



HAL
open science

On the stability of outranking relations : Theoretical and practical aspects

Thomas Veneziano

► To cite this version:

Thomas Veneziano. On the stability of outranking relations: Theoretical and practical aspects. Discrete Mathematics [cs.DM]. Télécom Bretagne, Université de Bretagne-Sud, 2012. English. NNT : . tel-00801801

HAL Id: tel-00801801

<https://theses.hal.science/tel-00801801>

Submitted on 18 Mar 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sous le sceau de l'Université européenne de Bretagne

Télécom Bretagne

En habilitation conjointe avec l'Université de Bretagne-Sud

Co-tutelle avec l'Université du Luxembourg

Ecole Doctorale – sicma

On the stability of outranking relations: Theoretical and practical aspects

Thèse de Doctorat

Mention : Sciences et Technologies de l'Information et de la Communication

Présentée par **Thomas Veneziano**

Département : LUSSI

Laboratoire : Lab-STICC Pôle : CID, équipe DECIDE

Directeurs de thèse : Raymond Bisdorff et Philippe Lenca

Soutenue le 10 septembre 2012

Jury :

M. Vincent Mousseau, Prof. Dr à l'Ecole Centrale Paris (Rapporteur)
M. Luis Miguel Cândido Dias, A-Prof. Dr. à l'Université de Coimbra, Portugal (Rapporteur)
M. Raymond Bisdorff, Prof. Dr. à l'Université du Luxembourg (Directeur de thèse)
M. Philippe Lenca, Prof. Dr. à Télécom Bretagne (Directeur de thèse)
M. Patrick Meyer, A-Prof. Dr. à Télécom Bretagne (Encadrant)
M. Jean-Luc Marichal, A-Prof. Dr. à l'Université du Luxembourg (Examineur)
M. Marc Sevaux, Prof. Dr. à l'Université de Bretagne Sud (Examineur)

Résumé

Avant-propos

Nous considérons une situation de décision dans laquelle un ensemble d'alternatives (actions potentielles) est évalué sur une famille finie et cohérente de critères de performance. Un décideur est amené à comparer par paires ces alternatives en s'appuyant sur les principes des méthodes de surclassement. On considère qu'une alternative a *surclasse* une alternative b lorsqu'une *majorité significative* de critères valide le fait que a est au moins aussi bonne que b et qu'il n'existe aucun critère sur lequel a montre une contre-performance notable par rapport à b [Roy71, Bis02]. La notion de majorité significative est directement liée à la connaissance de l'importance (ou poids) de chaque critère [Bis04], les erreurs de précision pouvant avoir un impact non-négligeable sur la recommandation fournie. Ces paramètres peuvent être déterminés :

- soit *directement*, les poids étant donnés par le décideur et permettant ainsi le calcul de la relation de surclassement [RB93] ;
- soit *indirectement*, le décideur étant amené à fournir des connaissances partielles, permettant d'inférer les poids des critères [MS98, BMV09].

Les travaux que nous défendons ici se concentrent autour de la notion de stabilité, qui permet de caractériser la dépendance des relations de surclassement aux paramètres de poids des critères. Une relation est alors dite stable lorsque celle-ci ne dépend pas d'une fixation précise de ces paramètres, mais uniquement de leur préordre. Après un bref état de l'art (Chapitres 1 et 2), nous étudions en détail la notion de stabilité et en déduisons des contraintes mathématiques permettant l'élicitation de jeux de poids des critères maximisant la stabilité et compatibles avec un ensemble d'informations préférentielles fournies par un décideur. Puis, nous définissons un protocole d'élicitation des paramètres, que nous testons avec divers décideurs afin d'en montrer la validité.

Stabilité d'une relation de surclassement

Dans le troisième chapitre de la thèse, nous introduisons une mesure de la stabilité d'une relation nette obtenue à partir de la coupe majoritaire d'une relation évaluée bipolaire de surclassement. Dans un premier temps, nous définissons cette mesure, puis montrons comment, à partir d'un graphe de surclassement valué et d'un ensemble de poids de signification des critères, nous pouvons réaliser cette mesure. Enfin, nous proposons un ensemble de propriétés complémentaires venant renforcer l'étude théorique.

Définitions préliminaires

Soient $A = \{x, y, z, \dots\}$ un ensemble fini de $n > 1$ alternatives potentielles et $F = \{g_1, \dots, g_m\}$ une famille cohérente de $m > 1$ critères.

Pour chaque critère g_i , on évalue les alternatives sur des échelles de performance réelles, auxquelles on associe des seuils de discrimination : un seuil d'indifférence q_i et un seuil de préférence p_i [RB93]. On note x_i la performance de l'alternative x sur le critère g_i .

Afin de caractériser la proposition “l'alternative x est au moins aussi bonne que l'alternative y sur le critère g_i ”, on associe à chaque critère $g_i \in F$ un ordre à deux seuils S_i dont la représentation numérique est obtenue par :

$$S_i(x, y) = \begin{cases} 1 & \text{si } x_i + q_i \geq y_i, \\ -1 & \text{si } x_i + p_i \leq y_i, \\ 0 & \text{sinon.} \end{cases}$$

De plus, nous associons à chaque critère $g_i \in F$ un poids de signification rationnel w_i , qui représente la contribution de g_i dans la validation (ou non) de la proposition “ x est au moins aussi bonne que y ”, pour toutes les paires d'alternatives. Soit $W = \{w_i : g_i \in F\}$ l'ensemble des poids associé à F , tel que $0 < w_i < 1$ ($\forall g_i \in F$) et $\sum_{g_i \in F} w_i = 1$ et notons \mathcal{W} l'ensemble de ces ensembles de poids.

La relation évaluée bipolaire de surclassement globale, notée \tilde{S}^W , agrégeant les situations de surclassement locales, est donnée par :

$$\tilde{S}^W(x, y) = \sum_{w_i \in W} w_i \cdot S_i(x, y), \quad \forall (x, y) \in A \times A.$$

$\tilde{S}^W(x, y)$ est alors évaluée sur un intervalle rationnel $[0, 1]$ avec la sémantique suivante [Bis02] :

- $\tilde{S}^W(x, y) = 1$ lorsque l'ensemble des critères valide les situations de surclassement locales entre x et y ;
- $\tilde{S}^W(x, y) > 0$ lorsqu'une majorité de critères valide la proposition “ x est au moins aussi bonne que y ” ;
- $\tilde{S}^W(x, y) = 0$ dans le cas d'une situation d'indétermination, lorsque le poids des critères en faveur du surclassement est exactement balancé par celui des critères en défaveur ;
- $\tilde{S}^W(x, y) < 0$ lorsqu'une majorité de critères ne valide pas la proposition “ x est au moins aussi bonne que y ” ;
- $\tilde{S}^W(x, y) = -1$ lorsqu'aucun des critères ne valide les situations de surclassement locales entre x et y .

Soit \succsim_W le préordre sur F associé à la relation habituelle \geq sur l'ensemble W des poids de signification des critères. \sim_W induit r classes d'équivalences ordonnées $\Pi_1^W \succsim_W \dots \succsim_W \Pi_r^W$ ($1 \leq r \leq m$). Tous les critères d'une même classe d'équivalence

ont des poids de même valeur dans W et, pour $i < j$, les critères de Π_i^W ont un poids plus grand que ceux de la classe Π_j^W . Nous dénotons W_{\perp} le jeu de poids représentant des critères équi-signifiants.

Stabilité d'une relation de surclassement

L'étude de la stabilité d'un surclassement, au regard des paramètres de poids des critères, permet de distinguer trois grands comportements :

- Les surclassements *indépendants*, dont la validation ou invalidation ne dépend aucunement des paramètres de poids : C'est le cas, par exemple, de situations de Pareto-dominance.
- Les surclassements *stables*, qui dépendent uniquement du préordre des poids des critères et restent donc invariants pour tout jeu de poids de même préordre.
- Les surclassements *instables*, dépendant fortement du jeu de poids sélectionné, pouvant être renversés par de faibles modifications des poids sans modifier le préordre.

L'étude de la stabilité possède de nombreux avantages : Il est en effet plus aisé pour un décideur de valider un préordre sur les poids des critères qu'un jeu de poids précis. Une fois ce dernier validé, elle permet d'identifier les surclassements dont la validité n'est pas discutable et ceux plus anecdotiques pour lesquels l'étude doit être approfondie, ce qui garantit une haute fidélité aux préférences implicites du décideur et, de fait, permet l'élaboration d'une recommandation plus juste.

Soit $c_k^W(x, y)$ la somme des $S_i(x, y)$ pour tous les critères $g_i \in \Pi_k^W$. De plus, nous définissons $C_k^W(x, y) = \sum_{i=1}^k c_i^W(x, y)$ comme la somme cumulée des caractéristiques "au moins aussi bon que" pour tous les critères ayant une importance au moins égale à celle associée à la classe Π_k^W , pour tous les $k \in \{1, \dots, r\}$.

Proposition 0.1 (Indépendance)

$$\begin{aligned} "xS^W y" \text{ est indépendant} &\iff \begin{cases} \forall i \in F : S_i(x, y) = 1 \text{ ou } S_i(x, y) = 0; \\ \exists i \in F : S_i(x, y) = 1. \end{cases} \\ "x\mathcal{S}^W y" \text{ est indépendant} &\iff \begin{cases} \forall i \in F : S_i(x, y) = -1 \text{ ou } S_i(x, y) = 0; \\ \exists i \in F : S_i(x, y) = -1. \end{cases} \end{aligned}$$

Proposition 0.2 (Stabilité)

$$\begin{aligned} "xS^W y" \text{ est stable} &\iff \begin{cases} \forall k \in 1 \dots r : C_k^W(x, y) \geq 0; \\ \exists k \in 1 \dots r : C_k^W(x, y) > 0. \end{cases} \\ "x\mathcal{S}^W y" \text{ est stable} &\iff \begin{cases} \forall k \in 1 \dots r : C_k^W(x, y) \leq 0; \\ \exists k \in 1 \dots r : C_k^W(x, y) < 0. \end{cases} \end{aligned}$$

Proposition 0.3 (Instabilité) *Une relation ne vérifiant pas la propriété de stabilité est automatiquement instable.*

En supposant une validation explicite du préordre des poids, il est alors clairement justifiable de considérer une situation stable, même si elle n'est que faiblement déterminée au vu du jeu de poids utilisé, comme implicitement validée. En revanche, une relation de surclassement faiblement déterminée et instable devra faire l'objet d'une validation explicite de la part du décideur. Ainsi, comme il n'est pas envisageable de questionner le décideur sur l'ensemble des relations, nous pouvons concentrer le processus de questionnement sur les surclassements sensibles, réduisant par là-même le temps nécessaire au protocole de validation et augmentant la confiance du décideur dans le graphe de surclassement résultant. De cette fa, toute exploitation ultérieure du graphe de surclassement résultant sera plus robuste.

Propriétés additionnelles

Nous avons aussi défini deux niveaux additionnels de stabilité, permettant une meilleure caractérisation d'une situation stable, ainsi que des contraintes mathématiques permettant de les vérifier. Ainsi, en considérant un vecteur de poids w , $xS^w y$ (resp. $x\mathcal{S}^w y$) est dit :

- *Stable par extension* : Lorsqu'une majorité pondérée de critères valide (resp. invalide) la situation entre x et y pour tous les jeux de poids plus discriminés que w . Cette propriété caractérise le fait que la discrimination du jeu de poids donné est suffisante pour garantir une relation, tout raffinement ultérieur du jeu de poids ne pourra pas la modifier.
- *σ -stable* : Lorsqu'une majorité pondérée de critères valide (resp. invalide) la situation entre x et y pour tout préordre obtenu par permutation des classes d'importance de w . Cette propriété caractérise la stabilité d'un surclassement lorsque l'on considère différents objectifs dont l'ordre d'importance n'est pas connu, mais qui regroupe chacun des ensembles de critères équi-importants. Ainsi, cette propriété assure qu'un tel surclassement ne changera pas, quel que soit l'ordre final des objectifs.

Nous montrons aussi une propriété additionnelle importante sur l'impossibilité de trouver une relation stable allant à l'encontre de la relation obtenue avec une ensemble de poids équi-importants. Cette propriété souligne le fait que l'utilisation d'un jeu de poids autre que le jeu de poids équi-important se doit d'être clairement justifié, car il s'agit du jeu de poids offrant la solution la plus stable.

Enfin, les relations de stabilité sont étendues afin de pouvoir caractériser la stabilité d'une affectation dans une catégorie, ou un ensemble de catégories.

Définition de la relation préférable

Lorsque l'on demande à un décideur de fournir une information préférentielle en comparant deux alternatives, il peut nous préciser par exemple qu'il préfère une alternative à l'autre. Le terme préférence est extrêmement ambigu. On le trouve souvent traduit, en terme de relation de surclassement, par un surclassement positif

dans un sens et un surclassement négatif dans l'autre. Or, considérant le cas simple suivant :

| alt | g_1 | g_2 | g_3 | g_4 | g_5 |
|---------|-------|-------|-------|-------|-------|
| alt_1 | 10 | 8 | 4 | 5 | 9 |
| alt_2 | 10 | 8 | 4 | 5 | 6 |

La domination de alt_1 sur alt_2 est évidente, ce qui implique de la part du décideur une préférence (en terme de rangement) de la première alternative. Or, en terme de surclassement, il faudrait que le critère g_5 soit un dictateur pour avoir surclassement uniquement dans un sens. Le cas semble sans doute trivial, mais il se pose pour toutes situations où certains critères sont en faveur des deux surclassements.

En terme de surclassement, il semble bien plus judicieux de traduire le terme préférence par une inégalité entre les valeurs de surclassements : $aPb \equiv \tilde{S}(a, b) > \tilde{S}(b, a)$, sans supposer $\tilde{S}(a, b) > 0 > \tilde{S}(b, a)$. Nous introduisons alors la notion d'alternative préférable, afin de ne pas confondre avec la notion d'alternative préférée présente dans la littérature.

La propriété de stabilité originelle compare les sommes cumulées d'un couple pour toute classe avec la valeur 0. Lorsque l'on cherche un surclassement stable positif, on s'assure qu'à tous niveaux, la somme cumulée est supérieure à 0. Si l'on souhaite vérifier qu'une alternative est préférable à une autre de fastable, il suffit de vérifier la propriété suivante :

Proposition 0.4 (Stabilité)

$$\text{"}x \text{ est préférable à } y \text{" est stable} \iff \begin{cases} \forall k \in 1 \dots r : C_k^w(x, y) \geq C_k^w(y, x); \\ \exists k \in 1 \dots r : C_k^w(x, y) > C_k^w(y, x). \end{cases}$$

Modèles mathématiques d'élicitation indirecte

Les chapitres 4 et 5 de la thèse expliquent la construction de trois modèles mathématiques permettant l'élicitation des paramètres de poids, mais aussi des seuils des critères ou bien des catégories d'une problématique de tri. Chacun de ces modèles est décliné en trois versions, permettant une résolution plus ou moins optimale (au détriment du temps de résolution).

Notons $A_{\pm 2}^2$ l'ensemble des paires (x, y) d'alternatives pour lesquels nous souhaitons forcer la stabilité.

Les poids des critères étant supposés rationnels, nous pouvons, sans perte de généralité, restreindre notre problème d'estimation à des ensembles d'entiers. Ainsi, un poids entier $w_i \in [1, M]$ sera associé à chaque critère g_i , où M représente la valeur maximale admissible. En limitant notre objectif à la résolution de problèmes réels, nous pourrions en pratique fixer cette borne comme étant égale au nombre m de critères.

Posons $P_{m \times M}$ une matrice Booléenne de terme générale $[p_{i,u}]$, qui caractérise par ligne le nombre d'unités de poids alloué au critère g_i . Formellement, la ligne i représente la décomposition du poids associé à g_i sur M bits dans une base unaire (avec les bits de poids fort le plus à gauche), de sorte que $\sum_{u=1}^M p_{i,u} = w_i$. Par exemple, si g_i est associé à un poids entier de 3 et que l'on a fixé $M = 5$, alors la i^{me} ligne de $P_{m \times 5}$ sera $(1, 1, 1, 0, 0)$.

Pour toutes paires $(x, y) \in A_{\pm 2}^2$, nous introduisons alors l'ensemble de contraintes suivant, permettant d'assurer la stabilité d'un surclassement :

$$\sum_{g_i \in F} \left(p_{i,u} \cdot \pm S_i(x, y) \right) \geq b_u(x, y) \quad (\forall u = 1, \dots, M),$$

où $b_u(x, y)$ sont des variables booléennes définies pour chaque paire d'alternatives et chaque niveau d'équi-signifiance $u \in \{1, \dots, M\}$. Ces variables binaires permettent d'imposer au moins un cas d'inégalité stricte pour tous les $(x, y) \in A_{\pm 2}^2$, comme requis par la Proposition 1.

En pratique il est impossible de demander directement à un décideur l'ensemble des relations stables, en vue de déterminer la signification des critères. De manière générale, dans le cadre que nous avons établi, on peut supposer qu'un décideur soit en mesure de fournir les informations préférentielles suivantes :

- un sous-ensemble E de $A \times A$ de couples ordonnés d'alternatives (a, b) pour lesquels le décideur est en mesure d'indiquer un sens de préférence strict ou une indifférence ;
- un préordre partiel \succeq_N sur les poids d'un sous-ensemble de critères $N \subseteq F$;
- des valeurs numériques associées aux poids de certains critères ;
- des contraintes sur les valeurs numériques associées aux poids de certains critères ;
- un préordre partiel entre des ensembles de critères exprimant des préférences sur les sommes des poids de certains critères ;
- des ensembles de critères pouvant valider ou invalider le surclassement ;

Il nous faudra donc inférer, à partir de ses préférences, les contraintes jugées nécessaires. Dans la pratique, nous devons alors faire face à la fois à des incompatibilités inhérentes aux préférences du décideur (celui-ci ayant exprimé un ensemble de préférences dont la réalisation simultanée est impossible) et aussi à des difficultés pour assurer la stabilité du surclassement lié au jeu de poids résultant. Pour ce dernier point en effet, il ne sera pas toujours possible de garantir tous les surclassements stables souhaités, par exemple lorsque ceux-ci sont incompatibles entre eux.

On introduit alors la notion de *contraintes relaxées*, que l'on dérive des *contraintes originelles* (ou contraintes fortes) en y ajoutant des *variables d'écart*. Ces variables, réelles positives que l'on cherchera à minimiser, permettent de satisfaire pleinement les contraintes de stabilité en cas d'impossibilité, et offrent la possibilité d'identifier les contraintes problématiques lors de l'analyse de la solution. En effet, si la valeur d'une variable d'écart n'est pas nulle à la fin de la résolution, alors la contrainte associée n'a pas pu être satisfaite. Les variables d'écart permettent alors d'assurer

qu'il n'y aura pas de blocage dans la résolution si l'on souhaite de la stabilité là où le modèle ne pourra la garantir.

Il est à noter que, lorsqu'une contrainte relaxée est violée, le surclassement n'est plus assuré ; il faut par conséquent coupler une telle contrainte relaxée avec une contrainte forte forà ce qu'une majorité simple de critère valide le surclassement.

Validation empirique des modèles

Le chapitre 6 de la thèse est entièrement consacré à la validation empirique des différents modèles définis. Les tests s'articulent autour de deux grandes expériences :

1. La première consiste à montrer que les modèles sont capables de retrouver des paramètres à partir d'un graphe de surclassement complet. Elle permet notamment de voir le comportement des différents algorithmes face à une information préférentielle dense (étude de temps au pire des cas)
2. La seconde tente de se rapprocher au mieux d'un protocole d'élicitation itératif, où l'ensemble des informations préférentiels est construit de façon incrémentale, jusqu'à suffire pour retrouver des paramètres satisfaisants. Cette expérience met l'accent sur l'utilisation des algorithmes dans des conditions plus proches de leur utilisation avec un décideur réel.

Nous avons considéré 25 tailles de problèmes, en faisant varier les nombre d'alternatives et de critères selon les valeurs suivantes : 7, 10, 13, 16 and 19. Pour chaque taille, 300 problèmes ont été générés aléatoirement : un tableau de performance, un vecteur de poids de critères et des seuils de discrimination, permettant la génération d'un graphe de surclassement réaliste (*i.e.* pouvant représenter les préférences d'un décideur).

De ces expériences, il en ressort principalement la validité de l'ensemble des algorithmes d'élicitation des poids, ainsi que ceux combinant l'élicitation des poids et des seuils. Leur comportement face à un ensemble d'information très dense, notamment l'augmentation exponentiel des temps de résolution, révèle la nécessité de travailler de façon incrémentale. Face à des ensembles d'informations préférentielles réduits, nous montrons qu'il sera tout à fait raisonnable de les utiliser en temps réel face à un véritable décideur.

De leur côté, les algorithmes de tri montre un comportement qui, bien que valide, n'est pas désirable lors d'un protocole réel. En effet, les profils des catégories créés ont tendance à être associés à des seuils de préférence et d'indifférence peu crédibles, rendant l'exploitation du résultat des tests théoriques peu concluants. Des travaux futurs pour améliorer leur opérationnalité pourraient être envisagés.

Construction d'un protocole d'élicitation des préférences

Dans le septième chapitre de la thèse, nous mettons en place un protocole d'élicitation des préférences, qui se déroule en trois temps.

Dans un premier temps, nous cherchons à établir un préordre grossier, en regroupant les critères selon des classes d'importance simples, mais peu précises (par exemple, les critères très importants, importants et moins importants). Une fois ce préordre validé, nous cherchons à le raffiner jusqu'à obtenir un préordre entièrement validé par le décideur (*i.e.* deux critères dans une même classe d'importance seront automatiquement associés au même poids). Enfin, nous cherchons à établir par questionnement indirect des poids précis, à partir du questionnement du décideur sur les surclassements instables non encore forcés par les informations préférentielles déjà exprimées.

Le premier temps est assez classique dans la littérature de l'aide multicritère à la décision et n'est abordé que succinctement. Il se décompose en une phase de collecte des informations (ensemble des alternatives potentielles et évaluation de ces alternatives), et une phase de définition d'un premier préordre à raffiner. Nous proposons en général de classer les critères selon qu'ils soient très importants, importants, ou bien moins importants, mais cela peut être réduit à deux classes si le décideur n'est pas en mesure de fournir de telles informations avec précision.

Le second temps est résumé par l'algorithme suivant :

Algorithm 1 Raffiner un préordre initial

Entrée :

A : Ensemble d'alternatives ; F : Ensemble des critères ; P : Tableau de performance ;
 \geq_{w_0} : Préordre initial.

Variables :

w : Vecteur des poids des critères ;

$seuils$: Certains seuils de préférence et d'indifférence. /* Certains sont donnés */

- 1: $contraintes \leftarrow \emptyset$
 - 2: $contraintes \leftarrow \text{AJOUTER_CONTRAINTES_PRÉORDRE}(\geq_{w_0})$
 - 3: **Répéter**
 - 4: $(a_1, a_2) \leftarrow \text{SELECTIONNER_COUPLE_ALTERNATIVES}(A)$
 - 5: $contraintes \leftarrow \text{AJOUTER_CONTRAINTES_PRÉFÉRENTIELLES}(a_1, a_2)$
 - 6: $\{w, seuils\} \leftarrow \text{RÉSOUTRE}(A, F, P, contraintes)$
 - 7: **Si** $\{w, seuils\} \equiv \emptyset$ **Alors**
 - 8: $\text{RÉSOUTRE_CONFLITS}(contraintes)$
 - 9: **Sinon**
 - 10: $\text{PRÉSENTER_PRÉORDRE}(\geq_w)$
 - 11: **Jusqu'à** ce que \geq_w soit validé
 - 12: $\text{VALIDER_SEUILS}(seuils)$
 - 13: **Retourner** $\{\geq_w, seuils\}$
-

Notons que la stabilité permet ainsi d'écartier du questionnement l'ensemble des relations stables par extension, puisque nous ne faisons que raffiner le préordre.

Deux versions de cette seconde étape ont été implémentés. La première considérait l'ensemble des seuils de discrimination comme connus et la seconde permettait d'éliciter simultanément ces paramètres, en plus des paramètres de poids des critères. Enfin, la validation du préordre pouvait se faire de deux façons différentes : Soit par validation du décideur dès que la discrimination du préordre courant était suffisante, soit lorsque le préordre courant donnait lieu à un graphe pour lequel tous les surclassements non forcés par les informations préférentielles du décideur étaient stables par extension, soit instables (*i.e.* dans les deux cas la stabilité n'était plus modifiable).

Enfin, le troisième temps peut être résumé par l'algorithme suivant :

Algorithm 2 Détermination des surclassements instables résiduels

Entrée :

A : Ensemble d'alternatives ; F : Ensemble des critères ; P : Tableau de performance ;
 \geq_w : Préordre obtenu à l'étape précédente.

Variables :

w^* : Vecteur des poids des critères ;

1: $contraintes \leftarrow \emptyset$

2: $contraintes \leftarrow \text{AJOUTER_CONTRAINTES_POIDS} (\geq_w)$

3: $contraintes \leftarrow \text{AJOUTER_CONTRAINTES_PRÉFÉRENTIELLES_INSTABLES_VALIDÉES}$

4: **Répéter**

5: $(a_1, a_2) \leftarrow \text{SELECTIONNER_COUPLE_INSTABLE} (A)$

6: **Si** la relation entre a_1 et a_2 est **non** nécessairement fixée **Alors**

7: $contraintes \leftarrow \text{AJOUTER_CONTRAINTES_PRÉFÉRENTIELLES} (a_1, a_2)$

8: $w^* \leftarrow \text{RÉSOUTRE} (A, F, P, seuils, contraintes)$

9: **Jusqu'à** ce que le décideur soit satisfait

10: **Retourner** w^*

Notons que le fait de forcer le préordre garantit que l'ensemble des relations stables ne changeront pas, de sorte qu'il est possible de les valider de façon implicite, mais aussi de montrer certains couples au décideur afin de s'assurer qu'il soit entièrement d'accord avec le préordre validé. Cependant, certains surclassements ne pouvant pas être stables ont pu être demandé par le décideur, c'est pourquoi il convient d'ajouter les contraintes permettant de les forcer (car ceux-ci ne sont pas stables et donc ne dépendent pas simplement du préordre).

Enfin, le chapitre 7 présente des outils permettant à un décideur une meilleure visualisation des données. En outre, nous faisons état de l'avantage de tels outils pour une vision globale de l'information, mais précisons aussi les biais possibles, notamment les effets de compensation entre critères dans les méthodes de surclassement.

Cas pratique

Le huitième et dernier chapitre de la thèse présente la résolution d'un cas pratique avec la méthode décrite aux chapitres précédents. Le sujet était la sélection d'une thèse pour poursuivre des études d'informatique et a été présenté à plusieurs étudiants de Master, mais aussi en première année de thèse.

Cette étude, dont les premières expériences ont permis l'élaboration et l'amélioration du modèle présenté dans la thèse, ont démontré l'utilité de la stabilité afin de simplifier le protocole de questionnement, mais aussi pour éviter de créer des biais dans la solution, qui ne seraient pas perceptibles par des décideurs face à une décision inhabituelle.

Le protocole a permis aussi de se rendre compte de la nécessité de tenir compte de l'élicitation simultanée des poids des critères et des seuils de discrimination. En effet, durant les premiers tests, les seuils étaient fixés préalablement avec le décideur. De par son manque d'expérience, une telle fixation s'est vite révélée laborieuse et sujette à des contradictions dans l'expression des informations préférentielles. Cela a donc permis de définir des modèles plus complexes et plus coûteux, mais plus opérationnels. De plus, nous avons pu constater que le temps restait très faible (de l'ordre de 2 secondes maximum pour chaque résolution) durant le protocole, ce qui était tout à fait concevable en temps réel.

Contents

| | |
|--|-----------|
| Introduction | 1 |
| I An overview of robust multiple criteria decision aid | 3 |
| 1 Multicriteria decision aid | 5 |
| 1.1 The decision aiding approach | 6 |
| 1.1.1 Making a decision | 6 |
| 1.1.2 Aiding the decision | 7 |
| 1.1.3 Involving the decision-maker | 9 |
| 1.2 Modeling the decision aid process | 10 |
| 1.2.1 Defining the fundamental decision objects | 10 |
| 1.2.2 Formulating the decision aid problem | 11 |
| 1.2.3 Modeling and exploiting preferences | 13 |
| 1.3 The main formal multicriteria decision philosophies | 14 |
| 1.3.1 Designing and exploiting an overall value function | 15 |
| 1.3.2 Designing and exploiting an outranking relation | 17 |
| 2 Preference elicitation processes | 21 |
| 2.1 Decision-makers' profiles and preferential information | 22 |
| 2.1.1 Decision-makers' profiles | 22 |
| 2.1.2 Expressing some preferential information | 24 |
| 2.1.3 Quality of the expressed preferential information | 26 |
| 2.2 Setting up an iterative preference elicitation process | 27 |
| 2.2.1 Principles of an iterative preference elicitation approach | 28 |
| 2.2.2 An overview of disaggregation approaches | 30 |
| 2.2.3 Knowing the potential pitfalls | 31 |
| 2.3 Analysing the robustness of preferential results | 32 |
| 2.3.1 Sensitivity analysis within multi attribute valued theory | 33 |
| 2.3.2 Credibility level cutting technique | 34 |
| 2.3.3 Dealing with imprecise but accurate information | 35 |
| 2.3.4 Stability of outranking relations | 36 |

| | | |
|-----------|--|-----------|
| II | On the stability of median-cut outranking relations | 39 |
| 3 | Stability of the median-cut outranking digraph | 43 |
| 3.1 | Preliminary definitions | 44 |
| 3.1.1 | Construction of a weighted outranking relation | 44 |
| 3.1.2 | Weights preorder | 46 |
| 3.1.3 | Defining the preferable relation | 47 |
| 3.2 | Defining the stability of valued outranking relations | 48 |
| 3.2.1 | Stability | 50 |
| 3.2.2 | Extensible stability | 51 |
| 3.2.3 | σ -stability | 54 |
| 3.3 | Additional properties | 55 |
| 3.3.1 | Limitation of the stability | 56 |
| 3.3.2 | Stability of the preferable relation | 57 |
| 3.3.3 | Stability within the context of the sorting problem | 57 |
| 3.3.4 | Checking the stability property with missing evaluations | 60 |
| 3.3.5 | Properties on the discrimination of the preorder | 61 |
| 4 | Stable elicitation of criteria weights | 63 |
| 4.1 | Stability constraints | 64 |
| 4.1.1 | Auxiliary variables and constraints | 64 |
| 4.1.2 | Modeling of the stability constraints | 65 |
| 4.1.3 | Constraint relaxation using slack variables | 69 |
| 4.2 | Taking into account decision-maker's preferences | 71 |
| 4.2.1 | Types of preferential information | 71 |
| 4.2.2 | Preferences on alternatives | 71 |
| 4.2.3 | Preferences on criteria | 72 |
| 4.3 | Mathematical programs | 74 |
| 4.3.1 | Control algorithm (ACON) | 75 |
| 4.3.2 | MILP with real relaxed stability constraints (STAB ₁) | 76 |
| 4.3.3 | MILP with boolean relaxed stability constraints (STAB ₂) | 77 |
| 5 | Elicitation of weights and other parameters | 79 |
| 5.1 | Elicitation of weights and thresholds | 81 |
| 5.1.1 | Modeling of the constraints on the thresholds | 81 |
| 5.1.2 | Additional preferential information | 88 |
| 5.1.3 | The complete models | 89 |

| | | |
|------------|--|------------|
| 5.2 | Elicitation of weights and categories profiles | 91 |
| 5.2.1 | Modeling of the constraints on the profiles | 92 |
| 5.2.2 | Ensuring a stable assignment of an alternative | 94 |
| 5.2.3 | The complete models | 95 |
| 5.2.4 | Stable assignment of the other alternatives | 97 |
| | | |
| III | A progressive method for a robust parameters elicitation | 99 |
| | | |
| 6 | Empirical validation of the algorithms | 103 |
| 6.1 | Parameters elicitation from a complete set of information | 106 |
| 6.1.1 | Elicitation of criteria weights | 106 |
| 6.1.2 | Elicitation of criteria weights and discrimination thresholds | 108 |
| 6.1.3 | Elicitation of criteria weights and categories profiles | 109 |
| 6.2 | Iterative recovering of the median-cut outranking relation | 112 |
| 6.2.1 | Iterative elicitation of criteria weights | 113 |
| 6.2.2 | Iterative elicitation of both criteria weights and thresholds | 116 |
| 6.3 | Impact of the stability constraints on the preference modeling | 120 |
| 6.3.1 | Impact on the weights preorder | 121 |
| 6.3.2 | Impact on the preference discriminating thresholds | 122 |
| | | |
| 7 | REWAT: Robust elicitation of the weights and thresholds | 127 |
| 7.1 | Designing a robust elicitation protocol | 128 |
| 7.1.1 | Stage I: Initialising the outranking preference model | 129 |
| 7.1.2 | Stage II: Validating the criteria weights preorder | 131 |
| 7.1.3 | Stage III: Tuning the numerical values of the criteria weights | 140 |
| 7.2 | Tools for supporting a robust elicitation | 143 |
| 7.2.1 | Dynamic pairwise performance comparison table | 144 |
| 7.2.2 | Display of the elicited weights preorder | 145 |
| | | |
| 8 | Case study: Applying for a Ph.D. thesis | 147 |
| 8.1 | Applying the REWAT process | 148 |
| 8.1.1 | Stage I: Initialising the outranking preference model | 148 |
| 8.1.2 | Stage II: Validating the criteria weights preorder | 151 |
| 8.1.3 | Stage III: Tuning the numerical values of the criteria weights | 155 |
| 8.2 | Critical review of the case study | 157 |
| 8.2.1 | Encountered difficulties | 157 |
| 8.2.2 | Perspectives for future methodological enhancement | 161 |

| | |
|--|------------|
| Summary of the main achievements | 165 |
| Perspectives | 167 |
| Concluding remarks | 171 |
| A Annex | 173 |
| A.1 Mathematical proof of Proposition 3.8 | 173 |
| A.2 Complete mathematical models | 175 |
| A.2.1 ACON | 175 |
| A.2.2 STAB ₁ | 176 |
| A.2.3 STAB ₂ | 177 |
| A.2.4 ACON' | 178 |
| A.2.5 STAB' ₁ | 179 |
| A.2.6 STAB' ₂ | 181 |
| A.2.7 ACON* | 183 |
| A.2.8 STAB ₁ * | 184 |
| A.2.9 STAB ₂ * | 185 |
| A.3 Data for the case study | 186 |
| A.4 Local concordance values for the case study | 187 |
| A.4.1 Initial local concordance values | 187 |
| A.4.2 After the comparison of alternatives SW ₂ and CA ₂ | 188 |
| A.4.3 After the comparison of alternatives NL ₂ and US ₂ | 189 |
| A.4.4 After the comparison of alternatives NL ₁ and US ₁ | 190 |
| A.4.5 After the comparison of alternatives CA ₂ and FR ₂ | 191 |
| A.4.6 After the comparison of alternatives FR ₂ and EN ₂ | 192 |
| A.4.7 Validation of the new preorder \succeq_{w^2} | 193 |
| A.4.8 Validation of the preorder and the thresholds | 194 |
| Bibliography | 195 |

List of Tables

| | | |
|-----|--|-----|
| 1.1 | Selection of a thesis | 10 |
| 3.1 | Comparing two cars | 47 |
| 3.2 | Performance table and associated outranking relation | 49 |
| 3.3 | Computation of the stability of some outranking situations | 51 |
| 3.4 | Extensible stability test for the proposition “ $dS^w a$ ” | 53 |
| 3.5 | Extensible stability | 53 |
| 3.6 | σ -stability | 55 |
| 3.7 | Stable assignments of some alternatives using pessimistic rule | 58 |
| 3.8 | Stable outrankings statements with missing local concordance values | 61 |
| 6.1 | Exp ₁ (1/3) – Stability increase and running time | 107 |
| 6.2 | Exp ₁ (2/3) – Stability increase and running time | 110 |
| 6.3 | Exp ₁ (3/3) – Stable sorting | 111 |
| 6.4 | Exp ₂ (1/3) – STAB ₂ iterative reconstruction, using the WDV heuristic | 115 |
| 6.5 | Exp ₂ (2/3) – STAB’ ₁ iterative reconstruction, using the RS heuristic | 118 |
| 6.6 | Exp ₂ (2/3) – STAB’ ₁ iterative reconstruction, using the RS heuristic | 119 |
| 6.7 | Exp ₁ – Median Kendall values | 121 |
| 6.8 | Median values for %ID | 124 |
| 6.9 | Median values for %ID (iterative recapture) | 125 |
| 7.1 | Selecting a new monthly package with a mobile phone | 144 |
| 7.2 | Dynamic performance table – 1/2 | 144 |
| 7.3 | Dynamic performance table – 2/2 | 145 |
| 7.4 | Displaying the current preorder – 1/4 | 145 |
| 7.5 | Displaying the current preorder – 2/4 | 146 |
| 7.6 | Displaying the current preorder – 3/4 | 146 |
| 7.7 | Displaying the current preorder – 4/4 | 146 |
| 8.1 | Performance table of the 9 selected alternatives | 150 |
| 8.2 | Initial preorder | 151 |
| 8.3 | Comparing the alternatives CA ₂ and SW ₂ | 152 |

Introduction

*“Don’t you understand that we need to be childish in order to understand?
Only a child sees things with perfect clarity, because it hasn’t developed all those
filters which prevent us from seeing things that we don’t expect to see.”*

[Douglas Adams]

Making a decision may be regarded as a trivial act, as we are dealing daily with so many alike situations. However, it requires the implementation of a cognitive process whose complexity is closely linked to each specific situation: Obviously, we do not make a decision in the same way when we select our clothes in the morning or when we apply for a new job. In addition, for complex decision situations, in day to day life or within a professional context, there is a need for formalising them in order to motivate one particular action, in a clear and understandable way, for every actors involved in the decision.

These formal models for making a decision are rapidly growing since the middle of the XXth century, especially with the increasing computation power that permits the solving of more and more complex problems.

Aiding a decision does not only provides a solution to a given problem, but also supports the decision process in the collection of every opportunities, in determining a clear and complete measure of their different implications. In the particular situation of facing different conflictual objectives, or contradictory interests, it defines the framework of *Multiple Criteria Decision Aid* in which our work is located.

Using a multicriteria decision aid method to provide a *recommendation* requires the determination of numerous parameters. These parameters are obtained during a preference elicitation process, to construct an evaluation model that summarises the subjective aspects related to the decision-maker’s perceptions or expectations, as for example the *criteria weights*, which appreciate the local role of each criterion in the construction of the overall evaluations.

Two different approaches exist to specify these values: – either via *direct* preference information, where the values are first assessed and then the overall evaluations are computed, or – via *indirect* preference information, where some a priori partial knowledge about the resulting evaluation model is used in order to infer plausible estimators of the parameters.

However, determining precisely the numerical values of these parameters is an important issue: As the final recommendation is highly depending on these parameters, it is of the highest importance to avoid impreciseness or to be able to measure their impact on the modeling of the preferences. In addition, when considering

an indirect elicitation of these parameters, according to an incomplete knowledge of the resulting evaluation model, the selection of one particular set of compatible parameters is also a very debatable question.

The work we present in this thesis takes part within the multicriteria decision aid methods, more specifically within the *outranking philosophy*. It focuses on the *stability concept*, which characterises the impact of some possible impreciseness of the criteria weights on the evaluation model.

The thesis is divided into three parts. The first one is a brief state of the art about the decision aid domain, that focuses on the preference elicitation processes and robustness concerns, in order to motivate our work. Chapter 1 presents the main concepts and contributions, but also details the different existing methodological approaches in the domain to handle a multicriteria decision problem. Then, Chapter 2 briefly studies the common framework of the preference elicitation processes and discusses the quality of the resulting evaluation model, in order to take into account the possible use of incomplete or imprecise information.

The second part of the thesis focuses on the stability concept, to give a theoretical framework to our work. At first, Chapter 3 gives a more intuitive formulation of the stability concept, as well as simple mathematical conditions and applications examples, but it also extends this concept by defining two additional degrees, for a sharper characterisation of the dependency. In Chapter 4, we define mathematical models to elicit a vector of weights that is compatible with a set of preferential information given by the decision-maker. These models take advantage of the stability concept, defined in the previous chapter, by searching for a compatible weights vector that maximises the stability of the resulting evaluation model. In Chapter 5, these models are enhanced in order to recover, in addition to the criteria weights, some other parameters, namely some unknown discrimination thresholds in a first time, then the categories profiles of a sorting problem.

Finally, the third part of the thesis intends to implement the mathematical models on real practical cases. Chapter 6 first validates the models empirically. In Chapter 7, we define a robust preference elicitation protocol, called REWAT, in order to construct iteratively an evaluation model that illustrates the decision-maker's expectations. Finally, Chapter 8 presents the use of the REWAT process on a real-case application.

Part I

An overview of robust multiple criteria decision aid

Multicriteria decision aid

“Man is man because he is free to operate within the framework of his destiny. He is free to deliberate, to make decisions, and to choose between alternatives.”

[Martin Luther King, Jr.]

Contents

| | | |
|------------|--|-----------|
| 1.1 | The decision aiding approach | 6 |
| 1.1.1 | Making a decision | 6 |
| 1.1.2 | Aiding the decision | 7 |
| 1.1.3 | Involving the decision-maker | 9 |
| 1.2 | Modeling the decision aid process | 10 |
| 1.2.1 | Defining the fundamental decision objects | 10 |
| 1.2.2 | Formulating the decision aid problem | 11 |
| 1.2.3 | Modeling and exploiting preferences | 13 |
| 1.3 | The main formal multicriteria decision philosophies | 14 |
| 1.3.1 | Designing and exploiting an overall value function | 15 |
| 1.3.2 | Designing and exploiting an outranking relation | 17 |

Abstract

In this first chapter is to define the multicriteria decision aid domain, to present its concepts and main contributions. We present a brief introduction of the multicriteria decision aid domain, followed by the definition of the necessary concepts. In a third section, we detail the different existing methodological approaches in the domain to handle a multicriteria decision problem.

1.1 The decision aiding approach

1.1.1 Making a decision

Making a decision could be considered as a trivial action, as we are constantly dealing with so many of these situations. Thinking about it, every act, including the will of not acting, is a decision: from the simplest daily decisions like taking or not an umbrella when leaving home [Poo92], to some much more complex and unusual situations like, for instance, the selection of a location for a new airport [MA90].

There are so many ways to deal with a decision, which are closely linked to each specific situation. For instance, we are not making a decision with a low degree of *emotional implication*, like choosing a cold drink in the fridge, as we are dealing with a critical choice that shall condition our future, like going and living in a foreign country, or thinking about divorcing [Wat93]. People directly involved in the decision also have an impact on the way we consider the decision : if we are the only person to decide (“should I take a bath or a shower?”) or if there are several persons (“where should the family go on holidays?”); At least, the private or the professional context modify the way we envision the decision, as well as our *experience* in making such decisions.

Some of these ways may be considered as *not rational*. Indeed, One can take an *instinctive* decision, a *random*, or *impulsive* one. Such decisions have some clear advantages: they are easy to take and they are not time-consuming. In the absence of a possible *regret* in the selection of a not completely satisfying option (*i.e.* when most of the considered options may be judged as relatively equivalent), they are particularly efficient.

A *delegated decision* is not a decision we directly take. In fact, it is taken by someone we have faith in. It is mainly some situations where we do not have any experience, that way, we have no idea about what options can be considered as good options. The difference with the random decision is that we have here the feeling that selecting one or another option may have a really different impact. One example is the situation where someone wants to buy his very first computer and does not know anything about it; he will be guided towards a model by a shop assistant and will base his decision upon the other’s opinion.

Last but not least, a *rational decision* consists in an understandable analysis of the different opportunities, as well as an in-depth evaluation of their implications. It is a standard when making a professional decision. For instance when searching for a new supplier, we have to compare the different prices of every possible suppliers, consider the different contracts,... Making a decision this way is much more time-consuming and has a cost, but it warrants, when successfully achieved, a justifiable decision, in a clear and transparent process (see for example [BeCNdSV01] for the advantages of a formal framework in decision involving multiple stakeholders). As it is the most formal approach, we shall focus on it. Notice also that such an approach brings to light some structures that can be used again for a further decision (some examples are given in [VV00]). In that sense, we can really aid the decision, helping

everyone to understand, analyse, explain or justify a problem or an option (for an overview of the classical decision theories, see [Tso06]).

1.1.2 Aiding the decision

Example 1 *To strengthen our clauses, let us envision the situation where a Master student is considering the opportunities for his future career. Indeed, as it is a brilliant student, two of his professors offer him a position for making a thesis. The first thesis is devoted to an interesting subject and the future Ph.D. student will work for the university where he made his studies; The second thesis is devoted to an even more interesting subject, but the student needs to relocate far from his family.*

We can easily notice the conflictual situation between the two theses that both have some advantages and drawbacks. For instance, our student shall consider the first thesis because his office will be the closest from his family. But in another hand, the second thesis is the most interesting one. In the absence of an opportunity that will be the best on every considered aspects, making a decision comes automatically to make a compromise, or a sacrifice, possibly the least important one. Notice that we assume in our work that no such case exists, otherwise the decision problem of selecting the best option is trivial. We also assume a similar hypothesis on the fact that it does not exist an opportunity which is the worst on every considered aspects.

The aim of decision aid is, as its name suggests, to help one person, or a group of people, to take the most satisfying decision. In the particular case of facing different conflictual objectives, or contradictory interests, the purpose is not only to provide a solution to a given problem, but it is also to guide the decision process, in the collection of every opportunities, in determining a clear and complete measure of their different implications for every persons involved in the decision (see for example [Roy87]), until the expression of a final recommendation (*i.e.* the output of a method). Such an approach defines the framework of *Multiple Criteria Decision Aid* (abbreviated to MCDA in the sequel of our work).

Literature on decision theory usually describes four types of approaches in the resolution of a given decision problem: descriptive, constructive, normative and prescriptive (see for instance [Roy93] or [BRT88]). Indeed, for some experts, the decision aid process is a *descriptive* tool, designed to help justifying a choice that is already existing in the minds of the persons involved in the decision. It intends to describe and model their behavior empirically when facing a decision (see for instance [Sch88]). For some others, it should be a *constructive* process that helps creating the preferences and the model simultaneously. Notice that such an approach is not of small importance on the final output, as the process guides the individual in the construction of his preferences (see for example [LPB85] or [Ros01]). In a *normative* approach, we try to define models, based on norms and standards, that may be used by anyone under the acceptance of the model hypotheses (see for instance [VNM54], [LR57] or [Wak89]). Finally, a *prescriptive* approach intends to provide readily usable mathematical tools for the search of the optimal solution

of a particular situation. For example, these approaches are predominant in the medical domain (when determining a cure for a patient, etc.). They make a lot of assumptions on the user's beliefs and should only be used in situations where it exists a consensus in the points of view to be considered, so as their relative importance (see for instance [Roy85] or [BS02]).

In practice, a decision analysis method may belong to more than one approach, such that it combines their different aspects in the resolution of a problem. For instance, [DT04] shows the interest of using a normative approach to impose some rationality principles, in a prescriptive approach, in order to ease the discussion between the different actors of the decision. In both ways, all the approaches intend to provide a formalized framework for a more objective understanding of the considered situation, strengthening the conviction in the given recommendation.

The premises of MCDA are usually accredited to the Marquis DE CONDORCET (1743–1794), who first promoted the use of a mathematical framework in social choice voting procedures, for a rational way of electing a candidate based on the preferences of a set of voters [Con85]. However, its real emergence can be dated back around the middle of the XXth century, with SAMUELSON's fundamental researches on the theory of revealed preferences [Sam38], the work from DANTZIG [Dan48, Dan51] and KANTOROVICH [Kan39] in linear programming, the stepping-stones of game theory and decision theory by VON NEUMANN and MORGENSTERN [VNM54] and NASH [NJ50, Nas51], the bases of social choice theory originated from ARROW's work [Arr63], but also the increasing interconnection between the mathematical and psychological aspects of the decision [LR57, Tve67, Fis70].

In the late '50s, SIMON [Sim56, Sim57] stated that in real-life decision problems, a fully-rational decision is bounded by human factors as, for example, the decision domain knowledge, or the cognitive capacities of the people involved in the decision. This bounded rationality theory stated that one may consider an alternative as a solution, not in a strict mathematical optimality, but if this alternative tends to be *satisfying enough*. This vision underlies the whole discourse presented in this work.

First methods taking into account the multidimensional aspects of a problem for its resolution appear by the beginning of the '60s, with the precursory work of CHARNES *et al.* on "Goal Programming" [CCF55, CC61], known as *multi-objective programming* today. By the end of the '60s, ROY introduced an innovative decision aid perspective, leading to the creation of the *outranking methods domain* [BRS66, Roy68]. In 1976, KEENEY and RAIFFA extended the valued theory to the multi-dimensional situations [KR76]. These two differentiated philosophies lead to the establishment of the so-called *European* and *American* schools, we shall present in the sequel of the chapter. We may highlight that some more recent work on *Rough sets approach* are searching to offer a general framework to unify these two trends using a rule-based system [GMS01].

In this study, we are focusing on decisions where a unique person is responsible of the decision act, namely the *decision-maker*¹ (DM). This hypothesis implies that

1. For a more efficient reading, we have avoided the question in regards to the decision-maker

we are not considering here the well-known problem of the search of a consensus among a set of decision-makers (for more details, see [MS01], [JF89] or the second chapter of [Adl10]). We also consider every decisions that may imply a regret *a posteriori*: As we already discussed, decision aid methods are only useful when one can perceive the selection of one option rather than another as implying different consequences.

1.1.3 Involving the decision-maker

Most of the time, a decision is strongly linked to the *preferences* of the decision-maker. For instance, considering the first example, if the student estimates that the improvement on the interest from the first thesis to the second one is not worth the increase of the remoteness, he shall prefer the first thesis. On the contrary, if he could conceive that working farther from his family for the duration of his Ph.D. is not so important compared to the fact that the thesis will be in better accordance with his wishes, he shall consider to apply for the second thesis.

It is the main difference with classical Operational Research methods, where a problem is solved through an optimisation method and the role of the decision-maker is confined to the delimitation of the problem and the validation of the solution, if it exists, called the *optimum*. In decision aid, the decision-maker's preferences must be correctly modeled, as they condition the output of the *decision analysis*. Thereby, as stated in SIMON's work, specifically his study on the decision-makers' behaviors in administrative organisations [Sim47], the decision should be seen as a *process* and not as an act, where the decision-maker has to be highly involved.

To help the decision-maker in his task, an *analyst* is in charge of supporting him. This individual will be the interface between the decision-maker plus his preferences and the mathematical modeling of the problem, by giving a clear formulation and a rational structuring of the problem, but mainly discussing with every actors, or stakeholders, of the decision process. We may notice that in some case, both the roles of decision-maker and analyst can be played by a single person.

Example 2 *Back to our example, after a short reflexion, our student may consider that the laboratory of the second thesis is too far from his family and he does not want to go and work so far away. In that case, the second thesis cannot be considered as an acceptable solution and must be taken apart. As there only remains one opportunity, the student should trivially consider it as the best solution. However, he decides to search for other applications and finds three other opportunities. Also, he extends the collected information in order to have a better understanding of the opportunities and their different implications. Data are summarised up in Table 1.1.*

gender. Thus, *he* and *his* will refer to this person in the sequel.

Table 1.1: Selection of a thesis

| | Interest | Dist. from family | Salary | Lab. reputation |
|----------|-------------------|-------------------|---------|-----------------|
| Thesis 1 | Interesting | 10 km | € 1 630 | Medium |
| Thesis 2 | Very Interesting | 900 km | € 1 900 | Very Good |
| Thesis 3 | A Few Interesting | 80 km | € 1 600 | Medium |
| Thesis 4 | Interesting | 90 km | € 1 655 | Good |

Once again, we can observe the different conflictual points of view to be considered in the decision. Notice also the variety of the evaluations: some evaluations are quantitative, while some others are qualitative.

1.2 Modeling the decision aid process

Basically, the decision aid process may be regarded as the series of four critical steps: The *problem situation*, the *problem formulation*, the implementation of an *evaluation model* and the expression of a *final recommendation* [MT97, PT99, ST03]. The first two steps help formulating the decision aid problem, while the last ones intend to model and exploit the decision-maker's preferences. Notice that these steps are not always sequential in a real case study. For an extensive presentation of the decision aid process, the reader should refer to [Tso07].

In this section, we define the formal, but abstract, framework on which a rational decision aid process relies. At first, we introduce the fundamental objects on which is based the multicriteria decision aid activity we shall refer to, all along the sequel of this work; Specific objects and concepts will be introduced later when necessary. Then, we briefly detail the main steps of such a process.

1.2.1 Defining the fundamental decision objects

We call *alternatives*, or *decision actions*, the formal definition of the considered potential options in the decision aid process. Most of these alternatives are real ones, as they can be implemented through the decision; some other may be fictitious, only created in order to help the process. All elements of this set have to be clearly identified and validated by the decision-maker (for instance, it only contains acceptable alternatives) [Roy85]. The set of alternatives is usually called A and its cardinality $m \geq 3$.

In the context of a multidimensional approach, the alternatives have to be evaluated according to different aspects, or characteristics, in order to allow the comparison of one alternative to another one on every dimensions. Consequently, we first define all the meaningful points of view that the decision-maker wants to consider (*i.e.* points of view having a significance on the global understanding of the problem) and then construct, on every considered point of view, a mathematical tool for evaluating and comparing the alternatives, called *criterion* [Bou90]. In [BMP⁺06]

the authors define as a criterion any dimension with which it is possible to associate a preference model, even a partial one, such that one should be able to make a choice along this single dimension. Basically, it is a function on the set of alternatives that acknowledges the local *performances* of the alternatives, based on the decision-maker's expectations, in a measuring preference *scale* that a decision-maker must fully understand. Then, two alternatives are compared on any point of view according to their evaluations on the associated criterion.

A criterion is not conceived only to give an objective evaluation of some aspects of the reality (as for example a *price* criterion on a quantitative scale that measure the price of the different alternatives), but can also account for the subjective points of view of the decision-maker. For instance, referring to our example, the student is considering a criterion *interest* for the thesis, which is the subjective evaluation of his thought about the subject and the domain of the thesis.

As the evaluation of the alternatives is often a complex task to perform with accuracy and preciseness, especially when the criterion is constructed without an expert of the considered point of view, one should not forget that it is most of the time an approximation of the reality, not a perfect and complete representation. As we shall detail in the sequel, the consequences of these imprecisenesses are important and have to be measured and minimised to their lowest possible impact.

The $n > 2$ considered criteria define a finite *family* F . This family is *coherent* [Roy00], as it considers all the dimensions, or the points of view, that the decision-maker takes into account in the decision and that have an effective impact on the decision analysis. Thus, the *exhaustiveness* is granted by the fact that two alternatives having exactly the same evaluation on the considered family of criteria must be considered as indifferent. The family is also *non-redundant* (see [Roy85, Bou90, Bis02]), as the family is considered as minimal, according to the preceding two conditions.

In the sequel of our work, the alternatives will be denoted as x , y , z or t . The letters i and j will refer to criteria and the evaluation of the alternative x on the criterion i will be denoted as x_i . All these evaluations are gathered in the *performance table*.

1.2.2 Formulating the decision aid problem

Problem situation

It is a prerequisite step that helps defining for any person, even remotely involved in the decision, a well determined role. For that, the analyst should ask the decision-maker, or the client (*i.e.* the person that has requested a decision support), some critical questions, like for instance:

- Who has a decision problem?
- Who are the different persons involved in the decision?
- Who is in charge of the final decision? Does he relies on some other persons?

- What are the common stakes? The personal stakes?
- Who is going to pay for the consequences of a decision?

Besides, this step structures the different actors' roles, their common or conflictual interests, allowing the analyst a better understanding of the situation, as well as the invested stakes. Thus, this last one can better apprehend what we expect from him and then really supports the decision aid process. In addition, it makes easier the communication between the different actors and may clarify their different positions.

Example 3 *Back to our example, we should question the student to know the different persons involved in the decision, for instance his parents or his girlfriend, and their degree of involvement (“My girlfriend’s opinion about the country is really important, as we will leave together”: In that case, it seems useless and time-consuming to discuss about some alternatives that will be invalidated by the student’s girlfriend).*

Problem formulation

Although it may appear as obvious, the problem formulation is a key step in the decision aid process, formalising the situation, in order to precisely define what is expected at the end of the process. Besides, it appeals to Simon’s limited rationality concept, such that the decision-maker’s concern are translated into a “formal” problem on which a decision aid method can be applied.

Indeed, for a given problem situation, we may formulate the problem in different manners that condition the whole process, especially the way the alternatives are selected and evaluated.

Example 4 *If we consider again our example, the analyst should ask the student about a clear expression of his expectations: does he want to find a reduced set of potential theses on which he may apply for? Or does he wants to select, among a set of accepted applications, the most interesting one? In that case, what does “interesting” mean? ...*

Notice that the problem formulation may be based on different *problem typologies*. Indeed, as stated in [Roy85], one may identify three main problem typologies a decision-maker can deal with, namely the *choice* problem, the *ranking* and the *sorting* one.

The *choice problem*, also denoted P_α , tries to identify a subset $A' \in A$, possibly the smallest, of the alternatives a decision-maker may consider as the best ones. Having more than one alternative in A' leads to the difficulties of ordering them. A related problem is the *portfolio problem* [BS02], where we consider the selection of the $k > 1$ best alternatives.

The *sorting problem* (P_β) is willing to assign the alternatives into predefined and ordered categories (for instance, the good, medium and bad ones). In that case,

the assignment is done by checking the alternatives evaluations accordance to some defined rules.

Finally, the *ranking problem* (P_γ) considers the assessment of a complete or partial preorder of the alternatives.

For an in-depth study of the different problem typologies, the reader should refer to [BeC96].

1.2.3 Modeling and exploiting preferences

Evaluation model

When working with multiple criteria, namely dealing with evaluation vectors, there is no objective definition of what an optimum vector is. At most, we can consider a vector that is the best on every dimensions to be the optimum, but this trivial situation is merely to exist and does not necessitate a decision support.

To our knowledge, the only objective definition has been introduced under the name of *efficiency* [Par06]. Indeed, an alternative x is considered as *efficient* if no other alternatives y are at least as good as x on every criteria and strictly better on one criterion. A not efficient alternative is said to be dominated. But this definition is not restrictive enough to be considered as a final recommendation, especially when the number of criteria increases [Ros91], and need to be exploited in a way that takes into account the decision-maker's subjective expectations.

In order to compare in a global manner any pairs of alternatives of a given set A , according to their performances on every criteria, we shall define a *Multiple Criteria Aggregation Procedure* (or MCAP for short). This procedure, which is most of the time a parametric procedure, associates with any pairs of alternatives exactly one of the following binary relations [Roy96, FGRS10]:

- *Indifference* (I): This reflexive and symmetric relation appears when there are clear and positive reasons that justify an equivalence between the two alternatives;
- *Strict preference* (P): It is a non-reflexive and asymmetric relation that indicates clear and positive reasons in favor of one (identified) of the two alternatives;
- *Weak preference* (Q): It is also a non-reflexive and asymmetric relation that corresponds to a situation where there are clear and positive reasons for invalidating a strict preference in favor of one (identified) of the two alternatives, but not enough reasons for deciding either the strict preference in favor of the other alternative or the indifference between them, leading to the inability to distinguish between the two previous relations;
- *Incomparability* (R): This non-reflexive and symmetric relation occurs when none of the three preceding relations can obtain a clear and positive support. According to ROY [Roy90], it allows the representation of the hesitations of

the decision-maker linked, for instance, to uncertainties, conflicts or contradictions.

Let us notice that some procedures may consider only a subset of these relations.

The two main approaches we shall describe briefly in section 1.3 particularly differ from the way these relations are constructed.

Finally, notice that it is the procedure, and not the decision-maker, that constructs the relations. As we shall present in the next chapter, the decision-maker may be asked to express his opinion about some relations, but this will only help in the tuning of the parameters.

Final recommendation

This last step allows to translate the formal result from the exploitation of the evaluation model into a clear and understandable recommendation for the decision-maker, but also the other actors of the decision.

Notice that the recommendation relies on some particularly strong hypothesis [Tso07]:

- The analyst is certain about the formal accuracy of the evaluation model;
- The decision-maker is certain about the fact that his preferences has been modeled with accuracy, *i.e.* the recommendation seems to be in accordance with his expectations and is satisfying enough;
- The recommendation should be legitimated, *i.e.* its acceptance has to be verified amongst the different actors and their reasons to accept or reject it must be taken into consideration by the analyst [LBO96]: A non legitimated recommendation is barely satisfying and does not deserve to be implemented.

Although legitimating a recommendation validates it within the process, it is highly necessary to validate at first the reliability of the recommendation within the evaluation model. Indeed, the different models intend to acknowledge an overall picture of the situation from the decision-maker's expressed partial information. Thus, it is mandatory to identify and to best limit the potentially resulting biases, which involves a perfect knowledge about the considered model from the analyst. That will be discussed in the next chapter.

1.3 The main formal multicriteria decision philosophies

The purpose of this section is to present the two main philosophies, or schools, for solving multicriteria decision problems, namely the *American school* and the *European school*. We present the main outlines of the two approaches, which are mostly composed of two steps: the construction of the evaluation model, which mainly results in defining the parameters of the considered method, and its exploitation with the aim of providing a final recommendation to the decision-maker.

These two philosophies may appear as quite similar, especially by the fact that they deal with information of the same kind, but they clearly differ in the way they envision the construction of the decision-maker's preference. We will not detail here the consideration of one particular philosophy, done by an analyst with the decision-maker's agreement: For further details, an interested reader may refer to SIMPSON [Sim96]. Simply notice that the analyst must ensure the complete match between the considered philosophy and the decision-maker's thought patterns.

1.3.1 Designing and exploiting an overall value function

In a very intuitive approach, the *Multiattribute Value Theory* (MAVT) [KR76] consists in avoiding the difficulty of the multidimensional evaluation by creating a unique criterion that aggregates every decision criteria in order to construct a numerical representation of the global value of each alternative, often called *score* or *value*, based on the decision-maker's preferences.

Such an approach, which assumes that the decision-maker's preferences can be specified as a weak order over the set of alternatives A (see *e.g.* [BP05, KLST71]), attempts to model the complete and transitive binary relation \succeq on A via an *overall value function* U [Roy71, KR76] such that, for every $(x, y) \in A^2$:

$$x \succeq y \iff U(x) \geq U(y), \quad \forall x, y \in A.$$

This overall value function may be of any kind, but the most studied one is the additive form [Fis64, Fis65, Fis70], the overall evaluation being equal to the sum of the whole *marginal value functions* u_i :

$$U(x) = \sum_{i \in F} u_i(x_i), \quad \forall x \in A,$$

where u_i is a function entirely determined by the criterion i . More particularly, its linear form of a weighted sum is fairly studied, where we associate with every criteria i a *weight* w_i :

$$U(x) = \sum_{i \in F} w_i \cdot x_i, \quad \forall x \in A.$$

One classical example is the evaluation of students in a class: Each course may represent one criterion for the evaluation and the applied coefficients define the linear additive value functions allowing the computation of a global score for each student, namely his average grade.

In that case, the weights parameters w_i are acting like *trade-offs* among the criteria, allowing to balance a locally weak evaluation on one criterion by a good performance on one or some others.

Two different approaches exist to specify the parameters of a given method: either via *direct* preference information, where the parameters are first assessed and then the aggregated relation is computed, or via *indirect* preference information, where some a priori partial knowledge of the resulting aggregated relation is used in order to infer plausible estimators of the parameters.

We shall remark that the additive form of U makes an important hypothesis on the independence of the criteria. Indeed, it is possible to have some interactions between some subsets of criteria. For instance, we may consider the use of a Choquet integral [Cho53] or a Sugeno integral [Sug74], that considers positive interactions on some criteria coalitions, when there is a reinforcement of the impact of one criterion with another one, or some negative ones when there is some redundancy in the different criteria. But, as the number of parameters increases, the complexity of the model, especially the tuning of its parameters, becomes a harder task, such that it does not seem realistic to ask the decision-maker to provide such parameters. Notice that in these cases, the weights are called *capacities* for the Choquet integral or *fuzzy measures* for the Sugeno integral.

Designing the evaluation model is a particularly demanding stage which assumes the endorsement of some hypotheses, which are often hard to obtain, especially the fact that we suppose the existence of a value function and its accordance to a specified form. Indeed, as this function may have any existing form, enforcing one model may result in a bad setting of the parameters, by interpreting the preference information of the decision-maker in an incorrect basis. As we will explain in the next chapter, it is then of high importance to verify *a posteriori* these hypotheses, by making a sensitivity analysis of the expressed value function before giving any recommendation, in order to provide only some solid and clearly established statements.

Another hard task is the translation of every evaluations into a numerical scale. Indeed, when considering qualitative scales (like “Bad/Medium/Good” for example), we have to associate precise numerical values to any of the labels. As these values are, by definition, imprecise (but accurate), this transformation may induce a bias in the exploitation (as we are doing precise computation with imprecise evaluations).

However, the asset of these methods is certain, that is to say that they produce a complete weak order of the alternatives without any incomparability between them (it is always possible to compare the relative position of two alternatives) which eases very much the exploitation. If we attempt to give a recommendation on the selection of the best compromises, we only need to consider the alternatives with the best scores. The ranking problem is solved directly and the sorting problem in predefined categories becomes as obvious (for instance, all the alternatives with a score between the given values v_1 and v_2 are ranked in the first category, ...).

As this work is not focusing on these methods, an interested reader may find some extensive explanations in the fourth part of [FGE05].

1.3.2 Designing and exploiting an outranking relation

In response to the assumptions of the valued methods, often considered as too difficult to ensure, the European School, lead by ROY's work [Roy68], suggests a different approach based on the use of less precise information, but with a stronger support. Indeed, instead of building complex value functions that rank every alternatives on a common scale, such an approach constructs a binary relation, called the *outranking relation*, by comparing the alternatives systematically by pairs.

The main purpose of these methods is not to provide a complete preorder on the alternatives, but to support the decision-maker on his preferences and his choices, in order to explicit them.

Literature on MCDA methods suggests different ways of constructing the outranking relations. Among the most famous ones, you can find the ELECTRE-like methods (see for example [KR76, RB93] with their detailed description) or the PROMETHEE-like methods (an extensive presentation can be found in [BM02]), and also the RUBIS method [BMR08]. Again, as it is not in the scope of this thesis, we will not detail them.

The outranking paradigm is the following: we consider that an alternative x *outranks* an alternative y when there is sufficient support amongst the criteria to validate the fact that x is at least as good as y . In a formal manner, it translates the fact that there is a *qualified majority* of weighted criteria on which x is performing at least as good as y and there is no criterion where y seriously outperforms x [RB93]. Notice that the outranking relation is neither transitive nor reflexive.

Unlike in MAVT, the outranking methods permit three types of alternatives comparisons, wich are: *preference*, *indifference* and also *incomparability*. According to ROY [Roy90], incomparability allows to represent decision-maker's hesitations which may result from phenomena like uncertainty, conflicts and/or contradictions. Most of the time, it results from the comparison of two alternatives stating some very contrasted advantages, describing two opportunities completely opposed.

An alternative x outranks another alternative y when x is at least as good as y . Logically, x and y are indifferent when both alternatives outranks the other one (namely x is at least as good as y and y is at least as good as x). In a similar manner, x is said to be preferred to y when x outranks y and y does not outrank x .

In order to measure the global accordance to the *at least as good as* statement, between any two alternatives x and y of A , with each criterion i is associated a *local concordance degree* $S_i(x, y)$ whose numerical representation is, most of the time, given by:

$$S_i(x, y) = \begin{cases} 1 & : \text{if } x \text{ is at least as good as } y \text{ on criterion } i, \\ 0 & : \text{if } x \text{ is not at least as good as } y \text{ on criterion } i, \\ f_i(x, y) \in]0; 1[& : \text{if we do not know exactly if } x \text{ is or is not} \\ & \text{at least as good as } y \text{ on criterion } i. \end{cases}$$

where f_i is a monotonically non-decreasing function (w.r.t. the difference $x_i - y_i$) which characterises the tendency of an undetermined situation to be validated or invalidated. Most of the time, it is a linear interpolation, from 0 to 1 (see *e.g.* an overview of the ELECTRE methods [FGRS10]), or a constant function equal to the median value 0.5 (see *e.g.* [Bis02, BMR08]), but it can have any shape. Notice that this local concordance may also be given on a scale valued between -1 and 1 ; In that case, 0 is the median value. Changing from one scale to another one is done by a simple translation.

Assuming the independence of the criteria, which is a necessary condition when considering an outranking approach, we compute the *global valued concordance relation*, denoted S^w , aggregating the partial *at least as good as* situations, as follow:

$$S^w(x, y) = \sum_{i \in F} w_i \cdot S_i(x, y), \quad \forall (x, y) \in A \times A.$$

where $W = (w_1, w_2, \dots, w_n)$ is the vector of the rational *importance weight* which represents the contribution of each criterion to the global *at least as good as* situation for every alternatives pairs. Notice that every weights are strictly positive. Also notice that they may be normalised (*i.e.* their sum is equal to 1), such that S^w will be evaluated in the rational interval $[0; 1]$ (or $[-1; 1]$, according to the chosen concordance scale).

The criteria importance weights may appear quite similar to the weights used in the valued theory, but the semantics is rather different. In fact, the outranking methods are inspired from voting methods, within the social choice theory, such that each criterion can be seen as a group of voters having the same opinion and the associated weight represents the strength of the group (for example, the number of voters). In value theory, the weights are trade-offs, where a loss in one criterion may be balanced by a better performance in one or some other criteria. Once more, these parameters may be specified via direct or indirect preference information, as we shall detail in the next chapter.

Also, unlike in MAVT, the evaluations do not necessary need to be numerical, as long as the decision-maker is able to pairwise compare the values. According to our example, as we may assume that a decision-maker clearly prefers a very interesting subject to an interesting one, we do not need to scale into numerical values these labels, which bypasses some difficulties related to the construction of value functions.

On such qualitative scales with few possible values, as the decision-maker's point of view impacts each of them so that they are considered as significantly different from the other ones, the local pairwise comparison is pretty easy, *i.e.* the local concordance $S_i(x, y)$ is equal to 1 if $x_i \geq y_i$ and otherwise equal to 0. But in a continuous scale (like for example on a criterion *price*), or a broader qualitative scale, one can legitimately argue that some small differences may not be significant. This is taken into account by considering some indifference $q_i \geq 0$ and preference $p_i > q_i$ discriminating thresholds in order to determine the interval definition of f_i .

To put it more simply, the discriminating thresholds define some zones on which a difference is, according to the decision-maker, either considered to be irrelevant (*i.e.* there is an indifference between the evaluations), or clearly significant. In between, we have to construct the function f_i as close as possible to the decision-maker's preferences.

Notice that these discriminating thresholds may have two different functions that condition their elicitation: Either they allow to represent the "objective" impreciseness of the tools used to construct the criteria, or they acknowledge for the "subjective" decision-maker's perception of what can be considered as a significant difference. In the first case, these thresholds are given by the person in charge of the construction of the criteria. In the second case, the analyst needs to discuss with the decision-maker in order to determine his preference assessment. That will be discussed later.

In absence of an incomparability situation, x is said to outrank y as soon as the concordance value reaches a defined level of acceptance. For instance, one may consider that x outranks y if at least half of the weighted criteria (*i.e.* a *median-cut*) is warranting this assumption, namely when $S^w(x, y) > 0.5$ if the weights are normalised on a $[0; 1]$ scale or $S^w(x, y) > 0$ on a $[-1; 1]$ scale. However, as we will detail in the next chapter, it may be higher (*i.e.* a *qualified majority*), mainly for taking into consideration some possible impreciseness on the parameters.

Considering such an approach assumes the decision-maker's agreement to the fact that there is no compensation between the criteria. This assumption is often easier to ensure when dealing with imprecise, but accurate, evaluations (for instance, some qualitative evaluations), such that it is difficult to provide some strong recommendations based on such imprecise compensations.

It is often considered that the construction of the outranking relation is an easier and more reliable step than the construction of a value function but, consequently, the difficulties arise in the exploitation phase when searching for providing the decision-maker an understandable and justifiable final recommendation. Indeed, it is hardly possible to show a decision-maker such a complex relation: we have to extract the desired information, namely the best compromise, or also a ranking or a sorting of the alternatives.

Moreover, this relation may be incomplete, when facing some incomparable statements, and not necessarily transitive [Roy90]. Thus, there is an automatic loss of information during the required exploitation of the outranking digraph, in order to provide some recommendations. But the construction of this relation must be an asset of the decision aid process which will lead to a positive and strong acceptance of the recommendation from the decision-maker. In consequence, it is essential to be able to measure the "quality" of this relation, namely its complete accordance to the decision-maker's expectations.

Preference elicitation processes

“Non puoi insegnare niente a un uomo. Puoi solo aiutarlo a scoprire ciò che ha dentro di sé.”¹

[Galileo Galilei]

Contents

| | | |
|------------|---|-----------|
| 2.1 | Decision-makers’ profiles and preferential information . . . | 22 |
| 2.1.1 | Decision-makers’ profiles | 22 |
| 2.1.2 | Expressing some preferential information | 24 |
| 2.1.3 | Quality of the expressed preferential information | 26 |
| 2.2 | Setting up an iterative preference elicitation process | 27 |
| 2.2.1 | Principles of an iterative preference elicitation approach . . . | 28 |
| 2.2.2 | An overview of disaggregation approaches | 30 |
| 2.2.3 | Knowing the potential pitfalls | 31 |
| 2.3 | Analysing the robustness of preferential results | 32 |
| 2.3.1 | Sensitivity analysis within multi attribute valued theory . . . | 33 |
| 2.3.2 | Credibility level cutting technique | 34 |
| 2.3.3 | Dealing with imprecise but accurate information | 35 |
| 2.3.4 | Stability of outranking relations | 36 |

Abstract

In this chapter, we briefly study the common framework of the preference elicitation processes in MCDA. First, we define the different profiles of decision-makers we may encounter, as well as the different types of preferential information they can provide. Then, we describe the general framework of an iterative preference elicitation process. Finally, we discuss the robustness of the resulting evaluation model, in order to take into account the possible use of incomplete or imprecise information. Besides, the stability concept of outranking relations is introduced and motivated within such a process.

1. “You cannot teach a man anything; you can only help him find it within himself.”

Introduction

In the previous chapter, we have shown the main theoretical frameworks of the multicriteria decision aid methods. In both approaches, a setting of different model parameters is needed in order to construct a reliable evaluation model.

Nevertheless, it is not an easy task and their elicitation from a human decision-maker may arise some well-known issues if done without enough precision. Indeed, a decision-maker may ignore his preference or have an imprecise knowledge of its expression in the chosen aggregation procedure. It is then essential to propose also some indirect approaches for eliciting the parameters, based on information a decision-maker is able to provide in a clear manner.

In addition, this knowledge is strongly correlated to the decision-maker's degree of experience in the considered decision domain, not to mention that the decision-makers are far from being experts in the employed multicriteria decision method.

The chapter is constructed as follows: First, we define the different profiles of the decision-makers we may encounter, as well as the different types of preferential information they can provide. Then, we describe the general framework of an iterative preference elicitation process. Finally, we discuss the robustness of the resulting evaluation model, in taking care of possibly incomplete or imprecise information. The stability concept of an outranking relation, which will be extensively studied in the next chapter, is introduced and motivated in such a preference elicitation process.

2.1 Decision-maker's profiles and provided preferential information

At first, we define the different decision-makers' profiles that we may encounter. Then, we list the different kinds of preferential information they can provide. Finally, we draw some critical views about the credibility we should consent to these information.

2.1.1 Decision-makers' profiles

Three decision-makers' profiles are usually considered, according to their knowledge of the domain of the considered decision and their habits in making such decision:

- The *naive decision-maker*: He does not have any particular knowledge about the considered decision domain and is not used to make such decisions;
- The *novice decision-maker*: He has pretty good knowledge of the decision domain but is confronted to an unusual decision.
- The *expert decision-maker*: He is an expert of the decision domain and is used to make such decisions [Sha88];

The naive decision-maker

A naive decision-maker is willing to make a decision in an unusual domain which he is not knowledgeable in. Although he is hardly considered in the literature, he is the most common decision-maker in day to day situations. For instance, we may quote the case of someone willing to buy his very first computer and who does not have any idea about the model that he may consider as a good one according to him.

The naive decision-maker has no particular knowledge about the decision domain and is not able to configure properly any parametric decision methods, nor to express some reliable preferential information. Most of the time, he may randomly select one option or focus on a very few criteria he may understand (for instance, he takes the first computer, the cheapest, the prettiest or the one he saw on television, *etc.*). In the best case, his choice will be strongly supported by the recommendations of some experts: a computer magazine comparing different models and selecting the best ones according to different profiles of users (*i.e.* a normative approach), or also the advices of a sales assistant (*i.e.* a delegated decision).

The novice decision-maker

The novice decision-maker is an expert, or has at least a good knowledge, of the considered decision domain, but he faces an unusual decision. A simple example may be a man in fond of cars, who wants to buy a new one after many years.

He is particularly at ease in talking about the domain and can easily compare justifiably two different alternatives, but his understanding of the different points of views is mainly implicit, such that he is uncomfortable in determining the impact of each criterion in the global evaluation. For instance, when considering the car example again, such a novice decision-maker will probably be able to define the overall relation between some couples of cars, but will not automatically express the relative importance of the criterion *Number of seats* compared to the criterion *Engine power*.

Notice that, when dealing with a novice decision-maker, the consideration of a constructive approach seems to be an appropriate approach, as it highlights and makes explicit his preferences.

The expert decision-maker

He is an expert in the considered decision domain and in actually making such decisions. We may quote, for instance, the case of a physician examining his patients and determining precisely of which disease they are suffering. He has an intuitive knowledge of the criteria to take into account (the pulse, the blood pressure, *etc.*), he is able to precisely measure, or evaluate, these characteristics (for instance, by measuring the temperature) and clearly understands the impact of the variations.

His experience and his knowledge of similar decisions previously made, strongly support him in stating his preferences. As proved in [LN80], such an expert processes information in his short term working memory, but he relies on the strategies compiled in his long term memory: when he deals with a small set of alternatives, most of the time he is able to select the best compromise without the need of any decision aid method. Using such a method would save time when dealing with a broader set of potential alternatives. Notice that a descriptive approach is well-adapted to such a decision-maker.

2.1.2 Expressing some preferential information

Basically, a decision-maker is willing to express some preferences over a decision situation. Let \mathcal{I} denote this set of information. Its elements can take the form of some constraints on the parameters of the given aggregation procedure, but they may also be some expectations on the result of the aggregation procedure. Such preferential information are usually and respectively called *input-oriented*, or *output-oriented*. In both case, this information can be seen as constraints that reduce the universe of admissible parameters for the setting of the method.

Input oriented preferential information

Let denote \mathcal{I}^{in} , included in \mathcal{I} , the subset of the input-oriented preferential information, which are some information directly expressed on the method parameters. Basically, they take one of the following aspects:

- An evaluation difference that is (resp. that is not) significant on a specific criterion (*i.e.* some bounds on the discriminating thresholds values);
- A marginal utility value;
- A value, or an interval of values, for a parameter;
- A ratio or a tradeoff between two criteria weights;
- A relative importance information between two criteria, or two coalitions of criteria (*i.e.* one is more important, or less important, than another).

Asking directly these parameters is putting a very strong hypothesis on the ability of a decision-maker to provide them, or to give some reliable ratios between them (see for instance the AHP method [Saa80]), which induces a clear and precise knowledge of his preferences, but also a complete understanding of the considered aggregation procedure and the semantic of the parameters.

A fair number of authors argue that the notion of relative importance only makes sense when related to a specific method (see for instance [Mou93], [Pod94, Pod02], [Vin89] and a debate held during the 71st meeting of the EWG on MCDA²). In every day language, a criterion, or a point of view, is said to be *important* when it plays a significant role in the construction of the overall preference statement. In a logical

2. March 25–27, 2010 in Torino, Italy. <http://www.mcda71.polito.it>

manner, a criterion is *more important* than another if it has a more decisive role in the overall preference.

To formalise this definition, a criterion i is said to be more important (according to every day language) than a criterion j when its marginal contribution to the construction of the overall preference is higher than the marginal contribution of criterion j .

This definition is hardly compatible with the concept of compensation between criteria, as the strength of a criterion, *i.e.* the associated marginal utility, in the construction of each overall score, is depending on the performance of the considered alternative: the higher a performance is, the higher the support of the criterion is.

Within the outranking methods, if we consider the evaluation of the local concordances with only three values, namely 1, -1 and 0, which corresponds respectively to an agreement, a disagreement or a neutral position on the “*at least as good as*” statement, the notion of importance of a criterion has the same semantics as in every day language: a positive concordant statement exactly translates the fact that the coalition of criteria in favor is more important than the coalition of criteria in disfavor. A criterion plays a role in supporting the statement or in refuting it, but always with the same importance (or strength), which is not the case, for instance, with an ELECTRE-like construction of the discriminating thresholds, due to the use of a linear interpolation, where some criteria may play a weakened role.

Working hypothesis 1 *To clarify our discourse, we assume in this thesis the following translation: “criterion i is more important, or more significant, than criterion j ” means that the decision-maker should positively answer the question “Would you be willing to consider the alternative b as a better compromise than a if b was not at least as good as a on criterion j , but better on criterion i , assuming similar performances on the other criteria?”.*

Output oriented preferential information

An output-oriented preferential information is any information on the expected result of the multicriteria aggregation procedure (MCAP). It can be a desired relative comparison between two alternatives (for instance, the decision-maker may express the fact that an alternative is strictly preferred to another), but also the assignment of some alternatives to certain categories within the context of a sorting problem. The set of such information is denoted \mathcal{I}^{out} and is included in \mathcal{I} .

These overall judgments between the alternatives assume that the decision-maker’s argumentation is in accordance with the underlying principles of the considered MCAP, as they condition the elicitation of its parameters.

In [RV85], the authors stated that a decision-maker who is confronted to the comparison of two given alternatives x and y , should be asked to express one of the following information (in accordance with a given and accepted aggregation procedure \mathcal{P}):

- A clear *indifference* I between the alternatives;
- A strict *preference* P of one of the alternatives he shall identify;
- A *weak preference* Q of one of the alternatives (he hesitates between an indifference or a preference, in favor of one alternative);
- An *incomparability* R , when the alternatives are considered too different to be compared;
- An *indetermination*, when he cannot express with enough conviction one of the preceding information.

In the context of a valued approach, the incomparability is not an option for the potential answer of the decision-maker. An indifference between two alternatives x and y will result in an equality between their scores, *i.e.* $U(x) = U(y)$, and a preference for alternative x will highlight the fact that $U(x)$ is greater than $U(y)$.

Working hypothesis 2 *In the context of an outranking approach, we have to translate these preferential information in terms of outranking relations between the alternatives. In [RV85], the authors give the following regular translations we shall consider in the sequel of our work:*

$$\begin{aligned}
 x P y &\iff x S y \quad \text{and} \quad y \bar{S} x \\
 x I y &\iff x S y \quad \text{and} \quad y S x \\
 x Q y &\iff x S y \\
 x R y &\iff x \bar{S} y \quad \text{and} \quad y \bar{S} x
 \end{aligned}$$

The weak preference relation is resulting from the fact that $Q \equiv P \cup I \equiv S$.

2.1.3 Quality of the expressed preferential information

To better describe the “quality” of the expressed information, we should make a clear distinction between a *precise information* and an *accurate information*:

Definition 2.1 (Preciseness) *An information, given by the decision-maker, is said to be precise when it constraints the value of one parameter, or the ratio between some parameters into reduced intervals (the intervals may be reduced to a unique value). An imprecise information is then a less restrictive constraint.*

Definition 2.2 (Accuracy) *An information may be also viewed as accurate when it can be considered in total accordance with the decision-maker’s mind. On the contrary, an inaccurate information is going against the decision-maker’s thoughts.*

An example of precise information can be the association of a criterion weight with a unique value (*e.g.* “the weight associated with criterion i is 0.2”), or the fact that two criteria must be associated with the same weight. An accurate information may be the clear consideration by the decision-maker of a criterion more important

than another one, without any precision on their relative importance degree. Notice that this last information is accurate but also imprecise. Also notice that we can have precise information that are inaccurate, when for instance a decision-maker is asked to give a precise value for a criterion weight, but he may not be totally confident about the expressed value.

In a quite intuitive manner, one can conceive that the more precise the information are, the more questionable their accuracy is. In that case, assuming the modeling of a decision problem, it is appropriate to consider that only an expert decision-maker is comfortable in the expression of precise and accurate preferential information, due to his experience in the domain on which he uses to take this particular decision.

For a novice decision-maker, the expression of precise preferential information on the parameters may appear quite arbitrary. Indeed, he may be able to provide an accurate partial preorder between some criteria (for instance, when comparing some cars, the fact that the *color* is less important than the *security*), but he probably cannot express the exact relative importance between two criteria (for instance, the security is three times more important than the color in the decision). Asking for such precise, but inaccurate, input-oriented information may result in the setting of a method that will not reflect the decision-maker's expectations. In consequence, it seems more advisable to focus on less precise information, but with an incontestable accuracy (*i.e.* a stronger support from the decision-maker).

2.2 Setting up an iterative preference elicitation process

In [Mou05], MOUSSEAU defines the preference elicitation process as a “*process that goes through an interaction between the decision-maker and the analyst (or a software) and leads the decision-maker to express preference information within the framework of a selected MCAP*”.

The preference elicitation is a part of the decision aid process, that allows to construct the evaluation model. Notice that it requires the explicit use of an aggregation procedure. Hence, the MCAP has to be selected before the preference elicitation process and should not be modified, nor questioned, during the process, unless the preference elicitation is restarted.

In this section, we show how it is possible to implement an iterative process for the preference elicitation, based on a constructive approach, by first defining the elicitation process in a formal way and then studying the behavior of the decision-maker in such a process.

This section sums up to a large extent the work of MOUSSEAU on the preference elicitation. For a more detailed discussion, we refer to [Mou03, Mou05].

2.2.1 Principles of an iterative preference elicitation approach

First, we briefly present two different approaches, called *aggregation approach* and *disaggregation approach*, and show how they are combined into an iterative process called *aggregation/disaggregation approach*.

The aggregation approach

Let us consider an MCAP \mathcal{P} , as well as a set A of alternatives evaluated on a coherent family F of criteria. An aggregation approach consists in inferring, from a set of input-oriented preference information only, a compatible set of parameters for \mathcal{P} that allows the construction of an evaluation model, *i.e.* the construction of a binary preference relation between the alternatives.

The framework of such an aggregation approach has been defined as follows by Mousseau [Mou03]:

- Defining the set A of alternatives;
- Defining the coherent family F of criteria;
- Selecting a multicriteria aggregation procedure \mathcal{P} ;
- Setting values for the parameters of \mathcal{P} ;
- Constructing the global preferences by application of \mathcal{P} ;
- Analysing the sensitivity of the preference relation in order to express recommendations.

Let us notice that this framework is not necessarily defining a sequential process, as we may observe some step backs, in order to refine, for instance, the sets of alternatives or criteria, or to test also different values for the parameters.

In such a case, the analyst must ask the decision-maker a large number of questions for a correct tuning of the parameters. In order to determine the trade-off between the criteria, for instance, he may ask questions like “How much do we have to increase the evaluation x_i of an alternative x on criterion i in order to compensate a loss of 1 unit on the evaluation x_j on criterion j ?”. The AHP method [Saa80] proposes to determine the parameters by asking the relative importance between the criteria (for instance, criterion i is 3 times more important than criterion j).

It is commonly stated that this approach requires a sensitivity analysis of its results, due to the possible impreciseness of the parameters and their effective impact on the MCAP result. We will discuss this point in section 2.3.

The disaggregation approach

When we are dealing with the difficulty of defining some method parameters in order to infer an evaluation model of the alternatives, we may consider a reversed approach, namely to start from a complete or partial evaluation model given by the decision-maker and to see if these information are compatible, or consistent, with a given MCAP \mathcal{P} .

The purpose of a *disaggregation approach* is then to determine a set of admissible values for the parameters of \mathcal{P} , from a set of overall judgments between the alternatives a decision-maker is willing to express (*i.e.* a set of output-oriented preferential information), assuming that his argumentation is in accordance with the underlying principles of \mathcal{P} . This information may be a ranking over a subset $A' \subset A$ of alternatives, some pairwise alternatives comparisons, or some assignments of alternatives to categories.

Notice that in the literature, these approaches are sometimes called *ordinal regression analysis* or *inverse analysis*.

In the vast majority of cases, these approaches use linear programming techniques (see for instance [JS82], [SY85] and [Sis85]). Most of the time, to be operational, they do not explore the whole admissible solution polytope, but they focus on one particular set of parameters that maximises a given objective function. We must remark that the number of optimal or near optimal solutions may be quite huge, such that an exhaustive search method (like for instance the reverse simplex method [DOH54], or the Mañas-Nedoma algorithms [MN68]) may be very time-consuming, and may require a considerable effort from the decision-maker to select one particular solution. As we shall discuss in section 2.3, it is then mandatory to discuss the reliability of this solution in order to legitimate the resulting recommendations.

The aggregation/disaggregation approach

An *aggregation/disaggregation approach* is a process in which we alternate between aggregation and disaggregation steps, using the output of the previous step for the next iteration. Intuitively, we generate a set of parameters from a disaggregation step, based on a part of the expected result, allowing to aggregate and reconstruct the complete preference relation (*i.e.* the valued or the outranking relation). Then, we discuss with the decision-maker his accordance with the relation and add, the case given, new preferential information on the comparisons he may not agree. We continue until a consensus is found.

The process can be seen like an iterative sequence of questions and answers whose purpose is to lead the decision-maker to express gradually a preferential information. This sequence may also be the opportunity to test the validity of some hypotheses, or even to come back to some previously given information.

Example 5 *We assume that an aggregation procedure is given and the set of alternatives, as well as the family of criteria, have been defined. For example, let us suppose that a decision-maker expresses the fact that an alternative a is preferred to an alternative b . Applying a disaggregation algorithm, we find some compatible parameters and can then start the aggregation procedure. By presenting the output to the decision-maker (for example the ranking of a valued method), he may disagree the fact that alternative a is preferred to alternative c . Then, we start a second disaggregation phase. The process continues until the decision-maker is satisfied by the recommendations.*

Note that the preference elicitation process is an important part of an aggregation/disaggregation process. Therefore, it is particularly important that the selection of the aggregation procedure must be done beforehand and should not be modified during the elicitation process, as it determines the necessary parameters, as well as their meaning, to be elicited [Vin89, Mou93, Pod94].

2.2.2 An overview of disaggregation approaches

The very first disaggregation approach is implemented by JACQUET-LAGREZE and SISKOS in the seminal UTA method [JS82, SGM05], using linear programming for assessing additive value functions from a partial subjective ranking of decision alternatives, in order to aggregate multiple criteria into a single composite criterion.

The UTA method enabled a large number of derivative methods to rise up. As examples, we may refer to UTADIS [DGJ80] or UTA* [SY85]; but also ADELAIS [SD89], that enables an interactive use of the UTA method. A review of the UTA multicriteria method and some improvements can be found in [DYZ90].

An interested reader may find in [JLS01] a review by JACQUET-LAGREZE and SISKOS on the first twenty years of the preference disaggregation methods.

When considering the non-additive value theory, we may also mention the work of ANGILELLA *et al.* [AGM10], as well as an overview by GRABISCH, KOJADINOVIC and MEYER [GKM08], concerning the different methods for the identification of Choquet integral capacities.

Considering some examples of disaggregation approaches in outranking methods, we can mention the ELECCALC system [KMN94], which estimates indirectly the parameters of the ELECTRE II method. In addition, different solutions for assessing the parameters of the ELECTRE TRI method have been developed: In [MS98] and [MFN01], the authors suggest an interactive approach for assessing the criteria weights from the assignments of some alternatives in defined categories; in order to decrease the computational difficulty of the resulting mixed integer linear program, MOUSSEAU and DIAS [MD04] propose a slight adaptation of the valued outranking relation used in ELECTRE III and ELECTRE TRI. These works are complemented with the implementation of mathematical programs for inferring veto-related parameters, developed by [DM06], and the work of [RD08], assessing criteria weights and cutting level parameters without having to pre-define categories. Let us also mention the IRIS software [DM03], an extension of the ELECTRE TRI method which allows the decision-maker to provide assignment examples and constraints on the weights and the cut levels of the valued outranking relation for sorting the alternatives. A more general approach is used in the recent theoretical work in [MMB08], where a mixed integer linear program is presented for assessing, at the same time, criteria weights, performance discrimination thresholds as well as potential performance values directly from a median-cut outranking relation.

2.2.3 Knowing the potential pitfalls

As we consider a constructive approach, the decision-maker constructs his preferences all along the process, based on his experience, his convictions, his values, or his beliefs, but he is also influenced by the elicitation process itself. Indeed, the questions we may ask, the alternatives we shall present to him will automatically focus his attention and his deliberation. It is then of the highest importance to know the potential pitfalls and how to best avoid them.

Helping the decision-maker in the expression of preferential information

In order to express his preferences in the comparison of some alternatives, one must present the decision-maker with some information. It can be a part of the result, as for example the impact of some expressed constraints on the comparison of some alternatives. Such information is also useful for improving the convictions of the decision-maker, in order to understand why an alternative has to be preferred to another one, or why an alternative cannot be assigned into one particular category.

Notice that the tools we use to display the information should be in accordance with the considered MCDA philosophy, at the risk of inducing the expression of inconsistent preferential information. For instance, in MAVT, one may easily consider to display the alternatives performances in a graphical way (using for instance some box plots). But, such visualisation tools may induce some implicit compensation by the decision-maker that are not desired in outranking methods.

We can also show him a set of compatible parameters with its expressed information. Although, he may not be able to tune precisely the parameters, he may however react, or verbalize its disagreement if, for example, the ratio between two criteria weights seems underestimated or overvalued.

Independency of the comparisons

It is important to notice that, when comparing two alternatives, a decision-maker should not be influenced in his preferences by any other alternatives. In fact, we should assume this principle as a logical way of thinking.

If the decision-maker acts according to the principles of the valued methods, this property is always granted. When using an outranking method, this property is ensured for the outranking graph, but may not be validated for the resulting recommendation. We can for instance cite the ELECTRE I method [BRS66], that may rank an alternative a before an alternative b if a outranks more alternatives than b , no matter if b outranks a . Also, a non-transitive outranking digraph may never be summed up in a true and fair ranking.

Working hypothesis 3 *As it is a particularly challenging issue out of the scope of this work, we shall concentrate on the construction of the outranking digraph and not its exploitation. In consequence, we shall consider that the outranking digraph*

has to be compatible with the given preferences information on the comparisons of alternatives.

Dealing with an inconsistent set of preferential information

In a constructive approach of the elicitation of the parameters, the fact that a set of preferential information cannot be fully represented in the given aggregation procedure is an important issue. Indeed, as the expressed information by the decision-maker leads to the construction of a set of mathematical constraints, a contradictory set of constraints may result in an empty set of admissible parameters.

Such a situation appears when some information are incoherent between them, or impossible to model in the chosen aggregation procedure (for instance, a non-transitive situation between three alternatives, when considering a value-oriented aggregation method). Before trying to get rid of this conflictual situation, the analyst should verify carefully the agreement of the expressed preferential information to the underlying principles of the given aggregation method. Also, these incoherences may be due to an evolution of the decision-maker's point of view during the process.

Most of the time, trying to resolve the conflict comes to the determination of a non-conflictual subset of constraints: we can for instance take the smallest possible subset out or, if the decision-maker expressed a confidence index for each statement, take the least confident constraints out. We may remove the one expressed first, if considering that the construction of the preferences leads the decision-maker to express more and more accurate preferences. As this is out of this thesis scope, we will not detail thoroughly the existing techniques. We recommend the reading of the articles [LL01], [Roo79], [MDF⁺03], and [MDF06] for resolving such a situation.

It is a complex combinatory problem when the constraints are of any shape. When considering linear constraints only, some authors (see for instance [VL81], [Chi94], or [TMJ96]) intend to define some *Irreducibly Inconsistent Systems* (IIS), which are some minimal subset of constraints for which an inconsistency arises: putting any constraint apart makes the inconsistency disappear (let us note that if there is more than one IIS, the global system may still remain inconsistent). Some others try to determine the minimal set of constraints to be discarded in order to obtain a consistent set (see for example [Chi96], [MKC00], [MDF⁺03]).

We must present these inconsistencies to the decision-maker, and provide him also with some tools that allow a better understanding of the situation, in order to come back to a coherent situation.

2.3 Analysing the robustness of preferential results

We have seen that the construction, as well as the exploitation, of an evaluation model is depending on the use of one particular set of parameters, which may be

compatible with the expressed preferential information. Intuitively, we may have to deal with incomplete information, such that it should be possible to consider different sets of parameters, both compatible with the given information, but providing distinct recommendations.

As an analyst, how much confidence can we show in a recommendation resulting from the selection of one particular set of compatible parameters? Why should we give preference to this set instead of the others? What may happen if the information are not completely accurate?

The concept of *robustness*, as defined by ROY, represents the dependency of a recommendation to some *impreciseness*, *uncertainties* or some *not well-known*, or *not well-defined* characteristics of a problem (see for instance [Roy98, Roy02, Roy10]). In addition, DIAS states that the general idea behind the robustness analysis approaches is to accept multiple model versions (or scenarios, or parameters sets) and to try to identify a solution that is seen as being good or acceptable in (almost) every model versions [Dia07].

Thus, in the sequel of this work, a recommendation will be called *robust* when it is valid for a large set of possible scenarios (to be defined) that are compatible with the given information.

In this section, we first describe three well-known tools for establishing robust recommendations, namely the *sensitivity analysis*, the *credibility cutting level techniques* and the concept of *necessary and possible statements*. Then, we position and motivate our work on the concept of *stability*, which is an additional tool for addressing the robustness concern.

Notice that we will not consider here the methods that determine robust conclusions without the decision-maker's involvement in expressing preferential information. This is the case with the SMAA method [LHS98] (see also the SMAA-2 method [LS01] or SMAA-3 method [HLMS98]), that intends to compute, for any alternative in the context of an MAVT approach, the percentage of scenarios on which the alternatives is the best one, according to every possible scenarios (*i.e.* every possible utility functions). This is particularly useful in real-life public political decision making processes, where the decision-maker should not adopt a subjective position.

2.3.1 Sensitivity analysis within multi attribute valued theory

The main idea of sensitivity analysis is to test the impact of small parameters changes on the resulting preference relation, most of the time focusing on the best alternative or the alternatives ranking within the context of MAVT, where it has been extensively studied. DANTZIG [Dan63] stated that it is a "*fundamental concept in the effective use and implementation of quantitative decision models, whose purpose is to assess the stability of an optimal solution under changes in the parameters*". ROY and BOUYSSOU [RB93] noted that it may be quite time-consuming, but it is a necessary task to construct, modify or justify some preferences before starting a critical discussion and establish a valuable recommendation. A methodology for

sensitivity analysis in multi-objective decision-making is depicted in [RI90].

It is easy to understand that the complex setting of a utility function makes questionable the provided result, not to mention the fact that a decision-maker may often be uncomfortable in expressing some exact parameters. Indeed, based on possibly imprecise parameters, can a resulting optimal alternative be truly considered as the best one?

A fair number of authors defined a sensitivity analysis to measure the impact of such imprecise information, but it considers independently each parameter, as their weights are tested separately around an “ideal” solution. For instance, in [TS97], the authors considered two closely related sensitivity analysis problems, namely to determine the most critical criterion (*i.e.* the one that modifies the best alternative or the ranking with the smallest change in its associated current weight) and how critical the performance measures of the alternatives are in the alternatives ranking. MAYSTRE *et al* [MPS94] suggested different techniques to take into account the dependencies of the parameters, but the resulting high combinatorial number of values to be considered makes the interpretation of the results very difficult. In [DC99], within the context of value functions, the authors developed an analysis software, VIP, that computes a minimal and maximal score for each alternative, under a set of linear constraints, allowing to consider interdependencies between the parameters.

An interested reader that wants to go further may for instance refer to [WB87], [FRI89], [Haz86], [Web87], [AP97], [SH92], [Bar92], [SH01], [MPY92], or [KCCJ93].

2.3.2 Credibility level cutting technique

In the context of the outranking methods, we also may have to deal with inaccurate, imprecise, uncertain or ill-determined data. In order to construct a highly reliable crisp outranking relation to be exploited further, we may consider as valid only the fuzzy outranking relations associated with a clear positive credibility higher than a defined cut level λ [FMR05]. In fact, this cut level, introduced in the ELECTRE methods, is often defined as the credibility index smallest value that is compatible with the assertion “ x outranks y ” (see for instance [DM06]). As its direct elicitation may be quite difficult, it is often elicited via a disaggregation procedure with the other parameters simultaneously.

Informally, it underlines the idea that a high concordance value (*i.e.* a *qualified majority*) has little chance to be called into question, contrary to a low concordance value, which seems too anecdotic to be taken into consideration. However, as we shall present in what follows, a high value is not always a warrant for stability with respect to imprecise weights parameters, such that a cutting technique may not be the most suitable technique.

More details about this credibility level cutting can be found in [FR94] and some examples of applications are given, for example, in [LLAC05], [BP07] and [CAM03].

2.3.3 Dealing with imprecise but accurate information

Basically, the sensitivity analysis questions the quality of a recommendation based on precise, but not fully accurate, input-oriented information. In order to deal with imprecise, but supposedly accurate, input-oriented information, one may think about defining for each constituent of the evaluation model a set of admissible values. For instance, DIAS and CLÍMACO [DC00] compute robust assignments, *i.e.* the best and the worst assignments for each alternative, according to some given constraints by multiple decision-makers on the criteria weights, using the ELECTRE TRI method [Yu92]. Then, the decision-maker may confirm some intervals, narrow down some others (which means the add of new preferential output-oriented information) or dismiss an interval (leading automatically to a conflict).

In a similar manner, in response to the difficult work with inaccurate or imprecise input-oriented preferential information, GRECO, MOUSSEAU and SŁOWINSKI [GMS08] take the position to consider only output-oriented preferential information, supposedly accurate, in an aggregation/disaggregation approach. Indeed, the authors assess utility functions from a set of pairwise comparisons on an interactively increasing subset of alternatives: This variant of the UTA method [JS82] takes care about the robustness of the preference modeling by considering *necessary* preferential statements (which are valid for all value functions compatible with the given information) and *possible* statements (when at least one of the value function validates the statement).

Basically, to compute such a property for any pair (x, y) of alternatives, we test the consistency of the currently expressed set of preferential information, enlarged with an information that enforce the fact that x is preferred to y or y is preferred to x alternatively. If there is only one consistent set, the associated additional preferential information is necessary. If both are consistent, the preference relation is possible in both sense, such that the decision-maker should be questioned again in order to reduce the set of possible statements.

Thus, the decision-maker is only questioned about expressing some preferential information that are not already granted, in order to restrict the number of compatible scenarios. The robustness is ensured by the fact that the process stops only when (almost) all relations between the alternatives are necessary, namely there is only one (or a very few) admissible scenario, without the need to consider one specific parameters set.

Note that this approach assumes that the decision-maker expresses comparisons of alternatives that stay accurate all along the process. Indeed, as we are not asking for any necessary judgement, there is no possibility in highlighting some changes in the decision-maker's way of thinking. In addition, it requires a longer questioning of the decision-maker, not to mention that he may not be able to answer to every questions, and the running time may become prohibitive for a real-time processing of the decision aid.

2.3.4 Stability of outranking relations

The *stability* concept tries to characterise the dependency of the outranking relations with respect to the possibly imprecise weights parameters. This concept has been originally introduced by BİSDORFF [Bis04] under the terminology of *ordinal concordance*. Basically, a situation is said to be *stable*, with respect to a given weights vector w , when it only depends on the relative importance between the criteria. Thus, a stable outranking situation is valid for every weights vectors having the same preorder as w , such that it can be considered as a robust information (assuming that the preorder is an accurate information, *i.e.* strongly supported by the decision-maker).

The stability allows to extend the efficiency relation. As we have said in the first chapter, the efficiency relation is the only objective relation between the alternatives, but it is not restrictive enough to be considered as a final recommendation. Thus, the stability allows to introduce an intermediate step between this objective relation and the completely subjective relation from a precise setting of the parameters, by considering the preorder as a subjective, but less questionable, information.

Like the previous concepts, the stability deals with imprecise but supposedly accurate information. However, it only considers input-oriented information, specifically a complete preorder. We can make a parallel with the necessary and possible concepts, as a stable situation is a situation that is necessary, according to the preorder. However, an unstable situation may be necessary, according to some already expressed preferential information, such that if we only focus on the stability, we may question the decision-maker about some implicit preferential information. In addition, contrary to DIAS and CLÍMACO, we need to define a complete preorder, which can be more difficult to obtain. But the stability property is easy to compute and may provide some highly supported recommendations, without the need of heavy mathematical computations. We will show in the construction of our iterative preference elicitation process how both concepts may be combined.

Finally, we should draw a parallel with the recent work, on valued theory, from PODINOVSKI [Pod11, Pod12], who analyses the sensitivity of the solution to a multicriteria choice problem, based on the expression of a partial weak order on the importance of the criteria. The author assumes the use of criteria with a common ordinal scale and computes for each alternative the strength of stability via a linear program. Dealing with the outranking principles in our case, there is no restriction in any particular scale for the criteria and we will consider only some stability degrees that can be verified without the need of linear programming.

Thesis intention

The work we shall present in this thesis is in keeping with the multicriteria decision aid methods, more specifically within the outranking philosophy. We focus on situations where a unique decision-maker is involved, though we will have some discussions showing that it is possible to consider the stability concept with multiple decision-makers. The decision-maker is considered as a novice one, *i.e.* he has a pretty good knowledge about the decision domain and he is facing an unusual decision. Besides, he understands and agrees on the outranking principles.

The primary objective of this thesis is to implement a constructive preference elicitation process which can be qualified as robust, taking advantage of the stability concept we detail in chapter 3. Then, in chapter 4 and 5, we show how can be implemented a disaggregation process taking this concept into account. Finally, we describe in chapter 7 our robust preference elicitation process named REWAT, which stands for *Robust Elicitation of Weights And Thresholds*, and use it in a real decision situation in the last chapter.

Notice that such an approach is motivated by the work with a novice decision-maker that is assumed to be able to express accurate overall preferential judgements, to have a comprehensive view about his preferences on the relative importance of the criteria, but also not being able to express directly some precise and accurate parameters.

Part II

On the stability of median-cut outranking relations: Definitions and models

In the second part of the thesis, we focus on the stability concept of outranking relations, which characterises the dependency of any outranking situations towards possible imprecisenesses of a considered vector of criteria weights. This concept, originally introduced under the terminology of *ordinal concordance*, is here given with a new perspective: We define simpler mathematical conditions in order to verify the presence or the absence of such a property and we extend the concept with two new degrees of dependency, supported by some examples of applications.

Chapter 3 is dedicated to the definition of the above mentioned stability concept and how it may be useful when looking to provide a robust recommendation to a multicriteria decision aid problem.

Chapter 4 intends to model some mathematical constraints for using this concept when eliciting a vector of weights from a decision-maker's set of preferential information. It appears as an answer to the difficulty for a decision-maker, especially when he is not an expert in the decision domain, to provide accurately and precisely such parameters. Based on different algorithmic choices, we define three mixed integer linear model for the elicitation of the weights.

Finally, chapter 5 extends the defined models to elicit simultaneously additional parameters, namely the discrimination thresholds and the category profiles in case of a sorting problem. These parameters are supposed to be given in the previous chapter, but some practical issues when dealing with a novice decision-maker have shown that this is not a highly reliable working hypothesis.

Notice that we are focusing on assessing the dependency of the outranking situations with respect to the vector of criteria weights only. That presupposes the accuracy and preciseness of the other parameters, directly given by the decision-maker or elicited via one of our defined mathematical models. In fact, we are not considering the possible impact of these parameters on the resulting outranking relation. Such considerations will be tackled in future work.

On the stability of the median-cut outranking digraph

“*There is nothing so stable as change.*”

[Bob Dylan]

Contents

| | |
|--|-----------|
| 3.1 Preliminary definitions | 44 |
| 3.1.1 Construction of a weighted outranking relation | 44 |
| 3.1.2 Weights preorder | 46 |
| 3.1.3 Defining the preferable relation | 47 |
| 3.2 Defining the stability of valued outranking relations | 48 |
| 3.2.1 Stability | 50 |
| 3.2.2 Extensible stability | 51 |
| 3.2.3 σ -stability | 54 |
| 3.3 Additional properties | 55 |
| 3.3.1 Limitation of the stability | 56 |
| 3.3.2 Stability of the preferable relation | 57 |
| 3.3.3 Stability within the context of the sorting problem | 57 |
| 3.3.4 Checking the stability property with missing evaluations | 60 |
| 3.3.5 Properties on the discrimination of the preorder | 61 |

Abstract

Within the context of the outranking methods, the fact that an alternative is “*at least as good as*” another one is depending on a clear setting of different parameters, especially the criteria weights. In this chapter, we carry on the work on the concept of *stability*, which intends to characterise accurately this dependency and measure the impact of some possible impreciseness in the weights. The more stable an outranking statement is, the less important a precise fixation of the weights becomes necessary. We then give an intuitive formulation, as well as simple mathematical conditions to compute the degree of stability of any outranking statement. Moreover, we give some practical hints on how the stability may be used to ease the resolution of a multiple criteria decision aid problem.

Introduction

We consider a decision situation in which a finite set of decision alternatives is evaluated on a finite set of criteria. A decision-maker is willing to express the weights of each criterion according to the outranking paradigm in order to assess the overall outranking relations between all pairs of alternatives.

However, precisely determining the numerical values of these weights is an important issue in multicriteria decision aid, when applying outranking methods [RM96] and also in multi attribute utility theory [Zel82], with a considerable impact on the recommendations. Being able to measure the dependency of the outranking relations with respect to the impreciseness and uncertainty related to these weights can be a helpful instrument for providing robust recommendations [Roy98, Roy02].

In this chapter, considering a vector of criteria weights, we characterise the stability of the resulting median-cut outranking relations, namely the dependency of each outranking statement with respect to the precise fixation of the weights. This work extends the one in [Bis04], by giving a more intuitive formulation of the stability concept and a simplified way of computing it, but also by allowing a sharper characterisation of the dependencies with two additional levels of stability. Finally, we discuss some additional properties and sketch the use of this concept in a preference elicitation process, in order to simplify the determination of some weights in best accordance with the decision-maker's mind, allowing to save time and to draw some robust recommendations.

The chapter is organized as follows: First, we introduce some required preliminary definitions, then we formally define the stability of any outranking statements, its extension, the way of computing it, as well as some important properties. For each level of stability, we present a small didactic example and we conclude by showing some perspectives on the consideration of the stability.

3.1 Preliminary definitions

3.1.1 Construction of a weighted outranking relation

Let $A = \{x, y, z, \dots\}$ be a finite set of $m \geq 3$ potential decision alternatives evaluated on a coherent finite family $F = \{1, \dots, n\}$ of $n \geq 3$ criteria. The alternatives are evaluated on performance scales and the performance of alternative x on criterion i is denoted x_i .

Between any two alternatives x and y of A , the marginal “*at least as good as*” situation $S_i(x, y)$ [Bis02, BMR08], with each criterion i , is characterized as follows:

$$S_i(x, y) = \begin{cases} 1 & \text{if } x_i \text{ is clearly at least as good as } y_i, \\ -1 & \text{if } x_i \text{ is clearly not at least as good as } y_i, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, considering a real performance scale, to which an indifference $q_i \geq 0$ and a preference $p_i > q_i$ discrimination threshold (for all i in F) are associated [RB93], the double threshold order $S_i(x, y)$ is given by:

$$S_i(x, y) = \begin{cases} 1 & \text{if } x_i + q_i \geq y_i, \\ -1 & \text{if } x_i + p_i \leq y_i, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $S_i(x, y) = 0$ corresponds to an undetermined situation where criterion i is not taking part in favor nor in disfavor of the overall judgement. This may be the case when there is not enough support to validate or invalidate the marginal “at least as good as” situation, but also when there is a lack of information (for instance, a missing evaluation). However, we present in section 3.3.4 a way to deal with evaluations in a more general way, namely by considering every possible local concordance values.

We associate furthermore with each criterion $i \in F$ a rational *importance weight* $w_i > 0$ which represents the contribution of criterion i to the *overall warrant or not* of the “at least as good as” preference situation between all pairs of alternatives. Let $w = (w_1, \dots, w_n)$ be the vector of relative importance weights associated with F and let \mathcal{W} be the set of such weights vectors. The *overall valued concordance relation*, denoted S^w , aggregating the partial “at least as good as” situations, is then given by:

$$S^w(x, y) = \sum_{i \in F} w_i \cdot S_i(x, y), \quad \forall (x, y) \in A \times A.$$

If a veto situation occurs in the comparison of a couple (x, y) of alternatives, such that it invalidates the outranking situation disregarding any criteria weights, the associated overall outranking relation is always trivially invalidated. As the purpose of our work is to study the dependency of the outranking relation to the weights, we may without loss of generality ignore the veto principle normally taken into account when dealing with classical outranking relation.

In the absence of a veto situation, the outranking relation, denoted $\tilde{S}^w(x, y)$, is equal to the concordance relation $S^w(x, y)$ [RB93] on which we focus our argumentation. An alternative x *outranks* (resp. *does not outrank*) an alternative y when $S^w(x, y) > 0$, (resp. $S^w(x, y) < 0$), *i.e.* when a weighted majority of criteria warrants (resp. does not warrant) the “at least as good as” preference situation between x and y [Bis02]. This situation is denoted xS^wy (resp. $x\mathcal{S}^wy$). $S^w(x, y) = 0$ indicates a balanced situation where the criteria warranting the “at least as good as” preference situation between x and y are exactly as important as those which do not warrant this situation. This balanced situation is denoted $x?^wy$.

Notice that we may use the notation $\neg(xS^wy)$ in the sequel of the thesis to denote the fact that the statement “ xS^wy ” is untrue, either because $x\mathcal{S}^wy$ or $x?^wy$.

3.1.2 Weights preorder

Let \succsim_w be the preorder¹ on F associated with the natural \succsim relation on the values of the weights w_i of the vector w . $=_w$ induces r ordered equivalence classes $\Pi_1^w >_w \dots >_w \Pi_r^w$ ($1 \leq r \leq n$). The criteria gathered in each equivalence class have the same importance weight in w and for any ranks $i < j$, those of Π_i^w have a higher importance weight than those of Π_j^w , the most important class being Π_1^w .

Let $c_k^w(x, y)$ be the sum of “at least as good as” characteristics $S_i(x, y)$ for all criteria $i \in \Pi_k^w$. Furthermore, let $C_k^w(x, y) = \sum_{i=1}^k c_i^w(x, y)$ be the cumulative sum of “at least as good as” characteristics for all criteria having importance at least equal to the one associated with Π_k^w , for all k in $\{1 \dots r\}$. Intuitively speaking, it is the set of the most important criteria, on which we may limit the decision if the other ones are insignificant.

Last but not least, we define $\Pi_k^{w+}(x, y)$ (resp. $\Pi_k^{w-}(x, y)$) as the set of criteria of the k^{th} class of equivalence supporting (resp. not supporting) the fact that x is performing “at least as good as” y .

Definition 3.1 (Preorder-compatible) *Two vectors $w, w' \in \mathcal{W}$ are said to be preorder-compatible if they induce the same preorder on the weights.*

Example 6 $w_1 = \{2; 7; 5; 2\}$ and $w_2 = \{3; 6; 4; 3\}$ are preorder-compatible.

Definition 3.2 (σ -preorder-compatible) *Two vectors $w, w' \in \mathcal{W}$ are said to be σ -preorder-compatible if \succsim_w is a permutation of the equivalence classes of $\succsim_{w'}$. For instance, assuming that w_i and w_j (resp. w'_i and w'_j) are any components of w (resp. w'), we verify the property as follows:*

$$\forall i, j \in F : w_i = w_j \iff w'_i = w'_j$$

Example 7 $w_1 = \{2; 2; 3; 3; 1\}$ and $w_2 = \{4; 4; 1; 1; 2\}$ are σ -preorder-compatible, associated with the permutation (132).

As we shall explain further on, such a property may be useful when trying to consider different objectives, all gathering some equi-important criteria, when the decision-maker is not able to sequence them in order of priority.

Definition 3.3 (Less discriminated weights vectors) *Let us consider w and w' in \mathcal{W} . w' is said to be less discriminated than w when its preorder $\succsim_{w'}$ is obtained by joining some adjacent classes in \succsim_w together, i.e. if and only if it respects the two following conditions:*

$$\begin{aligned} w_i = w_j &\implies w'_i = w'_j \quad \forall i, j \in F, \\ w_i > w_j &\implies w'_i \succsim w'_j \quad \forall i, j \in F. \end{aligned}$$

1. Classically, $>_w$ denotes the asymmetric part of \succsim_w , whereas $=_w$ denotes its symmetric part.

Definition 3.4 (More discriminated weights vectors) *In the same way, W' is said to be more discriminated than W when its preorder is obtained by splitting some equivalence classes, without modifying the inequalities between the classes, i.e. if it respects the following condition:*

$$\forall i, j \in F : w_i > w_j \implies w'_i > w'_j$$

3.1.3 Defining the preferable relation

We introduce the following relation:

Definition 3.5 (Preferable relation) *Let x and y be any two different alternatives of A . x is said to be preferable to y when x outranks y and when the overall valued concordance relation $S^w(x, y)$ is higher than $S^w(y, x)$, namely the credibility of “ x is at least as good as y ” is higher than the credibility of “ y is at least as good as x ”. In other terms, considering only this two alternatives for a choice problem, x could be put aside by the decision-maker with no regret to y 's advantage.*

This additional relation seemed to be necessary for us for a correct use of the method we will implement in the following chapters. Indeed, when comparing some alternatives couples, expressing a preference between the alternatives (i.e. a positive outranking statement in one sense and a negative one in the other sense) was not possible and expressing an indifference between them was considered as rather pointless.

Example 8 *We consider this simple didactic example of a person that wants to buy a car and evaluates a large set of alternatives according to five criteria: Price, Power, Equipment, Number of seats and Color. During the preference elicitation process, this person is asked to give his preferences in the comparison of two cars (see Table 3.1), in terms of indifference, preference, or incomparability.*

Table 3.1: Comparing two cars

| | ↓ Price | ↑ Power | ↑ Equipment | ↑ Nb. seats | ↑ Color |
|-------|----------|---------|------------------|-------------|------------------------|
| Car 1 | 8 000 € | 255 | Air-conditioning | 5 | Black (Good) |
| Car 2 | 10 000 € | 257 | Air-conditioning | 5 | Steel Gray (Very Good) |

We assume the fact that the decision-maker is considering the two evaluations on criterion Power as indifferent and has a clear preference for the steel-gray-colored car. We assume also that criterion Price, where the difference induces a clear local preference of the cheapest car, is not a dictator: This is easily verified when comparing some other cars on which this person can express a preference for a car

that is more expensive but better in every other criteria. In that case, as the criterion Price is clearly more important than the criterion Color, choosing the most expensive car will imply a higher regret than taking the cheapest one. However, it is mathematically impossible to express a preference for the first car (as criterion Price is not a dictator) and an indifference will not take into consideration the fact that the cheapest alternative is less regrettable than the other.

Notice that such a situation can be generalised to every alternatives comparisons where there are some criteria expressing a local indifference.

In this work, we will denote “ $xF^w y$ ” the fact that x is preferable to y , with respect to the criteria weights w . Notice that this strict order binary relation includes the complete preference relation, which leads to the following property:

Property 3.6 x is preferred to $y \implies x$ is preferable to y .

Proof: The proof is obvious: If x outranks y and y does not outrank x , it follows that $S^w(x, y) > 0 > S^w(y, x)$. \square

Notice that, in absence of an incomparability relation (*i.e.* no veto is raised), in the comparison of any alternatives couple, at least one of the alternative outranks the other one, such that there is always one preferable alternative, unless $S^w(x, y)$ and $S^w(y, x)$ being equal. In that last case, the alternatives x and y will be considered as indifferent.

3.2 Defining and computing the stability of valued outranking relations

Let $w \in \mathcal{W}$. The *stability* of \tilde{S}^w characterizes, for all $(x, y) \in A \times A$, the dependency of the associated median-cut outranking situation on the fixation of the weights [Bis04]. $xS^w y$ (resp. $x\mathcal{S}^w y$) is said to be:

- *Independent* (with respect to the weights): if a weighted majority of criteria warrants (resp. does not warrant) this outranking situation, for all vectors of weights in \mathcal{W} ;
- *Stable* (w.r.t. the weights): when a weighted majority of criteria warrants (resp. does not warrant) the outranking situation between x and y for any vector of weights preorder-compatible with w . This situation is only dependent on the preorder of w , not its precise numerical values;
- *Unstable* (w.r.t. the weights): if a weighted majority of criteria warrants (resp. does not warrant) this outranking situation for w but not for every vectors of weights preorder-compatible with w . The situation is depending on the preciseness of the numerical values of the weights.

We define two additional levels of stability, allowing to more precisely characterise a stable situation. $xS^w y$ (resp. $x\mathcal{S}^w y$) is then said to be:

- *Extensibly stable* (w.r.t. the weights): when a weighted majority of criteria warrants (resp. does not warrant) the situation between x and y for any vector of weights more discriminated than w . It characterises the stability of a given outranking statement by a basic preorder that can be refined.
- *σ -stable* (w.r.t. the weights): when a weighted majority of criteria warrants (resp. does not warrant) the situation between x and y for any vector of weights σ -preorder-compatible with w . It characterises the stability of an outranking statement when considering different objectives that gather some equi-important criteria. It ensures that a statement will not change, no matter what the order of the objectives is.

Example 9 *To illustrate our discourse, let us define an example with 4 alternatives and 9 criteria. For simplifying the explanation and without losing any specificity, we are not considering here any indifference or preference discrimination threshold. A vector of weights w is defined, inducing the following importance ordering: $\{g_1, g_2, g_3\} >_w \{g_4, g_5, g_6\} >_w \{g_7, g_8, g_9\}$. The performance table, on which every evaluations have to be maximized, is given in the left part of Table 3.2. The concordance relation, (or outranking relation, as we consider no veto threshold in that case) is given in the right part of Table 3.2, with normalised values between -1 and 1 .*

Table 3.2: Performance table and associated outranking relation

| w | g_1 | g_2 | g_3 | g_4 | g_5 | g_6 | g_7 | g_8 | g_9 | S^w | | | |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| | a | b | c | d | a | b | c | d | a | b | c | d | |
| a | 5 | 2 | 6 | 4 | 1 | 3 | 7 | 6 | 5 | | -0.12 | -0.22 | -0.12 |
| b | 3 | 4 | 5 | 7 | 2 | 6 | 4 | 5 | 8 | 0.12 | | -0.12 | -1.00 |
| c | 6 | 6 | 3 | 5 | 8 | 2 | 5 | 7 | 3 | 0.22 | 0.12 | | -0.33 |
| d | 4 | 7 | 6 | 8 | 6 | 7 | 6 | 6 | 9 | 0.56 | 1.00 | 0.33 | |

We can identify two issues for which the concept of stability proposes an answer. First, what level of reliability can I assign to these concordance values, knowing that I am not that confident in the precise fixation of the weights, but I am sure about the preorder of the criteria? Second, considering any non-well determined value of the concordance relation (i.e. close to a balanced situation, as for example the concordance value for the ordered pair (b, a)), has this situation been considered as faithfully reflecting the decision-maker’s mind or is it just an anecdotic situation created by a not fine-enough tuning of the parameters?

In the following subsections, we detail all the levels of stability and the way of testing their validations.

An outranking situation is *independent* from every vectors of weights when any criterion validates (resp. invalidates) the local “emphat least as good as” situations (i.e. when the first alternative dominates or is dominated by the second one):

Proposition 3.7 (Independency)

$$\text{“}xS^w y\text{” is independent} \iff \begin{cases} \forall i \in F : S_i(x, y) = 1 \text{ or } S_i(x, y) = 0; \\ \exists i \in F : S_i(x, y) = 1. \end{cases} \quad (3.1)$$

$$\text{“}x\mathcal{S}^w y\text{” is independent} \iff \begin{cases} \forall i \in F : S_i(x, y) = -1 \text{ or } S_i(x, y) = 0; \\ \exists i \in F : S_i(x, y) = -1. \end{cases} \quad (3.2)$$

$$\text{“}x?^w y\text{” is independent} \iff \forall i \in F : S_i(x, y) = 0. \quad (3.3)$$

Example 10 Back to our example, we easily verify that alternative d is at least as good as b on every criteria. In that case, d outranks b independently of any vector of weights.

The careful reader will notice that if a veto situation occurs in the comparison of a couple of alternatives (x, y) , such that it invalidates the outranking situation disregarding any criteria weights, it is then considered as independent.

3.2.1 Stability of the median-cut outrankings

According to a given vector of weights w , a positive (resp. negative) outranking situation $xS^w y$ (resp. $x\mathcal{S}^w y$) is said to be *stable* if and only if it is validated (resp. invalidated) for every vectors of weights w' which are preorder-compatible with w . Consequently, it only depends on the preorder of w and not the precise fixation of the weights. The following conditions give us a test for the stability of any situation:

Proposition 3.8 (Stability)

$$\text{“}xS^w y\text{” is stable} \iff \begin{cases} \forall k \in 1 \dots r : C_k^w(x, y) \geq 0; \\ \exists k \in 1 \dots r : C_k^w(x, y) > 0. \end{cases} \quad (3.4)$$

$$\text{“}x\mathcal{S}^w y\text{” is stable} \iff \begin{cases} \forall k \in 1 \dots r : C_k^w(x, y) \leq 0; \\ \exists k \in 1 \dots r : C_k^w(x, y) < 0. \end{cases} \quad (3.5)$$

$$\text{“}x?^w y\text{” is stable} \iff \forall k \in 1 \dots r : C_k^w(x, y) = 0. \quad (3.6)$$

Proof: The underlying idea of the proof is the following: By constructing the cumulated sums, we can ensure that the add of less important criteria in disfavor of the statement will always be compensated by more important criteria in favor, namely there will always be enough support to the proposed statement.

A complete proof of Proposition 3.8 is given in Annex A.1. \square

Any outranking situation that does not validate the stability proposition is then said to be *unstable*: Indeed, we can find vectors of weights in accordance with the associated preorder \geq_w that validate the situation, but also that invalidate it or that create a balanced situation. A precise and accurate fixation of the weights is then

necessary in order to avoid providing a false situation. If not, the reliability of such a situation is weak, all the more when the associated outranking value is weakly determined.

Table 3.3: Computation of the stability of some outranking situations

| Proposition | c_1^w | c_2^w | c_3^w | C_1^w | C_2^w | C_3^w | Denotation | | a | b | c | d |
|-------------|---------|---------|---------|---------|---------|---------|------------|---|----|----|----|----|
| $bS^w a$ | -1 | 3 | -1 | -1 | 2 | 1 | unstable | a | | -1 | -2 | -1 |
| $cS^w b$ | 1 | -1 | 1 | 1 | 0 | 1 | stable | b | +1 | | -2 | -3 |
| $aS^w b$ | 1 | -3 | 1 | 1 | 2 | -1 | unstable | c | +2 | +2 | | -2 |
| $cS^w d$ | -1 | -1 | -1 | -1 | -2 | -3 | stable | d | +2 | +3 | +2 | |

3: Independent outranking statement; 2: Stable; 1: Unstable.
 + (resp. -): Positive (resp. negative) outranking statement.

Example 11 *Back to our example, we now compute, in the left part of Table 3.3, the stability of some of the previous outranking situations. We have already seen that $S^w(b, a) = S^w(c, b) = 0.12$, but these two situations have quite different behaviors. Indeed, when looking at the computation details in Table 3.3, “ $cS^w b$ ” is stable, contrary to “ $bS^w a$ ”. If we consider the preorder \geq_w as a strong constraint validated by the decision-maker, any compatible vector of weights will ensure the validation of the outranking situation for the ordered pair (c, b) . Its weak value of concordance is not a weakly-determined one, contrary to the second outranking relation. For instance, one shall easily verify that, considering two vectors $w' = (6, 6, 6, 2, 2, 2, 1, 1, 1)$ and $w^* = (8, 8, 8, 3, 3, 3, 2, 2, 2)$, both preorder-compatible with w , we obtain $bS^{w'} a$ and $aS^{w^*} b$. Notice that a sensitivity analysis that is not taking into account the preorder of w may have considered both relations as potentially invalidated. Notice also that considering that an outranking relation is validated only if it is associated with a clear positive value may have invalidated both relations, whereas the first one is not anecdotic.*

Assuming an explicit validation of the preorder \geq_w , it is then clearly justifiable to consider a stable situation, even if it is not-well determined, as implicitly validated, whereas an unstable and not-well determined situation has to be explicitly validated by the decision-maker. As it is not possible to ask him to validate the complete set of outranking statements, it allows him to focus on sensitive outranking situations only, decreasing the time of the validation protocol and increasing his confidence in the final outranking digraph. As a result, any post-exploitation of the digraph will be more robust.

3.2.2 Extensible stability of the median-cut outrankings

Although the computation of the stability eases the validation of some outranking statements, as they only depend on the preorder of the weights, there is a strong hypothesis on the fact that two criteria in the same equivalence class have to be associated with the exact same weight. Indeed, we can easily imagine a situation where the criteria are gathered in the same equivalence class as the decision-maker

is unable to express a higher importance in favor of one or the other criterion, but having a doubt about the fact that these two criteria act exactly with the same importance.

Considering a stable outranking situation, we measure its dependency to a possible lack of discrimination in the associated vector of weights. A positive (resp. negative) outranking situation $xS^w y$ (resp. $x\mathcal{S}^w y$) is then said to be *extensibly stable* (w.r.t. w) if and only if it is validated (resp. invalidated) for every vectors of weights more discriminated than w . Intuitively, it informs on the fact that the discrimination inherent in the vector of weights is sufficient to describe the outranking situation in a stable manner; adding some discrimination will not have any impact on the situation.

For any pairs of alternatives (x, y) , let us define w^\mp as any vector of weights associated with the preorder $\Pi^{w^\mp}(x, y)$ defined as follows:

$$\Pi^{w^\mp}(x, y) = \Pi_1^{w^-}(x, y) > \Pi_1^{w^+}(x, y) > \dots > \Pi_r^{w^-}(x, y) > \Pi_r^{w^+}(x, y)$$

$\Pi^{w^\mp}(x, y)$ is in fact obtained by splitting the classes of importances of Π^w between the criteria against and the ones in favor of the situation. This preorder is the worst case we can create of a preorder more discriminated than Π^w , when considering the validation of an outranking situation, as the criteria in favor (resp. against) are the least (resp. most) possibly important. In a similar way, we also define w^\pm as the worst case when trying to invalidate an outranking situation, splitting each equivalence class and prioritizing the criteria in favor of the validation. It follows that:

Proposition 3.9 (Extensible stability)

$$\begin{aligned} "xS^{w^\mp} y" \text{ is stable} &\iff "xS^w y" \text{ is extensibly stable} \\ "x\mathcal{S}^{w^\pm} y" \text{ is stable} &\iff "x\mathcal{S}^w y" \text{ is extensibly stable} \end{aligned}$$

Proof. For proving the first equivalence, if we assume the fact that " $xS^{w^\mp} y$ " is stable, it is easy to verify that any vector of weights w' more discriminated than w and different from w^\mp will have either a criterion in favor of the outranking situation more important than it is in w^\mp or a criterion in disfavor with a lower importance than in w^\mp . Then, " $xS^{w'} y$ " will be also stable. On the other way, assuming that " $xS^w y$ " is extensibly stable, then it is stable for every vector of weights more discriminated than w , especially w^\mp .

The second equivalence is similarly verified. \square

A careful reader may remark that we are not giving any condition for testing the extensible stability of a balanced situation. In fact, we have the property that only the balanced situations that are independent of the weights can be extensibly stable.

Table 3.4: Extensible stability test for the proposition “ $dS^w a$ ”
 $\Pi^w = \{g_2\} > \{g_1, g_3\} > \{g_4, g_5, g_6\} > \{g_7, g_8, g_9\}$

| $\Pi^{w\mp}$ | Π_1^{w-} \emptyset | Π_1^{w+} $\{g_2\}$ | Π_2^{w-} $\{g_1\}$ | Π_2^{w+} $\{g_3\}$ | Π_3^{w-} \emptyset | Π_3^{w+} $\{g_4, g_5, g_6\}$ | Π_4^{w-} $\{g_7\}$ | Π_4^{w+} $\{g_8, g_9\}$ |
|--------------------|-----------------------------|---------------------------|---------------------------|---------------------------|-----------------------------|-------------------------------------|---------------------------|--------------------------------|
| $c_k^{w\mp}(d, a)$ | 0 | 1 | -1 | 1 | 0 | 3 | -1 | 2 |
| $C_k^{w\mp}(d, a)$ | 0 | 1 | 0 | 1 | 1 | 4 | 3 | 5 |

Table 3.5: Extensible stability

$\{g_1, g_2, g_3\} > \{g_4, g_5, g_6\} > \{g_7, g_8, g_9\}$ $\{g_2\} > \{g_1, g_3\} > \{g_4, g_5, g_6\} > \{g_7, g_8, g_9\}$

| | a | b | c | d |
|---|---|-----|---|-----|
| a | | - | - | - |
| b | + | | - | -ES |
| c | + | + | | - |
| d | + | +ES | + | |

| | a | b | c | d |
|---|-----|-----|-----|-----|
| a | | - | -ES | - |
| b | + | | - | -ES |
| c | +ES | + | | -ES |
| d | +ES | +ES | +ES | |

+/-: Positive/Negative median-cut outranking statement
 es: Extensibly stable outranking statement

This is easily proved, when supposing an extensibly stable balanced situation, which means that we can refine each equivalence class without modifying the balanced situation. But, if there exists criteria where $S_i(x, y) \neq 0$, it automatically means that they are compensated with other criteria in the same class. Splitting the equivalence class will result in a lack of balance. Consequently, the only extensibly stable balanced relation are relations where $S_i(x, y) = 0$ for every criteria i , which are independent from any vector of weights.

Example 12 *Returning to our example, let us suppose that the decision-maker did not provide a precise preorder, but only grouped the criteria according to whether he considers them as very important (g_1, g_2, g_3), important (g_4, g_5, g_6) or less important (g_7, g_8, g_9). Then, we associate some weights from 1 to 3 to the criteria, according to the initial preorder and compute the extensible stability relation in left part of Table 3.5 (An example of how to compute the extensible stability property is given in Table 3.4). At that time, considering the given information on the weights, there is too many uncertainties on the outranking statements. If the decision-maker is uneasy to discriminate more the preorder, the exploitation of the current outranking relation should have a very low degree of reliability, unless a sizable questioning on the not extensibly stable outrankings.*

In continuing the discussion, the decision-maker acknowledged that criterion g_2 is clearly the most important one. The new outranking relation and the associated extensible stability property are given in the right part of Table 3.5. We easily see that the number of extensibly stable relation increased, reducing the necessary questioning on the outrankings that are not fully validated by the preorder and its extension. Notice that if the decision-maker is certain that the preorder is correct, namely if two criteria with the same weights have the same exact importance, we will only need to question him on the remaining unstable outrankings.

In a validation process of the weights, if a decision-maker is not totally sure

that some criteria in the same importance classes are really equi-important, we can ensure that any possible refinement will lead to the same consequences for every extensibly stable outranking statements. He then can focus on the other outranking situations in order to refine the preorder (if necessary).

Also, such a procedure can be useful when considering multiple decision-makers, which would agree with a basic preorder, but they would wish to refine it in different manners. The extensible stability will highlight the conflictual situations and those which are not.

3.2.3 σ -stability of the median-cut outrankings

Let us assume now a different situation, where the criteria have been gathered under some more general objectives that a decision-maker does not want to order according to their importance, but with the property of equi-importance of all criteria under the same objective. This situation may happen when, for instance, a jury has to evaluate different candidates, based on some defined criteria: For each criterion, each member of the jury is giving an evaluation. A decision-maker may consider the evaluations of each member of the jury on one particular objective as equi-important, but is not able to order the different objectives.

According to a given vector of weights w , a positive (resp. negative) outranking situation $xS^w y$ (resp. $x\mathcal{S}^w y$) is then said to be σ -stable if and only if it is validated (resp. invalidated) for every vectors of weights which are σ -preorder-compatible with w . Intuitively, this property is warranted when the order of the equivalence classes is not important, *i.e.* when there are more criteria in each equivalence class that validate (resp. invalidate) the “*at least as good as*” relation.

The following proposition gives us a test for the σ -stability of any outranking statement:

Proposition 3.10 (σ -stability)

$$“xS^w y” \text{ is } \sigma\text{-stable} \iff \begin{cases} \forall k \in 1 \dots r : c_k^w(x, y) \geq 0; \\ \exists k \in 1 \dots r : c_k^w(x, y) > 0. \end{cases} \quad (3.7)$$

$$“x\mathcal{S}^w y” \text{ is } \sigma\text{-stable} \iff \begin{cases} \forall k \in 1 \dots r : c_k^w(x, y) \leq 0; \\ \exists k \in 1 \dots r : c_k^w(x, y) < 0. \end{cases} \quad (3.8)$$

$$“x?^w y” \text{ is } \sigma\text{-stable} \iff \forall k \in 1 \dots r : c_k^w(x, y) = 0. \quad (3.9)$$

Proof. As we study every possible permutations between the equivalence classes, each class can be considered as the most important one. As for all w and (x, y) , $C_1^w(x, y) = c_1^w(x, y)$, at least the most important class of every preorder has to verify the Proposition 3.8, *i.e.* $c_1^w(x, y) \geq 0$. According to the permutations, every equivalence classes have to verify the inequality. The condition is then a necessary one. On the other way, assuming that every $c_k^w(x, y)$ are greater or equal to 0, no matter the order we have between the classes, every cumulative sum $C_k^w(x, y)$

Table 3.6: σ -stability

| Testing some couples of alternatives | | | | Complete relation | | | | | |
|--------------------------------------|---------|---------|---------|---------------------|---|-----------|-----------|-----------|-----------|
| Proposition | c_1^w | c_2^w | c_3^w | σ -stability | | a | b | c | d |
| cS^wb | 1 | -1 | 1 | \times | a | | - | - | - |
| dS^wa | 1 | 3 | 1 | σ -stable | b | + | | - | $-\sigma$ |
| aS^wd | 1 | -3 | 1 | \times | c | + | + | | $-\sigma$ |
| cS^wd | -1 | -1 | -1 | σ -stable | d | $+\sigma$ | $+\sigma$ | $+\sigma$ | |

+/-: Positive/Negative median-cut outranking statement
 σ : σ -stable outranking statement

will remain greater or equal to 0. Then, the condition is enough and we get the equivalence. \square

The σ -stability level corresponds in fact to the verification of a group unanimity condition. Assuming that a situation is granted for every equivalence classes, no matter the relative importance of each class is, the situation will be always granted. Such a situation is highly reliable, as it suffers less from the subjectivity of the decision.

Example 13 Now, let us assume that our example is modeling the evaluations of a jury composed of three judges that evaluated a set of candidates, based on three criteria. The judges are unable to agree on the way to prioritize these criteria, but agrees on the fact that the importance of each judge on each criterion should be the same. Consequently, we group the evaluations in three classes (based on the fact they concern the same criterion), arbitrarily assign the weights 1, 2 and 3 to the classes as in Table 3.2, and compute the σ -stability property in right part of Table 3.6.

We easily observe that d outranks every other alternatives without the need to order the importance classes. Moreover, d is clearly preferred to b and c (as they will never outrank d under the working hypotheses). However, as the alternative a might outrank d , they both could be considered as indifferent. Again, without further information, we cannot rely on the statements that are not σ -stable. Nevertheless, considering for instance a best choice problem, we could rationally recommend d as the best alternative, but an in depth discussion on the importance of the criteria will be required to rank the other alternatives.

3.3 Additional properties on the stability

It is important to notice that any dominant, σ -stable, or extensibly stable situation, is a stable situation above all. Furthermore, if any dominant situation is also at the same time σ -stable and extensibly stable, the opposite is not always true. Indeed, one can consider the following situation: two alternatives a and b evaluated on three criteria g_1 , g_2 and g_3 , with the following local “at least as good as” evaluations: $S_1(a, b) = S_2(a, b) = 1$ and $S_3(a, b) = -1$. Considering $w = \{2, 1, 1\}$, we easily verify that “ $S^w(a, b) > 0$ ” is σ -stable and also extensibly stable, but it is clearly not a dominance situation.

3.3.1 Limitation of the stability

Let w_1 be the weights vector for which all the criteria weights equal 1. In the absence of incomparability statement, it follows:

Property 3.11 (Limitation of the stability)

$$x\mathcal{S}^{w_1}y \implies \nexists w \in \mathcal{W}, \text{ s.t. } xS^w y \text{ is stable} \quad (3.10)$$

$$xS^{w_1}y \implies \nexists w \in \mathcal{W}, \text{ s.t. } x\mathcal{S}^w y \text{ is stable} \quad (3.11)$$

Proof: Assuming that $x\mathcal{S}^{w_1}y$, namely $\tilde{S}^{w_1}(x, y) < 0$. If we want to find w such that $xS^w y$ in a stable manner, according to Proposition 3.8, we need to verify the fact that $C_k^w(x, y) > 0$, for all indices k , especially for $k = r$, the index of the least important class. But $C_r^w(x, y)$ is in fact the number of criteria (among the complete set of criteria) warranting a local “at least as good as” situation. $\tilde{S}^{w_1}(x, y) < 0.5$ implies that less than half of the criteria are in favor of the outranking situation. Then, C_k^w will be always strictly negative, disregarding any weights vector.

The proof is similar for the second property. \square

In other words, when more than half of the criteria are invalidating an outranking situation, it is impossible to find a vector of criteria weights that validates this situation in a stable manner. Similarly, when more than half of the criteria are validating an outranking situation, it is impossible to find a vector of criteria weights for warranting a stable invalidation of this situation. Indeed, it is simply impossible to warrant a stable outranking situation going against the one obtained with a vector of equi-important weights.

A careful reader may notice that, in a comparison where there is the same number of criteria in favor and in disfavor, we can always find a preorder warranting a stable validation and another one warranting a stable invalidation. For instance, one may consider a preorder with two classes, the most important class grouping all criteria in favor, or all criteria in disfavor.

This property emphasizes the importance of the vector w_1 of equi-important weights, as the resulting outranking digraph is entirely stable. Using a different set of parameters has to be clearly justified, because the decision-maker has a precise idea on the preorder of the weights (for example, if he is certain that two criteria have different importances), either he evidently disagrees an outranking situation from the equi-important digraph.

Also, it highlights the fact that we cannot only rely on the set of stable statements for giving some recommendations, as they will always go in the sense of those obtained with equi-important weights. Hereafter, we give some hints for a practical use of the stability.

Thus, when considering the construction of such a digraph, we may proceed as follows: we first try to validate the preorder of the weights, in a direct or indirect

manner, as it is easier to validate such a preorder instead of a precise vector of weights. Assuming its validation, every stable outrankings are implicitly validated and we do not need to question the decision-maker on their reliability. But we need to ensure the validity of the unstable statements. Also, one may consider either a direct or an indirect method for eliciting the precise importance of the weights. As we consider that it is not an easy task, especially for a novice decision-maker, to accurately tune the weights, we shall present in chapter 7 the construction of a progressive method for the elicitation of these parameters, based on an incremental set of preferential information on the alternatives.

3.3.2 Stability of the preferable relation

As we already explained at the beginning of this chapter, in section 3.1.3, during the elicitation protocol, we may face some preferential information like “ x is preferable to y ”. Legitimately, one would like to know if this relation is depending on the precise fixation of weights, *i.e.* unstable, or not. To do so, we consider the following proposition, slightly adapted from 3.8:

Proposition 3.12 (Stability of the preferable relation)

$$“x F^w y” \text{ is stable} \iff \begin{cases} \forall k \in 1 \dots r : C_k^w(x, y) \geq C_k^w(y, x); \\ \exists k \in 1 \dots r : C_k^w(x, y) > C_k^w(y, x). \end{cases} \quad (3.12)$$

Proof: If we consider the differences $C_k^w(x, y) - C_k^w(y, x)$, instead of $C_k^w(x, y)$ like in Proposition 3.8, the proof of Proposition 3.12 is the same as Proposition 3.8, the stability of the difference automatically induces the stability of the inequation $S^w(x, y) > S^w(y, x)$. \square

3.3.3 Stability within the context of the sorting problem

An important application is the characterisation of the stability of category assignments. In fact, within the context of outranking methods, we can easily define a *stable assignment* as follows:

Definition 3.13 (Stable assignment) *According to a given vector of weights w , assuming that the assignment of an alternative in a category is translated in terms of the conjunction of a set of outranking statements, we say that an alternative x is assigned to a category C_h in a stable manner if and only if every outrankings that need to be verified are ensured in a stable manner. Otherwise, we will say that the assignment of x in C_h is unstable.*

For instance, according to the pessimistic rule of ELECTRE TRI [Yu92] we recalled in the previous chapter, one says that x belongs to C_h if it outranks its lower profile b_{h-1} and does not outrank its upper profile b_h . A decision-maker will positively give more credit to a stable sorting, namely $x S^w b_{h-1}$ and $\neg(x S^w b_h)$ are stable².

2. “ $\neg(x S^w y)$ is stable” means that either $x S^w y$ or $x ?^w y$, but the statement is stable in any case.

In the sequel of the thesis, we shall only consider the optimistic and pessimistic rules for the sorting of the alternatives. However, there is no restriction to these rules and some other ones can be envisioned, provided they can be translated in terms of the conjunction of positive or negative outrankings.

If an assignment is unstable, it may be useful to determine an interval of categories $[C_h, C_k]$, in decreasing order of preference, on which the alternative is assigned in a stable manner: in other words, for each category C_l of the interval, and only for these categories, there exists a vector of weights w' , preorder-compatible with w , such that the alternative is assigned to C_l . This can be easily verified when considering the two assignment rules from ELECTRE TRI separately.

Property 3.14 (Stable assignment interval) *Considering the pessimistic rule for assigning the alternatives, we say that an alternative x is assigned to an interval $[C_h, C_k]$ in a stable manner when the statements xS^wb_{h-1} and $\neg(xS^wb_k)$ are stable, where b_k is the upper profile of C_k and b_{h-1} the lower profile of C_h , and when every other outrankings from x on any profile b_l that separates two adjacent categories of the interval are unstable.*

Example 14 *Table 3.7 represents the stable assignments of 5 alternatives into 4 categories, namely VERY GOOD, GOOD, MEDIUM and BAD. These categories are separated by 3 profiles: b_{GV} , b_{MG} and b_{BM} . When looking at the outranking statements between alternative a , for instance, and the profiles, we easily see that a is stably assigned to category GOOD, as it outranks its lower bound b_{MG} and does not outrank its upper bound b_{GV} , in a stable manner. Similarly, alternative b is assigned in a stable manner to the set of categories [MEDIUM, GOOD], as the statement " bS^wb_{MG} " is unstable. Finally, notice that d is sorted (when we are not considering the stability) in category VERY GOOD. But, as none of the considered outranking statements with the profiles is stable, a simple tuning of the weights could sort it into any other categories. In that case, d is assigned, in a stable manner, in the set of whole categories.*

Table 3.7: Stable assignments of some alternatives using pessimistic rule

| | b_{GV} | b_{MG} | b_{BM} | Stable assignment |
|-----|---------------------|---------------------|-----------------------|---|
| a | aS^wb_{GV} (sta.) | aS^wb_{MG} (sta.) | aS^wb_{BM} (sta.) | $a \in \text{GOOD}$ |
| b | bS^wb_{GV} (sta.) | bS^wb_{MG} (uns.) | bS^wb_{BM} (sta.) | $b \in [\text{MEDIUM}, \text{GOOD}]$ |
| c | cS^wb_{GV} (uns.) | cS^wb_{MG} (uns.) | cS^wb_{BM} (stable) | $c \in [\text{MEDIUM}, \text{VERY GOOD}]$ |
| d | dS^wb_{GV} (uns.) | dS^wb_{MG} (uns.) | dS^wb_{BM} (uns.) | $d \in [\text{BAD}, \text{GOOD}]$ |
| e | eS^wb_{GV} (sta.) | eS^wb_{MG} (sta.) | eS^wb_{BM} (sta.) | $e \in \text{VERY GOOD}$ |

Property 3.15 (Stable assignment interval) *According to the optimistic rule for assigning the alternatives, we construct the interval $[C_h, C_k]$, possibly reduced to a unique category, on which an alternative x is assigned in a stable manner as follows: b_{h-1} is the profile with the highest index such that xS^wb_{h-1} is stable and b_k is the lowest profile validating the outrankings b_kS^wx and $\neg(xS^wb_k)$ in a stable manner.*

Proof: First, as $xS^w b_{h-1}$ is stable, it obviously means that we cannot find a vector of weights preorder-compatible with w such that the alternative is assigned in a category below C_h . Also, $xS^w b_h$ is unstable, which means that it is possible to have $xS^{w'} b_h$ for a w' preorder-compatible with w . This induces the validation of the outranking $b_h S^{w'} x$. In consequence, the alternative should be assigned in C_h , considering w' .

On the other hand, if the outrankings $b_k S^w x$ and $\neg(xS^w b_k)$ are stable, we can ensure that the alternative cannot be assigned in a category with a higher index than C_k when considering the preorder of w . As both two outrankings are not stable for b_{k-1} and every profiles with a lower index, x can be possibly assigned in any category in between C_h and C_k (included). \square

As a general principle, considering an optimistic and pessimistic rule allow the definition of an interval of categories which runs from the pessimistic assignment to the optimistic one [RB93]. When the stability is taken into consideration, we already show that an interval is created around each assignment (pessimistic or optimistic). In that case, we proceed as follows to define a unique interval of stable assignments:

Definition 3.16 (Stable assignment interval) *When both optimistic and pessimistic assignment rules are considered, we say that an alternative x is assigned to an interval $[C_h, C_k]$, possibly reduced to a unique category, in a stable manner when C_h is the lowest category of the stable pessimistic assignment interval and C_k the highest category of the stable optimistic assignment interval.*

Property 3.17 (Concordant assignment) *Not taking into consideration the notion of stability, we should notice that the two assignment rules always assign an alternative in the same category under the three following hypotheses:*

- i. We are not applying any qualified majority on the outranking values: an outranking is considered validated (resp. invalidated) as soon as the concordance value is strictly positive (resp. negative). That is assumed for the whole thesis.*
- ii. We are considering the sorting of alternatives such that no veto is raised in their comparison with every profiles. The remaining alternatives will be sorted separately, as considering any veto bypasses the notion of stability.*
- iii. Every outrankings between one alternative and one profile are either validated or invalidated, but there cannot be some balanced situations. It follows that $\neg(xS^w y) \equiv xS^{w'} y$.*

Proof: Let assume that the pessimistic assignment rule sorted an alternative x into a category C_h . It follows that $xS^w b_{h-1}$ and $\neg(xS^w b_h)$. According to hypothesis (iii), we have $xS^{w'} b_h$. Due to the integrity of the concordance relation, it follows that $b_h S^w x$ (assuming there is no raised veto). We then have $b_h \succ x$. As $xS^w b_{h-1}$, $b_{h-1} \not\succeq x$. h is then the lowest index that verify the optimistic condition. x is then assigned to the same category C_h , irrespective to one of the two rules. \square

Property 3.18 *Under the hypothesis given in Property 3.17, for any alternative x , the associated stable optimistic assignment interval, designated by $[C_{l^\circ}, C_{u^\circ}]$, is the*

same interval as the stable pessimistic assignment one $[C_{lp}, C_{up}]$ (where p stands for pessimistic, o for optimistic, l for lower and u for upper).

Proof: By construction, we directly deduce that $C_{lp} = C_{lo}$, as both pessimistic or optimistic interval are lower bounded with the same condition, *i.e.* their index $lp = lo$ is the lowest index such that $xS^w b_{lp-1}$ is stable. Then, we prove that $C_u^p = C_u^o$. We know that, for every index $k \geq lo$, we have $xS^w b_k$, which implies that $b_k S^w x$. As soon as $xS^w b_k$ is stable, $b_k S^w x$ also is stable. Indeed, this second outranking is mandatory when the first one is true: as the first one is stable (*i.e.* true for every vector of weights preorder-compatible with w), this one has to be stable. Then, both intervals stops at the same index, namely $C_{uo} = C_{up}$. \square

When intending to sort the alternatives in accordance to some given parameters (weights, thresholds and profiles), we cannot ensure the last hypothesis. Of course, when there is an unstable balanced outranking (which is the vast majority of the balanced statements), it is still possible to raise the indeterminateness by slightly modifying the vector of weights such that the outranking will be strictly positive or strictly negative (with no preference for one or the other), but still unstable. We then can search for a stable interval assignment, which will not be impacted by the validation or invalidation of the outranking. Only the case with stable balanced situation shall be managed by considering the union of both optimistic and pessimistic stable intervals.

In addition, when the profiles are unknown and are elicited via an algorithm we shall detail in section 5.2, it seems to be advisable and not limitative to enforce the outrankings between the alternatives and the profiles in being either valid or invalid, but not indeterminate. In that case, it will not be necessary to consider both rules. We shall simply model constraint for ensuring the sorting of an alternative according to the pessimistic assignment rule and we shall present the stable assignment interval also considering only the pessimistic rule.

3.3.4 Checking the stability property with missing evaluations

For several reasons, some evaluations of the performance table may be unavailable, or the decision-maker may be uneasy in expressing some precise discriminating thresholds, such that there may be some missing local concordance indices. In that case, it is still possible to test the stability of an outranking situation between two alternatives x and y by considering all the possible values for the local concordance values $S_i(x, y)$, for every criteria i such as x_i or y_i is missing. If the statement is the same for each possible configuration, *i.e.* x always outranks (resp. does not outrank) y , and if the stability is warranted for both of them, we can ensure that the missing evaluations have no impact on the stability of the outranking statement.

Notice that we only need to test the *most optimistic* and *most pessimistic* scenarios, namely when all unknown $S_i(x, y)$ are replaced by 1 or -1 , when trying to validate a statement: If the resulting statements are both stable and positive (or both stable and negative), we do not have to make further tests, as the statements

will have the same behavior for every possible configurations. Otherwise, we have to consider the statement as an unstable one.

Example 15 *On table 3.8, we compute the stability property of three outrankings. For instance, we associate for the ordered pair of alternatives (a, b) two scenarios: $(a, b)^o$, the most optimistic one (i.e. the two missing local concordance are replaced by 1) and $(a, b)^p$, the most pessimistic one (i.e. with missing values replaced by -1). When computing the stability property for these two scenarios, we see that they have different behaviors (as one is stable and the other not). Consequently, we have to consider the outranking statement from a to b as unstable.*

Table 3.8: Stable outrankings statements with missing local concordance values

| Criterion: w: | Local concordance S_i | | | | | Cumul. sums | | | Stability | |
|------------------|-------------------------|-------|-------|-------|-------|-------------|---------|---------|----------------|---------------------------|
| | g_1 | g_2 | g_3 | g_4 | g_5 | C_1^w | C_2^w | C_3^w | | |
| (a, b) | 1 | ? | ? | 1 | -1 | | | | | |
| $(a, b)^p$ | 1 | -1 | -1 | 1 | -1 | 1 | -1 | -1 | $a?b$ (uns.) | $\implies a?b$ (unstable) |
| $(a, b)^o$ | 1 | 1 | 1 | 1 | -1 | 1 | 3 | 3 | aS^wb (sta.) | |
| (c, d) | -1 | ? | -1 | 0 | -1 | | | | | |
| $(c, d)^p$ | -1 | -1 | -1 | 0 | -1 | -1 | -3 | -4 | cS^wd (sta.) | $\implies cS^wd$ (stable) |
| $(c, d)^o$ | -1 | 1 | -1 | 0 | -1 | -1 | -1 | -2 | cS^wd (sta.) | |
| (e, f) | ? | 1 | ? | 1 | -1 | | | | | |
| $(e, f)^p$ | -1 | 1 | -1 | 1 | -1 | -1 | -1 | -1 | eS^wf (sta.) | $\implies e?f$ (unstable) |
| $(e, f)^o$ | 1 | 1 | 1 | 1 | -1 | 1 | 3 | 3 | eS^wf (sta.) | |

In a logical way, we can compute the stable assignment of an alternative in an interval of categories when there are some missing local concordance values between an alternative and a profile. We only have to compute the stability, according to the worst possible cases like in the previous paragraph for every unknown global outranking statements and compute the interval as explained ahead, in section 3.3.3.

3.3.5 Properties on the discrimination of the preorder

Property 3.19 *A stable outranking remains stable when considering a less discriminated preorder.*

Property 3.20 *An unstable outranking cannot become stable when considering a more discriminated preorder.*

Proof. Property 3.19 is obvious when noticing that the set of constraints on the cumulative sums to be verified in order to validate a stable outranking using any w' less discriminated than w is included in the set of constraints validating a stable outranking using w .

Property 3.20 is the contrapositive of Property 3.19. □

Properties 3.19 and 3.20, as well as Proposition 3.9 on the extensible stability, highlight the behavior of the outrankings when modifying the discrimination of a vector of weights: In fact, when increasing the discrimination, any outranking shall either become unstable or extensibly stable. In both case, as soon as one of these two situation is reached, it will not change if we keep increasing the discrimination. Notice that any outranking that are independent from the weights will not be impacted by any modification of the discrimination.

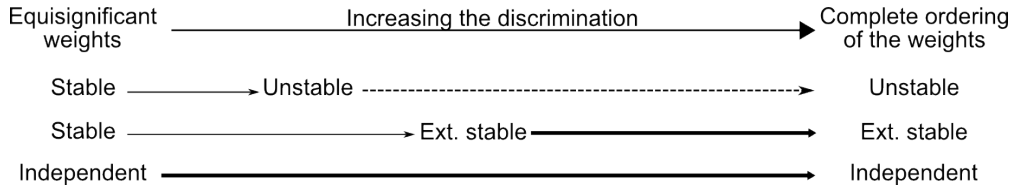


Figure 3.1: On the behavior of the stability when modifying the discrimination of a weight vector

Considering at the beginning of a study that every criteria are equi-important and discriminating two criteria only when there is a strong and clear higher importance of one criterion to another, could be a logical way for eliciting the criteria weights, as the resulting number of stable outrankings will be maximal. However, this stability may be reconsidered if the decision maker is uncertain on the equal importance of two or more criteria. In a similar manner, we may notice that too much discrimination will contribute to degrading the inherent stability of the outranking digraph, without needing so many information, not to mention the issues to ask the decision-maker for such an accurate information.

In consequence, and this will be detailed in chapter 7 on the construction of an protocol for the elicitation of the criteria weights, it seems to be advisable to start from the outranking digraph obtained with equi-important weights and to refine them by presenting him some outrankings that are stable but not extensibly stable. We shall present different assumptions in order to refine the preorder, helped with some examples of outranking to ease the discussion. When every considered outrankings will be extensibly stable or unstable, it will not be useless to go deeper in the refinement of the preorder.

Also, regarding the unstable outrankings that may appear during the protocol, they may be discussed with the decision-maker in order to be sure of their validation or invalidation.

Stable elicitation of criteria weights

*“Une chose ne vaut que par l’importance qu’on lui donne.”*¹

[André Gide]

Contents

| | | |
|------------|--|-----------|
| 4.1 | Stability constraints | 64 |
| 4.1.1 | Auxiliary variables and constraints | 64 |
| 4.1.2 | Modeling of the stability constraints | 65 |
| 4.1.3 | Constraint relaxation using slack variables | 69 |
| 4.2 | Taking into account decision-maker’s preferences | 71 |
| 4.2.1 | Types of preferential information | 71 |
| 4.2.2 | Preferences on alternatives | 71 |
| 4.2.3 | Preferences on criteria | 72 |
| 4.3 | Mathematical programs | 74 |
| 4.3.1 | Control algorithm (ACON) | 75 |
| 4.3.2 | MILP with real relaxed stability constraints (STAB ₁) | 76 |
| 4.3.3 | MILP with boolean relaxed stability constraints (STAB ₂) | 77 |

Abstract

In this chapter, we define mathematical models to elicit a vector of weights that is compatible with a set of preferential information given by the decision-maker. These models take advantage of the stability concept, defined in the previous chapter, by searching for a compatible weights vector that maximises the resulting number of stable statement.

1. “Something is worthy only if you think so.”

Introduction

In the previous chapter, we have defined some properties for characterising the stability level of any outranking statements induced by a given vector of weights.

Legitimately, one would like to set a mathematical model up for the weights elicitation that takes into account the stability concept in order to improve the overall stability of the resulting complete outranking relation: from partial global outranking statements confirmed by a decision-maker, one would compute a compatible vector of weights which maximises the total number of stable statements. Thus, it allows an easier validation of the outranking relation, as the validation of the elicited preorder automatically validates every stable statements.

Consequently, we present in this chapter such a mathematical model for the elicitation of criteria weights. First, we define a set of linear constraints that may ensure the stability of some global outranking statements. Then, we express the types of preferential information on alternatives a decision-maker can provide and how it can be integrated in the model. Finally, we enrich the model by potentially adding preferential information on the relative significance of the criteria.

4.1 Stability constraints

4.1.1 Auxiliary variables and constraints

As criteria significance weights are supposed to be rational, we can, without any loss of generality, restrict our assessment problem to integer weights vectors. Hence, an integer weight $w_i \in [1, M]$ will be associated with each criterion $i \in F$, M standing for the maximal admissible value. For the practical resolution of real decision problems, this bound may be set equal to the number m of criteria.

Let $\Omega_{m \times M}$ be a Boolean matrix with generic term $[\omega_{i,u}]$, characterizing, for each line i , the number of weights units allocated to criterion i . Formally, line i^{th} represents the decomposition of the weight associated with criterion i on M bits in a unary base, such that every non-zero values are grouped together in the right side of the matrix, in such a way that $\sum_{u=1}^M \omega_{i,u} = w_i$. For example, if criterion i is associated with an integer weights equal to 3, and if $M = 5$, then the i^{th} line of the matrix $\Omega_{m \times 5}$ will be $(0, 0, 1, 1, 1)$. Although this decomposition of the weights may be counter-intuitive, it takes all sense when we remember that we have to know on which equivalence class is any criterion. In this way, we can easily deduce the equivalence class of any weight by checking the index of the highest positive bit (to the left). In addition, the decomposition enables an instant reading of the cumulative sums C_k^w in which the criterion is taken into account, by looking in line i the indices k where $\omega_{i,u}$ is equal to 1.

As each criterion weight must be strictly positive, we easily deduce that at least one weight unit is allocated to each criterion, *i.e.* $\omega_{i,M} = 1$ for all $i \in F$. We obtain

the following constraint:

$$\sum_{i \in F} \omega_{i,M} = m. \quad (4.1)$$

The required cumulative semantics of $\Omega_{m \times M}$ is therefore achieved with the following set of constraints, that groups together the positive bits on the right side of the lines:

$$\omega_{i,u} \leq \omega_{i,u+1}, \quad \forall i \in F, \forall u = 1..M-1. \quad (4.2)$$

4.1.2 Modeling of the stability constraints

We characterise a constraint model for ensuring the stability of some desired outranking statements. Notice that we are not defining any constraints imposing an independent situation: Indeed, we can ignore such unanimous situations, positive or negative, as they concern a trivial pairwise comparison situation between Pareto dominant (resp. dominated) alternatives. The outranking situation is anyhow then unanimously warranted (resp. unwarranted), disregarding every possible significance of the criteria. Those denotations don't give us any specific information for the elicitation of the significance weights.

Notice also that we are not giving constraints for ensuring a σ -stability or extensible stability. We detail our reasons hereafter.

We define \mathfrak{S} (resp. $\overline{\mathfrak{S}}$ or $\mathfrak{S}^?$) as the set of ordered pairs (x, y) of alternatives such that the overall at least as good as situation between x and y has to be validated (resp. invalidated or in balance). Let us also denote \mathfrak{S}_2 (resp. $\overline{\mathfrak{S}}_2$ or $\mathfrak{S}_2^?$) the subset of pairs of alternatives on which we want to enforce a stability level.

Furthermore, let \mathfrak{F} define the set of alternatives pairs (x, y) such that x is preferable to y , namely when we want to ensure that $S^w(x, y)$ is strictly greater than $S^w(y, x)$. Thus, \mathfrak{F}_2 is the set of alternatives couples (x, y) on which we want to ensure that x is preferable to y in a stable manner.

Finally, we also model the enforcement of a stable balanced situation for any couple of alternatives. However, even if this enforcement may be useful when solving theoretical problems, it seems difficult for a decision-maker to express an accurate balanced situation. In practice, when a situation is balanced, or close to a balanced situation, the decision-maker will rather prefer not to express such a high constraining statement. Thus, these constraints will not be used in practical tests.

Ensuring a simple majority validation or invalidation

One may recall that \mathfrak{S}_2 is included in \mathfrak{S} , as well as \mathfrak{F}_2 is included in \mathfrak{F} . Which is the same for the constraints we want to model: ensuring the stability of a positive

(resp. negative) outranking automatically warrants its simple majority validation (resp. invalidation). Consequently, it may appear as redundant to define constraints ensuring a simple majority validation or invalidation. However, as stability cannot always be reached, we shall define in section 4.1.3 some relaxed versions of our constraints. In that case, it will be advisable to be able to impose the simple majority validation or invalidation.

Disregarding any desired level of stability, when a decision-maker expresses his preference on the validity (resp. invalidity) of an outranking statement, at least we have to ensure that the elicited vector of weights will validate the statement the same way.

In order to model the constraints enforcing the simple majority validation of the resulting vector of weights, we may formulate for all pairs $(x, y) \in \mathfrak{S}$ the following constraint:

$$\sum_{i \in F} \left(\left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \right) > 0, \quad \forall (x, y) \in \mathfrak{S}$$

where the factor $(\sum_{u=1}^M \omega_{i,u})$ represents the integer value of the estimated weight w_i of criterion i . As we are only dealing with integer values, the strict inequalities can be replaced by a large one when replacing the right-hand side of the equation by 1:

$$\sum_{i \in F} \left(\left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \right) \geq 1, \quad \forall (x, y) \in \mathfrak{S} \quad (4.3)$$

Similarly, for every pairs $(x, y) \in \overline{\mathfrak{S}}$, we may impose the following constraint:

$$\sum_{i \in F} \left(\left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \right) \leq -1, \quad \forall (x, y) \in \overline{\mathfrak{S}} \quad (4.4)$$

and for every pairs $(x, y) \in \mathfrak{S}^?$:

$$\sum_{i \in F} \left(\left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \right) = 0, \quad \forall (x, y) \in \mathfrak{S}^? \quad (4.5)$$

We then have an intuitive formulation of the constraints allowing to ensure the fact that an alternative x is preferable to an alternative y :

$$\sum_{i \in F} \left(\left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \right) \geq \sum_{i \in F} \left(\left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(y, x) \right) + 1, \quad \forall (x, y) \in \mathfrak{F}$$

For an easier understanding of the model, we slightly modify the writing of the constraint by factorizing them a bit. Consequently, the previous constraint is equivalent to the following one:

$$\sum_{i \in F} \left(\left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (S_i(x, y) - S_i(y, x)) \right) \geq 1, \quad \forall (x, y) \in \mathfrak{F} \quad (4.6)$$

Ensuring a stable majority validation or invalidation

Let us now translate Proposition 3.8 to a computable set of constraints.

Property 4.1 *When considering integer weights, Proposition 3.8 may be reformulated as:*

$$\begin{aligned} \text{"}xS^wy\text{" is } \sigma\text{-stable} &\iff \begin{cases} \forall u \in 1, \dots, \max w_i : C_u'^w(x, y) \geq 0; \\ \exists u \in 1, \dots, \max w_i : C_u'^w(x, y) > 0. \end{cases} \\ \text{"}x\mathcal{S}^wy\text{" is } \sigma\text{-stable} &\iff \begin{cases} \forall u \in 1, \dots, \max w_i : C_u'^w(x, y) \leq 0; \\ \exists u \in 1, \dots, \max w_i : C_u'^w(x, y) < 0. \end{cases} \\ \text{"}x?^wy\text{" is } \sigma\text{-stable} &\iff \forall u \in 1, \dots, \max w_i : C_u'^w(x, y) = 0. \end{aligned}$$

where $C_u'^w(x, y)$ is the sum of all $S_i(x, y)$ such that the significance weight $w_i \leq u$.

Proof: We easily verify that all constraints from Proposition 3.8 are present in the property, for all indices u such that it exists $w_i \in W$ equals to u . For all other values of u the constraints are redundant. \square

When remarking that $\omega_{i,u} = 1 \iff w_i \geq u$, we directly obtain:

$$C_u'^w(x, y) = \sum_{i \in F} \omega_{i,u} \cdot S_i(x, y).$$

In order to model the stability conditions, we introduce for all pairs $(x, y) \in \mathfrak{S}_2$ the following set of constraints:

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) \geq b_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{S}_2 \quad (4.7)$$

where the $b_u(x, y)$ are Boolean $(0, 1)$ variables for each pair of alternatives and each equi-importance level u in $\{1, \dots, M\}$ that allow us to impose at least one case

of strict inequality for each $(x, y) \in \mathfrak{S}_2 \cup \overline{\mathfrak{S}_2}$ as required in Proposition 4.1, via the following constraints:

$$\sum_{u=1}^M b_u(x, y) \geq 1, \quad \forall (x, y) \in \mathfrak{S}_2 \cup \overline{\mathfrak{S}_2} \quad (4.8)$$

Note that the ensuring a stable invalidation of the statements associated to a pair in $\overline{\mathfrak{S}_2}$ corresponds to a similar constraint with a reversed inequality and a negative $b_u(x, y)$:

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) \leq -b_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \overline{\mathfrak{S}_2} \quad (4.9)$$

Ensuring a stable balance situation is then easily modeled via the following constraints:

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) = 0, \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{S}_2^? \quad (4.10)$$

In a very similar way, we can ensure the stability of the statement “ x is preferable to y ” with the following set of constraints:

$$\sum_{i \in F} \omega_{i,u} \cdot (S_i(x, y) - S_i(y, x)) \geq b'_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{F}_2 \quad (4.11)$$

where the $b'_u(x, y)$ are Boolean variables defined for each ordered pairs of alternatives $(x, y) \in \mathfrak{F}_2$, imposing at least one case of strict inequality, via the constraint:

$$\sum_{u=1}^M b'_u(x, y) \geq 1, \quad \forall (x, y) \in \mathfrak{F}_2 \quad (4.12)$$

Enforcing an extensible stability

In the previous chapter, we have shown how useful can be the notion of extensible stability during the exploitation of the outranking relation, when the decision-maker is not totally sure that some criteria in the same importance classes have exactly the same importance.

However, trying to elicit a vector of criteria weights that best ensures the extensible stability of some desired outranking statements is resulting from a slightly different approach. In fact, trying to maximise the number of extensibly stable outranking statements will search for increasing the discrimination of the vector of criteria weights.

If we can ensure that it will not impact the stability of the outranking statements for which the decision-maker expressed a preference (and for which we modeled the stability constraints), the unforced outranking statements stability risks to be damaged by an unnecessary discriminated preorder, as explained in section 3.3.5.

In these conditions, it might not be advisable to model the constraints obliging the extensiveness of the stability of an outranking. Nevertheless, we will show how its use is integrated to the weight elicitation protocol, in chapter 7.

Enforcing a σ -stability

Even if this stability level brings us a more precise characterization of the behavior of any outranking, it is important to notice that its use among a real protocol for the elicitation of the importance weights is, to date, hardly possible. Indeed, one should easily perceive that such constraint is very restrictive and, imposing it on some outranking statements may quickly result in an impossible solving. If we can understand its implication in the validation of an outranking relation, when already having a vector of weights or at least a grouping of criteria with same importance, enforcing the computation of equivalence classes such as the relation obtains enough support for each of them doesn't appear relevant for us, from this moment.

In consequence, we are not giving here mathematical constraints ensuring the σ -stability of any outranking situation.

4.1.3 Constraint relaxation using slack variables

Ensuring the stability of the outranking statements validated by the decision-maker allows, the case given, to present him the overall picture of his preferences in a clearer manner (only considering the preorder), thus improving his understanding of the problem. However, as it may result in the inability of solving the problem, disregarding to the given preferential information, we have to introduce some *relaxed constraints*, obtained from the original constraints by adding boolean *slack variables*, which allow a mathematical resolution by relaxing the incompatible stability constraints with the underlying problem.

Let us now present the constraints in their first relaxed versions:

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) + s(x, y) \geq b_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{S}_2 \quad (4.13)$$

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) - s(x, y) \leq b_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \overline{\mathfrak{S}_2} \quad (4.14)$$

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) + s^+(x, y) - s^-(x, y) = 0, \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{S}_2^? \quad (4.15)$$

$$\sum_{i \in F} \omega_{i,u} \cdot (S_i(x, y) - S_i(y, x)) + s'(x, y) \geq b'_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{F}_2 \quad (4.16)$$

where $s(x, y)$ (resp. $s'(x, y)$) are real slack variables associated with the ordered pair (x, y) for enforcing the stability of the outranking (resp. preferable) relation. A careful reader may notice the fact that, for a constraint enforcing a balance situation, two slack variables are needed, $s^+(x, y)$ and $s^-(x, y)$ in order to know whether the outranking statement associated with the violated constraint has become positive (*i.e.* $s_l^{0+}(x, y) > 0$) or negative (*i.e.* $s_l^{0-}(x, y) > 0$).

We may highlight that an invalidated stability constraint no longer warrants the simple majority weight support of the considered outranking statements. Therefore, in order to ensure the weighted majority for the expressed preference information, we may associate with each relaxed stability constraint an original simple majority constraint.

Notice also that we are not going to relax the simple majority weight constraints (4.3). In case the decision-maker would express incompatible preferential information, we will resolve this inconsistency following the approach originally proposed in [LL01, Roo79] and further developed and adapted to MCDA by MOUSSEAU *et al.* [MDF⁺03, MDF06].

Minimizing the sum of the slack variables has a positive impact on the number of stable resulting outranking relation, but is not fully optimal. Indeed, the algorithm will prefer for example to set the value of three different slack variables up to 1 instead of setting only one up to 4, *i.e.* it will prefer to invalidate more constraints with lower values. We then propose a second model for the relaxed constraints that substitute the real variable $s(x, y)$ by a boolean one, $s_b(x, y)$, multiplied by the number of criteria², m . As a result, the relaxed constraints are slightly modified as follows:

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) + s_b(x, y) \cdot m \geq b_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{S}_2 \quad (4.17)$$

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) + s_b(x, y) \cdot m \leq -b_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \overline{\mathfrak{S}_2} \quad (4.18)$$

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) + (s^{b+}(x, y) - s^{b-}(x, y)) \cdot m = 0, \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{S}_2^? \quad (4.19)$$

$$\sum_{i \in F} \omega_{i,u} \cdot (S_i(x, y) - S_i(y, x)) + s'_b(x, y) \cdot m \geq b'_u(x, y), \quad \forall u = 1..M, \forall (x, y) \in \mathfrak{F}_2 \quad (4.20)$$

As we shall explain in the sequel, the second modeling is optimal, as we can minimise the number of violated constraints; however, as it needs the definition of a fair amount of new boolean variables, it may result in a great increase of the necessary time for the resolution of the mathematical program. Then, we shall define two models, one using real slack variables and the second one with boolean slack variables. We discuss in chapter 6 on the efficiency of both models, in terms of running time and stability.

2. Every slack variable is bounded by the number of criteria. In the worst case, $C_k^w(x, y) = -m$.

4.2 Taking into account decision-maker's preferences

4.2.1 Types of preferential information

We propose to integrate all further preferential information that a decision-maker can provide in our elicitation model. This information may take the form of:

- A subset $\mathfrak{S} \subseteq A \times A$ (resp. $\overline{\mathfrak{S}}$) of ordered pairs of alternatives (x, y) for which a decision-maker is able to express a valid (resp. invalid) outranking statement;
Example: x is at least as good as y , z is not at least as good as t ;
- A subset $\mathfrak{F} \subseteq A \times A$ of ordered pairs of alternatives (x, y) for which a decision-maker expressed the fact that x is preferable to y ;
- A partial preorder \succcurlyeq_n over the weights of a subset of criteria $N \subseteq F$;
Example: criterion 1 is more valuable than criterion 4;
- Some constraints over numerical values associated with some criteria weights;
Example: criterion 2 weight value is equal to 3, or is between 2 and 4;
- A partial preorder between some sets of criteria, expressing preferences about the sum of some criteria weights;
Example: the coalition of criteria 1 and 3 is more important than 2;
- Some sets of criteria able to validate or invalidate an outranking statement;
Example: when an alternative x is at least as good as y over criteria 1, 2 and 3, the decision-maker considers that x outranks y .

4.2.2 Preferences on alternatives

We assume that we do not have any outranking statement to be enforced to a balanced situation, *i.e.* $\mathfrak{S}^? = \emptyset$. Even if this situation has been easily modeled and can be useful in theoretical tests to recapture a set of parameter used to create the complete outranking relation, in practice, such hypotheses may be too restrictive for the elicitation of the parameters, especially as an indetermination is most of the time the result of a limited knowledge on the overall outranking relation.

In order to provide a solution as stable as possible to the decision-maker, we decide to associate for any couple of alternatives $(x, y) \in \mathfrak{S}$ the following linear constraints:

- an original simple majority constraint to warrant the outranking statement;
- a relaxed stability constraint to intend a stable outranking statement;

Understandably, we add the same kind of constraints for the pairs of alternatives in $\overline{\mathfrak{S}}$, but trying each time to invalidate the outranking statement. For the pairs in \mathfrak{F} , we proceed similarly, with enforcing the comparison with an original simple majority constraint and intend to stabilise it with a relaxed stability constraint.

Let us also notice the fact that, according to Proposition 3.8, when an outranking statement goes against the elementary outranking statement, it will not be possible to warrant stability. It is hence useless to keep active these stability constraints, the case given.

Finally, when considering the sorting problem, the decision-maker may express directly some assignments of an alternative into a defined category (bounded with known profiles) or a set of categories. In that case, we can easily translate these information in terms of outranking statements between the alternative and the profiles, according to the chosen assignment rule. For instance, considering the pessimistic assignment rule:

$$\begin{aligned} x \text{ is assigned to category } C_h &\implies \begin{cases} (x, b_{h+1}) \text{ is added to } \mathfrak{S}, \\ (x, b_h) \text{ is added to } \overline{\mathfrak{S}}. \end{cases} \\ x \text{ is assigned to the interval } [C_h, C_k] &\implies \begin{cases} (x, b_{h+1}) \text{ is added to } \mathfrak{S}, \\ (x, b_k) \text{ is added to } \overline{\mathfrak{S}}. \end{cases} \end{aligned}$$

Notice that, when the profiles are unknown, we cannot use this model as it now stands. We shall present in the next chapter, section 5.2, some additional constraints for dealing with the assignment of the alternatives in predefined categories, with no given profiles.

4.2.3 Preferences on criteria

Furthermore, direct numerical information on criteria weights provided, the case given, by the decision-maker, are easy to translate into linear constraints. Thus, if a decision-maker expresses the fact that the weights of criterion i is equal to an integer value u_i , or if he wants to restrict the value of the weight of criterion j between two integers v_i and v'_i , we add some of the following constraints:

$$\sum_{c=1}^M \omega_{i,c} = u_i, \quad \text{or} \quad \sum_{c=1}^M \omega_{j,c} \geq v_i \quad \text{and} \quad \sum_{c=1}^M \omega_{j,c} \leq v'_i \quad (4.21)$$

A decision-maker's statement "*criterion i is more important than criterion j* " will be taken into account by adding the constraint:

$$\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 \quad (4.22)$$

This formula can be generalised for subsets of criteria: If a subset of criteria H is more important than a subset K , then we add the following constraint:

$$\sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 \quad (4.23)$$

We can also model the fact that a subset H of criteria is, according to the decision-maker, sufficient to validate an outranking statement, namely the sum of its criteria weights is strictly greater than the sum of the remaining criteria weights:

$$\sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in F \setminus H} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 \quad (4.24)$$

By definition, ensuring a strictly higher importance to criterion i , with respect to criterion j in equation (4.22) will be warranted by the elicited preorder. However, the validation of equations (4.23) and (4.24) can be closely linked to the precise values of the weights.

Example 16 *Let us assume the following constraint, given by a decision-maker: the coalition with criteria 1 and 2 is more important than the coalition with criteria 3 and 4. If the algorithm returns, for instance, a vector of weights W such that its preorder gives $w_1 = w_2 > w_3 = w_4$, it is obvious that the constraint expressed by the decision-maker is warranted for every compatible vectors of weights with \geq_w . But if the algorithm returns a vector W' of weights such that $w_1 = 5$, $w_2 = 1$, $w_3 = 3$ and $w_4 = 2$, its preorder $\geq_{w'}$ gives $w_1 > w_3 > w_4 > w_2$. The constraint is then validated, as $w_1 + w_2 = 6$ and $w_3 + w_4 = 5$, but not in a stable manner. Indeed, we may consider the following weights: $w_1 = 5$, $w_2 = 1$, $w_3 = 4$ and $w_4 = 3$, compliant with $\geq_{w'}$, but that invalidates the initial constraint, as $w_1 + w_2 = 6$ and $w_3 + w_4 = 7$.*

When such a constraint is not warranted in a stable manner, we must be careful to keep considering these constraints when trying to tune the weights after the validation of the preorder, otherwise we might compute some vector of weights that are not in accordance with the complete expressed preferential information.

We then propose to model such a constraint in a stable manner.

Property 4.2 (Stable comparison between two criteria coalitions) *Let define $H, K \subset F$ such that the coalition of criteria in H is more important than the coalition of the criteria in K . To ensure the stability of the comparison of the two coalitions, we may replace constraint (4.23) by the following set of constraints:*

$$\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} \geq b_u(H, K), \quad \forall u = 1..M, \quad (4.25)$$

$$\sum_{u=1}^M b_u(H, K) \geq 1. \quad (4.26)$$

Proof: The proof is obvious when noticing that it is similar to ensure the validation of an outranking statement between two alternatives (a, b) when $S_i(a, b) = 1$ for each criterion $i \in H$, $S_i(a, b) = -1$ for each criterion $i \in K$ and $S_i(a, b) = 0$ for each

criterion $i \in F \setminus (H \cup K)$. $b_u(H, K)$ is a set of boolean variables created to ensure at least one strict inequality, as required in Proposition 4.1. \square

Of course, we should consider the relaxing of these constraints, for the same reasons as in section 4.1.3. In consequence, constraints (4.25) is relaxed, using real slack variables $s(H, K)$ as follows:

$$\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s(H, K) \geq b_u(H, K), \quad \forall u = 1..M, \quad (4.27)$$

and, in its second relaxed version, using boolean slack variables $s_b(H, K)$:

$$\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_b(H, K) \cdot m \geq b_u(H, K), \quad \forall u = 1..M. \quad (4.28)$$

Let us remind that a violated relaxed constraints does not ensure the validation of the original constraint. In consequence, when a decision-maker will express the fact that a coalition of criteria is more important than another one, we will model constraints (4.23) to ensure the simple validation of the constraints, constraints (4.27) or (4.28) to try to ensure the stability and constraint (4.26) for enforcing one necessary strict inequality in the stability constraints.

4.3 Mathematical programs

Solving the linear problem stated so far will naturally provide potentially many admissible criteria weights. Following our goal of not exploring the whole admissible solution polytope, we try, hence, to recover a vector of criteria weights w^* that ensures every given outranking statement by maximising the number of stable statements among them. We justify the research of a unique "optimal" admissible solution by the fact that these algorithms are intended to be integrated in an elicitation protocol, the decision-maker having, anytime, the possibility to give his opinion on the parameters. As a result, if w^* does not satisfy him, he would be able to provide additional preferential information which will be integrated to a new resolution (for instance, when the decision-maker considers that two equal weights should be different, the first being for him more important than the second, . . .)

From the constraints we defined in the previous sections, we may formulate two mathematical models allowing the elicitation of compatible weights of the criteria with the decision-maker's expressed preferential information, best maximising the stability of the given outranking statements:

- A first model, $STAB_1$, using the relaxed stability constraints with real slack variables;

- A second model, STAB_2 , using the relaxed stability constraints with boolean slack variables.

Compared with STAB_1 , STAB_2 presents the following enhancement: As it minimises the number of violated stability constraints, not the sum of real slack variables, it is optimal. However, it is using boolean variables that may lead to a longer running time. Then, we will have to check in Chapter 6 if the additional running time is “compensated” by a significant increase in terms of stability.

Notice that in real tests, we are considering a small set of given outranking statements only, as the decision-maker is not willing to express his preferences over the whole outranking relation. As we cannot use the stability constraints on a pairwise statement without knowing its validity, it may eventually result in a lack of stability. Recalling that potential equi-importance of the criteria naturally provides the kind of stability of the weights we are looking for, we hence adopt as a heuristic to minimise the overall sum of the criteria weights. Indeed, this heuristic, as it tends to reduce the number of equi-importance classes, will increase the number of stable statements that are not directly imposed by the decision-maker. In order to test the adequateness of this working hypothesis, we define a control algorithm ACON that only takes into account the simple majority constraints, minimises the sum of the weights and drops all further stability constraints.

In order to make the lecture easier, we recall \mathfrak{S} (resp. $\overline{\mathfrak{S}}$) as the set of ordered pairs of alternatives (x, y) such as the decision-maker expresses the fact that x outranks (resp. does not outrank) y . To avoid useless constraints, we are obviously not adding constraints trying to enforce the stability when it is impossible to reach it, according to Property 3.11 on the limitation of the stability.

To summarise, let us now present the three mixed integer linear programs. Their validity, as well as their respective behaviors, in terms of stability and running time, and how they can be successfully used in a real time elicitation protocol shall be presented in-depth in Chapter 6.

4.3.1 Control algorithm (ACON)

MILP ACON

Variables:

$$\omega_{i,u} \in \{0, 1\}$$

$$\forall i \in F, \forall u = 1..M$$

Objective function:

$$\min \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$s.t. \quad \sum_{i \in F} \omega_{i,1} = m$$

$$\omega_{i,u} \geq \omega_{i,u+1}$$

$$\forall i \in F, \forall u = 1..M - 1$$

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \geq 1$$

$$\forall (x, y) \in \mathfrak{S}$$

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \leq -1 \quad \forall (x, y) \in \overline{\mathfrak{S}}$$

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (S_i(x, y) - S_i(y, x)) \geq 1 \quad \forall (x, y) \in \mathfrak{F}$$

Constraints (informal) on the weights allowing to model decision-maker's preferences:

$$\sum_{c=1}^M \omega_{i,c} = v_i \quad \text{For some } i \in F$$

$$\sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v \quad \text{For some } i \in F$$

$$\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 \quad \text{For some } (i, j) \in F^2$$

$$\sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 \quad \text{For some } H, K \subset F$$

4.3.2 MILP considering real relaxed stability constraints (STAB₁)

For this model, we formulate a first objective function, which has to be minimised:

$$(F_1) \min : K_1 \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s(x, y) + K_2 \cdot \sum_{(x,y) \in \mathfrak{F}} s'(x, y) + K_3 \cdot \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

K_1 , K_2 and K_3 are parametric constants used to put in correct order the three sub-objectives. The first part of the objective function tends to minimise the sum of slack variables on simple stability constraints, the second part focuses on minimising the sum of slack variables on the stable enforcement of preferable relations and the last part cares about minimising the sum of the weights.

In order to prioritise the enforcement of the stability, as the highest value that can be reached by the third part of the objective function is $m \cdot M$, we set the values for the constants as follows: $K_3 = 1$, $K_1 = K_2 = m \cdot M$. It results the following mixed integer linear program:

MILP STAB₁

Variables:

$$\begin{array}{ll} \omega_{i,u} \in \{0, 1\}, & \omega_{i,0} = 0 & \forall i \in F, \forall u = 1..M \\ b_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M \\ b'_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M \\ s(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ s'(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{F} \\ b_u(H, K) \in \{0, 1\} & & \forall u = 1..M \text{ and some } H, K \subset F \\ s(H, K) \geq 0 & & \text{For some subsets } H, K \subset F \end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s(x, y) + m \cdot M \cdot \sum_{(x,y) \in \mathfrak{F}} s'(x, y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{aligned}
s.t. \quad & \sum_{i \in F} \omega_{i,1} = m && \\
& \omega_{i,u} \geq \omega_{i,u+1} && \forall i \in F, \forall u = 1..M-1 \\
& \sum_{u=1}^M b_u(x, y) \geq 1 && \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
& \sum_{u=1}^M b'_u(x, y) \geq 1 && \forall (x, y) \in \mathfrak{F} \\
\circ \quad & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \geq 1 && \forall (x, y) \in \mathfrak{S} \\
& \sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) + s(x, y) \geq b_u(x, y) && \forall (x, y) \in \mathfrak{S}, \forall u = 1..M \\
\circ \quad & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \leq -1 && \forall (x, y) \in \overline{\mathfrak{S}} \\
& \sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) - s(x, y) \leq -b_u(x, y) && \forall (x, y) \in \overline{\mathfrak{S}}, \forall u = 1..M \\
\circ \quad & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (S_i(x, y) - S_i(y, x)) \geq 1 && \forall (x, y) \in \mathfrak{F} \\
& \sum_{i \in F} \omega_{i,u} \cdot (S_i(x, y) - S_i(y, x)) + s'(x, y) \geq b'_u(x, y) && \forall (x, y) \in \mathfrak{F}, \forall u = 1..M
\end{aligned}$$

Constraints (informal) on the weights allowing to model decision-maker's preferences:

$$\begin{aligned}
& \sum_{c=1}^M \omega_{i,c} = v_i && \text{For some } i \in F \\
& \sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v && \text{For some } i \in F \\
& \sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 && \text{For some } (i, j) \in F^2 \\
\circ \quad & \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 && \text{For some } H, K \subset F \\
& \sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s(H, K) \geq b_u(H, K) && \forall u = 1..M \text{ and some } H, K \subset F \\
& \sum_{u=1}^M b_u(H, K) \geq 1 && \forall u = 1..M \text{ and some } H, K \subset F
\end{aligned}$$

4.3.3 MILP minimizing the boolean relaxed stability constraints (STAB₂)

We slightly modify the objective function defined for the previous model in order to take into account the boolean slack variables:

$$(F_2) \min : K_1 \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s_b(x, y) + K_2 \cdot \sum_{(x,y) \in \mathfrak{F}} s'_b(x, y) + K_3 \cdot \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

However, we easily see that the same values may be used for ordering the sub-objectives (which are the same as before). In consequence, we have again $K_3 = 1$ and $K_1 = K_2 = m \cdot M$.

It results the following mixed integer linear program:

MILP STAB₂

Variables:

$$\begin{array}{ll}
\omega_{i,u} \in \{0, 1\}, & \omega_{i,0} = 0 & \forall i \in F, \forall u = 1..M \\
b_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}}, \forall u = 1..M \\
b'_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M \\
s_b(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\
s'_b(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{F} \\
b_u(H, K) \in \{0, 1\} & & \forall u = 1..M \text{ and some } H, K \subset F \\
s_b(H, K) \geq 0 & & \text{For some } H, K \subset F
\end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{G} \cup \overline{\mathfrak{G}}} s_b(x, y) + m \cdot M \cdot \sum_{(x,y) \in \mathfrak{F}} s'_b(x, y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
s.t. & \sum_{i \in F} \omega_{i,1} = m \\
& \omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M-1 \\
& \sum_{u=1}^M b_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\
& \sum_{u=1}^M b'_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \geq 1 & \forall (x, y) \in \mathfrak{G} \\
& \sum_{i \in F} (\omega_{i,u} \cdot S_i(x, y)) + s_b(x, y) \cdot m \geq b_u(x, y) & \forall (x, y) \in \mathfrak{G}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \leq -1 & \forall (x, y) \in \overline{\mathfrak{G}} \\
& \sum_{i \in F} (\omega_{i,u} \cdot S_i(x, y)) - s_b(x, y) \cdot m \leq -b_u(x, y) & \forall (x, y) \in \overline{\mathfrak{G}}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (S_i(x, y) - S_i(y, x)) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
& \sum_{i \in F} \omega_{i,u} \cdot (S_i(x, y) - S_i(y, x)) + s'_b(x, y) \cdot m \geq b'_u(x, y) & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M
\end{array}$$

Constraints (informal) on the weights allowing to model decision-maker's preferences:

$$\begin{array}{ll}
& \sum_{c=1}^M \omega_{i,c} = v_i & \text{For some } i \in F \\
& \sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v & \text{For some } i \in F \\
& \sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 & \text{For some } (i, j) \in F^2 \\
\circ & \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 & \text{For some } H, K \subset F \\
& \sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_b(H, K) \cdot m \geq b_u(H, K) & \forall u = 1..M \text{ and some } H, K \subset F \\
& \sum_{u=1}^M b_u(H, K) \geq 1 & \forall u = 1..M \text{ and some } H, K \subset F
\end{array}$$

A simultaneous elicitation of criteria weights and other parameters

“Le problème actuel vient toujours de la solution précédente.”¹

[Descheneaux Daniel]

Contents

| | |
|---|-----------|
| 5.1 Elicitation of weights and thresholds | 81 |
| 5.1.1 Modeling of the constraints on the thresholds | 81 |
| 5.1.2 Additional preferential information | 88 |
| 5.1.3 The complete models | 89 |
| 5.2 Elicitation of weights and categories profiles | 91 |
| 5.2.1 Modeling of the constraints on the profiles | 92 |
| 5.2.2 Ensuring a stable assignment of an alternative | 94 |
| 5.2.3 The complete models | 95 |
| 5.2.4 Stable assignment of the other alternatives | 97 |

Abstract

In this chapter, we enhance the previous mathematical programs in order to recover, in addition to the criteria weights, some other parameters, namely some unknown discrimination thresholds in a first time, then the categories profiles of a sorting problem.

1. “The current issue is always resulting from the previous solution.”

Introduction

Previously, we have introduced some mathematical models assessing, from the decision-maker's preferential information, a compatible vector of weights that maximise the stability of some considered outrankings. Even so, these algorithms make an important hypothesis concerning the fact that the discrimination thresholds are given. If one can assume that, on some criteria, as these thresholds are implicitly linked to the defined scale (especially when using qualitative scales with a few admissible values, one could easily express a clear preference between two adjacent values), it is hardly possible to directly ask a novice decision-maker about precise determination of the thresholds for every criteria.

Some preliminary tests, which have been done during the development of the weights-elicitation protocol, have confirmed this issue. When looking at the evaluations of the alternatives on the criteria, a novice decision-maker is not fully aware of his preferences on each criterion. At best he may express a strict preference when two values are clearly distinguishable (the difference between them consequently imposes an upper bound on the associated preference threshold value) or he may also express an indifference when comparing two very close values.

Also, considering the sorting problem, similar remarks may be done when asking the decision-maker about expressing the profiles of the categories. Indeed, it is a difficult task a novice decision-maker cannot figure out with accuracy. In fact, their construction is hardly understandable for such a decision-maker, they have to be constructed so as only a weighted (and unknown) half of the criteria is necessary to consider an alternative globally at least as good as the profile (*i.e.* to sort the alternative in this category or in a higher one).

First, we then extend our mathematical models to recover, from a decision-maker's set of preferences on pairs of alternatives, the criteria weights and the discrimination indifference and preference thresholds of an outranking method, taking into account the stable constraints, as defined in chapter 3. Then, we show how to modify the new model in order to take into account the elicitation of the profiles of a set of predefined categories.

For these studies, we are still considering that the veto thresholds are given. Indeed, their integration in the defined mathematical models may highly increase the complexity and the computation time, whereas they are, most of the time, unnecessary. This hypothesis is easily conceivable in an elicitation protocol: if the difference on a criterion is consequent enough to invalidate the overall outranking relation, one can simply imagine the decision-maker capable of highlighting such situation, when comparing some pairs of alternatives. Furthermore, we shall present in Chapter 7 a step of the protocol allowing to determine the veto thresholds, even by questioning the decision-maker, or when solving conflicts that might appear.

5.1 Simultaneous elicitation of weights and discrimination thresholds

We are now taking an interest in extending the model by considering the discrimination thresholds as variables. Adding such variables induces that the double threshold order S_i becomes also variable and the constraints where it appears are no longer linear. We propose here a decomposition of these variables, adapted from [MMB08].

First and foremost, as the consideration of the stability may appear non natural and restricting, we justify it with quite similar argument than previously: Without taking into account this notion, either we elicit a unique vector of weights, or we compute all possible compatible scenarios with the expressed information. In the first case, the choice of one particular vector is, most of the time, not clearly justified and the discussion with the decision-maker is uneasy, due to the difficulty in measuring the impact of some possible modifications (we still have the possibility in modifying one value after another, but their dependencies are difficult to apprehend). In the second case, in order to reach a practical recommendation, it is necessary to have a long and not always easy questioning on a fair number of couples of alternatives which becomes more difficult because of the simultaneous elicitation of the discrimination thresholds.

Once again, taking into account the notion of stability in the elicitation of one specific vector of weights, simultaneously with some criteria discrimination thresholds, helps easing the discussion with the decision-maker (as the predominant discussion on the preorder is much easier). However, we cannot ignore that maximising the number of stable outrankings will have a not inconsiderable impact on the way the discrimination thresholds are fixed. We shall discuss in chapter 6 on these interactions, try to understand and measure it.

Finally, notice that this algorithm can be used when considering the sorting problem, with given profiles but some discrimination thresholds are unknown.

5.1.1 Modeling of the constraints on the thresholds

First, let us assume the fact that the preference direction associated with every criteria is always the maximisation of the value (the bigger a value is, the better it is considered). If the values on one criterion have to be minimised (for instance, in a cost criterion), we simply consider, when solving the mathematical model, the opposite values².

We have to differentiate three kinds of thresholds we want to deal with:

- *Constant thresholds*, such that the importance of the difference between two alternatives is constant along the whole scale of the associated criterion;

2. The evaluations are still presented to the decision-maker on their original scales

- *Proportional thresholds*, such that the importance of the difference is growing proportionally with the evaluations of the considered alternatives;
- *General thresholds*, such that the importance of the difference is any increasing function of the considered alternatives.

The kind of threshold has to be known for every criteria in F , in order to create the correct constraints in the mathematical model. As it is hardly possible to directly ask the decision-maker about this information, they should be given by an analyst, or elicited via a questioning of the decision-maker we shall present in chapter 7.

Modeling of the double threshold order

Let us remember that x_i and y_i represents the respective evaluation of alternatives x and y on criterion i . Also, let us define F_c , $F_\%$ and F_g as the decomposition of the family of criteria F , respectively in criteria with constant, proportional and general discrimination thresholds. Finally, let Max_i be the largest admissible value on each criterion i .

In the first case, let us associate a constant indifference threshold $q_i \geq 0$ and a constant preference threshold $p_i > 0$, $p_i > q_i$, for each involved criterion $i \in F_c$. The double thresholds order S_i is then given by:

$$S_i(x, y) = \begin{cases} 1 & \text{if } x_i + q_i \geq y_i, \\ -1 & \text{if } x_i + p_i \leq y_i, \\ 0 & \text{otherwise.} \end{cases} \quad \forall i \in F_c \quad (5.1)$$

In the second case, let us define $q_i \geq 0$ (resp. $p_i > 0$, $p_i > q_i$) the indifference (resp. preference) proportional discrimination threshold for each involved criterion $i \in F_\%$. The double thresholds order S_i is then given by:

$$S_i(x, y) = \begin{cases} 1 & \text{if } x_i + x_i \cdot q_i \geq y_i, \\ -1 & \text{if } x_i + x_i \cdot p_i \leq y_i, \\ 0 & \text{otherwise.} \end{cases} \quad \forall i \in F_\% \quad (5.2)$$

Lastly, let us define $q(x_i) \geq 0$ (resp. $p(x_i) > 0$, $p(x_i) > q(x_i)$) the indifference (resp. preference) general discrimination thresholds for each involved criterion $i \in F_g$ and each admissible value x_i . The double thresholds order S_i is then given by:

$$S_i(x, y) = \begin{cases} 1 & \text{if } x_i + q(x_i) \geq y_i, \\ -1 & \text{if } x_i + p(x_i) \leq y_i, \\ 0 & \text{otherwise.} \end{cases} \quad \forall i \in F_g \quad (5.3)$$

As we assume in the last case that the thresholds are any increasing function, we must ensure that the values are correctly ordered by adding the following set of constraints:

$$x_i > y_i \implies \begin{cases} q(x_i) \geq q(y_i) \\ p(x_i) \geq p(y_i) \end{cases} \quad \forall x, y \in A, \forall i \in F_g \quad (5.4)$$

Note that this general case includes the constant and proportional case. If both the decision-maker and the analyst does not have particular idea on the thresholds of a criterion, they should use such general type.

In the first place, let us model the integer variable $S_i(x, y) \in \{-1, 0, 1\}$ as the difference between two boolean variables $\alpha_i(x, y) \in \{0, 1\}$ and $\beta_i(x, y) \in \{0, 1\}$. This decomposition will be more efficient when using a mathematical solver than using ternary variables. For every criteria $i \in F$ and every pairs (x, y) of alternatives, the decomposition is the following:

$$S_i(x, y) = \alpha_i(x, y) - \beta_i(x, y) \quad (5.5)$$

Note that $S_i(x, y) = 1$ if $\alpha_i(x, y) = 1$ and $\beta_i(x, y) = 0$. In addition, $S_i(x, y) = -1$ if $\alpha_i(x, y) = 0$ and $\beta_i(x, y) = 1$. Finally, $S_i(x, y) = 0$ if $\alpha_i(x, y) = 0$ and $\beta_i(x, y) = 0$. Later, we shall explain why the case where $\alpha_i(x, y) = \beta_i(x, y) = 1$ is not possible.

In the constant case, the double thresholds order can be rewritten as follows, for every criteria $i \in F_c$ and every pairs $(x, y) \in A^2$:

$$\begin{cases} -2(1 - \alpha_i(x, y)) \cdot Max_i & \leq x_i - y_i + q_i < 2\alpha_i(x, y) \cdot Max_i \\ -2\beta_i(x, y) \cdot Max_i & \leq x_i - y_i + p_i < 2(1 - \beta_i(x, y)) \cdot Max_i \end{cases} \quad (5.6)$$

Max_i allows not having to normalise the value.

Proof: Note that, as $p_i > q_i \geq 0$, if $x_i - y_i + q_i \geq 0$, it implies $x_i - y_i + p_i > 0$. Then, right part of the first equation imposes $\alpha_i(x, y) = 1$ and right part of the second equation imposes $\beta_i(x, y) = 0$ ($\implies S_i(x, y) = 1$). If $x_i - y_i + p_i < 0$, it follows that $x_i - y_i + q_i < 0$. Right part of the first equation forces $\alpha_i(x, y) = 0$ and left part of the second one gives $\beta_i(x, y) = 1$ ($\implies S_i(x, y) = -1$). Otherwise, when $x_i - y_i + q_i < 0$ and $x_i - y_i + p_i \geq 0$, left part of the first equation imposes $\alpha_i(x, y) = 0$ and right part of the second one ensures $\beta_i(x, y) = 0$ ($\implies S_i(x, y) = 0$). \square

We may remark that this model avoids having $\alpha_i(x, y)$ and $\beta_i(x, y)$ simultaneously equal to 1. Indeed, when assuming $\alpha_i(x, y) = 1$, it automatically results that $x_i - y_i + q_i \geq 0$, which implies $x_i - y_i + p_i > 0$, and then $\beta_i(x, y) = 0$ (according to the left part of the second equation). In the same way, when $\beta_i(x, y) = 1$,

$x_i - y_i + p_i < 0$, which implies $x_i - y_i + q_i < 0$ and consequently, $\alpha_i(x, y) = 0$, according to the left part of the first equation. This remark will remain valid for the different kind of thresholds we model hereafter.

In the proportional case, the double threshold order is similarly rewritten as follows, for every criteria $i \in F_{\%}$ and every pairs $(x, y) \in A^2$:

$$\begin{cases} -2(1 - \alpha_i(x, y)) \cdot Max_i & \leq & x_i - y_i + x_i \cdot q_i & < & 2\alpha_i(x, y) \cdot Max_i \\ & & -2\beta_i(x, y) \cdot Max_i & \leq & x_i - y_i + x_i \cdot p_i & < & 2(1 - \beta_i(x, y)) \cdot Max_i \end{cases} \quad (5.7)$$

Proof: The proof is similar to the previous case with constant thresholds. \square

In the general case, the double threshold order is rewritten as follows, for every criteria $i \in F_g$ and every pairs $(x, y) \in A^2$:

$$\begin{cases} -2(1 - \alpha_i(x, y)) \cdot Max_i & \leq & x_i - y_i + q(x_i) & < & 2\alpha_i(x, y) \cdot Max_i \\ & & -2\beta_i(x, y) \cdot Max_i & \leq & x_i - y_i + p(x_i) & < & 2(1 - \beta_i(x, y)) \cdot Max_i \end{cases} \quad (5.8)$$

Proof: Again, the proof is similar to the case with constant thresholds. \square

In the sequel, we do not pay attention to the kind of thresholds, as the decomposition of the variable $S_i(x, y)$ is similar in any considered case and the constraints for imposing one or another kind have been previously defined. Also, let us notice that the decomposition is linked to the couple of alternatives (x, y) and the criterion i ; Consequently, we can use the same variables $\alpha_i(x, y)$ and $\beta_i(x, y)$ disregarding to the considered constraint.

Modification of the simple majority validation constraints

First, let us recall the constraints for ensuring a simple majority validation, or invalidation, we want to modify:

$$\begin{aligned} \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) & \geq 1, & \forall (x, y) \in \mathfrak{S} \\ \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) & \leq -1, & \forall (x, y) \in \overline{\mathfrak{S}} \end{aligned}$$

Using equation (5.5) to replace the double thresholds order variables S_i , the previous constraints can be rewritten as follows:

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (\alpha_i(x, y) - \beta_i(x, y)) \geq 1, \quad \forall (x, y) \in \mathfrak{S}$$

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (\alpha_i(x, y) - \beta_i(x, y)) \leq -1, \quad \forall (x, y) \in \overline{\mathfrak{S}}$$

As these constraints remain non-linear, we replace them by the following constraints, creating a new set of variables $w_i^1 \in [-M, M]$:

$$\sum_{i \in F} w_i^1(x, y) \geq 1, \quad \forall (x, y) \in \mathfrak{S} \quad (5.9)$$

$$\sum_{i \in F} w_i^1(x, y) \leq -1, \quad \forall (x, y) \in \overline{\mathfrak{S}} \quad (5.10)$$

and we add, for every indices i , the following linear constraints:

$$-\sum_{u=1}^M \omega_{i,u} \leq w_i^1(x, y) \leq \sum_{u=1}^M \omega_{i,u} \quad (5.11)$$

$$\sum_{u=1}^M \omega_{i,u} + (\alpha_i(x, y) - \beta_i(x, y)) \cdot M - M \leq w_i^1(x, y) \quad (5.12)$$

$$-\sum_{u=1}^M \omega_{i,u} + (\alpha_i(x, y) - \beta_i(x, y)) \cdot M + M \geq w_i^1(x, y) \quad (5.13)$$

$$-(\alpha_i(x, y) + \beta_i(x, y)) \cdot M \leq w_i^1(x, y) \leq (\alpha_i(x, y) + \beta_i(x, y)) \cdot M \quad (5.14)$$

Proof: Recalling that $\alpha_i(x, y)$ and $\beta_i(x, y)$ can not be simultaneously equal to 1, when $\alpha_i(x, y) = 1$ and $\beta_i(x, y) = 0$ (*i.e.* $S_i(x, y) = 1$), then $w_i^1(x, y) = \sum_{u=1}^M \omega_{i,u}$ (according to the first two constraints). When $\alpha_i(x, y) = 0$ and $\beta_i(x, y) = 1$ (*i.e.* $S_i(x, y) = -1$), it results that $w_i^1(x, y) = -\sum_{u=1}^M \omega_{i,u}$ (also based on the replacement of the variables in the two first constraints). Finally, when $\alpha_i(x, y) = 0$ and $\beta_i(x, y) = 0$, the third constraint imposes $w_i^1(x, y) = 0$. \square

Modification of the stability constraints

We remind the original constraint ensuring the stability of an outranking situation as the following:

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) \geq b_u(x, y), \quad \forall (x, y) \in \mathfrak{S}_2, \forall u = 1..M$$

$$\sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) \leq -b_u(x, y), \quad \forall (x, y) \in \overline{\mathfrak{S}}_2, \forall u = 1..M$$

Using equation (5.5) to replace the variables $S_i(x, y)$, the previous constraints are then equivalent to:

$$\begin{aligned} \sum_{i \in F} \omega_{i,u} \cdot (\alpha_i(x, y) - \beta_i(x, y)) &\geq b_u(x, y), \quad \forall (x, y) \in \mathfrak{S}_2, \forall u = 1..M \\ \sum_{i \in F} \omega_{i,u} \cdot (\alpha_i(x, y) - \beta_i(x, y)) &\leq -b_u(x, y), \quad \forall (x, y) \in \overline{\mathfrak{S}_2}, \forall u = 1..M \end{aligned}$$

We replace these stable non-linear constraints similarly to the simple ones. As a difference, we here consider each weight unit. We then have to create a set of double indices variables $w_{i,u}^2 \in [-1, 1]$:

$$\sum_{i \in F} w_{i,u}^2(x, y) \geq b_u(x, y), \quad \forall (x, y) \in \mathfrak{S}_2, \forall u = 1..M \quad (5.15)$$

$$\sum_{i \in F} w_{i,u}^2(x, y) \leq -b_u(x, y), \quad \forall (x, y) \in \overline{\mathfrak{S}_2}, \forall u = 1..M \quad (5.16)$$

and we add, for every couples of indices (i, u) , the following linear constraints:

$$-\omega_{i,u} \leq w_{i,u}^2(x, y) \leq \omega_{i,u} \quad (5.17)$$

$$\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) - 1 \leq w_{i,u}^2(x, y) \leq -\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) + 1 \quad (5.18)$$

$$-(\alpha_i(x, y) + \beta_i(x, y)) \leq w_{i,u}^2(x, y) \leq \alpha_i(x, y) + \beta_i(x, y) \quad (5.19)$$

Proof: The demonstration is similar to the validation of the construction of the variables $w_{i,u}^{\mathfrak{S}}$. When $\alpha_i(x, y) = 1$ and $\beta_i(x, y) = 0$ (*i.e.* $S_i(x, y) = 1$), then $w_i^2(x, y) = \omega_{i,u}$. When $\alpha_i(x, y) = 0$ and $\beta_i(x, y) = 1$ (*i.e.* $S_i(x, y) = -1$), it follows that $w_i^2(x, y) = -\omega_{i,u}$. Finally, when $\alpha_i(x, y) = \beta_i(x, y) = 0$, the third constraint forces $w_i^2(x, y) = 0$. \square

We must highlight that it is possible to replace the variables w_i^1 , created to ensure the simple majority validation of the overall outranking situation between two alternatives. Indeed, when noticing that:

$$\begin{aligned} w_i^1(x, y) &= \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \\ w_{i,u}^2(x, y) &= \omega_{i,u} \cdot S_i(x, y) \end{aligned}$$

we then easily discern the link between the variables $w_i^1(x, y)$ and $w_{i,u}^2(x, y)$:

$$w_i^1(x, y) = \sum_{u=1}^M w_{i,u}^2(x, y) \quad (5.20)$$

Consequently, it is not necessary to create variables $w_i^1(x, y)$ on constraints (5.9) to (5.14), when trying to ensure both simple majority and stability for an outranking situation of a pair of alternatives (x, y) . We only need to substitute them using the previous equation.

According to the fact that ensuring a certain degree of stability automatically validates the lower degrees, one may argue that it seems unnecessary to use simultaneously constraints enforcing a simple majority validation and the stability of an outranking situation. But, as we shall present later, the substitution of the variable can be done when considering relaxed constraints. Then, it will be possible to use the same variables $w_i^2(x, y)$ for relaxed constraints trying to enforce the stability of a situation and for original constraints ensuring the simple majority validation of the same situation.

Modifying of the relaxed constraints

As the modifications brought to the modeled constraints in the previous section only concern the rewriting of the double threshold order, the transformation of the product between $S_i(x, y)$ and some elements of the matrix Ω into linear constraints, there is no difficulty in modifying the relaxed constraints. Indeed, we simply need to add real or integer slack variables into constraints 5.15 and 5.16.

In consequence, considering real slack variables, constraints 5.15 and 5.16 are rewritten as follows:

$$\sum_{i \in F} w_{i,u}^2(x, y) + s(x, y) \geq b_u(x, y), \quad \forall (x, y) \in \mathfrak{S}_2, \forall u = 1..M \quad (5.21)$$

$$\sum_{i \in F} w_{i,u}^2(x, y) - s(x, y) \leq -b_u(x, y), \quad \forall (x, y) \in \overline{\mathfrak{S}_2}, \forall u = 1..M \quad (5.22)$$

and, when considering integer slack variables:

$$\sum_{i \in F} w_{i,u}^2(x, y) + s_b(x, y) \cdot M \geq b_u(x, y), \quad \forall (x, y) \in \mathfrak{S}_2, \forall u = 1..M \quad (5.23)$$

$$\sum_{i \in F} w_{i,u}^2(x, y) - s_b(x, y) \cdot M \leq -b_u(x, y), \quad \forall (x, y) \in \overline{\mathfrak{S}_2}, \forall u = 1..M \quad (5.24)$$

5.1.2 Additional preferential information

We propose to integrate in the inverse analysis model some additional preferential information on the discrimination thresholds a decision-maker may provide, in the context of our decision problem.

First of all, one should remark that it is simply not an option to directly ask the decision-maker about his preferences on the thresholds. Indeed, we can not really imagine to present him a huge amount of comparisons of evaluations and ask him for preference information³, especially when dealing with general thresholds and numerous parameters. Thus, we must collect some of these additional information along the protocol, when a decision-maker explicitly expresses the way he evaluates a global comparison of two alternatives. For instance, if he considers that an alternative is globally better than another, because he really feels clear preferences on some criteria he names, it may be useful to add constraints on the thresholds of these criteria in order to enforce the resulting parameter to be in accordance with his apparent preferences.

On a criterion $i \in F_c$ with constant thresholds, these information can take the form of:

- A lower bound q_i^{min} on the associated indifference threshold q_i , when the indifference threshold is for sure at least equal to q_i^{min} ;
ex.: On criterion i , one can strongly consider that a difference of 2 units is not relevant. Then, the indifference threshold q_i must be greater or equal to 2.
- An upper bound p_i^{max} associated to the preference threshold p_i , when the preference threshold is for sure at most equal to p_i^{max} ;
ex.: On criterion i , a difference of 2 units can induce a clear preference for a decision-maker. Then, p_i must be lower or equal to 2.

One may justifiably ask why we are not taking into account the other bounds on the values of the thresholds. We assume on our study, when confronted with a novice decision-maker (and even with an expert one), that it is not easy to ask for an upper bound on an indifference threshold, nor a lower bound on a preference threshold. Roughly, the decision-maker can not express an accurate balance situation, as it is the definition of indeterminateness. If he may express the fact that a difference between two values is clearly no longer an indifference, it shall automatically follow a clear preference between the two evaluations.

When q_i^{min} is defined, we add the following constraint to the model:

$$q_i \geq q_i^{min} \tag{5.25}$$

When p_i^{max} is defined, we add the following constraint to the model:

3. This goes against the hypothesis saying that a decision-maker is not able to give us precise thresholds and may be laborious

$$p_i \leq p_i^{max} \quad (5.26)$$

When dealing with proportional thresholds, we may obtain the same kind of information. The constraints we define in that case are the same as (5.25) and (5.26). Let us simply notice that the difference between two values should be expressed in term of percentage in order to help the decision-maker in staying coherent or clearly understanding how the information will be integrated in the model.

At last, when dealing with a criterion with general thresholds, if a decision-maker express his indifference between two values x_i and y_i , such that $x_i > y_i$, we add the following constraint:

$$q(y_i) \geq x_i - y_i \quad (5.27)$$

This equation necessarily entails $q(x_i) \geq x_i - y_i$, as the indifference threshold is an increasing function.

Alike, if he expresses his preference between two values x_i and y_i , such that $x_i > y_i$, we ensure this situation via the following constraint:

$$p(x_i) \leq x_i - y_i \quad (5.28)$$

Again, it automatically results that $p(y_i) \leq x_i - y_i$, as the preference threshold is also an increasing function.

5.1.3 The complete models

The outrankings are modeled just like in previous chapter, section 4.2, namely by ensuring at least the simple majority validation (or invalidation) with a strong constraint and a relaxed stability constraint. The kind of thresholds have to be known for every thresholds. In case some kinds are unknown, we shall use the general kind.

Also, we are not modifying the objective functions, as the considered objectives are the same, namely minimising the slack variables and the sum of the elicited criteria weights. However, one could consider the minimisation or maximisation of the elicited thresholds, with different purposes:

- Trying to maximise the indifference thresholds and minimising the preference ones can be interesting, as the algorithm will best reduce the number of local balanced situations, prioritizing some clear indifference or preference statements;

- On the contrary, trying to minimise the indifference thresholds and maximising the preference ones is considering to restrict the indifference and preference statements to the lowest possible level. In fact, with no need to consider that an evaluation is locally at least (or not at least) as good as another, the algorithm will not discriminate the evaluations;

We assume in our study that the selection of one or the other modeling is a tricky choice, with not enough perspective of the stability impact. We then decide not to take into account such considerations for the objective functions and keep them unchanged.

We define three algorithms to deal with the simultaneous elicitation of the criteria weights and discriminating thresholds, derived from the three algorithms given in the previous chapter:

- ACON': The control algorithm that does not take into account the stability constraints, simply minimising the sum of the weights;
- STAB'₁: The algorithm that takes into account the relaxed stability constraints using real slack variables;
- STAB'₂: The algorithm that considers integer slack variables.

In order to ease the reading, we are only giving here the algorithm for STAB'₁. A complete version of each algorithm can be found in Annex A.2.4, A.2.5 and A.2.6.

First stable algorithm with additional elicitation of the discrimination thresholds (STAB'₁)

We are only giving the extension STAB'₁ of the first stable algorithm STAB₁, defined with real slack variables, as the second one may be modified similarly.

MILP STAB'₁

Variables:

$$\begin{array}{ll}
 \omega_{i,u} \in \{0,1\}, & \omega_{i,0} = 0 & \forall i \in F, \forall u = 1..M \\
 \alpha_i(x,y), \beta_i(x,y) \in \{0,1\} & & \forall i \in F, \forall (x,y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\
 b_u(x,y) \in \{0,1\} & & \forall (x,y) \in \mathfrak{G} \cup \overline{\mathfrak{G}}, \forall u = 1..M \\
 b'_u(x,y) \in \{0,1\} & & \forall (x,y) \in \mathfrak{F}, \forall u = 1..M \\
 s(x,y) \geq 0 & & \forall (x,y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\
 s'(x,y) \geq 0 & & \forall (x,y) \in \mathfrak{F}
 \end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{G} \cup \overline{\mathfrak{G}}} s(x,y) + m \cdot M \cdot \sum_{(x,y) \in \mathfrak{F}} s'(x,y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
 s.t. & \sum_{g_i \in F} \omega_{i,1} = m \\
 & \omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M-1 \\
 & \sum_{u=1}^M b_u(x,y) \geq 1 & \forall (x,y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\
 & \sum_{u=1}^M b'_u(x,y) \geq 1 & \forall (x,y) \in \mathfrak{F}
 \end{array}$$

- $\sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \geq 1$ $\forall (x, y) \in \mathfrak{G}$
 $\sum_{i \in F} w_{i,u}^2(x, y) + s(x, y) \geq b_u(x, y)$ $\forall (x, y) \in \mathfrak{G}, \forall u = 1..M$
- $\sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \leq -1$ $\forall (x, y) \in \overline{\mathfrak{G}}$
 $\sum_{i \in F} w_{i,u}^2(x, y) - s(x, y) \leq -b_u(x, y)$ $\forall (x, y) \in \overline{\mathfrak{G}}, \forall u = 1..M$
- $\sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) - w_{i,u}^2(y, x) \right) \geq 1$ $\forall (x, y) \in \mathfrak{F}$
 $\sum_{i \in F} \left(w_{i,u}^2(x, y) - w_{i,u}^2(y, x) \right) + s'(x, y) \geq b'_u(x, y)$ $\forall (x, y) \in \mathfrak{F}, \forall u = 1..M$

Constraints on the coherence of the $w_{i,u}^2(x, y)$ variables:

$$\begin{aligned} -\omega_{i,u} &\leq w_{i,u}^2(x, y) \leq \omega_{i,u} && \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ \omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) - 1 &\leq w_{i,u}^2(x, y) && \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ w_{i,u}^2(x, y) &\leq -\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) + 1 && \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ -(\alpha_i(x, y) + \beta_i(x, y)) &\leq w_{i,u}^2(x, y) && \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ w_{i,u}^2(x, y) &\leq \alpha_i(x, y) + \beta_i(x, y) && \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \end{aligned}$$

Constraints on the coherence of the α_i and β_i variables:

$$\begin{aligned} -2(1 - \alpha_i(x, y)) \cdot \text{Max}_i &\leq x_i - y_i + q_i && \forall i \in F_c, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ x_i - y_i + q_i &< 2\alpha_i(x, y) \cdot \text{Max}_i && \forall i \in F_c, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ -2\beta_i(x, y) \cdot \text{Max}_i &\leq x_i - y_i + p_i && \forall i \in F_c, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ x_i - y_i + p_i &< 2(1 - \beta_i(x, y)) \cdot \text{Max}_i && \forall i \in F_c, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ -2(1 - \alpha_i(x, y)) \cdot \text{Max}_i &\leq x_i - y_i + x_i \cdot q_i && \forall i \in F_\%, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ x_i - y_i + x_i \cdot q_i &< 2\alpha_i(x, y) \cdot \text{Max}_i && \forall i \in F_\%, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ -2\beta_i(x, y) \cdot \text{Max}_i &\leq x_i - y_i + x_i \cdot p_i && \forall i \in F_\%, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ x_i - y_i + x_i \cdot p_i &< 2(1 - \beta_i(x, y)) \cdot \text{Max}_i && \forall i \in F_\%, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ -2(1 - \alpha_i(x, y)) \cdot \text{Max}_i &\leq x_i - y_i + q(x_i) && \forall i \in F_g, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ x_i - y_i + q(x_i) &< 2\alpha_i(x, y) \cdot \text{Max}_i && \forall i \in F_g, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ -2\beta_i(x, y) \cdot \text{Max}_i &\leq x_i - y_i + p(x_i) && \forall i \in F_g, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\ x_i - y_i + p(x_i) &< 2(1 - \beta_i(x, y)) \cdot \text{Max}_i && \forall i \in F_g, \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \end{aligned}$$

Constraints (informal) on the weights: See Section 4.3.2

Constraints (informal) on the thresholds allowing to model decision-maker's preferences:

$$\begin{aligned} q_i &\geq q_i^{\min} && \text{For some criteria } i \in F_c \cup F_\% \\ p_i &\leq p_i^{\max} && \text{For some criteria } i \in F_c \cup F_\% \\ q(y_i) &\geq x_i - y_i && \text{For some criteria } i \in F_g \\ p(x_i) &\leq x_i - y_i && \text{For some criteria } i \in F_g \end{aligned}$$

5.2 Simultaneous elicitation of criteria weights and category profiles

In the previous chapter, we explained how the model could be consistent with the sorting problem, when the profiles of the categories are defined. Indeed, as the profiles act like some potential alternatives, one may consider an assignment as the

expression of preference information on the outrankings between the alternative and the upper and lower profiles of the category.

Such an approach requires the preliminary definition of the profiles, as well as the discrimination thresholds, to make the computation of the double threshold order between the alternatives possible. Again, this is a complex task that a novice decision-maker may not afford with enough accuracy. Some preliminary tests, we shall discuss in the next part of the thesis, corroborate this assumption.

Consequently, we present an enhancement of the algorithms described previously in section 5.1 to allow them affecting the alternatives into predefined categories, when the profiles are unknown. The algorithm returns, in addition to the weights, the local concordance between the affected alternatives and the profiles.

Notice that the algorithm does not elicit criteria discrimination thresholds. In fact, their elicitation is not needed, as we directly compute the local concordance indices, allowing to determine whether or not the performance of an alternative, on every criteria, is at least as good as the performance of the profile. In addition, these values are fully correlated, as an improvement of k points for the performance of the profile in one criterion will be totally compensated by an increase of the same value of the corresponding indifference and preference thresholds. In that case, the algorithm sets the indifference threshold up to 0.

Notice also that the mathematical model we shall define does not sort the alternatives on which the decision-maker did not express preferential information. Indeed, without constraining their sorting, there is too many possibilities for the local concordance variables to consider the haphazard result as stable. Nevertheless, we propose hereafter a method for sorting these alternatives in a stable manner.

5.2.1 Modeling of the constraints on the profiles

Let B denote the set of p unknown ordered profiles (or bounds) b_1, \dots, b_p , defining $p + 1$ categories C_h , in a increasing order of preference, b_h being the lower limit of C_h and the upper limit of C_{h-1} . Let us remember that C_1 is the least preferred category and C_{p+1} the most preferred one.

Let $A' \subset A$ be the subset of the alternatives on which the decision-maker expresses a sorting information, namely an assignment of the alternatives in a category or in a set of categories. Let n' be the number of elements in A' . We shall detail in section 5.2.2 below the kind of preferential information he may express and the way of translating it into outrankings between the alternatives and the profiles.

We assume the use of one particular assignment rule, namely the pessimistic one, as we explained in section 3.3.3. In consequence, we propose to model constraints on the unknown outrankings between an alternative x and a profile b , namely by eliciting the local concordance variables $S_i(x, b)$.

Let $x \in A'$ be an alternative and $b \in B$ a profile. First, we model the local concordance variables $S_i(x, b) \in \{-1, 0, 1\}$ like the difference between two boolean

variables $\alpha_i(x, b)$ and $\beta_i(x, b)$, for every criterion i , as already given in equation (5.5) in section 5.1 above. $S_i(b, x) \in \{-1, 0, 1\}$ is defined in a similar manner.

Notice that the integrity of the variables α_i and β_i is warranted with the following constraint, that avoid both variables to be equal to 1 at the same time:

$$\alpha_i(x, b) + \beta_i(x, b) \leq 1, \forall x \in A', \forall b \in B, \forall i \in F \quad (5.29)$$

As the evaluations of the profiles are unknown, there is no restriction on the values of the variables S_i . However, we need to constrain them in order to elicit some meaningful values, by following these two logical integrity properties:

1. On each criterion i , the local concordance $S_i(x, b_h)$ of alternative x on profile b_h must be lower or equal than $S_i(x, b_k)$, for every indices $k < h$;
2. On each criterion i , if x_i is strictly greater than y_i , for every profile b , $S_i(x, b)$ must be at least as good as $S_i(y, b)$. Similarly, two alternatives having the same evaluation must be associated with the same local concordance value.

First property ensures that the profiles are correctly defined, with no cross between them. Indeed, if an alternative at least as good as a profile (on a given criterion i), we deduce logically that it has to be at least as good as all worse profiles also. It is simply warranted with the following mathematical constraints:

$$\alpha_i(x, b_h) \leq \alpha_i(x, b_{h-1}), \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F; \quad (5.30)$$

$$\beta_i(x, b_h) \geq \beta_i(x, b_{h-1}), \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F; \quad (5.31)$$

Second property warrants that every alternatives are compared similarly with the profiles. This is easily verified via the following sets of constraints:

$$\forall x, y \in A', \forall i \in F, x_i > y_i \implies \begin{cases} \alpha_i(x, b) \geq \alpha_i(y, b), \forall b \in B; \\ \beta_i(x, b) \leq \beta_i(y, b), \forall b \in B. \end{cases} \quad (5.32)$$

$$\forall x, y \in A', \forall i \in F, x_i = y_i \implies \begin{cases} \alpha_i(x, b) = \alpha_i(y, b), \forall b \in B; \\ \beta_i(x, b) = \beta_i(y, b), \forall b \in B. \end{cases} \quad (5.33)$$

We should notice that the resulting number of constraints derived from equations (5.32) and (5.33) is equal to $n' \cdot (n' - 1) \cdot m \cdot p$. For instance, with 7 criteria, 10 alternatives in A' and 4 profiles, it follows the definition of 2,520 constraints. In addition, the vast majority of these constraints are redundant. Indeed, due to the inherent transitivity of these constraints, if we sort the performances on each criterion and if we only create a constraint for each couple of adjacent performances only, we just need to define $2 \cdot (n' - 1) \cdot m \cdot p$ constraints, that is to say 504 constraints for our previous example.

Example 17 *Let consider a criterion i on which 4 alternatives are evaluated as follows: $t_i = 1$, $x_i = 2$, $z_i = 2$ and $y_i = 3$. In consequence, we only need to define constraints (5.32) for the couples (t, x) and (z, y) , and constraints (5.33) for the couple (x, z) .*

To ease the reading of the algorithms, we will keep the previous writing of the constraints, but we shall apply a preprocessing algorithm in order to create the necessary constraints only.

Finally, as we modeled the local concordance index the same way it has been modeled previously, the constraints allowing to enforce a positive or negative outranking statement (constraints (5.9) to (5.14)) can be used directly with no modification. The same applies to constraints ensuring the stability of a given outranking statement (constraints (5.15) to (5.19)).

5.2.2 Ensuring a stable assignment of an alternative

We are considering here the translation, in terms of outrankings, of the assignment of an alternative. As we previously did, the model shall impose, for every considered outranking statements, a simple majority validation (or invalidation) and try to ensure the stability by adding relaxed stability constraints.

When asked to sort an alternative x into some predefined categories, the decision-maker may formulate one of the following propositions:

1. "I am sure that x is in category C_h ";
Example: "x is a bad alternative".
2. "I am sure that x is in the category range going from C_k to C_l (included)";
Example: "x is a good or very good alternative".
3. "I am sure that x is at least in category C_h ";
Example: "x is for sure not a bad alternative".
4. "I am sure that x is at best in category C_h ";
Example: "x is clearly not a very good alternative".
5. "I am not sure / I do not know".

In the last case, when a decision-maker is unable to provide with accuracy any information on the assignment of an alternative, no constraint for the mathematical model should be deduced.

When he expresses the fact that an alternative x must belong to a predefined category C_h , bounded by a lower profile b_{h-1} and an upper one b_h , the alternative is added to the set A' and we simply translate the desired assignment in terms of outrankings, by imposing a positive outranking statement $xS^w b_{h-1}$ and a negative one, $xS^w b_h$.

We must highlight the fact that, according to the construction of the profiles, ensured by the constraints (5.30) on the integrity of the local concordance variables,

imposing the outranking statement “ $xS^w b_{h-1}$ ” automatically implies “ $xS^w b_k$ ”, for every indices $k \leq h - 1$. In the same way, “ $xS^w b_h$ ” implies “ $xS^w b_l$ ” for every indices $l \geq h$. Consequently, it is quite unnecessary to constrain the simple majority validation or invalidation of the outranking statement between the alternative and the other profiles (*i.e.* profiles that are not delimiting category C_h).

Even so, although we intend to maximise the number of stable outrankings when sorting the alternatives, it may occur some unstable ones in the final outranking relation, for instance when some relaxed constraints are violated. It is then advisable to add some relaxed stability constraints between x and the other profiles to narrow the set of categories down on which it is affected in a stable manner (as defined in section 3.3.3).

Situations 2, 3 and 4 can be similarly translated in the fact that an alternative is affected in one unknown category, from a continuous range of categories $[C_k, C_l]$, bounded by the lower profile b_{k-1} of C_k and the upper profile b_l of C_l . We then consider the set of categories like one “super” category, add the alternatives to the subset A' and translate the given information as we did for the first situation, by simply imposing $xS^w b_{k-1}$ and $xS^w b_l$.

Again, there is no need in imposing the simple majority of the outranking statements between the alternative and the profiles with indices lower than $k - 1$ or higher than l . But we shall also add some relaxed stability constraints between x and the other profiles.

Finally, as we did before, we could take into account some additional preferential information from the decision-maker, especially regarding the weights of the criteria, but also on the local behavior of an alternative on a profile. In the first case, we already modeled some useful constraints that can be used again without any modification. The second case is a bit more problematical and we decided on purpose not to model such information. Indeed, even though there is no difficulty in translating constraints like “ x is at least as good as a profile b on criterion i ”, namely by imposing $S_i(x, b) = 1$, it does not make sense when working with a novice decision maker. In fact, he might consider that the levels are independent and that an alternative is sorted in a category when it reaches a given level of acceptability on every criteria; but the principle of an outranking method is to validate a statement when we reach at least half of the condition, which is much more difficult to understand for a non expert of the decision. For these reasons, we are not asking the decision-maker about preferential information on the profile, as we consider this information as too sensitive to be expressed directly in an accurate and precise manner.

5.2.3 The complete models

As we did in the previous section, we define three algorithms for dealing with the sorting problem:

- ACON*: The control algorithm that assign the alternatives without taking into account the stability constraints, simply minimising the sum of the weights;

- STAB_1^* : The algorithm that takes into account the relaxed stability constraints using real slack variables;
- STAB_2^* : The algorithm that considers integer slack variables.

We are only giving here the algorithm for STAB_1^* . The complete models can be found in Annex A.2.7, A.2.8 and A.2.9.

First stable sorting algorithm (STAB_1^*)

MILP STAB_1^*

Variables:

$$\begin{array}{ll}
 \omega_{i,u} \in \{0, 1\} & \forall i \in F, \forall u = 1..M \\
 w_{i,u}^2(x, y) \in [-1, 1] & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
 \alpha_i(x, y), \beta_i(x, y) \in \{0, 1\} & \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
 b_u(x, y) \in \{0, 1\} & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M \\
 s(x, y) \geq 0 & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
 \end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s(x, y) + \sum_{g_i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
 \text{s.t.} & \sum_{g_i \in F} \omega_{i,1} = m \\
 & \omega_{i,u} \geq \omega_{i,u+1} \quad \forall i \in F, \forall u = 1..M - 1 \\
 & \sum_{u=1}^M b_u(x, y) \geq 1 \quad \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
 \circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \geq 1 \quad \forall (x, y) \in \mathfrak{S} \\
 & \sum_{i \in F} w_{i,u}^2(x, y) + s(x, y) \geq b_u(x, y) \quad \forall (x, y) \in \mathfrak{S}, \forall u = 1..M \\
 \circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \leq -1 \quad \forall (x, y) \in \overline{\mathfrak{S}} \\
 & \sum_{i \in F} w_{i,u}^2(x, y) - s(x, y) \leq -b_u(x, y) \quad \forall (x, y) \in \overline{\mathfrak{S}}, \forall u = 1..M
 \end{array}$$

Constraints on the coherence of the $w_{i,u}^2(x, y)$ variables:

$$\begin{array}{ll}
 -\omega_{i,u} \leq w_{i,u}^2(x, y) \leq \omega_{i,u} & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
 \omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) - 1 \leq w_{i,u}^2(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
 w_{i,u}^2(x, y) \leq -\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) + 1 & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
 -(\alpha_i(x, y) + \beta_i(x, y)) \leq w_{i,u}^2(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
 w_{i,u}^2(x, y) \leq \alpha_i(x, y) + \beta_i(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
 \end{array}$$

Constraints on the coherence of the α_i and β_i variables:

$$\begin{array}{ll}
 \alpha_i(x, b) + \beta_i(x, b) \leq 1 & \forall x \in A', \forall b \in B, \forall i \in F \\
 \alpha_i(x, b_h) \leq \alpha_i(x, b_{h-1}) & \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F \\
 \beta_i(x, b_h) \geq \beta_i(x, b_{h-1}) & \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F \\
 \alpha_i(x, b_h) \geq \alpha_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i \\
 \beta_i(x, b_h) \leq \beta_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i \\
 \alpha_i(x, b_h) = \alpha_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i \\
 \beta_i(x, b_h) = \beta_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i
 \end{array}$$

Constraints (informal) on the weights: See Section 4.3.2

5.2.4 Stable assignment of the other alternatives

Our algorithm simply elicits the local concordance variables associated with alternatives on which the decision-maker expressed an assignment. Thereby, without knowing the evaluations of the profiles and the criteria discrimination thresholds, we cannot directly sort the other alternatives. However, we show how it is possible to provide these alternatives a stable assignment in a set of adjacent categories, based on the elicited local concordance variables from the alternatives A' .

In fact, knowing for instance the evaluation z_i of an alternative $z \in A \setminus A'$, on a criterion i , we may distinguish five different cases for the evaluation of $S_i(z, b)$, for every profile $b \in B$:

1. $\exists x \in A'$ s.t. $x_i = z_i$: In that case, $S_i(z, b) = S_i(x, b)$;
2. $\exists x \in A'$ s.t. $x_i < z_i \wedge S_i(x, b) = 1$: Then, $S_i(z, b) = 1$;
3. $\exists x \in A'$ s.t. $x_i > z_i \wedge S_i(x, b) = -1$: Then, $S_i(z, b) = -1$;
4. $\exists x, y \in A'$ s.t. $x_i < z_i < y_i \wedge S_i(x, b) = S_i(y, b)$: Then, $S_i(z, b) = S_i(x, b)$;
5. Otherwise, let $x = \max\{x \in A' : x_i < z_i\}$ and $y = \min\{y \in A' : y_i > z_i\}$.
It follows that: $S_i(x, b) \leq S_i(z, b) \leq S_i(y, b)$.

In the first four cases, the variable is deduced automatically. In the last case, which is also the general case, we obtain a subset of acceptable values (from the original set $\{-1, 0, 1\}$) for the variable $S_i(z, b)$.

When $S_i(z, b)$ is clearly defined for each criterion i , as we know the weights of the criteria, we compute the associated outranking statement, so as its stability. Otherwise, as explained in section 3.3.4 on the stability of some outrankings with missing evaluations, we compute a worst and a best scenario, namely by considering the worst possible acceptable values in a first time and the best possible ones after. If the global outrankings are both stable and positive (resp. both stable and negative), the considered outranking is said to be stable and positive (resp. stable and negative). In any other configurations, it is said to be unstable.

Let us notice again that we are not searching for every possible assignments. From a subset of alternatives to be sorted, the algorithm returns a unique, but arguable, mathematical solution we extrapolate for the sorting of the other alternatives, chosen on purpose to minimise the dependency of the considered outrankings to the criteria weights.

Part III

Implementing a progressive method for a robust elicitation of the parameters

In this last part, we intend to implement our theoretical mathematical models on real practical cases.

For this purpose, we first validate the different algorithms in chapter 6 with large sets of benchmarks. Besides, we show how they guarantee a high degree of stability and therefore a high reliability of the resulting outranking relation. As the simultaneous elicitation brings a fair amount of variables, we also discuss about the running time of the algorithms.

In chapter 7, we define a robust preference elicitation protocol named REWAT, in order to construct iteratively an evaluation model that illustrates the decision-maker's expectations. Of course, in order to obtain valuable information for the determination of the parameters, we shall present in chapter 7 an interactive use of these models. In particular we intend to restrict the decision maker's intervention on a few pairs of alternatives and infer the outranking relation for those remaining.

Finally, chapter 8 presents the use of the REWAT process on a real-case application. In addition, it gives some critical attention to the defined process and it discusses about some perspectives for future enhancements.

Empirical validation of the algorithms

*“Les théories ont causé plus d’expériences que les expériences n’ont causé de théories.”*¹

[Joseph Joubert]

Contents

| | | |
|------------|---|------------|
| 6.1 | Parameters elicitation from a complete set of information | 106 |
| 6.1.1 | Elicitation of criteria weights | 106 |
| 6.1.2 | Elicitation of criteria weights and discrimination thresholds . | 108 |
| 6.1.3 | Elicitation of criteria weights and categories profiles | 109 |
| 6.2 | Iterative recovering of the median-cut outranking relation | 112 |
| 6.2.1 | Iterative elicitation of criteria weights | 113 |
| 6.2.2 | Iterative elicitation of both criteria weights and thresholds . . | 116 |
| 6.3 | Impact of the stability constraints on the preference modeling | 120 |
| 6.3.1 | Impact on the weights preorder | 121 |
| 6.3.2 | Impact on the preference discriminating thresholds | 122 |

Abstract

This chapter attempts to measure the behavior of the algorithms we have previously defined, in terms of running time and benefits of using stability constraints. We show that it is possible to use the algorithms to recover some parameters that are compatible with a given preference information set. In addition, we study the impact of taking into consideration some stability constraints on the elicited parameters.

1. “Theories created more experiences than the experiences created theories.”

This chapter attempts to measure the behavior of the algorithms we have previously defined, in terms of running time and benefits of using stability constraints. It also intends to define some practical hints we will consider we dealing with a real decision-maker. In order to do so, we consider the following experiments:

1. Starting from a complete set of preference information on the alternatives (namely a complete outranking relation or a complete sorting), obtained with an unknown vector of weights W , as well as some discrimination thresholds (known or unknown, according to the considered algorithm), we successfully compute another vector W^* , and also a set of discrimination thresholds when they are needed, compatible with the given information. Our objective is to first validate the models and analyse their behavior when dealing with a full set of information (*i.e.* a maximal number of constraints).
2. In the second experiment, we iteratively construct a set of preference information on the alternatives that allows to recover the complete median-cut outranking relation. It shows a reduced time for solving large instances with small sets of information, highlighting the possible use of the algorithms in real-time interactive processes for eliciting the parameters.

In the second experiment, we are considering a small set of given outranking statements, or a small set of sorting examples. As we cannot use the stability constraints on a statement without knowing its validity, it may result in a lack of stability. We then considered the hypothesis of minimising the sum of the weights, as it tends to reduce the number of equi-importance classes, as a heuristic for increasing the number of stable statements that are not directly imposed by the decision-maker. In order to test the validity of this assumption, we will consider ACON as a control algorithm in the first experiment, such that it only ensures the validation of the simple majority constraints of the given statements by the decision-maker, minimises the sum of the weights and throws apart all stability constraints.

We consider 25 different sizes of problems, by varying the numbers n of alternatives and m of criteria, according to the following values: 7, 10, 13, 16 and 19. For each size, we randomly generate 300 problems: a performance table (with a gaussian distribution of the generated values on each criterion, on a scale from 0 to 100), a vector of integer criteria weights and some associated discrimination thresholds, allowing to compute an initial outranking digraph, modeling a decision-maker's set of preferences. Notice that for each experiment, we will run the tests on the same problems, allowing to better compare the results. Finally, the problems are solved using CPLEX 11.0 on a machine with two Intel Xeon X5355 2,66 GHz processors and 4 cores each.

Without any loss of generality, we assume in these theoretical tests that we only consider some constant thresholds, in order to simplify the theoretical tests. Notice that the use of proportional thresholds should not have an impact on the running times, as the variables are the same. However, the use of general thresholds may result in an increase of the time.

Also, no veto thresholds have been defined in this empirical study, as an outrank-

ing situation on which a veto is raised is no longer fully depending on the weights and the discrimination thresholds. We detail in the next chapter how they can be taken into consideration when dealing with a real decision-maker.

Let us remember the defined algorithms and their specificities:

1. The algorithms defined in chapter 4, that elicit the criteria weights only:
 - ACON (see Annex A.2.1): A control algorithm, intended to verify that minimising the sum of the weights may have a beneficial impact on the number of stable outrankings;
 - STAB₁ (see Annex A.2.2): A first mathematical model that best ensures the stability of some considered outrankings by minimising the sum of some real slack variables;
 - STAB₂ (see Annex A.2.3): Similar than STAB₁, considering some integer slack variables.
2. The algorithms defined in section 5.1, that elicit both weights and thresholds:
 - ACON' (see Annex A.2.4): A control algorithm;
 - STAB'₁ (see Annex A.2.5): A mathematical model that best ensures the stability taking into account some real slack variables;
 - STAB'₂ (see Annex A.2.6): Similar to STAB'₁, considering some integer slack variables.
3. The algorithms defined in section 5.2, that elicit both criteria weights and categories profiles:
 - ACON* (see Annex A.2.7): A control algorithm;
 - STAB*₁ (see Annex A.2.8): Elicitation of the weights and "profiles" of a sorting problem, considering real slack variables on the stability constraints;
 - STAB*₂ (see Annex A.2.9): Similar to STAB*₁, considering some integer slack variables.

Beyond the fact that the control algorithms allow to validate the assumption that minimising the sum of the weights helps increasing the resulting number of stable statements, they will be also useful during some real-case applications. Indeed, assuming that the decision-maker expressed a set of preferential information which results in an impossible solving. As the stability constraints are always used in their relaxed forms, they cannot be responsible. Thus, as these algorithms are faster (because of a reduced number of constraints and variables), they will be considered when trying to solve a conflict. We shall detail this in the next chapter, in section 7.1.2. In addition, as we shall explain in the next chapter, assuming that the preorder of the weights is fixed and cannot be modified, it is useless to try to enforce the stability of the outranking statements. Thus, if the discrimination thresholds are given, we shall consider ACON for eliciting some precise weights; if not, ACON' will elicit both criteria weights and discrimination thresholds according to some expressed preferential information.

Finally, these tests have been ran before the consideration of the preferable relation in the mathematical models, such that it is not taken into account in this chapter for empirical validation. However, we have made fewer tests that have

successfully validated the models. We will not detail them in order to ease the reading, some future work will be envisioned to formally validate them.

6.1 Elicitation of the parameters based on a complete set of preference information

In this section, we are testing the validity of the different algorithms: From a complete set of preference information on the alternatives, namely a complete median-cut outranking relation or a complete sorting, we successfully recover some parameters that are compatible with the given information.

These experiments allow to measure the gain in terms of stability, as well as the running time when dealing with a great number of constraints.

6.1.1 Elicitation of criteria weights

First, we start from a complete given median-cut outranking digraph and successfully compute a vector of criteria weights recovering the relation, using the three defined versions of the initial algorithm, namely $STAB_1$, $STAB_2$ and $ACON$. We then measure the average percentage of stable statements in each case, highlighting the gain of using stability constraints.

Notice that, if a balanced situation occurs in the initial digraph (*i.e.* $x^?y$), we decide not to take it into account when creating the set of constraints, as we consider that a decision-maker would not express such a balanced situation and may refrain preferably from giving any judgement.

As a large value for the parameter M implies an exponential increase of the running time, we decided to fix it to 7 for each problem at the beginning and to increase it when a solution can not be reached with such a low parameter (about a 8% when $M = 7$, only a 4% when $M = 8, \dots$ ²), until a solution is found.

Notice that bounding this parameter may occur in finding a non-optimal solution, *i.e.* one could find a solution with a better number of stable statements when considering a higher parameter. To compare our solutions to the optimal ones, we run again all the problems, taking the number of criteria as the value of M . No real quality improvement were noticed: most of the time, the solutions were the same or improved by only 1 or 2 percents, for a running time ten to a hundred times longer. Let us simply remark that, for large instance that was already quite time consuming, we did not run again every problems, but only a few number, with the same conclusions. Thus, as there is no clear advantage in these considerations, we will not try again to improve the results for the following experiments.

2. These percentages are depending on the way we generate the problems and may be different with another generator. In any case, using a small parameter and increasing it if no solution is found will be always faster than using first a large parameter.

Table 6.1: Exp₁ (1/3) – Stability increase and running time

| m | n | Median percentage of stable outranking statements | | | | Median running time (s) and standard deviation | | | |
|----|----|---|------|-------------------|-------------------|--|--------|-------------------|-----------|
| | | Orig | ACON | STAB ₁ | STAB ₂ | STAB ₁ | | STAB ₂ | |
| 7 | 7 | 0.62 | 0.81 | 0.81 | 0.85 | 0.0 | ± 0.1 | 0.0 | ± 0.0 |
| 7 | 10 | 0.64 | 0.79 | 0.79 | 0.82 | 0.0 | ± 0.2 | 0.0 | ± 0.2 |
| 7 | 13 | 0.68 | 0.73 | 0.73 | 0.80 | 0.3 | ± 0.6 | 0.3 | ± 0.4 |
| 7 | 16 | 0.62 | 0.66 | 0.66 | 0.75 | 0.8 | ± 1.5 | 0.9 | ± 1.0 |
| 7 | 19 | 0.65 | 0.67 | 0.66 | 0.73 | 1.9 | ± 3.6 | 1.8 | ± 1.8 |
| 10 | 7 | 0.62 | 0.76 | 0.76 | 0.90 | 0.1 | ± 0.4 | 0.0 | ± 0.3 |
| 10 | 10 | 0.57 | 0.71 | 0.75 | 0.88 | 1.6 | ± 2.7 | 1.6 | ± 3.5 |
| 10 | 13 | 0.56 | 0.73 | 0.77 | 0.84 | 10.2 | ± 5.5 | 10.5 | ± 11.3 |
| 10 | 16 | 0.58 | 0.69 | 0.73 | 0.81 | 22.4 | ± 10.8 | 23.3 | ± 25.8 |
| 10 | 19 | 0.57 | 0.64 | 0.69 | 0.77 | 32.1 | ± 17.8 | 31.2 | ± 34.5 |
| 13 | 7 | 0.48 | 0.71 | 0.71 | 0.81 | 0.0 | ± 0.1 | 0.0 | ± 0.1 |
| 13 | 10 | 0.47 | 0.56 | 0.71 | 0.82 | 0.2 | ± 1.5 | 0.2 | ± 1.2 |
| 13 | 13 | 0.46 | 0.57 | 0.66 | 0.78 | 4.4 | ± 5.4 | 4.8 | ± 11.2 |
| 13 | 16 | 0.48 | 0.54 | 0.61 | 0.75 | 19.9 | ± 13.0 | 16.7 | ± 24.6 |
| 13 | 19 | 0.46 | 0.51 | 0.57 | 0.71 | 32.3 | ± 28.7 | 25.1 | ± 41.2 |
| 16 | 7 | 0.40 | 0.66 | 0.78 | 0.90 | 0.1 | ± 0.3 | 0.1 | ± 0.4 |
| 16 | 10 | 0.42 | 0.66 | 0.73 | 0.88 | 3.2 | ± 6.5 | 3.5 | ± 7.1 |
| 16 | 13 | 0.41 | 0.62 | 0.73 | 0.86 | 35.7 | ± 20.2 | 44.5 | ± 488.6 |
| 16 | 16 | 0.42 | 0.59 | 0.75 | 0.80 | 78.8 | ± 46.0 | 295.3 | ± 6,543.7 |
| 16 | 19 | 0.42 | 0.55 | 0.65 | 0.75 | 106.1 | ± 97.4 | 412.3 | ± 2,773.9 |
| 19 | 7 | 0.38 | 0.66 | 0.71 | 0.81 | 0.0 | ± 0.1 | 0.0 | ± 0.1 |
| 19 | 10 | 0.38 | 0.52 | 0.66 | 0.79 | 0.6 | ± 1.6 | 0.7 | ± 1.7 |
| 19 | 13 | 0.36 | 0.50 | 0.60 | 0.80 | 10.6 | ± 19.0 | 11.9 | ± 62.9 |
| 19 | 16 | 0.38 | 0.52 | 0.68 | 0.77 | 80.5 | ± 42.1 | 111.0 | ± 380.9 |
| 19 | 19 | 0.39 | 0.48 | 0.64 | 0.72 | 151.7 | ± 92.9 | 459.7 | ± 2,355.8 |

On its left part, Table 6.1 summarises, for each algorithm, the median percentage of stable outranking statements (according to the total number of outranking statements, $n^2 - n$). On right part of Table 6.1, we give the median running time of the algorithms. Notice that, as ACON always runs between 0.1 and 0.3 seconds, its running time is not represented.

One may remark that each method improves the average percentage of stable arcs compared to the initial outranking digraph (Orig). ACON, by simply minimising the sum of the weights, tends to minimise the number of equi-importance classes, inducing an increase of the stable statements of 18 percents in average. Then, such a consideration in the objective function will be useful for partially taking care of the stability when no preferential information is given on some pairs of alternatives, as we shall present in the next experiment.

As expected, the exact algorithm STAB₂ gives better result than STAB₁, with an increase of the running time. However, we can see that this time increase is only significant for a large set of alternatives. Indeed, when the number of alternatives is lower or equal to 13, the difference between the two algorithms is weak and clearly compensated by a gain of at least 10% of stable arcs. Also, as m is, most of the time, constant, the running time is more correlated to the number of pairs of alternatives than the number of criteria. Consequently, in the next experiment, as we consider

the construction of an incremental subset of pair of alternatives, we hope that the running time will be short enough for a real-time use of the exact algorithm STAB₂. We shall present the positive results afterwards.

Also notice that the standard deviations are quite high, especially on large instances, such that it seems difficult to predict in advance the necessary time for finding the optimal solution. For instance, considering the problems with 19 alternatives and 19 criteria, the fastest problems runs in 10 seconds and the longest in 7.5 hours. It should then be useful to consider solving these problems with heuristic methods.

We can point out the fact that if we consider the sorting problem, with given profiles and associated discrimination thresholds, there is no need in ensuring the complete median-cut outranking relation, but only the outranking statement between the alternatives and the profiles, reducing the set of preferential information, and obviously the running time.

To conclude this experiment, we underline that it is possible to reconstruct a complete outranking digraph, with a significant increase of the arcs stability. This can be useful when a decision-maker agrees with a digraph obtained via a certain outranking method, in order to reinforce the stability of the digraph before tackling the actual decision aid problem typology (*e.g.* selecting the best alternatives).

Notice that we are not discussing here about the truthfulness of the parameters, namely the fact that they are close, or not, to the initial parameters. We will try to apprehend empirically the impact of the use of stability constraints on the resulting parameters in section 6.3.

6.1.2 Elicitation of criteria weights and discrimination thresholds

We are now assuming that the criteria weights and discrimination thresholds are unknown and we want to elicit them according to a complete median-cut outranking digraph, using the three algorithms ACON', STAB'₁ and STAB'₂. The results are displayed in Table 6.2.

As a matter of course, one can easily distinguish the high increase of the median running times, compared with the previous experiment when eliciting the criteria weights only (see Table 6.1). This is explained by a more consequent number of variables. Notice that for large scale problems, presented in Table 6.2 with an asterisk (*), we only intended to solve a hundred problems (instead of 300), as their running time was too long. And, for the most sizable problems, such that we did not run a hundred tests, it appears to be not relevant to compute a median running time, especially as the standard deviation was too high to have a not arbitrarily value. Let us simply notice that some problems with 19 alternatives and 13 criteria took between 1 or 2 days for finding a solution, while the fastest problems of the same size only took about 1 hour.

These prohibitive times highlight the fact that the use of a complete set of preferential information does not allow a practical implementation of the algorithms.

However, to clarify our concern, it has to be noted that such considerations are mainly theoretical, as a decision-maker will hardly express a complete median-cut outranking relation³. Remember that the main objective of these tests is to validate the algorithm; we shall detail in the next section how to highly decrease the running time, without cutting down on the stability too much, by constructing an incremental set of preferential information that is enough to recapture the complete relation. Notice that future work would be dedicated in implementing good heuristic methods to solve these problems in efficient times.

Also, when comparing these results with the one in Table 6.1, we observe that the number of stable statements for the current algorithms increases of a few percentages again. This can be explained by a greater flexibility, as the algorithms may tune the thresholds and thus possibly improve the resulting stability. Of course, we will discuss in section 6.3.2 about the impact of considering some stability constraints on the elicited thresholds values.

Again, there is a significant increase of the number of stable arcs between the initial digraph and the ones obtained with our algorithms. ACON' improves this number of about a 14% (in average), STAB'₁ of about a 29% and STAB'₂, a 30%. Such a great improvement indicates that the high discrimination of the initial weights vectors is not needed: two weights should not be discriminated if there is no clear justification in discriminate them, as it creates many unstable arcs that will have to be validated "manually" (not automatically when validating the preorder). On the opposite way, two criteria should be associated with two different weights only either when there is a strong desire in differentiate them (for instance, if the decision-maker is firmly convinced that a criterion is more important than another one, or if there is institutional constraints that force the relation between some criteria) or when there is a concrete requirement to differentiate them (for instance, when it is needed to ensure a set of preferential information on the way some alternatives are compared).

Finally, contrary to the previous tests, the stability increase between the algorithms STAB'₁ and STAB'₂ is not relevant. In return, the median running time of STAB'₂ quickly becomes prohibitive; Hence, it will be more advisable to prefer the use of STAB'₁ during the next experiments.

6.1.3 Elicitation of criteria weights and categories profiles

Remind that the performances of the alternatives are generated on a scale from 0 to 100 on every criteria. In order to define a theoretical, but close to some real instances, problems, we assume the definition of 5 categories, from VERY BAD (VB) to VERY GOOD (VG), bounded by 4 profiles on which the performances are defined as follows: All the performances of the lowest profile (between VERY BAD and BAD) are equal to 25, all the performances for the second, third and fourth ones are respectively equal to 50, 60 and 70. Notice that these values have been defined empirically to have an homogeneous distribution of the alternatives on the categories. We then

3. It may occur when considering an alternatives subset, on which the DM is able to compare every pairs, in order to compute some parameters and to use them on a wider set of alternatives.

Table 6.2: Exp₁ (2/3) – Stability increase and running time

| m | n | Median percentage of stable outranking statements | | | | Median running time (s) and standard deviation | | |
|----|----|---|-------------------|--------------------------------|--------------------------------|--|--------------------------------|--------------------------------|
| | | Orig | ACON [*] | STAB [*] ₁ | STAB [*] ₂ | ACON [*] | STAB [*] ₁ | STAB [*] ₂ |
| 7 | 7 | 0.62 | 0.78 | 0.85 | 0.86 | 0 ± 0 | 0 ± 1 | 0 ± 2 |
| 7 | 10 | 0.64 | 0.79 | 0.84 | 0.85 | 0 ± 0 | 2 ± 7 | 1 ± 10 |
| 7 | 13 | 0.68 | 0.74 | 0.82 | 0.82 | 0 ± 0 | 23 ± 42 | 22 ± 243 |
| 7 | 16 | 0.62 | 0.68 | 0.73 | 0.73 | 1 ± 1 | 64 ± 3,077 | 121 ± 1,159 |
| 7 | 19 | 0.65 | 0.65 | 0.72 | 0.73 | 2 ± 3 | 139 ± 605 | * 2,324 ± 12,703 |
| 10 | 7 | 0.62 | 0.71 | 0.90 | 0.90 | 0 ± 0 | 2 ± 3 | 1 ± 7 |
| 10 | 10 | 0.57 | 0.71 | 0.88 | 0.88 | 0 ± 0 | 38 ± 70 | 57 ± 359 |
| 10 | 13 | 0.56 | 0.71 | 0.82 | 0.84 | 1 ± 0 | 296 ± 276 | * 1,631 ± 15,559 |
| 10 | 16 | 0.58 | 0.68 | 0.80 | — | 2 ± 1 | 1,059 ± 4,805 | |
| 10 | 19 | 0.57 | 0.68 | 0.77 | — | 4 ± 2 | 1,272 ± 2,055 | |
| 13 | 7 | 0.48 | 0.66 | 0.81 | 0.83 | 0 ± 0 | 1 ± 3 | 1 ± 2 |
| 13 | 10 | 0.47 | 0.58 | 0.83 | 0.83 | 0 ± 0 | 25 ± 122 | 24 ± 172 |
| 13 | 13 | 0.46 | 0.62 | 0.78 | 0.79 | 1 ± 2 | 581 ± 2,747 | * 1,843 ± 19,009 |
| 13 | 16 | 0.48 | 0.55 | 0.75 | — | 3 ± 6 | * 2,649 ± 5,335 | |
| 13 | 19 | 0.46 | 0.50 | 0.72 | — | 8 ± 12 | * 4,500 ± 23,189 | |
| 16 | 7 | 0.40 | 0.64 | 0.88 | 0.88 | 0 ± 0 | 5 ± 12 | 4 ± 14 |
| 16 | 10 | 0.42 | 0.63 | 0.84 | 0.86 | 0 ± 0 | 308 ± 592 | 269 ± 2,045 |
| 16 | 13 | 0.41 | 0.69 | 0.83 | — | 2 ± 2 | * 3,130 ± 8,628 | |
| 16 | 16 | 0.42 | 0.56 | 0.78 | — | 13 ± 14 | * 10,106 ± 55,603 | |
| 16 | 19 | 0.42 | 0.55 | — | — | 35 ± 21 | | |
| 19 | 7 | 0.38 | 0.66 | 0.83 | 0.83 | 0 ± 0 | 2 ± 9 | 1 ± 5 |
| 19 | 10 | 0.38 | 0.64 | 0.81 | 0.82 | 1 ± 1 | 177 ± 483 | 58 ± 934 |
| 19 | 13 | 0.36 | 0.60 | — | — | 2 ± 6 | | |
| 19 | 16 | 0.38 | 0.49 | — | — | 22 ± 23 | | |
| 19 | 19 | 0.39 | 0.48 | — | — | 68 ± 53 | | |

define the discrimination thresholds such that $q_i = 0$ and $p_i = 5$ on each criterion $i \in F$. The initial sorting that we want to recapture is obtained by computing the outranking statement between the alternatives and the profiles, according to the pessimistic rule from ELECTRE TRI, namely an alternative is assigned to a category if it outranks its lower profile and does not outrank its upper profile.

Left part of Table 6.3 summarises the median running time, as well as the standard deviation, for the three algorithms ACON^{*}, STAB^{*}₁ and STAB^{*}₂, according to the defined numbers of alternatives and criteria. First, we can observe that the two algorithms STAB^{*}₁ and STAB^{*}₂ are faster than STAB^{*}₁ and STAB^{*}₂ respectively (see Table 6.2), especially in large instances. That can be explained by the fact that the number of outranking statement to be considered is reduced (we are here only looking at the outranking statements between an alternative and a profile, not between two alternatives), but also by the fact that there is a higher liberty in the setting of the local concordance values.

This high degree of liberty can also explain another behavior of the three algorithms: When looking at the elicited vectors of weights, we observe that for more than a 99% of the instances, they found a vector of equi-important weights (*i.e.* all the weights are equal to 1) and a set of local concordance values that recapture the complete sorting with accuracy. Roughly speaking, the algorithms have enough

Table 6.3: Exp₁ (3/3) – Stable sorting

| m | n | Median running time (s) and standard deviation | | | Median percentage of local balanced situations | | | |
|----|----|---|---------------------|---------------------|---|-------|---------------------|---------------------|
| | | ACON* | STAB ₁ * | STAB ₂ * | Orig | ACON* | STAB ₁ * | STAB ₂ * |
| 7 | 7 | 0 ±1 | 1 ±1 | 1 ±0 | 5 % | 72 % | 80 % | 78 % |
| 7 | 10 | 6 ±5 | 5 ±2 | 2 ±1 | 5 % | 63 % | 76 % | 75 % |
| 7 | 13 | 15 ±11 | 10 ±4 | 7 ±3 | 5 % | 60 % | 75 % | 71 % |
| 7 | 16 | 38 ±15 | 23 ±9 | 15 ±6 | 5 % | 61 % | 73 % | 70 % |
| 7 | 19 | 65 ±20 | 32 ±26 | 26 ±14 | 5 % | 63 % | 71 % | 69 % |
| 10 | 7 | 0 ±2 | 3 ±2 | 1 ±1 | 5 % | 76 % | 84 % | 83 % |
| 10 | 10 | 4 ±7 | 13 ±8 | 8 ±3 | 5 % | 72 % | 85 % | 80 % |
| 10 | 13 | 21 ±15 | 31 ±10 | 20 ±9 | 5 % | 75 % | 84 % | 80 % |
| 10 | 16 | 67 ±24 | 67 ±105 | 45 ±29 | 5 % | 76 % | 84 % | 77 % |
| 10 | 19 | 151 ±56 | 97 ±156 | 106 ±56 | 5 % | 77 % | 84 % | 79 % |
| 13 | 7 | 0 ±2 | 5 ±4 | 2 ±1 | 5 % | 75 % | 87 % | 88 % |
| 13 | 10 | 7 ±9 | 30 ±22 | 9 ±7 | 5 % | 78 % | 89 % | 84 % |
| 13 | 13 | 34 ±22 | 67 ±24 | 38 ±25 | 5 % | 80 % | 89 % | 83 % |
| 13 | 16 | 98 ±41 | 124 ±45 | 96 ±42 | 5 % | 79 % | 88 % | 80 % |
| 13 | 19 | 86 ±45 | 206 ±341 | 168 ±133 | 5 % | 80 % | 88 % | 82 % |
| 16 | 7 | 1 ±3 | 5 ±9 | 2 ±2 | 5 % | 76 % | 90 % | 91 % |
| 16 | 10 | 15 ±17 | 47 ±21 | 10 ±10 | 5 % | 80 % | 91 % | 86 % |
| 16 | 13 | 37 ±26 | 120 ±45 | 57 ±34 | 5 % | 83 % | 91 % | 84 % |
| 16 | 16 | 38 ±20 | 212 ±69 | 148 ±58 | 5 % | 84 % | 91 % | 83 % |
| 16 | 19 | 134 ±66 | 317 ±708 | 289 ±138 | 5 % | 86 % | 91 % | 84 % |
| 19 | 7 | 1 ±5 | 7 ±9 | 2 ±1 | 5 % | 79 % | 92 % | 93 % |
| 19 | 10 | 11 ±16 | 71 ±34 | 11 ±13 | 5 % | 82 % | 92 % | 91 % |
| 19 | 13 | 21 ±22 | 186 ±80 | 54 ±54 | 5 % | 86 % | 93 % | 84 % |
| 19 | 16 | 53 ±30 | 365 ±161 | 247 ±123 | 5 % | 86 % | 92 % | 84 % |
| 19 | 19 | 126 ±103 | 489 ±193 | 337 ±211 | 5 % | 87 % | 92 % | 88 % |

liberty with the local concordance values such that they do not need to discriminate the weights. Remind that the equi-important vector of weights ensures the stability of every outranking statements, such that each alternative is sorted in a stable manner in the required category.

However, the elicited profiles have a not desired behavior: Indeed, the algorithms advantage the local balanced situations, *i.e.* when $S_i(x, b) = 0$, instead of positive or negative local “*at least as good as*” situations, between any alternative x and any profile b . In right part of Table 6.3, we display the median percentage of local balanced situations⁴, with respect to the three considered algorithms (between a 60% and a 93%) and compare them with the percentage of local balanced situation that occurs between the alternatives and the initially generated profiles (about a 5%). That can be explained by the fact that the algorithms try to reduce the discrimination to its lowest possible level: Intuitively, when ensuring that an alternative x outranks a profile b , we only need one criterion i such that $S_i(x, b) = 1$ and, for every other criteria j , $S_j(x, b) = 0$. Also notice that this percentage tends logically to increase when considering problems with more criteria, as there is more degrees of freedom.

4. According to the complete number of local concordance situations to be considered between the alternatives and the profiles.

As it stands, the sorting algorithms cannot be integrated into a progressive method for eliciting the parameters of an aggregation procedure, namely a vector of weights and the profiles that bounds the given categories. Indeed, as it considerably advantages the local balanced situations, it does not seem to be relevant, nor reliable, to use these elicited parameters to sort the alternatives on which the decision-maker did not express preferential information.

At the end of the thesis, we briefly show some work that should start in the near future to improve the behavior of these algorithms. Indeed, as there are still some substantial theoretical and practical work to provide before defining a stable sorting process in a real case application, we will not consider here a deeper study of these algorithms.

6.2 An iterative recovering of the median-cut outranking relation

We have shown that the algorithms were quite efficient in improving the inherent stability of an outranking relation, but also quite time-consuming, such that it is difficult to envision their use in a real-time protocol. In addition, it is hardly possible that a decision-maker expresses directly a complete set of preferential information, except in the situation where he perfectly knows the relations between a subset of alternatives that is used to find some compatible parameters he will consider in the resolution of the problem with a broader set of alternatives.

Thus, getting closer to a practical questioning protocol, we are here considering a context where a fictitious decision-maker is asked to give preferential information about selected couples of alternatives which aims at assessing iteratively enough information to infer compatible parameters that reconstruct his preferences (modeled by the initial complete outranking digraph, or with a few percentage of changes).

This experiment intends to show that the algorithms can be easily implemented in a real elicitation process when having to construct an incremental set of preferential information. We are not going into detail about the elicitation process, as it will be explained in chapter 7, but simply highlight the fact that the running times can be shortened such that, for most of the instances, an optimal resolution via a mathematical solver (in our case, CPLEX) will be possible in real-time applications. However, on large instances, we will show that a heuristic resolution of the problem will have to be envisioned to reach an acceptable running time for each iteration (obviously, a decision-maker will not wait 10 minutes between each iteration). Also, we are interested in trying to define an adequate number of couples to be selected, using $STAB_2$ or $STAB'_1$ for intermediate resolutions, as they previously gave us the best results in an acceptable time.

Let us state that, for the reasons that have been expressed in section 6.1.3, we will not intend an iterative recapture of a complete sorting of the alternatives using the algorithms $ACON^*$, $STAB_1^*$ or $STAB_2^*$.

6.2.1 Iterative elicitation of criteria weights

Considering a given set of alternatives, a set of criteria with their associated thresholds and a performance table, we iteratively elicit a vector of weights from an incremental set of preferential information until the agreement between the resulting median-cut outranking digraph and the initial one, modeling the decision-maker's preferences on the alternatives.

The algorithm is then the following:

Algorithm 1 Second theoretical experiment: Recapturing the weights

Input:

A : Alternatives set; F : criteria vector (with given thresholds); P : performance table; G : Outranking digraph, modeling DM's preferences on the alternatives.

Variables:

w : Elicited vector of weights.

```

1: constraints  $\leftarrow \emptyset$ 
2:  $w \leftarrow w_1$  /* The equi-important vector of weights */
3: while not IS_RECAPTURING_DIGRAPH( $w, G$ ) do
4:    $(a_1, a_2) \leftarrow$  SELECT_ALTERNATIVES_COUPLE( $A$ )
5:   constraints  $\leftarrow$  ADD_PREFERENTIAL_CONSTRAINTS( $G(a_1, a_2), G(a_2, a_1)$ )
6:    $w \leftarrow$  STAB2_SOLVE( $A, F, P, constraints$ )
7: return  $w$ 

```

The key steps of the algorithm are the selection of the alternatives couples at each iteration (step 4) and the loop-ending condition (step 3). For selecting a couple of alternatives, we test three natural heuristics:

- A *random selection* (RS);
- A selection of a couple among the *most represented class* (MRC) of couples having the same behavior on each criterion, in order to fix the greatest number of arcs at each iteration;
- A selection of the couple with the *worst determined concordance value* (WDV), *i.e.* arcs associated with values close to 0 (balanced situation) which are, in the absence of stability, anecdotic and very sensitive to criteria weights changes.

Let us note that these heuristics select the alternatives without the intervention of the decision-maker. Although this assumption is necessary for running our theoretical tests, it imposes to select a fair number of couples whose associated outranking statements are already warranted by the current weights. We will explain in the next chapter how a decision-maker can act on the alternatives selection, especially to call into question some outranking statements (for instance, if a set of parameters sort an alternative into a category which is, to his point of view, not the correct one).

For the loop-ending condition, we test a complete reconstruction, and also a 95% reconstruction (*i.e.* the algorithm stops when at least a 95% of the arcs, between the median-cut outranking digraph, obtained by the current vector of weights, and the

initial one are similar). This is motivated by the fact that, in a real-case applications, the decision-maker has a partial knowledge of the complete outranking relation. Thus, the algorithms will stop as soon as the decision-maker is happy enough with the resulting relation, or with its exploitation. If he has a clear disagreement, either about the way some alternatives are compared, or about the resulting recommendations, we will then not have to select arbitrarily some alternatives, but to consider the modification of the outrankings that are pointed out.

In step 5, we add to the set of preferential information the enforcement of the two outrankings statements that link the selected pair of alternatives. Notice that, when these information are already stably granted by the current weights vector, there is no need to compute a new vector: step 6 is then ignored and another pair of alternatives is selected.

Table 6.4 summarises, for each size of problems, the median number of selected couples (*i.e.* an idea on the number of questions we should ask the decision-maker before presenting him the final outranking relation) and the median number of effective resolutions. For the purpose of clarification, these values are rounded up to the whole nearest number. As the algorithm runs under a second for each iteration, even for the largest instances, we decided not to display the median running-time⁵. Thus, in a real-case application where we intend to elicit the criteria weights only, *i.e.* knowing the discrimination thresholds, we can easily solve the model and find the optimal solution at each iteration with a mathematical solver as the execution times are very short.

Also notice that we only present in Table 6.4 the results using the WDV heuristic, as RS and MRC heuristics results were not satisfying, as the number of selected couples were two to four times higher compared to the WDV heuristic, with a similar number of solver executions.

On the left part of Table 6.4, we detail the results for a complete reconstruction of the initial median cut outranking digraph. This experiment corresponds to an iterative version of the first one, without having to consider all couples of alternatives. Notice that the stability is slightly lower than for the first experiment, due to the fact that the algorithm only forces the stability for some selected couples, but tries to minimise the sum of the weights and so tends to reduce the weights vector discrimination. One can run again the first experiment at the end of the iterative protocol, once the decision-maker validates the outranking relation. Also notice that the WDV heuristic considers less than a 30% of the whole couples in the worst case (a large number of criteria and a few alternatives) and only a 6% in the best ones (few criteria and a large number of alternatives), helping us to find the weights parameters in a relatively fast questioning protocol.

On the right part of Table 6.4, we detail the results of a 95% reconstruction of the initial outranking digraph, running again the previous experiment by modifying the ending condition in order to stop the iterative process when at least a 95%

5. The number of CPLEX executions, in seconds, gives a good approximation of the complete execution time.

Table 6.4: Exp₂ (1/3) – STAB₂ iterative reconstruction, using the WDV heuristic

| m | n | nb pairs [†] | 100% reconstruction | | | | | | 95% reconstruction | | | | | |
|----|----|-----------------------|---------------------|------|-----------|------|--------------------|------------|--------------------|-----------|-----|--------------------|--|--|
| | | | nb select* | | nb solve* | | %stab [‡] | nb select* | | nb solve* | | %stab [‡] | | |
| 7 | 7 | 21 | 4 | 18 % | 3 | 6 % | 78 % | 2 | 10 % | 1 | 4 % | 82 % | | |
| 7 | 10 | 45 | 7 | 16 % | 2 | 5 % | 75 % | 5 | 10 % | 2 | 3 % | 78 % | | |
| 7 | 13 | 78 | 9 | 12 % | 3 | 3 % | 71 % | 5 | 6 % | 2 | 2 % | 77 % | | |
| 7 | 16 | 120 | 11 | 9 % | 3 | 2 % | 71 % | 6 | 5 % | 2 | 2 % | 77 % | | |
| 7 | 19 | 171 | 11 | 6 % | 3 | 2 % | 71 % | 5 | 3 % | 2 | 1 % | 79 % | | |
| 10 | 7 | 21 | 5 | 20 % | 1 | 4 % | 72 % | 2 | 9 % | 1 | 2 % | 70 % | | |
| 10 | 10 | 45 | 11 | 24 % | 1 | 3 % | 74 % | 4 | 9 % | 1 | 1 % | 72 % | | |
| 10 | 13 | 78 | 19 | 24 % | 2 | 2 % | 69 % | 6 | 8 % | 1 | 1 % | 71 % | | |
| 10 | 16 | 120 | 26 | 22 % | 2 | 2 % | 68 % | 7 | 6 % | 1 | 1 % | 70 % | | |
| 10 | 19 | 171 | 37 | 22 % | 2 | 1 % | 68 % | 9 | 6 % | 1 | 1 % | 70 % | | |
| 13 | 7 | 21 | 6 | 25 % | 3 | 10 % | 74 % | 3 | 14 % | 2 | 7 % | 78 % | | |
| 13 | 10 | 45 | 11 | 25 % | 4 | 8 % | 67 % | 6 | 13 % | 2 | 5 % | 72 % | | |
| 13 | 13 | 78 | 17 | 22 % | 5 | 6 % | 65 % | 8 | 10 % | 3 | 3 % | 74 % | | |
| 13 | 16 | 120 | 22 | 18 % | 6 | 5 % | 60 % | 9 | 7 % | 3 | 3 % | 72 % | | |
| 13 | 19 | 171 | 26 | 16 % | 7 | 4 % | 58 % | 8 | 5 % | 3 | 2 % | 73 % | | |
| 16 | 7 | 21 | 6 | 27 % | 1 | 5 % | 75 % | 3 | 14 % | 1 | 3 % | 74 % | | |
| 16 | 10 | 45 | 14 | 31 % | 2 | 4 % | 74 % | 6 | 13 % | 1 | 2 % | 74 % | | |
| 16 | 13 | 78 | 24 | 31 % | 3 | 4 % | 71 % | 10 | 12 % | 1 | 1 % | 73 % | | |
| 16 | 16 | 120 | 35 | 29 % | 4 | 3 % | 62 % | 15 | 13 % | 1 | 1 % | 71 % | | |
| 16 | 19 | 171 | 48 | 28 % | 4 | 4 % | 61 % | 22 | 13 % | 1 | 1 % | 70 % | | |
| 19 | 7 | 21 | 6 | 29 % | 3 | 12 % | 72 % | 4 | 19 % | 2 | 9 % | 73 % | | |
| 19 | 10 | 45 | 12 | 27 % | 4 | 9 % | 68 % | 7 | 14 % | 3 | 6 % | 72 % | | |
| 19 | 13 | 78 | 19 | 24 % | 6 | 8 % | 65 % | 8 | 10 % | 3 | 4 % | 73 % | | |
| 19 | 16 | 120 | 24 | 20 % | 8 | 6 % | 59 % | 10 | 8 % | 3 | 3 % | 71 % | | |
| 19 | 19 | 171 | 31 | 18 % | 9 | 5 % | 54 % | 11 | 6 % | 4 | 2 % | 71 % | | |

[†] nb pairs : Median number of alternatives pairs that can be considered

* nb select : Median number and percentage of alternatives pairs selected by the algorithm

* nb solve : Median number and percentage of CPLEX executions

[‡] %stab : Median percentage of stable statements in the resulting outranking digraphs

of the initial outranking digraph has been reconstructed. We can easily see the decrease of the number of selected pairs and the number of resolution. Notice that the stability is even better than for a complete reconstruction, as the desired median-cut outranking relation is less constrained.

The time-saving can be easily explained by the reduced number of constraints and variables that are taken into account. In return, the resulting stability rate is lower than with a complete of preferential information, as there is no optimal enforcement of the stability for the outranking relation that are not explicitly ensured.

Furthermore, this experiment highlights the fact that the decision-maker should play a role in the selection of the alternatives, for instance by considering the exploitation of a temporary outranking relation and by expressing some strong agreements or disagreements, in order to add the associated preferential information and compute a new set of parameters. Otherwise, we need to select a great number of couples, although the decision-maker cannot be always able to express an accurate opinion, such that the process becomes much longer and may be subject to an increased number of possible conflicts.

6.2.2 Iterative elicitation of both criteria weights and thresholds

In the first experiment, section 6.1.2, we observed an exponential increase of the running time for the algorithms that elicit both weights and thresholds. Such times are prohibitive for a real-case applications. However, in previous section 6.2.1, we highlighted a possible complete capture in clearly acceptable time, based on a reduced set of preferential information, inducing a reduced number of necessary constraints and (boolean) variables.

In consequence, we are here assuming an iterative recapture of the criteria weights and constant discrimination thresholds. To do so, we proceed as follows:

Algorithm 2 Second experiment: Recapturing the weights and the thresholds

Input:

A : Alternatives set; F : criteria vector (with unknown thresholds); P : performance table; G : Median-cut outranking digraph.

Variables:

w : Elicited vector of weights;

$Ind, Pref$: Elicited vectors of indifference and preference thresholds values.

```

1:  $constraints \leftarrow \emptyset$ 
2:  $w \leftarrow w_1$  /* The equi-important vector of weights */
3:  $\forall i \in F, ind_i = 0$  and  $pref_i = 0.01$  /* Default values */
4: while not IS_RECAPTURING_DIGRAPH( $w, Ind, Pref, G$ ) do
5:    $(a_1, a_2) \leftarrow SELECT\_ALTERNATIVES\_COUPLE(A)$ 
6:    $constraints \leftarrow ADD\_PREFERENTIAL\_CONSTRAINTS(G(a_1, a_2), G(a_2, a_1))$ 
7:    $w \leftarrow STAB'_1\_SOLVE(A, F, P, constraints)$ 
8: return  $w$ 

```

Notice that, in step 3, every indifference thresholds are, at the beginning, set to 0 and every preference thresholds to 0.01, namely any difference between two performances is considered to be significant. This is only required for checking if the median-cut outranking digraph, obtained with equi-important weights and without thresholds, is already in accordance with the initial digraph: it is a very seldom situation when considering a complete recapture, but if we accept a small percentage of changes, the chance to see such a situation is increased (about a 4% with our benchmarks).

Again, we will stop the algorithm (step 4) after a complete recapture of the median-cut outranking digraph, or a recapture of at least a 95% of its arcs.

We would point out that some previously defined heuristics for selecting a pair of alternatives (in step 5) will not be possible in that case. Indeed, without knowing the thresholds, it does not make any sense to try to group the pairs according to their local comparisons (which are variable). Also, as the thresholds are variable, the WDV heuristic does not seem to be quite advisable. In consequence, we will consider only the random selection heuristic (RS). We will define further some additional methods to select the alternatives couples in cooperation with the decision-maker.

Considerations on the execution times

First, we study the execution times of the algorithm STAB'_1 , according to the defined sizes of problems. Indeed, contrary to the algorithm STAB_2 , whose iterations with partial preferential information never went beyond one second, we can still observe a significant number of instances that cannot be solved using STAB'_1 in short time. The experiment is run over the 300 predefined instances, for every sizes of problems, but bounding the maximal execution time to 30 minutes (1,800 seconds). Notice that, as soon as the algorithm runs 100 experiments, we decided to stop if less than 20 instances have been successful (*i.e.* if more than 80 instances reached the time limit).

In Table 6.5, we summarise the observed median execution times of a complete recapture of the median-cut outranking digraph with an incremental set of preferential information on the alternatives, as well as the median time for the longest iteration of each instance (it was almost always the last iteration, as the considered set of information was the largest), but also the percentage of instances whose the running time of the longest iteration was less than 20 seconds, 1 minute or 2 minutes. We assume the fact that an iteration that would necessitate more than 2 minutes is useless in a real time application; for those time-consuming instances, it should be interesting to implement some heuristic models in the future. Notice that the median times are computed considering only the instances that was not stopped after 30 minutes and they are displayed only if at least a 20% of the problems has not been stopped: the last column of Table 6.5 gives us the percentage of problems that run under the time limit, and also the associated number of instances that run under 30 minutes during the first experiment, in parentheses.

We can highlight two important facts: First, just like the algorithm STAB_2 that elicits the criteria weights only, we observe that the use of a reduced set of preferential informations, and hence a reduced set of constraints and binary variables, has a positive impact on the running time of each iteration, compare with an execution with a complete set of preferential information.

Second, when we compare the percentage of instances on which an optimal solution is found under 30 minutes, between this second experiment and the first one (a recapture with a complete set of preferential information), we do not observe any significant improvement. In fact, for some sizes of problems, for instance with 13 criteria and 19 alternatives, the iterative recapture is clearly worst performing (only a 3% versus a 27% in the first experiment). This can be explained by the fact that we run the algorithm several times, especially on large instances as it will be explained next in Table 6.6, such that the complete necessary time becomes prohibitive. Thus, as it has been pointed out before, it is advisable to further implement some efficient heuristic methods to provide satisfying parameters in an acceptable time. Also, in order to reduce the problems complexity, it would be expedient in a real case application to most possibly take into consideration some accurate preferential information on the criteria or on the thresholds, namely to restrict the space of admissible parameters: Besides, we should only consider the elicitation of the thresholds that cannot be provided with accuracy by the decision-maker or the

Table 6.5: Exp₂ (2/3) – STAB₁ iterative reconstruction, using the RS heuristic

| m | n | Median running time (s) | | % of instances s.t. all iterations | | | Problems solved in less than 30min | |
|----|----|-------------------------|---------|------------------------------------|--------|--------|------------------------------------|-----------|
| | | Complete | Max it. | < 20sec | < 1min | < 2min | | |
| 7 | 7 | 2.32 | 0.02 | 100 % | 100 % | 100 % | 100 % | (100 %) |
| 7 | 10 | 2.92 | 0.06 | 100 % | 100 % | 100 % | 100 % | (100 %) |
| 7 | 13 | 7.84 | 0.85 | 100 % | 100 % | 100 % | 100 % | (100 %) |
| 7 | 16 | 22.36 | 7.69 | 74 % | 98 % | 100 % | 100 % | (98 %) |
| 7 | 19 | 159.43 | 38.07 | 36 % | 61 % | 75 % | 77 % | (99 %) |
| 10 | 7 | 4.70 | 0.08 | 100 % | 100 % | 100 % | 100 % | (100 %) |
| 10 | 10 | 12.71 | 1.88 | 89 % | 98 % | 98 % | 100 % | (100 %) |
| 10 | 13 | 116.16 | 34.36 | 40 % | 65 % | 81 % | 96 % | (100 %) |
| 10 | 16 | 212.16 | 30.87 | 52 % | 59 % | 65 % | 89 % | (81 %) |
| 10 | 19 | — | — | — | — | — | 19 % | (58 %) |
| 13 | 7 | 3.42 | 0.08 | 100 % | 100 % | 100 % | 100 % | (100 %) |
| 13 | 10 | 12.20 | 1.86 | 89 % | 98 % | 100 % | 100 % | (100 %) |
| 13 | 13 | 94.64 | 28.75 | 44 % | 59 % | 68 % | 86 % | (91 %) |
| 13 | 16 | 727.11 | 379.20 | 17 % | 22 % | 28 % | 51 % | (33 %) |
| 13 | 19 | — | — | — | — | — | 3 % | (27 %) |
| 16 | 7 | 6.46 | 0.27 | 100 % | 100 % | 100 % | 100 % | (100 %) |
| 16 | 10 | 32.06 | 15.20 | 54 % | 73 % | 82 % | 98 % | (98 %) |
| 16 | 13 | 887.03 | 468.37 | 2 % | 5 % | 7 % | 22 % | (16 %) |
| 16 | 16 | — | — | — | — | — | 2 % | (6 %) |
| 19 | 7 | 4.98 | 0.14 | 100 % | 100 % | 100 % | 100 % | (100 %) |
| 19 | 10 | 13.82 | 1.19 | 66 % | 74 % | 78 % | 96 % | (97 %) |
| 19 | 13 | — | — | — | — | — | 3 % | (5 %) |

analyst. Indeed, some of them may be explicitly given by the construction of the associated criterion, such that they can be set up during the elicitation process, reducing the complexity of the mathematical model and so the running time. We will discuss about this assumption in the next chapter, section 7.1.1.

To ease the reading of this chapter, notice that we are not displaying the result for a 95% recapture of the median-cut outranking digraph. However, apart from the instances with 19 alternatives and 19 criteria, there was always more than a 20% of the problems of a given size that runs under 30 minutes. As we will detail next in Table 6.6, this is due to the fact that the number of iterations is significantly reduced, so as the longest execution running time (as less preferential information need to be enforced).

Finally, notice again that these problems have been randomly generated. Thus, their complexity is for sure higher than real case problems, where the criteria are often correlated, although they are independent.

Iterative recovering of the parameters

Table 6.6 summarises, for each size of problems, the median number of couples to be selected, as well as the effective number of resolutions. Let us notice that we did not give any statistical results when less than a 100 instances (for a given size of problem) achieved in less than 30 minutes. Indeed, as the associated standard deviations are quite high, it appears not relevant to give some statistical results

Table 6.6: Exp₂ (2/3) – STAB₁ iterative reconstruction, using the RS heuristic

| m | n | nbp [†] | 100% reconstruction | | | 95% reconstruction | | | | | | |
|----|----|------------------|---------------------|-----------|--------------------|--------------------|-----------|--------------------|------|----|------|------|
| | | | nb select* | nb solve* | %stab [‡] | nb select* | nb solve* | %stab [‡] | | | | |
| 7 | 7 | 21 | 12 | 57 % | 3 | 14 % | 87 % | 10 | 48 % | 3 | 11 % | 83 % |
| 7 | 10 | 45 | 26 | 58 % | 4 | 9 % | 87 % | 16 | 34 % | 3 | 7 % | 84 % |
| 7 | 13 | 78 | 48 | 62 % | 6 | 8 % | 84 % | 20 | 26 % | 4 | 5 % | 84 % |
| 7 | 16 | 120 | 82 | 68 % | 7 | 6 % | 76 % | 22 | 18 % | 5 | 4 % | 81 % |
| 7 | 19 | 171 | 133 | 78 % | 10 | 6 % | 78 % | 27 | 16 % | 4 | 3 % | 82 % |
| 10 | 7 | 21 | 15 | 48 % | 5 | 24 % | 91 % | 11 | 52 % | 4 | 17 % | 89 % |
| 10 | 10 | 45 | 37 | 82 % | 8 | 18 % | 90 % | 20 | 43 % | 6 | 12 % | 89 % |
| 10 | 13 | 78 | 67 | 86 % | 11 | 14 % | 84 % | 32 | 40 % | 8 | 10 % | 90 % |
| 10 | 16 | 120 | 103 | 86 % | 14 | 12 % | 85 % | 29 | 24 % | 8 | 7 % | 89 % |
| 10 | 19 | 171 | — | — | — | — | — | 39 | 23 % | 9 | 6 % | 88 % |
| 13 | 7 | 21 | 15 | 48 % | 4 | 19 % | 83 % | 12 | 55 % | 4 | 17 % | 80 % |
| 13 | 10 | 45 | 38 | 84 % | 7 | 16 % | 86 % | 23 | 50 % | 6 | 12 % | 85 % |
| 13 | 13 | 78 | 68 | 87 % | 12 | 15 % | 82 % | 36 | 46 % | 7 | 9 % | 85 % |
| 13 | 16 | 120 | 108 | 90 % | 15 | 13 % | 81 % | 36 | 30 % | 9 | 7 % | 85 % |
| 13 | 19 | 171 | — | — | — | — | — | 46 | 27 % | 8 | 5 % | 85 % |
| 16 | 7 | 21 | 17 | 81 % | 6 | 29 % | 88 % | 13 | 59 % | 5 | 20 % | 85 % |
| 16 | 10 | 45 | 41 | 91 % | 10 | 22 % | 91 % | 24 | 53 % | 8 | 16 % | 88 % |
| 16 | 13 | 78 | — | — | — | — | — | 40 | 52 % | 8 | 11 % | 89 % |
| 16 | 16 | 120 | — | — | — | — | — | 30 | 25 % | 8 | 7 % | 86 % |
| 16 | 19 | 171 | — | — | — | — | — | 64 | 38 % | 11 | 7 % | 93 % |
| 19 | 7 | 21 | 17 | 81 % | 5 | 24 % | 84 % | 14 | 63 % | 5 | 20 % | 82 % |
| 19 | 10 | 45 | 41 | 91 % | 9 | 20 % | 86 % | 25 | 55 % | 7 | 16 % | 85 % |
| 19 | 13 | 78 | — | — | — | — | — | 32 | 41 % | 8 | 10 % | 84 % |
| 19 | 16 | 120 | — | — | — | — | — | 53 | 45 % | 11 | 9 % | 82 % |

[†] nbp : Median number of alternatives pairs that can be considered

* nb select : Median number and percentage of alternatives pairs selected by the algorithm

* nb solve : Median number and percentage of CPLEX executions

[‡] %stab : Median percentage of stable statements in the resulting outranking digraphs

based on the execution of less than 100 instances. Left part of the table presents the result of a complete recapture, and right part present a 95% recapture of the median-cut outranking digraph.

On the basis of the results, we may formulate some points of criticism about the efficiency of the algorithm and envision some necessary considerations for a practical use.

Indeed, we observe that the number of selected pairs of alternatives quickly becomes prohibitive when recapturing the complete median-cut outranking relation. This is also noticeable for a 95% recapture, as it does not seem to be reasonable to question the decision-maker on 40 or more couples of alternatives, especially when dealing with a novice decision-maker that needs a longer time to assert his preferences. That can be explained by the complete liberty of the parameters all along the iterations. Indeed, no bounds or constraints are defined on the parameters: every preorders are possible and the thresholds values are not restricted. When we are confronted to a real decision-maker, it seems to be more advisable to narrow this liberty down with some constraints that are clearly valid to the latter's point of view: For instance, he can express the fact that some criteria are for sure more important

than some other ones, but also that there is a clear preference of indifference between some alternatives performances. This will be detailed in the next chapter.

Finally, there is again an important difference between the number of selected couples of alternatives and the number of necessary elicitation. Being able to throw apart some couples from the questioning, as they are already warranted by the given preferential information, will help focusing the questioning on appropriate couples, namely those that needs to change the current parameters, and thus will try to best avoid asking for obvious preferences. Once again, without the setting of some highly validated constraints on the parameters, it is a difficult and very time-consuming task. For instance, we may consider to test the fact that an outranking statement can be positive or negative, according to the already expressed preferential information, otherwise it is necessary and should not be questioned. That means to run in background a fair number of iterations, which is time-consuming and not completely satisfactory. Notice that, in the next chapter, we will highlight how the concept of stability allows to reduce the set of outranking statements to be considered for the questioning, achieving to provide a robust recommendation in a satisfactory time.

6.3 Impact of the stability constraints on the preference modeling

In this section, we intend to deepen the impact study of taking into consideration the stability concept on the parameters settings, again in an empirical manner. We have already motivated its use, for direct or indirect elicitation, and detailed its advantages; to be exhaustive in the study, we should also understand and be able to measure the possible biases resulting from the use of the stability constraints on the elicited parameters. As enforcing the stability is not an explicit request from the decision-maker, it might have a not inconsiderable impact on the parameters, such that it would be necessary to validate them explicitly with him in order not to distort his point of view.

As a first step, we would like to compare the preorders, obtained by the different algorithms in the first experiment, to measure in a statistical manner the resulting distortion (or misrepresentation) that can be ascribed to the use of some stability constraints. We show, in section 6.3.1 that these constraints has a reduced impact on the preorders (with a significant impact on the stability), especially on large instances.

Then, in section 6.3.2, we try to measure the impact on the setting of the thresholds, to see if the algorithm $STAB'_1$ and $STAB'_2$, which elicit simultaneously both criteria weights and thresholds, have some specific behaviors that we could highlight (for instance, if they use to maximise or minimise the thresholds).

Finally, notice that it is a preliminary study of the impact of reinforcing the stability on the elicited parameters. Some further studies, especially a theoretical study of this impact, could be considered for future works.

Table 6.7: Exp₁ – Median Kendall values

| m | n | Median Kendall values between w^i and | | | | | |
|----|----|---|----------|----------|------------|-----------|-----------|
| | | w^{con} | w^{s1} | w^{s2} | w'^{con} | w'^{s1} | w'^{s2} |
| 7 | 7 | 0.59 | 0.60 | 0.59 | 0.65 | 0.68 | 0.69 |
| 7 | 10 | 0.75 | 0.75 | 0.74 | 0.79 | 0.79 | 0.79 |
| 7 | 13 | 0.83 | 0.83 | 0.82 | 0.84 | 0.82 | 0.82 |
| 7 | 16 | 0.87 | 0.88 | 0.87 | 0.91 | 0.92 | 0.89 |
| 7 | 19 | 0.89 | 0.89 | 0.89 | 0.91 | 0.91 | 0.89 |
| 10 | 7 | 0.43 | 0.42 | 0.44 | 0.45 | 0.45 | 0.44 |
| 10 | 10 | 0.65 | 0.64 | 0.65 | 0.66 | 0.69 | 0.69 |
| 10 | 13 | 0.76 | 0.75 | 0.75 | 0.77 | 0.81 | 0.81 |
| 10 | 16 | 0.84 | 0.83 | 0.83 | 0.82 | 0.81 | — |
| 10 | 19 | 0.90 | 0.89 | 0.89 | 0.92 | 0.90 | — |
| 13 | 7 | 0.34 | 0.36 | 0.37 | 0.43 | 0.40 | 0.38 |
| 13 | 10 | 0.54 | 0.55 | 0.55 | 0.55 | 0.55 | 0.55 |
| 13 | 13 | 0.70 | 0.68 | 0.68 | 0.74 | 0.74 | 0.74 |
| 13 | 16 | 0.80 | 0.80 | 0.79 | 0.81 | 0.79 | — |
| 13 | 19 | 0.85 | 0.85 | 0.84 | 0.88 | 0.86 | — |
| 16 | 7 | 0.30 | 0.32 | 0.32 | 0.32 | 0.40 | 0.41 |
| 16 | 10 | 0.45 | 0.45 | 0.46 | 0.50 | 0.49 | 0.51 |
| 16 | 13 | 0.63 | 0.64 | 0.64 | 0.65 | 0.56 | — |
| 16 | 16 | 0.74 | 0.71 | 0.72 | 0.80 | 0.75 | — |
| 16 | 19 | 0.82 | 0.82 | 0.81 | 0.85 | — | — |
| 19 | 7 | 0.25 | 0.29 | 0.27 | 0.27 | 0.29 | 0.29 |
| 19 | 10 | 0.43 | 0.41 | 0.42 | 0.45 | 0.43 | 0.45 |
| 19 | 13 | 0.55 | 0.54 | 0.55 | 0.58 | — | — |
| 19 | 16 | 0.68 | 0.67 | 0.67 | 0.71 | — | — |
| 19 | 19 | 0.77 | 0.79 | 0.78 | 0.76 | — | — |

6.3.1 Impact of the stability constraints on the weights preorder

Assuming an initially generated vector of weights and the vectors obtained in the first experiment with ACON, STAB₁ and STAB₂ when recapturing the associated complete median-cut outranking relation, we measure the correlation between the associated preorders, using the Kendall τ coefficient [Ken38] to highlight the impact of taking into account stability constraints. Let us define w^i as the initial vector of weights and w^{con} (resp. w^{s1} or w^{s2}) the one obtained by running ACON (resp. STAB₁ or STAB₂). The median results are detailed on left part of Table 6.7.

On the left part of Table 6.7, we can easily notice the positive correlation between the initial vector of criteria weights w^i and those obtained with the third algorithms, despite of the problem size. This tends to show that the algorithms have a reduced impact on the parameters, especially on large instance. In addition, the correlation value is almost the same, no matter the considered algorithm is, underlying the very limited impact resulting from the consideration of the stability concept.

In fact, the distortion is mainly due to the minimisation of the weights: Remind that the initial preorder is generated randomly, so there can be some discriminated weights that does not have any impact on the resulting median-cut outranking digraph. In consequence, minimising the sum of the weights reduces this unnecessary discrimination, but does not really distort the parameters.

The good correlations warrants that it is possible to recapture, with an acceptable degree of accuracy, the weights parameters by only considering some outranking statements as preferential information. In consequence, there is no particular need for a decision-maker to express local preferential information to reduce a potential bias. On the contrary, notice that the set of outranking statements to be considered is quite large, as shown in the second experiment, such that it seems advisable to take also into account every information on the comparison of some criteria importance that a decision-maker can strongly assert. Thus, in a real-case problem, we will present the elicited vector of weights to the decision-maker, such that he may validate or invalidate, when there is clear justification, the preorder.

Right part of Table 6.7 summarises the median Kendall's τ coefficient between the initial weights vectors w^i and the associated vectors w'^{con} , w'^{s1} and w'^{s2} , obtained with the algorithms ACON', STAB'₁ and STAB'₂ respectively. As the values are still quite high and very close to the ones obtained with the algorithms that elicit the criteria weights only, we can draw the same previous conclusions, namely the reduced impact of the stability constraints on the relative importance between the weights.

Finally, one may also want to compute the correlation between the initial weights vectors and the ones elicited via an iterative complete recapture (see the second experiment, section 6.2). As these vectors recapture the complete relation, the iterative versions of the algorithms work like a control algorithm, but with the stability enforcement of a reduced number of outranking statements only. Thus, the impact of the stability constraints should be reduced, compared with a recapture based on a complete set of preferential information. In any case, we computed the median Kendall's τ coefficient and the results confirmed this assumption. Indeed, the resulting table was very close to Table 6.7 such that, to avoid making this chapter more cumbersome, we will not display them.

6.3.2 Impact of the stability constraints on the preference discriminating thresholds

When we consider the simultaneous elicitation of the criteria weights and the discrimination thresholds, trying to maximise the stability, with respect to the elicited weights, may have an impact on the way the alternatives are compared locally. Indeed, we easily conceive that the algorithms make some choices on the elicited local concordances values for advantaging the stability. Roughly speaking, when trying to ensure the stability of a positive outranking statement, the algorithms may try, when possible, to favor the positive local concordances, as the stability will be "easier" to validate.

A significant difference between these values and the initial ones (*i.e.* the ones obtained in the initially generated problems) could indicate the definition of parameters that are compatible with the current problem, but that may be not generally applicable, for instance if we want to consider a broader set of alternatives. On the contrary, if these parameters are very close to the initial ones, it will be then possi-

6.3. Impact of the stability constraints on the preference modeling 123

ble to generalise their use for the automatic validation of the outranking statements between some additional alternatives.

Thus, we intend to observe, in an empirical manner, this impact by comparing the initial local concordance values to the ones obtained via the algorithms ACON', STAB'1, STAB'2. It helps in a better understanding of the algorithms behavior and gives some useful hints for using them cleverly in a real-case. Notice that we are not comparing the values of the thresholds directly, as similar comparisons may result from different values, but their impact on the local concordance values.

For this purpose, we define $\%ID(A, B)$ as the percentage of local concordance values that are identical between the two considered sets. A high percentage denotes a high degree of similar local comparisons. On the contrary, a low percentage shows a high dissimilarity in the way the alternatives are compared locally.

Let $S_i^1(x, y)$ (resp. $S_i^2(x, y)$) denote the value of the local concordance between alternative x and y elicited by the first (resp. second) algorithm to be considered (or also the initial vector of local concordance values). It follows:

$$ID(S^1, S^2, x, y, i) = \begin{cases} 1 & \text{if } S_i^1(x, y) = S_i^2(x, y); \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the index is computed as follows:

$$\%ID(S^1, S^2) = \frac{\sum_{(x,y) \in A^2} \sum_{i \in F} ID(S^1, S^2, x, y, i)}{n \times n \times m}$$

Table 6.8 summarises the median values for this index, as well as the maximal associated standard deviations. Notice that this index is computed for every instances on which a set of weights and thresholds has been found in the first experiment. Also, remind that for some sizes of problems, we only run 100 instances but, as we observed some really low standard deviations, we can consider the statistical values as significant.

The strong correlation on the local concordance values we observe, between the initial generated instances and the corresponding results from the different algorithms, shows that a complete set of preferential information on the way the alternatives are compared globally is enough to recapture the local preferences with accuracy. Contrary to the sorting algorithms, that only consider as input the global comparisons from the alternatives to the profiles, there is no particular need in validating the elicited local comparisons. Notice that the highest standard deviations (about a 4%) are associated with instances having reduced sets of alternatives and criteria. Thus, assuming that a decision-maker expressed a complete and accurate global information, we can ensure that the local concordance will also be elicited accurately, in an implicit manner.

Table 6.8: Median values for %ID

| Between the initial data and ACON' | | | | | | Between the initial data and STAB' ₁ | | | | | |
|------------------------------------|----------------------------|------|------|------|------|---|----------------------------|------|------|------|------|
| maximal standard deviation: 0.01 | | | | | | maximal standard deviation: 0.02 | | | | | |
| m | Number of alternatives n | | | | | m | Number of alternatives n | | | | |
| | 7 | 10 | 13 | 16 | 19 | | 7 | 10 | 13 | 16 | 19 |
| 7 | 0.98 | 0.98 | 0.98 | 0.98 | 0.99 | 7 | 0.99 | 0.98 | 0.99 | 0.99 | 0.99 |
| 10 | 0.99 | 0.98 | 0.98 | 0.99 | 0.98 | 10 | 0.98 | 0.98 | 0.99 | 0.99 | 1.00 |
| 13 | 0.99 | 0.98 | 0.98 | 0.99 | 0.99 | 13 | 0.98 | 0.99 | 0.99 | 0.99 | 1.00 |
| 16 | 0.99 | 0.98 | 0.98 | 0.99 | 0.99 | 16 | 0.98 | 0.99 | 0.99 | 0.99 | — |
| 19 | 0.99 | 0.98 | 0.98 | 0.99 | 0.99 | 19 | 0.98 | 0.99 | — | — | — |

| Between the initial data and STAB' ₂ | | | | | | Between ACON' and STAB' ₁ | | | | | |
|---|----------------------------|------|------|------|------|--------------------------------------|----------------------------|------|------|------|------|
| maximal standard deviation: 0.04 | | | | | | maximal standard deviation: 0.01 | | | | | |
| m | Number of alternatives n | | | | | m | Number of alternatives n | | | | |
| | 7 | 10 | 13 | 16 | 19 | | 7 | 10 | 13 | 16 | 19 |
| 7 | 0.94 | 0.95 | 0.99 | 1.00 | 1.00 | 7 | 0.98 | 0.98 | 0.98 | 0.99 | 0.99 |
| 10 | 0.94 | 0.98 | 0.99 | — | — | 10 | 0.98 | 0.98 | 0.99 | 0.99 | 0.99 |
| 13 | 0.95 | 0.99 | 0.99 | — | — | 13 | 0.98 | 0.98 | 0.99 | 0.99 | 0.99 |
| 16 | 0.96 | 0.99 | — | — | — | 16 | 0.98 | 0.98 | 0.99 | 0.99 | — |
| 19 | 0.96 | 0.99 | — | — | — | 19 | 0.98 | 0.98 | — | — | — |

| Between ACON' and STAB' ₂ | | | | | | Between STAB' ₁ and STAB' ₂ | | | | | |
|--------------------------------------|----------------------------|------|------|------|------|---|----------------------------|------|------|------|------|
| maximal standard deviation: 0.04 | | | | | | maximal standard deviation: 0.04 | | | | | |
| m | Number of alternatives n | | | | | m | Number of alternatives n | | | | |
| | 7 | 10 | 13 | 16 | 19 | | 7 | 10 | 13 | 16 | 19 |
| 7 | 0.95 | 0.95 | 0.98 | 0.99 | 0.99 | 7 | 0.95 | 0.96 | 0.99 | 1.00 | 1.00 |
| 10 | 0.94 | 0.98 | 0.99 | — | — | 10 | 0.95 | 0.98 | 0.99 | — | — |
| 13 | 0.95 | 0.99 | 0.98 | — | — | 13 | 0.95 | 0.99 | 0.99 | — | — |
| 16 | 0.96 | 0.99 | — | — | — | 16 | 0.96 | 0.99 | — | — | — |
| 19 | 0.96 | 0.99 | — | — | — | 19 | 0.96 | 0.99 | — | — | — |

As a matter of course, one would like to compare the correlation between the local concordances when considering a reduced set of preferential information (see the iterative recapture during the second experiment, section 6.2.2). Indeed, considering a reduced subset of preferential information that recapture the complete median-cut outranking relation, or at least a 95% of the outranking statements, do the elicited local concordances highly correlated to the initial ones?

In Table 6.9, we then compute the median index %ID between the initial instances and the associated results from the second experiment with the algorithm STAB'₁, for the complete recapture (to the left) or the 95% recapture (to the right). Again, we can observe a very strong correlation between the two sets of local concordance values, inducing the same conclusions as previously. We may add the fact that even for a 95% recapture, on which there can be some global outranking statements that are modified, the local concordance is elicited with a high degree of accuracy.

Finally, these results emphasise the fact that taking into account the concept of stability has a very limited impact on the way the alternatives are compared locally. In consequence, during a real-case application with a decision-maker, there is no particular need in validating these local concordance values. Thus, we will

6.3. Impact of the stability constraints on the preference modeling 125

Table 6.9: Median values for %ID (iterative recapture)

| Between the initial data and $STAB'_1$ Complete recapture maximal standard deviation: 0.02 | | | | | | Between the initial data and $STAB'_1$ 95% recapture maximal standard deviation: 0.04 | | | | | |
|--|----------------------------|------|------|------|------|---|----------------------------|------|------|------|------|
| m | Number of alternatives n | | | | | m | Number of alternatives n | | | | |
| | 7 | 10 | 13 | 16 | 19 | | 7 | 10 | 13 | 16 | 19 |
| 7 | 1.00 | 1.00 | 0.99 | 0.99 | 0.99 | 7 | 1.00 | 1.00 | 0.99 | 0.99 | 0.99 |
| 10 | 0.99 | 0.99 | 0.99 | 0.99 | — | 10 | 1.00 | 0.98 | 0.99 | 0.99 | 0.98 |
| 13 | 0.99 | 0.98 | 0.99 | 0.99 | — | 13 | 0.99 | 0.99 | 0.98 | 0.98 | 0.99 |
| 16 | 0.98 | 0.99 | — | — | — | 16 | 0.99 | 0.98 | 0.99 | 0.99 | 0.99 |
| 19 | 0.98 | 0.98 | — | — | — | 19 | 0.99 | 0.99 | 0.99 | 0.99 | — |

only provide some constraints on the thresholds values when the decision-maker expresses a clear and accurate opinion on the way two alternatives are compared locally.

REWAT: A progressive process for a robust elicitation of the weights and thresholds

“Celui qui a besoin d’un protocole n’ira jamais loin ; Les génies lisent peu, pratiquent beaucoup et se font d’eux-mêmes.”¹

[Denis Diderot]

Contents

| | | |
|------------|--|------------|
| 7.1 | Designing a robust elicitation protocol | 128 |
| 7.1.1 | Stage I: Initialising the outranking preference model | 129 |
| 7.1.2 | Stage II: Validating the criteria weights preorder | 131 |
| 7.1.3 | Stage III: Tuning the numerical values of the criteria weights | 140 |
| 7.2 | Tools for supporting a robust elicitation | 143 |
| 7.2.1 | Dynamic pairwise performance comparison table | 144 |
| 7.2.2 | Display of the elicited weights preorder | 145 |

Abstract

In this chapter, we explain how to set a robust preference elicitation protocol up, relying on the previously defined mathematical algorithms, that recovers a complete median-cut outranking digraph from decision-maker’s preferential information. In addition, we design some tools for supporting the decision-maker in a better understanding of the provisional parameters and their impact, as well as an easier expression of his preferences.

1. “The one who needs a protocol will never go farther; Geniuses read few, train a lot and oneself made.” I disagree.

Introduction

The conception of a robust indirect elicitation protocol, that relies on the stability concept to capture the weights of the considered criteria of an outranking method, can be broken down in three stages: First, the decision-maker defines an initial basic preorder with the analyst. Then, this one is refined by the incremental add of preferential information, until a complete and clear validation of the final preorder. Finally, the remaining unstable outranking statements are discussed, in order to find precisely the weights of the criteria.

In this chapter, we explain how to set such a protocol up, relying on the previously defined mathematical algorithms, when considering the complete construction of the median-cut outranking relation, not considering any particular problem typology. For this expressed purpose, we detail every necessary steps of the protocol. In addition, as such an approach is more specifically designed for a novice decision-maker, we design in section 7.2 some tools for supporting him in a better understanding of the provisional parameters and their impact, as well as an easier expression of his preferences.

7.1 Designing a robust elicitation protocol

In this section, we detail each step of the protocol, allowing to elicit a compatible set of parameters, namely a vector of weights and some discrimination thresholds, from given preferential information by the decision-maker, in order to recapture a complete outranking relation which is the closest possible translation of his preferences.

Notice that we are not considering here any particular problem typology, as we only focus on the construction of the outranking relation and not its exploitation. Such an approach can be considered when dealing with the best choice or the ranking problem, but also the sorting problem, provided that the profiles of the categories are given in the set of alternatives. However, we will detail in the next chapter a protocol for this last specific problem typology, when the categories are given but the profiles are unknown.

First, we detail some necessary preliminary steps, allowing to construct the problem, verify the decision-maker's accordance to the outranking principles and define an initial preorder. Then, we develop how this preorder is refined and validated, as well as the discrimination thresholds. Finally, we explain how the remaining unstable relations are validated with the decision-maker to provide a complete and highly reliable outranking relation.

We shall state that we are not going into details on the steps for the construction of a elicitation protocol, but only explicit how the stability concept is taken into account during the whole protocol.

7.1.1 Stage I: Initialising the outranking preference model

We are talking about the steps related to the elicitation protocol that are, “ideally”, completed once before iterating the process. First of all, we need to collect and to evaluate the alternatives. Then, we need to validate the principle of the method, as well as the chosen problem typology, and explain the stability concept to the decision-maker. Finally, we construct a first basic preorder that will be refined during the protocol.

Collecting and evaluating the alternatives

This important step, which conditions the whole following decision aid method, has been already studied in depth by [BMP⁺06].

Notice that there is a clear definition of the problem the decision-maker has to deal with, as it conditions the collect of the alternatives, as well as the construction of the coherent family of criteria for their evaluations.

Also notice that, according to the considered problem typology, the set of alternatives is more or less exhaustive. For instance, if we are searching for the best matching alternative (or a small set of the best ones), we only need to take into account the alternatives judged by the decision-maker as potentially the best ones. Of course, this also requires to consider a wider set, to be able to define which alternative could be pointed out as a good choice or not.

At the end of this step, we are assuming here that we know A as the set of alternatives to be considered, the set F of criteria on which the decision-maker will build his preferences exclusively and the evaluations of the alternatives on the criteria. In addition, we assume that the decision-maker knows the fact that the collected alternatives are the only opportunities: Looking at their evaluations, he becomes aware about the compromises he may have to do.

Validating the stable outranking approach

No surprise, we need to ensure that the decision-maker’s way of comparing the alternatives is in accordance with the outranking principles, more specifically with the multicriteria aggregation procedure described in section 3.1. Indeed, we must be careful in always verifying the compliance of the expressed opinions with these principles, especially not trying to compensate between the criteria, risking to obtain some incoherent results at the end.

Also, we have to present the concepts of stability and robustness to the decision-maker, in a way someone who is not familiar with the decision aid domain can apprehend. For this purpose, we first explain him the three different stages of the protocol. In any case, we explain that the purpose of the stability and the robustness is to take into consideration only when preferential information are accurate, even if imprecise, and to deduce only outranking statements with a strong reliability (as

they are stable with respect to the parameters, or given accurately by the decision-maker), in order to reduce the algorithmic choices to their lowest level.

Dealing with the discrimination thresholds

During some preliminary tests for attempting to define the criteria thresholds with a novice decision-maker, it appears that most of the time he was not very confident in doing such a task. In fact, dealing with him in a constructive approach of his overall preferences, the expressed discrimination thresholds on the criteria was most of the time given without accuracy and preciseness. In these conditions, even a small set of preferential information on the comparison of some alternatives often resulted in an impossible solving.

However, after few comparisons of some alternatives he was more confident in expressing the fact that a difference between two evaluations, on a same criterion, was either highly relevant, indicating a clear local preference of one alternative on the other, or highly not relevant, indicating a clear local indifference between both alternatives.

Also, we noticed a behavioral difference when dealing with ordinal criteria which are associated with a scale with a reduced number of levels. Indeed, such a scale is constructed so that the difference between two adjacent levels has a real significance. In that case, a decision-maker considers that the indifference threshold is set to 0 and the preference one to 1.

In consequence, for the criteria associated with an ordinal and reduced scale, the thresholds are given and fixed during the whole process. Of course, we ensure during the preliminary stage that the decision-maker clearly understands the semantics of each level and is in accordance with this principle. If a scale, according to the decision-maker, seems to have too many levels with no relevant difference between them, we then modify the scale.

Finally, on the other criteria, the thresholds are defined as some variables we have to elicit during the protocol. Notice that the kind of each threshold has to be known, namely constant, proportional or general thresholds, in order to create the correct constraints in the mathematical models. As it is hardly possible to directly ask the decision-maker about this information, they should be given by an analyst, or elicited via a questioning of the decision-maker about his preferences on relatively equal differences in the evaluation of alternatives along the whole scale. We already assumed that the second case is not advisable under the hypothesis of a novice decision-maker. In the first case, the analyst selects the most appropriate kind of thresholds, based on his experiment about the criterion: for instance, if he used to see decision-makers considering proportional thresholds on quantitative criterion with a relatively large scale (for instance, a criterion *price* for the evaluation of expensive alternatives), whereas on a reduced scale they consider constant thresholds, he can apply these principles on the criteria.

On the basis above mentioned, we decided to proceed as follows: the thresholds

associated with an ordinal and large scale, or a cardinal scale with a reduced range of values, will be considered as constant; on a cardinal scale with a broaden range of values, they will be considered as proportional. Finally, if we cannot solve the mathematical model, before trying to consider a contradiction between the preference information on couples of alternatives, we should check the coherence of the thresholds and modify the ones that may be incoherent.

Definition of a first basic preorder to be discriminated

In this step, the decision-maker is asked to sort the whole set of criteria into a very small set of ordered rough classes, according to their importance in his opinion. Such a partitioning of the criteria must not be too precise, as the decision-maker has to express some *strongly accurate* information. For instance, we consider the sorting of the criteria into three classes: the “very important” criteria, the “important” ones and the “less important” ones. We may also consider a partitioning into four classes, considering for instance an additional class for the “most important” criteria, but we recommend not to be more precise when dealing with novice decision-makers, especially as this step takes place in the very beginning of the protocol, when they are not fully conscious about their preferences. Notice that we always consider in our tests a three-classes partitioning, but the decision-maker will have the possibility to use only two classes (for instance to separate the important criteria to the less important ones), or also to specify a fourth class for the criteria that are clearly the most important ones.

This initial partial preorder must be considered as unchanging during the whole protocol and shall be only refined, namely by adding some discrimination between criteria in the same rough importance class, but without inverting the relation between two criteria in two different classes. For instance, any criterion that is judged as “very important” will always be associated with an importance weight strictly higher than any criterion considered as “important” or “less important”. Throughout the sequel of the process, if the decision-maker wants to change once this preorder according to a high reconsideration of his opinion, we should stop the process and restart it again.

In the sequel of the chapter, we will call this initial preorder \succsim_{w_0} .

7.1.2 Stage II: Validating the criteria weights preorder

The purpose of this second stage is to refine the given preorder \succsim_{w_0} , until the validation by the decision-maker of a more discriminated preorder \succsim_w . The main idea is to deduce a sufficient discrimination on the preorder, from a possibly reduced set of preference information on the alternatives. Notice that this stage does not necessitate to take into consideration the problem typology and the exploitation of the outranking digraph.

In consequence, at this stage and considering no particular problem typology, the algorithm is the following:

Algorithm 3 Second stage: Refining a given initial preorder

Input:

A : Set of Alternatives; F : set of criteria; P : Performance table;
 \geq_{w_0} : Initial preorder.

Variables:

w : Elicited vector of weights;
 $thresholds$: Some elicited ind. and pref. thresholds. /* Some are fixed */

- 1: $constraints \leftarrow \emptyset$
- 2: $constraints \leftarrow \text{ADD_CONSTRAINTS_FROM_PREORDER}(\geq_{w_0})$
- 3: **repeat**
- 4: $(a_1, a_2) \leftarrow \text{SELECT_ALTERNATIVES_COUPLE}(A)$
- 5: $constraints \leftarrow \text{ADD_PREFERENTIAL_CONSTRAINTS_ON}(a_1, a_2)$
- 6: $\{w, thresholds\} \leftarrow \text{STAB}'_1\text{-SOLVE}(A, F, P, constraints)$
- 7: **if** $\{w, thresholds\} \equiv \emptyset$ **then**
- 8: $\text{RESOLVING_CONFLICTS}(constraints)$
- 9: **else**
- 10: $\text{PRESENTING_PREORDER}(\geq_w)$
- 11: **until** \geq_w is validated
- 12: $\text{VALIDATE_THRESHOLDS}(thresholds)$
- 13: **return** $\{\geq_w, thresholds\}$

We detail here the not critical steps that only need a brief explanation. For instance, adding some constraints from the preorder \geq_{w_0} (step 2) means that we are enforcing some inequalities between the criteria in order to always elicit a vector of weights that is fulfilling the decision-maker's preferences, at each iteration: any criterion that is judged as "less important" in the decision will be forced to be associated with a weight strictly lower than those of every "important" criteria, and so on and so forth. Of course, two criteria in the same importance class of \geq_{w_0} will not be forced to have the same value, as we are trying to discriminate them when it is necessary.

Notice that we consider the use of STAB'_2 for the elicitation of the parameters (step 6), as it is the most appropriate MILP in that case, considering a small set of preferential information and the best in terms of stability. However, STAB'_1 could be considered, for instance if the resolution with STAB'_2 was too long (because of a too wide problem, or a too consequent set of preferential information).

In the following sub-sections, we detail the critical steps, namely the selection of the alternatives (step 4), the integration of additional preferential information (step 5), the validation of the parameters (the new preorder in step 11 and the discrimination thresholds in step 12) and the way we can solve some possible conflicts (step 8).

Selecting of a couple of alternatives

During that stage, we consider a initial preorder, given preliminarily by the decision-maker, to be refined by the incremental addition of preferential information. By definition, we know that an independent or extensibly stable outranking cannot

be affected by increasing the discrimination of the vector of weights. Thus, in a process where the discrimination thresholds are validated at the beginning, it should be logical not to question the validation or invalidation of such statements, as they are implicitly given by the preorder. However, such a consideration is not always possible when not every discrimination thresholds are given nor validated. In addition, as some thresholds are variable and may change from one iteration to another, these properties are not warranted during the whole process.

With a similar argumentation, we know that an unstable outranking cannot become stable when increasing the discrimination of a given preorder. As we first try to refine the preorder, it should not be advisable to question on unstable outrankings. But again, such property cannot always been verified when some thresholds values are missing.

Nevertheless, the decision-maker may express gradually some highly reliable bounds on the values of the thresholds (see section 7.2 on the design of graphical tools). For instance, if he considers that the difference (the value or the percentage of the difference, regarding the kind of thresholds) between two evaluations is not relevant, any smaller difference has to be considered similarly. If he considers that a difference is significant, so that the worse-performing alternative is, for sure, not at least as good as the other one on the considered criterion, every higher differences will also be considered as significant. Consequently, the number of outranking statements on which we can verify the extensible stability property, the independency or the lack of stability, is increasing along the second stage, allowing not having to consider a fair number of outranking statements, has they are either explicitly warranted with respect to the initial preorder \succeq_{w_0} , or unstable and not interesting for refining the preorder.

In the light of these considerations, we proceed as follows: for every ordered pairs (x, y) of alternatives, we compute the most optimistic and the most pessimistic outranking statements, $S_o^w(x, y)$ and $S_p^w(x, y)$ respectively, on which every missing local concordance values are set to 1 for the optimistic statement and -1 for the pessimistic one, as explained in section 3.3.4, according to the current preorder \succeq_w and the bounds on the thresholds that are defined by the decision-maker, not the elicited current thresholds. If both optimistic and pessimistic outranking statements are positive (resp. negative) and are at least extensibly stable (*i.e.* extensibly stable or independent), with respect to the current preorder, there is no need in questioning the decision-maker about such a statement which is already validated (resp. invalidated), no matter the lack of preciseness on the weights and the thresholds is. Also, if both optimistic and pessimistic statements are unstable, the outranking will be unstable for every possible settings of the missing values and we may not consider the associated ordered pair of alternatives during this stage.

Finally, as the decision-maker will be asked to compare two alternatives x and y , he will be potentially able to express his preferences on both outrankings statements from x to y or from y to x . In consequence, it may be advisable to consider the stability of the couple of outranking statements, instead of the stability of independent outranking statements, when selecting a pair of alternatives. Thus, as a matter of

priority, we consider the selection of the alternatives such that both $S^w(x, y)$ and $S^w(y, x)$ are not extensively stable, nor unstable. If there is no such couples, we consider the selection of an ordered pair that are not clearly extensively stable or unstable.

Notice that if every outrankings are at least extensively stable or unstable, there is no need in refining more the preorder and we shall then go to the next stage.

Questioning the decision-maker and integrating his preferences

Let (x, y) be a selected couple of alternatives on which the decision-maker will be questioned, in terms of the validation or invalidation of the outranking statements between x and y .

Also notice that in conjunction with the expression of global preference information on the comparison of some alternatives, the decision-maker will have the opportunity to express his feeling on the way they are compared locally. This is not mandatory and the decision-maker should not be directly asked to express such information. However, it helps reinforcing some bounds on the thresholds, in order to concentrate the elicitation of some thresholds in best accordance with the decision-maker's mind. It will be made easier with the graphical tools we defined for questioning him on the comparison of the alternatives. We show in section 7.2.1 how it is done.

Notice that if the decision-maker expresses some preferential information on the thresholds, we test if the outranking statement, according to the current preorder, becomes extensively stable or unstable, by computing again the most optimistic and pessimistic statements. If it becomes extensively stable, we do not need to question the decision-maker about his preference, as it is already implicitly validated by the current preorder. If it becomes unstable, we decide to ask him about his preferences, even if this is not the scope of this second stage. Indeed, the decision-maker is already comparing the alternatives, it would be not "user-friendly" to stop questioning him now and presenting him the same couple a few iterations after, during the last stage.

Thus, the decision-maker is asked to express his preference on the fact that x is or not at least as good as y , using one of the followings answers:

1. "I am certain that x is at least as good as y ";
2. "I am certain that x is not at least as good as y ";
3. "I do not know" / "I am not sure".

In the last case, no constraint will be integrated to the mathematical model. However, in the first (resp. second) case, we add the defined constraints ensuring the elicitation of a vector of weights that enforces a positive (resp. negative) outranking statement from x to y , possibly in a stable manner. When necessary, *i.e.* when we need to modify the preorder which does not validate or invalidate the outranking as the decision-maker would like, or when the local information were incomplete, we run a mathematical elicitation of some new parameters.

If the decision-maker validates a clear positive outranking statement for the ordered pair (x, y) , we decide to question him also on the overall outranking relation of the opposite ordered pair (y, x) , in the same previously defined conditions (*i.e.* the outranking statement is not extensively stable, nor unstable), and with the same possible answers. The expressed information is integrated similarly in the mathematical model.

As an analyst, we must be very careful when the decision-maker claims that an alternative does not outrank another. Indeed, such a situation may ensue from a negative concordance situation, but also because of a counter-performance on one or more criteria. In the last case, such a statement should not be constrained for the elicitation of stable weights, as it is an independent outranking statement that can go against the associated concordance relation (on which we base the elicitation of the parameters). We then simply record that situation, in order to present it in the final median-cut outranking relation.

Finally, if the decision-maker expressed two positives outranking statement, namely x outranks y and y outranks x , it may be advisable to ask him about their relative credibility. Thus, we ask him: “*For your liking, does x is preferable to y ?*”. Remind that x preferable to y means that the statement “ *x is at least as good as y* ” is more credible than “ *y is at least as good as x* ”, namely $S^w(x, y) > S^w(y, x)$. Again, he will be pleased to answer according to the following propositions:

1. “I am sure that x is preferable to y ”;
2. “I am sure that y is preferable to x ”;
3. “I do not know” / “I am not sure”.

In the first two cases, we simply add a constraint on the relative credibility of the statements, that enforce the fact that $S^w(x, y) > S^w(y, x)$ or $S^w(x, y) < S^w(y, x)$, and try to validate the constraint in a stable manner. Then, we run the mathematical problem and compute some new parameters.

Presenting and validating the preorder

We are considering here only the validation of the preorder, not the precise vector of weights. For the ease of the discussion, let us consider an initial preorder to discuss about the possible modifications. Let $\geq_w = \{g_1, g_2, g_3, g_4\}$.

Also, notice that we can validate independently the modification of each importance class of the previously validated preorder, as the inequalities between the classes are some strong and not modifiable constraints.

By construction, a preorder can be envision as a set of comparisons between every pairs of weights. Indeed, for every criteria in the same importance class, their weights are equal; and, for any pairs of criteria that are not in the same importance class, it exist a strict inequality between their weights. As our objective is to discriminate more the preorder, the equality relations between the weights are never forced during the second stage, contrary to some inequality relations. Notice

that, at the very beginning of the second stage, as the initial preorder has been highly validated, every inequality relations are forced. We show in the sequel that it is not always the case.

Thus, when a modification on the initial preorder is resulting from the addition of new preferential information, this can be translated into the modification of some equality relations into a set of new inequality relations.

Example 18 Let $\succsim_{w_0} = \{g_1, g_2, g_3, g_4\}$ be the initially validated preorder and let assume $\succsim_w = \{g_1, g_2\} > \{g_3, g_4\}$ as the current preorder to validate or not. The modifications can be translated into four new strict inequality constraints between the associated weights, which are the following: $w_1 > w_3$, $w_1 > w_4$, $w_2 > w_3$ and $w_2 > w_4$.

The decision-maker is then asked to validate or not each new constraint. For instance, when considering the constraint “ $w_i > w_j$ ”, he may express his preference using one of the following possible answers:

1. “I am highly confident that criterion g_i is more important than g_j ”;
2. “I am highly confident that criterion g_i is *not* more important than g_j ”;
3. “I am not sure” / “I do not know”;

In the first case, the constraint is added to the set of preferential information, to ensure the compliance of every incoming elicited parameters to the decision-maker’s wishes.

In the second case, we need to modify the preorder. The decision-maker is then asked to precise his objection, either by claiming that g_i is at most as important as g_j , *i.e.* “ $w_i \leq w_j$ ”, or telling that g_i is clearly less important than g_j . The resulting constraint is then added to the set of preferential information. Such a consideration may impact the rest of the preorder, as the algorithm may have to refine the preorder in a different way to warrants all the already expressed preferential information. Thus, we run the mathematical problem, elicit some new parameters and start again the discussion on the modifications of the preorder.

Finally, in the last case, if he is not able to express an accurate validation or invalidation of the resulting relation between criteria g_i and g_j , we decided not to enforce this constraint for the incoming elicitation. As these constraints may be inverted in the next elicitation, we will have in the future to question him again on the comparison of these two criteria.

When every new inequality constraints have been discussed, two situations can arise: either the validation of some constraints remains open, or the whole set of new constraints has been validated, inducing the validation of the importance classes of the current preorder. In the second case, the decision-maker is asked to validate the complete preorder, assuming that the current discrimination is enough for him and that two criteria in the same importance class clearly have the same importance in the global decision. If he agrees, the preorder is validated and we check the validity

of the thresholds (see below section) before taking the next stage. If he is not certain about that last assumptions, we iterate again the protocol, selecting some new pairs of alternatives in order to refine more the preorder.

Finally, let us notice that graphical tool for easing the validation of the preorder modifications will be presented in section 7.2.2.

Validating the thresholds

As we have already said, a novice decision-maker is sometimes uncomfortable in precisely validating some discrimination thresholds. However, such a precise information is not always necessary. Indeed, defining such parameters help expressing a local preferential information, namely a local concordance value, in order to accurately express a global preferential information, namely an outranking statement. When the lack of information on the way the evaluations are compared locally has no impact on a resulting stable outranking statement, that is to say when both most optimistic and pessimistic outranking statements² agree on the stability of the outranking statement, there is no need in discussing about the missing local concordance values. Notice that if a veto situation occurs for some ordered pairs of alternatives, the resulting outranking statement is also warranted irrespective of some possibly missing local concordance values.

Also, as the previous discussion on the comparison of some alternatives have defined some accurate bounds on the thresholds values, the lack of information on the way the evaluations are compared locally is restrained.

In consequence, we compute both most optimistic and pessimistic outranking values, as well as their associated stability, for every ordered pairs of alternatives such that there are some missing local concordance values, taking into account the validated preorder and all the given bounds on the thresholds. Remind that a local concordance value is considered to be missing when the bounds on the associated threshold cannot warrant a clear local indifference or a clear local preference. Then, we look at the critical outranking statements, namely that are depending on the lack of information: As the preorder is validated and will not change, every situations that are at least stable (namely both pessimistic and optimistic outranking statement are at least stable) are warranted without the need to refine the thresholds. In return, the other situations are sensitive to this lack of information, such that we have to discuss about the missing values with the decision-maker.

Among the sensitive situations, some may have been validated by the decision-maker in the previous steps. Indeed, he could have expressed the fact that an alternative outranks another one, without explicitly determining every local concordance values. The validation of such outranking statements is then depending on the criteria weights, but also on the elicited discrimination thresholds. Consequently, we have to validate the sensitive elicited concordance values as a matter

2. As a reminder, they are obtained by setting the missing concordance values both to the value 1, for the most optimistic case and -1 for the most pessimistic one (see section 3.3.4).

of priority, as the preorder could be brought into question. Logically, the number of such situations is bounded by the number of expressed outranking statements. Notice that, during the different tests we have done to validate our approach, this number was around one or two outrankings statements only.

Thus, for the sensitive outranking statements on which the decision-maker already expressed preferential information, we present him again the associated couples of alternatives, their evaluations, as well as the local concordance values and we try to validate the values elicited by the algorithm that are not enforced by the accurate thresholds bounds. Three cases can occur: Either he clearly validates the local concordance values, he is not fully confident but relies on the algorithm, or he clearly disagrees the value. In the first two cases, the threshold bounds are updated to be compliant with the validated information. In the last case, the local concordance values, as well as the associated thresholds bounds, are modified to be compliant with the decision-maker's expectations and we check if the desired outranking statement is still warranted, with respect to the current vector of weights. If not, we run the algorithm ACON', enforcing the complete preorder and taking into consideration the updated thresholds bounds, to see if it exists a precise vector of weights and some precise discrimination thresholds that are compliant with the complete current preferential information. If the set of preferential information becomes inconsistent, we shall discuss with the decision-maker about modifying either the global outranking statement, or the desired local concordance values. As it is again out of the scope, and as no such a situation occurs during our tests, we will not detail it hereafter, but this will have to be considered in further work.

Upon successful completion of this task, we compute again the most optimistic and pessimistic outranking statements, taking into account the updated thresholds bounds. For each criterion where there are still some missing local concordance values that may have an impact on the global outranking statements, we present the couples of evaluations in an increasing order of their differences and ask the decision-maker if he considers that the lower evaluation is, or not, clearly at least as good as the higher one. If he does not have any clear opinion, the associated local situation will be set to 0 (balanced local concordance value). Notice that as soon as he considers a difference to be significant, we stop questioning him, as the remaining differences on this criterion must be considered similarly.

Notice that, once the thresholds on a criterion have been updated, such that it clarifies some undetermined local comparisons, some critical statements that were depending on more than one missing value can be now considered as stable, *i.e.* with the additional information collected during the current step, both optimistic and pessimistic statements are positive (resp. negative) and stable now. Thus, as the remaining missing values have no impact anymore, this freshly non-critical statements are throw apart of the threshold validation step in order to save time.

Finally, when everything is done, we start the third phase of the protocol.

Solving the possible conflicts

First, let us notice that the stability constraints are only integrated to the model in their relaxed forms. Thus, it will not be possible that the algorithm does not find any compatible parameters due to these considerations.

Before considering a real conflict on an inconsistent set of preferential information, we must consider an increase of the parameter M , which limit the maximal value of a weight, but also the ratios between the weights. Indeed, trying to ensure an unstable outranking statement necessitates a precise fixation of the weights.

One may argue that we should try to find a vector of real weights, as the possible ratios between the weights are unbounded, instead of searching for some integer weights. But, let us remember that considering integer weights (or more precisely rational weights) is motivated by two main arguments: the first one is a technical one, as we need to define discrete weights for enforcing an indirect elicitation of stable outranking statements; the second one is more philosophical, as we consider that having the need of a too subtle tuning, that a decision-maker cannot seriously validate and maybe not fully understand, is not serving the decision aid process. Indeed, if a decision-maker is asked (by some stakeholders for instance), to justify his opinions, it appears more complicated to argue on the need of such a preciseness. Using integer or rational weights ensure that the difference between every values can always be quantified.

Thus, if the algorithm cannot find a set of compatible parameters, we first run the control algorithm $ACON'$, setting parameter M up to twice the number of criteria: Considering that the weights are scaled after the elicitation between 0 and m , that means that a difference between two consecutive weights of half a unit is significant, instead of one unit, which can be considered as an enough preciseness.

If the increase of the parameter M is enough to solve the conflict, we then consider the highest elicited weight as the new value for M along the sequel of the protocol. We run again $STAB'_2$ and compute a new set of parameters.

Failing that, the conflict is resulting from an inconsistent expressed set of preferential information. However, this set can be divided into two kinds of information: some local considerations on the parameters and some global outranking statements. Either the conflict is resulting from the fact that the local considerations are not in accordance with the global ones, namely an inaccurate expression of some local preferences, or it is resulting from a incompatible set of outranking statements to be validated collectively. We assume that the decision-maker could have expressed some preferential information on the weights that are not included in the preorder (for instance, the fact that a coalition of weights is more important than another one), that are clearly validated (they are not taken into consideration if he his not certain of their validity) and, in the scope of our thesis, we will not consider a possible relaxation of these constraints.

We then proceed as follows: First, we relax the constraints on the bounds of the thresholds and try to find a set of parameters, using $ACON'$, satisfying the whole set

of desired outrankings. If the resolution is successful, we discuss with the decision-maker about the bounds on the thresholds that are contradicting the given local information. For instance, if he expressed the fact that the indifference threshold q_i must be higher than a value v and the mathematical problem returns a value for q_i that is lower than v , we ask him if he is able to modify this information, considering it was not enough accurate, or not. The constraints he is ready to throw apart are then deleted and the ones he really believe in are added to the model and we recompute a new set of parameters. If the resolution is successful, we can continue the protocol.

In case we could not find any solution, without every constraints on the thresholds or only with the highly reliable constraints, we may consider searching for a maximal set of consistent outranking statements. As this is out of the scope of this thesis, one may refer to section 2.2.3 for a brief description and some related articles. Notice that during the tests, this situation only appeared once, but a solution has been found when the decision-maker became aware that one of the expressed outrankings was probably not accurate.

7.1.3 Stage III: Tuning the numerical values of the criteria weights

Assuming the validation of the preorder, as well as the criteria discrimination thresholds, we are now interested in verifying the validity of the remaining unstable statements, in order to construct a median-cut outranking digraph that is in best accordance with the decision-maker's mind.

This third stage runs as follows:

Algorithm 4 Third stage: Tuning the unstable outranking statements

Input:

A : Set of Alternatives; F : set of criteria; P : Performance table;

$\{\geq_w, thresholds\}$: From previous algorithm.

Variables:

w^* : Elicited vector of weights;

```

1:  $constraints \leftarrow \emptyset$ 
2:  $constraints \leftarrow ADD\_CONSTRAINTS\_ON\_WEIGHTS (\geq_w)$ 
3:  $constraints \leftarrow ADD\_UNSTABLE\_VALIDATED\_PREFERENTIAL\_CONSTRAINTS$ 
4: repeat
5:    $(a_1, a_2) \leftarrow SELECT\_UNSTABLE\_ALTERNATIVES\_COUPLE (A)$ 
6:   if the relation between  $a_1$  and  $a_2$  is not necessarily fixed then
7:      $constraints \leftarrow ADD\_PREFERENTIAL\_CONSTRAINTS\_ON (a_1, a_2)$ 
8:      $W^* \leftarrow A_{CON\_SOLVE} (A, F, P, thresholds, constraints)$ 
9:   until the decision-maker is happy enough
10: return  $W^*$ 

```

A careful reader may easily notice the similarity between the third stage algorithm and the second one. Indeed, the structure is quite similar: From an initial set of preferential information, we gradually increase this set by questioning the

decision-maker on some couples of alternatives and resolve each successive mathematical problem.

Again, there is some non-critical steps that are quite understandable. For instance, in order to take into account the already expressed preferential information in previous stages, the thresholds are given and fixed during this stage. Also, we must enforce the successively elicited vectors of weights w^* to be in accordance with the validated preorder \succsim_w , by adding in step 2 some simple constraints on the weights as follows:

$$\forall i, j \in F : \begin{cases} w_i > w_j \implies w_i^* > w_j^* \\ w_i = w_j \implies w_i^* = w_j^* \end{cases}$$

In step 8, we are considering the use of ACON to solve the mathematical problem, *i.e.* not to consider the stability anymore. In fact, as the preorder and the discrimination thresholds are fixed, we do not need to try to enforce the stability of previous outranking statements, as the stability is implicitly ensured with the parameters. Also, as we are dealing here with unstable outranking statements that could not become stable, such a consideration is unnecessary for this stage. Notice that the decision-maker may have expressed some preferential information that could have been not stably warranted, on some couples of alternatives and also on the comparison of the relative importance of two coalitions of weights. In that case, we need to explicitly keep these constraints (step 3) in order not to lose some information on the weights or to ask again the decision-maker about his preferences on some already given outranking statements.

Finally, we assumed the fact that the decision-maker is not able, or feels uncomfortable, to give accurate information on the degree of the difference between the importance of two criteria. For instance, we are not considering any information like “*criterion g_i is much more important than g_j* ”, or “ *g_i is about twice as more important as g_j* ”. Also, we assume a correct construction of the set of criteria, such that there is no criterion that could be of a very low importance compared with the other criteria, which is warranted in our protocol via the limitation of the weights to integer values from 1 to M . If M is set to the number m of criteria, the maximal possible ratio between two weights will be equal to m . Also, as already explained, to deal with a possible need of higher preciseness on the relative importance of the criteria, we will consider to set M up to $2m$. As the maximal possible ratio is then twice more important, we will force the weights of every criteria to be at least equal to 2, not only 1. This is motivated by the fact that we increase M not to take into account wider differences between the weights, but only to have a higher degree of preciseness that could be necessary, the weights being scaled after between 1 and m . As a general rule, considering $M = \kappa \cdot m$, every weights has to be higher or equal to κ , to have an importance of at least 1 after being scaled. Such constraints have been modeled in section 4.2.3.

Selecting the unstable couples of alternatives

Assuming the validation of the parameters, in order to tune the weights for a better setting of the unstable outranking statements, we should only focus on ordered pairs that are linked with such an unstable relation, as the stable ones are implicitly validated by the preorder and cannot be modified.

Let us notice that, according to the set of already expressed preferential information, some unstable outranking statements are *necessary*, *i.e.* their credibility has to be positive or (exclusive) negative. In order to avoid the construction of an incompatible set of preferential information, and also not to question the decision-maker about some information that are implicitly given, we may test if a selected relation is implicitly warranted or could be modified, before questioning him.

To do so, after having selected an ordered pair of alternatives and before questioning the decision-maker, we test with the algorithm ACON the consistency of the set of information plus the setting of the associated outranking in its opposite sense (*i.e.* an invalid statement if it is currently a valid one, and vice versa). If it is inconsistent, the outranking statement is in fact forced by the already expressed preferential information: We then simply select another couple and test it again.

Notice that, for small or medium-size problems, we consider a questioning of the whole unstable outrankings, throwing apart the necessary ones, but such a task becomes time consuming on large problems, not to mention that the decision-maker may not be always able to answer with accuracy.

Contrary to the previous stage, the questioning can be guided by taking into consideration the problem typology and the exploitation of the outranking relation. For instance, considering the best choice problem, the second step is focusing on the alternatives that could be considered as a recommendation (or amongst the recommended alternatives). Thus, if the study of the stable outrankings is sufficient to consider that an alternative cannot be recommended, there is no need to focus the questioning on trying to see if this alternative can be considered as a best one.

Finally, the questioning can be oriented by the decision-maker himself: For instance, when presenting him the set of the best matching alternatives, he may disagree on some results, which will be translated into some additional constraints on the outrankings. Notice that if he disagrees on a stable outranking statement, a conflict happens: either the preorder is brought into question, or a veto has to be raised. Also, confronted to the sorting problem, the decision-maker may disagree on some assignments, reconsidering the outranking relation between the alternatives and the profiles of the categories.

As the exploitation of the outranking digraph, especially in a robust concern, is out of the scope of this thesis, we shall consider in the practical case the questioning of all the outranking statements that are not necessary, in order to reconstruct the complete outranking relation in complete accordance with the decision-maker's mind. But we shall discuss about its exploitation and give some leads to be followed up in future work.

Questioning the decision-maker and integrating his preferences

The questioning of a selected couple (x, y) (step 7) is quite similar than in the second stage. Assuming that at least the outranking statement from x to y is unstable, we ask the decision-maker about his preferences by expressing again one of the following answers:

1. “I am certain that x is at least as good as y ”;
2. “I am certain that x is not at least as good as y ”;
3. “I do not know” / “I am not sure”.

In the first two cases, we add to the set of constraints the corresponding desired outranking statement and run the algorithm ACON.

More specifically, when dealing with a positive outranking statement from x to y , we also verify the opposite one from y to x , unless he is stable. As we did for the selection of the alternatives x and y , we verify if this opposite outranking is necessary or not. If not, we question the decision-maker and integrates his preferences as done previously.

Stopping the elicitation process

Finally, the algorithm is stopped when every outranking statements are either stable or necessary. In the situation where we consider the exploitation of the outranking digraph, the algorithm is stopped as soon as the decision-maker is happy enough with the exploitation. Again, as this latter considerations are out of the scope of this thesis, we will not detail more this step. However, with the practical case presented in the next chapter, we will show how the exploitation may be done, especially considering the *preferable* relation, highlighting some important considerations about the stability for future work.

7.2 Tools for supporting a robust elicitation

Using some graphical tools can improve the decision-maker’s understanding of the problem, by giving a simplified, but the most possibly objective, view of the situation. Relying on them, one should express preferential information with more confidence and have a better picture of their impact on the elicited parameters and the associated algorithmic results.

Such tools can be useful either for a better display of the input data, especially presenting the evaluations of the alternatives, or the elicited parameters.

In this section we present two adapted graphical tools for our purpose, namely one for helping to compare the alternatives and another one to ease the validation of the amendment on the preorder.

7.2.1 Dynamic pairwise performance comparison table

To collect accurate preferential information on the comparison of two alternatives, we have to present their evaluations in a way they can be easily compared in accordance with the outranking method principles.

For the ease of the presentation, we are giving here a small didactic performance table, considering the following situation: Someone wants to buy a new mobile phone, associated with a monthly package. We only show a small subset of the alternatives, on a subset of 5 criteria.

Table 7.1: Selecting a new monthly package with a mobile phone

| | ↓ Cost/month | ↓ Phone price | ↑ Calls | ↓ Binding period | ↑ 3g data | ... |
|---------|--------------|---------------|---------|------------------|-----------|-----|
| Offer 1 | 19 € | 150 € | 2h00 | 0 month | 0 MB | |
| Offer 2 | 21 € | 130 € | 2h00 | 24 months | 250 MB | |
| Offer 3 | 31 € | 105 € | 3h00 | 12 months | 1 000 MB | ... |
| Offer 4 | 25 € | 120 € | 1h30 | 12 months | 500 MB | |

⋮

Notice that on criterion *binding period*, there are only three possible values for every considered offers, namely 0, 12 or 24 months. After a brief discussion with the decision-maker, he agrees on the fact that there is a clear preference between each values. Then, the thresholds on this criterion are fixed, setting the indifference threshold up to 0 and the preference threshold up to 12.

Finally, we consider on every criteria some constant thresholds, as the scopes of the values are reduced, but this tool works also with proportional thresholds.

Dealing with the comparison of two selected alternatives, for instance the first and the second offer, the idea is to simply present their evaluations in a dynamic table, and ask the decision-maker about his preference, as explained in section 7.1.2. This table takes the following form:

Table 7.2: Dynamic performance table – 1/2

| | ↓ Cost/month | ↓ Phone price | ↑ Calls | ↓ Binding period | ↑ 3g data |
|----------------|--------------|---------------|---------|------------------|-----------|
| Highest | 36 € | 190 € | 3h00 | 24 months | 2 000 MB |
| Lowest | 15 € | 105 € | 1h30 | 0 month | 0 MB |
| Offer 1 | 19 € | 150 € | 2h00 | 0 month | 0 MB |
| Offer 2 | 21 € | 130 € | 2h00 | 24 months | 250 MB |
| Signif. diff.? | | | Indiff. | Pref.: Offer 1 | |

When comparing the two alternatives, the decision-maker has a complete view of their evaluations and how they are compared on each criterion. Indeed, the last row, called *significant difference*, translates the relative behavior between the evaluations, according to the expressed bounds of the decision-maker. For instance, on criterion *binding period*, according to the fixed thresholds, there is a clear local preference for the first offer. On criterion *calls*, as the values are equal, there is a clear

indifference between the values. On the other criteria, as the decision-maker did not expressed some constraints yet, we cannot presuppose about any local behavior.

Then, on criteria where the considered alternatives have close evaluations, (for instance on criteria *cost per month* or *phone price*), the analyst may ask the decision-maker if he considers a clear indifference between the values. If that is the case, for instance he is certain that a difference of 20 euros for the price of the phone is irrelevant, we add a lower bound on the indifference threshold of the associated criteria.

Finally, as the algorithm elicits at each iteration some discrimination thresholds, we can present their impact on the comparison of the alternatives. Then, we integrate them as follows:

Table 7.3: Dynamic performance table – 2/2

| | ↓ Cost/month | ↓ Phone price | ↑ Calls | ↓ Binding period | ↑ 3g data |
|----------------|------------------|---------------|---------|------------------|-----------|
| Highest | 36 € | 190 € | 3h00 | 24 months | 2 000 MB |
| Lowest | 15 € | 105 € | 1h30 | 0 month | 0 MB |
| Offer 2 | 21 € | 130 € | 2h00 | 24 months | 250 MB |
| Offer 4 | 25 € | 120 € | 1h30 | 12 months | 500 MB |
| Signif. diff.? | (Pref.: Offer 2) | Indiff. | (?) | Pref.: Offer 4 | (?) |

In parentheses, we show how the alternatives are compared locally by the elicited parameters, namely when there is a strict preference, an indifference or when we do not know if the smallest value is indifferent or not to the highest. Once again, the decision-maker may express a clear agreement or disagreement on an algorithmic choice: we then strengthen the bounds on the thresholds, according to the additional preferential information.

7.2.2 Display of the elicited weights preorder

For a didactic presentation of this simple graphical tools, let us consider a set $F = \{g_1, \dots, g_7\}$ of 7 criteria, on which the decision-maker expressed a first initial preorder $\geq_{w_0} = \{g_1, g_2, g_3\} >_{w_0} \{g_4, g_5, g_6, g_7\}$. It is then simply displayed as in Table 7.4:

Table 7.4: Displaying the current preorder – 1/4

$$\geq_{w_0} : \quad \boxed{g_1 \quad g_2 \quad g_3} > \boxed{g_4 \quad g_5 \quad g_6 \quad g_7}$$

The symbol $>$ indicates a strong inequality that has been already validated by the decision-maker, such that we have added preferential information on the relative importance of the associated weights. Notice that two criteria in the same class (here represented by a rectangle) are not forced to have the same weights, we only ensure the inequalities, until a complete and high validation of the preorder.

Assuming now that the add of some preferential information on the alternatives have modified the preorder. Table 7.5 is then presented to the decision-maker:

Table 7.5: Displaying the current preorder – 2/4

$$\begin{array}{l} \geq_{w_0} : \quad \boxed{g_1 \quad g_2 \quad g_3} > \boxed{g_4 \quad g_5 \quad g_6 \quad g_7} \\ \geq_{w_1} : \quad \boxed{g_1} >? \boxed{g_2 \quad g_3} > \boxed{g_4 \quad g_5 \quad g_6 \quad g_7} \end{array}$$

The symbol $>?$ indicates that the algorithm added some new discriminations on the preorder that have not been validated by the decision-maker. Let us suppose that he agrees on the fact that g_1 is more important than g_2 and g_3 . The inequality is then validated and we add constraints “ $g_1 > g_2$ ” and “ $g_1 > g_3$ ” to the set of preferential information.

Once again, after a few iterations, the discrimination of the preorder has been increased and we display the Table 7.6:

Table 7.6: Displaying the current preorder – 3/4

$$\begin{array}{l} \geq_{w_1} : \quad \boxed{g_1} > \boxed{g_2 \quad g_3} > \boxed{g_4 \quad g_5 \quad g_6 \quad g_7} \\ \geq_{w_2} : \quad \boxed{g_1} > \boxed{g_3} >? \boxed{g_2} > \boxed{g_5 \quad g_6} >? \boxed{g_4 \quad g_7} \end{array}$$

The decision-maker gives a strong agreement on the fact that g_3 is more important than g_2 : the inequality is validated and the resulting constraint “ $g_3 > g_2$ ” is added to the set of preferential information. On the contrary, for the last importance class, he is not certain about every inequalities that are induced. Indeed, g_5 is for sure a more important criterion than g_4 and g_7 , but he is not that confident in the fact that g_6 is also more important than g_4 and g_7 . Thus, we ensure the inequalities “ $g_5 > g_4$ ” and “ $g_5 > g_7$ ”, but we do not enforce any other constraints, such that g_6 could be, in a following iteration, less important or as much important as g_4 or g_7 . For instance, a few iterations after, the preorder can be changed again, as in Table 7.7. If the decision-maker validates the fact that g_5 is more important than g_6 , the inequality is then validated and, in consequence, the complete current preorder is validated. Either the decision-maker is happy enough with this preorder, namely he validates the fact that two criteria in a same importance class are associated with the same exact weight and we ensure the whole set of equalities and inequalities for the next iteration, or he wants to continue refining the preorder.

Table 7.7: Displaying the current preorder – 4/4

$$\begin{array}{l} \geq_{w_1} : \quad \boxed{g_1} > \boxed{g_2 \quad g_3} > \boxed{g_4 \quad g_5 \quad g_6 \quad g_7} \\ \geq_{w_2} : \quad \boxed{g_1} > \boxed{g_3} > \boxed{g_2} > \boxed{g_5} >? \boxed{g_4 \quad g_6 \quad g_7} \end{array}$$

Case study: Applying for a Ph.D. thesis

“Ce qu’il y a de plus profitable dans notre vie, généralement, personne ne nous l’a appris.”¹

[Georg Christoph Lichtenberg]

Contents

| | | |
|------------|--|------------|
| 8.1 | Applying the REWAT process | 148 |
| 8.1.1 | Stage I: Initialising the outranking preference model | 148 |
| 8.1.2 | Stage II: Validating the criteria weights preorder | 151 |
| 8.1.3 | Stage III: Tuning the numerical values of the criteria weights | 155 |
| 8.2 | Critical review of the case study | 157 |
| 8.2.1 | Encountered difficulties | 157 |
| 8.2.2 | Perspectives for future methodological enhancement | 161 |

Abstract

In this last chapter, we illustrate the use of our process in a practical case, dealing with a novice decision-maker. After a brief introduction of the problem, as well as a presentation of the test conditions, we present a complete execution with one decision-maker, highlighting the use of the stability concept. Finally, we give some critical attention to the defined process and discuss about some perspectives for future enhancements.

1. “Usually, nobody taught you what the most enjoyable things in life are.”

Summary of the case study

We consider a situation in which a student in master of Informatics wants to apply for a Ph.D. thesis. As a conscientious student, he wants to assess each opportunity. More precisely, he is interested in defining a reduced set of the best alternatives (to apply for the best opportunities) and, if possible a ranking between them (to choose the best one if he is accepted for more than one position).

We created in a first step a set of 22 fictitious — but realistic — alternatives (or Ph.D. applications) and create their evaluations on 14 attributes. Notice that, to be the most possibly exhaustive, we preliminary discussed with the decision-maker in order to add, if necessary, some other attributes that should be taken into consideration. When the occasion arises, we defined new attributes, with fictitious values. We give the complete evaluation table in Annex A.3.

The purpose of this chapter is to validate the usefulness of our process in real-case applications, so it has been tested (and enhanced) several times with different students in Master of Informatics who were doing a Research internship and really wanted to apply for a thesis for the next year, but also with 3 Ph.D. students currently working on their thesis. Confronted to this unusual decision, these students can be considered easily as novice decision-maker, as they all have some implicit knowledge on the domain of the decision and some relatively clear ideas about the criteria to be taken into account, as well as some ideas about their relative importance. Besides, they are able to express some global preferences, without an explicit knowledge about their local preferences.

For the ease of the reading, notice that we are presenting in the next section the discussion with only one of the decision-makers. This one is chosen as the questioning was not too long, such that we can present the complete protocol hereafter. We will discuss about the encountered issues, and also about the decision-makers' feeling during the process.

8.1 Applying the REWAT process

8.1.1 Stage I: Initialising the outranking preference model

Preliminary selection of the alternatives

As we already explained in the previous chapter, each student preliminary select the alternatives he considers as acceptable (*i.e.* potentially the best choices), among the whole set of possible alternatives. This is easily done by presenting him each alternative, with its attributes, and asking him if he wants to consider the alternative as a potential choice or not. For instance, some of them that did not want to go too far from their family put aside the foreign alternatives. In the case we define here, the student wanted to focus on making an interesting thesis and was ready to do foreign studies, provided that the domain of the thesis was at least interesting

(the “not very interesting” thesis far from his city was put aside). A subset of 8 alternatives we present hereafter is then retained.

On the construction of the criteria

In a first step, when looking at the available data (see Annex A.3), the student decides to adopt seven criteria on which he will lean his decision only:

- *Advisor Communication Ease* (ACE), based on the attribute Advisor Communication Language (ACL): an ordinal criterion with two values, VERY GOOD (for French-speaking advisor) and GOOD (for english-speaking advisor), to be maximised;
- *Advisor Research Domain* (ARD) (based on the attribute with the same name): an ordinal criterion based on the student’s interest of the potential advisor’s research domain, from value 0 (not interesting) to value 5 (very interesting), to be maximised;
- *Advisor Research Experience* (ARE): a criterion that intends to measure the experience of the potential future advisor, according to the available data;
- *Advisor Availability* (AA): a criterion that intends to quantify the availability of the future advisor, to be maximised. It is obtained by subtracting the number of current Ph.D. students to 10, namely a value of 10 means that the advisor does not have to take care about any other Ph.D. students, a value of 7 that he is already in charge of 3 students, . . . ;
- *University International Reputation* (UIR) (Based on the attribute with the same name): The ranking of the University according to the *Academic Ranking World of University* website², to be minimised (as the best universities are associated with the first places);
- *Country Purchasing Power* (CPP): the ratio between the month salary for a thesis and the median month salary in the country, to be maximised, in order to have an idea of the living conditions of a Ph.D. student in the foreign countries.
- *Trips Back Home* (TBH): An estimation of the number of trips back home per year, based on the distance, but also the travel price, to be maximised;

Notice that the construction of the criteria is done without the help of an expert of the considered points of view: The students are aware of the fact that, until we can find some clear indicators, for instance to find an accurate and understandable indicator of the actual purchasing power of a Ph.D. student in every considered countries, we need to construct them, based on the available data. In consequence, the use of an outranking method is motivated by the possible lack of preciseness in the evaluations of the alternatives performances, by making very hazardous any compensation between different criteria.

Also notice that the students were not limited in the construction of the criteria scales, although we guided them, such that some of them considered, for a same defined criterion, to construct either a qualitative scale or a quantitative one, building

2. <http://www.arwu.org>

some formulas to combine the available data. Most of the time, they constructed ordinal criteria, as it was difficult for them to apprehend the significance of a precise indicator. In this case, about the construction of the criterion ARE, the student decided to compute the ratio between the total number of citations and the number of articles of the potential future advisors. He is aware that someone with a few articles and some citations may have a better evaluation than someone with a great number of paper but a few citations, but he considers that this ratio is a good indicator of the “quality” of the 8 considered advisor’s work.

Table 8.1: Performance table of the 9 selected alternatives

| | ACE \uparrow | ARD \uparrow | ARE \uparrow | AA \uparrow | UIR \downarrow | CPP \uparrow | TBH \uparrow |
|-----------------|----------------|----------------|----------------|---------------|------------------|----------------|----------------|
| FR ₂ | VERY GOOD | OR (4) | 0.57 | 9 | 36 | 1.03 | 12 |
| SW ₂ | VERY GOOD | OR (4) | 2.18 | 7 | 31 | 0.86 | 6 |
| EN ₂ | GOOD | GT (3) | 11.03 | 9 | 162 | 1.36 | 6 |
| NL ₁ | GOOD | GT (3) | 3.81 | 6 | 123 | 1.20 | 6 |
| NL ₂ | GOOD | MCDA (5) | 0.40 | 9 | 123 | 1.20 | 6 |
| US ₁ | GOOD | GT (3) | 0.57 | 9 | 86 | 0.88 | 2 |
| US ₂ | GOOD | OR (4) | 7.81 | 3 | 66 | 0.88 | 2 |
| CA ₂ | VERY GOOD | MCDA (5) | 0.00 | 10 | 14 | 1.10 | 2 |
| Thresholds | constant | cons. | cons. | const. | const. | const. | const. |
| q_i | 0 | 0 | ? | ? | ? | ? | 0 |
| p_i | 1 | 1 | ? | ? | ? | ? | 1 |

OR: Operational research; GT: Graph theory.

On criteria ACE and ARD, associated with limited ordinal scales such that the difference between two adjacent values is always significative, the corresponding indifference and preference thresholds are set to 0 and 1 respectively. On criterion TBH, as there is only 3 well differentiated values, namely 2, 6 or 12 travels, the student easily validates a clear preference for 12 instead of 6 and for 6 instead of 2. Thus, we can consider constant indifference and preference thresholds set to 0 and 1, as they are enough to recapture the complete local concordance relation on this criterion, for the given alternatives. Notice that we could consider proportional thresholds, as it seems relevant to say that there is a clear preference to travel 2 times instead of 1, but the difference is not significative between 20 and 21 travels (for instance). Thus, we must be aware that if we want to subsequently consider additional alternatives with other evaluations on that criterion, these defined constant thresholds may not be satisfying and maybe we should consider some variable proportional thresholds.

On criterion UIR, we may also consider constant or proportional thresholds. We decide to use constant thresholds and to modify them into proportional one if the student expresses some information that will be contradictory.

Finally, we observe the fact that the alternative FR₂ outranks US₁ in an independent manner (FR₂ is at least as good as US₁ on every criteria). In a situation where we want to find a unique best alternative, such an alternative that is clearly dominated should be discarded. In this particular case, it seems more advisable to keep it during the elicitation process, as we are searching for a reduced set of the best alternatives. Of course, the student will not be questioned about the corresponding

statement.

Definition of an initial preorder to be refined

After the construction of the criteria and the performance table, the student is certain that the most important criterion is, in his mind, ARD. Then, the criteria ARE and UIR are judged as important. Finally, CPP is clearly pointed out as a less important criterion. In return, the student hesitates in qualifying the importance of the criteria ACE, AA and TBH, being either important or less important, but he highly agrees on the fact that they are less important than ARD. In consequence, we construct a first initial preorder \succsim_{w^i} with two importance classes as follows:

Table 8.2: Initial preorder

$$\succsim_{w^i} : \boxed{\text{ARD}} > \boxed{\text{ACE} \quad \text{ARE} \quad \text{AA} \quad \text{UIR} \quad \text{CPP} \quad \text{TBH}}$$

In addition with the fact that ARD will be associated with a weight strictly higher than those of every other criteria, we enforce the fact that ARE and UIR will be more important than CPP, for every elicited vectors of weights.

8.1.2 Stage II: Validating the criteria weights preorder

Notice that in Annex A.4, we display the local concordance relation for every ordered pairs of alternatives. We can observe the evolution of the local concordance values as the thresholds are refined, *i.e.* when the student gives additional information on the thresholds such that some undetermined situations becomes clearly positive or negative. Also, we compute both most pessimistic and optimistic outranking statements, as well as the associated stability (1 stand for an unstable relation, 2 for a stable one, 3d for an extensively stable one and 4 for an independent one). With an asterisk (*), we highlight the ordered pairs that can be considered in the current iteration: either the outrankings that are not warranted in an extensively stable manner and may change by refining the preorder or the local concordance values, or the unstable outrankings when trying to set some precise weights, according to a validated preorder. Notice that some outranking statements are given with a point (.) instead of an asterisk, as it concerns already questioned pairs of alternatives.

For information, we can point out the fact that it is not possible to have stable balanced outranking statements in the present case. Indeed, the most important criterion, ARD, is alone in the most important class of criteria. As every local concordance on this criterion always are clearly strictly positive or strictly negative, a slight modification of the weights associated to criterion ARD will validate (or invalidate) the balanced situations.

Refining the initial preorder

From this moment, we have defined the thresholds on criteria ACE, ARD and TBH, as well as the initial preorder \succsim_{w^i} , such that there is a lack of information on the way the alternatives are compared locally. Thus, for every pairs of alternatives, we compute in Annex A.4.1 both the most optimistic and pessimistic outranking values, as well as their associated resulting stability, with respect to \succsim_{w^i} .

The first couple of alternatives to be selected is SW_2 and CA_2 . The evaluations are displayed as explained in the previous chapter, section 7.2.1. Notice that we will display two rows starting with “Pref.?”³. The first row indicates the current local comparisons, computed with the current current bounds on the thresholds, or the current elicited thresholds (in italic); the second one is given by the decision-maker if he wants to modify the current bounds by adding accurate information.

Table 8.3: Comparing the alternatives CA_2 and SW_2

| | ↑ ACE | ↑ ARD | ↑ ARE | ↑ AA | ↓ UIR | ↑ CPP | ↑ TBH |
|---------|-----------|----------|--------|--------|--------|-------|--------|
| Highest | VERY GOOD | MCDA (5) | 11.03 | 10 | 14 | 1.36 | 12 |
| Lowest | GOOD | GT (3) | 0.00 | 3 | 162 | 0.86 | 2 |
| CA_2 | VERY GOOD | MCDA (5) | 0.00 | 10 | 14 | 1.10 | 2 |
| SW_2 | VERY GOOD | OR (4) | 2.18 | 7 | 31 | 0.86 | 6 |
| Pref.? | Indif. | CA_2 | (?) | (?) | (?) | (?) | SW_2 |
| Pref.? | Indif. | CA_2 | SW_2 | CA_2 | Indif. | (?) | SW_2 |

First, the student expresses some accurate bounds on the thresholds. For instance, the fact that he is certain that there is a clear preference for the second alternative on criterion ARE means that a difference of 2.18 units in this criterion is significant and the associated preference discrimination thresholds has to be lower or equal than this difference. After having added some new bounds on the thresholds, we compute the new local concordance relation in Annex A.4.2.

Notice that, according to the additional local information, the outranking statement from SW_2 to CA_2 is inevitably unstable, with respect to the initial preorder; precisely, both the most optimistic and pessimistic outranking values are unstable, such that no matter the lack of information on criterion CPP is, this information will remain unstable and will not help us in refining the preorder. However, as we have already presented the alternatives to the decision-maker, he is asked about his opinion on the corresponding outranking statement.

The decision-maker ensures that alternative CA_2 is at least as good as alternative SW_2 . He is not sure about the statement on the opposite pair, but is quite confident in the fact that CA_2 is preferable to SW_2 . In consequence, we ensure the stability of the statement “ $CA_2 S^w SW_2$ ” and the fact that $S^w(CA_2, SW_2)$ is higher than $S^w(SW_2, CA_2)$. We run the algorithm and obtain the following preorder \succsim_{w^1} :

$$\succsim_{w^1} : \boxed{\text{ARD}} > \boxed{\text{ARE} \quad \text{UIR}} >? \boxed{\text{ACE} \quad \text{AA} \quad \text{CPP} \quad \text{TBH}}$$

3. “Is there, or not, a significative preference for one of these two alternatives on each criterion?”

From this moment, the decision-maker does not feel at ease to validate the new inequalities. We then keep considering the initial preorder for the selection of the alternatives, *i.e.* to compute the most pessimistic and optimistic outranking statements.

A second couple is then selected, US_2 and NL_2 , and displayed to the student.

| | ↑ ACE | ↑ ARD | ↑ ARE | ↑ AA | ↓ UIR | ↑ CPP | ↑ TBH |
|---------|-----------|----------|--------|--------|--------|------------|--------|
| Highest | VERY GOOD | MCDA (5) | 11.03 | 10 | 14 | 1.36 | 12 |
| Lowest | GOOD | GT (3) | 0.00 | 3 | 162 | 0.86 | 2 |
| NL_2 | GOOD | MCDA (5) | 0.40 | 9 | 123 | 1.20 | 6 |
| US_2 | GOOD | OR (4) | 7.81 | 3 | 66 | 0.88 | 2 |
| Pref.? | Indif. | NL_2 | US_2 | NL_2 | (?) | (NL_2) | NL_2 |
| Pref.? | Indif. | NL_2 | US_2 | NL_2 | US_2 | NL_2 | NL_2 |

Again, the discrimination thresholds are refined when the student can express an accurate information. Annex A.4.3 presents the new local concordance relation. The student argues that US_2 is not at least as good as NL_2 and that NL_2 is at least as good as US_2 . These two statements are enforced in a stable manner, we run again the algorithm $STAB'_1$ and obtain a new elicited preorder \succeq_{w^2} :

$$\succeq_{w^2} : \boxed{ARD} > \boxed{ARE \quad UIR \quad TBH} >? \boxed{ACE \quad AA \quad CPP}$$

The student clearly validates the fact that TBH is more important than CPP: that will be ensured for the next elicitations. By contrast, he is not entirely certain about the fact that ACE and AA are less important than TBH, ARE or UIR, but does not have clear reasons to invalidate them. We then keep considering the initial preorder to be refined.

The alternatives NL_1 and US_1 are then presented to the student.

| | ↑ ACE | ↑ ARD | ↑ ARE | ↑ AA | ↓ UIR | ↑ CPP | ↑ TBH |
|---------|-----------|----------|--------|--------|-------|--------|--------|
| Highest | VERY GOOD | MCDA (5) | 11.03 | 10 | 14 | 1.36 | 12 |
| Lowest | GOOD | GT (3) | 0.00 | 3 | 162 | 0.86 | 2 |
| NL_1 | GOOD | GT (3) | 3.81 | 6 | 123 | 1.20 | 6 |
| US_1 | GOOD | GT (3) | 0.57 | 9 | 86 | 0.88 | 2 |
| Pref.? | Indif. | Indif. | NL_1 | US_1 | (?) | NL_1 | NL_1 |
| Pref.? | Indif. | Indif. | NL_1 | US_1 | (?) | NL_1 | NL_1 |

He indicates that NL_1 outranks US_1 , does not have particular opinion about the outranking statement of the opposite pair, but agrees on the fact that NL_1 is preferable to US_1 . As these information are already warranted in a stable manner with the current elicited vector of weights, we do not recompute a new one.

The alternatives FR_2 and CA_2 are selected to keep questioning the student.

| | ↑ ACE | ↑ ARD | ↑ ARE | ↑ AA | ↓ UIR | ↑ CPP | ↑ TBH |
|-----------------|-----------|-----------------|--------|--------|--------|--------|-----------------|
| Highest | VERY GOOD | MCDA (5) | 11.03 | 10 | 14 | 1.36 | 12 |
| Lowest | GOOD | GT (3) | 0.00 | 3 | 162 | 0.86 | 2 |
| CA ₂ | VERY GOOD | MCDA (5) | 0.00 | 10 | 14 | 1.10 | 2 |
| FR ₂ | VERY GOOD | OR (4) | 0.57 | 9 | 36 | 1.03 | 12 |
| Pref.? | Indif. | CA ₂ | (?) | (?) | (?) | (?) | FR ₂ |
| Pref.? | Indif. | CA ₂ | Indif. | Indif. | Indif. | Indif. | FR ₂ |

After having updated the thresholds, we observe that CA₂ outranks FR₂ in an extensively stable manner, with respect to the initial preorder (see Annex A.4.5). In that case, there is no need in asking the student about his preferences on this particular outranking statement, but we simply explain him that this situation is already warranted. He agrees on the assumption, expresses the fact that FR₂ outranks CA₂, and also that CA₂ is preferable to FR₂, to his point of view. In that case, one can easily conclude in the fact that criterion ARD is more important than criterion TBH, which is already enforced. Again, we do not compute a new vector of weights.

The fifth couple to be selected consists of the alternatives FR₂ and EN₂.

| | ↑ ACE | ↑ ARD | ↑ ARE | ↑ AA | ↓ UIR | ↑ CPP | ↑ TBH |
|-----------------|-----------------|-----------------|-----------------|--------|-----------------|-----------------|-----------------|
| Highest | VERY GOOD | MCDA (5) | 11.03 | 10 | 14 | 1.36 | 12 |
| Lowest | GOOD | GT (3) | 0.00 | 3 | 162 | 0.86 | 2 |
| FR ₂ | VERY GOOD | OR (4) | 0.57 | 9 | 36 | 1.03 | 12 |
| EN ₂ | GOOD | GT (3) | 11.03 | 9 | 162 | 1.36 | 6 |
| Pref.? | FR ₂ | FR ₂ | EN ₂ | Indif. | FR ₂ | (?) | FR ₂ |
| Pref.? | FR ₂ | FR ₂ | EN ₂ | Indif. | FR ₂ | EN ₂ | FR ₂ |

According to the student, the alternative FR₂ clearly outranks EN₂. In addition, he does not have a clear opinion on the opposite pair of alternatives, but he affirms that FR₂ is preferable to EN₂. The algorithm runs again, giving the same preorder. However, the student is now confident in the preorder and the resulting inequalities: It is then validated.

$$\geq_{w_2} : \boxed{\text{ARD}} > \boxed{\text{ARE} \quad \text{UIR} \quad \text{TBH}} > \boxed{\text{ACE} \quad \text{AA} \quad \text{CPP}}$$

To his point of view, the discrimination of the current preorder is enough, *i.e.* there is a clear significant difference between the importance of two criteria that are not in the same importance class and, for two criteria that are in the same class, there is no reasons for discriminating them. The preorder is then validated and enforced for the rest of the practical case.

Validating the thresholds

In Annex A.4.7, we compute the most optimistic and most pessimistic outranking values, as well as their associated stability, for every ordered pairs of alternatives, taking into account the new preorder and all given bounds on the thresholds. We

look at the situations, highlighted with an asterisk in Annex A.4.7, that are sensitive to the lack of information.

In this practical case, notice that there is no sensitive outranking statement that has been expressed by the student during the previous step. Namely, every given outranking statements are either stable (irrespective to the missing local information), or all their local concordance values are warranted by the accurate thresholds bounds. In consequence, refining the thresholds bounds, according to the sensitive outranking statements, will have no impact on the validated preorder (*i.e.* no inconsistencies can appear during this stage).

The three criteria, where there are some critical missing local concordance values, are the followings: ARE, UIR and CPP. For the first one, we ask the student if an evaluation of 0.57 is at least as good as an evaluation of 2.18. The student does not have clear opinion, the associated local concordance is then set to 0.

For the second criterion, UIR, the student hesitates when comparing the values 66 and 36, and also the values 66 and 31. On the contrary, he indicates a clear preference between 86 and 36, such that the preference thresholds on this criterion is lowered to 50, instead of 57.

Finally, for the third criterion CPP, the student expresses an indifference between 0.86 and 1.03, such that the indifference threshold is raised to 0.17, instead of 0.07.

8.1.3 Stage III: Tuning the numerical values of the criteria weights

Finding a precise setting for the criteria weights

From now, the preorder and the discrimination thresholds have been validated, such that we can compute, for all ordered pairs of alternatives, the outranking value and the associated stability (see Annex A.4.8). Remind that the unstable situations, highlighted by an asterisk, are the only ones that are not fully warranted by the preorder, *i.e.* on which we focus this last stage.

The current precise vector of weights w^2 is the followings: $w^2 = (1, 3, 2, 1, 2, 1, 2)$.

As the number of unstable statements to be verified is quite small (12 unstable statements), we test each of them to see if they are necessary or not, according to the already expressed preferential information. The statements “FR₂ outranks NL₂” and “SW₂ outranks NL₂” are necessary. On the contrary, the outranking statement from SW₂ to CA₂, which is currently positive, may be invalidated by another vector of weights, preorder-compatible with w^2 and compatible with the already expressed preferential information. As the decision-maker does not have any particular reason to enforce a positive or a negative outranking statement, no constraint is added to the model. The same applies for the comparison between EN₂ and SW₂, which can be possibly changed, but the decision-maker does not have any clear opinion.

The next statement to be considered, between the alternatives EN₂ and NL₂, is also necessary. The positive outranking statement from EN₂ and US₂ is not necessary,

the alternatives are then presented to the decision-maker who confirms the positive statement. This one is then added to the set of preferential information but, as it is already validated, no iteration of the algorithm is done.

Dealing with the not necessary outranking relation from EN_2 to CA_2 , the decision-maker clearly indicates that EN_2 is not at least as good as CA_2 . Using the algorithm ACON, a new vector of weights is then elicited: $w^3 = (1, 4, 2, 1, 2, 1, 2)$.

From this moment, every other unstable statements are necessary, even the two statements that were not necessary with respect to w^2 on which the student were not confident in expressing a preferential information.

The complete median-cut outranking relation is then recaptured. During the whole discussion, the decision-maker has been questioned of 9 couples of alternatives, 5 during the second stage and 4 during this last stage.

On the exploitation of the outranking relation

According to the validated thresholds and the last elicited vector of weights, w^3 , we now compute the complete outranking relation. Notice that the outranking values are given on the interval $[-13, 13]$, as 13 is the sum of the weights in w^3 . The values are not normalised, as it is useless for our concern. A positive (resp. negative) value indicates a clear positive (resp. negative) outranking statement.

| | FR ₂ | SW ₂ | EN ₂ | NL ₁ | NL ₂ | US ₁ | US ₂ | CA ₂ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| FR ₂ | | 11 | 7 | 9 | 5 | 13 | 9 | 5 |
| SW ₂ | 8 | | 6 | 9 | 2 | 12 | 9 | 2 |
| EN ₂ | -5 | -1 | | 11 | 3 | 9 | 1 | -1 |
| NL ₁ | -7 | -1 | 7 | | 3 | 9 | -3 | -3 |
| NL ₂ | 3 | 5 | 9 | 9 | | 11 | 5 | 7 |
| US ₁ | -5 | -7 | 3 | 3 | -1 | | 1 | -2 |
| US ₂ | 3 | 3 | 1 | 5 | -3 | 11 | | -4 |
| CA ₂ | 9 | 5 | 4 | 5 | 9 | 13 | 9 | |

We easily observe that four alternatives outranks every other alternatives: FR_2 , SW_2 , NL_2 and CA_2 . In consequence, they are presented to the student as the best potential alternatives. He agrees on this recommendation, namely he will only consider these opportunities for a future thesis application. From this moment, we successfully achieved the protocol defined in the previous chapter, as we have recaptured the complete median-cut outranking relation with accuracy.

Ranking the best alternatives according to the *preferable* relation

Going further in the use of the stability concept, we intend to order the alternatives according to the preferable relation.

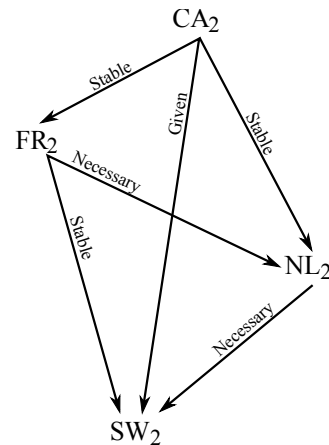


Figure 8.1: The resulting *preferable* relation

In Figure 8.1, we distinguish three different kinds of oriented arcs: - the *stable* ones, such that the preferable relation is only depending on the preorder of the weights; - the *given* ones, that are unstable but explicitly given by the decision-maker; - the *necessary* ones, unstable but implicitly enforced by the given preferential information. Notice that we could have also unstable and *non-necessary* arcs, such that the student should be questioned in order to validate their direction.

In this particular situation, as the resulting graph is transitive, we can logically rank the four best alternatives as follows: CA_2 , FR_2 , NL_2 , then SW_2 . According to the validated preorder and discrimination thresholds, as well as the expressed preferential information on the alternatives, the preferable relation is entirely determined, such that this recommendation is robust.

When we presented this recommendation to the student, he was satisfied. We then stopped the process.

8.2 Critical review of the case study

In this section, we give some critical attention to the defined process, especially about the difficulties experienced by the novice decision-makers, but also some considerations for further improvements.

8.2.1 Encountered difficulties

On the construction of the criteria

The criteria construction with a decision-maker is a critical and very time-consuming stage, especially when dealing with a novice decision-maker. Thus, in order to save time during the preliminary tests of the process, we defined in advance

a set of criteria without the decision-makers, on which they were questioned about the relevance of each criterion, namely they kept only the criteria they thought relevant for the decision aid. Besides, this preliminary set of criteria was the most possibly exhaustive, but some decision-makers wanted to consider some additional criteria. Furthermore, it was not always an easy task for them to apprehend the construction of some criteria, making the expression of preferential information more difficult. In consequence, we took the decision to construct the criteria with the decision-makers, according to the available attributes. The experiences were more time-consuming, but the results were more convincing, either for the decision-makers that were involved in the complete process, but also for us, as an analyst, for a better understanding of the novice decision-maker's behavior.

Most of the time, the students were more confident in evaluating the alternatives on ordinal scales instead of ratio scales, according to the available attributes. In fact, when facing a decision without an analyst or an expert of each point of view, and also with a novice decision-maker that has an implicit knowledge of his preferences, but not the required experiment for being able to apprehend the slight differences, it seems a bit illusory to construct some accurate quantitative indicators. Thus most of the time a ratio scale were considered only for criteria relying on a unique numerical attribute.

Dealing with direct or indirect elicitation of the thresholds

As we explained before, in some preliminary tests, we have taken care about the criteria construction without the decision-maker. As the algorithms for eliciting both criteria weights and discrimination thresholds were not modeled yet, that included to first define some associated thresholds, to be validated or modified by the students. It appeared to be a very questionable step, the decision-makers not having enough knowledge to set them in a satisfying manner at the beginning of the process, especially on quantitative scales. Thus, after only a few questions on the alternatives couples, some conflicts appeared, due to the inaccuracy of the thresholds.

That is why we modeled the algorithms ACON', STAB'₁ and STAB'₂, but keeping in mind to elicit the discrimination thresholds only when they cannot be expressed in an accurate and precise manner at the beginning of the process, for instance on large quantitative scales.

Besides, setting the thresholds that are "obvious" on criteria with small ordinal scales has been verified as a convincing assumption. Indeed, we have seen that the novice decision-makers expressed global preferential information in accordance with these thresholds all along the whole questioning.

Finally, in this constructive approach, our novice decision-makers were not confident in giving precise and accurate values for the thresholds, but were more trustworthy in comparing the alternatives locally, validating the assumption of not discussing about the values of the thresholds, but discussing about the resulting local concordance values.

Defining an initial preorder

We have made a strong hypothesis on the fact that a decision-maker is able to express accurately an initial preorder on the weights to be refined, at the beginning of the process.

Remind that the equi-important preorder, on which every criteria are associated with the same importance weight, allows every possible refining. Thus, it should be wise to start from this preorder, to let the decision-maker some time to have a better explicit feeling about his preferences over the criteria and also to add some discriminations only when it is necessary to recapture some expressed outranking statements. But this preorder only leave the independent outranking statement⁴ out, narrowing the consideration of the stability concept down and lengthening the process.

As a facilitator, we then intended to take into consideration only the relative importance constraints between the criteria that the decision-maker expressed with full confidence, namely to have clear motivations to start from a not equi-important preorder, to avoid possible conflicts during the process, or some modification of the initial preferential information, that would have invalidated the whole process.

However, sometimes we observed that the students came back on some strong assumptions: for instance, one was confident about the fact that the availability of the future advisor was more important than the distance from the family, but after considering a few couples of alternatives, he expressed the fact that he was not that much confident, such that we modified the initial preorder.

When we discussed with these students, it appeared that the criteria had two possibly different associated importances: one for the preliminary selection of the alternatives and another when considering the pairwise comparison of the acceptable alternatives only.

For instance, for one student, the five different research domain was acceptable and it was really important not to go too far away from their family, so they have only kept the “close-enough” opportunities, no matter the research domain is. In a logical manner, the student expressed in the initial preorder the fact that the distance was more important than the domain. But in the second stage, considering the acceptable alternatives, the criterion distance was important, but not that much important than the domain: Indeed, as the remaining alternatives was all at an acceptable distance, even if some were considered as clearly better than some other by the decision-maker, he was able to consider, without raising any veto, a further off alternatives associated with a more interesting domain preferable to a closer one with a less interesting domain.

To our point of view, this is not an inconsistent judgement: The importance of the criteria is depending on the problem we deal with, either to reduce the set of alternatives to the acceptable ones or to compare pairwise a set of acceptable

4. An extensively stable statement, with respect to an equi-important vector of weights, is automatically an independent statement.

alternatives. However, there is a direct risk in confusing both concepts when defining an initial preorder, strengthening the idea of starting from the least possibly discriminated preorder. Besides, that briefs us on the way we should construct this initial preorder, clearly explaining to the decision-maker that we consider the relative importance between the criteria in the comparison of the potential alternatives.

Decision-maker's ease at expressing preferential information

In order to reduce the number of questions during the process, we first considered to question the decision-makers in terms of *preference* or *indifference* of the non-ordered couples of alternatives, instead of questioning them about positive or negative outranking statements in both ways. However, it appeared that, apart from “obvious” situations, they were often uneasy in expressing accurate outranking indifferences or preferences.

In fact, we may clearly distinguish the outranking concepts of indifference and preference to their associated ordinary meanings: Indeed, in a semantical point of view, an indifference means that choosing one alternative does not lead to a stronger regret than choosing the other one. By opposition, a preference is the fact that there are more arguments in favor of the preferred alternative than in favor of the other one (assuming again a non-veto situation).

Returning to our practical cases, the decision-makers often had a strong feeling about one of the alternatives being less regrettable than the other one, but they admit that the other alternative was “not so bad”. For instance, let us consider the following alternatives CA_2 and FR_2 , on which the student has been questioned:

| | ↑ ACE | ↑ ARD | ↑ ARE | ↑ AA | ↓ UIR | ↑ CPP | ↑ TBH |
|--------|-----------|----------|--------|--------|--------|--------|--------|
| CA_2 | VERY GOOD | MCDA (5) | 0.00 | 10 | 14 | 1.10 | 2 |
| FR_2 | VERY GOOD | OR (4) | 0.57 | 9 | 36 | 1.03 | 12 |
| Pref.? | Indif. | CA_2 | Indif. | Indif. | Indif. | Indif. | FR_2 |

According to the last validated weights vector, both outranking statements are positive: it is then an outranking indifference. However, expressing an indifference between both alternative would have lost a critical information (the fact that CA_2 is less regrettable than FR_2), but expressing a preference would have made criterion ARD a dictator. This is, of course, a simplistic situation, that can be solved by considering the fact that criterion ARD is more important than criterion TBH (which is, in fact, already warranted by the preorder), but as soon as there are some criteria on which the alternative are indifferent (*i.e.* each alternative is at least as good as the other one), it is very difficult to express an accurate outranking indifference or preference. In consequence, these two outranking concepts are not satisfying enough to fully recapture the decision-maker's preferential information.

There are some strong motivations in the consideration of an indifference statement when a certain credibility is reached by the two outranking statements in an aggregation process that does not take into account the stability concept, such that

we must be careful in the exploitation of the outranking digraph and also in the expression of the recommendations. However, when dealing with preferential information on the alternatives, taking into consideration the stability concept, such that we can ensure that an outranking statement or the comparison between two outranking statements is not anecdotal, this is not really justified.

In consequence, in order to avoid any discussion on the way the terms “preference” and “indifference” should be envisioned, we decided to question the future testers in term of outranking statements, and also questioning them in terms of *preferable* alternative. Most of the time, they were confident in expressing the fact that one alternative outranked the other, but apart from the obvious situations, they were uneasy in giving an opinion on the opposite ordered pair, namely either a positive outranking statement (highlighting a indifference between the alternatives) or a negative one (highlighting a clear preference). However, they were confident in expressing their preference with the preferable relation.

8.2.2 Perspectives for future methodological enhancement

Taking into account the stability of a partial preorder

We have shown that the decision-maker can express some constraints between the criteria that cannot be translated directly in the preorder. For instance, he can ensure that a criterion is definitely more important than another, but he may not be able to sort them in the initial preorder (remember in the practical case that criterion CPP was described as less important than criterion ARE, but they were in the same importance class in the initial preorder). As these constraints are not given in the preorder, we can find an unstable statement that is already warranted in a robust manner (it is a necessary statement, according to all the constraints on the criteria). For instance, the fact that CA_2 is preferable to SW_2 was unstable, with respect to the preorder \succsim_{w_1} , but it was necessary according to the additional constraints on the criteria, *i.e.* the fact that ARE and UIR were more important than CPP. Thus, questioning the decision-maker is useless. A first idea should be to simply test the necessity of the unstable outranking statements before questioning the decision-maker, as it is done in the last stage. Some future work to extend the stability concept with a more general set of preferential information on the criteria (not only a preorder) will be envisioned.

Guiding the questioning around the preferable relation

As our process did not focus on the expression of alternatives comparisons in terms of preferable choices, the exploitation of such a graph at the end of the process may not be robust, namely some arcs may be unstable and not necessary. It was not the case in the presented resolution, but it could happen. In that case, drawing some robust recommendations is not always possible.

In addition, we have just explained that the novice decision-makers we worked with were more confident in talking about the fact that an alternative was preferable to another, instead of outranking relations. In consequence, an interesting approach would be to focus the questioning on this relation: “*According to the evaluations of two alternatives x and y , do you think that one is preferable to the other?*”. A positive answer, for instance if alternative x is preferable to y , automatically implies a positive outranking statement for the associated ordered pair, *i.e.* x outranks y , but does not assume the negative or positive outranking statement for the opposite pair. Remind that it was particularly demanding for the decision-makers to express such an accurate information, which is in fact useless if the preferable relation is clearly determined, *i.e.* every arcs are stable, given or necessary. From a mathematical point of view, everything are already modeled, such that the algorithms can work with such constraints, but there is still some theoretical work to do, as well as some empirical tests.

Conclusion and perspectives

Summary of the main achievements

Measuring the stability is an efficient tool for improving the understanding of the outranking relations' dependencies with respect to the chosen importance parameters. It ensures both a more stable validation, highlighting the assumptions induced by the parameters, and a more solid exploitation. The impact of an insecure fixation of the parameters is then measured, or even limited within the context of a stable indirect elicitation.

According to a given vector of weights (or a preorder), we have shown that the stability of any outranking statement is checked *via* simple and quite intuitive mathematical conditions. Thus, assuming an explicit validation of a preorder, it is clearly justifiable to consider a stable situation, even if it is not-well determined, as implicitly validated, whereas an unstable and not-well determined situation has to be explicitly validated by the decision-maker. In consequence, the process focuses on sensitive outranking situations only, decreasing the required time and increasing the decision-maker's confidence in the final recommendation, as he is more comfortable in validating a preorder than a precise vector of weights. As a result, any post-exploitation of the outranking relation will be more robust.

Also, we have extended the theoretical work on the stability concept with the definition of two additional levels of stability: – the σ -stability, which allows to work with non-ordered sets of equi-important criteria and – the extensible stability, which highlights the outranking statements that are depending on a possible lack of discrimination on the preorder, *i.e.* the outranking statements that may, or may not, become unstable by increasing the discrimination of the considered preorder.

In order to take advantage of the stability concept, we have presented some innovative mathematical models to determine importance weights of criteria from a decision-maker's set of preferential information, as well as discrimination thresholds or categories profiles. These models have been validated empirically, to underline their improvement on building the median-cut outranking relation, which induces a real gain on the credibility of eventual decision aid recommendations. In fact, the more stable a relation is, the less critical becomes the actual choice of precise numerical criteria importance weights. As a consequence, the robustness of a solution provided by a multiple criteria method exploiting this relation is clearly enhanced.

Besides, we have empirically demonstrated that the parameters perturbation due to the defined stability constraints is non-significant. Then, the risk of distorting the reality is narrowed when working with a decision-maker that expresses an accurate global opinion without being able to give some accurate or precise local information.

In addition, the stability concept has been adapted to the sorting problem, to characterise the stability of any alternative assignment and define an interval of categories on which an alternative is assigned in a stable manner. However, within the framework of a disaggregation approach, the models we have defined were not operational enough, such that we will envision to modify them at an early date.

Furthermore, we have created an iterative elicitation process, called REWAT, for taking interactively into account some of the decision-maker's a priori overall preferences and reconstruct an outranking digraph in best accordance with his expectations, with the use of a very reduced set of selected pairs and their resulting comparisons. We have shown that caring about stability may result in a powerful tool for focusing the attention of the criteria weight elicitation process on the most sensitive outranking statements: For instance, the implicit validation of some outranking statements, according to the extensible stability property, has improved the relevance of the selected pairs of alternatives and has decreased the complexity of the questioning protocol. This process has been successfully used in a practical case, especially with novice decision-makers.

Finally, notice that the preferable relation appeared as quite natural for the decision-makers, such that it opens new perspectives in the construction of a preference elicitation process focused on this relation, as well as a stable exploitation, when possible, of this relation in order to provide a robust recommendation.

Perspectives

“I think quotes are very dangerous things.”

[Kate Bush]

In this additional chapter, we draw a non exhaustive list of possible perspectives for carrying out our work on the theoretical and practical aspects of the stability concept. Notice that these ideas are presented in a informal manner, with no particular order of importance.

1. Extending the theoretical work on the stability concept

Characterising the unstable relations more precisely

According to proposition 3.11 on the limitation of the stability, we know that any situation that is not supported by a majority of criteria is inherently unstable. However, some of them have only little chance to be questioned, unless the use of unconvincing vectors of weights (for instance considering a huge weight ratio between the most important and the least important criteria).

In that case, we may take into account additional working hypotheses in order to warrant a *conditional stability* for some unstable situations. For instance, one may logically consider a *non dictatorship hypothesis*: Assuming that every criteria, except the most important one, validates a situation, we can obviously ensure that the situation will not be questioned, even if it is unstable. Notice that these properties should be checked with the use of simple mathematical conditions, not mathematical programs.

Finally, we may also provide the decision-maker with a SMAA-like tool that computes, for each unstable situation, a ratio between the weights vector that validate the situation and those who do not.

Extending the theoretical work to the preferable relation

During the practical cases, it appeared quite natural to deal with the preferable relation, such that we may extend the related theoretical work, for instance in order to define the mathematical properties for checking the additional levels of stability.

Also, several methods consider the exploitation of a valued outranking digraph instead of a cut digraph (see for instance the RUBIS method [BMR08] or also the ELECTRE III method [Roy78]). In that case, it is important to measure the stability

of the relative credibility between any pairs of oriented arcs, in order to compute some robust recommendations: If we cannot ensure that an alternative is preferable to another in a stable manner, we need to be careful during the exploitation and maybe try to validate each situation separately, or verify if they are not necessary, according to already expressed output oriented preferential information.

Working with multiple decision-makers

As a natural extension of our work, we may consider to enhance our protocol in order to deal with multiple decision-makers. For instance, in chapter 3, section 3.2.2, we have noticed the possible use of extensible stability when considering multiple decision-makers, which would agree with a basic preorder, but they would wish to refine it in different manners. In that situation, the extensible stability will highlight the conflictual situations and those which are not.

Dealing with a hierarchy of criteria

As it stands, the stability properties may only be verified when considering a unique set of criteria associated with a complete preorder. Intuitively, you would like to consider the criteria gathered under some objectives, such that there is a different preorder between the criteria under each objective, as well as a preorder between the objectives, but it seems meaningless to talk about the relative importance between two criteria under two different objectives.

In that case, is it possible to talk about the stability of an outranking situation, according to such a defined hierarchy?

2. Improving the behavior of the algorithms

Decreasing the algorithms running times

We have seen that the number of boolean variables is correlated to the value of the parameter M (*i.e.* the maximal admissible value for any elicited weight). In consequence, the smaller this parameter is, the smaller the necessary number of boolean variables is, such that it may result in a logical decrease of the running time. However, if this parameter is too low, the mathematical model cannot be solved. Thus, an idea to use the smallest possible parameter will be to start by setting up $M = 1$ and increase it by one unit each time the solver cannot find a solution. If the necessary time to say that a model cannot be solved is short enough, it could be time-saving, instead of starting with a parameter that could be too high.

Also, in order to decrease the algorithms running times which become prohibitive on large instances, it may be advisable to consider the use of heuristic technics. For instance, some current and preliminary work, considering the DCA method [LTT05], tend to show that it is possible to recover parameters with a satisfying resulting percentage of stable statements in an acceptable time.

Also, we have considered only the use of the commercial CPLEX solver. We may try to compare the resolution on different solvers, like for instance GLPK, SCIP, or also GUROBI.

Making operational the sorting elicitation algorithms

As it stands, the sorting algorithms cannot be integrated into a progressive method for eliciting the parameters of an aggregation procedure, namely a vector of weights and the profiles that bounds the given categories. Indeed, as it considerably advantages the local balanced situations, it does not seem to be relevant, nor reliable, to use these elicited parameters to sort the alternatives on which the decision-maker did not express preferential information.

Several different approaches for improving the behavior of the algorithms can be envisioned: The first one may consist in minimising the balanced situations, namely to best ensure that the variables $S_i(x, b)$ will differ from 0. Remind the decomposition of these variables as the difference of two boolean variables $\alpha_i(x, b)$ and $\beta_i(x, b)$ (see section 5.2.1), trying to minimise the balanced situation (*i.e.* $\alpha_i(x, b) = 0$ and $\beta_i(x, b) = 0$) is equivalent to maximise the sum of the $\alpha_i(x, b)$ and $\beta_i(x, b)$, for all alternatives x and profile b . Another possible improvement of the algorithms could be to try to construct some most possibly homogeneous profiles: On each criterion, we could try to ensure a certain percentage of alternatives that are at “*at least as good as*” each profile, or intend to have the most possibly homogeneous distribution within the performances of the profiles. However, such a consideration induces that we have a clear idea of how the profiles should be constructed in advance. Also, it complicates the models that may be much more time-consuming.

Furthermore, trying to impose some constraints on the construction of the profiles may have an impact on the stability we have not yet measured. Indeed, it is possible that the objective of maximising the stability and the one of constructing homogeneous profiles can be conflicting, such that we should find a compromise between these objectives. Also notice that in the experimental tests, the weights vectors were never constrained. Some additional tests when enforcing some constraints on the weights, namely an initial preorder to be refined, should be done.

Concluding remarks

In the introduction of the thesis, we have highlighted the difficulty to provide reliable (*i.e.* precise and accurate) parameters for a multicriteria decision aid problem and the fact that it may have a significant impact on the resulting recommendations.

With the consideration of the stability concept, we hope to have provided some answers to this concern, to better detect the sensitive outranking situations and to provide the analyst with tools for guiding the decision aid process. Also, we hope to have shown that the stability concept is an asset for recovering some robust parameters via a progressive process.

Notice that the use of a progressive process implies a discussion with the decision-maker. That is to say it is not appropriate when considering a naive decision-maker, nor in the case of automatic decisions generated by a software.

Also, we would like to remind that this work is bounded within the framework of the outranking methods, such that it is not suitable when the decision-maker is not expressing his preferences according to the outranking principles.

We believe that it is not advisable to impose one particular philosophy, namely a valued method or an outranking method, but it is the analyst's responsibility to consider the most appropriate one, according to the available data, the encoding of the performances, as well as the decision-maker's way of taking into account the multiple criteria. Based on our experience, we have become convinced that a novice decision-maker, who is evaluating the local performances of the alternatives, is very often more comfortable in expressing qualitative evaluations rather than quantitative ones. In that case, the use of an outranking method avoid the difficulty to associate precise real performances to the different levels of the scale. By contrast, when working with an expert decision-maker who has a more precise knowledge of the decision domain, but also when the criteria are constructed by some experts of each point of view, the analyst has to ensure the decision-maker's accordance to the outranking principles.

A.1 Mathematical proof of Proposition 3.8

We prove the equivalence (3.4), by showing that it is a necessary and sufficient condition. First, let us assume that $\forall k \in 1, \dots, r : C_k^w(x, y) \geq 0$ and also that $\exists k \in 1, \dots, r : C_k^w(x, y) > 0$. It is easy to verify that:

$$\begin{aligned} S^w(x, y) &= \sum_{k=1}^r c_k^w(x, y) \cdot w_k \\ &= c_1^w(x, y) \cdot w_1 + c_2^w(x, y) \cdot w_2 + \dots + c_r^w(x, y) \cdot w_r \\ &= C_1^w(x, y) \cdot w_1 + c_2^w(x, y) \cdot w_2 + \dots + c_r^w(x, y) \cdot w_r \end{aligned}$$

As $w_2 < w_1$ and $C_1^w(x, y) \geq 0$:

$$\begin{aligned} &\geq (C_1^w(x, y) \cdot w_2 + c_2^w(x, y) \cdot w_2) + \dots + c_r^w(x, y) \cdot w_r \\ &\geq C_2^w(x, y) \cdot w_2 + c_3^w(x, y) \cdot w_3 + \dots + c_r^w(x, y) \cdot w_r \end{aligned}$$

As $w_3 < w_2$, $C_2^w(x, y) \geq 0$ and $C_2^w(x, y) + c_3^w(x, y) = C_3^w(x, y)$:

$$\geq C_3^w(x, y) \cdot w_3 + c_4^w(x, y) \cdot w_4 + \dots + c_r^w(x, y) \cdot w_r$$

[..]

$$\begin{aligned} &\geq C_r^w(x, y) \cdot w_r \\ S^w(x, y) &\geq 0 \end{aligned}$$

As it exists at least one strictly positive cumulative sum, it results a strict inequality. Then $S^w(x, y) > 0$. The condition is sufficient.

Let us assume now that there exists a cumulative sum $C_l^w(x, y) < 0$. For the ease of the proof, let us also assume that there is only one strictly negative cumulative sum, *i.e.* $\forall k \neq l : C_k^w(x, y) \geq 0$. If the condition is untrue, it means that there are more criteria against the validation of the outranking situation. If we show that it is possible to find a vector of weights, which is compatible with the preorder, that invalidates the outranking situation, then it will be also possible if there are more

criteria against the validation. We can face two different cases: either $l = 1$, which means that the most important class has more criteria against the validation than in favor, or $l > 1$. In the first case, it is easy to imagine that associating a very large weights to the first class and a very small for all other class will result to a negative outranking value. In the second case, let us define w_1 (the weight associated to the most important class), w_l and w_{l+1} as follows:

$$w_1 = 1 + \frac{-C_l^w(x, y)}{\sum_{t \neq l} c_t}, \quad w_l = 1, \quad w_{l+1} = \frac{-C_l^w(x, y)}{\sum_{t \neq l} c_t},$$

We then compute the outranking value:

$$\begin{aligned} S^w(x, y) &= \sum_{k=1}^r c_k^w(x, y) \cdot w_k \\ &\text{As } \sum_{k=1}^{l-1} c_k^w(x, y) = C_{l-1}^w(x, y) \geq 0 \text{ and } \forall k = 2..l-1, w_k < w_1 \text{ and} \\ &\text{as } \sum_{k=l+1}^r c_k^w(x, y) = C_k^w(x, y) - C_{l-1}^w(x, y) > 0 \text{ and } \forall k = l+2..r, w_k < w_{l+1} : \\ S^w(x, y) &< C_{l-1}^w(x, y) \cdot w_1 + c_l^w(x, y) \cdot w_l + \sum_{k=l+1}^r c_k^w(x, y) \cdot w_{l+1} \\ &< C_{l-1}^w(x, y) \cdot \left(1 + \frac{-C_l^w(x, y)}{\sum_{t \neq l} c_t}\right) + c_l^w(x, y) + \sum_{k=l+1}^r c_k^w(x, y) \cdot \left(\frac{-C_l^w(x, y)}{\sum_{t \neq l} c_t}\right) \\ &< C_l^w(x, y) + \frac{-C_l^w(x, y) \cdot \sum_{t \neq l} c_t}{\sum_{t \neq l} c_t} = C_l^w(x, y) - C_l^w(x, y) \\ S^w(x, y) &< 0 \end{aligned}$$

The condition is a necessary one and we have the equivalence. Proof of equivalence (3.5) is similar when inverting criteria in favor and in disfavor. \square

A.2 Complete mathematical models

A.2.1 ACON

MILP ACON

Variables:

$$\omega_{i,u} \in \{0, 1\}$$

$$\forall i \in F, \forall u = 1..M$$

Objective function:

$$\min \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$s.t. \quad \sum_{g_i \in F} \omega_{i,1} = m$$

$$\omega_{i,u} \geq \omega_{i,u+1}$$

$$\forall i \in F, \forall u = 1..M-1$$

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \geq 1$$

$$\forall (x, y) \in \mathfrak{G}$$

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \leq -1$$

$$\forall (x, y) \in \overline{\mathfrak{G}}$$

$$\sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (S_i(x, y) - S_i(y, x)) \geq 1$$

$$\forall (x, y) \in \mathfrak{F}$$

Constraints (informal) on the weights:

$$\sum_{c=1}^M \omega_{i,c} = v_i$$

For some $i \in F$

$$\sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v$$

For some $i \in F$

$$\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1$$

For some $(i, j) \in F^2$

$$\sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1$$

For some $H, K \subset F$

A.2.2 STAB₁MILP STAB₁

Variables:

$$\begin{array}{ll}
\omega_{i,u} \in \{0, 1\}, & \omega_{i,0} = 0 & \forall i \in F, \forall u = 1..M \\
b_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M \\
b'_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M \\
s_2(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
s'_2(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{F} \\
b_u(H, K) \in \{0, 1\} & & \forall u = 1..M \text{ and some } H, K \subset F \\
s_2(H, K) \geq 0 & & \text{For some } H, K \subset F
\end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s_2(x, y) + m \cdot M \cdot \sum_{(x,y) \in \mathfrak{F}} s'_2(x, y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
s.t. & \sum_{i \in F} \omega_{i,1} = m \\
& \omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M-1 \\
& \sum_{u=1}^M b_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
& \sum_{u=1}^M b'_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \geq 1 & \forall (x, y) \in \mathfrak{S} \\
& \sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) + s_2(x, y) \geq b_u(x, y) & \forall (x, y) \in \mathfrak{S}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \leq -1 & \forall (x, y) \in \overline{\mathfrak{S}} \\
& \sum_{i \in F} \omega_{i,u} \cdot S_i(x, y) - s_2(x, y) \leq -b_u(x, y) & \forall (x, y) \in \overline{\mathfrak{S}}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (S_i(x, y) - S_i(y, x)) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
& \sum_{i \in F} \omega_{i,u} \cdot (S_i(x, y) - S_i(y, x)) + s'_2(x, y) \geq b'_u(x, y) & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M
\end{array}$$

Constraints (informal) on the weights allowing to model decision-maker's preferences:

$$\begin{array}{ll}
\sum_{c=1}^M \omega_{i,c} = v_i & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 & \text{For some } (i, j) \in F^2 \\
\circ \quad \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 & \text{For some } H, K \subset F \\
\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_2(H, K) \geq b_u(H, K) & \forall u = 1..M \text{ and some } H, K \subset F \\
\sum_{u=1}^M b_u(H, K) \geq 1 & \forall u = 1..M \text{ and some } H, K \subset F
\end{array}$$

A.2.3 STAB₂MILP STAB₂

Variables:

$$\begin{array}{ll}
\omega_{i,u} \in \{0, 1\}, & \omega_{i,0} = 0 & \forall i \in F, \forall u = 1..M \\
b_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}}, \forall u = 1..M \\
b'_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M \\
s_2^b(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\
s_2'^b(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F} \\
b_u(H, K) \in \{0, 1\} & & \forall u = 1..M \text{ and some } H, K \subset F \\
s_2^b(H, K) \in \{0, 1\} & & \text{For some } H, K \subset F
\end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{G} \cup \overline{\mathfrak{G}}} s_2^b(x, y) + m \cdot M \cdot \sum_{(x,y) \in \mathfrak{F}} s_2'^b(x, y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
s.t. & \sum_{i \in F} \omega_{i,1} = m \\
& \omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M-1 \\
& \sum_{u=1}^M b_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{G} \cup \overline{\mathfrak{G}} \\
& \sum_{u=1}^M b'_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \geq 1 & \forall (x, y) \in \mathfrak{G} \\
& \sum_{i \in F} (\omega_{i,u} \cdot S_i(x, y)) + s_2^b(x, y) \cdot m \geq b_u(x, y) & \forall (x, y) \in \mathfrak{G}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot S_i(x, y) \leq -1 & \forall (x, y) \in \overline{\mathfrak{G}} \\
& \sum_{i \in F} (\omega_{i,u} \cdot S_i(x, y)) - s_2^b(x, y) \cdot m \leq -b_u(x, y) & \forall (x, y) \in \overline{\mathfrak{G}}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M \omega_{i,u} \right) \cdot (S_i(x, y) - S_i(y, x)) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
& \sum_{i \in F} \omega_{i,u} \cdot (S_i(x, y) - S_i(y, x)) + s_2'^b(x, y) \cdot m \geq b'_u(x, y) & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M
\end{array}$$

Constraints (informal) on the weights allowing to model decision-maker's preferences:

$$\begin{array}{ll}
\sum_{c=1}^M \omega_{i,c} = v_i & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 & \text{For some } (i, j) \in F^2 \\
\circ \quad \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 & \text{For some } H, K \subset F \\
\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_2^b(H, K) \cdot m \geq b_u(H, K) & \forall u = 1..M \text{ and some } H, K \subset F \\
\sum_{u=1}^M b_u(H, K) \geq 1 & \forall u = 1..M \text{ and some } H, K \subset F
\end{array}$$

A.2.4 ACON'

MILP ACON'

Variables:

$$\begin{aligned} \omega_{i,u} &\in \{0, 1\} && \forall i \in F, \forall u = 1..M \\ w_i^1(x, y) &\in [-M, M] && \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ \alpha_i(x, y), \beta_i(x, y) &\in \{0, 1\} && \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \end{aligned}$$

Objective function:

$$\min \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{aligned} s.t. \quad \sum_{g_i \in F} \omega_{i,1} &= m \\ \omega_{i,u} &\geq \omega_{i,u+1} && \forall i \in F, \forall u = 1..M-1 \\ \sum_{i \in F} w_i^1(x, y) &\geq 1 && \forall (x, y) \in \mathfrak{S} \\ \sum_{i \in F} w_i^1(x, y) &\leq -1 && \forall (x, y) \in \overline{\mathfrak{S}} \\ \sum_{i \in F} (w_i^1(x, y) - w_i^1(y, x)) &\geq 1 && \forall (x, y) \in \mathfrak{F} \end{aligned}$$

Constraints on the coherence of the $w_i^1(x, y)$ variables:

$$\begin{aligned} -\sum_{u=1}^M \omega_{i,u} &\leq w_i^1(x, y) \leq \sum_{u=1}^M \omega_{i,u} && \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ \sum_{u=1}^M \omega_{i,u} + (\alpha_i(x, y) - \beta_i(x, y)) \cdot M - M &\leq w_i^1(x, y) && \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ w_i^1(x, y) &\leq -\sum_{u=1}^M \omega_{i,u} + (\alpha_i(x, y) - \beta_i(x, y)) \cdot M + M && \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ -(\alpha_i(x, y) + \beta_i(x, y)) \cdot M &\leq w_i^1(x, y) \leq (\alpha_i(x, y) + \beta_i(x, y)) \cdot M && \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \end{aligned}$$

Constraints on the coherence of the α_i and β_i variables:

$$\begin{aligned} -2(1 - \alpha_i(x, y)) \cdot Max_i &\leq x_i - y_i + q_i < 2\alpha_i(x, y) \cdot Max_i && \forall i \in F_c, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ -2\beta_i(x, y) \cdot Max_i &\leq x_i - y_i + p_i < 2(1 - \beta_i(x, y)) \cdot Max_i && \forall i \in F_c, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ -2(1 - \alpha_i(x, y)) \cdot Max_i &\leq x_i - y_i + x_i \cdot q_i < 2\alpha_i(x, y) \cdot Max_i && \forall i \in F_{\%}, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ -2\beta_i(x, y) \cdot Max_i &\leq x_i - y_i + x_i \cdot p_i < 2(1 - \beta_i(x, y)) \cdot Max_i && \forall i \in F_{\%}, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ -2(1 - \alpha_i(x, y)) \cdot Max_i &\leq x_i - y_i + q(x_i) < 2\alpha_i(x, y) \cdot Max_i && \forall i \in F_g, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\ -2\beta_i(x, y) \cdot Max_i &\leq x_i - y_i + p(x_i) < 2(1 - \beta_i(x, y)) \cdot Max_i && \forall i \in F_g, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \end{aligned}$$

Constraints (informal) on the weights:

$$\begin{aligned} \sum_{c=1}^M \omega_{i,c} &= v_i && \text{For some } i \in F \\ \sum_{c=1}^M \omega_{i,c} &\geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v && \text{For some } i \in F \\ \sum_{c=1}^M \omega_{i,c} &\geq \sum_{c=1}^M \omega_{j,c} + 1 && \text{For some } (i, j) \in F^2 \\ \sum_{i \in H} (\sum_{c=1}^M \omega_{i,c}) &\geq \sum_{j \in K} (\sum_{c=1}^M \omega_{j,c}) + 1 && \text{For some } H, K \subset F \end{aligned}$$

A.2.5 STAB'₁MILP STAB'₁*Variables:*

$$\begin{array}{ll}
\omega_{i,u} \in \{0, 1\}, & \omega_{i,0} = 0 & \forall i \in F, \forall u = 1..M \\
\alpha_i(x, y), \beta_i(x, y) \in \{0, 1\} & & \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
b_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M \\
b'_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M \\
s_2(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
s'_2(x, y) \geq 0 & & \forall (x, y) \in \mathfrak{F} \\
b_u(H, K) \in \{0, 1\} & & \forall u = 1..M \text{ and some } H, K \subset F \\
s_2(H, K) \geq 0 & & \text{For some } H, K \subset F
\end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s_2(x, y) + m \cdot M \cdot \sum_{(x,y) \in \mathfrak{F}} s'_2(x, y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
\text{s.t.} & \sum_{g_i \in F} \omega_{i,1} = m \\
& \omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M - 1 \\
& \sum_{u=1}^M b_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
& \sum_{u=1}^M b'_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \geq 1 & \forall (x, y) \in \mathfrak{S} \\
& \sum_{i \in F} w_{i,u}^2(x, y) + s_2(x, y) \geq b_u(x, y) & \forall (x, y) \in \mathfrak{S}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \leq -1 & \forall (x, y) \in \overline{\mathfrak{S}} \\
& \sum_{i \in F} w_{i,u}^2(x, y) - s_2(x, y) \leq -b_u(x, y) & \forall (x, y) \in \overline{\mathfrak{S}}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) - w_{i,u}^2(y, x) \right) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
& \sum_{i \in F} \left(w_{i,u}^2(x, y) - w_{i,u}^2(y, x) \right) + s'_2(x, y) \geq b'_u(x, y) & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M
\end{array}$$

Constraints on the coherence of the $w_{i,u}^2(x, y)$ variables:

$$\begin{array}{ll}
-\omega_{i,u} \leq w_{i,u}^2(x, y) \leq \omega_{i,u} & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) - 1 \leq w_{i,u}^2(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
w_{i,u}^2(x, y) \leq -\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) + 1 & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-(\alpha_i(x, y) + \beta_i(x, y)) \leq w_{i,u}^2(x, y) & \\
\leq \alpha_i(x, y) + \beta_i(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

Constraints on the coherence of the α_i and β_i variables:

$$\begin{array}{ll}
-2(1 - \alpha_i(x, y)) \cdot \text{Max}_i \leq x_i - y_i + q_i & \\
\leq 2\alpha_i(x, y) \cdot \text{Max}_i & \forall i \in F_c, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-2\beta_i(x, y) \cdot \text{Max}_i \leq x_i - y_i + p_i & \\
\leq 2(1 - \beta_i(x, y)) \cdot \text{Max}_i & \forall i \in F_c, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-2(1 - \alpha_i(x, y)) \cdot \text{Max}_i \leq x_i - y_i + x_i \cdot q_i & \\
\leq 2\alpha_i(x, y) \cdot \text{Max}_i & \forall i \in F_{\%}, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

$$\begin{aligned}
-2\beta_i(x, y) \cdot Max_i &\leq x_i - y_i + x_i \cdot p_i \\
&< 2(1 - \beta_i(x, y)) \cdot Max_i \quad \forall i \in F\%, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-2(1 - \alpha_i(x, y)) \cdot Max_i &\leq x_i - y_i + q(x_i) \\
&< 2\alpha_i(x, y) \cdot Max_i \quad \forall i \in F_g, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-2\beta_i(x, y) \cdot Max_i &\leq x_i - y_i + p(x_i) \\
&< 2(1 - \beta_i(x, y)) \cdot Max_i \quad \forall i \in F_g, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{aligned}$$

Constraints (informal) on the weights:

$$\begin{aligned}
\sum_{c=1}^M \omega_{i,c} &= v_i && \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} &\geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v && \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} &\geq \sum_{c=1}^M \omega_{j,c} + 1 && \text{For some } (i, j) \in F^2 \\
\circ \quad \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) &\geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 && \text{For some } H, K \subset F \\
\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_2(H, K) &\geq b_u(H, K) && \forall u = 1..M \text{ and some } H, K \subset F \\
\sum_{u=1}^M b_u(H, K) &\geq 1 && \forall u = 1..M \text{ and some } H, K \subset F
\end{aligned}$$

A.2.6 STAB'₂MILP STAB'₂*Variables:*

$$\begin{array}{ll}
\omega_{i,u} \in \{0, 1\}, & \omega_{i,0} = 0 & \forall i \in F, \forall u = 1..M \\
\alpha_i(x, y), \beta_i(x, y) \in \{0, 1\} & & \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
b_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M \\
b'_u(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M \\
s_2^b(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
s_2'^b(x, y) \in \{0, 1\} & & \forall (x, y) \in \mathfrak{F} \\
b_u(\mathbb{H}, \mathbb{K}) \in \{0, 1\} & & \forall u = 1..M \text{ and some } \mathbb{H}, \mathbb{K} \subset F \\
s_2^b(\mathbb{H}, \mathbb{K}) \in \{0, 1\} & & \text{For some } \mathbb{H}, \mathbb{K} \subset F
\end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s_2^b(x, y) + m \cdot M \cdot \sum_{(x,y) \in \mathfrak{F}} s_2'^b(x, y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
s.t. & \sum_{g_i \in F} \omega_{i,1} = m \\
& \omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M - 1 \\
& \sum_{u=1}^M b_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
& \sum_{u=1}^M b'_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \geq 1 & \forall (x, y) \in \mathfrak{S} \\
& \sum_{i \in F} w_{i,u}^2(x, y) + s_2^b(x, y) \cdot m \geq b_u(x, y) & \forall (x, y) \in \mathfrak{S}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \leq -1 & \forall (x, y) \in \overline{\mathfrak{S}} \\
& \sum_{i \in F} w_{i,u}^2(x, y) - s_2^b(x, y) \cdot m \leq -b_u(x, y) & \forall (x, y) \in \overline{\mathfrak{S}}, \forall u = 1..M \\
\circ & \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) - w_{i,u}^2(y, x) \right) \geq 1 & \forall (x, y) \in \mathfrak{F} \\
& \sum_{i \in F} \left(w_{i,u}^2(x, y) - w_{i,u}^2(y, x) \right) + s_2'^b(x, y) \cdot m \geq b'_u(x, y) & \forall (x, y) \in \mathfrak{F}, \forall u = 1..M
\end{array}$$

Constraints on the coherence of the $w_{i,u}^2(x, y)$ variables:

$$\begin{array}{ll}
-\omega_{i,u} \leq w_{i,u}^2(x, y) \leq \omega_{i,u} & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) - 1 \leq w_{i,u}^2(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
w_{i,u}^2(x, y) \leq -\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) + 1 & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-(\alpha_i(x, y) + \beta_i(x, y)) \leq w_{i,u}^2(x, y) & \\
& \leq \alpha_i(x, y) + \beta_i(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

Constraints on the coherence of the α_i and β_i variables:

$$\begin{array}{ll}
-2(1 - \alpha_i(x, y)) \cdot \text{Max}_i \leq x_i - y_i + q_i & \\
& < 2\alpha_i(x, y) \cdot \text{Max}_i & \forall i \in F_c, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-2\beta_i(x, y) \cdot \text{Max}_i \leq x_i - y_i + p_i & \\
& < 2(1 - \beta_i(x, y)) \cdot \text{Max}_i & \forall i \in F_c, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-2(1 - \alpha_i(x, y)) \cdot \text{Max}_i \leq x_i - y_i + x_i \cdot q_i & \\
& < 2\alpha_i(x, y) \cdot \text{Max}_i & \forall i \in F\%, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

$$\begin{aligned}
-2\beta_i(x, y) \cdot Max_i &\leq x_i - y_i + x_i \cdot p_i && \forall i \in F\%, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
&< 2(1 - \beta_i(x, y)) \cdot Max_i \\
-2(1 - \alpha_i(x, y)) \cdot Max_i &\leq x_i - y_i + q(x_i) && \forall i \in F_g, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
&< 2\alpha_i(x, y) \cdot Max_i \\
-2\beta_i(x, y) \cdot Max_i &\leq x_i - y_i + p(x_i) && \forall i \in F_g, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
&< 2(1 - \beta_i(x, y)) \cdot Max_i
\end{aligned}$$

Constraints (informal) on the weights:

$$\begin{aligned}
\sum_{c=1}^M \omega_{i,c} &= v_i && \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} &\geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v && \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} &\geq \sum_{c=1}^M \omega_{j,c} + 1 && \text{For some } (i, j) \in F^2 \\
\circ \quad \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) &\geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 && \text{For some } H, K \subset F \\
\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_2(H, K) &\geq b_u(H, K) && \forall u = 1..M \text{ and some } H, K \subset F \\
\sum_{u=1}^M b_u(H, K) &\geq 1 && \forall u = 1..M \text{ and some } H, K \subset F
\end{aligned}$$

A.2.7 ACON*

MILP ACON*

Variables:

$$\omega_{i,u} \in \{0, 1\}$$

$$w_{i,u}^2(x, y) \in [-1, 1]$$

$$\alpha_i(x, y), \beta_i(x, y) \in \{0, 1\}$$

$$b_u(x, y) \in \{0, 1\}$$

$$\forall i \in F, \forall u = 1..M$$

$$\forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

$$\forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

$$\forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M$$

Objective function:

$$\min \sum_{g_i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$s.t. \sum_{g_i \in F} \omega_{i,1} = m$$

$$\omega_{i,u} \geq \omega_{i,u+1}$$

$$\sum_{u=1}^M b_u(x, y) \geq 1$$

$$\circ \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \geq 1$$

$$\circ \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \leq -1$$

$$\forall i \in F, \forall u = 1..M - 1$$

$$\forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

$$\forall (x, y) \in \mathfrak{S}$$

$$\forall (x, y) \in \overline{\mathfrak{S}}$$

Constraints on the coherence of the $w_{i,u}^2(x, y)$ variables:

$$-\omega_{i,u} \leq w_{i,u}^2(x, y) \leq \omega_{i,u}$$

$$\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) - 1 \leq w_{i,u}^2(x, y)$$

$$w_{i,u}^2(x, y) \leq -\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) + 1$$

$$-(\alpha_i(x, y) + \beta_i(x, y)) \leq w_{i,u}^2(x, y) \leq \alpha_i(x, y) + \beta_i(x, y)$$

$$\forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

$$\forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

$$\forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

$$\forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

Constraints on the coherence of the α_i and β_i variables:

$$\alpha_i(x, b) + \beta_i(x, b) \leq 1$$

$$\alpha_i(x, b_h) \leq \alpha_i(x, b_{h-1})$$

$$\beta_i(x, b_h) \geq \beta_i(x, b_{h-1})$$

$$\alpha_i(x, b_h) \geq \alpha_i(y, b_h)$$

$$\beta_i(x, b_h) \leq \beta_i(y, b_h)$$

$$\alpha_i(x, b_h) = \alpha_i(y, b_h)$$

$$\beta_i(x, b_h) = \beta_i(y, b_h)$$

$$\forall x \in A', \forall b \in B, \forall i \in F$$

$$\forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F$$

$$\forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F$$

$$\forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i$$

$$\forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i$$

$$\forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i$$

$$\forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i$$

Constraints (informal) on the weights:

$$\sum_{c=1}^M \omega_{i,c} = v_i$$

$$\sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v$$

$$\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1$$

$$\circ \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1$$

$$\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_2(H, K) \geq b_u(H, K)$$

$$\sum_{u=1}^M b_u(H, K) \geq 1$$

For some criteria i For some criteria i For some couples (i, j) of criteriaFor some criteria subsets H and K $\forall u = 1..M$ and some H and K $\forall u = 1..M$ and some H and K

A.2.8 STAB₁^{*}MILP STAB₁^{*}

Variables:

$$\begin{array}{ll}
\omega_{i,u} \in \{0,1\} & \forall i \in F, \forall u = 1..M \\
w_{i,u}^2(x,y) \in [-1,1] & \forall i \in F, \forall u = 1..M, \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\alpha_i(x,y), \beta_i(x,y) \in \{0,1\} & \forall i \in F, \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
b_u(x,y) \in \{0,1\} & \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M \\
s_2(x,y) \geq 0 & \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s_2(x,y) + \sum_{g_i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
s.t. \quad \sum_{g_i \in F} \omega_{i,1} = m & \\
\omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M-1 \\
\sum_{u=1}^M b_u(x,y) \geq 1 & \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\circ \quad \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x,y) \right) \geq 1 & \forall (x,y) \in \mathfrak{S} \\
\sum_{i \in F} w_{i,u}^2(x,y) + s_2(x,y) \geq b_u(x,y) & \forall (x,y) \in \mathfrak{S}, \forall u = 1..M \\
\circ \quad \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x,y) \right) \leq -1 & \forall (x,y) \in \overline{\mathfrak{S}} \\
\sum_{i \in F} w_{i,u}^2(x,y) - s_2(x,y) \leq -b_u(x,y) & \forall (x,y) \in \overline{\mathfrak{S}}, \forall u = 1..M
\end{array}$$

Constraints on the coherence of the $w_{i,u}^2(x,y)$ variables:

$$\begin{array}{ll}
-\omega_{i,u} \leq w_{i,u}^2(x,y) \leq \omega_{i,u} & \forall i \in F, \forall u = 1..M, \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\omega_{i,u} + \alpha_i(x,y) - \beta_i(x,y) - 1 \leq w_{i,u}^2(x,y) & \forall i \in F, \forall u = 1..M, \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
w_{i,u}^2(x,y) \leq -\omega_{i,u} + \alpha_i(x,y) - \beta_i(x,y) + 1 & \forall i \in F, \forall u = 1..M, \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-(\alpha_i(x,y) + \beta_i(x,y)) \leq w_{i,u}^2(x,y) \leq \alpha_i(x,y) + \beta_i(x,y) & \forall i \in F, \forall u = 1..M, \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

Constraints on the coherence of the α_i and β_i variables:

$$\begin{array}{ll}
\alpha_i(x,b) + \beta_i(x,b) \leq 1 & \forall x \in A', \forall b \in B, \forall i \in F \\
\alpha_i(x, b_h) \leq \alpha_i(x, b_{h-1}) & \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F \\
\beta_i(x, b_h) \geq \beta_i(x, b_{h-1}) & \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F \\
\alpha_i(x, b_h) \geq \alpha_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i \\
\beta_i(x, b_h) \leq \beta_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i \\
\alpha_i(x, b_h) = \alpha_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i \\
\beta_i(x, b_h) = \beta_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i
\end{array}$$

Constraints (informal) on the weights:

$$\begin{array}{ll}
\sum_{c=1}^M \omega_{i,c} = v_i & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 & \text{For some } (i,j) \in F^2 \\
\circ \quad \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 & \text{For some } H, K \subset F \\
\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_2(H,K) \geq b_u(H,K) & \forall u = 1..M \text{ and some } H, K \subset F \\
\sum_{u=1}^M b_u(H,K) \geq 1 & \forall u = 1..M \text{ and some } H, K \subset F
\end{array}$$

A.2.9 STAB₂^{*}MILP STAB₂^{*}

Variables:

$$\begin{array}{ll}
\omega_{i,u} \in \{0, 1\} & \forall i \in F, \forall u = 1..M \\
w_{i,u}^2(x, y) \in [-1, 1] & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\alpha_i(x, y), \beta_i(x, y) \in \{0, 1\} & \forall i \in F, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
b_u(x, y) \in \{0, 1\} & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M \\
s_2^b(x, y) \geq 0 & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

Objective function:

$$\min \quad m \cdot M \cdot \sum_{(x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}} s_2^b(x, y) + \sum_{i \in F} \sum_{u=1}^M \omega_{i,u}$$

Constraints:

$$\begin{array}{ll}
s.t. \quad \sum_{g_i \in F} \omega_{i,1} = m & \\
\omega_{i,u} \geq \omega_{i,u+1} & \forall i \in F, \forall u = 1..M-1 \\
\sum_{u=1}^M b_u(x, y) \geq 1 & \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\circ \quad \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \geq 1 & \forall (x, y) \in \mathfrak{S} \\
\sum_{i \in F} w_{i,u}^2(x, y) + s_2^b(x, y) \cdot m \geq b_u(x, y) & \forall (x, y) \in \mathfrak{S}, \forall u = 1..M \\
\circ \quad \sum_{i \in F} \left(\sum_{u=1}^M w_{i,u}^2(x, y) \right) \leq -1 & \forall (x, y) \in \overline{\mathfrak{S}} \\
\sum_{i \in F} w_{i,u}^2(x, y) - s_2^b(x, y) \cdot m \leq -b_u(x, y) & \forall (x, y) \in \overline{\mathfrak{S}}, \forall u = 1..M
\end{array}$$

Constraints on the coherence of the $w_{i,u}^2(x, y)$ variables:

$$\begin{array}{ll}
-\omega_{i,u} \leq w_{i,u}^2(x, y) \leq \omega_{i,u} & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) - 1 \leq w_{i,u}^2(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
w_{i,u}^2(x, y) \leq -\omega_{i,u} + \alpha_i(x, y) - \beta_i(x, y) + 1 & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}} \\
-(\alpha_i(x, y) + \beta_i(x, y)) \leq w_{i,u}^2(x, y) \leq \alpha_i(x, y) + \beta_i(x, y) & \forall i \in F, \forall u = 1..M, \forall (x, y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}
\end{array}$$

Constraints on the coherence of the α_i and β_i variables:

$$\begin{array}{ll}
\alpha_i(x, b) + \beta_i(x, b) \leq 1 & \forall x \in A', \forall b \in B, \forall i \in F \\
\alpha_i(x, b_h) \leq \alpha_i(x, b_{h-1}) & \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F \\
\beta_i(x, b_h) \geq \beta_i(x, b_{h-1}) & \forall x \in A', \forall b_h \in B \setminus \{b_1\}, \forall i \in F \\
\alpha_i(x, b_h) \geq \alpha_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i \\
\beta_i(x, b_h) \leq \beta_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i > y_i \\
\alpha_i(x, b_h) = \alpha_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i \\
\beta_i(x, b_h) = \beta_i(y, b_h) & \forall b_h \in B, \forall x, y \in A', \forall i \in F, x_i = y_i
\end{array}$$

Constraints (informal) on the weights:

$$\begin{array}{ll}
\sum_{c=1}^M \omega_{i,c} = v_i & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq u \quad \text{and} \quad \sum_{c=1}^M \omega_{i,c} \leq v & \text{For some } i \in F \\
\sum_{c=1}^M \omega_{i,c} \geq \sum_{c=1}^M \omega_{j,c} + 1 & \text{For some } (i, j) \in F^2 \\
\circ \quad \sum_{i \in H} \left(\sum_{c=1}^M \omega_{i,c} \right) \geq \sum_{j \in K} \left(\sum_{c=1}^M \omega_{j,c} \right) + 1 & \text{For some } H, K \subset F \\
\sum_{i \in H} \omega_{i,u} - \sum_{j \in K} \omega_{j,u} + s_2(H, K) \geq b_u(H, K) & \forall u = 1..M \text{ and some } H, K \subset F \\
\sum_{u=1}^M b_u(H, K) \geq 1 & \forall u = 1..M \text{ and some } H, K \subset F
\end{array}$$

A.3 Data for the case study

| | Language | Domain | ANA | ANA5 | ANQ | ANS | ANC | LFR | UIR |
|-----------------|----------|--------|-----|------|------|-----|-----|-----|---------------|
| FR ₁ | French | ST | 108 | 22 | 844 | 21 | 4 | B | <12 (>300ème) |
| FR ₂ | French | OR | 7 | 7 | 4 | 1 | 1 | A+ | 39 (36ème) |
| FR ₃ | French | MCDA | 43 | 9 | 128 | 9 | 0 | A+ | 16 (187ème) |
| FR ₄ | French | DM | 33 | 8 | 65 | 13 | 2 | A+ | <12 (>300ème) |
| FR ₅ | French | MCDA | 12 | 6 | 24 | 1 | 1 | A | <12 (>300ème) |
| FR ₆ | French | MCDA | 17 | 8 | 25 | 2 | 1 | A+ | 18 (156ème) |
| FR ₇ | French | GT | 29 | 12 | 55 | 5 | 2 | B | <12 (>300ème) |
| LU ₁ | French | DM | 78 | 15 | 413 | 10 | 2 | NA | <12 (>300ème) |
| LU ₂ | English | OR | 75 | 19 | 153 | 11 | 5 | NA | <12 (>300ème) |
| SW ₁ | English | ST | 9 | 7 | 6 | 1 | 1 | NA | 42 (31ème) |
| SW ₂ | French | OR | 109 | 14 | 238 | 20 | 3 | NA | 42 (31ème) |
| EN ₁ | English | MCDA | 21 | 5 | 53 | 5 | 2 | NA | 16 (187ème) |
| EN ₂ | English | GT | 79 | 18 | 871 | 8 | 1 | NA | 18 (162ème) |
| NL ₁ | English | GT | 161 | 11 | 614 | 13 | 4 | NA | 21 (123ème) |
| NL ₂ | English | MCDA | 5 | 4 | 2 | 1 | 1 | NA | 21 (123ème) |
| US ₁ | English | GT | 23 | 12 | 13 | 2 | 1 | NA | 26 (86ème) |
| US ₂ | English | OR | 138 | 28 | 1078 | 24 | 7 | NA | 29 (66ème) |
| CA ₁ | French | GT | 18 | 10 | 22 | 3 | 0 | NA | 12 (299ème) |
| CA ₂ | French | MCDA | 6 | 6 | 0 | 1 | 0 | NA | 54 (14ème) |
| JP ₁ | French | MCDA | 148 | 20 | 536 | 14 | 4 | NA | 21 (132ème) |
| IN ₁ | French | ST | 29 | 6 | 163 | 6 | 1 | NA | 15 (211ème) |

| | Country (city) | Distance | EST | Salary | Med. salary |
|-----------------|-------------------------|--------------------|--------|--------|-------------|
| FR ₁ | France (Rennes) | 2h by train | 80€ | 1,600€ | 1,552€ |
| FR ₂ | France (Paris) | 5h by train | 120€ | 1,600€ | 1,552€ |
| FR ₃ | France (Toulouse) | 11h by train | 300€ | 1,600€ | 1,552€ |
| FR ₄ | France (Brest) | None | 0€ | 1,600€ | 1,552€ |
| FR ₅ | France (Brest) | None | 0€ | 1,600€ | 1,552€ |
| FR ₆ | France (Grenoble) | 2h30 by plane | 400€ | 1,600€ | 1,552€ |
| FR ₇ | France (Marseille) | 1h30 by plane | 120€ | 1,600€ | 1,552€ |
| LU ₁ | Luxembourg | 6h by train | 150€ | 2,400€ | 2,576€ |
| LU ₂ | Luxembourg | 6h by train | 150€ | 2,400€ | 2,576€ |
| SW ₁ | Switzerland (Lausanne) | 7h by plane | 240€ | 3,700€ | 4,300€ |
| SW ₂ | Switzerland (Lausanne) | 7h by plane | 240€ | 3,700€ | 4,300€ |
| EN ₁ | England (Liverpool) | 2h45 by plane | 300€ | 2,500€ | 1,835€ |
| EN ₂ | England (Londres) | 2h30 plane + train | 150€ | 2,500€ | 1,835€ |
| NL ₁ | Netherlands (Amsterdam) | 2h30 by plane | 300€ | 1,960€ | 1,629€ |
| NL ₂ | Netherlands (Amsterdam) | 2h30 by plane | 300€ | 1,960€ | 1,629€ |
| US ₁ | USA (Philadelphia) | 15h20 by plane | 750€ | 2,450€ | 2,788€ |
| US ₂ | USA (Washington) | 10h by plane | 680€ | 2,450€ | 2,788€ |
| CA ₁ | Canada (Quebec) | 17h by plane | 780€ | 1,800€ | 2,000€ |
| CA ₂ | Canada (Toronto) | 15h by plane | 850€ | 2,200€ | 2,000€ |
| JP ₁ | Japan (Tokyo) | 18h by plane | 1,100€ | 2,500€ | 3,500€ |
| IN ₁ | India (New Delhi) | 10h by plane | 1,200€ | 1,600€ | 48€ |

ANA : Advisor Number of published Articles

ANC : Number of current Ph.D. students

ANA5 : Number of Articles in the last 5 years

LFR : Laboratory French Reputation

ANQ : Advisor Number of Quotations

UIR : University International Ranking

ANS : Number of trained Ph.D. students

EST : Estimated transportation cost/year

ST : Security Theory

OR : Operational Research

MCDA : Multiple Criteria Decision Aid

DM : Data Mining

GT : Graph Theory

A.4 Local concordance values for the case study

A.4.1 Local concordance values – Initial preorder and thresholds

| | ACE(1) | ARD(2) | ARE(1) | AT(1) | UIR(1) | CPP(1) | TBH(1) | pess. | opt. | |
|---------------|--------|--------|--------|-------|--------|--------|--------|----------|----------|---|
| FR_02 - SW_02 | 1 | 1 | . | 1 | . | 1 | 1 | + 4 (2) | + 8 (4) | * |
| FR_02 - EN_02 | 1 | 1 | . | 1 | 1 | . | 1 | + 4 (2) | + 8 (4) | * |
| FR_02 - NL_01 | 1 | 1 | . | 1 | 1 | . | 1 | + 4 (2) | + 8 (4) | * |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | 1 | + 2 (1) | + 4 (1) | |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | + 8 (4) | + 8 (4) | |
| FR_02 - US_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) | |
| FR_02 - CA_02 | 1 | -1 | 1 | . | . | . | 1 | - 2 (2) | + 4 (1) | * |
| SW_02 - FR_02 | 1 | 1 | 1 | . | 1 | . | -1 | + 2 (2) | + 6 (3d) | * |
| SW_02 - EN_02 | 1 | 1 | . | . | 1 | . | 1 | + 2 (2) | + 8 (4) | * |
| SW_02 - NL_01 | 1 | 1 | . | 1 | 1 | . | 1 | + 4 (2) | + 8 (4) | * |
| SW_02 - NL_02 | 1 | -1 | 1 | . | 1 | . | 1 | 0 (1) | + 4 (1) | |
| SW_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | + 6 (3d) | + 8 (4) | |
| SW_02 - US_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) | |
| SW_02 - CA_02 | 1 | -1 | 1 | . | . | . | 1 | - 2 (2) | + 4 (1) | * |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | . | 1 | -1 | - 2 (2) | 0 (1) | * |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | . | 1 | 1 | 0 (1) | + 2 (1) | |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | + 8 (4) | |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | + 2 (1) | + 4 (1) | |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | + 8 (4) | |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | + 2 (1) | + 4 (1) | |
| EN_02 - CA_02 | -1 | -1 | 1 | . | . | 1 | 1 | - 2 (2) | + 2 (1) | * |
| NL_01 - FR_02 | -1 | -1 | 1 | . | . | 1 | -1 | - 4 (2) | 0 (1) | * |
| NL_01 - SW_02 | -1 | -1 | 1 | . | . | 1 | 1 | - 2 (2) | + 2 (1) | * |
| NL_01 - EN_02 | 1 | 1 | . | . | 1 | . | 1 | + 2 (2) | + 8 (4) | * |
| NL_01 - NL_02 | 1 | -1 | 1 | . | 1 | 1 | 1 | + 2 (1) | + 4 (1) | |
| NL_01 - US_01 | 1 | 1 | 1 | . | . | 1 | 1 | + 4 (2) | + 8 (4) | * |
| NL_01 - US_02 | 1 | -1 | . | 1 | . | 1 | 1 | 0 (1) | + 4 (1) | |
| NL_01 - CA_02 | -1 | -1 | 1 | . | . | 1 | 1 | - 2 (2) | + 2 (1) | * |
| NL_02 - FR_02 | -1 | 1 | . | 1 | . | 1 | -1 | 0 (1) | + 4 (2) | * |
| NL_02 - SW_02 | -1 | 1 | . | 1 | . | 1 | 1 | + 2 (2) | + 6 (3d) | * |
| NL_02 - EN_02 | 1 | 1 | . | 1 | 1 | . | 1 | + 4 (2) | + 8 (4) | * |
| NL_02 - NL_01 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) | |
| NL_02 - US_01 | 1 | 1 | . | 1 | . | 1 | 1 | + 4 (2) | + 8 (4) | * |
| NL_02 - US_02 | 1 | 1 | . | 1 | . | 1 | 1 | + 4 (2) | + 8 (4) | * |
| NL_02 - CA_02 | -1 | 1 | 1 | . | . | 1 | 1 | + 2 (2) | + 6 (3d) | * |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | . | . | -1 | - 4 (2) | 0 (1) | * |
| US_01 - SW_02 | -1 | -1 | . | 1 | . | . | -1 | - 6 (3d) | 0 (1) | * |
| US_01 - EN_02 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | + 6 (3d) | * |
| US_01 - NL_01 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | + 6 (3d) | * |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | -1 | 0 (1) | + 2 (1) | |
| US_01 - US_02 | 1 | -1 | . | 1 | . | 1 | 1 | 0 (1) | + 4 (1) | |
| US_01 - CA_02 | -1 | -1 | 1 | . | . | . | 1 | - 4 (2) | + 2 (1) | * |
| US_02 - FR_02 | -1 | 1 | 1 | . | . | . | -1 | - 2 (1) | + 4 (2) | * |
| US_02 - SW_02 | -1 | 1 | 1 | . | . | . | -1 | - 2 (1) | + 4 (2) | * |
| US_02 - EN_02 | 1 | 1 | . | . | 1 | . | -1 | 0 (1) | + 6 (3d) | * |
| US_02 - NL_01 | 1 | 1 | 1 | . | 1 | . | -1 | + 2 (2) | + 6 (3d) | * |
| US_02 - NL_02 | 1 | -1 | 1 | . | 1 | . | -1 | - 2 (2) | + 2 (1) | * |
| US_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | + 6 (3d) | + 8 (4) | |
| US_02 - CA_02 | -1 | -1 | 1 | . | . | . | 1 | - 4 (2) | + 2 (1) | * |
| CA_02 - FR_02 | 1 | 1 | . | 1 | 1 | 1 | -1 | + 4 (2) | + 6 (3d) | * |
| CA_02 - SW_02 | 1 | 1 | . | 1 | 1 | 1 | -1 | + 4 (2) | + 6 (3d) | * |
| CA_02 - EN_02 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | + 6 (3d) | * |
| CA_02 - NL_01 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | + 6 (3d) | * |
| CA_02 - NL_02 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | + 6 (3d) | * |
| CA_02 - US_01 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) | |
| CA_02 - US_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) | |

* : Situations that are not extensively stable

Selected alternatives: SW_02 - CA_02

A.4.2 After the comparison of alternatives SW₂ and CA₂

| | ACE(1) | ARD(2) | ARE(1) | AT(1) | UIR(1) | CPP(1) | TBH(1) | pess. | opt. |
|---------------|--------|--------|--------|-------|--------|--------|--------|----------|-----------|
| FR_02 - SW_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | +8 (4) |
| FR_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | +6 (3d) * |
| FR_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | +6 (3d) * |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | 1 | + 2 (1) | +4 (1) |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | + 8 (4) | +8 (4) |
| FR_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | +6 (3d) |
| FR_02 - CA_02 | 1 | -1 | 1 | . | . | . | 1 | - 2 (2) | +4 (1) * |
| SW_02 - FR_02 | 1 | 1 | 1 | . | 1 | . | -1 | + 2 (2) | +6 (3d) * |
| SW_02 - EN_02 | 1 | 1 | -1 | . | 1 | . | 1 | + 2 (2) | +6 (3d) * |
| SW_02 - NL_01 | 1 | 1 | . | 1 | 1 | . | 1 | + 4 (2) | +8 (4) * |
| SW_02 - NL_02 | 1 | -1 | 1 | . | 1 | . | 1 | 0 (1) | +4 (1) |
| SW_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | + 6 (3d) | +8 (4) |
| SW_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | +6 (3d) |
| SW_02 - CA_02 | 1 | -1 | 1 | -1 | 1 | . | 1 | 0 (1) | +2 (1) . |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | . | 1 | -1 | - 2 (2) | 0 (1) * |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | . | 1 | 1 | 0 (1) | +2 (1) |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | +8 (4) |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | + 2 (1) | +4 (1) |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | +8 (4) |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | + 2 (1) | +4 (1) |
| EN_02 - CA_02 | -1 | -1 | 1 | . | . | 1 | 1 | - 2 (2) | +2 (1) * |
| NL_01 - FR_02 | -1 | -1 | 1 | -1 | . | 1 | -1 | - 4 (2) | -2 (2) * |
| NL_01 - SW_02 | -1 | -1 | 1 | . | . | 1 | 1 | - 2 (2) | +2 (1) * |
| NL_01 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | 1 | + 2 (2) | +4 (2) * |
| NL_01 - NL_02 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | + 2 (1) | +2 (1) |
| NL_01 - US_01 | 1 | 1 | 1 | -1 | . | 1 | 1 | + 4 (2) | +6 (3d) * |
| NL_01 - US_02 | 1 | -1 | -1 | 1 | . | 1 | 1 | 0 (1) | +2 (1) |
| NL_01 - CA_02 | -1 | -1 | 1 | -1 | . | 1 | 1 | - 2 (2) | 0 (1) * |
| NL_02 - FR_02 | -1 | 1 | . | 1 | . | 1 | -1 | 0 (1) | +4 (2) * |
| NL_02 - SW_02 | -1 | 1 | . | 1 | . | 1 | 1 | + 2 (2) | +6 (3d) * |
| NL_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | +6 (3d) * |
| NL_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | +6 (3d) |
| NL_02 - US_01 | 1 | 1 | . | 1 | . | 1 | 1 | + 4 (2) | +8 (4) * |
| NL_02 - US_02 | 1 | 1 | -1 | 1 | . | 1 | 1 | + 4 (2) | +6 (3d) * |
| NL_02 - CA_02 | -1 | 1 | 1 | . | . | 1 | 1 | + 2 (2) | +6 (3d) * |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | . | . | -1 | - 4 (2) | 0 (1) * |
| US_01 - SW_02 | -1 | -1 | . | 1 | . | . | -1 | - 6 (3d) | 0 (1) * |
| US_01 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | +4 (2) * |
| US_01 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | +4 (2) * |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | -1 | 0 (1) | +2 (1) |
| US_01 - US_02 | 1 | -1 | -1 | 1 | . | 1 | 1 | 0 (1) | +2 (1) |
| US_01 - CA_02 | -1 | -1 | 1 | . | . | . | 1 | - 4 (2) | +2 (1) * |
| US_02 - FR_02 | -1 | 1 | 1 | -1 | . | . | -1 | - 2 (1) | +2 (2) * |
| US_02 - SW_02 | -1 | 1 | 1 | -1 | . | . | -1 | - 2 (1) | +2 (2) * |
| US_02 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | -1 | 0 (1) | +2 (2) * |
| US_02 - NL_01 | 1 | 1 | 1 | -1 | 1 | . | -1 | + 2 (2) | +4 (2) * |
| US_02 - NL_02 | 1 | -1 | 1 | -1 | 1 | . | -1 | - 2 (2) | 0 (1) * |
| US_02 - US_01 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | + 6 (3d) | +6 (3d) |
| US_02 - CA_02 | -1 | -1 | 1 | -1 | . | . | 1 | - 4 (2) | 0 (1) * |
| CA_02 - FR_02 | 1 | 1 | . | 1 | 1 | 1 | -1 | + 4 (2) | +6 (3d) * |
| CA_02 - SW_02 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | + 4 (2) | +4 (2) . |
| CA_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | +4 (2) * |
| CA_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | +4 (2) * |
| CA_02 - NL_02 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | +6 (3d) * |
| CA_02 - US_01 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | +8 (4) |
| CA_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | +6 (3d) |

* : Situations that are not extensively stable

Selected alternatives: NL_02 - US_02

A.4.3 After the comparison of alternatives NL₂ and US₂

| | ACE(1) | ARD(2) | ARE(1) | AT(1) | UIR(1) | CPP(1) | TBH(1) | peSS. | opt. |
|---------------|--------|--------|--------|-------|--------|--------|--------|----------|------------|
| FR_02 - SW_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| FR_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| FR_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | 1 | + 2 (1) | + 4 (1) |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | + 8 (4) | + 8 (4) |
| FR_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| FR_02 - CA_02 | 1 | -1 | 1 | . | . | . | 1 | - 2 (2) | + 4 (1) * |
| SW_02 - FR_02 | 1 | 1 | 1 | . | 1 | . | -1 | + 2 (2) | + 6 (3d) * |
| SW_02 - EN_02 | 1 | 1 | -1 | . | 1 | . | 1 | + 2 (2) | + 6 (3d) * |
| SW_02 - NL_01 | 1 | 1 | . | 1 | 1 | -1 | 1 | + 4 (2) | + 6 (3d) * |
| SW_02 - NL_02 | 1 | -1 | 1 | . | 1 | -1 | 1 | 0 (1) | + 2 (1) |
| SW_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| SW_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| SW_02 - CA_02 | 1 | -1 | 1 | -1 | 1 | . | 1 | 0 (1) | + 2 (1) |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | - 2 (2) | - 2 (2) * |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | + 8 (4) |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | + 2 (1) | + 4 (1) |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | -1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | -1 | 1 | 1 | + 2 (1) | + 2 (1) |
| EN_02 - CA_02 | -1 | -1 | 1 | . | -1 | 1 | 1 | - 2 (2) | 0 (1) * |
| NL_01 - FR_02 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | - 4 (2) | - 4 (2) * |
| NL_01 - SW_02 | -1 | -1 | 1 | . | -1 | 1 | 1 | - 2 (2) | 0 (1) * |
| NL_01 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | 1 | + 2 (2) | + 4 (2) * |
| NL_01 - NL_02 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | + 2 (1) | + 2 (1) |
| NL_01 - US_01 | 1 | 1 | 1 | -1 | . | 1 | 1 | + 4 (2) | + 6 (3d) * |
| NL_01 - US_02 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| NL_01 - CA_02 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | - 2 (2) | - 2 (2) * |
| NL_02 - FR_02 | -1 | 1 | . | 1 | -1 | 1 | -1 | 0 (1) | + 2 (2) * |
| NL_02 - SW_02 | -1 | 1 | . | 1 | -1 | 1 | 1 | + 2 (2) | + 4 (2) * |
| NL_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| NL_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| NL_02 - US_01 | 1 | 1 | . | 1 | . | 1 | 1 | + 4 (2) | + 8 (4) * |
| NL_02 - US_02 | 1 | 1 | -1 | 1 | -1 | 1 | 1 | + 4 (2) | + 4 (2) . |
| NL_02 - CA_02 | -1 | 1 | 1 | . | -1 | 1 | 1 | + 2 (2) | + 4 (2) * |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | . | . | -1 | - 4 (2) | 0 (1) * |
| US_01 - SW_02 | -1 | -1 | . | 1 | . | . | -1 | - 6 (3d) | 0 (1) * |
| US_01 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| US_01 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | -1 | 0 (1) | + 2 (1) |
| US_01 - US_02 | 1 | -1 | -1 | 1 | . | 1 | 1 | 0 (1) | + 2 (1) |
| US_01 - CA_02 | -1 | -1 | 1 | . | -1 | . | 1 | - 4 (2) | 0 (1) * |
| US_02 - FR_02 | -1 | 1 | 1 | -1 | . | . | -1 | - 2 (1) | + 2 (2) * |
| US_02 - SW_02 | -1 | 1 | 1 | -1 | . | . | -1 | - 2 (1) | + 2 (2) * |
| US_02 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | -1 | 0 (1) | + 2 (2) * |
| US_02 - NL_01 | 1 | 1 | 1 | -1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| US_02 - NL_02 | 1 | -1 | 1 | -1 | 1 | . | -1 | - 2 (2) | 0 (1) . |
| US_02 - US_01 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| US_02 - CA_02 | -1 | -1 | 1 | -1 | . | . | 1 | - 4 (2) | 0 (1) * |
| CA_02 - FR_02 | 1 | 1 | . | 1 | 1 | 1 | -1 | + 4 (2) | + 6 (3d) * |
| CA_02 - SW_02 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | + 4 (2) | + 4 (2) . |
| CA_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| CA_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| CA_02 - NL_02 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | + 6 (3d) * |
| CA_02 - US_01 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| CA_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |

* : Situations that are not extensively stable
 Selected alternatives: NL₀₁ - US₀₁

A.4.4 After the comparison of alternatives NL₁ and US₁

| | ACE(1) | ARD(2) | ARE(1) | AT(1) | UIR(1) | CPP(1) | TBH(1) | peess. | opt. |
|---------------|--------|--------|--------|-------|--------|--------|--------|----------|------------|
| FR_02 - SW_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| FR_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| FR_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | 1 | + 2 (1) | + 4 (1) |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | + 8 (4) | + 8 (4) |
| FR_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| FR_02 - CA_02 | 1 | -1 | 1 | . | . | . | 1 | - 2 (2) | + 4 (1) * |
| SW_02 - FR_02 | 1 | 1 | 1 | . | 1 | . | -1 | + 2 (2) | + 6 (3d) * |
| SW_02 - EN_02 | 1 | 1 | -1 | . | 1 | -1 | 1 | + 2 (2) | + 4 (2) * |
| SW_02 - NL_01 | 1 | 1 | . | 1 | 1 | -1 | 1 | + 4 (2) | + 6 (3d) * |
| SW_02 - NL_02 | 1 | -1 | 1 | . | 1 | -1 | 1 | 0 (1) | + 2 (1) |
| SW_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| SW_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| SW_02 - CA_02 | 1 | -1 | 1 | -1 | 1 | . | 1 | 0 (1) | + 2 (1) |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | - 2 (2) | - 2 (2) * |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | + 8 (4) |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | + 2 (1) | + 4 (1) |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | -1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | -1 | 1 | 1 | + 2 (1) | + 2 (1) |
| EN_02 - CA_02 | -1 | -1 | 1 | . | -1 | 1 | 1 | - 2 (2) | 0 (1) * |
| NL_01 - FR_02 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | - 4 (2) | - 4 (2) * |
| NL_01 - SW_02 | -1 | -1 | 1 | . | -1 | 1 | 1 | - 2 (2) | 0 (1) * |
| NL_01 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | 1 | + 2 (2) | + 4 (2) * |
| NL_01 - NL_02 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | + 2 (1) | + 2 (1) |
| NL_01 - US_01 | 1 | 1 | 1 | -1 | . | 1 | 1 | + 4 (2) | + 6 (3d) . |
| NL_01 - US_02 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| NL_01 - CA_02 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | - 2 (2) | - 2 (2) * |
| NL_02 - FR_02 | -1 | 1 | . | 1 | -1 | 1 | -1 | 0 (1) | + 2 (2) * |
| NL_02 - SW_02 | -1 | 1 | . | 1 | -1 | 1 | 1 | + 2 (2) | + 4 (2) * |
| NL_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| NL_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| NL_02 - US_01 | 1 | 1 | . | 1 | . | 1 | 1 | + 4 (2) | + 8 (4) * |
| NL_02 - US_02 | 1 | 1 | -1 | 1 | -1 | 1 | 1 | + 4 (2) | + 4 (2) . |
| NL_02 - CA_02 | -1 | 1 | 1 | . | -1 | 1 | 1 | + 2 (2) | + 4 (2) * |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | . | . | -1 | - 4 (2) | 0 (1) * |
| US_01 - SW_02 | -1 | -1 | . | 1 | . | . | -1 | - 6 (3d) | 0 (1) * |
| US_01 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | + 2 (2) | + 2 (2) * |
| US_01 - NL_01 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | + 2 (2) | + 2 (2) . |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | -1 | -1 | 0 (1) | 0 (1) |
| US_01 - US_02 | 1 | -1 | -1 | 1 | . | 1 | 1 | 0 (1) | + 2 (1) |
| US_01 - CA_02 | -1 | -1 | 1 | . | -1 | . | 1 | - 4 (2) | 0 (1) * |
| US_02 - FR_02 | -1 | 1 | 1 | -1 | . | . | -1 | - 2 (1) | + 2 (2) * |
| US_02 - SW_02 | -1 | 1 | 1 | -1 | . | . | -1 | - 2 (1) | + 2 (2) * |
| US_02 - EN_02 | 1 | 1 | -1 | -1 | 1 | -1 | -1 | 0 (1) | 0 (1) |
| US_02 - NL_01 | 1 | 1 | 1 | -1 | 1 | -1 | -1 | + 2 (2) | + 2 (2) * |
| US_02 - NL_02 | 1 | -1 | 1 | -1 | 1 | -1 | -1 | - 2 (2) | - 2 (2) . |
| US_02 - US_01 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| US_02 - CA_02 | -1 | -1 | 1 | -1 | . | . | 1 | - 4 (2) | 0 (1) * |
| CA_02 - FR_02 | 1 | 1 | . | 1 | 1 | 1 | -1 | + 4 (2) | + 6 (3d) * |
| CA_02 - SW_02 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | + 4 (2) | + 4 (2) . |
| CA_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| CA_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| CA_02 - NL_02 | 1 | 1 | . | 1 | 1 | . | -1 | + 2 (2) | + 6 (3d) * |
| CA_02 - US_01 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| CA_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |

* : Situations that are not extensively stable

Selected alternatives: FR_02 - CA_02

A.4.5 After the comparison of alternatives CA₂ and FR₂

| | ACE(1) | ARD(2) | ARE(1) | AT(1) | UIR(1) | CPP(1) | TBH(1) | pess. | opt. |
|---------------|--------|--------|--------|-------|--------|--------|--------|----------|------------|
| FR_02 - SW_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| FR_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| FR_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | 1 | + 2 (1) | + 4 (1) |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | + 8 (4) | + 8 (4) |
| FR_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| FR_02 - CA_02 | 1 | -1 | 1 | 1 | 1 | 1 | 1 | + 4 (1) | + 4 (1) . |
| SW_02 - FR_02 | 1 | 1 | 1 | . | 1 | . | -1 | + 2 (2) | + 6 (3d) * |
| SW_02 - EN_02 | 1 | 1 | -1 | . | 1 | -1 | 1 | + 2 (2) | + 4 (2) * |
| SW_02 - NL_01 | 1 | 1 | . | 1 | 1 | -1 | 1 | + 4 (2) | + 6 (3d) * |
| SW_02 - NL_02 | 1 | -1 | 1 | . | 1 | -1 | 1 | 0 (1) | + 2 (1) |
| SW_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | + 6 (3d) | + 8 (4) |
| SW_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| SW_02 - CA_02 | 1 | -1 | 1 | -1 | 1 | . | 1 | 0 (1) | + 2 (1) |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | - 2 (2) | - 2 (2) * |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | + 8 (4) |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | + 2 (1) | + 4 (1) |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | -1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | -1 | 1 | 1 | + 2 (1) | + 2 (1) |
| EN_02 - CA_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| NL_01 - FR_02 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | - 4 (2) | - 4 (2) * |
| NL_01 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| NL_01 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | 1 | + 2 (2) | + 4 (2) * |
| NL_01 - NL_02 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | + 2 (1) | + 2 (1) |
| NL_01 - US_01 | 1 | 1 | 1 | -1 | . | 1 | 1 | + 4 (2) | + 6 (3d) . |
| NL_01 - US_02 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | 0 (1) | 0 (1) |
| NL_01 - CA_02 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | - 2 (2) | - 2 (2) * |
| NL_02 - FR_02 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | + 2 (2) | + 2 (2) * |
| NL_02 - SW_02 | -1 | 1 | . | 1 | -1 | 1 | 1 | + 2 (2) | + 4 (2) * |
| NL_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | + 4 (2) | + 6 (3d) * |
| NL_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| NL_02 - US_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | + 6 (3d) | + 8 (4) |
| NL_02 - US_02 | 1 | 1 | -1 | 1 | -1 | 1 | 1 | + 4 (2) | + 4 (2) . |
| NL_02 - CA_02 | -1 | 1 | 1 | 1 | -1 | 1 | 1 | + 4 (2) | + 4 (2) * |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | . | . | -1 | - 4 (2) | 0 (1) |
| US_01 - SW_02 | -1 | -1 | . | 1 | . | 1 | -1 | - 4 (2) | 0 (1) * |
| US_01 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | + 2 (2) | + 2 (2) * |
| US_01 - NL_01 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | + 2 (2) | + 2 (2) . |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | -1 | -1 | 0 (1) | 0 (1) |
| US_01 - US_02 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | + 2 (1) | + 2 (1) |
| US_01 - CA_02 | -1 | -1 | 1 | 1 | -1 | . | 1 | - 2 (2) | 0 (1) * |
| US_02 - FR_02 | -1 | 1 | 1 | -1 | . | . | -1 | - 2 (1) | + 2 (2) * |
| US_02 - SW_02 | -1 | 1 | 1 | -1 | . | 1 | -1 | 0 (1) | + 2 (2) * |
| US_02 - EN_02 | 1 | 1 | -1 | -1 | 1 | -1 | -1 | 0 (1) | 0 (1) |
| US_02 - NL_01 | 1 | 1 | 1 | -1 | 1 | -1 | -1 | + 2 (2) | + 2 (2) * |
| US_02 - NL_02 | 1 | -1 | 1 | -1 | 1 | -1 | -1 | - 2 (2) | - 2 (2) . |
| US_02 - US_01 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |
| US_02 - CA_02 | -1 | -1 | 1 | -1 | . | . | 1 | - 4 (2) | 0 (1) * |
| CA_02 - FR_02 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | + 6 (3d) | + 6 (3d) |
| CA_02 - SW_02 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | + 4 (2) | + 4 (2) * |
| CA_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| CA_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | + 2 (2) | + 4 (2) * |
| CA_02 - NL_02 | 1 | 1 | 1 | 1 | 1 | . | -1 | + 4 (2) | + 6 (3d) * |
| CA_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | + 8 (4) | + 8 (4) |
| CA_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | + 6 (3d) | + 6 (3d) |

* : Situations that are not extensively stable
 Selected alternatives: FR_02 - EN_02

A.4.6 After the comparison of alternatives FR₂ and EN₂

| | ACE(1) | ARD(2) | ARE(1) | AT(1) | UIR(1) | CPP(1) | TBH(1) | | pess. | opt. |
|---------------|--------|--------|--------|-------|--------|--------|--------|--|----------|----------|
| FR_02 - SW_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | | + 6 (3d) | + 8 (4) |
| FR_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | 1 | | + 4 (2) | + 4 (2) |
| FR_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | 1 | | + 4 (2) | + 6 (3d) |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | 1 | | + 2 (1) | + 4 (1) |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | + 8 (4) | + 8 (4) |
| FR_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 6 (3d) | + 6 (3d) |
| FR_02 - CA_02 | 1 | -1 | 1 | 1 | 1 | 1 | 1 | | + 4 (1) | + 4 (1) |
| SW_02 - FR_02 | 1 | 1 | 1 | . | 1 | . | -1 | | + 2 (2) | + 6 (3d) |
| SW_02 - EN_02 | 1 | 1 | -1 | . | 1 | -1 | 1 | | + 2 (2) | + 4 (2) |
| SW_02 - NL_01 | 1 | 1 | . | 1 | 1 | -1 | 1 | | + 4 (2) | + 6 (3d) |
| SW_02 - NL_02 | 1 | -1 | 1 | . | 1 | -1 | 1 | | 0 (1) | + 2 (1) |
| SW_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | | + 6 (3d) | + 8 (4) |
| SW_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 6 (3d) | + 6 (3d) |
| SW_02 - CA_02 | 1 | -1 | 1 | -1 | 1 | . | 1 | | 0 (1) | + 2 (1) |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | | - 2 (2) | - 2 (2) |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) | 0 (1) |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | | + 6 (3d) | + 8 (4) |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | | + 2 (1) | + 4 (1) |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | -1 | 1 | 1 | | + 6 (3d) | + 6 (3d) |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | -1 | 1 | 1 | | + 2 (1) | + 2 (1) |
| EN_02 - CA_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) | 0 (1) |
| NL_01 - FR_02 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | | - 4 (2) | - 4 (2) |
| NL_01 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) | 0 (1) |
| NL_01 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | 1 | | + 2 (2) | + 4 (2) |
| NL_01 - NL_02 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | | + 2 (1) | + 2 (1) |
| NL_01 - US_01 | 1 | 1 | 1 | -1 | . | 1 | 1 | | + 4 (2) | + 6 (3d) |
| NL_01 - US_02 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | | 0 (1) | 0 (1) |
| NL_01 - CA_02 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | | - 2 (2) | - 2 (2) |
| NL_02 - FR_02 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | | + 2 (2) | + 2 (2) |
| NL_02 - SW_02 | -1 | 1 | . | 1 | -1 | 1 | 1 | | + 2 (2) | + 4 (2) |
| NL_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | | + 4 (2) | + 6 (3d) |
| NL_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 6 (3d) | + 6 (3d) |
| NL_02 - US_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | | + 6 (3d) | + 8 (4) |
| NL_02 - US_02 | 1 | 1 | -1 | 1 | -1 | 1 | 1 | | + 4 (2) | + 4 (2) |
| NL_02 - CA_02 | -1 | 1 | 1 | 1 | -1 | 1 | 1 | | + 4 (2) | + 4 (2) |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | . | . | -1 | | - 4 (2) | 0 (1) |
| US_01 - SW_02 | -1 | -1 | . | 1 | . | 1 | -1 | | - 4 (2) | 0 (1) |
| US_01 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | | + 2 (2) | + 2 (2) |
| US_01 - NL_01 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | | + 2 (2) | + 2 (2) |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | -1 | -1 | | 0 (1) | 0 (1) |
| US_01 - US_02 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | | + 2 (1) | + 2 (1) |
| US_01 - CA_02 | -1 | -1 | 1 | 1 | -1 | . | 1 | | - 2 (2) | 0 (1) |
| US_02 - FR_02 | -1 | 1 | 1 | -1 | . | . | -1 | | - 2 (1) | + 2 (2) |
| US_02 - SW_02 | -1 | 1 | 1 | -1 | . | 1 | -1 | | 0 (1) | + 2 (2) |
| US_02 - EN_02 | 1 | 1 | -1 | -1 | 1 | -1 | -1 | | 0 (1) | 0 (1) |
| US_02 - NL_01 | 1 | 1 | 1 | -1 | 1 | -1 | -1 | | + 2 (2) | + 2 (2) |
| US_02 - NL_02 | 1 | -1 | 1 | -1 | 1 | -1 | -1 | | - 2 (2) | - 2 (2) |
| US_02 - US_01 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | | + 6 (3d) | + 6 (3d) |
| US_02 - CA_02 | -1 | -1 | 1 | -1 | . | . | 1 | | - 4 (2) | 0 (1) |
| CA_02 - FR_02 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | | + 6 (3d) | + 6 (3d) |
| CA_02 - SW_02 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | | + 4 (2) | + 4 (2) |
| CA_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | | + 2 (2) | + 4 (2) |
| CA_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | | + 2 (2) | + 4 (2) |
| CA_02 - NL_02 | 1 | 1 | 1 | 1 | 1 | . | -1 | | + 4 (2) | + 6 (3d) |
| CA_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | + 8 (4) | + 8 (4) |
| CA_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 6 (3d) | + 6 (3d) |

A.4.7 Validation of the new preorder \geq_w^2

| | ACE(1) | ARD(3) | ARE(2) | AT(1) | UIR(2) | CPP(1) | TBH(2) | | pess. | opt. |
|---------------|--------|--------|--------|-------|--------|--------|--------|--|----------|------------|
| FR_02 - SW_02 | 1 | 1 | . | 1 | 1 | 1 | 1 | | + 8 (3d) | +12 (4) |
| FR_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | 1 | | + 6 (3d) | + 6 (3d) |
| FR_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | 1 | | + 6 (3d) | + 8 (3d) |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | . | 1 | | + 4 (1) | + 6 (1) * |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | +12 (4) | +12 (4) |
| FR_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) | + 8 (3d) |
| FR_02 - CA_02 | 1 | -1 | 1 | 1 | 1 | 1 | 1 | | + 6 (1) | + 6 (1) |
| SW_02 - FR_02 | 1 | 1 | 1 | . | 1 | . | -1 | | + 4 (3d) | + 8 (3d) |
| SW_02 - EN_02 | 1 | 1 | -1 | . | 1 | -1 | 1 | | + 4 (3d) | + 6 (3d) |
| SW_02 - NL_01 | 1 | 1 | . | 1 | 1 | -1 | 1 | | + 6 (3d) | +10 (3d) |
| SW_02 - NL_02 | 1 | -1 | 1 | . | 1 | -1 | 1 | | + 2 (1) | + 4 (1) * |
| SW_02 - US_01 | 1 | 1 | 1 | . | 1 | 1 | 1 | | +10 (3d) | +12 (4) |
| SW_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) | + 8 (3d) |
| SW_02 - CA_02 | 1 | -1 | 1 | -1 | 1 | . | 1 | | + 2 (1) | + 4 (1) * |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | | - 4 (3d) | - 4 (3d) |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) | 0 (1) |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | | + 8 (3d) | +12 (4) |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | . | 1 | 1 | | + 2 (1) | + 6 (1) |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | -1 | 1 | 1 | | + 8 (3d) | + 8 (3d) |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | -1 | 1 | 1 | | + 2 (1) | + 2 (1) |
| EN_02 - CA_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) | 0 (1) |
| NL_01 - FR_02 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | | - 6 (3d) | - 6 (3d) |
| NL_01 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) | 0 (1) |
| NL_01 - EN_02 | 1 | 1 | -1 | -1 | 1 | . | 1 | | + 4 (3d) | + 6 (3d) |
| NL_01 - NL_02 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | | + 4 (1) | + 4 (1) |
| NL_01 - US_01 | 1 | 1 | 1 | -1 | . | 1 | 1 | | + 6 (3d) | +10 (3d) |
| NL_01 - US_02 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | | - 2 (1) | - 2 (1) |
| NL_01 - CA_02 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | | - 2 (2) | - 2 (2) |
| NL_02 - FR_02 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | | + 2 (2) | + 2 (2) |
| NL_02 - SW_02 | -1 | 1 | . | 1 | -1 | 1 | 1 | | + 2 (2) | + 6 (3d) |
| NL_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | 1 | | + 6 (3d) | + 8 (3d) |
| NL_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) | + 8 (3d) |
| NL_02 - US_01 | 1 | 1 | 1 | 1 | . | 1 | 1 | | + 8 (3d) | +12 (4) |
| NL_02 - US_02 | 1 | 1 | -1 | 1 | -1 | 1 | 1 | | + 4 (2) | + 4 (2) |
| NL_02 - CA_02 | -1 | 1 | 1 | 1 | -1 | 1 | 1 | | + 6 (3d) | + 6 (3d) |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | . | . | -1 | | - 6 (3d) | 0 (1) * |
| US_01 - SW_02 | -1 | -1 | . | 1 | . | 1 | -1 | | - 8 (3d) | 0 (1) * |
| US_01 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | | + 2 (2) | + 2 (2) |
| US_01 - NL_01 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | | + 2 (2) | + 2 (2) |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | -1 | -1 | | 0 (1) | 0 (1) |
| US_01 - US_02 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | | + 2 (1) | + 2 (1) |
| US_01 - CA_02 | -1 | -1 | 1 | 1 | -1 | . | 1 | | - 2 (2) | 0 (1) * |
| US_02 - FR_02 | -1 | 1 | 1 | -1 | . | . | -1 | | - 2 (1) | + 4 (3d) * |
| US_02 - SW_02 | -1 | 1 | 1 | -1 | . | 1 | -1 | | 0 (1) | + 4 (3d) * |
| US_02 - EN_02 | 1 | 1 | -1 | -1 | 1 | -1 | -1 | | 0 (1) | 0 (1) |
| US_02 - NL_01 | 1 | 1 | 1 | -1 | 1 | -1 | -1 | | + 4 (3d) | + 4 (3d) |
| US_02 - NL_02 | 1 | -1 | 1 | -1 | 1 | -1 | -1 | | - 2 (2) | - 2 (2) |
| US_02 - US_01 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | | +10 (3d) | +10 (3d) |
| US_02 - CA_02 | -1 | -1 | 1 | -1 | . | . | 1 | | - 4 (2) | + 2 (1) * |
| CA_02 - FR_02 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | | + 8 (3d) | + 8 (3d) |
| CA_02 - SW_02 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | | + 4 (2) | + 4 (2) |
| CA_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | . | -1 | | + 2 (2) | + 4 (2) |
| CA_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | . | -1 | | + 2 (2) | + 4 (2) |
| CA_02 - NL_02 | 1 | 1 | 1 | 1 | 1 | . | -1 | | + 6 (3d) | + 8 (3d) |
| CA_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | +12 (4) | +12 (4) |
| CA_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) | + 8 (3d) |

* : Situations on which the missing local concordance values may have an impact

A.4.8 Validation of the preorder and the thresholds

| | ACE(1) | ARD(3) | ARE(2) | AT(1) | UIR(2) | CPP(1) | TBH(2) | | S |
|---------------|--------|--------|--------|-------|--------|--------|--------|--|-----------|
| FR_02 - SW_02 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | | +10 (3d) |
| FR_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | 1 | | + 6 (3d) |
| FR_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) |
| FR_02 - NL_02 | 1 | -1 | 1 | 1 | 1 | 1 | 1 | | + 6 (1) * |
| FR_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | +12 (4) |
| FR_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) |
| FR_02 - CA_02 | 1 | -1 | 1 | 1 | 1 | 1 | 1 | | + 6 (1) . |
| SW_02 - FR_02 | 1 | 1 | 1 | 0 | 1 | 1 | -1 | | + 7 (3d) |
| SW_02 - EN_02 | 1 | 1 | -1 | 0 | 1 | -1 | 1 | | + 5 (3d) |
| SW_02 - NL_01 | 1 | 1 | 0 | 1 | 1 | -1 | 1 | | + 8 (3d) |
| SW_02 - NL_02 | 1 | -1 | 1 | 0 | 1 | -1 | 1 | | + 3 (1) * |
| SW_02 - US_01 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | | +11 (3d) |
| SW_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) |
| SW_02 - CA_02 | 1 | -1 | 1 | -1 | 1 | 0 | 1 | | + 3 (1) . |
| EN_02 - FR_02 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | | - 4 (3d) |
| EN_02 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) * |
| EN_02 - NL_01 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | | +10 (3d) |
| EN_02 - NL_02 | 1 | -1 | 1 | 1 | 0 | 1 | 1 | | + 4 (1) * |
| EN_02 - US_01 | 1 | 1 | 1 | 1 | -1 | 1 | 1 | | + 8 (3d) |
| EN_02 - US_02 | 1 | -1 | 1 | 1 | -1 | 1 | 1 | | + 2 (1) * |
| EN_02 - CA_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) * |
| NL_01 - FR_02 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | | - 6 (3d) |
| NL_01 - SW_02 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | | 0 (1) * |
| NL_01 - EN_02 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | | + 6 (3d) |
| NL_01 - NL_02 | 1 | -1 | 1 | -1 | 1 | 1 | 1 | | + 4 (1) * |
| NL_01 - US_01 | 1 | 1 | 1 | -1 | 0 | 1 | 1 | | + 8 (3d) |
| NL_01 - US_02 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | | - 2 (1) * |
| NL_01 - CA_02 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | | - 2 (2) |
| NL_02 - FR_02 | -1 | 1 | 1 | 1 | -1 | 1 | -1 | | + 2 (2) |
| NL_02 - SW_02 | -1 | 1 | 0 | 1 | -1 | 1 | 1 | | + 4 (3d) |
| NL_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) |
| NL_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) |
| NL_02 - US_01 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | | +10 (3d) |
| NL_02 - US_02 | 1 | 1 | -1 | 1 | -1 | 1 | 1 | | + 4 (2) |
| NL_02 - CA_02 | -1 | 1 | 1 | 1 | -1 | 1 | 1 | | + 6 (3d) |
| US_01 - FR_02 | -1 | -1 | 1 | 1 | -1 | 1 | -1 | | - 4 (3d) |
| US_01 - SW_02 | -1 | -1 | 0 | 1 | -1 | 1 | -1 | | - 6 (3d) |
| US_01 - EN_02 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | | + 2 (2) |
| US_01 - NL_01 | 1 | 1 | -1 | 1 | 1 | -1 | -1 | | + 2 (2) |
| US_01 - NL_02 | 1 | -1 | 1 | 1 | 1 | -1 | -1 | | 0 (1) * |
| US_01 - US_02 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | | + 2 (1) * |
| US_01 - CA_02 | -1 | -1 | 1 | 1 | -1 | 0 | 1 | | - 1 (2) |
| US_02 - FR_02 | -1 | 1 | 1 | -1 | 0 | 1 | -1 | | + 2 (2) |
| US_02 - SW_02 | -1 | 1 | 1 | -1 | 0 | 1 | -1 | | + 2 (2) |
| US_02 - EN_02 | 1 | 1 | -1 | -1 | 1 | -1 | -1 | | 0 (1) * |
| US_02 - NL_01 | 1 | 1 | 1 | -1 | 1 | -1 | -1 | | + 4 (3d) |
| US_02 - NL_02 | 1 | -1 | 1 | -1 | 1 | -1 | -1 | | - 2 (2) |
| US_02 - US_01 | 1 | 1 | 1 | -1 | 1 | 1 | 1 | | +10 (3d) |
| US_02 - CA_02 | -1 | -1 | 1 | -1 | -1 | 0 | 1 | | - 3 (2) |
| CA_02 - FR_02 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | | + 8 (3d) |
| CA_02 - SW_02 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | | + 4 (2) |
| CA_02 - EN_02 | 1 | 1 | -1 | 1 | 1 | 0 | -1 | | + 3 (2) |
| CA_02 - NL_01 | 1 | 1 | -1 | 1 | 1 | 1 | -1 | | + 4 (2) |
| CA_02 - NL_02 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | | + 8 (3d) |
| CA_02 - US_01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | +12 (4) |
| CA_02 - US_02 | 1 | 1 | -1 | 1 | 1 | 1 | 1 | | + 8 (3d) |

* : unstable situations

Bibliography

- [Adl10] A. ADLA, *Aide à la facilitation pour une prise de décision collective: Proposition d'un modèle et d'un outil*, Ph.D. thesis, 2010. (Cited on page 9.)
- [AGM10] S. ANGILELLA, S. GRECO, AND B. MATARAZZO, *Non-additive robust ordinal regression: A multiple criteria decision model based on the choquet integral*, *European Journal of Operational Research* **201** (2010), no. 1, 277–288. (Cited on page 30.)
- [AP97] A.D. ATHANASSOPOULOS AND V.V. PODINOVSKI, *Dominance and potential optimality in multiple criteria decision analysis with imprecise information*, *Journal of the Operational research Society* **48** (1997), no. 2, 142–150. (Cited on page 34.)
- [Arr63] K.J. ARROW, *Social choice and individual values*, J. Wiley, New York (1951. 2nd edition, 1963). (Cited on page 8.)
- [Bar92] F.H. BARRON, *Selecting a best multiattribute alternative with partial information about attribute weights*, *Acta Psychologica* **80** (1992), no. 1, 91–103. (Cited on page 34.)
- [BeC96] C.A. BANA E COSTA, *Les problématiques de l'aide à la décision : vers l'enrichissement de la trilogie choix-tri-rangement*, *RAIRO / Recherche Opérationnelle* **30** (1996), no. 2, 191–216. (Cited on page 13.)
- [BeCNdSV01] C.A. BANA E COSTA, F. NUNES DA SILVA, AND J.C. VANSNICK, *Conflict dissolution in the public sector: A case-study*, *European Journal of Operational Research* **130** (2001), no. 2, 388–401. (Cited on page 6.)
- [Bis02] R. BISDORFF, *Logical foundation of multicriteria preference aggregation*, *Aiding Decisions with Multiple Criteria* (D. Bouyssou and al., eds.), Kluwer Academic Publishers, 2002, pp. 379–403. (Cited on pages 11, 18, 44 and 45.)
- [Bis04] ———, *Concordant outranking with multiple criteria of ordinal significance*, *4OR* **2** (2004), no. 4, 293–308. (Cited on pages 36, 44 and 48.)
- [BM02] J.P. BRANS AND B. MARESCHAL, *PROMÉTHÉE-GAIA : une méthodologie d'aide à la décision en présence de critères multiples*, ULB Institutional Repository (2002). (Cited on page 17.)
- [BMP⁺06] D. BOUYSSOU, T. MARCHANT, P. PERNY, M. PIRLOT, A. TSOUKIÀS, AND P. VINCKE, *Evaluation and decision models with multiple criteria: Stepping stones for the analyst*, vol. 86, Springer, 2006. (Cited on pages 10 and 129.)

- [BMR08] RAYMOND BISDORFF, PATRICK MEYER, AND MARC ROUBENS, *RUBIS: a bipolar-valued outranking method for the choice problem*, 4OR: A Quarterly Journal of Operations Research **6** (2008), 143–165, 10.1007/s10288-007-0045-5. (Cited on pages 17, 18, 44 and 167.)
- [Bou90] D. BOUYSSOU, *Building criteria: a prerequisite for MCDA*, Readings in multiple criteria decision aid (C.A. Bana e Costa, ed.), Springer-Verlag, Berlin, 1990, pp. 58–80. (Cited on pages 10 and 11.)
- [BP05] D. BOUYSSOU AND M. PIRLOT, *Conjoint measurement tools for MCDM*, Multiple Criteria Decision Analysis: State of the Art Surveys (2005), 73–112. (Cited on page 15.)
- [BP07] L. BASSON AND J. PETRIE, *A critical systems approach to decision support for process engineering*, Computers & chemical engineering **31** (2007), no. 8, 876–888. (Cited on page 34.)
- [BRS66] R. BENAYOUN, B. ROY, AND B. SUSSMAN, *ELECTRE : Une méthode pour guider le choix en présence de points de vue multiples*, Note de travail **49** (1966). (Cited on pages 8 and 31.)
- [BRT88] D.E. BELL, H. RAIFFA, AND A. TVERSKY, *Decision making: Descriptive, normative, and prescriptive interactions*, Cambridge University Press, 1988. (Cited on page 7.)
- [BS02] V. BELTON AND T.J. STUART, *Multiple criteria decision analysis: an integrated approach*, Springer, 2002. (Cited on pages 8 and 12.)
- [CAM03] A. COSTA, A.T. ALMEIDA, AND C.M.G. MIRANDA, *Multicriteria support to sort information systems portfolio*, Journal of Academy of Business and Economics **1** (2003), no. 2, 237–247. (Cited on page 34.)
- [CC61] A. CHARNES AND W.W. COOPER, *Management models and industrial applications of linear programming*, Management Models and Industrial Applications of Linear Programming, no. vol. 1, Wiley, 1961. (Cited on page 8.)
- [CCF55] A. CHARNES, W.W. COOPER, AND R.O. FERGUSON, *Optimal estimation of executive compensation by linear programming*, Management science **1** (1955), no. 2, 138–151. (Cited on page 8.)
- [Chi94] J.W. CHINNECK, *MINOS (IIS): infeasibility analysis using MINOS*, Computers & operations research **21** (1994), no. 1, 1–9. (Cited on page 32.)
- [Chi96] ———, *An effective polynomial-time heuristic for the minimum-cardinality IIS set-covering problem*, Annals of Mathematics and Artificial Intelligence **17** (1996), no. 1, 127–144. (Cited on page 32.)
- [Cho53] G. CHOQUET, *Theory of capacities*, Annales de l’institut Fourier, vol. 5, 1953, p. 87. (Cited on page 16.)
- [Con85] M. CONDORCET, *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix*, Paris: L’imprimerie royale (1785). (Cited on page 8.)

- [Dan48] G.B. DANTZIG, *Programming in a linear structure*, Washington, DC (1948). (Cited on page 8.)
- [Dan51] ———, *Application of the simplex method to a transportation problem*, *Activity analysis of production and allocation* **13** (1951), 359–373. (Cited on page 8.)
- [Dan63] ———, *Linear programming and extensions*, Princeton University, Princeton, NJ (1963). (Cited on page 33.)
- [DC99] L.C. DIAS AND J.N. CLÍMACO, *On computing ELECTRE’s credibility indices under partial information*, *Journal of Multi-Criteria Decision Analysis* **8** (1999), no. 2, 74–92. (Cited on page 34.)
- [DC00] L.C. DIAS AND J. CLÍMACO, *ELECTRE TRI for groups with imprecise information on parameter values*, *Group Decision and Negotiation* **9** (2000), 355–377. (Cited on page 35.)
- [DGJ80] J.M. DEVAUD, G. GROUSSAUD, AND E. JACQUET-LAGRÈZE, *UTADIS: Une methode de construction de fonctions d’utilite additives rendant compte de jugements globaux*, European working group on MCDA, Bochum, Germany, 1980. (Cited on page 30.)
- [Dia07] L.C. DIAS, *A note on the role of robustness analysis in decision-aiding processes*, *Annales du LAMSADE*, 2007, pp. 53–70. (Cited on page 33.)
- [DM03] L.C. DIAS AND V. MOUSSEAU, *IRIS: a DSS for multiple criteria sorting problems*, *Journal of Multi-Criteria Decision Analysis* **12** (2003), no. 4-5, 285–298. (Cited on page 30.)
- [DM06] ———, *Inferring ELECTRE’s veto-related parameters from outranking examples*, *European Journal of Operational Research* **170** (2006), no. 1, 172–191. (Cited on pages 30 and 34.)
- [DOH54] G.B. DANTZIG AND W. ORCHARD-HAYS, *The product form for the inverse in the simplex method*, *Mathematical Tables and Other Aids to Computation* (1954), 64–67. (Cited on page 29.)
- [DT04] L.C. DIAS AND A. TSOUKIÀS, *On the constructive and other approaches in decision aiding*, *Aide multicritère à la décision : Multiple criteria decision aiding* (C.H. Antunes, J.R. Figueira, and J. Clímaco, eds.), 2004, pp. 13–28. (Cited on page 8.)
- [DYZ90] D.K. DESPOTIS, D. YANNAKOPOULOS, AND C. ZOPOUNIDIS, *A review of the UTA multicriteria method and some improvements*, *Foundations of Computing and Decision Sciences* **15** (1990), no. 2, 63–76. (Cited on page 30.)
- [FGE05] J. FIGUEIRA, S. GRECO, AND M. EHRGOTT, *Multiple criteria decision analysis: state of the art surveys*, vol. 78, Springer Verlag, 2005. (Cited on page 16.)
- [FGRS10] J.R. FIGUEIRA, S. GRECO, B. ROY, AND R. SŁOWINSKI, *ELECTRE methods: Main features and recent developments*, *Handbook of Multicriteria Analysis* (2010), 51–89. (Cited on pages 13 and 18.)

- [Fis64] P.C. FISHBURN, *Decision and value theory*, no. 10, Wiley New York, 1964. (Cited on page 15.)
- [Fis65] ———, *Independence in utility theory with whole product sets*, Operations Research (1965), 28–45. (Cited on page 15.)
- [Fis70] ———, *Utility theory for decision making*, Tech. report, DTIC Document, 1970. (Cited on pages 8 and 15.)
- [FMR05] J. FIGUEIRA, V. MOUSSEAU, AND B. ROY, *ELECTRE methods*, Multiple criteria decision analysis: State of the art surveys (2005), 133–153. (Cited on page 34.)
- [FR94] J. FODOR AND M. ROUBENS, *Fuzzy preference modelling and multi-criteria decision support*, vol. 14, Springer, 1994. (Cited on page 34.)
- [FRI89] S. FRENCH AND D. RÍOS INSUA, *Partial information and sensitivity analysis in multi-objective decision making*, Lockett P. Improving Decision Making in Organizations (1989), 23–41. (Cited on page 34.)
- [GKM08] M. GRABISCH, I. KOJADINOVIC, AND P. MEYER, *A review of capacity identification methods for Choquet integral based multi-attribute utility theory: Applications of the Kappalab R package*, European Journal of Operational Research **186** (2008), 766–785. (Cited on page 30.)
- [GMS01] S. GRECO, B. MATARAZZO, AND R. SŁOWINSKI, *Rough sets theory for multicriteria decision analysis*, European journal of operational research **129** (2001), no. 1, 1–47. (Cited on page 8.)
- [GMS08] S. GRECO, V. MOUSSEAU, AND R. SŁOWINSKI, *Ordinal regression revisited: multiple criteria ranking using a set of additive value functions*, European Journal of Operational Research **191** (2008), no. 2, 415–435. (Cited on page 35.)
- [Haz86] G.B. HAZEN, *Partial information, dominance, and potential optimality in multiattribute utility theory*, Operations research (1986), 296–310. (Cited on page 34.)
- [HLMS98] J. HOKKANEN, R. LAHDELMA, K. MIETTINEN, AND P. SALMINEN, *Determining the implementation order of a general plan by using a multicriteria method*, 1998. (Cited on page 33.)
- [JF89] T.M. JELASSI AND A. FOROUGH, *Negotiation support systems: An overview of design issues and existing software*, Decision Support Systems **5** (1989), no. 2, 167–181. (Cited on page 9.)
- [JLS01] E. JACQUET-LAGRÈZE AND Y. SISKOS, *Preference disaggregation: 20 years of MCDA experience*, European Journal of Operational Research **130** (2001), no. 2, 233–245. (Cited on page 30.)
- [JS82] E. JACQUET-LAGRÈZE AND Y. SISKOS, *Assessing a set of additive utility functions for multicriteria decision making: the UTA method*, European Journal of Operational Research **10** (1982), 151–164. (Cited on pages 29, 30 and 35.)

- [Kan39] L.V. KANTOROVICH, *The mathematical method of production planning and organization*, Management Science **6** (1939), 363–422. (Cited on page 8.)
- [KCCJ93] W. KIRKWOOD CRAIG AND L. CORNER JAMES, *The effectiveness of partial information about attribute weights for ranking alternatives in multiattribute decision making*, Organizational behavior and human decision processes **54** (1993), no. 3, 456–476. (Cited on page 34.)
- [Ken38] M. G. KENDALL, *A new measure of rank correlation*, Biometrika **30** (1938), no. 1/2, 81–93. (Cited on page 121.)
- [KLST71] D.H. KRANTZ, R.D. LUCE, P. SUPPES, AND A. TVERSKY, *Foundations of measurement volume i: additive and polynomial representations*, vol. 28, academic press, 1971. (Cited on page 15.)
- [KMN94] L.N. KISS, J.M. MARTEL, AND R. NADEAU, *ELECCALC – an interactive software for modelling the decision maker’s preferences*, Decision Support Systems **12** (1994), no. 4-5, 311–326. (Cited on page 30.)
- [KR76] R.L. KEENEY AND H. RAIFFA, *Decisions with multiple objectives: Preferences and value tradeoffs*, J. Wiley, New York, 1976. (Cited on pages 8, 15 and 17.)
- [LBO96] M. LANDRY, C. BANVILLE, AND M. ORAL, *Model legitimisation in operational research*, European Journal of Operational Research **92** (1996), no. 3, 443–457. (Cited on page 14.)
- [LHS98] R. LAHDELMA, J. HOKKANEN, AND P. SALMINEN, *SMAA - Stochastic multiobjective acceptability analysis*, European Journal of Operational Research **106** (1998), no. 1, 137–143. (Cited on page 33.)
- [LL01] T. LEÓN AND V. LIERN, *A fuzzy method to repair infeasibility in linearly constrained problems*, Fuzzy Sets and Systems **122** (2001), no. 2, 237–243. (Cited on pages 32 and 70.)
- [LLAC05] J. LEYVA-LOPEZ AND M. AGUILERA-CONTRERAS, *A multiobjective evolutionary algorithm for deriving final ranking from a fuzzy outranking relation*, Evolutionary Multi-Criterion Optimization, Springer, 2005, pp. 235–249. (Cited on page 34.)
- [LN80] P.H. LINDSAY AND D.A. NORMAN, *Traitement de l’information et comportement humain: une introduction à la psychologie*, Editions Etudes vivantes, 1980. (Cited on page 24.)
- [LPB85] M. LANDRY, D. PASCOT, AND D. BRIOLAT, *Can DSS evolve without changing our view of the concept of problem?*, Decision Support Systems **1** (1985), no. 1, 25–36. (Cited on page 7.)
- [LR57] R.D. LUCE AND H. RAIFFA, *Games and decisions*, J. Wiley, New York, 1957. (Cited on pages 7 and 8.)
- [LS01] R. LAHDELMA AND P. SALMINEN, *SMAA-2: Stochastic multicriteria acceptability analysis for group decision making*, Operations Research **49** (2001), no. 3, 444–454. (Cited on page 33.)

- [LTT05] H.A. LE THI AND P.D. TAO, *The DC (difference of convex functions) programming and DCA revisited with DC models of real world nonconvex optimization problems*, Annals of Operations Research **133** (2005), no. 1, 23–46. (Cited on page 168.)
- [MA90] J-M. MARTEL AND B. AOUNI, *Méthode multicritère de choix d'un emplacement : Le cas d'un aéroport dans le nouveau québec*, Faculté des sciences de l'administration, Université Laval, 1990. (Cited on page 6.)
- [MD04] V. MOUSSEAU AND L.C. DIAS, *Valued outranking relations in ELECTRE providing manageable disaggregation procedures*, European Journal of Operational Research **156** (2004), no. 2, 467–482. (Cited on page 30.)
- [MDF⁺03] V. MOUSSEAU, L.C. DIAS, J. FIGUEIRA, C. GOMES, AND J.N. CLÍMACO, *Resolving inconsistencies among constraints on the parameters of an MCDA model*, European Journal of Operational Research **147** (2003), no. 1, 72–93. (Cited on pages 32 and 70.)
- [MDF06] V. MOUSSEAU, L.C. DIAS, AND J. FIGUEIRA, *Dealing with inconsistent judgments in multiple criteria sorting models*, 4OR **4** (2006), no. 2, 145–158. (Cited on pages 32 and 70.)
- [MFN01] V. MOUSSEAU, J. FIGUEIRA, AND J.-PH. NAUX, *Using assignment examples to infer weights for ELECTRE TRI method: Some experimental results*, European Journal of Operational Research **130** (2001), no. 2, 263–275. (Cited on page 30.)
- [MKC00] K.G. MURTY, S.N. KABADI, AND R. CHANDRASEKARAN, *Infeasibility analysis for linear systems, a survey*, Arabian Journal for Science and Engineering **25** (2000), no. 1; Part C, 3–18. (Cited on page 32.)
- [MMB08] P. MEYER, J.-L. MARICHAL, AND R. BISDORFF, *Disaggregation of bipolar-valued outranking relations*, Modelling, Computation and Optimization in Information Systems and Management Sciences, Second International Conference (H.A. Le Thi, P. Bouvry, and P.D. Tao, eds.), Communications in Computer and Information Science, vol. 14, Springer, 2008, pp. 204–213. (Cited on pages 30 and 81.)
- [MN68] M. MAÑAS AND J. NEDOMA, *Finding all vertices of a convex polyhedron*, Numerische Mathematik **12** (1968), no. 3, 226–229. (Cited on page 29.)
- [Mou93] V. MOUSSEAU, *Problèmes liés à l'évaluation de l'importance relative des critères en aide multicritère à la décision: Réflexions théoriques, expérimentations et implementations informatiques*, Ph.D. thesis, 1993. (Cited on pages 24 and 30.)
- [Mou03] ———, *Elicitation des préférences pour l'aide multicritère à la décision*, Habilitation à diriger des recherches, Université Paris-Dauphine (2003). (Cited on pages 27 and 28.)

- [Mou05] ———, *A general framework for constructive learning preference elicitation in multiple criteria decision aid*, Cahiers du LAMSADE **223** (2005). (Cited on page 27.)
- [MPS94] L.Y. MAYSTRE, J. PICTET, AND J. SIMOS, *Méthodes multicritères ELECTRE : description, conseils pratiques et cas d'application à la gestion environnementale*, vol. 8, PPUR presses polytechniques, 1994. (Cited on page 34.)
- [MPY92] H. MOSKOWITZ, P.V. PRECKEL, AND A. YANG, *Multiple-criteria robust interactive decision analysis (MCRID) for optimizing public policies*, European Journal of Operational Research **56** (1992), no. 2, 219–236. (Cited on page 34.)
- [MS98] V. MOUSSEAU AND R. SŁOWINSKI, *Inferring an ELECTRE TRI model from assignment examples*, Journal of Global Optimization **12** (1998), no. 2, 157–174. (Cited on page 30.)
- [MS01] N.F. MATSATSINIS AND A.P. SAMARAS, *MCDA and preference disaggregation in group decision support systems*, European Journal of Operational Research **130** (2001), no. 2, 414–429. (Cited on page 9.)
- [MT97] M. MORISIO AND A. TSOUKIÀS, *IUSWARE: A methodology for the evaluation and selection of software products*, Software Engineering. IEE Proceedings, vol. 144, IET, 1997, pp. 162–174. (Cited on page 10.)
- [Nas51] J. NASH, *Non-cooperative games*, The Annals of Mathematics **54** (1951), no. 2, 286–295. (Cited on page 8.)
- [NJ50] J.F. NASH JR, *The bargaining problem*, Econometrica: Journal of the Econometric Society (1950), 155–162. (Cited on page 8.)
- [Par06] V. PARETO, *Manuale di economia politica*, Societa Editrice, 1906. (Cited on page 13.)
- [Pod94] V.V. PODINOVSKI, *Criteria importance theory*, Mathematical Social Sciences **27** (1994), no. 3, 237–252. (Cited on pages 24 and 30.)
- [Pod02] ———, *The quantitative importance of criteria for MCDA*, Journal of Multi-Criteria Decision Analysis **11** (2002), no. 1, 1–15. (Cited on page 24.)
- [Pod11] ———, *Analyzing the stability of choice for the partial preference relation*, Scientific and Technical Information Processing **38** (2011), 377–383, 10.3103/S014768821105008X. (Cited on page 36.)
- [Pod12] ———, *Sensitivity analysis for choice problems with partial preference relations*, European Journal of Operational Research (2012). (Cited on page 36.)
- [Poo92] D. POOLE, *Decision-theoretic defaults*, Ninth Biennial Conference of the Canadian Society for Computational Studies of Intelligence, Morgan Kaufmann, San Francisco, 1992, pp. 190–197. (Cited on page 6.)

- [PT99] E. PASCHETTA AND A. TSOUKIÀS, *A real-world MCDA application: Evaluating software*, Technical report. Document du LAMSADE (1999), no. 113. (Cited on page 10.)
- [RB93] B. ROY AND D. BOUYSSOU, *Aide multicritère à la décision : Méthodes et cas*, Economica, Paris, 1993. (Cited on pages 17, 33, 45 and 59.)
- [RD08] C. ROCHA AND L.C. DIAS, *An algorithm for ordinal sorting based on ELECTRE with categories defined by examples*, Journal of Global Optimization **42** (2008), 255–277. (Cited on page 30.)
- [RI90] D. RÍOS INSUA, *Sensitivity analysis in multi-objective decision making*, Lecture notes in economics and mathematical systems **347** (1990). (Cited on page 34.)
- [RM96] B. ROY AND V. MOUSSEAU, *A theoretical framework for analysing the notion of relative importance of criteria*, Journal of Multi-Criteria Decision Analysis **5** (1996), 145–159. (Cited on page 44.)
- [Roo79] G. ROODMAN, *Post-infeasibility analysis in linear programming*, Management Science **25** (1979), no. 9, 916–922. (Cited on pages 32 and 70.)
- [Ros91] E. ROSINGER, *Beyond preference information based multiple criteria decision making*, European Journal of Operational Research (1991), no. 53, 217–227. (Cited on page 13.)
- [Ros01] J. ROSENHEAD, *Rational analysis for a problematic world*, vol. 126, Wiley Chichester, 1989. Second edition in 2001. (Cited on page 7.)
- [Roy68] B. ROY, *Classement et choix en présence de points de vue multiples (la méthode ELECTRE)*, Revue française d’Informatique et de Recherche Opérationnelle (RIRO) **2** (1968), 57–75. (Cited on pages 8 and 17.)
- [Roy71] B. ROY, *Problems and methods with multiple objective functions*, Mathematical Programming **1** (1971), no. 2, 239–266. (Cited on page 15.)
- [Roy78] B. ROY, *ELECTRE III : Un algorithme de classement fondé sur une représentation floue des préférences en présence de critères multiples*, Cahiers du CERO **20** (1978), no. 1, 3–24. (Cited on page 167.)
- [Roy85] B. ROY, *Méthodologie multicritère d’aide à la décision*, Economica, collection Gestion, 1985. (Cited on pages 8, 10, 11 and 12.)
- [Roy87] ———, *Des critères multiples en recherche opérationnelle : Pourquoi ?*, Documents du LAMSADE, (80) (1987). (Cited on page 7.)
- [Roy90] B. ROY, *The outranking approach and the foundations of ELECTRE methods*, Reading in multiple criteria decision aid (C.A. Bana e Costa, ed.), Springer-Verlag, Berlin, 1990, pp. 155–183. (Cited on pages 13, 17 and 19.)
- [Roy93] ———, *Decision science or decision-aid science?*, European Journal of Operational Research **66** (1993), no. 2, 184–203. (Cited on page 7.)

- [Roy96] ———, *Multicriteria methodology for decision aiding*, Springer, 1996. (Cited on page 13.)
- [Roy98] ———, *A missing link in OR-DA: robustness analysis*, Foundations of computing and decision sciences **23** (1998), no. 3, 141–160. (Cited on pages 33 and 44.)
- [Roy00] B. ROY, *A french-english decision aiding glossary*, Newsletter of the European Working Group Multicriteria Aid for Decisions (2000). (Cited on page 11.)
- [Roy02] B. ROY, *Robustesse de quoi et vis-à-vis de quoi mais aussi robustesse pourquoi en aide à la décision*, Newsletter of the European Working Group Multicriteria Aid for Decisions **3** (2002), no. 6. (Cited on pages 33 and 44.)
- [Roy10] ———, *Robustness in operational research and decision aiding: A multi-faceted issue*, European Journal of Operational Research **200** (2010), no. 3, 629–638. (Cited on page 33.)
- [RV85] M. ROUBENS AND P. VINCKE, *Preference modelling*, Lecture Notes in Economics and Mathematical Systems (1985), no. 250. (Cited on pages 25 and 26.)
- [Saa80] T. L. SAATY, *Analytic hierarchy process*, Encyclopedia of Biostatistics (1980). (Cited on pages 24 and 28.)
- [Sam38] P.A. SAMUELSON, *A note on the pure theory of consumer's behaviour*, *Economica* **5** (1938), no. 17, 61–71. (Cited on page 8.)
- [Sch88] G. SCHAFFER, *Savage revisited*, Decision Making: descriptive, normative and prescriptive interactions (D.E. Bell, H. Raiffa, and A. Tversky, eds.), Cambridge University Press, 1988, pp. 193–235. (Cited on page 7.)
- [SD89] J. SISKOS AND D.K. DESPOTIS, *A DSS oriented method for multi-objective linear programming problems*, Decision Support Systems **5** (1989), no. 1, 47–55. (Cited on page 30.)
- [SGM05] Y. SISKOS, E. GRIGOROUDIS, AND N.F. MATSATSINIS, *UTA methods*, Multiple Criteria Decision Analysis: State of the Art Surveys (J. Figueira, S. Greco, and M. Ehrgott, eds.), Springer Verlag, Boston, Dordrecht, London, 2005, pp. 297–344. (Cited on page 30.)
- [SH92] A.A. SALO AND R.P. HÄMÄLÄINEN, *Preference assessment by imprecise ratio statements*, Operations Research (1992), 1053–1061. (Cited on page 34.)
- [SH01] A.A. SALO AND R.P. HAMALAINEN, *Preference ratios in multiattribute evaluation (prime)-elicitation and decision procedures under incomplete information*, Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on **31** (2001), no. 6, 533–545. (Cited on page 34.)
- [Sha88] J. SHANTEAU, *Psychological characteristics and strategies of expert decision makers*, Acta Psychologica **68** (1988), no. 1-3, 203–215. (Cited on page 22.)

- [Sim47] H.A. SIMON, *Administrative behavior: A study of decision making processes in administrative organizations*, 1947. (Cited on page 9.)
- [Sim56] ———, *Rational choice and the structure of the environment.*, Psychological Review; Psychological Review **63** (1956), no. 2, 129. (Cited on page 8.)
- [Sim57] ———, *A behavioral model of rational choice*, models of man; social and rational. ed., pp. 241–260, J. Wiley, New York, 1957. (Cited on page 8.)
- [Sim96] L. SIMPSON, *Do decision makers know what they prefer?: MAVT and ELECTRE II*, Journal of the Operational Research Society (1996), 919–929. (Cited on page 15.)
- [Sis85] Y. SISKOS, *Analyses de régression et programmation linéaire*, Revue de statistique appliquée **23** (1985), no. 2, 41–55. (Cited on page 29.)
- [ST03] I. STAMELOS AND A. TSOUKIÀS, *Software evaluation problem situations*, European Journal of Operational Research **145** (2003), no. 2, 273–286. (Cited on page 10.)
- [Sug74] M. SUGENO, *Theory of fuzzy integrals and its applications*, Ph.D. thesis, Tokyo Institute of Technology, 1974. (Cited on page 16.)
- [SY85] Y. SISKOS AND D. YANNAKOPOULOS, *UTASTAR: An ordinal regression method for building additive value functions*, Investigaçao Operacional **5** (1985), no. 1, 39–53. (Cited on pages 29 and 30.)
- [TMJ96] M. TAMIZ, SJ MARDLE, AND DF JONES, *Detecting IIS in infeasible linear programmes using techniques from goal programming*, Computers & operations research **23** (1996), no. 2, 113–119. (Cited on page 32.)
- [TS97] E. TRIANTAPHYLLOU AND A. SÁNCHEZ, *A sensitivity analysis approach for some deterministic multi-criteria decision-making methods*, Decision Sciences **28** (1997), no. 1, 151–194. (Cited on page 34.)
- [Tso06] A. TSOUKIÀS, *De la théorie de la décision à l'aide à la décision*, D. Bouyssou, D. Dubois, M. Pirlot, & H. Prade, Concepts et méthodes pour l'aide à la décision **1** (2006). (Cited on page 7.)
- [Tso07] ———, *On the concept of decision aiding process: an operational perspective*, Annals of Operations Research **154** (2007), no. 1, 3–27. (Cited on pages 10 and 14.)
- [Tve67] A. TVERSKY, *Additivity, utility, and subjective probability*, Journal of Mathematical Psychology **4** (1967), no. 2, 175–201. (Cited on page 8.)
- [Vin89] P. VINCKE, *L'aide multicritère à la décision*, Editions de l'Université Libre de Bruxelles (1989). (Cited on pages 24 and 30.)
- [VL81] J.N.M. VAN LOON, *Irreducibly inconsistent systems of linear inequalities*, European Journal of Operational Research **8** (1981), no. 3, 283–288. (Cited on page 32.)

- [VNM54] J. VON NEUMANN AND O. MORGENSTERN, *Theory of games and economic behavior*, Princeton University Press, 1944. Second edition in 1947, third in 1954. (Cited on pages 7 and 8.)
- [VV00] P. VALLIN AND D. VANDERPOOTEN, *Aide à la décision: une approche par les cas*, Recherche **67** (2000), 02. (Cited on page 6.)
- [Wak89] P.P. WAKKER, *Additive representations of preferences: A new foundation of decision analysis*, vol. 4, Springer, 1989. (Cited on page 7.)
- [Wat93] P. WATZLAWICK, *The situation is hopeless but not serious (the pursuit of unhappiness)*, W. W. Norton & Company, 1993. (Cited on page 6.)
- [WB87] S.R. WATSON AND D.M. BUEDE, *Decision synthesis: the principles and practice of decision analysis*, Cambridge Univ Pr, 1987. (Cited on page 34.)
- [Web87] M. WEBER, *Decision making with incomplete information*, European Journal of Operational Research **28** (1987), no. 1, 44–57. (Cited on page 34.)
- [Yu92] W. YU, *ELECTRE TRI : Aspects méthodologiques et manuel d'utilisation*, Document-Université de Paris-Dauphine, LAMSADE (1992). (Cited on pages 35 and 57.)
- [Zel82] M. ZELENY, *Multiple criteria decision making*, McGraw-Hill, New York, 1982. (Cited on page 44.)