



HAL
open science

Hodge-Newton filtrations, cell decomposition and cohomology of certain p-adic moduli spaces

Xu Shen

► **To cite this version:**

Xu Shen. Hodge-Newton filtrations, cell decomposition and cohomology of certain p-adic moduli spaces. General Mathematics [math.GM]. Université Paris Sud - Paris XI, 2012. English. NNT : 2012PA112341 . tel-00764117

HAL Id: tel-00764117

<https://theses.hal.science/tel-00764117>

Submitted on 12 Dec 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



UNIVERSITÉ
PARIS-SUD 11



Faculté des
sciences
d'Orsay

N° d'ordre:

THÈSE

Présentée pour obtenir

LE GRADE DE DOCTEUR EN SCIENCES
DE L'UNIVERSITÉ PARIS-SUD XI

Spécialité: Mathématiques

par

Xu SHEN

Filtrations de Hodge-Newton, décomposition cellulaire et cohomologie de certains espaces de modules p -adiques

Soutenue le 6 décembre 2012 devant la Commission d'examen:

- M. Jean-François DAT
- M. Laurent FARGUES (Directeur de thèse)
- M. Jean-Marc FONTAINE
- M. Alain GENESTIER (Rapporteur)
- M. Farid MOKRANE
- M. Thomas ZINK (Rapporteur)



Thèse préparée au
Département de Mathématiques d'Orsay
Laboratoire de Mathématiques (UMR 8628), Bât. 425
Université Paris-Sud 11
91 405 Orsay CEDEX

Résumé

Dans cette thèse, nous étudions la géométrie analytique p -adique et la cohomologie ℓ -adique de certains espaces de Rapoport-Zink, en utilisant la théorie des filtrations de Harder-Narasimhan des schémas en groupes finis et plats élaborée par Fargues dans [26] et [27].

Cette thèse se compose de trois parties. La première partie traite de certains espaces de Rapoport-Zink non-basiques, qui satisfont à la condition que leur polygone de Newton et polygone de Hodge ont un point de contact non-trivial, qui est un point de rupture pour le polygone de Newton. Sous cette hypothèse, nous prouvons que ces espaces de Rapoport-Zink peuvent être décomposés en une somme directe d'espaces de modules des types de Rapoport-Zink associés à certains sous-groupes paraboliques appropriés, donc leurs cohomologie ℓ -adique sont des induites paraboliques et en particulier ne contiennent pas de représentations supercuspidales. Nous prouvons ces faits en démontrant d'abord un théorème sur la filtration de Hodge-Newton pour les groupes p -divisibles avec des structures additionnelles sur des anneaux de valuation complets de rang un et de caractéristique mixte $(0, p)$.

Dans la deuxième partie, nous considérons les espaces de Rapoport-Zink basiques de signature $(1, n - 1)$ pour les groupes unitaires associés à l'extension quadratique non ramifiée de \mathbb{Q}_p . On étudie l'action de Hecke sur ces espaces en détails. En utilisant la théorie des filtrations de Harder-Narasimhan des schémas en groupes finis et plats, et la stratification de Bruhat-Tits de la fibre spéciale réduite \mathcal{M}_{red} étudié dans [81], on trouve un certain domaine analytique compact \mathcal{D}_K telle que ses itérés dans le groupe $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ forme un recouvrement localement fini de tout l'espace \mathcal{M}_K . Nous appelons un tel phénomène une décomposition cellulaire localement finie.

Dans la troisième partie, nous démontrons une formule de Lefschetz pour ces espaces pour l'action des éléments semi-simples réguliers elliptiques, en tenant compte de l'action de ces éléments sur les cellules et en appliquant le théorème principal de Mieda dans [60]. De la même manière, nous pouvons aussi reprouver la formule de Lefschetz pour les espaces de Lubin-Tate précédemment obtenue par Strauch dans [77] et Mieda dans [60]. Cette formule de Lefschetz devrait caractériser la réalisation de correspondances de Jacquet-Langlands locales pour les groupes unitaires dans la cohomologie ℓ -adique de ces espaces de Rapoport-Zink, dès que certains problèmes correspondants de théorie des représentations auront été résolus.

Mots-clefs : groupes p -divisibles, espaces de Rapoport-Zink, filtration de Hodge-Newton, décomposition cellulaire, formule de Lefschetz.

HODGE-NEWTON FILTRATIONS, CELL DECOMPOSITION AND COHOMOLOGY OF CERTAIN p -ADIC MODULI SPACES

Abstract

In this thesis we study p -adic analytic geometry and ℓ -adic cohomology of some Rapoport-Zink spaces, using the theory of Harder-Narasimhan filtration of finite flat group schemes developed by Fargues in [26] and [27].

This thesis consists of three parts. The first part deals with some non-basic Rapoport-Zink spaces, which satisfy the condition that their Newton polygon and Hodge polygon have a non-trivial contact point, which is a breakpoint for the Newton polygon. Under this hypothesis, we prove these Rapoport-Zink spaces can be decomposed as a direct sum of smaller Rapoport-Zink spaces associated to some suitable parabolic subgroups, thus their ℓ -adic cohomology is parabolically induced and in particular contain no supercuspidal representations. We prove these facts by first proving a theorem about the Hodge-Newton filtration for p -divisible groups with additional structures over complete valuation rings of rank one and mixed characteristic $(0, p)$.

In the second part, we consider the basic Rapoport-Zink spaces with signature $(1, n - 1)$ for the unitary groups associated to the unramified quadratic extension of \mathbb{Q}_p . We study the Hecke action on these spaces in details. By using the theory of Harder-Narasimhan filtrations of finite flat group schemes, and the Bruhat-Tits stratification of the reduced special fiber \mathcal{M}_{red} studied in [81], we find some compact analytic domain \mathcal{D}_K such that its translates under the group $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ form a locally finite cover of the whole space \mathcal{M}_K . We call such a phenomenon a locally finite cell decomposition.

In the third part we prove a Lefschetz trace formula for these spaces for the action of regular semi-simple elliptic elements, by considering the action of these elements on the cells and applying Mieda's main theorem in [60]. In the same way we can also reprove the Lefschetz trace formula for Lubin-Tate spaces as previously obtained by Strauch in [77] and Mieda in [60]. This Lefschetz trace formula should characterize the realization of local Jacquet-Langlands correspondences for unitary groups in the ℓ -adic cohomology of these Rapoport-Zink spaces, as soon as some corresponding representation theoretic problems are solved.

Keywords : p -divisible groups, Rapoport-Zink spaces, Hodge-Newton filtration, cell decomposition, Lefschetz trace formula.

Remerciements

Je tiens tout d'abord à exprimer ma gratitude la plus profonde envers mon directeur de thèse, Laurent Fargues. C'est une expérience assez magique de travailler avec lui. C'est lui qui m'a guidé dans les études des variétés de Shimura, espaces de Rapoport-Zink, et groupes p -divisibles, quand j'étais trop jeune comme un étudiant M2. C'est lui qui m'a partagé avec générosité ses connaissances assez profondes, ses idées assez originales, et en particulier c'est lui qui m'a proposé ce sujet de thèse si intéressant. Avec sa gentillesse et sa grande patience, il a consacré beaucoup de temps à discuter et à répondre à mes questions. Je voudrais le remercier vivement aussi pour ses encouragements constants pendant la préparation de cette thèse.

Je remercie sincèrement Alain Genestier et Thomas Zink d'avoir accepté la tâche de rapporter cette thèse. Je remercie également Jean-François Dat, Jean-Marc Fontaine, et Farid Mokrane, qui m'ont fait l'honneur d'accepter de faire partie du jury de ma soutenance.

Je voudrais remercier Bertrand Rémy, qui a répondu à ma question sur les immeubles, et m'a donné une référence dont j'ai besoin dans la dernière partie de cette thèse. Je dois remercier Yoichi Mieda, qui a proposé quelques questions utiles. Je souhaite remercier Guy Henniart, qui a répondu mes questions sur la théorie des représentations. Je souhaite remercier Chia-Fu Yu, avec qui j'ai eu une discussion utile au printemps de 2010. Je voudrais remercier Jean-François Dat, qui m'avait guidé dans le premier automne après mon entrée en France. Je ne peux pas oublier l'encouragement de Michel Harris avant que je commence mes études comme un doctorant à Orsay. Je tiens à le remercier.

Cette thèse a été effectuée au département de mathématiques d'Orsay, et particulièrement à la bibliothèque mathématique Jacques Hadamard, qui ont fourni moi des conditions agréables de travail, j'en remercie tous ses membres, en particulier David Harari, Valérie Lavigne, Martine Thouvenot pour leur aide. Certains travaux de correction ont été faites au cours d'une séjour à l'Institut Fields. Je tiens à le remercier pour son hospitalité et pour me donner une atmosphère parfaite pour travailler. J'ai pu faire mes études Frances grâce au programme Erasmus (ALGANT), j'exprime ici ma gratitude à tous ceux qui y ont participé ; en particulier, mon ancien directeur en Chine, Fei Xu, qui m'a encouragé à étudier à l'étranger. Je tiens à remercier aussi Ye Tian, qui m'a guidé dans mes première études de l'arithmétique à Beijing, surtout qui m'a souligné l'importance des variétés de Shimura.

J'adresse également des remerciements sincères à tous mes amis en France et en Europe, qui m'ont apporté de la chaleur et m'ont rendu heureux : Ke Chen, Li Chen, Miaofen Chen, Zongbin Chen, Yiwen Ding, Marco De Ieso, Yong Hu, Yongquan Hu, Xun Jiang, Zhi Jiang, Arno Kret, Xiangyu Liang, Yongqi Liang, Chengyuan Lu, Prof. Xiaonan Ma, Yieh-Da Shieh, Shenghao Sun, Zhe Sun, Shun Tang, Chunhui Wang, Hao-ran Wang, Shanwen Wang, Heer Zhao, Yigeng Zhao. Je voudrais remercier en particulier Yongqi Liang, Zhe Sun, et Yigeng Zhao pour leur aide dans la vie.

Table des matières

Introduction générale	9
1 Hodge-Newton filtration for p-divisible groups with additional structures	23
1.1 Introduction	23
1.2 Harder-Narasimhan filtration of finite flat group schemes I	27
1.3 Polygons and inequalities	30
1.4 Admissible filtered isocrystals	37
1.5 Hodge-Newton filtration for p -divisible groups with additional structures	41
1.6 Application to the geometry and cohomology of some non-basic Rapoport-Zink spaces	47
1.7 Application to the cohomology of some Shimura varieties	57
2 Cell decomposition of some unitary group Rapoport-Zink spaces	59
2.1 Introduction	59
2.2 The unitary group Rapoport-Zink spaces and Hecke action	65
2.3 Harder-Narasimhan filtration of finite flat group schemes II	74
2.4 Harder-Narasimhan polygon of p -divisible groups	74
2.5 An algorithm for p -divisible groups with additional structures	77
2.6 The analytic domain \mathcal{C}	81
2.7 Bruhat-Tits stratification of \mathcal{M}_{red}	85
2.8 The analytic domain \mathcal{D}	88
2.9 Some unitary group Shimura varieties and the relatively compactness of \mathcal{D}	89
2.10 Locally finite cell decompositions	94
3 Cell decomposition and Lefschetz trace formula	103
3.1 Introduction	103
3.2 The locally finite cell decomposition of Lubin-Tate spaces	107
3.3 Lefschetz trace formula for Lubin-Tate spaces	112
3.4 Lefschetz trace formula for the unitary group Rapoport-Zink spaces . . .	116
Références	128

Introduction générale

Motivation

Cette thèse est consacrée à l'étude de la géométrie analytique p -adique de certains espaces de Rapoport-Zink à l'aide de la théorie des filtrations de Harder-Narasimhan des schémas en groupes finis et plats développée par Fargues dans [26] et [27], et en déduire des résultats sur la cohomologie de ces espaces. Elle se compose des articles [72],[73] et [74]. En particulier, nous obtenons par des méthodes locales des résultats sur la cohomologie de ces espaces compatibles avec les conjectures de Harris et de Kottwitz (Conjecture 5.2 dans [34], Conjecture 5.1 dans [70]). Afin de donner une exposition simplifiée, nous examinons dans cette introduction certaines variétés de Shimura particulières étudiées par Harris et Taylor dans [36] qui donnent tous les problèmes de modules locaux que nous allons étudier.

Soient k un corps quadratique imaginaire et D une algèbre à division centrale sur k de dimension n^2 , munie d'une involution $*$ de seconde espèce. Soit $h_0 : \mathbb{C} \rightarrow D_{\mathbb{R}}$ un homomorphisme de \mathbb{R} -algèbres, de telle sorte que $h_0(z)^* = h_0(\bar{z})$, et l'involution $x \mapsto h_0(i)^{-1}x^*h_0(i)$ est positive. Ces données définissent un groupe réductif de similitudes unitaires G sur \mathbb{Q} dont les points à valeur dans la \mathbb{Q} -algèbre R sont

$$G(R) = \{g \in (D \otimes_{\mathbb{Q}} R)^{\times} | g^*g \in R^{\times}\},$$

et un homomorphisme

$$h : \text{Res}_{\mathbb{C}|\mathbb{R}}\mathbb{G}_m \rightarrow G_{\mathbb{R}}.$$

Cela définit une donnée de Shimura. Pour tout sous-groupe ouvert compact $K \subset G(\mathbb{A}_f)$ suffisamment petit, on obtient une variété de Shimura Sh_K propre et lisse sur k associée à ces données. Nous supposons que le morphisme h est tel que $G_{\mathbb{R}} = GU(1, n-1)$. Considérons un nombre premier impair p qui est non ramifié dans k . Supposons que D est non ramifiée en p . Alors, pour tout premier \mathfrak{p} de k au-dessus de p , K est compact hyperspécial en \mathfrak{p} , il y a des modèles entiers propres et lisses de ces variétés sur l'anneau des entiers $O_{k_{\mathfrak{p}}}$ définis par Kottwitz (cf. [51]). Ce sont des espaces de modules de variétés abéliennes avec des structures additionnelles. La géométrie locale en \mathfrak{p} de ces variétés de Shimura est assez différente pour les deux cas : p décomposé ou inerte dans k .

Cas p décomposé dans k . Dans ce cas, $k_{\mathfrak{p}} = \mathbb{Q}_p$, $G_{\mathbb{Q}_p} \simeq GL_n \times \mathbb{G}_m$, et les polygones de Newton associés à la réduction en \mathfrak{p} , correspondent à des nombres entiers $0, \dots, n-1$. Pour chaque entier $0 \leq h \leq n-1$, le polygone concave associé consiste en la droite reliant le point $(0, 0)$ et $(n-h, 1)$, et celle reliant $(n-h, 1)$ et $(n, 1)$. Le groupe p -divisible associé à un point dans la strate \overline{Sh}_K^h , admet la suite locale-étale : $0 \rightarrow H^0 \rightarrow H \rightarrow H^{\text{ét}} \rightarrow 0$ avec $htH^{\text{ét}} = h$. Toutes les strates de Newton sont non vides : la dimension de \overline{Sh}_K^h est h . À chaque strate \overline{Sh}_K^h , on peut associer à un espace de Rapoport-Zink $\mathcal{M}^{n-h,h}$, où, pour la strate supersingulière $h = 0$, c'est l'espace de Rapoport-Zink associé à l'espace de Lubin-Tate \mathcal{M}_{LT}^n pour GL_n/\mathbb{Q}_p . On peut considérer le cas avec niveau K_p en p . Parmi d'autres faits, ce que nous savons au sujet de ces espaces sont les faits suivants :

- Pour chaque strate non basique, $\mathcal{M}^{n-h,h}$ est une induite parabolique. Ce fait est facile à voir, par la décomposition local-étale du groupe p -divisible. On est donc réduit à étudier les espaces de Lubin-Tate. On voit en particulier, la cohomologie de

ces strates non basiques ne contient pas de représentation supercuspidale (« astuce de Boyer », cf. [9], [36]).

- Pour \mathcal{M}_{LT}^n , il existe une décomposition cellulaire localement finie avec des cellules compactes. Il s’agit d’un cas particulier du théorème principal de [27]. (Voir aussi le premier chapitre de [25] pour une décomposition variante pour le tour de ces espaces.)
- Pour \mathcal{M}_{LT, K_p}^n , K_p suffisamment petit, il existe une formule de Lefschetz pour les éléments semi-simples réguliers elliptiques agissant sur la cohomologie de ces espaces. Il s’agit d’un résultat principal de [77], où Strauch a utilisé cette formule pour prouver la réalisation de la correspondance Jacquet-Langlands locale dans la cohomologie de ces espaces. Ceci est bien sûr impliqué par les résultats de Harris-Taylor, mais ici la méthode est locale et il n’est pas nécessaire d’utiliser les variétés de Shimura ci-dessus.

Cas p inerte dans k . Dans ce cas, $k_p \simeq \mathbb{Q}_{p^2}$, $G_{\mathbb{Q}_p}$ est le groupe des similitudes unitaires quasi-déployé associé à l’extension $\mathbb{Q}_{p^2}|\mathbb{Q}_p$. L’ensemble des polygones de Newton est décrit explicitement dans [10] 3.1. Ils correspondent à des nombres entiers $0 \leq r \leq \frac{n}{2}$. En particulier, $r = 0$ correspond au polygone basique, $r = 1$ correspond au polygone de Hodge (généralisé), et chaque polygone non basique a un segment central de pente $\frac{1}{2}$ (peut-être de longueur 0) coïncidant avec une partie dans le polygone de Hodge. Toutes les strates de Newton sont non vides : la dimension de la strate non basique correspond à r est $n - r$, tandis que la strate basique est de dimension $\lfloor \frac{n-1}{2} \rfloor$. Comme dans le cas ci-dessus, les strates non basiques sont en fait des feuilles au sens de [55], c’est à dire les groupes p -divisibles sont uniquement déterminées par leur p -torsion. La géométrie de la strate basique est plus compliquée que dans le cas p -décomposé ci-dessus. Cela a été étudié par Valloord et Wedhorn dans [81]. On peut associer à toutes les strates de Newton un espace de Rapoport-Zink, et on essaye d’étudier ces espaces. En particulier on se demande si nous avons des résultats analogues comme le cas p -décomposé ci-dessus :

- Pour les espaces non basiques, sont-ils des induites paraboliques à partir d’espaces de Rapoport-Zink plus petits ?
- Pour l’espace basique, existe-t-il une décomposition cellulaire localement finie avec des cellules compacts ?
- Pour l’espace basique, existe-t-il une formule de Lefschetz pour les éléments semi-simples réguliers elliptiques agissant sur la cohomologie ?

Dans cette thèse, nous allons donner des réponses positives à toutes les questions ci-dessus. En fait, on peut commencer directement avec quelques données locales, qui donnent des espaces de Rapoport-Zink, et on n’a pas besoin de commencer avec les variétés de Shimura.

Espaces de Rapoport-Zink

Les espaces de Rapoport-Zink sont définis dans [66], comme des espaces de modules de groupes p -divisibles généralisant les espaces de Lubin-Tate et de Drinfeld. Ils sont des analogues locaux des variétés de Shimura. En fait, il existe des liens entre eux. Les données pour la définition d’une variété de Shimura de type PEL donnent des données locales en p qui définissent des espaces de Rapoport-Zink pour chaque strate de Newton (comme on l’a vu précédemment dans un cas particulier), et Rapoport et Zink ont montré que ces espaces permettent d’uniformiser certains sous-espaces ouvertes rigides

analytiques de la variété de Shimura rigide analytique associée.

On se restreint aux espaces de Rapoport-Zink non ramifiés de type EL et PEL unitaires ou symplectiques. Une donnée non ramifiée simple locale de type EL ou PEL (unitaires ou symplectiques) définit un groupe réductif non ramifié G sur \mathbb{Q}_p , qui est soit une restriction de scalaires d'un groupe linéaire, soit un groupe de similitudes unitaires ou symplectiques. Notons $W = W(\overline{\mathbb{F}}_p)$, $L = W_{\mathbb{Q}}$. Soient $b \in B(G) = G(L)/\sim$ (classes de σ -conjugaison), et $\mu : \mathbb{G}_{m, \overline{\mathbb{Q}}_p} \rightarrow G_{\overline{\mathbb{Q}}_p}$ un cocaractère minuscule à conjugaison près, telles que $b \in B(G, \mu)$ (l'ensemble de Kottwitz, cf. [52]). À partir des données locales, on peut construire un espace de Rapoport-Zink formel $\widehat{\mathcal{M}}$ sur $\text{Spf}W$: l'ensemble des points à valeur dans $S \in \text{Nilp}W$ (la catégorie des schémas S au-dessus de W tels que p soit localement nilpotent sur S) est $\widehat{\mathcal{M}}(S) = \{(H, \rho)\}/\simeq$, où

- H est un groupe p -divisible avec des structures additionnelles (en un certain sens précis déterminé par les données locales),
- $\rho : \mathbb{H}_{\overline{S}} \rightarrow H_{\overline{S}}$ est une quasi-isogénie compatible avec les structures additionnelles (\mathbb{H} est le groupe p -divisible standard sur $\overline{\mathbb{F}}_p$ à isogénie près associé, et \overline{S} est le sous schéma fermé de S défini par p).

Dans les cas unitaires ce que nous considérons ici, $\mu(z) = (\text{diag}(\underbrace{z, \dots, z}_{n-1}, 1), z)$ pour

l'isomorphisme $G_{\overline{\mathbb{Q}}_p} \simeq GL_n \times \mathbb{G}_m$, $\widehat{\mathcal{M}}(S) = \{(H, \iota, \lambda, \rho)\}/\simeq$, où

- $\iota : \mathbb{Z}_{p^2} \rightarrow \text{End}(H)$ est un morphisme d'algèbres, tel que dans la décomposition induite $\text{Lie}H = (\text{Lie}H)_0 \oplus (\text{Lie}H)_1$, $\text{rg}(\text{Lie}H)_0 = 1$, $\text{rg}(\text{Lie}H)_1 = n - 1$ (dans ces cas $htH = n$), avec $(\text{Lie}H)_0$ (resp. $(\text{Lie}H)_1$) est le sous module à quel \mathbb{Z}_{p^2} agit par le plongement $\mathbb{Z}_{p^2} \hookrightarrow W$ triviale (resp. non triviale).
- $\lambda : (H, \iota) \rightarrow (H^D, \iota \circ *)$ est une polarisation (principale) (i.e. un isomorphisme tel que $\lambda^D = -\lambda$).

L'élément b définit un polygone de Newton, tandis que μ définit un polygone de Hodge. Les groupes p -divisibles associés aux points de $\widehat{\mathcal{M}}$ ont leur polygone de Newton et polygone de Hodge définis par b et μ respectivement. Le groupe J_b des quasi-isogénies de \mathbb{H} , qui est une forme intérieure d'un sous groupe de Levi de G , agit sur $\widehat{\mathcal{M}}$ naturellement. L'espace $\widehat{\mathcal{M}}$ est muni d'une donnée de descente de L à E , le corps reflex qui est le corps de définition de μ . Bien que non effective, cette donnée de descente est suffisante pour définir une action de Frobenius sur la cohomologie de ces espaces.

Fixons un modèle entier réductif de G et soit $G(\mathbb{Z}_p)$ le sous-groupe compact hyperspécial associé dans $G(\mathbb{Q}_p)$. On considère la fibre générique analytique $\mathcal{M} := (\widehat{\mathcal{M}})^{an}$ de $\widehat{\mathcal{M}}$. Il existe plusieurs versions de la géométrie analytique p -adique. Nous allons principalement travailler avec la version de Berkovich. Alors il existe une tour d'espaces analytiques $(\mathcal{M}_K)_K$, où K parcourant les sous groupes ouverts de $G(\mathbb{Z}_p)$. Le groupe $J_b(\mathbb{Q}_p)$ agit sur chaque \mathcal{M}_K pour tout K naturellement. De plus, $G(\mathbb{Q}_p)$ agit par correspondances de Hecke sur la tour $(\mathcal{M}_K)_K : \forall g \in G(\mathbb{Q}_p)$,

$$\begin{array}{ccc} & \mathcal{M}_{gKg^{-1} \cap K} & \xrightarrow[\simeq]{g} \mathcal{M}_{K \cap g^{-1}Kg} \\ & \swarrow & \searrow \\ \mathcal{M}_K & & \mathcal{M}_K, \end{array}$$

qui ne dépend que de la double classe KgK . Cette action de $G(\mathbb{Q}_p)$ commute à celle de

$J_b(\mathbb{Q}_p)$. Fixons un premier $l \neq p$. La cohomologie l -adique à support compact de \mathcal{M}_K

$$H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$$

est bien définie (cf. [23]) pour chaque entier $i \geq 0$. On obtient ainsi une $\overline{\mathbb{Q}}_l$ -représentation

$$\varinjlim_K H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$$

du groupe $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ pour chaque $i \geq 0$. Lorsque b est basique, J_b est une forme intérieure de G , et dans ce cas la conjecture de Kottwitz décrit la partie de la représentation virtuelle $\sum_{i \geq 0} (-1)^i \varinjlim_K H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ associée aux paramètres discrets de Langlands (cf. [70], Conjecture 5.1). Cette conjecture est un analogue local non-archimédéen de la conjecture correspondante pour la cohomologie des variétés de Shimura (cf. [50]).

Les points $x \in \mathcal{M}_K$ sont de la forme $(H_x/O_{\mathcal{H}(x)}, \iota_x, \lambda_x, \rho_x, \eta_x)$ (cas PEL), où $\mathcal{H}(x)$ est le corps résiduel complet du point x , et $O_{\mathcal{H}(x)}$ est son anneau de valuation (qui est de rang un). Nous allons utiliser la théorie des filtrations de Harder-Narasimhan des schémas en groupes finis et plats développée par Fargues dans [26] et [27] pour étudier ces espaces de Rapoport-Zink.

Filtration de Harder-Narasimhan des schémas en groupes finis et plats

Soient K maintenant un corps valué de caractéristique 0 pour une valuation v à valeurs dans \mathbb{R} étendant la valuation p -adique, O_K son anneau d'entiers. Dans [26], Fargues a démontré la catégorie \mathcal{C} des schémas en groupes finis et plats d'ordre une puissance de p admet une filtration de type de Harder-Narasimhan. Plus précisément, il y a deux fonctions additives

$$ht, deg : \mathcal{C} \rightarrow \mathbb{R},$$

où pour un groupe $G \in \mathcal{C}$, htG est la hauteur, et $deg(G) = \sum_i v(a_i)$ si le faisceau conormal $\omega_G \simeq \oplus_i O_K/a_i O_K$. La fonction deg satisfait certaines propriétés utiles. On pose

$$\mu = \frac{deg}{ht}.$$

On appelle un groupe $G \in \mathcal{C}$ semi-stable, si pour tous $0 \neq G' \subset G, G' \in \mathcal{C}, \mu(G') \leq \mu(G)$. Fargues a démontré que, pour tout groupe $G \in \mathcal{C}$ non nul, il possède une unique filtration par des sous groupes fermés finis et plats

$$0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_k = G$$

telle que :

1. pour tout i , G_{i+1}/G_i est semi-stable.
2. Si $i \geq 1$, $\mu(G_i/G_{i-1}) > \mu(G_{i+1}/G_i)$.

On peut définir un polygone concave $HN(G)$ de G à partir sa filtration de Harder-Narasimhan : c'est le polygone débutant en $(0, 0)$ et terminant en $(htG, degG)$ tel que si

la filtration est comme ci-dessus, alors ses pentes sont les $(\mu(G_i/G_{i-1}))_{1 \leq i \leq k}$ avec multiplicité $ht(G_i/G_{i-1})$ pour la pente $\mu(G_i/G_{i-1})$. On peut prouver $HN(G)$ est l'enveloppe concave des points $(htG', degG')$ lorsque G' parcourt les sous groupes de G .

Il y a beaucoup de propriétés utiles pour cette filtration, pour plus de détails voir [26]. Mentionnons qu'elle est compatible avec les structures additionnelles. On considère (G, ι, λ) un schéma en groupes finis et plats avec structures additionnelles, où $\iota : O_F \rightarrow End(G)$ pour une extension $F|\mathbb{Q}_p$ fini non ramifiée avec O_F son anneau d'entiers, $\lambda : G \rightarrow G^D$ est une polarisation. Alors, la filtration de Harder-Narasimhan de G est invariant sous l'action de ι et le polygone $HN(G)$ est symétrique. On peut définir un nouveau polygone

$$HN(G, \iota, \lambda) := \frac{1}{d}HN(G)(d \cdot),$$

où $d = [F : \mathbb{Q}_p]$ et on a considérée $HN(G)$ comme une fonction sur $[0, htG]$. Dans [26], Fargues défini également un polygone de Hodge $Hdg(G)$ de G , et a prouvé que l'on a l'inégalité

$$HN(G) \leq Hdg(G).$$

Dans la première section de cette thèse, nous allons définir un polygone de Hodge $Hdg(G, \iota, \lambda)$ de (G, ι, λ) , qui contient des informations des structures additionnelles, et si $(G, \iota, \lambda) = (H, \iota, \lambda)[p]$ provient de la p -torsion d'un groupe p -divisible avec structures additionnelles, alors ce polygone coïncide avec le polygone de Hodge de la fibre spéciale de (H, ι, λ) défini par Kottwitz. Nous avons également une preuve d'une inégalité (cf. Proposition 1.3.10)

$$HN(G, \iota, \lambda) \leq Hdg(G, \iota, \lambda),$$

qui raffine le cas sans structures additionnelles. Cette inégalité sera nécessaire pour prouver l'existence de la filtration de Hodge-Newton dans la suite.

Polygone de Harder-Narasimhan des groupes p -divisibles

Soient K comme avant, H un groupe p -divisible sur O_K . On peut donc étudier les groupes $H[p^n]$ pour $n \geq 1$. Notons $h = htH, d = dimH$. Alors On a $htH[p^n] = nh, degH[p^n] = nd$. Le polygone de Harder-Narasimhan de $H[p^n]$ est donc une fonction $[0, nh] \rightarrow [0, nd]$. Dans [27], Fargues a démontré que la limite

$$\lim_{n \rightarrow \infty} \frac{1}{n}HN(H[p^n])(n \cdot)$$

existe, est égale à $\inf_n \frac{1}{n}HN(H[p^n])(n \cdot)$ et définit une fonction

$$HN(H) : [0, h] \rightarrow [0, d]$$

telle que $HN(H)(0) = 0, HN(H)(h) = d$. On appelle $HN(H)$ le polygone de Harder-Narasimhan de H , puisque on peut prouver que c'est bien un polygone. Dans le cas où O_K est de valuation discrète et son corps résiduel k est parfait, on peut l'expliquer comme le polygone de Harder-Narasimhan de la représentation cristalline associée à H ou bien le polygone de Harder-Narasimhan du φ -module filtré associé, pour certaines fonctions de pente appropriées sur ces catégories. Dans le cas général, on peut utiliser le module de Hodge-Tate associée à H , cf. [27] pour plus de détails. On a

$$HN(H) \leq \frac{1}{n}HN(H[p^n])(n \cdot), \forall n \geq 1.$$

D'autre part, si on note $Newt(H_k)$ et $Hdg(H_k)$ le polygone de Newton et polygone de Hodge de H_k respectivement, et si O_K est de valuation discrète, il est facile de prouver que nous avons l'inégalité

$$HN(H) \leq Newt(H_k).$$

(Rappelons que on a toujours l'inégalité de Mazur $Newt(H_k) \leq Hdg(H_k)$). Dans le cas général, avec une certaine hypothèse technique sur H (qui est toujours satisfaite pour les groupes p -divisibles proviennent des points des espaces de Rapoport-Zink), un résultat principal de [27] dit que nous avons encore l'inégalité $HN(H) \leq Newt(H_k)$. En résumé, on a des inégalités

$$\begin{aligned} HN(H) &\leq Newt(H_k) \leq Hdg(H_k), \\ HN(H) &\leq \frac{1}{n}HN(H[p^n])(n\cdot) \leq HN(H[p]) \leq Hdg(H[p]) = Hdg(H_k). \end{aligned}$$

Passons au cas avec structures additionnelles. Soit $F|\mathbb{Q}_p$ une extension finie non ramifiée. Notons $d = [F : \mathbb{Q}_p]$. Soit (H, ι, λ) un groupe p -divisibles avec des structures additionnelles (de type PEL) : $\iota : O_F \rightarrow End(H)$ est une action de O_F et $\lambda : H \rightarrow H^D$ est une polarisation. Comme dans le cas des schémas en groupes finis et plats, on peut définir

$$HN(H, \iota, \lambda) = \frac{1}{d}HN(H)(d\cdot).$$

Alors on a des inégalités

$$\begin{aligned} HN(H, \iota, \lambda) &\leq Newt(H_k, \iota, \lambda) \leq Hdg(H_k, \iota, \lambda), \\ HN(H, \iota, \lambda) &\leq \frac{1}{n}HN(H[p^n], \iota, \lambda)(n\cdot) \leq HN(H[p], \iota, \lambda) \leq Hdg(H[p], \iota, \lambda) = Hdg(H_k, \iota, \lambda). \end{aligned}$$

Ici $Newt(H_k, \iota, \lambda) \leq Hdg(H_k, \iota, \lambda)$ est l'inégalité généralisée de Mazur (cf. [65]).

Filtration de Hodge-Newton et induite parabolique

Soit (H, ι, λ) un groupe p -divisibles avec des structures additionnelles (de type PEL) comme avant. Faisons l'hypothèse suivante : *Newt(H_k, \iota, \lambda) et Hdg(H_k, \iota, \lambda) possèdent un point x de contact non trivial, qui est un point de rupture de Newt(H_k, \iota, \lambda)*. Par symétrie, le point \hat{x} symétrique de x satisfait la même hypothèse. Nous supposons que x se trouve devant \hat{x} . Sous cette hypothèse, par l'inégalités $HN(H, \iota, \lambda) \leq Newt(H_k, \iota, \lambda) \leq Hdg(H_k, \iota, \lambda)$ et la théorie des φ -modules filtrés admissibles, on peut prouver que $HN(H, \iota, \lambda)$ passe aussi par les points x et \hat{x} . En utilisant l'inégalités

$$HN(H, \iota, \lambda) \leq \frac{1}{n}HN(H[p^n], \iota, \lambda)(n\cdot) \leq HN(H[p], \iota, \lambda) \leq Hdg(H[p], \iota, \lambda) = Hdg(H_k, \iota, \lambda),$$

on peut trouver des crans dans les filtration de Harder-Narasimhan de $H[p^n]$ correspondant aux points x et \hat{x} pour les $n \gg 0$. On peut prouver qu'ils sont compatibles et donc définissent des groupes p -divisibles.

Théorème 0.0.1. *Les notations sont comme ci-dessus. Il existe une unique filtration de groupes p -divisibles avec des structures additionnelles sur O_K*

$$(H_1, \iota) \subset (H_2, \iota) \subset (H, \iota),$$

telle que :

1. λ induit les isomorphismes

$$(H_1, \iota) \simeq ((H/H_2)^D, \iota'),$$

$$(H_2, \iota) \simeq ((H/H_1)^D, \iota'),$$

où ι' est l'action induite par ι .

2. Si k est parfait la filtration induite

$$(H_{1k}, \iota) \subset (H_{2k}, \iota) \subset (H_k, \iota)$$

est scindée.

3. les polygones de Newton (resp. Harder-Narasimhan, resp. Hodge) de (H_1, ι) , $(H_2/H_1, \iota)$ et $(H/H_2, \iota)$ sont les parties du polygone de Newton (resp. Harder-Narasimhan, resp. Hodge) de (H, ι, λ) avant x , entre x et \hat{x} , et après \hat{x} respectivement.

Ce théorème généralise les résultats de l'article [57]. Mais notre démonstration est différente.

Soit maintenant $(\mathcal{M}_K)_K$ une tour d'espaces de Rapoport-Zink non ramifiée simple, telle que le polygone de Newton \mathcal{P}_b associé à b et le polygone de Hodge \mathcal{P}_μ associé à μ satisfont l'hypothèse ci-dessus. C'est à dire que ils possèdent un point x de contact non triviale, qui est un point de rupture de \mathcal{P}_b . On peut trouver alors un sous groupe parabolique P et un sous groupe de Levi $M \subset P$ correspondants à x et \hat{x} . Mantovan a introduit dans [56] deux autres tours d'espaces analytiques $(\mathcal{P}_K)_{K \subset M(\mathbb{Z}_p)}$, $(\mathcal{F}_K)_{K \subset P(\mathbb{Z}_p)}$ pour les groupes M, P respectivement. La tour $(\mathcal{P}_K)_{K \subset M(\mathbb{Z}_p)}$ est en fait la tour d'espaces de Rapoport-Zink pour le groupe M avec les b', μ' induites. Ils sont l'espaces des modules des paires de groupes p -divisibles avec structures additionnelles. La tour $(\mathcal{F}_K)_{K \subset P(\mathbb{Z}_p)}$ est une tour d'espaces des modules de groupes p -divisibles filtrés avec structures additionnelles. De plus, pour tout sous groupe ouvert compact $K \subset G(\mathbb{Z}_p)$ si on note $\mathcal{F}_K := \mathcal{F}_{K \cap P(\mathbb{Z}_p)}$, $\mathcal{P}_K := \mathcal{P}_{K \cap M(\mathbb{Z}_p)}$, alors il y a un diagramme de morphismes d'espaces analytiques

$$\begin{array}{ccc} & \mathcal{F}_K & \\ s_K \nearrow & & \searrow \pi_{2K} \\ \mathcal{P}_K & & \mathcal{M}_K \\ \pi_{1K} \searrow & & \end{array}$$

où s_K est une immersion fermé correspondant à l'extension triviale, et π_{1K} induit un isomorphisme au niveau de la cohomologie l -adique à support compact. Le théorème d'existence et d'unicité de la filtration de Hodge-Newton implique que

$$\mathcal{M}_K \simeq \mathcal{M}_K \times_{\mathcal{M}} \mathcal{F} \simeq \coprod_{K \setminus G(\mathbb{Q}_p)/P(\mathbb{Q}_p)} \mathcal{F}_{K \cap P(\mathbb{Q}_p)}.$$

Passons à la cohomologie, on a le théorème suivante.

Théorème 0.0.2. *Les notations sont comme ci-dessus. On a une égalité de représentations virtuelles de $G(\mathbb{Q}_p) \times W_E$*

$$H(\mathcal{M}_\infty)_\rho = \text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H(\mathcal{P}_\infty)_\rho,$$

où pour une $\overline{\mathbb{Q}_l}$ -représentation lisse ρ de $J_b(\mathbb{Q}_p)$,

$$H(\mathcal{M}_\infty)_\rho = \sum_{i,j \geq 0} (-1)^{i+j} \varinjlim_K \text{Ext}_{J_b(\mathbb{Q}_p)}^j (H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}_l}(D_{\mathcal{M}})), \rho)$$

$D_{\mathcal{M}} = \dim \mathcal{M}$ et la définition de $H(\mathcal{P}_\infty)_\rho$ est identique.

Cette formule a été précédemment conjecturée par Harris dans [34] (Conjecture 5.2).

Corollaire 0.0.3. *Pour les variétés de Shimura introduites au début dans le cas p inerte dans k , pour tous les strates non basiques, considérons les représentations virtuelles de $G(\mathbb{A}_f) \times W_E$*

$$H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}_p}, R\Psi_\eta(\overline{\mathbb{Q}_l})) := \sum_{i,j \geq 0} (-1)^{i+j} \varinjlim_{K_p \times K^p} H_c^i(\overline{Sh}_{K_p \times K^p}^{(b)} \times \overline{\mathbb{F}_p}, R^j\Psi_\eta(\overline{\mathbb{Q}_l})).$$

Alors $H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}_p}, R\Psi_\eta(\overline{\mathbb{Q}_l}))$ peuvent être écrites comme certains induites paraboliques adaptées, donc en particulier ne contiennent pas de représentations supercuspidales de $G(\mathbb{Q}_p)$.

Remarquons que ces variétés de Shimura ne sont pas contenues dans les cas étudiés dans l'article [56]. Dans le cas p décomposé dans k , ce résultat a été obtenu par Harris et Taylor en utilisant l'astuce de Boyer pour ce cas particulier. Dans [23], Fargues a prouvé des résultats similaires dans des cas généraux en comparant les formules de Lefschetz pour la fibre spéciale et la fibre générique.

L'algorithme de descente pour les groupes p -divisibles basiques et correspondences de Hecke

Nous suivons les idées de [27]. Soit $K|\mathbb{Q}_p$ une extension valuée complète pour une valuation à valeurs dans \mathbb{R} étendant la valuation p -adique. Soit H un groupe p -divisible sur O_K . On dit que H est un groupe semi-stable, si $H[p]$ l'est. Il est équivalent que pour tout entier $n \geq 1$, $H[p^n]$ est semi-stable, ou encore pour tout sous schéma en groupes fini et plat G de H , $\mu(G) \leq \mu(H) := \frac{\dim H}{ht H}$. Il y a une notion supplémentaire de groupe p -divisible de type HN qui généralise la notion de groupe p -divisible semi-stable. Mais on se restreint à ce dernier cas afin d'obtenir un résultat complet. Donc on suppose H est basique, c'est à dire que $Newt(H_k)$ est la droite qui relie les points $(0, 0)$ et $(ht H, \dim H)$. Cela implique que $HN(H) = Newt(H_k)$ est la droite ci-dessous par l'inégalité de Fargues. Dans ce cas, l'algorithme de Fargues pour H est comme suivant pour produire des groupes p -divisibles qui sont de plus en plus proches d'un groupe semi-stable. On pose pour un entier $k \geq 1$

$$G_k = \text{le premier cran de la filtration de Harder-Narasimhan de } H[p^k].$$

Alors la suite $(G_k)_{k \geq 1}$ forme une suite croissante de groupes semi-stables de même pente. On pose

$$\mathcal{F}_H = \varinjlim_{k \geq 1} G_k \subset H,$$

comme un sous faisceau fppf de H . Sous notre hypothèse que H est basique, on peut prouver que il existe $k_0 \geq 1$ tel que $\mathcal{F}_H = G_{k_0}$ (cf. lemme 3,4, [27]). On note $\mu_{max}(H) = \mu_{max}(H[p])$, la pente maximale dans la filtration de Harder-Narasimhan de $H[p]$. L'algorithme est

$$H_1 = H, H_2 = H_1/\mathcal{F}_{H_1}, \dots, \text{ si } H_i \neq 0, \text{ on pose } H_{i+1} = H_i/\mathcal{F}_{H_i}, \dots$$

On a donc une suite de groupes p -divisibles $(H_i)_{i \geq 1}$ munie de morphismes $\phi_i : H_i \rightarrow H_{i+1}$ qui sont des isogénies avec comme noyaux $\ker \phi_i = \mathcal{F}_{H_i}$ des sous groupes finis et plats. De plus, si $H_{i+1} \neq 0$ on a

$$\mu_{max}(H_{i+1}) < \mu_{max}(H_i).$$

On dit que l'algorithme s'arrête en temps fini, si $H_i = 0$ pour $i \gg 0$. Dans ce cas, si $r \geq 1$ est l'entier tel que $H_r \neq 0, H_{r+1} = 0$, alors on a une suite d'isogénies

$$\begin{array}{c} \phi \\ \curvearrowright \\ H = H_1 \xrightarrow{\phi_1} H_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{r-1}} H_r, \end{array}$$

avec H_1, \dots, H_{r-1} ne sont pas semi-stables, tandis que H_r est semi-stable. On peut trouver le noyau $\ker \phi$ dans la filtration de Harder-Narasimhan de $H[p^N]$ pour $N \gg 0$. Un théorème relativement facile dit que si la valuation sur K est discrète alors l'algorithme s'arrête en temps fini. Le théorème principal de [27] dit que lorsque la dimension et la hauteur de H sont premières entre elles alors l'algorithme s'arrête toujours en temps fini.

Soit \mathcal{M} l'espace de Rapoport-Zink basique pour GL_h/\mathbb{Q}_p avec signature $(d, h-d)$. Le paragraphe ci-dessus a une explication en termes de correspondences de Hecke sur \mathcal{M} . Soit \mathcal{M}^{ss} le lieu semi-stable, qui est un domaine analytique fermé dans \mathcal{M} . Alors l'algorithme s'arrête en temps fini pour les points rigides (les points dans \mathcal{M}^{rig}) implique que comme ensemble

$$\mathcal{M}^{rig} = \bigcup_{T \in GL_h(\mathbb{Z}_p) \backslash GL_h(\mathbb{Q}_p) / GL_h(\mathbb{Z}_p)} T \cdot \mathcal{M}^{ss, rig}.$$

Nous avons la décomposition $\mathcal{M} = \prod_{i \in \mathbb{Z}} \mathcal{M}^i$ donnée par la hauteur de la quasi-isogénie universelle. Supposons $(d, h) = 1$. Sous cette hypothèse $J_b = D^\times$, où D est l'algèbre à divison centrale sur \mathbb{Q}_p d'invariant $\frac{d}{h}$. On notera Π un uniformisante de D . Posons $\mathcal{D} = \mathcal{M}^{ss} \cap \mathcal{M}^0$. Alors $\mathcal{M}^{ss} = \prod_{i \in \mathbb{Z}} \Pi^{-i} \mathcal{D}$. Le théorème principal de [27] est qu'il y a un recouvrement localement fini par des domaines analytiques fermés

$$\mathcal{M} = \bigcup_{\substack{T \in GL_h(\mathbb{Z}_p) \backslash GL_h(\mathbb{Q}_p) / GL_h(\mathbb{Z}_p) \\ 0 \leq i \leq h-1}} T \cdot \Pi^{-i} \mathcal{D}.$$

Nous appelons ce phénomène une décomposition cellulaire localement finie de \mathcal{M} . On remarque que lorsque $d = 1$, c'est à dire dans le cas de Lubin-Tate, le domaine \mathcal{D} (donc tous les domaines $T \cdot \Pi^{-i} \mathcal{D}$) est compact.

Décomposition cellulaire localement finie des espaces de Rapoport-Zink pour les groupes unitaires

Soit maintenant \mathcal{M} l'espace de Rapoport-Zink basique avec signature $(1, n-1)$ pour le groupe unitaire associé à une extension quadratique non ramifiée de \mathbb{Q}_p . C'est l'espace

basique du début de cette introduction. Nous essayons de prouver que \mathcal{M} admet une décomposition cellulaire localement fini. Mais il y a quelques difficultés pour généraliser la méthode ci-dessus dans ce cas. La première difficulté est que l'algorithme n'est pas bien compatible avec les correspondances de Hecke dans les cas avec des structures additionnelles. Plus précisément, si (H, ι, λ) est un groupe p -divisible avec structures additionnelles provenant d'un point $x \in \mathcal{M}$, l'action de Hecke sur x signifie que on prend les quotients pour sous-groupes totalement isotropes dans $H[p^m]$ pour certains m , avec les structures induites et modifie la quasi-isogénie. Bien que la filtration de Harder-Narasimhan soit compatible avec les structures additionnelles, nous ne pouvons pas garantir que le groupe \mathcal{F}_H produit par l'algorithme est un sous-groupe totalement isotrope. Notre idée est de continuer l'algorithme en quelque sorte (d'une façon non canonique) pour obtenir des sous-groupes totalement isotropes. Plus précisément, supposons que H n'est pas semi-stable et l'algorithme pour H s'arrête en temps fini, par exemple qui est le cas si $x \in \mathcal{M}^{rig}$. On a donc la suite

$$H = H_1 \xrightarrow{\phi_1} H_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{r-1}} H_r,$$

ϕ

avec H_1, \dots, H_{r-1} ne sont pas semi-stables, tandis que H_r est semi-stable. Notons N le plus petit entier tel que $\ker \phi \subset H[p^N]$ mais $\ker \phi \not\subset H[p^{N-1}]$. Alors on peut trouver la filtration de Harder-Narasimhan de $H[p^N]$ à partir de celle de $\ker \phi$ (qui peut être lue par l'algorithme). On a deux possibilités : $\ker \phi = (\ker \phi)^\perp$ ou $(\ker \phi)^\perp / \ker \phi$ est un groupe non nul semi-stable de pente $\frac{1}{2}$. Le premier cas est bon. Pour le deuxième cas, dans la section 2.5 de cette thèse, on va trouver des sous groupe totalement isotropes dans $(\ker \phi)^\perp / \ker \phi$ (à une extension finie de $\mathcal{H}(x)$ près). En prenant l'image réciproque, on trouve des sous groupe totalement isotropes dans $H[p^N]$. Motivés par cette construction, nous définissons un sous-espace $\mathcal{C} \subset \mathcal{M}$:

$\mathcal{C} = \{x \in \mathcal{M} \mid \exists H \text{ semi-stable et une isogénie } f : H_x \rightarrow H \text{ sur } O_K, K \mid \mathcal{H}(x) \text{ fini, telle que } p(\ker f) = 0\}$.

On peut prouver que c'est un domaine analytique fermé qui contient le lieu semi-stable \mathcal{M}^{ss} (cf. Proposition 2.6.2). On peut penser que \mathcal{C} est le lieu «presque semi-stable». Par construction, on a comme ensemble

$$\mathcal{M}^{rig} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot \mathcal{C}^{rig}.$$

Dans ce cas on a aussi une décomposition donnée par la hauteur de la quasi-isogénie universelle

$$\mathcal{M} = \coprod_{i \in \mathbb{Z}, \text{ in paire}} \mathcal{M}^i.$$

L'élément $p^{-1} \in J_b(\mathbb{Q}_p)$ agit sur \mathcal{M} de telle manière qu'il induit un isomorphisme $p^{-1} : \mathcal{M}^i \xrightarrow{\sim} \mathcal{M}^{i+2}$. Pour les groupes locaux de similitudes unitaires, nous avons $G \simeq J_b$ si n est impaire. Si n est paire, J_b est la forme intérieure non quasi-déployée de G . Dans ce dernier cas, on choisit un élément $g_1 \in J_b(\mathbb{Q}_p)$ tel que $v_p(\det g_1) = \frac{n}{2}$ pour la valuation v_p . Donc g_1 induit un isomorphisme $g_1 : \mathcal{M}^0 \rightarrow \mathcal{M}^1$. On pose $\mathcal{C}^i := \mathcal{C} \cap \mathcal{M}^i$, $\mathcal{C}' = \mathcal{C}^0$ si n est paire et $\mathcal{C}' = \mathcal{C}^0 \amalg \mathcal{C}^1$ si n est impaire. Alors

$$\mathcal{M}^{rig} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot (\mathcal{C}')^{rig}.$$

La deuxième difficulté dans notre cas unitaire est que, contrairement au cas d'espace de Rapoport-Zink basique pour GL_h/\mathbb{Q}_p avec signature $(d, h-d)$ et d et h sont premier entre eux, l'espace \mathcal{M}^0 est très large. En fait, on peut considérer le morphisme de spécialisation $sp : \mathcal{M}^0 \rightarrow \mathcal{M}_{red}^0$. L'espace \mathcal{M}_{red}^0 est quasi compact dans le cas sans structures additionnelles et $(d, h) = 1$, tandis que dans notre cas \mathcal{M}_{red}^0 admet une stratification indexée par les sommets dans l'immeuble $\mathcal{B}(J_b^{der}, \mathbb{Q}_p)$. D'après Volllaard et Wedhorn [81], à chaque sommet Λ , on peut associé un sous schéma fermé $\mathcal{M}_\Lambda \subset \mathcal{M}_{red}^0$, qui est projectif et lisse (une variété de Deligne-Lusztig généralisée). Les composantes irréductibles de \mathcal{M}_{red}^0 sont les \mathcal{M}_Λ avec $t(\Lambda)$ (la fonction de type) maximale. En outre, l'action de $J_b^{der}(\mathbb{Q}_p)$ sur ces sous schémas est compatible avec son action sur l'immeuble. Nous nous référons à [81] pour plus de détails concernant la géométrie de \mathcal{M}_{red}^0 . Les tubes au-dessus des composantes irréductibles nous donne un recouvrement localement fini par ouverts

$$\mathcal{M}^0 = \bigcup_{\Lambda, t(\Lambda)=t_{max}} sp^{-1}(\mathcal{M}_\Lambda).$$

On choisit un sommet Λ avec $t(\Lambda) = t_{max}$, et pose

$$\mathcal{D} = \mathcal{C}^0 \cap sp^{-1}(\mathcal{M}_\Lambda),$$

qui est un domaine analytique localement fermé.

Théorème 0.0.4. *Les notations sont comme ci-dessus. Le domaine analytique \mathcal{D} est relativement compact. De plus, on a recouvrements localement finis*

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} T.g\mathcal{D}$$

si n est impaire, et

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} T.g_1^j.g\mathcal{D}$$

si n est paire.

Le point clé est de prouver la finitude localement. C'est pourquoi nous avons besoin d'introduire \mathcal{D} , mais nous ne travaillons pas avec \mathcal{C}' . Dans la preuve nous verrons que nous pouvons élargir \mathcal{D} un peu pour obtenir un domaine fondamental compact.

Ce théorème admet des corollaires concernant les domaines de périodes p -adiques et variétés de Shimura.

Corollaire 0.0.5. *Soit $\pi : \mathcal{M} \rightarrow \mathcal{F}$ l'application période p -adique de Rapoport-Zink (cf. [66] chapter 5). Notons \mathcal{F}^a l'image de π , un ouvert de \mathcal{F} . Alors on a recouvrements localement finis*

$$\mathcal{F}^a = \bigcup_{g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)} g\pi(\mathcal{D})$$

si n est impaire, et

$$\mathcal{F}^a = \bigcup_{\substack{j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} g_1^j.g\pi(\mathcal{D})$$

si n est paire.

Corollaire 0.0.6. Soit $\widehat{Sh}_{K^p}^{b_0}$ le tube au-dessus la strate basique dans la variété de Shimura introduite au début. Écrivons l'uniformisation de Rapoport-Zink (cf. [66] chapitre 6)

$$\widehat{Sh}_{K^p}^{b_0} = I(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}_f^p)/K^p = \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p)/K^p} \Gamma_i \backslash \mathcal{M},$$

l'images de \mathcal{C}' dan $\Gamma_i \backslash \mathcal{M}$ comme \mathcal{E}'_i , et enfin $\mathcal{E}' = \coprod_i \mathcal{E}'_i$. Alors on a un recouvrement

$$\widehat{Sh}_{K^p}^{b_0} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)} T.\mathcal{E}'.$$

Décomposition cellulaire et formule de Lefschetz

Soient G et \mathcal{M} comme ci-dessous. Soient $K \subset G(\mathbb{Z}_p)$ un sous group ouvert compact, et \mathcal{D}_K l'image réciproque de \mathcal{D} sous la projection $\mathcal{M}_K \rightarrow \mathcal{M}$. On voit facilement que $KhK.\mathcal{D}_K$ ne dépend que l'image de KgK dans la projection $K \backslash G(\mathbb{Q}_p)/K \rightarrow G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K$. Donc dans le niveau K , on a les décomposition cellulaire

$$\mathcal{M}_K = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.g\mathcal{D}_K$$

si n impaire, et

$$\mathcal{M}_K = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.g_1^j g\mathcal{D}_K$$

si n est paire.

Pour mieux comprendre les actions des groupes sur les cellules, nous devons introduire un ensemble de paramètres plus naturel. Considérons

$$\mathcal{I}_K = (G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \times J_b(\mathbb{Q}_p)/Stab(\Lambda))/\mathbb{Q}_p^\times,$$

où le quotient par \mathbb{Q}_p^\times est par son action via le plongement $\mathbb{Q}_p^\times \rightarrow G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$, $z \mapsto (z, z^{-1})$. (En vertu de notre convention, (z, z^{-1}) agir trivialement sur \mathcal{M}_K .) Il y a une application

$$\begin{aligned} \varphi : \mathcal{I}_K &\longrightarrow \mathbb{Z} \\ [T, g'] &\mapsto -\frac{2}{n}(v_p(\det T) + v_p(\det g')), \end{aligned}$$

avec l'image \mathbb{Z} si n paire et $2\mathbb{Z}$ si n impaire. Notons $\mathcal{I}_K^i = \varphi^{-1}(i)$. Pour chaque $[T, g'] \in \mathcal{I}_K$, le cellule

$$\mathcal{D}_{[T, g'], K} := T.g'\mathcal{D}_K$$

est bien défini. Le théorème ci-dessus implique

$$\mathcal{M}_K = \bigcup_{[T, g'] \in \mathcal{I}_K} \mathcal{D}_{[T, g'], K},$$

$$\mathcal{M}_K^i = \bigcup_{[T,g'] \in \mathcal{I}_K^i} \mathcal{D}_{[T,g'],K}.$$

De plus, c'est sont des recouvrements localement finis. Plus important encore, il y a une métrique \bar{d} sur l'ensemble \mathcal{I}_K induite par la métrique sur le quotient par K de l'immeuble quotient $\mathcal{B} = \mathcal{B}(G \times J_b, \mathbb{Q}_p)/\mathbb{Q}_p^\times$. La finitude localement signifie que il existe un constant $c > 0$ tel que pour tout $[T, g'] \in \mathcal{I}_K$,

$$\{[T', g''] \in \mathcal{I}_K \mid \mathcal{D}_{[T',g''],K} \cap \mathcal{D}_{[T,g'],K} \neq \emptyset\} \subset \{[T', g''] \in \mathcal{I}_K \mid \bar{d}([T, g'], [T', g'']) \leq c\},$$

où l'ensemble dernier est fini. Si $\gamma = (h, g) \in G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ tel que $hKh^{-1} = K$, alors il agit sur \mathcal{I}_K par $[T, g'] \mapsto [Th, gg']$. L'action de γ sur \mathcal{M}_K est compatible avec son action sur \mathcal{I}_K :

$$\gamma(\mathcal{D}_{[T,g'],K}) = \mathcal{D}_{[Th,gg'],K}.$$

Si de plus $v_p(\text{deth}) + v_p(\text{det}g) = 0$, les actions de γ sur \mathcal{M}_K et \mathcal{I}_K stables \mathcal{M}_K^0 et \mathcal{I}_K^0 . Nous supposons encore de plus h, g sont semi-simples, réguliers, elliptiques. Sous ces hypothèse, en étudiant l'action de γ sur l'immeuble, nous pouvons trouver une chaîne croissante de sous-ensembles finis stable par γ , $(A_\rho)_\rho$, qui épuise l'ensemble \mathcal{I}_K^0 . On pose

$$U_\rho = \mathcal{M}_K^0 - \bigcup_{[T,g'] \in \mathcal{I}_K^0 - A_\rho} \mathcal{D}_{[T,g'],K}.$$

Grâce à la finitude localement du recouvrement et la compacité du \mathcal{D}_K , U_ρ est un ouvert relativement compact. En particulier, ses groupes de cohomologie sont de dimension finis.

Théorème 0.0.7. *Les notations sont comme ci-dessus. Il existe un nombre réel $\rho_0 > 0$ et un sous groupe ouvert compact K' de $G(\mathbb{Z}_p)$, telle que pour tout $\rho \geq \rho_0$ et tout sous groupe ouvert compact $K \subset K'$ normalisé par h , on a la formule suivante*

$$\text{Tr}(\gamma | H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \# \text{Fix}(\gamma | \mathcal{M}_K^0 \times \mathbb{C}_p).$$

Nous notons en particulier le côté droit est indépendante de ρ quand $\rho \gg 0$. La preuve est que par étudier l'action de γ sur les cellules on vérifie les conditions du théorème 3.13 de [60] détient. La méthode de la preuve fonctionne également pour le cas de Lubin-Tate, qui est plus simple. Ainsi, nous pouvons reprouver la formule de Lefschetz dans ce cas (cf. sous-section 3.3).

Si nous ne supposons que $\frac{2}{n}(v_p(\text{deth}) + v_p(\text{det}g))$ est paire dans le cas n paire. Alors l'ensemble des points fixés de γ sur $\mathcal{M}_K/p^\mathbb{Z}$ est non vide. Si $g \in J_b(\mathbb{Q}_p)$ est un élément régulier elliptique, pour tout $x \in \mathcal{F}^a(\mathbb{C}_p)$, on peut trouver un élément $h_{g,x} \in G(\mathbb{Q}_p)$ conjugué stablement à g par l'isomorphisme de comparaison. On a une bonne formule des points fixés.

Corollaire 0.0.8. *Après le choix de certaines des mesures de Haar appropriées, on a*

$$\text{Tr}(\gamma | H_c^*((\mathcal{M}_K/p^\mathbb{Z}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \sum_{x \in \text{Fix}(g | \mathcal{F}^a(\mathbb{C}_p))} \text{Vol}(G_{h_{g,x}}/p^\mathbb{Z}) O_{h_{g,x}} \left(\frac{1_{h^{-1}K}}{\text{Vol}(K)} \right).$$

Soit π une représentation supercuspidale de $G(\mathbb{Q}_p)$, on a un élément bien défini dans $\text{Groth}_{\overline{\mathbb{Q}}_l}(J_b(\mathbb{Q}_p))$:

$$H(\pi) = \sum_{j \geq 0} (-1)^j \text{Hom}_{G(\mathbb{Q}_p)} \left(\varinjlim_K H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi \right).$$

Corollaire 0.0.9. *Soit $g \in J_b(\mathbb{Q}_p)$ un élément semi-simple régulier elliptique. Supposons que π est de la forme $c\text{-Ind}_{K_\pi}^{G(\mathbb{Q}_p)} \lambda$, où K_π est un sous-groupe ouvert de $G(\mathbb{Q}_p)$ compact modulo le centre et λ est une représentation de dimension finie de K_π . On a une formule*

$$\text{tr}_{H(\pi)}(g) = \sum_{x \in \text{Fix}(g|_{\mathcal{F}^a(\mathbb{C}_p)})} \text{tr}_\pi(h_{g,x}).$$

Cette formule devrait confirmer la conjecture de Kottwitz, une fois les problèmes de classification des L -paquets pour les groupes des similitudes unitaires sont résolus.

1 Hodge-Newton filtration for p -divisible groups with additional structures

1.1 Introduction

The motivation of this section is to study the cohomology of some unitary group Shimura varieties, namely those introduced in [10],[81]. The fixed prime p is assumed to be inert in the quadratic field in the PEL data, so the local reductive groups at p are the quasi-split unitary similitude groups. The generic fibers of these Shimura varieties are the same with those of some special cases studied by Harris-Taylor in [36], where they proved the local Langlands correspondence for GL_n . The geometry of the special fibers of Harris-Taylor's Shimura varieties is simpler, since in fact one is reduced to the study of one dimensional p -divisible groups. For any non-basic Newton polygon strata, the one dimensional p -divisible group attached to any point in it admits the splitting local-étale exact sequence. This simple fact plus the theory of Katz-Mazur's "full set of sections" lead to the geometric fact that, any non-basic strata in Drinfeld levels is decomposed as some disjoint union of Igusa varieties of first kind defined there. Thus the cohomology of any non-basic strata can be written as a parabolic induction. This reduces the construction of local Langlands correspondence to the study of the basic strata and the corresponding Lubin-Tate spaces. There they got such a conclusion inspired by Boyer's trick in [9] originally for function fields case. Note in Harris-Taylor's case any non-basic Newton polygon has a nontrivial étale part, contained in the (generalized) Hodge polygon. For general PEL type Shimura varieties, there are also Newton polygon stratifications. Consider those Shimura varieties which satisfy the condition that, there is some non-basic Newton polygon admitting a nontrivial contact point with the Hodge polygon, and assume this contact point is a break point of the Newton polygon. Under this condition, we will wonder that, whether the cohomology of this non-basic strata contains no supercuspidal representations of the associated local reductive group, or even whether the cohomology is some parabolic induction. The methods of Harris-Taylor will hardly work, since in general one knows very little of the geometry of their special fibers in Drinfeld levels. In our cases above, one can draw the pictures of all Newton polygons as in [10], and find that any non-basic polygon has some nontrivial break contact points with the Hodge polygon.

In this section we will give a positive answer of the above consideration. The idea is proving the existence of a canonical filtration under the above condition, the so called "Hodge-Newton filtration"(see below), for p -divisible groups with additional structures, and then passing to their moduli-Rapoport-Zink spaces. This idea is due to Mantovan. In [56] Mantovan considered this question under the stronger condition that, the Newton polygon coincide with Hodge polygon up to or from on the nontrivial break contact point. Under this stronger condition, Mantovan and Viehmann had proven in [57] the existence of the Hodge-Newton filtration over characteristic 0 by lifting the corresponding one from characteristic p . Note both [57] and [56] restricted themselves in just the EL and PEL symplectic cases. In this section we generalize their results under the more natural condition as above, that is, the Newton polygon admits a nontrivial contact point with the Hodge polygon, and assume this contact point is a break point of the Newton polygon. We will consider also the PEL unitary case. In particular we can prove the desired results for the Shimura varieties in [10] and [81] mentioned above.

The notions of Hodge-Newton decomposition and Hodge-Newton filtration for F -crystals was first introduced by Katz in [46], where under the hypothesis that the Newton polygon and the Hodge polygon of a F -crystal possess a non-trivial contact point which is a break point of the Newton polygon, he proved that the F -crystal over a perfect field of characteristic p admits a decomposition, such that the two parts of the Newton (resp. Hodge) polygon divided by the point correspond to the Newton (resp. Hodge) polygon of the two sub- F -crystals. This can be viewed as a generalization of the multiplicative-bilocal-étale filtration for p -divisible groups. Katz also proved the existence of Hodge-Newton filtration for F -crystals over certain algebras over characteristic p .

In [57], Mantovan and Viehmann considered the case of F -crystals and p -divisible groups with actions of the integer ring of an unramified finite extension of \mathbb{Q}_p . They proved that under the stronger condition that the Newton and Hodge polygon coincide up to or from on the contact point, one can lift the Hodge-Newton filtration for p -divisible groups from characteristic p to characteristic 0. See theorem 10 of [57] for the precise statement. In [56], Mantovan used this result to prove that, the cohomology of the Rapoport-Zink spaces whose Newton and Hodge polygons satisfy this stronger condition contains no supercuspidal representation of the underlying reductive group defined by the local EL/PEL data.

We consider p -divisible groups with additional structures over complete valuation rings of rank one of mixed characteristic $(0, p)$, i.e. which are complete extensions of \mathbb{Z}_p for a valuation with values in \mathbb{R} . Here additional structures means an action of O_F , the integer ring of an unramified finite extension $F|\mathbb{Q}_p$, and a polarization compatible with this action. For precise definition see definition 1.3.1 in subsection 1.3. A typical such p -divisible group comes from a K -valued point of the simple unramified EL/PEL Rapoport-Zink spaces introduced in [23], chapter 2, where K is a complete extension of \mathbb{Q}_p for a rank one valuation. The first main result of this paper is the following. Here to simplify the exposition, we just state the theorem for the PEL cases.

Theorem 1.1.1. *Let $K|\mathbb{Q}_p$ be a complete discrete valuation field with residue field k perfect, (H, ι, λ) be a p -divisible group with additional structures over O_K . Assume the (HN) condition : the Newton polygon $\text{Newt}(H_k, \iota, \lambda)$ and Hodge polygon $\text{Hdg}(H_k, \iota, \lambda)$ possess a contact point x outside their extremal points, which is a break point for the polygon $\text{Newt}(H_k, \iota, \lambda)$. Denote by \hat{x} the symmetric point of x , and assume x lies before \hat{x} , i.e. the horizontal coordinate of x is smaller than that of \hat{x} . Then there are unique subgroups*

$$(H_1, \iota) \subset (H_2, \iota) \subset (H, \iota)$$

of (H, ι) as p -divisible groups with additional structures over O_K , such that

1. λ induces isomorphisms

$$(H_1, \iota) \simeq ((H/H_2)^D, \iota'),$$

$$(H_2, \iota) \simeq ((H/H_1)^D, \iota').$$

Here $(H/H_i)^D$ is the Cartier-Serre dual of H/H_i , and ι' is the action naturally induced by ι on $(H/H_i)^D$, for $i = 1, 2$;

2. the induced filtration of the p -divisible group with additional structures (H_k, ι) over k

$$(H_{1k}, \iota) \subset (H_{2k}, \iota) \subset (H_k, \iota)$$

is split;

3. the Newton (resp. Harder-Narasimhan, resp. Hodge) polygons of (H_1, ι) , $(H_2/H_1, \iota)$, and $(H/H_2, \iota)$ are the parts of the Newton (resp. Harder-Narasimhan, resp. Hodge) polygon of (H, ι, λ) up to x , between x and \hat{x} , and from \hat{x} on respectively.

When $x = \hat{x}$, then H_1 and H_2 coincide.

In fact, the above theorem holds for more general complete valuation rings of rank one (not necessary discrete) which are extensions of \mathbb{Z}_p , with some technical restriction to the so called “modular” p -divisible groups, see definition 1.5.6 and theorem 1.5.7 in section 1.5.

The proof of this theorem is quite different from that of Mantovan-Viehmann in [57]. Our ideas are that, firstly using explanation in terms of filtered isocrystals we get a Hodge-Newton filtration for p -divisible groups up to isogeny ; then using the theory of Harder-Narasimhan filtration for finite flat group schemes over O_K developed in [26],[27], we prove the existence and uniqueness of such a filtration for p -divisible groups. More precisely, we define Harder-Narasimhan polygons $HN(H, \iota, \lambda)$ for the p -divisible groups and finite flat group schemes with (PEL) additional structures studied here, by adapting the case without additional structures studied in loc. cit.. Then the crucial points are the following inequalities :

$$HN(H, \iota, \lambda) \leq Newt(H_k, \iota, \lambda) \leq Hdg(H_k, \iota, \lambda),$$

and

$$\begin{aligned} HN(H, \iota, \lambda) &\leq \frac{1}{m} HN(H[p^m], \iota, \lambda)(m \cdot) \leq HN(H[p], \iota, \lambda) \\ &\leq Hdg(H[p], \iota, \lambda) = Hdg(H_k, \iota, \lambda). \end{aligned}$$

Here $Newt(H_k, \iota, \lambda)$ and $Hdg(H_k, \iota, \lambda)$ are the Newton and Hodge polygons of the p -divisible group with additional structures (H_k, ι, λ) over k , and the inequality $Newt(H_k, \iota, \lambda) \leq Hdg(H_k, \iota, \lambda)$ is the generalized Mazur’s inequality, see [65]; $Hdg(H[p], \iota, \lambda)$ is the Hodge polygon for the finite flat group scheme $(H[p], \iota, \lambda)$ over O_K , defined in subsection 1.3 by adapting the Hodge polygon for the case without additional structures defined in [26], 8.2 to our situations. By using the explanation of the polygons for p -divisible groups in terms of the associated filtered isocrystals, from the above first line of inequalities we deduce that the Harder-Narasimhan polygon also passes the point \hat{x} , thus it is necessarily a break point for this polygon. Then by the second line of inequalities one can find a subgroup in the Harder-Narasimhan filtration of $H[p^n]$ for every n large enough. We will show these finite flat group schemes are compatible. Thus they define a p -divisible group H_2 , with its filtered isocrystal as the sub filtered isocrystal corresponding the point \hat{x} of that of H . Similarly there is a p -divisible group H_1 corresponding to the point x . One can then check that the statements in the theorem hold.

This theorem generalizes theorem 10 in [57]. With this generalization we can study the cohomology of some non-basic Rapoport-Zink spaces (\mathcal{M}_K) exactly as Mantovan did in [56], but here we can deal with a larger class of Rapoport-Zink spaces due to our weaker condition. Mantovan’s method is to introduce two other towers of moduli spaces $(\mathcal{P}_K), (\mathcal{F}_K)$ of p -divisible groups with additional structures, corresponding to the levi subgroup M and parabolic subgroup P respectively associated to the nontrivial break

contact point x . Here (\mathcal{P}_K) are in fact Rapoport-Zink spaces for the levi subgroup M , and (\mathcal{F}_K) are deformation spaces of filtered p -divisible group by quasi-isogenies analogous to Rapoport-Zink's definition. Fix a prime $l \neq p$. By studying some geometric aspects between the towers (\mathcal{P}_K) and (\mathcal{F}_K) , one finds

$$H(\mathcal{P}_\infty)_\rho = H(\mathcal{F}_\infty)_\rho,$$

where ρ is an admissible smooth $\overline{\mathbb{Q}}_l$ -representation of $J_b(\mathbb{Q}_p)$,

$$H(\mathcal{P}_\infty)_\rho = \sum_{i,j \geq 0} (-1)^{i+j} \varinjlim_K \text{Ext}_{J_b(\mathbb{Q}_p)}^j (H_c^i(\mathcal{P}_K \times \mathcal{C}_p, \overline{\mathbb{Q}}_l(D_{\mathcal{P}})), \rho)$$

($D_{\mathcal{P}}$ is the dimension of \mathcal{P}_K), and similarly one has $H(\mathcal{F}_\infty)_\rho, H(\mathcal{M}_\infty)_\rho$. Here the reason that we consider the formula of cohomology in this type is to apply Mantovan's formula in [55]. On the other hand, under our condition (HN), thanks to the existence of Hodge-Newton filtration one has

$$\mathcal{M}_K = \coprod_{K \setminus G(\mathbb{Q}_p)/P(\mathbb{Q}_p)} \mathcal{F}_{K \cap P(\mathbb{Q}_p)}.$$

This decomposition has the following application to monodromy representations.

In [15], Chen has constructed some determinant morphisms for the towers of simple unramified Rapoport-Zink spaces. Under the condition that there is no non-trivial contact point of the Newton and Hodge polygons, and assume the conjecture that

$$\pi_0(\widehat{\mathcal{M}}) \simeq \text{Im} \varkappa$$

for the morphism $\varkappa : \widehat{\mathcal{M}} \rightarrow \Delta$ constructed in [66] 3.52, Chen proved that the associated monodromy representation under this condition is maximal, and thus the geometric fibers of her determinant morphisms are exactly the geometric connected components, see théorème 5.1.2.1., and 5.1.3.1. of loc. cit..

Under our condition (HN), the existence of Hodge-Newton filtration implies also that, the monodromy representations associated to the local systems defined by Tate modules of p -divisible groups, factor through the parabolic subgroup.

Corollary 1.1.2. *Under the above notations, let \bar{x} be a geometric point of the Rapoport-Zink space \mathcal{M} , and \bar{y} be its image under the p -adic period morphism $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$. Then the monodromy representations*

$$\rho_{\bar{x}} : \pi_1(\mathcal{M}, \bar{x}) \longrightarrow G(\mathbb{Z}_p)$$

and

$$\rho_{\bar{y}} : \pi_1(\mathcal{F}^a, \bar{y}) \longrightarrow G(\mathbb{Q}_p)$$

factor through $P(\mathbb{Z}_p)$ and $P(\mathbb{Q}_p)$ respectively.

This confirms that the condition “there is no non-trivial contact point of the Newton and Hodge polygons” in the chapter 5 of [15] is necessary, see the remark in 5.1.5 of loc. cit..

We have the following theorem considering the cohomological application.

Theorem 1.1.3. *Assume the Newton polygon and the Hodge polygon associated to the simple unramified EL/PEL Rapoport-Zink space \mathcal{M} possess a contact point x outside their extremal points which is a break point for the Newton polygon. Then we have equality of virtual representations of $G(\mathbb{Q}_p) \times W_E$:*

$$H(\mathcal{M}_\infty)_\rho = \text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H(\mathcal{P}_\infty)_\rho.$$

In particular, there is no supercuspidal representations of $G(\mathbb{Q}_p)$ appear in the virtual representation $H(\mathcal{M}_\infty)_\rho$.

The proof is by adapting the corresponding construction and strategy in [56] to our cases. In particular, by combining with the main formula of the cohomology of Newton stratas of PEL-type Shimura varieties in [55], we have the following corollary.

Corollary 1.1.4. *For the Shimura varieties studied by [10],[81], the cohomology of any non-basic strata $H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l))$ can be written as some suitable parabolic induction of virtual representation of some parabolic subgroup of $G(\mathbb{Q}_p)$, thus it contains no supercuspidal representations of $G(\mathbb{Q}_p)$.*

This confirms Harris's conjecture 5.2 in [34] in our case, although the parabolic subgroup may be not the same as that defined in loc. cit. for the non-basic strata. Note these Shimura varieties are out of the cases studied in [56], corollary 42. On the other hand, as said above, the similar conclusion for the Shimura varieties with the same generic fibers as that studied in [10],[81], but with p splits in the quadratic field, whose non-basic stratas satisfy the stronger condition in [56], was obtained previously by Harris-Taylor in [36]. There the Hodge-Newton filtration is just the local-étale filtration of the p -divisible groups. The conclusion in the above corollary that any non-basic strata contains no supercuspidal representations, was once obtained by Fargues in [23] by using more complicated Lefschetz trace formula methods, initially proposed by Harris in [34]. Here our result is more precise and the proof is more natural. With a recent preprint [47], we remark that we can also apply our main result to the study of cohomology of non-proper Shimura varieties over characteristic 0, and to the geometric realization of local Langlands correspondences.

1.2 Harder-Narasimhan filtration of finite flat group schemes I

In this subsection we recall briefly the theory of Harder-Narasimhan filtration of finite flat group schemes which is presented in detail in [26], but see also [27].

Let $K|\mathbb{Q}_p$ be a complete rank one valuation field extension, O_K be the integer ring of K , and \mathcal{C} be the exact category of commutative finite flat group schemes with order some power of p over $\text{Spec}O_K$. For $G \in \mathcal{C}$, recall there is an operation of scheme theoretic closure which is the inverse of taking generic fibers, and which induces a bijection between the following two finite sets

$$\{\text{closed subgroups of } G_K\} \xrightarrow{\sim} \{\text{finite flat subgroups of } G \text{ over } O_K\}.$$

There are two additive functions

$$ht : \mathcal{C} \rightarrow \mathbb{N}$$

$$\deg : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0},$$

where htG is the height of $G \in \mathcal{C}$, and $\deg G$ is the valuation of the discriminant of G which is defined as

$$\deg G = \sum a_i, \text{ if } \omega_G = \bigoplus O_K/p^{a_i}O_K.$$

We will use the following properties of the function \deg .

Proposition 1.2.1 ([26], Corollaire 3). *1. Let $f : G \rightarrow G'$ be a morphism of finite flat group schemes over O_K such that it induces an isomorphism on their generic fibers. Then we have*

$$\deg G \leq \deg G'.$$

Moreover, f is an isomorphism if and only if $\deg G = \deg G'$.

2. If

$$0 \rightarrow G' \xrightarrow{u} G \xrightarrow{v} G''$$

is a sequence of finite flat group schemes, such that u is a closed immersion, $v \circ u = 0$, and the induced morphism $G/G' \rightarrow G''$ is an isomorphism on their generic fibers. Then we have

$$\deg G \leq \deg G' + \deg G'',$$

with the equality holds if and only if v is a fppf epimorphism, i.e. flat. In this case we have an exact sequence

$$0 \rightarrow G' \xrightarrow{u} G \xrightarrow{v} G'' \rightarrow 0.$$

See loc. cit. section 3 for more properties of the function \deg .

For a group scheme $0 \neq G \in \mathcal{C}$, we set

$$\mu(G) := \frac{\deg G}{htG},$$

and call it the slope of G . The basic properties of the slope function are as follow.

- One always has $\mu(G) \in [0, 1]$, with $\mu(G) = 0$ if and only if G is étale and $\mu(G) = 1$ if and only if G is multiplicative.
- If G^D is the Cartier dual of G then $\mu(G^D) = 1 - \mu(G)$.
- For a p -divisible group H of dimension d and height h over O_K , then for all $n \geq 1$ one has $\mu(H[p^n]) = \frac{d}{h}$.
- If

$$0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$$

is an exact sequence of non trivial groups in \mathcal{C} , then we have $\inf\{\mu(G'), \mu(G'')\} \leq \mu(G) \leq \sup\{\mu(G'), \mu(G'')\}$, and if $\mu(G') \neq \mu(G'')$ we have in fact $\inf\{\mu(G'), \mu(G'')\} < \mu(G) < \sup\{\mu(G'), \mu(G'')\}$.

- If $f : G \rightarrow G'$ is a morphism which induces an isomorphism between their generic fibers, then we have $\mu(G) \leq \mu(G')$, with equality holds if and only if f is an isomorphism.

– If

$$0 \longrightarrow G' \xrightarrow{u} G \xrightarrow{v} G''$$

is a sequence of non trivial groups such that u is a closed immersion, $u \circ v = 0$, and the morphism induced by v

$$G/G' \rightarrow G''$$

is an isomorphism in generic fibers, then we have

- $\mu(G) \leq \sup\{\mu(G'), \mu(G'')\}$;
- if $\mu(G') \neq \mu(G'')$ then $\mu(G) < \sup\{\mu(G'), \mu(G'')\}$;
- if $\mu(G) = \sup\{\mu(G'), \mu(G'')\}$ then $\mu(G) = \mu(G') = \mu(G'')$ and the sequence $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ is exact.

For a group $0 \neq G \in \mathcal{C}$, we call G semi-stable if for all $0 \subsetneq G' \subset G$ we have $\mu(G') \leq \mu(G)$. In [26], Fargues proved the following theorem.

Theorem 1.2.2 ([26], Théorème 2). *There exists a Harder-Narasimhan type filtration for all $0 \neq G \in \mathcal{C}$, that is a chain of finite flat subgroups in \mathcal{C}*

$$0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = G,$$

with the group schemes G_{i+1}/G_i are semi-stable for all $i = 0, \dots, r-1$, and

$$\mu(G_1/G_0) > \mu(G_2/G_1) > \cdots > \mu(G_r/G_{r-1}).$$

Such a filtration is then uniquely characterized by these properties.

So G is semi-stable if and only if its Harder-Narasimhan filtration is $0 \subsetneq G$. We can define a concave polygon $HN(G)$ of any $0 \neq G \in \mathcal{C}$ by its Harder-Narasimhan filtration, and call it the Harder-Narasimhan polygon of G . It is defined as function

$$HN(G) : [0, htG] \rightarrow [0, degG],$$

such that

$$HN(G)(x) = degG_i + \mu(G_{i+1}/G_i)(x - htG_i)$$

if $x \in [htG_i, htG_{i+1}]$. We will also identify $HN(G)$ with its graph, that is a polygon with starting point $(0, 0)$, terminal point $(htG, degG)$, and over each interval $[htG_i, htG_{i+1}]$ it is a line of slope $\mu(G_{i+1}/G_i)$. An important property of this polygon is that (proposition 7 in loc. cit.), for all finite flat subgroups $G' \subset G$, the point $(htG', degG')$ is on or below the polygon $HN(G)$, that is $HN(G)$ is the concave envelop of the points $(htG', degG')$ for all $G' \subset G$. We denote by $\mu_{max}(G)$ the maximal slope of $HN(G)$, and $\mu_{min}(G)$ the minimal slope of $HN(G)$.

We recall some useful facts.

Proposition 1.2.3 ([26], Proposition 8). *Let G_1 and G_2 be two finite flat group schemes over O_K . Suppose that $\mu_{min}(G_1) > \mu_{max}(G_2)$. Then we have*

$$Hom(G_1, G_2) = 0.$$

Proposition 1.2.4 ([26], Proposition 10). *Let $0 \rightarrow G' \rightarrow G \xrightarrow{v} G'' \rightarrow 0$ be an exact sequence of finite flat group schemes in \mathcal{C} . Suppose $\mu_{\min}(G') \geq \mu_{\max}(G'')$. If $0 = G'_0 \subsetneq G'_1 \subsetneq \cdots \subsetneq G'_{r'} = G'$ is the Harder-Narasimhan filtration of G' and $0 = G''_0 \subsetneq G''_1 \subsetneq \cdots \subsetneq G''_{r''} = G''$ that of G'' , then the Harder-Narasimhan filtration of G is*

$$0 = G'_0 \subsetneq G'_1 \subsetneq \cdots \subsetneq G'_{r'} = G' \subsetneq v^{-1}(G''_0) \subsetneq v^{-1}(G''_1) \subsetneq \cdots \subsetneq v^{-1}(G''_{r''}) = G$$

if $\mu_{\min}(G') > \mu_{\max}(G'')$, and

$$0 = G'_0 \subsetneq G'_1 \subsetneq \cdots \subsetneq G'_{r'-1} \subsetneq v^{-1}(G''_1) \subsetneq \cdots \subsetneq v^{-1}(G''_{r''}) = G$$

if $\mu_{\min}(G') = \mu_{\max}(G'')$. In particular the extension of two semi-stable groups of the same slope μ is semi-stable of slope μ .

The Harder-Narasimhan filtration of finite flat group schemes is compatible with additional structures. Firstly, the Harder-Narasimhan filtration of $0 \neq G \in \mathcal{C}$ is stable under $\text{End}(G)$. So if $\iota : R \rightarrow \text{End}(G)$ is some action of an O_K -algebra R , then every subgroup G_i in the Harder-Narasimhan filtration of G is a R -subgroup via ι . Secondly, if the Harder-Narasimhan filtration of G is

$$0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = G$$

with slopes $\mu_1 > \cdots > \mu_r$, then the Harder-Narasimhan filtration of the Cartier dual G^D of G is

$$0 = (G/G_r)^D \subsetneq (G/G_{r-1})^D \subsetneq \cdots \subsetneq (G/G_1)^D \subsetneq G^D$$

with slopes $1 - \mu_r > \cdots > 1 - \mu_1$. In particular, if $\lambda : G \xrightarrow{\sim} G^D$ is a polarization, then it induces isomorphisms

$$G_i \simeq (G/G_{r-i})^D, i = 1, \dots, r$$

and thus $\mu_i + \mu_{r+1-i} = 1, i = 1, \dots, r$.

1.3 Polygons and inequalities

We start by defining the reductive groups which we will work with. They are defined by the simple unramified EL/PEL data for defining some special Rapoport-Zink spaces as the chapter 2 of [23].

More precisely, let $F|\mathbb{Q}_p$ be a finite unramified extension of degree d , V be a finite dimensional F -vector space. In the EL case, let $G = \text{Res}_{F|\mathbb{Q}_p} GL(V)$, the Weil scalar restriction of the automorphism group of V as a F -vector space. In the PEL symplectic case, we assume further there is a hermitian symplectic pairing $\langle, \rangle : V \times V \rightarrow \mathbb{Q}_p$, which is such that there exists an autodual lattice Λ for \langle, \rangle in V . In the PEL unitary case, besides the above \langle, \rangle and Λ , we assume there is a non trivial involution $*$ on F , compatible with \langle, \rangle , which means that $\langle bu, v \rangle = \langle u, b^*v \rangle$ for all $b \in F, u, v \in V$. We define a reductive group G over \mathbb{Q}_p for these PEL cases, such that for all \mathbb{Q}_p -algebra R ,

$$G(R) = \{g \in \text{End}_{F \otimes R}(V_R) | gg^\# \in R^\times\},$$

here $\#$ is the involution on $\text{End}_F(V)$ induced by \langle, \rangle . Then we have $G \subset \text{Res}_{F|\mathbb{Q}_p} GSp(V, \langle, \rangle')$ (symplectic case) or $G \subset \text{Res}_{F_0|\mathbb{Q}_p} GU(V, \langle, \rangle')$ (unitary case), where

$F_0 = F^{*=1}$ and $\langle, \rangle' : V \times V \rightarrow F^{*=1}$ is a suitable pairing coming from \langle, \rangle in the PEL cases ($*$ may be trivial). The rational Tate module of a p -divisible group with additional structures (see definition 1.3.1 in the following) will naturally give rise such an EL data (F, V) or a PEL data $(F, *, V, \langle, \rangle)$ ($*$ may be trivial). Note these reductive groups G are unramified over \mathbb{Q}_p , and for the PEL cases there is a similitude morphism $c : G \rightarrow \mathbb{G}_m, g \mapsto gg^\#$.

Let \mathbb{D} be the pro-algebraic torus with character group \mathbb{Q} . We will be interested in the set

$$N(G) = (\text{Int}G(L) \setminus \text{Hom}_L(\mathbb{D}, G))^{(\sigma)},$$

where $L = \text{Frac}W(\overline{\mathbb{F}}_p)$, σ is the Frobenius of L over \mathbb{Q}_p . This set generalizes the classical notation of Newton polygon associated to an F -isocrystal, see the introduction of [12] or [65], section 1. Since the group G is unramified, we can choose a maximal torus T contained in a Borel subgroup B of G defined over \mathbb{Q}_p . Let $A \subset T$ be the maximal splitting torus contained in T , W (resp. W_0) be the absolute (resp. relative) Weyl group, then we have

$$\begin{aligned} N(G) &= (X_*(T)_{\mathbb{Q}}/W)^{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \\ &= (\overline{C} \cap X_*(T)_{\mathbb{Q}})^{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \\ &= X_*(A)_{\mathbb{Q}}/W_0 \\ &=: \overline{C}_{\mathbb{Q}}, \end{aligned}$$

where $\overline{C} \subset X_*(T)_{\mathbb{R}}$ is the Weyl chamber associated to B . Recall there is an order in $N(G)$, cf. [65] section 2, such that for all $x, x' \in \overline{C}_{\mathbb{Q}}$,

$$x \leq x' \Leftrightarrow x' - x = \sum_{\alpha \in \Delta_B} n_{\alpha} \alpha^{\vee}, n_{\alpha} \in \mathbb{Q}_{\geq 0}.$$

Here Δ_B denotes the set of simple roots determined by B , α^{\vee} denotes the co-root corresponding to α . Note $N(\cdot)$ is in fact an ordered set-valued functor on the category of connected reductive algebraic groups.

We want to make the elements in the above cone $N(G)$ “visible”, i.e. as polygons defined over some suitable interval. Let $n = \dim_F V$, then in the EL case after choosing a base of the F -vector space V , we have $\text{Res}_{F|\mathbb{Q}_p} GL(V) = \text{Res}_{F|\mathbb{Q}_p} GL_n$, and for this case we can explicitly calculate

$$N(G) = \{(x_i) \in \mathbb{Q}^n | x_1 \geq x_2 \geq \dots \geq x_n\} := \mathbb{Q}_+^n,$$

which we will identify with the set of concave polygons with rational slopes over the interval $[0, n]$, see [23] 2.1. For PEL symplectic case,

$$G \subset \text{Res}_{F|\mathbb{Q}_p} GSp(V, \langle, \rangle') \subset \text{Res}_{F|\mathbb{Q}_p} GL_n,$$

since $N(\cdot)$ is a functor on the category of connected reductive algebraic groups, we have an order preserving map

$$N(G) \longrightarrow N(\text{Res}_{F|\mathbb{Q}_p} GL_n),$$

which is injective, and the image corresponds to symmetric polygons, cf. [82]. For PEL unitary case,

$$G \subset \text{Res}_{F_0|\mathbb{Q}_p} GU(V, \langle, \rangle') \subset \text{Res}_{F_0|\mathbb{Q}_p} \text{Res}_{F|F_0} GL(V) = \text{Res}_{F|\mathbb{Q}_p} GL_n,$$

similarly we have an order preserving map

$$N(G) \longrightarrow N(\mathrm{Res}_{F|\mathbb{Q}_p} GL_n),$$

which is also injective, and the image also corresponds to symmetric polygons, cf. loc. cit..

After the preliminary on the reductive groups G , we define p -divisible groups with additional structures. Since p -divisible groups are closely related to finite flat group schemes, we will also consider the related notions for these group schemes.

Definition 1.3.1. *Let S be a formal scheme and $F|\mathbb{Q}_p$ be a finite unramified extension.*

By a p -divisible group with additional structures over S , we mean

- *in the EL case, a pair (H, ι) , where H is a p -divisible group over S , and $\iota : O_F \rightarrow \mathrm{End}(H)$ is a homomorphism of algebras ;*
- *in the PEL symplectic case, a triplet (H, ι, λ) , where H is a p -divisible group over S , $\iota : O_F \rightarrow \mathrm{End}(H)$ is homomorphism of algebras, $\lambda : (H, \iota) \rightarrow (H^D, \iota^D)$ is a polarization, i.e. an O_F -equivariant isomorphism of p -divisible groups. Here H^D is the Cartier-Serre dual of the p -divisible group H , $\iota^D : O_F \rightarrow \mathrm{End}(H^D) = \mathrm{End}(H)^{opp}$ is induced by ι , such that $\lambda^D = -\lambda$, under the identification $H = H^{DD}$;*
- *in the PEL unitary case, a triplet (H, ι, λ) , where H, ι is similar as the symplectic case, $\lambda : (H, \iota) \rightarrow (H^D, \iota^D \circ *)$ is a polarization, $*$ is a nontrivial involution on F . Here ι^D is as above, but such that $\lambda^D = \lambda$, under the identification $H = H^{DD}$.*

Similarly, one can define finite locally free (=flat, in the case S is noetherian or the spec of a local ring) group schemes with additional structures in the same way.

If (H, ι, λ) is a p -divisible group with additional structures in the PEL cases, then for all $n \geq 1$, $(H[p^n], \iota, \lambda)$ is a finite locally free group scheme with the naturally induced additional structures. Similar remark holds for the EL case.

In the rest of this subsection, let $K|\mathbb{Q}_p$ be a complete field extension for a rank one valuation, O_K be the ring of integers of K , k be the residue field, and $F|\mathbb{Q}_p$ be a finite unramified extension of degree d . For the PEL (unitary) case we also assume there is an involution $*$ on F . We shall mostly be interested only in p -divisible groups and finite flat group schemes with additional structures for $F|\mathbb{Q}_p$ over O_K , k , and \bar{k} , a fixed algebraic closure of k . Let \underline{H}/O_K denote a p -divisible group with additional structures over O_K for the EL case ($\underline{H} = (H, \iota)$) or PEL cases ($\underline{H} = (H, \iota, \lambda)$), then \underline{H}_k (resp. $\underline{H}_{\bar{k}}$) is a p -divisible group with additional structures over k (resp. \bar{k}). Kottwitz defined the Newton polygon $\mathrm{Newt}(\underline{H}_{\bar{k}})$ and the Hodge polygon $\mathrm{Hdg}(\underline{H}_{\bar{k}})$ as elements in $N(G)$, see [49],[52]. Here the reductive group G is defined by the rational Tate module of H with the induced additional structures as in the beginning of this section. Assume $htH = dn$, then via the injection

$$N(G) \hookrightarrow N(\mathrm{Res}_{F|\mathbb{Q}_p} GL_n),$$

one can explain them as polygons as following :

$$\begin{aligned} \mathrm{Newt}(\underline{H}_{\bar{k}}) : [0, n] &\longrightarrow [0, \dim H/d] \\ x &\mapsto \frac{1}{d} \mathrm{Newt}(H_{\bar{k}})(dx). \end{aligned}$$

Here $\mathrm{Newt}(H_{\bar{k}})$ is the concave Newton polygon of $H_{\bar{k}}$ defined by the Dieudonné-Manin

decomposition of its isocrystal. The Hodge polygon is

$$Hdg(\underline{H}_{\bar{k}}) = \frac{1}{d} \sum_{i \in \mathbb{Z}/d\mathbb{Z}} Hdg_i(\underline{H}_{\bar{k}}),$$

where

$$Hdg_i(\underline{H}_{\bar{k}}) : [0, n] \rightarrow [0, \dim H/d]$$

is the polygon defined by the relative position of $(M_i, V(M_{i+1}))$ in $M_{i\mathbb{Q}}$. Here M is the covariant Dieudonné module, V is the Verschiebung. Under the action of O_F , we have

$$M = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} M_i, M_i = \{m \in M \mid a \cdot m = \sigma^i(a)m, \forall a \in O_F\}, \forall i \in \mathbb{Z}/d\mathbb{Z}.$$

Note $Gal(F/\mathbb{Q}_p) = \{\sigma^i \mid i \in \mathbb{Z}/d\mathbb{Z}\}$. One can check that these two polygons don't depend on the choice of the algebraic closure. Thus we can define the Newton (resp. Hodge) polygon of \underline{H}_k by $Newt(\underline{H}_k) := Newt(\underline{H}_{\bar{k}})$ (resp. $Hdg(\underline{H}_k) := Hdg(\underline{H}_{\bar{k}})$).

Let \underline{H}/O_K be a p -divisible group with additional structures as above. We are going to define the Harder-Narasimhan polygon of \underline{H}/O_K , and compare this polygon with the above polygons. We first consider the case of finite flat group schemes with additional structures with order some power of p . We now use \underline{H} to denote such a finite flat group scheme with additional structures. Recall the underlying finite flat group scheme H/O_K admits a unique Harder-Narasimhan filtration. Let

$$HN(H) : [0, htH] \rightarrow [0, degH]$$

be the concave polygon associated to this filtration.

Definition 1.3.2. *Consider*

$$\begin{aligned} HN(H, \iota) : [0, htH/d] &\longrightarrow [0, degH/d] \\ x &\mapsto \frac{1}{d} HN(H)(dx) \end{aligned}$$

in the EL case, and $HN(H, \iota, \lambda) := HN(H, \iota)$ in the PEL cases, which is symmetric. $HN(H, \iota)$ and $HN(H, \iota, \lambda)$ are called the Harder-Narasimhan polygons of the finite flat group schemes \underline{H}/O_K with additional structures.

Now we define the Harder-Narasimhan polygon for p -divisible groups with additional structures. To be more concrete on notations, assume we are in the PEL cases, although all the following works for the EL case, which is simpler. Let $(H, \iota, \lambda)/O_K$ be a p -divisible group with additional structures, then we get a family of finite flat group schemes with additional structures $(H[p^m], \iota, \lambda)/O_K$.

Proposition 1.3.3. *The sequence of functions*

$$\begin{aligned} [0, htH/d] &\rightarrow [0, \dim H/d] \\ x &\mapsto \frac{1}{m} HN(H[p^m], \iota, \lambda)(mx) \end{aligned}$$

uniformly converge when $m \rightarrow \infty$ to a concave continuous ascending function

$$HN(H, \iota, \lambda) : [0, htH/d] \rightarrow [0, \dim H/d],$$

which is equal to

$$\inf_{m \geq 1} \frac{1}{m} HN(H[p^m], \iota, \lambda)(mx)$$

and moreover, $HN(H, \iota, \lambda)(0) = 0$, $HN(H, \iota, \lambda)(htH/d) = \dim H/d$.

Proof. Essentially the same with the proof of théorème 2 in [27], or one can easily deduce this from the results there, since by definition $HN(H[p^m], \iota, \lambda) = \frac{1}{d} HN(H[p^m], \iota, \lambda)(d \cdot)$. \square

Definition 1.3.4. We call the function $HN(H, \iota, \lambda)$ or its graph which we denote by the same symbol, the Harder-Narasimhan polygon of the p -divisible group with PEL additional structures (H, ι, λ) over O_K . Similarly we can define the Harder-Narasimhan polygon of p -divisible groups with EL structures.

In fact, we have the following inequality which can be also easily deduced from the corresponding result in [27] : for all $i \geq 1$, $x \in [0, i \frac{htH}{d}]$, and all $m \geq 1$,

$$\frac{1}{m} HN(H[p^{im}], \iota, \lambda)(mx) \leq HN(H[p^i], \iota, \lambda)(x).$$

For the p -divisible group with additional structures $(H_{\bar{k}}, \iota, \lambda)$ over \bar{k} , we have the Newton and Hodge polygons $Newt(H_{\bar{k}}, \iota, \lambda)$, $Hdg(H_{\bar{k}}, \iota, \lambda)$ respectively defined by Kottwitz. By Rapoport-Richartz, we have the generalized Mazur's inequality (see [65]) :

$$Newt(H_{\bar{k}}, \iota, \lambda) \leq Hdg(H_{\bar{k}}, \iota, \lambda).$$

We assume H/O_K is a “modular” p -divisible group in the sense of définition 25 in [27], see also definition 1.5.6 in section 1.5. The following theorem is one of the main theorems of [27], see also the introduction of section 2.

Theorem 1.3.5 ([27], Théorème 21). *Let K be as above and H/O_K be a p -divisible group over O_K . When the valuation ring O_K is not discrete we assume that H/O_K is “modular”, we have the following inequality*

$$HN(H) \leq Newt(H_k).$$

Thus one gets from their definitions the following generalization.

Proposition 1.3.6. *Let (H, ι, λ) be a p -divisible group with additional structures over O_K . When O_K is not discrete we assume that H/O_K is “modular”. Then we have the following inequality*

$$HN(H, \iota, \lambda) \leq Newt(H_k, \iota, \lambda).$$

Combined with the generalized Mazur's inequality we get the following corollary.

Corollary 1.3.7. *Under the above assumption, we have inequalities*

$$HN(H, \iota, \lambda) \leq Newt(H_k, \iota, \lambda) \leq Hdg(H_k, \iota, \lambda).$$

Now we return to the case of finite flat group schemes with additional structures (H, ι) or (H, ι, λ) . Fargues defined the Hodge polygon of H from ω_H , see [26] 8.2. We would like to generalize his definition to define a Hodge polygon of \underline{H} , which contains the information of the additional structures. First, assume F admits an imbedding in K , then we have also the decomposition

$$\omega_H = \bigoplus_{\tau: F \hookrightarrow K} \omega_{H, \tau}, \quad \omega_{H, \tau} = \{m \in \omega_H \mid a \cdot m = \tau(a)m, \forall a \in O_F\}, \forall \tau : F \hookrightarrow K.$$

Definition 1.3.8. *Under the above assumption,*

1.

$$Hdg_\tau : [0, htH/d] \rightarrow [0, degH/d]$$

is the polygon such that

$$Hdg_\tau(i) = deg(\omega_{H, \tau}) - \nu(Fitt_i \omega_{H, \tau}), 0 \leq i \leq htH/d,$$

where ν is the valuation on K such that $\nu(p) = 1$ and $Fitt_i M$ means the i -th Fitting ideal of an O_K -module M . In particular, $degM = \nu(Fitt_0 M)$.

2. In the EL case, the Hodge polygon of (H, ι) is

$$Hdg(H, \iota) = \frac{1}{d} \sum_{\tau: F \hookrightarrow K} Hdg_\tau : [0, htH/d] \rightarrow [0, degH/d].$$

In the PEL cases, the Hodge polygon of (H, ι, λ) is $Hdg(H, \iota, \lambda) := Hdg(H, \iota)$, which is then symmetric.

For the general case, choose a complete field $K' \supset K$, such that $F \hookrightarrow K'$, we define $Hdg(\underline{H}) := Hdg(\underline{H}_{O_{K'}})$. One can check this definition doesn't depend on the choice of K' .

Remark 1.3.9. *The above definition is compatible with the Hodge polygon defined by Kottwitz, in the sense that if \underline{H}/O_K is a p -divisible group with additional structures, then $Hdg(\underline{H}[p]) = Hdg(\underline{H}_k)$.*

Proposition 1.3.10. *Let \underline{H}/O_K be a finite flat group scheme with additional structures of order a power of p . Assume there is a p -divisible group with additional structures $\underline{\mathcal{G}}/O_K$ such that \underline{H} admits an imbedding $\underline{H} \hookrightarrow \underline{\mathcal{G}}$. Then we have*

$$HN(\underline{H}) \leq Hdg(\underline{H}).$$

Proof. We may assume we are in the EL case : $\underline{\mathcal{G}} = (\mathcal{G}, \iota)$, $\underline{H} = (H, \iota)$. We may also assume $F \hookrightarrow K$. Then $\forall \tau : F \hookrightarrow K$, we have an exact sequence of covariant Dieudonné modules :

$$0 \rightarrow \mathbb{D}(\mathcal{G}_{\bar{k}})_\tau \rightarrow \mathbb{D}((\mathcal{G}/H)_{\bar{k}})_\tau \rightarrow \mathbb{D}(H_{\bar{k}})_\tau \rightarrow 0.$$

To simplify notation, let $M = \mathbb{D}(\mathcal{G}_{\bar{k}})_\tau$, $M' = \mathbb{D}((\mathcal{G}/H)_{\bar{k}})_\tau$. Then the length of the $W(\bar{k})$ -module $\mathbb{D}(H_{\bar{k}})_\tau$ is the index $[M'_\tau : M_\tau]$ of M_τ in M'_τ . Consider the Lie algebras $Lie(\mathcal{G}_{\bar{k}})$ and $Lie((\mathcal{G}/H)_{\bar{k}})$ of $\mathcal{G}_{\bar{k}}$ and $(\mathcal{G}/H)_{\bar{k}}$ respectively, then we have also the decompositions induced by the O_F action :

$$Lie(\mathcal{G}_{\bar{k}}) = \bigoplus_{\tau} Lie(\mathcal{G}_{\bar{k}})_\tau, \quad Lie((\mathcal{G}/H)_{\bar{k}}) = \bigoplus_{\tau} Lie((\mathcal{G}/H)_{\bar{k}})_\tau.$$

By Diedonné theory we get the identities

$$\text{Lie}(\mathcal{G}_{\bar{k}})_{\tau} = M_{\tau}/VM_{\sigma\tau}, \text{Lie}((\mathcal{G}/H)_{\bar{k}})_{\tau} = M'_{\tau}/VM'_{\sigma\tau}.$$

Here σ is the Frobenius for the extension $F|\mathbb{Q}_p$, and V is the Verschiebung on M . Then the isogeny $\mathcal{G} \rightarrow \mathcal{G}/H$ induces

$$\text{rank}(\text{Lie}(\mathcal{G}_{\bar{k}})_{\tau}) = \text{rank}(\text{Lie}((\mathcal{G}/H)_{\bar{k}})_{\tau}),$$

which can be rewritten as

$$[M_{\tau} : VM_{\sigma\tau}] = [M'_{\tau} : VM'_{\sigma\tau}].$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_{\tau} & \longrightarrow & M'_{\tau} & \longrightarrow & \mathbb{D}(H_{\bar{k}})_{\tau} \longrightarrow 0 \\ & & \uparrow V & & \uparrow V & & \uparrow V \\ 0 & \longrightarrow & M_{\sigma\tau} & \longrightarrow & M'_{\sigma\tau} & \longrightarrow & \mathbb{D}(H_{\bar{k}})_{\sigma\tau} \longrightarrow 0, \end{array}$$

then

$$\begin{aligned} [M'_{\tau} : VM_{\sigma\tau}] &= [M'_{\tau} : M_{\tau}][M_{\tau} : VM_{\sigma\tau}] \\ &= [M'_{\tau} : VM'_{\sigma\tau}][VM'_{\sigma\tau} : VM_{\sigma\tau}]. \end{aligned}$$

Thus

$$[M'_{\tau} : M_{\tau}] = [VM'_{\sigma\tau} : VM_{\sigma\tau}] = [M'_{\sigma\tau} : M_{\sigma\tau}],$$

i.e. the lengths of these $W(\bar{k})$ -modules $\mathbb{D}(H_{\bar{k}})_{\tau}$ for any τ are the same. We can then conclude that $\mathbb{D}(H_{\bar{k}})_{\tau}$ is generated by $ht_{O_F}H = htH/d$ elements. So

$$\omega_{H_{\bar{k},\tau} = (\mathbb{D}(H_{\bar{k}})_{\tau}/V\mathbb{D}(H_{\bar{k}})_{\sigma\tau})^* = \omega_{H,\tau} \otimes \bar{k}$$

is generated by htH/d elements, where $M^* = \text{Hom}(M, W(\bar{k}))$ for a $W(\bar{k})$ -module M . By Nakayama lemma, $\omega_{H,\tau}$ is generated by htH/d elements.

Now for any subgroup $(H', \iota) \subset (H, \iota)$, we have an exact sequence

$$0 \rightarrow \omega_{H/H',\tau} \rightarrow \omega_{H,\tau} \rightarrow \omega_{H',\tau} \rightarrow 0.$$

By the basic properties of Fitting ideals,

$$\text{Fitt}_{ht_{O_F}H'}\omega_{H',\tau}\text{Fitt}_0\omega_{H/H',\tau} \subset \text{Fitt}_{ht_{O_F}H'}\omega_{H,\tau}.$$

But

$$\text{Fitt}_{ht_{O_F}H'}\omega_{H',\tau} = O_K$$

by the above paragraph, thus

$$\begin{aligned} \nu(\text{Fitt}_0\omega_{H/H',\tau}) &= \text{deg}\omega_{H/H',\tau} \\ &= \text{deg}\omega_{H,\tau} - \text{deg}\omega_{H',\tau} \\ &\geq \nu(\text{Fitt}_{ht_{O_F}H'}\omega_{H,\tau}), \end{aligned}$$

and

$$Hdg_{\tau}(ht_{O_F}H') = deg\omega_{H,\tau} - \nu(Fitt_{ht_{O_F}H'}\omega_{H,\tau}) \geq deg\omega_{H',\tau}.$$

We sum the above inequality over $\tau : F \hookrightarrow K$, then divide by d to get

$$\frac{1}{d} \sum_{\tau:F \hookrightarrow K} Hdg_{\tau}(ht_{O_F}H') \geq \frac{1}{d} deg\omega_{H'}.$$

But the left hand side is by definition $Hdg(H, \iota)(\frac{htH'}{d})$, and since $HN(H, \iota)$ is the convex hull $(\frac{htH'}{d}, \frac{degH'}{d})$ as (H', ι) varies as a subgroup of (H, ι) ([26], proposition 7), we therefore get

$$Hdg(H, \iota) \geq HN(H, \iota).$$

□

Combined with proposition 1.3.3, the remark below definition 1.3.4, proposition 1.3.10 and remark 1.3.9, we get

Corollary 1.3.11. *Let (H, ι, λ) be a p -divisible group with additional structures over O_K . Then for all $x \in [0, htH/d]$,*

$$\begin{aligned} HN(H, \iota, \lambda)(x) &\leq \frac{1}{m} HN(H[p^m], \iota, \lambda)(mx) \leq HN(H[p], \iota, \lambda)(x) \\ &\leq Hdg(H[p], \iota, \lambda)(x) = Hdg(H_k, \iota, \lambda)(x). \end{aligned}$$

1.4 Admissible filtered isocrystals

In this section, let $K|\mathbb{Q}_p$ be a complete field extension for a discrete valuation, with residue field k of O_K perfect, and $K_0 = \text{Frac}W(k)$. We will explain the inequalities in corollary 1.3.7 for the discrete valuation base case in terms of filtered isocrystals with additional structures.

First, we review the classical case, i.e. $G = GL_n$, there is no additional structures. Consider the category $\text{FillSoc}_{K|K_0}$ of filtered isocrystals over K . The objects are in the form of triplets $(V, \varphi, \text{Fil}^{\bullet}V_K)$, where

- (V, φ) is an isocrystal over k ;
- $\text{Fil}^{\bullet}V_K$ is a filtration of $V \otimes_{K_0} K$ such that $\text{Fil}^i V_K = 0$ for $i \gg 0$ and $\text{Fil}^i V_K = V_K$ for $i \ll 0$.

Recall we have three functions on this category :

$$ht, t_N, t_H : \text{FillSoc}_{K|K_0} \longrightarrow \mathbb{Z},$$

where for an object $(V, \varphi, \text{Fil}^{\bullet}V_K) \in \text{FillSoc}_{K|K_0}$,

$$ht(V, \varphi, \text{Fil}^{\bullet}V_K) = \dim_{K_0} V,$$

$$t_N(V, \varphi, \text{Fil}^{\bullet}V_K) = t_N(V, \varphi) = \sum_{\lambda \in \mathbb{Q}} \lambda \dim_{K_0} V_{\lambda}$$

= the (vertical coordinate of the) terminal point of the Newton polygon of (V, φ) ,

here $V = \bigoplus_{\lambda \in \mathbb{Q}} V_\lambda$ is the Dieudonné-Manin decomposition of V into isocline subspaces V_λ of slope λ ,

$$t_H(V, \varphi, \text{Fil}^\bullet V_K) = t_H(V_K, \text{Fil}^\bullet V_K) = \sum_{i \in \mathbb{Z}} \text{idim}_K(\text{gr}_{\text{Fil}^\bullet}^i V_K)$$

= the (vertical coordinate of the) terminal point of the Hodge polygon of $(V_K, \text{Fil}^\bullet V_K)$.

Consider the functions

$$\text{deg}_1 = t_H - t_N : \text{Fillsoc}_{K|K_0} \longrightarrow \mathbb{Z}$$

and $\mu_1 = \frac{\text{deg}_1}{ht}$, then the objects in $\text{Fillsoc}_{K|K_0}$ admit the Harder-Narasimhan filtration relative to the slope function μ_1 , see [27] 9.3. The abelian category of (weakly) admissible filtered isocrystals in the sense of Fontaine is then

$$\text{Fillsoc}_{K|K_0}^{ad} = \text{Fillsoc}_{K|K_0}^{\mu_1 - ss, 0},$$

which is equivalent to the category of crystalline representations of the Galois group $\text{Gal}(\overline{K}/K)$.

On this abelian category of (weakly) admissible filtered isocrystals, we have two functions :

$$ht, -t_N (= -t_H) : \text{Fillsoc}_{K|K_0}^{ad} \longrightarrow \mathbb{Z}.$$

Let $\mu = \frac{-t_N}{ht}$, then the objects in $\text{Fillsoc}_{K|K_0}^{ad}$ admit the Harder-Narasimhan filtration relative to the slope function μ . In section 9 of loc. cit. Fargues introduced a Harder-Narasimhan filtration of crystalline representations, by considering the larger category of Hodge-Tate representations and its link with categories of filtered vector spaces. For a p -divisible group H over O_K one has the equality of polygons $HN(H) = HN(V_p(H))$, where $V_p(H)$ is the rational Tate module. Recall the equivalence functors

$$\text{Fillsoc}_{K|K_0}^{ad} \begin{array}{c} \xrightarrow{V_{cris}} \\ \xleftarrow{D_{cris}} \end{array} \text{Rep}_{\mathbb{Q}_p}^{cris}(\text{Gal}(\overline{K}/K))$$

defined by

$$V_{cris}(N, \varphi, \text{Fil}^\bullet N_K) = \text{Fil}^0(N \otimes_{K_0} B_{cris})^{\varphi=id}$$

and

$$D_{cris}(V) = (V \otimes_{\mathbb{Q}_p} B_{cris})^{\text{Gal}(\overline{K}/K)}.$$

We have the fact that, the Harder-Narasimhan filtrations in $\text{Rep}_{\mathbb{Q}_p}^{cris}(\text{Gal}(\overline{K}/K))$ introduced by Fargues and in $\text{Fillsoc}_{K|K_0}^{ad}$ defined above coincide, cf. 9.4 of loc. cit..

If we denote the category of isocrystals over k by $\text{Isoc}(k)$, and use $\text{deg} = -t_N$, $ht = \text{dim}$, $\mu = \frac{-t_N}{\text{dim}}$ to develop the formalism of Harder-Narasimhan filtration, then since the functor of forgetting the filtration

$$\text{Fillsoc}_{K|K_0}^{ad} \longrightarrow \text{Isoc}(k)$$

$$(V, \varphi, \text{Fil}^\bullet V_K) \mapsto (V, \varphi)$$

is exact, and preserves the functions of ht and $-t_N$ on these two categories, we have the following inequality between concave Harder-Narasimhan polygons :

$$HN(V, \varphi, \text{Fil}^\bullet V_K) \leq HN(V, \varphi) =: \text{Newt}(V, \varphi).$$

On the other hand, if we denote the category of filtered vector spaces over K by $\text{FilVect}_{K|K}$, which admits the Harder-Narasimhan filtration for the functions

$$ht(V, \text{Fil}^\bullet V) = \dim_K V, \quad deg(V, \text{Fil}^\bullet V) = - \sum_{i \in \mathbb{Z}} i \cdot \dim_K (gr_{\text{Fil}^\bullet}^i V)$$

and $\mu = \frac{deg}{ht}$. We have an exact functor

$$\text{Fillsoc}_{K|K_0}^{ad} \longrightarrow \text{FilVect}_{K|K}$$

defined by composition of the forgetting functor and tensor product with K

$$\text{Fillsoc}_{K|K_0}^{ad} \rightarrow \text{FilVect}_{K|K_0} \rightarrow \text{FilVect}_{K|K}.$$

This functor preserves the height and degree functions, thus we have the inequality between concave Harder-Narasimhan polygons

$$HN(V, \varphi, \text{Fil}^\bullet V_K) \leq HN(V_K, \text{Fil}^\bullet V_K) =: Hdg(V_K, \text{Fil}^\bullet V_K).$$

In fact since $(V, \varphi, \text{Fil}^\bullet V_K)$ is admissible, we have the inequality by definition

$$Newt(V, \varphi) \leq Hdg(V_K, \text{Fil}^\bullet V_K).$$

To summarize, we get the following inequalities of concave polygons associated to an admissible filtered isocrystal $(V, \varphi, \text{Fil}^\bullet V_K)$:

$$HN(V, \varphi, \text{Fil}^\bullet V_K) \leq Newt(V, \varphi) \leq Hdg(V_K, \text{Fil}^\bullet V_K).$$

In particular, if H/O_K is a p -divisible group over O_K , we have the filtered isocrystal $(N, p^{-1}\varphi, \text{Fil}^\bullet N_K) \in \text{Fillsoc}_{K|K_0}^{ad, [-1, 0]}$ associated to it, and the exact sequence of K -vector spaces

$$0 \rightarrow \omega_{H^D, K} \rightarrow N_K \rightarrow \text{Lie}(H)_K \rightarrow 0.$$

Here $\text{Fil}^0 N_K = \omega_{H^D, K} = \text{Hom}(\text{Lie}(H^D)_K, K)$, $\text{Fillsoc}_{K|K_0}^{ad, [-1, 0]}$ is the full subcategory of $\text{Fillsoc}_{K|K_0}^{ad}$, consisting of objects $(V, \varphi, \text{Fil}^\bullet V_K)$ of the form that $\text{Fil}^{-1} V_K = V_K$, $\text{Fil}^0 V_K \subset V_K$, $\text{Fil}^1 V_K = 0$. We use the covariant isocrystal (N, φ) of H_k here, thus $\varphi = \mathbb{D}(V) \otimes K_0$ for the Verschiebung V of H_k . Under the covariant functor V_{cris} of Fontaine

$$V_{cris} : \text{Fillsoc}_{K|K_0}^{ad} \rightarrow \text{Rep}_{\mathbb{Q}_p}^{cris}(\text{Gal}(\overline{K}/K)),$$

we have the equality $V_{cris}(N, p^{-1}\varphi, \text{Fil}^\bullet N_K) = V_p(H)$ for the rational Tate module $V_p(H)$. Then we can identify the following various polygons :

$$\begin{aligned} HN(H) &= HN(N, p^{-1}\varphi, \text{Fil}^\bullet N_K) \\ Newt(H_k) &= Newt(N, p^{-1}\varphi) \\ Hdg(H_k) &= Hdg(N_K, \text{Fil}^\bullet N_K). \end{aligned}$$

Here the Hodge polygons in the two sides of the last equality are both the polygon which is the line of slope 1 between the points $(0, 0)$ and (d, d) , and the line of slope 0 between the points (d, d) and (h, d) (assume $\dim H = d$, $ht H = h$). Thus the above inequalities become the following

$$HN(H) \leq Newt(H_k) \leq Hdg(H_k).$$

Now we consider the cases with additional structures. Let G be a reductive group introduced at the beginning of section 1.3. Let k be an algebraically closed field of characteristic p , $L = \text{Frac}W(k)$. Then a pair (b, μ) is said to be admissible filtered isocrystal with G -structures, where $b \in G(L)$ and $\mu : \mathbb{G}_{mK} \rightarrow G_K$ is a co-character defined over a finite extension K of L , if for all $(V, \rho) \in \text{Rep}_{\mathbb{Q}_p} G$, $(V \otimes L, b\sigma, \text{Fil}_\mu^\bullet V_K)$ is an admissible filtered isocrystal in $\text{Fill}\text{soc}_{K|L}$. Note to check that (b, μ) is admissible, it suffices to check for a faithful representation (V, ρ) , $(V \otimes L, b\sigma, \text{Fil}_\mu^\bullet V_K)$ is admissible. Since the tensor product of two semi-stable admissible filtered isocrystal is semi-stable (see [27], section 9), for an admissible pair (b, μ) , we can define its Harder-Narasimhan polygon $\nu_{b,\mu} \in N(G)$, such that for any $(V, \rho) \in \text{Rep}_{\mathbb{Q}_p} G$, $\rho_*(\nu_{b,\mu}) \in N(GL(V))$ is the Harder-Narasimhan polygon of $(V \otimes L, b\sigma, \text{Fil}_\mu^\bullet V_K)$ we have just defined, as in [16],[30]. Similarly, we can define its Hodge polygon $\nu_\mu \in N(G)$ such that for any $(V, \rho) \in \text{Rep}_{\mathbb{Q}_p} G$, $\rho_*(\nu_\mu) \in N(GL(V))$ is the Hodge polygon of $(V_K, \text{Fil}_\mu^\bullet V_K)$. This was already done in the book [16]. On the other hand, Kottwitz has defined the Newton polygon $\nu_b \in N(G)$ such that for any $(V, \rho) \in \text{Rep}_{\mathbb{Q}_p} G$, $\rho_*(\nu_b) \in N(GL(V))$ is the Newton polygon of (V, φ) . Thus we have the following inequalities

$$\nu_{b,\mu} \leq \nu_b \leq \nu_\mu$$

as elements in the ordered sets $N(G)$. Via the injection $N(G) \hookrightarrow N(\text{Res}_{F|\mathbb{Q}_p} GL_n)$, we can view these inequalities as inequalities between polygons over $[0, n]$.

To fix notation, we will work in the PEL cases. The EL case can be treated in the same way, which is simpler. So let $(H, \iota, \lambda)/O_K$ be a p -divisible group with additional structures, then its associated filtered isocrystal $(N, p^{-1}\varphi, \text{Fil}^\bullet N_K)$ admits induced additional structures, which means that we have an action $\iota : F \rightarrow \text{End}(N, \varphi)$ and a perfect pairing $\langle, \rangle : N \times N \rightarrow \mathbb{Q}_p(1)$ such that $\text{Fil}^0 N_K$ is F -invariant and totally isotropic under the induced pairing \langle, \rangle on N_K . We denote it as $(N, p^{-1}\varphi, \text{Fil}^\bullet N_K, \iota, \langle, \rangle)$ considering its additional structures. This will then determine an admissible filtered isocrystal with additional structures (b, μ) for G , as in [65]. The inequalities

$$\nu_{b,\mu} \leq \nu_b \leq \nu_\mu$$

now translate as

$$\begin{aligned} HN(N, p^{-1}\varphi, \text{Fil}^\bullet N_K, \iota, \langle, \rangle) &\leq \text{Newt}(N, p^{-1}\varphi, \iota, \langle, \rangle) \\ &\leq \text{Hdg}(N_K, \text{Fil}^\bullet N_K, \iota, \langle, \rangle), \end{aligned}$$

where

$$\begin{aligned} HN(N, p^{-1}\varphi, \text{Fil}^\bullet N_K, \iota, \langle, \rangle) &= HN(N, p^{-1}\varphi, \text{Fil}^\bullet N_K, \iota) \\ &= \frac{1}{d} HN(N, p^{-1}\varphi, \text{Fil}^\bullet N_K)(d \cdot) \end{aligned}$$

$$\text{Newt}(N, p^{-1}\varphi, \iota, \langle, \rangle) = \frac{1}{d} \text{Newt}(N, p^{-1}\varphi)(d \cdot)$$

$$\text{Hdg}(N_K, \text{Fil}^\bullet N_K, \iota, \langle, \rangle) = \frac{1}{d} \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \sigma^i(\text{Hdg}(N_{1K}, \text{Fil}^\bullet N_{1K}))$$

via the injection $N(G) \hookrightarrow N(\text{Res}_{F|\mathbb{Q}_p} GL_n)$. Here under the action ι on N_K , we have the decomposition as K -vector spaces

$$N_K = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} N_{iK}, N_{iK} = \{x \in N_K | a \cdot x = \sigma^i(a)x\}, \forall i \in \mathbb{Z}/d\mathbb{Z},$$

$$\mathrm{Fil}^\bullet N_{iK} = \mathrm{Fil}^\bullet N_K \cap N_{iK},$$

Note the action of σ on the set of polygons $\{Hdg(N_{iK}, \mathrm{Fil}^\bullet N_{iK})\}_{i \in \mathbb{Z}/d\mathbb{Z}}$ is

$$\sigma(Hdg(N_{iK}, \mathrm{Fil}^\bullet N_{iK})) = Hdg(N_{i+1K}, \mathrm{Fil}^\bullet N_{i+1K}).$$

By definitions we have the following identities between various polygons

$$\begin{aligned} HN(H, \iota, \lambda) &= HN(N, p^{-1}\varphi, \mathrm{Fil}^\bullet N_K, \iota, \langle, \rangle) \\ Newt(H_k, \iota, \lambda) &= Newt(N, p^{-1}\varphi, \iota, \langle, \rangle) \\ Hdg(H_k, \iota, \lambda) &= Hdg(N_K, \mathrm{Fil}^\bullet N_K, \iota, \langle, \rangle). \end{aligned}$$

Thus the above inequalities explain these ones obtained in corollary 1.3.7 :

$$HN(H, \iota, \lambda) \leq Newt(H_k, \iota, \lambda) \leq Hdg(H_k, \iota, \lambda).$$

1.5 Hodge-Newton filtration for p -divisible groups with additional structures

We restrict ourselves to the PEL cases, the EL case can be treated in the same way, which is simpler. Let $K|\mathbb{Q}_p$ be a complete discrete valuation field with residue field k perfect. Let (H, ι, λ) be a p -divisible group with additional structures over O_K . We make the following basic assumption :

(HN) : *Newt(H_k, ι, λ) and $Hdg(H_k, \iota, \lambda)$ possess a contact point x outside their extremal points which is a break point for the polygon $Newt(H_k, \iota, \lambda)$.*

Since these two polygons are symmetric, the symmetric point \hat{x} of x satisfies the same assumption. Denote the coordinate of x by (x_1, x_2) . Without loss of generality, we may assume $x_1 \leq \frac{htH}{2d}$, and note the equality holds if and only if $x = \hat{x}$.

We now use the explanation of the various polygons in terms of the filtered isocrystal with additional structures $(N, p^{-1}\varphi, \mathrm{Fil}^\bullet N_K, \iota, \langle, \rangle)$ attached to (H, ι, λ) :

$$\begin{aligned} HN(H, \iota, \lambda) &= HN(N, p^{-1}\varphi, \mathrm{Fil}^\bullet N_K, \iota, \langle, \rangle) \\ Newt(H_k, \iota, \lambda) &= Newt(N, p^{-1}\varphi, \iota, \langle, \rangle) \\ Hdg(H_k, \iota, \lambda) &= Hdg(N_K, \mathrm{Fil}^\bullet N_K, \iota, \langle, \rangle). \end{aligned}$$

Then the break points x and \hat{x} correspond to decompositions of isocrystal with additional structures

$$\begin{aligned} (N, p^{-1}\varphi, \iota) &= (N_1, p^{-1}\varphi, \iota) \oplus (N'_1, p^{-1}\varphi, \iota), \\ (N, p^{-1}\varphi, \iota) &= (N_2, p^{-1}\varphi, \iota) \oplus (N'_2, p^{-1}\varphi, \iota), \end{aligned}$$

where the Newton polygon of $(N_1, p^{-1}\varphi, \iota)$ (resp. $(N_2, p^{-1}\varphi, \iota)$) corresponds to the part in the polygon of $(N, p^{-1}\varphi, \iota, \langle, \rangle)$ before x (resp. \hat{x}), and the Newton polygon of $(N'_1, p^{-1}\varphi, \iota)$ (resp. $(N'_2, p^{-1}\varphi, \iota)$) corresponds to the part in the polygon of $(N, p^{-1}\varphi, \iota, \langle, \rangle)$ after x (resp. \hat{x}). We consider the induced filtered isocrystal with additional structures $(N_1, p^{-1}\varphi, \mathrm{Fil}^\bullet N_{1K}, \iota)$, where $\mathrm{Fil}^\bullet N_{1K} = \mathrm{Fil}^\bullet N_K \cap N_{1K}$.

Proposition 1.5.1. *The underlying filtered isocrystal of $(N_1, p^{-1}\varphi, \text{Fil}^\bullet N_{1K}, \iota)$ is admissible.*

Proof. We just need to show the equality

$$t_N(N_1, p^{-1}\varphi) = t_H(N_{1K}, \text{Fil}^\bullet N_{1K}).$$

First, the admissibility of $(N, p^{-1}\varphi, \text{Fil}^\bullet N_K)$ implies that

$$t_H(N_{1K}, \text{Fil}^\bullet N_{1K}) \leq t_N(N_1, p^{-1}\varphi).$$

So we just need to show the inequality

$$t_H(N_{1K}, \text{Fil}^\bullet N_{1K}) \geq t_N(N_1, p^{-1}\varphi).$$

We denote by

$$N = \bigoplus_{i=1}^d N^i, N_1 = \bigoplus_{i=1}^d N_1^i$$

for the decomposition of N and N_1 by the action of F . Then each subspaces N_K^i and N_{1K}^i admit the induced filtration $\text{Fil}^\bullet N_K^i = \text{Fil}^\bullet N_K \cap N_K^i$, $\text{Fil}^\bullet N_{1K}^i = \text{Fil}^\bullet N_{1K} \cap N_{1K}^i$. By the property of Harder-Narasimhan polygons, we have for all $i = 1, \dots, d$

$$\deg(N_{1K}^i, \text{Fil}^\bullet N_{1K}^i) = \dim_K N_{1K}^i - \dim_K \text{Fil}^0 N_{1K}^i \leq \text{Hdg}(N_K^i, \text{Fil}^\bullet N_K^i) \left(\frac{\dim_K N_{1K}}{d} \right).$$

Thus

$$\begin{aligned} \deg(N_{1K}, \text{Fil}^\bullet N_{1K}) &= -t_H(N_{1K}, \text{Fil}^\bullet N_{1K}) \\ &= \dim_K N_{1K} - \dim_K \text{Fil}^0 N_{1K} \\ &\leq \sum_{i=1}^d (\dim_K N_{1K}^i - \dim_K \text{Fil}^0 N_{1K}^i) \\ &\leq \sum_{i=1}^d \text{Hdg}(N_K^i, \text{Fil}^\bullet N_K^i) \left(\frac{\dim_K N_{1K}}{d} \right) \\ &= -t_N(N_1, p^{-1}\varphi). \end{aligned}$$

Here the last equality comes from our assumption (HN) and the definition of $\text{Newt}(N, p^{-1}\varphi, \iota, \langle, \rangle)$ and $\text{Hdg}(N_K, \text{Fil}^\bullet N_K, \iota, \langle, \rangle)$. \square

Corollary 1.5.2. *With the above notation, the Hodge polygon $\text{Hdg}(N_{1K}, \text{Fil}^\bullet N_{1K}, \iota)$ equals to the part before the point x of the Hodge polygon $\text{Hdg}(N_K, \text{Fil}^\bullet N_K, \iota)$.*

Proof. Indeed, in the proof of proposition 1.5.1, we get for all $i = 1, \dots, d$

$$\dim_K N_{1K}^i - \dim_K \text{Fil}^0 N_{1K}^i = \text{Hdg}(N_K^i, \text{Fil}^\bullet N_K^i) \left(\frac{\dim_K N_{1K}}{d} \right).$$

Thus for all $i = 1, \dots, d$, $\text{Hdg}(N_{1K}^i, \text{Fil}^\bullet N_{1K}^i)$ is the part before the point $(\frac{\dim_K N_{1K}}{d}, \dim_K N_{1K}^i - \dim_K \text{Fil}^0 N_{1K}^i)$ in the polygon $\text{Hdg}(N_K^i, \text{Fil}^\bullet N_K^i)$. Then by definition we get the corollary. \square

Similarly for the point \hat{x} we have an admissible filtered isocrystal $(N_2, p^{-1}\varphi, \text{Fil}^\bullet N_{2K})$.

Since $(N_1, p^{-1}\varphi, \text{Fil}^\bullet N_{1K})$ is admissible,

$$\begin{aligned} \deg(N_1, p^{-1}\varphi, \text{Fil}^\bullet N_{1K}, \iota) &= -\frac{1}{d}t_N(N_1, p^{-1}\varphi) = \text{Newt}(N, p^{-1}\varphi, \iota, \langle, \rangle)\left(\frac{\dim_K N_{1K}}{d}\right) \\ &\leq \text{HN}(N, p^{-1}\varphi, \text{Fil}^\bullet N_K, \iota, \langle, \rangle)\left(\frac{\dim_K N_{1K}}{d}\right) \leq \text{Newt}(N, p^{-1}\varphi, \iota, \langle, \rangle)\left(\frac{\dim_K N_{1K}}{d}\right), \end{aligned}$$

thus all the inequalities above are in fact equalities. One has similar equalities for $(N_2, p^{-1}\varphi, \text{Fil}^\bullet N_{2K})$. We get the following important corollary.

Corollary 1.5.3. *The Harder-Narasimhan polygon of (H, ι, λ) also passes the points x and \hat{x} , which are thus also break points of $\text{HN}(H, \iota, \lambda)$. Moreover, for $i = 1, 2$, $(N_i, p^{-1}\varphi, \text{Fil}^\bullet N_{iK})$ appear in the Harder-Narasimhan filtration of $(N, p^{-1}\varphi, \text{Fil}^\bullet N_K)$.*

Theorem 1.5.4. *Let $K|\mathbb{Q}_p$ be a complete discrete valuation field with residue field k perfect, (H, ι, λ) be a p -divisible group with additional structures over O_K . Under the basic assumption (HN), there are unique subgroups*

$$(H_1, \iota) \subset (H_2, \iota) \subset (H, \iota)$$

of (H, ι) as p -divisible groups with additional structures over O_K , such that

1. λ induces isomorphisms

$$\begin{aligned} (H_1, \iota) &\simeq ((H/H_2)^D, \iota'), \\ (H_2, \iota) &\simeq ((H/H_1)^D, \iota'); \end{aligned}$$

2. the induced filtration of the p -divisible group with additional structures (H_k, ι) over k

$$(H_{1k}, \iota) \subset (H_{2k}, \iota) \subset (H_k, \iota)$$

is split;

3. the Newton (resp. Harder-Narasimhan, resp. Hodge) polygons of (H_1, ι) , $(H_2/H_1, \iota)$, and $(H/H_2, \iota)$ are the parts of the Newton (resp. Harder-Narasimhan, resp. Hodge) polygon of (H, ι, λ) up to x , between x and \hat{x} , and from \hat{x} on respectively.

When $x = \hat{x}$, then H_1 and H_2 coincide.

Proof. We use the formula

$$\text{HN}(H, \iota, \lambda) = \inf_{n \geq 1} \frac{1}{n} \text{HN}(H[p^n], \iota, \lambda)(n \cdot)$$

and the following inequalities

$$\text{HN}(H, \iota, \lambda) \leq \frac{1}{n} \text{HN}(H[p^n], \iota, \lambda)(n \cdot) \leq \text{HN}(H[p], \iota, \lambda) \leq \text{Hdg}(H_k, \iota, \lambda)$$

to deduce that, for $n \gg 0$ large enough, \hat{x} is a break point of the polygons $\frac{1}{n} \text{HN}(H[p^n], \iota, \lambda)(n \cdot)$. We fix such a n . Thus there exists a sub-group

$$H_n \subset H[p^n],$$

which appears in the Harder-Narasimhan filtration of $H[p^n]$, and admits an induced action by ι . We denote it as (H_n, ι) . Consider the family of finite flat group schemes with additional structures

$$(H_{2^k n}, \iota)_{k \geq 0} \subset (H[p^{2^k n}], \iota)_{k \geq 0}.$$

Since

$$\frac{1}{2}HN(H[p^{2^{k+1}n}], \iota, \lambda)(2 \cdot) \leq HN(H[p^{2^k n}], \iota, \lambda),$$

we have

$$\mu_{max}(H[p^{2^{k+1}n}]/H_{2^{k+1}n}) \leq \mu_{max}(H[p^{2^k n}]/H_{2^k n}) < \mu_{min}(H_{2^k n}),$$

thus

$$Hom(H_{2^k n}, H[p^{2^{k+1}n}]/H_{2^{k+1}n}) = 0$$

by proposition 1.2.3. In particular, the composition of

$$H_{2^k n} \hookrightarrow H[p^{2^k n}] \hookrightarrow H[p^{2^{k+1}n}] \rightarrow H[p^{2^{k+1}n}]/H_{2^{k+1}n}$$

is 0, i.e.

$$H_{2^k n} \subset H_{2^{k+1}n}.$$

Similarly, since

$$\mu_{min}(H_{2^{k+1}n}) \geq \mu_{min}(H_{2^k n}) > \mu_{max}(H[p^{2^k n}]/H_{2^k n}),$$

thus

$$Hom(H_{2^{k+1}n}, H[p^{2^k n}]/H_{2^k n}) = 0.$$

In particular, the composition of

$$H_{2^{k+1}n} \hookrightarrow H[p^{2^{k+1}n}] \xrightarrow{\times p^{2^k n}} H[p^{2^k n}] \rightarrow H[p^{2^k n}]/H_{2^k n}$$

is 0, i.e.

$$p^{2^k n}(H_{2^{k+1}n}) \subset H_{2^k n}.$$

Let C be the scheme theoretic closure in $H[p^{2^k n}]$ of

$$ker(H_{2^{k+1}n, K} \xrightarrow{p^{2^k n}} H_{2^k n, K}),$$

and D be the scheme theoretic closure in $H[p^{2^k n}]$ of

$$im(H_{2^{k+1}n, K} \xrightarrow{p^{2^k n}} H_{2^k n, K}).$$

Then $H_{2^k n} \subset C$, and we have a sequence

$$0 \rightarrow C \rightarrow H_{2^{k+1}n} \rightarrow D \rightarrow 0,$$

and

$$deg H_{2^{k+1}n} \leq deg C + deg D$$

with the equality holds if and only if the above sequence is exact, cf. proposition 1.2.1.

Note

$$\deg H_{2^{k+1}n} = 2\deg H_{2^k n}$$

$$ht H_{2^{k+1}n} = 2ht H_{2^k n}.$$

Let $a = ht H_{2^k n} \leq ht C$, since $ht C + ht D = ht H_{2^{k+1}n} = 2a$, we have $ht D \leq a$.

Consider the non-normalized Harder-Narasimhan polygon $HN(H[p^{2^k n}])$ of $H[p^{2^k n}]$, we have

$$\deg C \leq HN(H[p^{2^k n}])(ht C)$$

$$\deg D \leq HN(H[p^{2^k n}])(ht D),$$

thus

$$\begin{aligned} \deg C + \deg D &\leq HN(H[p^{2^k n}])(ht C) + HN(H[p^{2^k n}])(ht D) \\ &\leq 2HN(H[p^{2^k n}])(\frac{ht C + ht D}{2}) \\ &= 2HN(H[p^{2^k n}])(ht H_{2^k n}) \\ &= 2\deg H_{2^k n} \\ &= \deg H_{2^{k+1}n}. \end{aligned}$$

Thus we have

$$\deg H_{2^{k+1}n} = \deg C + \deg D$$

and

$$0 \rightarrow C \rightarrow H_{2^{k+1}n} \rightarrow D \rightarrow 0$$

is exact. We claim that

$$C = H_{2^k n}.$$

In fact, if $C \supsetneq H_{2^k n}$, we have $D \subsetneq H_{2^k n}$. Then

$$\deg C + \deg D < 2\deg H_{2^k n},$$

a contradiction! Thus $C = H_{2^k n}$. Similarly $D = H_{2^k n}$. Therefore we have an exact sequence

$$0 \rightarrow H_{2^k n} \rightarrow H_{2^{k+1}n} \xrightarrow{p^{2^k n}} H_{2^k n} \rightarrow 0.$$

Now consider

$$H_2 = \varinjlim_{k \geq 0} H_{2^k n}$$

as a fppf sheaf over O_K . It is of p -torsion by definition. It is also p -divisible, i.e. $H_2 \xrightarrow{p} H_2$ is an epimorphism, since $H_2 \xrightarrow{p^{2^k n}} H_2$ is. As $H_2[p] = H_{2^k n}[p]$, and $H_{2^k n} \xrightarrow{p^{2^k n-1}} H_{2^k n}[p]$ is an epimorphism, $H_{2^k n}$ is flat over O_K , we can deduce that $H_{2^k n}[p]$ is a finite flat group scheme (see [28]). Therefore, H_2 is a p -divisible group over O_K . By construction, it naturally admits the induced additional structures ι , and its filtered isocrystal is exactly $(N_2, p^{-1}\varphi, \text{Fil}^\bullet N_{2K}, \iota)$. Thus we get a sub p -divisible group with additional structures (H_2, ι) of (H, ι) . Over k , the exact sequence

$$0 \rightarrow H_{2k} \rightarrow H_k \rightarrow (H/H_2)_k \rightarrow 0$$

splits since $H_{2k} \subset H_k$ is a part of the slope filtration of H_k , and the Newton (resp. Harder-Narasimhan, resp. Hodge) polygons of (H_2, ι) and $(H/H_2, \iota)$ are the parts of the Newton (resp. Harder-Narasimhan, resp. Hodge) polygon of (H, ι, λ) up to \hat{x} and from \hat{x} on respectively.

Similarly, for the point x , we can construct a sub p -divisible group with additional structures (H_1, ι) of (H_2, ι) , and the Newton (resp. Harder-Narasimhan, resp. Hodge) polygons of (H_1, ι) and $(H_2/H_1, \iota)$ are the parts of the Newton (resp. Harder-Narasimhan, resp. Hodge) polygon of (H, ι, λ) up to x and between x and \hat{x} respectively. The polarization $\lambda : H \xrightarrow{\sim} H^D$ then induces the isomorphisms

$$(H_1, \iota) \simeq ((H/H_2)^D, \iota'),$$

$$(H_2, \iota) \simeq ((H/H_1)^D, \iota').$$

□

Remark 1.5.5. 1. For all $n \geq 1$, we know that the polygon $\frac{1}{n}HN(H[p^n], \iota, \lambda)(n \cdot)$ passes \hat{x} . From the proof of the above theorem, \hat{x} is a break point of this polygon for all n large enough (i.e. there exists $n_0 \gg 0$, for all $n \geq n_0$ \hat{x} is a break point of $\frac{1}{n}HN(H[p^n], \iota, \lambda)(n \cdot)$), and $H_2[p^n]$ is a subgroup in the Harder-Narasimhan filtration of $H[p^n]$. In fact by [26] lemme 7 and [27] lemme 3, we get that \hat{x} is a break point of $\frac{1}{n}HN(H[p^n], \iota, \lambda)(n \cdot)$ for all $n \geq 1$, and $H_2[p^n]$ is the subgroup in the Harder-Narasimhan filtration of $H[p^n]$ corresponding to \hat{x} for all $n \geq 1$. Similarly x is also a break point of $\frac{1}{n}HN(H[p^n], \iota, \lambda)(n \cdot)$ for all $n \geq 1$, and $H_1[p^n]$ is the subgroup in the Harder-Narasimhan filtration of $H[p^n]$ corresponding to x for all $n \geq 1$.

2. In the above proof, we just need the fact that the polygon $HN(H, \iota, \lambda)$ also passes the points x and \hat{x} , then based on this we can use the theory of Harder-Narasimhan filtration of finite flat group schemes to find H_1 and H_2 . Thus we can prove the theorem over a general complete rank one valuation ring $O_K|\mathbb{Z}_p$, once we can prove that under our assumption (HN) $HN(H, \iota, \lambda)$ also passes the points x and \hat{x} , see the following.

For the application to the cohomology of Rapoport-Zink spaces as in the next section, we will need a stronger version of the above theorem, namely the case $K|\mathbb{Q}_p$ is a complete field extension for a general rank one valuation, not necessarily discrete. For some technical reason we introduce some “reasonable” class of p -divisible groups over such bases.

Definition 1.5.6 ([27], Définition 25). Let $K|\mathbb{Q}_p$ be a complete field extension for a rank one valuation, O_K be the ring of integers. Suppose the residue field k of O_K is perfect. Let $\alpha : k \hookrightarrow O_K/pO_K$ be the Teichmüller section of the projection $O_K/pO_K \rightarrow k$. Let H be a p -divisible group over O_K , $H_k = H \otimes_{O_K} k$ be its special fiber. We say H is modular, if the identity map $H_k \rightarrow H_k$ lifts to a quasi-isogeny (not necessarily unique)

$$H_k \otimes_{k, \alpha} O_K/pO_K \rightarrow H \otimes_{O_K} O_K/pO_K.$$

When the residue field k is not necessarily perfect, a p -divisible group H over O_K is called modular, if for some algebraic closure \bar{K} of K , $H \otimes_{O_K} \bar{K}$ is modular in the above sense.

We refer the reader to the various lists of equivalent formulations in proposition 22 of loc. cit.. In particular, any p -divisible group over a complete discrete valuation ring is modular.

Theorem 1.5.7. *Let $K|\mathbb{Q}_p$ be a complete field extension for a rank one valuation, O_K be the ring of integers. Let (H, ι, λ) be a p -divisible group with additional (PEL) structures over O_K , with the underlying p -divisible group H modular. Assume (H, ι, λ) satisfies the assumption (HN), then the same conclusions as theorem 1.5.4 (1) and (3) hold for (H, ι, λ) , and if the residue field k of O_K is perfect, then the conclusion (2) of theorem 1.5.4 also holds for (H, ι, λ) .*

Proof. In fact, we just need to show under the above conditions, the Harder-Narasimhan polygon $HN(H, \iota, \lambda)$ also passes the contact point x of $Newt(H_k, \iota, \lambda)$ and $Hdg(H_k, \iota, \lambda)$. Then the other arguments in the proof of the above theorem work in the same way in this case, see the remark 1.5.5 (2).

To show $HN(H, \iota, \lambda)$ also passes the contact point x , we use the tool of Rapoport-Zink spaces, see the next subsection for some review of these spaces. We may assume $k = \bar{k}$ is algebraical closed. Consider the Rapoport-Zink space of EL type defined as the quasi-isogeny deformation space of (H_k, ι) . We use \mathcal{M} to denote the Berkovich analytic space over $K_0 = \text{Frac}W(k)$. Then (H, ι) defines a K -valued point of \mathcal{M} . For any finite extension $K'|K_0$, the valuation on K' is discrete, and the p -divisible groups (H', ι) over $O_{K'}$ associated to the points of $\mathcal{M}(K')$ satisfy the assumption (HN). Thus the Harder-Narasimhan polygons $HN(H', \iota)$ pass the point x . As the rigid points $\mathcal{M}^{rig} = \{y \in \mathcal{M} | [\mathcal{H}(y) : K_0] < \infty\}$ are dense in \mathcal{M} , and the function of Harder-Narasimhan polygon is semi-continuous over \mathcal{M} ([27] 13.3), we deduce that the Harder-Narasimhan polygons of the p -divisible groups associated any points $y \in \mathcal{M}$ pass x . In particular this holds for $HN(H, \iota, \lambda) = HN(H, \iota)$.

□

1.6 Application to the geometry and cohomology of some non-basic Rapoport-Zink spaces

The existence and uniqueness of Hodge-Newton filtration can be used to deduce that, the cohomology of simple unramified EL/PEL Rapoport-Zink spaces which satisfy the assumption (HN) contains no supercuspidal representations, as Mantovan did in [56], where her assumption was stronger and her results were just stated for the EL case and PEL symplectic case.

Let $(F, V, b, \mu)/(F, *, V, \langle, \rangle, b, \mu)$ be a simple EL/PEL data with $[F : \mathbb{Q}_p] = d$, $\dim_F V = n$, where $(F, V)/(F, *, V, \langle, \rangle)$ are as in the section 1.3 used to define the reductive group G . The remaining data (b, μ) consists of

- an element $b \in G(L)$ up to σ -conjugacy (thus we can view $b \in B(G)$, here $B(G)$ is the set of σ -conjugacy classes in $G(L)$), such that the associated isocrystal $(V \otimes L, b\sigma)$ has slopes in $[0, 1]$, thus coming from a p -divisible group Σ up to isogeny. Here $L = \text{Frac}W(\overline{\mathbb{F}}_p)$;
- a minscale co-character $\mu : \mathbb{G}_{m, \overline{\mathbb{Q}}_p} \rightarrow G_{\overline{\mathbb{Q}}_p}$, up to $G(\overline{\mathbb{Q}}_p)$ -conjugacy, such that

1. $b \in B(G, \mu)$ as an element in $B(G)$, thus the pair (b, μ) is admissible ([30]). Here $B(G, \mu)$ is the set defined by Kottwitz ([52]);
2. in the PEL case, $c \circ \mu = id$, $v_p(c(b)) = 1$, here v_p is the standard valuation on L and $c : G \rightarrow \mathbb{G}_m$, $c(x) = x^\# x \in R^\times$ for any \mathbb{Q}_p -algebra R and $x \in G(R)$ (see section 1.3).

We can make the form of μ more explicitly. Recall we assume $n = \dim_F V$. Fix an F -base of V and denote $I_F := \text{Hom}_{\mathbb{Q}_p}(F, \overline{\mathbb{Q}_p})$. In the EL case, $G = \text{Res}_{F|\mathbb{Q}_p} GL_n$ and thus μ is given by a collection of pairs of integer (p_τ, q_τ) such that $p_\tau + q_\tau = n$ for all $\tau \in I_F$:

$$\begin{aligned} \mu : \mathbb{G}_{m\overline{\mathbb{Q}_p}} &\rightarrow G_{\overline{\mathbb{Q}_p}} \simeq \prod_{\tau \in I_F} GL_{n\overline{\mathbb{Q}_p}} \\ z &\mapsto \prod_{\tau \in I_F} \text{diag}(\underbrace{z, \dots, z}_{p_\tau}, \underbrace{1, \dots, 1}_{q_\tau}). \end{aligned}$$

In the PEL unitary case, let $\Phi \subset I_F$ be a CM-type, i.e. $\Phi \amalg \Phi^* = I_F$ where $\Phi^* = \{\tau \circ * | \tau \in \Phi\}$. By definition the group G is such that

$$G_{\overline{\mathbb{Q}_p}} \simeq \left(\prod_{\tau \in \Phi} GL_{n\overline{\mathbb{Q}_p}} \right) \times \mathbb{G}_{m\overline{\mathbb{Q}_p}} \subset \left(\prod_{\tau \in I_F} GL_{n\overline{\mathbb{Q}_p}} \right) \times \mathbb{G}_{m\overline{\mathbb{Q}_p}}.$$

The μ is given by a collection of pairs of integer $(p_\tau, q_\tau)_{\tau \in I_F}$ such that $p_{\tau^*} = q_\tau$, $q_{\tau^*} = p_\tau$, and $p_\tau + q_\tau = n$ for all $\tau \in I_F$:

$$\begin{aligned} \mu : \mathbb{G}_{m\overline{\mathbb{Q}_p}} &\rightarrow G_{\overline{\mathbb{Q}_p}} \subset \left(\prod_{\tau \in I_F} GL_{n\overline{\mathbb{Q}_p}} \right) \times \mathbb{G}_{m\overline{\mathbb{Q}_p}} \\ z &\mapsto \prod_{\tau \in I_F} (\underbrace{z, \dots, z}_{p_\tau}, \underbrace{1, \dots, 1}_{q_\tau}) \times (z). \end{aligned}$$

For the PEL symplectic case, the group G is such that

$$G_{\overline{\mathbb{Q}_p}} = G\left(\prod_{\tau \in I_F} GSp_n \right),$$

where $G(\prod_{\tau \in I_F} GSp_n) \subset \prod_{\tau \in I_F} GSp_n$ is the subgroup which consists of elements in the product group with the same similitude for all $\tau \in I_F$. In this case the μ can be given by the following :

$$\begin{aligned} \mu : \mathbb{G}_{m\overline{\mathbb{Q}_p}} &\rightarrow G_{\overline{\mathbb{Q}_p}} = G\left(\prod_{\tau \in I_F} GSp_n \right) \\ z &\mapsto \prod_{\tau \in I_F} \text{diag}(\underbrace{z, \dots, z}_{\frac{n}{2}}, \underbrace{1, \dots, 1}_{\frac{n}{2}}). \end{aligned}$$

The element $b \in G(L)$ defines an isocrystal with additional structures $N_b : \text{Rep}_{\mathbb{Q}_p} G \rightarrow \text{Isoc}(\overline{\mathbb{F}_p})$ (cf. [16],[65]), in particular for the natural faithful representation V of G , $N_b(V) = (V \otimes L, b\sigma)$ is a usual isocrystal, whose Newton polygon after normalization by the action of F is just the image of the element $\nu_b \in N(G)$ defined by Kottwitz in $N(\text{Res}_{F|\mathbb{Q}_p} GL_F V)$, under the natural injection $N(G) \hookrightarrow N(\text{Res}_{F|\mathbb{Q}_p} GL_F V)$.

Here the normalization of a polygon \mathcal{P} over $[0, dn]$ ($n = \dim_F V, d = [F : \mathbb{Q}_p]$), we mean a polygon \mathcal{P}' over $[0, n]$ such that $\mathcal{P}'(x) = \frac{1}{d}\mathcal{P}(dx)$, for all $x \in [0, n]$. On the other hand, the conjugate class of μ defines a Hodge polygon (cf. [52],[65])

$$\nu_\mu := \bar{\mu} = \frac{1}{|\Gamma : \Gamma_\mu|} \sum_{\sigma \in \Gamma/\Gamma_\mu} \sigma(\mu) \in N(G).$$

We will view ν_b and $\bar{\mu}$ as polygons over $[0, n]$ by the natural injection $N(G) \hookrightarrow N(\text{Res}_{F|\mathbb{Q}_p} GL_F V)$. Note the above data defines a p -divisible group with additional structures Σ over $\overline{\mathbb{F}}_p$ up to isogeny. The polygons ν_b and $\bar{\mu}$ will be the Newton and Hodge polygons respectively of Σ . They are also the corresponding polygons of the p -divisible groups with additional structures classified by the Rapoport-Zink spaces associated to the above EL/PEL data which we review in the following.

The Rapoport-Zink spaces $\widehat{\mathcal{M}}$ associated to the simple unramified EL/PEL data $(F, V, b, \mu)/(F, *, V, \langle, \rangle, b, \mu)$ are formal schemes locally formally of finite type over $\text{Spf}W(\overline{\mathbb{F}}_p)$, as deformation spaces of p -divisible groups with additional structures by quasi-isogenies. More precisely, let $\text{Nilp}W(\overline{\mathbb{F}}_p)$ be the category of schemes over $W(\overline{\mathbb{F}}_p)$ over which p is locally nilpotent, then for any scheme $S \in \text{Nilp}W(\overline{\mathbb{F}}_p)$, in the EL case $\widehat{\mathcal{M}}(S) = \{(H, \iota, \beta)\} / \sim$; and in the PEL cases $\widehat{\mathcal{M}}(S) = \{(H, \iota, \lambda, \beta)\} / \sim$, where

- H/S is a p -divisible group;
- $\iota : O_F \rightarrow \text{End}(H)$ is an action such that

$$\det_{O_S}(a, \text{Lie}(H)) = \det(a, V_0), \forall a \in O_F,$$

here V_0 is the weight 0 subspace of $V_{\overline{\mathbb{Q}}_p}$ defined by μ ;

- $\beta : \Sigma_{\overline{S}} \rightarrow H_{\overline{S}}$ is an O_F -equivariant quasi-isogeny, here $\overline{S} \subset S$ is the closed subscheme defined by killing p ;
- in the PEL cases, $\lambda : H \rightarrow H^D$ is a polarization, compatible with the action ι , and whose pullback via β is the polarization on Σ up to a p power scalar multiple. Here as before H^D is the Cartier-Serre dual of H .
- \sim is the relation defined by isomorphisms of p -divisible groups with additional structures.

Let $J_b(\mathbb{Q}_p)$ be the group of self-quasi-isogenies of Σ as p -divisible group with additional structures over $\overline{\mathbb{F}}_p$, which is in fact the \mathbb{Q}_p -valued points of a reductive group J_b defined over \mathbb{Q}_p . Then there is an action of $J_b(\mathbb{Q}_p)$ on $\widehat{\mathcal{M}}$ defined by $\gamma \in J_b(\mathbb{Q}_p)$,

$$\gamma : \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}, (\underline{H}, \beta) \mapsto (\underline{H}, \beta \circ \gamma^{-1}).$$

Let E be the definition field of the conjugate class of μ , the so called reflex field, then there is a non-effective descent datum on $\widehat{\mathcal{M}}$ over O_E , for details see [66].

Let $\mathcal{M} = \widehat{\mathcal{M}}^{an}$ be the Berkovich analytic fiber of $\widehat{\mathcal{M}}$ over L . Then the local system \mathcal{T} over \mathcal{M} defined by the p -adic Tate module of the universal p -divisible group on $\widehat{\mathcal{M}}$ gives us a tower of Berkovich analytic spaces $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$, where for any open compact subgroup K of $G(\mathbb{Z}_p)$, \mathcal{M}_K is the finite étale covering of \mathcal{M} parameterizing the K -level structures, i.e. the classes modulo K of O_F -linear trivialization of \mathcal{T} by Λ . In particular $\mathcal{M} = \mathcal{M}_{G(\mathbb{Z}_p)}$. The action of $J_b(\mathbb{Q}_p)$ on \mathcal{M} then extends to each rigid analytic space \mathcal{M}_K , and the group $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ by Hecke correspondences.

We will be interested in the cohomology of the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ of Berkovich analytic spaces. Let $l \neq p$ be a prime number, \mathbb{C}_p be the completion of an algebraic closure of L , for any open compact subgroup $K \subset G(\mathbb{Z}_p)$, the l -adic cohomology with compact support

$$H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l(D_{\mathcal{M}})) = \varinjlim_U H_c^i(U \times \mathbb{C}_p, \overline{\mathbb{Q}}_l(D_{\mathcal{M}}))$$

was defined, for details see 4.2 of [23]. Here

$$D_{\mathcal{M}} = \dim \mathcal{M}_K = \begin{cases} \sum_{\tau \in I_F} p_{\tau} q_{\tau} & \text{EL case} \\ \frac{1}{2} \sum_{\tau \in I_F} p_{\tau} q_{\tau} & \text{PEL unitary case} \\ d \frac{n}{2} (\frac{n}{2} + 1) / 2 & \text{PEL symplectic case.} \end{cases}$$

Following Mantovan, we will consider the following groups

$$H^{i,j}(\mathcal{M}_{\infty})_{\rho} := \varinjlim_K \text{Ext}_{J_b(\mathbb{Q}_p)}^j(H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l(D_{\mathcal{M}})), \rho),$$

for any admissible $\overline{\mathbb{Q}}_l$ -representation ρ of $J_b(\mathbb{Q}_p)$. By [55], these groups vanish for almost all $i, j \geq 0$, and there is a natural action of $G(\mathbb{Q}_p) \times W_E$ on them. Moreover, as a representation of $G(\mathbb{Q}_p) \times W_E$, $H^{i,j}(\mathcal{M}_{\infty})_{\rho}$ is admissible/continuous. For any admissible $\overline{\mathbb{Q}}_l$ -representation ρ of $J_b(\mathbb{Q}_p)$, we define a virtual representation of $G(\mathbb{Q}_p) \times W_E$:

$$H(\mathcal{M}_{\infty})_{\rho} = \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathcal{M}_{\infty})_{\rho}.$$

To apply our results on the Hodge-Newton filtration, we make as before the following basic assumption :

(HN) : ν_b and $\bar{\mu}$ possess a contact point x outside their extremal points which is a break point for the polygon ν_b .

Thus in the PEL cases, we have a symmetric point \hat{x} of x , which satisfies also the above condition. In these cases, if $x = (x_1, x_2)$, we may assume $x_1 \leq n/2$.

By the assumption, we can choose decompositions $V = V^1 \oplus V^2$ (EL case) or $V = V^1 \oplus V^2 \oplus V^3$ (PEL cases), such that $N_b(V) = N_b(V^1) \oplus N_b(V^2)$ or $N_b(V) = N_b(V^1) \oplus N_b(V^2) \oplus N_b(V^3)$ is the decomposition of the isocrystals corresponding to the break point x or \hat{x} . In the PEL cases, when $x = \hat{x}$ then V^2 is trivial. Let Λ be a fixed lattice in V for the EL case and an auto-dual lattice for the PEL cases. Then we can choose decompositions $\Lambda = \Lambda^1 \oplus \Lambda^2$ (EL case) or $\Lambda = \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3$ (PEL cases), such that they induce the above decompositions for V .

Associated to the decompositions $V = \bigoplus_{i=1}^t V^i$, $t = 2$ or 3 , we have a Levi subgroup M of G over \mathbb{Q}_p , such that for all \mathbb{Q}_p -algebra R ,

$$M(R) = \{g \in G(R) | g \text{ stabilizes } V_{i,R}^i, \forall 1 \leq i \leq t\}.$$

Similarly, if we consider the filtrations $0 \subset V_1 \subset \dots \subset V_t = V$, $t = 2$ or 3 , where $V_i = \bigoplus_{1 \leq j \leq i} V^j$, $\forall 1 \leq i \leq t$, we can define a parabolic subgroup P of G over \mathbb{Q}_p , such that for all \mathbb{Q}_p -algebra R ,

$$P(R) = \{g \in G(R) | g \text{ stabilizes } V_{i,R}, \forall 1 \leq i \leq t\}.$$

Clearly, $M \subset P$. We denote by $P = MN$ the Levi decomposition of P , here N is the unipotent radical of P . By definition, we have $b \in M(L) \subset P(L) \subset G(L)$ up to σ -conjugacy. There is an element ω_b in the absolute Weyl group of G , such that $\omega_b \mu$ factors through M and up to σ -conjugacy $b \in B(M.\omega_b \mu)$, see [56] or [34]. The above choices of decompositions of lattices imply M, P are unramified.

Mantovan in [56] constructed two other type Rapoport-Zink spaces $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{F}}$ for the data $(M, b, \omega_b \mu)$ and $(P, b, \omega_b \mu)$ respectively. We briefly recall the definition of these spaces. Both are formal schemes of formally locally of finite type over $\text{Spf}W(\overline{\mathbb{F}}_p)$, and classify some type of p -divisible groups with additional structures. More precisely, for any $S \in \text{Nilp}W(\overline{\mathbb{F}}_p)$, $\widehat{\mathcal{P}}(S) = \{(H^i, \iota^i, \beta^i)_{1 \leq i \leq t}\} / \sim$ in the EL case, and $\widehat{\mathcal{P}}(S) = \{(H^i, \iota^i, \lambda^i, \beta^i)_{1 \leq i \leq t}\} / \sim$ in the PEL cases, where

- H^i/S are p -divisible groups;
- $\iota^i : O_F \rightarrow \text{End}(H^i)$ are actions of O_F on H^i ;
- $\beta^i : \Sigma_{\overline{S}}^i \rightarrow H_{\overline{S}}^i$ are quasi-isogenies, commuting with the action of O_F ;
- in the PEL cases, $\lambda^i : H^i \rightarrow (H^j)^D, i + j = t + 1$, are isomorphisms and $(\lambda^i)^D = -\lambda^j$; such that
 1. $\det_{O_S}(a, \text{Lie}(H^i)) = \det(a, V_0^i), \forall a \in O_F, 1 \leq i \leq t$;
 2. in the PEL cases, there exists $c \in \mathbb{Q}_p^\times$ such that $\lambda^i = c(\beta^{jD})^{-1} \circ \phi^i \circ (\beta^i)^{-1}$ for all i, j such that $i + j = t + 1$. Here $\phi^i : \Sigma^i \rightarrow (\Sigma^j)^D$ are the isomorphisms induced by the polarization $\phi : \Sigma \rightarrow \Sigma^D$.

- \sim is the relation defined by isomorphisms.

As the case of the tower of Rapoport-Zink spaces $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$, we may consider the Berkovich analytic fiber $\mathcal{P} = \widehat{\mathcal{P}}^{an}$ of $\widehat{\mathcal{P}}$, and use the local system provided by the universal Tate module on \mathcal{P} to construct a tower of Berkovich analytic spaces $(\mathcal{P}_K)_{K \subset M(\mathbb{Z}_p)}$ indexed by open compact subgroups $K \subset M(\mathbb{Z}_p)$. These spaces in fact can be decomposed as product of some smaller Rapoport-Zink spaces defined by the EL/PEL data $(F, V^i, b^i, (\omega_b \mu)^i) / (F, *, V^i, \langle, \rangle, b^i, (\omega_b \mu)^i)$, for more details see section 3 of [56]. There are natural actions of $J_b(\mathbb{Q}_p)$ on each spaces \mathcal{P}_K , and the group $M(\mathbb{Q}_p)$ acts on the tower $(\mathcal{P}_K)_{K \subset M(\mathbb{Z}_p)}$ as Hecke correspondences. Similarly, there is a non-effective descent datum on each of these spaces over E .

The filtration $0 \subset N_b(V_1) \subset \dots \subset N_b(V_t) = N_b(V)$ induce a filtration of p -divisible groups with additional structures over $\overline{\mathbb{F}}_p$:

$$0 \subset \Sigma_1 \subset \dots \subset \Sigma_t = \Sigma.$$

For any $S \in \text{Nilp}W(\overline{\mathbb{F}}_p)$, $\widehat{\mathcal{F}}(S) = \{(H, \iota, H_\bullet, \beta)\} / \sim$ in the EL case, and $\widehat{\mathcal{F}}(S) = \{(H, \iota, \lambda, H_\bullet, \beta)\} / \sim$ in the PEL cases, where

- H/S is a p -divisible group;
- $\iota : O_F \rightarrow \text{End}(H)$ is an action of O_F on H ;
- in the PEL cases, $\lambda : H \rightarrow H^D$ is an isomorphism compatible with the action of O_F ;
- $H_\bullet = (0 \subset H_1 \subset \dots \subset H_t = H)$ is an increasing filtration of H by O_F -sub- p -divisible groups over S , such that in the PEL cases λ induces isomorphisms $H_i \simeq (H/H_j)^D$ for $i + j = t + 1$;
- $\beta : \Sigma_{\overline{S}} \rightarrow H_{\overline{S}}$ is a quasi-isogeny of p -divisible groups with additional structures, and compatible with the filtration, i.e. $\beta(\Sigma_{j\overline{S}}) \subset H_{j\overline{S}}$ for any $j = 1, \dots, t$; satisfying the following conditions

1. the restrictions of β to the p -divisible subgroups defining the filtration

$$\beta_j : \Sigma_{j\bar{S}} \rightarrow H_{j\bar{S}}$$

are quasi-isogenies ;

- 2.

$$\det_{O_S}(a, \text{Lie}(H_j)) = \det(a, V_{0j}), \forall a \in O_F, j = 1, \dots, t.$$

– \sim is the relation defined by isomorphisms.

As usual, we consider the Berkovich analytic fiber $\mathcal{F} = \widehat{\mathcal{F}}^{an}$ of \mathcal{F} over L , and we can construct a tower of Berkovich analytic spaces $(\mathcal{F}_K)_{K \subset P(\mathbb{Z}_p)}$ indexed by open compact subgroups $K \subset P(\mathbb{Z}_p)$, see [56] definition 10. There are then natural actions of $J_b(\mathbb{Q}_p)$ on each spaces \mathcal{F}_K , and the group $P(\mathbb{Q}_p)$ acts on the tower $(\mathcal{F}_K)_{K \subset P(\mathbb{Z}_p)}$ as Hecke correspondences. Moreover, there is a non-effective descent datum of \mathcal{F}_K over E .

As the case of Rapoport-Zink spaces, we will consider the groups

$$H^{i,j}(\mathcal{P}_\infty)_\rho := \varinjlim_K \text{Ext}_{J_b(\mathbb{Q}_p)}^j(H_c^i(\mathcal{P}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l(D_{\mathcal{P}})), \rho)$$

$$H^{i,j}(\mathcal{F}_\infty)_\rho := \varinjlim_K \text{Ext}_{J_b(\mathbb{Q}_p)}^j(H_c^i(\mathcal{F}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l(D_{\mathcal{F}})), \rho)$$

for any admissible $\overline{\mathbb{Q}}_l$ -representation ρ of $J_b(\mathbb{Q}_p)$. Here $D_{\mathcal{P}}$ (resp. $D_{\mathcal{F}}$) is the dimension of \mathcal{P} (resp. \mathcal{F}). These groups vanish for almost all $i, j \geq 0$, and as $M(\mathbb{Q}_p) \times W_E$ and $P(\mathbb{Q}_p) \times W_E$ representations respectively they are both admissible/continuous, cf. [56] theorem 12. We consider the virtual representations

$$H(\mathcal{P}_\infty)_\rho = \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathcal{P}_\infty)_\rho$$

$$H(\mathcal{F}_\infty)_\rho = \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathcal{F}_\infty)_\rho.$$

We would like to compare these representations with

$$H(\mathcal{M}_\infty)_\rho = \sum_{i,j \geq 0} (-1)^{i+j} H^{i,j}(\mathcal{M}_\infty)_\rho.$$

This is achieved by considering the relations between the three towers of Berkovich analytic spaces : $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$, $(\mathcal{F}_K)_{K \subset P(\mathbb{Z}_p)}$, $(\mathcal{P}_K)_{K \subset M(\mathbb{Z}_p)}$. More precisely, we have the following diagram of morphisms of Berkovich analytic spaces :

$$\begin{array}{ccc} & \mathcal{F} & \\ s \nearrow & & \searrow \pi_2 \\ \mathcal{P} & & \mathcal{M} \\ \pi_1 \searrow & & \end{array}$$

where in the PEL cases

$$s : (H^i, \iota^i, \lambda^i, \beta^i)_{1 \leq i \leq t} \mapsto (\oplus_{i=1}^t H^i, \oplus_{i=1}^t \iota^i, \oplus_{i=1}^t \lambda^i, H_\bullet, \oplus_{i=1}^t \beta^i)$$

$$\pi_1 : (H, \iota, \lambda, H_\bullet, \beta) \mapsto (gr^i H, \iota^i, \lambda^i, \beta^i)_{1 \leq i \leq t}$$

$$\pi_2 : (H, \iota, \lambda, H_\bullet, \beta) \mapsto (H, \iota, \lambda, \beta),$$

here the filtration H_\bullet in the right hand side of the first arrow is the natural one, the $\iota^i, \lambda^i, \beta^i$ in the right hand side of the second arrow are induced by ι, λ, β on the graded pieces, for $i = 1, \dots, t$.

By construction, we have the following facts.

- Proposition 1.6.1** ([56], Prop. 14, 28, Theorem 36 (3)).
1. s is a closed immersion;
 2. π_1 is a fibration in balls;
 3. π_2 is a local isomorphism onto its image.

In fact, to find the relation between the cohomology groups $H(\mathcal{P}_\infty)_\rho$ and $H(\mathcal{F}_\infty)_\rho$, one has to consider the geometry between \mathcal{F}_K and $\mathcal{P}_K := \mathcal{P}_{K \cap M(\mathbb{Q}_p)}$ for any open compact subgroup $K \subset P(\mathbb{Z}_p)$. We extend the action of $M(\mathbb{Q}_p)$ on the tower $(\mathcal{P}_K)_{K \subset M(\mathbb{Z}_p)}$ to an action of $P(\mathbb{Q}_p)$ on this tower with the unipotent radical of $P(\mathbb{Q}_p)$ acts trivially. In this case, there are $J_b(\mathbb{Q}_p) \times P(\mathbb{Q}_p)$ -equivariant closed immersions

$$s_K : \mathcal{P}_K \longrightarrow \mathcal{F}_K$$

commute with the descent data, for $K \subset P(\mathbb{Z}_p)$ varies. Moreover, there are $J_b(\mathbb{Q}_p) \times P(\mathbb{Q}_p)$ -equivariant morphisms of analytic spaces

$$\pi_{1K} : \mathcal{F}_K \longrightarrow \mathcal{P}_K$$

commute with the descent data, for $K \subset P(\mathbb{Z}_p)$ varies, such that

$$\pi_{1K} \circ s_K = id_{\mathcal{P}_K}.$$

For $K \subsetneq P(\mathbb{Z}_p)$, π_{1K} are not necessarily fibrations in balls and their fibers may change. Mantovan's solution of this problem is that for each integer $m \geq 1$, she introduces a formal scheme $j_m : \widehat{\mathcal{F}}_m \rightarrow \widehat{\mathcal{F}}$ over $\widehat{\mathcal{F}}$, such that for any morphism of formal schemes $f : S \rightarrow \widehat{\mathcal{F}}$, the p^m -torsion subgroup $f^*\mathcal{H}[p^m]$ is split if and only if f factors through j_m . Here \mathcal{H} is the universal p -divisible group over $\widehat{\mathcal{F}}$. By definition, one has a formal model $\widehat{\pi}_1 : \widehat{\mathcal{F}} \rightarrow \widehat{\mathcal{P}}$ of π_1 . Then one has the fact that formal schemes $\widehat{\mathcal{F}}_m$ and $\widehat{\mathcal{F}}$ are isomorphic when considered as formal schemes over $\widehat{\mathcal{P}}$, via $\widehat{\pi}_1 \circ j_m$ and $\widehat{\pi}_1$ respectively, cf. [56] proposition 30 (2). Thus the formal schemes $\widehat{\mathcal{F}}_m$ can be viewed as some twisted version of $\widehat{\mathcal{F}}$. Let $K = K_m := \ker(P(\mathbb{Z}_p) \rightarrow P(\mathbb{Z}_p/p^m\mathbb{Z}_p))$ for the natural projection $P(\mathbb{Z}_p) \rightarrow P(\mathbb{Z}_p/p^m\mathbb{Z}_p)$, and \mathcal{F}_m be the analytic generic fiber of $\widehat{\mathcal{F}}_m$, then one can define a cover $f_{mK} : \mathcal{F}_{mK} \rightarrow \mathcal{F}_m$ by the pullback of $\mathcal{F}_K \rightarrow \mathcal{F}$, i.e.

$$\mathcal{F}_{mK} = \mathcal{F}_K \times_{\mathcal{F}, j_{m\eta}} \mathcal{F}_m.$$

Let $j_{mK} : \mathcal{F}_{mK} \rightarrow \mathcal{F}_K$ be the natural projection, we have the following cartesian diagram

$$\begin{array}{ccc} \mathcal{F}_{mK} & \xrightarrow{f_{mK}} & \mathcal{F}_m \\ \downarrow j_{mK} & & \downarrow j_{m\eta} \\ \mathcal{F}_K & \longrightarrow & \mathcal{F}. \end{array}$$

On the other hand one can also define a cover $f'_{mK} : \mathcal{F}'_{mK} \rightarrow \mathcal{F}_m$ by the pullback of $\mathcal{P}_K \rightarrow \mathcal{P}$ via $\pi_1 \circ j_{m\eta} : \mathcal{F}_m \rightarrow \mathcal{P}$, which is the same with $\pi_1 : \mathcal{F}_m \rightarrow \mathcal{P}$, i.e.

$$\mathcal{F}'_{mK} = \mathcal{P}_K \times_{\mathcal{P}, \pi_1} \mathcal{F}_m.$$

Let π_{1mK} be the natural projection $\mathcal{F}'_{mK} \rightarrow \mathcal{P}_K$, we have the following cartesian diagram

$$\begin{array}{ccc} \mathcal{F}'_{mK} & \xrightarrow{f'_{mK}} & \mathcal{F}_m \\ \downarrow \pi_{1mK} & & \downarrow \pi_1 \\ \mathcal{P}_K & \longrightarrow & \mathcal{P}. \end{array}$$

There are morphisms $\phi_K : \mathcal{F}_{mK} \rightarrow \mathcal{F}'_{mK}$ and $\varphi_K : \mathcal{F}'_{mK} \rightarrow \mathcal{F}_{mK}$ such that $\phi_K \circ \varphi_K = id_{\mathcal{F}'_{mK}}$ and $\pi_{1mK} \circ \phi_K = \pi_{1K} \circ j_{mK}$. Since π_1 is a fibration in balls, by the base change theorem for the cohomology with compact support of analytic spaces and proposition 1.6.1 (2), one has a quasi-isomorphism of cohomological complex

$$R\Gamma_c(\mathcal{F}'_{mK} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq R\Gamma_c(\mathcal{P}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l(-d))[-2d], \quad d = D_{\mathcal{F}} - D_{\mathcal{P}}.$$

In proposition 32 in loc. cit. Mantovan studied the relation between $\mathcal{F}_K, \mathcal{F}_{mK}$ and \mathcal{F}'_{mK} , from which she can deduce a quasi-isomorphism

$$R\Gamma_c(\mathcal{F}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq R\Gamma_c(\mathcal{F}'_{mK} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Thus one has the following proposition.

Proposition 1.6.2 ([56], Theorem 26). *For any admissible $\overline{\mathbb{Q}}_l$ -representation ρ of $J_b(\mathbb{Q}_p)$, we have an equality of virtual representations of $P(\mathbb{Q}_p) \times W_E$:*

$$H(\mathcal{P}_\infty)_\rho = H(\mathcal{F}_\infty)_\rho.$$

To find the relation between $H(\mathcal{M}_\infty)_\rho$ and $H(\mathcal{P}_\infty)_\rho$, we use our main result on the Hodge-Newton filtration. Under our basic assumption, which is weaker than that in [56], the results of last subsection on the existence and uniqueness of Hodge-Newton filtration tell us

Proposition 1.6.3. π_2 is bijective, thus it is an isomorphism of Berkovich analytic spaces.

Proof. This is a direct consequence of theorem 1.5.4, 1.5.7, and proposition 1.6.1 (3). \square

For an open compact subgroup $K \subset G(\mathbb{Z}_p)$, denote $\mathcal{F}_K := \mathcal{F}_{K \cap P(\mathbb{Q}_p)}$, then we have a natural morphism $\pi_{2K} : \mathcal{F}_K \rightarrow \mathcal{M}_K$ such that $\pi_{2G(\mathbb{Z}_p)} = \pi_2$ which is defined above. Let $\mathcal{P}_K := \mathcal{P}_{K \cap M(\mathbb{Q}_p)}$, we have also natural generalizations s_K, π_{1K} of s and π_1 respectively. Moreover, we have the following diagram in level K

$$\begin{array}{ccc} & \mathcal{F}_K & \\ s_K \nearrow & & \searrow \pi_{2K} \\ \mathcal{P}_K & & \mathcal{M}_K \\ \pi_{1K} \searrow & & \nearrow \end{array}$$

Corollary 1.6.4. *With the above notation, π_{2K} is a closed immersion, and we have isomorphisms*

$$\mathcal{M}_K \simeq \mathcal{M}_K \times_{\mathcal{M}} \mathcal{F} \simeq \coprod_{K \backslash G(\mathbb{Q}_p) / P(\mathbb{Q}_p)} \mathcal{F}_{K \cap P(\mathbb{Q}_p)}.$$

Proof. One argues exactly as [56] 8.2. □

Before passing to cohomological conclusion, we consider some application to the p -adic period morphism and monodromy representations. Recall that in [66] chapter 5 Rapoport-Zink had defined a p -adic period morphism

$$\pi : \mathcal{M} \longrightarrow \mathcal{F}^a := \mathcal{F}(G, \mu)^a \subset \mathcal{F}(G, \mu)^{an},$$

which is $G(\mathbb{Q}_p)$ -invariant and $J_b(\mathbb{Q}_p)$ -equivariant. Here \mathcal{F}^a is the image of π , which is an open subspace of $\mathcal{F}(G, \mu)^{an}$, the associated Berkovich analytic space of $\mathcal{F}(G, \mu) := G_L/P_{\mu L}$. Here P_{μ} is the parabolic subgroup defined by μ over the reflex field E , and $P_{\mu L}$ is its base change over L . The definition of π for rigid points is as follow. Associated to a rigid point $x \in \mathcal{M}(K)$ ($K|L$ is thus a finite extension) there is the p -divisible group with additional structures (H, ι, λ) over O_K and the quasi-isogeny $\rho : \Sigma_{O_K/pO_K} \rightarrow H_{O_K/pO_K}$, which defines an isomorphism

$$\rho_* : (V_L, b\sigma) \xrightarrow{\sim} (\mathbb{D}(H_k)_L, \varphi).$$

Let

$$\text{Fil}_{\pi(x)} V_K = \rho_*^{-1}(\omega_{H^D, K}) \subset V_K$$

for the Hodge filtration sequence

$$0 \rightarrow \omega_{H^D, K} \rightarrow \mathbb{D}(H_k)_K \rightarrow \text{Lie}(H)_K \rightarrow 0.$$

Then by definition $\pi(x) = \text{Fil}_{\pi(x)} V_K \in \mathcal{F}^a(K)$.

In our situation, we have also the p -adic period morphism which is still denoted by π by abuse of notation

$$\pi : \mathcal{P} \longrightarrow \mathcal{F}(M, \omega_b \mu)^a \subset \mathcal{F}(M, \omega_b \mu)^{an},$$

for the Rapoport-Zink \mathcal{P} . Let

$$\pi : \mathcal{F} \longrightarrow \mathcal{F}(G, \mu)^{an}$$

be the composition of $\pi_2 : \mathcal{F} \rightarrow \mathcal{M}$ and $\pi : \mathcal{M} \rightarrow \mathcal{F}(G, \mu)^{an}$, and $\mathcal{F}^a(P, \omega_b \mu)$ be its image, which is contained in the p -adic Schubert cell $\mathcal{F}^{an}(P, \omega_b \mu) = (P_L \omega_b P_{\mu L} / P_{\mu L})^{an} \subset \mathcal{F}^{an}(G, \mu)$. We have the following enlarged diagram :

$$\begin{array}{ccccc}
 & & \mathcal{F} & & \\
 & \swarrow \pi_1 & \downarrow & \searrow \pi_2 & \\
 \mathcal{P} & & \mathcal{F}^a(P, \omega_b \mu) & & \mathcal{M} \\
 \downarrow & \swarrow \pi'_1 & & \searrow \pi'_2 & \downarrow \\
 \mathcal{F}^a(M, \omega_b \mu) & & & & \mathcal{F}^a(G, \mu).
 \end{array}$$

Here ω_b is the element in the absolute Weyl group of G as above, which is contained in a double coset $:\bar{\omega}_b \in W_P \backslash W/W_{P_{\mu}}$, P_{μ} is the parabolic subgroup of G_E defining $\mathcal{F}(G, \mu)$. We have

1. π'_1 is a fibration in affine analytic spaces;
2. π'_2 is the identity, i.e. $\mathcal{F}^a(G, \mu)$ is contained in the p -adic Schubert cell $(P_L \omega_b P_{\mu L} / P_{\mu L})^{an}$.

Let \mathcal{T} be the \mathbb{Z}_p -local system over \mathcal{M} defined by the Tate module of the universal p -divisible group over $\widehat{\mathcal{M}}$, then it descends to a \mathbb{Q}_p -local system over the p -adic period domain \mathcal{F}^a . Let \bar{x} be a geometric point of \mathcal{M} and \bar{y} be its image under the p -adic period morphism $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$. Then by [18] theorem 4.2, these local systems define monodromy representations

$$\rho_{\bar{x}} : \pi_1(\mathcal{M}, \bar{x}) \longrightarrow G(\mathbb{Z}_p)$$

and

$$\rho_{\bar{y}} : \pi_1(\mathcal{F}^a, \bar{y}) \longrightarrow G(\mathbb{Q}_p)$$

respectively. Here $\pi_1(X, \bar{x})$ is the fundamental group defined by de Jong in loc. cit. for a Berkovich analytic space X and a geometric point \bar{x} of X . Then under our basic assumption (HN) and the notations above, the existence of Hodge-Newton filtration implies the following.

Corollary 1.6.5. *The monodromy representations $\rho_{\bar{x}}$ and $\rho_{\bar{y}}$ factor through $P(\mathbb{Z}_p)$ and $P(\mathbb{Q}_p)$ respectively.*

In [15], M. Chen has constructed some determinant morphisms for the towers of simple unramified Rapoport-Zink spaces. Under the condition that there is no non-trivial contact point of the Newton and Hodge polygons, and assume the conjecture that

$$\pi_0(\widehat{\mathcal{M}}) \simeq \text{Im} \varkappa$$

for the morphism $\varkappa : \widehat{\mathcal{M}} \rightarrow \Delta$ constructed in [66] 3.52, Chen proved that the associated monodromy representation under this condition is maximal, and thus the geometric fibers of her determinant morphisms are exactly the geometric connected components, see théorème 5.1.2.1 and 5.1.3.1 of loc. cit.. Our result confirms that the condition that “there is no non-trivial contact point of the Newton and Hodge polygons” is thus necessary, see the remark in 5.1.5 of loc. cit.. In the split cases considered in [57] and [56], their results already confirmed that the above condition is necessary.

Now we look at the cohomological consequence of the existence of Hodge-Newton filtration. Proposition 1.6.2 and corollary 1.6.4 together imply that

$$\begin{aligned} H(\mathcal{M}_\infty)_\rho &= \text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H(\mathcal{F}_\infty)_\rho \\ &= \text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H(\mathcal{P}_\infty)_\rho. \end{aligned}$$

We summarize as the following theorem.

Theorem 1.6.6. *Under the assumption (HN), we have an equality of virtual representations of $G(\mathbb{Q}_p) \times W_E$:*

$$H(\mathcal{M}_\infty)_\rho = \text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H(\mathcal{P}_\infty)_\rho.$$

In particular, there is no supercuspidal representations of $G(\mathbb{Q}_p)$ appear in the virtual representation $H(\mathcal{M}_\infty)_\rho$.

1.7 Application to the cohomology of some Shimura varieties

The above theorem generalizes the main result of [56]. As there, we can consider further the application to the cohomology of Newton strata of some more general PEL-type Shimura varieties. We will not pursue the full generalities here, but concentrate on the Shimura varieties studied in [10] and [81], since studying these varieties was one of the motivation of this thesis as said in the introduction. They are closely related to the Shimura varieties studied by Harris and Taylor, but the local reductive groups involved are unitary groups. Note also they are not in the class of Shimura varieties studied in section 11 of [56].

More precisely, let $Sh_{G(\mathbb{Z}_p) \times K^p} / O_{E_\nu}$ be a smooth PEL-type Shimura variety over the integer ring of E_ν as in [10] or [81], the local reflex field which is a quadratic unramified extension of \mathbb{Q}_p if $n \neq 2$ and \mathbb{Q}_p if $n = 2$. Let $\overline{Sh}_{G(\mathbb{Z}_p) \times K^p}$ be its special fiber, then we have the Newton polygon stratification

$$\overline{Sh}_{G(\mathbb{Z}_p) \times K^p} = \coprod_{b \in B(G, \mu)} \overline{Sh}_{G(\mathbb{Z}_p) \times K^p}^{(b)}.$$

Here in this special case, $G_{\mathbb{Q}_p}$ is isomorphic to a simple PEL unitary group in our notions by the Morita equivalence, and $B(G, \mu)$ is in bijection with the set of polygons defined in [81] (3.1), in particular, any non basic element b satisfy our assumption (HN).

Let $R^j \Psi_\eta(\overline{\mathbb{Q}}_l)$, $j \geq 0$ denote the l -adic nearby cycles of some fixed integral models of the Shimura varieties $Sh_{K_p \times K^p}$ with some level structures K_p at p , defined for example as in [55] for Drinfeld level structures, or the book [66] for parahoric level structures. Then we have also the Newton polygon stratification for the special fibers with level structures at p , and for each $b \in B(G, \mu)$, we have the virtual representation of $G(\mathbb{A}_f) \times W_{E_\nu}$

$$H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l)) := \sum_{i, j \geq 0} (-1)^{i+j} \varinjlim_{K_p \times K^p} H_c^i(\overline{Sh}_{K_p \times K^p}^{(b)} \times \overline{\mathbb{F}}_p, R^j \Psi_\eta(\overline{\mathbb{Q}}_l)).$$

Let $\widehat{Sh}_{G(\mathbb{Z}_p) \times K^p}$ be the p -adic completion of $Sh_{G(\mathbb{Z}_p) \times K^p}$, $\widehat{Sh}_{G(\mathbb{Z}_p) \times K^p}^{an}$ be the Berkovich analytic fiber of this formal scheme. For any open compact subgroup $K_p \subset G(\mathbb{Z}_p)$, let $\widehat{Sh}_{K_p \times K^p}^{an}$ be the étale covering of $\widehat{Sh}_{G(\mathbb{Z}_p) \times K^p}^{an}$ defined by trivializing the Tate module in the usual way. When K_p is a Drinfeld level structure subgroup or a parahoric subgroup, $\widehat{Sh}_{K_p \times K^p}^{an}$ has a formal model : the p -adic completion $\widehat{Sh}_{K_p \times K^p}$ of $Sh_{K_p \times K^p}$. Then the theory of formal vanishing cycles tells us we have the equality of cohomology

$$R\Gamma(\overline{Sh}_{K_p \times K^p} \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l)) = R\Gamma(\widehat{Sh}_{K_p \times K^p}^{an} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Thus we have the equality of virtual representations

$$\sum_{i, j \geq 0} (-1)^{i+j} \varinjlim_{K_p \times K^p} H^i(\overline{Sh}_{K_p \times K^p} \times \overline{\mathbb{F}}_p, R^j \Psi_\eta(\overline{\mathbb{Q}}_l)) = \sum_{i \geq 0} (-1)^i \varinjlim_{K_p \times K^p} H^i(\widehat{Sh}_{K_p \times K^p}^{an} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

If $Sh_{G(\mathbb{Z}_p) \times K^p}$ is proper, (for example if $B = V$ as the notations in [10] and [81], these Shimura varieties have the same generic fibers as that of the Shimura varieties of some special cases studied by [36],) then we have

$$\widehat{Sh}_{K_p \times K^p}^{an} = Sh_{K_p \times K^p}^{an},$$

where the later is the associated Berkovich analytic space of the Shimura varieties $Sh_{K_p \times K^p}$ over E_ν . Thus we have the equality of virtual representations

$$\begin{aligned} \sum_{i \geq 0} (-1)^i H^i(Sh_\infty \times \overline{E}_\nu, \overline{\mathbb{Q}}_l) &:= \sum_{i \geq 0} (-1)^i \varinjlim_K H^i(Sh_K \times \overline{E}_\nu, \overline{\mathbb{Q}}_l) \\ &= \sum_{i \geq 0} (-1)^i \varinjlim_K H^i(Sh_K^{an} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \\ &= \sum_{i, j \geq 0} (-1)^{i+j} \varinjlim_{K_p \times K^p} H^i(\overline{Sh}_{K_p \times K^p} \times \overline{\mathbb{F}}_p, R^j \Psi_\eta(\overline{\mathbb{Q}}_l)) \\ &= \sum_{b \in B(G, \mu)} H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l)). \end{aligned}$$

The main results of [55] tell us the cohomology of each Newton polygon strata can be computed in terms of the l -adic cohomology of the corresponding Rapoport-Zink spaces and Igusa varieties. More precisely, we have the formula

$$\begin{aligned} &H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l)) \\ &= \sum_{i, j, k \geq 0} (-1)^{i+j+k} \varinjlim_{K_p} \text{Ext}_{J_b(\mathbb{Q}_p)}^i (H_c^j(\mathcal{M}_{K_p} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l(D_{\mathcal{M}})), H_c^k(Ig_b, \overline{\mathbb{Q}}_l)), \end{aligned}$$

see [55] for the precise definition of the Igusa varieties and their cohomology. Thus the main results of this section imply in particular

Corollary 1.7.1. *For the Shimura varieties studied by [10],[81], for any non-basic strata, the cohomology group $H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l))$ can be written as some suitable parabolic induction of virtual representations, and thus contains no supercuspidal representations of $G(\mathbb{Q}_p)$.*

Finally note that in a recent preprint [47], Imai and Mieda have proven that, for non-proper Shimura varieties the supercuspidal parts (see [23] definition 7.1.4, 8.1.2 for example) of the compactly supported (or intersection) cohomology and nearby cycle cohomology are the same :

$$H_c^i(Sh_\infty \times \overline{E}_\nu, \overline{\mathbb{Q}}_l)_{cusp} = H_c^i(\overline{Sh}_\infty \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l))_{cusp}.$$

Since one has the equality of virtual representations

$$\sum_{i \geq 0} (-1)^i H_c^i(\overline{Sh}_\infty \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l)) = \sum_{b \in B(G, \mu)} H_c(\overline{Sh}_\infty^{(b)} \times \overline{\mathbb{F}}_p, R\Psi_\eta(\overline{\mathbb{Q}}_l)),$$

combined with Mantovan's formula above we find that, once a non-basic Newton polygon has a nontrivial contact point to the Hodge polygon, which is a break point for the Newton polygon, then there is no contribution of the cohomology of this non-basic strata to the supercuspidal part of the cohomology (of whichever kind) of the non-proper Shimura varieties. For example, this is the case for the two non-basic stratas for the Shimura varieties associated to $GS p_4$.

2 Cell decomposition of some unitary group Rapoport-Zink spaces

2.1 Introduction

The motivation of this section is the realization of local Langlands correspondences in the cohomology of Rapoport-Zink spaces, see [70] and [34]. We have seen these spaces in section 1.6. These spaces are local analogues of PEL type Shimura varieties, and they uniformize some parts of these type Shimura varieties when passing to formal completion and p -adic analyticification.

The most well known Rapoport-Zink spaces are the Lubin-Tate spaces and Drinfeld spaces. In [11], it is conjectured the cohomology of Lubin-Tate spaces realizes the local Langlands and Jacquet-Langlands correspondences for GL_n . This has been essentially proved by Harris-Taylor in [36], and completed by many other authors. In [21] and [25], Faltings and Fargues has established an isomorphism between the Lubin-Tate tower and the Drinfeld tower, and deduced also an isomorphism of the cohomology of the two towers. Thus the cohomology of the tower of Drinfeld spaces also realizes the local Langlands and Jacquet-Langlands correspondences for GL_n , as predicted originally by Drinfeld and partly realized by Harris [33].

The description of the cohomology of some other Rapoport-Zink spaces in terms of irreducible smooth representations of the underlying p -adic reductive groups, has been done successfully by Fargues in [23] and Shin in [75]. Both of them use global methods as that of Harris-Taylor, although their approaches are quite different : the former uses heavily rigid analytic geometry while the later is based on the stable trace formula. Their results are both about Rapoport-Zink spaces of EL type, and Fargues has also obtained results of the Rapoport-Zink spaces for $GU(3)$, based on the complete classification of automorphic representations for unitary groups in three variables in [69]. It would be nice if one can prove these local results by local methods. This will require a careful study of the geometry of Rapoport-Zink spaces, and then pass to cohomological applications. Some works in this direction are as [20], [77], [84].

In [25], the first step of the construction of an isomorphism between the towers of Lubin-Tate and Drinfeld, is by “ p -adicify” the Lubin-Tate tower. This p -adicify procedure is to glue some formal models of Gross-Hopkins’s fundamental domain.

To be precise, let $\widehat{\mathcal{M}}_{LT}$ be the formal Lubin-Tate space over $SpfW(\overline{\mathbb{F}}_p)$ for GL_n/\mathbb{Q}_p for simple in this introduction. Recall that for a scheme $S \in \text{Nilp}W$, a S -valued point of $\widehat{\mathcal{M}}_{LT}$ is given by a pair (H, ρ) , with H a one dimensional formal p -divisible group over S , and $\rho : \mathbf{H}_{\overline{S}} \rightarrow H_{\overline{S}}$ is a quasi-isogeny. Here $W = W(\overline{\mathbb{F}}_p)$, $\text{Nilp}W$ is the category of schemes S over $SpecW$ such that p is locally nilpotent over S , \overline{S} is the closed subscheme of S defined by p , and \mathbf{H} is the unique one dimensional formal p -divisible group of height n over $\overline{\mathbb{F}}_p$. This space decomposes as a disjoint union of open and closed formal subschemes according the height of quasi-isogeny. The associated p -adic Lubin-Tate space (in the sense of Berkovich) $\mathcal{M}_{LT} = \coprod_{i \in \mathbb{Z}} \mathcal{M}_{LT}^i$ admits an action by $GL_n(\mathbb{Q}_p) \times D^\times$, here D is the division algebra of invariant $\frac{1}{n}$ over \mathbb{Q}_p . The action of D^\times is just changing the quasi-isogeny, while the action of $GL_n(\mathbb{Q}_p)$ is a little complicated : it is defined by the Hecke correspondences, see [66] or section 2 of this article for details.

There is a p -adic period mapping

$$\pi : \mathcal{M}_{LT} \rightarrow \mathbf{P}^{n-1, an}$$

of p -adic analytic spaces over $L := W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$, and the fibers of this mapping is exact the Hecke orbits on \mathcal{M}_{LT} . This reveals the very difference of the theories of uniformization of Shimura varieties between the complex and p -adic situation. Moreover, since by de Jong [18] π is an étale covering of p -adic analytic spaces, its fibers, i.e. the p -adic Hecke orbits, are thus discrete. This is also quite different to the situation over positive characteristic for non basic Newton polygon strata and the prime to p Hecke orbits on Shimura varieties, see [13] and [14] for example.

The fundamental domain of Gross-Hopkins is then given by

$$\mathcal{D} := \pi^{-1}(C) \bigcap \mathcal{M}_{LT}^0,$$

where $C \subset \mathbb{P}^{n-1, an}$ is the closed subspace defined by

$$C = \{x = (x_0, \dots, x_{n-1}) \mid v_x\left(\frac{x_i}{x_0}\right) \geq 1 - \frac{i}{n} \forall i\},$$

v_x is the valuation on the complete residue field of x . Let $\Pi \in D^\times$ be an uniformizer such that it induces an isomorphism between the components $\Pi^{-1} : \mathcal{M}_{LT}^i \xrightarrow{\sim} \mathcal{M}_{LT}^{i+1}$, then the domain \mathcal{D} is such that we have a locally finite covering of the Lubin-Tate space

$$\mathcal{M}_{LT} = \bigcup_{\substack{T \in GL_n(\mathbb{Z}_p) \backslash GL_n(\mathbb{Q}_p) / GL_n(\mathbb{Z}_p) \\ i=0, \dots, n-1}} T \cdot \Pi^{-i} \mathcal{D}.$$

Note \mathcal{D} is closed, and more importantly (and non trivially) its underlying topological space is compact. The locally finiteness means that we can start from \mathcal{D} and its translations $T \cdot \Pi^{-i} \mathcal{D}$ to glue a Berkovich space, which is isomorphism our Lubin-Tate space \mathcal{M}_{LT} . We may call such \mathcal{M}_{LT} admits a cell decomposition, as an analogue of the classical situation. In next section we will use this locally finite cell decomposition and the compactness of \mathcal{D} , to deduce a Lefschetz trace formula for Lubin-Tate spaces, by applying Mieda's theorem 3.13 in [60].

In [26] Fargues has developed a theory of Harder-Narasimhan filtration for finite flat group schemes, and applied to the study of p -divisible groups in [27]. For details of Harder-Narasimhan filtration see [26] or subsection 2.2 in the following for a review. In particular we have notions of semi-stable finite flat group schemes and p -divisible groups over a complete rank one valuation ring $O_K | \mathbb{Z}_p$. The basic observation is that, the $\pi^{-1}(C)$ is exactly the semi-stable locus $\mathcal{M}_{LT}^{ss} \subset \mathcal{M}_{LT}$, that is the locus where the associated p -divisible groups are semi-stable. And Gross-Hopkins's fundamental domain is the semi-stable locus in the connected component \mathcal{M}_{LT}^0 . Motivated by this fact, Fargues has studied fundamental domains in the Rapoport-Zink spaces for GL_h / \mathbb{Q}_p with signature $(d, h-d)$, in particular there is no additional structures for the p -divisible groups considered.

There are two main ingredients in the article [27]. The first is an algorithm based the theory of Harder-Narasimhan filtrations of finite flat group schemes, which starts from any p -divisible groups over an O_K as above and produce new ones which are

more and more tend to be of HN-type, that is semi-stable for formal p -divisible groups whose special fiber is supersingular, see loc. cit. for the precise definition of p -divisible groups of HN-type. When the valuation K is discrete, the algorithm stops after finite times. Passing to the Shimura varieties which give locally the Rapoport-Zink spaces for GL_h/\mathbb{Q}_p with signature $(d, h-d)$, one can define a Hecke-equivariant stratification of the underlying topological space of these p -adic analytic Shimura varieties by Harder-Narasimhan polygons. The algorithm stops after finite times over complete discrete valuation rings means that, the Hecke orbits of the rigid points in the HN-type locus in each Harder-Narasimhan polygon strata, cover all the rigid points in the strata. For the basic polygon \mathcal{P}_{ss} that is the line between the point $(0, 0)$ and (h, d) (d is the dimension of p -divisible groups in the Rapoport-Zink spaces), the HN-type locus is exactly the semi-stable locus, and one has the statement as above.

The second main ingredient of [27] is the inequality

$$HN(H) \leq \text{Newt}(H_k)$$

between the concave Harder-Narasimhan and Newton polygons, here k is the residue field of K . The proof of this inequality for the case the valuation of K is discrete is easy, while for the general case it is quite involved : Fargues has used the notions of Hodge-Tate modules and Banch-Colmez spaces in p -adic Hodge theory, and in fact one has also to pose a mild condition on H in this case, which is naturally satisfied when H coming from a point in Rapoport-Zink spaces. The moduli consequences of this inequality are that, the basic Newton polygon strata of the p -adic Shimura varieties is contained in their basic Harder-Narasimhan polygon strata, and the Hecke orbit of the semi-stable locus in the basic Rapoport-Zink space cover at least all the rigid points.

For the case h and d are co-prime to each other, Fargues can prove that the Hecke orbit of the semi-stable locus in the basic Rapoport-Zink covers all the space. More precisely, the main theorem of [27] is the following.

Theorem 2.1.1 (Fargues, [27]). *Let $\mathcal{M}^{ss} \subset \mathcal{M}$ be the semi-stable locus in the basic p -adic Rapoport-Zink space $\mathcal{M} = \coprod_{i \in \mathbb{Z}} \mathcal{M}^i$ for GL_h/\mathbb{Q}_p with signature $(d, h-d)$, and $\mathcal{D} := \mathcal{M}^{ss} \cap \mathcal{M}^0$. Assume $(h, d) = 1$. Let $\Pi \in D^\times$ be a uniformizer in $J_b(\mathbb{Q}_p) = D^\times$, where D is the division algebra of invariant $\frac{d}{h}$ over \mathbb{Q}_p , such that Π induces isomorphisms $\Pi^{-1} : \mathcal{M}^i \rightarrow \mathcal{M}^{i+1}$. Then there is a locally finite covering of \mathcal{M}*

$$\mathcal{M} = \bigcup_{\substack{T \in GL_h(\mathbb{Z}_p) \backslash GL_h(\mathbb{Q}_p) / GL_h(\mathbb{Z}_p) \\ i=0, \dots, h-1}} T \cdot \Pi^{-i} \mathcal{D}.$$

For the case $d = 1$ we recover the cell decomposition of Lubin-Tate space.

The purpose of this section is to prove a similar result of cell decomposition for some unitary group Rapoport-Zink spaces.

More precisely, let $p > 2$ be a fixed prime number, \mathbb{Q}_{p^2} be the unramified extension of \mathbb{Q}_p of degree 2, and G is the quasi-split unitary similitude group defined a n -dimensional \mathbb{Q}_{p^2} hermitian space. The basic formal Rapoport-Zink space $\widehat{\mathcal{M}}$ for G with signature $(1, n-1)$ is the formal scheme formally locally of finite type over $SpfW$. A S -valued point of $\widehat{\mathcal{M}}$ for a $S \in \text{Nilp}W$ is given by $(H, \iota, \lambda, \rho)$, where H is a p -divisible group over S , $\iota : \mathbb{Z}_{p^2} \rightarrow \text{End}(H)$ is an action of \mathbb{Z}_{p^2} on H satisfying certain Kottwitz type

determinant conditions, $\lambda : H \rightarrow H^D$ is a polarization compatible with the action ι , and $\rho : \mathbf{H}_{\bar{S}} \rightarrow H_{\bar{S}}$ is a quasi-isogeny. For more details see the following section 2. One has a decomposition $\widehat{\mathcal{M}} = \coprod_{i \in \mathbb{Z}, \text{in even}} \widehat{\mathcal{M}}^i$, where $\widehat{\mathcal{M}}^i$ is the locus where the height of the quasi-isogenies is in .

The geometry of the reduced special fiber \mathcal{M}_{red} has been completely described by Vollaard-Wedhorn in [81]. It turns out each connected component \mathcal{M}_{red}^i admits a Bruhat-Tits stratification

$$\mathcal{M}_{red}^i = \coprod_{\Lambda \in \mathcal{B}(J_b^{der}, \mathbb{Q}_p)^0} \mathcal{M}_{\Lambda}^0,$$

where $\mathcal{B}(J_b^{der}, \mathbb{Q}_p)^0$ is the set of vertices in the Bruhat-Tits building of the derived subgroup J_b^{der} of J_b over \mathbb{Q}_p , and \mathcal{M}_{Λ}^0 is a locally closed subscheme. Recall J_b is the inner form of G associated to the local data to define the Rapoport-Zink space, and when n is odd we have in fact an isomorphism $G \simeq J_b$. There is a type function $t : \mathcal{B}(J_b^{der}, \mathbb{Q}_p)^0 \rightarrow [1, n]$, which takes values on all the odd integers between 1 and n , and the fibers of t are exactly the $J_b^{der}(\mathbb{Q}_p)$ -orbits in $\mathcal{B}(J_b^{der}, \mathbb{Q}_p)^0$. Let $t_{max} = n$ if n is odd and $t_{max} = n - 1$ if n is even. Then the irreducible components of \mathcal{M}_{red}^i are exactly these \mathcal{M}_{Λ} , the schematic closure of \mathcal{M}_{Λ}^0 , with $t(\Lambda) = t_{max}$.

Let $g_1 \in J_b(\mathbb{Q}_p)$ be an element such that it induces isomorphisms $g_1 : \widehat{\mathcal{M}}^i \xrightarrow{\sim} \widehat{\mathcal{M}}^{i+1}$ for n even and $g_1 : \widehat{\mathcal{M}}^i \xrightarrow{\sim} \widehat{\mathcal{M}}^{i+2}$ for n odd. The element $p^{-1} \in J_b(\mathbb{Q}_p)$ induces always isomorphisms $p^{-1} : \widehat{\mathcal{M}}^i \rightarrow \widehat{\mathcal{M}}^{i+2}$. We fix a choice of g_1 compatible with p^{-1} . Consider the connected component for $i = 0$ and fix a choice of Λ such that $t(\Lambda) = t_{max}$, and let $Stab(\Lambda)$ be the stabilizer group of Λ in $J_b^{der}(\mathbb{Q}_p)$. Let $\mathcal{M} = \coprod_{i \in \mathbb{Z}, \text{in even}} \mathcal{M}^i$ be the associated Berkovich analytic space of $\widehat{\mathcal{M}}$, and $sp : \mathcal{M} \rightarrow \mathcal{M}_{red}$ be the specialization map.

Theorem 2.1.2. *There is a closed analytic domain $\mathcal{C} \subset \mathcal{M}$, which contains the semi-stable locus \mathcal{M}^{ss} , such that if we set*

$$\mathcal{D} := \mathcal{C} \cap sp^{-1}(\mathcal{M}_{\Lambda}),$$

then \mathcal{D} is relatively compact. Moreover, we have a locally finite covering of \mathcal{M}

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} T.g\mathcal{D}$$

if n is odd, and

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} T.g_1^j g\mathcal{D}$$

if n is even.

Let $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$ be the p -adic period mapping (see section 2.2), $K \subset G(\mathbb{Z}_p)$ be an open compact subgroup, \mathcal{M}_K be the Rapoport-Zink space with level K and $\pi_K : \mathcal{M}_K \rightarrow \mathcal{M}$ be the natural projection, then we have the following corollaries.

Corollary 2.1.3. *We have a locally finite covering of \mathcal{F}^a*

$$\mathcal{F}^a = \bigcup_{g \in J_b^{\text{der}}(\mathbb{Q}_p)/\text{Stab}(\Lambda)} g\pi(\mathcal{D})$$

if n is odd, and

$$\mathcal{F}^a = \bigcup_{\substack{j=0,1 \\ g \in J_b^{\text{der}}(\mathbb{Q}_p)/\text{Stab}(\Lambda)}} g_1^j g\pi(\mathcal{D})$$

if n is even.

Corollary 2.1.4. *We have a locally finite covering of the analytic space \mathcal{M}_K*

$$\mathcal{M}_K = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ g \in J_b^{\text{der}}(\mathbb{Q}_p)/\text{Stab}(\Lambda)}} T.g\mathcal{D}_K$$

if n is odd, and

$$\mathcal{M}_K = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ j=0,1 \\ g \in J_b^{\text{der}}(\mathbb{Q}_p)/\text{Stab}(\Lambda)}} T.g_1^j g\mathcal{D}_K$$

if n is even.

Finally we have a corollary for Shimura varieties.

Corollary 2.1.5. *Let Sh_{K^p} be as the Shimura variety introduced in section 2.9, $\widehat{Sh}_{K^p}^{\text{an}}$ be the generic analytic fiber of its p -adic completion \widehat{Sh}_{K^p} , and $\widehat{Sh}_{K^p}^{\text{an},b_0}$ be the tube in $\widehat{Sh}_{K^p}^{\text{an}}$ over the basic strata $\overline{Sh}_{K^p}^{b_0}$, which is an open subspace. Let $\widehat{Sh}_{K_p \times K^p}^{\text{an}} \rightarrow \widehat{Sh}_{K^p}^{\text{an}}$ be the covering in level $K_p \subset G(\mathbb{Z}_p)$ (an open compact subgroup), and $\widehat{Sh}_{K_p \times K^p}^{\text{an},b_0}$ be the inverse image of $\widehat{Sh}_{K^p}^{\text{an},b_0}$. Denote $\mathcal{C}^i = \mathcal{C} \cap \mathcal{M}^i$ for each $i \in \mathbb{Z}$ such that i is even, $\mathcal{C}' = \mathcal{C}^0$ if n is odd and $\mathcal{C}' = \mathcal{C}^0 \amalg \mathcal{C}^1$ if n is even, \mathcal{C}'_{K_p} the inverse image of \mathcal{C}' in \mathcal{M}_{K_p} , \mathcal{E}'_{K_p} the image of \mathcal{C}'_{K_p} under the p -adic uniformization*

$$I(\mathbb{Q}) \backslash \mathcal{M}_{K_p} \times G(\mathbb{A}_f^p)/K^p \simeq \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p)/K^p} \mathcal{M}_{K_p}/\Gamma_i \simeq \widehat{Sh}_{K_p \times K^p}^{\text{an},b_0}.$$

1. *Let $\Gamma = \Gamma_i$ be one of the above discrete, torsion free, cocompact modulo center subgroups of $J_b(\mathbb{Q}_p)$, and $\Gamma^{\text{der}} = \Gamma \cap J_b^{\text{der}}(\mathbb{Q}_p)$, $D_{K_p} = D_{iK_p}$ be the image of \mathcal{D}_{K_p} under the morphism $\mathcal{M}_{K_p} \rightarrow \mathcal{M}_{K_p}/\Gamma$, then we have a covering*

$$\mathcal{M}_{K_p}/\Gamma = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K_p \\ g \in \Gamma^{\text{der}} \backslash J_b^{\text{der}}(\mathbb{Q}_p)/\text{Stab}(\Lambda)}} T.gD_{K_p}$$

if n is odd, and

$$\mathcal{M}_{K_p}/\Gamma = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K_p \\ j=0,1 \\ g \in \Gamma^{\text{der}} \backslash J_b^{\text{der}}(\mathbb{Q}_p)/\text{Stab}(\Lambda)}} T.g_1^j gD_{K_p}$$

if n is even.

2. Under the above notation, we have a covering

$$\mathcal{E}'_{K_p} = \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) / K^p} \bigcup_{g \in \Gamma^{der} \backslash J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)} g D_{iK_p}$$

if n is odd, and

$$\mathcal{E}'_{K_p} = \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) / K^p} \bigcup_{\substack{j=0,1 \\ g \in \Gamma^{der} \backslash J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} g_1^j g D_{iK_p}$$

if n is even. We have a covering

$$\widehat{Sh}_{K_p \times K^p}^{an, b_0} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / K_p} T \cdot \mathcal{E}'_{K_p}.$$

As one has seen, the first difficulty in our unitary group case is that, the geometry of the reduced special fiber of Rapoport-Zink space is more complicated than that for the case of GL_h/\mathbb{Q}_p with signature $(d, h-d)$, since for the case $(h, d) = 1$ considered above each connected component of the special fiber is already irreducible, see [78]. This is why we have to take the intersection of \mathcal{C} with the tube over a fixed irreducible component to have a locally finite cell decomposition. The second difficulty is that, the algorithm above when applied to the PEL type Rapoport-Zink spaces, for example the unitary group case considered here, is not well compatible with the action of Hecke correspondences. One has to modify it. This is why the semi-stable locus may be not enough and we find a closed domain $\mathcal{C} \supset \mathcal{M}^{ss}$.

On the other hand, the inequality

$$HN(H, \iota, \lambda) \leq \text{Newt}(H_k, \iota, \lambda)$$

between the Harder-Narasimhan and Newton polygons for p -divisible groups with additional structures still holds. In fact this can be easily deduced from Fargues's inequality $HN(H) \leq \text{Newt}(H_k)$, since the former polygons are just defined respectively by normalization of the later polygons.

The general strategy to prove the above theorem, is that using the modified algorithm and the above inequality to deduce first the equalities in the theorem hold for rigid points. For the rest points, by the equivalence of suitable categories between Berkovich spaces and rigid analytic spaces, it suffices to prove these coverings are locally finite, thus admissible. This last argument is different from that in [27] section 16.

In our unitary case, we have in fact that the underlying topological space of \mathcal{D} is locally compact, like the case of Lubin-Tate space. These two facts both come from the special phenomenon that, all the non basic Newton polygon has contacted points with the Hodge polygon, and thus one can deduce the Harder-Narasimhan polygon stratification and the Newton polygon stratification of the associated p -adic Shimura varieties coincide.

At this point we should note that, the Rapoport-Zink spaces for GSp_4 is quite similar with our unitary case. See [54] section 4 for a similar geometric description of the reduced special fiber. And the two non basic Newton polygons have contacted points with the Hodge polygon. In particular our method here will enable us to prove an analogue result of cell decomposition for the basic p -adic GSp_4 Rapoport-Zink spaces.

2.2 The unitary group Rapoport-Zink spaces and Hecke action

We consider here a special case of PEL type Rapoport-Zink spaces.

Let $p > 2$ be a fixed prime number. Let \mathbb{Q}_{p^2} be the unramified extension of \mathbb{Q}_p of degree 2 and denote by $*$ the nontrivial Galois automorphism of \mathbb{Q}_{p^2} over \mathbb{Q}_p . Let V be a finite dimensional \mathbb{Q}_{p^2} -vector space with $\dim_{\mathbb{Q}_{p^2}}(V) = n$. Let $\langle, \rangle : V \times V \rightarrow \mathbb{Q}_p$ be a \mathbb{Q}_p -valued skew-hermitian form, and G be the associated reductive group, i.e.,

$$G(R) = \{g \in GL_{\mathbb{Q}_{p^2} \otimes R}(V_R) \mid \exists c \in R^\times : \langle gv, gw \rangle = c \langle v, w \rangle, \forall v, w \in V_R := V \otimes R\}$$

for all \mathbb{Q}_p -algebra R . We remark that there exists a unique skew-hermitian form $\langle, \rangle' : V \times V \rightarrow \mathbb{Q}_{p^2}$ such that $\langle, \rangle = \text{Tr}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p} \circ \langle, \rangle'$. Moreover, if $\delta \in \mathbb{Q}_{p^2}^\times$ with $\delta^* = -\delta$, then $(,) := \delta \langle, \rangle'$ is a hermitian form, and G is just the unitary similitude group $GU(V, (,))$ of the hermitian space $(V, (,))$. Let \mathbb{Z}_{p^2} be the ring of integers of \mathbb{Q}_{p^2} . We assume that there exists a \mathbb{Z}_{p^2} -lattice Λ such that \langle, \rangle induces a perfect \mathbb{Z}_p -pairing on Λ . This implies that G is unramified over \mathbb{Q}_p and has a reductive model over \mathbb{Z}_p .

Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of \mathbb{Q}_p . Then there is a canonical imbedding

$$G_{\overline{\mathbb{Q}_p}} \subset (\text{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p} GL_{\mathbb{Q}_{p^2}}(V))_{\overline{\mathbb{Q}_p}} = GL(V \otimes_{\mathbb{Q}_{p^2}, \text{id}} \overline{\mathbb{Q}_p}) \times GL(V \otimes_{\mathbb{Q}_{p^2}, *} \overline{\mathbb{Q}_p}),$$

and we have an isomorphism

$$G_{\overline{\mathbb{Q}_p}} \simeq GL(V \otimes_{\mathbb{Q}_{p^2}, \text{id}} \overline{\mathbb{Q}_p}) \times \mathbb{G}_m.$$

Via this isomorphism, we fix a $G(\overline{\mathbb{Q}_p})$ -conjugate class of cocharacter

$$\begin{aligned} \mu : \mathbb{G}_{m, \overline{\mathbb{Q}_p}} &\longrightarrow G_{\overline{\mathbb{Q}_p}} \\ z &\mapsto (\text{diag}(z, \dots, z, 1), z). \end{aligned}$$

Let $L = W(\overline{\mathbb{F}_p})_{\mathbb{Q}_p}$, σ be the Frobenius relative the field extension L/\mathbb{Q}_p . Consider the set $B(G) = G(L)/\sim$ of σ -conjugate classes in $G(L)$, and the Kottwitz set $B(G, \mu) \subset B(G)$ ([52]). In our special case we can have an explicit description of the set $B(G, \mu)$ as a set of polygons, see [10] 3.1. We consider the basic element $b = b_0 \in B(G, \mu)$, and let J_b be the reductive group of automorphisms of the unitary isocrystal $(V_L, b\sigma, \iota, \langle, \rangle)$, which is then an inner form of G over \mathbb{Q}_p .

Associated to the above data $(\mathbb{Q}_{p^2}, *, V, \langle, \rangle, b, \mu)$, we have the Rapoport-Zink space $\widehat{\mathcal{M}}$ which is a formal scheme locally formally of finite type over $\text{Spf}O_L$. It is a moduli space of p -divisible groups with additional structures of the following type : for each $S \in \text{Nilp}O_L$, $\widehat{\mathcal{M}}(S) = \{(H, \iota, \lambda, \rho)\} / \simeq$, where

- H is a p -divisible group over S ;
- $\iota : \mathbb{Z}_{p^2} \rightarrow \text{End}(H)$ is an action of \mathbb{Z}_{p^2} on H satisfying locally

$$\text{Lie}(H) = \text{Lie}(H)_0 \oplus \text{Lie}(H)_1, \text{rank}_{O_S} \text{Lie}(H)_0 = 1, \text{rank}_{O_S} \text{Lie}(H)_1 = n - 1$$

where

$$\text{Lie}(H)_0 = \{x \in \text{Lie}(H) \mid \iota(a)x = ax\}, \text{Lie}(H)_1 = \{x \in \text{Lie}(H) \mid \iota(a)x = a^*x\};$$

- $\lambda : H \rightarrow H^D$ is a principal \mathbb{Z}_{p^2} -linear polarization, here H^D is the dual p -divisible group endowed with the \mathbb{Z}_{p^2} -action $\iota_{H^D}(a) = (\iota(a^*))^D$;
- $\rho : \mathbf{H}_{\bar{S}} \rightarrow H_{\bar{S}}$ is a quasi-isogeny, such that $\rho^D \circ \lambda \circ \rho$ is a \mathbb{Q}_p^\times -multiple of λ , here $\bar{S} = S \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$;
- $(H_1, \iota_1, \lambda_1, \rho_1) \simeq (H_2, \iota_2, \lambda_2, \rho_2)$ if there exists a \mathbb{Z}_{p^2} -linear isomorphism $\alpha : H_1 \rightarrow H_2$ such that $\rho_2 = \rho_1 \circ \alpha$, $\alpha^D \circ \lambda_2 \circ \alpha$ is a \mathbb{Z}_p^\times -multiple of λ_1 .

We call such a p -divisible group with additional structures $H = (H, \iota, \lambda)$ a unitary p -divisible group. For such a unitary p -divisible group H , we have

$$\text{rank}_{O_S} \text{Lie}(H) = n, \text{height}(H) = 2n.$$

The height of ρ is a multiple of n by [80] 1.7 or [15] and we obtain a decomposition

$$\widehat{\mathcal{M}} = \prod_{i \in \mathbb{Z}} \widehat{\mathcal{M}}^i,$$

where $\widehat{\mathcal{M}}^i$ is the open and closed formal subscheme of $\widehat{\mathcal{M}}$ where ρ has height in . Moreover, we have in fact

$$\widehat{\mathcal{M}}^i \neq \emptyset \Leftrightarrow in \text{ is even,}$$

and in this case there is an isomorphism $\widehat{\mathcal{M}}^i \cong \widehat{\mathcal{M}}^0$ induced the action of $J_b(\mathbb{Q}_p)$, see [80] or the section 2.6 below.

The standard p -divisible group $\mathbf{H} = (\mathbf{H}, \iota, \lambda)$ is definable over \mathbb{F}_{p^2} . We let

$$(\mathbf{M}, \mathbf{F}, \mathbf{V}, \mathbf{M} = \mathbf{M}_0 \oplus \mathbf{M}_1, \langle, \rangle)$$

denote its covariant Diéudonne module over $W(\mathbb{F}_{p^2}) = \mathbb{Z}_{p^2}$, where $\langle, \rangle : \mathbf{M} \times \mathbf{M} \rightarrow \mathbb{Z}_{p^2}$ is a perfect alternating \mathbb{Z}_{p^2} -bilinear pairing satisfying

$$\langle \mathbf{F}x, y \rangle = \langle x, \mathbf{V}y \rangle^\sigma, \langle ax, y \rangle = \langle x, a^*y \rangle$$

for all $x, y \in \mathbf{M}$, $a \in \mathbb{Z}_{p^2}$, here $\sigma = *$ is the Frobenius on $W(\mathbb{F}_{p^2}) = \mathbb{Z}_{p^2}$; the decomposition $\mathbf{M} = \mathbf{M}_0 \oplus \mathbf{M}_1$ is induced by the decomposition $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2}) \simeq W(\mathbb{F}_{p^2}) \times W(\mathbb{F}_{p^2})$ and the \mathbb{Z}_{p^2} -action on \mathbf{M} . The \mathbf{F} and \mathbf{V} are homogeneous of degree 1 with respect to the above decomposition and \mathbf{M}_0 and \mathbf{M}_1 are totally isotropic with respect to \langle, \rangle . The signature condition on the Lie algebra then imply

$$\dim_{\mathbb{F}_{p^2}}(\mathbf{M}_0/\mathbf{V}\mathbf{M}_1) = 1, \dim_{\mathbb{F}_{p^2}}(\mathbf{M}_1/\mathbf{V}\mathbf{M}_0) = n - 1.$$

We denote by $(\mathbf{N}, \mathbf{F}) = (\mathbf{M}, \mathbf{F}) \otimes \mathbb{Q}_{p^2}$ the isocrystal of \mathbf{H} . We can assume that \mathbf{H} is superspecial and that the isocrystal (\mathbf{N}, \mathbf{F}) is generated by the elements x such that $\mathbf{F}^2x = px$, see [81]. As \mathbf{F}^2 is \mathbb{Q}_{p^2} -linear, we have $\mathbf{F}^2 = \text{pid}_{\mathbf{N}}$ and therefore $\mathbf{F} = \mathbf{V}$. For $i = 1, 2$, let $\mathbf{N}_i = \mathbf{M}_i \otimes \mathbb{Q}_{p^2}$, then $\mathbf{N} = \mathbf{N}_0 \oplus \mathbf{N}_1$ and with respect to this decomposition \mathbf{F} is of degree 1. We fix an element $\delta \in \mathbb{Z}_{p^2}^\times$ such that $\delta^* = -\delta$ and define a nondegenerate hermitian form on the \mathbb{Q}_{p^2} -vector space \mathbf{N}_0 by

$$\{x, y\} := \delta \langle x, \mathbf{F}y \rangle.$$

Recall the reductive group J_b over \mathbb{Q}_p defined by the automorphisms of the unitary isocrystal $(\mathbf{N}, \mathbf{F}, \mathbf{N} = \mathbf{N}_0 \oplus \mathbf{N}_1, \langle, \rangle)$, which is an inner form of G . We have then an

isomorphism of J_b with the unitary similitude group $GU(\mathbf{N}_0, \{, \})$ of the hermitian space $(\mathbf{N}_0, \{, \})$. Thus for n odd, we have in fact an isomorphism

$$G \cong J_b;$$

while for n even, J_b is the non quasi-split inner form of G .

We now describe the group actions on the Rapoport-Zink space $\widehat{\mathcal{M}}$. First, we have a left action of $J_b(\mathbb{Q}_p)$ on $\widehat{\mathcal{M}} : \forall g \in J_b(\mathbb{Q}_p)$,

$$\begin{aligned} g : \widehat{\mathcal{M}} &\longrightarrow \widehat{\mathcal{M}} \\ (H, \iota, \lambda, \rho) &\mapsto (H, \iota, \lambda, \rho \circ g^{-1}), \end{aligned}$$

since J_b can be viewed as the group of self quasi-isogenies of $(\mathbf{H}, \iota, \lambda)$.

To consider the action of $G(\mathbb{Q}_p)$ on $\widehat{\mathcal{M}}$, we would rather to consider the associated Berkovich analytic space $\mathcal{M} = \widehat{\mathcal{M}}^{an}$ of $\widehat{\mathcal{M}}$. As always, by trivializing the Tate module of the universal p -divisible group over \mathcal{M} , we can define a tower of Berkovich analytic spaces $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$. These spaces are separated smooth good Berkovich analytic spaces over L . A point $x \in \mathcal{M}_K$ is given by $(H, \iota, \lambda, \rho, \eta K)$, where $\eta : V \xrightarrow{\sim} V_p(H)$ is the rigidification isomorphism such that $\eta(\Lambda) = T_p(H)$. Then $J_b(\mathbb{Q}_p)$ also acts on each space \mathcal{M}_K in the natural way. Moreover, $G(\mathbb{Q}_p)$ acts on this tower : for $g \in G(\mathbb{Q}_p)$ and $K \subset G(\mathbb{Z}_p)$ such that $g^{-1}Kg \subset G(\mathbb{Z}_p)$, we have an isomorphism

$$\begin{aligned} g : \mathcal{M}_K &\rightarrow \mathcal{M}_{g^{-1}Kg}, \\ (H, \iota, \lambda, \rho, \eta K) &\mapsto (H', \iota', \lambda', \rho', \eta'(g^{-1}Kg)), \end{aligned}$$

here $(H', \iota', \lambda', \rho', \eta')$ is defined as following. Assume first $g^{-1} \in M_n(\mathbb{Z}_{p^2})$. Then $\Lambda \supset g^{-1}(\Lambda)$. Since $\eta(\Lambda) = T_p(H)$ for the rigidification η , $\eta(\Lambda/g^{-1}(\Lambda))$ defines a finite flat subgroup of H . We take $H' := H/\eta(\Lambda/g^{-1}(\Lambda))$ with the naturally induced structures (ι', λ') on H' , and $\rho' = \pi(\text{mod } p) \circ \rho$ for the natural projection $\pi : H \rightarrow H'$. Finally there is a rigidification $\eta' = \eta \circ g : V \rightarrow V_p(H')$ such that the following diagram commutes :

$$\begin{array}{ccc} V & \xrightarrow{\eta} & V_p(H) \\ \downarrow g & & \downarrow \\ V & \xrightarrow{\eta'} & V_p(H'). \end{array}$$

For the general case, one can always find an integer $r \in \mathbb{Z}$ such that $p^r g^{-1} \in M_n(\mathbb{Z}_{p^2})$, then we can define $(H'', \iota'', \lambda'', \rho'', \eta'')$ as above for $p^{-r}g$. We set $H' = H'', \iota' = \iota'', \lambda' = \lambda'', \rho' = p^{-r} \rho'', \eta' = \eta''$.

For any open compact subgroups $K' \subset K \subset G(\mathbb{Z}_p)$, we denote $\pi_{K', K} : \mathcal{M}_{K'} \rightarrow \mathcal{M}_K$ the natural projection of forgetting levels, which is a finite étale morphism of degree K/K' . In particular, for $K \subset G(\mathbb{Z}_p)$ fixed, each $g \in G(\mathbb{Z}_p)$ defines a Hecke correspondence on \mathcal{M}_K by the following diagram :

$$\begin{array}{ccc} & \mathcal{M}_{gKg^{-1} \cap K} & \xrightarrow[\simeq]{g} & \mathcal{M}_{K \cap g^{-1}Kg} & \\ \pi_{gKg^{-1} \cap K, K} \swarrow & & & & \searrow \pi_{K \cap g^{-1}Kg, K} \\ \mathcal{M}_K & & & & \mathcal{M}_K \end{array}$$

and this Hecke correspondence depends only on the double coset KgK . Thus we get an "action" of $K \backslash G(\mathbb{Q}_p)/K$ on \mathcal{M}_K , which commutes with the (left) action of $J_b(\mathbb{Q}_p)$.

Definition 2.2.1. *Let $K \subset G(\mathbb{Z}_p)$ be an open compact subgroup. For any subset $A \subset |\mathcal{M}_K|$ of the underlying topological space $|\mathcal{M}_K|$, and any Hecke correspondence T defined by a coset KgK as above, we define the image of A under T by the set*

$$T.A = \pi_{K \cap g^{-1}Kg, Kg} \pi_{gKg^{-1} \cap K, K}^{-1}(A).$$

We call the set

$$\text{Hecke}(A) := \bigcup_{T \in K \backslash G(\mathbb{Q}_p)/K} T.A$$

the Hecke orbit of A .

Remark 2.2.2. 1. *By the above, the Hecke action of $G(\mathbb{Q}_p)$ on the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ is in fact a right action. So maybe we should better write the image of A under T as $A.T$. On the other hand, there is in general no composition law for the action of Hecke correspondences on \mathcal{M}_K , since the product (in the usual way) of double cosets $KgK \cdot KhK$ is in general not a single double coset. Therefore, we will write T on the left as $T.A$, and for two Hecke correspondences T_1, T_2 , $T_2.(T_1.A)$ should always be understood as the image of $T_1.A$ under T_2 .*

2. *More precisely, we have*

$$Kh_2K.(Kh_1K.A) = \bigcup_{KhK \subset Kh_1Kh_2K} KhK.A,$$

where the right hand side is the finite union over all the double cosets in Kh_1Kh_2K . In particular, $A \subset KhK.(Kh^{-1}K.A)$ and $\text{Hecke}(A) = \text{Hecke}(T.A)$ for any $T \in K \backslash G(\mathbb{Q}_p)/K$.

3. *Note that if A is an analytic domain, then so is $T.A$ for any $T \in K \backslash G(\mathbb{Q}_p)/K$.*

In the following we will mainly focus on the Hecke action on the space \mathcal{M} . We would like to describe the images of a point $x \in \mathcal{M}$ under the action of $G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ on \mathcal{M} explicitly. To this end we first recall the Cartan decomposition to describe the set $G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ explicitly. Let $A \subset G$ be a maximal \mathbb{Q}_p -split torus such that

$$A(\mathbb{Q}_p) = \left\{ \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \mid d_1 d_n^* = d_2 d_{n-1}^* = \cdots = \text{constant} \in \mathbb{Q}_p^\times \right\}.$$

Then the cocharacter group

$$X_*(A) = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 + a_n = a_2 + a_{n-1} = \cdots = \text{constant} \in \mathbb{Z}\},$$

and we denote the dominant coweights by

$$X_*(A)_+ = \{(a_1, \dots, a_n) \in X_*(A) \mid a_1 \geq \cdots \geq a_n\}.$$

The Cartan decomposition says that the following map is a bijection :

$$\begin{aligned} X_*(A)_+ &\longrightarrow G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \\ (a_1, \dots, a_n) &\mapsto G(\mathbb{Z}_p) \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} G(\mathbb{Z}_p). \end{aligned}$$

A point $x \in \mathcal{M}$ corresponds to a tuple $(H/O_{K=\mathcal{H}(x)}, \iota, \lambda, \rho)$, as an element of $\mathcal{M}(K) = \widehat{\mathcal{M}}(O_K)$. For

$$T = G(\mathbb{Z}_p) \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} G(\mathbb{Z}_p) \in G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p)/G(\mathbb{Z}_p),$$

we now give a moduli description of the finite set $T.x$. First assume $a_1 \leq 0$. By the definition of the Hecke correspondence T , we have

$$\begin{aligned} T.x = \{y \in \mathcal{M} \mid & (H_y, \iota_y, \lambda_y, \rho_y) \otimes_{O_{\mathcal{H}(y)}} O_K \simeq (H/G_y, \iota', \lambda', \pi \circ \rho), \\ & \text{where } G_y \subset H \text{ is a finite flat subgroup scheme, such that its} \\ & \text{geometric generic fiber } G_{y\overline{K}} \simeq \mathbb{Z}_{p^2}/p^{-a_1}\mathbb{Z}_{p^2} \oplus \cdots \oplus \mathbb{Z}_{p^2}/p^{-a_n}\mathbb{Z}_{p^2}, \\ & \iota', \lambda' \text{ are the naturally induced additional structures,} \\ & \pi : H_{O_K/pO_K} \rightarrow (H/G_y)_{O_K/pO_K} \text{ is the natural projection.} \} \end{aligned}$$

For the general T , note the action of an element $z \in \mathbb{Q}_p^\times \subset G(\mathbb{Q}_p)$, is the same as the action of $z \in \mathbb{Q}_p^\times \subset J_b(\mathbb{Q}_p)$, see [66] lemma 5.36. Since

$$T.x = p^{a_1}(p^{-a_1}T).x,$$

here the first scalar p^{a_1} is considered as an element of $G(\mathbb{Q}_p)$, we have the description of the set $(p^{-a_1}T).x$ as in the above way. Then we consider p^{a_1} as an element of $J_b(\mathbb{Q}_p)$ which just changes the quasi-isogeny. So we can describe the set $T.x$ explicitly in all cases.

We examine the effect of the group actions on connected components. First recall Rapoport-Zink (cf. [66],3.52) have defined generally a locally constant mapping

$$\varkappa : \widehat{\mathcal{M}} \rightarrow \Delta,$$

where $\Delta = \text{Hom}_{\mathbb{Z}}(X_{\mathbb{Q}_p}^*(G), \mathbb{Z})$ and $X_{\mathbb{Q}_p}^*(G)$ is the group of \mathbb{Q}_p -rational characters of G . This mapping satisfies that

$$\varkappa(gx) = \omega_J(g) + \varkappa(x)$$

for all $g \in J_b(\mathbb{Q}_p), x \in \widehat{\mathcal{M}}$. Here $\omega_J : J_b(\mathbb{Q}_p) \rightarrow \Delta$ is defined by $\langle \omega_J(x), \chi \rangle = v_p(i(\chi)(x))$ where $i : X_{\mathbb{Q}_p}^*(G) \rightarrow X_{\mathbb{Q}_p}^*(J_b)$ is the natural morphism between the two groups of \mathbb{Q}_p -rational characters. In our unitary group case, the similitude morphism $c : G \rightarrow \mathbb{G}_m$ defines the identification $\Delta = \mathbb{Z}$. The mapping

$$\begin{aligned} \varkappa : \widehat{\mathcal{M}} &\longrightarrow \mathbb{Z} \\ (H, \iota, \lambda, \rho) &\mapsto -\frac{ht\rho}{n}. \end{aligned}$$

The image of \varkappa is then \mathbb{Z} if n is even, and $2\mathbb{Z}$ if n is odd. In section 2.7 we will review the geometry of the reduced special fiber \mathcal{M}_{red} of $\widehat{\mathcal{M}}$. In particular we find \mathcal{M}_{red}^0 is connected and $\pi_0(\widehat{\mathcal{M}}) = im\varkappa$. Since one has the equalities of the sets of connected components

$$\pi_0(\widehat{\mathcal{M}}) = \pi_0(\mathcal{M}_{red}) = \pi_0(\mathcal{M}) = \pi_0(\mathcal{M} \times \mathbb{C}_p),$$

thus each analytic space \mathcal{M}^i for $i \in \mathbb{Z}$ such that in is even is connected, which is in fact geometrically connected, cf. [15] lemme 5.1.2.1.

Let $D = G/G^{der} = J_b/J_b^{der}$ be the co-center group. More explicitly, we have

$$D = \{(x, c) \in (Res_{\mathbb{Q}_{p^2}|\mathbb{Q}_p} \mathbb{G}_m) \times \mathbb{G}_m \mid N_{\mathbb{Q}_{p^2}|\mathbb{Q}_p}(x) = c^n\},$$

where $N_{\mathbb{Q}_{p^2}|\mathbb{Q}_p} : Res_{\mathbb{Q}_{p^2}|\mathbb{Q}_p} \mathbb{G}_m \rightarrow \mathbb{G}_m$ is the norm morphism. We have the determinant morphisms

$$\begin{aligned} det : G &\longrightarrow D \\ g &\mapsto (det_{\mathbb{Q}_{p^2}}(g), c(g)), \end{aligned}$$

and similarly for $det : J_b \rightarrow D$. In her doctoral thesis [15], Chen has associated to the torus D and the cocharacter $det\tilde{\mu}$ ($\tilde{\mu}$ is a variant of μ), a tower of analytic spaces $(\mathcal{M}(D, det\tilde{\mu})_K)_{K \subset D(\mathbb{Z}_p)}$ with mappings $\varkappa_{D, \tilde{\mu}} : \mathcal{M}(D, det\tilde{\mu})_K \rightarrow \Delta$. The geometric points are $\mathcal{M}(D, det\tilde{\mu})_K(\bar{L}) = D(\mathbb{Q}_p)/K$. By construction there is an action of $D(\mathbb{Q}_p) \times D(\mathbb{Q}_p)$ on each space $(\mathcal{M}(D, det\tilde{\mu})_K)$ such that on geometric points the action is just the left multiplication :

$$(a, b).xK = abxK, \forall (a, b) \in D(\mathbb{Q}_p) \times D(\mathbb{Q}_p), xK \in D(\mathbb{Q}_p)/K.$$

Moreover, via the morphism $(det, det) : G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \rightarrow D(\mathbb{Q}_p) \times D(\mathbb{Q}_p)$, she has constructed a $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ -equivariant determinant morphism of towers of analytic spaces

$$(\mathcal{M}_K)_K \longrightarrow (\mathcal{M}(D, det\tilde{\mu})_{detK})_{detK}$$

for K varies as open compact subgroup of $G(\mathbb{Z}_p)$, which is compatible with the mappings \varkappa and $\varkappa_{D, \tilde{\mu}}$. The main results of loc. cit. imply that the geometric fibers of the determinant morphism

$$\mathcal{M}_K \longrightarrow \mathcal{M}(D, det\tilde{\mu})_{detK}$$

are exactly the geometric connected components of \mathcal{M}_K .

For the case we are interested, $K = G(\mathbb{Z}_p)$, $detK = D(\mathbb{Z}_p)$, the set of geometric components of \mathcal{M} is the same with the set of its connected components, which is in bijection with $\mathcal{M}(D, \tilde{\mu})(\bar{L}) = D(\mathbb{Q}_p)/D(\mathbb{Z}_p)$. Via the mappings \varkappa and $\varkappa_{D, \tilde{\mu}}$ they are in turn bijection with $im\varkappa = im\varkappa_{D, \tilde{\mu}}$, which is thus \mathbb{Z} if n is even, and $2\mathbb{Z}$ if n is odd. Now the effect of the actions of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ on the connected components of \mathcal{M} translates on the last set is as following. First for $g \in J_b(\mathbb{Q}_p)$, we have $\omega_J(g) = v_p(c(g))$ and

$$g : \mathcal{M}^0 \xrightarrow{\sim} \mathcal{M}^{-v_p(c(g))}.$$

For

$$T = G(\mathbb{Z}_p) \left(\begin{array}{ccc} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{array} \right) G(\mathbb{Z}_p) \in G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p)/G(\mathbb{Z}_p),$$

let $g = diag(p^{a_1}, \dots, p^{a_n})$, then

$$det(g) = (p^{a_1+\dots+a_n}, c(g)) \in D(\mathbb{Q}_p), c(g)^n = p^{2(a_1+\dots+a_n)}$$

and

$$v_p(c(g)) = a_1 + a_n = v_p(c(g')) := v_p(c(T)), \forall g' \in T.$$

We have

$$T.\mathcal{M}^0 \subset \mathcal{M}^{-v_p(c(T))}$$

and in fact this is an equality $T.\mathcal{M}^0 = \mathcal{M}^{-v_p(c(T))}$. Let $\tilde{G} = \{g \in G(\mathbb{Q}_p) \mid c(g) \in \mathbb{Z}_p^\times\}$, $U_n = \{g \in G(\mathbb{Q}_p) \mid c(g) = 1\}$, then $G^{der}(\mathbb{Q}_p) \subset U_n \subset \tilde{G}$, and the Hecke correspondences associated to elements in \tilde{G} stabilize \mathcal{M}^0 .

We consider the p -adic period mapping

$$\pi : \mathcal{M} \rightarrow \mathcal{F}^a \subset \mathcal{F}^{wa} \subset \mathcal{F},$$

where $\mathcal{F} = (G_L/P_{\mu L})^{an} \simeq \mathbf{P}^{n-1,an}$ is the p -adic projective space over $L = W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$, \mathcal{F}^{wa} is the weakly admissible locus and \mathcal{F}^a is the image of π , see [38] for some discussion of these objects. We recall the definition of π in the following. The universal quasi-isogeny ρ induces an isomorphism

$$V_L \otimes_{O_L} O_{\mathcal{M}} \simeq LieE(\mathcal{H})^{an},$$

here $E(\mathcal{H})$ is the universal vector extension of $\mathcal{H}_{\overline{\mathcal{M}}}$ over $\widehat{\mathcal{M}}$, $\overline{\mathcal{M}} \subset \widehat{\mathcal{M}}$ is the closed subscheme defined by p . For a p -divisible group H/O_K , recall we have the exact sequence

$$0 \rightarrow \omega_{H^D, K} \rightarrow LieE(H)_K \rightarrow Lie(H)_K \rightarrow 0.$$

If $(H, \iota, \lambda, \rho)$ is associated to a point $x \in \mathcal{M}$, then ρ induces an isomorphism

$$\rho_* : \mathbf{M} \otimes K \xrightarrow{\sim} LieE(H)_K,$$

where $K = \mathcal{H}(x)$ is the complete residue field associated to x , the filtration $\rho_*^{-1}(\omega_{H^D, K}) \subset V_K = \mathbf{M} \otimes K$ defines a point in the Grassmannian $\mathcal{F} = (G_L/P_{\mu L})^{an} \simeq \mathbf{P}^{n-1,an}$, this is the image $\pi(x)$ of $x \in \mathcal{M}$. There is an action of $J_b(\mathbb{Q}_p)$ on \mathcal{F}^a and the mapping π is $J_b(\mathbb{Q}_p)$ -equivariant. In fact π is $G(\mathbb{Q}_p)$ -invariant for the Hecke action on \mathcal{M} , see the following proposition 2.2.3. Under the p -adic period mapping $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$, we have $\pi(\mathcal{M}^i) = \mathcal{F}^a$ for each $i \in \mathbb{Z}$ such that in is even, and \mathcal{F}^a is connected, cf. [15] lemme 5.1.1.1.

When $x \in \mathcal{M}$ is a rigid point, i.e. K is a finite extension of L thus in particular discrete with perfect residue field $\overline{\mathbb{F}}_p$, we have the unitary filtered isocrystal $(V_L, b\sigma, \text{Fil}^\bullet V_K, \iota, \langle, \rangle)$ associated to $(H, \iota, \lambda, \rho)$. Here the filtration on V_K is defined by

$$\text{Fil}^{-1}V_K = V_K, \text{Fil}^0V_K = \rho_*^{-1}(\omega_{H^D, K}), \text{Fil}^iV_K = 0, \forall i \neq -1, 0.$$

The filtered isocrystal $(V_L, b\sigma, \text{Fil}^\bullet V_K, \iota, \langle, \rangle)$ determines the isogeny class of $(H, \iota, \lambda, \rho)$. We have the following description of the Hecke orbit of a point $x \in \mathcal{M}$.

Proposition 2.2.3. *The Hecke orbit of a point $x \in \mathcal{M}$ is exactly the fiber $\pi^{-1}(\pi(x))$ of its image under the p -adic period mapping π .*

Proof. Let $x, y \in \mathcal{M}$ be two points, and $(H_1/O_{\mathcal{H}(x)}, \iota_1, \lambda_1, \rho_1), (H_2/O_{\mathcal{H}(y)}, \iota_2, \lambda_2, \rho_2)$ be the unitary p -divisible groups associated to x and y respectively. Then x, y in the same Hecke orbit if and only if there exists a finite extension K of both $\supset \mathcal{H}(x)$ and $\mathcal{H}(y)$, and a (unique) quasi-isogeny $\varphi : H_1 \rightarrow H_2$ over O_K lifting

$$\rho_2 \circ \rho_1^{-1} : H_{1O_K/pO_K} \xrightarrow{\rho_1^{-1}} \mathbf{H}_{O_K/pO_K} \xrightarrow{\rho_2} H_{2O_K/pO_K}.$$

Let $\mathbf{M} = \mathbb{D}(\mathbf{H})_{W(\overline{\mathbb{F}}_p) \rightarrow \overline{\mathbb{F}}_p}$ be the value on $W(\overline{\mathbb{F}}_p) \rightarrow \overline{\mathbb{F}}_p$ of the covariant crystal $\mathbb{D}(\mathbf{H})$. Then since

$$\begin{array}{ccc} \text{Spec} O_K/pO_K & \hookrightarrow & \text{Spec} O_K \\ \downarrow & & \downarrow \\ \text{Spec} \overline{\mathbb{F}}_p & \hookrightarrow & \text{Spec} W(\overline{\mathbb{F}}_p) \end{array}$$

is a PD-morphism, we have

$$\mathbb{D}(\mathbf{H} \otimes O_K/pO_K)_{O_K \rightarrow O_K/pO_K} = \mathbb{D}(\mathbf{H})_{O_K \rightarrow O_K/pO_K} = \mathbf{M} \otimes O_K.$$

The quasi-isogenies ρ_1, ρ_2 then induce isomorphisms

$$\rho_{i*} : \mathbf{M} \otimes K \xrightarrow{\sim} \text{Lie} E(H_i)_K, i = 1, 2.$$

The images of x, y under the p -adic period mapping $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$ by definition are

$$\rho_{i*}^{-1}(\text{Fil}_{H_i}) := \text{Fil}_i \subset \mathbf{M} \otimes K, i = 1, 2.$$

$\rho_2 \circ \rho_1^{-1}$ can be lifted to a quasi-isogeny $\widetilde{\rho_2 \circ \rho_1^{-1}} : H_1 \rightarrow H_2$, by the theory of Grothendieck-Messing, if and only if the homomorphism

$$\mathbb{D}(\rho_2 \circ \rho_1^{-1})_{O_K \rightarrow O_K/pO_K} : \mathbb{D}(H_1)_{O_K \rightarrow O_K/pO_K} \left[\frac{1}{p} \right] \xrightarrow{\sim} \mathbb{D}(H_2)_{O_K \rightarrow O_K/pO_K} \left[\frac{1}{p} \right]$$

send the Hodge filtration Fil_{H_1} to Fil_{H_2} . But this is equivalent to say $\text{Fil}_1 = \text{Fil}_2$, i.e. $\pi(x) = \pi(y)$. Thus the proposition holds. \square

Thus a point $y \in \mathcal{M}$ is in the Hecke orbit of $x \in \mathcal{M}$, if and only if there exist a finite extension K of both $\mathcal{H}(x)$ and \mathcal{H}_y , and an unique quasi-isogeny $H_y \rightarrow H_x$ over O_K lifting the quasi-isogeny $\rho_x \circ \rho_y^{-1}$ over O_K/pO_K . Note the last condition is equivalent to there exists an isogeny $H_y \rightarrow H_x$. If $x \in \mathcal{M}^{rig}$ is a rigid point, then one find easily that its Hecke orbit consist all of rigid points, i.e. $\text{Hecke}(x) \subset \mathcal{M}^{rig}$. In this case the Hecke orbit is determined by the filtered isocrystal $(V_L, b\sigma, \text{Fil}_{\pi(x)}^\bullet V_K, \iota, \langle, \rangle)$.

For a geometric point $\bar{x} \in \mathcal{M}(K), K = \overline{K}$, the geometric fiber $\pi^{-1}(\pi(\bar{x}))$ is then bijective to the set of cosets $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, see [66] proposition 5.37. By [18] and [15], $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$ is an étale covering map in the sense that, $\forall y \in \mathcal{F}^a$, there exists an open neighborhood $\mathcal{U} \subset \mathcal{F}^a$ such that $\pi^{-1}(\mathcal{U})$ is a disjoint union of spaces \mathcal{V}_i , each restriction map $\pi|_{\mathcal{V}_i} : \mathcal{V}_i \rightarrow \mathcal{U}$ is finite étale. In particular, the Hecke orbit $\pi^{-1}(\pi(x))$ is a discrete subspace of \mathcal{M} . Thus it makes sense to talk “fundamental domain” for the Hecke action of $G(\mathbb{Q}_p)$ on \mathcal{M} . In the following section, we will consider the action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ on \mathcal{M} and find some “fundamental domain” for this action.

We explain how is the Hecke action of $G(\mathbb{Q}_p)$ on a Hecke orbit $\pi^{-1}(\pi(x))$. We first look at the geometric Hecke orbits. Fix a geometric pint \bar{x} over x and denote their images under the π by \bar{y} and y respectively, let $\pi_1(\mathcal{F}^a, \bar{y})$ be the étale fundamental group of \mathcal{F}^a defined by de Jong in [18]. Then by definition there is an action of $\pi_1(\mathcal{F}^a, \bar{y})$ on the geometric Hecke orbit $\pi^{-1}(\bar{y})$. Let us fix a point in this orbit, say \bar{x} , then we have an identification $\pi^{-1}(\bar{y}) = G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$. Thus $\pi_1(\mathcal{F}^a, \bar{y})$ acts on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$. On this

set $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ we have two other group actions, namely the group $G(\mathbb{Q}_p)$ and the Galois group $\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y))$. The relation between these actions is as follow. Recall the \mathbb{Z}_p -local system \mathcal{T} defined by the universal étale unitary p -divisible group on the p -adic analytic space \mathcal{M} , it descends to a \mathbb{Q}_p -local system on \mathcal{F}^a , which we still denote by \mathcal{T} . Since \mathcal{F}^a is connected, by de Jong [18] theorem 4.2. we have the equivalence of categories

$$\begin{aligned} \mathbb{Q}_p - \text{Loc}_{\mathcal{F}^a} &\xrightarrow{\sim} \text{Rep}_{\mathbb{Q}_p}(\pi_1(\mathcal{F}^a, \bar{y})) \\ \mathcal{E} &\mapsto \mathcal{E}_{\bar{y}} \end{aligned}$$

by the monodromy representation functor. Here $\mathbb{Q}_p - \text{Loc}_X$ is the category of \mathbb{Q}_p -local systems over a Berkovich space X introduced in loc. cit. definition 4.1. One can translate the above equivalence to the case with additional structures by using Tannakian language. In particular the \mathbb{Q}_p -local system \mathcal{T} over \mathcal{F}^a defines a representation of the fundamental group :

$$\rho : \pi_1(\mathcal{F}^a, \bar{y}) \rightarrow G(\mathbb{Q}_p).$$

Then the above action of $\pi_1(\mathcal{F}^a, \bar{y})$ on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is compatible with the natural action of $G(\mathbb{Q}_p)$ on the quotient set, through the morphism ρ . On the other hand, there is the natural action of the Galois group $\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y))$ on the geometric Hecke orbit $\pi^{-1}(\bar{y})$, and the quotient set is the Hecke orbit of x

$$\pi^{-1}(\pi(x)) = \pi^{-1}(y) \simeq \pi^{-1}(\bar{y})/\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y)).$$

The point y in \mathcal{F}^a defines a morphism of fundamental groups

$$\pi_1(y, \bar{y}) = \text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y)) \rightarrow \pi_1(\mathcal{F}^a, \bar{y}),$$

and the action of $\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y))$ and $\pi_1(\mathcal{F}^a, \bar{y})$ is compatible on $\pi^{-1}(\bar{y})$ through the above morphism. Thus the three group $\pi_1(\mathcal{F}^a, \bar{y})$, $G(\mathbb{Q}_p)$ and $\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y))$ act compatibly on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ via the morphisms

$$\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y)) \rightarrow \pi_1(\mathcal{F}^a, \bar{y}) \rightarrow G(\mathbb{Q}_p).$$

Here although we will not need it in the following, we remark that the monodromy representations of geometric fundamental groups $\pi_1(\mathcal{M} \times \mathbb{C}_p, \bar{x})$ and $\pi_1(\mathcal{F}^a \times \mathbb{C}_p, \bar{y})$ factor through $G^{\text{der}}(\mathbb{Z}_p)$ and $G^{\text{der}}(\mathbb{Q}_p)$ respectively :

$$\pi_1(\mathcal{M} \times \mathbb{C}_p, \bar{x}) \longrightarrow G^{\text{der}}(\mathbb{Z}_p)$$

$$\pi_1(\mathcal{F}^a \times \mathbb{C}_p, \bar{y}) \longrightarrow G^{\text{der}}(\mathbb{Q}_p).$$

Moreover these monodromy representations are maximal in the sense that both images are dense in the targets respectively, cf. [15] théorème 5.1.2.1. Let Γ be the image of the later. Note that $\pi_1(\mathcal{F}^a \times \mathbb{C}_p, \bar{y})$ also acts on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ compatibly with the action of $G(\mathbb{Q}_p)$. By de Jong's description of \mathcal{M} in term of lattice in the \mathbb{Q}_p -local system \mathcal{T} over \mathcal{F}^a , one have the bijection

$$\pi_0(\mathcal{M}) \simeq \Gamma \backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p).$$

The action of the Hecke correspondences on the geometric orbit is then quite easy : with the fixed point \bar{x} in $\pi^{-1}(\bar{y})$ and the identification $\pi^{-1}(\bar{y}) = G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, a correspondence $G(\mathbb{Z}_p)gG(\mathbb{Z}_p)$ sends a coset $hG(\mathbb{Z}_p)$ to the set of cosets $\{h'G(\mathbb{Z}_p) \mid h'G(\mathbb{Z}_p) \subset$

$G(\mathbb{Z}_p)ghG(\mathbb{Z}_p)\}$. Thus the Hecke action is compatible with the natural action of $G(\mathbb{Q}_p)$ on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$, and thus compatible with the action of the Galois group $\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y))$. The Hecke action on $\pi^{-1}(\overline{y})/\text{Gal}(\overline{\mathcal{H}(y)}/\mathcal{H}(y))$ then induces the Hecke action on the orbit $\pi^{-1}(\pi(x))$.

2.3 Harder-Narasimhan filtration of finite flat group schemes II

In order to study the Rapoport-Zink space \mathcal{M} , from this subsection to the end of subsection 2.5, we will turn to the study of finite flat group schemes and p -divisible groups over complete valuation rings following the ideas in [26] and [27]. In this subsection we recall some further propositions in the theory of Harder-Narasimhan filtration of finite flat group schemes which we will need.

Let the notations be as in subsection 1.2. We recall some useful facts.

Proposition 2.3.1 ([26], Corollaire 6). *For $\mu \in [0, 1]$ fixed, the category of semi-stable finite flat group schemes of slope μ and the trivial group 0 is a sub abelian category of the category of fppf sheaves over $\text{Spec}O_K$.*

Proposition 2.3.2 ([26], Corollaire 7). *For a semi-stable group G , the kernel of multiplication by p^n is flat and semi-stable of slope $\mu(G[p^n]) = \mu(G)$. If $p^n G \neq 0$ then $p^n G$ is also semi-stable of slope $\mu(p^n G) = \mu(G)$.*

Finally we have the semi-continuity of the function HN for a family of finite flat group schemes.

Theorem 2.3.3 ([26], Théorème 3). *Let $K|\mathbb{Q}_p$ be a complete discrete valuation field extension, and \mathcal{X} be a formal scheme of formally locally of finite type over $\text{Spf}O_K$. Let G be a locally free finite group scheme over \mathcal{X} of constant height $h = \text{ht}(G)$. Then the map $x \mapsto HN(G_x)$ from the underlying topological space of the associated Berkovich analytic space \mathcal{X}^{an} to the space of Harder-Narasimhan polygons is continuous. Moreover, it is semi-continuous for the G -topology on \mathcal{X}^{an} defined by analytic domains in the following sense. If $\mathcal{P} : [0, h] \rightarrow \mathbb{R}$ is a fixed polygon such that the abscissas its break points are integers. Then*

$$\{x \in |\mathcal{X}^{an}| \mid HN(G_x) \geq \mathcal{P}\}$$

is a closed analytic domain in \mathcal{X}^{an} , whose associated rigid space is an admissible open in \mathcal{X}^{rig} . In particular if the degree function $x \mapsto \text{deg}G_x$ is constant on \mathcal{X}^{an} , then the semi-stable locus

$$\{x \in |\mathcal{X}^{an}| \mid G_x \text{ is semi-stable}\}$$

is a closed analytic domain of \mathcal{X}^{an} .

2.4 Harder-Narasimhan polygon of p -divisible groups

One can then use the theory of Harder-Narasimhan filtration of finite flat group schemes to study p -divisible groups, p -adic analytic Rapoport-Zink spaces and Shimura

varieties. Let H/O_K be a p -divisible group of dimension d and height h , where O_K as above. Then for any $n \geq 1$, we have a function $HN(H[p^n])$. We normalize it as a function

$$\begin{aligned} \frac{1}{n}HN(H[p^n])(n\cdot) : [0, h] &\longrightarrow [0, d] \\ x &\mapsto \frac{1}{n}HN(H[p^n])(nx). \end{aligned}$$

In [27], Fargues proved as $n \rightarrow \infty$ these functions uniformly convergent to a continue function, which we call the (normalized) Harder-Narasimhan polygon of H

$$HN(H) : [0, h] \rightarrow [0, d],$$

and in fact

$$HN(H)(x) = \inf_{n \geq 1} \frac{1}{n}HN(H[p^n])(nx)$$

for all $x \in [0, h]$. Moreover, this function is invariant when H varies in its isogeny class : $HN(H) = HN(H')$ for any p -divisible group H' isogenous to H . A not evident fact is that $HN(H)$ is in fact the polygon attached to a Harder-Narasimhan type filtration of the rational Hodge-Tate module, so it is really a polygon ! In the case the valuation on K is discrete, then it is the Harder-Narasimhan polygon of the crystalline representation defined by the rational Tate module $V_p(H)$ for suitably defined slope function, which in turn can also be formulated in the associated admissible filtered isocrystals, see section 1.4 and [27] sections 8, 10.

As mentioned in the introduction, one of the main results in [27] is the following inequality between the Harder-Narasimhan and Newton polygons (see Theorem 1.3.5)

$$HN(H) \leq Newt(H_k).$$

In fact when the base valuation ring O_K is not necessary discrete, one has to assume H is “modular” in the sense of définition 25 in loc. cit., see also definition 1.5.6, which is naturally satisfied for p -divisible groups coming from points in the Berkovich analytic Rapoport-Zink spaces. The proof of the above theorem for p -divisible groups over complete rank one discrete valuation $O_K|\mathbb{Z}_p$ with perfect residue field is easy. It comes from the fact that the reduction functor between the two categories of p -divisible groups up to isogenies

$$\text{pdiv}_{O_K} \otimes \mathbb{Q} \longrightarrow \text{pdiv}_k \otimes \mathbb{Q}$$

is exact and preserving the height and dimension functions. One can also rewrite these polygons in terms of the associated filtered isocystal and explain the inequality by the theory of filtered isocystals. For the non-discrete case, Fargues has used heavily p -adic Hodge theory and studied the Harder-Narasimhan filtration of the Banach-Colmez spaces. For more details see section 10, 11 of [27].

In the section 1.3 we have introduced Harder-Narasimhan polygons for p -divisible groups with additional structures, which include our unitary p -divisible groups in section 2.2 as a special case. Let $F|\mathbb{Q}_p$ be a finite unramified extension of degree d , (H, ι, λ) (resp. (H, ι)) be a p -divisible group with PEL (resp. EL) additional structures for $F|\mathbb{Q}_p$ over a complete rank one valuation ring $O_K|\mathbb{Z}_p$. Recall we define the Harder-Narasimhan polygon of (H, ι, λ) (resp. (H, ι)) as the normalization of $HN(H)$

$$HN(H, \iota, \lambda) \text{ (resp. } (H, \iota)) = \frac{1}{d}HN(H)(d\cdot)$$

as a function $[0, htH/d] \rightarrow [0, dimH/d]$, which we will also identify with its graph as a polygon in $[0, htH/d] \times [0, dimH/d]$. Under the above notations, and when O_K is not of discrete valuation we assume in addition that H is modular in the sense of definition 25 in [25], recall we have the basic inequality (see Proposition 1.3.6)

$$HN(H, \iota, \lambda) \leq Newt(H_k, \iota, \lambda).$$

Similarly conclusion holds for the EL case.

Finally we have also the semi-continuity of the function HN for a family of p -divisible groups. Fix a finite unramified extension $F|\mathbb{Q}_p$ of degree d , and we consider p -divisible groups with additional structures for $F|\mathbb{Q}_p$. The following proposition can be deduced directly from the case without additional structures, see [27] proposition 4. To fix notations we just state it for the PEL cases.

Proposition 2.4.1. *Let $K|\mathbb{Q}_p$ be a complete discrete valuation field, and \mathcal{X} be a formal scheme locally formally of finite type over $SpfO_K$. Let (H, ι, λ) be a p -divisible group with additional structures over \mathcal{X} of dimension $\frac{dn}{2}$ and height dn constant. Then the normalized Harder-Narasimhan function on the underlying topological space of \mathcal{X}^{an} is semi-continuous : if $\mathcal{P} : [0, n] \rightarrow [0, n/2]$ is a concave function such that $\mathcal{P}(0) = 0, \mathcal{P}(n) = n/2$, then the subset*

$$\{x \in |\mathcal{X}^{an}| \mid HN(H_x, \iota, \lambda) \geq \mathcal{P}\}$$

is closed.

In particular, with the above notation this proposition permit us to define a stratification of the underlying topological space of X . More precisely let Poly denotes the set of concave polygons starting from the point $(0,0)$ to the point $(n, n/2)$, such that the abscissas of its break points are integers. Then we have a stratification by Harder-Narasimhan polygons

$$X = \coprod_{\mathcal{P} \in \text{Poly}} X^{HN=\mathcal{P}},$$

where

$$X^{HN=\mathcal{P}} = \{x \in |X| \mid HN(H_x, \iota, \lambda) = \mathcal{P}\}$$

which is a locally closed subset of X by proposition 2.4.1. On the other hand there is a stratification by Newton polygons

$$X = \coprod_{\mathcal{P} \in \text{Poly}} X^{Newt=\mathcal{P}},$$

where

$$X^{Newt=\mathcal{P}} = \{x \in |X| \mid Newt(H_{xk(x)}, \iota, \lambda) = \mathcal{P}\} = sp^{-1}(X_{red}^{Newt=\mathcal{P}})$$

which is a locally closed analytic domain of X . Here $k(x)$ is the residue field of the complete valuation $O_{\mathcal{H}(x)}$ associated to x , X_{red} is the reduced special fiber of the formal scheme \mathcal{X} , $X_{red}^{Newt=\mathcal{P}}$ is the Newton polygon strata of X_{red} for the polygon \mathcal{P} , and $sp : X \rightarrow X_{red}$ is the specialization map, which is anti-continuous in the sense that $sp^{-1}(Y)$ is an open (resp. a closed) subset of X if Y is a Zariski closed (resp. open) subset of X_{red} .

Let $\mathcal{P}_{ss} \in \text{Poly}$ denote the line of slope $\frac{1}{2}$ between $(0,0)$ and $(n, \frac{n}{2})$. Then proposition 1.3.6 tells us we have the inclusion

$$X^{\text{Newt}=\mathcal{P}_{ss}} \subset X^{\text{HN}=\mathcal{P}_{ss}}.$$

Now we look at the unitary Rapoport-Zink space $\widehat{\mathcal{M}}$ introduced in section 2.2. Since it is basic, we have

$$\mathcal{M} = \mathcal{M}^{\text{Newt}=\mathcal{P}_{ss}}.$$

Thus there is just one Harder-Narasimhan strata, i.e. the whole space

$$\mathcal{M} = \mathcal{M}^{\text{HN}=\mathcal{P}_{ss}}.$$

In section 2.9 we will look at the Harder-Narasimhan stratification for some unitary p -adic Shimura varieties.

2.5 An algorithm for p -divisible groups with additional structures

In [27], Fargues introduced an algorithm for p -divisible groups over complete valuation rings of rank one which is an extension of \mathbb{Z}_p , to produce p -divisible groups more close to those of HN-type, see loc. cit. for the definition of p -divisible groups of HN-type. For the case which we are interested, it suffices to consider the formal p -divisible groups with special fibers supersingular and semi-stable p -divisible groups. Let $K|\mathbb{Q}_p$ be a complete field extension for a rank one valuation, and O_K be its ring of integer. Let H be a p -divisible group over O_K . Recall the following definition of Fargues ([27], lemme 2, définition 4) :

Definition 2.5.1. *H is called semi-stable if it satisfies one of the following three equivalent conditions :*

- $H[p]$ is semi-stable ;
- for all $n \geq 1$, $H[p^n]$ is semi-stable ;
- for all finite flat subgroup scheme $G \subset H$, $\mu(G) \leq \mu_H := \frac{\dim H}{ht H} (= \mu(H[p^n]), \forall n \geq 1)$.

For a finite flat group scheme G over O_K , let $\mu_{max}(G)$ be the maximal slope of the Harder-Narasimhan polygon $HN(G)$ of G , then it is semi-stable if and only if $\mu_{max}(G) = \mu_G$. Thus for the p -divisible group H , it is semi-stable if and only if one of the following two another conditions holds :

- $\mu_{max}(H[p]) = \mu_H$;
- for all $n \geq 1$, $\mu_{max}(H[p^n]) = \mu_H$.

For the p -divisible group H over O_K , for all $k \geq 1$, we set

$$G_k = \text{the first scan of the Harder-Narasimhan filtration of } H[p^k].$$

Then one has for all $i \geq j \geq 1$

$$G_j = G_i[p^j] \subset G_i, p^{i-j}G_i \subset G_j,$$

see [27] lemme 3 and the remark below there. In particular, the slopes $\mu(G_k) = \mu_{\max}(H[p^k])$ do not change when $k \geq 1$ varies. We denote $\mu_{\max}(H) := \mu_{\max}(H[p^k])$ for any $k \geq 1$, and one can find that

$$\mu_{\max}(H) = \sup\{\mu(G)|G \subset H\} = \sup\{\mu(G)|G \subset H[p]\}.$$

We have thus always $\mu_{\max}(H) \geq \mu_H$, and the equality holds if and only if H is semi-stable.

We set

$$\mathcal{F}_H = \varinjlim_{k \geq 1} G_k \subset H,$$

considered as a sub-fppf sheaf of H . Then we have for all $k \geq 1$, $\mathcal{F}_H[p^k] = G_k$ is a finite flat group scheme over O_K , and $\mathcal{F}_H = H$ if and only if H is semi-stable.

Suppose H is not semi-stable. Then lemme 4 of loc. cit. tells us there are two possibilities :

- \mathcal{F}_H is a finite flat group scheme over O_K , that is there exists some $k_0 \geq 1$ such that $\mathcal{F}_H = \mathcal{F}_H[p^{k_0}]$;
- there exists some integer $k_0 \geq 1$ such that $\mathcal{F}_H/\mathcal{F}_H[p^{k_0}]$ is a semi-stable sub- p -divisible group of $H/\mathcal{F}_H[p^{k_0}]$ with $\mu_{\mathcal{F}_H/\mathcal{F}_H[p^{k_0}]} = \mu_{\max}(H) > \mu_H$.

As said above, we will only be interested in formal p -divisible groups over O_K , such that their special fibers are supersingular. We call such a formal p -divisible group basic. From now on we will suppose that H is a basic modular (see definition 1.5.6) p -divisible group over O_K . Then we have only the first possibility for \mathcal{F}_H , i.e. it is a finite flat subgroup scheme of H . This comes from the facts $HN(H) \leq \text{Newt}(H_k)$ thus both of them are the line between $(0, 0)$ and (h, d) , and one can read off the Harder-Narasimhan polygon of H from the algorithm below.

The algorithm of Fargues for such a p -divisible group H , defines a sequence of p -divisible groups $(H_i)_{i \geq 1}$ by induction, with an isogeny $\phi_i : H_i \rightarrow H_{i+1}$ for each $i \geq 1$. For $i = 1$, we set $H_1 = H$, and if $H_i \neq 0$, we set

$$H_{i+1} = H_i/\mathcal{F}_{H_i},$$

and $\phi_i : H_i \rightarrow H_{i+1}$ is the natural projection; if $H_i = 0$ we set $H_{i+1} = 0$ and ϕ_i the trivial morphism. Then by construction,

$$\mu_{\max}(H_{i+1}) < \mu_{\max}(H_i) = \mu(\mathcal{F}_{H_i})$$

if $H_{i+1} \neq 0$. Note $\mu_H = \mu_{H_{i+1}} \leq \mu_{\max}(H_{i+1})$ if $H_{i+1} \neq 0$. The section 8 of [27] tells us that if the valuation on K is discrete, then the algorithm stops after finite times, i.e. $H_i = 0$ for i large enough. For the general valuation case, the main theorem of loc. cit. says if $(\dim H, htH) = 1$, then $H_i = 0$ for $i \gg 0$.

Until the end paragraph of this section we assume the valuation on K is discrete. Then by the above discussion, for a basic formal p -divisible group H over O_K , if it is not semi-stable, we have a sequence of p -divisible groups with each arrow between them an isogeny :

$$H = H_1 \xrightarrow{\phi_1} H_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_r} H_{r+1},$$

ϕ

with H_1, \dots, H_r not semi-stable and H_{r+1} semi-stable of slope $\mu = \frac{d}{h}$, and for each $i = 1, \dots, r$, the kernel of the isogeny $\ker(\phi_i)$ is the first scan of the Harder-Narasimhan filtration of $H_i[p^{n_i}]$ for some $n_i \gg 0$. There exists some $N \gg 0$ such that the kernel $\ker\phi$ of the composition of these isogenies ϕ is contained in $H[p^N]$. By construction, $\ker\phi$ is in fact a scan in the Harder-Narasimhan filtration of $H[p^N]$ and these $\ker\phi_i$'s are exactly the sub-quotient factors of the Harder-Narasimhan filtration of $\ker\phi$. If we denote $\mu_i = \mu(\ker\phi_i)$, then

$$\mu_1 > \mu_2 > \dots > \mu_r > \frac{d}{h}.$$

Now we consider p -divisible groups with additional structures. First, the above construction works totally in the EL case, that is p -divisible groups with actions of the integer ring O_F of some finite unramified extension $F|\mathbb{Q}_p$, since the Harder-Narasimhan filtration is of O_F -invariant, cf. subsection 1.2. Next, we consider PEL cases, that is a p -divisible group H over O_K , with action $\iota : O_F \rightarrow \text{End}(H)$ of the integer ring O_F of some finite unramified extension $F|\mathbb{Q}_p$, and a polarization $\lambda : H \rightarrow H^D$, such that ι and λ are compatible in the sense of definition 2.4.2. In particular, we can apply the unitary p -divisible groups studied above to this situation. So let (H, ι, λ) be a p -divisible group with (PEL) additional structures, such that H is basic. Assume H is not semi-stable, then we have a sequence of O_F -linear isogenies of p -divisible groups with actions of O_F :

$$H = H_1 \xrightarrow{\phi_1} H_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_r} H_{r+1},$$

ϕ

with H_1, \dots, H_r not semi-stable and H_{r+1} semi-stable of slope $\mu = \frac{1}{2}$, and there exists some $N \gg 0$ such that $E := \ker\phi \subset H[p^N]$ and $E \not\subset H[p^{N-1}]$. Then E is a scan in the Harder-Narasimhan filtration of $H[p^N]$. Let

$$0 = E_0 \subsetneq E_1 \subsetneq \dots \subsetneq E_r = E$$

be the Harder-Narasimhan filtration of E , then we have

$$\begin{aligned} E_i/E_{i-1} &\simeq \ker\phi_i \\ \mu_1 &> \dots > \mu_r > \frac{1}{2}, \end{aligned}$$

where $\mu_i := \mu(\ker\phi_i)$ for $i = 1, \dots, r$. The polarization λ on H now induces a polarization on $H[p^N] : \lambda : H[p^N] \xrightarrow{\sim} H[p^N]^D$. Thus there is a perfect pairing

$$H[p^N] \times H[p^N] \rightarrow \mu_{p^{2nN}}.$$

Let E_i^\perp be the orthogonal subgroup of $H[p^N]$ under this pairing, for $i = 1, \dots, r$. Since E_i is a scan of the Harder-Narasimhan filtration of $H[p^N]$, so is E_i^\perp by the compatibility of Harder-Narasimhan filtration with polarizations. Moreover, we have the following inclusions :

$$0 \subsetneq E_1 \subsetneq \dots \subsetneq E_{r-1} \subsetneq E_r = E \subset E^\perp = E_r^\perp \subsetneq E_{r-1}^\perp \subsetneq \dots \subsetneq E_1^\perp \subsetneq H[p^N],$$

and the equalities

$$\mu_i + \mu(E_{i-1}^\perp/E_i^\perp) = 1,$$

for $i = 1, \dots, r$. Recall that we have

$$\mu_1 > \mu_2 > \dots > \mu_r > \frac{1}{2}.$$

For all $k \geq N$, the finite flat group schemes $E_1, \dots, E_r \subset H[p^k]$ are the same and do not depend on k , but their orthogonal groups $E_1^\perp, \dots, E_r^\perp$ do depend on the group $H[p^k]$. Thus for $k \geq N$ varies, just the height of $ht(E^\perp/E)$ varies and $ht(E_i/E_{i-1}) = ht(E_{i-1}^\perp/E_i^\perp)$ do not change for $i = 1, \dots, r-1$.

There are two different cases : $E = E^\perp$ or $E \subsetneq E^\perp$.

1. $E = E^\perp$, i.e. there is no slope of $\frac{1}{2}$ in the Harder-Narasimhan filtration of $H[p^N]$. This case is good, since the polarization λ on H then induces a polarization $\lambda' : H/E \rightarrow (H/E)^D$, such that $\pi \circ \lambda' \circ \pi^D = p^N \lambda$, where $\pi : H \rightarrow H/E$ is the natural projection, i.e. we have the following commutative diagram :

$$\begin{array}{ccc} H & \xrightarrow{p^N \lambda} & H^D \\ \downarrow \pi & & \uparrow \pi^D \\ H/E & \xrightarrow{\lambda'} & (H/E)^D. \end{array}$$

Thus in this case we get a p -divisible group with naturally induced additional structures $(H/E, \iota', \lambda')$.

2. $E \subsetneq E^\perp$, i.e. E^\perp/E is a factor in the Harder-Narasimhan filtration of $H[p^N]$ of slope $\frac{1}{2}$. Note there is a natural perfect pairing

$$(E^\perp/E) \times (E^\perp/E) \rightarrow \mu_{p^{ht(E^\perp/E)}}.$$

Let $C := E^\perp/E$. We make the following claim :

Claim 2.5.2. *there is a filtration of sub-semi-stable groups of slope $\frac{1}{2}$*

$$0 \subsetneq C_1 \subsetneq \dots \subsetneq C_k \subset C_k^\perp \subsetneq \dots \subsetneq C_1^\perp \subsetneq C,$$

such that $p(C_k^\perp/C_k) = 0$.

In fact, if we let m be the minimal integer such that $p^m C = 0$, if $m = 1$ we are done ; so assume $m \geq 2$ now. Consider $0 \neq p^{m-1} C \subsetneq C$, which is also semi-stable of slope $\frac{1}{2}$. Then we have a filtration of semi-stable groups of slope $\frac{1}{2}$

$$0 \neq p^{m-1} C \subsetneq (p^{m-1} C)^\perp = C[p^{m-1}] \subsetneq C.$$

Now

$$ht((p^{m-1} C)^\perp/p^{m-1} C) < ht C,$$

and set $C' = (p^{m-1} C)^\perp/p^{m-1} C$, by induction we thus have the above claim.

Now we can translate the above filtration to a filtration of subgroups of $E^\perp \subset H[p^N]$, that is there exists a filtration

$$E \subsetneq E^1 \subsetneq \dots \subsetneq E^k \subset E^{k^\perp} \subsetneq \dots \subsetneq E^\perp,$$

such that $E^i/E = C^i \subset E^\perp/E = C$. Let $E' := E^k$, then since E'/E is semi-stable, H/E' is semi-stable. We still have two cases.

- (a) If $C_k = C_k^\perp$ that is $E' = E'^\perp$, this is still good in this case : we have the semi-stable p -divisible group with additional structures $(H/E', \iota', \lambda')$.
- (b) If $E' \subsetneq E'^\perp$, we have the following proposition.

Proposition 2.5.3. *Let the notation be as above, and (H, ι, λ) be a p -divisible group with additional structures. Assume that $E' \subsetneq E'^\perp$. Then after changing N to $N + 1$ if N is odd in the PEL unitary case, there exists some finite extension $K'|K$ and a totally isotrope subgroup $E'' = (E'')^\perp$ of $H[p^N]$ over $O_{K'}$.*

Proof. Let $V := (E'^\perp/E')(\overline{K})$, since $p(E'^\perp/E') = 0$, it is a \mathbb{F}_p -vector space equipped with an action of the Galois group $Gal(\overline{K}/K)$. Moreover, there is an induced \mathbb{F}_{p^d} -action $\iota : \mathbb{F}_{p^d} \rightarrow End(V)$ ($d = [F : \mathbb{Q}_p]$, $\iota : O_F \rightarrow End(H)$), so we can view V as a \mathbb{F}_{p^d} -vector space via ι . The pairing \langle, \rangle on E'^\perp/E' induces a hermitian form $V \times V \rightarrow \mathbb{F}_{p^d}$. By assumption $dim_{\mathbb{F}_{p^d}} V = 2m$ for some integer $m \geq 1$. Thus there exists a maximal totally isotrope subspace $W \subset V$, $W = W^\perp$, and a finite extension $K'|K$ such that W is stable by $Gal(\overline{K}/K')$. Then the schematic closure of W in E'^\perp/E' over $O_{K'}$ corresponds to a totally isotrope subgroup $E'' = (E'')^\perp$ of $H[p^N]$ over $O_{K'}$. \square

Let $K'|K$, and $E'' \subset H[p^N]$ be as above. Then the p -divisible group H/E'' over $O_{K'}$ admits naturally induced additional structures : $\iota' : O_F \rightarrow End(H/E'')$, $\lambda' : H/E'' \rightarrow (H/E'')^D$ such that the following diagram commutes :

$$\begin{array}{ccc} H & \xrightarrow{p^N \lambda} & H^D \\ \downarrow \pi & & \uparrow \pi^D \\ H/E'' & \xrightarrow{\lambda'} & (H/E'')^D. \end{array}$$

Recall H/E' is a semi-stable p -divisible group. Thus we have an isogeny $f : H/E'' \rightarrow H/E'$ of p -divisible groups over $O_{K'}$ such that $p(ker f) = 0$.

Now for the case that the valuation ring O_K is not discrete, let (H, ι, λ) be a basic p -divisible group with additional structures over O_K . Assume H is not semi-stable. We still have Fargues's algorithm

$$H = H_1 \xrightarrow{\phi_1} H_2 \rightarrow \cdots \xrightarrow{\phi_{i-1}} H_i \xrightarrow{\phi_i} \cdots ,$$

with

$$\mathcal{F}_H = ker \phi_1, \mathcal{F}_{H_2} = ker \phi_2, \cdots .$$

If the algorithm stops after finite times, i.e. there exists some r such that $H_r \neq 0$ and $H_i = 0$ for all $i \geq r+1$. In this case H_r is semi-stable, and we can continue our procedure as above to find the E , E' and E'' . Thus once the algorithm stops after finite times, we can continue as above and get a modified algorithm for the case with additional structures.

2.6 The analytic domain \mathcal{C}

We return to the study of the p -adic analytic unitary group Rapoport-Zink space \mathcal{M} we introduced in subsection 2.2. Let $K|L = W(\overline{\mathbb{F}_p})_{\mathbb{Q}}$ be a complete discrete valuation

field, then for any K -valued point $x \in \mathcal{M}(K)$, by the algorithm of last section, we can associate to it an another point $x' \in \mathcal{M}(K')$, where $K'|K$ is a finite extension, such that if (H_x, ι, λ) (resp. $(H_{x'}, \iota', \lambda')$) is the p -divisible group associated to x (resp. x'), then we have an isogeny $\phi : H_x \rightarrow H_{x'}$ of p -divisible groups over $O_{K'}$, satisfying the following commutative diagram :

$$\begin{array}{ccc} H_x & \xrightarrow{p^N \lambda} & H_x^D \\ \downarrow \phi & & \uparrow \phi^D \\ H_{x'} & \xrightarrow{\lambda'} & (H_{x'})^D, \end{array}$$

for some integer N , and there is a finite flat subgroup scheme $G \subset H_{x'}[p]$, such that $H_{x'}/G$ is semi-stable. Motivated by this, we introduce a subspace $\mathcal{C} \subset \mathcal{M}$ as follow.

Definition 2.6.1. *We define a subspace $\mathcal{C} \subset \mathcal{M}$ as*

$$\mathcal{C} = \{x \in \mathcal{M} \mid \exists \text{ finite extension } K'|K(x), \text{ and a finite flat subgroup } G \subset H_x[p] \text{ over } O_{K'}, \\ \text{such that } H_x/G \text{ is semi-stable over } O_{K'}\}.$$

Let $\mathcal{M}^{ss} \subset \mathcal{M}$ be the semi-stable locus, that is

$$\mathcal{M}^{ss} = \{x \in \mathcal{M} \mid H_x \text{ is semi-stable}\}.$$

Then we have the inclusion

$$\mathcal{M}^{ss} \subset \mathcal{C}.$$

Proposition 2.6.2. *The subset \mathcal{M}^{ss} and \mathcal{C} are closed analytic domains of \mathcal{M} .*

Proof. The fact that $\mathcal{M}^{ss} \subset \mathcal{M}$ is a closed analytic domain is an easy consequence of Theorem 2.3.3. So we concentrate here to prove that $\mathcal{C} \subset \mathcal{M}$ is a closed analytic domain.

Let \mathcal{N} be the basic Rapoport-Zink analytic space for $Res_{\mathbb{Q}_{p^2}|\mathbb{Q}_p} GL_n$ obtained by forgetting the polarization from \mathcal{M} . Then there is a natural closed immersion $\mathcal{M} \subset \mathcal{N}$. We fix such an imbedding. We have the inclusions $G(\mathbb{Q}_p) \subset GL_n(\mathbb{Q}_{p^2})$, and $G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \hookrightarrow GL_n(\mathbb{Z}_{p^2}) \setminus GL_n(\mathbb{Q}_{p^2})/GL_n(\mathbb{Z}_{p^2})$. We have in fact a $G(\mathbb{Q}_p)$ -equivariant imbedding of tower of analytic spaces $\mathcal{M}_{K \cap G(\mathbb{Z}_p)} \subset \mathcal{N}_K$ for open compact subgroups $K \subset GL_n(\mathbb{Q}_{p^2})$. We have $\mathcal{M}^{ss} = \mathcal{N}^{ss} \cap \mathcal{M}$. By definition, the subset \mathcal{C} is exactly the intersection with \mathcal{M} of some Hecke translations of the semi-stable locus $\mathcal{N}^{ss} \subset \mathcal{N}$:

$$\mathcal{C} = \left(\bigcup_{\underline{a}} T_{\underline{a}} \cdot \mathcal{N}^{ss} \right) \cap \mathcal{M},$$

where the index set is all $\underline{a} = (a_1, \dots, a_n) \in \{(a_1, \dots, a_n) \mid a_1 \geq \dots \geq a_n, \text{ and } a_i \in \{0, 1\}\}$, and

$$T_{\underline{a}} = GL_n(\mathbb{Z}_{p^2}) \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} GL_n(\mathbb{Z}_{p^2}) \in GL_n(\mathbb{Z}_{p^2}) \setminus GL_n(\mathbb{Q}_{p^2})/GL_n(\mathbb{Z}_{p^2}).$$

Now since $\mathcal{N}^{ss} \subset \mathcal{N}$ is a closed analytic domain, so is $\mathcal{C} \subset \mathcal{M}$ by the definition of Hecke correspondences. □

We observe the characterization of points in \mathcal{C} as follow.

Proposition 2.6.3. *A point $x \in \mathcal{M}$ is in \mathcal{C} if and only if the algorithm for the p -divisible group H_x associated to x stops after finite times, and $N_x = 1$, where N_x is the smallest integer such that $\ker \phi_x \subset H_x[p^{N_x}]$, ϕ_x is the composition of the isogenies when applying the algorithm to H_x .*

Proof. If the algorithm for H_x stops after finite times and $N_x = 1$, then by definition $x \in \mathcal{C}$. To prove the other direction, we have the following general lemma.

Lemma 2.6.4. *Let H/O_K be a basic p -divisible group over a complete rank one valuation ring $O_K|\mathbb{Z}_p$, and $G \subset H$ be a finite flat subgroup scheme. If H/G is semi-stable, then $\mathcal{F}_H \subset G$. In particular, if the sequence of isogenies of p -divisible groups*

$$H = H_1 \xrightarrow{\phi_1} H_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_r} H_{r+1}$$

ϕ

is such that H_1, \dots, H_r are not semi-stable (H_{r+1} may be semi-stable or may be not), then $\ker \phi \subset G$.

Proof. Let $0 \neq G' \subset H, G' \not\subset G$ be a finite flat subgroup not contained in G . Consider the morphism $\varphi : G' \rightarrow H[p^N]/G$ for $N \gg 0$. Then it is non zero. Let G''' (resp. G'') be the flattening schematic image (resp. kernel) of φ , then we have the following sequence which is exact in generic fiber :

$$0 \rightarrow G'' \rightarrow G' \rightarrow G''' \rightarrow 0.$$

Since H/G is semi-stable, and $G''' \subset H/G$ is a finite flat subgroup, by definition

$$\mu(G''') \leq \mu_{H/G} = \mu_H.$$

On the other hand we have $G'' \subset G$. If $G'' = 0$, then $\mu(G') \leq \mu(G''') \leq \mu_H$. If $G'' \neq 0$, $\mu(G') \leq \sup\{\mu(G''), \mu(G''')\} \leq \sup\{\mu_{\max}(G), \mu_H\}$. We have two cases :

1. if $\mu_{\max}(G) \leq \mu_H$, then since $\mu(G') \leq \mu_H$ for any $0 \neq G' \not\subset G$, we have H is semi-stable. In particular $\mathcal{F}_H = 0 \subset G$.
2. if $\mu_{\max}(G) > \mu_H$, then since $\mu(G') < \mu_{\max}(G)$, we have $\mu_{\max}(H) = \mu_{\max}(G)$, and $\mathcal{F}_H \subset G$ is the first scan of the Harder-Narasimhan filtration of G .

Thus the lemma holds. □

With the lemma above, we can now easily deduce the proposition. If $x \in \mathcal{C}$, then by the definition of \mathcal{C} , there exists a finite extension $K'|K = \mathcal{H}(x)$ and a finite flat subgroup $G \subset H[p]$ over $O_{K'}$, such that H/G is semi-stable over $O_{K'}$. The algorithm for H over $O_{K'}$ is just the base change of that over O_K . By the lemma, $\ker \phi \subset G \subset H[p]$. Thus the algorithm stops after finite times and $N_x = 1$. □

Example 2.6.5. For $n = 1$ the Rapoport-Zink space \mathcal{M} is trivial : each connected component \mathcal{M}^i is just a point. Thus in this case $\mathcal{C} = \mathcal{M}$. For $n \geq 2$, it is unluckily difficult to describe the domain \mathcal{C} explicitly. Here we calculate \mathcal{C} for the case $n = 2$. In this case each reduced special fiber \mathcal{M}_{red}^i is just a point, while the analytic space \mathcal{M}^i is of dimension 1. Let \mathcal{N} be the basic Rapoport-Zink analytic space for $Res_{\mathbb{Q}_p^2|\mathbb{Q}_p}GL_2$ obtained by forgetting the polarization from \mathcal{M} . We have

$$\mathcal{C} = \left(\bigcup_{\underline{a}} T_{\underline{a}} \cdot \mathcal{N}^{ss} \right) \cap \mathcal{M}.$$

There are 3 possibilities for the index $\underline{a} = (a_1, a_2) : (0,0), (0,-1), (-1,-1)$. Let (H, ι) be the p -divisible group associated to a point $x \in \mathcal{N}^{ss}$. The Hecke correspondences $T_{(0,0)}$ is the identity, and $T_{(-1,-1)} \cdot x$ is the quotient of H by $H[p]$ with its additional structure, thus $T_{(-1,-1)} \cdot x \in \mathcal{N}^{ss}$. For the Hecke correspondence $T_{(0,-1)}$, a point $y \in T_{(0,-1)} \cdot x$ corresponds to a height 2 finite flat subgroup $G \subset H[p]$, and the p -divisible group associated to y is the quotient $(H/G, \iota')$. Since H is semi-stable, we have $\mu(G) \leq \frac{1}{2}$, i.e. $\deg G \leq 1$. The arbitrary possibility of the choices of G will make $T_{(0,-1)} \cdot \mathcal{N}^{ss} \not\subseteq \mathcal{N}^{ss}$. Thus the inclusion $\mathcal{M}^{ss} \subset \mathcal{C}$ is strict.

Similarly we have a characterization of the points in the Hecke orbit of \mathcal{C} .

Proposition 2.6.6. A point $x \in \mathcal{M}$ is in $\bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot \mathcal{C}$ if and only if the algorithm above for the p -divisible group H_x associated to x stops after finite times.

For a Hausdorff paracompact strictly Berkovich analytic space X over a non-archimedean field, we denote by X^{rig} the associated rigid analytic space in the sense of Tate. As a set, $X^{rig} \subset X$ is the subset of rigid analytic points. Then the Hecke orbit of \mathcal{C}^{rig} covers \mathcal{M}^{rig} .

Proposition 2.6.7. We have the following equality of sets

$$\mathcal{M}^{rig} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot \mathcal{C}^{rig}.$$

Proof. By the inequality

$$HN(H, \iota, \lambda) \leq Newt(H_k, \iota, \lambda),$$

there is just one Harder-Narasimhan strata, i.e. the whole space $\mathcal{M} = \mathcal{M}^{HN=\mathcal{P}_{ss}}$. By the algorithm, we have the equality $\mathcal{M}^{HN=\mathcal{P}_{ss}, rig} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot \mathcal{C}^{rig}$. \square

We want some locally finite covering of \mathcal{M} . One may wonder whether the family of analytic domains $(T \cdot \mathcal{C})_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)}$ is such a locally finite covering of \mathcal{M} . Unfortunately, the analytic domain \mathcal{C} is so big that the union $\bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot \mathcal{C}$ is far from locally finite. We have to refine this family. Nevertheless, this family is indeed a covering of \mathcal{M} , although it is not locally finite. In the next subsection, we will review some basic facts about the geometry of reduced special fiber \mathcal{M}_{red} over $\overline{\mathbb{F}}_p$ obtained by Vollaard-Wedhorn in [80] and [81]. Then we will define some smaller analytic domain $\mathcal{D} \subset \mathcal{C}$ such that under the Hecke action and the action of $J_b(\mathbb{Q}_p)$, we can get some locally finite covering of \mathcal{M}^{rig} . Finally by some gluing arguments, and the equivalence

between the categories of Berkovich analytic spaces and rigid analytic spaces satisfying certain conditions, we can get an equality for all analytic points

$$\mathcal{M} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T.C.$$

That is, for all $x \in \mathcal{M}$, the algorithm for H_x stops after finite times.

2.7 Bruhat-Tits stratification of \mathcal{M}_{red}

We recall some of basic results of Vol्लाard-Wedhorn in [80] and [81]. First we remark that the descent data on \mathcal{M}_{red} is effective, i.e., there is a model \mathcal{M}'_{red} over \mathbb{F}_{p^2} of \mathcal{M}_{red} . The results of Vol्लाard-Wedhorn are rather about the scheme \mathcal{M}'_{red} , but we just state them for \mathcal{M}_{red} here. Recall that the formal Rapoport-Zink space $\widehat{\mathcal{M}}$ has a decomposition according the height of the universal quasi-isogeny :

$$\widehat{\mathcal{M}} = \prod_{i \in \mathbb{Z}, in \text{ even}} \widehat{\mathcal{M}}^i,$$

where $\widehat{\mathcal{M}}^i$ is the open and closed formal subscheme of $\widehat{\mathcal{M}}$ such that for any scheme $S \in \text{Nilp}O_L$, a S -valued point of $\widehat{\mathcal{M}}$ $(H, \iota, \lambda, \rho) \in \widehat{\mathcal{M}}^i(S)$ if $ht\rho = in$, which is not empty if and only if in is even. Let $\mathcal{M}^i = (\widehat{\mathcal{M}}^i)^{an}$, and \mathcal{M}_{red}^i be its reduced special fiber for such an i , we have decompositions

$$\mathcal{M} = \prod_{i \in \mathbb{Z}, in \text{ even}} \mathcal{M}^i, \quad \mathcal{M}_{red} = \prod_{i \in \mathbb{Z}, in \text{ even}} \mathcal{M}_{red}^i.$$

By Theorem 4.2.(1) of [81], \mathcal{M}_{red}^i is connected of pure dimension $[\frac{n-1}{2}]$. Thus \mathcal{M}^i is an connected analytic space of dimension $n - 1$. For each $i \in \mathbb{Z}$ such that in is even, there is a $g \in J_b(\mathbb{Q}_p)$ such that $g(\widehat{\mathcal{M}}^i) = \widehat{\mathcal{M}}^0$, in particular $\widehat{\mathcal{M}}^i$ is isomorphic to $\widehat{\mathcal{M}}^0$, and so are their analytic fibers $\mathcal{M}^i, \mathcal{M}^0$ and reduced special fibers $\mathcal{M}_{red}^i, \mathcal{M}_{red}^0$. So we just need consider the \mathcal{M}^0 and \mathcal{M}_{red}^0 . It turns out the geometry of \mathcal{M}_{red}^0 over $\overline{\mathbb{F}}_p$ is controlled by the Bruhat-Tits building $\mathcal{B}(J_b^{der}, \mathbb{Q}_p)$ of the derived subgroup J_b^{der} of J_b over \mathbb{Q}_p .

More precisely, for each $i \in \mathbb{Z}$ such that in is even, let

$$\mathcal{L}_i := \{\Lambda \subset \mathbf{N}_0 \text{ is a } \mathbb{Z}_{p^2}\text{-lattice} \mid p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i\Lambda^\vee\},$$

where $\Lambda^\vee = \{x \in \mathbf{N}_0 \mid \{x, \Lambda\} \subset \mathbb{Z}_{p^2}\}$. One can construct an abstract simplicial complex \mathcal{B}_i from \mathcal{L}_i : an m -simplex of \mathcal{B}_i is a subset $S \subset \mathcal{L}_i$ of $m + 1$ elements which satisfies the following condition. There exists an ordering $\Lambda_0, \dots, \Lambda_m$ of the elements of S such that

$$p^{i+1}\Lambda_m \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \dots \subsetneq \Lambda_m.$$

There is an obvious action of $J_b^{der}(\mathbb{Q}_p)$ on \mathcal{L}_i . By Theorem 3.6 of [80], for each fixed i , we have a natural $J_b^{der}(\mathbb{Q}_p)$ -equivariant isomorphism of \mathcal{B}_i with the associated simplicial complex of the Bruhat-Tits building $\mathcal{B}(J_b^{der}, \mathbb{Q}_p)$. Thus we can identify \mathcal{L}_i with the set of vertices of $\mathcal{B}(J_b^{der}, \mathbb{Q}_p)$. For $\Lambda \in \mathcal{L}_i$ the index $t(\Lambda) := [\Lambda : p^{i+1}\Lambda^\vee]$ of $p^{i+1}\Lambda^\vee$ in Λ is always an odd number with $1 \leq t(\Lambda) \leq n$, and for any odd number d with $1 \leq d \leq n$

there exists a $\Lambda \in \mathcal{L}_i$ such that $t(\Lambda) = d$. Moreover two lattices $\Lambda_1, \Lambda_2 \in \mathcal{L}_i$ are in the same $J_b^{der}(\mathbb{Q}_p)$ -orbit if and only if $t(\Lambda_1) = t(\Lambda_2)$. And the neighborhood vertices of $\Lambda \in \mathcal{L}_i$ in the building is exactly the set

$$\{\Lambda' \in \mathcal{L}_i \mid \Lambda' \subset \Lambda, \text{ or } \Lambda \subset \Lambda'\}.$$

If n is even, we choose and fix a $g_1 \in J_b(\mathbb{Q}_p)$ such that $g_1 : \widehat{\mathcal{M}}^0 \xrightarrow{\sim} \widehat{\mathcal{M}}^1$. We fix a bijection

$$\varphi_i : \mathcal{L}_0 \rightarrow \mathcal{L}_i$$

once for all, such that

$$\varphi_i(\Lambda) = \begin{cases} p^{\frac{i+1}{2}} g_1 \Lambda & i \text{ odd} \\ p^{\frac{i}{2}} \Lambda & i \text{ even.} \end{cases}$$

Now for each $i \in \mathbb{Z}$ such that in is even and each $\Lambda \in \mathcal{L}_i$, we define a subscheme \mathcal{M}_Λ of \mathcal{M}_{red}^i . First we associate two p -divisible groups H_{Λ^-} and H_{Λ^+} over \mathbb{F}_{p^2} with \mathbb{Z}_{p^2} -linear polarizations λ_{Λ^+} and λ_{Λ^-} respectively. To this end, set

$$\begin{aligned} \Lambda_0^+ &:= \Lambda \\ \Lambda_1^+ &:= \mathbf{V}^{-1}(\Lambda_0^+) \\ \Lambda^+ &:= \Lambda_0^+ \oplus \Lambda_1^+ \\ \Lambda^- &:= p^i(\Lambda^+)^\vee. \end{aligned}$$

Since $\mathbf{F} = \mathbf{V}$ the Λ^\pm are Dieudonné submodules of the isocrystal \mathbf{N} , and the pairing $p^{-i}\langle, \rangle$ on \mathbf{N} induces a perfect \mathbb{Z}_{p^2} -pairing on Λ^\pm . Thus Λ^\pm define unitary p -divisible groups H_{Λ^\pm} , with the \mathbb{Z}_{p^2} -linear polarizations λ_{Λ^\pm} and $p^{-i}\langle, \rangle$ induces an isomorphism

$$H_{\Lambda^+} \xrightarrow{\sim} H_{\Lambda^-}^D.$$

Moreover, we have \mathbb{Z}_{p^2} -linear quasi-isogenies

$$\rho_{\Lambda^\pm} : H_{\Lambda^\pm} \rightarrow \mathbf{H}$$

which are compatible with the polarizations on the two sides. We have the following commutative diagram :

$$\begin{array}{ccc} H_{\Lambda^+} & \xrightarrow{\sim} & H_{\Lambda^-}^D \\ \downarrow \rho_{\Lambda^+} & & \uparrow \rho_{\Lambda^-}^D \\ \mathbf{H} & \xrightarrow{\rho} & \mathbf{H}^D. \end{array}$$

By construction we have always $\Lambda^- \subset \Lambda^+$, which corresponds to the composition of quasi-isogenies

$$H_{\Lambda^-} \xrightarrow{\rho_{\Lambda^-}} \mathbf{H} \xrightarrow{\rho_{\Lambda^+}^{-1}} H_{\Lambda^+}.$$

We a fixed vertex $\Lambda \in \mathcal{L}_i$. For any $\overline{\mathbb{F}}_p$ -scheme S and a S -valued point $(H, \iota, \lambda, \rho) \in \mathcal{M}_{red}^i(S)$ we define quasi-isogenies

$$\begin{aligned} \rho_{H, \Lambda^+} &: H \xrightarrow{\rho^{-1}} \mathbf{H}_S \xrightarrow{(\rho_{\Lambda^+})_S^{-1}} (H_{\Lambda^+})_S, \\ \rho_{\Lambda^-, H} &: (H_{\Lambda^-})_S \xrightarrow{(\rho_{\Lambda^-})_S} \mathbf{H}_S \xrightarrow{\rho} H. \end{aligned}$$

Then one has that

$$ht(\rho_{H, \Lambda^+}) = ht(\rho_{\Lambda^-, H}) = t(\Lambda)$$

and that ρ_{H, Λ^+} is an isogeny if and only if $\rho_{\Lambda^-, H}$ is an isogeny.

Definition 2.7.1. We define the subfunctor $\mathcal{M}_\Lambda \subset \mathcal{M}_{red}^i$ as

$$\mathcal{M}_\Lambda(S) = \{(H, \iota, \lambda, \rho) \in \mathcal{M}_{red}^i(S) \mid \rho_{\Lambda^-, H} \text{ is an isogeny}\}$$

for any $\overline{\mathbb{F}}_p$ -scheme S .

Then the main theorems of [81] tell us the following facts.

Theorem 2.7.2. 1. \mathcal{M}_Λ is represented by a smooth projective closed subscheme of dimension $\frac{1}{2}(t(\Lambda) - 1)$ of \mathcal{M}_{red}^i , which we will also denote by \mathcal{M}_Λ . It is in fact a generalized Deligne-Lusztig variety for the maximal reductive quotient $\overline{J}_\Lambda^{der, red}$ over \mathbb{F}_p of the special fiber of the Bruhat-Tits group scheme J_Λ^{der} attached to the vertex $\Lambda \in \mathcal{B}(J_b^{der}, \mathbb{Q}_p)$.

2. for two lattices $\Lambda_1, \Lambda_2 \in \mathcal{L}_i$, $\mathcal{M}_{\Lambda_1} \subset \mathcal{M}_{\Lambda_2}$ if and only if $\Lambda_1 \subset \Lambda_2$. In this case $t(\Lambda_1) \leq t(\Lambda_2)$, and the equality holds if and only if $\Lambda_1 = \Lambda_2$.
3. for two lattices $\Lambda_1, \Lambda_2 \in \mathcal{L}_i$, the following assertions are equivalent :
 - $\Lambda_1 \cap \Lambda_2 \in \mathcal{L}_i$;
 - $\Lambda_1 \cap \Lambda_2$ contains a lattice of \mathcal{L}_i ;
 - $\mathcal{M}_{\Lambda_1} \cap \mathcal{M}_{\Lambda_2} \neq \emptyset$.

If these conditions are satisfied we have

$$\mathcal{M}_{\Lambda_1} \cap \mathcal{M}_{\Lambda_2} = \mathcal{M}_{\Lambda_1 \cap \Lambda_2},$$

where $\mathcal{M}_{\Lambda_1} \cap \mathcal{M}_{\Lambda_2}$ is the scheme-theoretic intersection in \mathcal{M}_{red}^i .

4. for two lattices $\Lambda_1, \Lambda_2 \in \mathcal{L}_i$, the following assertions are equivalent :
 - $\Lambda_1 + \Lambda_2 \in \mathcal{L}_i$;
 - $\Lambda_1 + \Lambda_2$ is contained in a lattice of \mathcal{L}_i ;
 - \mathcal{M}_{Λ_1} and \mathcal{M}_{Λ_2} are both contained in \mathcal{M}_Λ for some $\Lambda \in \mathcal{L}_i$.

If these conditions are satisfied, $\mathcal{M}_{\Lambda_1 + \Lambda_2}$ is the smallest subscheme of the form \mathcal{M}_Λ that contains \mathcal{M}_{Λ_1} and \mathcal{M}_{Λ_2} .

5. let $t_{max} = n$ if n is odd, and $t_{max} = n - 1$ if n is even, then the irreducible components of \mathcal{M}_{red}^i are exactly the subschemes \mathcal{M}_Λ with $t(\Lambda) = t_{max}$.
6. let

$$\begin{aligned} \mathcal{L}_\Lambda &:= \{\Lambda' \in \mathcal{L}_i \mid \Lambda' \subsetneq \Lambda\}, \\ \mathcal{M}_\Lambda^0 &:= \mathcal{M}_\Lambda \setminus \bigcup_{\Lambda' \in \mathcal{L}_\Lambda} \mathcal{M}_{\Lambda'}, \end{aligned}$$

then \mathcal{M}_Λ^0 is open and dense in \mathcal{M}_Λ , and we have a stratification of \mathcal{M}_{red}^i :

$$\mathcal{M}_{red}^i = \coprod_{\Lambda \in \mathcal{L}_i} \mathcal{M}_\Lambda^0.$$

Proof. These are the contents of lemma 3.2, theorem 3.10, corollary 3.11, theorem 4.1, 4.2, and proposition 4.3 of [81]. \square

Note that the stratification $\mathcal{M}_{red}^i = \coprod_{\Lambda \in \mathcal{L}_i} \mathcal{M}_\Lambda^0$ is $J_b^{der}(\mathbb{Q}_p)$ -equivariant in the sense that we have

$$g\mathcal{M}_\Lambda^0 = \mathcal{M}_{g\Lambda}^0,$$

for any $g \in J_b^{der}(\mathbb{Q}_p)$ and $\Lambda \in \mathcal{L}_i$. For an algebraically field $k|\mathbb{F}_{p^2}$ and a k -valued point $x \in \mathcal{M}_{red}^i(k)$, if we denote M the corresponding unitary Dieudonné module viewed as a lattice in \mathbf{N} via the quasi-isogeny $\rho_x^{-1} : H_x \rightarrow \mathbf{H}$, then we have the following equivalent assertions

- $x \in \mathcal{M}_\Lambda(k)$;
- $M \subset (\Lambda^+)_k$;
- $M_0 \subset (\Lambda)_k$;
- $(\Lambda^-)_k \subset M$.

2.8 The analytic domain \mathcal{D}

Recall that we have defined some closed analytic domains $\mathcal{M}^{ss} \subset \mathcal{C} \subset \mathcal{M}$, where for any complete valuation field extension $K|L = W(\overline{\mathbb{F}}_p)\mathbb{Q}$,

$$\begin{aligned} \mathcal{M}^{ss}(K) &= \{(H, \iota, \lambda, \rho) \in \mathcal{M}(K) \mid H \text{ is semi-stable}\} \\ \mathcal{C}(K) &= \{(H, \iota, \lambda, \rho) \in \mathcal{M}(K) \mid \exists \text{ finite extension } K'|K, \text{ and a finite flat subgroup } \\ &\quad G \subset H[p] \text{ over } O_{K'}, \text{ such that } H/G \text{ is semi-stable}\}. \end{aligned}$$

Now since we have the decomposition

$$\mathcal{M} = \coprod_{i \in \mathbb{Z}, \text{in even}} \mathcal{M}^i,$$

we set

$$\mathcal{C}^i := \mathcal{C} \cap \mathcal{M}^i,$$

which is still a closed analytic domain in \mathcal{M}^i and \mathcal{M} , and we have an induced decomposition of analytic spaces

$$\mathcal{C} = \coprod_{i \in \mathbb{Z}, \text{in even}} \mathcal{C}^i.$$

We choose an element $g_1 \in J_b(\mathbb{Q}_p)$ such that the action by g_1 on \mathcal{M} induces isomorphisms :

$$g_1 : \mathcal{M}^0 \xrightarrow{\sim} \begin{cases} \mathcal{M}^1 & n \text{ even} \\ \mathcal{M}^2 & n \text{ odd.} \end{cases}$$

For example let

$$g_1 = \text{diag}(\underbrace{p^{-1}, \dots, p^{-1}}_{\frac{n}{2}}, \underbrace{1, \dots, 1}_{\frac{n}{2}})$$

if n is even and $g_1 = p^{-1} \in J_b(\mathbb{Q}_p)$ if n is odd. We have then

$$\mathcal{C}^i = \begin{cases} g_1^{\frac{i}{2}} \mathcal{C}^0 & n \text{ even, } i \text{ even} \\ p^{-\frac{i+1}{2}} g_1 \mathcal{C}^0 & n \text{ even, } i \text{ odd} \\ g_1^{\frac{i}{2}} \mathcal{C}^0 & n \text{ odd.} \end{cases}$$

Note the element $p^{-1} \in J_b(\mathbb{Q}_p)$ induces an isomorphism $p^{-1} : \mathcal{M}^0 \xrightarrow{\sim} \mathcal{M}^2$. We denote by $g_2 := p^{-1} g_1^{-1}$ if n is even, i.e. the above g_1 is such that $p^{-1} = \begin{cases} g_1 g_2 & n \text{ even} \\ g_1 & n \text{ odd.} \end{cases}$ Set

$$\mathcal{C}' = \begin{cases} \mathcal{C}^0 \amalg g_1 \mathcal{C}^0 & n \text{ even} \\ \mathcal{C}^0 & n \text{ odd,} \end{cases} \text{ then we have}$$

$$\mathcal{C} = \prod_{i \in \mathbb{Z}} p^{-i} \mathcal{C}', \text{ and } \pi(\mathcal{C}) = \pi(\mathcal{C}'),$$

where $\pi : \mathcal{M} \rightarrow \mathcal{F}^a \subset \mathbf{P}^{n-1, an}$ is the p -adic period mapping over L . Thus the Hecke orbits of the two analytic domains are the same :

$$\bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot \mathcal{C} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T \cdot \mathcal{C}'.$$

Using the geometric description of the reduced special fiber \mathcal{M}_{red} in last section, we have the following covering of \mathcal{M}^0 by open subsets :

$$\mathcal{M}^0 = \bigcup_{\Lambda \in \mathcal{L}_0, t(\Lambda) = t_{max}} sp^{-1}(\mathcal{M}_\Lambda),$$

where $sp : \mathcal{M}^0 \rightarrow \mathcal{M}_{red}^0$ is the specialization map. We have equally the Bruhat-Tits stratification of \mathcal{M}^0 by locally closed analytic subspaces

$$\mathcal{M}^0 = \prod_{\Lambda \in \mathcal{L}_0} sp^{-1}(\mathcal{M}_\Lambda^0).$$

By definition, a point $x \in sp^{-1}(\mathcal{M}_\Lambda)$ if and only if the composition

$$(H_{\Lambda^-})_{O_K/pO_K} \xrightarrow{\rho_{\Lambda^-}} \mathbf{H}_{O_K/pO_K} \xrightarrow{\rho} H_{xO_K/pO_K}$$

is an isogeny, and $x \in sp^{-1}(\mathcal{M}_\Lambda^0)$ if and only if the above composition is an isogeny and it does not factor through $H_{\Lambda'}$ for any $\Lambda' \subsetneq \Lambda$. Recall that there is a natural isogeny $\iota_{\Lambda, \Lambda'} : H_{\Lambda^-} \rightarrow H_{\Lambda'^-}$ corresponding the inclusion $\Lambda' \subset \Lambda$, and we have the compatibility $\rho_{\Lambda^-} = \rho_{\Lambda'^-} \circ \iota_{\Lambda, \Lambda'}$.

We fix a choice $\Lambda = \Lambda_0 \in \mathcal{L}_0$ with $t(\Lambda) = t_{max}$.

Definition 2.8.1. We define an analytic domain in \mathcal{M}^0

$$\mathcal{D} := \mathcal{C} \cap sp^{-1}(\mathcal{M}_\Lambda) = \mathcal{C}^0 \cap sp^{-1}(\mathcal{M}_\Lambda),$$

which is locally closed.

2.9 Some unitary group Shimura varieties and the relatively compactness of \mathcal{D}

In the following we prove that the underlying topological space $|\mathcal{D}|$ of \mathcal{D} is relatively compact, that is the topological closure $\overline{|\mathcal{D}|}$ in $|\mathcal{M}^0|$ (or $|\mathcal{C}^0|$) is compact. For this, we will use some unitary group Shimura varieties (those studied in [10] and [81]) and the theory of p -adic uniformization. The PEL data $(B, *, \mathbb{V}, \langle, \rangle, h, O_B, \Lambda)$ for defining these Shimura varieties are as following.

- B is a simple \mathbb{Q} -algebra such that $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_m(\mathbb{C})$ and $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq M_m(\mathbb{Q}_{p^2})$, for some integer m .
- $*$ is a positive involution on B .
- \mathbb{V} is a non-trivial finitely generated left B -module such that $n = \dim_{\mathbb{Q}}(\mathbb{V})/2m$.
- $\langle, \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{Q}$ is a nondegenerate skew-hermitian \mathbb{Q} -valued form. Let $G := GU_B(\mathbb{V}, \langle, \rangle)$ denote the reductive algebraic group over \mathbb{Q} of B -linear symplectic similitudes of $(\mathbb{V}, \langle, \rangle)$.
- $h : \text{Res}_{\mathbb{C}|\mathbb{R}} G_m \rightarrow G_{\mathbb{R}}$ is a homomorphism of real algebraic groups such that it defines a Hodge structure of type $\{(-1, 0), (0, -1)\}$ on \mathbb{V} and $\langle \cdot, h(\sqrt{-1}) \cdot \rangle : \mathbb{V}_{\mathbb{R}} \times \mathbb{V}_{\mathbb{R}} \rightarrow \mathbb{R}$ is symmetric and positive definite.
- O_B is a $*$ -invariant $\mathbb{Z}_{(p)}$ -order of B such that $O_B \otimes \mathbb{Z}_p$ is a maximal order of $B_{\mathbb{Q}_p}$. We can and we do fix an isomorphism $B_{\mathbb{Q}_p} \simeq M_m(\mathbb{Q}_{p^2})$ such that $O_B \otimes \mathbb{Z}_p$ is identified with $M_m(\mathbb{Z}_{p^2})$.
- Λ is an O_B -invariant \mathbb{Z}_p -lattice of $\mathbb{V}_{\mathbb{Q}_p}$ such that the alternating form on Λ induced by \langle, \rangle is a perfect \mathbb{Z}_p -form.

The first condition implies that the center \mathbb{K} of B is a quadratic imaginary extension of \mathbb{Q} and p is inert in \mathbb{K} . The derived subgroup G^{der} is an inner form of the quasi-split special unitary group $SU(n)$ for the extension $\mathbb{K}|\mathbb{Q}$. The assumption $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_m(\mathbb{C})$ implies $G_{\mathbb{R}}$ is isomorphic to the group of unitary similitudes $GU(r, s)$ of an hermitian form of signature for some nonnegative integers r and s such that $r + s = n$. We will assume $r = 1, s = n - 1$. The reflex field E will be \mathbb{K} if $1 \neq n - 1$ i.e. $n \neq 2$ and \mathbb{Q} if $n = 2$. Up to Morita equivalence, the localization of the above PEL data at p then induces the local PEL data for defining the Rapoport-Zink space $\widehat{\mathcal{M}}$.

For a sufficient small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$, the associated Shimura variety Sh_{K^p} over the integer ring O_{E_p} of the local field E_p (p is inert in E) is the moduli space of abelian varieties with additional structures in the following sense. For any O_{E_p} -scheme S , $Sh_{K^p}(S) = \{(A, \iota, \lambda, \eta)\} / \simeq$ where

- A is an abelian scheme over S of relative dimension equal to $\dim_{\mathbb{K}} \mathbb{V}$.
- $\iota : O_B \otimes \mathbb{Z}_p \rightarrow \text{End}(A) \otimes \mathbb{Z}_{(p)}$ is a nonzero homomorphism of $\mathbb{Z}_{(p)}$ -algebras, such that the induced action of O_B on the Lie algebra $Lie(A)$ satisfies $\text{rank}_{O_S} Lie(A)_1 = m, \text{rank}_{O_S} Lie(A)_2 = (n - 1)m$, where $Lie(A)_1$ (resp. $Lie(A)_2$) is the subsheaf of $Lie(A)$ that $O_{\mathbb{K}_p}$ acts via the natural inclusion $O_{\mathbb{K}_p} \subset O_B \otimes \mathbb{Z}_p$ (the composition of the nontrivial automorphism $*$ and the natural inclusion).
- $\lambda : A \rightarrow A^D$ is a principal $O_B \otimes \mathbb{Z}_p$ -linear polarization, such that the involution $*$ on B is compatible with the Rosati involution on $\text{End}(A)_{\mathbb{Q}}$ induced by λ , under the homomorphism $B \rightarrow \text{End}(A) \otimes \mathbb{Q}$.
- $\eta : \mathbb{V} \otimes \mathbb{A}_f^p \xrightarrow{\sim} H_1(A, \mathbb{A}_f^p) \bmod K^p$ is a K^p -level structure.
- $(A_1, \iota_1, \lambda_1, \eta_1) \simeq (A_2, \iota_2, \lambda_2, \eta_2)$, if there exists an O_B -linear isogeny $\phi : A_1 \rightarrow A_2$ of degree prime to p such that $\phi^*(\lambda_2) = a\lambda_1, \phi \circ \eta_1 = \eta_2$ for some $a \in \mathbb{Q}^{\times}$.

Note n is divisible by m and in fact the rank of \mathbb{V} as a B -module is $\frac{n}{m}$. In particular if $n = m$, the Shimura varieties Sh_{K^p} for K^p varies are all proper over O_{E_p} .

Now let $S \in \text{Nilp} \mathbb{Z}_{p^2}$. To any S -valued point $(A, \iota, \lambda, \eta) \in Sh_{K^p}(S)$, we attach to it a unitary p -divisible group of signature $(1, n - 1)$ as follow. Let $H' = A[p^{\infty}]$ be the p -divisible group of A . Then $O_B \otimes \mathbb{Z}_p = M_m(\mathbb{Q}_{p^2})$ acts on H' . By Morita equivalence, the functors $H' \mapsto O_{\mathbb{Z}_{p^2}}^m \otimes_{M_m(\mathbb{Q}_{p^2})} H'$ and $H \mapsto O_{\mathbb{Z}_{p^2}}^m \otimes_{\mathbb{Z}_{p^2}} H$ are mutually quasi-inverse between the category of p -divisible groups H' over S with a left $M_m(\mathbb{Q}_{p^2})$ -action and the category of p -divisible groups with a \mathbb{Z}_{p^2} -action over S . We set $H := O_{\mathbb{Z}_{p^2}}^m \otimes_{M_m(\mathbb{Q}_{p^2})} A[p^{\infty}]$,

and denote its \mathbb{Z}_{p^2} -action still by ι . The principal polarization λ on A then induces a \mathbb{Z}_{p^2} -linear principal polarization on H which we still denote by λ .

To state the link with the Rapoport-Zink space $\widehat{\mathcal{M}}$, we take the base change to $\text{Spec}O_L$ ($L = W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}, O_L = W(\overline{\mathbb{F}}_p)$) of the Shimura variety, which we still denote by Sh_{K^p} by abuse of notation. The special fiber \overline{Sh}_{K^p} of Sh_{K^p} over $\overline{\mathbb{F}}_p$ then admits the Newton polygon stratification.

$$\overline{Sh}_{K^p} = \coprod_{b \in B(G, \mu)} \overline{Sh}_{K^p}^b.$$

The Kottwitz set $B(G, \mu)$ of all Newton polygons can be written down explicitly as in [10] 3.1. In particular, one finds that every non-basic polygon has contacted points with the μ -ordinary polygon. This key special phenomenon will at last lead to the relatively compactness of our analytic domain \mathcal{D} in the p -adic analytic Rapoport-Zink space \mathcal{M} . As shown in [10], each strata $\overline{Sh}_{K^p}^b$ is non-empty, and any non-basic strata is in fact a leaf in the sense of [55]. The basic strata, which we denote by $\overline{Sh}_{K^p}^{b_0}$, was studied in [80] and [81], by studying the reduced special fiber \mathcal{M}_{red} of the Rapoport-Zink space $\widehat{\mathcal{M}}$ and the uniformization of $\overline{Sh}_{K^p}^{b_0}$ by \mathcal{M}_{red} .

Let $\widehat{Sh}_{K^p}^{b_0}$ be the completion of \overline{Sh}_{K^p} along $\overline{Sh}_{K^p}^{b_0}$, then the chapter 6 of [66] tell us there is an isomorphism of formal schemes over $\text{Spf}O_L$:

$$\widehat{Sh}_{K^p}^{b_0} \simeq I(\mathbb{Q}) \backslash \widehat{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \simeq \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p)/K^p} \widehat{\mathcal{M}}/\Gamma_i.$$

Note the group G satisfies Hasse principal : $\ker^1(\mathbb{Q}, G) = 1$ (cf. [81]). Here I is an inner form of G over \mathbb{Q} , which is anisotropic modulo center and such that $I_{\mathbb{Q}_p} = J_b, I_{\mathbb{A}_f^p} = G_{\mathbb{A}_f^p}$. Note the index set $I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p)/K^p$ is finite, and if $g_1, \dots, g_k \in G(\mathbb{A}_f^p)$ is a set of representatives, then $\Gamma_i = I(\mathbb{Q}) \cap g_i K^p g_i^{-1}$ for $i = 1, \dots, k$. The subgroups $\Gamma_i \subset J_b(\mathbb{Q}_p)$ are discrete and cocompact modulo center. Since K^p is sufficiently small, Γ_i is torsion free for all i .

We can describe the isomorphism

$$I(\mathbb{Q}) \backslash \widehat{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \rightarrow \widehat{Sh}_{K^p}^{b_0}$$

as follows. Let $(\mathbf{A}, \iota, \lambda, \eta)$ be a $\overline{\mathbb{F}}_p$ -valued point in $\overline{Sh}_{K^p}^{b_0}$, and the isomorphism we construct will depend on such a choice. Let $(\mathbf{H}, \iota, \lambda)$ be the unitary p -divisible group associated to this abelian variety as above. We take $(\mathbf{H}, \iota, \lambda)$ as the standard unitary p -divisible group for defining $\widehat{\mathcal{M}}$. For any $S \in \text{Nilp}O_L$, there is a map

$$\widehat{\mathcal{M}}(S) \times G(\mathbb{A}_f^p)/K^p \rightarrow Sh_{K^p}(S)$$

$$((H, \iota, \lambda, \rho), gK^p) \mapsto (A, \iota', \lambda', \eta g^{-1} K^p),$$

such that there is an unique quasi-isogeny $\mathbf{A}_{\overline{S}} \rightarrow A_{\overline{S}}$ compatible with additional structures inducing ρ when taking p -divisible groups. This map factor through the action of $I(\mathbb{Q})$ and functorially for S . It induces an closed immersion of formal algebraic spaces

$$I(\mathbb{Q}) \backslash \widehat{\mathcal{M}} \times G(\mathbb{A}_f^p)/K^p \rightarrow \widehat{Sh}_{K^p},$$

and one can prove that the left hand side is in fact a formal scheme and the image is $\widehat{Sh}_{K^p}^{b_0}$. Thus there is an isomorphism of formal schemes.

We denote $\widehat{Sh}_{K^p}^{an,b} := (\widehat{Sh}_{K^p}^b)^{an}$ the Berkovich analytic space associated to the formal scheme $\widehat{Sh}_{K^p}^b$, the completion of Sh_{K^p} along the strata $\overline{Sh}_{K^p}^b$, which is also the tube $sp^{-1}(\overline{Sh}_{K^p}^b)$ of $\overline{Sh}_{K^p}^b$ in $\widehat{Sh}_{K^p}^{an} \subset Sh_{K^p}^{an}$. Here $sp : \widehat{Sh}_{K^p}^{an} \rightarrow \overline{Sh}_{K^p}$ is the specialization map, and $Sh_{K^p}^{an}$ is the p -adic analyticfication of the generic fiber $Sh_{K^p} \times L = Sh_{G(\mathbb{Z}_p) \times K^p}$ over L , the last \subset is a closed immersion. It is an isomorphism if and only if Sh_{K^p} is proper over $Spec O_L$. We have the Newton polygon stratification of $\widehat{Sh}_{K^p}^{an}$ by locally closed subspaces :

$$\widehat{Sh}_{K^p}^{an} = \coprod_{b \in B(G, \mu)} \widehat{Sh}_{K^p}^{an,b}.$$

Analogous to the case of Rapoport-Zink spaces, there is a tower of analytic spaces $(\widehat{Sh}_{K_p \times K^p}^{an})_{K_p \subset G(\mathbb{Q}_p)}$ indexed by open compact subgroups $K_p \subset G(\mathbb{Q}_p)$, together with a family of closed immersions $(\widehat{Sh}_{K_p \times K^p}^{an})_{K_p} \subset (Sh_{K_p \times K^p}^{an})_{K_p}$, such that $\widehat{Sh}_{G(\mathbb{Z}_p) \times K^p}^{an} = \widehat{Sh}_{K^p}^{an}$. Then $G(\mathbb{Q}_p)$ acts on this tower and this gives the p -adic Hecke correspondence on each $\widehat{Sh}_{K_p \times K^p}^{an}$. The family of closed immersions $(\widehat{Sh}_{K_p \times K^p}^{an})_{K_p} \subset (Sh_{K_p \times K^p}^{an})_{K_p}$ is Hecke-equivariant, here the $G(\mathbb{Q}_p)$ action on the right hand side is the p -adic analyticfication of the Hecke action of $G(\mathbb{Q}_p)$ on $(Sh_{K_p \times K^p}^{an})_{K_p \subset G(\mathbb{Z}_p)}$. By taking the inverse images under the natural projection $\widehat{Sh}_{K_p \times K^p}^{an} \rightarrow \widehat{Sh}_{K^p}^{an}$, we can defined subspaces $\widehat{Sh}_{K_p \times K^p}^{an,b} \subset \widehat{Sh}_{K_p \times K^p}^{an}$, which are Hecke-invariant, thus we have $G(\mathbb{Q}_p)$ -equivariant stratifications

$$\widehat{Sh}_{K_p \times K^p}^{an} = \coprod_{b \in B(G, \mu)} \widehat{Sh}_{K_p \times K^p}^{an,b}.$$

Now pass to the p -adic analytic side, we have a family of isomorphisms of analytic spaces

$$I(\mathbb{Q}) \setminus \mathcal{M}_{K^p} \times G(\mathbb{A}_f^p)/K^p \simeq \coprod_{i \in I(\mathbb{Q}) \setminus G(\mathbb{A}_f^p)/K^p} \mathcal{M}_{K^p}/\Gamma_i \simeq \widehat{Sh}_{K_p \times K^p}^{an,b_0}.$$

These isomorphisms are Hecke-equivariant for the action of $G(\mathbb{Q}_p)$ on the two sides. If we let K^p varies, then they are Hecke-equivariant for the action of $G(\mathbb{A}_f)$ on the two sides.

We now look at the Harder-Narasimhan stratification of $\widehat{Sh}_{K_p \times K^p}^{an}$, see section 2.4. For a unitary p -divisible group (H, ι, λ) over a complete valuation ring $O_K | \mathbb{Z}_p$ of rank one, recall we have the Harder-Narasimhan polygon

$$HN(H, \iota, \lambda) := \frac{1}{2}HN(H)(2\cdot) = \lim_{k \rightarrow \infty} \frac{1}{2k}HN(H[p^k])(2k\cdot)$$

as a function $[0, n] \rightarrow [0, n/2]$, which we also identify with its graph. For a point $x \in \widehat{Sh}_{K_p \times K^p}^{an}$, denote by $(A_x, \iota_x, \lambda_x, \eta_{px} \times \eta_x^p)$ the abelian scheme over $O_K := O_{\mathcal{H}(x)}$ associated to x , let $(H_x = O_{\mathbb{Z}_p}^m \otimes_{M_m(\mathbb{Q}_p)} A_x[p^\infty], \iota_x, \lambda_x)$ be as above the unitary p -divisible group obtained after Mrita equivalence from $(A_x, \iota_x, \lambda_x, \eta_{px} \times \eta_x^p)$. Let $HN(x) := HN(H_x, \iota_x, \lambda_x)$. Thus we have defined a function

$$HN : \widehat{Sh}_{K_p \times K^p}^{an} \rightarrow \text{Poly},$$

here Poly denotes the set of concave polygons in $[0, n] \times [0, n/2]$ bounded by the μ -ordinary Hodge polygon. We identify the set $B(G, \mu)$ with a finite subset of Poly by associating each $b \in B(G, \mu)$ its polygon. By proposition 2.4.1, this function HN is semi-continuous.

Definition 2.9.1. For each $\mathcal{P} \in \text{Poly}$, we define the subset

$$\widehat{Sh}_{K_p \times K^p}^{an, HN=\mathcal{P}} := HN^{-1}(\mathcal{P}) = \{x \in \widehat{Sh}_{K_p \times K^p}^{an} \mid HN(x) = \mathcal{P}\},$$

which is then a locally closed subset.

We thus obtain a Harder-Narasimhan stratification of the underlying topological space $|\widehat{Sh}_{K_p \times K^p}^{an}|$ by locally closed subset

$$|\widehat{Sh}_{K_p \times K^p}^{an}| = \coprod_{\mathcal{P} \in \text{Poly}} \widehat{Sh}_{K_p \times K^p}^{an, HN=\mathcal{P}}.$$

Let \mathcal{P}_{ss} be the basic element in Poly, then the strata $\widehat{Sh}_{K_p \times K^p}^{an, HN=\mathcal{P}_{ss}}$ is an open subset, thus there is an analytic structure on it so that the inclusion $\widehat{Sh}_{K_p \times K^p}^{an, HN=\mathcal{P}_{ss}} \subset \widehat{Sh}_{K_p \times K^p}^{an}$ is an open immersion. For general Harder-Narasimhan strata, there is in general no obvious analytic structure on it. But, fortunately, in our case we have the following strong conclusion.

Proposition 2.9.2. The Harder-Narasimhan stratification and the Newton polygon stratification for $\widehat{Sh}_{K_p \times K^p}^{an}$ coincide.

Proof. This comes from the fact that, for a unitary p -divisible group (H, ι, λ) over O_K , we have the inequalities

$$HN(H, \iota, \lambda) \leq \text{Newt}(H_k, \iota, \lambda) \leq \text{Hodge}(H_k, \iota, \lambda),$$

and if there is a contact point x of the Newton polygon $\text{Newt}(H_k, \iota, \lambda)$ and the Hodge polygon $\text{Hodge}(H_k, \iota, \lambda)$, then the Harder-Narasimhan polygon $HN(H, \iota, \lambda)$ also passes at x , see corollary 1.5.3. If one draw all the possible Newton polygons in our cases, then one finds immediately the proposition holds. \square

The underling topological space of $\widehat{Sh}_{K^p}^{an}$ is compact. We now consider the image \mathcal{E} of the subspace $\mathcal{C} \subset \mathcal{M}$ under the p -adic uniformization morphism

$$\prod_{i=1}^k \mathcal{M}/\Gamma_i \simeq \widehat{Sh}_{K^p}^{an, b_0} = \widehat{Sh}_{K^p}^{an, HN=\mathcal{P}_{ss}} \subset \widehat{Sh}_{K^p}^{an}.$$

Since \mathcal{C} is $J_b(\mathbb{Q}_p)$ -stable, we have $\mathcal{E} \simeq \coprod_{i=1}^k \mathcal{C}/\Gamma_i$.

Proposition 2.9.3. The subset \mathcal{E} is a closed analytic domain in $\widehat{Sh}_{K^p}^{an}$, thus it is compact.

Proof. Let $\mathcal{H}/\widehat{Sh}_{K^p}$ be the p -divisible group associated to the universal Abelian scheme \mathcal{A} after applying the Morita equivalence. Then we can describe the locus $\mathcal{E} \subset \widehat{Sh}_{K^p}^{an}$ by

$$\mathcal{E} = \{x \in \widehat{Sh}_{K^p}^{an} \mid \exists \text{ finite extension } K' \mid \mathcal{H}(x), \text{ and a finite flat group } G \subset \mathcal{H}_x[p] \\ \text{over } O_{K'}, \text{ such that } \mathcal{H}_x/G \text{ is semi-stable over } O_{K'}.\}$$

For simplifying notation, we denote $\mathcal{X} = \widehat{Sh}_{K^p}$ and $X = \widehat{Sh}_{K^p}^{an}$. By forgetting the polarization, we can construct a tower of analytic spaces $(X_K)_{K \subset GL_n(\mathbb{Z}_{p^2})}$ with $X = X_{GL_n(\mathbb{Z}_{p^2})}$, and an action of $GL_n(\mathbb{Z}_{p^2})$ (not $GL_n(\mathbb{Q}_{p^2})$!) on this tower. Let $\pi : Y = X_{Id+pM_n(\mathbb{Z}_{p^2})} \rightarrow X$ be the natural finite étale morphism. Then Y classifies the level structures $\eta : (\mathbb{Z}/p\mathbb{Z})^{2n} \xrightarrow{\sim} \pi^* \mathcal{H}^{an}[p]$. After a possible admissible formal blow-up we can find a p -adic admissible formal model \mathcal{Y} of Y , and a morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ such that $f^{an} = \pi$. Consider the finite flat formal group scheme $f^* \mathcal{H}[p]$ over \mathcal{Y} . Then for any subgroup $M \subset (\mathbb{Z}/p\mathbb{Z})^{2n}$, there exists a finite flat formal subgroup scheme $\mathcal{G}_M \subset f^* \mathcal{H}[p]$ such that $\mathcal{G}_M^{an} = \eta(M)$. Let $\mathcal{H}_M := f^* \mathcal{H} / \mathcal{G}_M$ be the p -divisible group over \mathcal{Y} , then the following finite union

$$F := \bigcup_M \{y \in Y \mid \mathcal{H}_{M,y} / \mathcal{O}_{\mathcal{H}(y)} \text{ is semi-stable}\}$$

is a closed analytic domain of Y since each one on the right hand side is. Now the subset $\mathcal{E} \subset X$ is exactly the image $\pi(F)$ of F under the finite étale morphism $\pi : Y \rightarrow X$, so it is a closed analytic domain. \square

Corollary 2.9.4. *The underlying topological space $|\mathcal{D}|$ of \mathcal{D} is relatively compact.*

Proof. Since \mathcal{E} is a finite disjoint union of the form \mathcal{C}/Γ , therefore each \mathcal{C}/Γ is compact by the above proposition. Since \mathcal{M}_Λ is an irreducible component of \mathcal{M}_{red} , and $\mathcal{D} = \mathcal{C} \cap sp^{-1}(\mathcal{M}_\Lambda) \subset sp^{-1}(\mathcal{M}_\Lambda)$, we may chose K^p sufficiently small such that one associated Γ satisfies that $\forall id \neq \gamma \in \Gamma, \gamma \mathcal{D} \cap \mathcal{D} = \emptyset$. So we have a topological imbedding

$$\mathcal{D} \hookrightarrow \mathcal{C}/\Gamma,$$

since the right hand side is compact, we can conclude. \square

2.10 Locally finite cell decompositions

We will construct a locally finite covering of the unitary Rapoport-Zink space \mathcal{M} from the locally closed analytic domain \mathcal{D} . Recall we have fixed a choice $\Lambda \in \mathcal{L}_0$ with $t(\Lambda) = t_{max}$. Let $Stab(\Lambda) \subset J_b^{der}(\mathbb{Q}_p)$ be the stabilizer subgroup of Λ in $J_b^{der}(\mathbb{Q}_p)$. Then by our definition, \mathcal{D} is stable under the action of $Stab(\Lambda)$. Recall in section 8 we introduced the analytic domains $\mathcal{C}', \mathcal{C}^0$. We have the following covering of \mathcal{C}^0

$$\mathcal{C}^0 = \bigcup_{g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)} g\mathcal{D},$$

which is locally finite, i.e. for any $g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)$, there are finite $g' J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)$ such that $g\mathcal{D} \cap g'\mathcal{D} \neq \emptyset$, since

$$\mathcal{M}_{red}^0 = \bigcup_{g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)} g\mathcal{M}_\Lambda$$

is a locally finite union of its irreducible components.

We thus have the following equalities

$$\bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T.\mathcal{C} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T.\mathcal{C}'$$

If n is odd, this equals to

$$= \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)} T.\mathcal{C}^0 = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{\text{der}}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g\mathcal{D};$$

if n is even, the above equals to

$$= \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1}} T.g_1^j \mathcal{C}^0 = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{\text{der}}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g_1^j g\mathcal{D}.$$

We would like to prove the last unions in the above two cases are locally finite. For this, it suffices to prove the following union

$$\bigcup_{\substack{T \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p) \\ g \in J_b^{\text{der}}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g\mathcal{D}$$

is locally finite. To this end, we just need to prove the following holds

$$\#\{(T, g) \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p) \times J_b^{\text{der}}(\mathbb{Q}_p) / \text{Stab}(\Lambda) \mid T.g\mathcal{D} \cap \mathcal{D} \neq \emptyset\} < \infty.$$

This comes from the following several propositions.

Proposition 2.10.1. *The Bruhat-Tits stratification of the analytic space*

$$\mathcal{M}^0 = \coprod_{\Lambda \in \mathcal{L}_0} sp^{-1}(\mathcal{M}_\Lambda^0)$$

by locally closed spaces is invariant under the Hecke action of $G^{\text{der}}(\mathbb{Q}_p)$, i.e., for each tube $sp^{-1}(\mathcal{M}_\Lambda^0)$, we have

$$T.sp^{-1}(\mathcal{M}_\Lambda^0) \subset sp^{-1}(\mathcal{M}_\Lambda^0)$$

for any $T \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p)$.

Proof. We just check that $T.sp^{-1}(\mathcal{M}_\Lambda) \subset sp^{-1}(\mathcal{M}_\Lambda)$, for any $T \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p)$, $\Lambda \in \mathcal{L}_0$. The case for $sp^{-1}(\mathcal{M}_\Lambda^0)$ is similar. Assume that

$$T = G^{\text{der}}(\mathbb{Z}_p) \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} G^{\text{der}}(\mathbb{Z}_p) \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p),$$

$$a_1 \geq \dots \geq a_n, a_1 + a_n = a_2 + a_{n-1} = \dots = 0,$$

then $a_1 \geq 0$, $a_n - a_1 = -2a_1 \leq 0$, and we consider the Hecke correspondence

$$p^{-a_1}T = G^{der}(\mathbb{Z}_p) \left(\begin{array}{cccc} 1 & & & \\ & p^{a_2 - a_1} & & \\ & & \ddots & \\ & & & p^{a_n - a_1} \end{array} \right) G^{der}(\mathbb{Z}_p).$$

We just need to check that

$$p^{-a_1}T.sp^{-1}(\mathcal{M}_\Lambda) \subset p^{-a_1}sp^{-1}(\mathcal{M}_\Lambda),$$

here the p^{-a_1} on the right hand side is considered as an element of $J_b(\mathbb{Q}_p)$: it induces an isomorphism

$$p^{-a_1} : \mathcal{M}^0 \rightarrow \mathcal{M}^{2a_1},$$

under which the image $p^{-a_1}sp^{-1}(\mathcal{M}_\Lambda)$ of $sp^{-1}(\mathcal{M}_\Lambda)$ is $sp^{-1}(\mathcal{M}_{p^{a_1}\Lambda})$. Assume $x \in sp^{-1}(\mathcal{M}_\Lambda)$, the associated unitary p -divisible group $(H, \iota, \lambda, \rho : \mathbf{H}_{O_K/pO_K} \rightarrow H_{O_K/pO_K})$ is such that the composition

$$(H_{\Lambda^-})_{O_K/pO_K} \xrightarrow{\rho_{\Lambda^-}} \mathbf{H}_{O_K/pO_K} \xrightarrow{\rho} H_{O_K/pO_K}$$

is an isogeny. Let $y \in p^{-a_1}T.x$ be a any point such that the associated unitary p -divisible group is $(H/E, \iota, \lambda, \mathbf{H}_{O_K/pO_K} \xrightarrow{\rho} H_{O_K/pO_K} \xrightarrow{\pi} (H/E)_{O_K/pO_K})$, where $E \subset H$ is a finite flat subgroup scheme such that its geometric generic fiber

$$E_{\bar{\eta}} \simeq \mathbb{Z}_{p^2}/p^{a_1 - a_2}\mathbb{Z}_{p^2} \oplus \cdots \oplus \mathbb{Z}_{p^2}/p^{a_1 - a_n}\mathbb{Z}_{p^2}.$$

Since

$$\begin{aligned} (p^{a_1}\Lambda)^+ &= p^{a_1}\Lambda^+ \\ (p^{a_1}\Lambda)^- &= p^{2a_1}(p^{a_1}\Lambda^+)^{\vee} = p^{a_1}\Lambda^-, \end{aligned}$$

and the quasi-isogeny $\rho_{p^{a_1}\Lambda^-} : H_{p^{a_1}\Lambda^-} \rightarrow \mathbf{H}$ is given by the composition $\rho_{\Lambda^-} \circ \phi :$

$$H_{p^{a_1}\Lambda^-} \xrightarrow{\phi} H_{\Lambda^-} \xrightarrow{\rho_{\Lambda^-}} \mathbf{H},$$

where the first is the isogeny induced by the natural inclusion $p^{a_1}\Lambda^- \subset \Lambda^-$, thus its composition $\pi \circ \rho \circ \rho_{\Lambda^-} \circ \phi$ with

$$\pi \circ \rho : \mathbf{H}_{O_K/pO_K} \xrightarrow{\rho} H_{O_K/pO_K} \xrightarrow{\pi} (H/E)_{O_K/pO_K}$$

is an isogeny. That is $y \in sp^{-1}(\mathcal{M}_{p^{a_1}\Lambda}) \subset \mathcal{M}^{2a_1}$. So we have

$$p^{-a_1}T.sp^{-1}(\mathcal{M}_\Lambda) \subset p^{-a_1}sp^{-1}(\mathcal{M}_\Lambda) = sp^{-1}(\mathcal{M}_{p^{a_1}\Lambda}).$$

□

Recall we have the description

$$\mathcal{D} = \left(\bigcup_a T_a \cdot \mathcal{N}^{ss} \right) \cap sp^{-1}(\mathcal{M}_\Lambda),$$

for our fixed $\Lambda \in \mathcal{L}_0$ with $t(\Lambda) = t_{max}$ and the closed immersion $\mathcal{M} \subset \mathcal{N}$. We prove the following proposition, then it will be clear that the locally finiteness holds for \mathcal{M} .

Proposition 2.10.2. *Let \tilde{J}_b be the associated inner form of $\text{Res}_{\mathbb{Q}_p^2|\mathbb{Q}_p} GL_n$ for \mathcal{N} , and $U \subset \mathcal{N}_{red}$ be an open compact subset such that $\tilde{J}_b(\mathbb{Q}_p)U = \mathcal{N}_{red}$, cf. [23] 2.4. Let $Z = \mathbb{Q}_p^\times$ be the center of $GL_n(\mathbb{Q}_p^2)$ and $\tilde{J}_b(\mathbb{Q}_p)$. Set*

$$\mathcal{D}' := \mathcal{N}^{ss} \cap sp^{-1}(U),$$

then we have

$$\#\{[T] \in (GL_n(\mathbb{Z}_{p^2}) \backslash GL_n(\mathbb{Q}_{p^2})/GL_n(\mathbb{Z}_{p^2}))/Z | [T].\mathcal{D}'/Z \cap \mathcal{D}'/Z \neq \emptyset\} < \infty.$$

Proof. Recall that the criterion of quasi-compactness of an open subset $U \subset \mathcal{N}_{red}$ ([23] critère de quasicompacité 2.4.14) : U is quasi-compact if and only if there exist a Diedonné lattice $M \in \mathcal{N}_{red}(\bar{k})$ and an integer N , such that

$$U(\bar{k}) \subset \{M' \in \mathcal{N}_{red}(\bar{k}) | p^N M \subset M' \subset p^{-N} M\},$$

or equivalently, there exists an integer N , such that the universal quasi-isogeny ρ^{univ} satisfies that $p^N \rho^{univ}$ and $p^N (\rho^{univ})^{-1}$ are isogenies. The formal Rapoport-Zink space $\hat{\mathcal{N}}$ decomposes as disjoint union according the height of the universal quasi-isogeny :

$$\hat{\mathcal{N}} = \coprod_{i \in \mathbb{Z}} \hat{\mathcal{N}}^i,$$

where the height of the universal quasi-isogeny is $2i$ over $\hat{\mathcal{N}}^i$. Let $\tilde{\mathcal{N}} := \coprod_{i=0}^{n-1} \hat{\mathcal{N}}^i$, which is in bijection with the quotient $\hat{\mathcal{N}}/Z$. Let us denote by $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{N}}_{red}$ the analytic generic fiber and reduced special fiber respectively of $\tilde{\mathcal{N}}$. Then there is a metric function

$$d : \tilde{\mathcal{N}}_{red}(\bar{k}) \times \tilde{\mathcal{N}}_{red}(\bar{k}) \rightarrow \mathbb{N}$$

defined as

$$d((H_1, \iota_1, \rho_1), (H_2, \iota_2, \rho_2)) = q(\rho_1^{-1} \circ \rho_2) + q(\rho_2^{-1} \circ \rho_1),$$

here $q(\rho) = htp^{n(\rho)}\rho$ and $n(\rho)$ is the smallest integer such that $p^{n(\rho)}\rho$ is an isogeny for a quasi-isogeny ρ . Then an open subset $U \subset \tilde{\mathcal{N}}_{red}$ is quasi-compact if and only if there exist an integer N , such that $d(x, y) \leq N$ for all points $x, y \in U$.

To prove the proposition we may assume $U \subset \tilde{\mathcal{N}}_{red}$. Let $(H_1, \iota_1, \rho_1), (H_2, \iota_2, \rho_2)$ be the p -divisible groups associated to two points $x_1, x_2 \in U$. Let $M_1 = \rho_{1*}^{-1}(\mathbb{D}(H_{1\bar{k}}))$, $M_2 = \rho_{2*}^{-1}(\mathbb{D}(H_{2\bar{k}}))$, and $inv(M_1, M_2) = (a_1, \dots, a_n) \in \mathbb{Z}_+^n$ be their relative invariant. Then one check that easily

$$d((H_1, \iota_1, \rho_1), (H_2, \iota_2, \rho_2)) = a_1 - a_n.$$

So it is bounded by some fixed integer N dependent only by U .

Now $\mathcal{D}' \subset \tilde{\mathcal{N}} \subset \mathcal{N}$, and the Hecke correspondences on \mathcal{N} induce an action of the set $(GL_n(\mathbb{Z}_{p^2}) \backslash GL_n(\mathbb{Q}_{p^2})/GL_n(\mathbb{Z}_{p^2}))/Z$ on $\tilde{\mathcal{N}}$. let $x \in \mathcal{D}', y \in \mathcal{D}' \cap [T].x$ for some $[T] \in (GL_n(\mathbb{Z}_{p^2}) \backslash GL_n(\mathbb{Q}_{p^2})/GL_n(\mathbb{Z}_{p^2}))/Z$, $T.\mathcal{D}' \cap \mathcal{D}' \neq \emptyset$. Since by Cartan decomposition we have the bijection

$$GL_n(\mathbb{Z}_{p^2}) \backslash GL_n(\mathbb{Q}_{p^2})/GL_n(\mathbb{Z}_{p^2}) \xrightarrow{\sim} \mathbb{Z}_+^n.$$

If $T = GL_n(\mathbb{Z}_{p^2}) \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} GL_n(\mathbb{Z}_{p^2})$, then

$$zT = GL_n(\mathbb{Z}_{p^2}) \begin{pmatrix} p^{a_1+v_p(z)} & & \\ & \ddots & \\ & & p^{a_n+v_p(z)} \end{pmatrix} GL_n(\mathbb{Z}_{p^2}).$$

Thus $[T]$ corresponds to a class $[(a_1, \dots, a_n)] \in \mathbb{Z}_+^n/\mathbb{Z}$ where the action of \mathbb{Z} on \mathbb{Z}_+^n is the natural translation. Let $(H/O_K, \iota, \rho)$ be the p -divisible group associated to x over $O_{K=\mathcal{H}(x)}$. Then the p -divisible group associated to y over O_K is $(H/G, \iota', p^{-a_1}\pi \circ \rho)$, here $G \subset H$ is a finite flat group scheme such that $G_{\bar{K}} \simeq \mathbb{Z}/p^{a_1-a_2}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p^{a_1-a_n}\mathbb{Z}$. Moreover, since both H and H/G are semi-stable, G is therefore semi-stable, [27] lemme 11. By the following lemma, the covariant Dieudonné module

$$\mathbb{D}(G_{\bar{k}}) \simeq W/p^{a_1-a_2}W \oplus \dots \oplus W/p^{a_1-a_n}W.$$

The relative invariant of $\mathbb{D}(H_{\bar{k}})$ and $\mathbb{D}((H/G)_{\bar{k}})$ is then just $(0, a_2 - a_1, \dots, a_n - a_1)$, and by the above $a_1 - a_n \leq N$. This plus the fact that $a_1 \geq \dots \geq a_n, a_i \in \mathbb{Z}, i = 1, \dots, n$ with some easy combination argument imply that there are finite possibilities of the class $[(a_1, \dots, a_n)] \in \mathbb{Z}_+^n/\mathbb{Z}$, thus finite possibilities of $[T] \in (GL_n(\mathbb{Z}_{p^2}) \setminus GL_n(\mathbb{Q}_{p^2})/GL_n(\mathbb{Z}_{p^2}))/Z$. This finishes the proof. \square

Lemma 2.10.3. *Let G be a semi-stable finite flat group scheme over O_K . Suppose that $G_{\bar{K}} \simeq \mathbb{Z}/p^{a_1}\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p^{a_n}\mathbb{Z}$, then we have*

$$\mathbb{D}(G_{\bar{k}}) = W/p^{a_1}W \oplus \dots \oplus W/p^{a_n}W,$$

here $\mathbb{D}(G_{\bar{k}})$ is the covariant Dieudonné module of $G_{\bar{k}}$.

Proof. If $pG = 0$ then the above is evident. We assume that $p^2G = 0$ here, the general case follows by induction. Under this assumption, there exists a $1 \leq k \leq n$ such that $a_1 = \dots = a_k = 2, a_{k+1} = \dots = a_n = 1$, and $G_{\bar{K}} \simeq (\mathbb{Z}/p^2\mathbb{Z})^k \oplus (\mathbb{Z}/p\mathbb{Z})^{n-k}$. Since G is semi-stable, the following sequence

$$0 \longrightarrow G[p] \longrightarrow G \xrightarrow{p} pG \longrightarrow 0$$

is exact. So we get an exact sequence

$$0 \longrightarrow G[p]_{\bar{k}} \longrightarrow G_{\bar{k}} \longrightarrow (pG)_{\bar{k}} \longrightarrow 0,$$

and passing to (covariant) Dieudonné module we get an exact sequence

$$0 \longrightarrow \mathbb{D}(G[p]_{\bar{k}}) \longrightarrow \mathbb{D}(G_{\bar{k}}) \longrightarrow \mathbb{D}((pG)_{\bar{k}}) \longrightarrow 0.$$

We have $htG[p] = n, ht(pG) = k, htG = n + k$. Assume that $\mathbb{D}(G_{\bar{k}}) \simeq (W/p^2W)^{k'} \oplus (W/pW)^{n-k'}$ for some $1 \leq k' \leq n$. Then we have $dim_{\mathbb{F}_p}\mathbb{D}(G[p]_{\bar{k}}) = n, dim_{\mathbb{F}_p}\mathbb{D}((pG)_{\bar{k}}) = k'$. Since

$$htG[p] = dim_{\mathbb{F}_p}\mathbb{D}(G[p]_{\bar{k}}), ht(pG) = dim_{\mathbb{F}_p}\mathbb{D}((pG)_{\bar{k}}),$$

i.e. $k = k'$, the lemma follows. \square

Remark 2.10.4. *The above proposition and its proof hold generally for all EL Rapoport-Zink spaces.*

Proposition 2.10.5. *The union*

$$\bigcup_{T \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p)} T \cdot \mathcal{D}$$

is locally finite.

Proof. Since $\mathcal{C} = (\bigcup_{\underline{a}} T_{\underline{a}} \mathcal{N}^{\text{ss}}) \cap \mathcal{M}$ and $\mathcal{D} = \mathcal{C} \cap sp^{-1}(\mathcal{M}_{\Lambda})$, we can choose some open compact subset $U \supset \mathcal{M}_{\Lambda}$ in \mathcal{N}_{red} such that $\tilde{J}_b(\mathbb{Q}_p)U = \mathcal{N}_{\text{red}}$. Then $\mathcal{D} \subset \bigcup_{\underline{a}} T_{\underline{a}} \cdot \mathcal{D}'$ for $\mathcal{D}' := \mathcal{N}^{\text{ss}} \cap sp^{-1}(U)$. Denote $\mathcal{D}'' = \bigcup_{\underline{a}} T_{\underline{a}} \cdot \mathcal{D}'$, which is a finite union of closed analytic domains. By the above proposition we know that there are only finite $T \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p)$ such that $T \cdot \mathcal{D}' \cap \mathcal{D}' \neq \emptyset$. Therefore, there are also only finite $T \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p)$ such that $T \cdot \mathcal{D}'' \cap \mathcal{D}'' \neq \emptyset$. This implies the number of $T \in G^{\text{der}}(\mathbb{Z}_p) \backslash G^{\text{der}}(\mathbb{Q}_p) / G^{\text{der}}(\mathbb{Z}_p)$ such that $T \cdot \mathcal{D} \cap \mathcal{D} \neq \emptyset$ is finite. \square

Corollary 2.10.6. *The unions*

$$\bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{\text{der}}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T \cdot g \mathcal{D}$$

for n odd and

$$\bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{\text{der}}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T \cdot g_1^j g \mathcal{D}$$

for n even are both locally finite.

Remark 2.10.7. *By the proofs of the above two propositions, we see that if*

$$T = G(\mathbb{Z}_p) \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix} G(\mathbb{Z}_p)$$

corresponds to the point

$$(a_1, \dots, a_n) \in X_*(A)_+ \subset \mathbb{Z}_+^n$$

by the Cartan decomposition (see section 2.2), then the set

$$\{T' \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \mid T \cdot \mathcal{D} \cap T' \cdot \mathcal{D} \neq \emptyset\}$$

corresponds to the set of points in some neighborhood of $(a_1, \dots, a_n) \in X_(A)_+ \subset \mathbb{Z}_+^n$ (for the natural topology).*

With the notations above, we now can state the main theorem of this section. The proof is based on some gluing arguments, and the following basic observation : let $Y \subset X$ be two Hausdorff paracompact strictly analytic spaces over a complete non-archimedean field k , such that the inclusion of Y as a subspace of X induces the identity of their associating rigid analytic spaces $Y^{rig} = X^{rig}$, then we have $Y = X$. Here we require that the analytic Grothendieck topologies are the same. Note if one just has the equality of the underlying sets $|Y^{rig}| = |X^{rig}|$, one can not deduce $Y = X$ and in fact there are many counter examples.

Theorem 2.10.8. *We have a locally finite covering of the Berkovich analytic space \mathcal{M} :*

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g\mathcal{D}$$

if n is odd, and

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g_1^j g\mathcal{D}$$

if n is even.

Proof. Take an open quasi-compact subset $U \subset \mathcal{M}_{red}^0$ such that $U \cap \mathcal{M}_\Lambda \neq \emptyset$ and $J_b(\mathbb{Q}_p).U = \mathcal{M}_{red}$. Since U is quasi-compact, it intersects with only finite number irreducible components \mathcal{M}_{Λ_i} , $\Lambda_i \in \mathcal{L}_0$, $t(\Lambda_i) = t_{max}$, $i = 1, \dots, k$ with $\Lambda = \Lambda_1$. Let $g_i \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)$ be such that $g_i(\Lambda) = \Lambda_i$. Then we have the inclusion

$$\mathcal{D}' := \mathcal{C} \cap sp^{-1}(U) \subset \bigcup_{i=1}^k g_i \mathcal{D}.$$

Note that \mathcal{D}' is a closed analytic domain of \mathcal{M} . It is compact since \mathcal{D} is relatively compact. So we have equalities of locally finite coverings

$$\bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g\mathcal{D} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g\mathcal{D}'$$

if n is odd; and

$$\bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g_1^j g\mathcal{D} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g_1^j g\mathcal{D}'$$

if n is even. Since \mathcal{D}' is closed, and the above analytic covering $(T.g\mathcal{D}')_{T,g}$ or $(T.g_1^j g\mathcal{D}')_{j,T,g}$ obtained by translations of \mathcal{D}' is locally finite, by [2] we can glue them into a sub-analytic space $\mathcal{M}' \subset \mathcal{M}$, such that the underlying set of \mathcal{M}' is given by the union as above. On the other hand, the rigid covering $(T.g\mathcal{D}'^{rig})_{T,g}$ or $(T.g_1^j g\mathcal{D}'^{rig})_{j,T,g}$ of \mathcal{M}'^{rig} by admissible open subsets can always be glued as a rigid space \mathcal{M}'_0 , which is the associated rigid space of \mathcal{M}' : $\mathcal{M}'^{rig} = \mathcal{M}'_0$. Then these rigid coverings are admissible since the analytic coverings are locally finite, so from the equalities as sets

$$\mathcal{M}^{rig} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p) / \text{Stab}(\Lambda)}} T.g\mathcal{D}'^{rig}$$

if n is odd; and

$$\mathcal{M}^{rig} = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} T.g_1^j g \mathcal{D}'^{rig}$$

if n is even, we have the equality of $\mathcal{M}'_0 = \mathcal{M}^{rig}$ as rigid spaces. By the equivalence of the category of Hausdorff paracompact strictly analytic Berkovich spaces and the category of quasi-separated quasi-paracompact rigid analytic spaces, we must have the equality $\mathcal{M}' = \mathcal{M}$.

□

Remark 2.10.9. *The above argument also works for theorem 27 in [27].*

We have the following corollary when applying the theorem to the p -adic period domain \mathcal{F}^a .

Corollary 2.10.10. *Let $\pi : \mathcal{M} \rightarrow \mathcal{F}^a \subset \mathbf{P}^{n-1,an}$ be the p -adic period mapping, then we have a locally finite covering*

$$\mathcal{F}^a = \bigcup_{g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)} g\pi(\mathcal{D})$$

if n is odd, and

$$\mathcal{F}^a = \bigcup_{\substack{j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} g_1^j g\pi(\mathcal{D})$$

if n is even.

We look at the cases with level structures. Let $K \subset G(\mathbb{Z}_p)$ be an open compact subgroup and $\pi_K : \mathcal{M}_K \rightarrow \mathcal{M}$ be the natural projection, which is a $J_b(\mathbb{Q}_p)$ -equivariant finite étale surjection and also compatible with the Hecke actions. Denote $\mathcal{D}_K = \pi_K^{-1}(\mathcal{D})$, then $g\mathcal{D}_K = \pi_K^{-1}(g\mathcal{D})$ for all $g \in J_b(\mathbb{Q}_p)$, and

$$Kh_1K.g\mathcal{D}_K = Kh_2K.g\mathcal{D}_K$$

for $Kh_1K, Kh_2K \in K \backslash G(\mathbb{Q}_p) / K$ having the same image under the projection

$$K \backslash G(\mathbb{Q}_p) / K \rightarrow G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / K.$$

The last equality holds since any $h \in G(\mathbb{Z}_p)$ acts trivially on \mathcal{M} , therefore $Khh_1K.\pi_K^{-1}(g\mathcal{D}) = Kh_1K.\pi_K^{-1}(g\mathcal{D})$.

Corollary 2.10.11. *We have a locally finite covering of the analytic space \mathcal{M}_K*

$$\mathcal{M}_K = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / K \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} T.g\mathcal{D}_K$$

if n is odd, and

$$\mathcal{M}_K = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / K \\ j=0,1 \\ g \in J_b^{der}(\mathbb{Q}_p) / Stab(\Lambda)}} T.g_1^j g\mathcal{D}_K$$

if n is even.

We will consider some cohomological application of this corollary in the next section.

Finally we have a corollary for Shimura varieties.

Corollary 2.10.12. *Let Sh_{K^p} be as the Shimura variety introduced in subsection 2.9, $\widehat{Sh}_{K^p}^{an}$ be the generic analytic fiber of its p -adic completion \widehat{Sh}_{K^p} , and $\widehat{Sh}_{K^p}^{an, b_0}$ be the tube in $\widehat{Sh}_{K^p}^{an}$ over the basic strata $\overline{Sh}_{K^p}^{b_0}$, which is an open subspace. Let $\widehat{Sh}_{K_p \times K^p}^{an} \rightarrow \widehat{Sh}_{K^p}^{an}$ be the covering in level $K_p \subset G(\mathbb{Z}_p)$ (an open compact subgroup), and $\widehat{Sh}_{K_p \times K^p}^{an, b_0}$ be the inverse image of $\widehat{Sh}_{K^p}^{an, b_0}$. Denote \mathcal{C}'_{K_p} the inverse image of \mathcal{C}' in \mathcal{M}_{K_p} , \mathcal{E}'_{K_p} the image of \mathcal{C}'_{K_p} under the p -adic uniformization*

$$I(\mathbb{Q}) \backslash \mathcal{M}_{K_p} \times G(\mathbb{A}_f^p)/K^p \simeq \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p)/K^p} \mathcal{M}_{K_p}/\Gamma_i \simeq \widehat{Sh}_{K_p \times K^p}^{an, b_0}.$$

1. Let $\Gamma = \Gamma_i$ be one of the above discrete, torsion free, cocompact modulo center subgroups of $J_b(\mathbb{Q}_p)$, and $\Gamma^{der} = \Gamma \cap J_b^{der}(\mathbb{Q}_p)$, $D_{K_p} = D_{iK_p}$ be the image of \mathcal{D}_{K_p} under the morphism $\mathcal{M}_{K_p} \rightarrow \mathcal{M}_{K_p}/\Gamma$, then we have a covering

$$\mathcal{M}_{K_p}/\Gamma = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K_p \\ g \in \Gamma^{der} \backslash J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.gD_{K_p}$$

if n is odd, and

$$\mathcal{M}_{K_p}/\Gamma = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K_p \\ j=0,1 \\ g \in \Gamma^{der} \backslash J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.g_1^j g D_{K_p}$$

if n is even.

2. Under the above notation, we have a covering

$$\mathcal{E}'_{K_p} = \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p)/K^p} \bigcup_{g \in \Gamma^{der} \backslash J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)} gD_{iK_p}$$

if n is odd, and

$$\mathcal{E}'_{K_p} = \coprod_{i \in I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p)/K^p} \bigcup_{\substack{j=0,1 \\ g \in \Gamma^{der} \backslash J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} g_1^j g D_{iK_p}$$

if n is even. We have a covering

$$\widehat{Sh}_{K_p \times K^p}^{an, b_0} = \bigcup_{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K_p} T.\mathcal{E}'_{K_p}.$$

3 Cell decomposition and Lefschetz trace formula

3.1 Introduction

In this section we would like to find some cohomological application of our locally finite cell decomposition for the unitary group Rapoport-Zink spaces \mathcal{M}_K . By studying the action of regular semi-simple elliptic elements on the cells, we will verify the conditions of theorem 3.13 in [60] hold, thus deduce a Lefschetz trace formula. In the same way we can also reprove (in a rather different way) the Lefschetz trace formula for Lubin-Tate spaces which was proved previously by Strauch ([77], theorem 3.3.1) and Mieda ([60], example 4.21). To motivate the idea, we treat first the Lubin-Tate case, which is simpler.

Let p be a prime number, F be a finite extension of \mathbb{Q}_p , O be the ring of integers of F , and $\pi \in O$ be a uniformizer in O . We denote \widehat{F}^{nr} as the completion of the maximal unramified extension of F , and \widehat{O}^{nr} its ring of integers. For any integer $n \geq 1$, we consider the general linear group GL_n as well as its inner form D^\times over F , where D is the central division algebra over F with invariant $\frac{1}{n}$ and D^\times is the reductive group defined by inverse elements in D . Recall a formal O -module is a p -divisible group with an O -action over a base over O , such that the induced action on its Lie algebra is the canonical action of O . We consider the formal Lubin-Tate space $\widehat{\mathcal{M}} = \coprod_{i \in \mathbb{Z}} \widehat{\mathcal{M}}^i$ over \widehat{O}^{nr} : for any scheme $S \in \text{Nilp}\widehat{O}^{nr}$, $\widehat{\mathcal{M}}(S) = \{(H, \rho)\} / \simeq$, where

- H is a formal O -module over S ,
- $\rho: \mathbb{H}_{\overline{S}} \rightarrow H_{\overline{S}}$ is a quasi-isogeny.

Here $\text{Nil}\widehat{O}^{nr}$ is the category of schemes over \widehat{O}^{nr} on which π is locally nilpotent, \mathbb{H} is the unique (up to isomorphism) formal O -module over $\overline{\mathbb{F}}_p$ with O -height n , and \overline{S} is the closed subscheme defined by π of $S \in \text{Nilp}\widehat{O}^{nr}$. For $i \in \mathbb{Z}$, $\widehat{\mathcal{M}}^i$ is the open and closed subspace of $\widehat{\mathcal{M}}$ such that the quasi-isogenies ρ have O -height i . There is a natural (left) action of D^\times on $\widehat{\mathcal{M}}$ by $\forall b \in D^\times, b: \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}, (H, \rho) \mapsto (H, \rho \circ b^{-1})$. This action induces non-canonical isomorphisms

$$\widehat{\mathcal{M}}^i \simeq \widehat{\mathcal{M}}^0,$$

while one knows that there is a non-canonical isomorphism

$$\widehat{\mathcal{M}}^0 \simeq \text{Spf}(\widehat{O}^{nr}[[x_1, \dots, x_{n-1}]])$$

Let $\mathcal{M} = \widehat{\mathcal{M}}^{an} = \coprod_{i \in \mathbb{Z}} \mathcal{M}^i$ be the Berkovich analytic fiber of $\widehat{\mathcal{M}}$. By trivializing the local system over \mathcal{M} defined by the Tate module of p -divisible group, we have the Lubin-Tate tower $(\mathcal{M}_K)_{K \subset GL_n(O)}$ over \widehat{F}^{nr} , and the group $GL_n(F)$ acts (on right) on this tower through Hecke correspondences. When $K = K_m := \ker(GL_n(O) \rightarrow GL_n(O/\pi^m O))$ for some integer $m \geq 0$, there is a regular model $\widehat{\mathcal{M}}_m$ of \mathcal{M}_{K_m} by introducing Drinfeld structures on O -modules. We will not use these models and we will work always on the Berkovich spaces \mathcal{M}_K . Note there are natural actions of D^\times on each \mathcal{M}_K , which commute with the Hecke action.

Fix a prime $l \neq p$, let $\overline{\mathbb{Q}}_l$ (resp. $\overline{\mathbb{Q}}_p$) be a fixed algebraic closure of \mathbb{Q}_l (resp. \mathbb{Q}_p), and \mathbb{C}_p be the completion of $\overline{\mathbb{Q}}_p$. For each $i \geq 0$, we consider the cohomology with compact support

$$H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_U \varprojlim_n H_c^i(U \times \mathbb{C}_p, \mathbb{Z}/l^n \mathbb{Z}) \otimes \overline{\mathbb{Q}}_l,$$

where the inductive limit is taken over all relatively compact open subsets $U \subset \mathcal{M}_K$, see [23] section 4 and [42]. We have

$$H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \bigoplus_{j \in \mathbb{Z}} H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l),$$

where

$$\dim_{\overline{\mathbb{Q}}_l} H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) < \infty$$

by theorem 3.3 in [42]. In fact we have also the usual l -adic cohomology groups $H^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ which are Poincaré dual to those $H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$, and (cf. [77] lemma 2.5.1)

$$H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \neq 0 \Leftrightarrow n-1 \leq i \leq 2(n-1),$$

$$H^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \neq 0 \Leftrightarrow 0 \leq i \leq n-1.$$

The groups

$$\varinjlim_K H_c^i(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$$

are natural smooth representations of $GL_n(F) \times D^\times \times W_F$ (W_F is the Weil group of F), and the local Langlands and Jacquet-Langlands correspondences between the three groups were proved realized in these groups, see [11] and [36].

In [77] Strauch had proven a Lefschetz trace formula for regular elliptic elements action on the Lubin-Tate spaces. More precisely, we consider

$$H_c^*(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \sum_i (-1)^i H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Let $\gamma = (g, b) \in GL_n(F) \times D^\times$ such that g, b are both regular elliptic elements, $gKg^{-1} = K$, $v_p(\det g) + v_p(Nrd b) = 0$ ($Nrd : D^\times \rightarrow F^\times$ is the reduced norm and v_p is the valuation on F), then we have an automorphism

$$\gamma : \mathcal{M}_K^j \rightarrow \mathcal{M}_K^j,$$

which induces morphism on cohomology groups

$$\gamma : H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \rightarrow H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

We define

$$Tr(\gamma | H_c^*(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) := \sum_i (-1)^i Tr(\gamma | H_c^i(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

Strauch proved the following trace formula

Theorem 3.1.1 ([77], Theorem 3.3.1). *Under the above assumptions and notations, we have*

$$Tr(\gamma | H_c^*(\mathcal{M}_K^j \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#Fix(\gamma | \mathcal{M}_K^j \times \mathbb{C}_p).$$

By applying the p -adic period mapping

$$\mathcal{M} \rightarrow \mathbf{P}^{n-1, an},$$

Strauch obtained a nice fixed points number formula for the quotient space $\mathcal{M}_K/\pi^{\mathbb{Z}}$ (theorem 2.6.8 in loc. cit.)

$$\#\text{Fix}(\gamma|(\mathcal{M}_K/\pi^{\mathbb{Z}})(\mathbb{C}_p)) = n\#\{h \in GL_n(F)/\pi^{\mathbb{Z}}K|h^{-1}g_b h = g^{-1}\},$$

which can be rewritten as some suitable orbit integral, see [60] proposition 3.3. This Lefschetz trace formula enable Strauch to prove the Jacquet-Langlands correspondence between smooth representations of $GL_n(F)$ and D^\times is realized the cohomology of the tower $(\mathcal{M}_K)_K$, not involving with Shimura varieties as in [36], see section 4 of [77].

There are two main ingredients in Strauch's proof of the above theorem. The first is some careful approximation theorems of Artin in this special (affine) case and the second is Fujiwara's theorem of specialization of local terms ([31] proposition 1.7.1). In general case one has no sufficient approximation theorems, thus his method can be hardly generalized. In [59] Mieda proved a general Lefschetz trace formula for some open smooth adic spaces by totally working in rigid analytic geometry, and verified his conditions in the special case of Lubin-Tate spaces hold, thus he can reprove the above Lefschetz trace formula. Both Strauch and Mieda worked in the category of adic spaces, and study the action of γ on the boundary strata (outside the corresponding Berkovich space) of the analytic generic fiber of $\widehat{\mathcal{M}}_m$. Their boundary stratas are at last linked to the theory of generalized canonical subgroups (cf. [24] section 7), thus their approach can hardly be generalized.

In this section we work directly in Berkovich spaces. We will consider Fargues's locally finite cell decomposition of Lubin-Tate spaces, cf. [25] chapter 1 and [26], and the locally finite cell decomposition of the unitary group Rapoport-Zink spaces which we obtained in the last section. By studying the action of γ on the cells, we verify the conditions in Mieda's theorem of Lefschetz trace formula hold, by the dictionary between the equivalent categories of Hausdorff strictly Berkovich k -analytic spaces and adic spaces which are taut and locally of finite type over $\text{Spa}(k, k^0)$. (k is a complete non-archimedean field and k^0 is its ring of integers.) Thus we can reprove the above theorem, by different method. The advantage of our method is that, once we know there exists a locally finite cell decomposition, with the fundamental domain compact, then by studying the action on the cells we will easily verify Mieda's theorem applies.

In the following until the end of this introduction, let the notations be as in section 2. For the definition of the subspaces $U_\rho \subset \mathcal{M}_K$, see subsection 3.4.

Theorem 3.1.2. *For the fixed $\gamma = (h, g) \in G(\mathbb{Z}_p) \times \text{Stab}(\Lambda)$ with h, g both regular, elliptic, and $v_p(\det h) + v_p(\det g) = 0$, there exist a sufficient small open compact subgroup $K' \subset G(\mathbb{Z}_p)$ and a sufficient large number $\rho_0 \gg 0$, such that for all open compact normal subgroups $K \subset G(\mathbb{Z}_p)$ contained in K' and all $\rho \geq \rho_0$, we have the Lefschetz trace formula*

$$\text{Tr}(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\text{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p),$$

which is well defined and finite. Since the right hand side is independent of ρ , we can define

$$\text{Tr}(\gamma|H_c^*(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) := \text{Tr}(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l))$$

for $\rho \gg 0$, and thus

$$\text{Tr}(\gamma|H_c^*(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\text{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p).$$

We have a nice fixed points number formula for the quotient space $\mathcal{M}_K/p^{\mathbb{Z}}$. Note if $g \in J_b(\mathbb{Q}_p)$ is a regular elliptic semi-simple element, for any $x \in \text{Fix}(g|\mathcal{F}^a(\mathbb{C}_p))$, there is an element $h_{g,x} \in G(\mathbb{Q}_p)$ which is conjugate to g over $\overline{\mathbb{Q}_p}$ defined by the comparison isomorphism

$$V_p(H_y) \otimes_{\mathbb{Q}_p} B_{dR} \xrightarrow{\sim} V_L \otimes_L B_{dR},$$

where $y \in \pi^{-1}(x)$ is any point in the fiber of the p -adic period mapping $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$.

Corollary 3.1.3. *Let the notations be as in the above theorem. If n is even we assume that $\frac{2}{n}(v_p(\text{deth}) + v_p(\text{det}g))$ is even. Fix compatible Haar measures on $G(\mathbb{Q}_p)$ and the centralizer of $h_{g,x}$, $G_{h_{g,x}} := \{h' \in G(\mathbb{Q}_p) | h'h_{g,x}h'^{-1} = h_{g,x}\}$. Denote the characteristic function of $h^{-1}K$ by $1_{h^{-1}K}$ and the volume of K under the fixed Haar measure by $\text{Vol}(K)$. Then we have the following formula*

$$\text{Tr}(\gamma | H_c^*((\mathcal{M}_K/p^{\mathbb{Z}}) \times \mathbb{C}_p, \overline{\mathbb{Q}_l})) = \sum_{x \in \text{Fix}(g|\mathcal{F}^a(\mathbb{C}_p))} \text{Vol}(G_{h_{g,x}}/p^{\mathbb{Z}}) O_{h_{g,x}}\left(\frac{1_{h^{-1}K}}{\text{Vol}(K)}\right),$$

where $\text{Vol}(G_{h_{g,x}}/p^{\mathbb{Z}})$ is the volume of $G_{h_{g,x}}/p^{\mathbb{Z}}$ by the induced Haar measure on $G(\mathbb{Q}_p)/p^{\mathbb{Z}}$,

$$O_{h_{g,x}}\left(\frac{1_{h^{-1}K}}{\text{Vol}(K)}\right) = \int_{G(\mathbb{Q}_p)/G_{h_{g,x}}} \frac{1_{h^{-1}K}}{\text{Vol}(K)}(z^{-1}h_{g,x}z) dz$$

is the orbit integral of $\frac{1_{h^{-1}K}}{\text{Vol}(K)}$ over the conjugate class of $h_{g,x}$.

Assume π is a supercuspidal representation of $G(\mathbb{Q}_p)$, we consider

$$H(\pi) = \sum_{j \geq 0} (-1)^j \text{Hom}_{G(\mathbb{Q}_p)}\left(\varinjlim_K H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}_l}), \pi\right).$$

Assume that $\text{Hom}_{G(\mathbb{Q}_p)}(\varinjlim_K H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}_l}), \pi)$ is of finite length, which should be always the case, then $H(\pi)$ is a well defined element in $\text{Groth}_{\overline{\mathbb{Q}_l}}(J_b(\mathbb{Q}_p))$.

Corollary 3.1.4. *Let $g \in J_b(\mathbb{Q}_p)$ be a regular elliptic semi-simple element. Assume that π is of the form $\pi = c - \text{Ind}_{K_\pi}^{G(\mathbb{Q}_p)} \lambda$, for some open compact modulo center subgroup $K_\pi \subset G(\mathbb{Q}_p)$ and some finite dimensional representation λ of K_π . Then we have*

$$\text{tr}_{H(\pi)}(g) = \sum_{x \in \text{Fix}(g|\mathcal{F}^a(\mathbb{C}_p))} \text{tr}_\pi(h_{g,x}).$$

As remarked in the introduction 2.1 and the above, we should also prove an analogous Lefschetz trace formula for basic Rapoport-Zink spaces for $GS p_4$, by their corresponding locally finite cell decomposition and the compactness of the fundamental domain. The Lefschetz trace formula for these Rapoport-Zink spaces for unitary groups or $GS p_4$, and (for n even in the unitary case) should enable us to prove the realization of local Jacquet-Langlands correspondence between irreducible smooth representations of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ in the cohomology of these Rapoport-Zink spaces. These will be the contents of our future works. On the other hand, for the non-basic Rapoport-Zink spaces in these cases, our previous results in section 1 say that their cohomology is essentially a parabolic induction.

3.2 The locally finite cell decomposition of Lubin-Tate spaces

In [25] and [27] Fargues found some locally finite cell decompositions of \mathcal{M}_K . The parameter set of cells in [25] is the set of vertices of some Bruhat-Tits building, and these cells for K varies form in fact a cell decomposition of the tower $(\mathcal{M}_K)_K$ but not for a fixed space \mathcal{M}_K . Therefore we will mainly follow the construction in [27], where the parameter set is essentially some set of Hecke correspondences. To consider the group actions on these cells, we will relate the parameter set with a Bruhat-Tits building by borrowing some ideas from [25].

First consider the case without level structures. Fix a uniformizer $\Pi \in D^\times$, then

$$\Pi^{-1} : \mathcal{M}^i \xrightarrow{\sim} \mathcal{M}^{i+1}.$$

Let \mathcal{M}^{ss} be the semi-stable locus in \mathcal{M} , i.e. the locus where the associated p -divisible groups are semi-stable in the sense of [27] definition 4, see also definition 2.5.1, which is a closed analytic domain in \mathcal{M} . Let $\mathcal{D} = \mathcal{M}^{ss,0} := \mathcal{M}^{ss} \cap \mathcal{M}^0$, then $\mathcal{M}^{ss} = \prod_{i \in \mathbb{Z}} \Pi^{-i} \mathcal{D}$ and \mathcal{D} is the compact fundamental domain of Gross-Hopkins, see [25] 1.5. The main results of [27] for our special case say that we have a locally finite covering

$$\mathcal{M} = \bigcup_{\substack{GL_n(O) \backslash GL_n(F) / GL_n(O) \\ i=0, \dots, n-1}} T \cdot \Pi^{-i} \mathcal{D},$$

where $T.A$ is the image under the Hecke correspondence T for a subset A , which is an analytic domain if A is. In the following we shall actually work with one component \mathcal{M}^0 , so we consider its induced cell decomposition

$$\mathcal{M}^0 = \bigcup_{\substack{T \in GL_n(O) \backslash GL_n(F) / GL_n(O) \\ i=0, \dots, n-1}} ((T \cdot \Pi^{-i} \mathcal{D}) \cap \mathcal{M}^0).$$

For $T \in GL_n(O) \backslash GL_n(F) / GL_n(O), i = 0, \dots, n-1$,

$$(T \cdot \Pi^{-i} \mathcal{D}) \cap \mathcal{M}^0 \neq \emptyset \Leftrightarrow -v_p(\det T) + i = 0,$$

in which case

$$T \cdot \Pi^{-i} \mathcal{D} \subset \mathcal{M}^0.$$

Here $(v_p : F^\times \rightarrow \mathbb{Z}$ is the valuation of F^\times such that $v_p(\pi) = 1$) the composition $v_p \circ \det : GL_n(F) \rightarrow \mathbb{Z}$ factors through $GL_n(O) \backslash GL_n(F) / GL_n(O) \rightarrow \mathbb{Z}$. Thus we have

$$\mathcal{M}^0 = \bigcup_{\substack{T \in GL_n(O) \backslash GL_n(F) / GL_n(O) \\ i=0, \dots, n-1 \\ -v_p(\det T) + i = 0}} T \cdot \Pi^{-i} \mathcal{D}.$$

Let $K \subset GL_n(O)$ be an open compact subgroup, $\pi_K : \mathcal{M}_K \rightarrow \mathcal{M}$ be the natural projection. We set

$$\mathcal{D}_K = \pi_K^{-1}(\mathcal{D}),$$

which is a compact analytic domain in \mathcal{M}_K^0 . Since the group $GL_n(O)$ acts trivially on \mathcal{M} , any element in this group will stabilize \mathcal{D}_K . Thus for two Hecke correspondences

$T_1, T_2 \in K \backslash GL_n(F)/K$ having the same image under the projection $K \backslash GL_n(F)/K \rightarrow GL_n(O) \backslash GL_n(F)/K$, we have $T_1 \Pi^{-i} \mathcal{D}_K = T_2 \Pi^{-i} \mathcal{D}_K$ ($\Pi^{-i} \mathcal{D}_K = \pi_K^{-1}(\Pi^{-i} \mathcal{D})$ since π_K is D^\times -equivariant). Therefore, we have the following locally finite cell decomposition in level K

$$\mathcal{M}_K = \bigcup_{\substack{T \in GL_n(O) \backslash GL_n(F)/K \\ i=0, \dots, n-1}} T \cdot \Pi^{-i} \mathcal{D}_K.$$

We will denote the cells $T \cdot \Pi^{-i} \mathcal{D}_K$ by

$$\mathcal{D}_{T,i,K},$$

which are compact analytic domains. For any $T \in GL_n(O) \backslash GL_n(F)/K, i \in \mathbb{Z}$, we denote also $\mathcal{D}_{T,i,K} = T \cdot \Pi^{-i} \mathcal{D}_K$. Since the action of F^\times on \mathcal{M}_K through $F^\times \rightarrow GL_n(F), z \mapsto z$ is the same as the action of it on \mathcal{M}_K through $F^\times \rightarrow D^\times, z \mapsto z, (z, z^{-1}) \in GL_n(F) \times D^\times$ acts trivially on \mathcal{M}_K . We have

$$\mathcal{D}_{T,i,K} = \mathcal{D}_{Tz, i+nv_p(z), K}.$$

If $g \in GL_n(F)$ is an element such that $gKg^{-1} = K$, and $b \in D^\times$ is an arbitrary element, set

$$\gamma := (g, b).$$

Then automorphism $\gamma : \mathcal{M}_K \rightarrow \mathcal{M}_K$ naturally induces an action of γ on the set of cells of \mathcal{M}_K :

$$\gamma(\mathcal{D}_{T,i,K}) = \mathcal{D}_{Tg, i-v_p(Nrdb), K}.$$

Here $Nrd : D^\times \rightarrow F^\times$ is the reduced norm.

For the component \mathcal{M}_K^0 , for $T \in GL_n(O) \backslash GL_n(F)/K, i = 0, \dots, n-1$,

$$(T \cdot \Pi^{-i} \mathcal{D}_K) \cap \mathcal{M}_K^0 \neq \emptyset \Leftrightarrow -v_p(\det T) + i = 0,$$

in which case

$$T \cdot \Pi^{-i} \mathcal{D}_K \subset \mathcal{M}_K^0.$$

Thus we have a locally finite cell decomposition

$$\begin{aligned} \mathcal{M}_K^0 &= \bigcup_{\substack{T \in GL_n(O) \backslash GL_n(F)/K \\ i=0, \dots, n-1}} ((T \cdot \Pi^{-i} \mathcal{D}_K) \cap \mathcal{M}_K^0) \\ &= \bigcup_{\substack{T \in GL_n(O) \backslash GL_n(F)/K \\ i=0, \dots, n-1 \\ -v_p(\det T) + i = 0}} \mathcal{D}_{T,i,K}. \end{aligned}$$

In fact for any $i \in \mathbb{Z}, T \in GL_n(O) \backslash GL_n(F)/K$ such that $-v_p(\det T) + i = 0$, we have $\mathcal{D}_{T,i,K} \subset \mathcal{M}_K^0$ with the convention above. However one can always by multiplying with some $z \in F^\times$ to reduce to the cases $0 \leq i \leq n-1$. Let $\gamma = (g, b) \in GL_n(F) \times D^\times$ be such that $gKg^{-1} = K, v_p(\det g) + v_p(Nrdb) = 0$, then the action of γ on \mathcal{M}_K induces $\gamma : \mathcal{M}_K^0 \rightarrow \mathcal{M}_K^0$. In this case γ acts on the set of cells of \mathcal{M}_K^0 as in the same way as above.

To understand better the parameter set of cells of \mathcal{M}_K^0 , we look at some ideas from [25]. Consider the embedding $\mathbb{G}_m \rightarrow GL_n \times D^\times, z \mapsto (z, z^{-1})$ of algebraic groups

over F . Let $\mathcal{B}(GL_n \times D^\times, F)$ be the (extended) Bruhat-Tits building of $GL_n \times D^\times$, and $\mathcal{B} = \mathcal{B}(GL_n \times D^\times, F)/F^\times$ be its quotient by the action of F^\times through the above embedding. The set \mathcal{B}^0 of vertices of \mathcal{B} , which we define by the quotient of vertices in $\mathcal{B}(GL_n \times D^\times, F)$, can be described as the set of equivalent classes

$$\{(\Lambda, M) \mid \Lambda \subset F^n \text{ is an } O\text{-lattice, } M \subset D \text{ is an } O_D^\times\text{-lattice}\} / \sim,$$

where

$$(\Lambda_1, M_1) \sim (\Lambda_2, M_2) \Leftrightarrow \exists i \in \mathbb{Z}, \Lambda_2 = \Lambda_1 \pi^i, M_2 = \pi^{-i} M_1,$$

see [25] 1.5. We can understand \mathcal{B} in this way : the (extended) Bruhat-Tits building of GL_n over F is the product $\mathcal{B}(PGL_n, F) \times \mathbb{R}$, while the (extended) Bruhat-Tits building of D^\times over F is $\mathcal{B}(D^\times, F) \simeq \mathbb{R}$, then by construction

$$\mathcal{B} = \mathcal{B}(GL_n \times D^\times, F)/F^\times \simeq (\mathcal{B}(PGL_n, F) \times \mathbb{R} \times \mathbb{R}) / \sim,$$

where $(x, s, t) \sim (x', s', t') \Leftrightarrow x = x', s - s' = t' - t = nr$ for some $r \in \mathbb{Z}$. Thus any point $[x, s, t]$ in \mathcal{B} can be written uniquely in the form $[x, s', t']$ for $x \in \mathcal{B}(PGL_n, F)$, $s' \in \mathbb{R}$, $t' \in [0, n)$. The elements $(g, b) \in GL_n(F) \times D^\times$ act on \mathcal{B} by $\forall [x, s, t] \in \mathcal{B}$,

$$(g, b)[x, s, t] = [g^{-1}x, s + v_p(\det g), t + v_p(Nrdb)].$$

If we consider the right action of $GL_n(F)$ on $\mathcal{B}(PGL_n, F)$ by $xg := g^{-1}x$, then we can also write $(g, b)[x, s, t] = [xg, s + v_p(\det g), t + v_p(Nrdb)]$.

On the other hand, consider the action of F^\times on $GL_n(O) \setminus GL_n(F) \times D^\times / O_D^\times$ by $z(GL_n(O)g, dO_D^\times) = (GL_n(O)gz, z^{-1}dO_D^\times)$, $\forall z \in F^\times$, then the quotient set

$$(GL_n(O) \setminus GL_n(F) \times D^\times / O_D^\times) / F^\times$$

is naturally identified with the set \mathcal{B}^0 after fixing the vertex $[O^n, O_D] \in \mathcal{B}^0$. For an element $[GL_n(O)g, dO_D^\times]$, the associated point in \mathcal{B}^0 can be written as $[GL_n(O)F^\times g, v_p(\det g), v_p(\det d)]$. Here $GL_n(O)F^\times g \in \mathcal{B}(PGL_n, F)$ by fixing the homothety class of O^n . Now let $K \subset GL_n(O)$ be an open compact subgroup, then the set

$$\mathcal{I}_K := (GL_n(O) \setminus GL_n(F) / K \times D^\times / O_D^\times) / F^\times$$

can be identified with the image \mathcal{B}^0/K of \mathcal{B}^0 in the quotient space \mathcal{B}/K . If $\gamma = (g, b) \in GL_n(F) \times D^\times$ such that $gKg^{-1} = K$, then γ acts on the set \mathcal{I}_K by $[T, d] \mapsto [Tg, bd]$. There are two natural projection maps $\mathcal{I}_K \rightarrow (GL_n(O) \setminus GL_n(F) / K) / F^\times$ and $\mathcal{I}_K \rightarrow (D^\times / O_D^\times) / F^\times \simeq \mathbb{Z}/n\mathbb{Z}$. There is as well as a map

$$\begin{aligned} GL_n(O) \setminus GL_n(F) / K \times D^\times / O_D^\times &\longrightarrow \mathbb{Z} \\ (T, d) &\mapsto -v_p(\det T) - v_p(Nrdd). \end{aligned}$$

Let $(GL_n(O) \setminus GL_n(F) / K \times D^\times / O_D^\times)^0$ be the inverse image of 0 under this map. Since the action of F^\times does not change the values of the above map, it factors through $\mathcal{I}_K \rightarrow \mathbb{Z}$. In fact there is a well defined continuous map

$$\begin{aligned} \varphi : \mathcal{B} &\longrightarrow \mathbb{R} \\ [x, s, t] &\mapsto -s - t, \end{aligned}$$

with each fiber stable under the action of K . The above map is induced by φ . For the γ as above with further condition that $v_p(\det g) + v_p(Nrdb) = 0$, it stabilizes the subset

$$\mathcal{I}_K^0 := (GL_n(O) \setminus GL_n(F)/K \times D^\times / O_D^\times)^0 / F^\times$$

for the above action. For the map φ above, we see that \mathcal{I}_K^0 is identified with the quotient set $\varphi^{-1}(0)^0 / K$ of vertices in $\varphi^{-1}(0)$.

For any element $[T, d] \in \mathcal{I}_K$, the cell $[T, d]\mathcal{D}_K$ is well defined, which is what we denoted by $\mathcal{D}_{T, -v_p(Nrdd), K}$ above. As before we denote $[T, d]\mathcal{D}_K$ as

$$\mathcal{D}_{[T, d], K}.$$

Then we can rewrite the cell decompositions as

$$\begin{aligned} \mathcal{M}_K &= \bigcup_{[T, d] \in \mathcal{I}_K} \mathcal{D}_{[T, d], K}, \\ \mathcal{M}_K^0 &= \bigcup_{[T, d] \in \mathcal{I}_K^0} \mathcal{D}_{[T, d], K}. \end{aligned}$$

For $\gamma = (g, b) \in GL_n(F) \times D^\times$ as above, it acts on the cells in the way compatible with its action on \mathcal{I}_K :

$$\gamma(\mathcal{D}_{[T, d], K}) = \mathcal{D}_{[Tg, bd], K}.$$

Recall there is a metric $d(\cdot, \cdot)$ on \mathcal{B} , so that $(GL_n(F) \times D^\times) / F^\times$ acts on it by isometries. If $d'(\cdot, \cdot)$ is the metric on $\mathcal{B}(PGL_n, F)$, then for two points $[x, s, t], [x', s', t']$ with $x, x' \in \mathcal{B}(PGL_n, F)$, $s, s' \in \mathbb{R}$, $t, t' \in [0, n)$ we have

$$d([x, s, t], [x', s', t']) = \sqrt{d'(x, x')^2 + (s - s')^2 + (t - t')^2}.$$

The group K acts on \mathcal{B} through the natural morphisms $K \rightarrow GL_n(F) \times D^\times \rightarrow (GL_n(F) \times D^\times) / F^\times$. There is an induced metric $\bar{d}(\cdot, \cdot)$ on the quotient space \mathcal{B}/K :

$$\bar{d}(xK, yK) := \inf_{k, k' \in K} d(xk, yk') = \inf_{k \in K} d(xk, y) = \inf_{k \in K} d(x, yk), \quad \forall xK, yK \in \mathcal{B}/K,$$

the last two equalities come from $d(xk, yk') = d(xk(k')^{-1}, y) = d(x, yk'k^{-1})$. Since K is compact, one checks it easily that this is indeed a metric on \mathcal{B}/K . With this metric, $\mathcal{I}_K, \mathcal{I}_K^0$ are both infinity discrete subset of \mathcal{B}/K , and any closed ball in \mathcal{B}/K contains only finitely many elements of \mathcal{I}_K and \mathcal{I}_K^0 . We will directly work with the induced metric space

$$\mathcal{I}_K = \mathcal{B}^0 / K.$$

For $\gamma = (g, b) \in GL_n(F) \times D^\times$ with $gKg^{-1} = K$, one can check by definition of \bar{d} that the above action of γ on \mathcal{I}_K is isometric :

$$\bar{d}(\gamma x, \gamma x) = \bar{d}(x, x), \quad \forall x \in \mathcal{I}_K.$$

Note that for $[T_1, d_1], [T_2, d_2] \in \mathcal{I}_K$, $\mathcal{D}_{[T_1, d_1], K} \cap \mathcal{D}_{[T_2, d_2], K} \neq \emptyset$ implies that $v_p(\det T_1) + v_p(Nrdd_1) = v_p(\det T_2) + v_p(Nrdd_2)$. If we write $[T_1, d_1] = [x_1K, s_1, t_1], [T_2, d_2] = [x_2K, s_2, t_2]$ with $x_1, x_2 \in \mathcal{B}(PGL_n, F)$, $s_1, s_2 \in \mathbb{Z} \subset \mathbb{R}$, $t_1, t_2 \in [0, n) \cap \mathbb{Z}$ (i.e. $\exists r_1, r_2 \in$

\mathbb{Z} , s.t. $v_p(\det T_i) = s_i + nr_i, v_p(Nrdd_i) = t_i + nr_i, i = 1, 2$), then $s_1 + t_1 = s_2 + t_2, s_1 - s_2 = t_2 - t_1 \in [1 - n, n - 1]$, the distance

$$\bar{d}([T_1, d_1], [T_2, d_2]) = \inf_{k \in K} \sqrt{d'(x_1, x_2 k)^2 + 2(s_1 - s_2)^2}$$

just depends on $\bar{d}'(x_1 K, x_2 K)$ for the induced metric \bar{d}' on $\mathcal{B}(PGL_n, F)$ defined in the same way as \bar{d} . By the construction of the locally finite cell decomposition of \mathcal{M}_K , we have the following proposition.

Proposition 3.2.1. *There exists a constant $c > 0$, such that for any $[T_1, d_1], [T_2, d_2] \in \mathcal{I}_K$ with $\bar{d}([T_1, d_1], [T_2, d_2]) > c$, we have*

$$\mathcal{D}_{[T_1, d_1], K} \cap \mathcal{D}_{[T_2, d_2], K} = \emptyset.$$

Proof. We need to prove that, there exists a constant $c > 0$, such that for any $[T, d] \in \mathcal{I}_K$, and any $[T', d'] \in \{[T', d'] \in \mathcal{I}_K | \mathcal{D}_{[T', d'], K} \cap \mathcal{D}_{[T, d], K} \neq \emptyset\}$, we have $\bar{d}([T, d], [T', d']) \leq c$. This just comes from the construction of the above locally finite cell decomposition of \mathcal{M}_K , and the definition of \bar{d} . We just indicate some key points. First, for any fixed choice of fundamental domain V_K in \mathcal{B} for the action of K , by definition $\forall x, y \in V_K, d(x, y) \geq \bar{d}(xK, yK)$. Next, By the proof of proposition 24 of [27], and the Cartan decomposition $GL_n(O) \backslash GL_n(F)/GL_n(O) \simeq \mathbb{Z}_+^n = \{(a_1, \dots, a_n) \in \mathbb{Z}^n | a_1 \geq \dots \geq a_n\}$, for any fixed the Hecke correspondence $T \in GL_n(O) \backslash GL_n(F)/GL_n(O), i \in \mathbb{Z}$, the finite set

$$A_{[T, i]} := \{[T', j] \in (GL_n(O) \backslash GL_n(F)/GL_n(O) \times \mathbb{Z})/F^\times | T \cdot \Pi^{-i} \mathcal{D} \cap T' \cdot \Pi^{-j} \mathcal{D} \neq \emptyset\}$$

is such that $\forall [T', j] \in A_{[T, i]}$ we have $v_p(\det T') + j = v_p(\det T) + i$; and if T corresponds to the point $(a_1, \dots, a_n) \in \mathbb{Z}_+^n$, then for $j \in \mathbb{Z}/n\mathbb{Z}$ fixed, the set $T' \in GL_n(O) \backslash GL_n(F)/GL_n(O)$ with $[T', j] \in A_{[T, i]}$, correspond to the points $(a'_1, \dots, a'_n) \in \mathbb{Z}_+^n$ such that $\sum_{k=1}^n a'_k = \sum_{k=1}^n a_k + i - j \pmod{n\mathbb{Z}}, |a_k - a'_k| \leq C$ for all $k = 1, \dots, n$, and $C > 0$ is a constant doesn't depend on $[T, i]$. From these two points one can easily deduce the proposition for $K = GL_n(O)$, and the general case will be obtained as soon as the case $K = GL_n(O)$ holds. \square

We remark that, in [24] Fargues defined an O_{D^\times} -invariant continuous map of topological spaces

$$\mathcal{M}^0 \longrightarrow \mathcal{B}(PGL_n, F)/GL_n(O),$$

and identified the image of \mathcal{D} under this map. However, this map depends quite on our special case, and in general there is no such a map from Rapoport-Zink spaces to Bruhat-Tits buildings. For any open compact subgroup $K \subset GL_n(O)$, there is also a continuous map $\mathcal{M}^0 \rightarrow \mathcal{B}(PGL_n, F)/K$, and we have a commutative diagram of continuous maps between topological spaces

$$\begin{array}{ccc} \mathcal{M}_K^0 & \longrightarrow & \mathcal{B}(PGL_n, F)/K \\ \downarrow & & \downarrow \\ \mathcal{M}^0 & \longrightarrow & \mathcal{B}(PGL_n, F)/GL_n(O). \end{array}$$

These maps are Hecke equivariant, thus compatible with the cell decomposition of \mathcal{M}_K^0 .

3.3 Lefschetz trace formula for Lubin-Tate spaces

In this subsection $\gamma = (g, b) \in GL_n(F) \times D^\times$ is an element such that both g and b are regular elliptic semi-simple, $gKg^{-1} = K$ and $v_p(\det g) + v_p(Nrdb) = 0$. Since γ is regular elliptic, the set of γ -fixed vertices $(\mathcal{B}^0)^\gamma$ is non empty, cf. [71]. Let \hat{o} be a fixed choice of point in $(\mathcal{B}^0)^\gamma$, and $o \in \mathcal{I}_K$ be its image in the quotient space. One can take the above choice of \hat{o} so that $\hat{o} \in \varphi^{-1}(0)^0, o \in \mathcal{I}_K^0$. Then $\gamma(o) = o$ by the action $\gamma : \mathcal{I}_K^0 \rightarrow \mathcal{I}_K^0$. For any real number $\rho > 0$, we consider the subset of \mathcal{I}_K^0

$$A_\rho = \{x \in \mathcal{I}_K^0 \mid \bar{d}(o, x) \leq \rho\},$$

which is a finite set for any fixed ρ . Moreover since $\gamma(o) = o$ and \bar{d} is γ -isometric, we have $\gamma(A_\rho) = A_\rho$.

Definition 3.3.1. For any finite set $A \subset \mathcal{I}_K^0$, we define two subspaces of \mathcal{M}_K^0

$$\begin{aligned} V_A &= \bigcup_{[T,d] \in A} \mathcal{D}_{[T,d],K}, \\ U_A &= \mathcal{M}_K^0 - \bigcup_{[T,d] \notin A} \mathcal{D}_{[T,d],K}. \end{aligned}$$

Proposition 3.3.2. U_A is an open subspace of \mathcal{M}_K^0 , while V_A is a compact analytic domain, and $U_A \subset V_A$.

Proof. Since $\mathcal{M}_K^0 - U_A = \bigcup_{[T,d] \notin A} \mathcal{D}_{[T,d],K}$, which is a locally finite union of closed subsets, therefore it is closed, and U_A is open. V_A is a finite union of compact analytic domains thus so is itself. The inclusion simply comes from the fact $\mathcal{M}_K^0 = V_A \cup (\mathcal{M}_K^0 - U_A)$. \square

When $\rho \rightarrow \infty$, the finite sets A_ρ exhaust \mathcal{I}_K^0 . For any $\rho \geq 0$, we denote $U_\rho = U_{A_\rho}, V_\rho = V_{A_\rho}$. Since U_ρ is relatively compact, we can compute the cohomology of \mathcal{M}_K^0 as

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_{\rho} H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Moreover, for $\rho \gg 0$ large enough, the cohomology groups $H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ is constant and bijective to $H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$, see proposition 3.3.5.

For the γ above, we consider the action $\gamma : \mathcal{M}_K^0 \rightarrow \mathcal{M}_K^0$. Since $\gamma(A_\rho) = A_\rho$,

$$\gamma(U_\rho) = U_\rho, \gamma(V_\rho) = V_\rho.$$

γ thus acts also on the cells contained in V_ρ : $\gamma(\mathcal{D}_{[T,d],K}) = \mathcal{D}_{[Tg,bd],K}$. Passing to cohomology, γ induces an automorphism

$$\gamma : H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \rightarrow H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Consider

$$H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \sum_i (-1)^i H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$$

as an element in some suitable Grothendieck group, and the trace of γ

$$Tr(\gamma | H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \sum_i (-1)^i Tr(\gamma | H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

Let $\text{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p)$ be the set of fixed points of γ on $\mathcal{M}_K^0 \times \mathbb{C}_p$, then each fixed point is simple since the p -adic period mapping is étale (cf. [77] theorem 2.6.8).

We will use the cell decomposition of \mathcal{M}_K^0 , to verify that the action of γ satisfies the conditions of Mieda's theorem 3.13 [59], thus deduce a Lefschetz trace formula. In fact we will use a Berkovich version of loc. cit. Recall that, if k is a complete non-archimedean field and k^0 is its ring of integers, then the category of Hausdorff strictly k -analytic spaces is equivalent to the category of adic spaces which are taut and locally of finite type over $\text{spa}(k, k^0)$, see [41] chapter 8. If X is a Hausdorff strictly k -analytic space, we denote by X^{ad} the associated adic space, which is taut and locally of finite type over $\text{spa}(k, k^0)$.

Theorem 3.3.3. *Let the notations and assumptions be as above. There exist an open compact subgroup $K' \subset GL_n(O)$ and a real number ρ_0 , such that for all open compact subgroup $K \subset K'$ which is normalized by g and all $\rho \geq \rho_0$, we have*

$$\text{Tr}(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\text{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p).$$

For ρ sufficiently large, the left hand side is just $\text{Tr}(\gamma|H_c^*(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l))$.

Proof. Since $g \in GL_n(F)$ is elliptic, we first note the following fact : for any sufficiently small open compact subgroup $K \subset GL_n(O)$ such that $gKg^{-1} = K$, we have

$$\bar{d}(x, \gamma x) \rightarrow \infty, \text{ when } x \in \mathcal{I}_K^0, \bar{d}(o, x) \rightarrow \infty.$$

In fact, since $o, x \in \mathcal{I}_K^0$, write $o = [o'K, -s, s], x = [x'K, -t, t]$ with $o', x' \in \mathcal{B}(PGL_n, F)^0$, then $\gamma(x) = [x'gK, v_p(\det g) - t, v_p(Nrdb) + t] = [x'gK, -t', t']$. If we denote the metric on $\mathcal{B}' = \mathcal{B}(PGL_n, F)$ by $d'(\cdot, \cdot)$ and the induced metric on \mathcal{B}'/K by \bar{d}' as before, then we just need to prove that

$$\bar{d}'(x'K, x'gK) \rightarrow \infty, \text{ when } x'K \in (\mathcal{B}')^0/K, \bar{d}'(o'K, x'K) \rightarrow \infty.$$

To prove this statement, we first work with \mathcal{B}' itself by not the quotient. Since g is elliptic, the fixed points set $(\mathcal{B}')^g$ is nonempty and compact. Moreover, for K sufficiently small, $(\mathcal{B}')^g = (\mathcal{B}')^{g'}$ for any $g' \in gK$ (cf. the proof of lemma 12 in [71]). For $o' \in (\mathcal{B}')^g$ fixed, a simple triangle inequality shows that $d'(x', (\mathcal{B}')^g) \rightarrow \infty$ when $d'(x', o') \rightarrow \infty$, since $(\mathcal{B}')^g$ is compact. On the other hand, for any automorphism σ of \mathcal{B}' with $(\mathcal{B}')^\sigma \neq \emptyset$, there exists a constant $0 < \theta \leq \pi$ which just depends on \mathcal{B}' and σ , such that

$$d'(x', \sigma x') \geq 2d'(x', (\mathcal{B}')^\sigma) \sin\left(\frac{\theta}{2}\right),$$

see [70] proposition 2.3. In particular, $d'(x', x'g') \rightarrow \infty$ when $d'(o', x') \rightarrow \infty$ for any $g' \in gK$. As K is compact this deduces the above statement.

For ρ sufficiently large,

$$\begin{aligned} \mathcal{M}_K^0 - U_\rho &= \bigcup_{[T, d] \in \mathcal{I}_K^0 - A_\rho} \mathcal{D}_{[T, d], K} \\ V_\rho - U_\rho &= \bigcup_{[T, d] \in A_\rho - A_{\rho-c}} F_{[T, d]}, \end{aligned}$$

where for $[T, d] \in A_\rho$,

$$F_{[T,d]} = \mathcal{D}_{[T,d],K} \cap (\mathcal{M}_K^0 - U_\rho),$$

which is nonempty if and only if $[T, d] \in A_\rho - A_{\rho-c}$ by the above proposition, in which case $F_{[T,d]}$ is a compact analytic domain in $\mathcal{D}_{[T,d],K} \subset V_\rho$. For K sufficiently small, ρ sufficiently large and $[T, d] \in \mathcal{I}_K^0 - A_{\rho-c}$, by the lemma $\bar{d}([T, d], \gamma([T, d])) > c$, thus

$$\mathcal{D}_{[T,d],K} \cap \gamma(\mathcal{D}_{[T,d],K}) = \emptyset, \quad F_{[T,d]} \cap \gamma(F_{[T,d]}) = \emptyset \quad (\text{for } [T, d] \in A_\rho - A_{\rho-c}).$$

To apply Mieda's theorem, we pass to adic spaces. We have the locally finite cell decomposition of the adic space $(\mathcal{M}_K^0)^{ad}$:

$$(\mathcal{M}_K^0)^{ad} = \bigcup_{[T,d] \in \mathcal{I}_K^0} \mathcal{D}_{[T,d],K}^{ad}$$

where each cell $\mathcal{D}_{[T,d],K}^{ad}$ is an open quasi-compact subspace, $\mathcal{D}_{[T_1,d_1],K}^{ad} \cap \mathcal{D}_{[T_2,d_2],K}^{ad} \neq \emptyset \Leftrightarrow \mathcal{D}_{[T_1,d_1],K} \cap \mathcal{D}_{[T_2,d_2],K} \neq \emptyset$, and the action of γ on $(\mathcal{M}_K^0)^{ad}$ induces an action on the cells in the same way as the case of Berkovich analytic spaces. By [41] 8.2, U_ρ^{ad} is an open subspace of $(\mathcal{M}_K^0)^{ad}$, which is separated, smooth, partially proper. On the other hand, $V_\rho^{ad} = \bigcup_{[T,d] \in A_\rho} \mathcal{D}_{[T,d],K}^{ad}$ is a quasi-compact open subspace. Consider the closure $\overline{V_\rho^{ad}} = \overline{\bigcup_{[T,d] \in A_\rho} \mathcal{D}_{[T,d],K}^{ad}}$ of V_ρ^{ad} in $(\mathcal{M}_K^0)^{ad}$, which is a proper pseudo-adic space. We know that $\overline{V_\rho^{ad}}$ (resp. $\overline{\mathcal{D}_{[T,d],K}^{ad}}$) is the set of all specializations of the points in V_ρ^{ad} (resp. $\mathcal{D}_{[T,d],K}^{ad}$). Moreover γ induce automorphisms $\gamma : \overline{V_\rho^{ad}} \rightarrow \overline{V_\rho^{ad}}, V_\rho^{ad} \rightarrow V_\rho^{ad}, U_\rho^{ad} \rightarrow U_\rho^{ad}$. Since $V_{\rho-c}^{ad} \subset U_\rho^{ad} \subset V_\rho^{ad}$, we have $\overline{V_\rho^{ad}} - V_\rho^{ad} = \bigcup_{[T,d] \in A_\rho - A_{\rho-c}} (\overline{\mathcal{D}_{[T,d],K}^{ad}} - \mathcal{D}_{[T,d],K}^{ad})$. Note

$$\mathcal{D}_{[T_1,d_1],K}^{ad} \cap \mathcal{D}_{[T_2,d_2],K}^{ad} \neq \emptyset \Leftrightarrow \overline{\mathcal{D}_{[T_1,d_1],K}^{ad}} \cap \overline{\mathcal{D}_{[T_2,d_2],K}^{ad}} \neq \emptyset.$$

For $[T, d] \in A_\rho - A_{\rho-c}$, let $W_{[T,d]} = \overline{\mathcal{D}_{[T,d],K}^{ad}} - \mathcal{D}_{[T,d],K}^{ad}$. By the paragraph above, for $\rho \gg 0$ we have $\gamma(W_{[T,d]}) \cap W_{[T,d]} = \emptyset$. One sees the conditions of theorem 3.13 of [59] hold for V_ρ^{ad} and its compactification $\overline{V_\rho^{ad}}$, i.e.

$$Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\text{Fix}(\gamma|V_\rho^{ad} \times \mathbb{C}_p) = \#\text{Fix}(\gamma|V_\rho \times \mathbb{C}_p).$$

Here and in the following $V_\rho^{ad} \times \mathbb{C}_p := V_\rho^{ad} \times \text{spa}(\mathbb{C}_p, O_{\mathbb{C}_p})$, and similar notations for other adic spaces. By [42] proposition 2.6 (i) and lemma 3.4, we have

$$Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(U_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) + Tr(\gamma|H_c^*((V_\rho^{ad} - U_\rho^{ad}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

By the paragraph above one can see it easily by the induction argument of the proof of proposition 4.10 in [59] that $Tr(\gamma|H_c^*((V_\rho^{ad} - U_\rho^{ad}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = 0$. Thus we can conclude by Huber's comparison theorem on compactly support cohomology of Berkovich spaces and adic spaces,

$$Tr(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(U_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\text{Fix}(\gamma|V_\rho \times \mathbb{C}_p).$$

But as reason above for $\rho \gg 0$ there is no fixed points of γ outside $V_\rho \times \mathbb{C}_p$,

$$\#\text{Fix}(\gamma|V_\rho \times \mathbb{C}_p) = \#\text{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p).$$

The last statement in the theorem comes from the following proposition 3.3.5.

□

Remark 3.3.4. *In fact we can use V_ρ to compute the cohomology of \mathcal{M}_K^0 directly when passing to adic spaces :*

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^i((\mathcal{M}_K^0)^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_{\rho} H_c^i(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \quad \forall i \geq 0,$$

here the second equality comes from proposition 2.1 (iv) of [42]. Here we prefer to transfer back the results to Berkovich spaces, so we insist on working with the open subspaces U_ρ .

In fact, the formal models $\widehat{\mathcal{M}}_K^0$ are algebraizable : they are the formal completion at closed points of some Shimura varieties as in [36], or one can find the algebraization directly as in theorem 2.3.1 in [77]. So we have for all integer $i \geq 0$

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq (\varprojlim_r H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \mathbb{Z}/l^r\mathbb{Z})) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l,$$

and similarly for the cohomology without compact support. We have the following proposition.

Proposition 3.3.5. *Let the notations be as above. Then for $\rho \gg 0$ and all integer $i \geq 0$, we have bijections*

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^i(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Proof. This comes from the description of V_ρ and Huber's theorem 2.9 in [43]. Recall the fundamental domain $\mathcal{D} \subset \mathcal{M}^0$ is associated to an admissible open subset $\mathcal{D}^{rig} \subset (\mathcal{M}^0)^{rig}$. On the rigid analytic space $(\mathcal{M}^0)^{rig}$ there is a natural coordinate x_1, \dots, x_{n-1} , such that for $x = (x_1, \dots, x_{n-1}) \in (\mathcal{M}^0)^{rig}$, the Newton polygon of multiplication by π on the formal group law associated to π -divisible group H_x is the convex envelope of the points $(q^i, v(x_i))_{0 \leq i \leq n}$, where $x_0 = 0, x_n = 1, q = \#O/\pi$, cf. [25] 1.1.5. Under this coordinate

$$\mathcal{D}^{rig} = \{x = (x_1, \dots, x_{n-1}) \in (\mathcal{M}^0)^{rig} \mid v(x_i) \geq 1 - \frac{i}{n}, i = 1, \dots, n-1\},$$

cf. loc. cit. 1.4. Thus after base change to \mathbb{C}_p it is isomorphic to a closed ball. In [24] section 5 Fargues had described the Newton polygons of the points in a Hecke orbit. In particular at level $K = GL_n(O)$ the admissible open subsets $V_\rho^{rig} \times \mathbb{C}_p$ are locally described by closed balls. Then this is also the case for any level K . Now pass to adic spaces, $V_\rho^{ad} \times \mathbb{C}_p$ are quasi-compact open subsets and locally described by $\mathbb{B}_{\epsilon_\rho} = \{z \in (\mathcal{M}^0)^{ad} \times \mathbb{C}_p \mid |x_i(z)| \leq \epsilon_\rho\}$. Since $U_\rho^{ad} \times \mathbb{C}_p, (\mathcal{M}^0)^{ad} \times \mathbb{C}_p$ can be described as unions of ascending chains of quasi-compact open subsets locally in the above forms, by theorem 2.9 of [43] and the comparison theorem 8.3.5 of [41] one can conclude. \square

Let $\gamma = (g, b) \in GL_n(F) \times D^\times$, with g, b both regular elliptic and $gKg^{-1} = K$. For the quotient space $\mathcal{M}_K/\pi^\mathbb{Z}$ we have a nice fixed points formula by considering the p -adic period mapping, which is non zero if and only if $v_p(\text{deg}g) + v_p(Nrdb)$ is divisible by n . Fix compatible Haar measures on $GL_n(F)$ and G_{g_b} (see below), we can also write it as some suitable orbit integral ([77] theorem 2.6.8, [59] proposition 3.3).

$$\begin{aligned} \#\text{Fix}(\gamma | (\mathcal{M}_K/\pi^\mathbb{Z}) \times \mathbb{C}_p) &= n \#\{h \in GL_n(F)/\pi^\mathbb{Z}K \mid h^{-1}g_b h = g^{-1}\} \\ &= n \text{Vol}(G_{g_b}/\pi^\mathbb{Z}) \int_{GL_n(F)/G_{g_b}} \frac{1_{g^{-1}K}}{\text{Vol}(K)} (z^{-1}g_b z) dz, \end{aligned}$$

where $g_b \in GL_n(F)$ is an element stably conjugate to $b \in D^\times$, G_{g_b} is the centralizer of g_b in $GL_n(F)$, $\text{Vol}(K)$ (resp. $\text{Vol}(G_{g_b}/\pi^\mathbb{Z})$) is the volume of K (resp. $G_{g_b}/\pi^\mathbb{Z}$ for the induced Haar measure), and $1_{g^{-1}K}$ is the characteristic function of $g^{-1}K$.

3.4 Lefschetz trace formula for the unitary group Rapoport-Zink spaces

In this subsection we study some cohomological applications of the locally finite cell decomposition of the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ studied in section 2, as in the same way for Lubin-Tate spaces.

We first review some basic facts. Fix a prime $l \neq p$. Let $\overline{\mathbb{Q}}_l$ (resp. $\overline{\mathbb{Q}}_p$) be a fixed algebraic closure of \mathbb{Q}_l (resp. \mathbb{Q}_p), and \mathbb{C}_p be the completion of $\overline{\mathbb{Q}}_p$ for its valuation which extends that of \mathbb{Q}_p . For any open compact subgroup $K \subset G(\mathbb{Z}_p)$ and integer $j \geq 0$, the j -th cohomology with compact support of $\mathcal{M}_K \times \mathbb{C}_p$ with coefficient in $\overline{\mathbb{Q}}_l$ is

$$H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_U \varprojlim_n H_c^j(U \times \mathbb{C}_p, \mathbb{Z}/l^n\mathbb{Z}) \otimes \overline{\mathbb{Q}}_l,$$

where the inductive limit is taken over all relatively compact open subsets $U \subset \mathcal{M}_K$, see [23] section 4. Recall in section 2.2 we introduced the group $\Delta = \text{Hom}(X_{\mathbb{Q}_p}^*(G), \mathbb{Z}) \simeq \mathbb{Z}$ and there is a mapping

$$\varkappa : \widehat{\mathcal{M}} \rightarrow \mathbb{Z},$$

with the image Δ' is \mathbb{Z} if n is even and $2\mathbb{Z}$ if n is odd. This mapping satisfies that

$$\varkappa(gx) = \omega_J(g) + \varkappa(x)$$

for all $g \in J_b(\mathbb{Q}_p), x \in \widehat{\mathcal{M}}$. Here $\omega_J : J_b(\mathbb{Q}_p) \rightarrow \Delta$ is defined by $\langle \omega_J(x), \chi \rangle = v_p(i(\chi)(x))$ where $i : X_{\mathbb{Q}_p}^*(G) \rightarrow X_{\mathbb{Q}_p}^*(J_b)$ is the natural morphism between the two groups of \mathbb{Q}_p -rational characters. We denote

$$J_b^1 = \bigcap_{\chi \in X_{\mathbb{Q}_p}^*(G)} \ker |i(\chi)|,$$

which we consider as a subgroup of $J_b(\mathbb{Q}_p)$. Here

$$\begin{aligned} |i(\chi)| : J_b &\longrightarrow \mathbb{Z} \\ x &\mapsto v_p(i(\chi)(x)). \end{aligned}$$

We have a decomposition

$$\mathcal{M}_K = \coprod_{i \in \mathbb{Z}, \text{ in even}} \mathcal{M}_K^i$$

as the case of \mathcal{M} by considering the height of quasi-isogenies, and in fact $\mathcal{M}_K^i = \pi_K^{-1}(\mathcal{M}^i)$ for the projection $\pi_K : \mathcal{M}_K \rightarrow \mathcal{M}$. The group $J_b(\mathbb{Q}_p)$ acts transitively on Δ' and \mathcal{M}_K^0 is stable under the group J_b^1 for the action of $J_b(\mathbb{Q}_p)$. We have the equalities for cohomology groups

$$H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \bigoplus_{i \in \mathbb{Z}, \text{ in even}} H_c^j(\mathcal{M}_K^i \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \text{c-Ind}_{J_b^1}^{J_b(\mathbb{Q}_p)} H_c^j(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l),$$

for the last equality see lemme 4.4.10 of [23]. In fact there are also actions of $G(\mathbb{Q}_p)$ and W_E on Δ' , where E is the reflex local field. Let $(G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E)^1$ be the

subgroup of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ which acts trivially on Δ' . Let K varies as open compact subgroup of $G(\mathbb{Z}_p)$, we have equalities of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E$ -representations

$$\begin{aligned} \varinjlim_K H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) &= \bigoplus_{i \in \mathbb{Z}, \text{ in even}} \varinjlim_K H_c^j(\mathcal{M}_K^i \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \\ &= \text{c-Ind}_{(G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E)^1}^{G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W_E} \varinjlim_K H_c^j(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \end{aligned}$$

see remarque 4.4.11 of loc. cit. In the following we will forget the action of W_E and just consider the cohomology groups as $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ -representations.

The dimension of $H_c^j(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ as $\overline{\mathbb{Q}}_l$ -vector space is infinite. However, as J_b^1 -representation, it is of finite type, see loc. cit. proposition 4.4.13. As in section 2.10, we fix a $\Lambda \in \mathcal{L}_0$ such that $t(\Lambda) = t_{max}$. Recall the subscheme $\mathcal{M}_\Lambda \subset \mathcal{M}_{red}^0$, which is an irreducible component of \mathcal{M}_{red}^0 , and the set of all irreducible components of \mathcal{M}_{red}^0 is exactly $\{g\mathcal{M}_\Lambda = \mathcal{M}_{g\Lambda} | g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)\}$. We have a locally finite covering by open subsets

$$\mathcal{M}^0 = \bigcup_{g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)} gsp^{-1}(\mathcal{M}_\Lambda).$$

We have $J_b^{der}(\mathbb{Q}_p) \subset J_b^1$, and the action of $J_b^{der}(\mathbb{Q}_p)$ on \mathcal{L}_0 naturally extends to an action of J_b^1 , and we still denote by $Stab(\Lambda)$ the stabilizer of Λ in J_b^1 . We set $U := \pi_K^{-1}(sp^{-1}(\mathcal{M}_\Lambda)) \subset \mathcal{M}_K^0$, then by theorem 3.3 (ii) of [41]

$$\dim_{\overline{\mathbb{Q}}_l} H_c^j(U \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) < \infty.$$

We have a J_b^1 -equivariant spectral sequence

$$E_1^{p,q} = \bigoplus_{\substack{\alpha \in J_b^1/Stab(\Lambda) \\ \#\alpha = -p+1}} H_c^q(U(\alpha) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \Rightarrow H_c^{p+q}(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l),$$

where $p \leq 0, 0 \leq q \leq n-1, U(\alpha) = \bigcap_{g \in \alpha} gU$. The J_b^1 action on $E_1^{p,q}$ is

$$\forall h \in J_b^1, h : H_c^q(U(\alpha) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \xrightarrow{\sim} H_c^q(hU(\alpha) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Denote $K_\alpha = \bigcap_{g \in \alpha} gStab(\Lambda)g^{-1}$, then $H_c^q(U(\alpha) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ is a smooth $\overline{\mathbb{Q}}_l$ -representation of K_α , and $E_1^{p,q}$ can be rewritten as

$$E_1^{p,q} = \bigoplus_{[\alpha] \in J_b^1 \setminus (J_b^1/K)^{-p+1}} \text{c-Ind}_{K_\alpha}^{J_b^1} H_c^q(U(\alpha) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Since $(gU)_{g \in J_b^1/Stab(\Lambda)}$ is a locally finite covering of \mathcal{M}_K ,

$$\#\{[\alpha] \in J_b^1 \setminus (J_b^1/Stab(\Lambda))^{-p+1} | U(\alpha) \neq \emptyset\} < \infty,$$

i.e. the above direct sum has just finitely many non zero terms.

Let $\gamma = (h, g) \in G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ be a fixed element with both h and g regular elliptic. Then there is a fundamental system of neighborhoods of 1 in $G(\mathbb{Q}_p)$ consisting of open compact subgroups $K \subset G(\mathbb{Z}_p)$ which are normalized by h . From now on let

$K \subset G(\mathbb{Z}_p)$ be a sufficiently small open compact subgroup such that $hKh^{-1} = K$. Consider the locally finite cell decomposition of \mathcal{M}_K^0

$$\mathcal{M}_K^0 = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ g' \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} ((T.g'\mathcal{D}_K) \cap \mathcal{M}_K^0)$$

if n is odd, and

$$\mathcal{M}_K^0 = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ j=0,1 \\ g' \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} ((T.g_1^j g' \mathcal{D}_K) \cap \mathcal{M}_K^0)$$

if n is even. Here by replacing \mathcal{D} by \mathcal{D}' in the proof of theorem 2.10.8 we can assume \mathcal{D} is compact. Thus the notation \mathcal{D} in this section is a little different from that in previous sections. If n is odd, for any $g' \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda), T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K$,

$$(T.g'\mathcal{D}_K) \cap \mathcal{M}_K^0 \neq \emptyset \Leftrightarrow v_p(detT) = 0,$$

in which case we have

$$T.g'\mathcal{D}_K \subset \mathcal{M}_K^0.$$

If n is even, for any $g' \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda), j = 0, 1, T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K$,

$$(T.g_1^j g' \mathcal{D}_K) \cap \mathcal{M}_K^0 \neq \emptyset \Leftrightarrow -\frac{2}{n}v_p(detT) + j = 0,$$

in which case we have

$$T.g_1^j g' \mathcal{D}_K \subset \mathcal{M}_K^0.$$

Thus we can rewrite the locally finite cell decomposition of \mathcal{M}_K^0 as

$$\mathcal{M}_K^0 = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ v_p(detT)=0 \\ g' \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.g'\mathcal{D}_K$$

if n is odd, and

$$\mathcal{M}_K^0 = \bigcup_{\substack{T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \\ j=0,1, -\frac{2}{n}v_p(detT)+j=0 \\ g' \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.g_1^j g' \mathcal{D}_K$$

if n is even. We will write the cells $T.g'\mathcal{D}_K$ and $T.g_1^j g' \mathcal{D}_K$ as $\mathcal{D}_{T,g',K}$ and $\mathcal{D}_{T,j,g',K}$ respectively. If n is odd, for any $T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K, j \in 2\mathbb{Z}, g' \in J_b^{der}(\mathbb{Q}_p)$ such that $-\frac{2}{n}v_p(detT) + j = 0$, we denote also

$$\emptyset \neq \mathcal{D}_{T,j,g',K} := T.p^{\frac{-j}{2}} g' \mathcal{D}_K \subset \mathcal{M}_K^0.$$

If n is even, for any $T \in G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K, j \in \mathbb{Z}, g' \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)$ such that $v_p(detT) + j = 0$, we denote also

$$\emptyset \neq \mathcal{D}_{T,j,g',K} := \begin{cases} T.p^{\frac{-j}{2}} g' \mathcal{D}_K & (j \text{ even}) \\ T.p^{\frac{1-j}{2}} g_1 g' \mathcal{D}_K & (j \text{ odd}), \end{cases}$$

which is a compact analytic domain in \mathcal{M}_K^0 . Since $(z, z^{-1}) \in G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ acts trivially on \mathcal{M}_K for any $z \in \mathbb{Q}_p^\times$, with these notations we have

$$\mathcal{D}_{T,g',K} = \mathcal{D}_{Tz,z^{-1}g',K}, \quad \mathcal{D}_{T,j,g',K} = \mathcal{D}_{Tz,j+2v_p(z),g',K}, \quad \forall z \in \mathbb{Q}_p^\times$$

For the $\gamma = (h, g)$ above we suppose further $v_p(\det h) + v_p(\det g) = 0$. Then $\gamma(\mathcal{M}_K^0) = \mathcal{M}_K^0$. To describe the action of γ on the cells, we have to introduce some more natural parameter set of cells.

Consider the product $G \times J_b$ as reductive group over \mathbb{Q}_p , then \mathbb{G}_m acts on it through the imbedding $z \mapsto (z, z)$. Let $\mathcal{B}(G \times J_b, \mathbb{Q}_p)$ be the (extended) Bruhat-Tits building of $G \times J_b$ over \mathbb{Q}_p , and $\mathcal{B} = \mathcal{B}(G \times J_b, \mathbb{Q}_p)/\mathbb{Q}_p^\times$ be its quotient by the action of \mathbb{Q}_p^\times through the embedding above. More precisely, if $\mathcal{B}(G^{ad}, \mathbb{Q}_p)$ (resp. $\mathcal{B}(J_b^{ad}, \mathbb{Q}_p)$) is the Bruhat-Tits building of the adjoint group G^{ad} (resp. J_b^{ad}) over \mathbb{Q}_p , which is isomorphic to the Bruhat-Tits building of the derived group G^{der} (resp. J_b^{der}) over \mathbb{Q}_p , then the (extended) Bruhat-Tits building of G (resp. J_b) over \mathbb{Q}_p is $\mathcal{B}(G^{ad}, \mathbb{Q}_p) \times \mathbb{R}$ (resp. $\mathcal{B}(J_b^{ad}, \mathbb{Q}_p)$) over \mathbb{Q}_p . The quotient building \mathcal{B} is

$$\mathcal{B} \simeq (\mathcal{B}(G^{ad}, \mathbb{Q}_p) \times \mathbb{R} \times \mathbb{R} \times \mathcal{B}(J_b^{ad}, \mathbb{Q}_p))/\sim,$$

where

$$(x, s, t, y) \sim (x', s', t', y') \Leftrightarrow$$

$$x = x', y = y', s - s' = t' - t = r(n \text{ odd}) \text{ or } 2r(n \text{ even}), \text{ for some } r \in \mathbb{Z}.$$

Any point of \mathcal{B} can be written in the form $[x, s, t, y] = [x, s', t', y]$ where $s' \in \mathbb{R}, t' \in [0, 1)(n \text{ odd})$ or $[0, 2)(n \text{ even})$ are uniquely determined. The action of $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ on \mathcal{B} is given by

$$\forall (h, g) \times G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p), (h, g)[x, s, t, y] = [h^{-1}x, s + \frac{2}{n}v_p(\det h), t + \frac{2}{n}v_p(\det g), gy].$$

If we consider the right action of $G(\mathbb{Q}_p)$ on $\mathcal{B}(G^{ad}, \mathbb{Q}_p)$ by $xh := h^{-1}x$, we can write $(h, g)[x, s, t, y] = [xh, s + \frac{2}{n}v_p(\det h), t + \frac{2}{n}v_p(\det g), gy]$. The sets of vertices of $\mathcal{B}(G^{ad}, \mathbb{Q}_p)$ and $\mathcal{B}(J_b^{ad}, \mathbb{Q}_p)$ can be described as in section 2.7, as certain sets of lattices in \mathbb{Q}_p^n . By fixing a choice $(\mathbb{Z}_{p^2}^n, \Lambda)$ with $t(\Lambda) = t_{max}$, we can identify the following set

$$(G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)/\text{Stab}_{J_b^{der}(\mathbb{Q}_p)}(\Lambda))/\mathbb{Q}_p^\times$$

with a subset of the set of vertices \mathcal{B}^0 (the quotient by \mathbb{Q}_p^\times of vertices in $\mathcal{B}(G \times J_b, \mathbb{Q}_p)$), such that the projections to $\mathcal{B}(G^{ad}, \mathbb{Q}_p)^0$ and $\mathcal{B}(J_b^{ad}, \mathbb{Q}_p)^0$ are vertices of types determined by $\mathbb{Z}_{p^2}^n, \Lambda$ respectively. In the following we will simply denote $\text{Stab}_{J_b^{der}(\mathbb{Q}_p)}(\Lambda)$ by $\text{Stab}(\Lambda)$ as in section 2.10. For any open compact subgroup $K \subset G(\mathbb{Z}_p)$, we can identify

$$\mathcal{I}_K := (G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p)/K \times J_b(\mathbb{Q}_p)/\text{Stab}(\Lambda))/\mathbb{Q}_p^\times$$

with a subset of $\mathcal{B}^0/K \subset \mathcal{B}/K$. We can write an element of \mathcal{I}_K in the form $[T, g'] = [x, \frac{2}{n}v_p(\det T), \frac{2}{n}v_p(\det g'), y]$ for some $x \in \mathcal{B}(G^{ad}, \mathbb{Q}_p)^0, y \in \mathcal{B}(J_b^{ad}, \mathbb{Q}_p)^0$ uniquely determined by T and g' . There is a map

$$\begin{aligned} G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p)/K \times J_b(\mathbb{Q}_p)/\text{Stab}(\Lambda) &\longrightarrow \mathbb{Z} \\ (T, g') &\mapsto -\frac{2}{n}(v_p(\det T) + v_p(\det g')), \end{aligned}$$

with image \mathbb{Z} if n is even and $2\mathbb{Z}$ if n is odd. Let $(G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \times J_b(\mathbb{Q}_p)/\text{Stab}(\Lambda))^0$ be the inverse image of 0, then \mathbb{Q}_p^\times acts on this subset. We denote

$$\mathcal{I}_K^0 := (G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/K \times J_b(\mathbb{Q}_p)/\text{Stab}(\Lambda))^0 / \mathbb{Q}_p^\times.$$

In fact there is a well defined map

$$\begin{aligned} \varphi : \mathcal{B} &\longrightarrow \mathbb{R} \\ [x, s, t, y] &\mapsto -s - t, \end{aligned}$$

with each fiber stable for the action of K . Then we have $\mathcal{I}_K^0 \subset \varphi^{-1}(0)^0/K$.

Now for any $[T, g'] \in \mathcal{I}_K$, the subset

$$\mathcal{D}_{[T, g'], K} := T.g'\mathcal{D}_K \subset \mathcal{M}_K$$

is well defined, which is a compact analytic domain. If $[T, g'] \in \mathcal{I}_K^0$,

$$\mathcal{D}_{[T, g'], K} \subset \mathcal{M}_K^0.$$

We can rewrite the cell decomposition of \mathcal{M}_K and \mathcal{M}_K^0 as

$$\begin{aligned} \mathcal{M}_K &= \bigcup_{[T, g'] \in \mathcal{I}_K} \mathcal{D}_{[T, g'], K}, \\ \mathcal{M}_K^0 &= \bigcup_{[T, g'] \in \mathcal{I}_K^0} \mathcal{D}_{[T, g'], K}. \end{aligned}$$

For $\gamma = (h, g) \in G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ such that $hKh^{-1} = K$, it induces an action on \mathcal{I}_K by $[T, g'] \mapsto [Th, gg']$. On the other hand the automorphism $\gamma : \mathcal{M}_K \rightarrow \mathcal{M}_K$ induces an action of γ on the cells compatible with its action on the parameter set above :

$$\gamma(\mathcal{D}_{[T, g'], K}) = \mathcal{D}_{[Th, gg'], K}.$$

If we assume as above $v_p(\text{deth}) + v_p(\text{det}g) = 0$ then γ acts on \mathcal{I}_K^0 by $[T, g'] \mapsto [Th, gg']$, and the automorphism $\gamma : \mathcal{M}_K^0 \rightarrow \mathcal{M}_K^0$ induces a compatible action on the cells as above.

Recall there is a metric d on the building \mathcal{B} . If we denote the metrics on $\mathcal{B}(G^{ad}, \mathbb{Q}_p)$ and $\mathcal{B}(J_b^{ad}, \mathbb{Q}_p)$ by d_1, d_2 respectively, for $[x, s, t, y], [x', s', t', y'] \in \mathcal{B}$ with $t, t' \in [0, 1)$ (n odd) or $[0, 2)$ (n even), we have

$$d([x, s, t, y], [x', s', t', y']) = \sqrt{d_1(x, x')^2 + d_2(y, y')^2 + (s - s')^2 + (t - t')^2}.$$

It induces a metric \bar{d} on the quotient space \mathcal{B}/K :

$$\bar{d}(xK, yK) = \inf_{k, k' \in K} d(xk, yk') = \inf_{k \in K} d(xk, y) = \inf_{k \in K} d(x, yk).$$

Since $K \subset G(\mathbb{Z}_p)$ is compact, one checks it easily this is indeed a metric. For any fixed $\rho > 0$ and $o \in \mathcal{B}^0/K$, the closed ball $\mathcal{B}(o, \rho)$ of \mathcal{B}/K contains only finitely many points of the discrete subsets $\mathcal{B}^0/K, \mathcal{I}_K, \mathcal{I}_K^0$. For the $\gamma = (h, g) \in G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ with $hKh^{-1} = K$, the action of γ on the parameter set \mathcal{I}_K is isometric :

$$\bar{d}(\gamma x, \gamma x) = \bar{d}(x, x), \quad \forall x \in \mathcal{I}_K.$$

Note for any $[T_1, g_1], [T_2, g_2] \in \mathcal{I}_K$, $\mathcal{D}_{[T_1, g_1], K} \cap \mathcal{D}_{[T_2, g_2], K} \neq \emptyset$ implies that $v_p(\det T_1) + v_p(\det g_1) = v_p(\det T_2) + v_p(\det g_2)$. If we write

$$[T_1, g_1] = [x_1 K, s_1, t_1, y_1], [T_2, g_2] = [x_2 K, s_2, t_2, y_2]$$

with

$x_1, x_2 \in \mathcal{B}(G^{ad}, \mathbb{Q}_p), y_1, y_2 \in \mathcal{B}(J_b^{ad}, \mathbb{Q}_p), s_1, s_2 \in \mathbb{Z} \subset \mathbb{R}, t_1, t_2 \in \{0\} (n \text{ odd}) \text{ or } \{0, 1\} (n \text{ even})$, (i.e. $\exists r_1, r_2 \in \mathbb{Z}, s.t. \frac{2}{n} v_p(\det T_i) = s_i + nr_i, v_p(\det g_i) = t_i + nr_i, i = 1, 2$), then $s_1 + t_1 = s_2 + t_2, s_1 - s_2 = t_2 - t_1 \in [-1, 1]$, the distance

$$\bar{d}([T_1, g_1], [T_2, g_2]) = \inf_{k \in K} \sqrt{d_1(x_1, x_2 k)^2 + d_2(y_1, y_2)^2 + 2(s_1 - s_2)^2}$$

just depends on $\bar{d}_1(x_1 K, x_2 K)$ and $d_2(y_1, y_2)$: for the induced metric \bar{d}_1 on $\mathcal{B}(G^{ad}, \mathbb{Q}_p)/K$ defined in the same way as \bar{d} , and the metric d_2 on $\mathcal{B}(J_b^{ad}, \mathbb{Q}_p)$.

Proposition 3.4.1. *There exists a constant $c > 0$, which depends only on the locally finite cell decomposition of \mathcal{M}_K , such that for any $[T_1, g_1], [T_2, g_2] \in \mathcal{I}_K$, if $\bar{d}([T_1, g_1], [T_2, g_2]) > c$ then we have*

$$\mathcal{D}_{[T_1, g_1], K} \cap \mathcal{D}_{[T_2, g_2], K} = \emptyset.$$

Proof. We need to prove that, there exists a constant $c > 0$, such that for any $[T, g] \in \mathcal{I}_K$, and any $[T', g'] \in \{[T', g'] \in \mathcal{I}_K | \mathcal{D}_{[T', g'], K} \cap \mathcal{D}_{[T, g], K} \neq \emptyset\}$, we have $\bar{d}([T, g], [T', g']) \leq c$. This proposition is in fact implicitly contained in last section. We just indicate some key points here. First, if $V_K \subset \mathcal{B}$ is any fixed fundamental domain for the action of K , then $\forall x, y \in V_K$ by definition we have $\bar{d}(xK, yK) \leq d(x, y)$. Next, in the proof of proposition 2.10.2 (and also in remark 2.10.7), we see the $T \in G(\mathbb{Z}_p) \setminus G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ such that $\mathcal{D} \cap T \cdot \mathcal{D} \neq \emptyset$ corresponds to elements $(a_1, \dots, a_n) \in X_*(A)_+ \subset \mathbb{Z}_+^n$ such that $a_i \leq C$ for $i = 1, \dots, n$ and C is a constant independent of T . Also by results of Vollaard-Wedhorn which we reviewed in section 2.7, the vertices $\Lambda' = g' \Lambda \in \mathcal{B}(J_b^{der}, \mathbb{Q}_p)^0$ such that $\mathcal{D} \cap g' \mathcal{D} \neq \emptyset$ satisfy $\Lambda' \cap \Lambda \neq \emptyset$, i.e. they share some common neighborhood, therefore there exists some constant C' independent of Λ, Λ' such that $d(\Lambda, \Lambda') \leq C'$ for the metric d on $\mathcal{B}(J_b^{der}, \mathbb{Q}_p)$. Then one can easily deduce the proposition for the case $K = G(\mathbb{Z}_p)$, and the general case will follow as soon as this case holds. \square

From now on we assume $\gamma = (h, g) \in G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ such that both h and g are regular elliptic semi-simple, $hKh^{-1} = K$ and $v_p(\det h) + v_p(\det g) = 0$. Recall the γ -fixed vertices $(\mathcal{B}^0)^\gamma$ is non empty (cf. [71]), we fix a choice of γ -fixed vertex \hat{o} , and let o be its image in \mathcal{B}^0/K . We can take a choice $\hat{o} \in \varphi^{-1}(0)^0$ so that $o \in \varphi^{-1}(0)^0/K$ (see the above φ). Then $\gamma(o) = o$ for the induced action $\gamma : \mathcal{B}^0/K \rightarrow \mathcal{B}^0/K$. For any $\rho > 0$, we consider the subset in \mathcal{I}_K^0 defined by intersection of \mathcal{I}_K^0 with the closed balls of radius ρ with center o in \mathcal{B}^0/K :

$$A_\rho = \{x \in \mathcal{B}^0/K | \bar{d}(o, x) \leq \rho\} \cap \mathcal{I}_K^0,$$

which is a finite set, and $\gamma(A_\rho) = A_\rho$ since $\gamma(o) = o$, \bar{d} is γ -isometric, and $\gamma(\mathcal{I}_K^0) = \mathcal{I}_K^0$. For any finite set $A \subset \mathcal{I}_K^0$, we consider

$$V_A = \bigcup_{[T, g'] \in A} \mathcal{D}_{[T, g'], K},$$

$$U_A = \mathcal{M}_K^0 - \bigcup_{[T, g'] \notin A} \mathcal{D}_{[T, g'], K}.$$

Proposition 3.4.2. *With notations as above, U_A is an open subspace of \mathcal{M}_K^0 , V_A is a compact analytic domain of \mathcal{M}_K^0 , and we have $U_A \subset V_A$.*

Proof. By our assumption in this section, \mathcal{D}_K is compact, thus it is clear that V_A as a finite union of compact analytic domains, is still a compact analytic domain. Since the union of closed subsets $\bigcup_{[T,g'] \notin A} \mathcal{D}_{[T,g],K}$ is locally finite, one can check easily it is closed in \mathcal{M}_K^0 . Thus U_A is open. Finally the inclusion $U_A \subset V_A$ holds since $\mathcal{M}_K^0 = (\mathcal{M}_K^0 - U_A) \cup V_A$ by the cell decomposition of \mathcal{M}_K^0 . \square

For any $\rho > 0$, denote

$$U_\rho = U_{A_\rho}, \quad V_\rho = V_{A_\rho}.$$

Then

$$\gamma(U_\rho) = U_\rho, \quad \gamma(V_\rho) = V_\rho$$

since $\gamma(A_\rho) = A_\rho$. By the above proposition, each U_ρ is a locally compact open subspace of \mathcal{M}_K^0 . We have

$$H_c^j(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_{\rho} H_c^j(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l),$$

with

$$\dim_{\overline{\mathbb{Q}}_l} H_c^j(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) < \infty.$$

We have an induced action of γ on cohomology of U_ρ

$$\gamma : H_c^j(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \rightarrow H_c^j(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Denote

$$H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \sum_{j \geq 0} (-1)^j H_c^j(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l),$$

as an element in some suitable Grothendieck group. We consider the trace of γ on this Euler-Poincaré characteristic

$$Tr(\gamma | H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \sum_{j \geq 0} (-1)^j Tr(\gamma | H_c^j(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

We consider also

$$H_c^*(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \sum_{j \geq 0} (-1)^j H_c^j(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l),$$

as an element in the Grothendieck group of smooth representations of $(G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p))^1$. Let $\text{Fix}(\gamma | \mathcal{M}_K^0 \times \mathbb{C}_p)$ be the set of fixed points of γ on $\mathcal{M}_K^0 \times \mathbb{C}_p$, then each fixed point is simple since the p -adic period mapping $\mathcal{M}_K^0 \rightarrow \mathcal{F}^a$ is étale, and the fixed points of g on \mathcal{F}^a are all simple.

As last subsection, we will use our result of cell decomposition of \mathcal{M}_K^0 , to verify that the action of γ satisfies the conditions of Mieda's theorem 3.13 [60], thus deduce a Lefschetz trace formula for these unitary Rapoport-Zink spaces.

Theorem 3.4.3. *For the fixed $\gamma = (h, g) \in G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$ such that both h and g are regular elliptic, $v_p(\det h) + v_p(\det g) = 0$, there exist a sufficient small open compact subgroup $K' \subset G(\mathbb{Z}_p)$ and a sufficient large number $\rho_0 \gg 0$, such that for all open compact subgroup $K \subset K'$ which is normalized by h and all $\rho \geq \rho_0$, we have the Lefschetz trace formula*

$$\mathrm{Tr}(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\mathrm{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p),$$

which is well defined and finite. Since the right hand side is independent of ρ , we can define

$$\mathrm{Tr}(\gamma|H_c^*(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) := \mathrm{Tr}(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l))$$

for $\rho \gg 0$, and thus

$$\mathrm{Tr}(\gamma|H_c^*(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\mathrm{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p).$$

Proof. Since h is elliptic, for sufficiently small open compact normal subgroup $K \subset G(\mathbb{Z}_p)$,

$$\bar{d}(x, \gamma x) \rightarrow \infty, \text{ when } x \in \mathcal{I}_K^0, \bar{d}(o, x) \rightarrow \infty.$$

In fact, since $o, x \in \varphi^{-1}(0)^0/K$, write $o = [o_1K, -s, s, o_2], x = [x_1K, -t, t, x_2]$ with $o_1, x_1 \in \mathcal{B}(G^{ad}, \mathbb{Q}_p), o_2, x_2 \in \mathcal{B}(J_b^{ad}, \mathbb{Q}_p), s, t \in \{0\}(n \text{ odd})$ or $\{0, 1\}(n \text{ even})$, then

$$\gamma(x) = [x_1hK, \frac{2}{n}v_p(\det h) - t, \frac{2}{n}v_p(\det g) + t, gx_2] = [x_1hK, -t', t', gx_2]$$

with $t' \in \{0\}(n \text{ odd})$ or $\{0, 1\}(n \text{ even})$,

$$\begin{aligned} \bar{d}(x, \gamma x) &= \inf_{k \in K} \sqrt{d_1(x_1, x_1hk)^2 + d_2(x_2, gx_2)^2 + 2(t - t')^2}, \\ \bar{d}(o, x) &= \inf_{k \in K} \sqrt{d_1(x_1k, o_1)^2 + d_2(x_2, o_2)^2 + 2(t - s)^2}. \end{aligned}$$

To prove the above statement, we first work with $\mathcal{B}' := \mathcal{B}(G^{ad}, \mathbb{Q}_p) \times \mathcal{B}(J_b^{ad}, \mathbb{Q}_p)$. Denote the metric on \mathcal{B}' by d' . Since h, g are elliptic, the fixed points set $(\mathcal{B}')^\gamma$ is nonempty and compact. Moreover, for K sufficiently small, $(\mathcal{B}')^{(h, g)} = (\mathcal{B}')^{(h', g)}$ for any $h' \in hK$ (cf. the proof of lemma 12 in [71]). For $o' = (o_1, o_2) \in (\mathcal{B}')^{(h, g)}$ fixed, a simple triangle inequality shows that $d'(x', (\mathcal{B}')^{(h, g)}) \rightarrow \infty$ when $x' = (x_1, x_2) \in \mathcal{B}', d'(o', x') = \sqrt{d_1(x_1, o_1)^2 + d_2(x_2, o_2)^2} \rightarrow \infty$, since $(\mathcal{B}')^{(h, g)}$ is compact. On the other hand, for any automorphism σ of \mathcal{B}' with $(\mathcal{B}')^\sigma \neq \emptyset$, there exists a constant $0 < \theta \leq \pi$ which just depends on \mathcal{B}' and σ , such that

$$d'(x', \sigma x') \geq 2d'(x', (\mathcal{B}')^\sigma) \sin\left(\frac{\theta}{2}\right),$$

see [70] proposition 2.3. In particular, $d'(x', \gamma' x') \rightarrow \infty$ when $d'(o', x') \rightarrow \infty$ for any $h' \in hK, \gamma' = (h', g)$. As K is compact this deduces the above statement.

We have

$$\begin{aligned} \mathcal{M}_K^0 - U_\rho &= \bigcup_{[T, g'] \in \mathcal{I}_K^0 - A_\rho} \mathcal{D}_{[T, g'], K} \\ V_\rho - U_\rho &= \bigcup_{[T, g'] \in A_\rho - A_{\rho-c}} F_{T, g'}, \end{aligned}$$

where for any $[T, g'] \in A_\rho$,

$$F_{T, g'} = \mathcal{D}_{[T, g'], K} \cap (\mathcal{M}_K^0 - U_\rho),$$

which is nonempty if and only if $[T, g'] \in A_\rho - A_{\rho-c}$ by proposition 3.4.1, in which case $F_{T, g'}$ is a compact analytic domain in $\mathcal{D}_{[T, g'], K} \subset V_\rho$. By the above lemma, there exists a sufficiently large $\rho_0 \gg 0$, such that for any $\rho \geq \rho_0$, $[T, g'] \in \mathcal{I}_K^0 - A_{\rho-c}$, we have $\bar{d}([T, g'], \gamma([T, g'])) > c$, and thus by proposition 3.4.1

$$\mathcal{D}_{[T, g'], K} \cap \gamma(\mathcal{D}_{[T, g'], K}) = \emptyset, \quad F_{T, g'} \cap \gamma(F_{T, g'}) = \emptyset \quad (\text{for } [T, g'] \in A_\rho - A_{\rho-c}).$$

To apply Mieda's theorem, we pass to adic spaces. We have the locally finite cell decomposition

$$(\mathcal{M}_K^0)^{ad} = \bigcup_{[T, g'] \in \mathcal{I}_K^0} \mathcal{D}_{[T, g'], K}^{ad},$$

where each cell $\mathcal{D}_{[T, g'], K}^{ad}$ is a quasi-compact open subspace of $(\mathcal{M}_K^0)^{ad}$, $\mathcal{D}_{[T_1, g_1], K}^{ad} \cap \mathcal{D}_{[T_2, g_2], K}^{ad} \neq \emptyset \Leftrightarrow \mathcal{D}_{[T_1, g_1], K} \cap \mathcal{D}_{[T_2, g_2], K} \neq \emptyset$, and the action of γ on $(\mathcal{M}_K^0)^{ad}$ induces the action of the cells in the same way as the case of Berkovich spaces. By [41] 8.2, U_ρ^{ad} is an open subspace of $(\mathcal{M}_K^0)^{ad}$, which is separated, smooth, and partially proper. On the other hand $V_\rho^{ad} = \bigcup_{[T, g'] \in A_\rho} \mathcal{D}_{[T, g'], K}^{ad}$ is a quasi-compact open subspace. Consider the closure $\overline{V_\rho^{ad}} = \bigcup_{[T, g'] \in A_\rho} \overline{\mathcal{D}_{[T, g'], K}^{ad}}$ of V_ρ^{ad} in $(\mathcal{M}_K^0)^{ad}$, which is a proper pseudo-adic space and contained in the quasi-compact space V_ρ^{ad} . We know that $\overline{V_\rho^{ad}}$ (resp. $\overline{\mathcal{D}_{[T, g'], K}^{ad}}$) is the set of all the specializations of the points in V_ρ^{ad} (resp. $\mathcal{D}_{[T, g'], K}^{ad}$). Moreover, γ induces automorphisms $\gamma : \overline{V_\rho^{ad}} \rightarrow \overline{V_\rho^{ad}}, V_\rho^{ad} \rightarrow V_\rho^{ad}, U_\rho^{ad} \rightarrow U_\rho^{ad}$. Since $V_{\rho-c}^{ad} \subset U_\rho^{ad} \subset V_\rho^{ad}$, we have $\overline{V_\rho^{ad}} - V_\rho^{ad} = \bigcup_{[T, g'] \in A_\rho - A_{\rho-c}} (\overline{\mathcal{D}_{[T, g'], K}^{ad}} - \mathcal{D}_{[T, g'], K}^{ad})$. Note

$$\mathcal{D}_{[T_1, g_1], K}^{ad} \cap \mathcal{D}_{[T_2, g_2], K}^{ad} \neq \emptyset \Leftrightarrow \overline{\mathcal{D}_{[T_1, g_1], K}^{ad}} \cap \overline{\mathcal{D}_{[T_2, g_2], K}^{ad}} \neq \emptyset.$$

For $[T, g'] \in A_\rho - A_{\rho-c}$, let $W_{T, g'} = \overline{\mathcal{D}_{[T, g'], K}^{ad}} - \mathcal{D}_{[T, g'], K}^{ad}$. By the paragraph above, for $\rho \gg 0$ we have $\gamma(W_{T, g'}) \cap W_{T, g'} = \emptyset$. One sees the conditions of theorem 3.13 in [60] for V_ρ^{ad} and its compactification $\overline{V_\rho^{ad}}$ hold, i.e.

$$Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\text{Fix}(\gamma|V_\rho^{ad} \times \mathbb{C}_p) = \#\text{Fix}(\gamma|V_\rho \times \mathbb{C}_p).$$

By [42] proposition 2.6 (i) and lemma 3.4, we have

$$Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(U_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) + Tr(\gamma|H_c^*((V_\rho^{ad} - U_\rho^{ad}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

By the paragraph above and the induction argument as the proof of proposition 4.10 in [59], one has $Tr(\gamma|H_c^*((V_\rho^{ad} - U_\rho^{ad}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = 0$. By the comparison theorem on compactly support cohomology of Berkovich spaces and adic spaces, we can conclude

$$Tr(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(U_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\text{Fix}(\gamma|V_\rho \times \mathbb{C}_p).$$

As the reason above, for $\rho \gg 0$, γ permutes the cells $\mathcal{D}_{[T, g'], K}$ for $[T, g'] \notin A_\rho$, we have

$$\text{Fix}(\gamma|V_\rho \times \mathbb{C}_p) = \text{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p).$$

The theorem is thus proved. \square

Similarly we have a remark as the case of Lubin-Tate that

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^i((\mathcal{M}_K^0)^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_{\rho} H_c^i(V_{\rho}^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \forall i \geq 0,$$

and we can work totally in the framework of adic spaces when considering cohomology. But here we have chosen to transfer back the result to Berkovich spaces, so we insist on working with the open subspaces U_{ρ} .

We have a nice fixed points number formula for the quotient space $\mathcal{M}_K/p^{\mathbb{Z}}$. Note if $g \in J_b(\mathbb{Q}_p)$ is a regular elliptic semi-simple element, for any $x \in \text{Fix}(g|\mathcal{F}^a(\mathbb{C}_p))$, there is a element $h_{g,x} \in G(\mathbb{Q}_p)$ which is conjugate to g over $\overline{\mathbb{Q}}_p$ defined by the comparison isomorphism

$$V_p(H_y) \otimes_{\mathbb{Q}_p} B_{dR} \xrightarrow{\sim} V_L \otimes_L B_{dR},$$

where $y \in \pi^{-1}(x)$ is any point in the fiber of the p -adic period mapping $\pi : \mathcal{M} \rightarrow \mathcal{F}^a$.

Corollary 3.4.4. *Let the notations be as in the above theorem. If n is even we assume that $\frac{2}{n}(v_p(\text{deth}) + v_p(\text{det}g))$ is even. Fix compatible Haar measures on $G(\mathbb{Q}_p)$ and the centralizer of $h_{g,x}$, $G_{h_{g,x}} := \{h' \in G(\mathbb{Q}_p) | h'h_{g,x}h'^{-1} = h_{g,x}\}$. Denote the characteristic function of $h^{-1}K$ by $1_{h^{-1}K}$ and the volume of K under the fixed Haar measure by $\text{Vol}(K)$. Then we have the following formula*

$$\text{Tr}(\gamma | H_c^*((\mathcal{M}_K/p^{\mathbb{Z}}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \sum_{x \in \text{Fix}(g|\mathcal{F}^a(\mathbb{C}_p))} \text{Vol}(G_{h_{g,x}}/p^{\mathbb{Z}}) O_{h_{g,x}}\left(\frac{1_{h^{-1}K}}{\text{Vol}(K)}\right),$$

where $\text{Vol}(G_{h_{g,x}}/p^{\mathbb{Z}})$ is the volume of $G_{h_{g,x}}/p^{\mathbb{Z}}$ by the induced Haar measure on $G(\mathbb{Q}_p)/p^{\mathbb{Z}}$,

$$O_{h_{g,x}}\left(\frac{1_{h^{-1}K}}{\text{Vol}(K)}\right) = \int_{G(\mathbb{Q}_p)/G_{h_{g,x}}} \frac{1_{h^{-1}K}}{\text{Vol}(K)}(z^{-1}h_{g,x}z) dz$$

is the orbit integral of $\frac{1_{h^{-1}K}}{\text{Vol}(K)}$ over the conjugate class of $h_{g,x}$.

Proof. We just need count the number of the fixed geometric points set $\text{Fix}(\gamma | (\mathcal{M}_K/p^{\mathbb{Z}}) \times \mathbb{C}_p)$. This can be done by considering the map

$$\mathcal{M}_K/p^{\mathbb{Z}} \rightarrow \mathcal{F}^a \subset \mathbf{P}^{n-1,an}$$

induced the p -adic period mapping, as in theorem 2.6.8 of [77] and [62]. In particular we have

$$\#\text{Fix}(\gamma | (\mathcal{M}_K/p^{\mathbb{Z}}) \times \mathbb{C}_p) = \sum_{x \in \text{Fix}(g|\mathcal{F}^a(\mathbb{C}_p))} \#\{h' \in G(\mathbb{Q}_p)/p^{\mathbb{Z}}K | h'^{-1}h_{g,x}h' = h^{-1}\}.$$

One can then write this number easily in the form as in the corollary. □

For any irreducible smooth representation π of $J_b(\mathbb{Q}_p)$, and any integers $i, j \geq 0$, although the $\overline{\mathbb{Q}}_l$ -vector space

$$H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$$

is of infinity dimension, we have (cf. [23] corollarie 4.3.11)

$$\dim_{\overline{\mathbb{Q}}_l} \text{Ext}_{J_b(\mathbb{Q}_p)}^i(H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi) < \infty,$$

and there are finitely many i, j for which $\text{Ext}_{J_b(\mathbb{Q}_p)}^i(H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi) \neq 0$. Thus we can consider the element

$$J(\pi) := \sum_{i,j \geq 0} (-1)^{i+j} \varinjlim_K \text{Ext}_{J_b(\mathbb{Q}_p)}^i(H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi) \in \text{Groth}_{\overline{\mathbb{Q}}_l}(G(\mathbb{Q}_p)),$$

where $\text{Groth}_{\overline{\mathbb{Q}}_l}(G(\mathbb{Q}_p))$ is the Grothendieck group of smooth $\overline{\mathbb{Q}}_l$ -representations of $G(\mathbb{Q}_p)$. We can thus define a morphism

$$J : \text{Groth}_{\overline{\mathbb{Q}}_l}(J_b(\mathbb{Q}_p)) \rightarrow \text{Groth}_{\overline{\mathbb{Q}}_l}(G(\mathbb{Q}_p))$$

by linearly extension of the above map. If π is supercuspidal, $\varinjlim_K \text{Ext}_{J_b(\mathbb{Q}_p)}^i(H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi) = 0$ unless $j = 0$, in which case it is

$$J(\pi) = \varinjlim_K \text{Hom}_{J_b(\mathbb{Q}_p)}(H_c^*(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi),$$

for

$$H_c^*(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \sum_{j \geq 0} (-1)^j H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

In the following we will work in the opposite direction, that is assume π is now a supercuspidal representation of $G(\mathbb{Q}_p)$, and consider

$$H(\pi) = \sum_{j \geq 0} (-1)^j \text{Hom}_{G(\mathbb{Q}_p)}(\varinjlim_K H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi).$$

Assume that $\text{Hom}_{G(\mathbb{Q}_p)}(\varinjlim_K H_c^j(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \pi)$ is of finite length, which should be always the case, then $H(\pi)$ is a well defined element in $\text{Groth}_{\overline{\mathbb{Q}}_l}(J_b(\mathbb{Q}_p))$.

Corollary 3.4.5. *Let π be a supercuspidal representation of $G(\mathbb{Q}_p)$, $g \in J_b(\mathbb{Q}_p)$ be a regular elliptic semi-simple element. Assume that π is of the form $\pi = c - \text{Ind}_{K_\pi}^{G(\mathbb{Q}_p)} \lambda$, for some open compact modulo center subgroup $K_\pi \subset G(\mathbb{Q}_p)$ and some finite dimensional representation λ of K_π . Then we have*

$$\text{tr}_{H(\pi)}(g) = \sum_{x \in \text{Fix}(g|_{\mathcal{F}^a(\mathbb{C}_p)}} \text{tr}_\pi(h_{g,x}).$$

Proof. One computes exactly as in [62] or the proof of theorem 4.1.3 in [77], using theorem 3.4.3 and corollary 3.4.4. Here we just indicate a point when using the method of [62]. As the notations there, let $T = p^{\mathbb{Z}}$ considered as a subgroup of $G(\mathbb{Q}_p)$, and $K_0 \triangleleft K_\pi$ be an open compact normal subgroup such that λ is trivial on TK_0 . Let $\Xi_\pi = K_\pi/TK_0$ and Ξ_π^e be the subset of classes which contain an elliptic representative. We have a natural bijection $H_c^*((\mathcal{M}_{K_0}/T) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^*(\mathcal{M}_{K_0}^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ if n is odd, and $H_c^*((\mathcal{M}_{K_0}/T) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq \bigoplus_{i=0,1} H_c^*(\mathcal{M}_{K_0}^i \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ if n is even. Then

$$\begin{aligned} \text{tr}_{H(\pi)}(g) &= \text{tr}(g|_{\text{Hom}_{K_\pi}(H_c^*((\mathcal{M}_{K_0}/T) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \lambda)}) \\ &= \frac{1}{\#\Xi_\pi} \sum_{h \in \Xi_\pi^e} \text{tr}((h, g)|_{H_c^*(U_{\rho_h} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)}) \text{tr}(h^{-1}|\lambda) \end{aligned}$$

if n is odd, and

$$= \frac{1}{\#\Xi_\pi} \sum_{h \in \Xi_\pi^e} \text{tr}((h, g) | \oplus_{i=0,1} H_c^*(U_{\rho_h}^i \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) \text{tr}(h^{-1} | \lambda)$$

if n is even. The remaining computations are just using theorem 3.4.3 and corollary 3.4.4. \square

As [77] and [62], we will hope to use the above corollary to prove the realization of the local Jacquet-Langlands correspondence for smooth representations of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ (for n even) in the cohomology of our Rapoport-Zink spaces. By the methods of loc. cit., we are just reduced to problems of classification of L -packets of $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$, and characterization of the local Jacquet-Langlands correspondence between smooth representations of them. Considering the recent progress on classification of L -packets (global and local) for unitary groups, for example see [63] for the local case which concerns us, it seems that these can be achieved soon.

Références

- [1] V.G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1990.
- [2] V.G. Berkovich, *Étale cohomology for non-archimedean analytic spaces*, Publ. Math. IHES 78 (1993), 5-161.
- [3] V.G. Berkovich, *Vanishing cycles for formal schemes*, Invent. Math. 115 (1994), 539-571.
- [4] V.G. Berkovich, *Vanishing cycles for formal schemes II*, Invent. Math. 125 (1996), 367-390.
- [5] P. Berthelot, *Cohomologie rigide et cohomologie rigide à support propre, Première partie*, Prépublication IRMAR 96-03, 1996.
- [6] S. Bosch, *Lectures on formal and rigid geometry*, <http://www.math.uni-muenster.de/sfb/about/publ/heft378.pdf>
- [7] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean analysis*, volume 261 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1984.
- [8] S. Bosch, W. Lütkebohmert, *Formal and rigid geometry I, Rigid spaces*, Math. Ann. 295(2) (1993), 291-317.
- [9] P. Boyer, *Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale*, Invent. Math. 138 (1999), 573-629.
- [10] O. Bültel, T. Wedhorn, *Congruence relations for Shimura varieties associated to some unitary groups*, J. Inst. Math. Jussieu 5 (2006), 229-261.
- [11] H. Carayol, *Non-abelian Lubin-Tate theory*, in "Automorphic Forms, Shimura varieties, and L-Functions" Vol 2, Perspect. Math. 11, Academic Press, 1990, 15-39.
- [12] C.-L. Chai, *Newton polygons as lattice points*, Amer. Journal of Math., Vol 122, No 5 (2000), 967-990.
- [13] C.-L. Chai, *Hecke orbits on Siegel modular varieties*, in Geometric methods in algebra and number theory, Progr. in Math. 235, Birkhäuser, Boston, MA, 2004, 71-107.
- [14] C.-L. Chai, *Hecke orbits as Shimura varieties in positive characteristic*, in Proc. ICM Madrid 2006, Vol 2, 295-312, European Math. Soc. 2006.
- [15] M. Chen, *Le morphisme déterminant pour les espaces de modules de groupes p -divisibles*, Ph.D. thesis of Université Paris-Sud XI, 2011.
- [16] J.-F. Dat, S. Orlik, M. Rapoport, *Period Domains over finite and local fields*, Cambridge Tracts in Mathematics, 183. Cambridge University Press, Cambridge, 2010.
- [17] A.J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Inst. Hautes Études Sci. Publ. Math. 82 (1995), 5-96.
- [18] A.J. de Jong, *Étale fundamental groups of non-Archimedean analytic spaces*, Compo. Math. 97 (1995), 89-118.
- [19] A.J. de Jong, M. van der Put, *Étale cohomology of rigid analytic spaces*, Doc. Math. 1, No. 01 (1996) 1-56.
- [20] G. Faltings, *The trace formula and Drinfeld's upper halfplane*, Duke Math. 76 (1994), No.2, 467-481.

- [21] G. Faltings, *A relation between two moduli spaces studied by V.G. Drinfeld*, in “Algebraic Number Theory and Algebraic Geometry”, Contemp. Math. 300, Amer. Math. Soc., Providence, 2002, 115-129.
- [22] G. Faltings, *Coverings of p -adic period domains*, Journal für die reine und angewandte Mathematik 643 (2010), 111-139.
- [23] L. Fargues, *Cohomologie des espaces de modules de groupes p -divisibles et correspondances de Langlands locales*, in “Variétés de Shimura, espaces de Rapoport-Zink de correspondances de Langlands locales”, Astérisque 291 (2004), 1-199.
- [24] L. Fargues, *Application de Hodge-Tate duale d’un groupe de Lubin-Tate, immeuble de Bruhat-Tits du groupe linéaire et filtrations de ramification*, Duke Math. 140, No.3, (2007), 499-590.
- [25] L. Fargues, *L’isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques*, in L’isomorphisme entre les tours de Lubin-Tate et de Drinfeld, volume 262 of Progr. Math. Birkhäuser, Basel, 2008, 1-325.
- [26] L. Fargues, *La filtration de Harder-Narasimhan des schémas en groupes finis et plats*, Journal für die reine und angewandte Mathematik 645 (2010), 1-39.
- [27] L. Fargues, *Théorie de la réduction pour les groupes p -divisibles*, prépublications, [http ://www-irma.u-strasbg.fr/~fargues/Prepublications.html](http://www-irma.u-strasbg.fr/~fargues/Prepublications.html)
- [28] Notes of a course about the filtrations of finite group schemes given by L. Fargues in the Galois Trimester at IHP, 2010. [http ://www-irma.u-strasbg.fr/~fargues/Notes.html](http://www-irma.u-strasbg.fr/~fargues/Notes.html)
- [29] J.-M. Fontaine, *Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate*, Astérisque 65 (1979), 3-80.
- [30] J.-M. Fontaine, M. Rapoport, *Existence de filtrations admissibles sur des isocristaux*, Bull. Soc. Math. de France 133 (2005), no.1, 73-86.
- [31] K. Fujiwara, *Theory of tubular neighborhood in étale topology*, Duke Math. 80, No.1, (1995), 15-57.
- [32] K. Fujiwara, *Rigid geometry, Lefschetz-Verdier trace formula, and Deligne’s conjecture*, Invent. Math. 127 (1997).
- [33] M. Harris, *Supercuspidal representations in the cohomology of Drinfeld upper half spaces; elaboration of Carayol’s program*, Invent. Math. 129 (1997), No.1, 75-119.
- [34] M. Harris, *Local Langlands correspondences and vanishing cycles on Shimura varieties*, Proceedings of the E. C. M., (2000) 407-427.
- [35] M.Harris, *The local Langlands correspondence : notes of (half) a course at the IHP spring 2000*, in “Formes Automorphique I”, Astérisque 298, 2005, 17-145.
- [36] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Ann. of Math. Stud. 151, Princeton Univ. Press, 2001.
- [37] U. Hartl, *On period spaces of p -divisible groups*, C. R. Math. Acad. Sci. Paris Ser. I 346 (2008), 1123-1128.
- [38] U. Hartl, *On a conjecture of Rapoport-Zink*, Preprint, arXiv :math/0605254.
- [39] R. Huber, *Continuous valuations*, Math. Z. 212 (1993), no.3, 455-477.
- [40] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. 217 (1994), no.4, 513-551.

-
- [41] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Braunschweig 1996.
- [42] R. Huber, *A comparison theorem for l -adic cohomology*, Compositio Math. 112 (1998), no.2, 217-235.
- [43] R. Huber, *A finiteness result for the compactly supported cohomology of rigid analytic varieties*, J. Algebraic Geom. 7 (1998), no.2, 313-357.
- [44] B.H. Gross, M.J. Hopkins, *Equivariant vector bundles on the Lubin-Tate moduli space*, In Topology and representation theory, volume 158 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1994, 23-88.
- [45] D. Kazhdan, *Cuspidal geometry of p -adic groups*, J. Analyse Math. 47 (1986), 1-36.
- [46] N.M. Katz, *Slope filtration of F -cystals*. In *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I*, Astérisque 63 (1979), 113-163.
- [47] N. Imai, Y. Mieda, *Compactly supported cohomology and nearby cycle cohomology of open Shimura varieties of PEL type*, Preprint.
- [48] T. Ito, Y. Mieda, *Cuspidal representations in the l -adic cohomology of the Rapoport-Zink space for $GSp(4)$* , Preprint.
- [49] R.E. Kottwitz, *Isocrystals with additional structures*, Compositio Math. 56(2) (1985), 201-220.
- [50] R.E. Kottwitz, *Shimura varieties and λ -adic representations*, in “Automorphic Forms, Shimura varieties, and L-Functions” Vol 1, Perspect. Math. 10, Academic Press, 1990, 161-209.
- [51] R.E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. 5 (1992), 373-444.
- [52] R.E. Kottwitz, *Isocrystals with additional structures II*, Compositio Math. 109(3) (1997), 255-339.
- [53] S. Kudla, M. Rapoport, *Special cycles on unitary Shimura varieties I : unramified local theory*, Invent. Math. 184, 2011.
- [54] S. Kudla, M. Rapoport, *Special cycles on Siegel threefold and derivatives of Eisenstein series*, Ann. Scient. Éc. Norm. Sup. 33 (2000), 695-756.
- [55] E. Mantovan, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. 129 (2005), 573-610.
- [56] E. Mantovan, *On non-basic Rapoport-Zink spaces*, Ann. Sci. Éc. Norm. Sup. 41(5) (2008), 671-716.
- [57] E. Mantovan, E. Viehmann, *On the Hodge-Newton filtration for p -divisible O -modules*, Math. Z. 266 (2010), no. 1, 193-205.
- [58] W. Messing, *The crystals associated to Barsotti-Tate groups : with applications to abelian schemes*, LNM 264, Springer, 1972.
- [59] Y. Mieda, *Non-cuspidality outside the middle degree of l -adic cohomology of the Lubin-Tate tower*, Adv. Math. 225 (2010), No.4, 2287-2297.
- [60] Y. Mieda, *Lefschetz trace formula for open adic spaces*, Preprint, arXiv :1101.1720.
- [61] Y. Mieda, *Lefschetz trace formula and l -adic cohomology of Lubin-Tate tower*, Mathematical Research Letters 19 (2012), 95-107.

- [62] Y. Mieda, M. Strauch, *Jacquet-Langlands character relations in the cohomology of a Rapoport-Zink tower for $GSp(4)$* , working paper, <http://mypage.iu.edu/~mstrauch/papers/index.htm>
- [63] C. Moeglin, *Classification et changement de base pour les séries discrètes des groupes unitaires p -adiques*, Pacific J. Math. 233 (2007), no.1, 159-204.
- [64] M. Rapoport, *Non-archimedean period domains*, in “Proceedings of the International Congress of Mathematicians” Vol 1, (Zürich, 1994), Birkhäuser, Basel, 1995, 423-434.
- [65] M. Richartz, M. Rapoport, *On the classification and specialization of F -isocrystals with additional structure*. Compositio Math. 103(2) (1996), 153-181.
- [66] M. Rapoport, T. Zink, *Period spaces for p -divisible groups*, Ann. of Math. Stud. 141, Princeton Univ. Press, 1996.
- [67] B. Rémy, A. Thuillier, A. Werner, *Bruhat-Tits theory from Berkovich’s point of view. I : realizations and compactifications of buildings*, Ann. Sci. de l’ENS 43 (2010), 461-554.
- [68] B. Rémy, A. Thuillier, A. Werner, *Bruhat-Tits buildings and analytic geometry*, Preprint, arXiv :1110.1362.
- [69] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Ann. of Math. Stud. 123, Princeton Univ. Press, 1990.
- [70] G. Rousseau, *Exercices métriques immobiliers*, Indag. Mathem., N.S., 12(3) (2001), 383-405.
- [71] P. Schneider, U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Publ. Math. IHES, 85 (1997), 97-191.
- [72] X. Shen, *On the Hodge-Newton filtration for p -divisible groups with additional structures*, Preprint, arXiv :1203.2541.
- [73] X. Shen, *On the Lefschetz trace formula for Lubin-Tate spaces*, Preprint, arXiv :1203.2539.
- [74] X. Shen, *Cell decomposition of some unitary group Rapoport-Zink spaces*, Preprint, arXiv :1203.2537.
- [75] S.-W. Shin, *On the cohomology of Rapoport-Zink spaces of EL type*, Preprint, to appear in Amer. J. Math.
- [76] S. Stevens, *The supercuspidal representations of p -adic classical groups*, Invent. Math. 172 (2008), 289-352.
- [77] M. Strauch, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. 217 (2008), No.3, 889-951.
- [78] E. Viehmann, *Moduli spaces of p -divisible groups*, J. Algebraic Geom., 17(2) (2008), 341-374.
- [79] E. Viehmann, *The global structure of moduli spaces of polarized p -divisible groups*, Doc. Math. 13, (2008), 825-852.
- [80] I. Vollaard, *The supersingular locus of the Shimura variety of $GU(1,s)$* , Canad. J. Math. 62 (3), (2010), 668-720.
- [81] I. Vollaard, T. Wedhorn, *The supersingular locus of the Shimura variety of $GU(1,n-1)$ II*, Invent. Math. 184 (2011), 591-627.

-
- [82] T. Wedhorn, *Ordinariness in good reductions of Shimura varieties of PEL-type*, Ann. Sci. de l'ENS 32, (1999), 575-618.
- [83] T. Wedhorn, *Adic spaces*, notes available at <http://www2.math.uni-paderborn.de/fileadmin/Mathematik/People/wedhorn/Lehre/AdicSpaces.pdf>
- [84] T. Yoshida, *On non-abelian Lubin-Tate theory via vanishing cycles*, in “Algebraic and arithmetic structures of moduli spaces (Sapporo 2007)”, Vol. 58, Adv. Stud. Pure Math., 361-402. Math. Soc. Japan, Tokyo, 2010.