.** *Let  $m \in \mathbb{N}^*$ ,  $R_M > 0$ ,  $1 \leq j \leq N$  and  $x_j \in \Gamma_j^{in}$ . We assume that  $\Gamma_j$  is of class  $\mathcal{C}^{m,1}$  if  $m \geq 1$  and Lipschitz otherwise. Let  $\mathbb{1}_{\Gamma_0}g \in H^{-\frac{1}{2}}(\partial\Omega)$  and assume that  $q$  satisfies (6.2).*

*Then, there exists  $R > 0$  such that the solution  $(u, p)$  of system (6.1) belongs to  $H^{m+1}(B_R(x_j) \cap \Omega) \times H^m(B_R(x_j) \cap \Omega)$ . Furthermore, if we assume that  $q \leq R_M$ , there exists a constant  $C(R_M) > 0$  such that*

$$\|u\|_{H^{m+1}(B_R(x_j) \cap \Omega)} + \|p\|_{H^m(B_R(x_j) \cap \Omega)} \leq C(R_M) \|\mathbb{1}_{\Gamma_0}g\|_{H^{-\frac{1}{2}}(\partial\Omega)}.$$

The proof of Proposition 6.4 relies on classical argument and is similar to the proof of Proposition 6.3.

Then, we deduce from Proposition 6.4 local Hölder regularity on  $(u, p)$  solution of system (6.1) near the boundary  $\Gamma_j$ , as long as we stay away from the junction between two different boundary conditions.

**Corollary 6.5.** *Let  $1 \leq j \leq N$  and  $x_j \in \Gamma_j^{in}$ . We assume that  $\Gamma_j$  is of class  $\mathcal{C}^{2,1}$ . Let  $\mathbb{1}_{\Gamma_0}g \in H^{-\frac{1}{2}}(\partial\Omega)$  and assume that  $q$  satisfies (6.2).*

*Then, there exists  $R > 0$  and  $0 < \beta < 1$  such that the solution  $(u, p)$  of system (6.1) belongs to  $\mathcal{C}^{1,\beta}(\overline{B_R(x_j) \cap \Omega}) \times \mathcal{C}^{0,\beta}(\overline{B_R(x_j) \cap \Omega})$  for all  $j = 1, \dots, N$ .*



*Proof of Corollary 6.5.* Let  $1 \leq j \leq N$ . Thanks to Proposition 6.4, there exists  $R_j > 0$  such that  $(u, p) \in H^3(B_{R_j}(x_j) \cap \Omega) \times H^2(B_{R_j}(x_j) \cap \Omega)$ . Thanks to Proposition 6.3, we know that there exists a connected open set  $\tilde{\Omega} \subset \Omega$  such that  $\bigcup_{j=1}^N (B_{R_j}(x_j) \cap \Omega) \subset \tilde{\Omega}$  and  $(u, p) \in H^3(\tilde{\Omega}) \times H^2(\tilde{\Omega})$ . Then, since for all  $m \in \mathbb{N}$  such that  $2(m-1) \leq d < 2m$ , there exists  $0 < \lambda < 1$  such that  $H^m(\tilde{\Omega}) \hookrightarrow \mathcal{C}^{0,\lambda}(\tilde{\Omega})$  (see [1]), we deduce that there exists  $0 < \beta < 1$  such that  $H^2(\tilde{\Omega}) \subset \mathcal{C}^{0,\beta}(\tilde{\Omega})$ , which implies that  $(u, p) \in \mathcal{C}^{1,\beta}(\overline{B_R(x_j) \cap \Omega}) \times \mathcal{C}^{0,\beta}(\overline{B_R(x_j) \cap \Omega})$  for all  $j = 1, \dots, N$ , with  $R = \min_{1 \leq j \leq N} R_j$ .  $\square$

Proposition 6.6 states regularity result in the restriction to  $\Omega$  of a neighborhood of any point  $x_0 \in \Gamma_0^{in}$ .

**Proposition 6.6.** *Let  $m \in \mathbb{N}$  and  $x_0 \in \Gamma_0^{in}$ . We assume that  $\Gamma_0$  is of class  $\mathcal{C}^{m,1}$  if  $m \geq 1$  and Lipschitz otherwise. Let  $\mathbb{1}_{\Gamma_0} g \in H^{m-\frac{1}{2}}(\partial\Omega)$  and assume that  $q$  satisfies (6.2).*

*Then, there exists  $R > 0$  such that the solution  $(u, p)$  of system (6.1) belongs to  $H^{m+1}(B_R(x_0) \cap \Omega) \times H^m(B_R(x_0) \cap \Omega)$ . Furthermore, there exists a constant  $C(R_M) > 0$  such that*

$$\|u\|_{H^{m+1}(B_R(x_0) \cap \Omega)} + \|p\|_{H^m(B_R(x_0) \cap \Omega)} \leq C(R_M) \|\mathbb{1}_{\Gamma_0} g\|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

### 6.3 Unique continuation estimate

In this section, we state some unique continuation estimates for the Stokes system which will be useful in the next subsection to prove the Lipschitz stability estimate. They are obtained as corollaries of unique continuation estimates proved in Chapter 3.

To transmit information from a part of the boundary  $\Gamma \subset \Gamma_0$  to a relatively compact open  $\hat{\omega}$  set included in  $\Omega$ , we use Proposition 3.22. Note that in Proposition 3.22, regularity is needed both on the solution of the Stokes system (3.1) and on the open set. This is the reason why we assume that  $\bar{\Gamma}$  does not intersect  $\bar{\Gamma}_l$  and  $\bar{\Gamma}_{out}$  in Proposition 6.7.

**Proposition 6.7.** *Assume that  $\Gamma_0$  is of class  $\mathcal{C}^\infty$ . Let  $R_M > 0$ ,  $M_1 > 0$ ,  $\Gamma \subseteq \Gamma_0$  be a non empty open subset of the boundary of  $\Omega$  such that  $(\bar{\Gamma} \cap \bar{\Gamma}_l) \cup (\bar{\Gamma} \cap \bar{\Gamma}_{out}) = \emptyset$  and  $\hat{\omega} \subset \Omega$  be a relatively compact open set. Let  $\mathbb{1}_{\Gamma_0} g \in H^{\frac{3}{2}}(\partial\Omega)$  be such that  $\|\mathbb{1}_{\Gamma_0} g\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq M_1$  and assume that  $q$  satisfies (6.2) and  $q \leq R_M$ .*

*Then, there exist constants  $C(R_M, M_1) > 0$ ,  $0 < \delta < 1$  such that for all  $(u, p)$  solution of system (6.1), the following inequality is satisfied:*

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq C(R_M, M_1) \left( \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right)^\delta. \quad (6.7)$$

*Proof of Proposition 6.7.* Thanks to local regularity results stated in Subsection 6.2.2, we know that there exists a connected open set  $\tilde{\Omega} \subset \Omega$  of class  $\mathcal{C}^\infty$  such that  $\Gamma \subset \partial\tilde{\Omega}$ ,  $\hat{\omega} \subset \tilde{\Omega}$  and such that the solution  $(u, p)$  of (6.1) belongs to  $H^3(\tilde{\Omega}) \times H^1(\tilde{\Omega})$ . Moreover, there exists a constant  $C(R_M, M_1) > 0$  such that:

$$\|u\|_{H^3(\tilde{\Omega})} + \|p\|_{H^2(\tilde{\Omega})} \leq C(R_M, M_1). \quad (6.8)$$

We apply Proposition 3.22: it allows us to transmit information from  $\Gamma$  to  $\hat{\omega}$ . Then, there exists  $\sigma > 0$  and  $C > 0$  such that, for all  $\tilde{\epsilon} > 0$ ,

$$\begin{aligned} & \|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \\ & \leq \frac{C}{\tilde{\epsilon}} \left( \|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) + \tilde{\epsilon}^\sigma \left( \|u\|_{H^1(\tilde{\Omega})} + \|p\|_{H^1(\tilde{\Omega})} \right). \end{aligned} \quad (6.9)$$

Note that it is the  $H^1$  norms of  $u$  and  $p$  on  $\Gamma$  which appear in the first term in the right hand-side of (6.9). In order to replace them with the  $L^2$  norms of  $u$  and  $p$  on  $\Gamma$ , we use an interpolation inequality: there exists  $c > 0$  such that:

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left( \|u\|_{L^2(\Gamma)}^{\frac{1}{3}} \|u\|_{H^{\frac{3}{2}}(\Gamma)}^{\frac{2}{3}} + \|p\|_{L^2(\Gamma)}^{\frac{1}{3}} \|p\|_{H^{\frac{3}{2}}(\Gamma)}^{\frac{2}{3}} \right).$$

Let  $\bar{\epsilon} > 0$ . If we write

$$\|u\|_{L^2(\Gamma)}^{\frac{1}{3}} \|u\|_{H^{\frac{3}{2}}(\Gamma)}^{\frac{2}{3}} = \left( \frac{1}{\bar{\epsilon}} \|u\|_{L^2(\Gamma)} \right)^{\frac{1}{3}} \left( \bar{\epsilon}^{\frac{1}{2}} \|u\|_{H^{\frac{3}{2}}(\Gamma)} \right)^{\frac{2}{3}},$$

and

$$\|p\|_{L^2(\Gamma)}^{\frac{1}{3}} \|p\|_{H^{\frac{3}{2}}(\Gamma)}^{\frac{2}{3}} = \left( \frac{1}{\bar{\epsilon}} \|p\|_{L^2(\Gamma)} \right)^{\frac{1}{3}} \left( \bar{\epsilon}^{\frac{1}{2}} \|p\|_{H^{\frac{3}{2}}(\Gamma)} \right)^{\frac{2}{3}},$$

according to Young inequality and to the trace injection  $H^2(\tilde{\Omega}) \hookrightarrow H^{\frac{3}{2}}(\Gamma)$ , we obtain:

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left( \bar{\epsilon}^{\frac{1}{3}} \left( \|u\|_{H^3(\tilde{\Omega})} + \|p\|_{H^2(\tilde{\Omega})} \right) + \frac{1}{\bar{\epsilon}} \left( \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} \right) \right). \quad (6.10)$$

Combining inequalities (6.9), and (6.10) with  $\tilde{\epsilon}$  and  $\bar{\epsilon}$  suitably chosen, we obtain that there exists  $s > 0$  and  $C > 0$  such that, for all  $\epsilon > 0$ :

$$\begin{aligned} & \|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \\ & \leq \frac{C}{\epsilon} \left( \|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) + \epsilon^s \left( \|u\|_{H^3(\tilde{\Omega})} + \|p\|_{H^2(\tilde{\Omega})} \right). \end{aligned} \quad (6.11)$$

We conclude by using Lemma 3.17 and inequality (6.8).  $\square$

Then, Lemma 6.8 allows to transmit information inside the domain  $\Omega$ , from a relatively compact open set include in  $\Omega$  to another.

**Lemma 6.8.** [Three balls inequality] Let  $\rho > 0$  and  $q \in \mathbb{R}^d$ . There exist  $C > 0$ ,  $\alpha > 0$  such that for all function  $(u, p) \in H^1(B_{8\rho}(q)) \times H^1(B_{8\rho}(q))$  solution of

$$\begin{cases} -\Delta u + \nabla p & = 0, \\ \operatorname{div} u & = 0, \end{cases} \quad (6.12)$$

in  $B_{8\rho}(q)$  the following inequality is satisfied:

$$\begin{aligned} & \|u\|_{H^1(B_{3\rho}(q))} + \|p\|_{L^2(B_{3\rho}(q))} \\ & \leq C \left( \|u\|_{H^1(B_\rho(q))} + \|p\|_{L^2(B_\rho(q))} \right)^\alpha \left( \|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))} \right)^{1-\alpha}, \end{aligned} \quad (6.13)$$

with  $\alpha = \frac{g(3\rho) - g(\frac{7}{2}\rho)}{g(\frac{\rho}{3}) - g(\frac{7}{2}\rho)}$  and with  $g(r) = e^{-\lambda r^2}$ , for  $\lambda$  large enough.

In the following, in order to refer to this inequality, we will say that the three balls inequality associated to  $q$  and  $\rho$  is satisfied, with the associated constants  $C > 0$  and  $\alpha > 0$ .

**Remark 6.9.** Lemma 6.8 is the counterpart, in the case of the Stokes system, of the so-called three balls inequality for the Laplacian. We refer to [20] or [57] for a three balls inequality for the Laplacian. Note that in [58], C.-H. Lin, G. Uhlmann and J.-N. Wang have obtained an optimal three balls inequality for the Stokes system involving only the velocity in the  $L^2$  norm. From this inequality, they derive an upper bound on the vanishing order of any non trivial solution  $u$  to the Stokes system.

The ideas of the proof are the same as those developed in the proofs of Proposition 3.19 and Lemma 3.26. Nevertheless, we write the proof of Lemma 6.8 for the sake of completeness.

*Proof of Lemma 6.8.* Let us remember that, for  $0 < r < r'$ ,  $A_q(r, r')$  is defined in Notation 3.25. We apply Lemma 3.26 with  $r_1 = \frac{\rho}{3}$ ,  $r_2 = \frac{\rho}{2}$ ,  $r_3 = 3\rho$ ,  $r_4 = \frac{7}{2}\rho$  and  $r_5 = 4\rho$ . We get the existence of  $C > 0$ ,  $c_1 = g(\frac{\rho}{3}) - g(3\rho) > 0$  and  $c_2 = g(3\rho) - g(\frac{7}{2}\rho) > 0$  and  $h_1 > 0$ , such that for all  $h \in (0, h_1)$  and for all functions  $(u, p) \in H^1(B_{8\rho}(q)) \times H^1(B_{8\rho}(q))$  solution of (6.12), we have

$$\begin{aligned} & \|u\|_{H^1(A_q(\frac{\rho}{2}, 3\rho))} + \|p\|_{H^1(A_q(\frac{\rho}{2}, 3\rho))} \\ & \leq C \left( e^{c_1/h} (\|u\|_{H^1(B_{\frac{\rho}{2}}(q))} + \|p\|_{H^1(B_{\frac{\rho}{2}}(q))}) + e^{-c_2/h} (\|u\|_{H^1(B_{4\rho}(q))} + \|p\|_{H^1(B_{4\rho}(q))}) \right). \end{aligned} \quad (6.14)$$

Then, Cacciopoli inequality (see Lemma 3.30) allows to get rid of the  $L^2$  norm of  $\nabla p$ : there exists  $C > 0$  (which depends on  $\rho$ ) such that

$$\|\nabla p\|_{L^2(B_{\frac{\rho}{2}}(q))} \leq C \|p\|_{L^2(B_{\rho}(q))} \quad \text{and} \quad \|\nabla p\|_{L^2(B_{4\rho}(q))} \leq C \|p\|_{L^2(B_{8\rho}(q))}. \quad (6.15)$$

In addition, the following inequality is trivially satisfied:

$$\begin{aligned} & \|u\|_{H^1(B_{\frac{\rho}{2}}(q))} + \|p\|_{L^2(B_{\frac{\rho}{2}}(q))} \\ & \leq C \left( e^{c_1/h} (\|u\|_{H^1(B_{\rho}(q))} + \|p\|_{L^2(B_{\rho}(q))}) + e^{-c_2/h} (\|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))}) \right). \end{aligned} \quad (6.16)$$

Then, combining inequalities (6.14), (6.15) and (6.16), we obtain:

$$\begin{aligned} & \|u\|_{H^1(B_{3\rho}(q))} + \|p\|_{L^2(B_{3\rho}(q))} \\ & \leq C \left( e^{c_1/h} (\|u\|_{H^1(B_{\rho}(q))} + \|p\|_{L^2(B_{\rho}(q))}) + e^{-c_2/h} (\|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))}) \right). \end{aligned} \quad (6.17)$$

Note that for  $h \geq h_1$ , inequality (6.17) remains true by the continuous injection  $H^1(B_{8\rho}(q)) \hookrightarrow H^1(B_{3\rho}(q))$ . We conclude by using Lemma 3.17.  $\square$

We are going to apply Lemma 6.8 to a sequence of balls with decreasing radius. To this aim, we need to know the behavior of the constants when we pass from a ball to another.

**Lemma 6.10.** *Let  $\rho > 0$ ,  $(\bar{q}, q) \in \mathbb{R}^d \times \mathbb{R}^d$ , and  $\mu \in (0, 1)$ . We denote by  $\rho = \mu\bar{\rho}$ . We assume that the three balls inequality (6.13) associated to  $\bar{q}$  and  $\bar{\rho}$  holds for some constants  $C > 0$  and  $\alpha > 0$ .*

*Then, for all functions  $(u, p) \in H^1(B_{8\rho}(q)) \times H^1(B_{8\rho}(q))$  solution of (6.12) in  $B_{8\rho}(q)$ , the following inequality is satisfied:*

$$\begin{aligned} & \|u\|_{H^1(B_{3\rho}(q))} + \|p\|_{L^2(B_{3\rho}(q))} \\ & \leq \tilde{C} \left( \|u\|_{H^1(B_\rho(q))} + \|p\|_{L^2(B_\rho(q))} \right)^\alpha \left( \|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))} \right)^{1-\alpha}, \end{aligned} \quad (6.18)$$

where  $\tilde{C} = \frac{C}{\mu} > 0$ . In other words, the three balls inequality associated to  $q$ ,  $\rho$  is satisfied with the associated constants  $\frac{C}{\mu} > 0$  and  $\alpha > 0$ .

*Proof of Lemma 6.10.* This lemma is inspired from [20] where L. Bourgeois and J. Dardé are concerned with the operator  $P_k = -\Delta - k$ , with  $k \in \mathbb{R}$ , and use similar techniques.

Let  $m \in \mathbb{N}^*$ . By performing the change of variables

$$\begin{aligned} B_{m\bar{\rho}}(\bar{q}) & \rightarrow B_{m\rho}(q), \\ x & \rightarrow q + (x - \bar{q})\mu, \end{aligned}$$

we get:

$$\int_{B_{m\rho}(q)} |u(x)|^2 + |\nabla u(x)|^2 dx = \mu^d \int_{B_{m\bar{\rho}}(\bar{q})} |u(q + \mu(x - \bar{q}))|^2 + |\nabla u(q + \mu(x - \bar{q}))|^2 dx.$$

Let us denote by  $\bar{u}(x) = u(q + \mu(x - \bar{q}))$  and  $\bar{p}(x) = \mu p(q + \mu(x - \bar{q}))$ . Noticing that  $\nabla \bar{u}(x) = \mu \nabla u(q + \mu(x - \bar{q}))$ , we get:

$$\int_{B_{m\rho}(q)} |u(x)|^2 + |\nabla u(x)|^2 dx = \mu^d \int_{B_{m\bar{\rho}}(\bar{q})} |\bar{u}(x)|^2 dx + \frac{1}{\mu^2} |\nabla \bar{u}(x)|^2 dx.$$

Moreover, since

$$\int_{B_{m\rho}(q)} |p(x)|^2 dx = \mu^{d-2} \int_{B_{m\bar{\rho}}(\bar{q})} |\bar{p}(x)|^2 dx,$$

and since  $0 < \mu < 1$ , we obtain:

$$\begin{aligned} & \mu^{\frac{d}{2}} \left( \|\bar{u}\|_{H^1(B_{m\bar{\rho}}(\bar{q}))} + \|\bar{p}\|_{L^2(B_{m\bar{\rho}}(\bar{q}))} \right) \\ & \leq \|u\|_{H^1(B_{m\rho}(q))} + \|p\|_{L^2(B_{m\rho}(q))} \\ & \leq \mu^{\frac{d}{2}-1} \left( \|\bar{u}\|_{H^1(B_{m\bar{\rho}}(\bar{q}))} + \|\bar{p}\|_{L^2(B_{m\bar{\rho}}(\bar{q}))} \right). \end{aligned}$$

Observe that  $(\bar{u}, \bar{p})$  is solution in  $B_{8\bar{\rho}}(\bar{q})$  of system (6.12):

$$-\Delta \bar{u}(\bar{x}) + \nabla \bar{p}(\bar{x}) = \mu^2 (-\Delta u(x) + \nabla p(x)) = 0,$$

for  $\bar{x} \in B_{8\bar{\rho}}(\bar{q})$  and where  $x = q + (\bar{x} - \bar{q})\mu \in B_{8\rho}(q)$ .

Thus,  $(\bar{u}, \bar{p})$  satisfies (6.13) for  $C > 0$  and  $\alpha > 0$ . We deduce that:

$$\begin{aligned} & \|u\|_{H^1(B_{3\rho}(q))} + \|p\|_{L^2(B_{3\rho}(q))} \leq \mu^{\frac{d}{2}-1} \left( \|\bar{u}\|_{H^1(B_{3\bar{\rho}}(\bar{q}))} + \|\bar{p}\|_{L^2(B_{3\bar{\rho}}(\bar{q}))} \right) \\ & \leq \mu^{\frac{d}{2}-1} C \left( \|\bar{u}\|_{H^1(B_{\bar{\rho}}(\bar{q}))} + \|\bar{p}\|_{L^2(B_{\bar{\rho}}(\bar{q}))} \right)^\alpha \left( \|\bar{u}\|_{H^1(B_{8\bar{\rho}}(\bar{q}))} + \|\bar{p}\|_{L^2(B_{8\bar{\rho}}(\bar{q}))} \right)^{1-\alpha} \\ & \leq \frac{C}{\mu} \left( \|u\|_{H^1(B_\rho(q))} + \|p\|_{L^2(B_\rho(q))} \right)^\alpha \left( \|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))} \right)^{1-\alpha}. \end{aligned}$$

□

## 6.4 Lipschitz stability estimate

In this section, we state and prove the Lipschitz stability estimate of Theorem 6.11.

**Theorem 6.11.** *Assume that  $\Gamma_0$  is of class  $C^\infty$  and  $\Gamma_i$  is of class  $C^{2,1}$  for  $i = 1, \dots, N$ . Let  $m > 0$ ,  $R_M > 0$ ,  $M_1 > 0$ ,  $\Gamma \subseteq \Gamma_0$  be a non empty open subset of the boundary of  $\Omega$  such that  $(\bar{\Gamma} \cap \bar{\Gamma}_l) \cup (\bar{\Gamma} \cap \bar{\Gamma}_{out}) = \emptyset$  and let  $g \in H^{\frac{3}{2}}(\Gamma_0)$  be non identically zero on  $\Gamma_0$  and such that  $\|g\|_{H^{\frac{3}{2}}(\Gamma_0)} \leq M_1$ . We assume that  $q_k$  satisfies (6.2) with  $q_i = q_i^k$  be such that  $q_i^k \leq R_M$  for  $i = 1, \dots, N$  and  $k = 1, 2$ . Let us denote by  $(u_k, p_k)$  the solution of (6.1) associated to  $q = q^k$  for  $k=1,2$ . We assume that there exists  $x_j \in \Gamma_j^{in}$  such that  $|u_2(x_j)| > m$ , for all  $j = 1, \dots, N$ .*

*Then, there exists  $C(R_M, M_1, N, m) > 0$  such that*

$$\begin{aligned} & \|q^1 - q^2\|_{L^\infty(\Gamma_{out})} \\ & \leq C(R_M, M_1, N, m) \left( \|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right). \end{aligned} \quad (6.19)$$

**Remark 6.12.** *Since  $g$  is not identically zero on  $\Gamma_0$ , Corollary 3.2 ensures that, for all  $j = 1, \dots, N$ , there exists  $x_j \in \Gamma_j^{in}$  such that  $u_2(x_j) \neq 0$ . We notice however that the constant involved in the estimate (6.19) depends on  $u_2$  through the constant  $m$ . Finding a uniform lower bound for a solution  $u$  of system (6.1) remains an open question.*

Let us state two preliminary results which will be very useful in the proof of Theorem 6.11.

**Lemma 6.13.** *Let  $A \in \mathbb{R}$ ,  $\mu \in \mathbb{R}^*$  and  $(\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ . If, for all  $k \in \mathbb{N}^*$ , we have*

$$\beta_k \leq \frac{1}{\mu^{k-1}} (\beta_{k-1})^\alpha A^{1-\alpha}, \quad (6.20)$$

*then*

$$\beta_k \leq \frac{1}{\mu^\iota} \beta_0^{\alpha^k} A^{1-\alpha^k},$$

*where  $\iota = \sum_{j=1}^{k-1} j \alpha^{k-1-j}$ .*

This lemma is proved in [20]. For the sake of completeness, we write it.

*Proof of Lemma 6.13.* We rewrite inequality (6.20) under the form:

$$\frac{\beta_k}{A} \leq \frac{1}{\mu^{k-1}} \left( \frac{\beta_{k-1}}{A} \right)^\alpha.$$

By iterating the above inequality, we get:

$$\frac{\beta_k}{A} \leq \frac{1}{\mu^\iota} \left( \frac{\beta_0}{A} \right)^{\alpha^k},$$

where  $\iota = \sum_{j=1}^{k-1} j \alpha^{k-1-j}$ . □

**Lemma 6.14.** *Let  $A, B, C_1$  and  $D$  be positive numbers and  $0 < \alpha < 1$ . Assume that*

$$D \leq C_1 A^\alpha B^{1-\alpha}.$$

*Then, for all  $\epsilon > 0$*

$$D \leq \frac{c}{\epsilon} A + \epsilon^s B,$$

*where  $c = C_1^{\frac{1}{\alpha}}$  and  $s = \frac{\alpha}{1-\alpha}$ .*

*Proof of Lemma 6.14.* Let  $\epsilon > 0$ . We rewrite  $C_1 A^\alpha B^{1-\alpha} = \left(\frac{C_1^{1/\alpha}}{\epsilon} A\right)^\alpha \epsilon^\alpha B^{1-\alpha}$ . Then it is sufficient to apply Young inequality:

$$D \leq \alpha \frac{C_1^{1/\alpha}}{\epsilon} A + (1-\alpha) \epsilon^\alpha B.$$

Since  $0 < \alpha < 1$ , the desired inequality follows.  $\square$

We are now able to prove Theorem 6.11.

*Proof of Theorem 6.11.* We follow the approach developed in [74] in the case of the Laplace equation.

We consider:

$$(w, \pi) = \left( \frac{u_1 - u_2}{\sum_{j=1}^N |q_j^1 - q_j^2|}, \frac{p_1 - p_2}{\sum_{j=1}^N |q_j^1 - q_j^2|} \right). \quad (6.21)$$

According to Proposition 6.2,  $(w, \pi)$  belongs to  $V_{\Gamma_l} \times L^2(\Omega)$ . Since for  $k = 1, 2$ ,  $q^k$  is piecewise constant,  $(w, \pi)$  is solution of:

$$\left\{ \begin{array}{ll} -\Delta w + \nabla \pi & = 0, & \text{in } \Omega, \\ \operatorname{div} w & = 0, & \text{in } \Omega, \\ w & = 0, & \text{on } \Gamma_l, \\ \frac{\partial w}{\partial n} - \pi n & = 0, & \text{on } \Gamma_0, \\ \frac{\partial w}{\partial n} - \pi n + q^1 w & = \frac{(q^2 - q^1)}{\sum_{j=1}^N |q_j^1 - q_j^2|} u_2, & \text{on } \Gamma_{out}. \end{array} \right. \quad (6.22)$$

Observe that if we do not assume that  $q^k$  is piecewise constant, we get additional terms which depend on the derivative of  $q_k$  in system (6.22).

Let  $1 \leq i \leq N$  and  $x_i \in \Gamma_i^{in}$  be such that  $|u_2(x_i)| > m$ . Let us give an idea of how information goes from  $\Gamma$  to a neighborhood  $\mathcal{V}_i$  of  $x_i$ . Thanks to Proposition 6.7, information goes from  $\Gamma$  to a relatively compact open set  $B_0$  included in  $\Omega$ . Then, we use a sequence of balls  $(B_k)_{k \in \mathbb{N}}$  with decreasing radius to approach the boundary near  $x_i$ , taking into account Lemma 6.10. Finally, we use the boundary condition on  $\Gamma_i$  and the local Hölder regularity of the solution on  $\mathcal{V}_i$ . We refer to Figure 6.1 for an illustration.

**Step 1 :** Since the open set  $\Omega$  satisfies the cone property, there exists  $\xi_i \in \mathbb{R}^d$ ,  $|\xi_i| = 1$  and  $R_0 > 0$  such that the cone  $\mathcal{C} = \{x \in \mathbb{R}^d / (x-x_i) \cdot \xi_i > |x-x_i| \cos \theta \text{ and } |x-x_i| \leq R_0\}$  is included in  $\Omega$ .

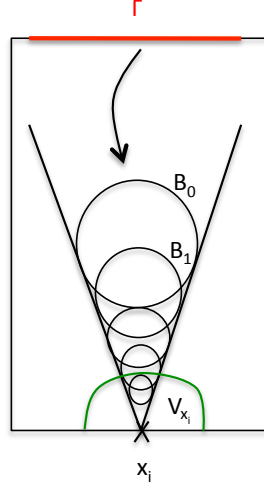


Figure 6.1: Figure illustrating how informations spread in the proof of Theorem 6.11.

We are going to construct a sequence of balls  $(B_{\rho_k}(\zeta_{i,k}))_{k \in \mathbb{N}}$  with decreasing radius and whose center is converging through  $x_i$ .

For this sequence of balls, we will prove that there exists  $0 < \alpha < 1$ ,  $0 < \mu < 1$  and  $C > 0$  such that for all  $k \in \mathbb{N}$  and for all  $\epsilon > 0$ , the following estimate is satisfied:

$$\begin{aligned} & \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ & \leq \frac{e^{\frac{c}{\alpha^k} \log\left(\frac{1}{\mu^{k-1}}\right)}}{\epsilon} \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + C\epsilon^{\alpha^k} \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right), \end{aligned} \quad (6.23)$$

where  $\tilde{\Omega}_i \subset \Omega$  is an open set such that  $\Gamma \subset \partial\tilde{\Omega}_i$  and  $\mathcal{C} \subset \tilde{\Omega}_i$  and such that  $(w, \pi) \in H^3(\tilde{\Omega}_i) \times H^2(\tilde{\Omega}_i)$ .

We consider

$$\mathcal{C}' = \{x \in \mathbb{R}^d / (x - x_i) \cdot \xi_i > |x - x_i| \cos \theta' \text{ et } |x - x_i| \leq R_0\},$$

with

$$\theta' = \arcsin(t \sin \theta). \quad (6.24)$$

The parameter  $t$  belongs to  $(0, 1)$  and will be specified later on. Note that we have  $\mathcal{C}' \subset \mathcal{C} \subset \Omega$ . We denote by  $\zeta_{i,0} = x_i + \frac{R_0}{2}\xi_i$ ,  $d_0 = |\zeta_{i,0} - x_i| = \frac{R_0}{2}$  and  $\rho_0 = d_0 \sin \theta'$ . Let  $k \in \mathbb{N}^*$ . We define the sequence of balls by induction:

$$\zeta_{i,k+1} = x_i + \mu|\zeta_{i,k} - x_i|\xi_i, \quad d_{k+1} = |\zeta_{i,k+1} - x_i|, \quad \rho_{k+1} = d_{k+1} \sin(\theta'),$$

with

$$\mu = \frac{1 - \sin \theta'}{1 + \sin \theta'} \iff \sin \theta' = \frac{1 - \mu}{1 + \mu}. \quad (6.25)$$

We refer to Figure 6.2 for an illustration of this construction. This construction implies that  $d_{k+1} = \mu d_k$  and  $\rho_{k+1} = \mu \rho_k$ . We choose the parameter  $t$  involved in (6.24) such that  $B_{8\rho_k}(\zeta_{i,k}) \subset \mathcal{C} \subset \Omega$  for all  $k \in \mathbb{N}$ , that is to say, using (6.24),  $8\rho_k = 8d_k \sin \theta' \leq \sin \theta d_k \iff t \leq \frac{1}{8}$ .

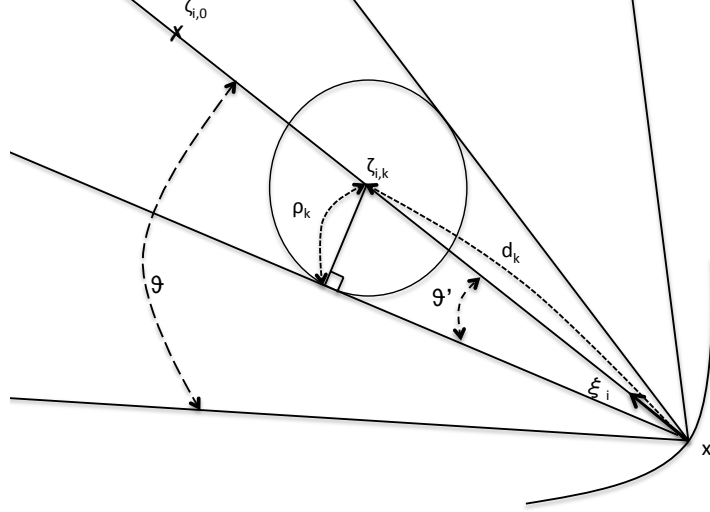


Figure 6.2: Figure illustrating the construction of the sequence of balls with decreasing radius and whose center is approaching  $x_i$ .

Next, we apply Lemma 6.8 with  $\rho = \rho_0$  and  $q = \zeta_0$ : there exists  $C > 0$  and  $\alpha > 0$  such that

$$\begin{aligned} & \|w\|_{H^1(B_{3\rho_0}(\zeta_0))} + \|\pi\|_{L^2(B_{3\rho_0}(\zeta_0))} \\ & \leq C \left( \|w\|_{H^1(B_{\rho_0}(\zeta_0))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_0))} \right)^\alpha \left( \|w\|_{H^1(B_{8\rho_0}(\zeta_0))} + \|\pi\|_{L^2(B_{8\rho_0}(\zeta_0))} \right)^{1-\alpha}. \end{aligned}$$

Taking into account Lemma 6.10, since  $\rho_{k-1} = \mu^{k-1}\rho_0$ , it follows that, for all  $k \in \mathbb{N}^*$ :

$$\begin{aligned} & \|w\|_{H^1(B_{3\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{3\rho_{k-1}}(\zeta_{i,k-1}))} \\ & \leq \frac{C}{\mu^{k-1}} \left( \|w\|_{H^1(B_{\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{\rho_{k-1}}(\zeta_{i,k-1}))} \right)^\alpha \left( \|w\|_{H^1(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} \right)^{1-\alpha}. \end{aligned} \quad (6.26)$$

By construction, we have:

$$B_{\rho_k}(\zeta_{i,k}) \subseteq B_{3\rho_{k-1}}(\zeta_{i,k-1}). \quad (6.27)$$

Indeed, we have  $\zeta_{i,k} - \zeta_{i,k-1} = \mu(d_k - d_{k-1})\xi_i = \frac{\mu}{\sin\theta'}(\rho_k - \rho_{k-1})\xi_i = \frac{\mu}{\sin\theta'}(1 - \mu)\rho_{k-1}\xi_i$  and using (6.24), we obtain  $\zeta_{i,k} - \zeta_{i,k-1} = -\mu(1 + \mu)\rho_{k-1}\xi_i$ . Then, if  $x \in B_{\rho_k}(\zeta_{i,k})$ , we have:

$$|x - \zeta_{i,k-1}| \leq |x - \zeta_{i,k}| + |\zeta_{i,k-1} - \zeta_{i,k}| \leq \rho_k + \mu(1 + \mu)\rho_{k-1} \leq 3\rho_{k-1}.$$

By combining (6.26) and (6.27) we deduce that there exists  $C > 0$  and  $0 < \alpha < 1$  such that for all  $k \in \mathbb{N}^*$ , we have:

$$\begin{aligned} & \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ & \leq \frac{C}{\mu^{k-1}} \left( \|w\|_{H^1(B_{\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{\rho_{k-1}}(\zeta_{i,k-1}))} \right)^\alpha \left( \|w\|_{H^1(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} \right)^{1-\alpha}. \end{aligned} \quad (6.28)$$

Let  $\tilde{\Omega}_i \subset \Omega$  be an open set such that  $\Gamma \subset \partial\tilde{\Omega}_i$ ,  $C \subset \tilde{\Omega}_i$  and such that  $(w, \pi) \in H^3(\tilde{\Omega}_i) \times H^2(\tilde{\Omega}_i)$ . Note that such an open set exists thanks to local regularity results stated in



Subsection 6.2.2. We have, for all  $k \in \mathbb{N}^*$ :

$$\|w\|_{H^1(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} \leq \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}.$$

Thus, we can rewrite inequality (6.28) as:

$$\begin{aligned} & \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ & \leq \frac{C}{\mu^{k-1}} \left( \|w\|_{H^1(B_{\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{\rho_{k-1}}(\zeta_{i,k-1}))} \right)^\alpha \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right)^{1-\alpha}. \end{aligned}$$

We now apply Lemma 6.13 with  $\beta_k = \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}$  and  $A = C \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right)$ , to obtain:

$$\begin{aligned} & \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ & \leq \frac{1}{\mu^\iota} \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right)^{\alpha^k} \left( C \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right) \right)^{1-\alpha^k}, \end{aligned}$$

where  $\iota = \sum_{j=1}^{k-1} j \alpha^{k-1-j}$ . Note that  $\iota \leq (k-1) \left( \sum_{j=0}^{k-2} \alpha^j \right) \leq \frac{k-1}{1-\alpha}$ . To summarize, the following inequality is satisfied:

$$\begin{aligned} & \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ & \leq \frac{1}{\mu^{\frac{k-1}{1-\alpha}}} \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right)^{\alpha^k} \left( C \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right) \right)^{1-\alpha^k}. \end{aligned}$$

Let  $\epsilon > 0$ . By Lemma 6.14 with  $A = \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}$ ,  $B = C \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right)$ ,  $C_1 = \frac{1}{\mu^{\frac{k-1}{1-\alpha}}}$ , and  $D = \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}$ , we obtain:

$$\begin{aligned} & \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ & \leq \frac{c_k}{\epsilon} \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + C \epsilon^{s_k} \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right), \end{aligned}$$

with  $s_k = \frac{\alpha^k}{1-\alpha^k}$  and  $c_k = \left( \frac{1}{\mu^{\frac{k-1}{1-\alpha}}} \right)^{1/\alpha^k}$ . Note that  $s_k \geq \alpha^k$  implies, for  $0 < \epsilon < 1$  that:

$$\begin{aligned} & \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ & \leq \frac{e^{\frac{c}{\alpha^k} \log\left(\frac{1}{\mu^{k-1}}\right)}}{\epsilon} \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + C \epsilon^{\alpha^k} \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right). \end{aligned}$$

Let us remark that the previous inequality is clearly satisfied for  $\epsilon \geq 1$  by continuity of the injection  $H^1(\tilde{\Omega}_i) \hookrightarrow H^1(B_{\rho_k}(\zeta_{i,k}))$ . Thus, we obtain inequality (6.23).

**Step 2** : Taking into account local Hölder regularity of the solution and inequality (6.23), and then optimizing the resulting inequality, we will prove the desired Lipschitz stability estimate (6.19).

Let us recall that thanks to Corollary 6.5, there exists  $R > 0$  and  $0 < \beta < 1$  such that  $(w, \pi) \in \mathcal{C}^{1,\beta}(\overline{B_R(x_i)} \cap \tilde{\Omega}) \times \mathcal{C}^{0,\beta}(\overline{B_R(x_i)} \cap \tilde{\Omega})$  for all  $i = 1, \dots, N$ .

Let  $0 < \epsilon' < \min(R, 1) = \epsilon'_0$ . If  $d_k + \rho_k < \epsilon'$ , that is to say  $\mu^k(d_0 + \rho_0) < \epsilon'$ , we have  $B_{\rho_k}(\zeta_{i,k}) \subset B_{\epsilon'}(x_i)$ . Let  $k_0 = k_0(\epsilon')$  be the smallest  $k$  which satisfies this inequality. We have:

$$\left| \frac{\log((d_0 + \rho_0)/\epsilon')}{\log(1/\mu)} \right| \leq k_0 < \left\lfloor \frac{\log((d_0 + \rho_0)/\epsilon')}{\log(1/\mu)} \right\rfloor + 1. \quad (6.29)$$

Since  $(w, \pi)$  is solution of system (6.22), we obtain, using the boundary condition on  $\Gamma_i$ :

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} |u_2(x_i)| \leq \left| \frac{\partial w}{\partial n}(x_i) \right| + |\pi(x_i)| + |q_i^1| |w(x_i)| \leq |\nabla w(x_i)| + |\pi(x_i)| + R_M |w(x_i)|. \quad (6.30)$$

Let  $y_i \in B_{\rho_{k_0}}(\zeta_{i,k_0})$ . Using the Hölder regularity of  $w$ ,  $\nabla w$  and  $\pi$  on  $B_{\rho_{k_0}}(\zeta_{i,k_0}) \subset B_{\epsilon'}(x_i) \subset B_R(x_i)$ , we have:

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} |u_2(x_i)| \leq |\nabla w(y_i)| + |\pi(y_i)| + R_M |w(y_i)| + C |x_i - y_i|^\beta.$$

Moreover, since  $|u_2(x_i)| > m$ , we obtain that, for all  $y$  in  $B_{\rho_{k_0}}(\zeta_{i,k_0})$ ,

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} m \leq |\nabla w(y_i)| + |\pi(y_i)| + R_M |w(y_i)| + C \epsilon'^\beta.$$

Let us denote by  $\omega_d$  the volume of the unit ball in  $\mathbb{R}^d$ . By integrating in  $L^2$  norm the previous inequality in  $B_{\rho_{k_0}}(\zeta_{i,k_0})$ , we obtain:

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} \leq \frac{C(m, R_M)}{\omega_d^{\frac{1}{2}} \rho_{k_0}^{\frac{d}{2}}} \left( \|w\|_{H^1(B_{\rho_{k_0}}(\zeta_{i,k_0}))} + \|\pi\|_{L^2(B_{\rho_{k_0}}(\zeta_{i,k_0}))} \right) + C \epsilon'^\beta.$$

The previous inequality together with (6.23) yields:

$$\begin{aligned} \frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} &\leq \frac{C(m, R_M) e^{\frac{c}{\alpha^{k_0}} \log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\rho_{k_0}^{\frac{d}{2}} \epsilon} \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) \\ &\quad + \frac{C(m, R_M)}{\rho_{k_0}^{\frac{d}{2}}} \epsilon^{\alpha^{k_0}} \left( \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right) + C \epsilon'^\beta. \end{aligned}$$

By summing up for  $i = 1, \dots, N$ , we obtain:

$$\begin{aligned} 1 &\leq \frac{C(m, R_M, N) e^{\frac{c}{\alpha^{k_0}} \log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\rho_{k_0}^{\frac{d}{2}} \epsilon} \sum_{i=1}^N \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) \\ &\quad + \frac{C(m, R_M, N)}{\rho_{k_0}^{\frac{d}{2}}} \epsilon^{\alpha^{k_0}} \left( \|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} \right) + C(N) \epsilon'^\beta, \end{aligned}$$

where  $\tilde{\Omega} = \bigcup_{i=1}^N \tilde{\Omega}_i$ . Moreover,  $\mu^{k_0-1}(d_0 + \rho_0) \geq \epsilon'$  and  $\rho_{k_0} = \mu^{k_0} \rho_0$  imply that  $\rho_{k_0} \geq \mu^{\frac{\rho_0}{d_0 + \rho_0}} \epsilon'$ . It follows:

$$\begin{aligned} 1 &\leq \frac{C(m, R_M, N) e^{\frac{c}{\alpha^{k_0}} \log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\epsilon'^{\frac{d}{2}} \epsilon} \sum_{i=1}^N \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) \\ &\quad + \left( \|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} + C(N) \right) \left( \frac{\epsilon^{\alpha^{k_0}}}{\epsilon'^{\frac{d}{2}}} + \epsilon'^\beta \right). \quad (6.31) \end{aligned}$$

Let us denote by

$$E = \left( \|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} + C(N) \right). \quad (6.32)$$

We simplify the last term in the right hand-side in (6.31) by choosing  $\epsilon > 0$  such that  $\frac{\epsilon^{\alpha k_0}}{\epsilon'^{\frac{d}{2}}} = \epsilon'^{\beta}$ . Since  $\alpha^{k_0} < 1$ , we obtain, for all  $\epsilon'$  small enough

$$1 \leq C(m, R_M, N) \frac{e^{\frac{c}{\alpha^{k_0}} \log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\epsilon'^{(\beta+d)/\alpha^{k_0}}} \sum_{i=1}^N \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + 2E\epsilon'^{\beta}. \quad (6.33)$$

Since  $\frac{1}{\alpha^{k_0}} = e^{k_0 \log(1/\alpha)}$ , we obtain, using (6.29):

$$\frac{1}{\alpha^{k_0}} < e^{\log(1/\alpha) \left( \frac{\log((d_0+\rho_0)/\epsilon')}{\log(1/\mu)} + 1 \right)} = \frac{1}{\alpha} \left( \frac{d_0 + \rho_0}{\epsilon'} \right)^{\gamma_0},$$

where  $\gamma_0 = \frac{\log(1/\alpha)}{\log(1/\mu)}$ . Furthermore, since  $\frac{1}{\mu^{k_0-1}} \leq \frac{d_0+\rho_0}{\epsilon'}$  by definition of  $k_0$ , we have

$$\log\left(\frac{1}{\mu^{k_0-1}}\right) < \log\left(\frac{d_0 + \rho_0}{\epsilon'}\right).$$

Then,

$$\begin{aligned} \frac{e^{\frac{c}{\alpha^{k_0}} \log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\epsilon'^{(\beta+d)/\alpha^{k_0}}} &= e^{\frac{1}{\alpha^{k_0}} \left( c \log\left(\frac{1}{\mu^{k_0-1}}\right) + (\beta+d) \log\left(\frac{1}{\epsilon'}\right) \right)} \\ &\leq e^{\frac{1}{\alpha} \left( \frac{d_0+\rho_0}{\epsilon'} \right)^{\gamma_0} \left( c \log\left(\frac{d_0+\rho_0}{\epsilon'}\right) + (\beta+d) \log\left(\frac{1}{\epsilon'}\right) \right)} \leq e^{\frac{c}{\epsilon'^{\gamma_0}} \log\left(\frac{1}{\epsilon'}\right)}. \end{aligned} \quad (6.34)$$

To summarize, for  $\gamma > \gamma_0$  and for  $0 < \epsilon' < \epsilon'_0$ , we obtain from (6.33) and (6.34):

$$1 \leq C(m, R_M, N) \left( e^{c/\epsilon'^{\gamma}} \sum_{i=1}^N \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + E\epsilon'^{\beta} \right).$$

By denoting  $\frac{1}{s} = \epsilon'^{\gamma}$ , it can be rewritten for all  $s > s_0 = \frac{1}{\epsilon'_0}$  as:

$$1 \leq C(m, R_M, N) \left( e^{cs} \left( \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + E \left( \frac{1}{s} \right)^{\beta/\gamma} \right).$$

By applying Proposition 6.7, there exist  $C(M_1, m, R_M, N) > 0$  and  $0 < \delta < 1$  such that for all  $s > s_0$ :

$$1 \leq C(M_1, m, R_M, N) \left( e^{cs} \left( \|w\|_{L^2(\Gamma)} + \|\pi\|_{L^2(\Gamma)} + \left\| \frac{\partial w}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial \pi}{\partial n} \right\|_{L^2(\Gamma)} \right)^{\delta} + E \left( \frac{1}{s} \right)^{\beta/\gamma} \right). \quad (6.35)$$

Note that the previous inequality remains true for  $0 < s < s_0$ , eventually by increasing the constant  $C(N)$  involved in (6.32).

We now look for the lower bound of this inequality with respect to  $s$ . We denote by

$$\Lambda = \left( \|w\|_{L^2(\Gamma)} + \|\pi\|_{L^2(\Gamma)} + \left\| \frac{\partial w}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial \pi}{\partial n} \right\|_{L^2(\Gamma)} \right)^{\delta}. \quad (6.36)$$

and  $f(s) = e^{cs} \Lambda + \tilde{d}E \left(\frac{1}{s}\right)^{\beta/\gamma}$ , with  $\tilde{d} \geq 1$ . Let us study the function  $f$  in  $\mathbb{R}_+^*$ . We have:

$$\begin{cases} \lim_{s \rightarrow 0} f(s) = +\infty, \\ \lim_{s \rightarrow \infty} f(s) = +\infty. \end{cases}$$

So since  $f$  is continuous on  $\mathbb{R}_+^*$ ,  $f$  reaches its minimum at a point  $s_0 > 0$ . At this point,

$$f'(s_0) = 0 \Leftrightarrow \Lambda = \frac{E\tilde{d}\beta}{c\gamma} \frac{e^{-cs_0}}{s_0^{\frac{\beta}{\gamma}+1}}, \text{ thus } f(s_0) = \frac{\beta}{c\gamma} \frac{E\tilde{d}}{s_0^{\frac{\beta}{\gamma}+1}} + \frac{E\tilde{d}}{s_0^{\frac{\beta}{\gamma}}}.$$

Hence, (6.35) leads to:

$$1 \leq \frac{E\tilde{d}C(M_1, m, R_M, N)}{s_0^\lambda} \left( \frac{\beta}{c\gamma} + 1 \right), \quad (6.37)$$

where  $\lambda = \frac{\beta}{\gamma}$  if  $s_0 \geq 1$  and  $\lambda = 1 + \frac{\beta}{\gamma}$  otherwise. But:

$$\frac{1}{\Lambda} = \frac{c\gamma}{E\tilde{d}\beta} s_0^{\frac{\beta}{\gamma}+1} e^{cs_0} \leq \frac{c\gamma}{E\tilde{d}\beta} e^{(\frac{\beta}{\gamma}+1+c)s_0},$$

which can be written as follows:

$$\frac{1}{s_0} \leq \frac{\frac{\beta}{\gamma} + 1 + c}{\ln \left( \frac{E\tilde{d}\beta}{c\gamma\Lambda} \right)},$$

for all  $\tilde{d}$  large enough. Taking into account (6.37) leads to:

$$1 \leq \frac{E\tilde{d}C(M_1, m, R_M, N)}{\ln \left( \frac{E\tilde{d}\beta}{c\gamma\Lambda} \right)^\lambda}.$$

Thanks to local regularity stated in Subsection 6.2.2, we know that there exists a constant  $C(M_1, R_M) > 0$  such that:

$$\|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} \leq C(M_1, R_M),$$

which, remembering the definition (6.32), leads to  $E \leq C(M_1, R_M, N)$ . Thus, by studying the variation of the function  $f_y(x) = \frac{x}{(\ln(\frac{x}{y}))^\lambda}$  on  $(y, +\infty)$ , for  $y = \frac{c\gamma\Lambda}{\tilde{d}\beta}$  (see the end of the proof of Theorem 4.16 for more details about it), we obtain the existence of two positive constants  $C(M_1, m, R_M, N)$  and  $C_1(M_1, R_M, N)$  such that

$$1 \leq \frac{C(M_1, m, R_M, N)}{\left( \ln \left( \frac{C_1(M_1, R_M, N)}{\Lambda} \right) \right)^\lambda}.$$

Remembering the definition (6.36) of  $\Lambda$ , the previous inequality is equivalent to

$$\|w\|_{L^2(\Gamma)} + \|\pi\|_{L^2(\Gamma)} + \left\| \frac{\partial w}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial \pi}{\partial n} \right\|_{L^2(\Gamma)} \geq C(M_1, m, R_M, N).$$

By replacing  $(w, \pi)$  by (6.21), we obtain:

$$\begin{aligned} & \sum_{j=1}^N |q_j^1 - q_j^2| \\ & \leq C(m, R_M, M_1, N) \left( \|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right), \end{aligned}$$

which concludes the proof of Theorem 6.11, since  $\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = (p_1 - p_2)n$  on  $\Gamma$ .  $\square$

## 6.5 Conclusion

We have proved a Lipschitz stability estimate for the inverse problem of identifying a Robin coefficient on some non accessible part of the boundary from measurements available on another part of the boundary, for  $(u, p)$  solution of the stationary Stokes system with mixed Dirichlet, Neumann and Robin boundary conditions and under the *a priori* assumption that the Robin coefficient is piecewise constant. It seems difficult to extend this stability estimate to the case of the non stationary problem as we did in Chapter 5 because, due to the mixed boundary conditions, we do not have global regularity on the solution of system (6.1).

One may wonder if the stability estimate presented in this chapter still stands when we replace the Robin boundary condition on  $\Gamma_{out}$ ,  $\frac{\partial u}{\partial n} - pn + qu = 0$  by the boundary condition used by A. Quarteroni and A. Veneziani in [69], namely  $\frac{\partial u}{\partial n} - pn + R(u \cdot n)n = 0$ . In fact, we are able to perform the same reasoning as in the proof of Theorem 6.11 above and obtain a Lipschitz stability estimate provided that there exists  $i \in \{1, \dots, N\}$  and  $x_i \in \Gamma_i^{in}$  such that  $u_1(x_i) \cdot n(x_i) \neq 0$ . Note that the unique continuation theorem of C. Fabre and G. Lebeau (see Theorem 3.1) does not allow us to obtain the existence of such an  $x_i$ . The case of dissipative boundary condition will be discussed in Chapter 7.

E. Sincich solves a similar problem for the Laplace equation in [74]. She proved in addition that the constant involved in the Lipschitz stability estimate behaves exponentially with respect to the number  $N$  of portions considered. It would be interesting to see if we can generalize this result to our problem. This will be done in future research.





## Chapter 7

# Conclusions and perspectives





In this Chapter, we come back to the resolution of the initial problem introduced in Chapter 1. We investigate, both theoretically and numerically, the inverse problem of identifying resistances defined on some non accessible part of the boundary from measurements available on another part of the boundary. This chapter is divided into two sections: the first one is dedicated to the theoretical resolution of the inverse problem for the Stokes system with dissipative boundary conditions and the second one to the numerical resolution of the initial inverse problem. This last chapter leads to new perspectives of work.

## 7.1 Inverse problem for the Stokes system with dissipative boundary conditions

### 7.1.1 Introduction

Let  $\Omega \subset \mathbb{R}^d$ , with  $d = 2, 3$ , be a Lipschitz bounded connected open set such that  $\partial\Omega = \Gamma_l \cup \Gamma_0 \cup \left(\bigcup_{i=1}^N \Gamma_i\right)$ .

We assume that  $\Gamma_i$  is contained in an hyperplane of  $\mathbb{R}^d$  and meets  $\Gamma_l$  at angle  $\frac{\pi}{2}$  and that  $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset$  for  $i \neq j$ ,  $(i, j) \in \{0, \dots, N\}$ . We refer to Figure 7.1 for an example of such an open set in dimension 2.

Let us denote by  $\Gamma_{out} = \bigcup_{i=1}^N \Gamma_i$  and  $\mathbf{n}$  the exterior to  $\Omega$ . We consider the following Stokes system:

$$\left\{ \begin{array}{ll} u_t - \Delta u + \nabla p & = 0, \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} u & = 0, \quad \text{in } (0, T) \times \Omega, \\ u & = 0, \quad \text{on } (0, T) \times \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = -P_0 n, \quad \text{on } (0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + R_i \left( \int_{\Gamma_i} u \cdot n \right) n & = 0, \quad \text{on } (0, T) \times \Gamma_i, \text{ for } i = 1, \dots, N, \\ u(0) & = u_0 \quad \text{in } \Omega, \end{array} \right. \quad (7.1)$$

with  $R = (R_i)_{1 \leq i \leq N} \in (\mathbb{R}_+)^N$ .

System (7.1) is similar to system (5.1) except that we have replaced the Robin boundary conditions on  $\Gamma_{out}$ ,  $\frac{\partial u}{\partial n} - pn + qu = 0$ , by **dissipative boundary conditions**,  $\frac{\partial u}{\partial n} - pn + R_i \left( \int_{\Gamma_i} u \cdot n \right) = 0$ , on  $\Gamma_i$  for  $i = 1, \dots, N$ . The other difference lies on the fact that homogeneous Dirichlet boundary conditions on one part of the boundary have been added.

The aim of this chapter is to **discuss the uniqueness and stability properties** for the identification problem of the resistances  $R = (R_i)_{1 \leq i \leq N}$  from measurements available on  $\Gamma \subseteq \Gamma_0$  for  $(u, p)$  solution of system (7.1). New difficulties occur with dissipative boundary conditions.

Let us explain the structure of this section. In Subsection 7.1.2, we present the main differences which arise from imposing the dissipative boundary conditions and propose new perspectives of work. Finally, in Subsection 7.1.3, we prove a **Lipschitz stability estimate** in a particular case, when  $N = 1$ .

We recall that  $\mathbf{V}_{\Gamma_l}$  and  $\mathbf{H}_{\Gamma_l}$  are defined in Chapter 2 by (2.3) and (2.4).

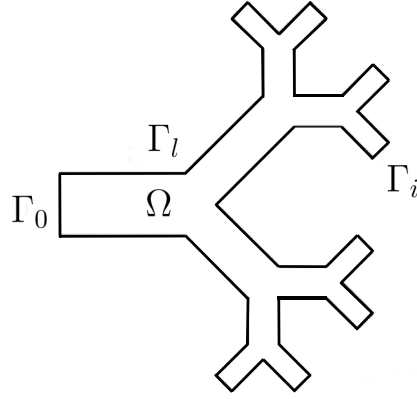


Figure 7.1: Example of open set  $\Omega \subset \mathbb{R}^2$  in consideration in Section 7.1.

### 7.1.2 Discussion

#### Identifiability result

Let us discuss the question of uniqueness. The idea is to apply the same reasoning as in the proof of Proposition 5.8.

Let  $x_0 \in \Gamma_0$ ,  $r > 0$  and  $(u_k, p_k)$  be solution of system (7.1) associated to  $R^k = (R_i^k)_{1 \leq i \leq N}$  for  $k = 1, 2$ . We assume that  $u_1 = u_2$  on  $\mathcal{B}(x_0, r) \cap \Gamma_0$ .

Let us remark that, under some regularity assumption on the data, one can prove that  $(u_k, p_k) \in (L^2(0, T; H^1(\Omega))) \cap H^1(0, T; H_{\Gamma_l}) \times L^2(0, T; L^2(\Omega))$  for  $k = 1, 2$ . The proof of this result is contained in [7] where the same result is proved for the Navier–Stokes system. Thus, we have enough regularity on  $(u_k, p_k)$  to apply Corollary 5.6.

We denote by  $v = u_1 - u_2$  and  $\pi = p_1 - p_2$ . Let us notice that  $(v, \pi)$  is the solution of the following problem:

$$\left\{ \begin{array}{ll} v_t - \Delta v + \nabla \pi & = 0, \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} v & = 0, \quad \text{in } (0, T) \times \Omega, \\ v & = 0, \quad \text{on } (0, T) \times \Gamma_l, \\ \frac{\partial v}{\partial n} - \pi n & = 0, \quad \text{on } (0, T) \times \Gamma_0, \\ \frac{\partial v}{\partial n} - \pi n + R_i^1 \left( \int_{\Gamma_i} u_1 \cdot n \right) n - R_i^2 \left( \int_{\Gamma_i} u_2 \cdot n \right) n & = 0, \quad \text{on } (0, T) \times \Gamma_i, \text{ for } i = 1, \dots, N, \\ v(0) & = 0 \quad \text{in } \Omega. \end{array} \right.$$

By assumption,  $v = 0$  and  $\frac{\partial v}{\partial n} - \pi n = 0$  on  $(0, T) \times \mathcal{B}(x_0, r) \cap \Gamma_0$ . According to Corollary 5.6, it implies that  $u_1 = u_2$  and  $p_1 = p_2$  in  $\Omega$ . Consequently, we deduce from

$$\begin{aligned} \frac{\partial u_1}{\partial n} - p_1 n + R_i^1 \left( \int_{\Gamma_i} u_1 \cdot n \right) n &= 0, \quad \text{on } (0, T) \times \Gamma_i, \\ \frac{\partial u_1}{\partial n} - p_1 n + R_i^2 \left( \int_{\Gamma_i} u_1 \cdot n \right) n &= 0, \quad \text{on } (0, T) \times \Gamma_i, \end{aligned}$$

that

$$(R_i^1 - R_i^2) \left( \int_{\Gamma_i} u_1 \cdot n \right) = 0, \quad \text{in } (0, T), \quad (7.2)$$

for  $i = 1, \dots, N$ . Then, we can not perform the same reasoning by contradiction as in the proof of Proposition 5.8 because we can not deduce from  $\int_{\Gamma_i} u_1 \cdot n = 0$  that  $u_1 = 0$  and  $p_1 = 0$  in  $(0, T) \times \Omega$ . To conclude that  $R_i^1 = R_i^2$  for  $i \in \{1, \dots, N\}$ , we need to know the existence of  $t_0 \in (0, T)$  such that  $\int_{\Gamma_i} u_1(t_0) \cdot n \neq 0$ . This assumption has to be added in the hypotheses satisfied by our solution unless we are able to prove that, in some specific case, this always holds.

### Stability estimates

Let us now investigate if we can extend the stability estimates given in Chapter 4. We start with the stationary problem:

$$\left\{ \begin{array}{ll} -\Delta u + \nabla p & = 0, & \text{in } \Omega, \\ \operatorname{div} u & = 0, & \text{in } \Omega, \\ u & = 0, & \text{on } \Gamma_l, \\ \frac{\partial u}{\partial n} - pn & = -P_0 n, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} - pn + R_i \left( \int_{\Gamma_i} u \cdot n \right) n & = 0, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \\ u(0) & = u_0 & \text{in } \Omega, \end{array} \right. \quad (7.3)$$

Let  $(u_k, p_k)$  be solution of the stationary system associated to (7.3) with  $R_i = R_i^k$  for  $i = 1, \dots, N$  and  $k = 1, 2$ . We denote by  $(v, \pi) = (u_1 - u_2, p_1 - p_2)$ .

Thanks to the boundary conditions satisfied on  $\Gamma_i$ , we have:

$$(R_i^2 - R_i^1) \left( \int_{\Gamma_i} u_1 \cdot n \right) = R_i^2 \left( \int_{\Gamma_i} v \cdot n \right) + \frac{\partial v}{\partial n} - \pi n, \text{ for } i = 1, \dots, N. \quad (7.4)$$

Let  $\mathcal{K} \subseteq \Gamma_i$  be a non empty set. By integrating (7.4) in  $L^2$  norm on  $\mathcal{K}$ , we obtain:

$$|R_i^1 - R_i^2| \left| \int_{\Gamma_i} u_1 \cdot n \right| \leq |R_i^2| \left| \int_{\Gamma_i} v \cdot n \right| + \frac{1}{|\mathcal{K}|} \left\| \frac{\partial v}{\partial n} \right\|_{L^2(\mathcal{K})} + \frac{1}{|\mathcal{K}|} \|\pi\|_{L^2(\mathcal{K})}, \quad (7.5)$$

for  $i = 1, \dots, N$ , where  $|\mathcal{K}|$  denotes the measure of  $\mathcal{K}$  in  $\mathbb{R}^{d-1}$ .

The term in front of the difference between the resistances is  $\left| \int_{\Gamma_i} u_1 \cdot n \right|$ : this term has to be different from zero. As for the identifiability, we need an additional assumption on the non nullity of the flux on  $\Gamma_i$ .

Moreover,  $\left| \int_{\Gamma_i} v \cdot n \right|$  appears in the right hand-side of (7.5). Consequently, an estimate of  $v$  over all  $\Gamma_i$  is needed.

It is not possible to extend directly the stability estimates obtained in Chapters 4 and 6 for two reasons:

- due to the mixed Dirichlet and Neumann boundary conditions, we do not have local (and thus global) regularity on  $(u, p)$  solution of the stationary system associated to (7.1) near the corners,
- the open set  $\Omega$  is not enough regular.

We will thus consider modified dissipative boundary conditions for which the solution is more regular.

Let  $\tau = (\tau_1, \dots, \tau_{d-1})$  be  $d - 1$  vectors such that  $(n, \tau)$  is an orthogonal basis of  $\mathbb{R}^d$ .

Let us introduce the following Stokes system:

$$\left\{ \begin{array}{ll} -\Delta u + \nabla p & = 0, \quad \text{in } \Omega, \\ \operatorname{div} u & = 0, \quad \text{in } \Omega, \\ u & = 0, \quad \text{on } \Gamma_l, \\ \left( \frac{\partial u}{\partial n} - pn \right) \cdot n & = -P_0, \quad \text{on } \Gamma_0, \\ \left( \frac{\partial u}{\partial n} - pn + R_i \left( \int_{\Gamma_i} u \cdot n \right) n \right) \cdot n & = 0, \quad \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \\ u \cdot \tau_k & = 0, \quad \text{on } \Gamma_i, \text{ for } i = 0, \dots, N \text{ and } k = 1, \dots, d - 1. \end{array} \right. \quad (7.6)$$

Let us state this regularity result:

**Proposition 7.1.** *Let  $R_M > 0$ ,  $(R_i)_{1 \leq i \leq N} \in (\mathbb{R}_+)^N$  and  $P_0$  be a constant. Then, system (7.6) admits a unique solution  $(u, p) \in H^3(\Omega) \times H^2(\Omega)$ . Moreover, if  $R_i \leq R_M$  for all  $i = 1, \dots, N$ , then*

$$\|u\|_{H^3(\Omega)} + \|p\|_{H^2(\Omega)} \leq C(R_M, P_0). \quad (7.7)$$

*Proof of Proposition 7.1.* In a classical way, one can prove that there exists a unique weak solution  $(u, p) \in V_{\Gamma_l} \times L^2(\Omega)$  of system (7.6) and a constant  $C(P_0) > 0$  such that

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(P_0). \quad (7.8)$$

Thanks to regularity result for the Laplacian with mixed Neumann and Dirichlet boundary conditions in non smooth domain (see [49]), there exists a unique  $\hat{p} \in H^2(\Omega)$  solution of

$$\left\{ \begin{array}{ll} \Delta \hat{p} & = 0, \quad \text{in } \Omega, \\ \frac{\partial \hat{p}}{\partial n} & = 0, \quad \text{on } \Gamma_l, \\ \hat{p} & = P_0, \quad \text{on } \Gamma_0, \\ \hat{p} & = R_i \left( \int_{\Gamma_i} u \cdot n \right), \quad \text{on } \Gamma_i, \text{ for } i = 1, \dots, N, \end{array} \right. \quad (7.9)$$

and a constant  $C(R_M) > 0$  such that

$$\|\hat{p}\|_{H^2(\Omega)} \leq C(R_M) \left| \int_{\Gamma_i} u \cdot n \right| + C(P_0). \quad (7.10)$$

Note, thanks to Lemma 2.4, there exists  $C > 0$  such that  $\left| \int_{\Gamma_i} u \cdot n \right| \leq C \|u\|_{L^2(\Omega)}$ . Combining it with (7.8) and (7.10), we obtain

$$\|\hat{p}\|_{H^2(\Omega)} \leq C(P_0, R_M). \quad (7.11)$$

We denote by  $\bar{p} = p - \hat{p}$ . Note that  $(u, \bar{p})$  is solution of system

$$\left\{ \begin{array}{ll} -\Delta u + \nabla p & = -\nabla \hat{p}, \quad \text{in } \Omega, \\ \operatorname{div} u & = 0, \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \Gamma_l, \\ \left( \frac{\partial u}{\partial n} - pn \right) \cdot n & = 0, \quad \text{on } \Gamma_i, \text{ for } i = 0, \dots, N, \\ u \cdot \tau_k & = 0, \quad \text{on } \Gamma_i, \text{ for } i = 0, \dots, N \text{ and } k = 1, \dots, d - 1. \end{array} \right. \quad (7.12)$$

Then, we conclude by using regularity result for system (7.12) stated in Proposition 2.22 and by using inequality (7.11).  $\square$

Thanks to the previous proposition, we have a regularity result on  $(u, p)$  solution of system (7.6) that could enable us to obtain stability estimate by applying stability estimates for the unique continuation properties of the Stokes system proved in Chapter 3. Nevertheless, the domain is not enough regular, and even if we have local regularity, the proof of such an estimate would require that  $\Gamma_i \subsetneq \Gamma$  with  $\Gamma$  of class  $C^\infty$  because of the need of estimating  $\left| \int_{\Gamma_i} u \cdot n \right|$ . Thus, if we were able to prove that the stability estimates for the unique continuation properties of the Stokes system proved in Chapter 3 hold for Lipschitz domain, then we would obtain a stability estimate for the resistances  $R = (R_i)_{1 \leq i \leq N}$  from measurements available on  $\Gamma \subseteq \Gamma_0$ . To be more precise, we have to extend to Lipschitz domain the stability estimate of Proposition 3.21 which spreads information from the interior of the domain up to the boundary. This is exactly what L. Bourgeois and J. Dardé did in the case of the Laplace equation in [20]. It would be interesting to see if their method can be extended to the case of Stokes equations. Note that the techniques used in [20] are similar to those used in Chapter 6: they use a sequence of balls whose radius is decreasing and whose center is approaching the boundary of the domain.

Let us now come back to the system (7.3) with standard dissipative boundary conditions. Another natural issue concerns the extension of the Lipschitz stability estimate of Chapter 6. Let us investigate this case.

We consider:

$$(w, \pi) = \left( \frac{u_1 - u_2}{\sum_{j=1}^N |R_j^1 - R_j^2|}, \frac{p_1 - p_2}{\sum_{j=1}^N |R_j^1 - R_j^2|} \right). \quad (7.13)$$

Since for  $k = 1, 2$ ,  $R^k$  is piecewise constant on  $\Gamma_{out}$ ,  $(w, \pi)$  is solution of:

$$\left\{ \begin{array}{ll} -\Delta w + \nabla \pi & = 0, & \text{in } \Omega, \\ \operatorname{div} w & = 0, & \text{in } \Omega, \\ w & = 0, & \text{on } \Gamma_l, \\ \frac{\partial w}{\partial n} - \pi n & = 0, & \text{on } \Gamma_0, \\ \frac{\partial w}{\partial n} - \pi n + R_i^2 \left( \int_{\Gamma_i} w \cdot n \right) n & = \frac{(R_i^2 - R_i^1)}{\sum_{j=1}^N |R_j^1 - R_j^2|} \left( \int_{\Gamma_i} u_1 \cdot n \right) n, & \text{on } \Gamma_i, \text{ for } i = 1, \dots, N. \end{array} \right. \quad (7.14)$$

Let  $\Gamma \subset \partial\Omega$  be a non empty part of the boundary. We recall that we denote by

$$\Gamma^{in} = \{x \in \Gamma / d(x, \overline{\partial\Omega \setminus \Gamma}) > 0\}. \quad (7.15)$$

Let  $x_i \in \Gamma_i^{in}$ . Using the boundary condition on  $\Gamma_i$ , we obtain:

$$\frac{|R_i^1 - R_i^2|}{\sum_{j=1}^N |R_j^1 - R_j^2|} \left| \int_{\Gamma_i} u_1 \cdot n \right| \leq \left| \frac{\partial w}{\partial n}(x_i) \right| + |\pi(x_i)| + |R_i^2| \left| \int_{\Gamma_i} w \cdot n \right|.$$

Due to the fact that the integral in the right hand-side is carried out on the whole part  $\Gamma_i$ , it is not possible to use the local Hölder regularity and then to perform directly the same reasoning as in the proof of Theorem 6.11. Nevertheless, minor drafting changes resulting of this non-local term allow us to conclude in the particular case where  $N = 1$ . We refer to Subsection 7.1.3.

Let us conclude this subsection with a new approach concerning the resolution of our inverse problem for non stationary Stokes system which could open new perspectives of research. Let  $(u, p)$  be solution of system (7.3). Thanks to the *change of variables* in pressure introduced below in the proof of Proposition 7.1, the resistances we want to estimate appear now in the equation through  $\hat{p}$ . Indeed,  $(u, \bar{p}) = (u, p - \hat{p})$  is solution to

$$\left\{ \begin{array}{ll} u_t - \Delta u + \nabla \bar{p} &= -\nabla \hat{p}, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u &= 0, & \text{in } (0, T) \times \Omega, \\ u &= 0, & \text{on } (0, T) \times \Gamma_l, \\ \frac{\partial u}{\partial n} - \bar{p}n &= 0, & \text{on } (0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial n} - \bar{p}n &= 0, & \text{on } (0, T) \times \Gamma_i, \text{ for } i = 1, \dots, N, \\ u(0) &= u_0 & \text{in } \Omega, \end{array} \right.$$

where  $\hat{p}$  is the solution of (7.9) (which depends on  $R$ ). Thus, the classical ‘‘Bukhgeim and Klibanov method’’ could allow us to solve the inverse problem we are interested in.

### 7.1.3 Stability estimates in a particular case

In the special case where  $N = 1$ , we are going to prove a Lipschitz stability estimate. Note that one could also obtain a logarithmic stability estimate as we did in the proof of Theorem 4.16 by using moreover the fact that  $\int_{\Gamma_0} u \cdot n = \int_{\Gamma_1} u \cdot n$  for  $u \in V_{\Gamma_l}$  and the fact that  $x \leq \frac{1}{\log(\frac{1}{x})^b}$  for  $0 < b < 1$  and  $x$  small enough.

**Theorem 7.2.** *We assume that  $N = 1$ ,  $\Gamma_0$  is of class  $C^\infty$  and  $\Gamma_1$  is of class  $C^{2,1}$ . Let  $0 < R_m < R_M$ ,  $R^k \in \mathbb{R}$  be such that  $R_m \leq R^k \leq R_M$  for  $k = 1, 2$  and  $P_0$  be a non-zero constant. We denote by  $(u_k, p_k)$  the solution of system (7.6) with  $R_1 = R^k$  for  $k = 1, 2$ .*

*Then, there exists  $C(R_M, R_m, P_0) > 0$  such that*

$$|R^1 - R^2| \leq C(R_M, R_m, P_0) \left( \|u_1 - u_2\|_{L^2(\Gamma_0)} + \|p_1 - p_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma_0)} \right),$$

*if  $\|u_1 - u_2\|_{L^2(\Gamma_0)} + \|p_1 - p_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma_0)}$  is small enough.*

**Remark 7.3.** *Let  $P_0 \in \mathbb{R}^*$ . Note that thanks to the continuity of the solution with respect to the data, there exists  $C(R_m, R_M, P_0) > 0$  such that  $\left| \int_{\Gamma_0} u_{R_1} \cdot n \right| > C(R_m, R_M, P_0)$ , for all  $(u_{R_1}, p_{R_1})$  solution of system (7.6) with  $R_m \leq R_1 \leq R_M$ .*

*Consequently, since  $N = 1$  and using the divergence-free, there exists  $C(R_m, R_M, P_0) > 0$  such that  $\left| \int_{\Gamma_1} u_{R_1} \cdot n \right| > C(R_m, R_M, P_0)$ , for all  $(u_{R_1}, p_{R_1})$  solution of system (7.6) with  $R_m \leq R_1 \leq R_M$ .*

*Proof of Theorem 7.2.* We recall that  $(w, \pi)$  is defined in (7.13). We resume the proof of Theorem 6.11 and we highlight the differences.

The first step, based on a sequence of balls which approach the boundary, does not involve the boundary conditions, thus it is still valid here. Let us now look at the second step. Using the boundary conditions on  $\Gamma_1$ , we obtain for  $x_1 \in \Gamma_1^{in}$ :

$$\left| \int_{\Gamma_1} u_1 \cdot n \right| \leq \left| \frac{\partial w}{\partial n}(x_1) \right| + |\pi(x_1)| + |R^2| \left| \int_{\Gamma_1} w \cdot n \right|. \quad (7.16)$$

Thus, by taking into account Remark 7.3, the main difference between (7.16) and (6.30) is, as we have noted previously, the presence of the non-local term  $\left| \int_{\Gamma_1} w \cdot n \right|$ , that we will keep unchanged in our computations. In doing so, we resume computations from (6.30) to (6.35), line by line, and we obtain that there exists  $0 < \delta < 1$  and  $\theta > 0$  such that for all  $s > 0$ :

$$1 \leq C(P_0, R_m, R_M) \left( e^{cs} \left( \|w\|_{L^2(\Gamma_0)} + \|\pi\|_{L^2(\Gamma_0)} + \left\| \frac{\partial w}{\partial n} \right\|_{L^2(\Gamma_0)} + \left\| \frac{\partial \pi}{\partial n} \right\|_{L^2(\Gamma_0)} \right)^\delta + \left( \frac{1}{s} \right)^\theta \right) + C(R_M) \left| \int_{\Gamma_1} w \cdot n \right|,$$

We now look for the lower bound of this inequality with respect to  $s$ . We denote by

$$\Lambda = \left( \|w\|_{L^2(\Gamma_0)} + \|\pi\|_{L^2(\Gamma_0)} + \left\| \frac{\partial w}{\partial n} \right\|_{L^2(\Gamma_0)} + \left\| \frac{\partial \pi}{\partial n} \right\|_{L^2(\Gamma_0)} \right)^\delta. \quad (7.17)$$

and  $f(s) = e^{cs} \Lambda + \frac{1}{s^\theta} + \left| \int_{\Gamma_1} w \cdot n \right|$ . Let us study the function  $f$  in  $\mathbb{R}_+^*$ . We have:

$$\begin{cases} \lim_{s \rightarrow 0} f(s) = +\infty, \\ \lim_{s \rightarrow \infty} f(s) = +\infty. \end{cases}$$

So since  $f$  is continuous on  $\mathbb{R}_+^*$ ,  $f$  reaches its minimum at a point  $s_0 > 0$ . At this point,

$$f'(s_0) = 0 \Leftrightarrow \Lambda = \frac{\theta e^{-cs_0}}{c s_0^{\theta+1}}, \text{ thus } f(s_0) = \frac{\theta}{c} \frac{1}{s_0^{\theta+1}} + \frac{1}{s_0^\theta} + \left| \int_{\Gamma_1} w \cdot n \right|.$$

Hence:

$$1 \leq \frac{C(P_0, R_m, R_M)}{s_0^\lambda} \left( \frac{\theta}{c} + 1 \right) + \left| \int_{\Gamma_1} w \cdot n \right|, \quad (7.18)$$

where  $\lambda = \theta$  if  $s_0 \geq 1$  and  $\lambda = 1 + \theta$  otherwise. But:

$$\frac{1}{\Lambda} = \frac{c}{\theta} s_0^{\theta+1} e^{cs_0} \leq \frac{c}{\theta} e^{(\theta+1+c)s_0},$$

which can be written as follows:

$$\frac{1}{s_0} \leq \frac{\theta + 1 + c}{\log\left(\frac{\theta}{c\Lambda}\right)},$$

for  $\Lambda$  small enough. Taking into account (7.18) leads to:

$$1 \leq \frac{C(P_0, R_m, R_M)}{\log\left(\frac{\theta}{c\Lambda}\right)^\lambda} + \left| \int_{\Gamma_1} w \cdot n \right|. \quad (7.19)$$

Note that, by using the fact that  $u_k$  is divergence-free and the homogeneous Dirichlet boundary condition on  $\Gamma_l$ , we have

$$\int_{\Gamma_0} w \cdot n = - \int_{\Gamma_1} w \cdot n,$$

for  $k = 1, 2$ . Thus, inequality (7.19) can be rewritten as:

$$1 \leq \frac{C(P_0, R_m, R_M)}{\log\left(\frac{\theta}{c\Lambda}\right)^\lambda} + \Lambda^{\frac{1}{\delta}}.$$



Then, we use the fact that

$$x^{\frac{1}{\delta}} \leq \frac{1}{\log(\frac{1}{x})^\lambda}, \text{ for all } x \text{ small enough.}$$

We obtain:

$$1 \leq \frac{C(P_0, R_m, R_M)}{\log(\frac{\theta}{c\Lambda})^\lambda},$$

for  $\Lambda$  small enough. Remembering the definition (7.17) of  $\Lambda$ , the previous inequality is equivalent to

$$\|w\|_{L^2(\Gamma_0)} + \|\pi\|_{L^2(\Gamma_0)} + \left\| \frac{\partial w}{\partial n} \right\|_{L^2(\Gamma_0)} + \left\| \frac{\partial \pi}{\partial n} \right\|_{L^2(\Gamma_0)} \geq C(P_0, R_m, R_M).$$

By replacing  $(w, \pi)$  by (7.13), we obtain:

$$\begin{aligned} & |R^1 - R^2| \\ & \leq C(R_m, R_M, P_0) \left( \|u_1 - u_2\|_{L^2(\Gamma_0)} + \|p_1 - p_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right\|_{L^2(\Gamma_0)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma_0)} \right), \end{aligned}$$

for  $\|u_1 - u_2\|_{L^2(\Gamma_0)} + \|p_1 - p_2\|_{L^2(\Gamma_0)} + \left\| \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right\|_{L^2(\Gamma_0)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma_0)}$  small enough, which concludes the proof of Theorem 7.2, since  $\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = (p_1 - p_2)n$  on  $\Gamma_0$ .  $\square$

### 7.1.4 Comments

This section is a preliminary work on inverse problem for the Stokes system with dissipative boundary conditions. Note that, since we have enough global regularity on  $(u, p)$  solution of system (7.6), we can extend the stability estimate of Theorem 7.2 to the associated non stationary Stokes system, exactly as we did in Chapter 5. Anyway, this is not very satisfactory from the modeling perspective: on the one hand, we assume that  $P_0$  does not depend on time and on the other hand, we need measurements in infinite time.

## 7.2 Numerical resolution of the inverse problem for the Navier–Stokes system with dissipative boundary conditions

We are interested in this section in recovering numerically some parameters (air-flow resistance  $R$  and stiffness constant of the diaphragm  $k$ ) which appear in the multi-scale model (1.2) of the lungs introduced in Chapter 1, from measurements of the volume and the flow at the mouth. So far, we were interested in recover theoretically the resistances  $R_i$  and now we will explore numerically the identification on the complete model (1.2). The long-term goal is to get reliable and powerful new tools to aid medical diagnosis. This is a preliminary work.

As a first step, we work with synthetic data: we generate numerically a flow volume loop and volume curve: this curves will be our references curves. We want to recover parameters that have allowed to generate our reference curves. To do so, we will

use a genetic algorithm as an optimization algorithm to minimize the difference between the reference curve and the curves generated numerically with random parameters settings. For the simulation of the direct problem, we use the method described in [35] by A. Devys, C. Grandmont, B. Grec, B. Maury and D. Yacoubi. We present this method in Subsection 7.2.1. Then, in Subsection 7.2.2, we focus on the optimization algorithm used to recover the parameters. Finally, we present the identification results in Subsection 7.2.3.

### 7.2.1 Numerical method for the direct problem

Let us explain the numerical method developed in [35].

Firstly, using the facts that  $P_a = \frac{m}{S}\ddot{x} + \frac{k}{S}x - \frac{f_{ext}}{S}$  and  $S\dot{x} = -\int_{\Gamma_0} u \cdot n$ , the coupled system (1.2) can be rewritten as a Navier–Stokes system with non standard boundary conditions. Indeed, by setting  $\bar{p} = p - P_a$  in (1.2), we obtain:

$$\left\{ \begin{array}{ll} \rho u_t + \rho(u \cdot \nabla)u - \mu \Delta u + \nabla \bar{p} = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0, T) \times \Omega, \\ u = 0, & \text{on } (0, T) \times \Gamma_l, \\ \mu \frac{\partial u}{\partial n} - \bar{p}n = -P_0 n - \frac{f_{ext}}{S}n - \frac{m}{S^2} \frac{d}{dt} \left( \int_{\Gamma_0} u \cdot n \right) n, & \\ & - \frac{k}{S^2} \left( \int_0^t \int_{\Gamma_0} u \cdot n - Sx_0 \right) n \quad \text{on } (0, T) \times \Gamma_0, \\ \mu \frac{\partial u}{\partial n} - \bar{p}n = -R_i \left( \int_{\Gamma_i} u \cdot n \right) n, & \text{on } (0, T) \times \Gamma_i. \end{array} \right.$$

This system leads, after time discretization, to a Stokes system with non dissipative boundary conditions that can not be easily implemented in most FEM software, in particular in FreeFem++. Indeed, the unusual boundary conditions coming from the resistances and the mass-spring time discretization modify the standard bilinear form associated to a Stokes problem with mixed Dirichlet-Neumann boundary conditions. The idea is to pre-compute a set of  $N + 1$  solutions with Neumann boundary conditions on each  $\Gamma_i$ , for  $i = 1, \dots, N$  and then to define the solution as a linear combination of these solutions and of a corrector term. This correction term solves also a Stokes problem with standard boundary conditions, the coefficients of the linear combination being calculated so that the solution satisfies the dissipative boundary conditions.

### 7.2.2 Presentation of the optimization algorithm

The optimization algorithm used is a genetic algorithm that was written by Laurent Dumas in Scilab (see [36]). Genetic algorithms are global optimization algorithms which do not need the computation of the gradient of the function cost. The principle is directly inspired by the evolutionary theory of Darwin. The three main principles from this theory that are used are:

- natural selection,
- genetic mutation,
- genetic recombination (chromosomal crossovers).

Let  $N_{par} \in \mathbb{N}^*$  be the number of elements in our population and  $p \in \mathbb{N}^*$  be the number of parameters to estimate. Let us consider a function  $f : \mathbb{R}^p \rightarrow \mathbb{R}$  which we want to minimize. We consider an initial population consisting of  $N_{par}$  elements  $\{X_1, \dots, X_{N_{par}}\}$  randomly selected in a bounded set of  $\mathbb{R}^p$  in which we find reasonable to seek the parameters. We evaluate  $f$  exactly or in an approximated way (using interpolation) at each point  $X_i$ ,

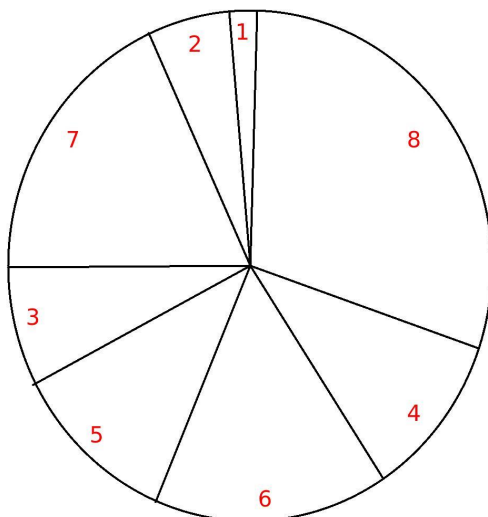


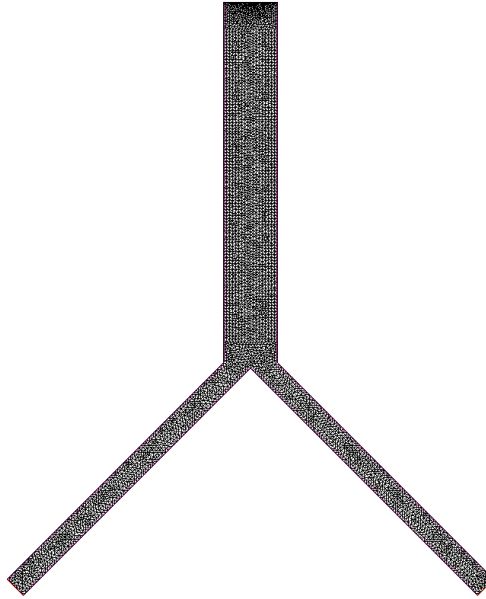
Figure 7.2: Each individual is represented by a portion proportional to its adaptation to the problem.

the number of exact evaluation being one parameter of the algorithm. We assign to each individual  $X_i$ , for  $i = 1, \dots, N_{par}$ , a score between 1 and  $N_{par}$  inversely proportional to the value of the function at this point. In other words, the higher the score is, the smallest  $f$  at this point is. If the score of an individual is large, we will say that the individual is well suited to the problem. We carry a selection within this population: for each individual, the probability of being selected is proportional to its adaptation to the problem. One can visualize the principle as a roulette on which each individual is represented by a portion proportional to its adaptation. We refer to Figure 7.2 for an example of such a roulette with a population consisting of 8 elements. For each selected individual, principles of crossover and mutation are applied successively. The principle of mutation is to replace a member of the population by a new one randomly chosen in a neighborhood of that member. By this way, we obtain a new population corresponding to the next generation and we start again with this new population. After a certain number of iterations, the algorithm stops and gives the setting that minimizes  $f$ .

Let us say a few words of the advantages of genetic algorithms. First of all, it is very easy to implement numerically since it does not require to compute the gradient. Moreover, the algorithm is easily parallelizable because it consists in several evaluations of a function. In addition, if the set wherein parameters are sought is sufficiently large, the algorithm will not converge towards local extrema. Note however that there is no theoretical results concerning convergence. Moreover, since the initial population is generated randomly, computations have to be launched several times to get a mean value of the optimum.

### 7.2.3 Presentation of results

We numerically solve the direct problem on a domain consisting of a single generation. We refer to Figure 7.3 for an illustration of the domain. We used a triangulation composed of  $N_{Vertices} = 38307$  and  $N_{Triangles} = 75512$ .

Figure 7.3: Domain  $\Omega$ .

In the sequel, we take the physiological data proposed in [35].

- $m = 0,3 \text{ kg}$ , total mass of the lung,
- $S = 0,011 \text{ m}^2$ , surface of the moving box,
- $E = 3,32 \cdot 10^5 \text{ N} \cdot \text{m}^{-5}$ , the lung elastance,
- $k_0 = E \times S^2 = 40,172 \text{ N} \cdot \text{m}^{-1}$ , the stiffness of the spring,
- $R_i = 1,33 \cdot 10^5 \text{ Pa} \cdot \text{s} \cdot \text{m}^{-3}$ , the resistance at the outlet  $\Gamma_i$ .

We will play mainly on two parameters of the genetic algorithm:

$N_{par}$  = number of individuals in the population at each generation,  
 $N_{exact}$  = number of allowed exact evaluations.

Thereafter, we will refer to the parameters we want to recover as reference parameters and to the parameters given by the optimization algorithm as obtained parameter. We test a number of different options. In each case, we compare the results obtained by minimizing the volume curve and by minimizing the flow-volume loop. In the first case, we minimize the difference between the reference volume curve and those generate numerically by the optimization algorithm, in  $L^2$  norm on  $(0, T)$ . In the second case, we cannot proceed in the same way as previously because the value on the abscissa (namely, the volume at a time  $t$ ) may be different. That is why we interpolate, using spline functions, the curve obtained numerically, in the points where we know the value of the reference curve.

We search the stiffness constant in the interval  $[30, 60]$  and the resistances in the interval  $[110000, 140000]$  and the number of generations considered in the algorithm before it stops is equal to 20.

- **Estimation of the stiffness constant**

We fix:

$$N_{par} = 8,$$

$$N_{exact} = 50.$$

We refer to the following tabular for the results obtained in each case.

	Volume curve	Flow-volume loop
parameter reference	40.172	40.172
obtained parameter	40.175608	40.140012

• **Estimation of one resistance**

We fix:

$$N_{par} = 8,$$

$$N_{exact} = 50.$$

We refer to the following tabular for the results obtained in each case.

	Volume curve	Flow-volume loop
obtained parameter	133000	133000
obtained parameter	132995.22	132989.59

• **Estimation of two resistances**

We fix:

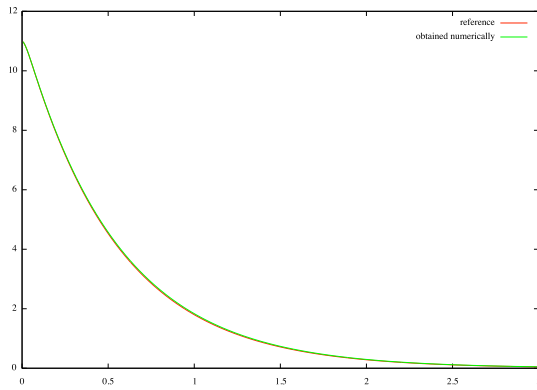
$$N_{par} = 8,$$

$$N_{exact} = 50.$$

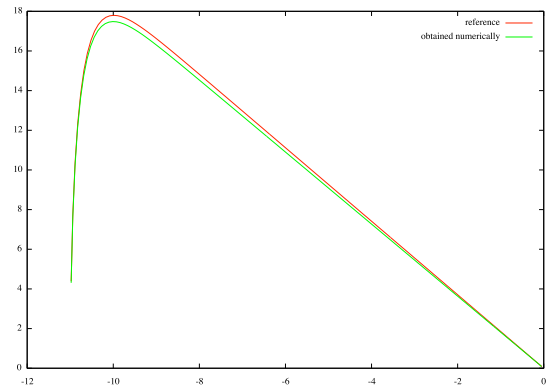
We refer to the following tabular for the results obtained when minimizing the volume curve.

	Volume curve	
reference parameters	133000	123000
parameters obtained	134846.11	121444.9

Increasing  $N_{par}$  and  $N_{exact}$  allows to get accurate results. We fix:



(a) Figure illustrating the volume reference curve and the one obtained numerically.



(b) Figure illustrating the flow-volume reference curve and the one obtained numerically.

Figure 7.4: References curves and those obtained numerically when we want to estimate the resistances (the same on both sides) and the stiffness constant.

$$N_{par} = 10,$$

$$N_{exact} = 70.$$

The following tabular summarizes the result obtained when minimizing the flow volume loop:

	Flow-volume loop	
reference parameters	133000	123000
parameters obtained	131654.7	124181.23

• **Estimation of resistances (the same on both sides) and stiffness constant**

We fix

$$N_{par} = 10,$$

$$N_{exact} = 50.$$

	Volume curve		Flow-volume loop	
reference parameters	133000	40.172	133000	40.172
parameters obtained	129246.69	39.746233	125890.82	39.410234

We refer to Figure 7.4 for an illustration.

**• Estimation of two different resistances and stiffness constant**

$$N_{par} = 8,$$

$$N_{exact} = 50.$$

	Volume curve		
reference parameters	40.172	123000	133000
obtained parameter	40.4016	133577.82	126444.24

	Flow-volume loop		
reference parameters	40.172	123000	133000
obtained parameter	40.01386	123882.17	129023.7

**7.2.4 Comments**

This is a preliminary work concerning the numerical resolution of the inverse problem we are interested in. We obtain good results, especially for the stiffness parameter. Nevertheless, when the number of parameters to estimate increases, the parameters identification is very costly (this is one of the drawbacks of genetic algorithm). The resolution of the direct problem may be too long (about 10 minutes). An extension would be to use real data as reference curve. This would need in a preliminary to enrich the direct model in order to reproduce different profiles.

It is interesting to note that from the numerical point of view, it seems enough to know the volume and the flow during a finite time to recover one or more parameters, whereas in the theoretical results that we have proved, we need more measurements (the pressure and the normal derivative of the pressure are required) during an infinite time.







## Appendix A

# Simultaneous straightening of the boundary and the Laplace operator

We are going to illustrate the simultaneous straightening of the boundary and the Laplace operator in dimension  $n=3$ .

Let  $\phi$  be a function of class  $C^\infty(\mathbb{R}^2)$  such that

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / x_3 > \phi(x_1, x_2)\}.$$

We denote by

$$\begin{aligned} a(x_1, x_2) &= \partial_{x_1}\phi(x_1, x_2), \quad b(x_1, x_2) = \partial_{x_2}\phi(x_1, x_2), \\ d(x_1, x_2) &= \sqrt{1 + a(x_1, x_2)^2 + b(x_1, x_2)^2}, \\ n &= \begin{pmatrix} a/d \\ b/d \\ -1/d \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \end{aligned}$$

Up to a change of basis, we can assume that  $x_0 = (0, 0, \phi(0, 0)) \in \partial\Omega$ . We consider the following application:

$$\Phi : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \phi(x_1, x_2) \end{pmatrix} - x_3 n(x_1, x_2).$$

The jacobian matrix of  $\Phi$  at point  $(x_1, x_2, x_3)$  is equal to:

$$J(x_1, x_2, x_3) = \begin{pmatrix} 1 - x_3 \frac{\partial n_1}{\partial x_1}(x_1, x_2) & -x_3 \frac{\partial n_1}{\partial x_2}(x_1, x_2) & -n_1(x_1, x_2) \\ -x_3 \frac{\partial n_2}{\partial x_1}(x_1, x_2) & 1 - x_3 \frac{\partial n_2}{\partial x_2}(x_1, x_2) & -n_2(x_1, x_2) \\ a(x_1, x_2) - x_3 \frac{\partial n_1}{\partial x_1}(x_1, x_2) & b(x_1, x_2) - x_3 \frac{\partial n_2}{\partial x_2}(x_1, x_2) & -n_3(x_1, x_2) \end{pmatrix}.$$

Note that  $\det(J(x_1, x_2, 0)) = d(x_1, x_2) > 0$ . Thus, thanks to the inverse function theorem, there exist  $V$  a neighborhood of 0 such that  $\Phi$  is a  $C^\infty$  diffeomorphism from  $V$  to  $\Phi(V)$ . This diffeomorphism straighten the boundary:  $\{x_3 = 0\} \cap V = \partial\Omega \cap \Phi(V)$  and  $\{x_3 > 0\} \cap V = \Omega \cap \Phi(V)$ . Let us check that it also straightens the operator. We have

$$\nabla u(y) = (J^{-1}(x))^T \nabla v(x),$$

where we denote by  $v = u \circ \Phi$  and  $x = \Phi^{-1}(y)$ . In other words, we have

$$\begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \\ \partial_{y_3} \end{pmatrix} = (J^{-1}(x))^T \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}.$$

Assume that  $J^{-1}(x) = (a_{ij}(x))_{1 \leq i, j \leq 3}$ . We have

$$\Delta = \sum_{i=1}^3 \partial_{y_i} \partial_{y_i} = \sum_{i=1}^3 \sum_{j, k=1}^3 a_{ji}(x) a_{ki}(x) \partial_{x_j} \partial_{x_k} + \text{lower order terms}.$$

One can easily check that, for  $1 \leq j, k \leq 3$ , the term in front of the derivative  $\partial_{x_j} \partial_{x_k}$  corresponds to the element  $b_{jk}$  of the matrix  $J^{-1}(x) (J^{-1}(x))^T$ . Let us compute this matrix. The matrix  $J^T(x)J(x)$  is of the form  $\begin{pmatrix} a(x) & b(x) & 0 \\ c(x) & d(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}$  because the two first vectors of  $J$  are orthogonal to the third one and the norm of the third one is equal to one.

Consequently,  $J^{-1}(x) (J^{-1}(x))^T = (J^T(x)J(x))^{-1}$  is also of the form  $\begin{pmatrix} \tilde{a}(x) & \tilde{b}(x) & 0 \\ \tilde{c}(x) & \tilde{d}(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

We deduce that the coefficient in front of  $\partial_{x_3} \partial_{x_3}$  is equal to one and the coefficient in front of each of the cross terms  $\partial_{x_1} \partial_{x_3}$ ,  $\partial_{x_3} \partial_{x_1}$ ,  $\partial_{x_2} \partial_{x_3}$  and  $\partial_{x_3} \partial_{x_2}$  is equal to zero. Finally, since the ellipticity assumption on the operator is preserved by diffeomorphism (see [22]), the Laplace operator is transported near the boundary in an operator which satisfies the assumptions of Proposition 3.9.

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## Appendix B

# Some properties of harmonic function

We detail here some properties of harmonic function required to construct the weight functions in the proof of Theorem 4.8.

Let  $d \in \mathbb{N}^*$  and  $\Omega \subset \mathbb{R}^d$  an open set. We refer to [45] and [33] for details about results stated in this appendix.

Let us begin by recalling the definition of a harmonic function.

**Definition B.1.** *Let  $u \in \mathcal{C}^2(\Omega)$ . The function  $u$  is said harmonic in  $\Omega$  if it satisfies:*

$$\Delta u = 0 \text{ in } \Omega.$$

Then we state an existence and regularity result.

**Theorem B.2.** *Let  $0 < \alpha < 1$  and  $\Omega$  be an open set of class  $\mathcal{C}^{2,\alpha}$ ,  $f \in \mathcal{C}^\alpha(\overline{\Omega})$  and  $\phi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ . Then, the following Dirichlet problem:*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

*has a unique solution which belongs to  $\mathcal{C}^{2,\alpha}(\overline{\Omega})$ .*

**Theorem B.3** (Weak maximum principle). *Assume that  $\Omega$  is bounded. Let  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$  satisfying  $\Delta u \geq 0$  in  $\Omega$ . Then, we have:*

$$u(x) \leq \sup_{y \in \partial\Omega} u(y), \forall x \in \Omega.$$

**Theorem B.4** (Strong maximum principle). *Assume that  $\Omega$  is a connected open set. Let  $u \in \mathcal{C}^2(\Omega)$  satisfying  $\Delta u \geq 0$  in  $\Omega$ . Assume that there exists  $y \in \Omega$  such that  $u(y) = \sup_{\Omega} u$ . Then,  $u$  is constant.*

**Corollary B.5** (Comparison principle). *Let  $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$  be such that  $u$  is an harmonic function in  $\Omega$  and  $\Delta v \geq \Delta u$  in  $\Omega$  and  $u \geq v$  on  $\partial\Omega$ . Then,  $v \leq u$  in  $\Omega$ .*

We will also need the following Hopf Lemma which requires some geometrical condition on the open  $\Omega$ .

**Definition B.6.** We say that  $\Omega$  satisfies an inner sphere condition in  $x_0 \in \partial\Omega$  if there exists a ball  $B \subset \Omega$  with  $x_0 \in \partial B$ .

**Remark B.7.** In particular, if  $\Omega$  is of class  $\mathcal{C}^1$ , it satisfies this condition.

**Lemma B.8** (Hopf Lemma). Assume that  $\Delta u \geq 0$  in  $\Omega$  and let  $x_0 \in \partial\Omega$  be such that

- (i)  $u$  is continuous in  $x_0$ ,
- (ii)  $u(x_0) > u(x)$  for all  $x \in \Omega$ ,
- (iii)  $\partial\Omega$  satisfies an inner sphere condition in  $x_0$ .

Then,  $\frac{\partial u}{\partial n}(x_0) > 0$ .

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## Appendix C

# Proof of Lemma 2.36

In this appendix, we give a proof of Lemma 2.36.

Let  $n \in \mathbb{N}$  and  $\varpi \subset \mathbb{R}^n$ . Let us begin by recalling the following property of  $H^1(\varpi)$ :

$$h \in H^1(\varpi) \iff \forall i \in \{1, \dots, n\}, \exists C > 0, \forall \phi \in \mathcal{C}_c^\infty(\varpi), \left| \int_{\varpi} h \cdot \partial_i \phi \right| \leq C \|\phi\|_{L^2(\varpi)}. \quad (\text{C.1})$$

Indeed, if  $h$  belongs to  $H^1(\varpi)$ , the right hand-side property of equivalence (C.1) is satisfied by definition of  $H^1$  space. Reciprocally, if the right hand-side property of equivalence (C.1) is satisfied, then the linear form  $l : \phi \rightarrow \int_{\varpi} h \cdot \partial_i \phi$  is continuous on  $L^2(\varpi)$ . Thanks to Hilbert space representation theorem, there exists a function, denoted by  $-\partial_i h$ , such that  $l(\phi) = -\int_{\varpi} \partial_i h \cdot \phi$ . Thus,  $h \in H^1(\varpi)$ .

We are now able to prove Lemma 2.36 using equivalence (C.1). Let  $i = 1, 2$  and  $\phi \in \mathcal{C}_c^\infty(K_R \cup \tilde{K}_R)$ . We have:

$$\int_{K_R \cup \tilde{K}_R} W \cdot \partial_i \phi = \int_{K_R} w \partial_i \phi + \int_{\tilde{K}_R} \tilde{w} \partial_i \phi. \quad (\text{C.2})$$

Performing integration by parts, we get

$$\int_{K_R} w \partial_i \phi = - \int_{K_R} \partial_i w \cdot \phi + \int_{(x_d=0) \cap K_R} w \phi n_i, \quad (\text{C.3})$$

and

$$\int_{\tilde{K}_R} \tilde{w} \partial_i \phi = - \int_{\tilde{K}_R} \partial_i \tilde{w} \cdot \phi + \int_{(x_d=0) \cap \tilde{K}_R} \tilde{w} \phi \tilde{n}_i. \quad (\text{C.4})$$

By gathering inequalities (C.2), (C.3) and (C.4) and using the assumption (2.54), we get:

$$\int_{K_R \cup \tilde{K}_R} W \cdot \partial_i \phi = \int_{K_R} g \cdot \phi,$$

where  $g \in L^2(K_R \cup \tilde{K}_R)$  is equal to  $-\partial_i w$  in  $K_R$  and to  $-\partial_i \tilde{w}$  on  $\tilde{K}_R$ . The function  $W$  satisfy the right hand-side property of equivalence (C.1), therefore it implies that  $W \in H^1(K_R \cup \tilde{K}_R)$ .



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