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à l'Université La Rochelle
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présentée par

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Méthodes variationnelles : applications à l'analyse d'image et au modèle de Frenkel-Kontorova

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*A mes parents et à toi Monzer.
Cette thèse je la vous dédie.*

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RÉSUMÉ

Cette thèse est décomposée en deux parties. La première partie est consacrée à l'étude de la restauration d'image et la seconde partie est consacrée à l'étude d'un modèle de Frenkel-Kontorova par des méthodes issues du calcul variationnel et des équations aux dérivées partielles. Au chapitre 1, nous présentons les questions essentielles que nous traiterons dans cette thèse, puis on fait des rappels sur quelques propriétés des espaces des fonctions à variation bornée BV , sur les espaces d'Orlicz et sur les modèles de Frenkel-Kontorova. Au chapitre 2, nous montrons que les problèmes de minimisation non convexe (restauration d'image) contenant des termes de régularisation sous-linéaires sont *mal posés*, en général. Au chapitre 3, nous étudions un modèle de restauration avec un terme de régularisation à croissance non standard, inspiré du modèle proposé par Blomgren *et al.* : le module du gradient est élevé à une puissance qui dépend elle-même du gradient, i.e. de la forme $\int_{\Omega} |\nabla u|^{p(|\nabla u|)} dx$. Dans notre cadre plus général, on montre un résultat de semi-continuité inférieure pour la topologie faible- \star dans un espace d'Orlicz adéquat. Ceci permet d'établir un résultat d'existence et d'unicité de la solution. Au chapitre 4, nous étudions un modèle de Frenkel-Kontorova, dont on montre l'existence d'au moins une solution de type onde progressive.

ABSTRACT

This thesis is divided into two parts. The first one is devoted to the study of image restoration and the second part is devoted to the study of a Frenkel-Kontorova model using variational techniques and partial differential equations theory. In chapter 1, we present the key issues that we will discuss in this thesis, and recall the definitions and some properties of spaces of functions of bounded variation BV , Orlicz spaces and Frenkel-Kontorova model. In chapter 2, we show that the non-convex minimization problems (image restoration) involving sublinear regularizing term are ill-posed, in general. In chapter 3, we study a model of restoration with nonstandard increasing regularizing terms inspired by the model proposed by Blomgren *et al.* : i.e, the regularization term is of the form $\int_{\Omega} |\nabla u|^{p(|\nabla u|)} dx$. In our more general context, we show a lower semi-continuity result in the weak- \star topology in an adequate Orlicz space. This allows existence and uniqueness result of the solution. In Chapitre 4, we study a Frenkel-Kontorova model and show the existence of at least a traveling wave type solution.

Chapitre 1

Introduction

Le cadre général de cette thèse est le calcul variationnel dans des espaces fonctionnels de type Sobolev, Orlicz ou BV. Ce travail est formé de deux parties. Dans la première partie on s'intéresse principalement à la minimisation d'énergies issues de l'analyse d'image. On distingue deux types d'énergie :

- (i) Energies à croissance sous-linéaire à l'infini et qui conduisent à des problèmes mal-posés, en général.
- (ii) Energies à croissance sur-linéaire non-standard qui conduisent à des problèmes bien-posés dans des espaces d'Orlicz adéquats.

Dans la seconde partie, on étudie un modèle de Frenkel-Kontorova, décrivant le mouvement d'une chaîne de particules interagissant entre elles et soumises à un potentiel périodique. On montre que le problème possède des solutions de type onde progressive, à énergie potentielle finie, dans des espaces de Sobolev bien choisis.

1.1 Introduction à la première partie : analyse d'image.

Cette partie concerne deux classes de problèmes de minimisation issus de l'analyse d'image :

1. Minimisation dans BV : régularisation sous-linéaires

2. Minimisation dans les espaces d'Orlicz : régularisation non-standard sur-linéaires.

La restauration d'image est l'un des objectifs majeurs du traitement d'image : on cherche à reconstruire l'image originale à partir d'une image observée qui a subi des dégradations de type *flou* et *bruit*. Typiquement, on observe une image $f : \Omega \rightarrow \mathbb{R}$, $f \in L^2(\Omega)$, (image en niveaux de gris, définie sur un domaine $\Omega \subset \mathbb{R}^2$), et on cherche à retrouver l'image originale u dont f est une version dégradée vérifiant

$$f = R \cdot u + \eta,$$

où R est un opérateur compact de $L^2(\Omega)$ (qui peut représenter la convolution de u par un noyau $r \in L^2(\Omega)$, par exemple) et η est un bruit Gaussien.

Une approche classique pour résoudre ce problème est la minimisation d'une énergie formée d'un terme d'attache à la donnée f , augmenté d'un terme de régularisation.

Il est bien connu que les problèmes de minimisation avec des termes de régularisation sous-linéaires sont mal posés dans les espaces de Sobolev.

Une première généralisation aux cas des espaces des fonctions à variation bornée a été obtenue par Aubert *et al.* dans [10] dans le cas où la fonction de régularisation est bornée ainsi que la non *présence simultanée* de la partie singulière de la dérivée distributionnelle et de l'opérateur R .

Dans le chapitre 2, nous allons montrer le caractère mal-posé de ce type de problème dans la situation générale où le terme régularisant est sous-linéaire, mais pas nécessairement borné. De plus, nous considérons la présence d'un opérateur compact R simultanément avec une définition très générale de la variation totale non-linéaire $\int \Phi(|Du|)$ incluant à la fois les parties régulière et singulière.

Dans le chapitre 3, nous allons étudier le problème de minimisation suivant

$$\inf_{u \in W} \frac{1}{2} \|f - u\|_{L^2(\Omega)} + \lambda \int_{\Omega} \Psi(|Du|) dx,$$

où f est une image observée, u est l'image à reconstruire et

$$W = \left\{ u : \Omega \rightarrow \mathbb{R} : u \in L^2(\Omega) \text{ et } \int_{\Omega} \Psi(|Du|) dx < \infty \right\}$$

est l'espace naturel dans lequel notre énergie est bien définie.

Pour la démonstration de l'existence d'au moins un minimiseur, on utilise les méthodes directes du calcul des variations. Le seul obstacle auquel on doit faire face réside dans la semi-continuité inférieure du terme régularisant pour la topologie faible- \star de W .

1.2 Introduction à la seconde partie : modèle de Frenkel-Kontorova

Cette partie est détaillée dans le chapitre 4, où l'on va étudier l'existence de solutions spéciales pour les modèles de réseaux étendus. Plus précisément, on va étudier le cas du modèle de Frenkel-Kontorova, qui décrit le mouvement d'une chaîne de particules (ou atomes), $(q_n)_{n \in \mathbb{Z}}$, où chaque particule interagit avec ses deux voisines, cette chaîne étant soumise à un potentiel.

Les équations du mouvement d'un tel système est donnée par :

$$q_n'' = \gamma(q_{n+1} + q_{n-1} - 2q_n) + W'(q_n) \quad n \in \mathbb{Z}, \quad (1.2.1)$$

où q_n représente la position de la n -ième particule, (ou le déplacement de la n -ième particule de sa position d'équilibre), $\gamma > 0$ est une constante mesurant le couplage (qui est supposé linéaire) entre deux particules voisines et W est le potentiel qui est supposé 1-périodique, analytique au voisinage de 0 tel que $W(0) = 0 = W'(0) < W''(0)$.

Dans ce papier, on va s'intéresser aux solutions de type onde progressive, *i. e.*, des solutions satisfaisant la propriété suivante :

$$q_n(t) = u(n - c t) \quad n \in \mathbb{Z}, \quad (1.2.2)$$

où $c \in \mathbb{R}$ est la célérité de l'onde et $t > 0$ représente le temps. En posant $x = n - c t$, $\gamma = 1$ et remplaçant $q_n(t)$ par $u(x)$ dans (1.2.1), on obtient

$$c^2 u''(x) = u(x+1) + u(x-1) - 2u(x) + W'(u(x)) \quad \text{pour tout } x \in \mathbb{R}. \quad (1.2.3)$$

Pour montrer l'existence d'au moins une solution $u \in C^2(\mathbb{R})$, on va supposer $c > 1$ et que le potentiel périodique W satisfait les propriétés suivantes

$$\begin{cases} W \in C^2(\mathbb{R}), \\ W(a+1) = W(a) & \text{pour tout } a \in \mathbb{R}, \\ W(a) > 0 = W(0) = W'(0) & \text{pour tout } a \in \mathbb{R} \setminus \mathbb{Z}. \end{cases} \quad (1.2.4)$$

L'équation (1.2.3) étant variationnelle, on va lui associer une énergie définie sur un espace adéquat dont les points critiques sont les solutions faibles.

Pour cela, soit H l'espace défini par

$$H = \left\{ u \text{ mesurable telle que } u' \in L^2(\mathbb{R}) \text{ et } W(u) \in L^1(\mathbb{R}) \right\}.$$

Sur H , on définit l'opérateur D par :

$$D(u)(x) := u\left(x + \frac{1}{2}\right) - u\left(x - \frac{1}{2}\right) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} u'(t) dt.$$

On peut vérifier que tout point critique de la fonctionnelle

$$J(u) = \int_{\mathbb{R}} \left(\frac{c^2}{2} (u')^2 - \frac{1}{2} (D(u))^2 + W(u) \right) dx$$

sur H est une solution de (1.2.3).

Pour $R > 1$ fixé, on remplace H par H_R défini pour $\ell \in \mathbb{Z}$ par

$$H_R = \left\{ u \in H_{loc}^1(\mathbb{R}) \text{ tel que } u(x + 2R) - u(x) = \ell \right\}$$

et J par J_R définie par

$$J_R(u) = \int_{-R}^R \left(\frac{c^2}{2} (u')^2 - \frac{1}{2} (D(u))^2 + W(u) \right) dx.$$

En utilisant des méthodes variationnelles classiques, on montre, pour tout $R > 1$, l'existence d'au moins un $u_R \in H_R$ tel que

$$J_R(u_R) = \inf_{v \in H_R} J_R(v).$$

Ainsi, on obtient une suite $(u_R)_R$, puis on fait tendre R vers l'infini. La fonction limite $u \in H$ de u_R n'est pas nécessairement une ℓ -transition, i.e. $\lim_{x \rightarrow +\infty} u(x) - \lim_{x \rightarrow -\infty} u(x) = \ell$. Ensuite, on va prouver que u minimise J sur H , pour cela on montre $J(u) = E(\ell)$, où pour tout $k \in \mathbb{Z}$, on définit E par

$$E(k) = \inf_{\psi \in C_c^\infty(\mathbb{R})} \{ J(k\varphi + \psi) : \varphi \in C^\infty(\mathbb{R}), \varphi = 0 \text{ sur }]-\infty, -1] \text{ et } \phi = 1 \text{ sur }]-1, +\infty[\}.$$

De plus, on va montrer que la solution trouvée, u , décroît exponentiellement à l'infini, i.e. il existe deux constantes $C, \lambda > 0$ telles que

$$|f(x) - \lim_{x \rightarrow +\infty} u(x)| \leq C e^{-\lambda x} \quad \text{pour tout } x \geq 0$$

et

$$|f(x) - \lim_{x \rightarrow -\infty} u(x)| \leq C e^{\lambda x} \quad \text{pour tout } x \leq 0.$$

En outre, Pour un bon choix de c , $c \geq \sqrt{\frac{25}{24}}$, on va montrer que :

- L'entier 1 est stable, c'est à dire que

$$E(1) \leq E(1-p) + E(p), \quad \text{pour tout } p \in \mathbb{Z} \setminus \{0, 1\},$$

- La solution trouvée u est une 1-transition, c'est à dire $\lim_{x \rightarrow +\infty} u(x) - \lim_{x \rightarrow -\infty} u(x) = 1$.

1.3 Rappels généraux

1.3.1 Espace de BV

Dans cette sous-section, nous décrivons certaines propriétés, utiles pour la suite, sur la théorie des fonctions à variation bornée [4, 11]. Soit Ω un ouvert borné de \mathbb{R}^N ($N \geq 2$) de frontière Lipschitzienne et $C_c^1(\Omega, \mathbb{R}^2)$ l'espace des fonctions C^1 à support compact dans Ω à valeurs dans \mathbb{R}^2 .

Definition 1.3.1 Soit $u \in L^1(\Omega)$. On dit que u est une fonction à variation bornée dans Ω si la dérivée au sens des distributions de u est représentée par une mesure de Radon finie $Du = (D_1u, \dots, D_Nu)$ dans Ω , c.à-d.

$$\int_{\Omega} u \frac{\partial \phi}{\partial x_j} = - \int_{\Omega} \phi d(D_j u), \quad \forall \phi \in C_c^1(\Omega; \mathbb{R}), \quad j = 1, \dots, N. \quad (1.3.1)$$

L'espace de toutes les fonctions à variation bornée dans Ω est noté par $BV(\Omega)$; et la variation totale de la mesure Du est notée par $|Du|(\Omega)$. Une définition équivalente de l'espace $BV(\Omega)$ est donnée par :

Definition 1.3.2 Soit $u \in L^1(\Omega)$. On définit la variation totale de la fonction u sur Ω par :

$$VT(u) := \sup \left\{ \int_{\Omega} u \operatorname{div} g \, dx; \quad g \in C_c^1(\Omega; \mathbb{R}^2) \text{ et } |g|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Proposition 1.3.3 Soit $u \in L^1(\Omega)$. Alors on a l'équivalence suivante $u \in BV(\Omega) \iff VT(u) < +\infty$. De plus, $VT(u) = |Du|(\Omega)$ pour tout $u \in BV(\Omega)$.

Notations : Pour tous $u, v \in BV(\Omega)$, $\alpha \in \mathbb{R}$, on a clairement

1. $|D(u+v)|(\Omega) \leq |Du|(\Omega) + |Dv|(\Omega)$, et
2. $|D(\alpha u)|(\Omega) = |\alpha| |Du|(\Omega)$.

Donc, la variation totale $|Du|(\Omega)$ est une semi-norme de $BV(\Omega)$. Pour la semi-norme, nous nous servons aussi de la notation

$$|u|_{BV(\Omega)} = \int_{\Omega} |Du|(\Omega).$$

Muni de la norme

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |u|_{BV(\Omega)}, \quad u \in BV(\Omega),$$

l'espace $BV(\Omega)$ est un espace de Banach.

Exemple 1.3.4 *Pour $u \in W^{1,1}(\Omega)$, la dérivée au sens des distributions Du existe et appartient à $L^1(\Omega)$. Et, pour tout $\phi \in C_c^1(\Omega, \mathbb{R}^N)$, telle que $\|\phi\|_{\infty} \leq 1$, on a*

$$\int_{\Omega} u \operatorname{div} \phi \, dx := - \int_{\Omega} Du \cdot \phi \, dx \leq \int_{\Omega} |Du| \, dx < 1.$$

Donc,

$$W^{1,1}(\Omega) \subset BV(\Omega).$$

En particulier, comme Ω est borné,

$$W^{1,p}(\Omega) \subset BV(\Omega), \quad \forall 1 \leq p \leq +\infty.$$

Remarque 1.3.5 *Dans le cas où $u = \chi_A$, la fonction caractéristique de l'ensemble $A \subset \mathbb{R}^N$, alors*

$$\int_{\Omega} |Du| = \operatorname{Per}_{\Omega}(A).$$

Si de plus ∂A est suffisamment régulier on peut démontrer que dans ce cas cette définition correspond avec la notion de périmètre ($N = 2$) ou d'aire ($N = 3$) classique. Ainsi l'espace BV semble bien adapté à notre problème. Examinons quelques unes de ses propriétés fondamentales. Quand u est différentiable, on a

$$\int_{\Omega} |Du| = \int_{\Omega} |\nabla u|.$$

Nous résumons ici les propriétés principales de $BV(\Omega)$ que l'on rencontre en analyse d'image avec des méthodes variationnelles. Nous supposons que le bord de Ω est régulier. L'espace des mesures bornées, réelles ou vectorielles, de Radon sur Ω sera noté

$\mathcal{M}(\Omega)$. De plus, on dit qu'une suite de mesures $(\mu_n)_n \subset \mathcal{M}(\Omega)$ converge faiblement vers $\mu \in \mathcal{M}(\Omega)$, et on note $\mu_n \rightharpoonup \mu$, si

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \phi \, d\mu_n = \int_{\Omega} \phi \, d\mu, \quad \forall \phi \in C_c(\Omega, \mathbb{R}).$$

Alors l'espace BV vérifie :

1. Semi-continuité inférieure :

Soit $(u_n)_n \in BV(\Omega)$ une suite telle que $u_n \rightarrow u$ dans $L^1_{loc}(\Omega)$. Alors $u \in BV(\Omega)$ et

$$|Du|(\Omega) \leq \liminf_{n \rightarrow +\infty} |Du_n|(\Omega).$$

2. Trace :

L'opérateur de trace

$$tr = \begin{cases} u \mapsto u|_{\partial\Omega} \\ BV(\Omega) \longrightarrow L^1(\partial\Omega, \mathcal{H}^{N-1}) \end{cases}$$

est linéaire et continu pour la topologie forte de $BV(\Omega)$. Ici, \mathcal{H}^{N-1} désigne la mesure $(N-1)$ -dimensionnelle de Hausdorff.

3. Topologie faible- \star sur $BV(\Omega)$:

La topologie forte de $BV(\Omega)$ ne possède pas de bonnes propriétés de compacité. On lui préfère une topologie affaiblie notée : $BV - \omega^*$ et définie par

$$u_n \rightharpoonup^{BV-\omega^*} u \iff u_n \longrightarrow^{L^1(\Omega)} u \text{ et } Du_n \rightharpoonup Du \text{ dans } \mathcal{M}(\Omega).$$

4. Compacité :

Si $(u_n)_n$ est une suite bornée dans $BV(\Omega)$ alors elle est relativement compacte dans $L^p(\Omega)$ pour $1 \leq p < \frac{N}{N-1}$, $N \geq 2$. Donc, il existe $u \in L^p(\Omega)$ et une suite extraite $(u_{\phi(n)})$ telle que $u_{\phi(n)} \rightarrow u$ dans $L^p(\Omega)$ pour $1 \leq p < \frac{N}{N-1}$, $N \geq 2$. De plus, il existe une suite extraite $(u_{\psi(n)})$ telle que $u_{\psi(n)} \rightharpoonup u$ dans $BV - \omega^*$. L'espace $BV(\Omega)$ s'injecte continûment dans $L^\infty(\Omega)$ si $N = 1$ et dans $L^{\frac{N}{N-1}}(\Omega)$ si $N \geq 2$.

5. Décomposition :

Si $u \in BV(\Omega)$ alors la mesure Du peut être décomposée en la somme d'une mesure régulière et d'une mesure singulière

$$Du = \nabla u \, dx + D_s u$$

où $\nabla u \in L^1(\Omega)$ est la dérivée de Radon-Nikodym par rapport à la mesure de Lebesgue et $D_s u \perp dx$. Il est possible d'aller encore plus loin en ce qui concerne cette décomposition. Dans [4], Ambrosio a en effet démontré que la partie singulière $D_s u$ pouvait être décomposée en une composante de "saut" et une composante de "Cantor". Afin de définir ces deux notions, nous introduisons, pour tout $x \in \Omega$, la boule $B(x, r)$ centrée en x et de rayon r . Nous définissons d'abord la limite supérieure approchée et la limite inférieure approchée par

$$u^+(x) = \inf \left\{ t \in \overline{\mathbb{R}} : \lim_{r \rightarrow 0} \frac{dx(\{u > t\} \cap B(x, r))}{r^n} = 0 \right\},$$

$$u^-(x) = \inf \left\{ t \in \overline{\mathbb{R}} : \lim_{r \rightarrow 0} \frac{dx(\{u < t\} \cap B(x, r))}{r^n} = 0 \right\}.$$

Si $u \in L^1(\Omega)$ alors, pour presque tout $x \in \Omega$, on a

$$\lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |u(x) - u(y)| dy = 0.$$

Les points de Ω tels que cette égalité est vérifiée sont appelés les points de Lebesgue (on dit qu'en ces points, la fonction u est approximativement continue). En effet, on a alors

$$u(x) = \lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy,$$

et

$$u(x) = u^+(x) = u^-(x).$$

Que se passe-t-il alors pour les autres points ? On définit S_u l'ensemble des points de saut comme étant le complémentaire pour la mesure de Hausdorff \mathcal{H}^{N-1} des points de Lebesgue.

$$S_u := \{x \in \Omega : u^+(x) > u^-(x)\},$$

où \mathcal{H}^{N-1} est la mesure de Hausdorff $(N-1)$ -dimensionnelle. Il est connu que l'ensemble S_u est rectifiable, et on peut définir un vecteur normal $n_u(x)$ pour $x \in \Omega$, \mathcal{H}^{N-1} presque partout.

Enfin on définit l'ensemble C_u des points de Cantor comme étant l'ensemble complémentaire des deux ensembles : S_u et l'ensemble des points de Lebesgue. La version améliorée de la décomposition de Lebesgue de la mesure D_u pour une fonction à variation bornée u , due à L. Ambrosio, est donnée par :

$$Du = \nabla u dx + (u^+ - u^-)n_u \mathcal{H}_{|S_u}^{N-1} + C_u,$$

où $J_u = (u^+ - u^-)n_u \mathcal{H}_{|S_u}^{N-1}$ est la partie saut de Du . Nous pouvons alors décrire la variation totale comme

$$\int_{\Omega} |Du| = \int_{\Omega} |\nabla u| dx + \int_{S_u} |u^+ - u^-| d\mathcal{H}^{N-1} + \int_{\Omega - S_u} |Cu|.$$

6. Décomposition de Lebesgue et fonctions convexes :

Soit $\Phi : \mathbb{R} \rightarrow \mathbb{R}^+$ paire, convexe, croissante et linéaire à l'infini. On appelle Φ^∞ la fonction de récession de Φ définie par :

$$\Phi^\infty(z) := \lim_{s \rightarrow \infty} \frac{\Phi(sz)}{s}.$$

Pour tout $u \in BV(\Omega)$, on définit :

$$\int_{\Omega} \Phi(|Du|) = \int_{\Omega} \Phi(|\nabla u|) dx + \Phi^\infty(1) \int_{S_u} |u^+ - u^-| d\mathcal{H}^{N-1} + \Phi^\infty(1) \int_{\Omega - S_u} |Cu|.$$

Notons que si $\Phi(s) = s$, alors on retrouve la variation totale de u , c.à-d. $\int_{\Omega} |Du|$. Le résultat important dans ce cadre est :

La fonctionnelle : $u \mapsto \int_{\Omega} \Phi(|Du|)$ est sci pour la topologie $BV - \omega^*$.

D'où, si l'on dispose d'une suite $(u_n)_n$ bornée dans $BV(\Omega)$ alors en combinant les propriétés 4. et 6., il existe une suite extraite, encore notée par $(u_n)_n$, et $u \in BV(\Omega)$ tels que $u_n \rightharpoonup u$ dans $BV - \omega^*$, en outre, on a :

$$\int_{\Omega} \Phi(|Du|) \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \Phi(|Du_n|).$$

Pour un domaine borné de \mathbb{R}^n . On pose

$$u_{\Omega} = \left(\frac{1}{|\Omega|} \int_{\Omega} u(x) dx \right) \mathbb{1}_{\Omega}, \text{ avec } u \in L^1(\Omega).$$

Theorem 1.3.6 (Inégalité de Poincaré)

Soit Ω un domaine borné de \mathbb{R}^n à frontière Lipschitzienne, il existe une constante C qui dépend uniquement de Ω telle que

$$\int_{\Omega} |u - u_{\Omega}| dx \leq C |Du|(\Omega), \quad \forall u \in BV(\Omega).$$

Theorem 1.3.7 (Inégalité de Poincaré-Wirtinger)

Le point 4. et le théorème précédent impliquent l'inégalité de Sobolev-Poincaré (aussi connue par l'inégalité de Poincaré-Wirtinger) :

$$\|u - u_{\Omega}\|_p \leq C |Du|(\Omega), \quad \forall u \in BV(\Omega), \quad 1 \leq p \leq \frac{N}{N-1}.$$

Aussi, la constante $C > 0$ dépend seulement de Ω .

1.3.2 Espace d'Orlicz

Soit (G, μ) un espace mesuré. Les espaces $L^p(G, \mu)$ sont définis par :

$$L^p(G, \mu) = \left\{ f : G \rightarrow \mathbb{R}, \int_G |f(t)|^p d\mu(t) < \infty \right\}.$$

Munis de la norme

$$\|f\|_p = \left(\int_G |f(t)|^p d\mu(t) \right)^{\frac{1}{p}},$$

les espaces $(L^p(G, \mu), \|\cdot\|_p)$ sont des espaces de Banach. Dans leur définition même, il apparaît la fonction homogène $\Phi(u) = |u|^p$. Les espaces d'Orlicz constituent une classe d'espaces plus large que les espaces L^p . Leur construction est basée sur la généralisation de cette fonction Φ . L'objet de cette sous-section est de présenter les bases de la théorie des espaces d'Orlicz utiles pour le Chapitre 3 de ce mémoire. Les livres de Krasnosel'skii-Ya [32] et celui de Rao et Ren [36] sont des références complètes sur ce thème.

Fonctions de Young

Une généralisation de la fonction définie par $\Phi(u) = |u|^p$, est donnée par la classe de fonctions suivantes. Soit $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ une fonction convexe du type

$$\Phi(t) = \int_0^t \varphi(s) ds \quad \text{pour tout } t > 0,$$

où $\varphi : [0, +\infty) \rightarrow \mathbb{R}$ est une fonction croissante et continue à droite satisfaisant les propriétés suivantes :

$$\varphi(s) > 0, \quad \forall s > 0, \quad \varphi(0) = 0 \quad \text{et} \quad \lim_{s \rightarrow \infty} \varphi(s) = +\infty. \quad (1.3.2)$$

Définissons

$$\tilde{\varphi}(s) = \sup_{\varphi(t) \leq s} t.$$

Il est clair que $\tilde{\varphi}$ satisfait aussi les propriétés (1.3.2) et que l'on peut trouver φ à l'aide de $\tilde{\varphi}$ par

$$\varphi(t) = \sup_{\tilde{\varphi}(s) \leq t} s.$$

La fonction $\tilde{\Phi}$ donnée par

$$\tilde{\Phi}(s) = \int_0^s \tilde{\varphi}(\sigma) d\sigma$$

est dite la complémentaire de Φ . De plus, les fonctions complémentaires Φ et $\tilde{\Phi}$ satisfont l'inégalité de Young suivante :

$$st \leq \tilde{\Phi}(s) + \Phi(t) \quad \text{Pour tout } s, t \in \mathbb{R}, \quad (1.3.3)$$

Dans 1.3.3, l'égalité est atteinte si et seulement si $t = \tilde{\varphi}(s)$ ou $s = \varphi(t)$. D'après cette dernière inégalité, on trouve que

$$\tilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\}.$$

Definition 1.3.8 Soient Φ_1 et Φ_2 sont deux fonctions de Young, On dit que Φ_1 domine Φ_2 globalement s'il existe une constante $k > 0$ telle que

$$\Phi_2(t) \leq \Phi_1(kt) \quad \text{pour tout } t \geq 0. \quad (1.3.4)$$

Similairement, Φ_1 domine Φ_2 au voisinage de l'infini s'ils existent deux constantes k et $t_0 > 0$ telles que l'inégalité (1.3.4) reste vraie pour tout $t \geq t_0$.

Deux fonctions sont dites équivalentes globalement (ou au voisinage de l'infini) si chacune domine l'autre globalement (ou au voisinage de l'infini).

Proposition 1.3.9 Soient Φ_1 et Φ_2 deux fonctions de Young. Si Φ_1 et Φ_2 possèdent respectivement les complémentaires $\tilde{\Phi}_1$ et $\tilde{\Phi}_2$. Alors Φ_1 domine Φ_2 globalement (ou au voisinage de l'infini) si et seulement si $\tilde{\Phi}_2$ domine $\tilde{\Phi}_1$ globalement (ou au voisinage de l'infini).

Par conséquent, Φ_1 et Φ_2 sont équivalentes si et seulement si $\tilde{\Phi}_1$ et $\tilde{\Phi}_2$ sont équivalentes.

Definition 1.3.10 Soient Φ_1 et Φ_2 sont deux fonctions de Young. Si Φ_1 domine Φ_2 au voisinage de l'infini et Φ_1 et Φ_2 ne sont pas équivalentes au voisinage de l'infini, alors on dit que Φ_2 croît essentiellement plus lentement que Φ_1 au voisinage de l'infini. C'est le cas si et seulement si pour tout $k > 0$

$$\lim_{t \rightarrow \infty} \frac{\Phi_2(kt)}{\Phi_1(t)} = 0.$$

Definition 1.3.11 On dit qu'une fonction de Young, Φ , satisfait la condition Δ_2 globalement s'il existe une constante $k > 0$ telle que pour tout $t > 0$,

$$\Phi(2t) \leq k\Phi(t), \quad (1.3.5)$$

pour simplifier, on écrit $\Phi \in \Delta_2$ globalement.

Similairement, on dit que Φ satisfait la condition Δ_2 au voisinage de l'infini, on écrit $\Phi \in \Delta_2$ au voisinage de l'infini, s'il existent deux constantes t_0 et $k > 0$ telles que l'inégalité (1.3.5) reste vraie pour tout $t \geq t_0$.

Espace d'Orlicz

Soit Ω un domaine de \mathbb{R}^n , avec $n \geq 1$, et Φ une fonction de Young.

Definition 1.3.12 L'ensemble $K^\Phi(\Omega)$ de toutes les classes d'équivalence, modulo l'égalité p.p. dans Ω , des fonctions mesurables u définies sur Ω et satisfaisant

$$\int_{\Omega} \Phi(|u(x)|) dx < \infty,$$

est appelé classe d'Orlicz.

Definition 1.3.13 (Δ -régulier)

Soit Φ est une fonction de Young. On dit que la paire (Φ, Ω) est Δ -régulier si

- (a) Φ satisfait la condition Δ_2 globalement, ou bien
- (b) Φ satisfait la condition Δ_2 au voisinage de l'infini et Ω est de mesure finie.

Remarque 1.3.14 $K^\Phi(\Omega)$ est un espace vectoriel si et seulement si la paire (Φ, Ω) est Δ -régulier.

Definition 1.3.15 L'espace d'Orlicz $L^\Phi(\Omega)$ est défini comme étant l'enveloppe linéaire de la classe d'Orlicz $K^\Phi(\Omega)$, c.à d. qu'il est le plus petit espace vectoriel contenant $K^\Phi(\Omega)$.

$L^\Phi(\Omega)$ muni de la norme

$$\|u\|_{\Phi} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}$$

est un espace de Banach.

Si Φ et $\tilde{\Phi}$ sont deux fonctions de Young complémentaires, alors on a une version généralisée de l'inégalité de Hölder

$$\left| \int_{\Omega} u(x)v(x) dx \right| \leq 2\|u\|_{\Phi}\|v\|_{\tilde{\Phi}} \quad \text{pour tous } u \in L^{\Phi}(\Omega) \text{ et } v \in L^{\tilde{\Phi}}(\Omega).$$

Notons $E^{\Phi}(\Omega)$ la fermeture dans $L^{\Phi}(\Omega)$ de l'espace des fonctions u qui sont bornées dans Ω et ayant un support compact dans $\overline{\Omega}$.

$E^{\Phi}(\Omega)$ est le plus grand sous-espace linéaire de $K^{\Phi}(\Omega)$. On a

$$E^{\Phi}(\Omega) \subseteq K^{\Phi}(\Omega) \subseteq L^{\Phi}(\Omega).$$

Lemma 1.3.16 *Pour tout $v \in L^{\tilde{\Phi}}(\Omega)$, la fonction linéaire L_v définie par*

$$L_v(u) = \int_{\Omega} u(x)v(x) dx,$$

est dans $[L^{\Phi}(\Omega)]'$. En notant par $\|L_v\|$ sa norme dans cet espace, on obtient

$$\|v\|_{\tilde{\Phi}} \leq \|L_v\| \leq 2\|v\|_{\tilde{\Phi}}.$$

Remarque 1.3.17 *Ce dernier lemme reste vrai également si L_v est limitée à agir sur $E^{\Phi}(\Omega)$.*

Theorem 1.3.18 *L'espace dual $[E^{\Phi}(\Omega)]'$ de $E^{\Phi}(\Omega)$ est isomorphe et homéomorphe à $L^{\tilde{\Phi}}(\Omega)$.*

Theorem 1.3.19 *$L^{\Phi}(\Omega)$ est réflexif si et seulement si les deux paires (Φ, Ω) et $(\tilde{\Phi}, \Omega)$ sont Δ -réguliers.*

Espaces d'Orlicz-Sobolev

Definition 1.3.20 *Soit Ω un domaine de \mathbb{R}^n et Φ une fonction de Young. L'espace d'Orlicz-Sobolev, $W^{m,\Phi}(\Omega)$, est l'ensemble de toute les (classes d'équivalence de) fonctions $u \in L^{\Phi}(\Omega)$ dont les dérivées au sens des distributions $D^{\alpha}u$ sont aussi dans $L^{\Phi}(\Omega)$ pour tout α tel que $|\alpha| \leq m$.*

L'espace $W^{m,\Phi}(\Omega)$ muni de la norme

$$\|u\|_{m,\Phi} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_\Phi,$$

est un espace de Banach.

Théorème d'injections pour les espaces d'Orlicz-Sobolev

Soit A une fonction de Young. On introduit les hypothèses suivantes :

$$\int_0^1 \frac{A^{-1}(t)}{t^{(n+1)/n}} dt < \infty, \quad (\text{H1})$$

$$\int_0^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}} dt = \infty, \quad (\text{H2})$$

Définissons la conjuguée de Sobolev, A_* , de A par :

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}} d\tau, \quad t \geq 0. \quad (\text{H3})$$

Theorem 1.3.21 *Soit Ω un domaine borné, ayant la propriété du cône dans \mathbb{R}^n , si (H1) et (H2) sont satisfaites, alors l'injection suivante*

$$W^{1,A}(\Omega) \rightarrow L^{A_*}(\Omega)$$

est continue.

De plus, si B est une fonction de Young qui croît essentiellement plus lentement que A_ au voisinage de l'infini, alors l'injection suivante*

$$W^{1,A}(\Omega) \rightarrow L^B(\Omega)$$

est compacte.

Theorem 1.3.22 *Soit Ω un domaine ouvert de \mathbb{R}^n dont le bord $\partial\Omega$ est lipschitzien, Soit $\Phi \in \Delta_2$, alors l'injection suivante :*

$$W^{1,\Phi}(\Omega) \rightarrow L^\Phi(\Omega)$$

est compacte.

1.3.3 Modèle de Frenkel-Kontorova (FK)

Pour un bon aperçu sur le modèle de Frenkel-Kontorova (FK), le livre de Braun et Kivshar [16, 17] et l'article [13] de Floria et Mazo constituent deux bonnes références. Un modèle simple et classique de FK décrit une chaîne de particules interagissant avec ses voisines et soumise à un potentiel périodique. Ce modèle est représenté par le Hamiltonien suivant :

$$\mathcal{H} = K + U, \quad (1.3.6)$$

où K est l'énergie cinétique,

$$K = \frac{m}{2} \sum_n \left(\frac{dq_n}{dt} \right)^2,$$

m étant la masse de la particule, et l'énergie potentielle U est divisée en deux parties

$$U = U_{sub} + U_{int},$$

où d'une part U_{sub} caractérise une interaction de la chaîne avec un potentiel externe périodique, prenant la forme simple :

$$U_{sub} = \frac{\varepsilon}{2} \sum_n \left[1 - \cos \left(\frac{2\pi q_n}{a} \right) \right],$$

où ε est l'amplitude et a est la période. D'autre part, U_{int} prend en considération l'interaction harmonique des particules voisines dans la chaîne,

$$U_{int} = \frac{g}{2} \sum_n (q_{n+1} - q_n - a_0)^2, \quad (1.3.7)$$

où g est la constante d'élasticité et a_0 est la distance à l'équilibre du potentiel interatomique.

En renormalisant

$$q_n \rightarrow \frac{2\pi}{a} q_n \quad \text{and} \quad t \rightarrow \frac{2\pi}{a} \sqrt{(\varepsilon/2m)} t,$$

le Hamiltonien (1.3.6)-(1.3.7) prend la forme

$$H = \frac{\mathcal{H}}{(\varepsilon/2)} = \sum_n \left\{ \frac{1}{2} \left(\frac{dq_n}{dt} \right)^2 + (1 - \cos q_n) + \frac{1}{2} g (q_{n+1} - q_n - a_0)^2 \right\}, \quad (1.3.8)$$

où $a_0 \rightarrow a_0(2\pi/a)$ et $g \rightarrow g \frac{(a/2\pi)^2}{(\varepsilon/2)}$.

Dans une telle forme renormalisée, le Hamiltonien (1.3.8) décrit une chaîne harmonique d'atomes de masses unité, soumise à un potentiel extérieur sinusoïdal de la

période $a = 2\pi$ et d'amplitude $\varepsilon = 2$.

De l'Hamiltonien (1.3.8) découle l'équation de mouvement

$$\frac{d^2 q_n}{dt^2} + \sin q_n - g(q_{n+1} + q_{n-1} - 2q_n) = 0.$$

Dans le chapitre 4, nous étudions l'équation

$$\frac{d^2 q_n}{dt^2} + W'(q_n) - (q_{n+1} + q_{n-1} - 2q_n) = 0,$$

où W est une fonction 1-périodique.

Chapitre 2

Sublinear minimization problems

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Abstract

It is well known that minimization problems involving sublinear regularization terms are ill-posed in Sobolev spaces. Extended results to the space of bounded variation functions (BV) were recently showed in the special case of *bounded* regularization terms and *unblurred* data. In this paper, general sublinear regularization is studied in BV spaces. We show in a general context that such problems are indeed ill-posed, except for very special cases which will be described. Our results are then optimal in the sense that linear regularization leads to well-posed minimization problems in BV spaces. Applications to image restoration and image decomposition are presented.

Keywords : Total variation, sublinear regularization, image decomposition.

Mathematics Subject Classification (2000) : 35J, 35L, 35Q, 49J, 49N.

2.1 Introduction

In this work, minimization problems with sublinear regularization terms are considered on the space of bounded variation functions. More precisely, we are interested

by ill-posedness of minimization problems of type :

$$\inf_{u \in BV(\Omega)} \left\{ \mathcal{N}(f - Ru) + \int_{\Omega} \Phi(|Du|) \right\},$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 2$, f is a given function in $L^{\frac{N}{N-1}}(\Omega)$ and R is a compact operator on $L^{\frac{N}{N-1}}(\Omega)$ or just the (non compact) identity operator. \mathcal{N} is a given norm on Lebesgue spaces or on distributions subsets, this norm will be specified in the sequel.

Such minimization problems appear in image processing in the special case $N = 2$. In such frameworks, Ω may denote the image domain, f the observed image and R is a convolution operator, *i. e.*, there is a kernel $r \in L^2(\mathbb{R}^2)$ such that :

$$Ru(x) := \int_{\Omega} r(x-y)u(y) dy,$$

which expresses that the original image is blurred. The norm \mathcal{N} is introduced to minimize oscillations between the observed image and the optimal function. The regularization term $\int_{\Omega} \Phi(|Du|)$ allows the selection of the optimal solution among more *regular* functions and not among all functions on which the norm \mathcal{N} is defined.

The function $\Phi : [0, +\infty[\rightarrow [0, +\infty[$ is a given sublinear function (at infinity), that is $\lim_{s \rightarrow +\infty} \Phi(s)/s = 0$, the measure Du denotes the generalized gradient of the function u and $BV(\Omega)$ is the space of functions of bounded variation, a brief description of this space is given below.

In image restoration, this kind of problems occurs in the situation where an ideal unknown image \hat{u} is corrupted by a compact operator (blur) and an additive noise, that is $f = R\hat{u} + \eta$, η is the additive noise. A well-known mathematical method to recover \hat{u} is to minimize a certain norm of the noise $\mathcal{N}(f - Ru)$ added to a regularization term, $\int_{\Omega} \Phi(|Du|)$, which penalizes functions having high oscillations. To penalize noise but not edges, one judicious variational idea is to choose Φ quadratic in the homogeneous regions of the image and sublinear in regions with high gradients (edges) [11]. Indeed, such regularizations induce an isotropic smoothing (Laplacian) effect in flat regions and preserves edges thanks to their limited cost (the exponent of Φ for high gradients is less than one).

The energy considered in [24] is of the form :

$$H(u) = \int_I (f(x) - u(x))^2 dx + \int_I \frac{\alpha[u'(x)]^2}{1 + \alpha[u'(x)]^2} dx,$$

where I is a real open interval, $f \in L^\infty(I)$ and α is a positive real parameter. Notice here that the considered regularizing term in [24] is bounded ($0 \leq \Phi \leq 1$).

The authors showed that minimizers of H are achieved in the Sobolev space $W^{1,2}(I)$ only in the case where the observed image f is constant and that the unique solution is f itself. The key idea is that the sublinearity of the regularizing term allows the choice of minimizing sequences $(u_n)_n$ such that

$$\lim_{n \rightarrow +\infty} \|u_n - f\|_{L^2(I)} = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \int_I \frac{\alpha [u'_n(x)]^2}{1 + \alpha [u'_n(x)]^2} dx = 0.$$

The purpose of this paper is to generalize such ideas in the framework of $BV(\Omega)$ in image restoration and decomposition problems [35] for all sublinear regularizing terms, in the presence of blur operator, for higher space dimensions and for any observed image in $L^2(\Omega)$

In [10], the authors considered the special cases :

- The presence of a blur operator R , but the regularization term depends only on the regular part of the measure Du , that is $\int_\Omega \Phi(|Du|) = \int_\Omega \Phi(|\nabla u|) dx$, where $Du = \nabla u dx + D^s u$, is the Lebesgue decomposition of the measure Du .
- The regularization term $\int_\Omega \Phi(|Du|)$ takes into account both regular and singular parts of Du , but Φ bounded, $f \in L^\infty(\Omega) \subset L^2(\Omega)$ and R is the identity operator on $L^2(\Omega)$.

In this paper, we extend the results in [10] without any restriction on the regularization function Φ nor on the observed image f . We suppose however that the operator R is compact, which is a natural assumption for convolution operators, which belong to a special class of Hilbert-Schmidt integral operators.

In our opinion, this work provides a complete picture about this large class of minimization problems :

- If Φ is superlinear at infinity, that is, $\lim_{s \rightarrow +\infty} \Phi(s)/s = +\infty$, the theory is well known in the framework of Sobolev spaces.
- If Φ is convex and linear at infinity, that is, $\lim_{s \rightarrow +\infty} \Phi(s)/s \in]0, +\infty[$, the theory is well known in the framework of $BV(\Omega)$. Indeed, the associated Sobolev space $W^{1,1}(\Omega)$ is not adapted for such growth conditions. The convexity assumption implies the weak- $*$ lower semicontinuity in $BV(\Omega)$ of the map $u \mapsto \int_\Omega \Phi(|D^a u|) dx + c \int_\Omega |D^s u|$, where $c = \lim_{s \rightarrow +\infty} \Phi(s)/s$ and $D^a u$, $D^s u$ denote the absolutely continuous and the singular parts of the measure Du .

- If Φ is sublinear at infinity, that is, $\lim_{s \rightarrow +\infty} \Phi(s)/s = 0$, we show here that the minimization problem is ill-posed, except for data $f = R \cdot \mathbf{1}_\Omega$, up to a multiplicative constant, where $\mathbf{1}_\Omega$ is the indicatrice function of Ω .

2.2 Functions of Bounded Variation (BV)

For the reader's convenience, we recall some basic definitions and classical results on bounded variation functions. Let Ω be a bounded domain of \mathbb{R}^N , $N \geq 2$, and $u \in L^1(\Omega)$. We say that $u \in BV(\Omega)$ if its distributional derivative Du is an \mathbb{R}^N -valued Radon measure with bounded total variation on Ω ; this total variation will be denoted by $\int_\Omega |Du|$. Endowed with the norm

$$\|u\| := \|u\|_{L^1(\Omega)} + \int_\Omega |Du|$$

the set $BV(\Omega)$ has the Banach space structure. Moreover, $BV(\Omega)$ is embedded compactly in $L^p(\Omega)$ for $1 \leq p < N/(N-1)$ and continuously in $L^{\frac{N}{N-1}}(\Omega)$.

Let us recall also the Lebesgue decomposition for distributional derivatives of functions with bounded variation [5]. Let $u \in L^1(\Omega)$, we say that u has an approximate limit at $x \in \Omega$ if there exists $\tilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - \tilde{u}(x)| dy = 0,$$

where $B_r(x)$ is the ball centered at x of radius $r > 0$ and $|B_r(x)|$ is its volume. The set \mathcal{C}_u of all points where u admits an approximate limit is a Borel set, moreover, almost every point of Ω belongs to \mathcal{C}_u . For every $x \in \mathcal{C}_u$, if $u(x) = \tilde{u}(x)$, we say that u is approximatively continuous at x .

On the other hand, a point $x \in \Omega \setminus \mathcal{C}_u$ is an approximate jump for u if there are two real numbers $u^+(x)$ and $u^-(x)$ and $\nu_u(x) \in \mathbb{S}^{N-1}$ such that

$$\lim_{r \rightarrow 0} \frac{1}{|B_r^\pm(x, \nu_u(x))|} \int_{B_r^\pm(x, \nu_u(x))} |u(y) - u^\pm(x)| dy = 0,$$

where \mathbb{S}^{N-1} is the unit sphere of \mathbb{R}^N , $B_r^+(x, \nu_u(x)) = \{y \in B_r(x) : \langle y - x, \nu_u(x) \rangle > 0\}$ and $B_r^-(x, \nu_u(x)) = \{y \in B_r(x) : \langle y - x, \nu_u(x) \rangle < 0\}$. The set \mathcal{J}_u of all approximate jump points of u is a Borel set.

Let $x \in \mathcal{C}_u$, we say that u is approximately derivable at x if there exists a vector $\nabla u(x) \in \mathbb{R}^N$ such that

$$\lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - \tilde{u}(x) - \langle \nabla u(x), y - x \rangle| dy = 0.$$

The vector $\nabla u(x)$ is called the approximate gradient of u at x . The set of all points of Ω where u is approximately derivable is a Borel set and will be denoted by \mathcal{D}_u .

We recall that $\tilde{u} : \mathcal{C}_u \rightarrow \mathbb{R}$, $u^+, u^- : \mathcal{J}_u \rightarrow \mathbb{R}$, $\nu_u : \mathcal{J}_u \rightarrow \mathbb{S}^{N-1}$ and $\nabla u : \mathcal{D}_u \rightarrow \mathbb{R}^N$ are all Borel functions.

Finally, for every $u \in BV(\Omega)$, the measure Du can be decomposed as : $Du = D^a u + D^s u$, where the absolutely continuous part $D^a u := \nabla u \mathcal{L}^N \llcorner_{\mathcal{D}_u}$, and the singular part $D^s u = D^{\mathcal{J}} u + D^{\mathcal{T}} u$, is the sum of the *jump* part and the *Cantor* part of $D^s u$ respectively. More precisely,

$$D^{\mathcal{J}} u := (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner_{\mathcal{J}_u} \quad \text{and} \quad D^{\mathcal{T}} u := Du \llcorner_{\mathcal{C}_u \setminus \mathcal{D}_u}.$$

Notice that the Hausdorff dimension of the support of $D^{\mathcal{T}} u$ is strictly less than N but strictly greater than $N - 1$. Therefore, in the sequel, the measure Du is decomposed as

$$Du = \nabla u dx + (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner_{\mathcal{J}_u} + Du \llcorner_{\mathcal{C}_u \setminus \mathcal{D}_u},$$

where we used the notation dx for the N -dimensional Lebesgue measure.

Finally, we define the sublinear total variation $\Phi(|Du|)$ in a general way :

Definition 2.2.1 *Let $u \in BV(\Omega)$ and Du its distributional derivative, which is an \mathbb{R}^N -valued Radon measure. The sublinear total variation $\Phi(|Du|)$ is defined by :*

$$\Phi(|Du|) = \Phi(|\nabla u|) dx + \Phi_1(u^+ - u^-) d\mathcal{H}^{N-1} + \Phi_2(|D^{\mathcal{T}} u|),$$

where Φ_1 and Φ_2 are any nonnegative functions satisfying $\Phi_i(s) = 0$ if and only if $s = 0$, $i = 1, 2$.

2.3 Sublinear Regularization in Image Restoration

2.3.1 The Deblurring-Denoising case

This case is motivated by image restoration. More precisely, in the special case $N = 2$, let Ω be a bounded domain in \mathbb{R}^2 and $f \in L^2(\Omega)$ be an observed image, which is the result of a compact transformation R (blur) and corrupted by an additive zero-mean Gaussian noise $\eta \in L^2(\Omega)$ both performed on the ideal image \hat{u} ; that is, $f = R\hat{u} + \eta$. It is well-known since the seminal work of Tikhonov and Arsenin [39], that a good approximation of \hat{u} is provided by the minimizer of a certain (strictly) convex energy

$$T(u) := \frac{\lambda}{2} \int_{\Omega} (f - Ru)^2 dx + \int_{\Omega} \Psi(|\nabla u|) dx,$$

on an adequate Sobolev space, where the function Ψ has to be chosen to realize some desired smoothing effects. The parameter $\lambda > 0$ can be interpreted as the Lagrange multiplier with respect to the constraint on the variance of η or as a regularizing weight.

In what follows, consider the functional

$$J(u) := \frac{\lambda}{2} \int_{\Omega} (f - Ru)^2 dx + \int_{\Omega} \Phi(|Du|) \tag{2.3.1}$$

defined on the more realistic functional space $BV(\Omega)$, where $\lambda > 0$ is a given parameter and the regularization term $\int_{\Omega} \Phi(|Du|)$ is defined as in Definition 2.2.1.

It is clear that we can assume, without loss of generality,

$$\Phi(0) = 0.$$

The sublinearity at infinity of the function Φ will be assumed, that is, we set the hypothesis :

$$(H_1) \quad \lim_{s \rightarrow +\infty} \frac{\Phi(s)}{s} = 0.$$

We quote here that in [10], the authors consider the special case Φ is bounded on \mathbb{R} and $R = Id_{L^2(\Omega)}$, the identity operator $L^2(\Omega)$.

In this first subsection, the operator R is assumed to be compact from $L^2(\Omega)$ to itself, it is not necessarily a Hilbert-Schmidt operator.

Now, we introduce the following subspace of $L^\infty(\Omega)$

$$X := \{u : \Omega \longrightarrow \mathbb{R} \text{ s. t. } u \text{ is piecewise constant}\}. \quad (2.3.2)$$

The algebra $X \subset BV(\Omega)$ is dense in $L^2(\Omega)$, with respect to the norm $\|\cdot\|_{L^2(\Omega)}$. In particular, X is dense in $(BV(\Omega), \|\cdot\|_{L^2(\Omega)})$. Then the closure of $R(X)$ with respect to the $L^2(\Omega)$ norm is not other than the closure of $R(L^2(\Omega))$ with respect to the $L^2(\Omega)$ norm, since $R(\overline{X}) \subset \overline{R(X)}$. That is

$$\overline{R(X)} = \overline{\text{Im}(R)}, \quad \text{where } \text{Im}(R) \text{ is the range of } R.$$

It is clear that $\overline{\text{Im}(R)}$ is a (convex) and closed subspace of $L^2(\Omega)$, then the orthogonal projection operator on $\overline{\text{Im}(R)}$, *i.e.*, $\Pi_{\overline{\text{Im}(R)}} : L^2(\Omega) \longrightarrow L^2(\Omega)$, is uniquely defined.

Before stating our results, we introduce some technical tools and lemmas.

For every positive integer n , let $h_n = 2^{-n}$ and the sequence of squares $C_{i,j}^n := [i h_n, (i+1) h_n] \times [j h_n, (j+1) h_n]$, $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

Let the subsets of \mathbb{R}^2

$$\Omega_+^n := \bigcup_{\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : C_{i,j}^n \cap \Omega \neq \emptyset\}} C_{i,j}^n \quad \text{and} \quad \Omega_-^n := \bigcup_{\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : C_{i,j}^n \subset \Omega \neq \emptyset\}} C_{i,j}^n.$$

and the sets of indices

$$\mathbb{I}_+^n := \{(k, l) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } C_{k,l}^n \cap \Omega \neq \emptyset\} \quad \text{and} \quad \mathbb{I}_-^n := \{(k, l) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } C_{k,l}^n \subset \Omega\}. \quad (2.3.3)$$

It is clear that $\Omega_-^n \subset \Omega \subset \Omega_+^n$, for every integer n and that

$$\bigcup_{n=0}^{+\infty} \Omega_-^n = \Omega \quad \text{and} \quad \bigcap_{n=0}^{+\infty} \Omega_+^n = \overline{\Omega}.$$

We introduce the subspace

$$X_n^+ := \{u \in L^2(\Omega_+^n) : u \text{ is constant on every } C_{i,j}^n \subset \Omega_+^n\}$$

and

$$X_n := \{u|_\Omega : u \in X_n^+\},$$

where $u|_\Omega$ denotes the restriction of u to Ω .

It is obvious that X_n is a strictly increasing sequence of finite dimension subspaces of $X \subset L^\infty(\Omega)$ and that

$$\overline{\bigcup_{n \in \mathbb{N}} X_n} = L^2(\Omega).$$

Moreover, since R is continuous from $L^2(\Omega)$ to itself then

$$\overline{\bigcup_{n \in \mathbb{N}} R(X_n)} = \overline{\text{Im}(R)}.$$

Hence, we get the following lemma :

Lemma 2.3.1 *For every $g \in \overline{\text{Im}(R)}$, there is a sequence $(g_n)_n$ such that $g_n \in X_n \subset L^\infty(\Omega)$, for every n , and $(g_n)_n$ converges to g in $L^2(\Omega)$.*

Lemma 2.3.2 *The hypothesis (H_1) is equivalent to : for every $n \in \mathbb{N}^*$, there exists $a_n > 0$ such that $0 \leq \Phi(s) \leq a_n + \frac{s}{n^2}$.*

Proof. Assume that (H_1) holds true and let $n \in \mathbb{N}^*$. Then there is $A_n > 0$ such that for every $s \geq A_n$ we have $\frac{\Phi(s)}{s} \leq \frac{1}{n^2}$. If we set $a_n := \max_{s \in [0, A_n]} \Phi(s)$, it follows that $0 \leq \Phi(s) \leq a_n + \frac{s}{n^2}$, for every $s \geq 0$. Conversely, for every $n \in \mathbb{N}^*$ (fixed), one has $\lim_{s \rightarrow +\infty} \frac{\Phi(s)}{s} \leq \frac{1}{n^2}$. Hence $\lim_{s \rightarrow +\infty} \frac{\Phi(s)}{s} \leq 0$, which implies (H_1) since $\Phi \geq 0$. \square

Remark 2.3.3 *In the previous lemma, the sequence $(A_n)_n$ can be choosed an increasing one, in this situation the sequence $(a_n)_n$ is non decreasing. If the sequence $(a_n)_n$ is bounded, this means that the function Φ is bounded and this case was studied in [10]. Here, we are concerned with the more complex situation when $(a_n)_n$ is not bounded. Then, without loss of generality, we may assume in the sequel that $a_n \rightarrow +\infty$ as $n \rightarrow +\infty$, (if not, we can replace a_n by $a_n + 2^n$, for example).*

Theorem 2.3.4 *Assume that the operator R is compact from $L^2(\Omega)$ to itself and the hypothesis (H) is satisfied. Then, for every $u \in BV(\Omega)$, there exists a sequence $(w_n) \in BV(\Omega)$ such that*

$$\begin{cases} w_n & \longrightarrow u \text{ in } L^1(\Omega) \\ \int_{\Omega} \Phi(|Dw_n|) & \longrightarrow 0 \end{cases}$$

as n goes to $+\infty$. Consequently, if the infimum of J on $BV(\Omega)$ is achieved at \hat{u} then

$$\int_{\Omega} \Phi(|D\hat{u}|) = 0.$$

In particular, we get \hat{u} is constant.

Proof. Step 1. Let $u \in BV(\Omega)$, then applying Theorem 3.9, p. 122 of [5], we can find a sequence $(u_n)_n \subset C^\infty(\Omega) \cap BV(\Omega)$ such that

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \left| \int_{\Omega} |\nabla u_n| dx - \int_{\Omega} |Du| \right| = 0.$$

For any integer p and any $(i, j) \in \mathbb{I}_-^p$, defined by (2.3.3), let $x_{i,j}^p$ be the center of the square $C_{i,j}^p \subset \Omega_+^p$.

For every fixed integer n , consider the sequence $(v_n^p)_p \subset X_p$ defined by

$$v_n^p := \sum_{(i,j) \in \mathbb{I}_-^p} u_n(x_{i,j}^p) \mathbb{1}_{C_{i,j}^p},$$

where $\mathbb{1}_{C_{i,j}^p}$ denotes the characteristic function of $C_{i,j}^p$. It is clear that

$$\lim_{p \rightarrow +\infty} \|v_n^p - u_n\|_{L^1(\Omega)} = 0.$$

Then for every integer n , there is $\theta(n) > \theta(n-1)$, such that

$$\|v_n^{\theta(n)} - u_n\|_{L^1(\Omega)} \leq \frac{1}{2^n}.$$

By construction, the map θ is strictly increasing. If we set $v_n := v_n^{\theta(n)}$, we get

$$\lim_{n \rightarrow +\infty} \|v_n - u_n\|_{L^1(\Omega)} = 0.$$

At this stage, let introduce the parameter

$$\delta_k := \min \left(h_k, \frac{1}{a_k} \right), \quad (2.3.4)$$

where a_k is defined in Lemma 2.3.2.

Step 2. For any integer k and $(i, j) \in \mathbb{I}_-^k$, let $D_{i,j}^k \subset C_{i,j}^k$ defined by

$$D_{i,j}^k := [i h_k + \delta_k^\beta, (i+1) h_k - \delta_k^\beta] \times [j h_k + \delta_k^\beta, (j+1) h_k - \delta_k^\beta],$$

where $\beta > 2$. Using elementary geometrical properties, we can construct a sequence of *continuous* functions $(w_n)_n$ such that $w_n = v_n$ on each square $D_{i,j}^{\theta(n)}$ and w_n is affine on $C_{i,j}^{\theta(n)} \setminus D_{i,j}^{\theta(n)}$. Therefore, the slope of these affine functions are controlled by the inverse of $\delta_{\theta(n)}^\beta$ multiplied by the jump of v_n from a rectangle to its neighbors (see Figure 1).

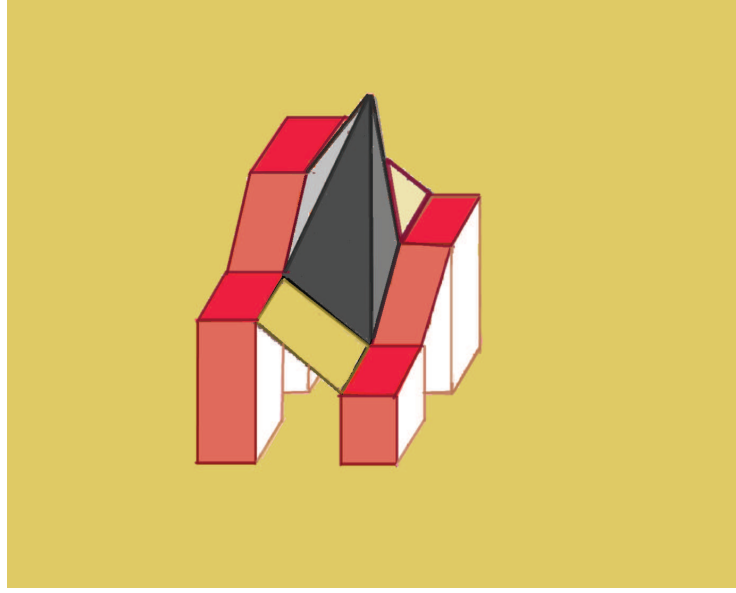


FIGURE 2.3.1.1 – Continuous approximations w_n of v_n s.t. $D^s w_n \equiv 0$ (Dw_n has no singular part)

We show that the the sequence $(\nabla w_n)_n$ is bounded in $L^1(\Omega)$. Indeed, for every $g \in BV(\Omega)$ and $(i, j) \in \mathbb{I}_-^n$, set

$$\Delta_1 g(x_{i,j}^n) := g((i+1)h_n, jh_n) - g(ih_n, jh_n) \quad \text{and} \quad \Delta_2 g(x_{i,j}^n) := g(ih_n, (j+1)h_n) - g(ih_n, jh_n),$$

where $g((i+1)h_n, jh_n) := g(ih_n, jh_n)$ if $(i+1, j) \notin \mathbb{I}_-^n$ and $g(ih_n, (j+1)h_n) := g(ih_n, jh_n)$ if $(i, j+1) \notin \mathbb{I}_-^n$.

$$\begin{aligned} \int_{\Omega} |\nabla w_n| dx &= \int_{\Omega} \left[\bigcup_{(i,j) \in \mathbb{I}_-^{\theta(n)} (C_{i,j}^{\theta(n)} \setminus D_{i,j}^{\theta(n)}) \right] |\nabla w_n| dx \\ &\leq \sum_{(i,j) \in \mathbb{I}_-^{\theta(n)}} (|\Delta_1 w_n(x_{i,j}^{\theta(n)})| h_{\theta(n)} + |\Delta_2 w_n(x_{i,j}^{\theta(n)})| h_{\theta(n)}) \\ &= \sum_{(i,j) \in \mathbb{I}_-^{\theta(n)}} (|\Delta_1 u_n(x_{i,j}^{\theta(n)})| h_{\theta(n)} + |\Delta_2 u_n(x_{i,j}^{\theta(n)})| h_{\theta(n)}) \\ &= \sum_{(i,j) \in \mathbb{I}_-^{\theta(n)}} \left(\frac{|\Delta_1 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} + \frac{|\Delta_2 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} \right) h_{\theta(n)}^2. \end{aligned}$$

Moreover, since

$$\lim_{p \rightarrow +\infty} \sum_{(i,j) \in \mathbb{I}_-^p} \left(\frac{|\Delta_1 u_n(x_{i,j}^p)|}{h_p} + \frac{|\Delta_2 u_n(x_{i,j}^p)|}{h_p} \right) h_p^2 = \int_{\Omega} \left(\left| \frac{\partial u_n}{\partial x_1} \right| + \left| \frac{\partial u_n}{\partial x_2} \right| \right) dx \quad (2.3.5)$$

$$\leq \sqrt{2} \int_{\Omega} |\nabla u_n| dx \quad (2.3.6)$$

$$\leq c_0 \quad (2.3.7)$$

it follows that

$$\int_{\Omega} |\nabla w_n| dx \leq \sum_{(i,j) \in \mathbb{I}_-^{\theta(n)}} \left(\frac{|\Delta_1 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} + \frac{|\Delta_2 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} \right) h_{\theta(n)}^2 \leq c_1,$$

where c_0, c_1 are positives constants independent of n .

We claim now that $\lim_{n \rightarrow +\infty} \|w_n - v_n\|_{L^1(\Omega)} = 0$. Indeed,

$$\begin{aligned} \|w_n - v_n\|_{L^1(\Omega)}^2 &= \int_{\Omega \cap \bigcup_{(i,j) \in \mathbb{I}_-^{\theta(n)}} (C_{i,j}^{\theta(n)} \setminus D_{i,j}^{\theta(n)})} |w_n - v_n| \\ &\leq \sum_{(i,j) \in \mathbb{I}_-^{\theta(n)}} \left(|\Delta_1 v_n(x_{i,j}^{\theta(n)})| h_{\theta(n)}^2 + |\Delta_2 v_n(x_{i,j}^{\theta(n)})| h_{\theta(n)}^2 \right) \\ &= h_{\theta(n)} \sum_{(i,j) \in \mathbb{I}_-^{\theta(n)}} \left(\frac{|\Delta_1 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} + \frac{|\Delta_2 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} \right) h_{\theta(n)}^2. \end{aligned}$$

Using (2.3.5) we obtain that

$$\sum_{(i,j) \in \mathbb{I}_-^{\theta(n)}} \left(\frac{|\Delta_1 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} + \frac{|\Delta_2 u_n(x_{i,j}^{\theta(n)})|}{h_{\theta(n)}} \right) h_{\theta(n)}^2 \leq c_2$$

and we get $\lim_{n \rightarrow +\infty} \|w_n - v_n\|_{L^1(\Omega)} = 0$ and obviously

$$\lim_{n \rightarrow +\infty} \|w_n - u\|_{L^1(\Omega)} = 0.$$

Therefore, the sequence $(w_n)_n$ is uniformly bounded in $BV(\Omega)$. Using the continuous embedding $BV(\Omega) \subset L^2(\Omega)$, $(w_n)_n$ is uniformly bounded in $L^2(\Omega)$. Then, up to a subsequence, $(w_n)_n$ converges weakly to u in $L^2(\Omega)$. Moreover, since the operator R is compact from $L^2(\Omega)$ to itself, it follows that

$$\lim_{n \rightarrow +\infty} \|R w_n - R u\|_{L^2(\Omega)} = 0. \quad (2.3.8)$$

Concerning the sublinear term :

$$\begin{aligned} \int_{\Omega} \Phi(|Dw_n|) &= \int_{\Omega} \Phi(|\nabla w_n|) dx = \int_{\Omega \cap \bigcup_{(i,j) \in \mathbb{I}_{-}^{\theta(n)}(C_{i,j}^{\theta(n)} \setminus D_{i,j}^{\theta(n)})} \Phi(|\nabla w_n|) dx \\ &\leq \int_{\Omega \cap \bigcup_{(i,j) \in \mathbb{I}_{-}^{\theta(n)}(C_{i,j}^{\theta(n)} \setminus D_{i,j}^{\theta(n)})} \left(a_{\theta(n)} + \frac{|\nabla w_n|}{n^2} \right) dx. \end{aligned}$$

Using the facts that the cardinal of \mathbb{I}_k is proportional to h_k^{-2} and $\delta_k = \min(a_k^{-1}, h_k)$, for any positive integer k , we get for the first term

$$\begin{aligned} \int_{\Omega \cap \bigcup_{(i,j) \in \mathbb{I}_{-}^{\theta(n)}(C_{i,j}^{\theta(n)} \setminus D_{i,j}^{\theta(n)})} a_{\theta(n)} dx &= 2a_{\theta(n)} \sum_{(i,j) \in \mathbb{I}_{-}^{\theta(n)}} h_{\theta(n)} \delta_{\theta(n)}^{\beta} \\ &\leq a_{\theta(n)} h_{\theta(n)} \delta_{\theta(n)}^{\beta} \frac{C(\Omega)}{h_{\theta(n)}^2} \\ &\leq C(\Omega) \delta_{\theta(n)}^{\beta-2} \longrightarrow 0 \text{ as } n \rightarrow +\infty \text{ since } \beta > 2. \end{aligned}$$

Finally, the second term gives

$$\int_{\Omega \cap \bigcup_{(i,j) \in \mathbb{I}_{-}^{\theta(n)}(C_{i,j}^{\theta(n)} \setminus D_{i,j}^{\theta(n)})} \frac{|\nabla w_n|}{n^2} dx \leq \frac{c_3}{n^2} \longrightarrow 0 \text{ as } n \rightarrow +\infty.$$

We conclude that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \Phi(|Dw_n|) = 0.$$

Now, if $\hat{u} \in BV(\Omega)$ is a minimizer of the functional J on $BV(\Omega)$. Consider a sequence $\hat{w}_n \in BV(\Omega)$ such that

$$\begin{cases} \hat{w}_n & \longrightarrow \hat{u} \text{ in } L^1(\Omega) \\ \int_{\Omega} \Phi(|D\hat{w}_n|) & \longrightarrow 0 \end{cases}$$

as n tends to $+\infty$. It follows that

$$\lim_{n \rightarrow +\infty} J(\hat{w}_n) = \|f - R\hat{u}\|_{L^2(\Omega)} \geq J(\hat{u}).$$

Whence $\int_{\Omega} \Phi(|D\hat{u}|) = 0$, which achieves the proof. \square

At this stage, we show that the minimization problem

$$\text{Find } u \in BV(\Omega) \text{ such that } J(u) = \inf_{v \in BV(\Omega)} J(v) \quad (2.3.9)$$

is ill-posed. For this, we will assume that the operator R does not annihilate constant functions. More precisely, we assume that :

$$(H_2) \quad R \cdot \mathbf{1}_\Omega \neq 0 \quad (2.3.10)$$

Notice that hypothesis (H_2) is very natural in applications and was introduced first in [?]. We show in the following that Problem (2.3.9) is ill-posed in general, unless the data f and the compact operator R are linked by a binding relation on the support of the function $R \cdot \mathbf{1}_\Omega$.

Theorem 2.3.5 *Assume that the hypotheses of Theorem 4.1.4 and (H_2) are satisfied. Then Problem (2.3.9) has a solution if the data f satisfies*

$$f(x) = t_R^*(\Omega, f) \times (R \cdot \mathbf{1}_\Omega)(x) \text{ on the set } \{x \in \Omega : R \cdot \mathbf{1}_\Omega(x) \neq 0\},$$

where $t_R^*(\Omega, f)$ is defined by (2.3.11) and $\mathbf{1}_\Omega$ is the indicatrice function of Ω .

Proof. If Problem (2.3.9) admits a solution \hat{u} . From Theorem 4.1.4, the function $\hat{u} = t_R^*(\Omega, f) \cdot \mathbf{1}_\Omega$, where the constant $t_R^*(\Omega, f)$ realizes the minimum of the real function $t \mapsto \int_\Omega (f - t R \cdot \mathbf{1}_\Omega)^2 dx$. Then

$$t_R^*(\Omega, f) = \frac{\int_\Omega f \times (R \cdot \mathbf{1}_\Omega) dx}{\int_\Omega (R \cdot \mathbf{1}_\Omega)^2 dx}. \quad (2.3.11)$$

Suppose that there exists $\omega \subset \{x \in \Omega : R \cdot \mathbf{1}_\Omega(x) \neq 0\}$ such that $t_R^*(\omega, f) \neq t_R^*(\Omega, f)$, where $t_R^*(\omega, f)$ is defined by

$$t_R^*(\omega, f) = \frac{\int_\omega f \times (R \cdot \mathbf{1}_\Omega) dx}{\int_\omega (R \cdot \mathbf{1}_\Omega)^2 dx}.$$

Then $\int_\omega (f - t_R^*(\omega, f) R \cdot \mathbf{1}_\Omega)^2 dx < \int_\omega (f - t_R^*(\Omega, f) R \cdot \mathbf{1}_\Omega)^2 dx$. Let $\tilde{u} = t_R^*(\omega, f) \mathbf{1}_\omega + t_R^*(\Omega, f) \mathbf{1}_{\Omega \setminus \omega}$. Let $(w_n)_n \in BV(\Omega)$ such that

$$\begin{cases} w_n & \longrightarrow \tilde{u} \text{ in } L^1(\Omega) \\ \int_\Omega \Phi(|Dw_n|) & \longrightarrow 0 \end{cases}$$

Then

$$\lim_{n \rightarrow +\infty} J(w_n) = \int_\Omega (f - R\tilde{u})^2 dx < J(\hat{u}),$$

which leads to a contradiction. It follows that for every $\omega \subset \{x \in \Omega : R \cdot \mathbf{1}_\Omega(x) \neq 0\}$ we get $t_R^*(\omega) = t_R^*(\Omega, f)$. Now, let $a \in \{x \in \Omega : R \cdot \mathbf{1}_\Omega(x) \neq 0\}$, for $\omega = B(a, \varepsilon)$, the ball centered at a and of radius ε , we get

$$\lim_{\varepsilon \rightarrow 0} t_R^*(B(a, \varepsilon), f) = \frac{f(a) \times (R \cdot \mathbf{1}_\Omega)(a)}{(R \cdot \mathbf{1}_\Omega)^2(a)} = t_R^*(\Omega, f).$$

Therefore, for every $a \in \{x \in \Omega : R \cdot \mathbf{1}_\Omega(x) \neq 0\}$,

$$f(a) = t_R^*(\Omega, f)(R \cdot \mathbf{1}_\Omega)(a),$$

which ends the proof. \square

Remark 2.3.6 *In the case where the operator R is the identity, the constant*

$$t_{Id}^*(\Omega, f) = \frac{\int_\Omega f \, dx}{\int_\Omega \mathbf{1}_\Omega \, dx},$$

which is no other than the mean of f on Ω . Therefore, $t_R^(\Omega, f)$ gives a natural generalization, in our context, taking into account of the presence of the compact operator R .*

At this stage, we can state the following result which generalizes the results obtained in [24] for every Φ sublinear, for $f \in L^2(\Omega)$ and in the presence of a compact operator R , in the framework of Sobolev spaces.

Consider the energy $K : W^{1,1}(\Omega) \rightarrow [0, +\infty[$ defined by

$$K(u) := \frac{\lambda}{2} \int_\Omega (f - Ru)^2 \, dx + \int_\Omega \Phi(|\nabla u|) \, dx.$$

Then we have the following

Corollary 2.3.7 *For every $u \in W^{1,1}(\Omega)$, there exists a sequence $(w_n) \in W^{1,1}(\Omega)$ such that*

$$\begin{cases} w_n & \longrightarrow u \text{ in } L^1(\Omega) \\ \int_\Omega \Phi(|\nabla w_n|) & \longrightarrow 0 \end{cases}$$

as n goes to $+\infty$. Consequently, the infimum of K on $W^{1,1}(\Omega)$ is achieved if the data f satisfies

$$f(x) = t_R^*(\Omega, f) \times (R \cdot \mathbf{1}_\Omega)(x) \text{ on the set } \{x \in \Omega : R \cdot \mathbf{1}_\Omega(x) \neq 0\},$$

where $t^(\Omega, f)$ is defined by (2.3.11).*

Proof. Using similar arguments as in Theorem 4.1.4, it follows that $\int_{\Omega} \Phi(|\nabla \hat{u}|) dx = 0$ and then \hat{u} is constant. We end the proof in the same way as in the previous corollary. \square

2.3.2 The Pure Denoising case

In the previous results, we assumed that the operator R is compact on $L^2(\Omega)$, which translates the presence of the blur in the observed image. In the case of pure noisy image (unblurred), the operator R is the identity operator of $L^2(\Omega)$ which is not compact. Then, we consider the following energy

$$E(u) = \frac{\lambda}{2} \|f - u\|_2^2 + \int_{\Omega} \Phi(|Du|),$$

defined on $BV(\Omega)$, with the same assumptions on Ω , f and Φ . Notice that in [10], only the following special case has been investigated : " Φ is bounded and the observed image f is bounded" . In the following result, we consider the general case : Φ sublinear and $f \in L^2(\Omega)$.

Theorem 2.3.8 *Consider the minimization problem*

$$\inf_{u \in BV(\Omega)} E(u), \quad (2.3.12)$$

where the regularizing function Φ is sublinear and the data $f \in L^2(\Omega)$. Then the following assertions hold true :

- i) $\inf_{u \in BV(\Omega)} E(u) = 0$,
- ii) Problem (2.3.12) has a solution if and only if the data $f \in BV(\Omega)$ and $\int_{\Omega} \Phi(|Df|) = 0$. In this situation, the unique solution is f itself.

Proof. i) Let $(v_n)_n \in X_n$ such that $\lim_{n \rightarrow +\infty} \|v_n - f\|_2 = 0$. For any integer k and $(i, j) \in \mathbb{I}_k$, let $D_{i,j}^k \subset C_{i,j}^k$ defined by

$$D_{i,j}^k := [i h_k + \delta_k^\beta, (i+1) h_k - \delta_k^\beta] \times [j h_k + \delta_k^\beta, (j+1) h_k - \delta_k^\beta],$$

where $\beta > 2$. Using elementary geometrical properties, we can construct a sequence of continuous functions $(w_n)_n$ such that $w_n = v_n$ on each square $D_{i,j}^n$ and w_n is affine on

$C_{i,j}^n \setminus D_{i,j}^n$. Therefore, the slope of these affine functions are controlled by the inverse of δ_n^β multiplied by the jump of v_n from a rectangle to its neighbors (Figure 1).

$$\begin{aligned}
\int_{\Omega} (w_n - v_n)^2 dx_1 dx_2 &= \int_{\Omega \cap \bigcup_{(i,j) \in \mathbb{I}_n} (C_{i,j}^n \setminus D_{i,j}^n)} (w_n - v_n)^2 dx_1 dx_2 \\
&= \sum_{(i,j) \in \mathbb{I}_n} \int_{C_{i,j}^n \setminus D_{i,j}^n} \left(\frac{|\Delta_1 v_n(x_{i,j})|^2}{\delta_n^2} x_1^2 + \frac{|\Delta_2 v_n(x_{i,j})|^2}{\delta_n^2} x_2^2 \right) dx_1 dx_2 \\
&\leq c_1 \sum_{(i,j) \in \mathbb{I}_n} (|\Delta_1 v_n(x_{i,j})|^2 + |\Delta_2 v_n(x_{i,j})|^2) \times \delta_n \\
&\leq \frac{8c_1 \delta_n}{h_n^2} \sum_{(i,j) \in \mathbb{I}_n} v_n^2(x_{i,j}) \times h_n^2 \\
&\leq 8c_1 h_n^{\beta-2} \|v_n\|_2^2.
\end{aligned}$$

Using the fact that $(v_n)_n$ is (uniformly) bounded in $L^2(\Omega)$, we get

$$\lim_{n \rightarrow +\infty} \|w_n - v_n\|_2 = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|w_n - f\|_2 = 0.$$

On the other hand, following the same arguments in Section 1 of Theorem 4.1.4, we show that the real-valued sequence $(\int_{\Omega} |Dw_n|)_n$ is bounded and Section 2 shows that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \Phi(|Dw_n|) = 0. \quad \text{Therefore, } \lim_{n \rightarrow +\infty} E(w_n) = 0 \text{ and consequently } \inf_{u \in BV(\Omega)} E(u) = 0.$$

ii) This is a direct consequence of i). \square

In the following section, we will show that similar results to image restoration hold true in image decomposition [35].

2.4 Sublinear Regularization in image decomposition

The space $G(\mathbb{R}^2)$, more adapted to capture oscillating patterns in a minimization problems, was introduced by Y. Meyer [35, 29]. This space was proposed as an alternative to the space $L^2(\Omega)$ which appears in the Rudin-Osher-Fatemi model and which is not sensitive to oscillations. For example, on the interval $[0, 2\pi]$, the two functions $\sin(x)$ and $\sin(nx)$, have the same $L^2([0, 2\pi])$ -norm while the second function is more oscillating than the first one, for any large integer n . The space $G(\mathbb{R}^2)$ consists to the following set of *distributions* :

$$G(\mathbb{R}^2) := \operatorname{div}([L^\infty(\mathbb{R}^2)]^2).$$

Endowed with the norm $\|v\|_{G(\mathbb{R}^2)} := \inf\{\|\boldsymbol{\xi}\|_\infty : \boldsymbol{\xi} \in [L^\infty(\mathbb{R}^2)]^2, v = \operatorname{div}(\boldsymbol{\xi})\}$, the space $G(\mathbb{R}^2)$ is a Banach space. In this section, we focus ourselves to the case $N = 2$, in particular Ω is a bounded domain in \mathbb{R}^2 . To guarantee that functions (*distributions*) of $G(\Omega)$ have well-defined traces on the boundary $\partial\Omega$ and a zero mean value on Ω , the authors in [9] gave the following characterization :

$$G(\Omega) := \{v = \operatorname{div}(\boldsymbol{\xi}) : \boldsymbol{\xi} \in L^\infty(\Omega, \mathbb{R}^2), \boldsymbol{\xi} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

where \mathbf{n} is the outward unit normal vector to $\partial\Omega$. The space $G(\Omega)$ is endowed with a similar norm

$$\|v\|_{G(\Omega)} := \inf\{\|\boldsymbol{\xi}\|_\infty : \boldsymbol{\xi} \in [L^\infty(\Omega)]^2, v = \operatorname{div}(\boldsymbol{\xi}), \boldsymbol{\xi}|_{\partial\Omega} = 0\}.$$

In the same paper, using a result due to J. Bourgain and H. Brézis, the authors gave a more simple characterization of $G(\Omega)$:

$$G(\Omega) = \left\{ v \in L^2(\Omega); \int_\Omega v(x) dx = 0 \right\},$$

and for a given $v \in G(\Omega)$, there is $\boldsymbol{\xi} \in [L^\infty(\Omega)]^2$ with $\boldsymbol{\xi}|_{\partial\Omega} = 0$ such that $v = \operatorname{div}(\boldsymbol{\xi})$. Moreover, the continuity dependence of the data holds true, that is, there is a constant $C(\Omega) > 0$ such that

$$\|v\|_{G(\Omega)} \leq \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \leq C(\Omega)\|v\|_{L^2(\Omega)}. \quad (2.4.1)$$

In this context, it is proved in [9] that the decomposition problem (texture + geometry)

$$\inf_{\substack{(u, v) \in BV(\Omega) \times G(\Omega) \\ u + v = f}} \left\{ \|v\|_{G(\Omega)} + \int_\Omega |Du| \right\}$$

has a solution. We mention here that the used regularization term $\int_\Omega |Du|$ is not sublinear. We will show that the sublinearity of Φ implies that every minimizing solution still satisfies the strong condition : $\int_\Omega \Phi(|Du|) = 0$.

Now, consider the nonconvex decomposition/minimization problem :

$$\inf_{\substack{(u, v) \in BV(\Omega) \times G(\Omega) \\ Ru + v = f}} \left\{ \|v\|_{G(\Omega)} + \int_\Omega \phi(|\nabla u|) dx \right\}, \quad (2.4.2)$$

where f is a given function in $L^2(\Omega)$ and R is a compact operator on $L^2(\Omega)$. For this introduce the equivalent formulation of (2.4.2)

$$\inf_{u \in BV_f(\Omega)} F(u), \quad (2.4.3)$$

where $BV_f(\Omega) := \{u \in BV(\Omega) : f - Ru \in G(\Omega)\}$ and

$$F(u) := \|f - Ru\|_{G(\Omega)} + \int_{\Omega} \Phi(|Du|).$$

Theorem 2.4.1 *If the infimum of F on $BV_f(\Omega)$ is achieved at \hat{u} then \hat{u} is the constant function given by :*

$$\hat{u} = \frac{\int_{\Omega} f(x) dx}{\int_{\Omega} R \cdot \mathbf{1}_{\Omega}(x) dx} \mathbf{1}_{\Omega}.$$

Proof. Let $\hat{u} \in BV_f(\Omega)$ be a solution of Problem (2.4.3) and $\hat{v} \in G(\Omega)$ defined by $\hat{v} := f - R\hat{u}$. Applying Theorem 4.1.4, we can construct a sequence $u_n \in BV(\Omega)$ such that

$$\left\{ \begin{array}{l} u_n \longrightarrow \hat{u} \text{ in } L^1(\Omega) \\ Ru_n \longrightarrow R\hat{u} \text{ in } L^2(\Omega) \\ \int_{\Omega} \Phi(|Du_n|) \longrightarrow 0. \end{array} \right.$$

In what follows, we will construct a modified sequence $(\tilde{u}_n) \in BV_f(\Omega)$ having the same convergence properties as (u_n) . Indeed, set $v_n := f - Ru_n \in L^2(\Omega)$. On one hand, this sequence satisfies

$$\left| \int_{\Omega} v_n dx \right| = \left| \int_{\Omega} (v_n - \hat{v}) dx \right| = \left| \int_{\Omega} (Ru_n - R\hat{u}) dx \right| \leq \sqrt{|\Omega|} \|Ru_n - R\hat{u}\|_{L^2(\Omega)}.$$

Then

$$\lim_{n \rightarrow +\infty} \int_{\Omega} v_n dx = 0.$$

On the other hand, set

$$\tilde{u}_n := u_n + \frac{\int_{\Omega} v_n(x) dx}{\int_{\Omega} R \mathbf{1}_{\Omega}(x) dx} \mathbf{1}_{\Omega} \in BV(\Omega),$$

where $\mathbb{1}_\Omega$ denotes the indicatrice function of the set Ω . Then, the sequence \tilde{u}_n converges clearly to \hat{u} in $L^1(\Omega)$ and

$$f - R\tilde{u}_n = v_n - \frac{\int_{\Omega} v_n(x) dx}{\int_{\Omega} R \mathbb{1}_\Omega(x) dx} R \mathbb{1}_\Omega,$$

which satisfies

$$\int_{\Omega} (f - R\tilde{u}_n) dx = 0 \quad \text{and then} \quad \tilde{u}_n \in BV_f(\Omega).$$

Finally, by construction, we have $D\tilde{u}_n = Du_n$ which means that

$$\int_{\Omega} \Phi(|D\tilde{u}_n|) \longrightarrow 0.$$

We conclude that

$$\lim_{n \rightarrow +\infty} F(\tilde{u}_n) = \|f - R\hat{u}\|_{G(\Omega)}.$$

Therefore, we get necessarily

$$\int_{\Omega} \Phi(|D\hat{u}|) = 0.$$

Writing $\hat{u} = \hat{t}_R(\Omega, f)\mathbb{1}_\Omega$ and using the fact that $\int_{\Omega} (f - R\hat{u})(x) dx = 0$, we get the result. \square

Chapitre 3

Minimization of convex functional in Orlicz spaces

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Abstract

In this chapter, we consider a minimization problem with superlinear regularizing term, in the framework of Orlicz spaces. More precisely, we are interested to the minimization of functionals of the form :

$$E(u) := \int_{\Omega} (f - u)^2 dx + \frac{\lambda}{2} \int_{\Omega} \Psi(|Du|) dx,$$

where Ψ is a Young and Δ_2 function. We show that this this superlinearity condition leads to existence and uniqueness of a minimizing solution in the Orlicz's space

$$W := \left\{ u : \Omega \rightarrow \mathbb{R}; u \in L^2(\Omega) \text{ and } |Du| \in L^{\Psi}(\Omega) \right\}.$$

3.1 Introduction

In the previous chapter, we showed that minimizing problems with sublinear regularizing terms are ill posed, in general, in the spaces of functions of bounded variation.

On the other hand, it is well known that minimizing problems with convex regularizing terms and having linear behavior (at infinity) are well posed in the spaces of functions of bounded variation. In this chapter, we complete the picture by showing that minimizing problems with superlinear Young and Δ_2 regularizing terms are well posed in the framework of Orlicz spaces.

Consider the restoration model with additive noise :

$$f = u + \text{noise}. \quad (3.1.1)$$

The energy associated to Equation (3.1.1) is then

$$J(u) = \int_{\Omega} (f - u)^2 dx, \quad (3.1.2)$$

where Ω is a bounded domain in \mathbb{R}^2 . Since this is an ill-conditioned problem, then we need to add a regularizing term to the energy. It is usually of the form $\int_{\Omega} \Psi(|Du|) dx$ which represents the diffusion. The energy takes then the form

$$E(u) = \frac{\lambda}{2} \int_{\Omega} (f - u)^2 dx + \int_{\Omega} \Psi(|Du|) dx. \quad (3.1.3)$$

In this chapter, we generalize the case $\Psi(s) = s^p$, $p > 1$, by considering a large class of superlinear functions Ψ at infinity. We show the existence and uniqueness of solutions for such problem in Orlicz spaces. In what follows, we consider the energy

$$E(u) := \frac{\lambda}{2} \int_{\Omega} (f - u)^2 dx + \int_{\Omega} \Psi(|Du|) dx, \quad (3.1.4)$$

on the space

$$W := \left\{ u : \Omega \rightarrow \mathbb{R}; \quad u \in L^2(\Omega) \text{ and } |Du| \in L^{\Psi}(\Omega) \right\},$$

and we prove the following theorem

Theorem 3.1.1 (Existence and uniqueness of a minimizer of E on W)

Assume that the function $\Psi: \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a Young and Δ_2 function. Then E admits a unique minimizer u on W.

In the particular case were $\Psi(s) = s^{p(s)}$, p is such that Ψ is a Young and Δ_2 function

Corollary 3.1.2 (Application to Image processing)

Assume p is non-increasing satisfying, for every $t > 0$, $1 < p(t) \leq p(0) < +\infty$ and the function $\Psi(s) = s^{p(s)}$ is a Young and Δ_2 function. Then E admits at least one minimizer u on W .

Remark 3.1.3 Note that, such Ψ is a Young function, if the exponent function p is convex, i.e. satisfies the following inequality

$$p''(s) \ln(s) + 2\frac{p'(s)}{s} - \frac{p(s)}{s^2} + \left(p'(s) \ln(s) + \frac{p^2(s)}{s}\right)^2 \geq 0.$$

Strategy of the proof

The difficulty is in the lower semi-continuity of the regularizing term $\int_{\Omega} \Psi(|Du|) dx$. Many works have dealt with the semi-continuity of vectorial and quasi-convex functionals in these spaces, for instance see M. Forcardi [28], and A. Verde and G. Zecca [7] that prove the semi-continuity of the functional $\int_{\Omega} f(Du(x)) dx$, where Ω is a bounded set in \mathbb{R}^N , with $n \geq 2$, $f: \mathbb{R}^N \rightarrow \mathbb{R}$ is a quasi-convex function satisfying non standard growth conditions such

$$c_1\Psi(|z|) \leq f(z) \leq c_2(1 + \Phi(|z|)),$$

where Ψ and Φ are suitable functions and $c_1, c_2 > 0$, see [7, Section 2] and $u: \Omega \rightarrow \mathbb{R}^N$. The proof of [7, Theorem 3.1] could not be directly applied to our case. Indeed, in [7, Theorem 3.1] and since $\Psi \neq f$, we don't need the intermediate functions Φ , Γ and $\tilde{\Gamma}$. Moreover, the condition on the dimension $N \leq 2$ is needed in this case to ensure compact embedding. Up to our knowledge, nobody cover our problem, since we are in the case $N = 1$ and $\Psi: \mathbb{R}^+ \rightarrow \mathbb{R}$ and $f = \Psi$. For this purpose, we will adapt [7, Theorem 3.1] to our case. More precisely, we will prove

Theorem 3.1.4 Let Ω be a bounded open set in \mathbb{R}^2 and Ψ a real Young function verifying the Δ_2 -condition. Then the functional

$$I(u, \Omega) = \int_{\Omega} \Psi(|Du|) dx$$

is sequentially lower semicontinuous weak \star in $W^{1,\Psi}(\Omega)$, i.e. for every sequence $u_n \rightharpoonup u$ weak \star in $W^{1,\Psi}(\Omega)$, we have

$$\liminf_n \int_{\Omega} \Psi(|Du_n|) dx \geq \int_{\Omega} \Psi(|Du|) dx. \quad (3.1.5)$$

In addition, we need to show an inequality that controls the norm $\|u\|_\Psi$ by the term $\int_\Omega \Psi(|Du|) dx$. This is done in Lemma 3.3.9.

Organization of this paper

In sections 2 and 3 we recall basic properties of Young and quasi-convex functions and Orlicz-Sobolev spaces. In Section 4, we prove the lower semi-continuity of $u \mapsto \int \Psi(|Du|)$ in the weak \star -topology of the associated Orlicz-Sobolev space. Finally, the last section is devoted to show the main result Theorem.

3.2 Young and quasi-convex functions

This section is devoted to recall some definitions and well known properties on Young and quasi-convex functions (see for instance Adams [1, 32]).

Let $\Phi: [0, +\infty) \rightarrow [0, +\infty)$ be a Young function, i.e. a convex function of the type

$$\Phi(t) = \int_0^t \varphi(s) ds \quad \text{for every } t > 0,$$

where $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ is non-decreasing, right-continuous function satisfying the following properties

$$\varphi(s) > 0 \quad \forall s > 0, \quad \varphi(0) = 0, \quad \lim_{s \rightarrow \infty} \varphi(s) = +\infty. \quad (3.2.1)$$

Define

$$\tilde{\varphi}(s) = \sup_{\varphi(t) \leq s} t.$$

It is easy to check that $\tilde{\varphi}$ satisfies properties (3.2.1) and we can recover φ from $\tilde{\varphi}$ via

$$\varphi(t) = \sup_{\tilde{\varphi}(s) \leq t} s.$$

The function $\tilde{\Phi}$ given by

$$\tilde{\Phi}(s) = \int_0^s \tilde{\varphi}(\sigma) d\sigma$$

is said to be complementary to Ψ . Moreover the pair Φ and Ψ satisfies Young's inequality :

$$st \leq \tilde{\Phi}(s) + \Phi(t) \quad \text{for every } s, t \in \mathbb{R},$$

the equality holds if and only if $t = \tilde{\varphi}(s)$ or $s = \varphi(t)$. From the last inequality we can find

$$\tilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\}.$$

Definition 3.2.1 (*The Δ_2 class of functions*)

We say that a Young function Φ belongs to the class Δ_2 , $\Phi \in \Delta_2$, if there exists a constants $\ell > 0$ such that

$$\Phi(2t) \leq \ell\Phi(t), \quad \forall t \geq 0. \quad (3.2.2)$$

Definition 3.2.2 (*Δ_2 regular pair*)

We say that the pair (Ψ, Ω) is Δ_2 regular if either

- (a) Ψ belongs to Δ_2 globally, or
- (b) Ψ belongs to Δ_2 near infinity and Ω has finite volume.

In the sequel, we take Φ a young function that belongs to Δ_2 .

Lemma 3.2.3 (*Properties of functions in Δ_2*)

For all $t \geq 0$, Φ satisfies the following inequalities :

$$\Phi(t) \leq t\varphi(t) \leq \ell\Phi(t), \quad (3.2.3)$$

and

$$\Phi(\lambda t) \leq \lambda^\ell \Phi(t), \quad \forall t \geq 0, \forall \lambda \geq 1. \quad (3.2.4)$$

Lemma 3.2.4 (*Sub-additivity of functions of Δ_2*)

For every a, b such that $0 \leq a \leq b$ we have

$$\Phi(a) + \Phi(b) \leq \Phi(a + b).$$

Using this lemma, we can prove that $\ell \geq 2$.

Lemma 3.2.5 *For all $a, b \in \mathbb{R}$, Φ satisfies the following inequality*

$$|\Phi(|a|) - \Phi(|b|)| \leq \varphi(|a| + |b|)|a - b|.$$

Definition 3.2.6 (Quasi-convex functions)

We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is quasiconvex if and only if for all $\xi \in \mathbb{R}^n$, all bounded domain $\Omega \subset \mathbb{R}^n$ and for every $\varphi \in \mathcal{D}(\Omega)$ we have

$$\int_{\Omega} f(\xi + D\varphi) dx \geq f(\xi) \text{mes}(\Omega).$$

Remarque 3.2.7 *By Jensen's inequality we can see that every convex function is quasi-convex.*

3.3 Some properties of Orlicz-Spaces

In this section we recall some basic definitions and properties of Orlicz spaces. For further results on Orlicz spaces we refer the reader to Adams [1], [3, Ch.8] and Rao and Ren [36].

Let Ψ be a Young function. The Orlicz class is

$$K^{\Psi}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} ; u \text{ measurable and } \int_{\Omega} \Psi(|u(x)|) dx < +\infty \right\}.$$

The Orlicz space $L^{\Psi}(\Omega)$ is defined to be the linear hull of the Orlicz class $K^{\Psi}(\Omega)$, that is, the smallest vector space containing $K^{\Psi}(\Omega)$. Thus $K^{\Psi}(\Omega) \subset L^{\Psi}(\Omega)$; these sets being equal if and only if (Ψ, Ω) is Δ_2 regular.

The norm on $L^{\Psi}(\Omega)$ is defined by

$$\|u\|_{\Psi} := \inf \left\{ k > 0; \int_{\Omega} \Psi \left(\frac{|u(x)|}{k} \right) dx \leq 1 \right\}.$$

Endowed with this norm, L^Ψ is a Banach space. Moreover, For every $f \in L^\Psi(\Omega)$, we have the following estimate

$$\|f\|_{L^\Psi(\Omega)} \leq 1 + \int_{\Omega} \Psi(|f|) dx. \quad (3.3.1)$$

We will denote by $E^{\tilde{\Psi}}(\Omega)$ the dual space of $L^\Psi(\Omega)$.

Remarque 3.3.1 *If $\Psi \in \Delta_2$ then $E^\Psi(\Omega) = L^{\tilde{\Psi}}(\Omega)$. The space $L^\Psi(\Omega)$ is reflexive if and only if both Ψ and $\tilde{\Psi}$ belong to Δ_2 .*

Definition 3.3.2 (Strong Convergence in $L^\Psi(\Omega)$)

We say that a sequence $(u_n)_n$ converges strongly in $L^\Psi(\Omega)$ to u , if and only if

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \Psi(|u_n - u|) dx = 0.$$

Definition 3.3.3 (Weak \star -convergence in $L^\Psi(\Omega)$)

We say that a sequence $(u_n)_n$ weak \star -converges in $L^\Psi(\Omega)$ to u , which we will denote by weak \star - $L^\Psi(\Omega)$, if and only if

$$\lim_{n \rightarrow +\infty} \int_{\Omega} u_n v dx = \int_{\Omega} u v dx, \quad \text{for every } v \in E^{\tilde{\Psi}}(\Omega).$$

Moreover, $u_n \rightharpoonup u$ weak \star in $W^{1,\Psi}(\Omega)$ if and only if $(u_n)_n$ and $(D_i u_n)_n$, $i = 1, 2$, converge weak \star in $L^\Psi(\Omega)$ to u and $D_i u$ respectively.

Lemma 3.3.4 (Weak \star -convergence implies boundedness in $L^\Psi(\Omega)$)

If $u_n \rightharpoonup u$ weak \star - $L^\Psi(\Omega)$ then $(u_n)_n$ is bounded in the strong topology of $L^\Psi(\Omega)$.

Proof. Let $u_n \rightharpoonup 0$ weak \star - $L^\Psi(\Omega)$, then $\int_{\Omega} u_n v dx \rightarrow 0$ for all $v \in E^{\tilde{\Psi}}(\Omega)$. Consequently, $u_n v \rightarrow 0$ a.e. in Ω and $\forall v \in E^{\tilde{\Psi}}(\Omega)$. Thus $u_n \rightarrow 0$ a. e. and there exists a positif constant M such that $|u_n(x)| \leq M$ a. e. Now, since Ψ is non decreasing then $\int_{\Omega} \Psi(|u_n(x)|) \leq \Psi(M)|\Omega|$.

If $\Psi(M)|\Omega| \leq 1$ the proof is terminated.

If $\Psi(M)|\Omega| \geq 1$, then

$$\int_{\Omega} \Psi \left(\frac{|u_n(x)|}{\Psi(M)|\Omega|} \right) \leq \frac{1}{\Psi(M)|\Omega|} \int_{\Omega} \Psi(|u_n(x)|) \leq \frac{\Psi(M)|\Omega|}{\Psi(M)|\Omega|} = 1.$$

□

Let A and B be two Young functions. We recall the following embedding results due to Adams (see [1, Theorem 8.23] and [2]).

Theorem 3.3.5 *Let Ω be an open subset of \mathbb{R}^2 with $\partial\Omega$ lipschitzian, let $A \in \Delta_2$, then the embedding*

$$W^{1,A}(\Omega, \mathbb{R}) \hookrightarrow L^A(\Omega, \mathbb{R})$$

is compact.

Theorem 3.3.6 *(see [1])*

Let Ω have finite volume and suppose that the Young-function B increases essentially more slowly than A near infinity. Then any bounded subset S of $L^A(\Omega)$ which is precompact in $L^1(\Omega)$ is also precompact in $L^B(\Omega)$.

We deduce the following result

Corollary 3.3.7 *(see [1])*

Let Ω be an open bounded set in \mathbb{R}^2 , and let A and B be Young functions. If B increases essentially more slowly than A near infinity, then the imbedding

$$W^{1,A}(\Omega) \hookrightarrow L^B(\Omega)$$

is compact.

Proof. Since A dominates B near infinity and $|\Omega| < \infty$, then the embedding $L^A(\Omega) \longrightarrow L^B(\Omega)$ is continuous, (see [1, theorem 8.12]).

Since Ω has finite volume we have

$$W^{1,A}(\Omega) \hookrightarrow W^{1,1}(\Omega) \hookrightarrow L^1(\Omega),$$

the latter embedding being compact, (see [1, theorem 6.2]). A bounded subset S of $W^{1,A}(\Omega)$ is bounded in $L^A(\Omega)$, bounded in $W^{1,1}(\Omega)$ and precompact in $L^1(\Omega)$ and hence precompact in $L_B(\Omega)$. \square

Theorem 3.3.8 [25, Proposition 1.15] *If Ω is bounded then $L^\Psi(\Omega) \subset L^1(\Omega)$.*

We terminate this section by a Poincaré-type inequality that will be needed for the main result. We provide the proof for completeness.

Lemma 3.3.9 (Poincaré - Wirtinger inequality in $W^{1,\Psi}(\Omega)$)

There exists a positive constant C depending on Ω and Ψ such that, for every function $u \in W^{1,\Psi}(\Omega)$

$$\|u(x) - (u)_\Omega\|_{L^\Psi} dx \leq C \|Du(x)\| dx, \quad (3.3.2)$$

where $(u)_\Omega$ is the average of u over Ω .

Proof.

Reasoning by contradiction. Then there exists a sequence $(u_n)_n \in W^{1,\Psi}(\mathbb{R})$ such that

$$\|u_n - (u_n)_\Omega\|_{L^\Psi} > n \|Du_n\|_{L^\Psi}. \quad (3.3.3)$$

Set, for $n \geq 1$,

$$v_n = \frac{u_n - (u_n)_\Omega}{\|u_n - (u_n)_\Omega\|_{L^\Psi}}.$$

Then $(v_n)_\Omega = 0$ and $\|v_n\|_{L^\Psi} = 1$. Therefore $(v_n)_n$ is bounded in $W^{1,\Psi}(\Omega)$. Using Theorem 3.3.5 there exists a subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, and an element $v \in W^{1,\Psi}(\mathbb{R})$ such that

$$v_n \rightarrow v \quad \text{strongly in } L^\Psi(\Omega).$$

Consequently $\|v\|_{L^\Psi} = 1$, $(v)_\Omega = 0$ and by (3.3.3) we have

$$\|Dv_n\|_{L^\Psi} \leq \frac{1}{n} \|v_n - (v_n)_\Omega\|_{L^\Psi} \leq \frac{1}{n} \|v_n\|_{L^\Psi} = \frac{1}{n}.$$

Using the lower semi continuity of $f: \mapsto \|Df\|_{L^\Psi}$ in $L^\Psi(\Omega)$ we get, $Dv = 0$ (a.e.) hence $v = \text{constant}$ (a.e.). But, $(v)_\Omega = 0$, hence, $v = 0$. This contradict the fact that $\|v\|_{L^\Psi}(\Omega) = 1$. \square

3.4 Proof of Theorem 3.1.3

Let $u, (u_n)_n \in W^{1,\Psi}(\Omega)$ such that $u_n \rightharpoonup u$ weak \star in $W^{1,\Psi}(\Omega)$.

We will start our proof by a first step where we assume that u is an affine function.

Step 1 : Assume first that u is affine, i.e. there exists $z_0 \in \mathbb{R}$ such that $Du(x) = z_0$, for all $x \in \Omega$. Fix an open set Ω_0 which is compactly contained in Ω , set $K = \text{dist}(\bar{\Omega}_0, \partial\Omega)/2$ and let h be a positive integer. For $i = 1, \dots, h$, define

$$\Omega_i = \left\{ x \in \Omega : \text{dist}(x, \Omega_0) < \frac{i}{h}K \right\}$$

and choose scalar function $\phi_i \in C_0^1(\Omega_i)$ such that

$$0 \leq \phi_i \leq 1; \quad \phi_i = \begin{cases} 1 & \text{in } \Omega_{i-1} \\ 0 & \text{in } \Omega - \Omega_i, \end{cases} \quad \text{and } |D\phi_i| \leq \frac{h+1}{K}. \quad (3.4.1)$$

Consider the functions $v_n = u_n - u$, and $\nu_{i,n} = \phi_i(x)v_n(x)$. For every n and i the function $\nu_{i,n}$ belongs to $W_0^{1,\Psi}(\Omega)$. Thus

$$\begin{aligned} I(u, \Omega) &= \int_{\Omega} \Psi(|Du|) dx = \Psi(z_0)|\Omega| \leq \int_{\Omega} \Psi(z_0 + D\nu_{i,n}) dx \\ &\leq \int_{\Omega_{i-1}} \Psi(|Du_n|) dx + \int_{\Omega_i - \Omega_{i-1}} \Psi(|z_0 + D\nu_{i,n}|) dx + \int_{\Omega - \Omega_i} \Psi(|z_0|) dx \\ &\leq \int_{\Omega} \Psi(|Du_n|) dx - \int_{\Omega - \Omega_{i-1}} \Psi(|Du_n|) dx + \int_{\Omega_i - \Omega_{i-1}} \Psi(|z_0 + Du_{i,n}|) dx \\ &\quad - \int_{\Omega_i - \Omega_0} \Psi(|Du|) dx + \Psi(|z_0|)|\Omega - \Omega_0|. \end{aligned}$$

Now, summing up with respect to $i = 1, \dots, h$, we get

$$\begin{aligned} h \int_{\Omega} \Psi(|Du|) dx &\leq h\Psi(|z_0|)|\Omega - \Omega_0| + \sum_{i=1}^h \int_{\Omega_i - \Omega_{i-1}} \Psi(|z_0 + D\nu_{i,n}|) dx + h \int_{\Omega} \Psi(|Du_n|) dx \\ &\quad - \sum_{i=1}^h \int_{\Omega - \Omega_{i-1}} \Psi(|Du_n|) dx - \sum_{i=1}^h \int_{\Omega_i - \Omega_0} \Psi(|Du|) dx. \end{aligned} \quad (3.4.2)$$

On the other hand, we have

$$\begin{aligned} \int_{\Omega_i - \Omega_{i-1}} |\Psi(|z_0 + D\nu_{i,n}|) - \Psi(|\phi_i Du_n|)| dx &\leq \int_{\Omega_i - \Omega_{i-1}} \int_0^1 \varphi(t|z_0 + D\nu_{i,n}| + (1-t)|\phi_i Du_n|) \\ &\quad |z_0 + D\nu_{i,n} - \phi_i Du_n| dt dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\Omega_i - \Omega_{i-1}} \int_0^1 \varphi(|z_0 + D\nu_{i,n}| + |\phi_i Du_n|) \\
&\quad (|z_0 + D\nu_{i,n}| + |\phi_i Du_n|) dt dx \\
&\leq \ell \int_{\Omega_i - \Omega_{i-1}} \Psi(|z_0 + D\nu_{i,n}| + |\phi_i Du_n|) dx \\
&\leq \ell \int_{\Omega_i - \Omega_{i-1}} \Psi(|z_0| + |D\nu_{i,n}| + |\phi_i Du_n|) dx \\
&\leq \ell \int_{\Omega_i - \Omega_{i-1}} \Psi(|z_0| + |\phi_i Dv_n| + |D\phi_i v_n| + |\phi_i Du_n|) dx \\
&\leq \frac{\ell}{4} \int_{\Omega_i - \Omega_{i-1}} \{\Psi(4|z_0|) + \Psi(4|\phi_i Dv_n|)\} \\
&\quad + \frac{\ell}{4} \int_{\Omega_i - \Omega_{i-1}} \{\Psi(4|D\phi_i v_n|) + \Psi(4|\phi_i Du_n|)\} dx \\
&\leq \frac{\ell}{4} \Psi(|4z_0|) |\Omega_i - \Omega_{i-1}| + \int_{\Omega_i - \Omega_{i-1}} \Psi(4|\phi_i Dv_n|) \\
&\quad + \int_{\Omega_i - \Omega_{i-1}} \Psi(4|D\phi_i v_n|) + \int_{\Omega_i - \Omega_{i-1}} \Psi(4|\phi_i Du_n|) dx.
\end{aligned}$$

(3.4.1) with (3.2.3) implies

$$\begin{aligned}
\int_{\Omega_i - \Omega_{i-1}} |\Psi(|z_0 + D\nu_{i,n}|) - \Psi(|\phi_i Du_n|)| dx &\leq \ell(4)^{\ell-1} \{\Psi(|z_0|) |\Omega - \Omega_i| \\
&\quad + \int_{\Omega_i - \Omega_{i-1}} \Psi(|Dv_n|) \\
&\quad + \left(\frac{h+1}{K}\right)^\ell \int_{\Omega_i - \Omega_{i-1}} \Psi(|v_n|) \\
&\quad + \int_{\Omega_i - \Omega_{i-1}} \Psi(|Du_n|) dx \}
\end{aligned}$$

Now, summing up with respect to $i = 1, \dots, h$, we obtain

$$\begin{aligned}
\sum_{i=1}^h \int_{\Omega_i - \Omega_{i-1}} |\Psi(|z_0 + D\nu_{i,n}|) - \Psi(|\phi_i Du_n|)| dx &\leq c |\Omega - \Omega_0| + c_1 \int_{\Omega - \Omega_0} \Psi(|Dv_n|) \\
&\quad + c_2 \int_{\Omega - \Omega_0} \Psi(|v_n|) \\
&\quad + c_1 \int_{\Omega - \Omega_0} \Psi(|Du_n|) dx,
\end{aligned}$$

where $c = \ell(4)^{\ell-1} \Psi(|z_0|)$, $c_1 = \ell(4)^{\ell-1}$ and $c_2 = \ell(4)^{\ell-1} \left(\frac{h+1}{K}\right)^\ell$.

Since $u_n \rightharpoonup u$ weak \star in $W^{1,\Psi}(\Omega)$, the sequence $v_n = u_n - u$ is bounded in the

strong topology of $W^{1,\Psi}(\Omega)$. Therefore there exists a positive constant C_1 such that

$$\limsup_n \sum_{i=1}^h \int_{\Omega_i - \Omega_{i-1}} |\Psi(|z_0 + D\nu_{i,n}|) - \Psi(|\phi_i Du_n|)| dx \leq C_1 \quad (3.4.3)$$

Inequality (3.4.3) implies that

$$\begin{aligned} \liminf_n \sum_{i=1}^h \int_{\Omega_i - \Omega_{i-1}} \Psi(|z_0 + D\nu_{i,n}|) &\leq C_1 + \liminf_n \sum_{i=1}^h \int_{\Omega_i - \Omega_{i-1}} \Psi(|\phi_i Du_n|) dx \\ &\leq C_1 + \liminf_n \sum_{i=1}^h \int_{\Omega_i - \Omega_{i-1}} \Psi(|Du_n|) dx \end{aligned} \quad (3.4.4)$$

Since $\Psi(x) \geq 0$ for every $x \geq 0$, then we deduce that

$$-\sum_{i=1}^h \int_{\Omega - \Omega_{i-1}} \Psi(|Du_n|) dx - \sum_{i=1}^N \int_{\Omega_i - \Omega_0} \Psi(|Du|) dx \leq 0 \quad (3.4.5)$$

So, by (3.4.2), (3.4.4) and (3.4.5) it follows that

$$h \int_{\Omega} \Psi(|Du|) dx \leq h\Psi(|z_0|)|\Omega - \Omega_0| + C_1 + (h+1) \liminf_n \int_{\Omega} \Psi(|Du_n|) dx.$$

Then, by dividing by h we have

$$\int_{\Omega} \Psi(|Du|) dx \leq \Psi(|z_0|)|\Omega - \Omega_0| + \frac{C_1}{h} + \frac{h+1}{h} \liminf_n \int_{\Omega} \Psi(|Du_n|) dx.$$

We obtain the result as $h \rightarrow \infty$ and $\Omega_0 \rightarrow \Omega$.

Step 2 : Removing the assumption that u is affine. The fact that $u_n \rightarrow u$ weak \star in $\overline{W^{1,\Psi}(\Omega)}$ implies $u_n \rightarrow u$ strongly in $L^1_{loc}(\Omega)$ and $\liminf_n \int_{\Omega} \Psi(|Du_n|) dx < +\infty$.

By extracting subsequences, not relabeled for convenience, we have that

$$\liminf_n \int_{\Omega} \Psi(|Du_n|) dx = \lim_n \int_{\Omega} \Psi(|Du_n|) dx.$$

Moreover, we can assume the existence of a positive and finite Radon measure, μ , such that

$$\mu = \lim_{n \rightarrow \infty} dx \llcorner \Psi(|Du_n|). \quad (3.4.6)$$

This latter limit is to be considered in the sense of measures, i.e., for every $v \in C_0(\Omega, \mathbb{R}^N)$ there holds

$$\lim_{n \rightarrow \infty} \int_{\Omega} v \Psi(|Du_n|) dx = \int_{\Omega} v \Psi(|Du|) d\mu.$$

We are going to show that for a.e. $x \in \Omega$ there holds

$$\frac{d\mu}{dx}(x) := \lim_{\varepsilon \rightarrow 0^+} \frac{\mu(B(x, \varepsilon))}{|B(x, \varepsilon)|} \geq \Psi(|Du|). \quad (3.4.7)$$

Indeed, if (3.4.7) holds, we have that for any $v \in C_0(\Omega, \mathbb{R}^N)$ such that $0 \leq v \leq 1$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \Psi(|Du_n|) dx \geq \lim_{n \rightarrow \infty} \int_{\Omega} v \Psi(|Du_n|) dx = \lim_{n \rightarrow \infty} \int_{\Omega} v d\mu \geq \lim_{n \rightarrow \infty} \int_{\Omega} v \Psi(|Du|) dx.$$

Thus the lower semi-continuity inequality follows by letting v increase to 1 and applying Levi's theorem.

To prove (3.4.7) we recall that there exists a set $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$, and that if $x \in \Omega_0$ the quantity

$$\frac{d\mu}{dx}(x) \text{ are finite,} \quad (3.4.8)$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \int_{B(x, \varepsilon)} |u(y) - u(x) - Du(x)(y - x)| dy = 0. \quad (3.4.9)$$

Let $x_0 \in \Omega_0$ and let $\varepsilon_k \rightarrow 0$ be such that $\mu(\partial B(x, \varepsilon_k)) = 0$, we get

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\mu(B(x, \varepsilon_k))}{|B(x, \varepsilon_k)|} &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{B(x_0, \varepsilon_k)} \Psi(|Du_n|) dx \\ &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{|B(0, 1)|} \int_{B(0, 1)} \Psi(|Du_{n,k}|) dx, \end{aligned}$$

where for every $y \in B(0, 1)$

$$u_{n,k} = \frac{1}{\varepsilon_k} (u_n(x_0 + \varepsilon_k y) - u(x_0)).$$

Notice that $(u_{n,k}) \in W^{1, \Psi}(\Omega)$ and $(\Psi(|Du_{n,k}|))$ are $L^1(\Omega)$ norm bounded. Indeed, by the choice of x_0 we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{B(0, 1)} \Psi(|Du_{n,k}|) dx &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\varepsilon_k} \int_{B(x_0, \varepsilon_k)} \Psi(|Du_n|) dx \\ &= |B(0, 1)| \frac{d\mu}{dx}(x_0) < +\infty. \end{aligned} \quad (3.4.10)$$

By taking into account the strong convergence $u_n \rightarrow u$ in $L_{\Psi}(\Omega)$ and (3.4.9) for $x = x_0$ and setting $u_0(x) = Du(x_0)x$, we get

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \|u_{n,k} - u_0\|_{L^1(B(0, 1))} = 0.$$

Thus $(u_{n,k})$ has a subsequence $v_k = u_{n_k,k}$ which converges strongly in $L^1(B(0,1))$ to the affine function u_0 . By (3.4.10) $(v_k)_k$ satisfies

$$v_k \rightarrow u_0 \text{ strongly in } L^1_{loc}(\Omega) \text{ and } \liminf_{n \rightarrow \infty} \int_{\Omega} \Psi(|Du_n|) dx < +\infty,$$

by Step 1, Inequality (3.4.7) follows, i.e.

$$\frac{d\mu}{dx}(x_0) = \lim_{k \rightarrow \infty} \frac{1}{|B(0,1)|} \int_{B(0,1)} \Psi(|Dv_k|) dx \geq \Psi(|Du(x_0)|).$$

□

3.5 Proof of Theorem 3.1.1

In this section, we will show the existence of at least one minimizer for the energy

$$E(u) := \int_{\Omega} (f - u)^2 dx + \frac{\lambda}{2} \int_{\Omega} \Psi(|Du|) dx,$$

on the space W , where Ψ is a Young and Δ_2 function. Recall that in this case, Ψ satisfies the following property $\lim_{t \rightarrow +\infty} \frac{\Psi(t)}{t} = +\infty$ and

$$W := \left\{ u : \Omega \rightarrow \mathbb{R}; u \in L^2(\Omega) \text{ and } |Du| \in L^{\Psi}(\Omega) \right\}.$$

Proof of Theorem 3.1.1

Let $(u_n)_n$ be a minimizer sequence of E on W , then there exists a positive constant M such that $E(u_n) \leq M$ for every n . Then, we have :

$$\begin{aligned} \|u_n\|_{L^2(\Omega)}^2 &\leq \|f - u_n\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 \\ &\leq \frac{2M}{\lambda} + \|f\|_{L^2(\Omega)}^2 =: N, \end{aligned}$$

$$\int_{\Omega} \Psi(|Du_n|) dx \leq M,$$

and

$$\int_{\Omega} \Psi((u_n)_{\Omega}) dx = \int_{\Omega} \Psi\left(\frac{1}{|\Omega|} \int_{\Omega} u_n\right) dx$$

$$\begin{aligned}
&\leq \int_{\Omega} \Psi \left(\frac{1}{|\Omega|} \|u_n\|_{L^1(\Omega)} \right) dx \\
&\leq |\Omega| \Psi \left(\frac{C_1}{|\Omega|} \|u_n\|_{L^2(\Omega)} \right) \\
&\leq |\Omega| \Psi \left(\frac{C_1}{|\Omega|} N \right) =: N_1,
\end{aligned}$$

where, we have used the fact that Ψ is increasing, Ω is bounded and the imbedding $L^2(\Omega) \subset L^1(\Omega)$ is continuous, i.e., there exists a positive constant $C_1 > 0$ such that $\|u\|_{L^1(\Omega)} \leq C_1 \|u\|_{L^2(\Omega)}$, for every $u \in L^2(\Omega)$.

Consequently, $(u_n)_n$ and $(|Du_n|)_n$ are uniformly bounded in $L^2(\Omega)$ and in $L^\Psi(\Omega)$ respectively. Thus there exist a subsequence still denoted by $(u_n)_n$ and an element $u \in W$ such that

- (i) $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$, and
- (ii) $|Du_n| \rightharpoonup |Du|$ weak \star in $L^\Psi(\Omega)$.

On the other hand, we have

$$\begin{aligned}
\|u_n\|_{L^\Psi(\Omega)} &\leq \|u_n - (u_n)_\Omega\|_{L^\Psi(\Omega)} + \|(u_n)_\Omega\|_{L^\Psi(\Omega)} \\
&\leq C \|Du_n\|_{L^\Psi(\Omega)} + \|(u_n)_\Omega\|_{L^\Psi(\Omega)} \\
&\leq C \left(1 + \int_{\Omega} \Psi(|Du_n|) \right) + N_1 \\
&\leq C(1 + M) + N_1.
\end{aligned}$$

where we have used Inequality (3.3.2) in the second line and Inequality (3.3.1) in the third line.

Consequently, $(u_n)_n$ is also bounded in $W^{1,\Psi}(\Omega)$. Thus, using Theorem 3.1.4 and the weak lower semi-continuity of the norm of $L^2(\Omega)$, we deduce that

$$E(u) = \liminf_n E(u_n).$$

Hence $E(u) = \inf_{v \in W} E(v)$. □

We will start showing the following lemma

Lemma 3.5.1 *If p is non increasing and for every $t > 0$, $1 < p(t) \leq p(0) < +\infty$, then the function $\Psi(s) = s^{p(s)}$ belongs to Δ_2 near infinity.*

Proof. For every $t > 0$, we have

$$\Psi(2t) = (2t)^{p(2t)} = 2^{p(2t)} t^{p(2t)}$$

$$\leq 2^{p^{(0)}} t^{p^{(0)}} = 2^{p^{(0)}} \Psi(t) \quad \forall t \geq 1.$$

□

Proof of Corollary 3.1.2

This proof is given by direct application of Lemma 3.5.1 and Theorem 3.1.1.

□

An example for such Ψ is $\Psi(s) = \frac{2+\ln(1+s)}{1+\ln(1+s)}$.

Chapitre 4

Supersonic traveling wave in a Frenkel-Kontorova model

Abstract

In this paper, we would solve a Frenkel-Kontorova model, and prove that we can build a solution admits a ℓ -transition.

Keywords : Traveling wave, Frenkel-Kontorova model, variational calculus.

4.1 Introduction

4.1.1 Setting of the problem

This paper is concerned with traveling waves in a lattice. The dynamics of the lattice is described by the Frenkel-Kontorova model. In this model, each particle interacts with nearest neighbors and is submitted to a fixed periodic potential. Corresponding

equations of motion are of the form

$$\ddot{q}_n = q_{n+1} + q_{n-1} - 2q_n + W'(q_n), \quad n \in \mathbb{Z}. \quad (4.1.1)$$

Here $q_n(t)$ is the position of the n^{th} particle at the time t . We assume in the whole paper that the function W satisfies the following assumption

$$\begin{cases} W \in C^2(\mathbb{R}), \\ W(a+1) = W(a) & \text{for every } a \in \mathbb{R}, \\ W(a) > 0 = W(0) = W'(0) & \text{for every } a \in \mathbb{R} \setminus \mathbb{Z}, \\ W''(0) > 0. \end{cases} \quad (4.1.2)$$

The equation for a traveling wave is obtained by looking for solutions of the form

$$q_n(t) = u(n - ct),$$

where c is a prescribed wave speed. This means that u is solution of

$$c^2 u''(x) = u(x+1) + u(x-1) - 2u(x) + W'(u(x)) \quad \text{for every } x \in \mathbb{R}. \quad (4.1.3)$$

That is to say, we are dealing with a second order forward-backward differential difference equation. Equation (4.1.3) is supplemented by the following conditions at infinity :

$$\begin{cases} u(\pm\infty) \in \mathbb{Z}, \\ u(+\infty) - u(-\infty) \in 1 + 2\mathbb{Z}. \end{cases} \quad (4.1.4)$$

4.1.2 First properties and definitions

We define the following exponential decay property

Definition 4.1.1 (Exponential decay property)

We say that a function u has the exponential decay property if there exist constants $C > 0$, $\lambda > 0$ such that

$$|u(x) - k_+| \leq Ce^{-\lambda x} \quad \text{for } x \geq 0, \text{ where } k_+ = \lim_{x \rightarrow +\infty} u(x) \in \mathbb{R},$$

and

$$|u(x) - k_-| \leq Ce^{\lambda x} \quad \text{for } x \leq 0, \text{ where } k_- = \lim_{x \rightarrow -\infty} u(x) \in \mathbb{R}.$$

We show the following result

Theorem 4.1.2 (Exponential decay)

If $u \in L^\infty(\mathbb{R})$ is solution of (4.1.3) with its limits $u(\pm\infty) \in \mathbb{Z}$, then u has the exponential decay property.

Define the energy associated by Equation (4.1.3) as following

$$J(u) = \int_{\mathbb{R}} \left[\frac{c^2}{2}(u')^2 - \frac{1}{2}(Du)^2 + W(u(x)) \right] dx,$$

where

$$Du(x) := u\left(x + \frac{1}{2}\right) - u\left(x - \frac{1}{2}\right).$$

J is defined for all $u \in H$, with

$$H := \left\{ u \in H_{loc}^1(\mathbb{R}); u' \in L^2(\mathbb{R}) \text{ and } W(u) \in L^1(\mathbb{R}) \right\}.$$

Note that each minimizer, if it exists, of J on H is solution of (4.1.3). For $k \in \mathbb{Z} \setminus \{0\}$ and $\varphi \in C^\infty(\mathbb{R})$ such that

$$\varphi(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ 1 & \text{if } x \geq 1, \end{cases}$$

and

$$1 - \varphi(-x) = \varphi(x), \tag{4.1.5}$$

set

$$E(k) = \inf_{\psi \in C_c^\infty(\mathbb{R})} J(k\varphi + \psi).$$

By direct calculation we can show that E is independent of the choice of φ . We show that E satisfies the following proposition

Proposition 4.1.3 (Properties of the energy E)

If $c > 1$, then the energy E satisfies the following properties :

(i) $E(0) = 0$.

(ii) $E(k) = E(-k) > 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

(iii) $E(k) \leq E(k-p) + E(p) \quad \forall k \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \forall p \in \mathbb{Z} \setminus \{0, k\}$.

4.1.3 Main results

Our first main result is :

Theorem 4.1.4 (Existence of a solution)

Let $c > 1$, then there exists at least one solution $u \in C^2(\mathbb{R})$ of (4.1.3)-(4.1.4).

Remarque 4.1.5 *The potential W , appearing in equation (4.1.3), is opposed to the usual W for phase transitions and we construct a transition between two physically unstable phases.*

Now, let us state the following definitions

Definition 4.1.6 (k -transition)

We say a function u is a k -transition if and only if it satisfies the following conditions

$$\left\{ \begin{array}{l} u \in H, \\ u(\pm\infty) \in \mathbb{Z}, \\ u(+\infty) - u(-\infty) = k, \\ J(u) = E(k), \\ u \text{ solve (4.1.3)}. \end{array} \right.$$

Definition 4.1.7 (Stability)

We say that k is stable if and only if $E(k) < E(k-p) + E(p)$ for every $p \in \mathbb{Z} \setminus \{0, k\}$.

Then we have the following results

Theorem 4.1.8 (Relation between stability and transition)

For $c > 1$, if k is stable, then there exists a k -transition.

Theorem 4.1.9 (Existence of a stable odd integer)

For $c > 1$, there exists $k \in 1 + 2\mathbb{Z}$ which is stable.

Theorem 4.1.10 ($k = 1$ is stable)

For every $c \geq \sqrt{\frac{25}{24}}$, the integer 1 is stable and then there exists a 1-transition u .

Remark 4.1.11 Theorem 4.1.4 can be seen as a corollary of Theorem 4.1.9 and Theorem 4.1.8.

Remark 4.1.12 It would be interesting to study the limit $c \rightarrow +\infty$, and to study the uniqueness of the solutions.

4.1.4 Strategy of the proofs

Our approach relies on avoiding a direct study of equation (4.1.3), but instead recasts the problem of finding a solution of a variational problem. This consists in looking for minimizers of the functional J on H . For a fixed $R > 0$, we replace H by H_R defined for $\ell \in \mathbb{Z}$ by

$$H_R := \left\{ u \in H_{loc}^1(\mathbb{R}) \text{ such that } u(x + 2R) - u(x) = \ell \right\},$$

and J by J_R defined by

$$J_R(u) = \int_{-R}^R \left[\frac{c^2}{2}(u')^2 - \frac{1}{2}(Du)^2 + W(u) \right] dt \quad (4.1.6)$$

and we look for minimizers of the problem

$$\inf_{v \in H_R} J_R(v). \quad (4.1.7)$$

Existence of a ℓ -transition.

We first build “ ℓ -transition” u_R on interval of length $2R$, and then take the limit as R goes to infinity.

The limit function u of u_R is not necessary a ℓ -transition.

In the simplest case, we may have the situation sketched on the picture 4.1.4.1, where the ℓ -transition is splitted in two smaller transitions ℓ_1 and ℓ_2 .

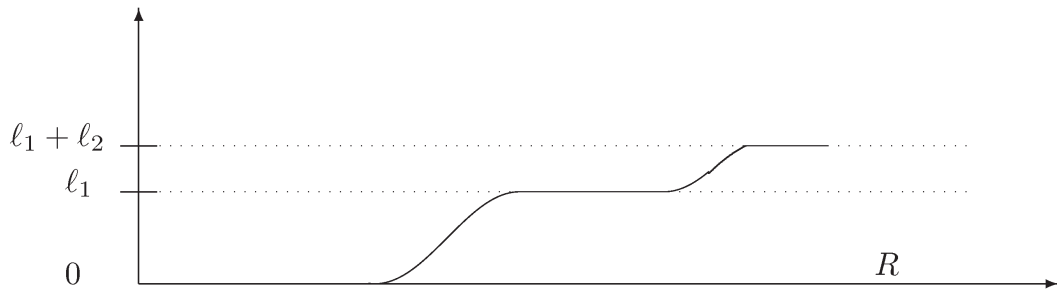


FIGURE 4.1.4.1 – Splitting of an ℓ -transition in $\ell_1 + \ell_2$

We show that this situation can only occur if

$$E(\ell_1) + E(\ell_2) \leq E(\ell). \quad (4.1.8)$$

In particular, if ℓ is stable, then it is impossible.

Indeed (4.1.8) can be shown using an argument similar to the concentration-compactness argument of Lions, (see [34]).

Existence of a stable odd integer ℓ .

Firstly, we choose a $\ell \in 1 + 2\mathbb{Z}$ such that

$$E(\ell) = \inf_{k \in 1+2\mathbb{Z}} E(k).$$

We assume by contradiction that we have a splitting of ℓ in a finite sequence of integers, i.e.

$$\ell = \ell_1 + \dots + \ell_N \quad \text{with } \ell_i \in \mathbb{Z} \text{ for all } i = 1, \dots, N.$$

Then there is at least one integer $\ell_{i_0} \in 1 + 2\mathbb{Z}$ such that

$$E(\ell_{i_0}) + E(\ell - \ell_{i_0}) \leq E(\ell).$$

From the properties of the energy E (Proposition 4.1.3) and the definition of the stability of ℓ (Definition 4.1.7), we deduce that

$$E(\ell_{i_0}) = E(\ell) \text{ and } E(\ell - \ell_{i_0}) = 0,$$

and then $\ell_{i_0} = \ell$.

Justification of the simple scenario.

First, we have a BV estimation for $c > 1$ which claims that

$$\int_{[-R,R]} \left| \frac{d}{dx} \beta(u_R) \right| dx \leq C, \quad (4.1.9)$$

with C is independent of R and

$$\beta(v) = \int_0^v \sqrt{W(w)} dw.$$

Because of the bound (4.1.9), the solution has to be close to constants on large intervals and, in order to minimize the energy, these constants have to be integers.

We get a control on the length of the transition between two integers k_1 and k_2 using the Cleaning Lemma, (see Lemma 4.5.2).

Indeed the Cleaning Lemma states that if u is close to an integers k on two intervals I_1 and I_2 of lengths bigger than 2, then u is close to this integer k on the convex hull of $I_1 \cup I_2$.

Using these argument, we can pass to the limit $R \rightarrow +\infty$, and show the existence of a ℓ -transition.

4.1.5 Organization of the article

In section 2, we prove exponential decay properties and in particular Theorem 4.1.2. In Section 3, we fix R and construct a minimizer u_R for problem (4.1.7) (see Proposition 4.3.4). In Section 4, we show boundedness of the sequence $(u_R)_R$. In Section 5, we prove a cleaning lemma for u_R (see Lemma 4.5.2), stating that if u_R is small on two

intervals I and J with lengths ≥ 2 then u_R is also small on the convex hull of $I \cup J$. Then, in Section 6, we study the distance from u_R to \mathbb{Z} and prove some bounds on the jump of u_R , uniformly with respect to R (see Proposition 4.6.1). Section 7 is devoted to the proof of Proposition 4.1.3 and to show the convergence of the sequence $(u_R)_R$ to a function u which will be a solution of equation (4.1.3) (see Proposition 4.7.1). In particular Theorem 4.1.4 is a straightforward corollary of Proposition 4.7.1. In Section 8, we prove Theorem 4.1.8, 4.1.9 and 4.1.10. In the appendix, we put some additional results a first integral for the solution, u , of equation (4.1.3) (see Lemma 4.9.1).

Notation

For a real number a , we denote by $\lfloor a \rfloor$ the floor integer part of a .

4.2 An exponential decay property

Proposition 4.2.1 (*Exponential decay*)

There exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$, then there exist two constants $\lambda, \kappa > 0$ such that for any solution $u \in L^\infty(\mathbb{R})$ of (4.1.3), we have the following properties :

(i) If $|u(x)| \leq \delta$ for every $x \geq 0$, then

$$|u(x)| \leq \kappa \delta e^{-\lambda x} \quad \text{for } x \geq 0.$$

(ii) If $|u(x)| \leq \delta$ for every $x \leq 0$, then

$$|u(x)| \leq \kappa \delta e^{-\lambda|x|} \quad \text{for } x \leq 0.$$

(iii) If $|u(x)| \leq \delta$ for every $x \in I$, where I is a bounded interval in \mathbb{R} , then

$$|u(x)| \leq \kappa \delta e^{-\lambda \text{dist}(x, \partial I)} \quad \text{for } x \in I,$$

where ∂I is the boundary of I .

In what follows we denote, for $r \in \mathbb{R}$,

$$M_r(u) := \sup_{x \geq r} |u(x)|.$$

Lemma 4.2.2 (Basic estimate)

There exist $\delta_0 > 0$, $\mu \in (0, 1)$ and $L > 0$ such that for all $u \in L^\infty(\mathbb{R})$ solution of equation (4.1.3), we have

$$\text{If } M_0(u) \leq \delta_0, \quad \text{then} \quad M_{r+L}(u) \leq \mu M_r(u) \quad \text{for all } r \geq 0.$$

Proof. Suppose that the result of this lemma is false. Then for every sequences

$$\begin{cases} L_n \rightarrow +\infty, \\ \delta_n \rightarrow 0, \\ \mu_n \rightarrow 1, \end{cases}$$

there exists a solution $u_n \in L^\infty(\mathbb{R})$, of equation (4.1.3), such that

$$M_0(u_n) \leq \delta_n \quad \text{and} \quad M_{L_n+r_n}(u_n) > \mu_n M_{r_n}(u_n) \quad \text{for some } r_n \geq 0.$$

Step 1. Case where the supremum $M_{L_n+r_n}(u_n)$ is not reached at infinity

In this case there exists $x_n \in [L_n + r_n, +\infty)$ such that

$$M_{L_n+r_n}(u_n) = |u_n(x_n)| =: \varepsilon_n.$$

Then, we have

$$\delta_n \geq M_0(u_n) \geq M_{L_n+r_n}(u_n) = \varepsilon_n \rightarrow 0. \quad (4.2.1)$$

Set

$$v_n(x) := \varepsilon_n^{-1} u_n(x + x_n),$$

then v_n is solution of the following equation

$$c^2 v_n''(x) = v_n(x+1) + v_n(x-1) - 2v_n(x) + \varepsilon_n^{-1} W'(v_n \varepsilon_n). \quad (4.2.2)$$

By the definition of v_n , we have

$$|v_n(0)| = 1.$$

On the other hand, we have

$$\begin{aligned} M_{-L_n}(v_n) &= \varepsilon_n^{-1} \sup_{x \geq -L_n} |u_n(x + x_n)| \\ &\leq \varepsilon_n^{-1} \sup_{y \geq r_n} |u_n(y)| \end{aligned}$$

$$\begin{aligned}
&= \varepsilon_n^{-1} M_{r_n}(u_n) \\
&< \varepsilon_n^{-1} \frac{M_{L_n+r_n}(u_n)}{\mu_n} = \frac{1}{\mu_n}.
\end{aligned}$$

Hence, we have

$$|v_n(x)| \leq \frac{1}{\mu_n} \quad \text{for every } x \in [-L_n, +\infty). \quad (4.2.3)$$

Then (4.2.1) implies

$$\varepsilon_n^{-1} W'(v_n \varepsilon_n) = W''(0) \cdot v_n + o_{\varepsilon_n}(1) \quad \text{on } [-L_n, +\infty), \quad (4.2.4)$$

since $W'(0) = 0$, with $o_{\varepsilon_n}(1) \rightarrow 0$ when n goes to infinity.

Equations (4.2.2), (4.2.4) and (4.2.3), imply the existence of a constant $C_1 > 0$ such that $|v_n''(x)| \leq C_1$ for every $x \in [-L_n, +\infty)$. Consequently there exist a subsequence, still denoted by $(v_n)_n$, and an element $v \in W^{2,\infty}(\mathbb{R})$ such that $v_n \rightarrow v$ uniformly on every compact set of \mathbb{R} .

Passing to the limit in (4.2.3) and using (4.2.2), we get

$$\sup_{x \in \mathbb{R}} |v(x)| \leq |v(0)| = 1.$$

We also have that v is solution of :

$$c^2 v''(x) = v(x+1) + v(x-1) - 2v(x) + W''(0) \cdot v \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (4.2.5)$$

Let us apply the Fourier transform to equation (4.2.5), then we obtain :

$$\phi(\xi) \hat{v}(\xi) = 0, \quad (4.2.6)$$

with $\phi(\xi) = c^2 \xi^2 + (e^{i\xi} + e^{-i\xi} - 2) + W''(0)$ and \hat{v} is the Fourier transform to v and $\xi \in \mathbb{R}$.

Because $W''(0) > 0$, we show easily that $\phi > 0$. This implies that $v = 0$.

This contradict the fact that $|v(0)| = 1$.

Step 2. Case where the supremum $M_{L_n+r_n}(u_n)$ is “reached at infinity”

In this case, there exists $x_n \in [L_n + r_n, +\infty)$ such that

$$\varepsilon_n := |u_n(x_n)| \geq \frac{n}{n+1} M_{L_n+r_n}(u_n).$$

□

Proof of Proposition 4.2.1

In the sequel, we will show only statement (i). Statements (ii) and (iii) can be proved in the same manner.

Let $\delta \in (0, \delta_0) > 0$, such that $|u(x)| \leq \delta$ for every $x \geq 0$. Then by Lemma 4.2.2, there exists $\mu \in (0, 1)$ and there exists $L > 0$ such that

$$M_{L+r} \leq \mu M_r \quad \text{for all } r \geq 0.$$

where M_r stand for $M_r(u_n)$.

Since $\mu \in (0, 1)$, then there exists $\sigma > 1$ such that $\mu = \sigma^{-1}$.

Fix r such that $L \leq r$, and let $x \in \mathbb{R}$ such that $x \geq 0$.

For all $x \geq 0$, there is $n \in \mathbb{N}$ such that

$$-n \leq \frac{-x+r}{L}. \tag{4.2.7}$$

On the one hand, we have

$$M_{nL} \leq \mu M_{(n-1)L} \leq \mu^2 M_{(n-2)L} \leq \dots \leq \mu^n M_0 \leq \mu^n \delta.$$

If $x \in [nL, (n+1)L]$ for some $n \geq 0$, we have

$$\begin{aligned} \mu^n &= \exp(n \ln(\mu)) \\ &\leq \exp\left(\frac{x \ln(\mu)}{L} - \ln(\mu)\right). \end{aligned}$$

This implies that $|u(x)| \leq \beta \delta e^{-\lambda x}$, where $\lambda = -\frac{\ln(\mu)}{L} > 0$ and $\beta = \frac{1}{\mu} > 0$. □

4.3 Resolution of problem (4.1.7)

Let us define $\Omega_R = (-R, R)$.

Lemma 4.3.1 (*Control of the term $(Du(x))^2$*)

For all $u \in H_R$, we have the following inequality :

$$\int_{\Omega_R} \left(u\left(x + \frac{1}{2}\right) - u\left(x - \frac{1}{2}\right)\right)^2 dx \leq \int_{\Omega_R} (u'(x))^2 dx. \quad (4.3.1)$$

Proof. For every $u \in H_R$, we have :

$$\begin{aligned} \int_{\Omega_R} \left(u\left(x + \frac{1}{2}\right) - u\left(x - \frac{1}{2}\right)\right)^2 dx &\leq \int_{\Omega_R} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} u'(x+t) dt\right)^2 dx \\ &\leq \int_{\Omega_R} \int_{-\frac{1}{2}}^{\frac{1}{2}} |u'(x+t)|^2 dt dx \\ &= \int_{\Omega_R} (u'(x))^2 dx, \end{aligned}$$

where we have used Cauchy-Schwarz inequality in the second line, Fubini theorem and the $2R$ -periodicity of u' in the third line. \square

Lemma 4.3.2 (*Poincaré-Wirtinger inequality in H_R*)

For every every $R > 0$, there exists a constant $C_R > 0$, such that

$$\int_{\Omega_R} \left(u(x) - \frac{1}{|\Omega_R|} \int_{\Omega_R} u dx\right)^2 dx \leq C_R \int_{\Omega_R} (u'(x))^2 dx \text{ for every } u \in H_R. \quad (4.3.2)$$

Proof. See [19, Theorem 1, page 275]. \square

Lemma 4.3.3 (*Coercivity of J_R*)

For all $c > 1$, the functional J_R is coercive on H_R for the semi norm $\|u\|_{H_R} = \|u'\|_{L^2(\mathbb{R})}$, and satisfies $J_R \geq 0$ on H_R .

Proof. For all $u \in H_R$, we have :

$$J_R(u) := \frac{c^2}{2} \int_{\Omega_R} (u'(x))^2 dx - \frac{1}{2} \int_{\Omega_R} (Du(x))^2 dx + \int_{\Omega_R} W(u(x)) dx$$

$$\begin{aligned}
&\geq \frac{1}{2}(c^2 - 1) \int_{\Omega_R} (u'(x))^2 dx + \int_{\Omega_R} W(u(x)) dx \\
&\geq \frac{1}{2}(c^2 - 1) \int_{\Omega_R} (u'(x))^2 dx,
\end{aligned} \tag{4.3.3}$$

where we have used Lemma 4.3.1 in the second line and the fact that $W \geq 0$ in the third line. We deduce that :

$$J_R(u) \rightarrow \infty \quad \text{as } \|u\|_{H_R} \rightarrow \infty.$$

□

Proposition 4.3.4 (Existence of a minimizer for problem (4.1.7))

Let $c > 1$, then J_R has at least one minimizer u_R on H_R . Moreover u_R is solution of (4.1.3) and $u_R \in C^2(\mathbb{R})$.

Proof. Let $(u_n)_n$ be a minimizing sequence of J_R in H_R , i.e. $(u_n)_n \subset H_R$ and $J_R(u_n) \rightarrow \inf_{v \in H_R} J_R(v) =: \alpha \geq 0$.

Step 1 : Extraction of a subsequence

Since $J_R(u_n) \rightarrow \alpha$, then there exists a constant $M > 0$ such that :

$$|J_R(u_n)| \leq M \quad \text{for every } n \in \mathbb{N}.$$

On the one hand, using (4.3.3) we have

$$\frac{1}{2}(c^2 - 1) \int_{\Omega_R} (u'_n(x))^2 dx \leq J_R(u_n) \leq M,$$

Hence, we obtain the following bound :

$$\int_{\Omega_R} (u'_n(x))^2 dx \leq \frac{2M}{c^2 - 1} := C_1.$$

Moreover, by Poincaré-Wirtinger inequality (Lemma 4.3.2), we have :

$$\int_{\Omega_R} \left(u_n(x) - \frac{1}{|\Omega_R|} \int_{\Omega_R} u_n \right)^2 dx \leq C_R \int_{\Omega_R} (u'_n(x))^2 dx \leq C_R C_1.$$

For every $n \in \mathbb{N}$, we define $\tilde{u}_n := u_n - \left\lfloor \frac{1}{|\Omega_R|} \int_{\Omega_R} u_n \right\rfloor \in H_R$, then

$$0 \leq \frac{1}{|\Omega_R|} \int_{\Omega_R} \tilde{u}_n \leq 1, \quad \text{and} \quad J_R(\tilde{u}_n) = J_R(u_n).$$

Consequently the sequence $(\tilde{u}_n)_n$ is bounded in $H^1(\Omega_R)$, implying that there is a subsequence of $(\tilde{u}_n)_n$, still denoted by $(\tilde{u}_n)_n$, and an element $u \in H^1(\Omega_R)$, such that :

$$\tilde{u}_n \rightharpoonup u =: u_R \quad \text{weakly in } H^1(\Omega_R). \quad (4.3.4)$$

Since $|\Omega_R|$ is bounded, then by Rellich-Kondrachev theorem, we have :

$$\tilde{u}_n \rightarrow u \quad \text{strongly in } L^2(\Omega_R), \text{ up to a subsequence.} \quad (4.3.5)$$

Hence, by the Lebesgue inverse theorem, we have

$$\tilde{u}_n(x) \rightarrow u(x) \quad \text{for a.e. } x \in \Omega_R, \text{ up to a subsequence,}$$

and particularly, we have

$$\lim_{n \rightarrow \infty} [\tilde{u}_n(x + 2R) - \tilde{u}_n(x)] = u(x + 2R) - u(x) = \ell,$$

which implies that $u \in H_R$.

Step 2 : $\mathbf{J}_R(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{H}_R} \mathbf{J}_R(\mathbf{v})$

From relation (4.3.4) and by lower semi-continuity, we get

$$\int_{\Omega_R} (u'(x))^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega_R} (u'_n(x))^2 dx. \quad (4.3.6)$$

Property (4.3.5) and the periodicity property of $D\tilde{u}_n$ imply :

$$\int_{\Omega_R} (Du(x))^2 dx = \lim_{n \rightarrow \infty} \int_{\Omega_R} (D\tilde{u}_n(x))^2 dx. \quad (4.3.7)$$

On the other hand, since W is a bounded and continuous function on \mathbb{R} , then Lebesgue dominated convergence theorem implies :

$$\lim_{n \rightarrow \infty} \int_{\Omega_R} W(\tilde{u}_n(x)) \, dx = \int_{\Omega_R} W(u(x)) \, dx. \quad (4.3.8)$$

Relations (4.3.6), (4.3.7) and (4.3.8) imply :

$$J_R(u) \leq \liminf_{n \rightarrow \infty} J_R(\tilde{u}_n) = \inf_{v \in H_R} J_R(v).$$

Since $u \in H_R$, then $J_R(u) = \inf_{v \in H_R} J_R(v)$.

Step 3 : conclusion

By classical arguments, we get

$$-c^2 u'' - D^2 u + W'(u) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}). \quad (4.3.9)$$

From the fact that $\int_{\Omega_R} u'^2 \, dx \leq C$, we deduce that $u \in L_{loc}^\infty(\mathbb{R})$ and then from (4.3.9), we get $u \in W_{loc}^{2,\infty}(\mathbb{R})$, which by bootstrap implies $u \in C^2(\mathbb{R})$. \square

4.4 Uniform bounds on $(u_R)_R$

In this section, we make R tend to infinity. Henceforth, we will denote u_R the solution given in Proposition 4.3.4.

Lemma 4.4.1 (*Bound on the energy*)

Let u_R be the function constructed in Proposition 4.3.4, then there exists a constant $M > 0$, independent on $R \geq 1$, such that :

$$J_R(u_R) \leq M. \quad (4.4.1)$$

Proof. Define the function v_R as follows :

$$v_R(x) = \begin{cases} 0 & \text{if } -R \leq x \leq 0, \\ \ell x & \text{if } 0 < x \leq 1, \\ \ell & \text{if } 1 < x \leq R, \\ v_R(x+2R) - v_R(x) = \ell & \text{if } x \notin \Omega_R. \end{cases}$$

Since $v_R \in H_R$, and u_R is a minimizer of J_R on H_R , then

$$\begin{aligned} J_R(u_R) &\leq J_R(v_R) \\ &\leq \frac{c^2}{2} \int_{\Omega_R} (v'_R)^2 dx + \int_{\Omega_R} W(v_R) dx \\ &= \frac{c^2}{2} \int_0^1 (v'_R)^2 dx + \int_0^1 W(v_R) dx \\ &\leq \frac{c^2 \ell^2}{2} + \|W\|_{L^\infty(\mathbb{R})} =: M. \end{aligned}$$

This ends the proof. \square

Lemma 4.4.2 (Bound on the total variation of $\beta(u_R)$)

Let us define $\beta(v) := \int_0^v \sqrt{W(x)} dx$. Then

$$TV(\beta(u_R), \Omega_R) := \int_{\Omega_R} \left| \frac{d}{dx} \beta(u_R(x)) \right| dx \leq \frac{M}{\sqrt{2(c^2 - 1)}} := M', \quad (4.4.2)$$

where M is the constant given in (4.4.1).

Proof. We have

$$\begin{aligned} J_R(u_R) &\geq \frac{c^2 - 1}{2} \int_{\Omega_R} (u'_R)^2 dx + \int_{\Omega_R} W(u_R(x)) dx \\ &\geq \sqrt{2(c^2 - 1)} \int_{\Omega_R} |u'_R| \sqrt{W(u_R)} dx \\ &= \sqrt{2(c^2 - 1)} \int_{\Omega_R} \left| \frac{d}{dx} \beta(u_R) \right| dx, \end{aligned} \quad (4.4.3)$$

where, in the first line, we have used (4.3.3), and in the second line we have used Young's inequality. We conclude to (4.4.2) using (4.4.1). \square

Lemma 4.4.3 (Bounds on W)

There exist $\gamma_2 > \gamma_1 > 0$ such that

$$\gamma_1 [\text{dist}(a, \mathbb{Z})]^2 \leq W(a) \leq \gamma_2 [\text{dist}(a, \mathbb{Z})]^2. \quad (4.4.4)$$

The proof is left to the reader, using (4.1.2).

Lemma 4.4.4 (Bounds on β)

There exist two positive constants C_1 and C_2 such that

$$C_1 \min(v^2, 1) \leq |\beta(v)| \leq C_2 v^2 \quad \text{for all } v \in \mathbb{R}. \quad (4.4.5)$$

Proof**Step 1 : Proof of the lower bound**

For $0 \leq v \leq \frac{1}{2}$, we have

$$\begin{aligned} \beta(v) &= \int_0^v \sqrt{W(y)} \, dy \\ &\geq \sqrt{\gamma_1} \int_0^v \text{dist}(y, \mathbb{Z}) \, dy \\ &\geq \sqrt{\gamma_1} \int_0^v y \, dy \\ &\geq \sqrt{\gamma_1} \frac{v^2}{2}. \end{aligned}$$

Now, since β is nondecreasing, if $v \geq \frac{1}{2}$, then $\beta(v) \geq \beta(\frac{1}{2}) \geq \frac{\sqrt{\gamma_1}}{8}$.

Hence, $|\beta(v)| \geq \min(\frac{\sqrt{\gamma_1}}{2} v^2, \frac{\sqrt{\gamma_1}}{8}) \geq C_1 \min(v^2, 1)$, with $C_1 = \frac{\gamma_1}{8}$. If $v \leq 0$, we get the same estimate.

Step 2 : proof of the upper bound

If $0 \leq v \leq 1$, then $\text{dist}(y, \mathbb{Z}) \leq |y|$, and therefore, proceeding as in step 1, we obtain

$$|\beta(v)| \leq \sqrt{\gamma_2} \frac{v^2}{2}.$$

Furthermore, for $v \geq 1$ we have

$$\begin{aligned} \beta(v) &= \int_0^v \sqrt{W(y)} \, dy \\ &= \int_0^{\lfloor v \rfloor} \sqrt{W(y)} \, dy + \int_{\lfloor v \rfloor}^v \sqrt{W(y)} \, dy \\ &\leq (1 + \lfloor v \rfloor) \int_0^1 \sqrt{W(y)} \, dy \\ &\leq 2v \int_0^1 \sqrt{W(y)} \, dy \leq 2v^2 \int_0^1 \sqrt{W(y)} \, dy. \end{aligned}$$

As a conclusion, we get for all $v \geq 0$:

$$\beta(v) \leq C_2 v^2, \quad \text{with } C_2 = \max\left(\frac{\sqrt{\gamma_2}}{2}, 2 \int_0^1 \sqrt{W(y)} \, dy\right).$$

If $v \leq 0$, we get the same estimate.

□

4.5 Cleaning lemma

Lemma 4.5.1 (*Cleaning lemma for β*)

Let u_R be the function given in Proposition 4.3.4, $\delta \in (0, 1]$ and a, b are reals numbers such that : $-R \leq a - 1 \leq a + 1 \leq b - 1 \leq b + 1 \leq R$.

If

$$\sup_{x \in [a-1, a+1] \cup [b-1, b+1]} |u_R(x)| \leq \delta, \tag{4.5.1}$$

then there exists a positive constant C , independent of δ and R , such that

$$\int_I \left| \frac{d}{dx} \beta(u_R(x)) \right| dx \leq C\delta^2 \quad \text{where } I = [a, b] \subset \Omega_R, \quad (4.5.2)$$

with β defined in Lemma 4.4.2 above.

Proof.

Set

$$E_I(u) := \frac{c^2}{2} \int_I u'^2 dx - \frac{1}{2} \int_I (Du)^2 dx + \int_I W(u(x)) dx \quad \text{for every } u \in H_R.$$

Step 1 : Bound on $E_I(u_R)$

We will prove

$$E_I(u_R) \geq \int_I \left[\frac{c^2 - 1}{2} u_R'^2 + W(u_R) \right] dx - O(\delta^2). \quad (4.5.3)$$

Let

$$\tilde{u}(x) := \begin{cases} 0 & \text{if } -\infty \leq x \leq a - 1, \\ u_R(a)(x - (a - 1)) & \text{if } a - 1 \leq x \leq a, \\ u_R(x) & \text{if } a \leq x \leq b, \\ -u_R(b)(x - (b + 1)) & \text{if } b \leq x \leq b + 1, \\ 0 & \text{if } b + 1 \leq x \leq +\infty, \end{cases}$$

Then, we get :

$$\begin{aligned} J(\tilde{u}) &\geq \int_{\mathbb{R}} \left[\frac{c^2 - 1}{2} (\tilde{u}')^2 + W(\tilde{u}(x)) \right] \\ &\geq \int_I \left[\frac{c^2 - 1}{2} (u_R')^2 + W(u_R(x)) \right] \end{aligned} \quad (4.5.4)$$

Notice that

$$\frac{1}{2} \int_{[a,b]} [(D\tilde{u})^2 - (Du_R)^2] dx = \frac{1}{2} \int_{[a, a+\frac{1}{2}] \cup [b-\frac{1}{2}, b]} [(D\tilde{u})^2 - (Du_R)^2] dx$$

$$\begin{aligned}
&\geq -\frac{1}{2} \int_{[a, a+\frac{1}{2}] \cup [b-\frac{1}{2}, b]} (Du_R)^2 dx \\
&\geq -2\delta^2,
\end{aligned} \tag{4.5.5}$$

where we have used the hypothesis (4.5.1).

On the other hand, we have

$$\begin{aligned}
J(\tilde{u}) - E_I(u_R) &= \int_{(-\infty, a) \cup (b, +\infty)} \left[\frac{c^2}{2} \tilde{u}^2 - \frac{1}{2} (D\tilde{u})^2 + W(\tilde{u}) \right] dx - \frac{1}{2} \int_{[a, b]} [(D\tilde{u})^2 - (Du_R)^2] dx \\
&\leq \int_{(a-1, a) \cup (b, b+1)} \left[\frac{c^2}{2} \tilde{u}^2 + W(\tilde{u}) \right] dx - \frac{1}{2} \int_{[a, b]} [(D\tilde{u})^2 - (Du_R)^2] dx \\
&\leq C^2 \delta^2 + 2\gamma_2 \delta^2 + 2\delta^2 \\
&= O(\delta^2),
\end{aligned}$$

where we have used hypothesis (4.5.1), the fact that $W(a) \leq \gamma_2 a^2$ and (4.5.5).

Joint to (4.5.4), this shows (4.5.3).

Step 2 : Upper bound on $\int_I \left| \frac{d}{dx} \beta(u_R(x)) \right| dx$

In this Step, we will show that

$$\int_I \left| \frac{d}{dx} \beta(u_R(x)) \right| dx \leq \frac{1}{\sqrt{2(c^2 - 1)}} (O(\delta^2) + E_I(u_R)). \tag{4.5.6}$$

From (4.5.3), we have

$$\begin{aligned}
E_I(u_R) + O(\delta^2) &\geq \int_I \left(\frac{c^2 - 1}{2} u_R^2 + W(u_R) \right) \\
&\geq \sqrt{2(c^2 - 1)} \int_I \left| \frac{d}{dx} \beta(u_R(x)) \right| dx,
\end{aligned}$$

where we have used (4.4.3).

This show (4.5.6).

Step 3 : Bounds on $E_I(u_R)$ in terms of δ^2

We will define a new candidate \bar{u} on $I = [a, b]$, as follows :

$$\bar{u}(x) := \begin{cases} u_R(x) & \text{if } x \in \Omega_R \setminus [a, b], \\ -u_R(a)(x - (a + 1)) & \text{if } x \in (a, a + 1), \\ 0 & \text{if } x \in (a + 1, b - 1), \\ u_R(b)(x - (b - 1)) & \text{if } x \in (b - 1, b), \end{cases}$$

Since u_R minimizes J_R , then

$$J_R(u_R) \leq J_R(\bar{u}). \quad (4.5.7)$$

We want to use this inequality to estimate E_I .

We have

$$J_R(u_R) = E_I(u_R) + E_{\Omega_R \setminus I}(u_R) \quad (4.5.8)$$

and

$$J_R(\bar{u}) = E_I(\bar{u}) + E_{\Omega_R \setminus I}(\bar{u}) \quad (4.5.9)$$

By relations (4.5.7), (4.5.8) and (4.5.9), we obtain the following inequality

$$\begin{aligned} E_I(u_R) &\leq J_R(\bar{u}) - E_{\Omega_R \setminus I}(u_R) \\ &\leq E_I(\bar{u}) + E_{\Omega_R \setminus I}(\bar{u}) - E_{\Omega_R \setminus I}(u_R) \\ &= E_I(\bar{u}) - \frac{1}{2} \int_{\Omega_R \setminus I} [(D\bar{u})^2 - (Du_R)^2] \\ &= E_I(\bar{u}) - \frac{1}{2} \int_{[a-\frac{1}{2}] \cup [b, b+\frac{1}{2}]} [(D\bar{u})^2 - (Du_R)^2] \\ &\leq E_I(\bar{u}) + \frac{1}{2} \int_{[a-\frac{1}{2}] \cup [b, b+\frac{1}{2}]} (Du_R)^2 \\ &\leq O(\delta^2). \end{aligned}$$

Therefore $E_I(u_R) \leq O(\delta^2)$.

Step 4 : Conclusion

Finally, by using steps 2 and 3, we obtain

$$\int_I \left| \frac{d}{dx} \beta(u_R) \right| dx \leq C\delta^2.$$

□

Lemma 4.5.2 (Cleaning lemma for u_R)

There exists $\delta_0 \in (0, 1]$, such that the following holds :

Let a and b be real numbers such that $-R \leq a - 1 \leq a + 1 \leq b - 1 \leq b + 1 \leq R$. If the function u_R given by Proposition 4.3.4, satisfies

$$\sup_{x \in [a-1, a+1] \cup [b-1, b+1]} |u_R(x)| \leq \delta, \quad \text{for some } \delta \in (0, \delta_0)$$

then there exists a constant $\nu \geq 1$ independent of R and δ such that

$$|u_R(x)| \leq \nu\delta \quad \text{on} \quad I = [a, b]. \quad (4.5.10)$$

Proof.

By the Cleaning lemma 4.5.1, we have

$$\int_I \left| \frac{d}{dx} \beta(u_R) \right| dx \leq C\delta^2,$$

with C is the constant of lemma 4.5.1.

Moreover, for every $x \in [a, b]$ we have

$$\beta(u_R(x)) = \beta(u_R(a)) + \int_a^x \frac{d}{dy} (\beta(u_R(y))) dy.$$

Hence, using (4.4.5), we get

$$|\beta(u_R(x))| \leq C_2\delta^2 + C\delta^2 = C_3\delta^2,$$

where $C_3 = C_2 + C$. Using (4.4.5) again, we obtain

$$C_1 \min(u_R^2(x), 1) \leq |\beta(u_R(x))| \leq C_3 \delta^2.$$

Consequently, $\min(u_R^2(x), 1) \leq \frac{C_3}{C_1} \delta^2$.

Hence, for $\delta \leq \min(1, \sqrt{C_3/C_1})$, we have

$$|u_R(x)| \leq \nu \delta \quad \forall x \in I,$$

with $\nu = \sqrt{C_3/C_1} \geq 1$.

□

4.6 Bounds on the jumps of u_R

Proposition 4.6.1 (Bound on the "jumps" of u_R for finite R)

There exist $\bar{N} \in \mathbb{N}$ and $\delta_0 > 0$, such that for all $\delta \in (0, \delta_0)$, there exists $L_\delta > 0$, such that for all R large enough, there exists u_R a solution of (4.1.3) constructed in Proposition 4.3.4, such that there exist an integer $K \in \mathbb{N} \setminus \{0\}$ and a finite sequence of integers $(k_i)_{i=1, \dots, K}$, with $k_i \in \mathbb{Z}$ such that :

- i. $K \leq \bar{N}$,
- ii. $k_{K+1} = k_1 + \ell$,
- iii. $\bar{N} \geq \sum_{i=1, \dots, K} |k_{i+1} - k_i|$,
- vi. $\sup_{\Omega_R} u_R - \inf_{\Omega_R} u_R \leq \bar{N}$.

Moreover, there exists a sequence of intervals $(\tilde{I}_i)_{i=1, \dots, K}$, such that :

- a. $\tilde{I}_i \subset \bar{\Omega}_R$ for $i = 1, \dots, K$,
- b. $\sup \tilde{I}_i \leq \inf \tilde{I}_{i+1}$, for $i = 1, \dots, K - 1$,
- c. $|u_R(x) - k_i| \leq C\delta$ on \tilde{I}_i ,

d. $\left| \Omega_R \setminus \bigcup_{i=1}^K \tilde{I}_i \right| \leq L_\delta$ and then $\left| \bigcup_{i=1}^K \tilde{I}_i \right| \geq 2R - L_\delta$.

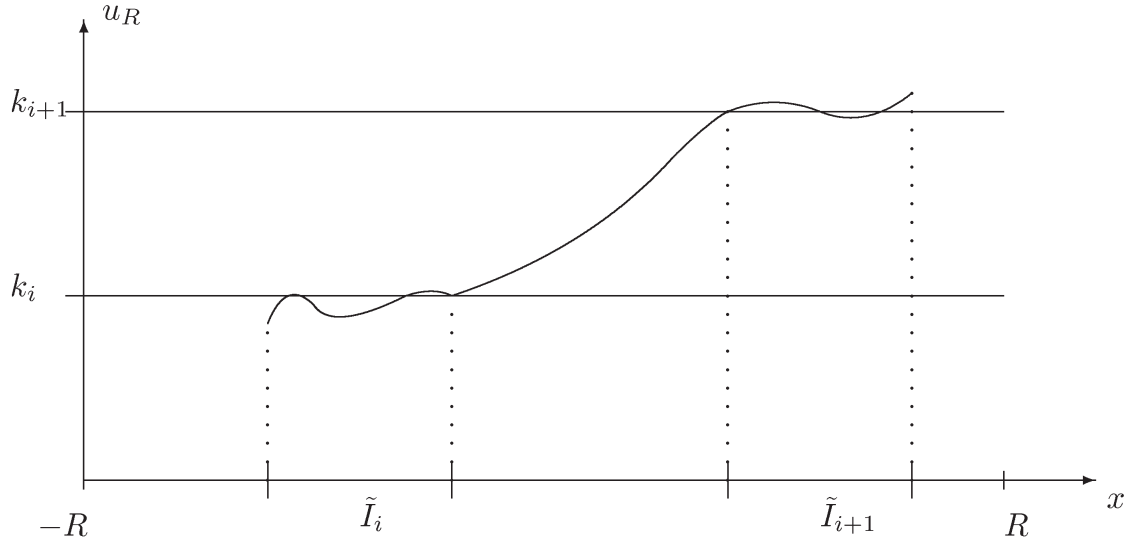


FIGURE 4.6.0.1 – Here k_i and k_{i+1} are two integers.

Proof.

Define the following sets

$$E_\delta := \left\{ x \in \Omega_R ; \text{dist} (u_R(x), \mathbb{Z}) \leq \frac{\delta}{2} \right\},$$

and

$$F_\delta := \Omega_R \setminus E_\delta = \left\{ x \in \Omega_R ; \text{dist} (u_R(x), \mathbb{Z}) > \frac{\delta}{2} \right\}.$$

Step 1 : $|E_\delta| \geq 2R - C_\delta$

Let m_δ be a positive constant such that $W \geq m_\delta > 0$ on $\mathbb{R} \setminus (\mathbb{Z} + B_{\frac{\delta}{2}}(0))$, then we have :

$$m_\delta |F_\delta| \leq \int_{F_\delta} W(u_R) dx \leq \int_{\Omega_R} W(u_R) dx \leq J_R(u_R) \leq M.$$

where M independent of R , is the constant of Lemma 4.4.1. Notice that we used (4.3.3) and $c > 1$.

Hence $|F_\delta| \leq \frac{M}{m_\delta} =: C_\delta$, notice that F_δ is not necessarily an interval.

Consequently, $|E_\delta| \geq 2R - C_\delta$.

Step 2 : (Bound on the oscillation of u_R)

By Lemma 4.4.2, we have

$$TV(\beta(u_R), \Omega_R) \leq M'.$$

Then

$$\max_{\Omega_R} \beta(u_R) - \min_{\Omega_R} \beta(u_R) \leq M'.$$

Since β is nondecreasing then we have

$$\max_{\Omega_R} \beta(u_R) - \min_{\Omega_R} \beta(u_R) = \beta\left(\max_{\Omega_R}(u_R)\right) - \beta\left(\min_{\Omega_R}(u_R)\right).$$

Using the 1-periodicity of W and the definition of β , we get

$$\left[\max_{\Omega_R} u_R - \min_{\Omega_R} u_R \right] \int_0^1 \sqrt{W(v)} dv \leq \int_{\min_{\Omega_R} u_R}^{\max_{\Omega_R} u_R} \sqrt{W(v)} dv \leq M'.$$

Hence

$$\max_{\Omega_R} u_R - \min_{\Omega_R} u_R \leq 1 + \frac{M'}{\int_0^1 \sqrt{W(v)} dv} = 1 + \frac{M'}{\beta(1)}.$$

Step 3 : Definition of the set E^k and existence of E^{k_0} of large measure

Let k be an integer, and

$$E^k := \left\{ x \in \Omega_R \ ; \ |u_R(x) - k| \leq \frac{\delta}{2} \right\}.$$

By step 2, there is a finite number of integers k , such that $E^k \neq \emptyset$.

Let \mathbb{K} be the set of such k and set

$$N := \text{card}(\mathbb{K}), \quad \text{and } E^k \neq \emptyset \quad \text{if } k \in \mathbb{K}.$$

In particular we have

$$\begin{aligned} M' \geq TV(\beta(u_R), \Omega_R) &\geq \inf_{(e_j)_j} \sum_{j=1}^{N-1} \left| \beta(\bar{k}_{j+1} + e_{j+1}) - \beta(\bar{k}_j + e_j) \right| \\ &\geq (N-1) \left[\beta\left(1 - \frac{\delta}{2}\right) - \beta\left(\frac{\delta}{2}\right) \right], \end{aligned}$$

with $\{\bar{k}_1, \dots, \bar{k}_N\} = \mathbb{K}$ and $|e_j| \leq \frac{\delta}{2}$ and $\bar{k}_1 \leq \dots \leq \bar{k}_N$. Then

$$N \leq 1 + \frac{M'}{\beta\left(1 - \frac{\delta}{2}\right) - \beta\left(\frac{\delta}{2}\right)}. \quad (4.6.1)$$

Furthermore, we find that

$$E_\delta = \bigcup_{k \in \mathbb{K}} E^{\bar{k}},$$

then by step 1

$$|E_\delta| = \sum_{\bar{k} \in \mathbb{K}} |E^{\bar{k}}| \geq 2R - C_\delta.$$

Note that there is at least one set $E^{\bar{k}_0} \geq \frac{2R - C_\delta}{N}$.

Step 4 : For k fixed, covering of E^k by intervals I_i^k each of length 2

We define the finite sequence $(x_i^k)_{i=1, \dots, N_\delta^k}$, for $1 \leq i \leq N_\delta^k$, as follows :

1. $x_0^k = \inf \{x \in E^k, x \geq -R\},$

2. $x_{i+1}^k := \inf \{x \in E^k, x_i^k + 2 \leq x \leq R\}$.

Let us define the following set $I_i^k = [x_i^k, x_{i+1}^k + 2]$, for every $1 \leq i \leq N_\delta^k$.

We have also

$$E^k \subset \bigcup_{i=1, \dots, N_\delta^k} I_i^k.$$

Notice that $I_i^k \subset \bar{\Omega}_R$ for $i = 1, \dots, N_\delta^k - 1$.

Step 5 : Dichotomy on the I_i^k

For every interval I_i^k , there are two possibilities :

a. there is an element $y_i^k \in I_i^k$ such that : $|u_R(y_i^k) - k| > \delta$, hence if $u_R(y_i^k) \geq u_R(x_i^k)$

$$\begin{aligned} \beta(u_R(y_i^k)) - \beta(u_R(x_i^k)) &\geq \int_{u_R(x_i^k)}^{u_R(y_i^k)} \sqrt{W(v)} \, dv \\ &\geq \int_{k+|u_R(x_i^k)-k|}^{k+(u_R(y_i^k)-k)} \sqrt{W(v)} \, dv \\ &\geq \int_{k+\frac{\delta}{2}}^{k+\delta} \sqrt{W(v)} \, dv \\ &\geq \sqrt{\gamma_1} \left[\frac{v^2}{2} \right]_{\frac{\delta}{2}}^{\delta} \\ &\geq C\delta^2, \end{aligned}$$

with $C = \frac{3}{8}\sqrt{\gamma_1}$, where we have used Lemma 4.4.3 with $\delta < \frac{1}{2}$. The case $u_R(y_i^k) < u_R(x_i^k) \leq k + \frac{\delta}{2}$ is similar.

Therefore $TV(\beta(u_R), I_i^k) \geq |\beta(u_R(y_i^k)) - \beta(u_R(x_i^k))| \geq C\delta^2$.

b. otherwise $|u_R(x) - k| \leq \delta$ for every $x \in I_i^k$.

Then we define two subsets of indices following the dichotomy :

$$M_a^k := \left\{ i \in \{1, \dots, N_\delta^k\}, \exists y_i^k \in I_i^k \text{ such that } |u_R(y_i^k) - k| > \delta \right\},$$

$$M_b^k := \left\{ i \in \{1, \dots, N_\delta^k\} \text{ such that } |u_R(x) - k| \leq \delta \quad \forall x \in I_i^k \right\},$$

and the associated sets

$$E_\alpha^k = \bigcup_{i \in M_\alpha^k} I_i^k \quad \text{where } \alpha = a, b.$$

Then we have : $E_\delta^k \subset E_a^k \cup E_b^k$.

Step 6 : Total variation and $|E_a^k| \leq C'_\delta$

$$TV(\beta(u_R), \Omega_R) \geq \sum_{i \in M_a^k \setminus \{N_\delta^k\}} TV(\beta(u_R), I_i^k) \geq C\delta^2 (\text{card}(M_a^k) - 1),$$

then $\text{card}(M_a^k) \leq 1 + \frac{M'}{C\delta^2}$. Consequently, we have :

$$\begin{aligned} |E_a^k| &\leq \sum_{i \in M_a^k} |I_i^k| \\ &= 2\text{card}(M_a^k) \\ &\leq 2 \left(1 + \frac{M'}{C\delta^2} \right) =: C'_\delta. \end{aligned}$$

Step 7 : Replacement of E^k by a large interval

Let us consider the convex hull : $\hat{E}_b^k = \text{conv} \left(\bigcup_{i \in M_b^k} I_i^k \right) = \text{conv}(E_b^k)$.

If $\text{card}(M_b^k) \geq 2$, just take the first and last interval, and then apply the Cleaning lemma 4.5.2 to obtain :

$$|u(x) - k| \leq \nu\delta \quad \text{on } \hat{E}_b^k, \quad (4.6.2)$$

where ν given in Lemma 4.5.2.

If $\text{card}(M_b^k) = 1$ then (4.6.2) is still true.

Step 8 : Definition of L_δ

Let $R_\delta = \Omega_R \setminus \bigcup_{k \in \mathbb{K}} \hat{E}_b^k$.

We have $\Omega_R = \bigcup_{k \in \mathbb{K}} E^k \subset \bigcup_{k \in \mathbb{K}} (E_a^k \cup \hat{E}_b^k)$.

Then $R_\delta \subset \Omega_R \cap \left(\cup_{k \in \mathbb{K}} E_a^k\right)$. As by step 6, $|E_a^k| \leq C'_\delta$ thus

$$|R_\delta| \leq \text{card}(\mathbb{K})C'_\delta \leq NC'_\delta =: L_\delta,$$

with N independent on R , bounded in (4.6.1).

Step 9 : Conclusion

Let us define the finite sequence $(k_i)_{1 \leq i \leq K}$, $\tilde{I}_i = \hat{E}_b^{k_i} \cap \Omega_R \neq \emptyset$ such that $\sup \tilde{I}_i \leq \inf \tilde{I}_{i+1}$ (for δ small enough) and $\left(\cup_{k \in \mathbb{K}} \hat{E}_b^k\right) \cap \Omega_R = \cup_{i=1, \dots, K} \tilde{I}_i$. Notice that $R_\delta = \Omega_R \setminus \cup_{i=1, \dots, K} \tilde{I}_i$.

We have for $R \geq 1$ and $k_{K+1} = \ell + k_1$:

$$\begin{aligned} 2M' &\geq TV(\beta(u_R), (-R, 3R)) \\ &\geq \inf_{(e_j)_j} \sum_{j=1}^K |\beta(k_{j+1} + e_{j+1}) - \beta(k_j + e_j)| \\ &\geq \sum_{j=1}^K \max\left(0, |\beta(k_j + 1) - \beta(k_j)| - 2C_2\nu^2\delta^2\right) \\ &\geq \sum_{j=1}^K \max\left(0, \beta(1)|k_{j+1} - k_j| - 2C_2\nu^2\delta^2\right), \end{aligned}$$

where we have used in the third line the fact that $|e_j| \leq \nu\delta$ and in the fourth line we have used (4.4.5).

Because $|k_{j+1} - k_j| \geq 1$ for $1 \leq j \leq K-1$ and for $2C_2\nu^2\delta^2 \leq \frac{\beta(1)}{2}$, we get $K \leq 1 + \frac{4M'}{\beta(1)}$.

Notice that we may have $k_{K+1} = k_K$ for $k_{K+1} = k_1 + \ell$.

Moreover,

$$\sum_{j=1}^K |k_{j+1} - k_j| \leq \frac{2M'}{\beta(1)} + \frac{K}{2} \leq 1 + \frac{4M'}{\beta(1)}.$$

Then, we can take

$$\bar{N} := \max\left(2M', 1 + \frac{4M'}{\beta(1)}\right)$$

□

4.7 The limit as R goes to infinity

4.7.1 The energy E

Proof of Proposition 4.1.3

Proof of (i).

From (4.3.3), we have for all $\psi \in C_c^\infty(\mathbb{R})$

$$J(\psi) \geq \frac{c^2 - 1}{2} \int_{\mathbb{R}} (\psi')^2 dx + \int_{\mathbb{R}} W(\psi) dx \geq 0 = J(0).$$

Therefore $0 = J(0) \leq \inf_{\psi \in C_c^\infty(\mathbb{R})} J(\psi) = E(0)$ and then $E(0) = 0$.

Proof of (ii).

Let $k \in \mathbb{Z}$ and $\psi \in C_c^\infty(\mathbb{R})$. Notice that if $u \in H$, then

$$J(u(-\cdot)) = J(u) \quad \text{and} \quad J(u + l) = J(u) \text{ if } l \in \mathbb{Z}.$$

Therefore

$$\begin{aligned} J(-k\varphi + \psi) &= J(-k\varphi(-\cdot) + \psi(-\cdot)) \\ &= J(k(-\varphi(-\cdot)) + \psi(-\cdot)) \\ &= J(k(1 - \varphi(-\cdot)) + \psi(-\cdot)) \\ &= J(k\varphi + \psi(-\cdot)), \end{aligned}$$

where, we have used (4.1.5).

Thus

$$E(-k) = \inf_{\psi} J(-k\varphi + \psi) = \inf_{\psi(-\cdot)} J(k\varphi + \psi(-\cdot)) = E(k).$$

Proof of (iii).

Let $k \in \mathbb{Z}$. We know that $E(k) = \inf_{\psi \in C_c^\infty(\mathbb{R})} J(k\varphi + \psi)$.

Then for all $\delta > 0$ and $j \in \mathbb{Z}$ there exist $\psi_\delta^j \in C_c^\infty(\mathbb{R})$ such that

$$|E(j) - J(j\varphi + \psi_\delta^j)| \leq \delta. \quad (4.7.1)$$

Let

$$\psi^k = (k-p)\varphi(\cdot+n) + \psi_\delta^{k-p}(\cdot+n) + p\varphi(\cdot-n) + \psi_\delta^p(\cdot-n) - k\varphi \in C_c^\infty(\mathbb{R}),$$

then for $p \in \mathbb{Z}$, we have

$$\begin{aligned} E(k) &\leq J(k\varphi + \psi^k) \\ &\leq J\left((k-p)\varphi(\cdot+n) + \psi_\delta^{k-p}(\cdot+n) + p\varphi(\cdot-n) + \psi_\delta^p(\cdot-n)\right) \\ &\leq J\left(k\varphi(\cdot+n) + \psi_\delta^{k-p}(\cdot+n)\right) + J\left(p\varphi(\cdot-n) + \psi_\delta^p(\cdot-n)\right) \\ &\leq E(k-p) + E(p) + 2\delta, \end{aligned}$$

where we choose n in the third line, large enough such that for $|a| \leq \frac{1}{2}$

$$\text{supp}\left(\psi_\delta^{k-p}(\cdot+n+a)\right) \cap \text{supp}\left(\psi_\delta^{k-p}(\cdot-n-a)\right) = \emptyset$$

and in the last line we applied (4.7.1).

Let tend δ to zero in the last inequality, we deduce that

$$E(k) \leq E(k-p) + E(p).$$

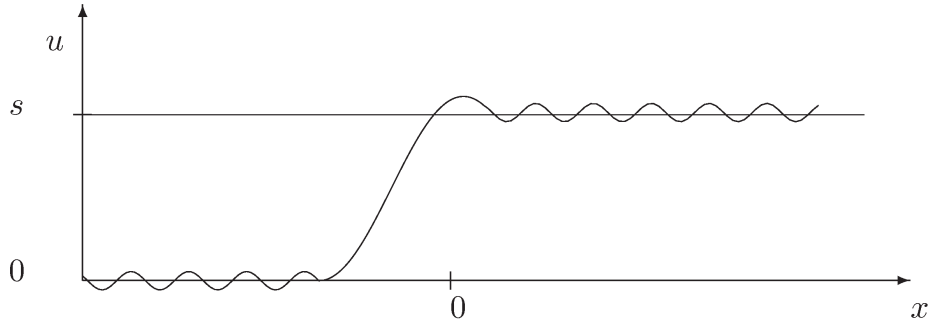
□

4.7.2 Limit as $R \rightarrow +\infty$

Proposition 4.7.1 (*The limit $R \rightarrow \infty$ and bound on the energy*)

For any $\ell \in \mathbb{Z} \setminus \{0\}$, then there is $s \in \mathbb{Z} \setminus \{0\}$, such that there exists a solution u of (4.1.3)-(4.1.4), with $u(+\infty) - u(-\infty) = s$, and up to a subsequence $u_R \rightarrow u$ as $R \rightarrow +\infty$, when u_R is given in Proposition 4.3.4.

Moreover, if $\ell \in 1 + 2\mathbb{Z}$, then we can choose $s \in 1 + 2\mathbb{Z}$. The solution u is pictured on Figure 4.7.2.1.

FIGURE 4.7.2.1 – Schematic graph of the function u .**Proof of Theorem 4.1.4.**

Apply Proposition 4.7.1.

Proof of Proposition 4.7.1.**Step 1 : Normalization of u_R .**

We fix $\ell \in \mathbb{Z} \setminus \{0\}$. We first notice that

$$\left| \bigcup_{i=1}^K \tilde{I}_i \right| \geq 2R - L_\delta.$$

Then there exists at least one interval \tilde{I}_{i_0} such that

i. $|\tilde{I}_{i_0}| \geq \frac{2R - L_\delta}{\bar{N}} \rightarrow +\infty$ as $R \rightarrow +\infty$, with $K \leq \bar{N}$.

ii. $|u - k_{i_0}| \leq C\delta$ on \tilde{I}_{i_0} .

iii. For $k_{K+1} = k_1 + \ell$, then

$$\ell = \sum_{i=1}^K (k_{i+1} - k_i). \quad (4.7.2)$$

By periodicity $\tilde{I}_{i_0} + 2R$ is another interval of length going to $+\infty$ as $R \rightarrow +\infty$, such that $|u - (k_{i_0} + \ell)| \leq C\delta$ on $\tilde{I}_{i_0} + 2R$.

Calling z_{i_0} the center of \tilde{I}_{i_0} and shifting z_{i_0} to $-R$, we get a solution still denoted by u_R as on the picture (see Figure 4.7.2.2).

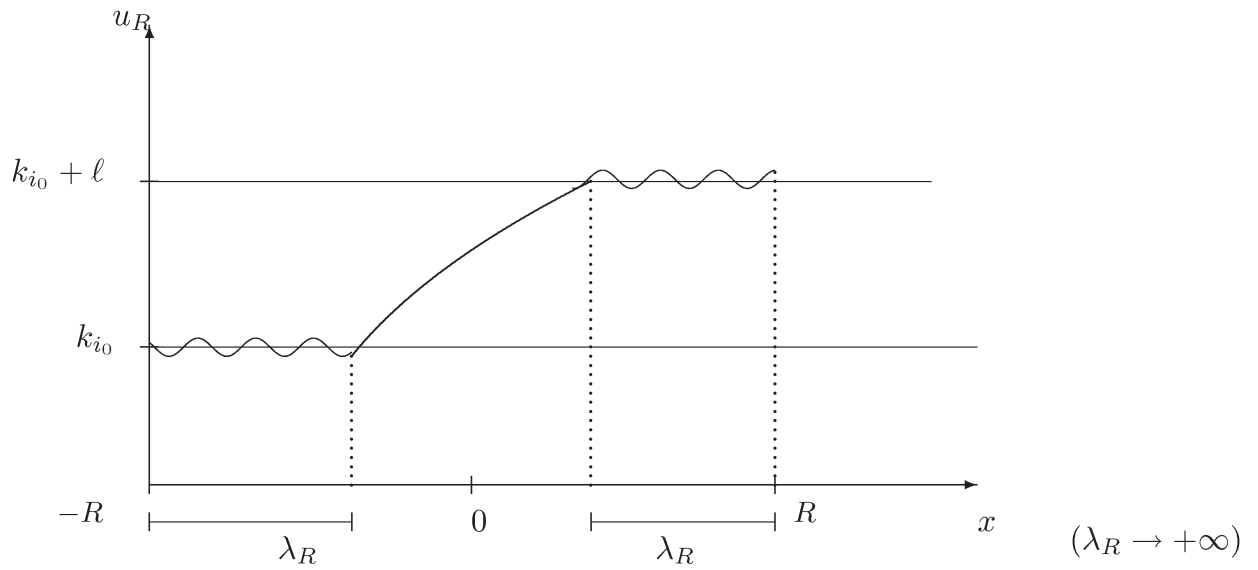


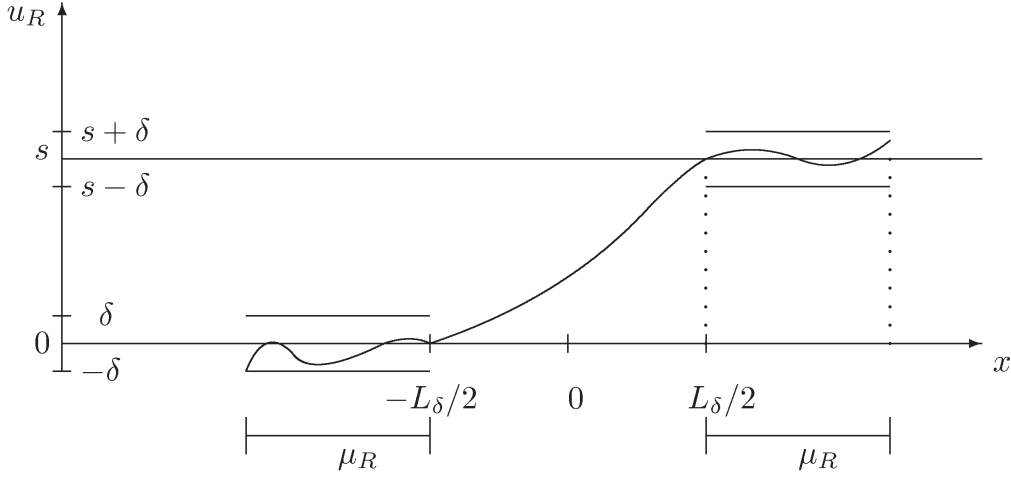
FIGURE 4.7.2.2 – Schematic graph of the function u_R

We may have the other intervals $I \subset (-R, R)$ of length going to $+\infty$, such that $|u - k| \leq C\delta$ on I .
 Recall that

$$|\{x \in \Omega_R, \text{dist}(u(x), \mathbb{Z}) > C\delta\}| \leq L_\delta.$$

Step 2 : Definition of s

Therefore, there exists at least some $s \in \mathbb{Z} \setminus \{0\}$, with $s = s_{i_0}$ is fixed, such that up to extract a subsequence and to replace u_R by $u_R(x + a) + n$ with $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have a solution u_R of (4.1.3) like on the picture (see Figure 4.7.2.3).

FIGURE 4.7.2.3 – Graph of the “ s -transition u_R ”

with

$$\begin{cases} |u_R| \leq C\delta & \text{on } (-\mu_R - L_\delta/2, -L_\delta/2) \\ |u_R - s| \leq C\delta & \text{on } (L_\delta/2, L_\delta/2 + \mu_R) \\ |u_R| \leq \bar{N} + C\delta & \text{on } (-\mu_R - L_\delta/2, \mu_R + L_\delta/2), \end{cases} \quad (4.7.3)$$

with $\mu_R \rightarrow +\infty$ as $R \rightarrow +\infty$.

Notice that because of (4.7.2), if $\ell = 1 \pmod{2}$ then there exists $i_0 \in \{1, \dots, K\}$ such that $s_{i_0} = k_{i_0} + \dots + k_{i_0} = 1 \pmod{2}$ and then we can choose $s = 1 \pmod{2}$. Therefore, we can pass to the limit in (4.1.3) and get a solution u of (4.1.3) satisfying (4.7.3) with $\mu_R = +\infty$.

From Proposition 2.1, we deduce that $u(-\infty) = 0$ and $u(+\infty) = s$. \square

Proposition 4.7.2 (*ℓ is not stable*)

Let u be the function constructed in Proposition 4.7.1. Under assumption of Proposition 4.7.1, we have

$$J(u) = E(s), \quad (4.7.4)$$

and

$$E(s) + E(\ell - s) \leq E(\ell). \quad (4.7.5)$$

Consequently if $s \neq \ell$, then ℓ is not stable.

Proof.

We will prove (4.7.5) in the six first steps and (4.7.4) in the seventh step.

Step 1 : Preliminaries

Let φ the function defined in the introduction (see subsection 1.2). Then we set

$$\varphi_R(x) := \sum_{k \geq 0} \varphi(x - 2kR) + \sum_{k < 0} (\varphi(x - 2kR) - 1), \quad (4.7.6)$$

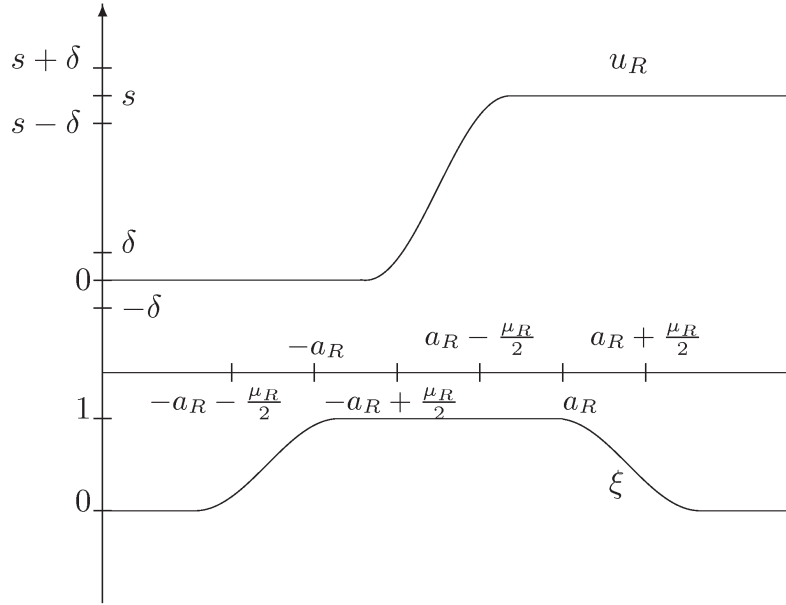
and check that $\ell\varphi_R \in H_R$.

Step 2 : $J_R(\ell\varphi_R + \psi) = J(\ell\varphi + \psi)$ for all $\psi \in C_c^\infty(\mathbb{R})$.

Straightforward, because $\varphi_R = \varphi$ on $[-R - 1, R + 1]$ for $R \geq 2$.

Step 3 : Splitting of u_R in “two phase transitions” \tilde{u}_R and \hat{u}_R

We use the same notations as in Step 2 of the proof of Proposition 4.7.1. In particular, we are in the situation of Figure 4.7.2.3.

FIGURE 4.7.2.4 – Possible graphs for u_R and ξ .

Let $a_R = \frac{\mu_R + L\delta}{2}$. Let $\xi \in C_c^\infty(\mathbb{R})$ such that

$$\xi = 1 \text{ on } \left[-a_R + \frac{1}{2}, a_R - \frac{1}{2}\right] \quad \text{and} \quad \text{supp}(\xi) \subset \left[-a_R - \frac{1}{2}, a_R + \frac{1}{2}\right].$$

Set

$$\xi_R(x) = \sum_{k \geq 0} \xi(x - 2kR) + \sum_{k \leq -1} \xi(x - 2kR)$$

and

$$\begin{cases} \tilde{\varphi}_R^s &= s\phi_R, \\ \tilde{\varphi}_R^{\ell-s} &= (\ell - s)\phi_R(\cdot - R), \\ \varphi_R^\ell &= \tilde{\varphi}_R^s + \tilde{\varphi}_R^{\ell-s}. \end{cases}$$

Then we can write u_R as

$$u_R = \tilde{u}_R + \hat{u}_R, \tag{4.7.7}$$

where

$$\tilde{u}_R = \xi_R (u_R - \varphi_R^\ell) + \tilde{\varphi}_R^s \quad \text{and} \quad \hat{u}_R = (1 - \xi_R) (u_R - \phi_R^\ell) + \hat{\varphi}_R^{\ell-s}.$$

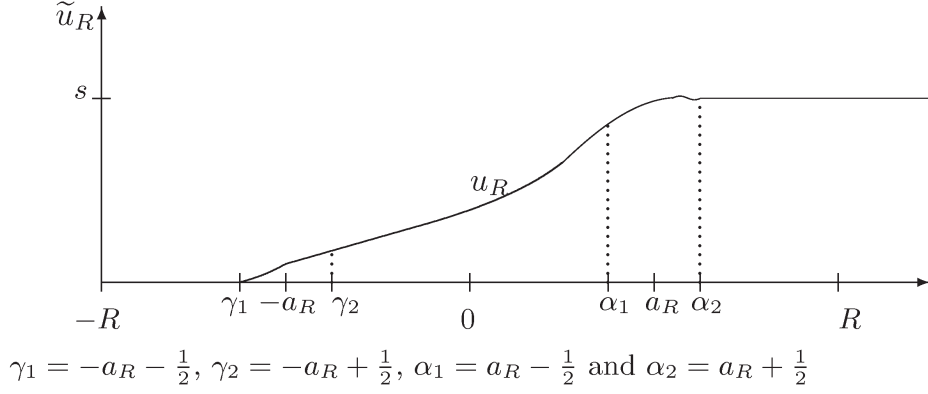


FIGURE 4.7.2.5 – Schematic graph for \tilde{u}_R .

We have

$$\Lambda(\tilde{u}_R + \hat{u}_R) - \Lambda(\tilde{u}_R) - \Lambda(\hat{u}_R) = c^2 \tilde{u}'_R \hat{u}'_R - D\tilde{u}_R D\hat{u}_R + W(\tilde{u}_R + \hat{u}_R) - W(\tilde{u}_R) - W(\hat{u}_R),$$

with

$$\left[\text{supp}(\tilde{u}'_R) \cup \text{supp}(D\tilde{u}_R) \cup \text{supp}(W(\tilde{u}_R)) \right] \cap \bar{\Omega}_R \subset [-a_R - 1, a_R + 1] =: \tilde{I}_R,$$

and

$$\left[\text{supp}(\hat{u}'_R) \cup \text{supp}(D\hat{u}_R) \cup \text{supp}(W(\hat{u}_R)) \right] \cap \bar{\Omega}_R \subset \bar{\Omega}_R \setminus [-a_R - 1, a_R + 1] =: \hat{I}_R.$$

Therefore

$$J_R(u_R) - J_R(\tilde{u}_R) - J_R(\hat{u}_R) = \int_{\tilde{I}_R \cap \hat{I}_R} c^2 \tilde{u}'_R \hat{u}'_R - D\tilde{u}_R D\hat{u}_R + W(\tilde{u}_R + \hat{u}_R) - W(\tilde{u}_R) - W(\hat{u}_R).$$

Let us define $I_- = [-a_R - 1, a_R + 1]$, $I_+ = [a_R - 1, a_R + 1]$. Then $\tilde{I}_R \cap \hat{I}_R = I_- \cup I_+$.

From (4.7.3) and the exponential decay property of u_R (see Proposition 4.2.1 (iii)) on $[-L_\delta/2 - \mu_R, -L_\delta/2]$ and on $[L_\delta/2, L_\delta/2 + \mu_R]$ and using the PDE (4.1.3) satisfied by u_R , we get that

$$|u'_R|, |Du_R|, |W(u_R)| \leq C'\delta e^{-\lambda\frac{\mu_R}{2}} \quad \text{on } I_\pm.$$

Therefore, we can easily get

$$J_R(\tilde{u}_R) + J_R(\hat{u}_R) \leq J_R(u_R) + C\delta e^{-\lambda\frac{\mu_R}{2}}. \quad (4.7.8)$$

Step 4 : $E(s) \leq J(\tilde{u})$

We define

$$\tilde{u}(x) = \begin{cases} u_R(-R) = 0 & \text{for } x \leq -R, \\ u_R(x) & \text{for } -R \leq x \leq R, \\ u_R(\cdot + R) = s & \text{for } x \geq R. \end{cases}$$

Notice that \tilde{u} is constant on $(-\infty, -R + 1]$ and on $[R - 1, +\infty)$. Therefore, $J(\tilde{u}) = J_R(u_R)$.

Moreover, $\tilde{u} \in s\varphi + C_c^\infty(\mathbb{R})$. Therefore

$$E(s) \leq J(\tilde{u}).$$

Step 5 : $E(\ell - s) \leq J(\hat{u})$

We define

$$\hat{u}(x) = \begin{cases} \hat{u}_R(-R + R) = 0 & \text{for } x \leq -R, \\ \hat{u}_R(x + R) & \text{for } -R \leq x \leq R, \\ \hat{u}_R(R + R) = \ell - s & \text{for } x \geq R. \end{cases}$$

Similarly, we have

$$\begin{aligned} E(\ell - s) \leq J(\hat{u}) &= \int_0^{2R} \Lambda(\hat{u}_R)(x) dx \\ &= J_R(\hat{u}_R), \end{aligned}$$

because \hat{u}_R is linear + $2R$ -periodic.

Step 6 : Conclusion

From (4.7.8) and Steps 4 and 5, we have

$$\begin{aligned} E(s) + E(\ell - s) &\leq J(\tilde{u}) + J(\hat{u}) \\ &= J_R(\tilde{u}_R) + J_R(\hat{u}_R) \\ &\leq J_R(u_R) + C\delta e^{-\lambda\frac{\mu R}{2}} \\ &\leq J_R(\ell\varphi_R + \psi) + C\delta e^{-\lambda\frac{\mu R}{2}} \\ &= J_R(\ell\varphi + \psi) + C\delta e^{-\lambda\frac{\mu R}{2}}, \end{aligned}$$

because $\delta > 0$ is arbitrarily small, we get for all $\psi \in C_c^\infty(\mathbb{R})$

$$E(s) + E(\ell - s) \leq J(\ell\varphi + \psi). \quad (4.7.9)$$

Consequently

$$E(s) + E(\ell - s) \leq E(\ell).$$

Step 7.1 : $J_R(u_R) \leq E(s) + E(\ell - s) + \eta$ for $R \geq R(\eta)$

For every $\eta > 0$ there exists $\psi_\eta^s \in C_c^\infty(\mathbb{R})$ such that

$$J(s\varphi + \psi_\eta^s) \leq E(s) + \eta,$$

and there exists $\psi_\eta^{\ell-s} \in C_c^\infty(\mathbb{R})$ such that

$$J((\ell - s)\varphi + \psi_\eta^{\ell-s}) \leq E(\ell - s) + \eta.$$

For $R > 0$ large enough ($R \geq 2$)

$$\text{supp } \psi_\eta^s \subset [-R/4, R/4],$$

and

$$\text{supp } \psi_\eta^{\ell-s} \subset [-R/4, R/4].$$

We set

$$\begin{aligned}\tilde{u}^s(x) &= s\varphi_R(x) + \sum_{k \in \mathbb{Z}} \psi_\eta^s(x - 2kR), \\ \hat{u}^{\ell-s}(x+R) &= (\ell-s)\psi_R(x) + \sum_{k \in \mathbb{Z}} \psi_\eta^{\ell-s}(x - 2kR),\end{aligned}$$

and

$$\bar{u} = \tilde{u}^s + \hat{u}^{\ell-s} \in H_R.$$

Therefore, $(\tilde{u}^s)'(x+a)(\hat{u}^{\ell-s})'(x) = 0$ for every $a \in [0, 1]$.

$$\begin{aligned}J_R(u_R) &\leq J_R(\bar{u}) \\ &= J_R(\tilde{u}^s) + J_R(\hat{u}^{\ell-s}(\cdot + R)) \\ &= J_R(s\varphi + \psi_\eta^s) + J_R((\ell-s)\varphi + \psi_\eta^{\ell-s}) \\ &\leq E(s) + E(\ell-s) + 2\eta,\end{aligned}$$

for any $R \geq R(\eta)$.

Step 7.2 : $J_R(\tilde{u}_R) \leq E(s) + \eta$ for $R \geq R(\eta)$

We recall that

$$J_R(\tilde{u}_R) + J_R(\hat{u}_R) \leq J_R(u_R) + C\delta e^{-\lambda \frac{\mu R}{2}}.$$

This implies that

$$J_R(\tilde{u}_R) + J_R(\hat{u}_R) \leq E(s) + E(\ell-s) + 2\eta + C\delta e^{-\lambda \frac{\mu R}{2}},$$

for $R \geq R(\delta, \eta)$.

On the other hand, we have

$$E(s) \leq J(\tilde{u}) = J_R(\tilde{u}_R) \quad \text{and} \quad E(\ell-s) \leq J(\hat{u}) = J_R(\hat{u}_R).$$

This implies

$$0 \leq J_R(\tilde{u}_R) - E(s) \leq 2\eta$$

and

$$0 \leq J_R(\hat{u}_R) - E(\ell-s) \leq 2\eta.$$

Step 7.3 : $J(u) \leq E(s)$

For any $A > 0$, and $a_R - 1 \geq A$, we get in particular :

$$J_A(\tilde{u}_R) \leq E(s) + 2\eta,$$

$$\tilde{u}_R = u_R \quad \text{on } [-A, A]$$

and

$$D(\tilde{u}_R) = D(u_R) \quad \text{on } [-A, A].$$

This implies that

$$J_A(u_R) \leq E(s) + 2\eta.$$

Passing to the limit $R \rightarrow +\infty$, we get

$$J_A(u) \leq E(s) + 2\eta.$$

Because $\eta > 0$ is arbitrarily small, we get

$$J_A(u) \leq E(s).$$

This implies (with $A \rightarrow \infty$) that

$$J(u) \leq E(s).$$

□

4.8 Stability

Lemma 4.8.1 *For every function $u \in H_{loc}^1(\mathbb{R})$, such that $(u')^2$ has exponential decay at infinity. Then we have*

$$\int_{\mathbb{R}} [D(u)(x)]^2 dx \leq \int_{\mathbb{R}} (u')^2 dx.$$

Proof. Let $u \in H_{loc}^1(\mathbb{R})$, such that $(u')^2$ has exponential decay at infinity. For every $x \in \mathbb{R}$, we have $Du(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} u'(s) ds$. Therefore

$$\begin{aligned} \int_{\mathbb{R}} [Du(x)]^2 dx &= \int_{\mathbb{R}} dx \left(\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} u'(s) ds \right)^2 \\ &\leq \int_{\mathbb{R}} dx \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} (u'(s))^2 ds \end{aligned}$$

$$\leq \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} dx \int_{\mathbb{R}} (u'(s))^2 ds = \int_{\mathbb{R}} (u'(s))^2 ds,$$

where in the second line, we used Holder's inequality. In the third line, we used Fubini's theorem.

□

Proof of Theorem 4.1.10

To show that the integer 1 is stable, we have to prove the following inequality

$$E(1) < E(s) + E(1-s) \quad \text{for every } s \in \mathbb{Z} \setminus \{0, 1\}. \quad (4.8.1)$$

Step 1 : Lower bound on $E(s)$

Using Lemma 4.8.1, we get for any $u \in s\varphi + C_c^\infty(\mathbb{R})$

$$J(u) \geq \frac{(c^2 - 1)}{2} \int_{\mathbb{R}} (u'(x))^2 dx + \int_{\mathbb{R}} W(u(x)) dx.$$

By Cauchy-Schwarz inequality, we obtain

$$J(u) \geq \sqrt{2(c^2 - 1)} \int_{-\infty}^{+\infty} |u'(x)| \sqrt{W(u(x))} dx.$$

Making the change of variable $y = u(x)$, we obtain

$$J(u) \geq \sqrt{2(c^2 - 1)} \int_0^{|s|} |\beta'(y)| dy,$$

where $\beta(y) = \int_0^y \sqrt{W(v)} dv$.

If $|s| \neq 0$, then setting $z = \frac{y}{|s|}$, we get

$$J(u) \geq |s| \sqrt{2(c^2 - 1)} \beta(1).$$

Therefore

$$E(s) \geq |s| \sqrt{2(c^2 - 1)} \beta(1).$$

Step 2 : Upper bound of $E(1)$

Let $\phi \in C^2(\mathbb{R})$ be a function satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = 1$, and $\frac{c^2}{2} (\phi'(x))^2 = W(\phi(x))$. This function exists because of our assumption (4.1.2).

Then, necessarily

$$\begin{aligned} J(\phi) &< \frac{c^2}{2} \int_{\mathbb{R}} (\phi'(x))^2 dx + \int_{\mathbb{R}} W(\phi(x)) \\ &= c\sqrt{2} \int_{-\infty}^{+\infty} |\phi'(x)| \sqrt{W(\phi(x))} dx = c\sqrt{2} \int_0^1 \beta'(u) du \\ &= c\sqrt{2}\beta(1). \end{aligned}$$

Using the fact that ϕ can be approximated by $\varphi + \psi$ with $\psi \in C_c^\infty(\mathbb{R})$, we deduce that for any $\delta > 0$, there exists $\psi^\delta \in C_c^\infty(\mathbb{R})$ such that

$$|J(u_R) - J(\varphi + \psi^\delta)| \leq \delta.$$

Therefore,

$$\begin{aligned} E(1) &\leq J(\varphi + \psi^\delta) \\ &\leq J(\varphi) + \delta, \end{aligned}$$

this implies

$$E(1) \leq J(\varphi) \leq c\sqrt{2}\beta(1).$$

Step 3. Conclusion

Equality in (4.8.1) never holds for $s = -1, 0, 1, 2$. Therefore, inequality (4.8.1) is satisfied if $c\sqrt{2}\beta(1) \leq (|s| + |1 - s|)\beta(1)\sqrt{2(c^2 - 1)}$, for $s \in \mathbb{Z} \setminus \{-1, 0, 1, 2\}$, if $c \leq \sqrt{c^2 - 1}(|s| + |1 - s|)$, i.e. if $c \leq s\sqrt{c^2 - 1}$.

Consequently (4.8.1) is true if $c \geq \sqrt{\frac{25}{24}}$.

□

Theorem 4.8.2 (*Relation between stability and transition*)

If $\ell \in \mathbb{Z}$ is stable with respect to E then the solution u , constructed in Proposition 4.7.1, is a ℓ -transition.

Proof of Theorem 4.1.8. Apply Theorem 4.8.2.

Proof of Theorem 4.8.2.

Let $\ell \in \mathbb{Z} \setminus \{0\}$ stable. Construct an u_R such that $u_R(x + 2R) = u_R(x) + \ell$. By Proposition 4.7.1 and Proposition 4.7.2 there exists a solution u which has a s -transition with $s = \ell$.

□

Set

$$I = \arg \inf_{k \in 1+2\mathbb{Z}} E(k).$$

Note that by Step 1 of the proof of Theorem 4.1.10, we have $I \neq \emptyset$.

Lemma 4.8.3 (*Stability of the elements of I*)

k is stable for every $k \in I$.

Proof.

Let $k \in I$, we want to show that for any $\ell \in \mathbb{Z} \setminus \{0, k\}$

$$E(k) < E(k - \ell) + E(\ell). \quad (4.8.2)$$

By the definition of E , we have

$$E(k) \leq E(k - l) + E(l) \quad \forall l \in \mathbb{Z} \setminus \{0, k\}. \quad (4.8.3)$$

By definition of I , for any $k' \in 1 + 2\mathbb{Z}$, we have

$$E(k') \geq \inf_{p \in 1+2\mathbb{Z}} E(p) = E(k). \quad (4.8.4)$$

Let $l \in \mathbb{Z} \setminus \{0\}$.

Case 1 : $l \in 2\mathbb{Z} \setminus \{0\}$

Then $E(l) > 0$ and taking $k' = k - l$ in (4.8.4), we get $E(k - l) \geq E(k)$. Then (4.8.1) implies (4.8.2).

Case 2 : $l \in 1 + 2\mathbb{Z} \setminus \{k\}$

Then $E(k - l) > 0$ and taking $k' = l$ in (4.8.4), we get $E(l) \geq E(k)$. Similarly (4.8.3) implies (4.8.2), and then it shows that k is stable.

□

Proof of Theorem 4.1.9. Apply Lemma 4.8.3.

Set

$$J = \arg \inf_{k \in 2\mathbb{Z} \setminus \{0\}} E(k).$$

Lemma 4.8.4 (*Stability of elements of J*)

If

$$\inf_{k \in 2\mathbb{Z} \setminus \{0\}} E(k) < 2 \left(\inf_{k \in 1+2\mathbb{Z}} E(k') \right), \quad (4.8.5)$$

then any $k \in J$ is stable.

Proof. Let $k \in J$ and $l \in \mathbb{Z}$, then we have two possibilities.

Case 1 : $l \in 2\mathbb{Z} \setminus \{0, k\}$

This implies that $0 < E(k) \leq E(l)$. On the other hand, we have also $k - l \in 2\mathbb{Z} \setminus \{0\}$. Hence

$$E(k) < E(k - l) + E(l).$$

Case 2 : $l \in 1 + 2\mathbb{Z} \setminus \{0\}$

Then $k - l \in 1 + 2\mathbb{Z} \setminus \{0\}$ and

$$E(l) \geq E(k') \text{ and } E(k - l) \geq E(k') \quad \text{for all } k' \in I. \quad (4.8.6)$$

If

$$E(k) = E(k - l) + E(l), \quad (4.8.7)$$

then by equations (4.8.6), we have $E(k) \geq 2E(k')$ for all $k' \in I$. This contradicts assumption (4.8.5). Consequently $E(k) < E(k - l) + E(l)$.

This shows that k is stable.

□

4.9 Appendix

4.9.1 First integral

Lemma 4.9.1 (*First integral for the solutions of equation (4.1.3)*)

For all $c \in \mathbb{R}$, every solution u of equation (4.1.3) satisfies the following first integral :

$$\sum_{i \in \mathbb{Z}} c^2 \frac{u'^2(x+i)}{2} - W(u(x+i)) + \frac{1}{2} (u(x+i+1) - u(x+i))^2 = \text{constant}.$$

Proof. Let u a solution of equation (4.1.3), and set :

$$P := \sum_{i \in \mathbb{Z}} c^2 \frac{u'^2(x+i)}{2} + W(u(x+i)) + \frac{1}{2} (u(x+i+1) - u(x+i))^2,$$

then

$$\begin{aligned} P' &= \sum_{i \in \mathbb{Z}} c^2 u'(x+i) u''(x+i) - W'(u(x+i)) u'(x+i) \\ &\quad + (u(x+i+1) - u(x+i)) (u'(x+i+1) - u'(x+i)) \\ &= \sum_{i \in \mathbb{Z}} [u(x+i+1) + u(x+i-1) - 2u(x+i) + W'(u(x+i))] u'(x+i) \\ &\quad - W'(u(x+i)) u'(x+i) + [u(x+i+1) - u(x+i)] [u'(x+i+1) - u'(x+i)] \\ &= \sum_{i \in \mathbb{Z}} [u(x+i-1) - u(x+i)] u'(x+i) + [u(x+i+1) - u(x+i)] u'(x+i+1) \\ &= 0, \end{aligned}$$

where we used equation (4.1.3) in the second line.

Hence $P = \text{constant}$, which completes the proof.

□

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