



Institut de Recherche  
sur les Phénomènes  
Hors Équilibre



# Harmonic forcing in rotating flows: zonal flow, inertial waves, instabilities

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01/02/2013

# Geo-/Astrophysical context

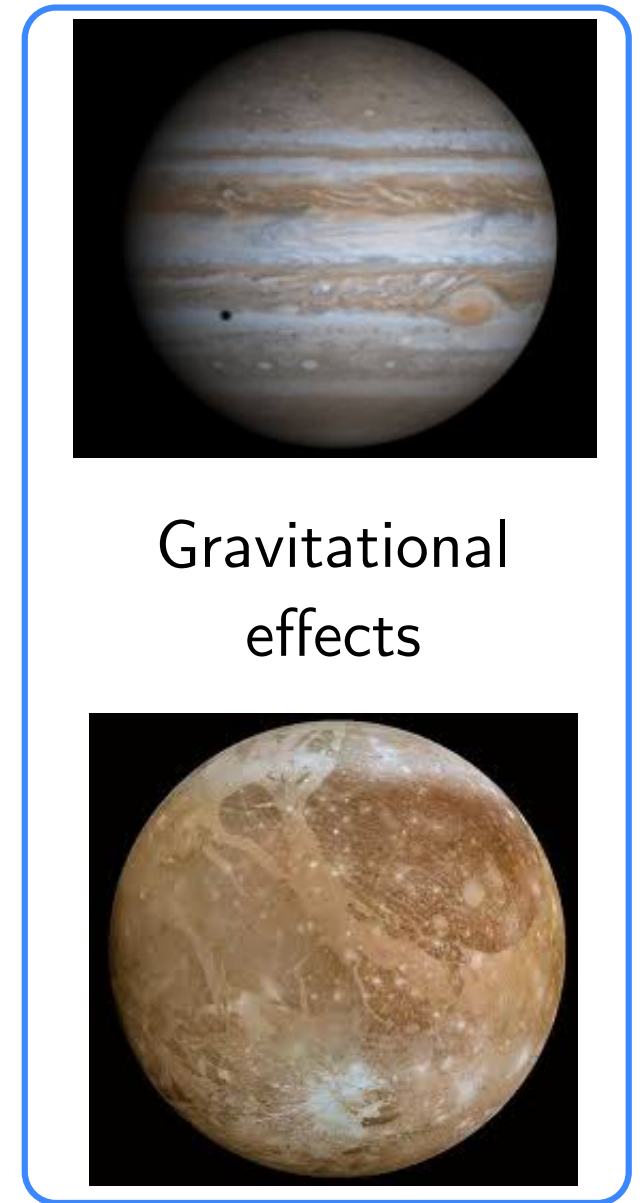
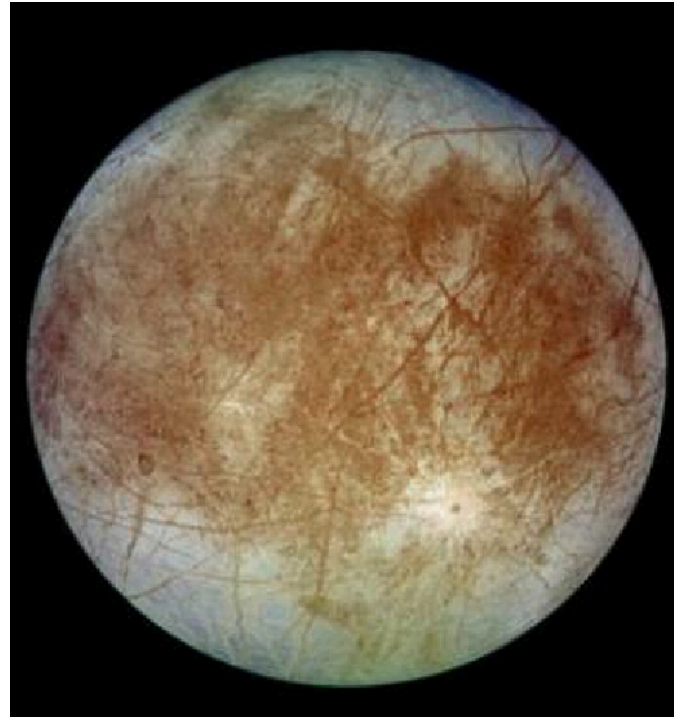
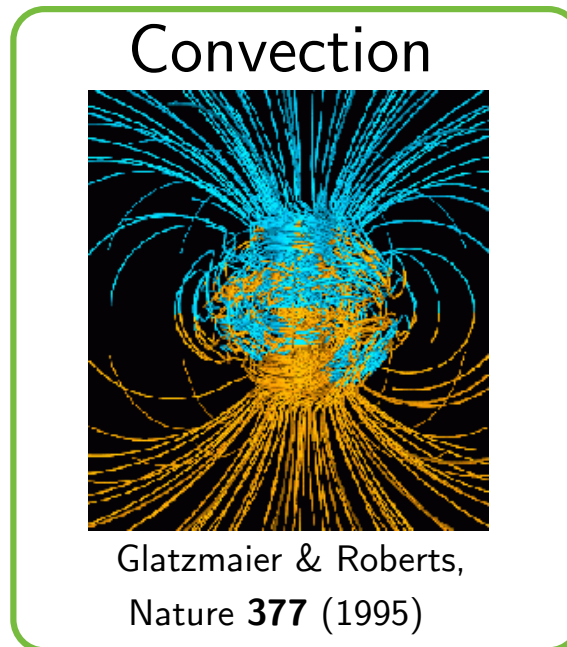
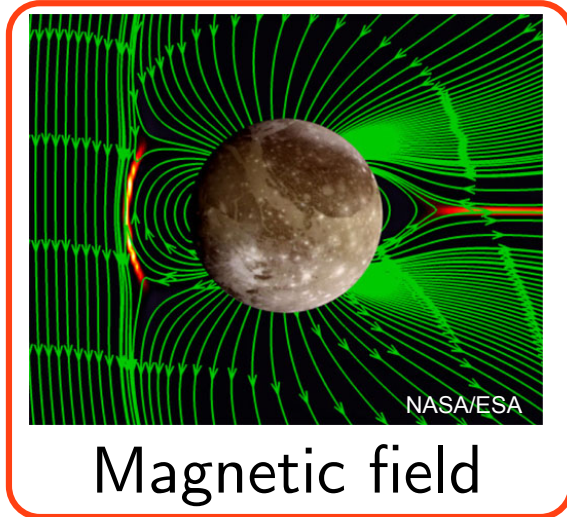
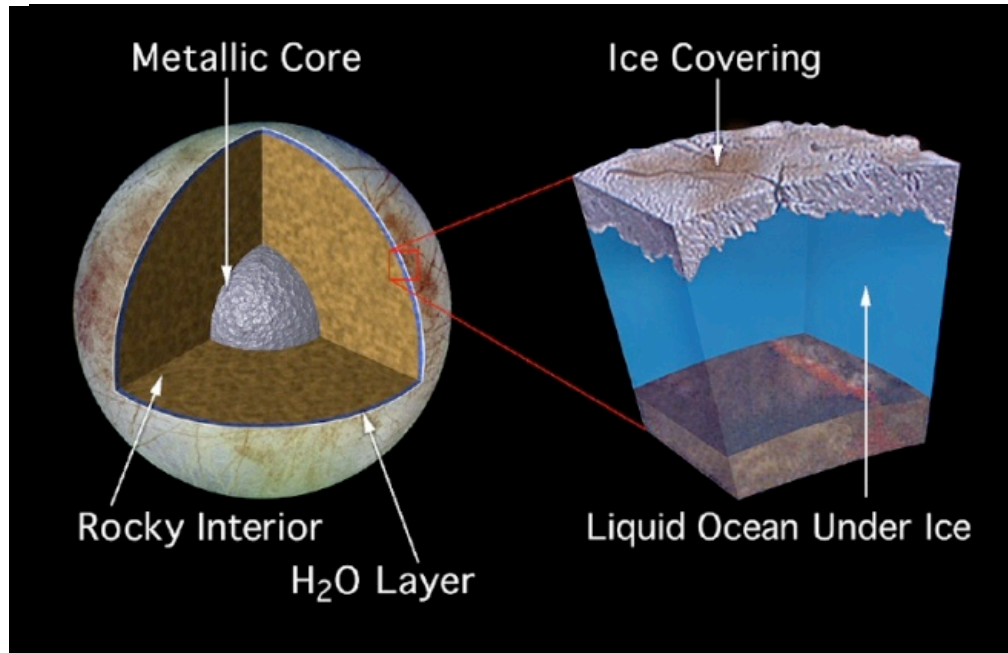


Image Credits: NASA/JPL



# Applications



Generation of zonal winds in atmosphere and core  
(alternative mechanism for patterns, dynamo,...)



Investigation of internal structure  
(Europa Jupiter System Mission :  
exploration of Jupiter's moons)

**Model** : a rotating incompressible fluid enclosed in a container  
**Study** of the generated flow  
**Evaluate** the role of each process

Image Credits: NASA/JPL

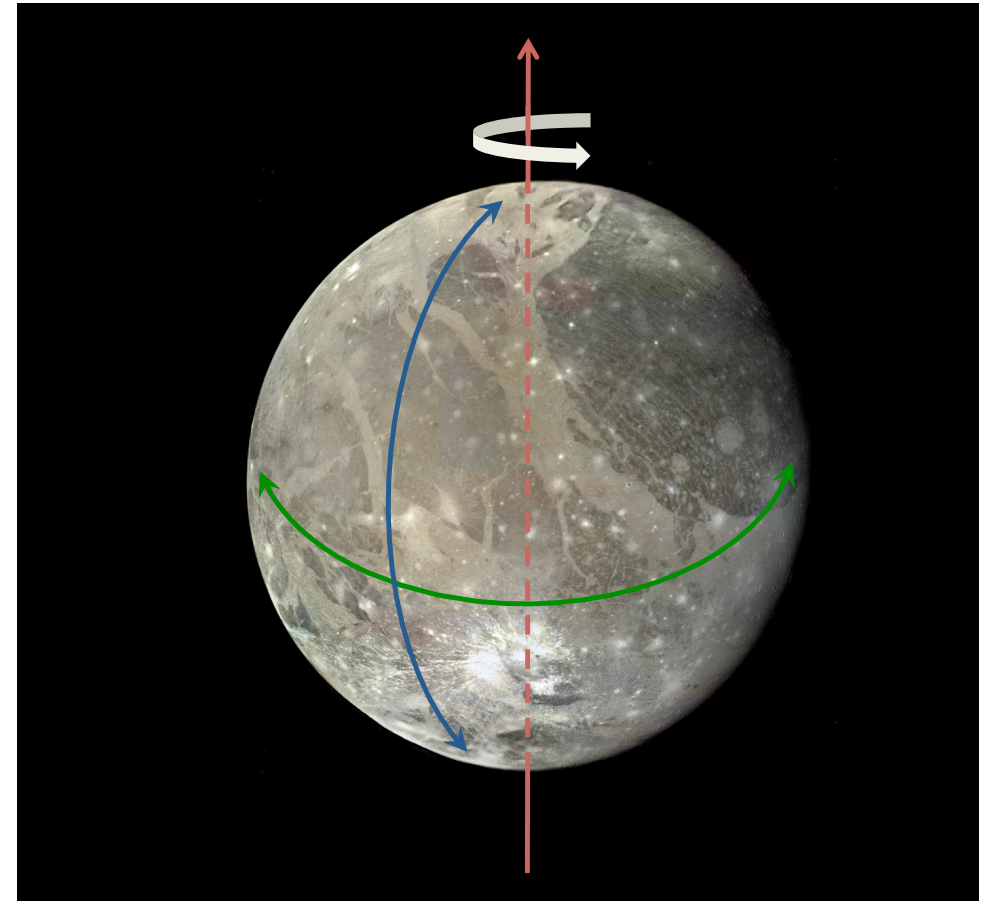
# Longitudinal and latitudinal libration

Rotation of a satellite at the mean angular velocity  $\Omega_0$

Longitudinal libration: small oscillation of the **rotation rate** of container

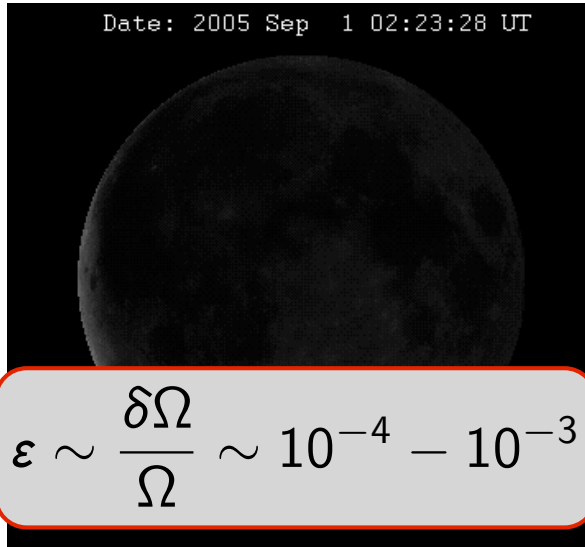
$$\Omega(t) = \Omega_0 + \delta\Omega \cos(\omega t)$$

Latitudinal libration: small oscillation of the **rotation axis** of container



# Harmonic Forcings

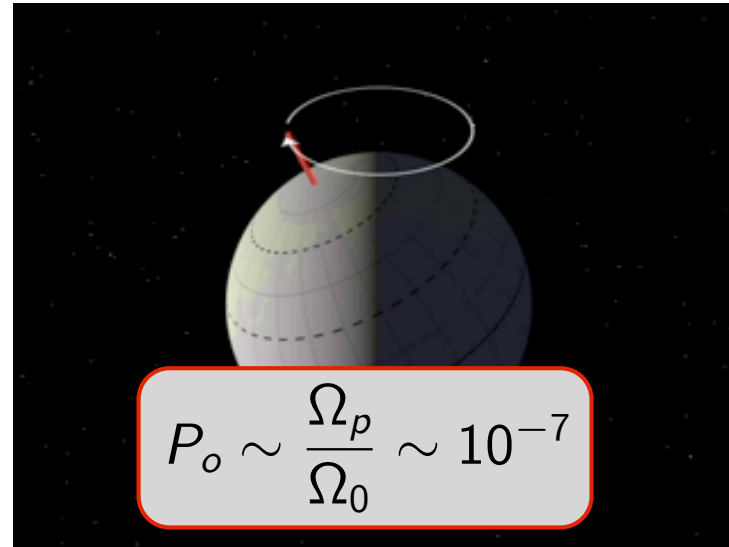
Animation Credits: NASA/JPL



$$\varepsilon \sim \frac{\delta\Omega}{\Omega} \sim 10^{-4} - 10^{-3}$$

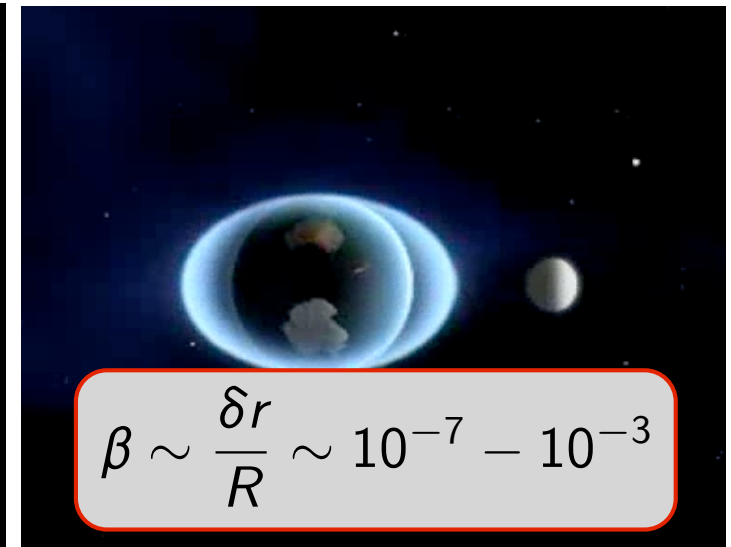
**Longitudinal**/latitudinal  
libration (m=0)

$$\Omega(t) = \Omega_0 + \delta\Omega \cos(\omega t)$$



$$P_o \sim \frac{\Omega_p}{\Omega_0} \sim 10^{-7}$$

Precession (m=1)



$$\beta \sim \frac{\delta r}{R} \sim 10^{-7} - 10^{-3}$$

**Tides** (m=2)

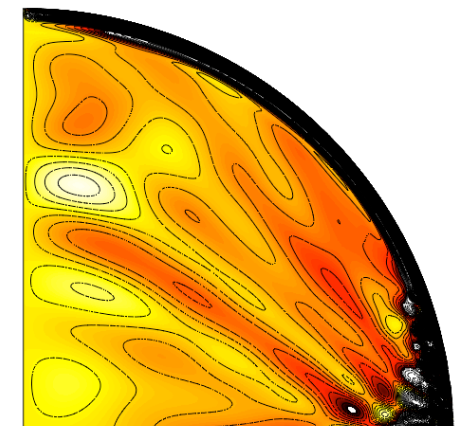
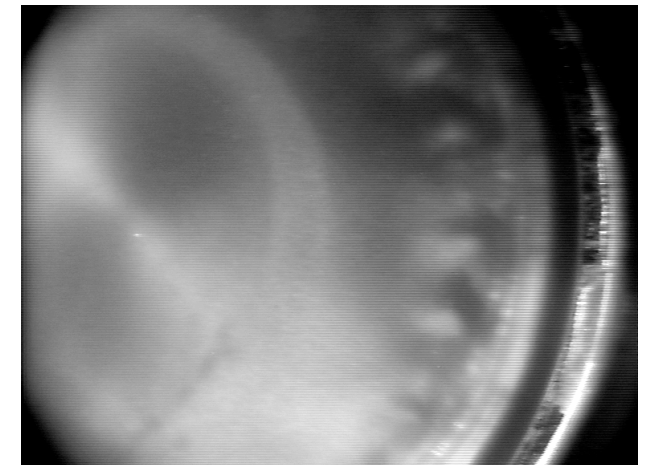
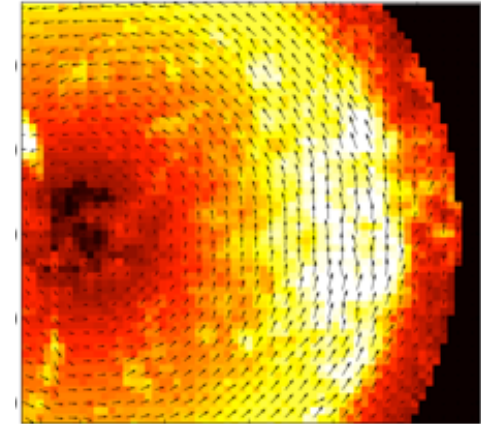
$$R_{ext} = 1 + \frac{\beta}{2} \cos(2\Phi)$$

Small amplitudes  $\longrightarrow$  Negligible forcings?

But can drive inertial waves & instabilities...

Fluid flows driven by harmonic forcing ?

- 1 - Theory of rotating flows
- 2 - Methods of investigations
- 3 - Mean zonal flow
- 4 - Resonance of inertial modes
- 5 - Spontaneous generation of inertial waves





- 1 - Theory of rotating flows
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# Navier-Stokes equation

Navier-Stokes equation in the rotating frame at  $\Omega = \Omega_0 \mathbf{k}$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \Omega_0 \times \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \Delta \mathbf{u}$$

Timescale:  $\Omega_0^{-1}$

Lengthscale:  $R$

Velocity scale:  $U$

associated with  
the forcing  $(\varepsilon, \beta)$

Dimensionless form:

$$\frac{\partial \mathbf{u}}{\partial t} + R_o (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \mathbf{k} \times \mathbf{u} = -\nabla p + E \Delta \mathbf{u}$$

$$R_o = \frac{\text{NL effects}}{\text{Coriolis force}} = \frac{U}{\Omega_0 R}$$

$$E = \frac{\text{viscous effects}}{\text{Coriolis force}} = \frac{\nu}{\Omega_0 R^2}$$

Geo-/astrophysical applications:  $R_o \ll 1$  &  $E \ll 1$

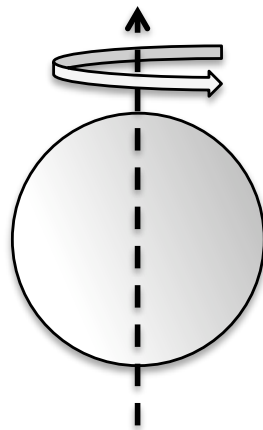
# From planets to laboratory

## Amplitude of the forcing $R_0 \sim \varepsilon, \beta$

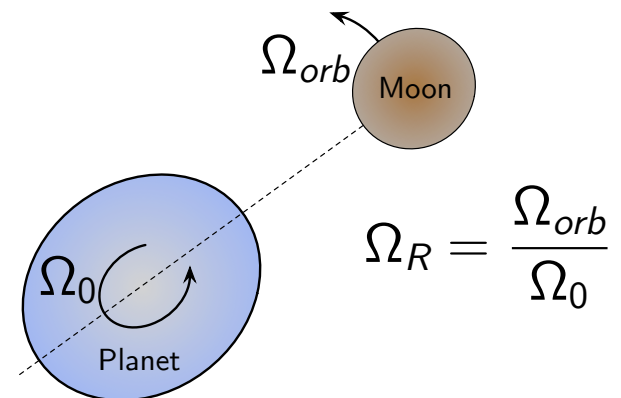
	Libration	Tides
Planets	$\varepsilon \sim 10^{-4} - 10^{-3}$	$\beta \sim 10^{-6} - 10^{-4}$
Laboratory	$\varepsilon \sim 0.02 - 0.1$	$\beta \sim 0.02 - 0.08$

## Frequency of the forcing

	Libration	Tides
Planets	$\omega \sim 10^{-3} - 1$	$\Omega_R \sim 0 - 2$
Laboratory	$\omega \sim 10^{-2} - 1$	$\Omega_R \sim 0 - 1$



$$\Omega(t) = \Omega_0 [1 + \varepsilon \cos(\omega t)]$$



$$R_{ext} = 1 + \frac{\beta}{2} \cos(2\Phi)$$

# From planets to laboratory

## Amplitude of the forcing $R_0 \sim \varepsilon, \beta$

	Libration	Tides
Planets	$\varepsilon \sim 10^{-4} - 10^{-3}$	$\beta \sim 10^{-6} - 10^{-4}$
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## Frequency of the forcing

	Libration	Tides
Planets	$\omega \sim 10^{-3} - 1$	$\Omega_R \sim 0 - 2$
Laboratory	$\omega \sim 10^{-2} - 1$	$\Omega_R \sim 0 - 1$

## Ekman number

$$E = \frac{\nu}{\Omega_0 R^2}$$

Planets

$$E \sim 10^{-15} - 10^{-12}$$

Laboratory

$$E \sim 10^{-6} - 10^{-4}$$



# Inertial waves

$$\frac{\partial \mathbf{u}}{\partial t} + R_o (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + E \Delta \mathbf{u} - 2 \mathbf{k} \times \mathbf{u}$$

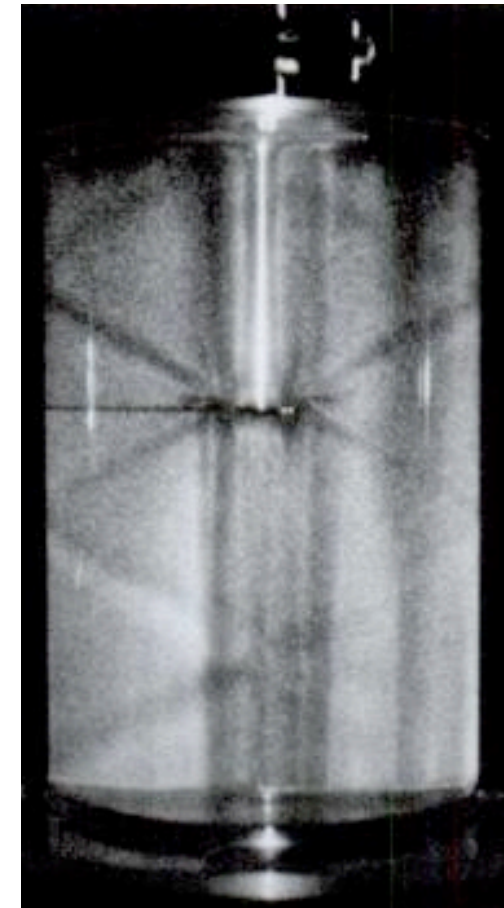
Coriolis force: **restoring force**

## Plane inertial waves

$$\mathbf{u} = \mathbf{u}_0 e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$\text{Dispersion relation: } \omega = 2 \frac{k_z}{|\mathbf{k}|} = 2 \cos \gamma$$

No inertial waves for  $\omega > 2$



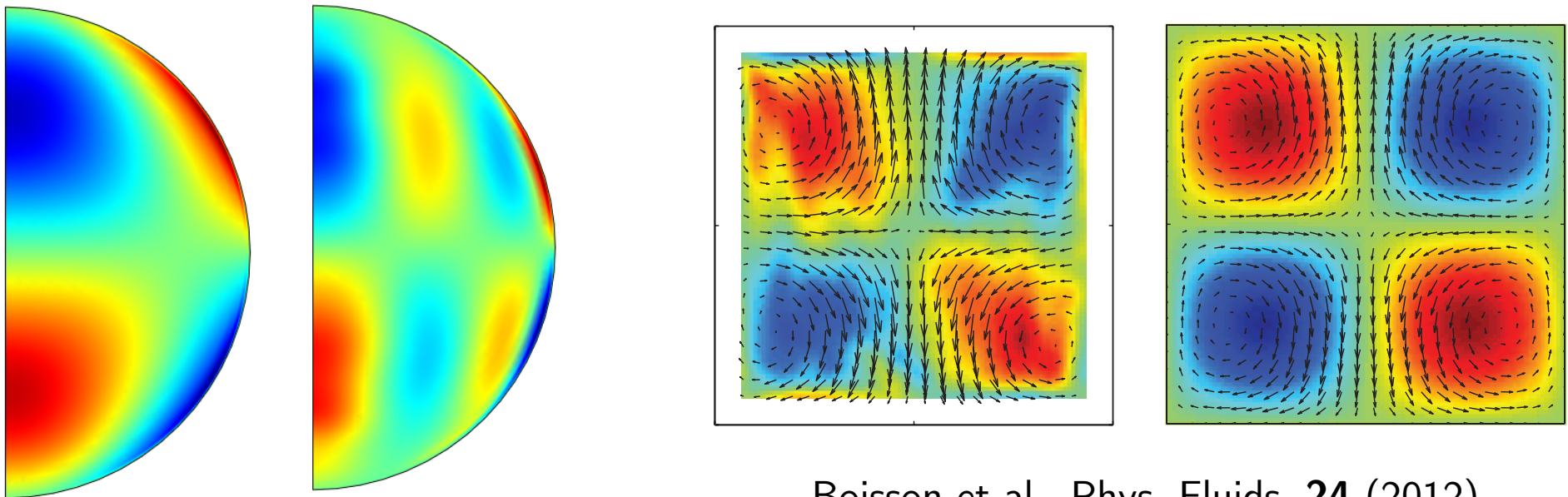
Görtler (1957)

Inertial waves are usually damped by viscosity but can be forced...

Bounded volume (sphere, cylinder)

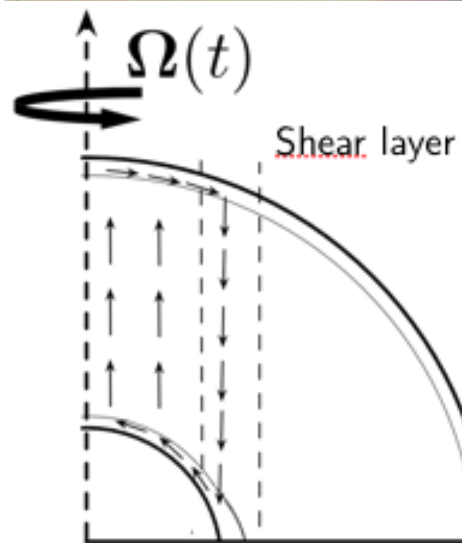
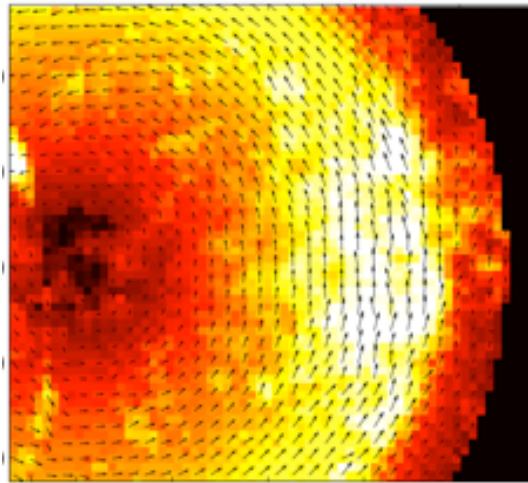
wave becomes phase-coherent,  
multiple reflection : **resonance**

## inertial modes

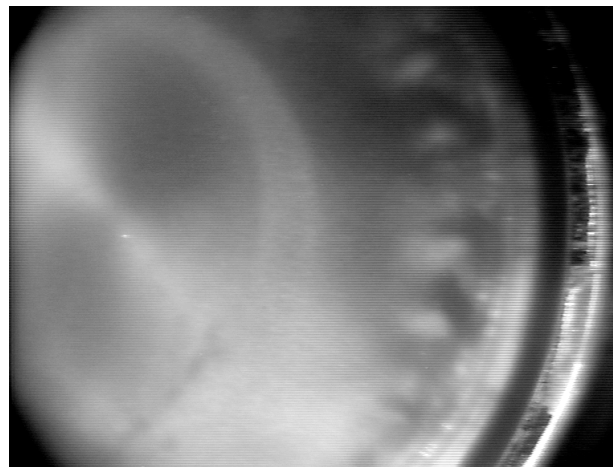
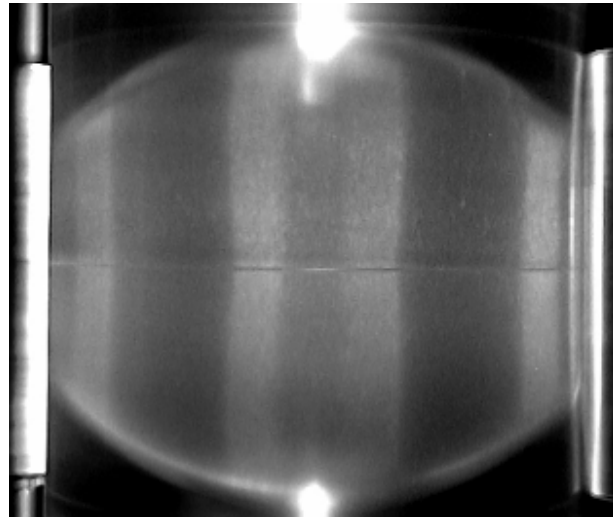


Boisson et al., Phys. Fluids, **24** (2012)

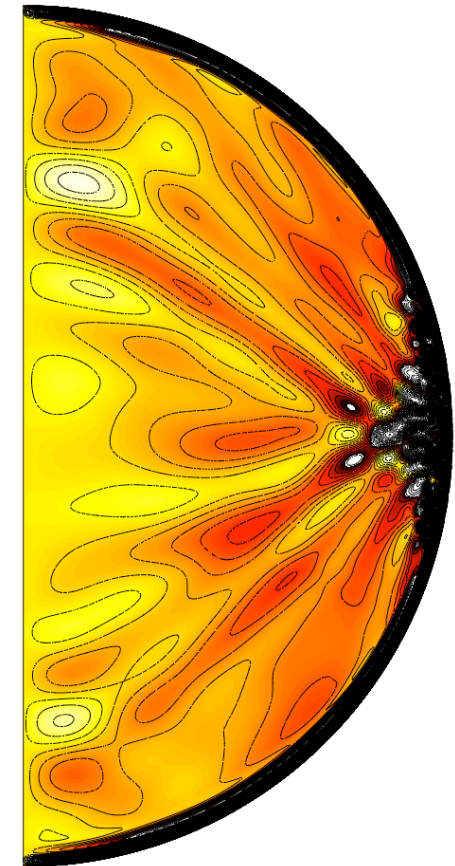
# Key features



Mean zonal flow generation by Ekman layers



Resonance of inertial modes and intense zonal jet



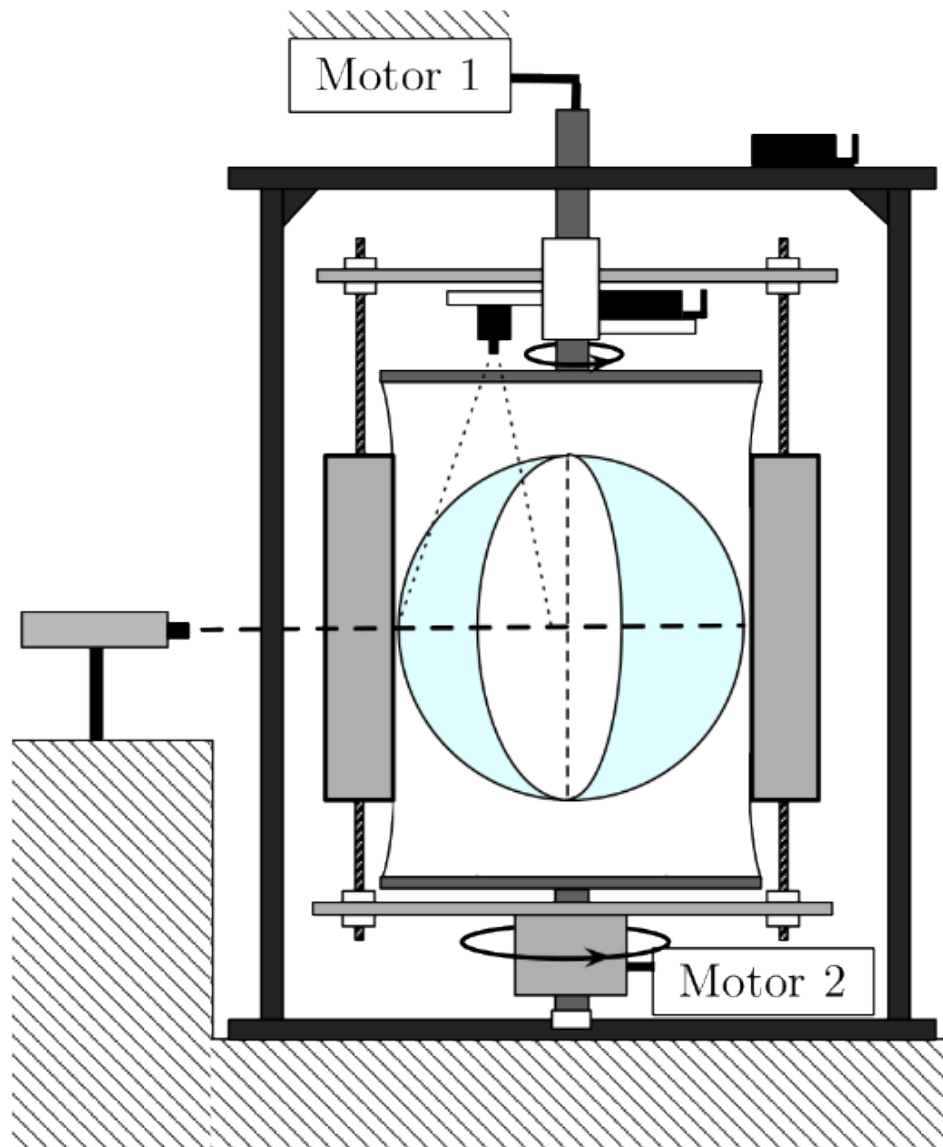
Instabilities and spontaneous generation of inertial waves

- 1 - Theory of rotating flows
- 2 - Methods of investigations
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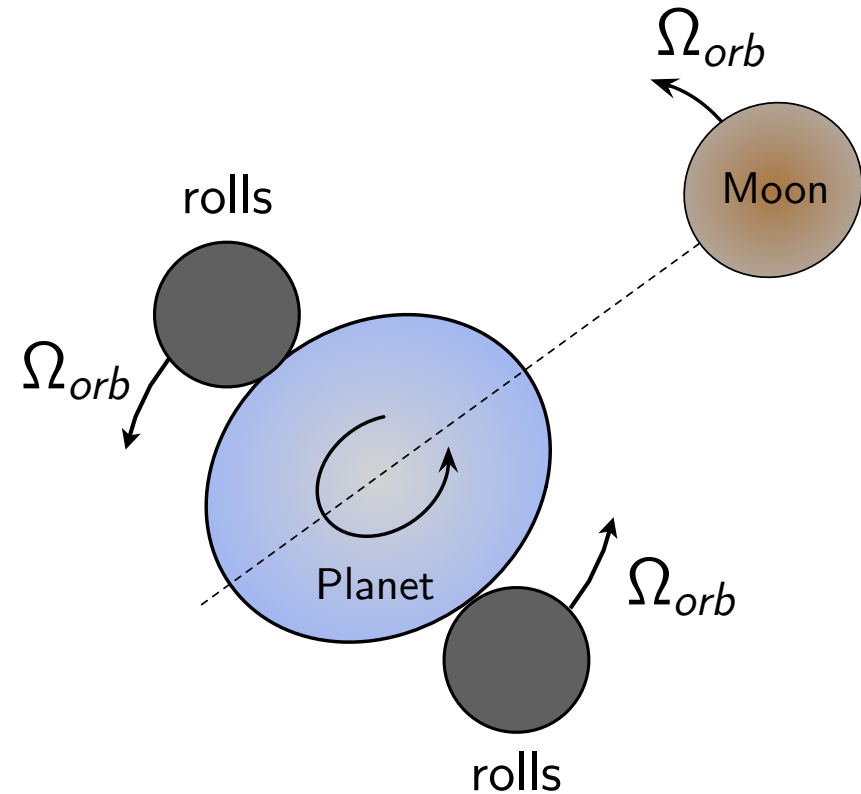


# From planets to the laboratory

Malkus W.V.R., GAFD, **48** (1989)

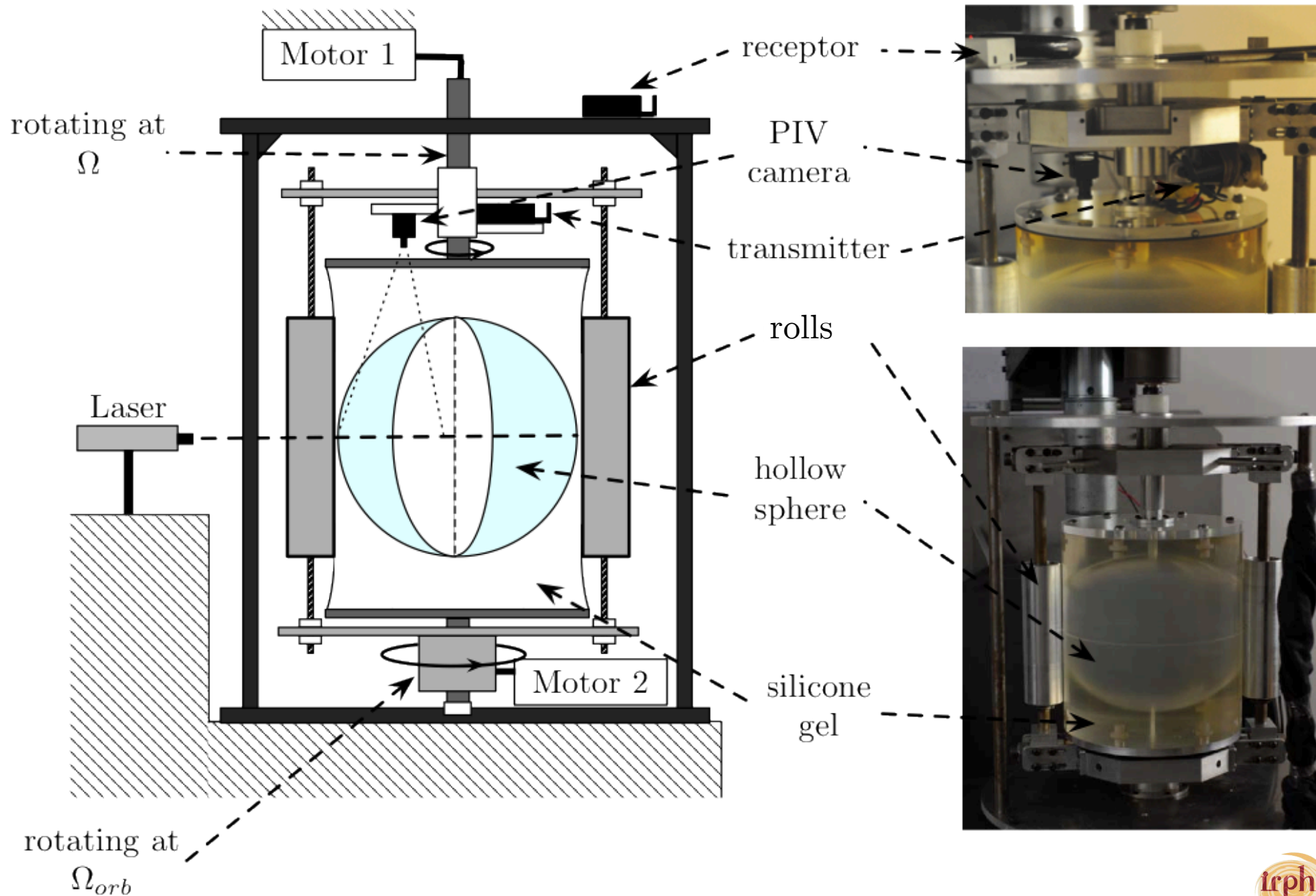


$$R_{ext} = 1 + \frac{\beta}{2} \cos(2\Phi)$$

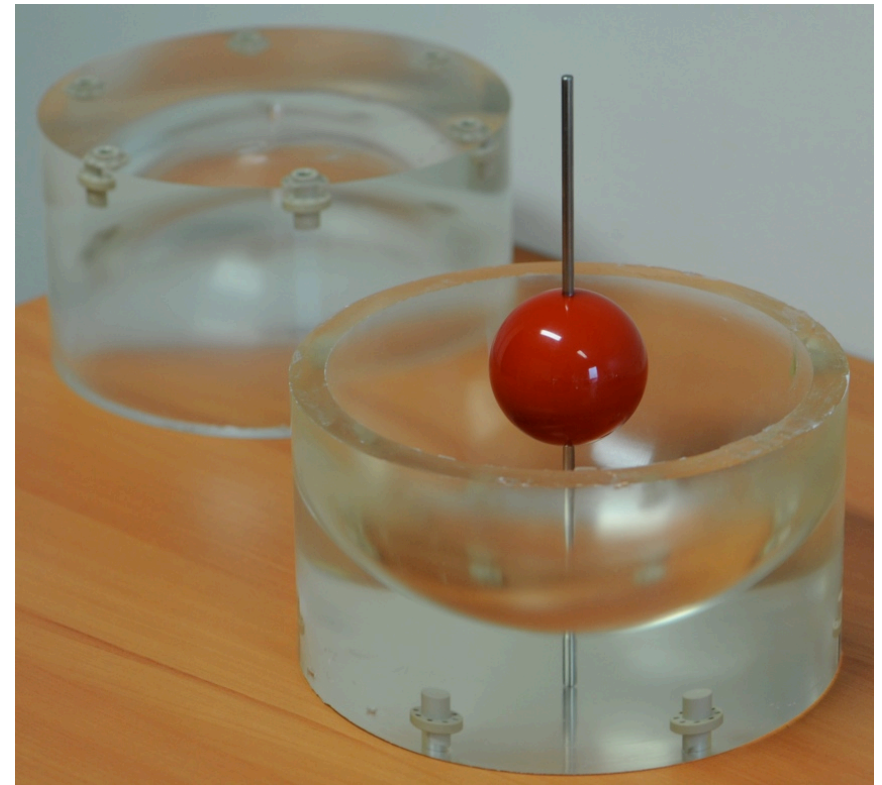
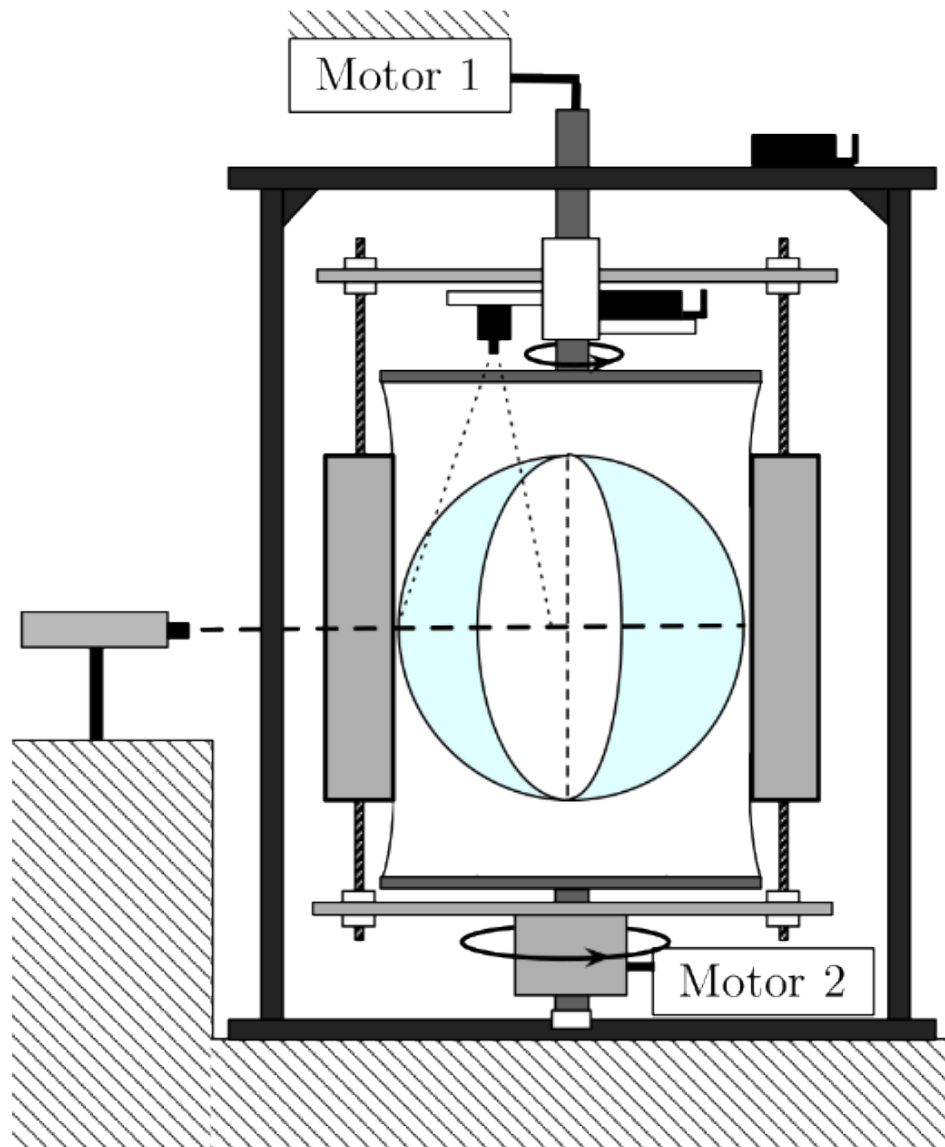


$$\Omega_R = \frac{\Omega_{orb}}{\Omega_0}$$

# Experimental setup

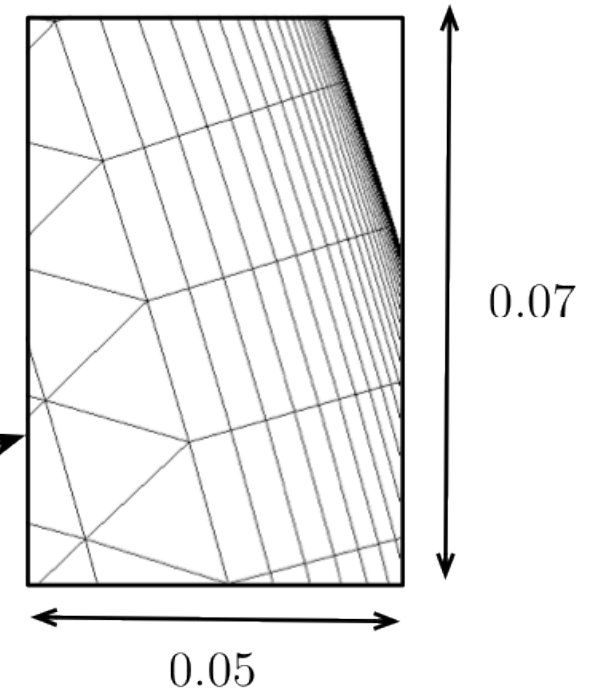
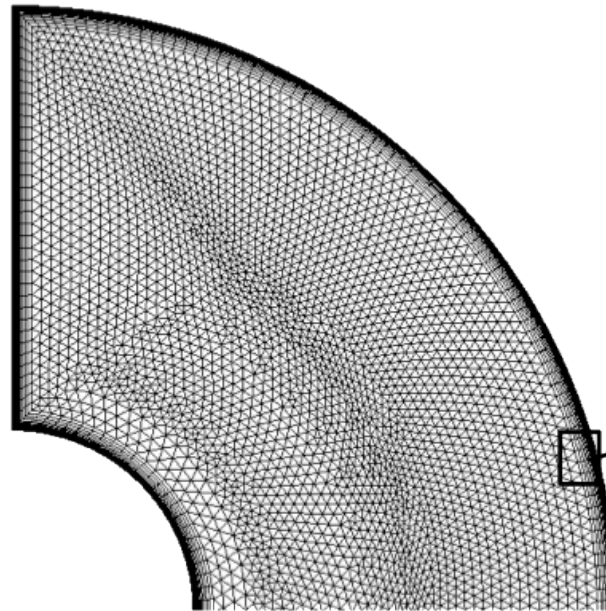
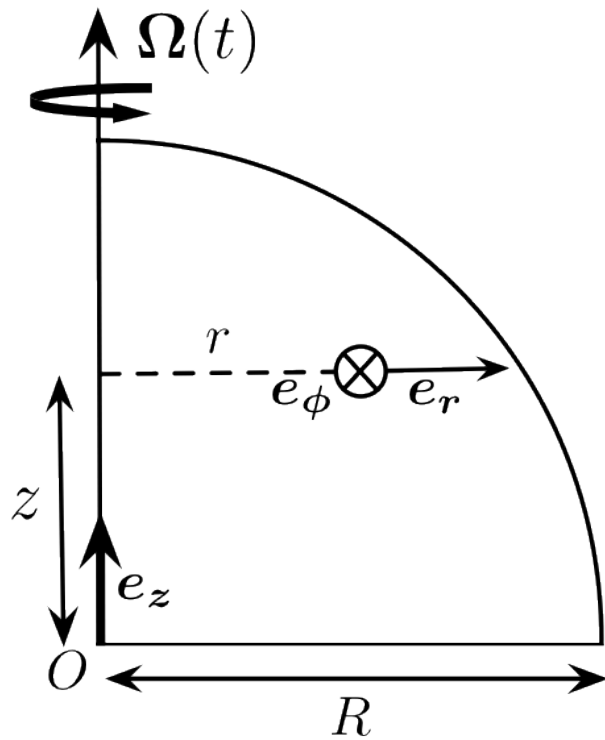


# From sphere to spherical shell

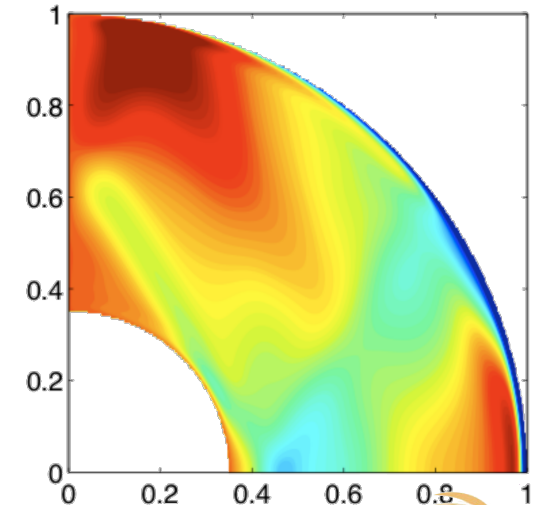
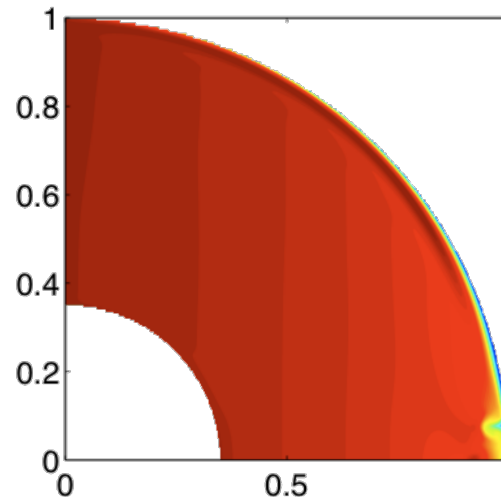


Addition of an inner core

# Numerical method



finite element  
axisymmetric  
fully nonlinear  
time-stepping

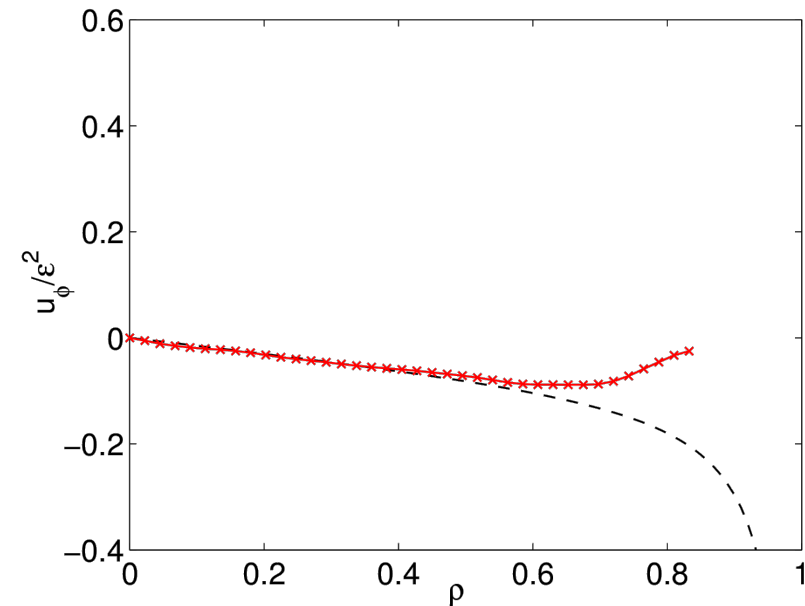
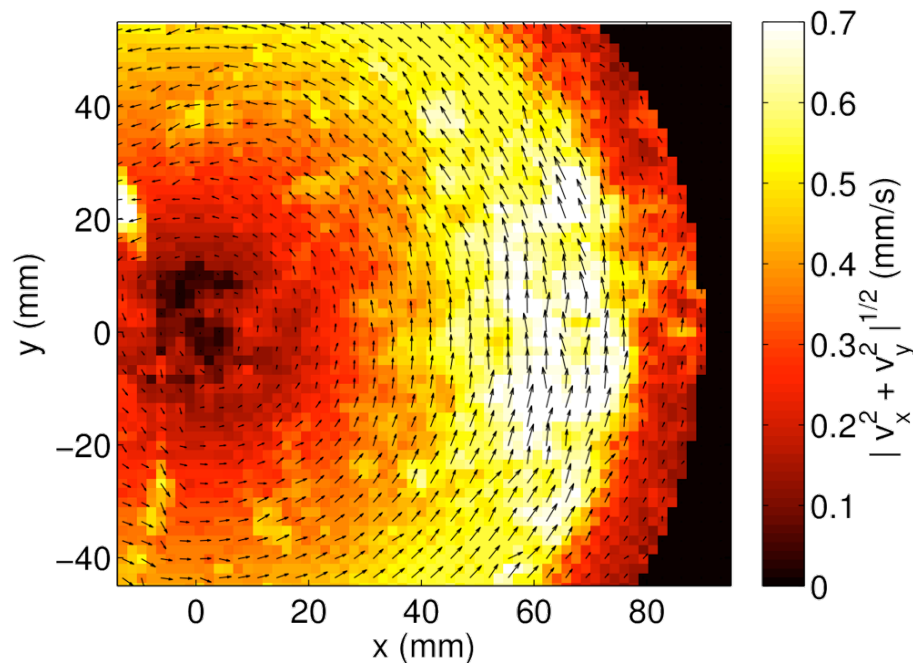




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# Experimental observations

Longitudinal libration in a sphere  $\Omega(t) = 1 + \varepsilon \cos(\omega t)$   $E = \frac{\nu}{\Omega_0 R^2}$



$$E = 1.15 \times 10^{-5}$$

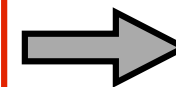
$$\omega = 0.1$$

$$\varepsilon = 0.08$$

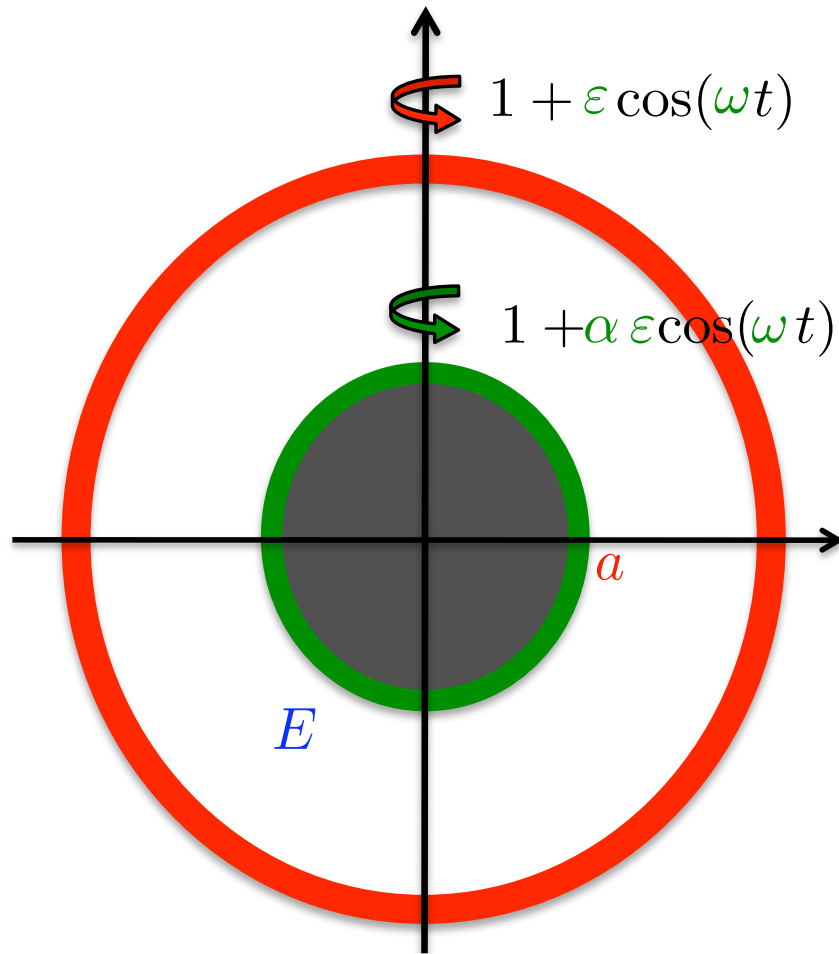
$$u_\theta = r\Omega_0 + \varepsilon^2 u_{\theta,2}$$

solid-body  
rotation

mean zonal  
flow



**Estimation?  
Calculation?**



## Geometry: spherical shell

$$a = \frac{R_i}{R_e}$$

$R_e$  external radius

$R_i$  internal radius

## Fluid: rotating, incompressible, homogeneous, viscous

$$E = \frac{\nu}{\Omega_0 R_e^2}$$

$\nu$  kinematic viscosity

$\Omega_0$  mean rotation rate

## Longitudinal libration

$\omega_l$  frequency of libration

$\delta \Omega_e$  amplitude of libration of the outer sphere

$\delta \Omega_i$  amplitude of libration of the inner sphere

$$\omega = \frac{\omega_l}{\Omega_0}$$

$$\varepsilon = \frac{\delta \Omega_e}{\Omega_0}$$

$$\alpha = \frac{\delta \Omega_i}{\delta \Omega_e}$$

# Framework of the asymptotic analysis

$\varepsilon \ll 1$   $\longrightarrow$  weakly nonlinear analysis

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2$$

$$\mathbf{u}_0 = r \mathbf{e}_\phi$$

Solid body rotation

$$\mathbf{u}_1 = \tilde{\mathbf{u}}_1 e^{i\omega t}$$

Harmonic flow induced by libration

$$\mathbf{u}_2 = \tilde{\mathbf{u}}_2 e^{2i\omega t} + \bar{\mathbf{u}}_2$$

Double-harmonic flow + Mean flow correction

$$\begin{array}{ccc} e^{i\omega t} & & e^{2i\omega t} \\ e^{-i\omega t} & \xrightarrow{(\mathbf{u} \cdot \nabla) \mathbf{u}} & e^{-2i\omega t} \\ & & e^0 = 1 \end{array}$$

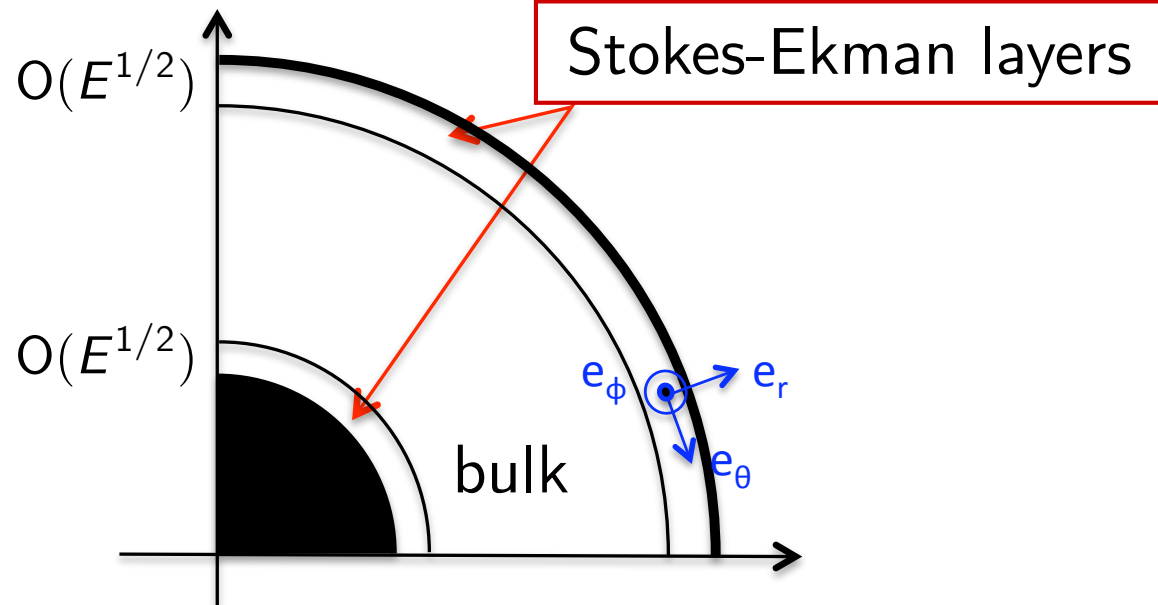
$E \ll 1$   $\longrightarrow$  Boundary layer analysis

$\omega > 2$   $\longrightarrow$  No inertial wave

# Harmonic flow structure

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2$$

$$\mathbf{u}_1 = \tilde{\mathbf{u}}_1 e^{i\omega t}$$



In the Stokes-Ekman layer

$$\tilde{\mathbf{u}}_1 \sim \frac{\sin \theta}{4} [e^{-\lambda_+ \tilde{r}} + e^{-\lambda_- \tilde{r}}] \mathbf{e}_\phi$$

$$\tilde{r} = \frac{1-r}{\sqrt{E}}$$

$$\lambda_{\pm} = (1+i) \sqrt{\frac{\omega}{2} \pm \cos \theta}$$

In the bulk

$$\tilde{\mathbf{u}}_1 = O(\sqrt{E})$$

# Second harmonic & mean flow correction

Sauret et Le Dizès, JFM, (in press)

Generated by the non-linear interactions

$$\mathbf{u}_2 = \tilde{\mathbf{u}}_2 e^{2i\omega t} + \bar{\mathbf{u}}_2$$

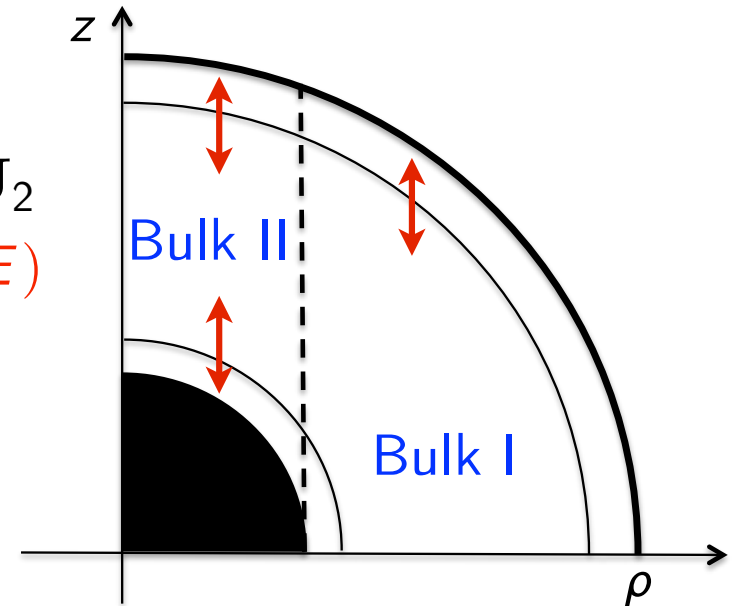
In the bulk:

$$\cancel{\frac{\partial \mathbf{U}_2}{\partial t}} + (\mathbf{U}_1 \cdot \nabla) \mathbf{U}_1 + \boxed{2\mathbf{k} \times \mathbf{U}_2 = -\nabla p_2} + E \Delta \mathbf{U}_2$$

$= O(E)$                        $= O(1)$                        $= O(E)$

$$\frac{\partial \mathbf{U}_2}{\partial z} = 0$$

Taylor-Proudman theorem



Non linear interactions in the Ekman layers



Stationary flux in the bulk



Equilibrium of the flux: induced the mean zonal flow

# Second harmonic & mean flow correction

Sauret et Le Dizès, JFM, (in press)

Generated by the non-linear interactions

$$u_{\phi_2} = \frac{\chi_2}{\rho} \quad u_{z_2} = \frac{\sqrt{E} \partial \Psi_2}{\rho \partial \rho} \quad \omega_{z_2} = \frac{1}{\rho} \frac{\partial \chi_2}{\partial \rho}$$

$$\mathbf{u}_2 = \tilde{\mathbf{u}}_2 e^{2i\omega t} + \bar{\mathbf{u}}_2$$

## Solution in Bulk I

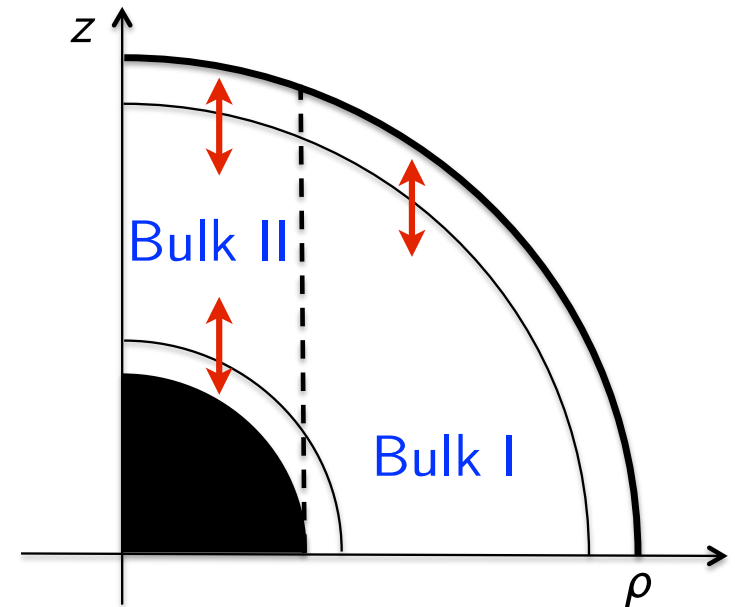
$$\chi_2(\rho) = \mathcal{F}(\rho; \omega)$$

$$\Psi_2(\rho) = 0$$

## Solution in Bulk II

$$\chi_2(\rho) = \frac{a^2 a^2 (1 - \rho^2)^{1/4} \mathcal{F}(\rho/a; \omega) + [1 - (\rho/a)^2]^{1/4} \mathcal{F}(\rho; \omega)}{(1 - \rho^2)^{1/4} + [1 - (\rho/a)^2]^{1/4}}$$

$$\Psi_2(\rho) = \frac{\mathcal{F}(\rho; \omega) - a^2 a^2 \mathcal{F}(\rho/a; \omega)}{2(1 - \rho^2)^{1/4} + 2[1 - (\rho/a)^2]^{1/4}}$$



# Second harmonic & mean flow correction

Sauret et Le Dizès, JFM, (in press)

Generated by the non-linear interactions

$$\mathbf{u}_2 = \tilde{\mathbf{u}}_2 e^{2i\omega t} + \bar{\mathbf{u}}_2$$

$$u_{\phi_2} = \frac{\mathcal{X}_2}{\rho} \quad u_{z_2} = \frac{\sqrt{E} \partial \Psi_2}{\rho \partial \rho} \quad \omega_{z_2} = \frac{1}{\rho} \frac{\partial \mathcal{X}_2}{\partial \rho}$$

$$\begin{aligned} \mathcal{F}(\sin \theta; \omega) = & \left[ \frac{(s_+^3 + s_-^3) \sin^4 \theta}{32 s_+^2 s_-^2} + \frac{(s_- - s_+) \cos \theta \sin^2 \theta}{8} \right] \left[ \frac{s_+ - \sqrt{\cos \theta}}{s_+^2 + \cos \theta} - \frac{s_- - \sqrt{\cos \theta}}{s_-^2 - \cos \theta} \right] \\ & + \frac{\sin^2 \theta (s_+ + \sqrt{\cos \theta})}{32 s_+^3 (\cos^2 \theta + 4 s_+^4)} (2 s_+^2 - 2 s_+ \sqrt{\cos \theta} + \cos \theta) (\sin^2 \theta + 4 s_+^2 \cos \theta) \\ & + \frac{\sin^2 \theta (s_- + \sqrt{\cos \theta})}{32 s_-^3 (\cos^2 \theta + 4 s_-^4)} (2 s_-^2 - 2 s_- \sqrt{\cos \theta} + \cos \theta) (\sin^2 \theta - 4 s_-^2 \cos \theta) \\ & - \frac{\sin^2 \theta}{32 s_-^3 s_+^3 (s_-^2 + s_+^2) [s_-^2 + s_+^2 + 2 s_+ \sqrt{\cos \theta} + \cos \theta]^2} \\ & \left[ (s_-^2 + s_+^2) [s_-^6 + 7 s_+^2 s_-^4 + 8 s_-^7 s_+^3 - 7 s_-^2 s_+^4 + 16 s_-^5 s_+^5 \right. \\ & - s_+^6 + 4 s_+^3 s_-^3 (-3 + 2 s_+^4)] + [4 s_-^6 s_+ - s_-^4 s_+^3 - 16 s_-^2 s_+^5 - 3 s_+^7 \\ & + 4 s_-^5 s_+^2 (-3 + 16 s_+^4) + s_+^4 s_-^3 (-19 + 32 s_+^4) + s_-^7 (-1 + 32 s_+^4)] \sqrt{\cos \theta} \\ & + [4 s_-^8 s_+^2 - 3 s_+^6 + s_-^6 (1 - 20 s_+^4) + s_-^2 s_+^4 (-11 + 4 s_+^4) \\ & + 4 s_+^3 s_-^3 (-3 + 16 s_+^4) + s_-^5 (-4 s_+ + 64 s_+^5) + s_-^4 (s_+^2 - 20 s_+^6)] \cos \theta \\ & - [4 s_-^7 s_+^2 + 4 s_-^6 s_+^3 + s_+^5 + 56 s_-^4 s_+^5 + s_-^3 s_+^2 (3 - 76 s_+^4) \\ & + s_-^5 (1 - 8 s_+^4) + 3 s_+^3 s_-^2 (1 - 4 s_+^4)] \cos^{3/2} \theta \\ & + [-s_-^8 - 4 s_-^6 s_+^2 - 4 s_+^3 s_-^5 - 48 s_-^4 s_+^4 + 60 s_+^5 s_-^3 + 20 s_-^2 s_+^6 + s_+^8] \cos^2 \theta \\ & + [s_-^7 - 4 s_-^6 s_+ + 8 s_-^5 s_+^2 - 11 s_-^4 s_+^3 + 31 s_+^4 s_-^3 + 20 s_-^2 s_+^5 + 3 s_+^7] \cos^{5/2} \theta \\ & + [-s_-^6 + 4 s_-^5 s_+ - s_+^2 s_-^4 + 12 s_+^3 s_-^3 + 11 s_-^2 s_+^4 + 3 s_+^6] \cos^3 \theta \\ & + (s_-^5 + 3 s_-^3 s_+^2 + 3 s_+^3 s_-^2 + s_+^5) \cos^{7/2} \theta \left. \right] \\ & + \frac{\sin^2 \theta}{4}. \end{aligned}$$

Solution in Bulk I

$$\mathcal{X}_2(\rho) = \mathcal{F}(\rho; \omega)$$

$$\Psi_2(\rho) = 0$$

with  $s_{\pm} = \sqrt{\frac{\omega}{2} \pm \cos \theta}$

Valid for  $\omega > 2$



# Stewartson layer structure

## External Outer Layer $E^{1/4}$

smoothing of azimuthal velocity

$$u_{\phi_2} \sim \frac{\mathcal{F}(a; \omega)}{a} - a \delta \Omega_2 e^{-\tilde{\rho}}$$

$$\tilde{\rho} = (\rho - a) / \left[ (1 - a^2)^{3/8} E^{1/4} \right]$$

## Internal Outer Layer $E^{2/7}$

smoothing of axial vorticity

$$\omega_{z_2} = E^{-1/4} \frac{\delta \Omega_2 a}{(1 - a^2)^{3/8}}$$

$$\bar{\rho} = (a - \rho) \left[ a / (2 E^2 (1 - a^2)^2) \right]^{1/7}$$

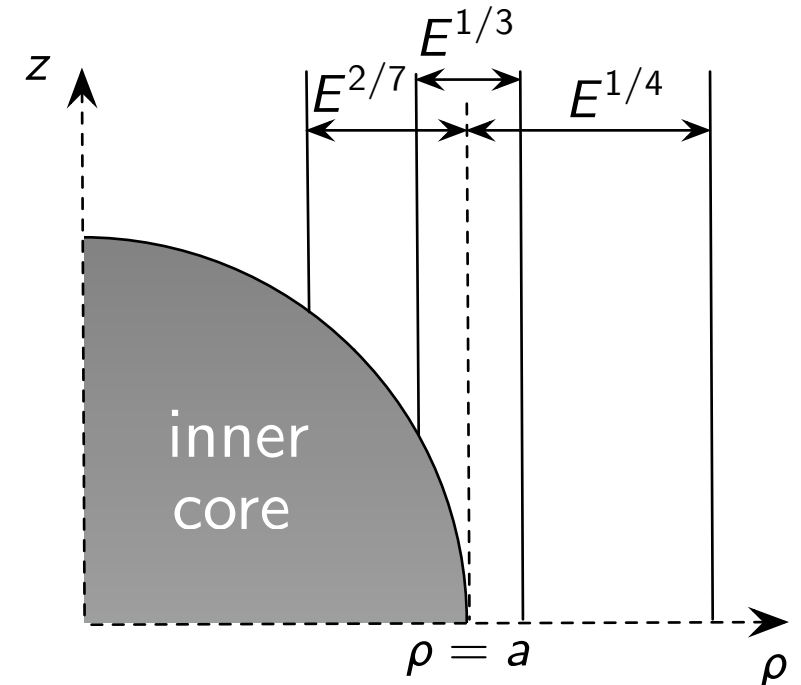
## Inner Layer $E^{1/3}$

smoothing of axial velocity

$$u_{z_2} \sim E^{5/42} \frac{a_0 \delta \Omega_2}{a^{81/28} (1 - a^2)^{25/84}} \mathbf{J} \left( \frac{z}{\sqrt{1 - a^2}}, \frac{\rho - a}{E^{1/3} (1 - a^2)^{1/6}} \right)$$

Stewartson, JFM, **26** (1966)

Sauret et Le Dizès, JFM, (in press)



# Second harmonic & mean flow correction

Sauret et Le Dizès, JFM, (in press)

Generated by the non-linear interactions

$$\mathbf{u}_2 = \tilde{\mathbf{u}}_2 e^{2i\omega t} + \bar{\mathbf{u}}_2$$

$$u_{\phi_2} = \frac{\mathcal{X}_2}{\rho} \quad u_{z_2} = \frac{\sqrt{E} \partial \Psi_2}{\rho \partial \rho} \quad \omega_{z_2} = \frac{1}{\rho} \frac{\partial \mathcal{X}_2}{\partial \rho}$$

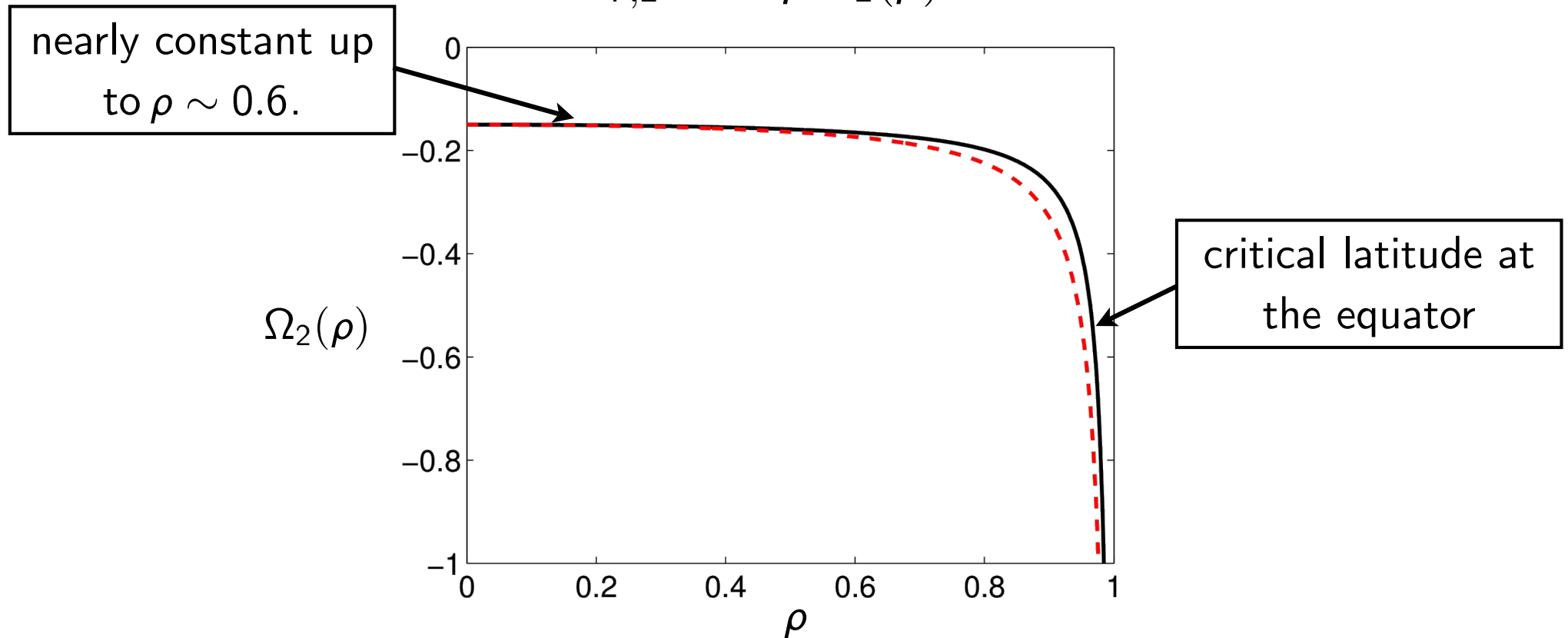
$$\begin{aligned} \mathcal{F}(\sin \theta; \omega) = & \frac{\sin^2 \theta}{32} \left[ 8 + \frac{(l_+ + \sqrt{\cos \theta})(4l_+^2 \cos \theta + \sin^2 \theta)}{l_+^3 (2l_+^2 + 2l_+ \sqrt{\cos \theta} + \cos \theta)} + \frac{(l_- + \sqrt{\cos \theta})(4l_-^2 \cos \theta + \sin^2 \theta)}{l_-^3 (2l_-^2 + 2l_- \sqrt{\cos \theta} + \cos \theta)} \right. \\ & - \frac{4(l_-^3 + l_-^2 \sqrt{\cos \theta} + (l_- + l_+) \cos \theta)}{l_+ l_- (l_- + \sqrt{\cos \theta})} + \frac{l_-^4 \sin^2 \theta (l_- + \sqrt{\cos \theta} + (l_+^3 + l_-^3)/l_-^4 \cos \theta)}{\cos \theta l_+^3 l_-^3 (l_- + \sqrt{\cos \theta})} \\ & - \frac{4(l_+^3 + l_+^2 \sqrt{\cos \theta} + (l_+ + l_-) \cos \theta)}{l_- l_+ (l_+ + \sqrt{\cos \theta})} + \frac{l_+^4 \sin^2 \theta (l_+ + \sqrt{\cos \theta} + (l_-^3 + l_+^3)/l_+^4 \cos \theta)}{\cos \theta l_-^3 l_+^3 (l_+ + \sqrt{\cos \theta})} \\ & + \frac{1}{\cos \theta l_-^3 l_+^3 (l_- + l_+) (l_- + l_+ + \sqrt{\cos \theta})^2} \\ & \left. \left( 4l_-^2 l_+^2 (l_- - l_+)^2 \cos^{5/2} \theta [1 + 2(l_- + l_+) \sqrt{\cos \theta}] \right. \right. \\ & + \cos \theta [l_- l_+ (l_- + l_+) (2(2l_-^2 - 3l_- l_+ + 2l_+^2) \sin^2 \theta + 4l_-^5 l_+ - 8l_-^3 l_+^3 + 4l_- l_+^5) \\ & + 2(l_-^5 + l_+^5) \sin^2 \theta] + (l_- + l_+)^2 (l_-^4 + l_+^4) \sin^2 \theta [(l_- + l_+) + 2\sqrt{\cos \theta}] \\ & \left. \left. + \cos^{3/2} \theta [8l_+^2 l_-^2 (l_-^2 - l_+^2)^2 + (l_-^4 + l_+^4 + l_- l_+ (l_- - l_+)^2) \sin^2 \theta] \right) \right]. \end{aligned}$$

$$l_{\pm} = \sqrt{\cos \theta \pm \frac{\omega}{2}}$$

Valid for  $\omega < 2 \cos \theta$

Sauret et Le Dizès, JFM, (in press)

$$u_{\phi,2} = \varepsilon^2 \rho \Omega_2(\rho)$$



Busse, JFM, **650** (2010)

$$\omega \ll \varepsilon \ll 1$$

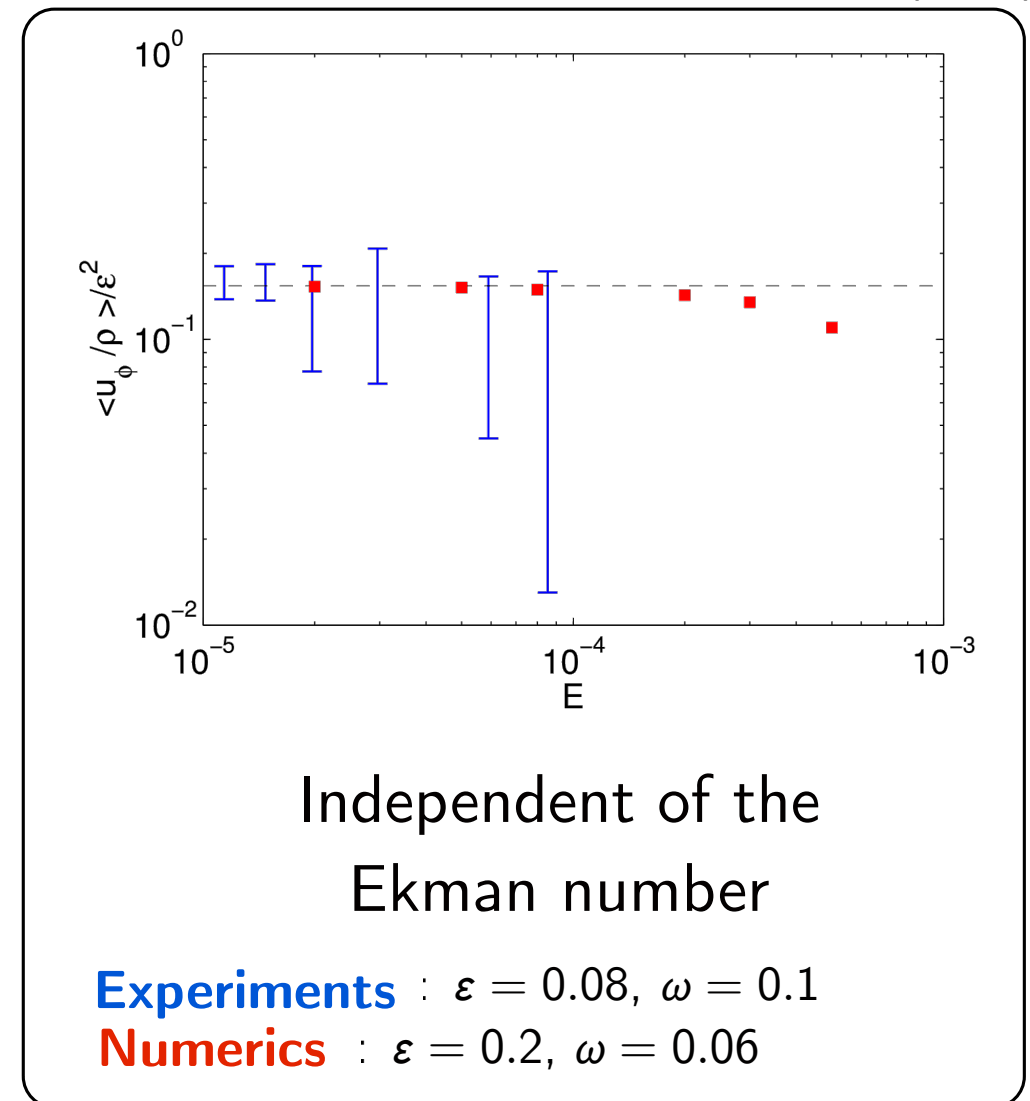
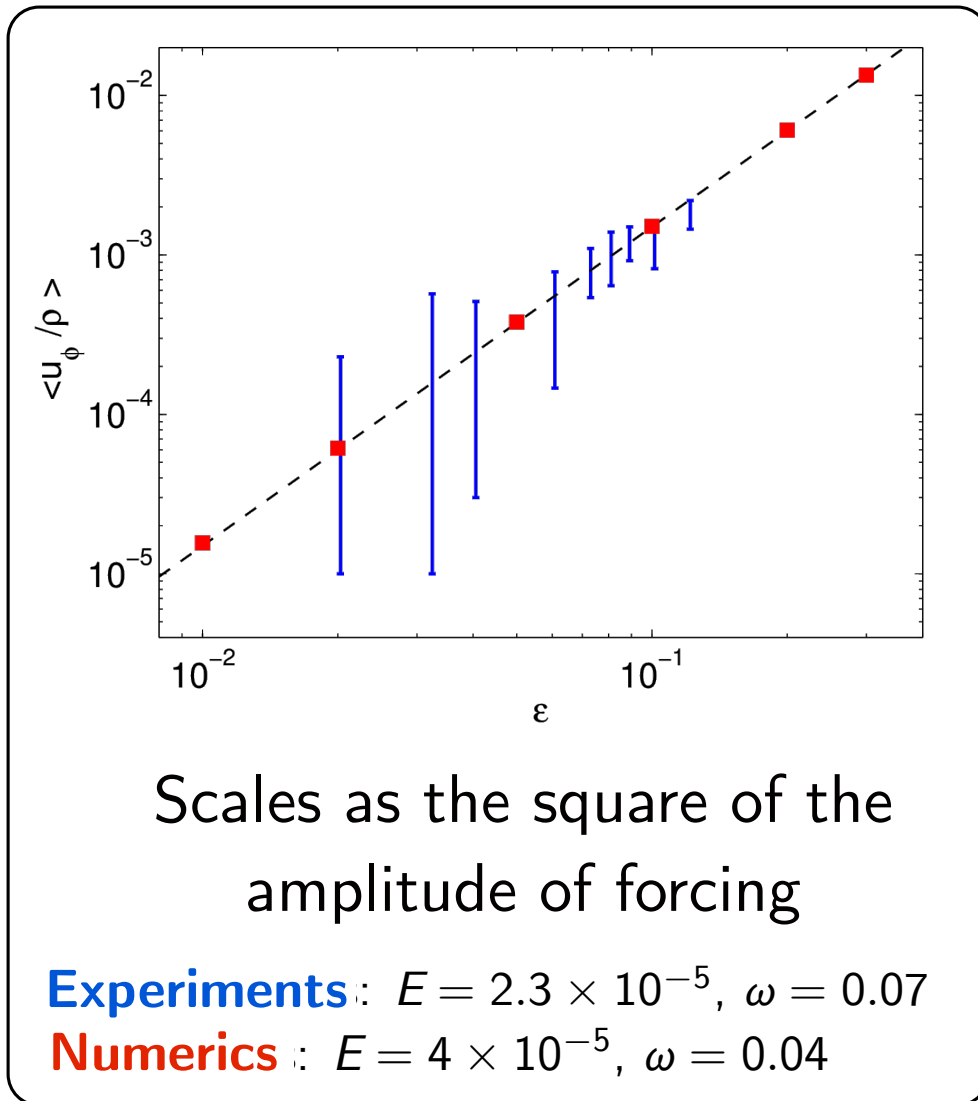
Sauret et Le Dizès, JFM, (in press)

$$\varepsilon \ll \omega \ll 1$$

## Deviation to solid-body rotation

# Comparison and scalings $\omega \rightarrow 0$

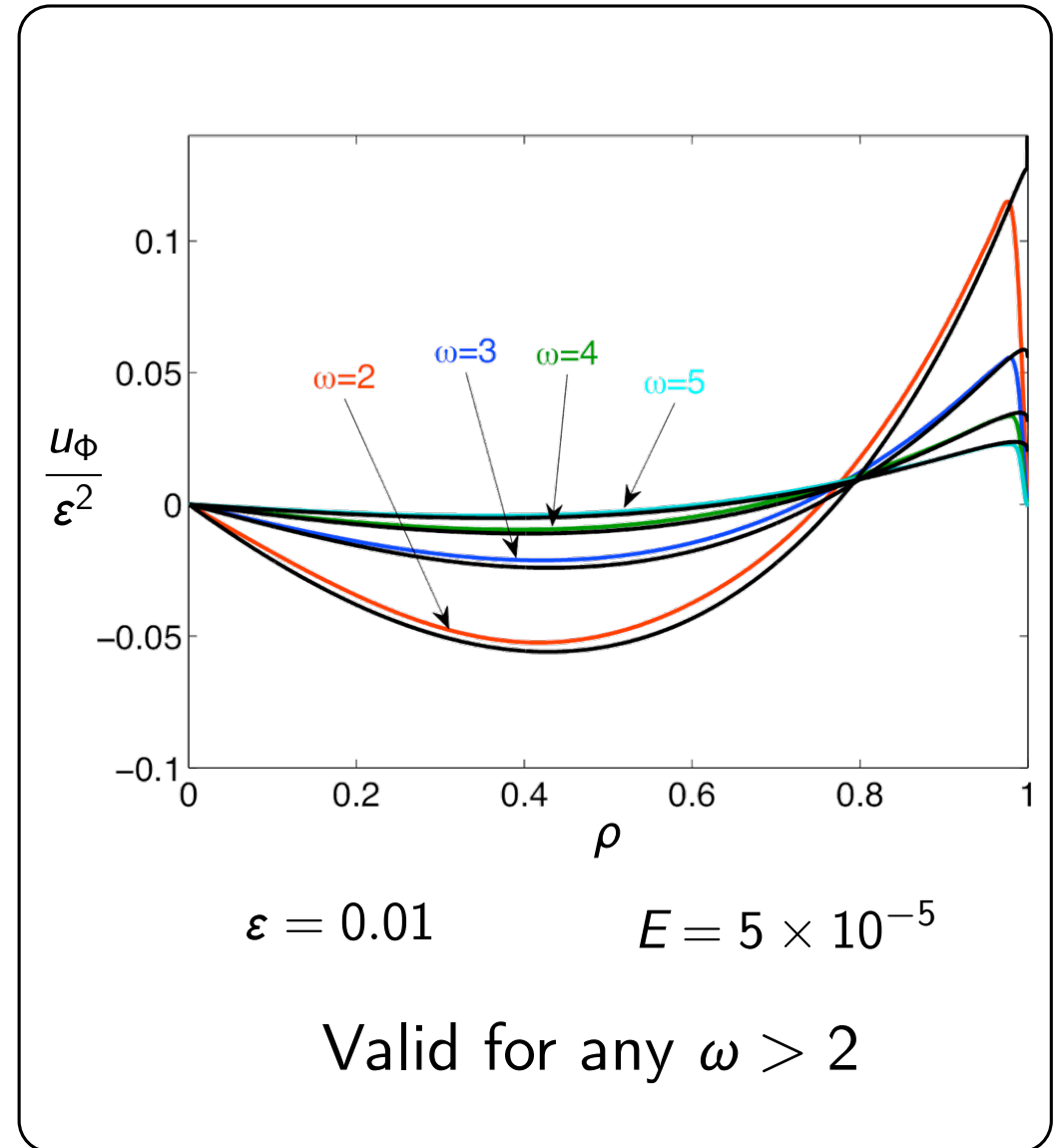
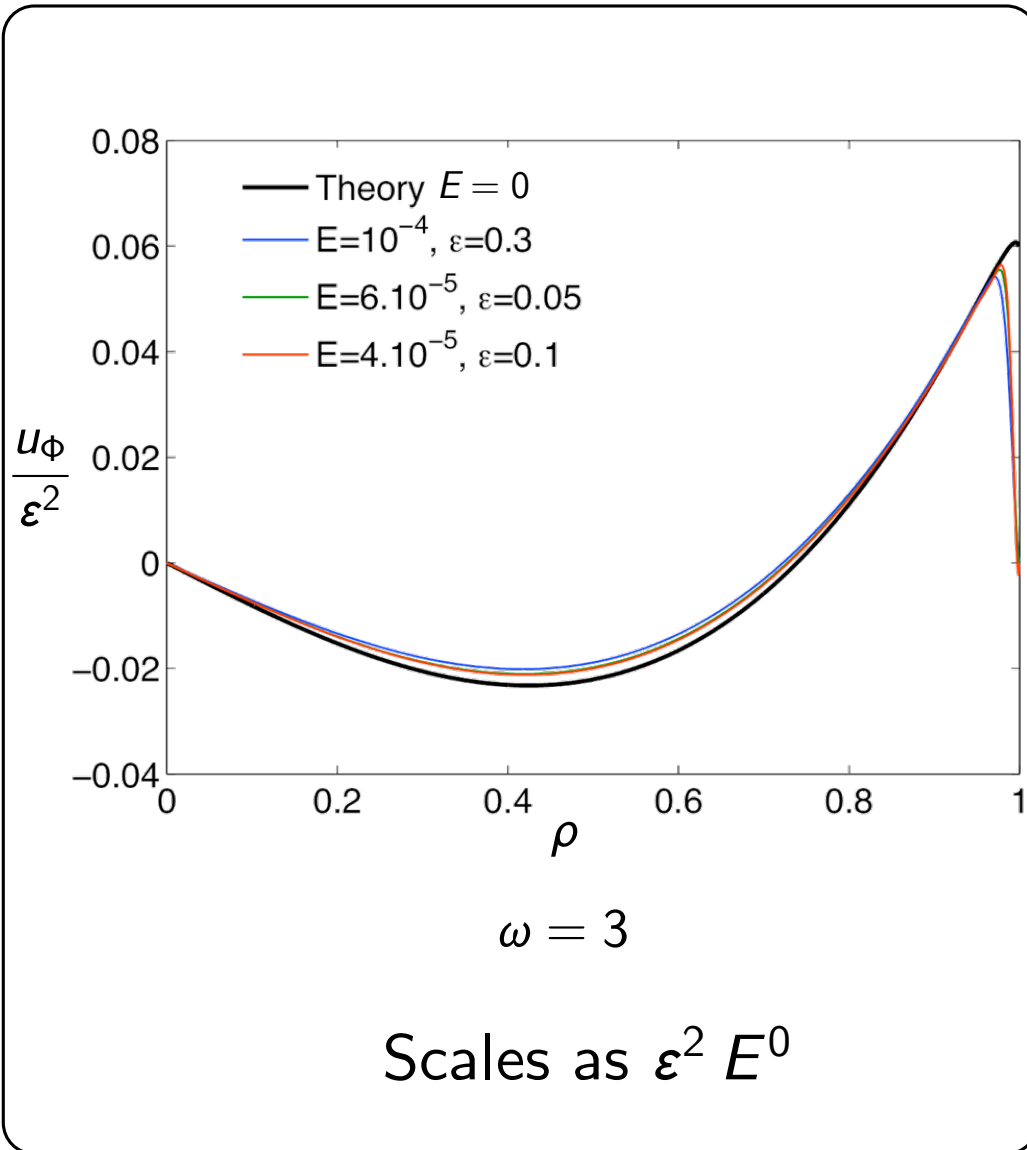
Sauret et al., JFM, **662** (2010)



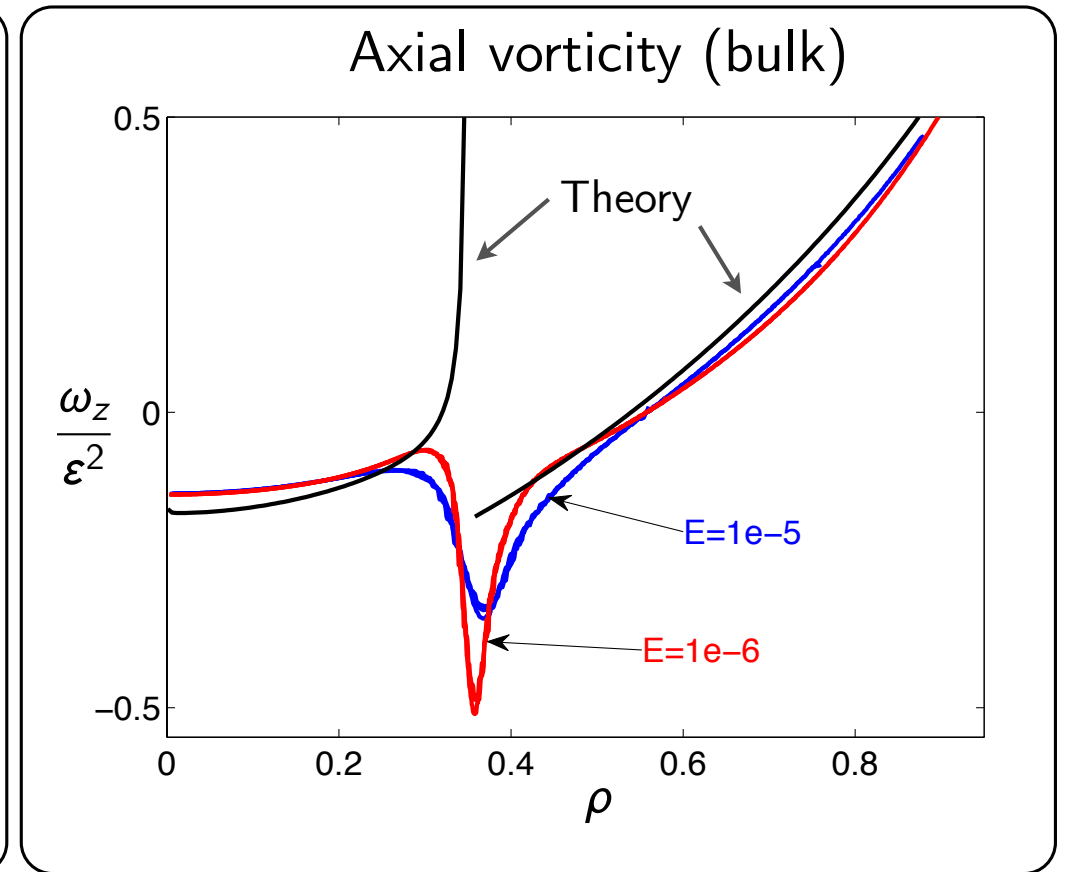
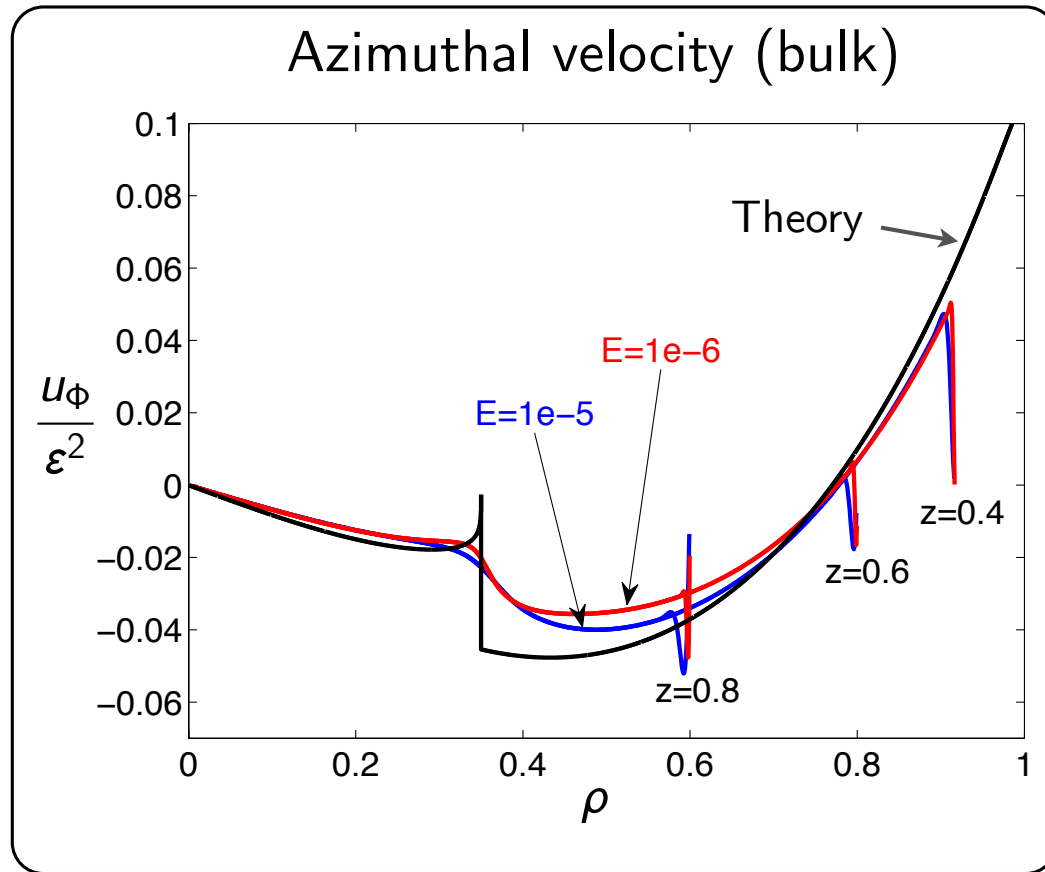
$$u_{\phi,2} = \varepsilon^2 E^0 \rho \Omega_2(\rho)$$

# Comparison and scalings $\omega > 2$

Sauret et Le Dizès, JFM, (in press)



# Spherical shell - bulk

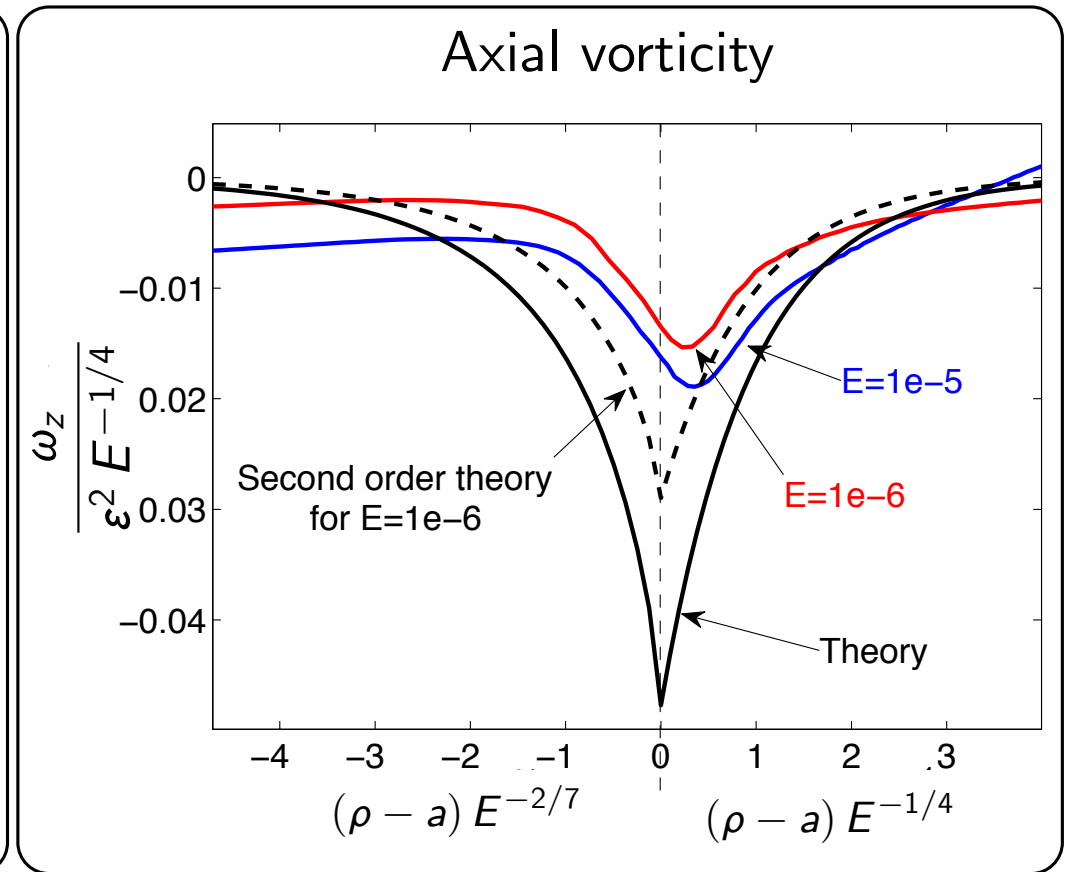
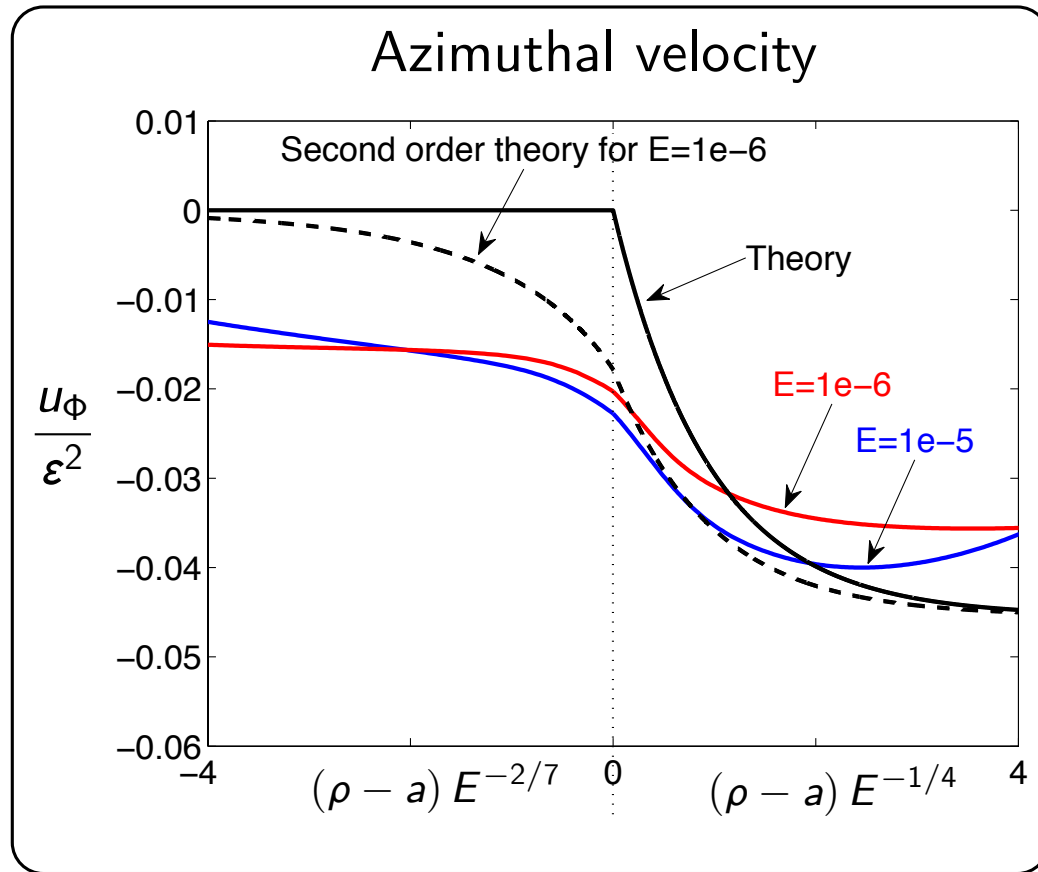


$$\begin{aligned} a &= 0 \\ \varepsilon &= 2.2 \times 10^{-3} \\ \omega &= 2.2 \\ a &= 0.35 \end{aligned}$$

Calkins et al., Phys. Fluids **22** (2010)

Sauret et Le Dizès, JFM, (in press)

# Spherical shell - Stewartson layers



$$a = 0$$

$$\varepsilon = 2.2 \times 10^{-3}$$

$$\omega = 2.2$$

$$a = 0.35$$

Calkins et al., Phys. Fluids **22** (2010)

Sauret et Le Dizès, JFM, (in press)





# Analogy between azimuthal and time forcing

Sauret A. (submitted to Physica D)

Libration ( $m=0$ ) forces a flow of azimuthal wavenumber  $m=0$

Tides ( $m=2$ ) forces a flow of azimuthal wavenumber  $m=2$

Solid body rotation

Harmonic flow

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2$$

Double-harmonic flow  
+ Mean flow correction

$$\begin{array}{c} e^{i\omega t} \\ e^{-i\omega t} \end{array} \xrightarrow{(\mathbf{u} \cdot \nabla) \mathbf{u}} \begin{array}{c} e^{2i\omega t} \\ e^{-2i\omega t} \\ \boxed{e^0 = 1} \end{array}$$

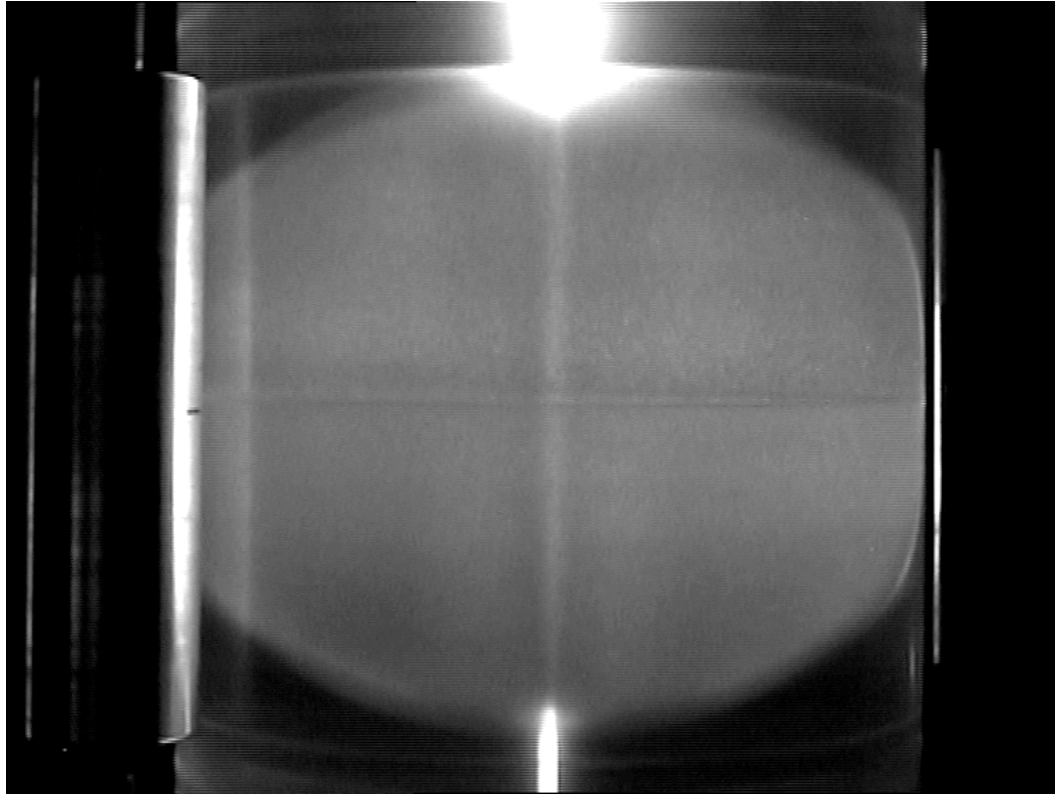
$$\begin{array}{c} e^{2i\theta} \\ e^{-2i\theta} \end{array} \xrightarrow{(\mathbf{u} \cdot \nabla) \mathbf{u}} \begin{array}{c} e^{4i\theta} \\ e^{-4i\theta} \\ \boxed{e^0 = 1} \end{array}$$

Mean zonal flow (axisymmetric & steady)



- 1 - Theory of rotating flows
- 2 - Methods of investigations
- 3 - Mean zonal flow
- 4 - Resonance of inertial modes**
- 5 - Spontaneous generation of inertial waves

# Kalliroscope visualisation



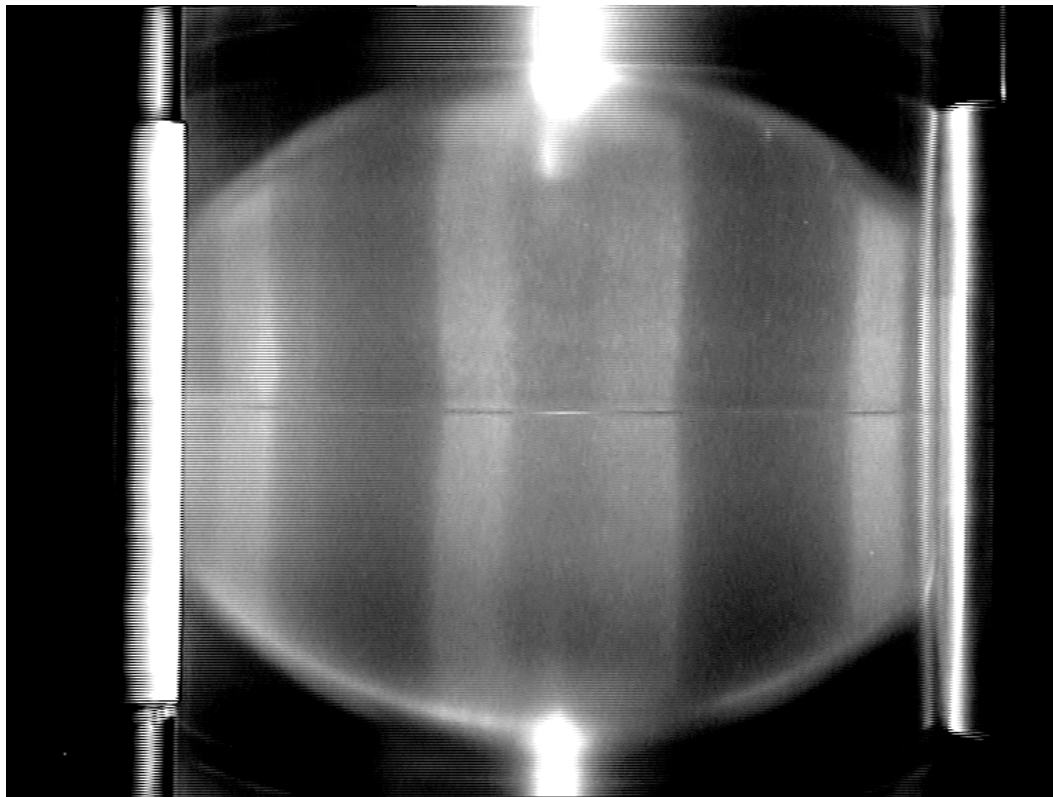
Solid-body  
rotation

$$E = 9.46 \times 10^{-6}, \beta = 0, \Omega_R = 0$$

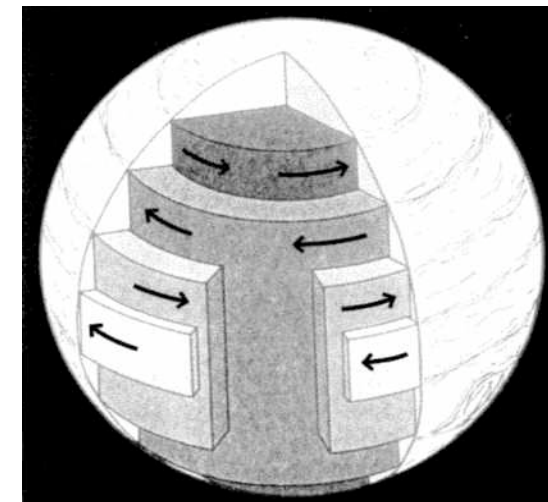
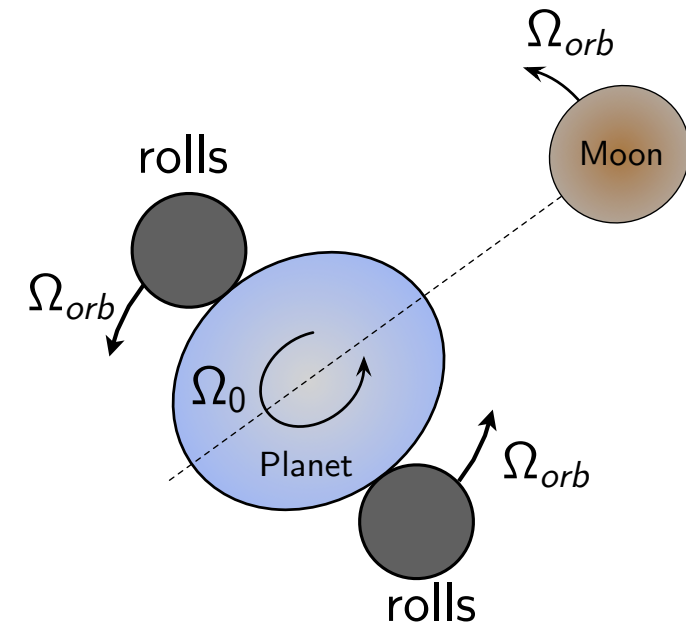
# Kalliroscope visualisation

$$R_{ext} = 1 + \frac{\beta}{2} \cos(2\Phi)$$

$$E = \frac{\nu}{\Omega_0 R^2} \quad \Omega_R = \frac{\Omega_{orb}}{\Omega_0}$$

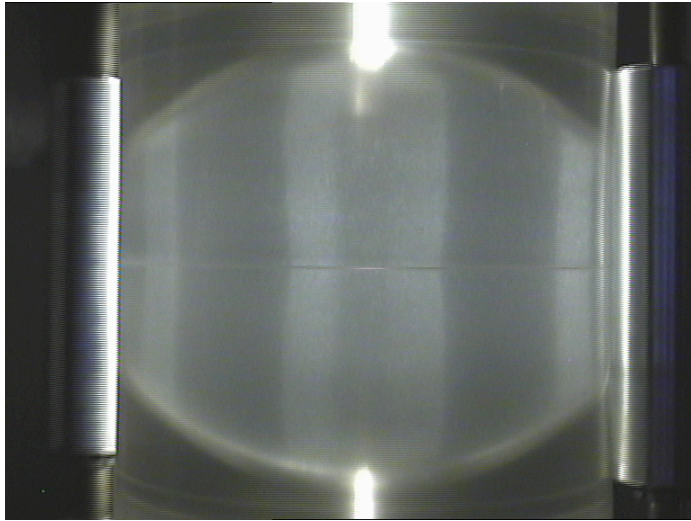


$$E = 1.4 \times 10^{-5}, \beta = 0.05, \Omega_R = 0.18$$

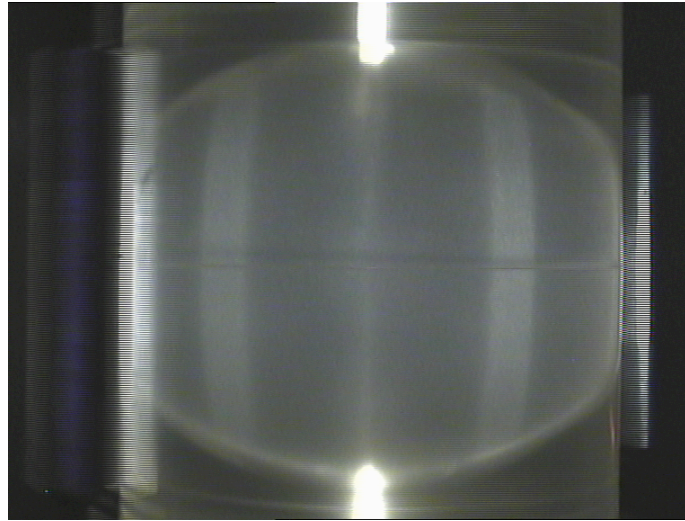


Presence of a differential rotation

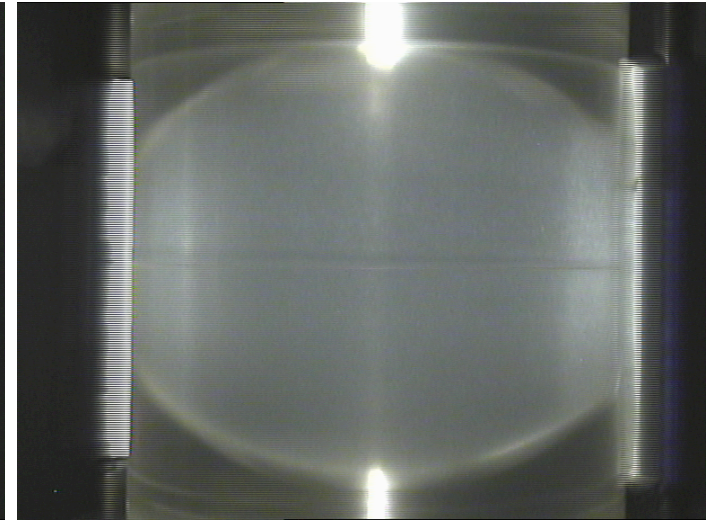
# Influence of the frequency



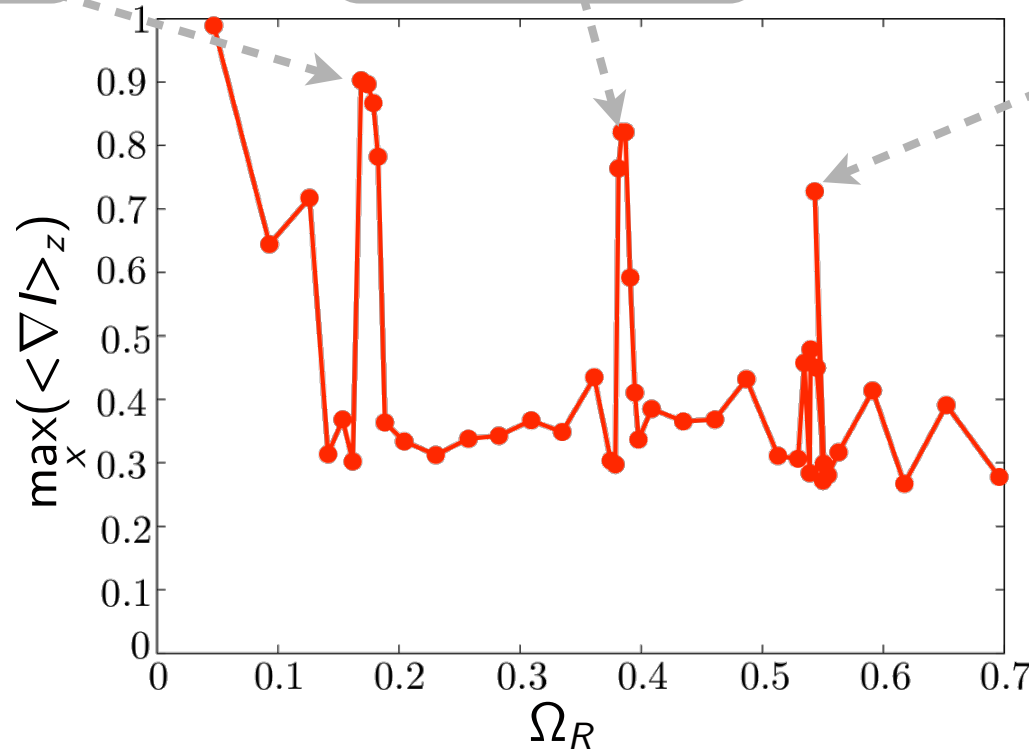
$\Omega_R = 0.178$



$\Omega_R = 0.384$



$\Omega_R = 0.55$



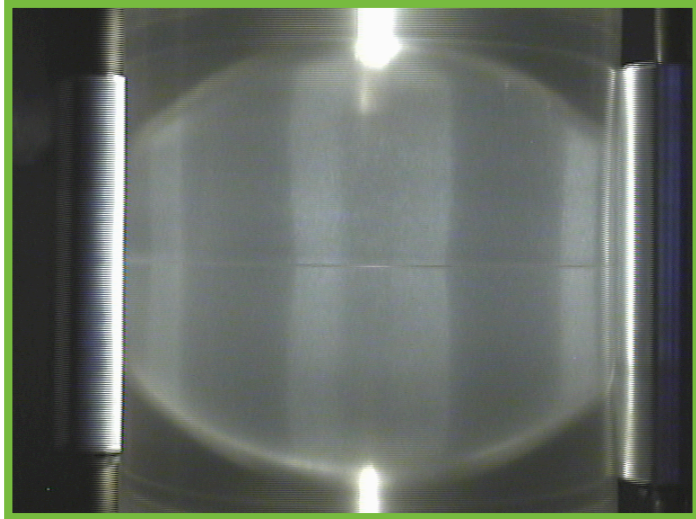
$$\Omega_R = \frac{\Omega_{orb}}{\Omega_{spin}}$$

Morize et al.,  
PRL (2010)

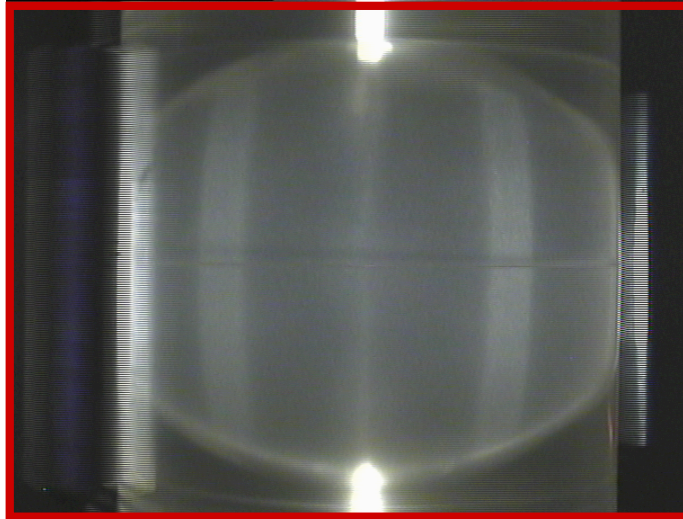
$E = 1.45 \times 10^{-5}$   
 $\beta = 0.04$



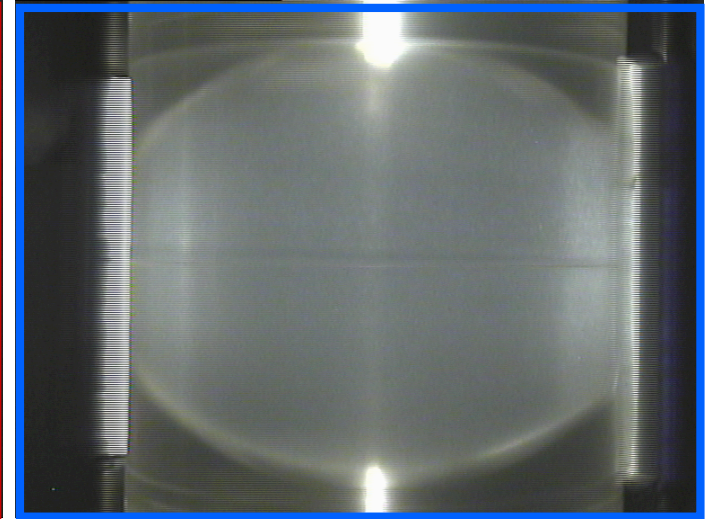
# Modes of the sphere



$$\Omega_R = 0.178$$

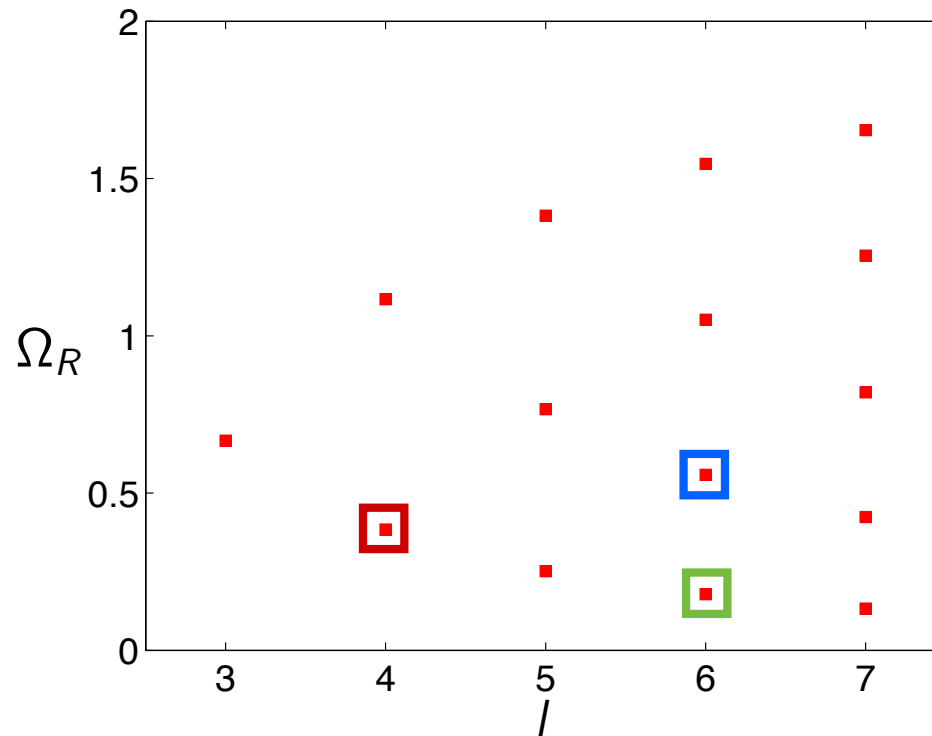


$$\Omega_R = 0.384$$



$$\Omega_R = 0.55$$

$$\Omega_R = \frac{\omega}{2}$$



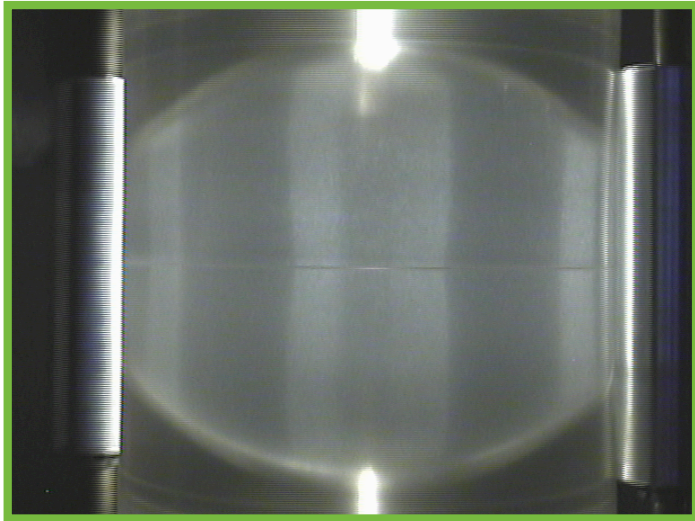
Dispersion  
relation for  $m=2$

$$E = 1.45 \times 10^{-5}$$

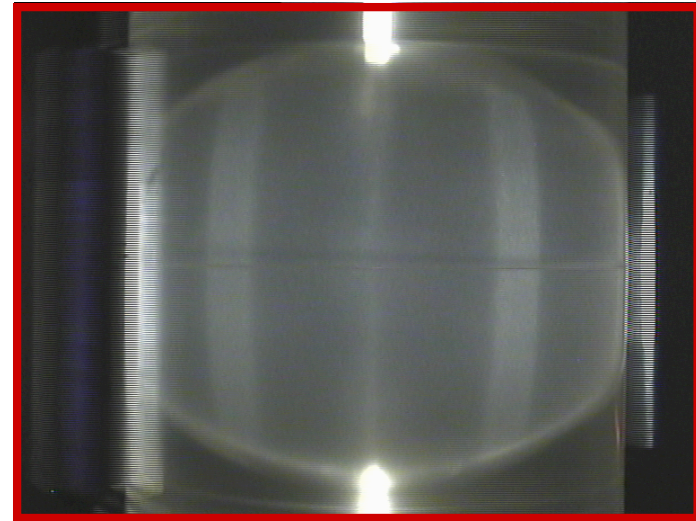
$$\beta = 0.04$$



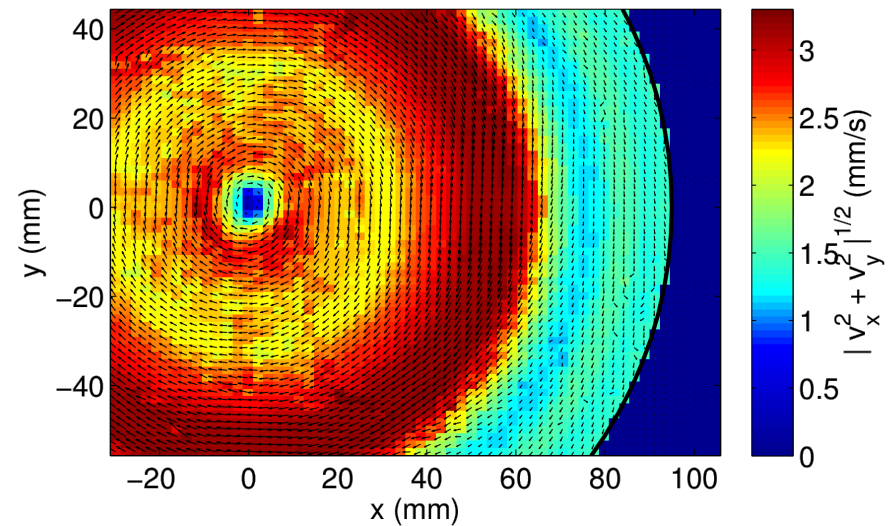
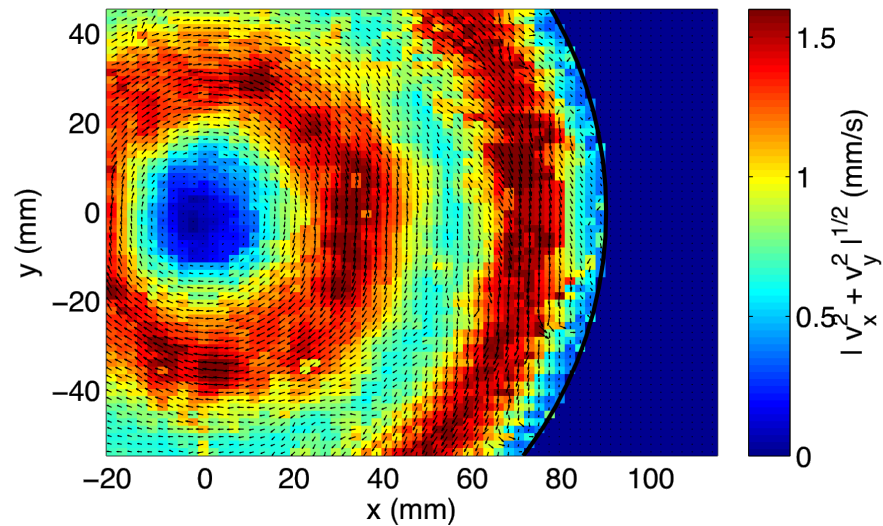
# PIV Measurements



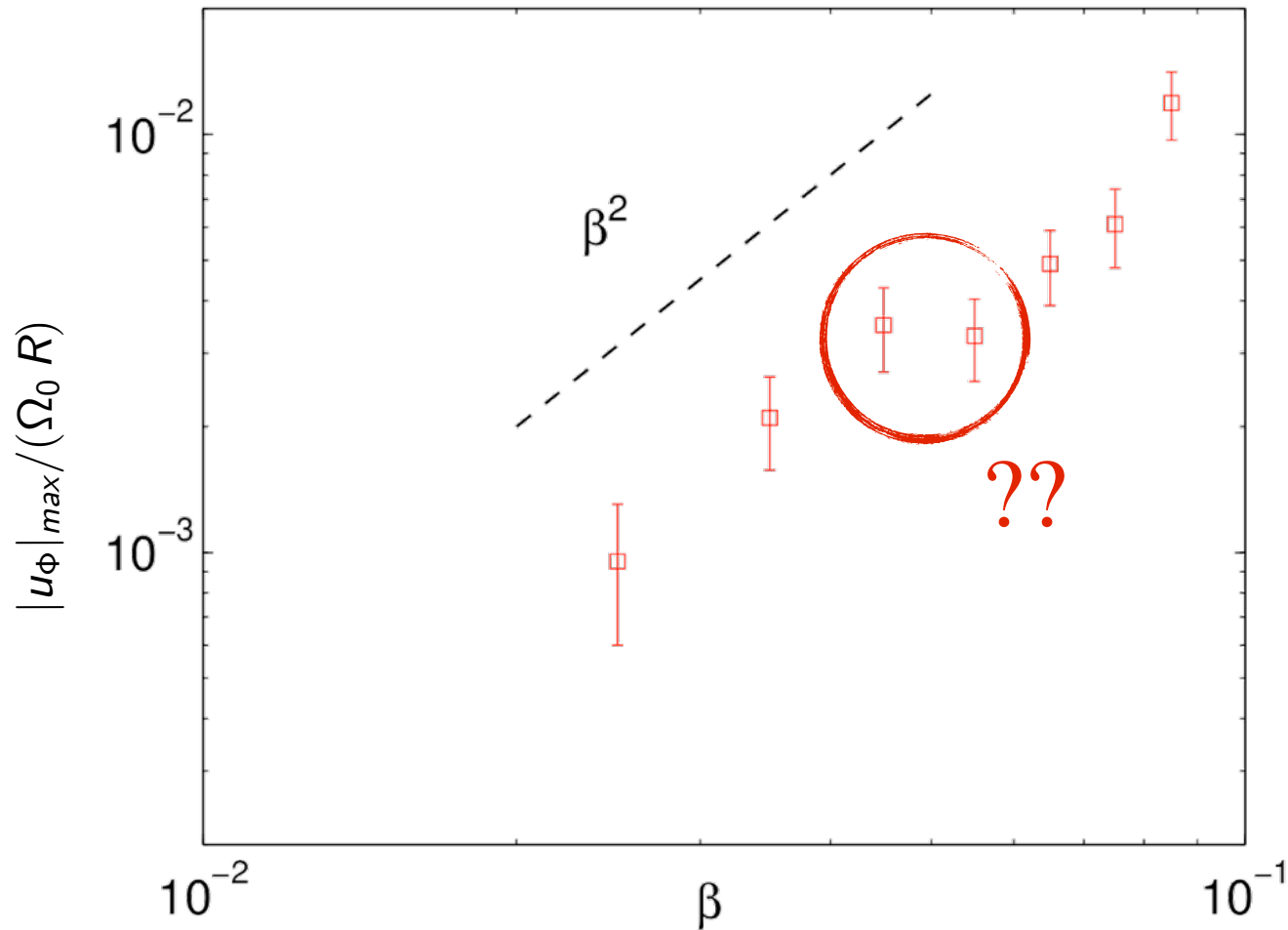
$$\Omega_R = 0.178$$



$$\Omega_R = 0.384$$



# Influence of tides amplitude



$$E = 9.46 \times 10^{-6}$$

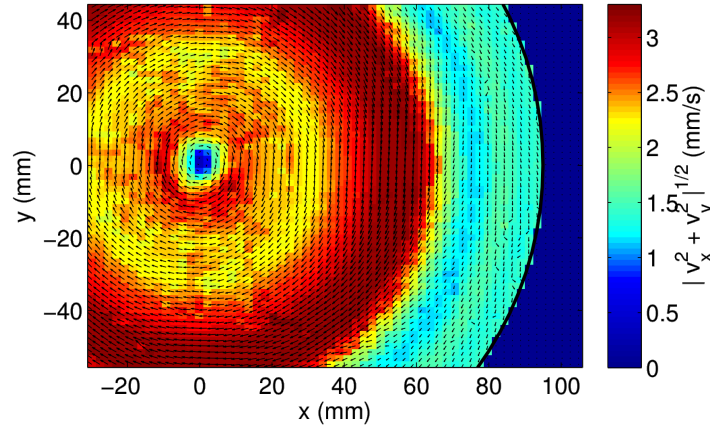
$$\Omega_R = 0.384$$

$$R_{\text{ext}} = 1 + \frac{\beta}{2} \cos(2\Phi)$$

Zonal flow scales as the square of the tidal deformation

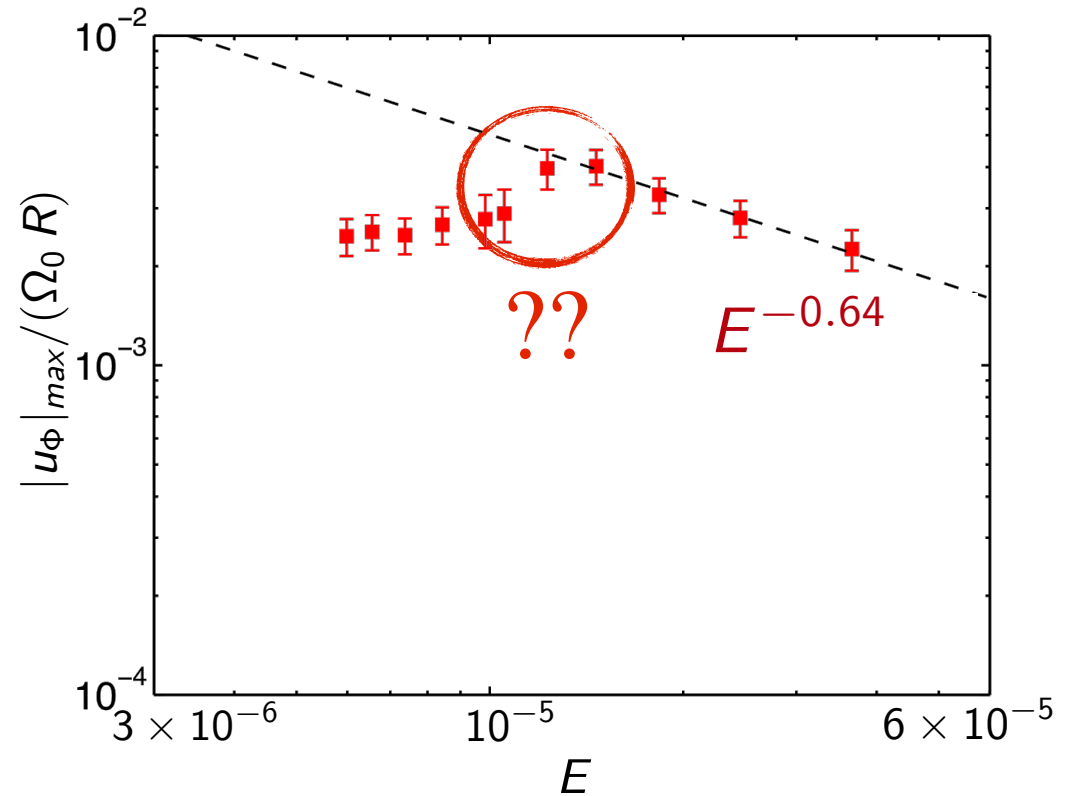
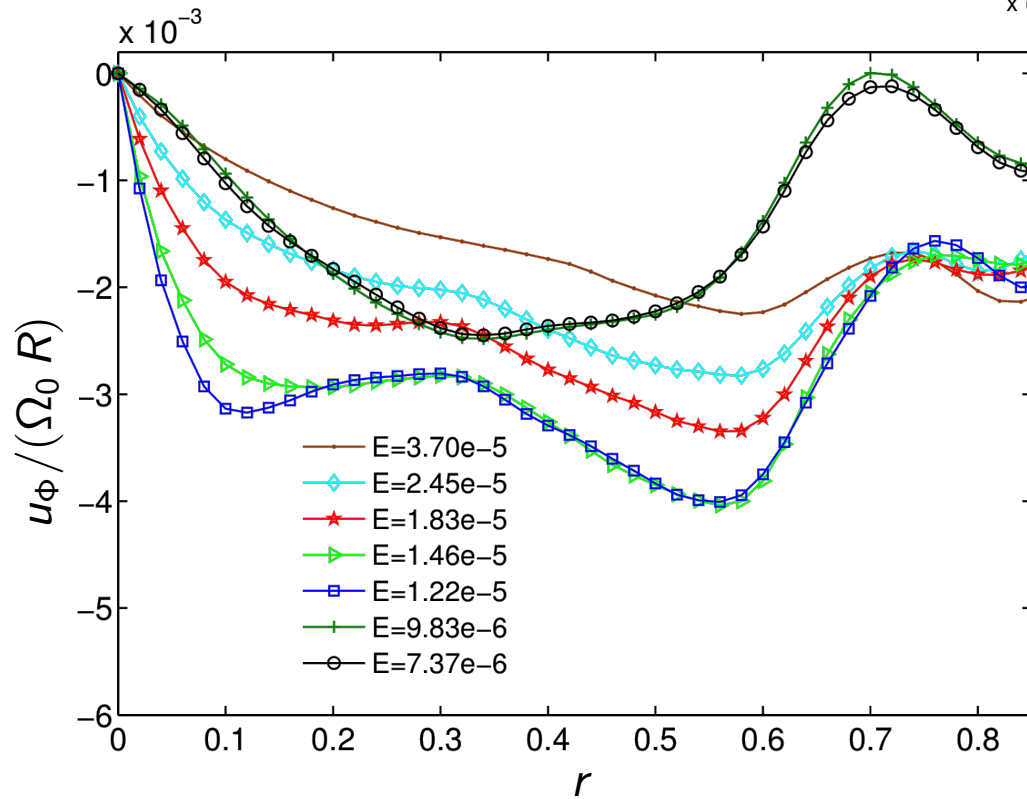
$$u_\phi \propto \beta^2$$

# Influence of the Ekman number

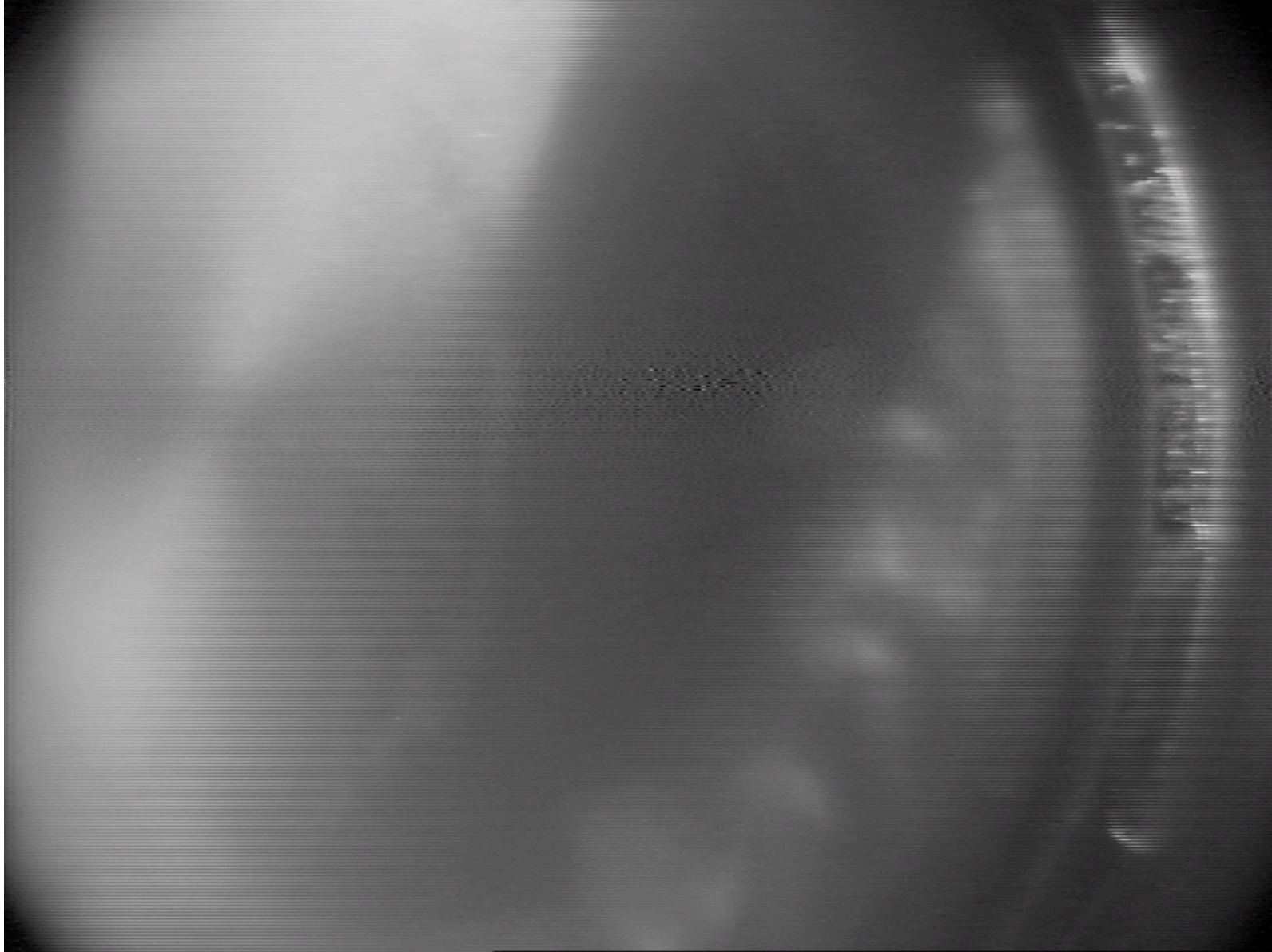


$$\beta = 0.045$$

$$\Omega_R = 0.384$$



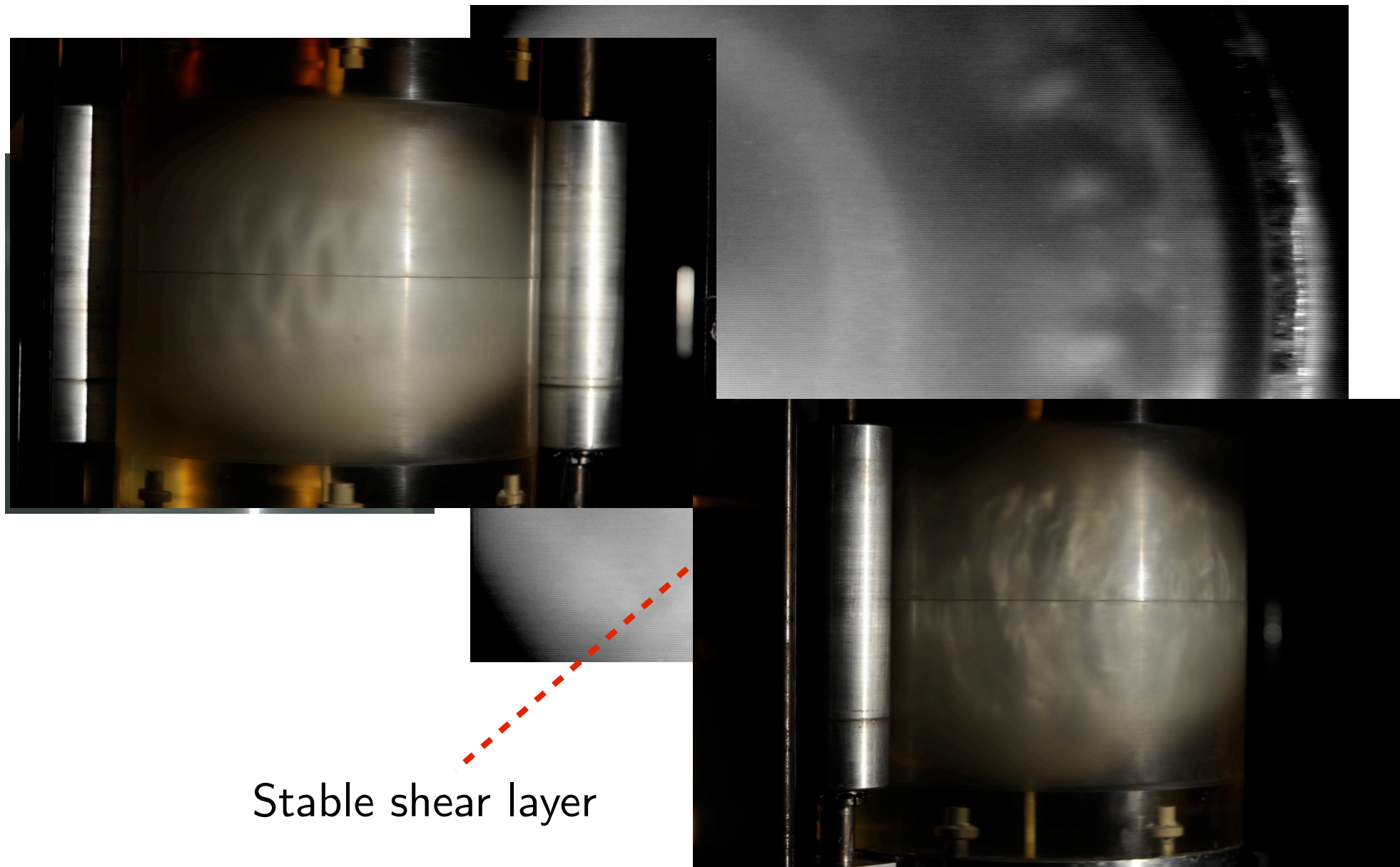
# Shear instability



$$\Omega_R = 0.384, \beta = 0.045 \text{ \& } E = 9.46 \times 10^{-6}$$

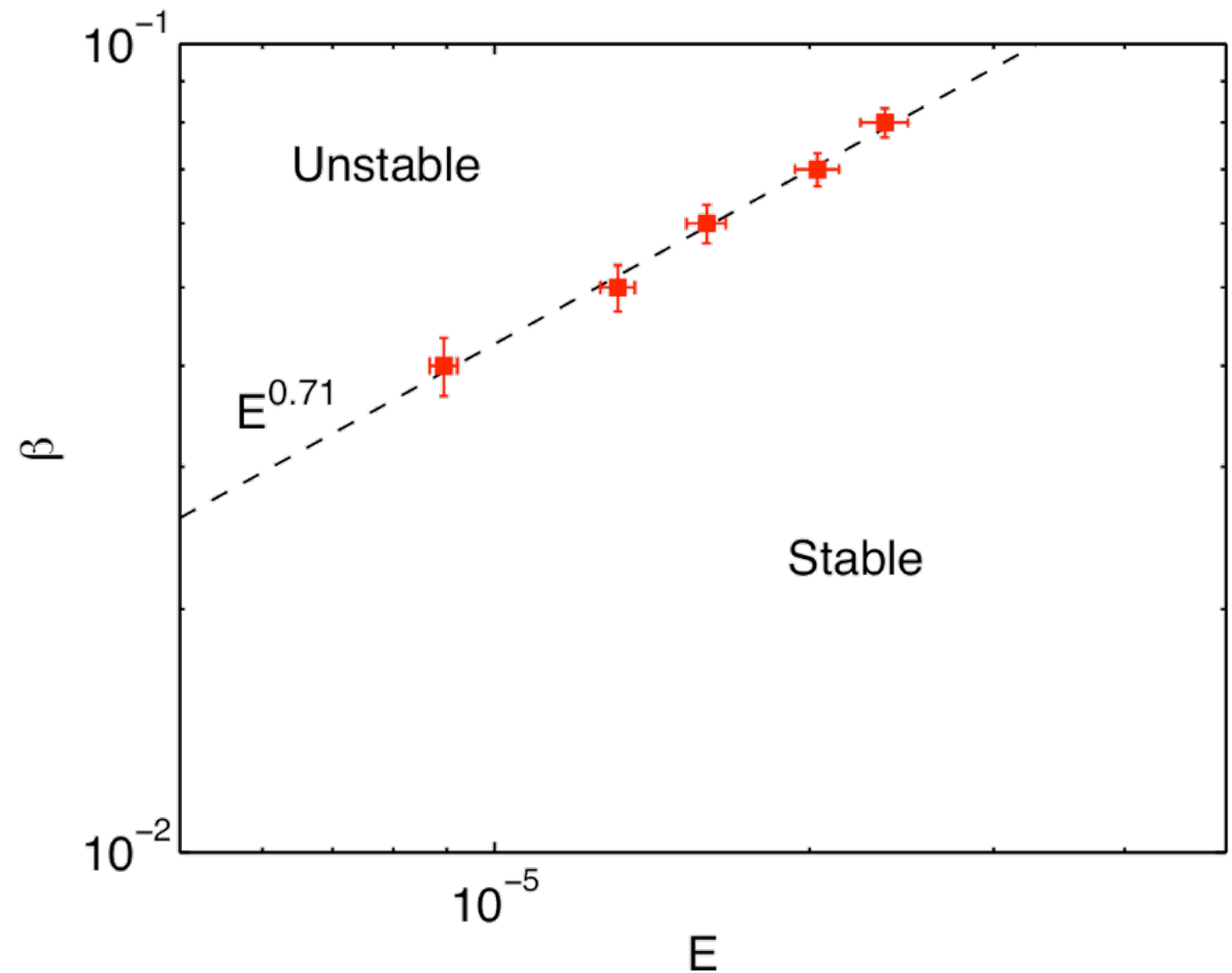
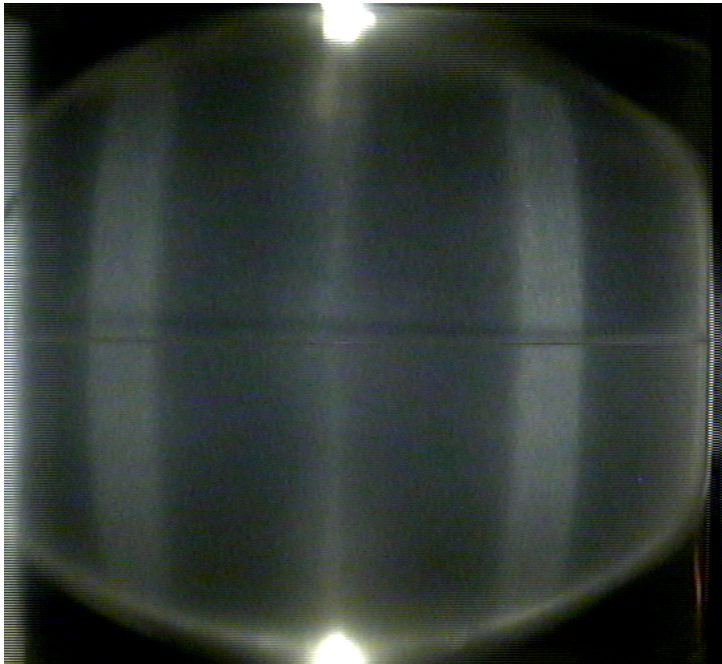


# Shear instability



# Shear instability - scaling

$$\Omega_R = 0.384$$

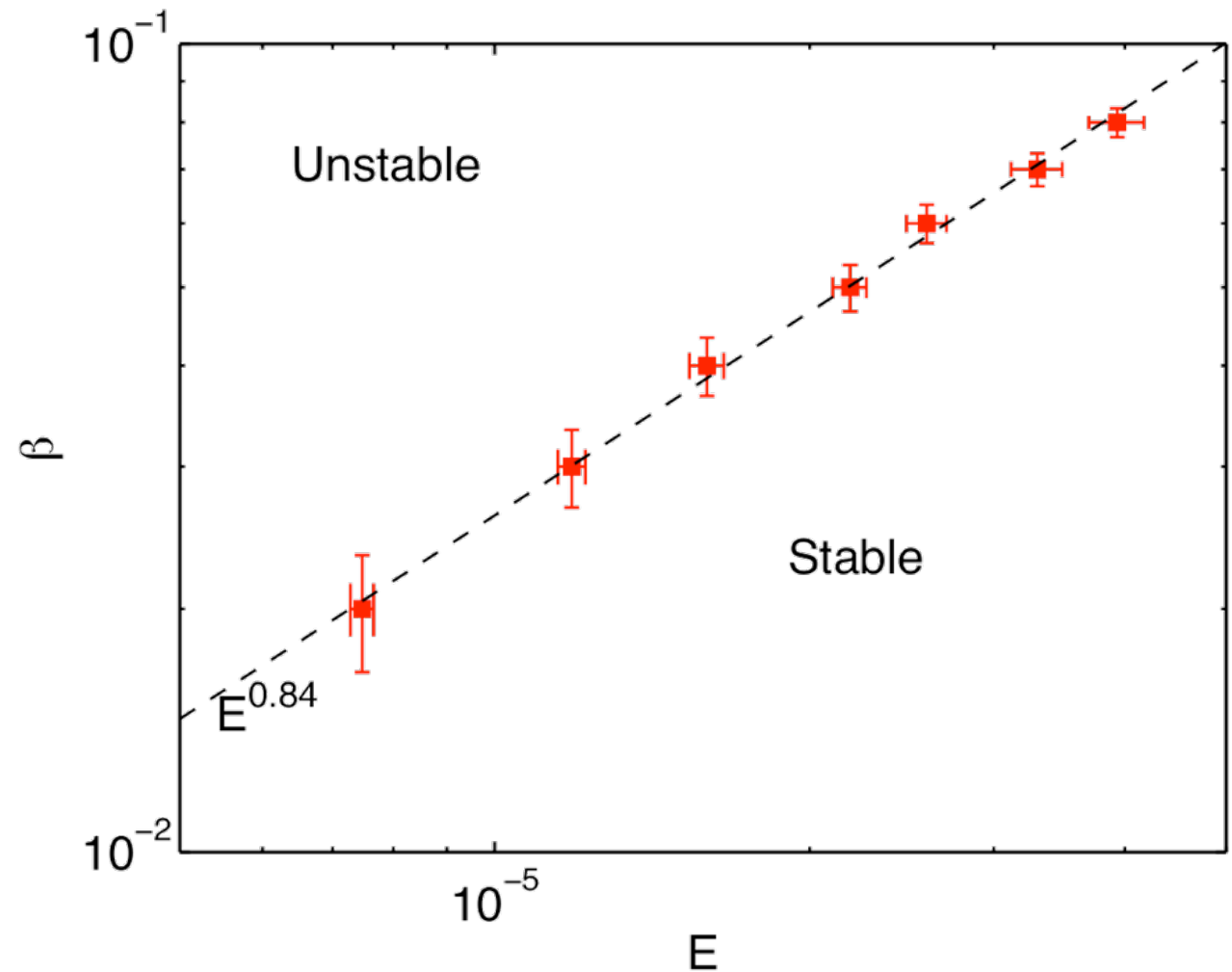
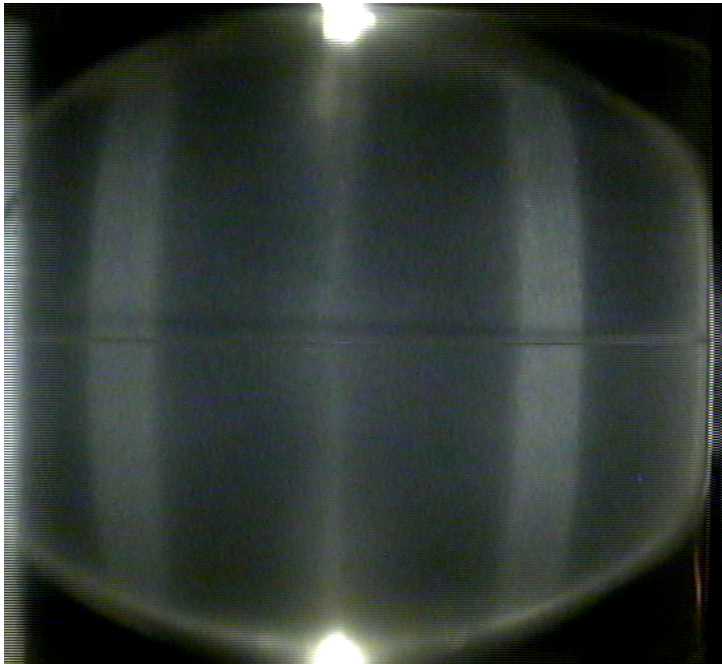


Critical deformation:  $\beta_c \propto E^{0.71}$



# Shear instability - scaling

$$\Omega_R = 0.384$$



Critical deformation:  $\beta_c \propto E^{0.84}$

# Shear instability - scaling

Inviscid criterion:  $\sigma_{NV} \sim \omega_z \sim \frac{\partial U_\phi}{\partial \rho}$

$$\omega_z = \frac{U_\phi}{\rho} + \frac{\partial U_\phi}{\partial \rho}$$

Viscous damping term:  $-k^2 E$

Noir et al., JFM **437** (2001)

Kida, JFM **680** (2011)

Thickness of the shear layer:  $\rho - \rho_c \sim E^{1/5} \tilde{\rho}$

Experimental results:  $U_\phi \sim \rho + \beta^2 E^{-a} \Omega_2(\rho)$

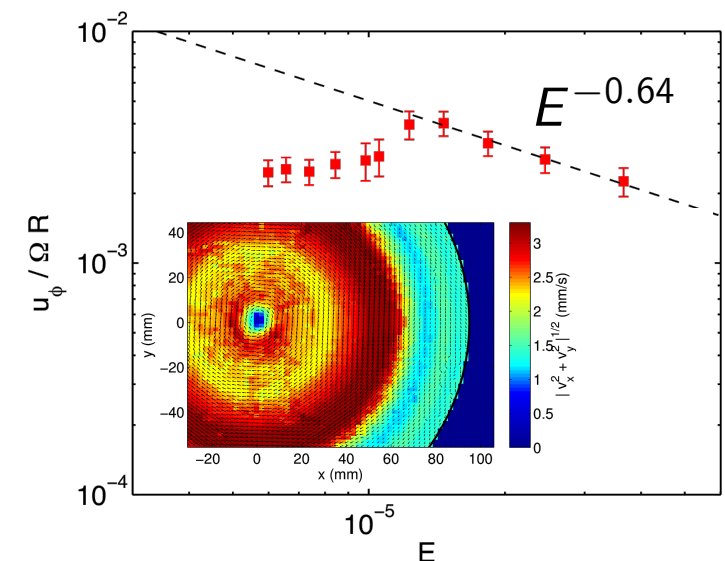
$$\omega_z \sim k^2 E$$



$$\beta_c^2 E^{-a} \delta^{-1} \sim k^2 E \quad k \sim \delta^{-1} \sim E^{-1/5}$$

with  $a \simeq -0.64$

$$\beta_c \sim E^{0.72}$$



# Shear instability - scaling

Inviscid criterion:  $\sigma_{NV} \sim \omega_z \sim \frac{\partial U_\phi}{\partial \rho}$

$$\omega_z = \frac{U_\phi}{\rho} + \frac{\partial U_\phi}{\partial \rho}$$

Viscous damping term:  $-k^2 E$

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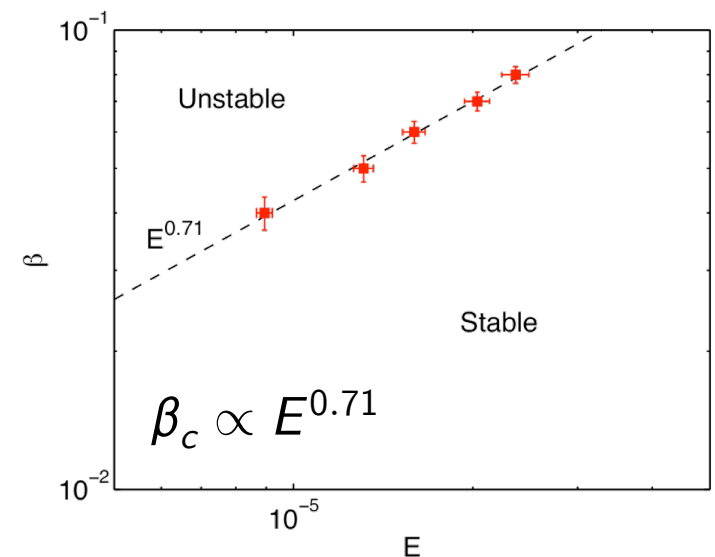
$$\omega_z \sim k^2 E$$



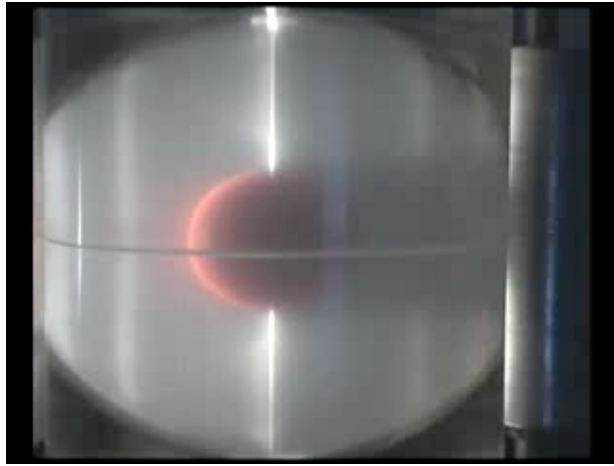
$$\beta_c^2 E^{-a} \delta^{-1} \sim k^2 E \quad k \sim \delta^{-1} \sim E^{-1/5}$$

with  $a \simeq -0.64$

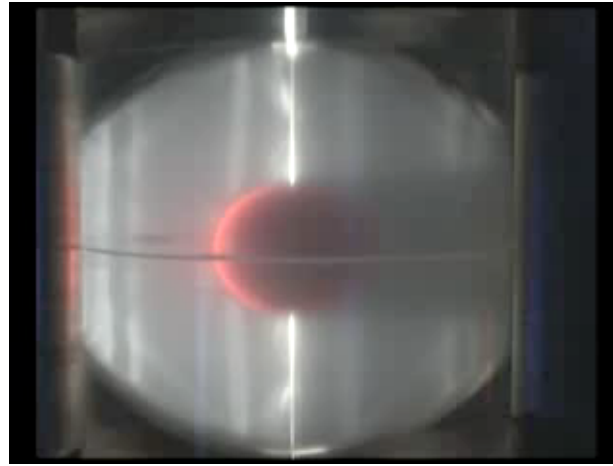
$$\beta_c \sim E^{0.72}$$



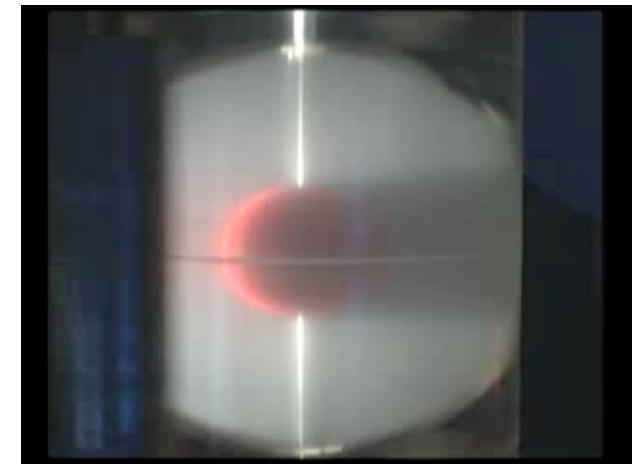
# Presence of an inner core



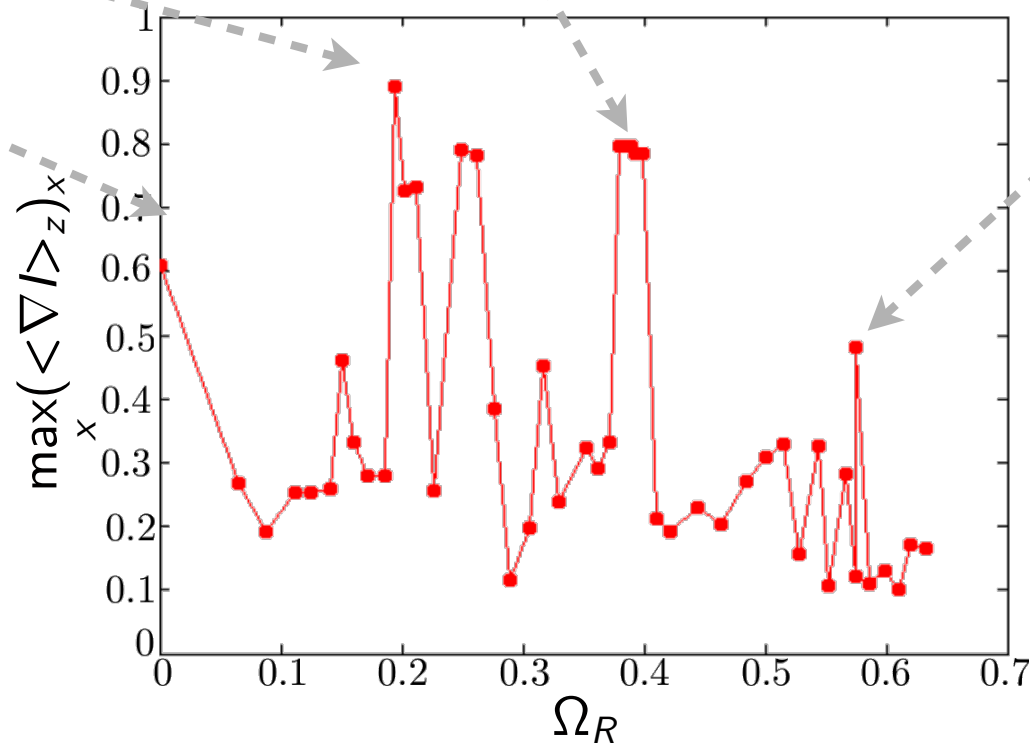
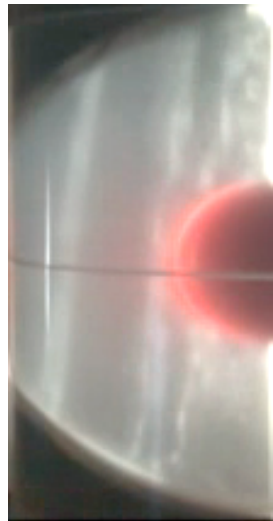
$\Omega_R = 0.19$



$\Omega_R = 0.39$



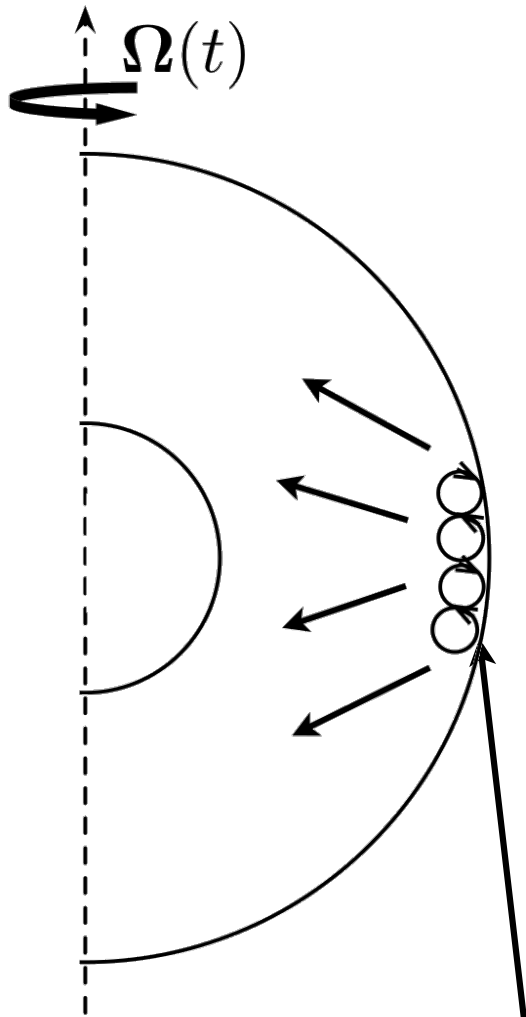
$\Omega_R = 0.55$



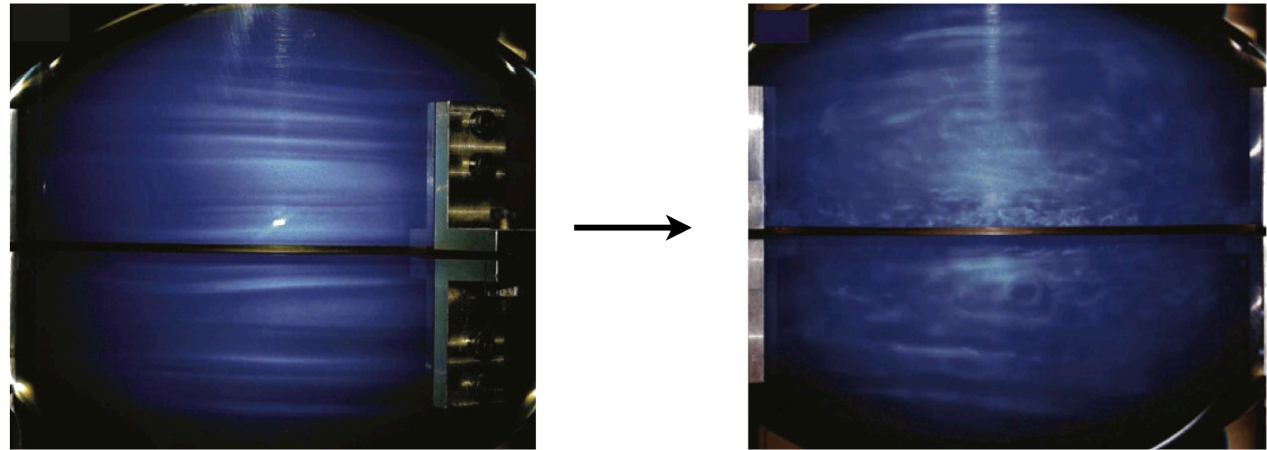
- 1 - Theory of rotating flows
- 2 - Methods of investigations
- 3 - Mean zonal flow
- 4 - Resonance of inertial modes
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# Centrifugal instability

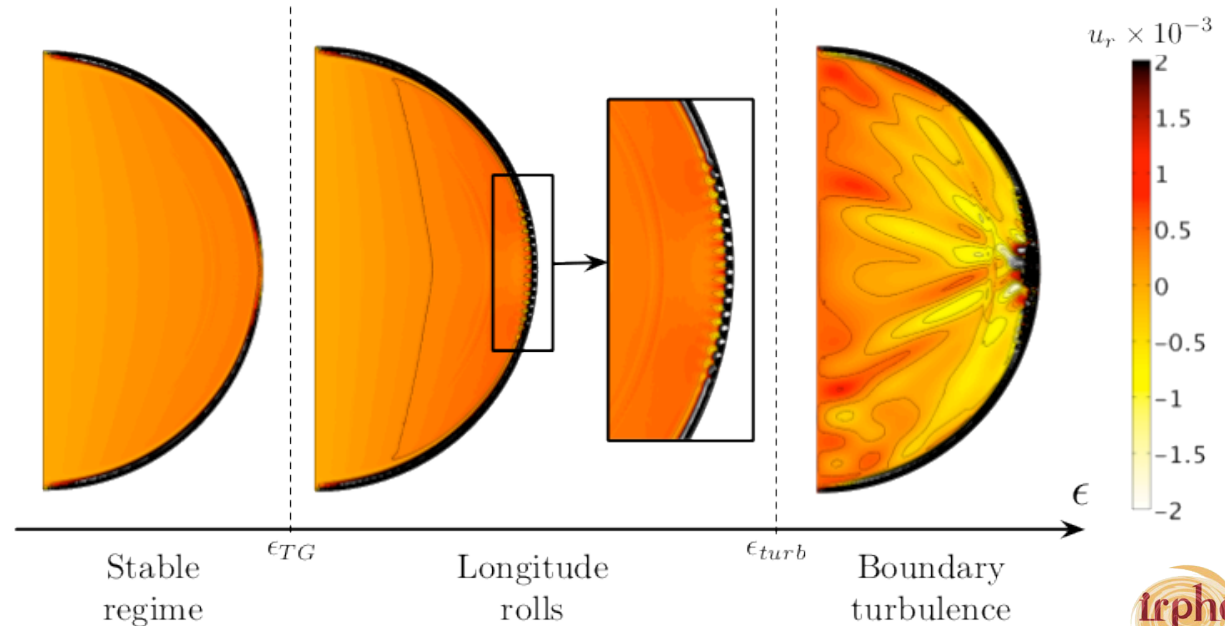
Longitudinal libration:  $\Omega(t) = 1 + \varepsilon \cos(\omega t)$



Instability near the outer boundary  
Taylor-Görtler vortices

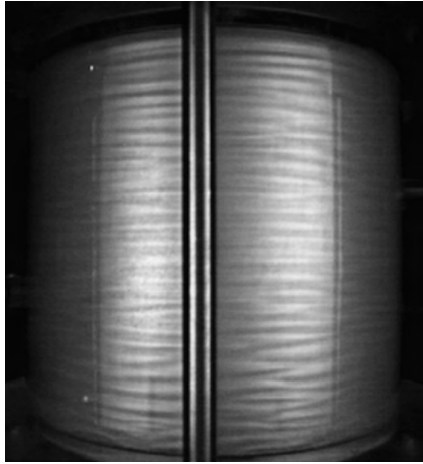


Noir et al., PEPI (2009)

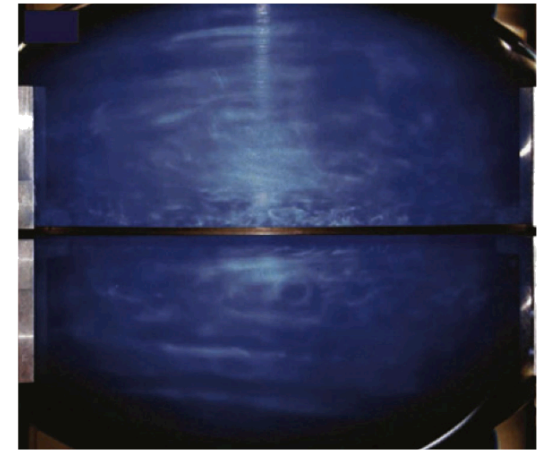
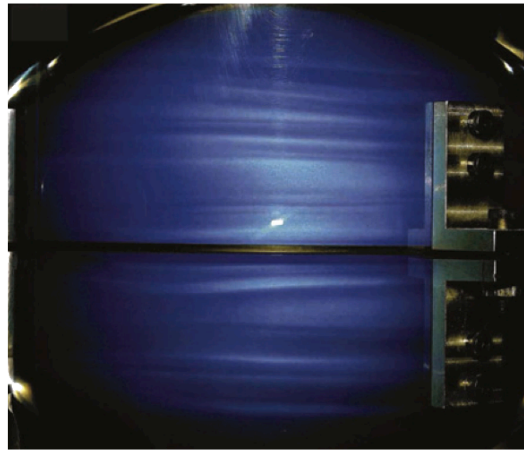


# Centrifugal instability

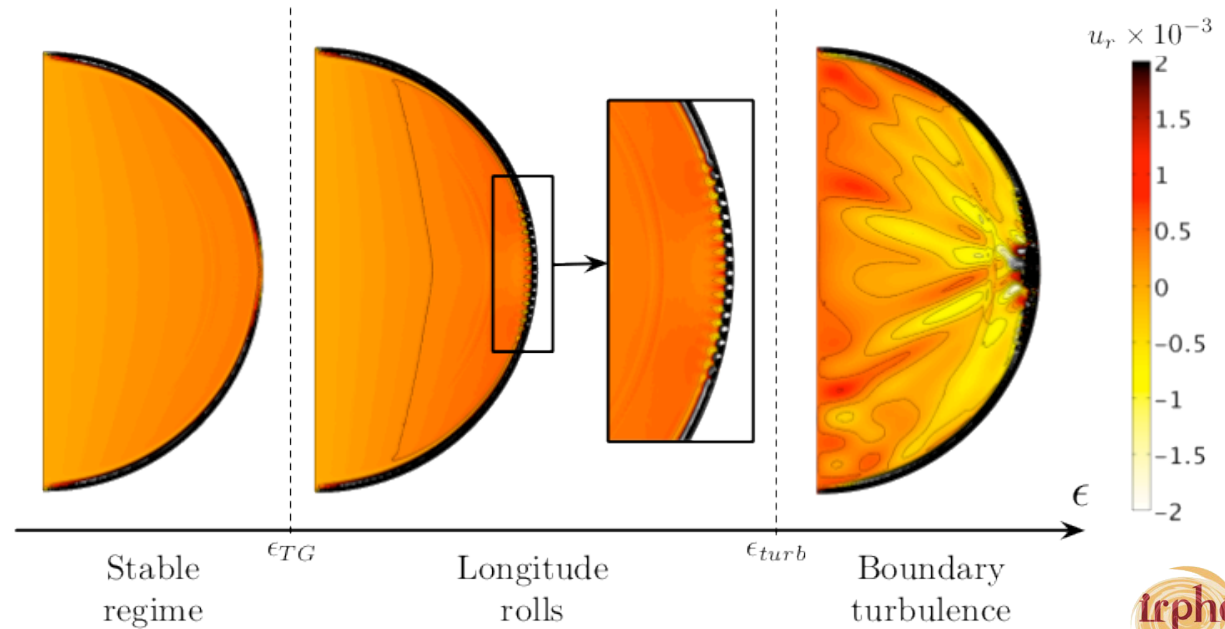
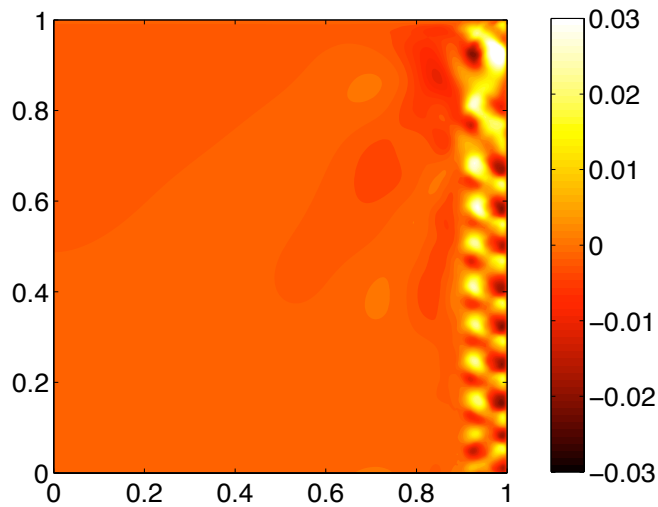
Longitudinal libration:  $\Omega(t) = 1 + \varepsilon \cos(\omega t)$



Noir et al., PEPI (2010)



Noir et al., PEPI (2009)



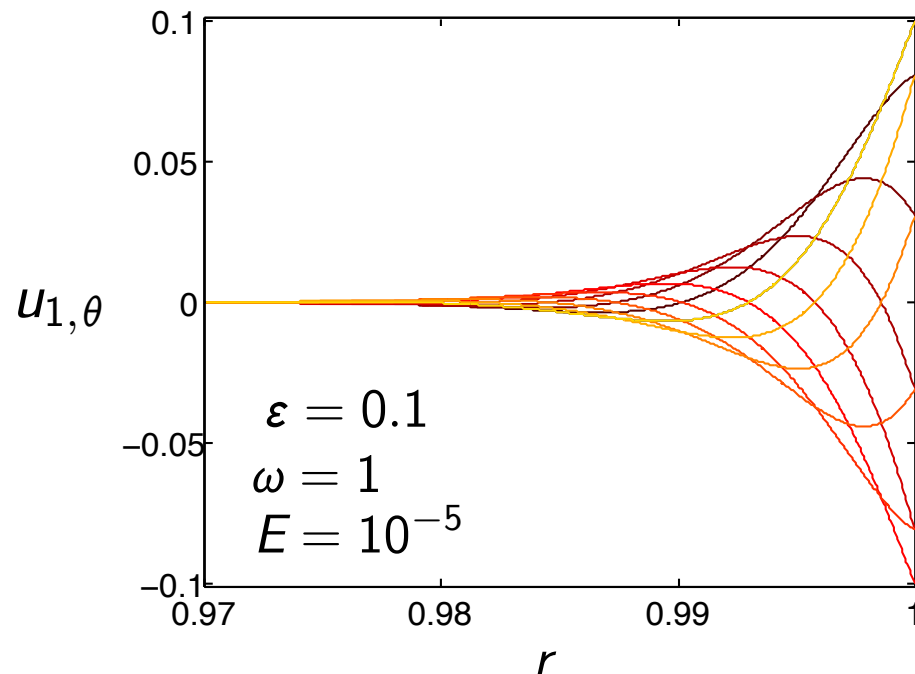


# Analytical determination of the threshold

Experiments (in a cylinder): at the threshold the instability is **axisymmetric**

Oscillatory flow (Stokes layer):

$$u_{1,\theta} = \varepsilon \exp\left(-\frac{1-r}{\sqrt{E}} \sqrt{\frac{\omega}{2}}\right) \cos\left(\omega t - \frac{1-r}{\sqrt{E}} \sqrt{\frac{\omega}{2}}\right)$$



Unstable at some instants

Centrifugal instability

Frozen state

Non viscous analysis

→ Rayleigh criterion

$$\rightarrow \Phi = \frac{d}{dr} \left( r^2 u_{\theta}^{(tot)2} \right) < 0$$

Viscous analysis

# Threshold

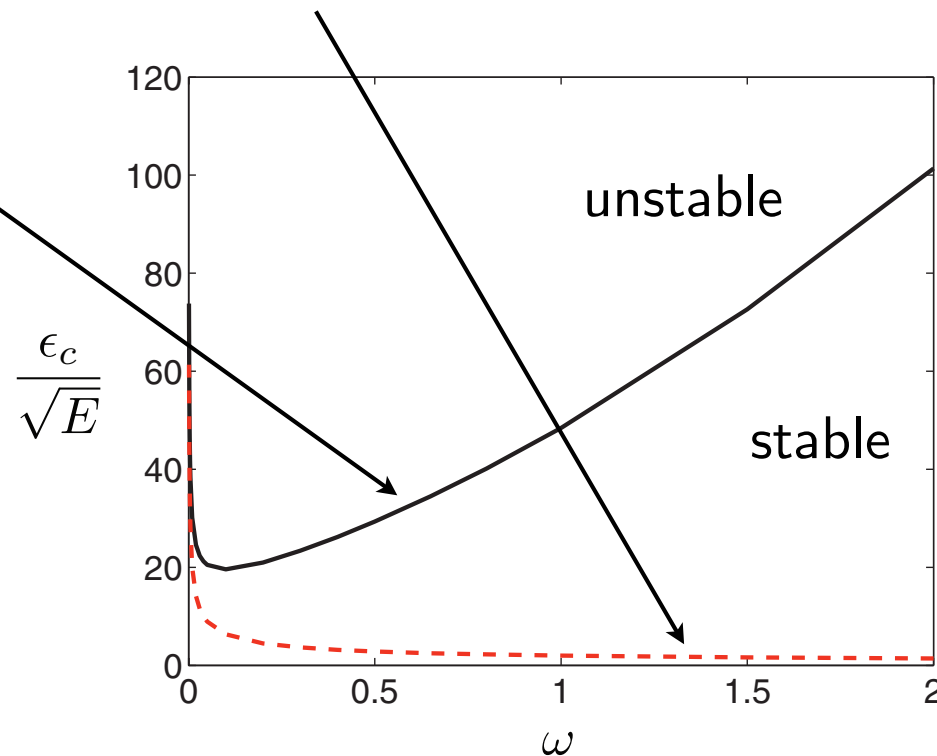
Experiments (in a cylinder): at the threshold the instability is **axisymmetric**

Oscillatory flow (Stokes layer):

$$u_{1,\theta} = \varepsilon \exp\left(-\frac{1-r}{\sqrt{E}} \sqrt{\frac{\omega}{2}}\right) \cos\left(\omega t - \frac{1-r}{\sqrt{E}} \sqrt{\frac{\omega}{2}}\right)$$

Centrifugal instability  $\longrightarrow$  Rayleigh criterion

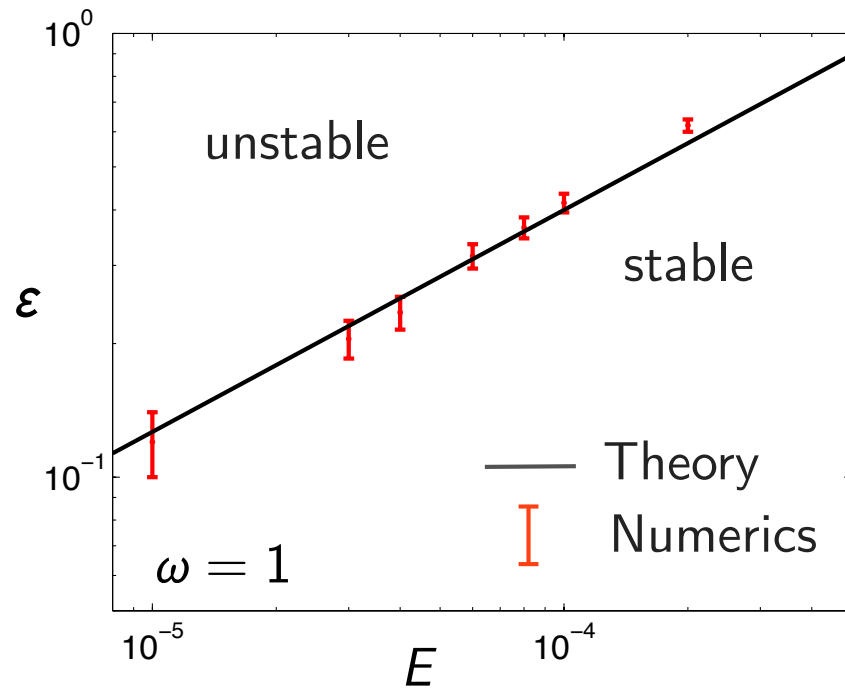
Viscous stability analysis



Sauret et al., Phys. Fluids, **24** (2012)

# Systematic study in a cylinder

Sauret et al., Phys. Fluids, **24** (2012)



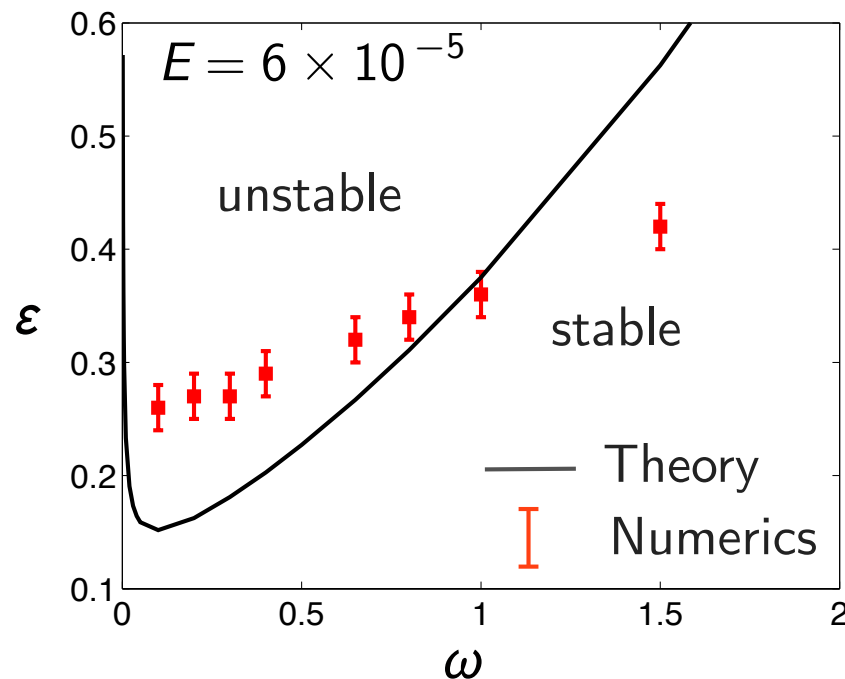
Scales as  $\varepsilon_c \propto E^{1/2}$

Qualitative dependence with  $\omega$

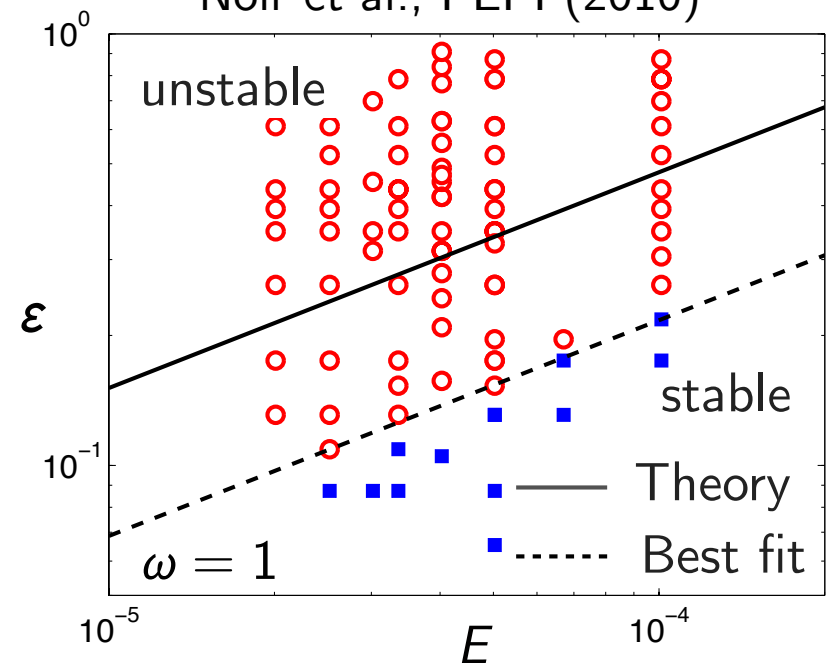
→ quasi-steady approach

→ non linear effects

→ axial flow (Ekman pumping)

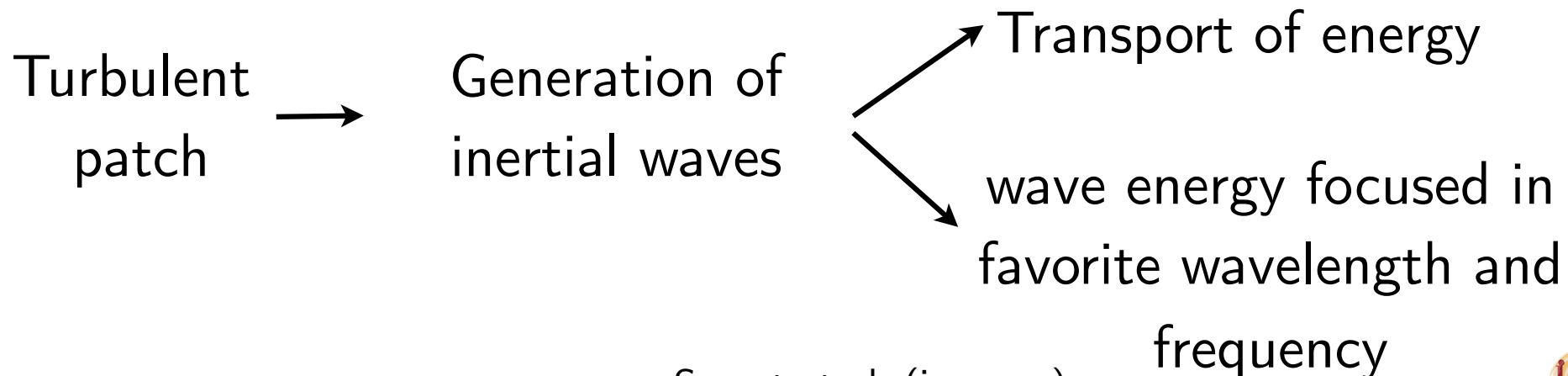
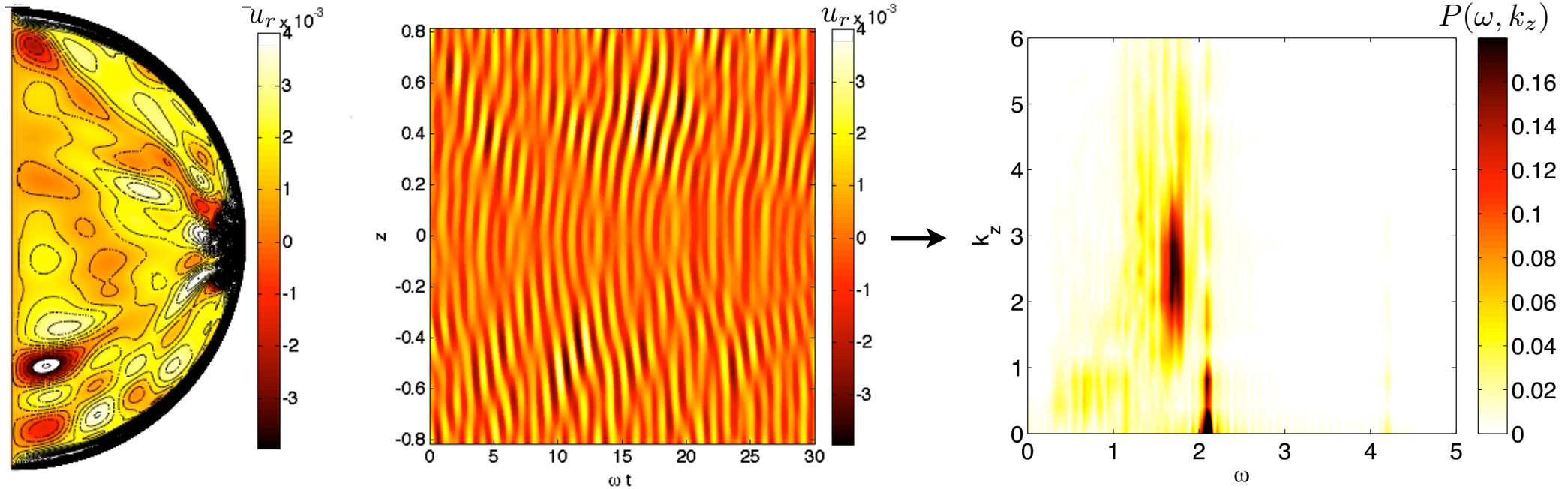


Noir et al., PEPI (2010)



# Spontaneous generation of inertial waves

$$E = 5 \times 10^{-5}, \quad \omega = 2.1, \quad \varepsilon = 0.7$$



Sauret et al. (in prep.)

# Preliminary mechanism

**Localized source:**  $P_{init}(z, t) = f(z) g(t)$

**Propagation:**  $p(x, z, t) = \frac{1}{4\pi^2} \int d\omega \int dk_z \hat{f}(k_z) \hat{g}(\omega) \exp [i (k_x x + k_z z + \omega t)]$

Dispersion relation:  $\omega = 2 \frac{k_z}{\sqrt{k_x^2 + k_z^2}} \longrightarrow k_x^2 = k_z^2 \left( \frac{4}{\omega^2} - 1 \right)$

Evanescent waves:  $|\omega| > 2$       Propagative waves:  $\omega \in [-2; 2]$

$$p(x, z, t) \simeq \frac{1}{2\pi} \int_{-2}^2 d\omega f \left( z - x \sqrt{\frac{4}{\omega^2} - 1} \right) \hat{g}(\omega) e^{-i\omega t}$$

function localized

around  $\omega = 1/\sqrt{1 + z^2/x^2}$

Spectrum of the  
initial excitation

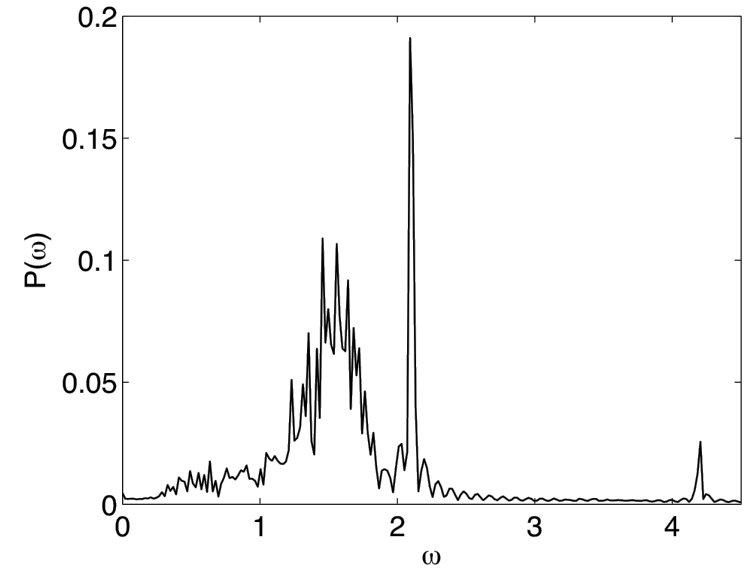
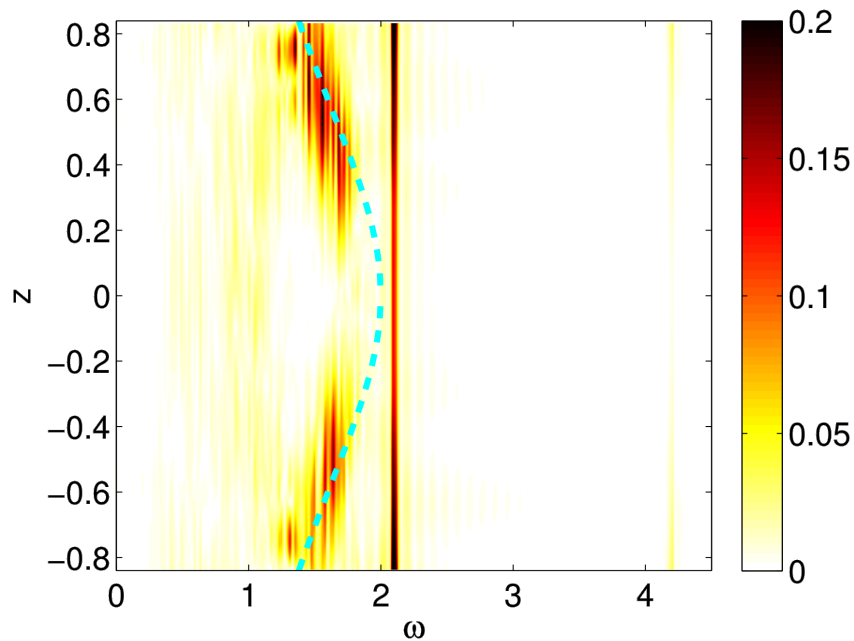
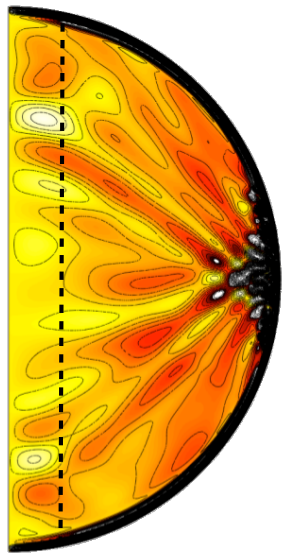
Selection of the frequency during the propagation

Generic mechanism

# Focalisation & selection

$u_r$  at  $r = 0.2$

$\omega = 2.1, \varepsilon = 0.8, E = 4 \times 10^{-5}$



**Propagation:** 
$$\omega_{max} = \frac{1}{\sqrt{1 + \frac{z^2}{(1-0.2)^2}}}$$

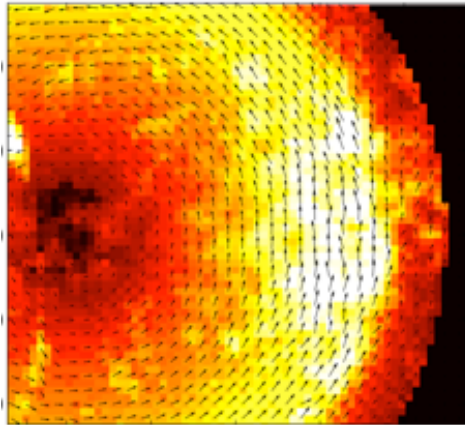
**Maximum radial flux** of axial momentum

Dohan & Sutherland, *Dyn. Atmos. Oceans*, **40** (2005)

Least amount of **viscous damping**

Taylor & Sarkar, *J. Fluid Mech.*, **590** (2007)

## Mean zonal flow

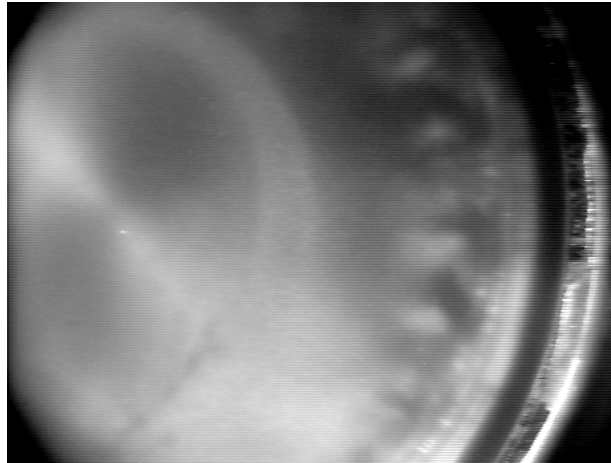


Analytical determination of the **mean zonal flow** in a shell

Experimental and numerical **validation**

**Analogy** between azimuthal and time forcing

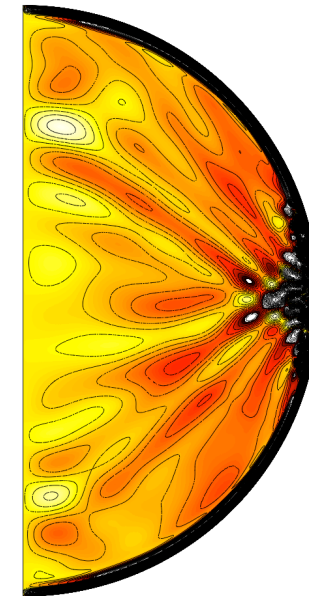
## Inertial waves



Experimental measurements of the **flow at resonance**

Characterization of the subsequent **shear instability**

## Instabilities

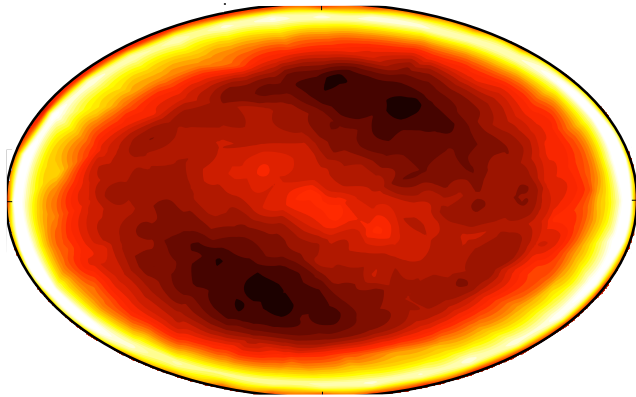


Analytical study of the **threshold** and validation

Numerical evidence of the **spontaneous generation of inertial waves**



## Mean zonal flow

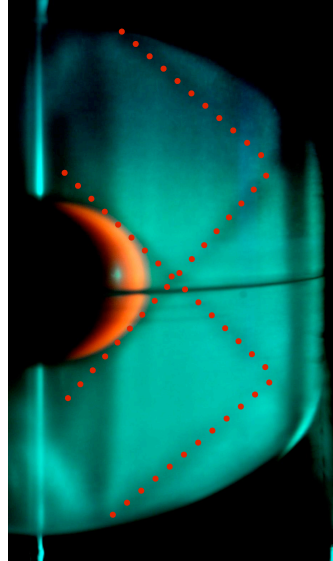


**Topographic & viscous**  
coupling

**Critical latitude** new  
scalings and resolution

Presence of **the inner**  
**core**

## Inertial waves

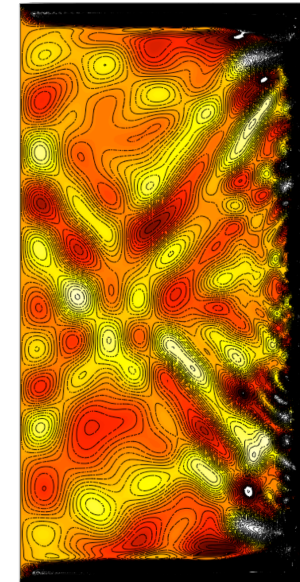


**Experimental**  
measurements in a shell

More accurate **PIV**  
**measurements**

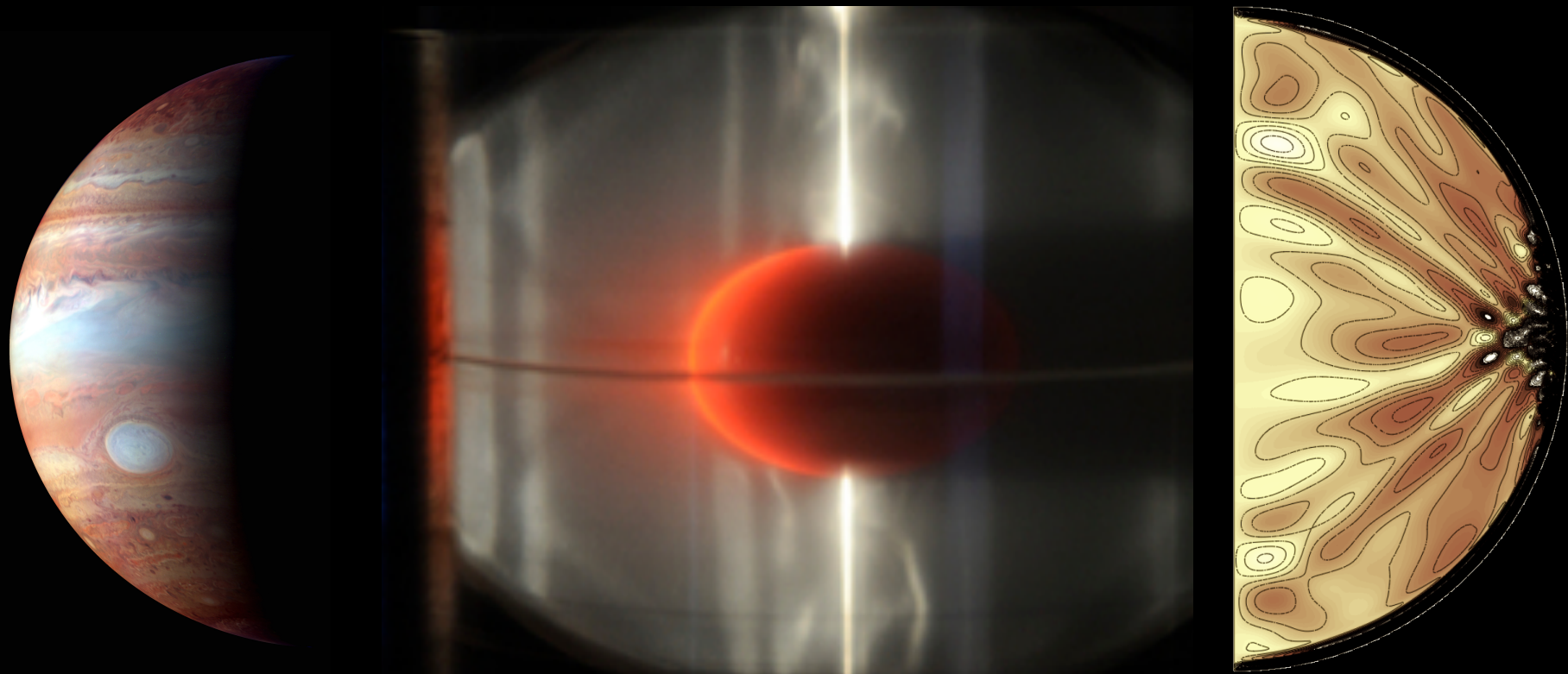
Shear instability with  
**others forcings**

## Instabilities



**Characterization** of the  
excited frequency

**Experimental** evidence  
and quantification



**Thank you for your attention**