Analyse de quelques problèmes elliptiques et paraboliques semi-linéaires

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Cette thése est dédiée à mes parents et ma femme.

谨以此论文献给我的父母和太太。

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Résumé

Cette thèse porte sur l'analyse de quelques problèmes elliptiques ou paraboliques semi-linéaires. Elle se compose de deux parties indépendantes. Dans la première partie, on considère le système de réaction-diffusion-advection suivant

$$
(P^\varepsilon) \qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + \dfrac{1}{\varepsilon^2} f(u) & \text{dans } \Omega \times (0,T] \\ v_t = -\lambda m v & \text{dans } \Omega \times (0,T] \\ m_t = d\Delta m + u - m & \text{dans } \Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega \\ v(x,0) = v_0(x) & x \in \Omega \\ m(x,0) = m_0(x) & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial m}{\partial \nu} = 0 & \text{sur } \partial\Omega \times (0,T] \end{cases}
$$

qui est un modèle d'haptotaxie, mécanisme lié à la dissémination de tumeurs cancéreuses. Le résultat principal concerne la convergence de la solution du système (P^ε) vers la solution d'un problème à frontière libre (P^0) qui est bien définie.

Dans la seconde partie, on considère une classe générale d'équations elliptiques du type Hénon

$$-\Delta u = |x|^\alpha f(u) \qquad \text{dans} \quad \Omega \subset \mathbb{R}^N,$$

où $\alpha > -2$. On examine deux cas classiques : $f(u) = e^u$, $|u|^{p-1}u$ et deux autres cas : $f(u) = u_+^p$ puis $f(u)$ nonlinéarité générale. En étudiant les solutions stables en dehors d'un ensemble compact (en particulier, solutions stables et solutions avec indice de Morse fini) avec différentes méthodes, on obtient des résultats de classification.

Mots-clefs

Système de réaction-diffusion-advection, Haptotaxie, Problème à frontière libre, Principe de comparaison,
Théorème de Liouville, Équation du type Hénon, Stabilité, Solution avec indice de Morse fini, Comportement asymptotique.

Analysis of some semi-linear elliptic and parabolic problems

Abstract

This thesis focuses on the analysis of some semi-linear elliptic and parabolic problems, which is divided into two main parts. In the first part, we consider the following reaction-diffusion-taxis system

$$(P^\varepsilon)\qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + \dfrac{1}{\varepsilon^2} f(u) & \text{in } \Omega \times (0,T] \\ v_t = -\lambda m v & \text{in } \Omega \times (0,T] \\ m_t = d\Delta m + u - m & \text{in } \Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega \\ v(x,0) = v_0(x) & x \in \Omega \\ m(x,0) = m_0(x) & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T] \end{cases}$$

which is a haptotaxis model - a mechanism related to the spread of cancer cells. The main result concerns the convergence of the solution of System (P^ε) to the solution of a free boundary problem (P^0), where system (P^0) is well-posed.

In the second part, we consider a general class of Hénon type elliptic equations

$$-\Delta u = |x|^\alpha f(u) \qquad \text{in} \quad \Omega \subset \mathbb{R}^N$$

with $\alpha > -2$. We investigate two classical cases $f(u) = e^u$, $|u|^{p-1}u$ and two others cases $f(u) = u_+^p$, or $f(u)$ a general function. By studying the solutions which are stable outside a compact set (in particular, stable solutions and finite Morse index solutions) with different methods, we establish some classification results.

Keywords

Reaction-diffusion-advection system, Haptotaxis, Free boundary problem, Comparison principle,
Liouville-type Theorem, Hénon equation, Stability, Finite Morse index solution, Asymptotic behavior.

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Introduction générale et résultats principaux

Cette thèse porte sur l'analyse de quelques problèmes elliptiques ou paraboliques semi-linéaires. Elle se compose de deux parties indépendantes.

Le but de la première partie est d'étudier un système de réaction-diffusion-advection apparaissant dans un modèle de biomathématiques, plus précisément un modèle d'haptotaxie, phénomène intervenant dans la dissémination de cellules cancéreuses.

Cette première partie correspond au Chapitre 1, qui est constitué de l'article suivant :

1. The singular limit of a haptotaxis model with bistable growth, en collaboration avec Elisabeth Logak, *Communications on Pure and Applied Analysis*, **11(1)** (2012), 209–228.

La deuxième partie correspond aux Chapitres 2 et 3, dans lesquels on étudie les solutions stables ou localement stables de quelques équations elliptiques du type Hénon. Cette seconde partie est constituée essentiellement de 2 articles :

2. Some Liouville theorems for Hénon type elliptic equations, en collaboration avec Dong Ye, *Journal of Functional Analysis*, **262** (2012), 1705-1727 ;
3. Classification of finite Morse index solutions for Hénon type elliptic equation $-\Delta u = |x|^\alpha u_+^p$, en collaboration avec Salem Rebhi, soumis.

0.1 Limite singulière d'un modèle d'haptotaxie

0.1.1 Modélisation en dynamique des populations

En dynamique des populations, on décrit classiquement l'évolution de la densité $u = u(x,t) \in \mathbb{R}$ d'une espèce biologique (bactéries, cellules tumorales, etc...) par une équation de réaction-diffusion-advection qui peut s'écrire sous la forme générale suivante :

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\overrightarrow{V} u) + f(u). \tag{1}$$

Le premier terme de l'équation représente la diffusion, avec un coefficient de mobilité $D(u) \geq 0$. On supposera par la suite $D(u) = D > 0$, en l'absence de pression démographique.

Le dernier terme rend compte de la "croissance" de cette population qui résulte du cycle de vie : division cellulaire, mutation, production, dégradation. L'hypothèse standard est que $f(u)$ est une fonction non linéaire, de type monostable (e.g. $f(u) = u(1-u)$, on parle alors de "croissance logistique") ou bistable (e.g. $f(u) = u(1-u)(u-a)$ avec $0 < a < 1$), ce dernier cas modélisant des effets de coopération et de compétition instra-spécifique.

Enfin, le terme d'advection $\nabla \cdot (\overrightarrow{V} u)$ est caractéristique des systèmes vivants, car il décrit les interactions avec l'environnement dans lequel l'espèce de densité u réside. L'interaction implique souvent le mouvement vers ou loin d'un stimulus externe et une telle réponse est appelée "taxie", ce qui signifie "mouvement dirigé". La "taxie" peut être positive ou négative, selon qu'il y ait attirance ou répulsion vis à vis du stimulus externe. Ainsi $\overrightarrow{V} = \pm \nabla \chi(v)$, où v représente la densité de l'espèce attractive ou répulsive (nutrient, protéine, etc...) et où χ est la fonction de sensibilité de la "taxie", qui est régulière et vérifie

$$\forall v > 0, \ \ \chi(v) > 0, \ \ \chi'(v) > 0. \tag{2}$$

Ici on supposera que v est la concentration d'un "attracteur biochimique" de sorte que l'on considérera une équation du type

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla \chi(v)) + f(u). \tag{3}$$

Par ailleurs, la densité v dépend elle-même de u, ce qui conduit à un système couplé.

En particulier, le système le plus étudié dans cette classe correspond à un modèle de chimiotaxie ("chemotaxis"). Ce mécanisme est invoqué pour expliquer la formation de "patterns", de structures complexes dans les colonies de bactéries : v est alors la densité d'un nutrient, qui diffuse dans l'environnement et qui attire les bactéries - c'est la chimiotaxie - et que par ailleurs les bactéries sécrètent elles-mêmes. Le modèle de Keller-Segel correspondant à ce "feedback" s'écrit

$$\left\{ \begin{array}{rcll} u_t & = & \nabla.(d \nabla u - u \nabla \chi(v)) & \text{dans } \ \Omega \times [0,T] \\ v_t & = & \Delta v + u - \gamma v & \text{dans } \ \Omega \times [0,T] \\ \dfrac{\partial u}{\partial \nu} & = & \dfrac{\partial v}{\partial \nu} = 0 & \text{sur } \ \partial\Omega \times [0,T]. \end{array} \right.$$

Ce système a fait l'objet de nombreux travaux mathématiques, qui ont analysé le phénomène de blow-up de la solution u (explosion en temps fini). On pourra consulter ([42],[19]) et les références de ces travaux pour une description de quelques-uns de ces résultats.

Une variante du système de Keller-Segel a été proposée par Mimura et Tsujikawa dans [47]. Il s'agit d'un modèle de chimiotaxie avec croissance, où l'on fait l'hypothèse que f est de type bistable, donné par

$$f(u) = u(1-u)\left(u - \frac{1}{2}\right) \qquad \text{pour tout} \;\; u \in \mathbb{R}$$

et vérifie en particulier $\int_0^1 f(u)du = 0$. En introduisant un paramètre $\varepsilon > 0$, correspondant à la taille de la zone de transition séparant les bactéries de leur environnement, le système s'écrit

$$(C_\epsilon) \qquad \begin{cases} u_t = \Delta u - \nabla.(u\nabla\chi(v)) + \dfrac{1}{\varepsilon^2} f(u) & \text{dans } \Omega \times (0,T] \\ 0 = \Delta v + u - \gamma v & \text{dans } \Omega \times (0,T], \end{cases}$$

avec une condition au bord de Neumann homogène pour u et des conditions initiales appropriées. On notera qu'ici v est couplé à u par une équation elliptique, linéaire. Ce système a été étudié dans [9, 10, 2]. Il est établi dans [9, 10] que, dans la limite singulière $\varepsilon \to 0$, la solution $(u^\varepsilon, v^\varepsilon)$ de (C_ϵ) converge vers (u_0, v_0) sur $[0,T]$ pour certain temps $T > 0$, où $u_0 = \chi_{\Omega_t}$ est la fonction caractéristique d'un domaine mobile $\Omega_t \subset\subset \Omega$. L'interface est alors par définition $\Gamma_t = \partial\Omega_t$. Si n est le vecteur normal extérieur sur Γ_t et V_n la vitesse normale de Γ_t, alors, en notant κ la courbure moyenne en chaque point de Γ_t, la loi de mouvement de l'interface est la solution du problème à frontière libre suivant,

$$(C_0) \qquad \begin{cases} V_n = -(N-1)\kappa + \dfrac{\partial\chi(v^0)}{\partial n} & \text{sur } \;\Gamma_t, \; t \in (0,T] \\ 0 = \Delta v^0 + u^0 - \gamma v^0 & \text{dans } \;\Omega \times (0,T]. \end{cases}$$

Alfaro [2] a étendu les résultats de [9, 10] à des conditions initiales très générales et a obtenu des estimations optimales sur le temps de génération d'interface et sur la taille de la zone de transition.

0.1.2 Le modèle d'haptotaxie

Dans cette thèse on considère un modèle d'haptotaxie (ou "haptotaxis"). Ce mécanisme intervient dans la première phase d'une invasion tumorale, au moment où les cellules cancéreuses adhèrent à la matrice extracellulaire (ou "Extra-Cellular Matrix", notée ECM). Cette étape précède la dissémination et la prolifération de ces cellules vers d'autre organes via le système sanguin (angiogénèse). Un processus similaire intervient lors de la cicatrisation des plaies. De manière générale, l'haptotaxie se caractérise par le mouvement dirigé des cellules en réponse aux propriétés du milieu extra-cellulaire, suivant le gradient de concentration de molécules adhésives (enzymes, etc...) fixées dans la matrice extra-cellulaire, et qui sont elles-mêmes immobiles.

C'est là que réside la différence esssentielle avec la chimiotaxie : il n'y a pas de diffusion de l'attracteur biochimique.

Dans le modèle que nous considérons, l'invasion de cellules cancéreuses est associée à la dégradation de l'ECM par des enzymes protéolytiques (ou "Matrix Degrading Enzymes", notées MDE). Ces enzymes sont sécrétées par les cellules cancéreuses elles-mêmes. La dégradation crée ainsi les gradients spatiaux qui dirigent la migration des cellules tumorales par haptotaxie.

On passe d'abord en revue la littérature mathématique liée au modèle d'haptotaxie.

Un modèle hybride utilisant des équations aux dérivées partielles et des automates cellulaires a été proposé par Anderson [4]. Il fait intervenir 4 constituants : les cellules cancéreuses, l'ECM, les MDE et l'oxygène. L'existence globale et l'unicité pour le modèle d'Anderson ont été établies en dimension $N \leq 3$ par Walker et Webb dans [61].

Dans le cas d'une non-linéarité logistique, l'existence globale a été obtenue en dimension $N = 3$ (voir [46] et ses références bibliographiques).

Un autre modèle combinant haptotaxie et chimiotaxie avec une non-linéarité logistique a été proposé par Chaplain, Lolas dans [13] . L'existence globale d'une solution de ce système a été prouvée par Tao, Wang [59] et Tao [58] en dimension $N \leq 2$. Par ailleurs, Tao, Winkler [60] ont établi récemment un résultat d'existence globale pour un système analogue avec diffusion dégénérée.

Notre modèle est une version simplifiée de celui d'Anderson faisant intervenir 3 composantes, l'oxygène n'apparaissant pas dans le modèle. De plus, on a introduit une non-linéarité bistable. Précisément on s'intéresse au modèle d'évolution suivant

$$
(P^\varepsilon) \qquad \begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + \dfrac{1}{\varepsilon^2} f(u) & \text{dans } \Omega \times (0,T] \\
v_t = -\lambda m v & \text{dans } \Omega \times (0,T] \\
m_t = d\Delta m + u - m & \text{dans } \Omega \times (0,T] \\
u(x,0) = u_0(x) & x \in \Omega \\
v(x,0) = v_0(x) & x \in \Omega \\
m(x,0) = m_0(x) & x \in \Omega \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial m}{\partial \nu} = 0 & \text{sur } \partial\Omega \times (0,T],
\end{cases}
$$

où $u(x,t)$ représente la concentration de cellules cancéreuses, $v(x,t)$ la concentration de molécules d'ECM et $m(x,t)$ la densité de MDE. La première équation est du type (3) : la migration des cellules cancéreuses est dûe à la diffusion (marche aléatoire), à l'haptotaxie et à la prolifération, que nous modélisons par $f(u)$. On a normalisé la diffusivité des cellules cancéreuses à 1. La deuxième équation modélise la dégradation de l'ECM par les MDE au taux $\lambda > 0$, et sans diffusion d'ECM. Enfin, dans la troisième équation, on suppose que l'évolution de la concentration des MDE se fait par diffusion (avec une diffusivité constante $d > 0$), par production (à partir des cellules cancéreuses) et par dégradation (à un taux normalisé à 1).

Ce système est considéré dans un domaine borné régulier $\Omega \subset \mathbb{R}^N (N \geq 2)$. On note $\Omega_T = \Omega \times [0, T]$ avec $T > 0$, ν le vecteur normal extérieur sur $\partial\Omega$ et on suppose $\varepsilon > 0$. La fonction de sensibilité d'haptotaxie est régulière et vérifie (2).

Nos travaux se situent dans la lignée des travaux [9, 10, 2]. Nous nous sommes intéressés à la limite singulière du système (P^ε) quand $\varepsilon \to 0$. Notons qu'à la différence de ces travaux, v^ε est couplé à u^ε par l'intermédiaire de m^ε et via une équation différentielle ordinaire au lieu du couplage elliptique ou parabolique rencontré dans les modèles de chimiotaxie.

0.1.3 Hypothèses et résultats principaux

On fait ici les hypothèses suivantes sur les données initiales.

(H1) $v_0 \in C^2(\overline{\Omega})$ est une fonction positive et régulière dans $\overline{\Omega}$, elle satisfait la condition aux limites de Neumann homogène

$$\frac{\partial v_0}{\partial \nu} = 0 \quad \text{sur} \quad \partial\Omega. \tag{4}$$

(H2) u_0 et m_0 sont positives et de classe C^2 dans $\overline{\Omega}$. On fixe une constante $C_0 > 1$ telle que

$$\|u_0\|_{C^2(\overline{\Omega})} + \|v_0\|_{C^2(\overline{\Omega})} + \|m_0\|_{C^2(\overline{\Omega})} \leq C_0. \tag{5}$$

(H3) L'ensemble ouvert Ω_0 défini par

$$\Omega_0 := \left\{ x \in \Omega, u_0(x) > \frac{1}{2} \right\}$$

est convexe et $\Omega_0 \subset\subset \Omega$.

(H4) $\Gamma_0 := \partial\Omega_0$ est une hypersurface C^∞ sans bord.

Sous ces hypothèses, on a

$$u_0 > \frac{1}{2} \text{ dans } \Omega_0, \;\; 0 \leq u_0 < \frac{1}{2} \text{ dans } \Omega \setminus \overline{\Omega}_0.$$

On remarque aussi qu'il résulte de (4) et du fait que $\dfrac{\partial m^\varepsilon}{\partial \nu} = 0$ sur $\partial\Omega \times [0, T]$ que

$$\frac{\partial v^\varepsilon}{\partial \nu} = 0 \text{ sur } \partial\Omega \times [0, T]. \tag{6}$$

Énonçons à présent nos principaux résultats. Tout d'abord, l'existence d'une unique solution positive $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ au problème (P^ε) pour $\varepsilon > 0$ petit est établie en utilisant les estimations a priori et le théorème du point fixe de Schauder.

Théorème 0.1. *Supposons que (u_0, v_0, m_0) satisfont les hypothèses $(H1)-(H4)$. Alors il existe $\varepsilon_0 > 0$ tel que pour tout $0 < \varepsilon < \varepsilon_0$, le problème (P^ε) a une solution unique $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ sur Ω_T pour tout $T > 0$. De plus, cette solution vérifie $0 \leq u^\varepsilon \leq C_0$ dans Ω_T.*

Ensuite, nous sommes intéressés au comportement asymptotique de $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ quand $\varepsilon \to 0$. Nous montrons que quand $\varepsilon \to 0$, $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ converge vers la solution d'un problème à frontière libre où le mouvement d'interface est lié à la courbure moyenne et au terme d'haptotaxie, qui est un terme non-local. Plus précisément, la limite asymptotique du problème (P^ε) quand $\varepsilon \to 0$ est donnée par le problème à frontière libre (P^0) suivant

$$(P^0) \begin{cases} u^0(x,t) = \chi_{\Omega_t}(x) = \begin{cases} 1 \text{ dans } \Omega_t, \ t \in [0,T] \\ 0 \text{ dans } \Omega \setminus \overline{\Omega}_t, \ t \in [0,T] \end{cases} & \\ v_t^0 = -\lambda m^0 v^0 & \text{dans } \Omega \times (0,T] \\ m_t^0 = d\Delta m^0 + u^0 - m^0 & \text{dans } \Omega \times (0,T] \\ V_n = -(N-1)\kappa + \dfrac{\partial \chi(v^0)}{\partial n} & \text{sur } \Gamma_t = \partial\Omega_t, t \in (0,T] \\ \Gamma_t|_{t=0} = \Gamma_0 & \\ v^0(x,0) = v_0(x) & x \in \Omega \\ m^0(x,0) = m_0(x) & x \in \Omega \\ \dfrac{\partial m^0}{\partial \nu} = 0 & \text{sur } \partial\Omega \times (0,T], \end{cases}$$

où $\Omega_t \subset\subset \Omega$ est un domaine mobile, $\Gamma_t = \partial\Omega_t$ est l'interface limite, n est le vecteur normal extérieur sur Γ_t, V_n est la vitesse normale de Γ_t et κ la courbure moyenne en chaque point de Γ_t.

La dérivation de l'équation de mouvement de l'interface dans le problème (P^0) à partir du problème (P^ε) peut être obtenue par un argument formel analogue à celui expliqué dans [9].

On montre d'abord que le problème (P^0) est bien posé localement en temps et on obtient l'existence et l'unicité d'une solution régulière du problème (P^0) jusqu'à un certain temps $T > 0$.

Théorème 0.2. *Soit $\Gamma_0 = \partial\Omega_0$, où $\Omega_0 \subset\subset \Omega$ est un domaine de classe $C^{2+\alpha}$ avec $\alpha \in (0,1)$. Alors il existe $T > 0$ tel que le problème (P^0) ait une solution unique (u^0, v^0, m^0, Γ) sur $[0,T]$ avec*

$$\Gamma = (\Gamma_t \times \{t\})_{t\in[0,T]} \in C^{2+\alpha,(2+\alpha)/2} \ \textit{et} \ v^0|_\Gamma \in C^{1+\alpha,(1+\alpha)/2}.$$

De plus, si v_0 est régulière, $(\Gamma_t \times \{t\})_{t\in(0,T]}$ est de classe C^∞ dans les deux variables. Si v_0 est régulière et Γ_0 est $C^{k+\alpha}$ pour certain $k \in \mathbb{N}$, $k \geq 2$, alors Γ est de classe $C^{k+\alpha,(k+\alpha)/2}$.

La preuve du théorème ci-dessus repose sur l'utilisation d'un argument de point fixe dans un espace de Hölder approprié. Notons que la courbure est un terme régularisant, qui rend l'équation du mouvement parabolique en coordonnées locales. Le terme d'haptotaxie induit un effet non-local dans la vitesse de l'interface et doit être estimée avec soin. Enfin, il y a un effet régularisant en temps de l'équation différentielle ordinaire vérifiée par v^0.

Afin de démontrer la convergence de $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ vers (u^0, v^0, m^0), nous établissons dans une première étape une propriété de génération d'interface.

Théorème 0.3. *Supposons que* (u_0, v_0, m_0) *satisfont les hypothèses* $(H1)-(H4)$. *Soit* $0 < \eta < \frac{1}{4}$ *une constante quelconque et soit* $\mu = f'(\frac{1}{2}) = \frac{1}{4}$. *Alors il existe* $\varepsilon_0 > 0$ *et* $M_0 > 0$ *tel que, pour tout* $\varepsilon \in (0, \varepsilon_0]$ *et au temps* $t^* = \mu^{-1}\varepsilon^2|\ln\varepsilon|$, *on a*

(a) pour tout $x \in \Omega$,

$$0 \leq u^\varepsilon(x, t^*) \leq 1 + \eta,$$

(b) pour tout $x \in \Omega$ *tel que* $|u_0(x) - \frac{1}{2}| \geq M_0\varepsilon$,

$$\text{si } u_0(x) \geq \frac{1}{2} + M_0\varepsilon, \text{ alors } u^\varepsilon(x, t^*) \geq 1 - \eta,$$

$$\text{si } u_0(x) \leq \frac{1}{2} - M_0\varepsilon, \text{ alors } 0 \leq u^\varepsilon(x, t^*) \leq \eta.$$

La démonstration repose sur la construction d'une sous-solution et d'une sur-solution à l'EDO bistable. En fait, ce théorème prouve la formation rapide d'une couche de transition autour d'un voisinage de Γ_0 dans un intervalle de temps très petit, de l'ordre $\varepsilon^2|\ln\varepsilon|$. Autrement dit, après ce court laps de temps, la solution devient proche de 1 ou 0 sauf dans un petit voisinage de Γ_0. De plus, la largeur de la couche de transition autour de Γ_0 est de l'ordre ε, et il s'agit là d'une estimation optimale.

Nous pouvons à présent énoncer le résultat principal.

Théorème 0.4. *Supposons que* (u_0, v_0, m_0) *satisfont les hypothèses* $(H1)-(H4)$. *Soit* $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ *la solution du problème* (P^ε) *et soit* (u^0, v^0, m^0, Γ) *avec* $\Gamma = (\Gamma_t \times \{t\})_{t\in[0,T]}$ *la solution du problème* (P^0) *sur* $[0, T]$. *Alors, quand* $\varepsilon \to 0$, $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ *converge vers* (u^0, v^0, m^0) *p.p. dans* $\bigcup_{0<t\leq T}((\Omega \setminus \Gamma_t) \times t)$. *Plus précisément,*

$$\lim_{\varepsilon\to 0} u^\varepsilon(x, t) = u^0(x, t) \text{ p.p. dans } \bigcup_{0<t\leq T} ((\Omega \setminus \Gamma_t) \times t)$$

et pour tout $\alpha \in (0, 1)$,

$$\lim_{\varepsilon\to 0} \|v^\varepsilon - v^0\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} = 0,$$

$$\lim_{\varepsilon\to 0} \|m^\varepsilon - m^0\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} = 0.$$

En réécrivant le problème (P^ε) comme une équation d'évolution non-locale u^ε, on peut lui associer un principe de comparaison. Grâce à la construction de sous-et sur-solutions convenables sur $[t^*, T]$ avec $t^* = \mu^{-1}\varepsilon^2|\ln\varepsilon|$, on montre alors la convergence.

La difficulté principale réside dans le fait que la fonction v^ε est couplée à u^ε par une équation différentielle ordinaire, qui ne régularise pas en espace. Il a fallu obtenir des estimations sur les dérivées spatiales de v^ε en utilisant l'expression exacte de la solution de cette équation différentiellle, qui dépend de l'intégrale en temps de m^ε, elle-même solution d'une EDP elliptique.

De plus, comme dans [2], nous avons estimé la distance entre l'interface Γ_t solution de problème (P^0) et l'ensemble $\Gamma_t^\varepsilon := \{x \in \Omega, u^\varepsilon(x, t) = 1/2\}$.

Théorème 0.5. *Il existe $C > 0$ tel que*

$$\Gamma_t^\varepsilon \subset \aleph_{C\varepsilon}(\Gamma_t) \quad pour \;\; 0 \leq t \leq T \; et \; \varepsilon \in (0, \varepsilon_0],$$

où $\varepsilon_0 > 0$ est défini dans le théorème 0.3, et où $\aleph_r(\Gamma_t) := \{x \in \Omega, \; dist(x, \Gamma_t) < r\}$ est le voisinage tubulaire de Γ_t de rayon $r > 0$.

0.2 Solutions stables des équations elliptiques du type Hénon

La deuxième partie de ma thèse est consacrée à l'étude des solutions avec indice de Morse fini de l'équation elliptique du type Hénon suivante

$$-\Delta u = |x|^\alpha f(u) \quad \text{dans} \quad \Omega, \tag{7}$$

où $\alpha > -2$, $\Omega \subset \mathbb{R}^N$ avec $N \geq 2$. On va considérer quatre situations : $f(u) = e^u$, $|u|^{p-1}u$, u_+^p ou $f(u)$ est une fonction sur-linéaire assez générale.

L'équation

$$(H) \qquad -\Delta u = |x|^\alpha u^p$$

avec $p > 1$ et $\alpha > 0$ a été introduite par Hénon [41] comme un modèle pour étudier les grappes sphériquement symétriques d'étoiles. Beaucoup de chercheurs ont consacré leurs travaux aux solutions positives de l'équation (H), [48, 57, 5] ont examiné les solutions positives du problème avec condition de Dirichlet au bord et, [38, 32, 7] ont considéré les solutions positives de l'équation (H) dans l'espace euclidien entier $\mathbb{R}^N$.

Pour le cas $\alpha = 0$, le résultat optimal du type Liouville a été établi par Gidas et Spruck dans leur célèbre article [38]. À savoir, (H) n'a pas de solution positive si et seulement si $p < \underline{p} := \frac{N+2}{N-2}$.

Le cas $\alpha \neq 0$ est moins bien compris. Nous rappelons que d'après [38], pour $\alpha \leq -2$, (H) n'a pas de solution positive dans n'importe quel domaine contenant l'origine, c'est pourquoi nous pouvons nous limiter au cas $\alpha > -2$. Dans le cas des solutions radiales, on a le résultat ci-dessous.

Théorème 0.6 (voir [38] et [7])**.** *Soient $N \geq 2$ et $\alpha > -2$. Alors*

(1) (H) n'a pas de solution positive radiale dans $\mathbb{R}^N$ si $1 < p < \frac{N+2+2\alpha}{N-2}$.

(2) (H) possède des solutions positives, bornées et radiales dans $\mathbb{R}^N$ si $p \geq \frac{N+2+2\alpha}{N-2}$.

L'exposant de Hardy-Sobolev

$$\overline{p} := \frac{N+2+2\alpha}{N-2} \quad (= \infty \text{ si } N = 2)$$

joue un rôle critique dans le cas radial ci-dessus et cela appuie une conjecture naturelle suivante.

Conjecture 0.7. *Soient $N \geq 2$ et $\alpha > -2$. Alors (H) n'a pas de solution positive dans $\mathbb{R}^N$ si $1 < p < \overline{p}$.*

La condition $p < \overline{p}$ est la meilleure possible en raison du premier résultat du Théorème 0.6. Cependant, en dehors du cas radial, le meilleur résultat disponible de non-existence jusqu'à présent est le suivant.

Théorème 0.8 (Théorème 1.7, [7]). *Soient $N \geq 2$ et $\alpha > -2$. Alors*

(1) (H) n'a pas de solution positive dans $\mathbb{R}^N$ si $1 < p < \min(\underline{p}, \overline{p})$.

(2) Le résultat de (1) reste valable si $1 < p < \frac{N+\alpha}{N-2}$.

En particulier, le point (1) du Théorème 0.8 implique la Conjecture 0.7 pour $\alpha \in (-2, 0)$, car dans ce cas là $\overline{p} < \underline{p}$. Cependant, cette conjecture est encore un problème ouvert pour $\alpha > 0$.

Tout récemment, Phan et Souplet ont montré dans [49] la non-existence de solution positive bornée dans $\mathbb{R}^N$ en dimension 3. Fazly et Ghoussoub [37] ont obtenu ensuite un résultat similaire pour un système couplé d'équation d'Hénon.

Au lieu de la positivité, nous supposons la stabilité de la solution et nous voulons établir un phénomène similaire et de nouvelles applications. D'un point de vue mathématique, (7) est un très bon exemple pour étudier des phénomènes de la théorie de points critiques appliquée aux équations elliptiques nonlinéaires. En fait, comme mentionné dans la Conjecture 0.7 ci-dessus, l'exposant critique classique est $\overline{p}$. Avec la stabilité, notre analyse révèle l'existence d'un nouvel exposant critique $p(N, \alpha)$ qui est plus grand que $\overline{p}$.

Dans la suite, la solution qu'on étudie est toujours une solution faible de l'équation (7) dans le sens suivant.

Définition 0.9. On dit que u est une solution faible de l'équation (7) dans Ω (borné ou non) si $u \in H^1_{loc}(\Omega)$, $|x|^\alpha f(u) \in L^1_{loc}(\Omega)$ et

$$\int_\Omega \Big[\nabla u \cdot \nabla \psi - |x|^\alpha f(u)\psi\Big] dx = 0 \quad \text{pour tout } \psi \in C^1_c(\Omega).$$

Afin d'énoncer nos résultats plus clairement, nous introduisons d'abord les notions de stabilité et d'indice de Morse.

On commence par la solution stable. En fait, une solution u est stable, si et seulement si la seconde variante de l'énergie associée à l'équation (7) est non-négative, où l'énergie associée est

$$\mathfrak{E}_\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega |x|^\alpha F(u) dx, \quad \text{avec } F(u) \text{ une primitive de } f(u).$$

Plus précisément, comme expliqué dans [29], fixant u, nous étudions la variation de l'énergie le long d'une direction $\psi \in C^1_c(\Omega)$, c'est-à-dire qu'on considère la fonction $E(t) := \mathfrak{E}_\Omega(u + t\psi)$ pour $t \in \mathbb{R}$. D'après l'argument de la stabilité classique, $t = 0$ est un point stable de E si $E'(0) = 0$ et $E''(0) \geq 0$. On trouve que u est la solution de (7) si et seulement si $E'(0) = 0$ (ça implique que la dérivée de Fréchet de $\mathfrak{E}_\Omega$ en u est nulle), et on a besoin

$$E''(0) = \int_\Omega \Big[|\nabla\psi|^2 - |x|^\alpha f'(u)\psi^2\Big]dx \geq 0 \quad \text{pour tout } \psi \in C_c^1(\Omega).$$

Définition 0.10. Soit u la solution faible de l'équation (7) dans Ω, on dit que u est stable dans Ω si $|x|^\alpha f'(u) \in L^1_{loc}(\Omega)$ et

$$Q_u(\psi) := \int_\Omega \Big[|\nabla\psi|^2 - |x|^\alpha f'(u)\psi^2\Big]dx \geq 0 \quad \text{pour tout } \ \psi \in C_c^1(\Omega). \tag{8}$$

Si $|x|^\alpha f'(u) \geq 0$, (8) reste aussi valable pour tout $\psi \in H_0^1(\Omega)$ par l'argument de densité.

Il y a de nombreux exemples de solutions stables. Par exemple, un minimiseur local de l'énergie, une solution minimale, une solution monotone, etc. La solution stable joue un rôle important dans le traitement de l'unicité et la symétrie de la solution et d'autres propriétés qualitatives. On peut généraliser logiquement la notion de stabilité en exigeant que (8) ne vaille que pour les fonctions de test à support loin d'un ensemble compact.

Définition 0.11. Soit $K \subset\subset \Omega$ un sous ensemble compact de Ω. Soit u la solution faible de l'équation (7) dans Ω. On dit que u est stable en dehors d'un ensemble compact si $|x|^\alpha f'(u) \in L^1_{loc}(\Omega\backslash K)$ et

$$\int_\Omega \Big[|\nabla\psi|^2 - |x|^\alpha f'(u)\psi^2\Big]dx \geq 0 \quad \text{pour tout } \ \psi \in C_c^1(\Omega\backslash K).$$

Plus brièvement, u est solution faible sur Ω et stable sur $\Omega\backslash K$.

Cette définition englobe une classe fondamentale et essentielle de solutions : les solutions avec indice de Morse fini.

Définition 0.12. L'indice de Morse d'une solution u sur Ω - $ind_\Omega(u)$, est défini comme la dimension maximale de tous les sous espaces X de $C_c^1(\Omega)$ tel que $Q_u(\psi) < 0$ pour tout $\psi \in X\backslash\{0\}$. Il est clair que u est stable dans Ω si et seulement si $ind_\Omega(u) = 0$.

On peut remarquer que toutes les solutions avec indice de Morse fini sont solutions stables en dehors d'un ensemble compact. En effet, pour $\ell = \mathrm{ind}_\Omega(u) \geq 0$, il existe $X = \mathrm{span}\{\varphi_1, \cdots, \varphi_\ell\} \subset C_c^1(\Omega)$ tel que $Q_u(\varphi) < 0$ pour tout $\varphi \in X\backslash\{0\}$, alors $Q_u(\psi) \geq 0$ pour tout $\psi \in C_c^1(\Omega\backslash\mathcal{K})$, où $\mathcal{K} = \bigcup_j \mathrm{supp}(\varphi_j)$.

La compréhension des solutions avec indice de Morse fini est très importante, il semble naturel de relier l'indice de Morse d'une solution avec certaines propriétés qualitatives. Par exemple, cela peut être utile pour montrer l'existence de l'infinité de solutions et aussi pour donner une estimation uniforme des solutions (voir [6] et les références qui y sont). Dans [6], après l'obtention du résultat de type Liouville pour les solutions avec indice de Morse fini d'une équation dans l'espace tout entier et dans un demi-espace, Bahri et Lions ont montré l'équivalence entre le fait qu'une famille de solutions soit uniformément bornée et le fait que les indices de Morse de ces solutions soient uniformément bornés. Ceci implique que le travail de classification des solutions avec indice de Morse fini est essentiel, c'est aussi la source de motivation de notre étude.

A l'aide des notions de base ci-dessus, nous sommes prêts à exhiber nos résultats principaux.

0.2.1 Équation du type Hénon avec l'exponentielle

On considère tout d'abord une équation du type Hénon suivante

$$-\Delta u = |x|^\alpha e^u \quad \text{dans} \quad \Omega \subset \mathbb{R}^N, \tag{9}$$

avec $\alpha > -2$ et $N \geq 2$.

Cette équation classique se pose dans de nombreux problèmes physiques et mathématiques. Par exemple, quand $N = 2$, elle est cruciale dans le problème de la courbure gaussienne prescrite et le modèle de Chern-Simons Higgs.

Le cas autonome, i.e. lorsque $\alpha = 0$, a été étudié récemment. Il est démontré par Farina [36] que $-\Delta u = e^u$ n'a pas de solution stable dans $\mathbb{R}^N$ pour $2 \leq N \leq 9$. De plus, quand $N = 2$, il a prouvé aussi que la solution classique de $-\Delta u = e^u$ avec indice de Morse fini dans $\mathbb{R}^2$ vérifie $e^u \in L^1(\mathbb{R}^2)$ donc elle doit être la solution classée par Chen et Li [14], qui est

$$u(x) = \ln \left[\frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2} \right] \quad \text{avec} \quad \lambda > 0 \text{ et } x_0 \in \mathbb{R}^2.$$

Ensuite, Dancer et Farina ont prouvé dans [22] que $-\Delta u = e^u$ admet les solutions classiques qui sont stables en dehors d'un ensemble compact si et seulement si $N \geq 10$.

Une question naturelle est de se demander si des résultats similaires peuvent être observés pour le cas non-autonome, i.e. lorsque $\alpha \neq 0$. Dans [34], Esposito a étudié les solutions stables de (9) dans $\mathbb{R}^N$ et il a donné les résultats de classification en supposant que $\alpha \geq 0$ et que e^u est borné.

Bien que nous empruntions de nombreuses idées de ces travaux antérieurs, nous avons essayé de traiter ce problème dans un cadre plus général.

1. Nous étudions des solutions faibles qui ne sont pas supposées a priori localement bornées. De plus, nous travaillons avec tout $\alpha > -2$, donc nous avons moins de régularité à priori. Si $\alpha < 0$ et $0 \in \Omega$, toute solution faible de (9) ne peut pas être une solution classique, en raison de la singularité à l'origine.
2. Nous ne classifions pas seulement les solutions stables, mais aussi les solutions qui sont stables en dehors d'un ensemble compact pour (9), ce qui n'a pas été considéré dans [34].

Tout d'abord, nous pouvons affirmer que la restriction $\alpha > -2$ est nécessaire, à cause du résultat de non-existence suivant.

Proposition 0.13. *Si* $\alpha \leq -2$, (9) *n'admet pas de solution faible pour tout domaine* $\Omega \subset \mathbb{R}^N$ *contenant* 0.

Ensuite, on peut obtenir une estimation du type intégral pour les solutions stables dans Ω, qui est l'extension du résultat obtenu dans [36].

Proposition 0.14. *Soit* Ω *un domaine (borné ou non) dans* $\mathbb{R}^N$ *avec* $N \geq 2$. *Soit* u *une solution faible de* (9) *qui est stable, avec* $\alpha > -2$. *Alors, pour tout entier* $m \geq 5$ *et tout réel* $\beta \in (0, 2)$, *il existe* $C > 0$ *dépendant de* m, α *et* β *tel que*

$$\int_\Omega |x|^\alpha e^{(2\beta+1)u}\psi^{2m} dx \leq C \int_\Omega |x|^{-2\beta\alpha} \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1} dx, \tag{10}$$

pour toutes les fonctions $\psi \in C_c^\infty(\Omega)$ *vérifiant* $\|\psi\|_\infty \leq 1$.

Il faut remarquer que, dans la preuve de la Proposition 0.14, nous empruntons la stratégie de la preuve de la Proposition 5 dans [36], mais nous devons l'adapter au cas d'une solution faible. L'estimation (10) résulte de la condition de stabilité avec les fonctions de test adaptées. Ici, u n'est plus supposé être borné, donc $e^{\beta u}\varphi$ n'est pas, a priori, une fonction de test licite (pour tout $\beta \in \mathbb{R}$), même avec $\varphi \in C_c^\infty(\Omega)$. Notre idée est de considérer des troncatures convenables de $e^{\beta u}$ afin d'obtenir la fonction de test désirée.

Le premier résultat de classification concerne les solutions stables dans l'espace entier.

Théorème 0.15. *Soient* $\alpha > -2$ *et* $\Omega = \mathbb{R}^N$. *Pour* $2 \leq N < 10 + 4\alpha$, *il n'existe pas de solution faible et stable de* (9).

Théorème 0.15 est optimal. En effet, pour $\alpha > -2$ et $N \geq 10 + 4\alpha$, (9) possède des solutions faibles qui sont radiales et stables dans $\mathbb{R}^N$, générées par

$$U(x) = -(2+\alpha)\ln|x| + \ln[(2+\alpha)(N-2)], \tag{11}$$

où $|x|$ dénote la norme euclidienne. La stabilité de U est la conséquence directe de l'inéqualité de Hardy classique,

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\psi^2}{|x|^2} dx \quad \text{pour tout } \psi \in H^1(\mathbb{R}^N).$$

Pour le cas de la solution stable en dehors d'un compact, en obtenant une estimation de décroissance rapide à l'infini, on peut démontrer le résultat suivant de classification.

Théorème 0.16. *Soient* $\alpha > -2$ *et* $\Omega = \mathbb{R}^N$. *Pour* $2 < N < 10 + 4\alpha^-$ *avec* $\alpha^- = \min(\alpha, 0)$, (9) *n'a pas de solution faible qui est stable en dehors d'un ensemble compact. En particulier, toute solution faible de* (9) *dans* $\mathbb{R}^N$ *admet un indice de Morse infini pour* $2 < N < 10 + 4\alpha^-$.

Le Théorème 0.16 est optimal pour $\alpha \le 0$, mais on ne sait pas s'il reste valable quand $\alpha > 0$ et $10 \le N < 10 + 4\alpha$.

Pour le cas de la dimension 2, i.e. $N = 2$, on montre que toute solution stable en dehors d'un ensemble compact est sûrement une solution énergétique, c'est-à-dire que $|x|^\alpha e^u \in L^1(\mathbb{R}^2)$ et Prajapat et Tarantello ont déjà classé dans [52] toutes ces solutions.

Théorème 0.17. *Soient $\alpha > -2$ et $\Omega = \mathbb{R}^2$. Soit u une solution faible de* (9) *qui est stable en dehors d'un ensemble compact. Alors*

$$\int_{\mathbb{R}^2} |x|^\alpha e^u dx = 4\pi(\alpha + 2).$$

De plus, si $\alpha \notin 2\mathbb{N}$, on a $u(x) = U_*(\varepsilon x) + (\alpha + 2)\ln \varepsilon$ où $\varepsilon > 0$ et

$$U_*(x) = \ln 2 + 2 \ln \frac{\alpha + 2}{1 + |x|^{\alpha+2}}.$$

Si $\alpha \in 2\mathbb{N}$, soient θ l'angle de x en coordonnées polaires et $k = \frac{\alpha+2}{2}$, soit

$$U_{**}(x) = 2 \ln \frac{2k}{1 + |x|^{2k} - 2|x|^k \cos(k\theta - \theta_0) \tanh \xi} + \ln \frac{2}{\cosh^2 \xi}$$

avec $\xi, \theta_0 \in \mathbb{R}$. On a $u(x) = U_{**}(\varepsilon x) + (\alpha + 2)\ln \varepsilon$ où $\varepsilon > 0$.

0.2.2 Équation du type Hénon avec croissance polynomiale

Ici on considère le deuxième type d'équation d'Hénon

$$-\Delta u = |x|^\alpha |u|^{p-1} u \quad \text{dans} \quad \Omega \subset \mathbb{R}^N, \tag{12}$$

avec $\alpha > -2$, $p > 1$ et $N \ge 2$.

La motivation pour étudier (12) dans les domaines non bornés provient de problème posé à la fois en physique et en géométrie. Par exemple, quand $N = 3$, il apparait dans l'étude de la structure stellaire dans l'astrophysique ; quand $N \ge 3$ et $p = \frac{N+2}{N-2}$, il est crucial dans l'étude des problèmes de géométrie conforme, voir [11, 12].

La compréhension des solutions stables en dehors d'un ensemble compact de (12) dans le cas autonome a été réalisée récemment. Farina a classé complètement toutes les solutions classiques avec indice de Morse fini de (12) avec $\alpha = 0$ dans $\mathbb{R}^N$ pour $1 < p < p_{JL}$, où $p_{JL} = p(N, 0)$ est l'exposant de Joseph-Lundgren (voir [43] et (13) au-dessous). Plus précisément, (12) avec $\alpha = 0$ admet des solutions classiques non-triviales dans $\mathbb{R}^N$ qui sont stables en dehors d'un ensemble compact, si et seulement si $N \ge 3$ et $p = \frac{N+2}{N-2}$, ou $N \ge 11$ et $p \ge p_{JL}$. De plus, nous voulons mentionner que, pour le cas autonome, les solutions positives radiales ont été étudiées dans [63] et les solutions faibles sont étudiées dans [23].

Quand $\alpha \neq 0$, l'équation (12) a été considéré par Dancer, Du et Guo dans [21], ils ont montré : soient $\alpha > -2$ et $u \in H^1_{loc} \cap L^\infty_{loc}(\mathbb{R}^N)$ une solution stable de (12) dans $\mathbb{R}^N$ avec $1 < p < p(N, \alpha)$ et

$$p(N,\alpha) = \begin{cases} \infty, & \text{si } N \leq 10 + 4\alpha \\ \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) + 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}, & \text{si } N > 10 + 4\alpha. \end{cases} \tag{13}$$

Alors $u \equiv 0$. D'autre part, pour $N > 10 + 4\alpha$ et $p \geq p(N, \alpha)$, (12) admet une famille de solutions stables qui sont positives et radiales dans $\mathbb{R}^N$.

Ils ont aussi étudié les solutions *positives* dans le domaine perforé $\Omega \setminus \{0\}$ ou dans le domaine extérieur $\mathbb{R}^N \setminus \Omega$ et ils ont obtenu des résultats intéressants sur le comportement asymptotique quand $|x|$ tend vers 0 ou ∞, pour les solutions qui sont stables près de l'origine ou en dehors d'un ensemble compact. De plus, ils ont obtenu quelques résultats de classification. Dans [34], Esposito a étudié les solutions stables de (12) dans $\mathbb{R}^N$ et il a aussi montré quelques résultats de type Liouville en supposant que $\alpha \geq 0$ et $|u|$ est borné.

Comme pour le problème (9), on a essayé de traiter le problème (12) dans un cadre plus général. En effet,

1. On n'ajoute pas de condition supplémentaire et on suppose simplement que u est localement bornée ou bornée.
2. On travaille avec tout $\alpha > -2$, et on classifie non seulement les solutions stables, mais aussi les solutions avec indice de Morse fini pour (12), ce qui n'a pas été considéré dans [21] ou [34].
3. On n'impose pas de condition de signe en u et on prouve le comportement asymptotique en 0 (resp. en l'infini) pour des solutions faibles qui sont stables près de l'origine (resp. près de l'infini, i.e., en dehors d'un ensemble compact). Enfin on considère également des solutions avec indice de Morse fini dans le demi-espace.

Il a été montré dans [21] que, quand $\alpha \leq -2$, (12) n'admet pas de solution *positive* dans tous les domaines perforés $B(0, R)\backslash\{0\}$. A notre connaissance, on ne sait pas si la condition $\alpha > -2$ est nécessaire pour avoir une solution faible changeant de signe pour (12). Cependant, il est raisonnable de supposer que $\alpha > -2$. L'argument de départ qui garantit les résultats désirés du type Liouville est toujours une estimation a priori résultant de la définition de la stabilité.

Proposition 0.18. *Soit Ω un domaine (borné ou non) dans $\mathbb{R}^N$ avec $N \geq 2$. Soit u la solution faible de* (12) *qui est stable, avec $p > 1$ et $\alpha > -2$. Alors, pour tout réel $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ et tout entier $m \geq \max\left(\frac{p+\gamma}{p-1}, 2\right)$, on a*

$$\begin{aligned} &\int_\Omega \left(|\nabla(|u|^{\frac{\gamma-1}{2}}u)|^2 + |x|^\alpha |u|^{\gamma+p}\right)|\psi|^{2m}dx \\ &\leq C\int_\Omega |x|^{-\frac{(\gamma+1)\alpha}{p-1}}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}}dx \end{aligned} \tag{14}$$

pour toutes les fonctions de test $\psi \in C_c^\infty(\Omega)$ *vérifiant* $\|\psi\|_\infty \leq 1$*, où la constante* C *ne dépend que de* p, m, γ *et* α*.*
De même, si on suppose que la solution faible de (12) u *appartient à* $H^1_{loc}(\Omega)$ *telle que* $u = 0$ *sur* $\partial\Omega$ *et* u *est stable en dehors d'un ensemble compact* $\mathcal{K} \subset \Omega$*, alors l'estimation* (14) *est aussi vraie pour toutes les fonctions de test* $\psi \in C_c^\infty(\mathbb{R}^N \backslash \mathcal{K})$ *vérifiant* $\|\psi\|_\infty \leq 1$*.*

La preuve de la proposition ci-dessus suit les grandes lignes de la démonstration de la Proposition 4 et 6 dans [35] (voir aussi la Proposition 1.7 dans [21]), avec des techniques de troncature pour avoir une fonction de test licite.

Nous donnons d'abord les résultats du type Liouville dans l'espace entier.

Théorème 0.19. *Soient* $\alpha > -2$ *et* u *une solution faible et stable de* (12) *dans* $\mathbb{R}^N$ *avec* $1 < p < p(N, \alpha)$*. Alors* $u \equiv 0$*.*

Le Théorème 0.19 est optimal d'après le résultat de Dancer-Du-Guo [21]. De plus, nous pouvons classifier les solutions avec indice de Morse fini dans $\mathbb{R}^N$.

Théorème 0.20. *Soient* $\alpha > -2$ *et* $N \geq 2$*. Soit* u *une solution faible de* (12) *dans* $\mathbb{R}^N$ *avec indice de Morse fini. Suppose de plus que*

$$1 < p < p(N, \alpha^-) \quad \text{et} \quad p \neq \frac{N + 2 + 2\alpha}{N - 2}, \tag{15}$$

alors $u \equiv 0$*.*

Le Theorème 0.20 est optimal pour $\alpha \in (-2, 0]$. Comme dans le cas exponentiel, on ne sait pas si le même résultat est aussi valable pour $\alpha > 0$, $p(N, 0) \leq p < p(N, \alpha)$ et $N > 10$. D'ailleurs, quand $N \geq 3$ et $p = \frac{N+2+2\alpha}{N-2}$ avec $\alpha > -2$, on sait que (12) possède les solutions positives et radiales, données par

$$V(x) = \lambda^{\frac{N-2}{2}} \left(\frac{\sqrt{(N + \alpha)(N - 2)}}{1 + \lambda^{2+\alpha}|x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}} \quad \text{avec} \quad \lambda > 0.$$

Au vu de l'inégalité de Cwikel-Lieb-Rozenbljum (voir Lemma 3.1.1 dans Chapitre 3), V admet un indice de Morse fini.

Pour montrer le Théorème 0.20 dans le cas sous-critique $p < \min(\overline{p}, p(N, \alpha^-))$, on utilise l'identité de Pohozaev classique. Comme on a moins de régularité, pour estimer les termes au bord associées, on a besoin de l'estimation de décroissance pour u et ∇u.

Théorème 0.21. *Soient* $N \geq 2$, $1 < p < p(N, 0)$ *et* u *une solution faible de* (12) *avec indice de Morse fini. Si le domaine* Ω *contient* $B(0, r) \setminus \{0\}$ *(resp.* $\mathbb{R}^N \setminus B(0, r)$*), alors on a* $u \in C^{2,\beta}_{loc}(\Omega \setminus \{0\})$ *pour certain* $\beta \in (0, 1)$*. De plus*

$$u(x) = O\left(|x|^{-\frac{2+\alpha}{p-1}}\right), \quad \nabla u(x) = O\left(|x|^{-1-\frac{2+\alpha}{p-1}}\right),$$

quand $|x| \to 0$ *(resp.* $|x| \to \infty$*).*

La preuve du Théorème 0.20 pour le cas sur-critique $\overline{p} < p < p(N, \alpha^-)$ n'est pas seulement basée sur la Proposition 0.18, mais aussi sur le comportement asymptotique des solutions près de l'origine et près de l'infini. Dans tous les travaux antérieurs, les auteurs ont besoin seulement de prouver la décroissance rapide des solutions à l'infini, car leurs solutions u sont régulières en l'origine. Pour nous, $x = 0$ est un point singulier, nous avons besoin aussi de démontrer la décroissance rapide des solutions prés de $x = 0$.

Théorème 0.22. *Soit u une solution faible de* (12) *avec indice de Morse fini dans Ω contenant* 0*. Suppose que p satisfait*

$$1 < p < p(N, \alpha^-).$$

Alors $u \in C(\Omega) \cap C^2(\Omega \setminus \{0\})$ et on a l'estimation de décroissance rapide suivante

$$\lim_{|x| \to 0} |x|^{1+\frac{2+\alpha}{p-1}} |\nabla u(x)| = 0.$$

Nous avons besoin de plus de restrictions sur p pour obtenir la décroissance rapide quand $|x|$ tend vers ∞.

Théorème 0.23. *Suppose que u est une solution faible, stable dans $\mathbb{R}^N \setminus K$, avec un ensemble compact $K \subset \mathbb{R}^N$ et $N \geq 3$. Suppose que $\alpha > -2$ et p satisfait*

$$\underline{p}(N, \alpha) < p < p(N, \alpha^-)$$

où

$$\underline{p}(N, \alpha) := \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) - 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}.$$

Alors on a

$$\lim_{|x| \to \infty} |x|^{\frac{2+\alpha}{p-1}} |u(x)| = \lim_{|x| \to \infty} |x|^{1+\frac{2+\alpha}{p-1}} |\nabla u(x)| = 0.$$

Finalement, pour le problème sur le demi-espace de (12) avec la condition de Dirichlet au bord, comme $|u|^{p-1}u$ est une fonction impaire, on trouve que v, le prolongement impair de u, est bien une solution faible, en procédant de la même manière que pour le Théorème 0.20 (voir aussi Farina [35]). On obtient le résultat suivant :

Théorème 0.24. *Soient $\alpha > -2$ et u une solution faible avec indice de Morse fini du problème*

$$-\Delta u = |x|^\alpha |u|^{p-1} u \;\; \textit{dans} \;\; \mathbb{R}^N_+ := \{x \in \mathbb{R}^N, x_N > 0\}, \quad u = 0 \;\; \textit{sur} \;\; \partial\mathbb{R}^N_+.$$

Alors $u \equiv 0$ sous les hypothèses (15).

0.2.3 Équation du type Hénon avec une non-linéarité générale

Dans cette dernière partie on étudie l'équation du type Hénon générale

$$-\Delta u = |x|^\alpha f(u) \quad \text{dans } \mathbb{R}^N, \tag{16}$$

avec $N \geq 2$ et $\alpha \geq -2$. Ici f est une fonction non-triviale qui vérifie

$$f \in C^2(I, \mathbb{R}) \cap C^0(\overline{I}, \overline{\mathbb{R}}); f \geq 0, f' \geq 0 \text{ et } f \text{ est convexe dans } I, \tag{17}$$

où $I = (a, b) \subset \mathbb{R}$ est un intervalle ouvert (borné ou non) et $\overline{I} = [a, b] \subset \overline{\mathbb{R}}$. Notre but est toujours de classifier les solutions stables.

Remarque 0.25. Si $f(u) = u_+^p$, on sait que $u_+^p \in C^2$ seulement si $p \geq 2$. Alors, quand on étudiera le cas particulier $f(u) = u_+^p$ avec tout $p > 1$, on n'a pas besoin de la condition $f \in C^2$.

Dupaigne et Farina ont étudié le cas autonome dans [30, 31] et ils ont montré des résultats du type Liouville. Notre but est de savoir si des résultats similaires existent pour $\alpha > -2$. Ici la présence du potentiel $|x|^\alpha$ soulève pas mal de difficultés, mais nous pouvons aussi emprunter les idées principales utilisées dans [30].

Comme dans [30], nous introduisons encore l'ensemble de zéros $Z(f)$ de la fonction $f(u)$:

$$Z(f) = \{u \in \overline{I} : f(u) = 0\}.$$

En observant les hypothèses sur f, nous constatons que seulement quatre cas peuvent se présenter :

1. $Z(f) = \emptyset$. Alors $f(u) \geq C > 0$.
2. $Z(f) = \{-\infty\}$. Par exemple $f(u) = e^u$ avec $I = \mathbb{R}$.
3. $Z(f) = \{c\}$ pour $c \in I$. Par exemple $f(u) = u^p$ avec $p > 1$ et $I = \mathbb{R}_+$.
4. $Z(f) = [c, d]$ pour $c, d \in I$. Par exemple $f(u) = (u - d)_+^p$ avec $p > 2$ et $I = \mathbb{R}$.

On ne considère pas le premier cas $Z(f) = \emptyset$. En fait, on peut toujours supposer $f(u) \equiv 1$ car $f(u) \geq C > 0$, alors (16) devient $-\Delta u = |x|^\alpha$, dont l'existence d'une solution stable est évidente et il n'y a pas de résultat de type Liouville dans ce cas. À partir de maintenant, on suppose que

$$Z(f) \neq \emptyset. \tag{18}$$

Avant l'étude de l'équation générale (16), nous considérons tout d'abord un exemple - une équation spéciale du type Hénon dans $\mathbb{R}^N$ ou dans $\mathbb{R}_+^N$ avec condition de Dirichlet au bord. Plus précisément, on considère

$$-\Delta u = |x|^\alpha u_+^p \quad \text{dans } \mathbb{R}^N \tag{19}$$

et

$$-\Delta u = |x|^\alpha u_+^p \ \text{ dans } \ \mathbb{R}_+^N, \quad u = 0 \ \text{ sur } \ \partial\mathbb{R}_+^N. \tag{20}$$

Ici $p > 1$, $\alpha > -2$, $N \geq 2$ et $u_+ := \max(u, 0)$ est la partie positive de u.

Dans [54], le cas autonome avec $\alpha = 0$ est déjà étudié, Rebhi a montré que la solution classique stable de (19) (resp. (20)) est $u \equiv c$ (resp. $u \equiv cx_N$) avec $c \leq 0$ si $1 < p < p_{JL}$, où $p_{JL} = p(N, 0)$ représente toujours l'exposant de Joseph-Lundgren. De plus, il a aussi prouvé que pour (19), la seule solution classique stable en dehors d'un ensemble compact si $1 < p < p_{JL}$ et $p \neq \frac{N+2}{N-2}$ est constante négative.

Nous voulons considérer les solutions faibles de (19) et (20), qui ne sont pas censées être localement bornée avec tout $\alpha > -2$. Les techniques que nous avons utilisées pour l'équation (9) ou l'équation (12) ne marchent pas automatiquement pour l'équation (19) ou l'équation (20). Par conséquent, nous allons utiliser la représentation par intégrale des solutions et nous avons besoin des propriétés des fonctions de Green associées.

Comme avant, nous commençons par une estimation du type intégral pour les solutions "localement" stables.

Proposition 0.26. *Soient Ω un domaine dans $\mathbb{R}^N$ avec $N \geq 2$. Soit u une solution faible de* (19)*, stable dans Ω avec $p > 1$ et $\alpha > -2$. Alors pour tout $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ et tout entier $m \geq \max\{\frac{p+\gamma}{p-1}, 2\}$, il existe une constante $C(p, m, \gamma, \alpha) > 0$ telle que*

$$\int_\Omega \left(|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{p+\gamma}\right)\psi^{2m} \leq C\int_\Omega |x|^{-\frac{(\gamma+1)\alpha}{p-1}}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}} \tag{21}$$

pour toutes les fonctions $\psi \in C_c^2(\Omega)$ vérifiant $|\psi| \leq 1$ in Ω.
De même, si on suppose que la solution faible de (19) *u appartient à $H^1_{loc}(\Omega)$ telle que $u = 0$ sur $\partial\Omega$ et u est stable en dehors d'un ensemble compact $\mathcal{K} \subset \Omega$, l'estimation* (21) *reste valable pour toutes les fonctions $\psi \in C_c^2(\mathbb{R}^N\backslash\mathcal{K})$ vérifiant $\|\psi\|_\infty \leq 1$.*

Nous classifions d'abord les solutions faibles stables de (19) et (20). En utilisant la Proposition 0.26 et en étudiant la fonction de Green associée, on a

Théorème 0.27. *Soient $\alpha > -2$, $N \geq 2$ et $1 < p < p(N, \alpha)$. Alors toute solution faible stable de* (19) *ou* (20) *satisfait $u \leq 0$. Plus précisément, pour* (19)*, u est une constante négative et $u \equiv cx_N$ avec $c \leq 0$ pour* (20)*.*

A nouveau, le Théorème 0.27 est optimal. En effet, les solutions stables positives de (12) sont aussi les solutions stables de (19), quand $p \geq p(N, \alpha)$ et $N > 10+4\alpha$.

Ensuite nous établissons un résultat plus délicat sur les solutions faibles avec indice de Morse fini. Dans le cas $N \geq 3$, on a

Théorème 0.28. *Soient $\alpha > -2$, $N \geq 3$ et $1 < p < p(N, \alpha^-)$ avec $p \neq \frac{N+2+2\alpha}{N-2}$. Soit u une solution faible de (19) ou (20) qui est avec indice de Morse fini, alors supp(u_+) est compact. Plus précisément, pour (19), on a soit $u \equiv 0$, soit*

$$u(x) = \frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{|y|^\alpha u_+^p(y)}{|x-y|^{N-2}} dy + c \quad \textit{avec} \quad c < 0.$$

Pour (20), on a soit $u \equiv 0$, soit

$$u(x) = \int_{\mathbb{R}_+^N} |y|^\alpha u_+^p(y) G(x,y) dy + c x_N \quad \textit{avec} \quad c < 0.$$

Ici pour tout $x \in \mathbb{R}_+^N$, $G(x,y)$ est la fonction de Green dans $\mathbb{R}_+^N$ avec condition de Dirichlet au bord :

$$\begin{cases} -\Delta_y G(x,y) = \delta_x(y) & \text{dans} \quad \mathbb{R}_+^N, \\ G(x,y) = 0 & \text{si} \quad y \in \partial\mathbb{R}_+^N, \\ G(x,y) \to 0 & \text{quand} \quad |y| \to \infty. \end{cases} \tag{22}$$

Quand $\alpha > 0$, on ne sait pas si cette classification reste valable pour $p(N, 0) \leq p < p(N, \alpha)$ et $N > 10 + 4\alpha$.

Dans le cas de la dimension 2, on obtient un résultat similaire.

Théorème 0.29. *Soient $\alpha > -2$, $p > 1$ et $N = 2$. Soit u une solution faible de (19) ou (20) qui est avec indice de Morse fini, alors on a supp(u_+) est compact. Plus précisément, pour (19), on a soit u est une constante négative, soit*

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| |y|^\alpha u_+^p(y) dy + c \quad \textit{avec} \quad c \in \mathbb{R}.$$

Pour (20), on a soit $u \equiv 0$, soit

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}_+^2} \log\left(\frac{|\overline{x} - y|}{|x-y|}\right) |y|^\alpha u_+^p(y) dy + c x_2 \quad \textit{avec} \quad c < 0,$$

òu $\overline{x} = (x_1, -x_2)$.

Ensuite on étudie les solutions u qui sont stables près de l'infini dans l'espace entier ou dans le demi-espace.

$$-\Delta u = |x|^\alpha u_+^p \quad \text{dans } B_R^c := \mathbb{R}^N \setminus B(0,R) \tag{23}$$

et

$$-\Delta u = |x|^\alpha u_+^p \quad \text{dans } B_R^c \cap \mathbb{R}_+^N \quad \text{et} \quad u = 0 \quad \text{sur } \partial B_R^c \cap \mathbb{R}_+^N. \tag{24}$$

On ne sait pas classifier les solutions de (23) ou (24), mais on peut déduire l'estimation de décroissance rapide pour u en utilisant la méthode de Farina et l'argument modifié de Pucci-Serrin.

Proposition 0.30. *Soient $\alpha > -2$, $N \geq 3$, $\frac{N+2+2\alpha}{N-2} < p < p(N, \alpha^-)$ et u une solution faible stable de* (23) *ou* (24). *Alors on a*

$$\lim_{|x|\to+\infty} |x|^{\frac{2+\alpha}{p-1}} u_+(x) = 0. \tag{25}$$

Nous revenons maintenant à l'équation générale (16). Notons

$$z := \sup Z(F).$$

Le nombre z joue un rôle crucial dans notre étude comme dans [30]. Nous suivons [30] et donnons quelques définitions.

Définition 0.31. On définit la fonction $q(u)$ comme

$$q(u) = \frac{f'^2}{ff''}(u) = \frac{(\ln f)'}{(\ln f')'}(u) \quad \text{pour tout } u \in \mathbb{R} \text{ et } u > z.$$

De plus les fonctions $\overline{q_0}$, $\underline{q_0}$ and $\overline{q_\infty}$ sont définie comme

$$\overline{q_0} = \limsup_{u\to z^+} q(u), \quad \underline{q_0} = \liminf_{u\to z^+} q(u), \quad \overline{q_\infty} = \limsup_{u\to b^-} q(u).$$

Voici notre résultat principal sur l'équation générale (16).

Théorème 0.32. *Soit $\alpha > -2$ et suppose que f satisfait* (17) *et* (18). *Soit $u \in L^\infty_{loc}(\Omega)$ une solution faible stable de* (16). *Alors u est constante si*

$$0 < \underline{q_0} \leq \overline{q_0} < +\infty, \quad 0 < \overline{q_\infty} < +\infty, \quad et \quad N = 2 \tag{26}$$

ou bien si

$$\overline{q_0} < +\infty, \quad \frac{2(2+\alpha)}{N-2}\left(1 + \frac{1}{\sqrt{\overline{q_0}}}\right) > \frac{1}{\underline{q_0}} \tag{27}$$

et

$$\overline{q_\infty} < +\infty, \quad \frac{2(2+\alpha)}{N-2}\left(1 + \frac{1}{\sqrt{\overline{q_\infty}}}\right) > \frac{1}{\overline{q_\infty}}, \quad et \quad N \geq 3. \tag{28}$$

On peut remarquer que la condition (28) est equivalente à

$$\begin{cases} 0 < \overline{q_\infty} < +\infty, & N < 10 + 4\alpha; \\ 1 < \overline{q_\infty} < +\infty, & N = 10 + 4\alpha; \\ q(N,\alpha) < \overline{q_\infty} < +\infty, & N > 10 + 4\alpha, \end{cases} \tag{29}$$

où $q(N, \alpha)$ est l'exposant conjugué de $p(N, \alpha)$ défini par (13).

Dans beaucoup de cas, le Théorème 0.32 est optimal. Par exemple, si $f(u) = e^u$, $|u|^{p-1}u$ ou u_+^p. En fait, on a déjà montré que l'équation $-\Delta u = |x|^\alpha e^u$ n'a pas de solution faible stable pour $N < 10 + 4\alpha$ et qu'il existe des solutions stables dans $\mathbb{R}^N$ pour $N \geq 10 + 4\alpha$. En définissant $-\infty \in \overline{\mathbb{R}}$ comme une constante on peut dire que pour $N < 10 + 4\alpha$ l'unique solution faible stable est $u \equiv -\infty$. Pour le cas $f(u) = |u|^{p-1}u$ ou $f(u) = u_+^p$, la remarque est similaire.

On considère maintenant la solution avec indice Morse fini de (16). Malheureusement, même avec les conditions du Théorème 0.32, on ne peut pas montrer de résultats du type Liouville sur la solution de (16) avec indice de Morse fini pour f générale. Nous le voyons par un simple contre-exemple en utilisant un résultat de Ni [48] et l'extension harmonique :

Soient $\alpha > 0$, $1 \geq p < \frac{N+2+2\alpha}{N-2}$ et $N \geq 2$. Alors l'équation $-\Delta u = |x|^\alpha u_+^p$ admet des solutions radiales avec u_+ de support compact. Évidemment, elles sont avec indice de Morse fini dans $\mathbb{R}^N$.

Il y a encore beaucoup de problèmes ouverts sur les équations du type Hénon (même dans le cas autonome $\alpha = 0$). J'en présente quelques-uns dans la section 2.5 du Chapitre 2.

Première partie

Analysis of some Reaction-Diffusion-Advection Problem in Biological Mathematics

Chapitre 1

The singular limit of a haptotaxis model with bistable growth

ABSTRACT. We consider a model for haptotaxis with bistable growth and study its singular limit. This yields an interface motion where the normal velocity of the interface depends on the mean curvature and on some nonlocal haptotaxis term. We prove the result for general initial data after establishing a result about generation of interface in a small time.

The most part of this chapter has been published in Communications on Pure and Applied Analysis, (11) 2012, 209–228.

Solutions stables en dehors d'un ensemble compact de deux équations de type Hénon

RÉSUMÉ. On considère un modèle d'haptotaxie avec la prolifération bistable et étudie sa limite singulière. Ca implique un mouvement d'interface, où la vitesse normale de l'interface dépend de la courbure moyenne et d'un terme d'haptotaxie non-local. On montre notre résultat pour les données initiales générales en établissant une propriété sur la génération de l'interface en petit temps.

La majeure partie de ce chapitre a été publiée dans Communications on Pure and Applied Analysis, (11) 2012, 209–228.

1.1 Introduction

In this chapter we consider a model for haptotaxis with growth. Haptotaxis is the directed motion of cells by migration up a gradient of cellular adhesion sites located in the extracellular matrix (ECM). This process appears in tumor invasion and is involved in the first stage of proliferation. It also plays an important role in wound healing.

The basic mechanism involves 3 main cellular components : the tumor cells, the Extracellular Matrix (ECM), and some Matrix Degrading Enzymes (MDE). Tumor cells migrate in response to gradients of some ECM proteins. Those ECM proteins are degraded by MDE, these enzymes being produced by tumor cells themselves. Moreover, both tumor cells and MDE diffuse in the cellular medium but ECM proteins do not diffuse.

This mechanism is reminiscent of chemotaxis, which is accounting for the directed migration of biological individuals (e.g. bacteria) towards higher gradients of some chemical substance. Chemotaxis often works as an aggregating mechanism, which is reflected in the blow-up of solutions of the Keller-Segel model, a phenomenon that has been widely studied in the recent years. However there is a major difference between chemotaxis and haptotaxis : since ECM proteins do not diffuse, instead of the elliptic or parabolic coupling appearing in chemotaxis, the haptotaxis model involves an ODE coupling between the concentration of ECM proteins and the MDE concentration. This is also the case in angiogenesis models but models for haptotaxis involve at least 3 equations, whereas angiogenesis is a coupled system of 2 equations (cf [19]).

We now give a brief review of the mathematical literature related to haptotaxis modelling. The relevant variables are the tumor cells concentration, the Extracellular Matrix concentration (ECM), the Matrix Degrading Enzymes concentration (MDE) as well as the oxygen concentration. A hybrid model using PDEs and cellular automata has been proposed by Anderson [4], involving 4 components : tumor cells, ECM, MDE and Oxygen. Global existence for Anderson's model has been established in dimension $N \leq 3$ in [61]. Our model is a simpler version from this model involving 3 components where we introduce a bistable nonlinearity to model the role of changes in oxygen concentration. For a similar model of haptotaxis with a logistic nonlinearity, global existence has been proved in dimension $N \leq 3$ (see [46] and the references therein). Finally Chaplain, Lolas (see [13] and the references therein) proposed a combined chemotaxis-haptotaxis model with logistic source. Recent results ([59], [58]) show global existence for this model in dimension $N \leq 2$. Complex patterns in haptotaxis models are obtained numerically in [61], and also in [33] and [18].

Our starting point is the haptotaxis model proposed in [61]. In this paper, the authors prove global well-posedness in dimension $N \leq 3$ for a large class of initial data, a result which strongly emphasizes the difference with Keller-Segel chemotaxis odel, where we do not explicitely consider the oxygen concentration as a variable. Instead we replace it by a bistable nonlinearity in the equation for the cell concentration. Precisely let us denote by $u(x,t)$ the tumor cell concentration, by $v(x,t)$ the ECM concentration and by $m(x,t)$ the MDE concentration. We study the initial value Problem (P^ε)

$$(P^\varepsilon) \qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + \dfrac{1}{\varepsilon^2} f(u) & \text{in } \Omega \times (0,T] \\ v_t = -\lambda m v & \text{in } \Omega \times (0,T] \\ m_t = d\Delta m + u - m & \text{in } \Omega \times (0,T] \\ u(x,0) = u_0(x) & x \in \Omega \\ v(x,0) = v_0(x) & x \in \Omega \\ m(x,0) = m_0(x) & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T], \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $\Omega_T = \Omega \times [0,T]$ with $T > 0$, ν is the exterior normal vector on $\partial\Omega$ and $\lambda > 0$, $d > 0$ and $\varepsilon > 0$ are strictly positive constants.

The haptotaxis sensitivity function χ is smooth and satisfies

$$\forall v > 0, \;\; \chi(v) > 0, \;\; \chi'(v) > 0.$$

The growth term f is bistable and is given by

$$\forall u \in \mathbb{R}, \;\; f(u) = u(1-u)(u - \frac{1}{2})$$

so that $\int_0^1 f(u)du = 0$.

We make the following assumptions about the initial data.

1. v_0 is a nonnegative and smooth function in $\overline{\Omega}$ and satisfies the homogeneous Neumann boundary condition
$$\frac{\partial v_0}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.1}$$
2. u_0 and m_0 are nonnegative C^2 functions in $\overline{\Omega}$ and we fix a constant $C_0 > 1$ such that
$$||u_0||_{C^2(\overline{\Omega})} + ||v_0||_{C^2(\overline{\Omega})} + ||m_0||_{C^2(\overline{\Omega})} \leq C_0. \tag{1.2}$$
3. The open set Ω_0 defined by
$$\Omega_0 := \{x \in \Omega, u_0(x) > 1/2\}$$
is connected and $\Omega_0 \subset\subset \Omega$.
4. $\Gamma_0 := \partial\Omega_0$ is a smooth hypersurface without boundary.

Under these assumptions, Ω_0 is a domain enclosed by the initial interface Γ_0 and

$$u_0 > 1/2 \text{ in } \Omega_0, \;\; 0 \leq u_0 < 1/2 \text{ in } \Omega \setminus \overline{\Omega}_0.$$

Firstly the existence of a unique nonnegative solution $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ to Problem (P^ε) for $\varepsilon > 0$ small enough is established in Section 2. Note that it follows from (1.1) and $\frac{\partial m^\varepsilon}{\partial \nu} = 0$ on $\partial\Omega \times [0, T]$ that

$$\frac{\partial v^\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0, T]. \tag{1.3}$$

We are interested in the asymptotic behavior of $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ as $\varepsilon \to 0$. In this singular limit, we show that the solutions converge to the solutions of a free boundary problem where the interface motion is driven by mean curvature plus an haptotaxis term. Precisely the asymptotic limit of Problem (P^ε) as $\varepsilon \to 0$ is given by the following free boundary Problem (P^0)

$$(P^0)\begin{cases} u^0(x,t) = \chi_{\Omega_t}(x) = \begin{cases} 1 \text{ in } \ \Omega_t, t \in [0,T] \\ 0 \text{ in } \ \Omega \setminus \overline{\Omega}_t, t \in [0,T] \end{cases} & \\ v_t^0 = -\lambda m^0 v^0 & \text{in } \Omega \times (0,T] \\ m_t^0 = d\Delta m^0 + u^0 - m^0 & \text{in } \Omega \times (0,T] \\ V_n = -(N-1)\kappa + \dfrac{\partial \chi(v^0)}{\partial n} & \text{on } \Gamma_t = \partial\Omega_t, t \in (0,T] \\ \Gamma_t|_{t=0} = \Gamma_0 & \\ v^0(x,0) = v_0(x) & x \in \Omega \\ m^0(x,0) = m_0(x) & x \in \Omega \\ \dfrac{\partial m^0}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T], \end{cases}$$

where $\Omega_t \subset\subset \Omega$ is a moving domain, $\Gamma_t = \partial\Omega_t$ is the limit interface, n is the exterior normal vector on Γ_t, V_n is the normal velocity of Γ_t in the exterior direction and κ is the mean curvature at each point of Γ_t. We first establish the well-posedness of Problem (P^0) locally in time in Section 3. Our main result is then to prove rigorously the convergence of $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ to (u^0, v^0, m^0) for initial data satisfying the above assumptions. In a first step, we establish the following generation of interface property.

Theorem 1.1.1. *Assume that* (u_0, v_0, m_0) *satisfy the hypotheses* 1-2-3-4. *Let* $0 < \eta < 1/4$ *be an arbitrary constant and define* $\mu = f'(1/2) = 1/4$. *Then there exist* $\varepsilon_0 > 0$ *and* $M_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0]$, *we have that at time* $t^* = \mu^{-1}\varepsilon^2|\ln\varepsilon|$,
(a) for all $x \in \Omega$,

$$0 \le u^\varepsilon(x, t^*) \le 1 + \eta,$$

(b) for all $x \in \Omega$ *such that* $|u_0(x) - \frac{1}{2}| \ge M_0\varepsilon$,

$$\text{if } u_0(x) \ge \frac{1}{2} + M_0\varepsilon, \text{ then } u^\varepsilon(x,t^*) \ge 1 - \eta,$$

$$\text{if } u_0(x) \le \frac{1}{2} - M_0\varepsilon, \text{ then } 0 \le u^\varepsilon(x,t^*) \le \eta.$$

The main result reads as follows.

Theorem 1.1.2. *Assume that* (u_0, v_0, m_0) *satisfy the hypotheses* 1-2-3-4. *Let* $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ *be the solution of Problem* (P^ε) *and let* (u^0, v^0, m^0, Γ) *with* $\Gamma = (\Gamma_t \times \{t\})_{t\in[0,T]}$ *be the smooth solution of the free boundary Problem* (P^0) *on* $[0,T]$. *Then, as* $\varepsilon \to 0$, *the solution* $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ *converges to* (u^0, v^0, m^0) *almost everywhere in* $\bigcup_{0<t\leq T}((\Omega \setminus \Gamma_t) \times t)$. *More precisely,*

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = u^0(x,t) \text{ a.e. in } \bigcup_{0<t\leq T} ((\Omega \setminus \Gamma_t) \times t),$$

and for all $\alpha \in (0,1)$,

$$\lim_{\varepsilon\to 0} ||v^\varepsilon - v^0||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} = 0,$$

$$\lim_{\varepsilon\to 0} ||m^\varepsilon - m^0||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} = 0.$$

We actually prove a stronger convergence result concerning u^ε.

Corollary 1.1.3. *Assume that* (u_0, v_0, m_0) *satisfy the hypotheses* 1-2-3-4. *Then for any* $t \in (0,T]$, *for any* $x \in \Omega \setminus \Gamma_t$,

$$\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = \chi_{\Omega_t}(x) = \begin{cases} 1 & \text{for } x \in \Omega_t \\ 0 & \text{for } x \in \Omega \setminus \overline{\Omega_t} \end{cases} \tag{1.4}$$

Moreover as in [2], we also obtain the following estimate of the distance between the interface Γ_t solution of Problem (P^0) and the set

$$\Gamma_t^\varepsilon := \{x \in \Omega, u^\varepsilon(x,t) = 1/2\}.$$

Theorem 1.1.4. *There exists* $C > 0$ *such that*

$$\Gamma_t^\varepsilon \subset \aleph_{C\varepsilon}(\Gamma_t) \text{ for } 0 \leq t \leq T,$$

where $\aleph_r(\Gamma_t) := \{x \in \Omega, dist(x, \Gamma_t) < r\}$ *is the tubular neighborhood of* Γ_t *of radius* $r > 0$.

Corollary 1.1.5. $\Gamma_t^\varepsilon \to \Gamma_t$ *as* $\varepsilon \to 0$, *uniformly in* $t \in [0,T]$ *in the sense of the Hausdorff distance.*

The organization of this chapter is as follows. In section 2 we show preliminarily the formal derivation of the interface motion euqation, the proof follows the main line explained in [9]. In section 3 we prove some a priori estimates, and prove the existence of a unique global solution for Problem (P^ε). Section 4 is devoted to establishing a comparison principle for Problem (P^ε). In section 5 we prove the well-posedness of the free boundary problem (P^0) and obtain the existence of a smooth unique solution up to some time $T > 0$. In section 6 we establish the property of generation of interface. Finally in section 7 we prove the convergence of the solution of Problem (P^ε) to the solution of Problem (P^0).

1.2 Preliminary - Formal derivation of the interface motion equation

This work has been done in [9] for a similar chemotaxis model, but in order to give an intuitive feeling of the way how Problem (P^0) arises from Problem (P^ε), we rewrite the proof here also for completing our work.

For the given functions v^ε who converging to v^0, we consider the equation of u^ε in Problem (P^ε)

$$u_t^\varepsilon = \Delta u^\varepsilon - \nabla \cdot (u^\varepsilon \nabla \chi(v^\varepsilon)) + \frac{1}{\varepsilon^2} u(1-u)(u-\frac{1}{2}) \tag{1.5}$$

and show heuristically how to derive from (1.5) the motion equation

$$V_n = -(N-1)\kappa + \frac{\partial \chi(v^0)}{\partial n} \text{on } \Gamma_t = \partial\Omega_t, t \in (0,T] \tag{1.6}$$

as ε tends to 0. We take $u^\varepsilon = u$ for simplicity. Rewrite (1.5) as

$$L^\varepsilon u = u_t - \Delta u + \nabla u \cdot \nabla \chi(v) - \frac{1}{\varepsilon^2}\left[u(1-u)(u-\frac{1}{2}) - \varepsilon^2 \Delta(\chi(v))u\right] = 0. \tag{1.7}$$

Note that for $|\eta|$ small enough, the equation $u(1-u)(u-\frac{1}{2}) - \eta u = 0$ has three solutions

$$u_-(\eta) < u_0(\eta) < u_+(\eta) \quad \text{with } u_-(\eta) = 0 \text{ and } u_+(0) = 1.$$

We now define by $U(z,\eta)$ the traveling wave solution associated to this bistable nonlinearity, i.e. the unique solution of the following problem

$$\begin{cases} U_{zz} + c(\eta)U_z + U(1-U)(U-\frac{1}{2}) - \eta U = 0 \\ U(-\infty,\eta) = u_+(\eta),\ U(0,\eta) = \frac{1}{2},\ U(+\infty,\eta) = u_-(\eta) = 0, \end{cases} \tag{1.8}$$

where $c(\eta)$ is the traveling wave velocity and satisfies $c(\eta) = c_1\eta(1+c_2\eta)+O(|\eta|^3)$ for some constants c_1, c_2 and for η close to 0. By looking at the signed distance function $\tilde{d}(x,t)$ defined by (1.40) in Section 7, we have that

1. $\tilde{d}(x,t) < 0$ for $x \in \Omega_t$ and $\tilde{d}(x,t) > 0$ for $x \in \Omega\backslash\Omega_t$.
2. $\tilde{d}(x,t) = 0$ on Γ_t and that $|\nabla\tilde{d}(x,t)| = 1$ in a neighborhood of Γ_t.

Now we can take the logic ansatz that for ε small enough, u can be approximated by

$$\hat{u}(x,t) := U(\frac{\tilde{d}(x,t)}{\varepsilon}, \varepsilon^2 \Delta\chi(v)). \tag{1.9}$$

The direct computation shows that

$$\begin{aligned}
&\hat{u}_t = \tilde{U}_z\frac{\tilde{d}_t}{\varepsilon} + \varepsilon^2\tilde{U}_\eta(\Delta\chi(v))_t = \tilde{U}_z\frac{\tilde{d}_t}{\varepsilon} + O(\varepsilon^2),\\
&\nabla\hat{u} = \tilde{U}_z\frac{\nabla\tilde{d}}{\varepsilon} + \varepsilon^2\tilde{U}_\eta\nabla(\Delta\chi(v)) = \tilde{U}_z\frac{\nabla\tilde{d}}{\varepsilon} + O(\varepsilon^2), \qquad (1.10)\\
&\nabla\hat{u}\cdot\nabla\chi(v) = \tilde{U}_z\frac{\nabla\tilde{d}\cdot\nabla\chi(v)}{\varepsilon} + O(\varepsilon^2)\\
\text{and}\quad &\Delta\hat{u} = \tilde{U}_{zz}\frac{|\nabla\tilde{d}|^2}{\varepsilon^2} + \tilde{U}_z\frac{\Delta\tilde{d}}{\varepsilon} + O(\varepsilon),
\end{aligned}$$

where $\tilde{U}_z, \tilde{U}_{zz}$ and $\tilde{U}_\eta$ are the values of the functions U_z, U_{zz} and U_η at the point $(z,\eta) = (\frac{\tilde{d}(x,t)}{\varepsilon}, \varepsilon^2\Delta\chi(v))$.

It follows from (1.7), (1.8) and (1.10) that

$$\begin{aligned}
L^\varepsilon\hat{u} =& \frac{\tilde{U}_{zz}}{\varepsilon^2}\left(1-|\nabla\tilde{d}|^2\right)\\
&+\frac{\tilde{U}_z}{\varepsilon}\left(\tilde{d}_t - \Delta\tilde{d} + \nabla\tilde{d}\cdot\nabla\chi(v)\right) \qquad (1.11)\\
&+c_1\tilde{U}_z\Delta\chi(v)\\
&+O(\varepsilon).
\end{aligned}$$

Since $L^\varepsilon u = 0$, we need to cancel the higher order terms in (1.11). For x close to Γ, the first term is equal to 0 since $|\nabla\tilde{d}| = 1$. Setting the second term be 0, we get

$$-\tilde{d}_t = -\Delta\tilde{d} + \nabla\tilde{d}\cdot\nabla\chi(v).$$

Now we consider (1.11) on Γ_t, since the exterior normal vector $n = \frac{\nabla\tilde{d}}{|\nabla\tilde{d}|} = \nabla\tilde{d}$, it follows that the mean curve en $x\in\Gamma_t$, $\kappa = \frac{1}{N-1}div(\nabla\tilde{d}) = \frac{1}{N-1}\Delta\tilde{d}$. Moreover we have the normal velocity of Γ_t, $V_n = \frac{-\tilde{d}_t}{|\nabla\tilde{d}|} = -\tilde{d}_t$, therefore (1.11) can be written as

$$V_n = -(N-1)\kappa + \frac{\partial\chi(v)}{\partial n},$$

where we use the fact that $\nabla\tilde{d}\cdot\nabla\chi(v) = n\cdot\nabla\chi(v) = \frac{\partial\chi(v)}{\partial n}$. The above equation is exactly the interface motion equation (1.6).

We note that the third term in $L^\varepsilon\hat{u}$ does not vanish on Γ_t but converges pointwise to 0 away from Γ_t therefore it converges weakly to 0 as a function of x. Assuming that Γ_t is a classical solution to (1.6), we can obtain that $L^\varepsilon\hat{u}(\cdot,t)$ converge weakly to 0 in the sense of Radon measures, for every $t\in[0,T]$.

1.3 Existence and uniqueness of the global solution to Problem (P^ε)

1.3.1 Some a priori estimates

We need some preparing works. For a given $T > 0$ and a given nonnegative function $u_0 \in C^2(\overline{\Omega})$, we define

$$X_T = \{u \in C^0(\overline{\Omega}_T), \quad 0 \le u \le C_0 \text{ in } \Omega_T \text{ and } u(x,0) = u_0(x)\},$$

where $C_0 > 1$ is the constant defined in (1.2). It is convenient to rewrite Problem (P^ε) as an evolution equation for u with a nonlocal coefficient $H(u) = v$, namely

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(H(u))) + \dfrac{1}{\varepsilon^2} f(u) & \text{in } \Omega \times (0,T] \\ u(x,0) = u_0(x), & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{1.12}$$

where for a given function $u = u(x,t) \in X_T$, we define $H(u) = v$ as the first component of the unique solution (v,m) of the auxiliary problem

$$\begin{cases} v_t = -\lambda m v & \text{in } \Omega \times (0,T] \\ m_t = d\Delta m + u - m & \text{in } \Omega \times (0,T] \\ v(x,0) = v_0(x), & x \in \Omega \\ m(x,0) = m_0(x), & x \in \Omega \\ \dfrac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases} \tag{1.13}$$

The functions v_0 and m_0 are given and satisfy 1-2. We give below some a priori estimates on the solution to Problem (P^ε) and state the related properties of H.

Lemma 1.3.1. *For $u \in X_T$, let (v,m) be the solution of Problem* (1.13) *and let $H : X_T \to C^2(\overline{\Omega}_T)$ be the operator defined by $H(u) = v$. Then there exists $C > 0$ only depending on T and Ω such that*

(a) for all $(u_1, u_2) \in X_T^2$ with $0 \le u_1 \le u_2$ in Ω_T, the solution (v_i, m_i) of Problem (1.13) *for $i = 1,2$ satisfies*

$$0 \le m_1 \le m_2 \text{ and } 0 \le v_2 \le v_1 \text{ in } \Omega_T$$

so that the operator H is non-increasing on X_T.

(b) for all $u \in X_T$,

$$||m||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} \le CC_0 \text{ and } \sup_{(x,t)\in\overline{\Omega}_T} \Big| \int_0^t \Delta m(x,s)ds \Big| \le CC_0.$$

(c) for all $u \in X_T$, the function $v = H(u)$ satisfies

$$||v||_{C^0(\overline{\Omega}_T)} \le C_0 \quad \text{ and } \quad ||\nabla v||_{C^0(\overline{\Omega}_T)} + ||\Delta v||_{C^0(\overline{\Omega}_T)} \le CC_0^3.$$

Proof of Lemma 1.3.1. To prove property (a), let $(u_1, u_2) \in X_T^2$ with $0 \leq u_1 \leq u_2$ in Ω_T. Since for $i = 1, 2$

$$(m_i)_t - d\Delta m_i + m_i = u_i \geq 0 \text{ in } \Omega_T,$$

with

$$m_i|_{t=0} = m_0 \geq 0 \text{ and } \frac{\partial m_i}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T],$$

we deduce from the standard maximum principle that $0 \leq m_1 \leq m_2$ in Ω_T.

Next solving the equation $v_t = -\lambda m v$ we get that

$$v_i(x,t) = v_0(x) e^{-\lambda \int_0^t m_i(x,s)ds} \tag{1.14}$$

for all $(x,t) \in \Omega_T$ and $i = 1, 2$, so that $v_1 \geq v_2 \geq 0$ in Ω_T, which proves (a) and shows that H is non-increasing on X_T.

In order to prove (b), note that m satisfies the linear parabolic equation

$$\begin{cases} m_t = d\Delta m + u - m & \text{in } \ \Omega \times (0, T] \\ m(x, 0) = m_0(x) & x \in \Omega \\ \dfrac{\partial m}{\partial \nu} = 0 & \text{on } \ \partial\Omega \times (0, T] \end{cases} \tag{1.15}$$

with $0 \leq u \leq C_0$ in Ω_T and $0 \leq m_0 \leq C_0$ in Ω. Thus it follows from the maximum principle and from standard parabolic estimates that there exists a constant $C > 0$ only depending on T and Ω such that

$$0 \leq m \leq C_0 \text{ in } \Omega_T, \quad ||m||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} \leq CC_0. \tag{1.16}$$

For any fixed $x \in \Omega$, we integrate the equation $m_t - d\Delta m + m = u$ on $[0, t]$ and obtain that

$$\int_0^t \Delta m(x,s)ds = \frac{1}{d}[m(x,t) - m_0(x) + \int_0^t (m(x,s) - u(x,s))ds]$$

so that in view of (1.16) there exists a constant $C > 0$ such that

$$\forall (x,t) \in \Omega_T, \quad |\int_0^t \Delta m(x,s)ds| \leq CC_0 \tag{1.17}$$

which completes the proof of (b).

In order to prove (c), note that by (1.14), $v \geq 0$ and $v_t \leq 0$ in Ω_T so that

$$0 \leq v(x,t) \leq v_0(x) \leq C_0 \text{ for all } (x,t) \in \Omega_T. \tag{1.18}$$

Since for all $(x,t) \in \Omega_T$

$$\nabla v(x,t) = \nabla v_0(x) e^{-\lambda \int_0^t m(x,s)ds} - \lambda v(x,t) \Big(\int_0^t \nabla m(x,s)ds \Big), \tag{1.19}$$

it follows that there exists $C > 0$ such that

$$\begin{aligned}|\nabla v(x,t)| \leq &|\nabla v_0(x)| + \lambda v(x,t)| \int_0^t \nabla m(x,s)ds| \\ \leq &CC_0^2.\end{aligned} \tag{1.20}$$

Since for all $(x,t) \in \Omega_T$

$$\begin{aligned}\Delta v(x,t) =&\Delta v_0(x) e^{-\lambda \int_0^t m(x,s)ds} - 2\lambda \nabla v_0(x).\Big(\int_0^t \nabla m(x,s)ds\Big) e^{-\lambda \int_0^t m(x,s)ds} \\ &+\lambda^2 v(x,t)\Big|\int_0^t \nabla m(x,s)ds\Big|^2 - \lambda v(x,t)\Big(\int_0^t \Delta m(x,s)ds\Big),\end{aligned} \tag{1.21}$$

it follows that

$$\forall (x,t) \in \Omega_T, \quad |\Delta v(x,t)| \leq CC_0^3 + \lambda C_0 |\int_0^t \Delta m(x,s)ds| \tag{1.22}$$

with $C > 0$ a suitable constant. By (1.17), we conclude that there exists $C > 0$ such that

$$\forall (x,t) \in \Omega_T, \quad |\Delta v(x,t)| \leq CC_0^3$$

and obtain the property (c), which completes the proof of Lemma 1.3.1.

1.3.2 Existence of unique global solution to Problem (P^ε)

We prove the existence of a unique solution $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ to Problem (P^ε) on Ω_T for $\varepsilon > 0$ small enough.

Lemma 1.3.2. *Assume that (u_0, v_0, m_0) satisfy the hypotheses* 1-2-3-4. *Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, Problem (P^ε) has a unique solution $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ on $\Omega \times [0,T]$ for any $T > 0$. This solution satisfies $0 \leq u^\varepsilon \leq C_0$ in Ω_T.*

The above lemma is similar to Lemma 4.2 in [10] and we just sketch the proof. It relies on Schauder's fixed point theorem and on the a priori estimates on Problem (P^ε) obtained in Lemma 1.3.1.

Let $T > 0$ be arbitrarily fixed and for all $u \in X_T$, let $v = H(u)$ be defined as above. By the estimates of v in Lemma 1.3.1, there exists $C > 0$ such that

$$0 \leq v \leq C_0, \quad |\nabla v| + |\Delta v| \leq CC_0^3 \text{ in } \Omega_T. \tag{1.23}$$

Let $\tilde{u}$ be the unique solution of

$$\begin{cases}\tilde{u}_t = \Delta \tilde{u} - \nabla \cdot (\tilde{u} \nabla \chi(v)) + \dfrac{1}{\varepsilon^2} f(\tilde{u}) & \text{in } \Omega \times (0,T] \\ \tilde{u}(x,0) = u_0(x) & x \in \Omega \\ \dfrac{\partial \tilde{u}}{\partial \nu} = 0 & \text{on } \partial\Omega.\end{cases} \tag{1.24}$$

The key point of the proof is to show that for $0 < \varepsilon < \varepsilon_0$ small enough, we have

$$0 \le \tilde{u} \le C_0 \text{ in } \Omega_T.$$

This follows from the fact that C_0 is a supersolution for equation (1.24) for $\varepsilon > 0$ small enough. Precisely, using that $f(C_0) < 0$ since $C_0 > 1$ and (1.23), we have that there exists $\tilde{C}_0 > 0$ depending only on C_0 such that

$$\begin{aligned} & C_0\Delta(\chi(v)) - \frac{1}{\varepsilon^2} f(C_0) \\ = \;& C_0(\chi'(v)\Delta v + \chi''(v)|\nabla v|^2) - \frac{1}{\varepsilon^2} f(C_0) \\ \ge \;& -\tilde{C}_0 - \frac{1}{\varepsilon^2} f(C_0) \ge 0 \end{aligned}$$

for $\varepsilon > 0$ small enough. (Note that if χ' and χ'' are uniformly bounded on $\mathbb{R}$, one may choose $\tilde{C}_0 = CC_0^6$, with some constant $C > 0$ independant of C_0.) Moreover $\tilde{u} \in C^{\alpha,\alpha/2}(\overline{\Omega_T})$ for some $\alpha \in (0,1)$. Hence $u \to \tilde{u}$ maps X_T into itself and defines a compact operator. A fixed point of this operator obtained by Schauder's theorem is then a solution to Problem (P^ε). The uniqueness of solution follows from the a priori estimates on Problem (P^ε). For the details of the proof, we refer to [10] and [25].

1.4 A comparison principle for Problem (P^ε)

We first recall the definition of a pair of sub- and super-solutions similar to the one proposed in [10].

Definition 1.4.1. Let $(u_\varepsilon^-, u_\varepsilon^+)$ be two smooth functions with $0 \le u_\varepsilon^- \le u_\varepsilon^+$ in Ω_T and $\frac{\partial u_\varepsilon^-}{\partial \nu} \le 0 \le \frac{\partial u_\varepsilon^+}{\partial \nu}$ on $\partial\Omega \times (0,T)$. By definition, $(u_\varepsilon^-, u_\varepsilon^+)$ is a pair of sub- and super-solutions in Ω_T if for any $v = H(u)$, with $u_\varepsilon^- \le u \le u_\varepsilon^+$ in Ω_T, we have

$$L_v[u_\varepsilon^-] \le 0 \le L_v[u_\varepsilon^+] \quad \text{ in } \Omega_T,$$

where the operator L_v is defined by

$$L_v[\phi] = \phi_t - \Delta\phi + \nabla \cdot (\phi \nabla \chi(v)) - \frac{1}{\varepsilon^2} f(\phi).$$

Note that in Lemma 2.2, $(0, C_0)$ is a pair of sub- and super-solutions of Problem (P^ε) for $\varepsilon > 0$ small enough. It is then proved in [10] that the following comparison principle holds.

Proposition 1.4.2. *Let a pair of sub- and super-solutions $(u_\varepsilon^-, u_\varepsilon^+)$ in Ω_T be given. Assume that*

$$\forall x \in \Omega, \;\; u_\varepsilon^-(x,0) \le u_0(x) \le u_\varepsilon^+(x,0),$$

with (u_0, v_0, m_0) satisfying the hypotheses 1-2*. Then there exists a unique solution $(u^\varepsilon, v^\varepsilon, m^\varepsilon)$ of Problem (P^ε) with*

$$\forall (x,t) \in \Omega_T, \;\; u_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le u_\varepsilon^+(x,t).$$

1.5 Well-posedness of Problem (P^0)

We establish here the existence and uniqueness of a smooth solution to the free boundary Problem (P^0) locally in time.

Theorem 1.5.1. *Let $\Gamma_0 = \partial\Omega_0$, where $\Omega_0 \subset\subset \Omega$ is a $C^{2+\alpha}$ domain with $\alpha \in (0,1)$. Then there exists a time $T > 0$ such that Problem (P^0) has a unique solution (u^0, v^0, m^0, Γ) on $[0,T]$ with*

$$\Gamma = (\Gamma_t \times \{t\})_{t\in[0,T]} \in C^{2+\alpha,(2+\alpha)/2} \textit{ and } v^0|_\Gamma \in C^{1+\alpha,(1+\alpha)/2}.$$

Moreover, if v_0 is smooth, $(\Gamma_t \times \{t\})_{t\in(0,T]}$ is C^∞ in both variables. Also, if v_0 is smooth and Γ_0 is $C^{k+\alpha}$ for some $k \in \mathbb{N}$, $k \geq 2$, then Γ is $C^{k+\alpha,(k+\alpha)/2}$.

The proof of this theorem is using a contraction fixed-point argument in suitable Hölder spaces (see Section 2 in [10]). We show here how it can actually be obtained using the result established in Theorem 2.1 in [10] and some additional properties that we state and prove below.

First we introduce some notations used in [10] that were originally introduced in [17]. We assume that Γ_0 is parametrized by some smooth $(N-1)$-dimensional compact manifold $\mathcal{M}$ without boundaries which divides $\mathbb{R}^N$ into two pieces. We denote by $\vec{N}(s)$ the outward normal vector to $\mathcal{M}$ at $s \in \mathcal{M}$ and define

$$\begin{array}{rcl} X : \mathcal{M} \times (-L,+L) & \to & \mathbb{R}^N \\ (s, s_N) & \mapsto & X(s, s_N) \end{array}$$

where

$$X(s, s_N) = s + s_N \vec{N}(s).$$

If $L > 0$ is chosen small enough, X is a C^∞-diffeomorphism from $\mathcal{M} \times (-L,+L)$ onto a tubular neighborhood of $\mathcal{M}$ that we denote by $\mathcal{M}^L$. We assume that $\Gamma_0 \subset \mathcal{M}^{\frac{L}{2}}$ and is given by

$$\Gamma_0 = \{X(s, s_N),\ s_N = \Lambda_0(s),\ s \in \mathcal{M}\}$$

and that Ω_0 is the connected component of $\Omega \setminus \Gamma_0$ which contains

$$\{x = X(s, s_N),\ s_N < \Lambda_0(s),\ s \in \mathcal{M}\}.$$

According to the regularity hypothesis on Γ_0 in Theorem 1.5.1, Λ_0 is a $C^{2+\alpha}$ function with

$$||\Lambda_0||_{C^0(\mathcal{M})} < \frac{L}{2}.$$

Let $T > 0$ be a fixed constant that will be chosen later. We parametrize the interface $\Gamma = (\Gamma_t)_{t\in[0,T]}$ as follows

$$\Gamma_t = \{X(s, s_N),\ s_N = \Lambda(s,t),\ s \in \mathcal{M}\}, \tag{1.25}$$

where $\Lambda : \mathcal{M} \times [0,T] \to (-L,+L)$ is a function. By definition, we will say that Γ is $C^{k+\alpha,\frac{k+\alpha}{2}}$ if the function Λ satisfies

$$\Lambda \in C^{k+\alpha,\frac{k+\alpha}{2}}(\mathcal{M} \times [0,T]).$$

For any function $v(x,t)$ defined in $\overline{\Omega_T}$, we consider the restriction of v and of ∇v on the interface Γ and we associate to v the functions $w(s,t)$ and $\vec{h}(s,t)$ defined on $\mathcal{M} \times [0,T]$ by

$$w(s,t) = v(X(s,\Lambda(s,t)),t), \tag{1.26}$$
$$\vec{h}(s,t) = \nabla v(X(s,\Lambda(s,t)),t). \tag{1.27}$$

Next we split Problem (P^0) into two subproblems (p_a) and (p_b), where Problem (p_a) is given by

$$(p_a) \qquad \begin{cases} V_n = -(N-1)\kappa + \chi^{'}(w)\vec{h}\cdot\vec{n} \text{ on } \; \Gamma_t = \partial\Omega_t, \; t \in (0,T] \\ \Gamma_t|_{t=0} = \Gamma_0 \end{cases}$$

and Problem (p_b) is given by

$$(p_b) \qquad \begin{cases} v_t = -\lambda m v & \text{in } \; \Omega \times (0,T] \\ m_t - d\Delta m + m = u & \text{in } \; \Omega \times (0,T] \\ \dfrac{\partial m}{\partial \nu} = 0 & \text{on } \; \partial\Omega \times (0,T] \\ u(x,t) = \chi_{\Omega_t}(x) = \begin{cases} 1 \text{ in } \; \Omega_t, t \in [0,T] \\ 0 \text{ in } \; \Omega \setminus \overline{\Omega}_t, t \in [0,T] \end{cases} \end{cases}$$

Note that the difference with the free boundary problem considered in [10] lies in Problem (p_b). Let us consider

$$\forall (x,t) \in \Omega_T, \;\; M(x,t) = \int_0^t m(x,s)ds \tag{1.28}$$

The restrictions of M and ∇M on Γ are denoted $a(s,t)$ and $\vec{b}(s,t)$ and defined on $\mathcal{M} \times [0,T]$ by

$$a(s,t) = M(X(s,\Lambda(s,t)),t), \tag{1.29}$$
$$\vec{b}(s,t) = \nabla M(X(s,\Lambda(s,t)),t). \tag{1.30}$$

Note that using (1.14) and (1.19) we have that

$$w(s,t) = v_0(X(s,\Lambda(s,t)))e^{-\lambda a(s,t)}$$

and

$$\vec{h}(s,t) = \nabla v_0(X(s,\Lambda(s,t)))e^{-\lambda a(s,t)} - \lambda w(s,t)\vec{b}(s,t),$$

so that w has the same regularity as a and $\vec{h}$ has the same regularity as $\vec{b}$, assuming that v_0 is smooth. We deduce from Problem (p_b) that M satisfies

$$\begin{cases} -d\Delta M + M = g(x,t) & \text{in } \Omega \times (0,T] \\ \dfrac{\partial M}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{1.31}$$

where

$$g(x,t) = \int_0^t \chi_{\Omega_s}(x)ds + m_0(x) - m(x,t).$$

Problem (1.31) has been considered in [10] but with a right-hand-side $g(.,t) = \chi_{\Omega_t}$. Here the function $g(x,t)$ is continuous in time, its regularity being the one of a time-integral of χ_{Ω_t}. Thus we can use Theorem 2.2 in [10] and obtain the same (or better) regularity for $(a, \vec{b})$ in the case that we consider here. Let us state this result in the lemma below.

Lemma 1.5.2. *Let* $\Gamma = (\Gamma_t \times \{t\})_{t\in[0,T]}$ *be given by* (1.25) *with*

$$\Lambda \in C^{k+\alpha,\frac{k+\alpha}{2}}(\mathcal{M}\times[0,T])$$

for some $k \in \mathbb{N}$, $k \geq 2$ *and* $\alpha \in (0,1)$. *Let* M *satisfy* (1.31) *and let* a *and* $\vec{b}$ *be associated to* M *by (1.29) and (1.30) respectively. Then*

$$a \in C^{k+\alpha,\frac{k+\alpha}{2}}(\mathcal{M}\times[0,T])$$

and

$$\vec{b} \in [C^{k+\alpha',\frac{k+\alpha'}{2}}(\mathcal{M}\times[0,T])]^n \text{ for all } 0 < \alpha' < \alpha.$$

By the argument in [10] we know then that Problem (p_a) defines a mapping $(w, \vec{h}) \to \Lambda$ and Problem (p_b) defines a mapping $\Lambda \to (w, \vec{h})$ with the proper regularity in Hölder spaces. Therefore the composition of these two mappings defines a contraction in some closed ball for $T > 0$ small enough. The unique fixed point of this contraction is the solution to Problem (P^0) on $[0,T]$. This completes the proof of Theorem 1.5.1.

1.6 Generation of interface

In this section we establish the rapid formation of transition layers in a neighborhood of Γ_0 within a short time interval of order $\varepsilon^2|\ln\varepsilon|$. The width of the transition layer around Γ_0 is of order ε. After a short time the solution u^ε becomes close to 1 or 0 except in a small neighborhood of Γ_0. The precise result is stated in Theorem 1.1.1.

The proof of Theorem 1.1.1 relies on the construction of a suitable pair of sub- and super-solutions involving the solution of the bistable problem (P^ε). We refer to the proof of Theorem 3.1 in [2] in the simple case $\delta = 0$ though the equations are different. For the completeness of this part, we give the sketches of this proof.

In order to construct such a pair of sub- and super-solutions for (P^ε), we define the function $Y(\tau,\xi)$ as the solution of the following ODE

$$\begin{cases} Y_\tau(\tau,\xi) = f\left(Y(\tau,\xi)\right) \quad \text{for} \quad \tau > 0, \\ Y(0,\xi) = \xi \end{cases} \tag{1.32}$$

where $f(u) = u(u-\frac{1}{2})(1-u)$ and $\xi \in (-2C_0, 2C_0)$ with C_0 the constant chosen in (1.2). Note that the bistable function $f(u)$ has 3 zeros : 0, $\frac{1}{2}$ and 1, moreover, $f(u)$ is strictly positive in $(-\infty, 0) \cup (\frac{1}{2}, 1)$ and is strictly negative in $(0, \frac{1}{2}) \cup (1, +\infty)$. By differentiating (1.32) with respect to ξ then to τ, we get some basic properties of Y.

1. $Y_\xi > 0$ for all $\xi \in (-2C_0, 2C_0)\backslash\{0, \frac{1}{2}, 1\}$.
2. We have $Y_\xi(\tau, \xi) = \frac{f(Y(\tau,\xi))}{f(\xi)}$.

Define now a function

$$A(\tau, \xi) = \frac{f'(Y(\tau, \xi)) - f'(\xi)}{f(\xi)}.$$

With the above properties of Y in our hands, we can check that

$$A(\tau, \xi) = \int_0^\tau f''(Y(\tau, \xi))\, Y_\xi(s, \xi) ds, \quad \forall \xi \in (-2C_0, 2C_0)\backslash\{0, \frac{1}{2}, 1\}.$$

We show now some estimates of Y, its derivative and A. We distinguish two situations. Consider firstly the case when the initial value ξ is far from the stable equilibria 0 and 1, i.e. when $\xi \in (\eta, 1-\eta)$. Similar as Lemma 3.5 and Corollary 3.6 in [2], we have

Lemma 1.6.1. *Let $\eta \in (0, 1/4)$ be an arbitrary constant. There exist some constants $C_i(\eta) > 0$, $i = 1, \cdots, 5$, such that,*

1. *when $\xi \in (1/2, 1-\eta)$, then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval $(1/2, 1-\eta)$, we have*

$$\begin{aligned} & C_1 e^{\tau/4} \le Y_\xi(\tau, \xi) \le C_2 e^{\tau/4}, \\ & C_3 e^{\tau/4}(\xi - 1/2) \le Y(\tau, \xi) - 1/2 \le C_4 e^{\tau/4}(\xi - 1/2) \\ \text{and} \quad & |A(\tau, \xi)| \le C_5(e^{\tau/4} - 1). \end{aligned} \tag{1.33}$$

2. *when $\xi \in (\eta, 1/2)$, then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval $(\eta, 1/2)$, the estimates (1.33) also hold true.*

Consider next the case when $\xi \le \eta$ or $\xi \ge 1 - \eta$. We can prove

Lemma 1.6.2. *Let $\eta \in (0, 1/4)$ and $M > 0$ be arbitrary. Then exists positive constant $C_6(M)$ such that,*

1. *when $\xi \in [1-\eta, 1+M]$, then, for every $\tau > 0$, we have $Y(\tau, \xi)$ remains in the interval $[1-\eta, 1+M]$ and*

$$|A(\tau, \xi)| \le C_6 \tau. \tag{1.34}$$

2. *when $\xi \in [-M, \eta]$, then, for every $\tau > 0$, we have $Y(\tau, \xi)$ remains in the interval $[-M, \eta]$ and (1.34) holds as well.*

With all the above preparing works, we now use Y to construct a pair of sub- and super-solutions for Problem (P^ε). Set $\mu = 1/4$ and

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 r(\frac{t}{\varepsilon^2})\right),$$

where the function $r(\tau)$ is given by $r(\tau) = C_7(e^{\tau/4} - 1)$ with C_7 a positive constant chosen later. We can prove the well-defines of $w_\varepsilon^\pm$ by using the process of the proof of Lemma 3.9 in [2], that is

Lemma 1.6.3. *There exist $\varepsilon_0 > 0$ and $C_7 > 0$ such that for all $\varepsilon \in (0, \varepsilon)$, the functions w_ε^- and w_ε^+ are respectively sub- and super-solutions for Problem (P^ε) in the domain $\{(x,t)|x \in \Omega$ and $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]\}$. This means that, for all $x \in \Omega$ and $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$, we have*

$$w_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le w_\varepsilon^+(x,t). \tag{1.35}$$

In order to prove Theorem 1.1.1 we also need present a key estimate on the function Y after a time of order $\tau \sim |\ln\varepsilon|$, which is a simple version of Lemma 3.11 in [2].

Lemma 1.6.4. *Let $0 < \eta < 1/4$ be arbitrary. There exist $\varepsilon_0(\eta) > 0$ and $C_8(\eta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

1. *for all $\xi \in (-2C_0, 2C_0)$,*

$$-\eta \le Y(\mu^{-1}|\ln\varepsilon|, \xi) \le 1 + \eta, \tag{1.36}$$

2. *for all $\xi \in (-2C_0, 2C_0)$ such that $|\xi - \frac{1}{2}| \ge C_8\varepsilon$,*

$$\text{if } \xi \ge \frac{1}{2} + C_8\varepsilon, \text{ then } Y(\mu^{-1}|\ln\varepsilon|, \xi) \ge 1 - \eta, \tag{1.37}$$

$$\text{if } \xi \le \frac{1}{2} - C_8\varepsilon, \text{ then } Y(\mu^{-1}|\ln\varepsilon|, \xi) \le \eta. \tag{1.38}$$

We are definitely ready to prove Theorem 1.1.1. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (1.35) we obtain

$$\begin{aligned}
&Y\left(\mu^{-1}|\ln\varepsilon|, u_0(x) - \varepsilon^2 r\left(\mu^{-1}|\ln\varepsilon|\right)\right)\\
\le &u^\varepsilon\left(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|\right)\\
\le &Y\left(\mu^{-1}|\ln\varepsilon|, u_0(x) + \varepsilon^2 r\left(\mu^{-1}|\ln\varepsilon|\right)\right).
\end{aligned} \tag{1.39}$$

The definition of $r(\tau)$ implies

$$\varepsilon^2 r\left(\mu^{-1}|\ln\varepsilon|\right) = C_7\varepsilon(1-\varepsilon) \in (\frac{C_7}{2}\varepsilon, \frac{3C_7}{2}\varepsilon)$$

for ε_0 small enough, therefore $u_0(x) \pm \varepsilon^2 r\left(\mu^{-1}|\ln\varepsilon|\right) \in (-2C_0, 2C_0)$. Hence the result (a) of Theorem 1.1.1 is a direct consequence of (1.36) and (1.39).

To prove the result (b) of Theorem 1.1.1, remark that when $u_0(x) \geq \frac{1}{2} + M_0\varepsilon$, we have

$$\begin{aligned} u_0(x) - \varepsilon^2 r\left(\mu^{-1}|\ln\varepsilon|\right) &\geq \frac{1}{2} + M_0\varepsilon - \frac{3C_7}{2}\varepsilon \\ &\geq \frac{1}{2} + C_8\varepsilon \quad \text{if we choose } M_0 \text{ large enough.} \end{aligned}$$

Combining (1.39) and the first inequality in (1.37) we get the first inequality in (b) of Theorem 1.1.1. Similarly the second inequality in (b) follows from (1.39) and the second inequality in (1.37), which completes the proof of Theorem 1.1.1. Finally note that the positivity of u^ε in Theorem 1.1.1 is always established in Lemma 1.3.2.

1.7 Convergence from Problem (P^ε) to (P^0)

We split the present section into 2 main parts. In a first step we establish the desired convergence of u^ε to u^0 then prove Corollary 1.1.3. In a second step we give some quick proofs of Theorem 1.1.2 as well as Theorem 1.1.4 and Corollary 1.1.5.

1.7.1 Convergence of u^ε to u^0

In what follows, we first construct a pair of sub- and super-solution $u_\varepsilon^\pm$ for Problem (P^ε) in order to control the function u^ε on $[t^*, T]$, where t^* is the time appearing in Theorem 1.1.1. By the comparison principle it then follows that, if $u_\varepsilon^-(x,0) \leq u^\varepsilon(x,t^*) \leq u_\varepsilon^+(x,0)$, then $u_\varepsilon^-(x,t) \leq u^\varepsilon(x,t+t^*) \leq u_\varepsilon^+(x,t)$ for all $(x,t) \in \Omega_T$. As a result, if both u_ε^+ and u_ε^- converge to u^0, the solution u^ε also converge to u^0 for all $(x,t) \in \Omega_T \setminus \Gamma$.

1.7.1.1 Construction of sub- and super-solutions for Problem (P^ε)

The construction of sub- and super-solutions requires to define the signed distance function to the limit interface.

Definition 1.7.1. Let $\Gamma = \bigcup_{0\leq t\leq T}(\Gamma_t \times t)$ be the solution of the limit geometric motion Problem (P^0). The signed distance function $\tilde{d}(x,t)$ is defined by

$$\tilde{d}(x,t) = \begin{cases} dist(x,\Gamma_t) & \text{for } \ x \in \Omega \setminus \Omega_t \\ -dist(x,\Gamma_t) & \text{for } \ x \in \Omega_t, \end{cases} \tag{1.40}$$

where $dist(x,\Gamma_t)$ is the distance from x to the hyperface Γ_t in Ω. Note that $\tilde{d}(x,t) = 0$ on Γ and that $|\nabla\tilde{d}(x,t)| = 1$ in a neighborhood of Γ.

In fact, rather than working with the function $\tilde{d}(x,t)$, we need a modified signed distance function d defined as follows.

Definition 1.7.2. Let $d_0 > 0$ be small enough such that $\tilde{d}(x,t)$ is smooth in

$$\{(x,t) \in \overline{\Omega} \times [0,T], |\tilde{d}(x,t)| < 3d_0\}$$

with $|\nabla \tilde{d}(x,t)| = 1$ if $|\tilde{d}(x,t)| < 2d_0$ and with for all $t \in [0,T]$,

$$dist(\Gamma_t, \partial\Omega) > 4d_0.$$

We define the modified signed distance function $d(x,t)$ by

$$d(x,t) = \zeta(\tilde{d}(x,t)),$$

where $\zeta(s)$ is a smooth increasing function on $\mathbb{R}$ given by

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leq 2d_0 \\ -3d_0 & \text{if } s \leq -3d_0 \\ 3d_0 & \text{if } s \geq 3d_0. \end{cases} \tag{1.41}$$

Note that $|\nabla d| = 1$ in the region

$$\{(x,t) \in \overline{\Omega} \times [0,T], \ \ |d(x,t)| < 2d_0\}.$$

It follows that at $x \in \Gamma_t$, the exterior normal vector is $n(x,t) = \nabla d(x,t)$ and that the normal velocity and the mean curvature are given respectively by

$$V_n(x,t) = -d_t(x,t), \ \ \kappa = \frac{1}{N-1}\Delta d(x,t).$$

Since the motion law in Problem (P^0) is given by

$$V_n = -(N-1)\kappa + \frac{\partial \chi(v^0)}{\partial n} \text{ on } \Gamma_t,$$

we deduce that

$$d_t - \Delta d + \nabla d.\nabla \chi(v^0) = 0 \text{ on } \Gamma_t = \{x \in \Omega, \ \ d(x,t) = 0\}. \tag{1.42}$$

In view of Theorem 1.5.1 and Lemma 1.5.2, the interface Γ_t is smooth as well as v^0, assuming that Γ^0 and v_0 are smooth. Thus the functions d_t, ∇d, Δd are Lipschitz continuous near Γ_t. Moreover, the restrictions of $\nabla \chi(v^0)$ to $\overline{\Omega_t}$ and to $\Omega \setminus \Omega_t$ respectively are Lipschitz continuous near Γ_t. Therefore from the mean value theorem applied separately on both sides of Γ_t, it follows that there exists $N_0 > 0$ such that

$$\forall (x,t) \in \Omega_T, \ \ |d_t - \Delta d + \nabla d.\nabla \chi(v^0)| \leq N_0 |d(x,t)|. \tag{1.43}$$

Note also that by construction, $\nabla d(x,t) = 0$ in a neighborhood of $\partial\Omega$.

As in [2], the sub- and super-solutions $u_\varepsilon^\pm$ are defined by

$$u_\varepsilon^\pm = U_0\left(\frac{d(x,t) \mp \varepsilon p(t)}{\varepsilon}\right) \pm q(t), \tag{1.44}$$

where $U_0(z)$ is the unique solution of the stationary problem

$$\begin{cases} U_0'' + f(U_0) = 0 \\ U_0(-\infty) = 1, U_0(0) = \frac{1}{2}, U_0(+\infty) = 0 \end{cases} \tag{1.45}$$

and

$$p(t) = -e^{-\beta t/\varepsilon^2} + e^{Lt} + K, \tag{1.46}$$
$$q(t) = \sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}), \tag{1.47}$$

with $L > 0$ and $K > 1$ to be chosen later.

Note that $q = \varepsilon^2 \sigma p_t$ and that the unique solution U_0 of Problem (1.45) has the following properties.

Lemma 1.7.3. *There exist $C > 0$ and $\lambda_0 > 0$ such that the following estimates hold*

$$0 < U_0(z) \le Ce^{-\lambda_0|z|} \quad \textit{for } z \ge 0,$$
$$0 < 1 - U_0(z) \le Ce^{-\lambda_0|z|} \quad \textit{for } z \le 0.$$

In addition, U_0 is strictly decreasing and

$$\forall z \in \mathbb{R}, \ \ |U_0'(z)| + |U_0''(z)| \le Ce^{-\lambda_0|z|}.$$

The proof of Lemma 1.7.3 is given in [10]. We also note that

$$u_\varepsilon^-(x,t) \le U_0(\frac{d(x,t)}{\varepsilon}) \le u_\varepsilon^+(x,t)$$

and that $p(t)$ is bounded for all $0 < \varepsilon < \varepsilon_0$ and $t \in [0,T]$, $\lim_{\varepsilon \to 0} q(t) = 0$ for all $t > 0$. Therefore it follows from the definition of $u_\varepsilon^\pm(x,t)$ that for all $t \in (0,T]$,

$$\lim_{\varepsilon \to 0} u_\varepsilon^\pm(x,t) = \chi_{\Omega_t}(x) = \begin{cases} 1 & \text{for all } (x,t) \in \Omega_t \\ 0 & \text{for all } (x,t) \in \Omega \setminus \overline{\Omega_t} \end{cases} \tag{1.48}$$

The key result of this section is the following lemma.

Lemma 1.7.4. *There exist $\beta > 0, \sigma > 0$ such that for all $K > 1$, we can find $\varepsilon_0 > 0$ and $L > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $(u_\varepsilon^-,\ u_\varepsilon^+)$ is a pair of sub- and super-solutions for Problem (P^ε) in $\overline{\Omega} \times [0,T]$.*

1.7.1.2 Well defined of the sub- and super-solutions

We are ready to prove Lemma 1.7.4. First note that for all $(x,t) \in \overline{\Omega}_T$,

$$u_\varepsilon^-(x,t) \le U_0(\frac{d(x,t)}{\varepsilon}) - q(t) \le U_0(\frac{d(x,t)}{\varepsilon}) + q(t) \le u_\varepsilon^+(x,t).$$

Next, since $\nabla d = 0$ in a neighborhood of $\partial\Omega$, we have that $\frac{\partial u_\varepsilon^\pm}{\partial \nu} = 0$ on $\partial\Omega \times [0,T]$.

Let v be such that $v = H(u)$ with $u_\varepsilon^- \le u \le u_\varepsilon^+$ in Ω_T, we show below that

$$L_v[u_\varepsilon^-] \le 0 \le L_v[u_\varepsilon^+],$$

where the operator L_v is defined by

$$L_v[\phi] = \phi_t - \Delta\phi + \nabla(\phi\nabla\chi(v)) - \frac{1}{\varepsilon^2}f(\phi).$$

Here we just consider the inequality $L_v[u_\varepsilon^+] \ge 0$, because the proof of the other inequality $L_v[u_\varepsilon^-] \le 0$ is obtained by similar arguments. A direct computation gives

$$(u_\varepsilon^+)_t = U_0^{'}(\frac{d_t}{\varepsilon} - p_t) + q_t,$$

$$\nabla u_\varepsilon^+ = U_0^{'}\frac{\nabla d}{\varepsilon},$$

$$\Delta u_\varepsilon^+ = U_0^{''}\frac{|\nabla d|^2}{\varepsilon^2} + U_0^{'}\frac{\Delta d}{\varepsilon},$$

where the value of the function U_0 and its derivatives are taken at the point $\frac{d(x,t)-\varepsilon p(t)}{\varepsilon}$. Moreover the bistable function has the expansions

$$f(u_\varepsilon^+) = f(U_0) + qf^{'}(U_0) + \frac{1}{2}q^2 f^{''}(\theta),$$

where $\theta(x,t)$ is a function satisfying $U_0 < \theta < u_\varepsilon^+$. Hence, combining all the above, we obtain that

$$L_v[u_\varepsilon^+] = (u_\varepsilon^+)_t - \Delta u_\varepsilon^+ + \nabla u_\varepsilon^+\nabla\chi(v) + u_\varepsilon^+\Delta\chi(v) - \frac{1}{\varepsilon^2}f(u_\varepsilon^+) := E_1 + E_2 + E_3 + E_4$$

where

$$E_1 = -\frac{1}{\varepsilon^2}q[f^{'}(U_0) + \frac{1}{2}qf^{''}(\theta)] - U_0^{'}p_t + q_t,$$

$$E_2 = \frac{U_0^{''}}{\varepsilon^2}(1 - |\nabla d|^2),$$

$$E_3 = \frac{U_0^{'}}{\varepsilon}(d_t - \Delta d + \nabla d \cdot \nabla\chi(v_0))$$

$$\text{and } E_4 = \frac{U_0^{'}}{\varepsilon}\nabla d \cdot \nabla(\chi(v) - \chi(v^0)) + u_\varepsilon^+\Delta\chi(v).$$

We first recall some useful inequalities established in [2].

Since $f^{'}(0) = f^{'}(1) = -\frac{1}{2}$, we can find $0 < b < 1/2$ and $m > 0$ such that

$$\text{if } U_0(z) \in [0,b] \cup [1-b,1] \text{ then } f^{'}(U_0(z)) \le -m.$$

Furthermore, since the region $\{z \in \mathbb{R}, U_0(z) \in [b, 1-b]\}$ is compact and $U_0^{'} < 0$ on $\mathbb{R}$, there exists a constant $a_1 > 0$ such that

$$\text{if } U_0(z) \in [b, 1-b] \text{ then } U_0^{'}(z) \le -a_1.$$

Now we define

$$F = \sup_{-1\le t\le 2} (|f(t)| + |f^{'}(t)| + |f^{''}(t)|),$$

$$\beta = \frac{m}{4}, \tag{1.49}$$

and choose σ which satisfies

$$0 < \sigma < min(\sigma_0, \sigma_1, \sigma_2), \tag{1.50}$$

where $\sigma_0 = \frac{a_1}{m+F}$, $\sigma_1 = \frac{1}{\beta+1}$, $\sigma_2 = \frac{4\beta}{F(\beta+1)}$. Hence we obtain that

$$\forall z \in \mathbb{R}, -U_0^{'}(z) - \sigma f^{'}(U_0(z)) \ge 4\sigma\beta.$$

For $K > 1$ arbitrary, we prove below that $L_{v^\varepsilon}[u_\varepsilon^+] \ge 0$ provided that the constants $\varepsilon_0 > 0$ and $L > 0$ are appropriately chosen. From now on, we suppose that the following inequality is satisfied

$$\varepsilon_0^2 L e^{LT} \le 1. \tag{1.51}$$

Then given any $\varepsilon \in (0, \varepsilon_0]$, since $0 < \sigma < \sigma_1$, we have $0 < q(t) < 1$ for all $t \ge 0$. Since $0 < U_0 < 1$, it follows that for all $(x,t) \in \overline{\Omega}_T$,

$$-1 < u_\varepsilon^{\pm}(x,t) < 2. \tag{1.52}$$

The estimates of the terms E_1, E_2 and E_3 are similar to the estimates in [2] and we obtain that

$$E_1 \ge \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + C_1^{'} L e^{Lt},$$

with $C_1 = \sigma\beta^2 > 0$, $C_1^{'} = 2\sigma\beta > 0$. Next assuming as in [2] that

$$e^{LT} + K \le \frac{d_0}{2\varepsilon_0}, \tag{1.53}$$

we have that

$$|E_2| \le \frac{16C}{(e\lambda_0 d_0)^2}(1 + ||\nabla d||_\infty^2) = C_2,$$

with $C_2 > 0$, $C > 0$ and $\lambda_0 > 0$ given in Lemma 1.7.3.

For E_3, we use (1.43) and obtain that

$$|E_3| \le C_3(e^{Lt} + K) + C_3^{'},$$

where $C_3 = N_0 C > 0$ and $C_3^{'} = \frac{N_0 C}{\lambda_0} > 0$, with $N_0 > 0$ given by (1.43).

In order to estimate the term E_4, we need to prove the following inequalities involving the function v and the differences $v - v^0$, $m - m^0$ that we state in the next lemma.

Lemma 1.7.5. *Let u be any function satisfying*

$$u_\varepsilon^- \le u \le u_\varepsilon^+ \text{ in } \Omega_T$$

and let (v, m) be the corresponding solution of Problem (1.13) with $v = H(u)$. Then there exists $C > 0$ depending on T and Ω such that for all $(x,t) \in \Omega_T$,

$$|v(x,t)| + |\nabla v(x,t)| + |\Delta v(x,t)| \le C \tag{1.54}$$

$$|(m - m^0)(x,t)| + |\nabla d(x,t).\int_0^t \nabla(m - m^0)(x,s)ds| \le C\varepsilon p(t) \tag{1.55}$$

$$|(v - v^0)(x,t)| + |\nabla(v - v^0)(x,t)| \le C\varepsilon p(t), \tag{1.56}$$

where (v^0, m^0) are given by the solution of Problem (P^0).

We prove this lemma in the next part. Let us carry on with the proof of Lemma 1.7.4. Write

$$\nabla d \cdot \nabla(\chi(v) - \chi(v^0)) = \chi^{'}(v)\nabla d \cdot \nabla(v - v^0) + (\chi^{'}(v) - \chi^{'}(v^0))\nabla d \cdot \nabla v^0. \tag{1.57}$$

Since v^0 is bounded in $C^{1+\alpha^{'},\frac{1+\alpha^{'}}{2}}$ for any $\alpha^{'} \in (0,1)$, there exists $C > 0$ such that

$$||v^0||_{L^\infty(\Omega_T)} + ||\nabla v^0||_{L^\infty(\Omega_T)} \le C,$$

which combined with (1.57), yields that

$$|\nabla d \cdot \nabla(\chi(v) - \chi(v^0))| \le ||\chi^{'}||_\infty |\nabla d \cdot \nabla(v - v^0)| + C||\nabla d||_\infty ||\chi^{''}||_\infty |v - v^0|, \tag{1.58}$$

where the L^∞-norms of $\chi^{'}$ and $\chi^{''}$ are considered on the interval $(-C, C)$. Therefore, since χ is smooth and $||\nabla d||_\infty$ is bounded, it follows from (1.56) that for all $(x,t) \in \Omega_T$, there exists $C > 0$ such that

$$|\nabla d \cdot \nabla(\chi(v) - \chi(v^0))| \le C\varepsilon p(t). \tag{1.59}$$

Moreover, using the smoothness of χ and inequality (1.54), we obtain that there exists $C' > 0$ such that

$$|\Delta\chi(v)| \le C'. \tag{1.60}$$

Hence by (1.59), (1.60) and the fact that $|u_\varepsilon^+(x,t)| \le 2$, we obtain that for all $(x,t) \in \Omega_T$,

$$|E_4| \le \frac{C}{\varepsilon} C\varepsilon p(t) + 2C'.$$

Finally substituting the expression for p in (1.46), we obtain that there exist $C_4 > 0$, $C_4^{'} > 0$ such that

$$|E_4| \le C_4 + C_4^{'} e^{Lt}.$$

We collect the above four estimates of E_1, E_2, E_3 and E_4 and obtain that

$$\begin{aligned} L_v[u_\varepsilon^+] \ge& \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + C_1^{'} L e^{Lt} - C_2 \\ &- C_3(e^{Lt} + K) - C_3^{'} - C_4 - C_4^{'} e^{Lt} \\ =& \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + (LC_1^{'} - C_3 - C_4^{'})e^{Lt} - C_6, \end{aligned} \tag{1.61}$$

where $C_6 = C_2 + C_3K + C_3' + C_4 > 0$ is a positive constant. Now we set

$$L := \frac{1}{T}\ln\frac{d_0}{4\varepsilon_0},$$

where $\varepsilon_0 > 0$ is small enough so that (1.51) and (1.53) are satisfied and that

$$LC_1' - C_3 - C_4'' \geq \frac{1}{2}LC_1' \text{ and } LC_1' > 4C_6.$$

Therefore

$$L_v[u_\varepsilon^+] \geq \frac{1}{2}LC_1' - C_6 > C_6 > 0.$$

The proof of Lemma 1.7.4 is now completed, with the constants $\beta > 0$, $\sigma > 0$ given in (1.49), (1.50).

1.7.1.3 Some a priori estimates – Proof of Lemma 1.7.5

We prove here the useful tool Lemma 1.7.5. Lemma 1.7.5 gives the key estimate in the proof that $u_\varepsilon^\pm$ are sub-super solutions. It plays the same role as Lemma 4.9 in [10] and Lemma 4.2 in [2]. However the proof is markedly different since the coupling between u and v is given by a system with an ODE and a parabolic equation versus an elliptic equation in the two above references.

First note that (1.54) is established exactly as in Lemma 1.3.1 (c). Concerning the second inequality (1.55), let us recall the following properties of U_0 given in [1].

Lemma 1.7.6. *For all given $a \in \mathbb{R}$ and $z \in \mathbb{R}$, we have the inequality :*

$$|U_0(z+a) - \chi_{]-\infty,0]}(z)| \leq Ce^{-\lambda_0|z+a|} + \chi_{]-a,a]}(z).$$

Define $w(x,t) = m(x,t) - m^0(x,t)$, then w satisfies

$$\begin{cases} w_t - d\Delta w + w = h & \text{in } \Omega_T \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ w(x,0) = 0, & x \in \Omega, \end{cases} \tag{1.62}$$

with $h = u - u^0$ satisfying

$$u_\varepsilon^- - u^0 \leq h \leq u_\varepsilon^+ - u^0 \text{ in } \Omega_T.$$

From the definition of $u_\varepsilon^\pm$ in (1.44) and from Lemma 1.7.6 for $z = \frac{d(x,t)}{\varepsilon}$ and $a = \pm p(t)$, we deduce that for all $(x,t) \in \Omega_T$,

$$|h(x,t)| \leq C(e^{-\lambda|d(x,t)/\varepsilon + p(t)|} + e^{-\lambda|d(x,t)/\varepsilon - p(t)|}) + \chi_{\{|d(x,t)| \leq \varepsilon p(t)\}} + q(t). \tag{1.63}$$

We decompose the proof in two steps.

First step. We establish that there exists $C > 0$ such that

$$\forall (x,t) \in \Omega_T, \ \ |w(x,t)| \leq C\varepsilon p(t). \tag{1.64}$$

Let us define for all $(x,t) \in \Omega_T$,

$$h_1(x,t) = q(t),$$

$$h_2(x,t) = C(e^{-\lambda|d(x,t)/\varepsilon + p(t)|} + e^{-\lambda|d(x,t)/\varepsilon - p(t)|})\chi_{\{|d(x,t)|>d_0\}},$$

$$h_3(x,t) = C(e^{-\lambda|d(x,t)/\varepsilon + p(t)|} + e^{-\lambda|d(x,t)/\varepsilon - p(t)|})\chi_{\{|d(x,t)|\leq d_0\}} + \chi_{\{|d(x,t)|\leq \varepsilon p(t)\}}$$

and denote by $(w_i)_{i=1,2,3}$ the solutions of the three following auxiliary problems

$$(A_i) \begin{cases} (w_i)_t - d\Delta w_i + w_i = h_i & \text{in } \Omega_T \\ \dfrac{\partial w_i}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ w_i(x,0) = 0, & x \in \Omega. \end{cases}$$

Since

$$\forall (x,t) \in \Omega_T, \ \ |h(x,t)| \leq h_1(x,t) + h_2(x,t) + h_3(x,t), \tag{1.65}$$

we deduce from the maximum principle that

$$\forall (x,t) \in \Omega_T, \ \ |w(x,t)| \leq w_1(x,t) + w_2(x,t) + w_3(x,t). \tag{1.66}$$

We now establish estimates for w_i, with $i = 1, 2, 3$.

Concerning w_1, note that since $h_1 = q(t)$, the unique solution of Problem (A_1) is $w_1 = w_1(t)$ with

$$\begin{cases} w_1' + w_1 = q(t) \\ w_1(0) = 0 \end{cases} \tag{1.67}$$

so that, since $q(t) = \varepsilon^2 \sigma p'(t)$,

$$w_1(t) = e^{-t}\int_0^t e^s q(s)ds \leq \varepsilon^2\sigma(p(t) - p(0))$$

which implies that there exists $C > 0$ and $\varepsilon_0 > 0$ such that for all $t \in [0,T]$ and for $0 < \varepsilon \leq \varepsilon_0$

$$0 \leq w_1(t) \leq C\varepsilon p(t). \tag{1.68}$$

Concerning w_2, note first that in view of the definition of $p(t)$ and the inequality (1.53), we have that for all $t \in [0,T]$

$$0 < K - 1 \leq p(t) \leq \frac{d_0}{2\varepsilon_0} \tag{1.69}$$

so that in particular choosing $\varepsilon > 0$ small enough,

$$\varepsilon p(t) \leq d_0/2 \text{ for all } t \in [0,T]. \tag{1.70}$$

Thus in view of the definition of h_2, using (1.69), there exists a constant $C' > 0$ such that for all $(s,t) \in [0,T]^2$ and for all $y \in \Omega$,

$$\begin{aligned} 0 \leq h_2(y,s) &\leq 2Ce^{-\lambda_0(d_0/\varepsilon - p(s))} \\ &\leq 2Ce^{-\lambda_0 d_0/2\varepsilon} \\ &\leq \frac{4C}{\lambda_0 d_0 e}\varepsilon \\ &\leq \frac{4C}{\lambda_0 d_0 e(K-1)}\varepsilon p(s) \\ &\leq C_1 \varepsilon p(s) \leq C'\varepsilon p(t). \end{aligned} \tag{1.71}$$

Thus by standard parabolic estimates, we obtain that for all $(x,t) \in \Omega_T$ and for some $C > 0$,

$$|w_2(x,t)| + |\nabla w_2(x,t)| \leq C\varepsilon p(t). \tag{1.72}$$

Concerning w_3, it follows from (1.70) that $h_3(y,s)$ is supported in $\{|d(y,s)| \leq d_0\}$. Moreover by linearity we may suppose that the function h_3 satisfies one of the three following assumptions :

$$(H_1) \qquad |h_3(y,s)| \leq \chi_{\{|d(y,s)| \leq \varepsilon p(s)\}}$$

$$(H_2^{\pm}) \qquad |h_3(y,s)| \leq e^{-\lambda_0 |d(y,s)/\varepsilon \pm p(s)|}$$

Then under respectively assumptions (H_1), $(H_2^{\pm})$, we define a function $\tilde{h}$ on $\mathbb{R} \times [0,T]$, respectively by

$$\tilde{h}(r,s) = \begin{cases} \chi_{\{|r| \leq \varepsilon p(s)\}} \\ e^{-\lambda_0 |r/\varepsilon \pm p(s)|} \end{cases}$$

so that

$$\forall (y,s) \in \Omega \times [0,T], \;\; |h_3(y,s)| \leq \tilde{h}(d(y,s),s)\chi_{|d(y,s)| \leq d_0}. \tag{1.73}$$

Under either of the assumptions (H_1) or $(H_2^{\pm})$, there exists a constant $C > 0$ such that for all $(s,t) \in [0,T]^2$

$$0 \leq \int_{-d_0}^{d_0} \tilde{h}(r,s)dr \leq C\varepsilon p(t). \tag{1.74}$$

Let $\varphi(x,t) = e^t w_3(x,t)$, then in view of Problem (A_3), the function φ satisfies

$$\begin{cases} \varphi_t - d\Delta\varphi = f & \text{in } \Omega_T \\ \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{1.75}$$

where $f(x,t) = e^t h_3(x,t)$ and $\varphi(x,0) = w_3(x,0) = 0$ for all $x \in \Omega$. We establish now that there exists a constant $C > 0$ such that

$$\forall (x,t) \in \Omega_T, \;\; 0 \leq \varphi(x,t) \leq C\varepsilon p(t). \tag{1.76}$$

As in [3], the solution $\varphi(x,t)$ of Problem (1.75) can be expressed as

$$\varphi(x,t) = \int_0^t \int_{|d(y,s)|\leq d_0} G_p(x,y,t-s) f(y,s) dyds,$$

with $G_p(x,y,t)$ being the Green function associated to the Neumann boundary value problem in Ω for the parabolic operator $\varphi_t - d\Delta\varphi$. Thus for all $(x,t) \in \Omega_T$,

$$0 \leq \varphi(x,t) \leq \int_0^t \int_{|d(y,s)|\leq d_0} G_p(x,y,t-s) e^s \tilde{h}(d(y,s),s) dyds. \tag{1.77}$$

Next we recall the following important property of G_p which is established in [3].

Lemma 1.7.7 (Alfaro-Hilhorst-Matano, [3])**.** *Let Γ be a closed hypersurface in Ω and denote by $d(y)$ the signed distance function associated with Γ. Then there exists constants $C, d_0 > 0$ such that for any function $\eta(r) \geq 0$ on $\mathbb{R}$, it holds that*

$$\int_{|d|\leq d_0} G_p(x,y,t)\eta(d(y))dy \leq \frac{C}{\sqrt{t}} \int_{-d_0}^{d_0} \eta(r)dr \text{ for } 0 < t \leq T.$$

Moreover as pointed out in [3], the above inequality is uniform with respect to $x \in \Omega$, $t \in [0,T]$ and to smooth variations of Γ. Applying this inequality to our case, we deduce that there exists $C > 0$ such that for all $(x,y) \in \Omega^2$ and for all $0 \leq s < t \leq T$,

$$\int_{|d(y,s)|\leq d_0} G_p(x,y,t-s)\tilde{h}(d(y,s),s)dy \leq \frac{C}{\sqrt{t-s}} \int_{-d_0}^{d_0} \tilde{h}(r,s)dr. \tag{1.78}$$

In view of (1.77) and of (1.74), it follows that for all $x \in \Omega$ and for all $t \in [0,T]$,

$$\begin{aligned} 0 \leq \varphi(x,t) &\leq C \int_0^t \int_{|d(y,s)|\leq d_0} G_p(x,y,t-s)\tilde{h}(d(y,s),s)dyds \\ &\leq C' \int_0^t \frac{1}{\sqrt{t-s}} \int_{-d_0}^{d_0} \tilde{h}(r,s)drds \\ &\leq C' \int_0^t \frac{1}{\sqrt{t-s}} \varepsilon p(t) ds \leq 2C'\varepsilon p(t)\sqrt{T} \end{aligned}$$

which yields inequality (1.76). Coming back to w_3, we deduce that for all $(x,t) \in \Omega_T$,

$$|w_3(x,t)| = |e^{-t}\varphi(x,t)| \leq C\varepsilon p(t). \tag{1.79}$$

The inequality (1.64) directly follows from inequalities (1.68), (1.72), (1.79) and (1.66).

Second step. We establish below that there exists $C > 0$ such that

$$\forall (x,t) \in \Omega_T, \ \ |\nabla d(x,t).\nabla W(x,t)| \leq C\varepsilon p(t), \tag{1.80}$$

where we define

$$\forall (x,t) \in \Omega_T, \ \ W(x,t) = \int_0^t w(x,s)ds. \tag{1.81}$$

Note that time integration on $[0,t]$ of the equation for w in Problem (1.62) gives

$$w(x,t) - w(x,0) - d\Delta W(x,t) + W(x,t) = H(x,t),$$

where for all $(x,t) \in \Omega_T$,

$$H(x,t) = \int_0^t h(x,s)ds.$$

Since $w(x,0) = 0$, we obtain that $W(,t)$ is solution of the following elliptic problem for any fixed $t \in [0,T]$,

$$\begin{cases} -d\Delta W(.,t) + W(.,t) = H(.,t) - w(.,t) & \text{in } \Omega \\ \dfrac{\partial W}{\partial \nu}(.,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.82}$$

Let us denote by (E_f) the following elliptic problem with right-hand-side $f = f(x)$,

$$(E_f) \begin{cases} -d\Delta \phi + \phi = f & \text{in } \Omega \\ \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that by linearity

$$\forall (x,t) \in \Omega_T, \;\; W(x,t) = A(x,t) - B(x,t), \tag{1.83}$$

where $A(.,t)$ is the solution of Problem $(E_{H(.,t)})$ and $B(.,t)$ is the solution of Problem $(E_{w(.,t)})$. Concerning B, it follows from standard elliptic estimates in view of (1.64) that

$$|B(x,t)| + |\nabla B(x,t)| \leq C\varepsilon p(t). \tag{1.84}$$

Concerning A, let us define for any $t \in [0,T]$ the function

$$a(.,t) = \frac{\partial A}{\partial t}(.,t)$$

which is the solution of Problem $(E_{h(.,t)})$. Using the inequalities (1.65), (1.71) and (1.73), we obtain that there exists $C \geq 1$ such that for all $(y,s) \in \Omega \times [0,T]$,

$$|h(y,s)| \leq C(\varepsilon p(t) + q(t)) + \tilde{h}(d(y,s),s)\chi_{|d(y,s)| \leq d_0}, \tag{1.85}$$

with $\tilde{h}$ satisfying (1.74). Note that the elliptic problem (E_f) is the same as the one appearing in the chemotaxis-growth system studied in [10] and in [2], with the right-hand-side satisfying (1.74). Therefore the results stated in Lemma 4.2 in [2] and in Lemma 4.9 in [10] apply and prove that there exists a constant $C > 0$ such that for all $(x,t) \in \Omega_T$,

$$|\nabla d(x,t).\nabla a(x,t)| \leq C(\varepsilon p(t) + q(t))$$

and consequently, since $A(x,t) = \int_0^t a(x,s)ds$ and $q(t) = \varepsilon^2 \sigma p'(t)$, we deduce that there exists a constant $C > 0$ such that for all $(x,t) \in \Omega_T$,

$$|\nabla d(x,t).\nabla A(x,t)| \leq C\varepsilon p(t).$$

In view of (1.83) and (1.84), this completes the proof of (1.80). Therefore inequality (1.55) is now established.

In order to prove inequality (1.56), note that using (1.55) we obtain that for all $(x,t) \in \Omega_T$,

$$\begin{aligned} |(v-v^0)(x,t)| &= |v_0(x)e^{-\lambda\int_0^t m(x,s)ds} - v_0(x)e^{-\lambda\int_0^t m^0(x,s)ds}| \\ &\leq C|v_0(x)|| \int_0^t (m-m^0)(x,s)ds| \leq C'\varepsilon p(t), \end{aligned} \tag{1.86}$$

where $C' > 0$ is a suitable constant.

Next using (1.19) for ∇v and ∇v^0 we have similarly that for all $(x,t) \in \Omega_T$,

$$\begin{aligned} &|\nabla d(x,t) \cdot \nabla(v-v^0)(x,t)| \leq C|e^{-\lambda\int_0^t m(x,s)ds} - e^{-\lambda\int_0^t m^0(x,s)ds}| \\ &+C|v(x,t)\nabla d(x,t) \cdot \int_0^t \nabla m(x,s)ds - v^0(x,t)\nabla d(x,t) \cdot \int_0^t \nabla m^0(x,s)ds| \\ &\leq C'\varepsilon p(t) + C|v(x,t)||\nabla d(x,t) \cdot \int_0^t \nabla(m-m^0)(x,s)ds| \\ &+C|v(x,t)-v^0(x,t)||\nabla d(x,t) \cdot \int_0^t \nabla m^0(x,s)ds|, \end{aligned}$$

where $C, C' > 0$ are suitable constants. Using (1.86), (1.55) and the upper bounds on $|v|$ and $|\nabla m^0|$, we deduce that (1.56) is satisfied. This completes the proof of Lemma 5.6.

1.7.1.4 Convergence of u^ε to u^0

It's time to prove Corollary 1.1.3. Indeed, the pointwise convergence of u^ε to u^0 in $\bigcup_{0<t\leq T}((\Omega \setminus \Gamma_t) \times t)$ when $\varepsilon \to 0$ follows directly from Lemma 1.7.4 and from (1.48). So we are done.

1.7.2 Other convergence results

With Corollary 1.1.3 in hands, we give briefly the demonstrations of Theorem 1.1.2, Theorem 1.1.4 and Corollary 1.1.5 in this subsection and finish this chapter.

1.7.2.1 Proof of Theorem 1.1.2

The rest work is to show the convergence from m^ε to m^0 and from v^ε to v^0.

Note that $w^\varepsilon = m^\varepsilon - m^0$ is a solution of Problem (1.62) with the right-hand-side h^ε satisfying

$$|h^\varepsilon(x,t)| \leq h_1(x,t) + h_2(x,t) + h_3(x,t)$$

with h_i, $i = 1, 2, 3$ defined as in the proof of Lemma 1.7.5. This shows that for any $p > 1$, there exists $C_p > 0$ such that

$$||h^\varepsilon||_{L^p(\Omega_T)} \leq C\varepsilon^{\frac{1}{p}}.$$

It follows then from standard parabolic estimates and Sobolev inequalities that for any $\alpha \in (0,1)$ there exist $p \in (1,+\infty]$ and $C > 0$ such that

$$\begin{aligned} ||w^\varepsilon||_{C^{1+\alpha,1+\alpha/2}(\overline{\Omega}_T)} \leq & C||u^\varepsilon - u^0||_{L^p(\Omega_T)} \\ \leq & C||h^\varepsilon||_{L^p(\Omega_T)} \leq C_p \varepsilon^{\frac{1}{p}}. \end{aligned} \tag{1.87}$$

Thus for any $\alpha \in (0,1)$,

$$\lim_{\varepsilon \to 0} ||m^\varepsilon - m^0||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} = 0.$$

The expression of v^ε and ∇v^ε in (1.14) and (1.19) then show that

$$\lim_{\varepsilon \to 0} ||v^\varepsilon - v^0||_{C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_T)} = 0$$

which completes the proof of Theorem 1.1.2.

1.7.2.2 Proof of Theorem 1.1.4 and Corollary 1.1.5

The proofs are exactly the same as the proofs of Theorem 1.5 and Corollary 1.6 in [2] respectively, we omit the details here.

Deuxième partie

Classification of Solutions for Some Hénon Type Elliptic Equations

Chapitre 2

Stable solutions of two classical Hénon equations

ABSTRACT. We investigate the nonlinear elliptic equations $-\Delta u = |x|^\alpha e^u$ and $-\Delta u = |x|^\alpha |u|^{p-1}u$ in $\Omega \subset \mathbb{R}^N$ with $\alpha > -2$, $p > 1$ and $N \geq 2$. In particular, we prove some Liouville type theorems for weak solutions stable in $\mathbb{R}^N$, and outside a compact set (so for finite Morse index solutions) in the low dimensional Euclidean spaces or half spaces.

The most part of this chapter has been published in Journal of Functional Analysis, (262) 2012, 1705–1727.

Solutions stables de deux équations du type Hénon

RÉSUMÉ. On considère les équations elliptiques non-linéaires $-\Delta u = |x|^\alpha e^u$ et $-\Delta u = |x|^\alpha |u|^{p-1}u$ dans $\Omega \subset \mathbb{R}^N$ avec $\alpha > -2$, $p > 1$ et $N \geq 2$. En particulier, on démontre quelques résultats de type Liouville pour les solutions faibles stables dans $\mathbb{R}^N$, ou bien en dehors d'un ensemble compact (en particulier, pour les solutions avec indice de Morse fini) dans des espaces euclidiens ou dans des demi-espaces de petite dimension.

La majeure partie de ce chapitre a été publiée dans Journal of Functional Analysis, (262) 2012, 1705–1727.

2.1 Introduction

In this chapter, we consider two Hénon type elliptic equations as

$$-\Delta u = |x|^{\alpha} e^{u} \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{2.1}$$

and

$$-\Delta u = |x|^{\alpha} |u|^{p-1} u \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{2.2}$$

where $\alpha > -2$, $p > 1$ and $N \geq 2$.

The study of stable solutions in the autonomous case, i.e. when $\alpha = 0$ has been studied recently, Farina classified completely in [35] all stable solutions and finite Morse index classical solutions of (2.2) in $\mathbb{R}^N$ for $1 < p < p_{JL}$, where

$$p_{JL} = p(N, 0) = \begin{cases} \infty, & \text{if } N \leq 10 \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}, & \text{if } N > 10 \end{cases} \tag{2.3}$$

stands for the Joseph-Lundgren exponent (see p_{JL} in [43] and $p(N, \alpha)$ in Theorem 2.1.3 below). More precisely, he proved that

Theorem 2.1.1 (Farina, [35])**.** *Let u be a classical stable solution of $-\Delta u = |u|^{p-1}u$ in $\mathbb{R}^N$ with $1 < p < p(N, 0)$, then $u \equiv 0$.*

Theorem 2.1.2 (Farina, [35])**.** *Let u be a classical solution stable outside a compact set of $-\Delta u = |u|^{p-1}u$ in $\mathbb{R}^N$ with $1 < p < p(N, 0)$ and $p \neq \frac{N+2}{N-2}$, then $u \equiv 0$.*

Moreover we need to mention that, for the autonomous case of (2.2), the positive radial solutions have been studied by Wang in [63] and the weak solutions are studied by Davila-Dupaigne-Farina in [23].

For the exponential case, it was shown by Farina in [36] that $\Delta u + e^u = 0$ has no stable classical solution in $\mathbb{R}^N$ for $2 \leq N \leq 9$. He proved also that any classical solution of $\Delta u + e^u = 0$ with finite Morse index in $\mathbb{R}^2$ verifies $e^u \in L^1(\mathbb{R}^2)$, so it must be a solution classified by Chen & Li [14], that is

$$u(x) = \ln \left[\frac{32\lambda^2}{(4 + \lambda^2 |x - x_0|^2)^2} \right] \quad \text{with} \quad \lambda > 0, \ x_0 \in \mathbb{R}^2.$$

Finally, when $N \geq 3$, Dancer & Farina proved in [22] that the equation (2.1) with $\alpha = 0$ admits classical entire solution which are stable outside a compact set, if and only if $N \geq 10$.

A natural question is to ask if similar results can be observed for the nonautonomous case, i.e. when $\alpha \neq 0$. The equation (2.2) has been considered by Dancer, Du & Guo in [21], they proved

Theorem 2.1.3 (Dancer-Du-Guo, [21]). *Let $\alpha > -2$ and $u \in H^1_{loc} \cap L^\infty_{loc}(\mathbb{R}^N)$ be a stable solution of* (2.2) *in $\mathbb{R}^N$ with $1 < p < p(N, \alpha)$, where*

$$p(N,\alpha) = \begin{cases} \infty, & \text{if } N \le 10 + 4\alpha \\ \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) + 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}, & \text{if } N > 10 + 4\alpha. \end{cases}$$

Then $u \equiv 0$. On the other hand, for $N > 10 + 4\alpha$ and $p \ge p(N, \alpha)$, (2.2) *admits a family of stable positive radial solutions in $\mathbb{R}^N$.*

Dancer, Du and Guo have also studied *positive* solutions either in a punctured domain $\Omega \setminus \{0\}$ or in an exterior domain $\mathbb{R}^N \setminus \Omega$, they obtained interesting results on asymptotic behavior when $|x|$ tends to 0 or ∞, for solutions which are stable respectively near the origin or outside a compact set. Then they used these estimates and [8] to get some classification results, see more details in [21] and also [28]. In [34], Esposito studied the stability of solutions to the Hénon type equations (2.1), (2.2) in $\mathbb{R}^N$ and proved some Liouville results with $\alpha \ge 0$ and respectively bounded e^u or $|u|$. It is also worthy to mention that in [27, 34, 24], positive entire solutions of $\Delta u = |x|^\alpha u^{-p}$ $(p > 0)$, with finite Morse index or stable in an exterior domain are classified.

Although we borrow many ideas from the previous works, we try to handle the problems in more general setting by three folds.

1. In all previous works for (2.1) and (2.2), the authors considered solutions with locally or globally bounded e^u or $|u|$. Here we deal with weak solutions which are not supposed *a priori* to be locally upper or lower bounded. For example, if $\alpha < 0$ and $0 \in \Omega$, any weak solution of (2.1) or (2.2) cannot be a classical solution, due to the singularity at the origin.
2. We work with general $\alpha > -2$, and classify not only stable solution but also entire solution stable out of a compact set for (2.1), or with finite Morse index for (2.2) under suitable conditions on p, which were not considered in [21, 34].
3. For the equation (2.2), we do not impose any sign condition for u and prove the fast decay behavior near 0 (resp. ∞) for weak solutions which are stable near the origin (resp. outside a compact set) with suitable exponent p. Finally we consider also finite Morse index solutions in the half space $\mathbb{R}^N_+$.

In order to state our results more accurately, let us precise the meaning of weak solution and recall some basic notions. For simplicity, we assume always that Ω is a regular domain in $\mathbb{R}^N$ and $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is a C^1 function for almost every $x \in \Omega$.

Definition 2.1.4. 1. We say that u is a weak solution of $-\Delta u = f(x, u)$ in domain $\Omega \subset \mathbb{R}^N$ (bounded or not), if $u \in H^1_{loc}(\Omega)$ verifies $f(x, u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega \Big[\nabla u \cdot \nabla \psi - f(x,u)\psi\Big] dx = 0, \quad \forall\, \psi \in C^1_c(\Omega). \tag{2.4}$$

Here and in the following $C_c^k(\Omega)$ denotes the set of C^k functions with compact support in Ω.

2. Let u be a weak solution of $-\Delta u = f(x,u)$ in Ω, we say that u is stable if $\partial_u f(x,u) \in L^1_{loc}(\Omega)$ and

$$Q_u(\psi) := \int_\Omega \left[|\nabla\psi|^2 - \partial_u f(x,u)\psi^2\right]dx \geq 0 \quad \text{for all } \psi \in C_c^1(\Omega). \tag{2.5}$$

3. The Morse index of a solution u, $\text{ind}(u)$ is defined as the maximal dimension of all subspaces X of $C_c^1(\Omega)$ such that $Q_u(\psi) < 0$ for any $\psi \in X\backslash\{0\}$.
4. A weak solution u of $-\Delta u = f(x,u)$ in Ω is said sable outside a compact set $\mathcal{K}$, if $Q_u(\psi) \geq 0$ for any $\psi \in C_c^1(\Omega\backslash\mathcal{K})$.

Remark 2.1.5. 1. When $\partial_u f(x,u) \geq 0$, (2.5) holds for any $\psi \in H_0^1(\Omega)$ by density argument. It is the case for the equations (2.1) and (2.2).

2. u is stable if and only if $\text{ind}(u) = 0$.
3. Any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \Omega$. Indeed, for $\ell = \text{ind}(u) \geq 0$, there exists $X = \text{span}\{\varphi_1, \cdots, \varphi_\ell\} \subset C_c^1(\Omega)$ such that $Q_u(\varphi) < 0$ for any $\varphi \in X\backslash\{0\}$, so $Q_u(\psi) \geq 0$ for all $\psi \in C_c^1(\Omega\backslash\mathcal{K})$, where $\mathcal{K} = \bigcup_j \text{supp}(\varphi_j)$.

We should mention that when $\alpha = 0$, it is proved in [23] that any weak solution of (2.2) with finite Morse index and $p < p_{JL}$ is indeed in $C^2(\Omega)$, therefore all results for the special case with $\alpha = 0$ in (2.2) are well-known thanks to [35]. Moreover, many other interesting results for solutions stable outside a compact set can be found in [35] for the autonomous case of equation (2.2).

So our work concerns only the *nonautonomous* case $\alpha \neq 0$, which is different. Furthermore, the restriction on $\alpha > -2$ is necessary, seeing the following nonexistence result.

Proposition 2.1.6. *For $\alpha \leq -2$,* (2.1) *admits no weak solution for any domain $\Omega \subset \mathbb{R}^N$ containing* 0*.*

It was proved in [21] that when $\alpha \leq -2$, (2.2) admits no *positive* solution over any punctured domain $B_R\backslash\{0\}$. Here and after, we denote by $B_r(x)$ the ball of radius $r > 0$ and center x in $\mathbb{R}^N$, $B_r^+(x) = B_r(x) \cap \mathbb{R}^N_+$, and simply by B_r, B_r^+ when $x = 0$. However, as far as we are aware, it is not known if the condition $\alpha > -2$ is necessary to have a sign-changing weak solution to (2.2) around the origin.

From now on, we assume that $\alpha > -2$. Our main objective is to classify weak solutions of (2.1) or (2.2) in $\mathbb{R}^N$, which are stable outside a compact set or with finite Morse index.

Theorem 2.1.7. *Let $\alpha > -2$ and $\Omega = \mathbb{R}^N$. For $2 \leq N < 10 + 4\alpha$, there is no weak stable solution of* (2.1)*.*

Theorem 2.1.8. *Let $\alpha > -2$, $\Omega = \mathbb{R}^N$ and $2 < N < 10 + 4\alpha^-$ where $\alpha^- = \min(\alpha, 0)$. Then* (2.1) *has no weak solution which is stable outside a compact set. In particular, any weak solution of* (2.1) *in $\mathbb{R}^N$ has infinite Morse index if $2 < N < 10 + 4\alpha^-$.*

Theorem 2.1.7 is sharp. Indeed, for $N \geq 10 + 4\alpha$ and $\alpha > -2$, (2.1) possesses radial stable weak solutions in $\mathbb{R}^N$,

$$U(x) = -(2+\alpha)\ln|x| + \ln[(2+\alpha)(N-2)]$$

where $|x|$ denotes the Euclidean norm. The stability of U is a direct consequence of the classical Hardy inequality,

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 dx \geq \frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{\psi^2}{|x|^2}dx, \quad \forall\, \psi \in H^1(\mathbb{R}^N). \tag{2.6}$$

But we do not know if Theorem 2.1.8 holds under the assumptions of Theorem 2.1.7, i.e. when $\alpha > 0$ and $10 \leq N < 10 + 4\alpha$.

When $N = 2$ and $\alpha > -2$, it is not difficult to see that any finite Morse index solutions is indeed an energy solution, that is $|x|^\alpha e^u \in L^1(\mathbb{R}^2)$, Prajapat & Tarantello [52] have already classified all such solutions.

Theorem 2.1.9. *Let $\alpha > -2$, $\Omega = \mathbb{R}^2$ and u be a weak solution of* (2.1) *which is stable outside a compact set, then*

$$\int_{\mathbb{R}^2} |x|^\alpha e^u dx = 4\pi(\alpha+2). \tag{2.7}$$

Furthermore, if $\alpha \notin 2\mathbb{N}$, we have $u(x) = U_*(\varepsilon x) + (\alpha+2)\ln\varepsilon$ where $\varepsilon > 0$ and

$$U_*(x) = \ln 2 + 2\ln\frac{\alpha+2}{1+|x|^{\alpha+2}}.$$

If $\alpha \in 2\mathbb{N}$, let θ be the angle of x in polar coordinates, $k = \frac{\alpha+2}{2}$ and

$$U_{**}(x) = 2\ln\frac{2k}{1+|x|^{2k} - 2|x|^k\cos(k\theta-\theta_0)\tanh\xi} + \ln\frac{2}{\cosh^2\xi}$$

with $\xi, \theta_0 \in \mathbb{R}$. We have then $u(x) = U_{**}(\varepsilon x) + (\alpha+2)\ln\varepsilon$ where $\varepsilon > 0$.

For equation (2.2), we have similar Liouville type results.

Theorem 2.1.10. *Let $\alpha > -2$, $N \geq 2$. The result of Theorem 2.1.3 holds true for weak solution of* (2.2) *in $\mathbb{R}^N$. Moreover, let u be a weak solution of* (2.2) *in $\mathbb{R}^N$ with finite Morse index. Assume that*

$$1 < p < p(N, \alpha^-) \quad \text{and} \quad p \neq \frac{N+2+2\alpha}{N-2}, \tag{2.8}$$

then $u \equiv 0$.

We note that though Dancer, Du and Guo have considered *positive* solutions to (2.2) with finite Morse index, in punctured domains or in exterior domains under similar conditions on p. Using results in [8], it was proved that either these solutions are of fast decay, or up to a suitable scaling, they converge uniformly to a positive function on S^{N-1}, they did not consider the Liouville type result for entire solutions of (2.2) in $\mathbb{R}^N$ with finite Morse index.

Theorem 2.1.10 is sharp for $\alpha \in (-2, 0]$. However, we don't know if it holds true for $\alpha > 0$, $p(N, 0) \leq p < p(N, \alpha)$ and $N > 10$. On the other hand, let $N \geq 3$, $p = \frac{N+2+2\alpha}{N-2}$ be the critical exponent and $\alpha > -2$, it is well known that (2.2) possesses radial positive entire solutions, given by

$$V(x) = \lambda^{\frac{N-2}{2}} \left(\frac{\sqrt{(N+\alpha)(N-2)}}{1 + \lambda^{2+\alpha}|x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}}, \quad \lambda > 0. \tag{2.9}$$

Clearly, V has finite Morse index. Indeed, since $ind_{\mathbb{R}^N}(V) = \sup_{n \in \mathbb{N}^*} ind_{B_n}(V)$ where $ind_{B_n}(V)$ is the number of negative Dirichlet eigenvalue of the operator $-\Delta - p|x|^\alpha V^{p-1}$ in B_n, we can find that V has finite Morse index by using the following Cwikel-Lieb-Rozenbljum formula (see [35] and the references therein [20], [45] and [55])

Lemma 2.1.11 (Cwikel-Lieb-Rozenbljum formula)**.** *Let M be a non-negative potential belonging to $L^{N/2}(B_n)$ with $N \geq 3$. Then the number of negative Dirichlet eigenvalue of the operator $-\Delta - M$ in B_n is bounded by*

$$\left(\frac{4e}{N(N-2)} \right)^{N/2} \frac{1}{\omega_{N-1}} \int_{B_n} M^{N/2}.$$

Finally we consider the half-space problem of (2.2) with the Dirichlet boundary condition.

Theorem 2.1.12. *Let $\alpha > -2$ and u be a weak solution of*

$$-\Delta u = |x|^\alpha |u|^{p-1} u \ \text{ in } \ \mathbb{R}^N_+ = \{x \in \mathbb{R}^N, x_N > 0\}, \quad u = 0 \ \text{ on } \ \partial\mathbb{R}^N_+.$$

Suppose that u has finite Morse index in $\mathbb{R}^N_+$. Then $u \equiv 0$ under the assumption (2.8)*.*

Our proofs are based on some *a priori* estimations using the stability condition (2.5), in the spirit of Proposition 4 in [35] (see also Proposition 1.7 of [21] and Proposition 5 of [36]), but we need to be more careful with the weak solutions since they are not supposed to be bounded *a priori*. Another important ingredient to handle (2.2) is the asymptotic behavior of solutions near the origin and near infinity.

Theorem 2.1.13. *Let u be a weak solution of* (2.2) *in Ω containing* 0*. Suppose that u has finite Morse index and p satisfies*

$$1 < p < p(N, \alpha^-).$$

Then $u \in C(\Omega) \cap C^2(\Omega \setminus \{0\})$ and the following fast decay estimate holds :

$$\lim_{|x|\to 0} |x|^{1+\frac{2+\alpha}{p-1}} |\nabla u(x)| = 0. \tag{2.10}$$

For the fast decay estimate as $|x|$ goes to ∞, we need more restriction on the exponent p.

Theorem 2.1.14. *Suppose that u is a weak stable solution of (2.2) in $\mathbb{R}^N \setminus \mathcal{K}$ where $\mathcal{K} \subset \mathbb{R}^N$ is a compact set, $N \geq 3$. Assume that $\alpha > -2$ and p satisfies*

$$\underline{p}(N,\alpha) < p < p(N,\alpha^-).$$

where

$$\underline{p}(N,\alpha) := \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) - 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}.$$

Then we have

$$\lim_{|x|\to\infty} |x|^{\frac{2+\alpha}{p-1}} |u(x)| = \lim_{|x|\to\infty} |x|^{1+\frac{2+\alpha}{p-1}} |\nabla u(x)| = 0. \tag{2.11}$$

The estimates (2.11) were proved in [21, Theorems 1.6] only for positive solutions. We note that the Harnack type argument used in [21] cannot work with sign-changing solutions. Moreover, we will explain how the critical exponents $p(N,\alpha)$ and $\underline{p}(N,\alpha)$ appear in preliminaries.

The organization of this chapter is as follows. We state some preparing works and useful tools in section 2 and deal with equations (2.1), (2.2) respectively in sections 3 and 4 to show our main results. Some further remarks and open questions are exposed in section 5. In the following, C or C_i denotes some generic positive constant.

2.2 Preliminaries

We firstly explain how $p(N,\alpha)$ and $\underline{p}(N,\alpha)$ appear. For $\alpha > -2$ and $N \geq 2$, define the function $f(p)$ as

$$f(p) = p\frac{\alpha+2}{p-1}\left(N-2-\frac{\alpha+2}{p-1}\right) \quad \text{with} \quad p > 1.$$

When $N = 2$, we have $f(p) < 0$ for all $p > 1$. Suppose then $N \geq 3$. Evidently, the function f has the following properties :

1. $f(\frac{N+\alpha}{N-2}) = 0$ and $f(\infty) = \lim_{p\to\infty} f(p) = (\alpha+2)(N-2)$.
2. When $\alpha + 2 \geq N - 2$, we have $f'(p) > 0$ for all $p > 1$.

3. When $\alpha + 2 < N - 2$, there exist

$$p_\star = \frac{N+\alpha}{N-4-\alpha}$$

such that

$$f'(p) > 0 \text{ for } p \in (1, p_\star), \quad f'(p_\star) = 0 \quad \text{and} \quad f'(p) < 0 \text{ for } p > p_\star.$$

Since $1 < \frac{N+\alpha}{N-2} < \frac{N+2+2\alpha}{N-2} < p_\star$ and $f(\frac{N+2+2\alpha}{N-2}) = \frac{N+2+2\alpha}{N-2}\frac{(N-2)^2}{4} > \frac{(N-2)^2}{4}$, we find that the equation $f(p) = \frac{(N-2)^2}{4}$ always has a solution $p_1(\alpha) \in (\frac{N+\alpha}{N-2}, \frac{N+2+2\alpha}{N-2})$. Moreover, $f(p) = \frac{(N-2)^2}{4}$ is equivalent to

$$((N-2)(N-4\alpha-10))p^2+(-2(N-2)^2+4(\alpha+2)(\alpha+N))p+(N-2)^2 = 0 \quad (2.12)$$

From this, we get

$$p_1(\alpha) = \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) - 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}$$

if $N \neq 10 + 4\alpha$ and $p_1 = \frac{4}{3}$ if $N = 10 + 4\alpha$. Moreover, when $N \leq 10 + 4\alpha$, f has the property

$$f(p) < \frac{(N-2)^2}{4} \text{ for } 1 < p < p_1 \quad \text{and} \quad f(p) > \frac{(N-2)^2}{4} \text{ for } p > p_1.$$

When $N > 10+4\alpha$, the equation (2.12) has the second root $p_2(\alpha) \in (\frac{N+2+2\alpha}{N-2}, \infty)$, given by

$$p_2(\alpha) = \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) + 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}$$

and we obtain

$$f(p) < \frac{(N-2)^2}{4} \text{ for } p \in (1, p_1)\cup(p_2, \infty) \quad \text{and} \quad f(p) > \frac{(N-2)^2}{4} \text{ for } p \in (p_1, p_2).$$

Therefore, the numbers

$$p(N, \alpha) := \begin{cases} \infty, & \text{if } N \leq 10 + 4\alpha \\ p_2, & \text{if } N > 10 + 4\alpha \end{cases}$$

and

$$\underline{p}(N, \alpha) := p_1$$

serve as the critical powers for the equation (2.2).

Next, we need to state the following result which is widely used in this chapter and the next chapter.

Lemma 2.2.1. *Let $\alpha > -2$ and $N \geq 2$. If $1 < p < p(N,\alpha)$, there exists $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ such that*

$$N - \frac{2(p+\gamma) + (\gamma+1)\alpha}{p-1} < 0. \tag{2.13}$$

Proof. Lemma 2.2.1 has been proved in [21, 28], however, for the sake of completeness, we give this proof here. Consider the function

$$\Lambda(N,p,\gamma,\alpha) := N(p-1) - (2+\alpha)\gamma - 2p - \alpha \tag{2.14}$$

and define

$$\gamma(p) = 2p + 2\sqrt{p(p-1)} - 1, \quad \Gamma(p) = \frac{(2+\alpha)\gamma(p) + 2p + \alpha}{p-1}.$$

Rewrite $\Gamma(p)$ in the form

$$\Gamma(p) = 2(\alpha+2)\left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{1}{p-1}}\right) + 2,$$

clearly $\Gamma(p)$ is strictly decreasing in p for $p > 1$, with $\Gamma(1) = +\infty$ and $\Gamma(+\infty) = 10 + 4\alpha$. Therefore, when $N \leq 10 + 4\alpha$,

$$\Lambda(N,p,\gamma(p),\alpha) = (p-1)(N - \Gamma(p)) < 0 \quad \text{for all} \quad p > 1.$$

When $N > 10 + 4\alpha$, there exists a unique $p^\star \in (1,\infty)$ such that

$$\Gamma(p^\star) = N \tag{2.15}$$

and

$$(p - p^\star)\Lambda(N,p,\gamma(p),\alpha) > 0 \quad \text{for} \quad p \in (1,\infty), p \neq p^\star.$$

(2.15) is equivalent to

$$\left(\frac{N-2}{2+\alpha} - 2\right)p^\star - \frac{N-2}{2+\alpha} = 2\sqrt{p^\star(p^\star - 1)},$$

which can imply that $p^\star$ satisfies

$$p^\star > \frac{N-2}{2+\alpha}\left(\frac{N-2}{2+\alpha} - 2\right)^{-1} > \frac{N+2+2\alpha}{N-2} > \underline{p}(N,\alpha)$$

and $p^\star$ also verifies the quadratic equation (2.12), thus we necessarily have $p^\star = p(N,\alpha)$. Therefore,

$$\begin{aligned} &\Lambda(N,p,\gamma(p),\alpha) = 0 \quad \text{for} \quad p = p(N,\alpha); \\ &\Lambda(N,p,\gamma(p),\alpha) < 0 \quad \text{for} \quad 1 < p < p(N,\alpha). \end{aligned} \tag{2.16}$$

By continuity, (2.16) implies the existence of $\gamma \in [1, \gamma(p))$ satisfying (2.13). $\square$

In the rest of this section, we will collect some known results and some basic inequalities which will be used frequently in the following.

We recall first a classification result obtained by Prajapat and Tarantello, which will be used in the two dimensional case for the exponential growth type equation. They considered the following problem

$$-\Delta u = |x|^\alpha e^u \quad \text{in } \mathbb{R}^2 \text{ with the condition } \int_{\mathbb{R}^2} |x|^\alpha e^u dx < \infty \tag{2.17}$$

with $\alpha > -2$. Note that by the result of Chen-Li in [15], the condition

$$\int_{\mathbb{R}^2} |x|^\alpha e^u(x) dx = 4\pi(\alpha+2) \tag{2.18}$$

imposes a severe restriction on the solvability of (2.17). Set now

$$v(x) = u(x) - \ln \int |x|^\alpha e^{u(x)} dx,$$

it follows from (2.18) that (2.17) becomes to the modified equation

$$-\Delta v = 4\pi(\alpha+2)|x|^\alpha e^v \quad \text{in } \mathbb{R}^2 \text{ and } \int_{\mathbb{R}^2} |x|^\alpha e^v dx = 1. \tag{2.19}$$

Then Theorem 1.1 in [52] tells us

Theorem 2.2.2 (Prajapat-Tarantello, [52])**.** *Let $\alpha > -2$, $\Omega = \mathbb{R}^2$ and u be a solution of* (2.19)*.*

1. *If $\alpha \notin 2\mathbb{N}$, we have $u(x) = U_*(\varepsilon x) + (\alpha+2)\ln\varepsilon$ where $\varepsilon > 0$ and*

$$U_*(x) = \ln 2 + 2\ln \frac{\alpha+2}{1+|x|^{\alpha+2}}.$$

2. *If $\alpha \in 2\mathbb{N}$, let θ be the angle of x in polar coordinates, $k = \frac{\alpha+2}{2}$ and*

$$U_{**}(x) = 2\ln \frac{2k}{1+|x|^{2k} - 2|x|^k \cos(k\theta - \theta_0)\tanh\xi} + \ln \frac{2}{\cosh^2\xi}$$

with $\xi, \theta_0 \in \mathbb{R}$. We have then $u(x) = U_{**}(\varepsilon x) + (\alpha+2)\ln\varepsilon$ where $\varepsilon > 0$.

Therefore, in order to get Theorem 2.1.9, we need only to prove the feature $|x|^\alpha e^u \in L^1(\mathbb{R}^2)$ for the weak solution u of (2.1) which is stable outside a compact set. We will show briefly its proof in next section.

For the Hénon type equation with power growth, some slow decay estimates for the weak solutions of (2.2) with finite Morse index on punctured domains or exterior domains is desired, which is very useful in the demonstration of Theorem 2.1.10. In order to get these slow decay estimates, we need the delicate doubling lemma showed in [51].

Lemma 2.2.3 (Poláčik-Quittner-Souplet, [51])**.** *Let (X,d) be a complete metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Gamma = \Sigma \backslash D$. Finally let $M : D \to (0,\infty)$ be bounded on compact subsets of D and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)dist(y,\Gamma) > 2k,$$

then there exists $x \in D$ such that

$$M(x)dist(x,\Gamma) > 2k, \quad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \textit{for all } z \in D \cap \overline{B}_X(x, kM^{-1}(x)).$$

Nest we state the basic formula. In order to prove the Liouville type result for finite Morse index solutions of (2.2) in the subcritical case, we have to show the expression of classical Pohozaev identity for $-\Delta u = |x|^\alpha f(u)$. Indeed, we multiply $-\Delta u = |x|^\alpha f(u)$ by $x \cdot \nabla u := \sum_{i=1}^N x_i \frac{\partial u}{\partial x_i}$ and integrate by parts over Ω in both left hand side and right hand side, then by direct computation we find

Lemma 2.2.4 (Pohozaev identity, [50])**.** *Let u be a classical solution of $-\Delta u = |x|^\alpha f(u)$ in Ω where $\Omega \subset \mathbb{R}^N \backslash \{0\}$ is a compact set. Then we have*

$$\begin{aligned}
&(N+\alpha)\int_\Omega |x|^\alpha F(u)dx - \frac{N-2}{2}\int_\Omega |\nabla u|^2 dx \\
=& \int_{\partial\Omega} |x|^\alpha F(u)\langle x,\nu\rangle d\sigma + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\langle x, \nabla u\rangle d\sigma - \int_{\partial\Omega} \frac{|\nabla u|^2}{2}\langle x,\nu\rangle d\sigma,
\end{aligned}$$

where $F(u)$ is a primitive of f.

2.3 Liouville type theorems for $-\Delta u = |x|^\alpha e^u$

Consider $-\Delta u = |x|^\alpha e^u$. First we show the necessity to working with $\alpha > -2$ and a quick proof of Theorem 2.1.9, then we prove Theorems 2.1.7 and 2.1.8.

2.3.1 Nonexistence of weak solution for $\alpha \leq -2$

We prove Proposition 2.1.6. Arguing by contradiction, assume that $\alpha \leq -2$ and u is a weak solution of (2.1) with $0 \in \Omega$. Let $B_R \subset \Omega$. Define v to be the average of u over spheres centered at the origin, i.e.

$$v(r) := \overline{u(r,\theta)} = \frac{1}{\omega_N}\int_{S^{N-1}} u(r,\theta)d\theta \quad \text{with } \omega_N = |S^{N-1}|.$$

By Jensen's inequality, there holds

$$-\Delta v = \overline{|x|^\alpha e^u} = r^\alpha \overline{e^u} \geq r^\alpha e^v,$$

which means that
$$-\frac{1}{r^{N-1}}(r^{N-1}v')' \geq r^\alpha e^v. \tag{2.20}$$
Note that
$$-r^{N-1}\omega_N v'(r) = -\int_{\partial B_r} \frac{\partial v}{\partial \nu} d\sigma = -\int_{B_r} \Delta v dx = \int_{B_r} |x|^\alpha e^u dx > 0,$$
thus $v'(r) < 0$ for any $r \in (0, R)$ and v is decreasing in $(0, R)$. As $r^\alpha e^v \in L^1_{loc}(\Omega)$ and $0 \in \Omega$, we must have $\alpha > -N$. Furthermore, integrate (2.20) in $[r_1, r]$ with $0 < r_1 < r < R$,
$$-r^{N-1}v'(r) \geq \int_{r_1}^r s^{N-1+\alpha} e^{v(s)} ds - r_1^{N-1} v'(r_1) \geq \int_{r_1}^r s^{N-1+\alpha} e^{v(s)} ds.$$
Tending r_1 to 0, we get
$$-r^{N-1}v'(r) \geq \int_0^r s^{N-1+\alpha} e^{v(s)} ds \geq \frac{r^{N+\alpha} e^{v(r)}}{N+\alpha}. \tag{2.21}$$
Hence $-e^{-v}v' \geq Cr^{1+\alpha}$, which yields
$$e^{-v(r)} > e^{-v(r)} - e^{-v(r_1)} \geq C\int_{r_1}^r s^{1+\alpha} ds \quad \forall\, 0 < r_1 < r.$$
As $\alpha \leq -2$, a contradiction occurs by tending r_1 to 0 since the last term goes to ∞. So Proposition 2.1.6 is proved. □

Remark 2.3.1. We can remark that the above proof works for $-\Delta u = |x|^\alpha g(u)$ with a positive, convex and nondecreasing nonlinearity g.

2.3.2 Two dimensional case

Here we prove Theorem 2.1.9. Assume that u is stable outside B_{R_0}. Fix $\phi \in C_c^\infty(\mathbb{R})$ verifying $\phi(t) = 1$ if $|t| \leq 1$ and $\phi(t) = 0$ if $|t| \geq 2$. For any $R \geq 4R_0$, let $\psi_R(x) = \phi(R^{-1}|x|) - \phi(R_0^{-1}|x|)$, then $\text{supp}(\psi_R) \subset B_{R_0}^c$, ψ_R is fixed in B_{2R_0} and
$$\psi_R \equiv 1 \text{ in } B_R \setminus B_{2R_0}, \quad |\nabla \psi_R(x)| \leq CR^{-1} \text{ in } B_R^c.$$
Using ψ_R as the test function in (2.5), there holds
$$\begin{aligned}\int_{B_R\setminus B_{2R_0}} |x|^\alpha e^u dx &\leq \int_{\mathbb{R}^2} |x|^\alpha e^u \psi_R^2 dx \\ &\leq \int_{\mathbb{R}^2} |\nabla \psi_R|^2 dx \\ &\leq \int_{B_{2R_0}} |\nabla \psi_R|^2 dx + \int_{B_{2R}\setminus B_R} |\nabla \psi_R|^2 dx \leq C\end{aligned}$$
Tending R to ∞, we get $|x|^\alpha e^u \in L^1(\mathbb{R}^2)$ thus we arrive at the problem (2.17). By Theorem 2.2.2, all classification results in Theorem 2.1.9 follow straightforwardly. □

2.3.3 Main technical tool

As already mentioned, our proof of Liouville type results is based on the estimate $Q_u(\psi) \geq 0$ with suitable test function. In fact, we have the following estimate which is an extension of result in [36].

Proposition 2.3.2. *Let Ω be a domain (bounded or not) in $\mathbb{R}^N$, $N \geq 2$. Let u be a weak and stable solution of* (2.1) *with $\alpha > -2$. Then for any integer $m \geq 5$ and any $\beta \in (0,2)$, there exists $C > 0$ depending on m, α and β such that*

$$\int_\Omega |x|^\alpha e^{(2\beta+1)u}\psi^{2m} dx \leq C\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1} dx, \qquad (2.22)$$

for all function $\psi \in C_c^\infty(\Omega)$ verifying $\|\psi\|_\infty \leq 1$.

Proof of Proposition 2.3.2. We use some ideas in [36, 17], but we need to pay more attention with weak solution. As u is not assumed to be bounded, $e^{\beta u}\varphi$ is not, *a priori*, a licit test function (for any $\beta > 0$), even with $\varphi \in C_c^\infty(\Omega)$. Our idea is to consider suitable truncations of $e^{\beta u}$ and proceed as in [17]. Let $\beta > 0$, $k \in \mathbb{N}$, $k \geq \beta^{-1}$ and

$$\zeta_k(t) = \begin{cases} e^{\beta t}, & \text{if } t \leq k \\ \frac{e^{\beta k}}{k} t, & \text{if } t \geq k. \end{cases}$$

We choose also another Lipschitz function η_k and a differentiable function ξ_k such that $\eta_k' = \zeta_k'^2$, $\xi_k' = \eta_k$ in $\mathbb{R}$. Let

$$\eta_k(t) = \begin{cases} \frac{\beta}{2} e^{2\beta t}, & \text{if } t \leq k \\ \frac{e^{2\beta k}}{k^2}(t-k) + \frac{\beta}{2} e^{2\beta k}, & \text{if } t \geq k \end{cases}$$

and

$$\xi_k(t) = \begin{cases} \frac{e^{2\beta t}}{4}, & \text{if } t \leq k \\ \frac{e^{2\beta k}}{2k^2}(t-k)^2 + \frac{\beta}{2} e^{2\beta k}(t-k) + \frac{e^{2\beta k}}{4}, & \text{if } t \geq k. \end{cases}$$

Since $u \in H^1_{loc}(\Omega)$, clearly $\zeta_k(u), \eta_k(u) \in H^1_{loc}(\Omega)$ for any $k \in \mathbb{N}$.

Applying now (2.5) with the test function $\zeta_k(u)\varphi$, where $\varphi \in C_c^\infty(\Omega)$, we get

$$\begin{aligned} \int_\Omega |x|^\alpha e^u \zeta_k^2(u)\varphi^2 dx \leq & \int_\Omega |\nabla(\zeta_k(u)\varphi)|^2 dx \\ = & \int_\Omega |\nabla(\zeta_k(u))|^2\varphi^2 dx \\ & + \int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx \\ & - \int_\Omega \frac{\zeta_k^2(u)}{2}\Delta(\varphi^2) dx. \end{aligned}$$

So $|x|^\alpha e^u \zeta_k^2(u) \in L^1_{loc}(\Omega)$. On the other hand,

$$\begin{aligned}
\int_\Omega |\nabla(\zeta_k(u))|^2 \varphi^2 dx &= \int_\Omega \zeta_k'^2(u)|\nabla u|^2 \varphi^2 dx\\
&= \int_\Omega (\nabla \eta_k(u)\cdot \nabla u)\varphi^2 dx\\
&= \int_\Omega \nabla u \cdot \nabla\left(\eta_k(u)\varphi^2\right) dx - \int_\Omega \eta_k(u)\nabla u\cdot\nabla(\varphi^2)dx\\
&= \int_\Omega |x|^\alpha e^u \eta_k(u)\varphi^2 dx - \int_\Omega \eta_k(u)\nabla u\cdot\nabla(\varphi^2)dx\\
&= \int_\Omega |x|^\alpha e^u \eta_k(u)\varphi^2 dx + \int_\Omega \xi_k(u)\Delta(\varphi^2)dx.
\end{aligned}$$

For the third line, we used the following argument : as $|x|^\alpha e^u > 0$, (2.4) is valid for any $\psi \in H_0^1(\Omega)$ by density argument. The above estimates imply then

$$\begin{aligned}
&\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\varphi^2 dx\\
\leq& \int_\Omega |x|^\alpha e^u \eta_k(u)\varphi^2 dx + \int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx + \int_\Omega \left[\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right]\Delta(\varphi^2)dx.
\end{aligned}$$

Moreover, direct calculation shows that

$$\eta_k(t) \leq \left(\frac{\beta}{2} + \frac{1}{4k}\right)\zeta_k^2(t) \quad \text{in } \mathbb{R}.$$

Therefore

$$\begin{aligned}
\left(1 - \frac{\beta}{2} - \frac{1}{4k}\right)\int_\Omega |x|^\alpha e^u \zeta_k^2(u)\varphi^2 dx &\leq \int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx\\
&\quad + \int_\Omega \left[\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right]\Delta(\varphi^2)dx.
\end{aligned} \tag{2.23}$$

Set now $\varphi = \psi^m$ with $\psi \in C_c^\infty(\Omega)$ satisfying $\|\psi\|_\infty \leq 1$ and $m \in \mathbb{N}^*$. Using $\zeta_k(u) \leq e^{\beta u}$ and Hölder's inequality, there holds

$$\begin{aligned}
&\int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx\\
=& m^2 \int_\Omega \zeta_k^2(u)\psi^{2(m-1)}|\nabla\psi|^2 dx\\
\leq& C\int_\Omega \left[e^u\zeta_k^2(u)\right]^{\frac{2\beta}{2\beta+1}}\psi^{2(m-1)}|\nabla\psi|^2 dx\\
\leq& C\left(\int_\Omega |x|^\alpha e^u \zeta_k^2(u)|\psi|^{\frac{(m-1)(2\beta+1)}{\beta}}dx\right)^{\frac{2\beta}{2\beta+1}}\left(\int_\Omega |x|^{-2\beta\alpha}|\nabla\psi|^{4\beta+2}dx\right)^{\frac{1}{2\beta+1}}.
\end{aligned}$$

Take $m \geq 5$ so that $(m-1)(2\beta+1) \geq 2m\beta$ for any $\beta \in (0,2)$. As $\|\psi\|_\infty \leq 1$, we obtain

$$\int_\Omega \zeta_k^2(u)|\nabla\varphi|^2 dx \leq C\left(\int_\Omega |x|^\alpha e^u\zeta_k^2(u)\psi^{2m}dx\right)^{\frac{2\beta}{2\beta+1}}\left(\int_\Omega |x|^{-2\beta\alpha}|\nabla\psi|^{4\beta+2}dx\right)^{\frac{1}{2\beta+1}}. \tag{2.24}$$

Furthermore, for any $\beta \in (0,2)$ there exists $C > 0$ depending only on β such that

$$e^{\frac{2\beta}{2\beta+1}u} \geq Ce^{\frac{2\beta}{2\beta+1}k}\left[1+(u-k)^2\right] \quad \forall\, u \geq k,$$

because $e^t \geq 1 + t + \frac{t^2}{2}$ for $t \geq 0$. We deduce then for any $\beta \in (0,2)$, there exists $C > 0$ (independent of $k \in \mathbb{N}^*$) satisfying

$$\left|\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right| \leq C\left[e^u\zeta_k^2(u)\right]^{\frac{2\beta}{2\beta+1}} \quad \text{in } \mathbb{R}.$$

As $\Delta(\varphi^2) = \Delta(\psi^{2m}) = 2m\psi^{2m-1}\Delta\psi + 2m(2m-1)\psi^{2m-2}|\nabla\psi|^2$, applying once again Hölder's inequality, we get

$$\begin{aligned}&\left|\int_\Omega \left[\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right]\Delta(\varphi^2)dx\right| \\ &\leq C\left(\int_\Omega |x|^\alpha e^u\zeta_k^2(u)|\psi|^{\frac{(m-1)(2\beta+1)}{\beta}}dx\right)^{\frac{2\beta}{2\beta+1}}\left[\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1}dx\right]^{\frac{1}{2\beta+1}}.\end{aligned} \tag{2.25}$$

Proceeding as above, fixing $m \geq 5$ so that $(m-1)(2\beta+1) \geq 2m\beta$ for any $\beta \in (0,2)$, we have $|\psi|^{\frac{(m-1)(2\beta+1)}{\beta}} \leq |\psi|^{2m}$ since $\|\psi\|_\infty \leq 1$. Then (2.25) can be rewritten as

$$\begin{aligned}&\left|\int_\Omega \left[\xi_k(u) - \frac{\zeta_k^2(u)}{2}\right]\Delta(\varphi^2)dx\right| \\ &\leq C\left(\int_\Omega |x|^\alpha e^u\zeta_k^2(u)\psi^{2m}dx\right)^{\frac{2\beta}{2\beta+1}}\left[\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1}dx\right]^{\frac{1}{2\beta+1}}.\end{aligned} \tag{2.26}$$

Combining (2.23)-(2.26),

$$\begin{aligned}&\left(1 - \frac{\beta}{2} - \frac{1}{4k}\right)\int_\Omega |x|^\alpha e^u\zeta_k^2(u)\psi^{2m}dx \\ &\leq C\left(\int_\Omega |x|^\alpha e^u\zeta_k^2(u)\psi^{2m}dx\right)^{\frac{2\beta}{2\beta+1}}\left[\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1}dx\right]^{\frac{1}{2\beta+1}},\end{aligned}$$

which means that there exists $C > 0$ independent of k such that

$$\int_\Omega |x|^\alpha e^u\zeta_k^2(u)\psi^{2m}dx \leq C\int_\Omega |x|^{-2\beta\alpha}\left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\beta+1}dx,$$

provided $1 - \frac{\beta}{2} - \frac{1}{4k} > \delta > 0$. Fix $\beta \in (0,2)$, tending $k \to \infty$, the proof of (2.22) is completed by the monotone convergence Theorem. □

Remark 2.3.3. In [62], the author considered the regularity of weak stable solutions for (2.1) with $\alpha = 0$. He obtained higher intergrability similar to (2.22) by using the simple cut-off function $u_k = \min(k, u)$. However, several arguments as the iteration process in Step 3 of the proof for [62, Lemma 2.1], are not valid for the nonautonomous equation.

2.3.4 Classification of Stable solution

In this subsection we prove Theorem 2.1.7. Suppose that (2.1) admits a weak and stable solution u with $\Omega = \mathbb{R}^N$ and $N < 10 + 4\alpha$. Fix $m \geq 5$ and choose $\beta \in (0,2)$ such that $N - 2(2\beta+1) - 2\beta\alpha < 0$.

For every $R > 0$, consider the function

$$\phi_R(x) = \phi\left(R^{-1}|x|\right), \tag{2.27}$$

where $\phi \in C_c^\infty(\mathbb{R})$ is defined as

$$0 \leq \phi \leq 1 \text{ in } \mathbb{R} \quad \phi(t) = 1 \text{ if } |t| \leq 1 \text{ and } \phi(t) = 0 \text{ if } |t| \geq 2. \tag{2.28}$$

Applying Proposition 2.3.2 with $\psi = \phi_R$,

$$\int_{B_R} |x|^\alpha e^{(2\beta+1)u} dx \leq \int_{\mathbb{R}^N} |x|^\alpha e^{(2\beta+1)u} \phi_R^{2m} dx \leq C R^{N-2(2\beta+1)-2\beta\alpha}, \quad \forall\, R > 0.$$

Taking $R \to \infty$, we have

$$\int_{\mathbb{R}^N} |x|^\alpha e^{(2\beta+1)u} dx = 0,$$

which is impossible, so we are done. $\square$

2.3.5 Classification of finite Morse index solution

We are now ready to prove Theorem 2.1.8. We argue always by contradiction. Suppose that (2.1) admits a weak solution u which is stable outside a compact set $\mathcal{K}$ in $\mathbb{R}^N$. There exists $R_0 > 0$ such that $\mathcal{K} \subset B_{R_0}$, therefore we can apply Proposition 2.3.2 with $\Omega = \mathbb{R}^N \backslash \overline{B_{R_0}}$. We claim :

Lemma 2.3.4. – *For any $\beta \in (0,2)$ and any $R > 4R_0$, it holds*

$$\int_{3R_0 < |x| < R} |x|^\alpha e^{(2\beta+1)u} dx \leq A + BR^{N-2(2\beta+1)-2\beta\alpha} \tag{2.29}$$

where $A, B > 0$ are independent of R.

– *For any $\beta \in (0,2)$ and any $B_{2r}(y) \subset \mathbb{R}^N \setminus \overline{B_{R_0}}$, it holds*

$$\int_{B_r(y)} |x|^\alpha e^{(2\beta+1)u} dx \leq C r^{N-2(2\beta+1)-2\beta\alpha}, \tag{2.30}$$

where $C > 0$ is independent of r and y.

Proof of Lemma 2.3.4. For $R > 4R_0$, let ϕ and ϕ_R be defined as (2.28) and (2.27) in the previous proof and define $\psi_R = \phi_R - \phi_{R_0}$. Notice that $\psi_R \in$

$C_c^\infty(\mathbb{R}^N\backslash\mathcal{K})$ and $0 \le \psi_R \le 1$ in $\mathbb{R}^N$ and ψ_R is a fixed function η_0 in B_{4R_0}. Hence Proposition 2.3.2 (applied with $m=5$ and ψ_R) implies that for all $R > 4R_0$,

$$\begin{aligned}
&\int_{\{3R_0<|x|<R\}} |x|^\alpha e^{(2\beta+1)u} dx \\
\le &\int_{\mathbb{R}^N\backslash\mathcal{K}} |x|^\alpha e^{(2\beta+1)u}\psi_R^2 dx \\
\le &\, C\int_{\mathbb{R}^N\backslash\mathcal{K}} |x|^{-2\beta\alpha}\left(|\nabla\psi_R|^2 + |\psi_R||\Delta\psi_R|\right)^{2\beta+1} dx \\
\le &\, C\int_{R_0\le|x|\le 4R_0} |x|^{-2\beta\alpha}\left(|\nabla\eta_0|^2 + \eta_0|\Delta\eta_0|\right)^{2\beta+1} dx \\
&\quad + C\int_{R\le|x|\le 2R} |x|^{-2\beta\alpha}\left(|\nabla\psi_R|^2 + |\psi_R||\Delta\psi_R|\right)^{2\beta+1} dx \\
\le &\, A + BR^{N-2(2\beta+1)-2\beta\alpha}.
\end{aligned}$$

As the constants A, B depend only on η_0, R_0, ϕ and β, we get the claim (2.29). Using Proposition 2.3.2 with the test function $\phi_r(x-y)$, we obtain easily the estimate (2.30). $\square$

Consider now $\Gamma(\beta) = N - 4\beta - 2 - 2\beta\alpha^-$. As $3 \le N < 10 + 4\alpha^-$, $\Gamma(0) > 0$ and $\Gamma(2) < 0$, so there exist $\beta_1 \in (0,2)$ and $\varepsilon_0 \in (0,2)$ such that

$$2\beta_1 + 1 \ge 2\beta_1 + 1 + \beta_1\alpha^- > \theta := \frac{N}{2-\varepsilon_0} > \frac{N}{2}.$$

Let $|y| > 4R_0$ and $R = \frac{|y|}{4}$, so $B_{2R}(y) \subset \mathbb{R}^N \setminus \overline{B_{R_0}}$. By Hölder's inequality and (2.30),

$$\begin{aligned}
&\int_{B_R(y)} \left(|x|^\alpha e^u\right)^\theta dx \\
\le &\left(\int_{B_R(y)} |x|^\alpha e^{(2\beta_1+1)u} dx\right)^{\frac{\theta}{2\beta_1+1}} \left(\int_{B_R(y)} |x|^{\frac{2\beta_1\alpha\theta}{2\beta_1+1-\theta}} dx\right)^{\frac{2\beta_1+1-\theta}{2\beta_1+1}} \\
\le &\, C\left(R^{N-2(2\beta_1+1)-2\beta_1\alpha}\right)^{\frac{\theta}{2\beta_1+1}} \left(R^{N+\frac{2\beta_1\alpha\theta}{2\beta_1+1-\theta}}\right)^{\frac{2\beta_1+1-\theta}{2\beta_1+1}} \\
= &\, CR^{N-2\theta},
\end{aligned}$$

that is,

$$\int_{B_R(y)} \left(|x|^\alpha e^u\right)^\theta dx \le CR^{N-2\theta}, \quad \forall\, |y| > 4R_0 \ \text{ and } \ R = \frac{|y|}{4}. \tag{2.31}$$

We claim now

Lemma 2.3.5.

$$\lim_{|x|\to+\infty} |x|^{2+\alpha} e^{u(x)} = 0. \tag{2.32}$$

Proof of Lemma 2.3.5. To prove that, we need a well-known result of Serrin in [56] (see also Theorem 7.1.1 in [53]) :

Lemma 2.3.6. *Let $\theta = \frac{N}{2-\varepsilon_0}$, $\varepsilon_0 \in (0,2)$, $q \in (1,\infty]$ and $\delta > 0$. For any weak solution of $-\Delta\eta = a(x)\eta$ in $B_{2R}(y) \subset \mathbb{R}^N$, if $R^{\varepsilon_0}\|a(x)\|_{L^\theta(B_{2R}(y))} \le \delta$, there holds*

$$\|\eta\|_{L^\infty(B_R(y))} \le CR^{-\frac{N}{q}}\|\eta\|_{L^q(B_{2R}(y))} \tag{2.33}$$

where C is a constant depending only on N, q, θ and δ.

Moreover, the estimate (2.33) holds also for any weak nonnegative function verifying $-\Delta\eta \le a(x)\eta$ in $B_{2R}(y) \subset \mathbb{R}^N$ assuming that $R^{\varepsilon_0}\|a(x)\|_{L^\theta(B_{2R}(y))} \le \delta$.

Set

$$\beta_2 = \frac{N-2}{2(2+\alpha)}, \quad \lambda = \frac{2\beta_2+1}{2} = \frac{N+\alpha}{2(2+\alpha)} > 0 \quad \text{and} \quad w = e^{\lambda u}.$$

Then $\beta_2 \in (0,2)$ since $3 \le N < 10+4\alpha$ and $N - 2(2\beta_2+1) - 2\beta_2\alpha = 0$. Take $\beta = \beta_2$ in (2.29) and tending R to ∞, we obtain

$$\int_{|x|\ge 3R_0} |x|^\alpha w^2 dx < \infty. \tag{2.34}$$

We have also

$$-\Delta w - \lambda|x|^\alpha e^u w = -\lambda^2 w|\nabla u|^2 \le 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Let $|y| > 8R_0$ and $R = \frac{|y|}{8}$. Using the estimate (2.31), as $\theta = \frac{N}{2-\varepsilon_0}$,

$$R^{\varepsilon_0}\|\lambda|x|^\alpha e^u\|_{L^\theta(B_{2R}(y))} \le \lambda R^{\varepsilon_0}\left(CR^{N-2\theta}\right)^{\frac{1}{\theta}} = C'R^{\varepsilon_0+\frac{N}{\theta}-2} = C'. \tag{2.35}$$

Applying Serrin's result with $\eta = w$, $a(x) = \lambda|x|^\alpha e^u$, $q = 2$ and $\delta = C'$ of (3.55), by (2.34),

$$\begin{aligned} w(y) &\le CR^{-\frac{N}{2}}\|w\|_{L^2(B_{2R}(y))} \\ &\le CR^{-\frac{N}{2}}R^{-\frac{\alpha}{2}}\||x|^{\frac{\alpha}{2}}w\|_{L^2(B_{2R}(y))} = o\left(R^{-\frac{N+\alpha}{2}}\right) \quad \text{as } |y| \to \infty. \end{aligned}$$

Consequently

$$e^{u(y)} = w(y)^{\frac{1}{\lambda}} = o\left(R^{-\frac{N+\alpha}{2\lambda}}\right) = o\left(R^{-2-\alpha}\right) \quad \text{as } |y| \to \infty,$$

hence the claim (2.32) holds true. □

To finish the proof of Theorem 2.1.8, consider v, the average of u over spheres. Fix $M > 0$ large satisfying $\alpha + 2 - \frac{2}{(N-2)M} > 0$ (recall that $\alpha > -2$). By (2.32), there exists $R_M > 0$ such that

$$-\Delta v(r) = \overline{r^\alpha e^u} \le \frac{1}{Mr^2}, \quad \forall\, r \ge R_M.$$

Integrating from R_M to r, we deduce then

$$v'(r) \ge -\frac{C}{r^{N-1}} - \frac{1}{(N-2)Mr}, \quad \forall\, r \ge R_M.$$

As $N \geq 3$, there exists $R' > R_M$ such that

$$v'(r) \geq -\frac{2}{(N-2)Mr}, \quad \forall\, r \geq R'.$$

Integrating on $[R', r]$, we get

$$r^{2+\alpha} e^{v(r)} \geq C r^{\alpha+2-\frac{2}{(N-2)M}}, \quad \forall\, r \geq R',$$

which yields

$$\sup_{|x|=r} \left(|x|^{2+\alpha} e^{u(x)} \right) = r^{2+\alpha} \sup_{|x|=r} e^{u(x)} \geq r^{2+\alpha} e^{v(r)} \geq C r^{\alpha+2-\frac{2}{(N-2)M}} \to \infty,$$

which contradicts (2.32). Therefore the proof of Theorem 2.1.8 is completed. □

2.4 Liouville type theorems for $-\Delta u = |x|^\alpha |u|^{p-1} u$

Here we consider the equation (2.2) in $\mathbb{R}^N$ or $\mathbb{R}^N_+$. As for Theorem 2.1.7, the basic argument is always the *a priori* estimates resulting from the stability condition (2.5).

Proposition 2.4.1. *Let Ω be a domain (bounded or not) in $\mathbb{R}^N$, $N \geq 2$. Let u be a weak stable solution of* (2.2) *with $p > 1$ and $\alpha > -2$. Then for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geq \max\left(\frac{p+\gamma}{p-1}, 2\right)$, there holds*

$$\begin{aligned} &\int_\Omega \left(|\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 + |x|^\alpha |u|^{\gamma+p} \right) |\psi|^{2m} dx \\ &\leq C \int_\Omega |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla \psi|^2 + |\psi||\Delta\psi| \right)^{\frac{p+\gamma}{p-1}} dx \end{aligned} \tag{2.36}$$

for all test function $\psi \in C_c^\infty(\Omega)$ verifying $\|\psi\|_\infty \leq 1$, where the constant C depends on p, m, γ and α.

Similarly, if we suppose that the weak solution of (2.2) *u belongs to $H^1_{loc}(\Omega)$ such that $u = 0$ on $\partial\Omega$ and u is stable outside a compact set $\mathcal{K} \subset \Omega$, then the estimate* (2.36) *holds for all test function $\psi \in C_c^\infty(\mathbb{R}^N \backslash \mathcal{K})$ verifying $\|\psi\|_\infty \leq 1$.*

Proof of Proposition 2.4.1. The proof follows the main lines of the demonstration of Proposition 1.7 in [21] or Proposition 6 in [35], with small modifications. As for Proposition 2.3.2, we need to consider truncations of the weak solution u, but the calculation is easier here (see also [23]).

Set just $\zeta_k(t) = \max(-k, \min(t, k))$, $k \in \mathbb{N}$ and use the test function $|\zeta_k(u)|^{\frac{\gamma-1}{2}} u\varphi \in H^1_0(\Omega)$ in (2.5), with $\varphi \in C_c^\infty(\Omega)$ and $\varphi \in C_c^\infty(\mathbb{R}^N \backslash \mathcal{K})$ respectively. The rest of proof can be proceeded as for Proposition 1.7 in [21], and by taking k tends to ∞. We omit the details. □

Assume for example that u is a weak solution of (2.2), stable outside a compact set of $\mathbb{R}^N$, i.e. there exists $R_0 > 0$ such that (2.36) holds true with $\Omega = \mathbb{R}^N \backslash \overline{B_{R_0}}$. Let $\alpha > -2$ and $p > 1$, using (2.36), we can prove

Lemma 2.4.2. *Let $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$. We have*

1. *Let $r > 4R_0$, then*

$$\int_{3R_0<|x|<r} \left[|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^{\alpha}|u|^{\gamma+p}\right] dx \le A + Br^{N-\frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \tag{2.37}$$

where A and B are constants independent of $r > 2R_0$.

2. *Let now $B_{2r}(y) \subset \mathbb{R}^N \backslash \overline{B_{R_0}}$, then*

$$\int_{B_r(y)} \left[|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^{\alpha}|u|^{\gamma+p}\right] dx \le Cr^{N-\frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \tag{2.38}$$

where $C > 0$ is independent of y and r.

Proof of Lemma 2.4.2. The proof is very similar to that for (2.29) and (2.30), we rewrite briefly the main idea here. Since (2.36) holds true with $\Omega = \mathbb{R}^N \backslash \overline{B_{R_0}}$, we need now choose a suitable test function. Fixing ϕ be as (2.28) and setting $\phi_r(x) = \phi(r^{-1}|x|)$. Since $\psi_r := \phi_r - \phi_{R_0} \in C_c^{\infty}(\mathbb{R}^N \backslash \mathcal{K})$ and $0 \le \psi_R \le 1$ in $\mathbb{R}^N$, it can be used as the test function, by Proposition 2.4.1 with some $m \ge \max\left(\frac{p+\gamma}{p-1}, 2\right)$, we have for $r > 4R_0$,

$$\begin{aligned}
&\int_{3R_0<|x|<r} \left[|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^{\alpha}|u|^{\gamma+p}\right] dx \\
\le &\int_{\mathbb{R}^N \backslash \overline{B_{R_0}}} \left[|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^{\alpha}|u|^{\gamma+p}\right] \psi_r^2 dx \\
\le &\, C\int_{\mathbb{R}^N \backslash \overline{B_{R_0}}} |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla\psi_r|^2 + |\psi_r||\Delta\psi_r|\right)^{\frac{p+\gamma}{p-1}} dx \\
\le &\, C\int_{R_0\le|x|\le 4R_0} |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla\eta_0|^2 + \eta_0|\Delta\eta_0|\right)^{\frac{p+\gamma}{p-1}} dx \\
&+ C\int_{r\le|x|\le 2r} |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla\psi_r|^2 + |\psi_r||\Delta\psi_r|\right)^{\frac{p+\gamma}{p-1}} dx \\
\le &\, A + Br^{N-\frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}.
\end{aligned}$$

As the constants A, B is independent on $r > 4R_0$, we get the claim (2.37). Note that the above η_0 is the function ψ_R fixed in B_{4R_0}. Moreover, using Proposition 2.4.1 with the test function $\phi_r(x-y)$, we obtain easily the desired estimate (2.38). □

2.4.1 Slow decay estimate and proof of Theorem 2.1.13

Before proving the fast decay (2.10), we prove a regularity result when $p < p(N, 0)$, and also slow decay estimates for solutions on punctured domains or at infinity.

Theorem 2.4.3. *Let $N \geq 2$, $1 < p < p(N,0)$ and u be a weak solution to* (2.2) *with finite Morse index. If the domain Ω contains $B_r \setminus \{0\}$ (resp. $\mathbb{R}^N \setminus B_r$), then there hold $u \in C^{2,\beta}_{loc}(\Omega \setminus \{0\})$ for some $\beta \in (0,1)$ and*

$$u(x) = O\left(|x|^{-\frac{2+\alpha}{p-1}}\right), \quad \nabla u(x) = O\left(|x|^{-1-\frac{2+\alpha}{p-1}}\right), \quad \text{for } |x| \to 0 \ (\text{resp. } |x| \to \infty). \tag{2.39}$$

Moreover, when $1 < p < p(N, \alpha^-)$ and $0 \in \Omega$, there exists $\beta \in (0,1)$ such that $u \in C^{0,\beta}_{loc}(\Omega)$.

Proof of Theorem 2.4.3. First, since each single point is of zero capacity in $\mathbb{R}^N$ when $N \geq 2$, all arguments for [23] (see also [27]) are still valid although the equation is different, that is

Lemma 2.4.4. *Let $N \geq 2$ and u be a weak solution to* (2.2) *with finite Morse index. For every $x_0 \in \Omega$, there exists $r_0 > 0$ such that u is stable in $B_{4r_0}(x_0)$.*

Similar to (2.38), by using standard cut-off function $\psi \in C^2_0(B_{4r_0}(x_0))$ with (2.36), we obtain $|x|^\alpha |u|^{\gamma+p} \in L^1(B_{3r_0}(x_0))$ for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$.

When $x_0 \neq 0$ and $p < p(N,0)$, we can just follow the proof of Theorem 2.1 in [21] and claim that

$$\int_{B_{2r_0}(x_0)} \left(|x|^\alpha |u|^{p-1}\right)^{\frac{N}{2-\varepsilon}} dx < C \quad \text{for some } \varepsilon \in [0, \varepsilon_0] \text{ and } r_0 < \frac{|x_0|}{4}, \tag{2.40}$$

where C is a constant depending on α, r_0, p and ε_0. We rewrite here the proof of (2.40) in order to explain this part clearly.

Recall the function $\Lambda(N, p, \gamma, \alpha)$ defined in (2.14), and recall the property (2.16) :

$$\Lambda(N, p, \gamma(p), \alpha) < 0 \quad \text{for} \quad 1 < p < p(N, \alpha).$$

Taking $\alpha = 0$ we have $\Lambda(N, p, \gamma(p), 0) < 0$ for $1 < p < p(N,0)$. By continuity we can fix $\gamma_\star \in (1, \gamma(p))$ such that

$$\Lambda(N, p, \gamma_\star, 0) < 0 \quad \text{for} \quad 1 < p < p(N,0),$$

which is equivalent to

$$\frac{p + \gamma_\star}{(p-1)N/2} > 1. \tag{2.41}$$

It follows from (2.41) that we can find $\varepsilon_0 = \varepsilon_0(p) > 0$ sufficiently small so that

$$\frac{p + \gamma_\star}{(p-1)\theta} > 1 \quad \text{for all} \quad \theta \in [\frac{N}{2}, \frac{N}{2 - \varepsilon_0}].$$

Fix such a θ and set

$$\xi = \frac{p + \gamma_*}{(p-1)\theta}.$$

We are ready to prove (2.40). By Hölder's inequality and (2.38),

$$\begin{aligned}
&\int_{B_{2r_0}(x_0)} \left(|x|^\alpha |u|^{p-1}\right)^\theta dx \\
\le&\left(\int_{B_{2r_0}(x_0)} |x|^\alpha |u|^{\gamma_\star+p}\right)^{1/\xi} \left(\int_{B_{2r_0}(x_0)} |x|^{\frac{\alpha(\theta\xi-1)}{\xi-1}}\right)^{(\xi-1)/\xi} \\
\le& CR^{\left(N-\frac{(2+\alpha)\gamma_\star+2p+\alpha}{p-1}\right)\frac{1}{\xi}} R^{\left(N+\frac{\alpha(\theta\xi-1)}{\xi-1}\right)\frac{\xi-1}{\xi}} \\
=& CR^{N-2\theta},
\end{aligned} \tag{2.42}$$

this implies (2.40) if we take $\theta = \frac{N}{2-\varepsilon}$.

Applying now Lemma 2.3.6, since $-\Delta u = |x|^\alpha |u|^{p-1}u$ and $u \in L^2_{loc}(\Omega)$, we get $u \in L^\infty(B_{r_0}(x_0))$. As x_0 can be any point in $\Omega \setminus \{0\}$, it means that $u \in L^\infty_{loc}(\Omega \setminus \{0\})$.

When $x_0 = 0$, to get the above estimate (2.40), in the usage of the Hölder inequality (see (2.42)), an extra condition is needed, that is,

$$N + \frac{\alpha(\theta\xi - 1)}{\xi - 1} > 0. \tag{2.43}$$

Taking $\theta = \frac{N}{2}$, (2.43) means that

$$\Lambda(N, p, \gamma_*, \alpha) < 0. \tag{2.44}$$

(2.44) holds true only if $\Lambda(N, p, \gamma(p), \alpha) < 0$, or equivalently only if $1 < p < p(N, \alpha)$. To conclude, when $1 < p < p(N, \alpha^-)$, there exists $r_0 > 0$ such that $|x|^\alpha |u|^{p-1} \in L^{\frac{N}{2-\varepsilon_0}}(B(0, 2r_0))$. Using again Lemma 2.3.6 we get $u \in L^\infty(B_{r_0})$.

Combining the above two cases we get that $u \in L^\infty_{loc}(\Omega)$ when $1 < p < p(N, \alpha^-)$. Hence $|x|^\alpha |u|^{p-1}u \in L^q_{loc}(\Omega)$ for some $q > \frac{N}{2}$ because $\alpha > -2$. We get $u \in C^{0,\beta}_{loc}(\Omega)$ by classical regularity theory.

Moreover, since $u \in L^\infty_{loc}(\Omega \setminus \{0\})$ when $1 < p < p(N, 0)$, the classical regularity theory also implies $u \in C^{2,\beta}_{loc}(\Omega \setminus \{0\})$.

The key argument to prove (2.39) is a uniform estimate inspired by the interesting work of Phan & Souplet [49].

Lemma 2.4.5. *Let $1 < p < p(N, 0)$ and $\beta > 0$. Assume that $c \in C^{0,\beta}(\overline{B}_1)$ verifies $\|c\|_{C^{0,\beta}(\overline{B}_1)} \le C_1$ and $c(x) \ge C_2 > 0$ in B_1. Then there exists a constant $C > 0$ depending only on β, C_1, C_2, p and N, such that any stable solution u to $-\Delta u = c(x)|u|^{p-1}u$ in B_1 satisfies*

$$|u(x)|^{\frac{p-1}{2}} + |\nabla u(x)|^{\frac{p-1}{p+1}} \le \frac{C}{1 - |x|}, \quad \forall\, x \in B_1.$$

Proof of Lemma 2.4.5. Indeed, arguing by contradiction, we can repeat exactly the proof of Lemma 2.1 in [49]. We here rewrite briefly its demonstration to explain Phan-Souplet's technic.

Suppose that there exist sequences c_k, u_k verifying all the properties mentioned in Lemma 2.4.5 and the points $y_k \in B_1$ such that the functions $M_k := |u_k|^{(p-1)/2} + |\nabla u_k|^{(p-1)/(p+1)}$ satisfy

$$M_k(y_k) > \frac{2k}{1-|y_k|}.$$

By Doubling Lemma 2.2.3 with $D = B_1$ and $\Gamma = \partial B_1$, there exists $x_k \in B_1$ such that

$$M_k(x_k) \geq M_k(y_k), \qquad M_k(x_k) > \frac{2k}{1-|x_k|}$$

and

$$M_k(z) \leq 2M_k(x_k) \quad \text{for all } z \in \{z \in B_1 \,:\, |z - x_k| \leq k\lambda_k\} \tag{2.45}$$

where $\lambda_k := M_k^{-1}(x_k) \to 0$ as $k \to \infty$ since $M_k(x_k) \geq M_k(y_k) > 2k$.

Consider now

$$v_k(y) = \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y) \quad \text{and} \quad \bar{c}_k(y) = c_k(x_k + \lambda_k y).$$

Note that $|v_k|^{(p-1)/2}(0) + |\nabla v_k|^{(p-1)/(p+1)}(0) = 1$ and

$$\left[|v_k|^{(p-1)/2} + |\nabla v_k|^{(p-1)/(p+1)}\right](y) \leq 2 \quad \text{for } |y| \leq k \text{ due to (2.45).} \tag{2.46}$$

Moreover we observe that v_k satisfies the following equation

$$-\Delta v_k = \bar{c}_k(y)|v_k|^{p-1}v_k \quad \text{in } B_k. \tag{2.47}$$

On the other hand, since $x_k + \lambda_k y \in B_1$ for $|y| \leq k$, we have $C_2 \leq \bar{c}_k \leq C_1$, then for every $R > 0$ and $k \geq k_0(R)$ large enough, we have

$$|\bar{c}_k(y) - \bar{c}_k(z)| \leq C_1|\lambda_k(y-z)|^\alpha \leq C_1|y-z|^\alpha \quad \text{for } |y|, |z| \leq R. \tag{2.48}$$

By Ascoli's theorem, there exists $\bar{c}$ in $C(\mathbb{R}^N)$ with $\bar{c} \geq C_2$ such that, after extracting a subsequence, $\bar{c}_k \to \bar{c}$ in $C_{loc}(\mathbb{R}^N)$. Moreover the first inequality in (2.48) implies that $|\bar{c}_k(y) - \bar{c}_k(z)| \to 0$ as $k \to \infty$ since $\lambda_k \to 0$, therefore the function $\bar{c}$ is actually a constant $C > 0$.

Secondly, for all $R > 0$ and $1 < q < \infty$, it follows from (2.47), (2.46) and the interior elliptic L^q estimates that the sequence v_k is uniformly bounded in $W^{2,q}(B_R)$. Using standard embedding and interior elliptic Schauder estimate, after extracting a subsequence, we have that $v_k \to v$ in $C^2_{loc}(\mathbb{R}^N)$. Hence we obtain that v is a stable solution of

$$-\Delta v = C|v|^{p-1}v \quad \text{in } \mathbb{R}^N \quad \text{and} \quad |v|^{(p-1)/2}(0) + |\nabla v|^{(p-1)/(p+1)}(0) = 1.$$

Since $1 < p < p(N, 0)$, this contradicts Farina's Liouville Theorem 2.1.1 and this completes the proof of Lemma 2.4.5. $\square$

We carry on the proof of Theorem 2.4.3. Using the scaling argument, the proof of (2.39) is the same as for [49, Theorem 1.2]. Look at the situation near the origin. Applying Lemma 2.4.4, u is stable in $B_R(x_0)$ when $|x_0| = 2R > 0$ is small. Define

$$U(y) = R^{\frac{2+\alpha}{p-1}} u(x_0 + Ry) \quad \text{with } y \in B_1. \tag{2.49}$$

Therefore U is a stable solution of

$$-\Delta U = c(y)|U|^{p-1}U \ \text{ in } B_1 \quad \text{with} \quad c(y) = \left|\frac{x_0}{R} + y\right|^\alpha.$$

We can check that $1 \leq c(y) \leq 3$ in B_1 and $\|c\|_{C^{1,\alpha}(\overline{B}_1)} \leq C_\alpha$, hence $|U(0)| + |\nabla U(0)| \leq C$ by Lemma 2.4.5, which is equivalent to say

$$|u(x_0)| + R|\nabla u(x_0)| \leq CR^{-\frac{2+\alpha}{p-1}}, \quad \text{for } |x_0| = 2R \text{ small enough.}$$

So we are done. $\square$

Remark 2.4.6. The estimate (2.39) was proved in Theorems 5.1 and 5.2 of [21] with different method.

Proof of Theorem 2.1.13. Thanks to Theorem 2.4.3, we need only to consider (2.10), which is also a direct consequence of scaling argument. Let $|x_0| > 0$ be small such that $B_{2|x_0|} \subset \Omega$. Define $V(y) = u(x_0 + Ry)$ in B_1, where $2R = |x_0|$. As $u \in C(\Omega)$, we have $\|\Delta V\|_\infty \leq CR^{2+\alpha}$, so $|\nabla V(0)| \leq CR^{2+\alpha}$ by standard elliptic theory. Hence $|\nabla u(x_0)| \leq C|x_0|^{1+\alpha}$, we get easily (2.10) since $\alpha > -2$. $\square$

2.4.2 Classification of finite Morse index solution in $\mathbb{R}^N$ for subcritical case

We here prove Theorem 2.1.10 for subcritical p, that is

$$1 < p < \frac{N + 2 + 2\alpha}{N - 2} \quad \text{and} \quad p < p(N, 0). \tag{2.50}$$

Let $\gamma = 1$, so

$$N - \frac{(2+\alpha)\gamma + 2p + \alpha}{p - 1} = N - \frac{2p + 2 + 2\alpha}{p - 1} < 0.$$

Consequently, since $u \in C(\mathbb{R}^N)$ by Theorem 2.4.3 and taking $r \to \infty$ in (2.37), we have $\nabla u \in L^2(\mathbb{R}^N)$ and $|x|^{\frac{\alpha}{p+1}} u \in L^{p+1}(\mathbb{R}^N)$.

Let $\phi_R(x) = \phi\left(R^{-1}|x|\right)$, where $\phi \in C_c^\infty(-2, 2)$ is a cut-off function such that $0 \leq \phi \leq 1$ in $\mathbb{R}$ and $\phi(t) = 1$ for $|t| \leq 1$. As $u\phi_R \in H_0^1 \cap L^\infty$ is compactly supported, by density argument, we can use it as test function in (2.2), to obtain

$$\int_{\mathbb{R}^N} |\nabla u|^2 \phi_R dx - \int_{\mathbb{R}^N} |x|^\alpha |u|^{p+1} \phi_R dx = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \Delta\phi_R dx. \tag{2.51}$$

By Hölder's inequality and the estimate (2.38), we get

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} |u|^2\Delta\phi_R dx\right| \\ \leq &\left[\int_{R<|x|<2R}\left(|x|^{\frac{\alpha}{p+1}}|u|\right)^{p+1}dx\right]^{\frac{2}{p+1}}\left[\int_{R<|x|<2R}\left(|x|^{-\frac{2\alpha}{p+1}}|\Delta\phi_R|\right)^{\frac{p+1}{p-1}}dx\right]^{\frac{p-1}{p+1}} \\ \leq &\, CR^{N-\frac{2p+2+2\alpha}{p-1}}. \end{aligned} \tag{2.52}$$

As $u \in C^2(\mathbb{R}^N \setminus \{0\})$ by Theorem 2.4.3, we can apply the Pohozaev identity (Lemma 2.2.4) to u in $\Omega_{\varepsilon,R} := B_R \setminus \overline{B_\varepsilon}$ with $R > \varepsilon > 0$, then

$$\begin{aligned} &\frac{N+\alpha}{p+1}\int_{\Omega_{\varepsilon,R}}|x|^\alpha|u|^{p+1}dx - \frac{N-2}{2}\int_{\Omega_{\varepsilon,R}}|\nabla u|^2dx \\ = &\int_{\partial\Omega_{\varepsilon,R}}|x|^\alpha\langle x,\nu\rangle|u|^{p+1}d\sigma + \int_{\partial\Omega_{\varepsilon,R}}\frac{\partial u}{\partial\nu}\langle x,\nabla u\rangle d\sigma - \int_{\partial\Omega_{\varepsilon,R}}\frac{|\nabla u|^2}{2}\langle x,\nu\rangle d\sigma. \end{aligned} \tag{2.53}$$

Using (2.39) and $p < \frac{N+2+2\alpha}{N-2}$,

$$\int_{\partial B_R}\left(|x||\nabla u|^2 + |x|^{\alpha+1}|u|^{p+1}\right)d\sigma \leq CR^{N-2-\frac{2(2+\alpha)}{p-1}} \to 0, \quad \text{if } R\to\infty. \tag{2.54}$$

On the other hand, we have $-\Delta u = O(|x|^\alpha)$ near the origin and $u \in C(\mathbb{R}^N)$. As $\alpha > -2$, by regularity theory and Sobolev embedding we can claim that $\nabla u \in L^q_{loc}(\mathbb{R}^N)$ for some $q > 2$. Applying again Hölder's inequality,

$$\begin{aligned} &\frac{1}{\varepsilon}\int_0^\varepsilon\left(\int_{\partial B_s}|x||\nabla u|^2d\sigma\right)ds \\ \leq &\int_{B_\varepsilon}|\nabla u|^2dx \leq \varepsilon^\sigma\|\nabla u\|_{L^q(B_\varepsilon)}, \quad \text{where } \sigma > 0. \end{aligned}$$

Thus there exists a sequence $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty}\int_{\partial B_{\varepsilon_j}}|x||\nabla u|^2d\sigma = 0. \tag{2.55}$$

Take $\varepsilon = \varepsilon_j$ in (2.53) then tend R and j to ∞, it follows from (2.54) and (2.55) that

$$\frac{N+\alpha}{p+1}\int_{\mathbb{R}^N}|x|^\alpha u^{p+1}dx - \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx = 0.$$

Combining with (2.51) and (2.52), we have

$$\left(\frac{N-2}{2} - \frac{N+\alpha}{p+1}\right)\int_{\mathbb{R}^N}|x|^\alpha|u|^{p+1}dx = 0.$$

As $\frac{N-2}{2} - \frac{N+\alpha}{p+1} < 0$ for subcritical p, we conclude that $u \equiv 0$ under the assumption (2.50). □

2.4.3 Fast decay behavior at infinity

We show here Theorem 2.1.14 which is wildly using later. Consider first

$$p \geq \frac{N+2+2\alpha}{N-2} \quad \text{and} \quad p < p(N,\alpha^-). \tag{2.56}$$

Recall again the property (2.16) :

$$\Lambda(N,p,\gamma(p),\alpha) < 0 \quad \text{for} \quad 1 < p < p(N,\alpha).$$

Moreover, we have

$$\Lambda(N,p,1,\alpha) = (N-2)p - (N+2) - 2\alpha \geq 0 \quad \text{for} \quad p \geq \frac{N+2+2\alpha}{N-2}$$

Therefore, under the assumption (2.56), due to $p(N,\alpha^-) \leq p(N,\alpha)$, there exists $\gamma_1 \in [1,\gamma(p))$ with $\gamma(p) = 2p + 2\sqrt{p(p-1)} - 1)$ such that $\Lambda(N,p,\gamma_1,\alpha) = 0$, that is

$$N - \frac{(2+\alpha)\gamma_1 + 2p + \alpha}{p-1} = 0.$$

Set now

$$\beta_0 = \frac{\gamma_1 + p}{2} > 1 \quad \text{and} \quad \omega(x) = |u(x)|^{\beta_0} \geq 0,$$

Using (2.37) with $\gamma = \gamma_1$ and letting $r \to \infty$, we have

$$\int_{|x|>3R_0} |x|^\alpha \omega^2 dx < \infty. \tag{2.57}$$

and

$$-\Delta\omega - \beta_0 |x|^\alpha |u|^{p-1}\omega = -\beta_0(\beta_0 - 1)|u|^{\beta_0-2}|\nabla u|^2 \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).$$

Moreover, as $p < p(N,\alpha^-) \leq p(N,0)$, it follows from (2.38) that there exists $\varepsilon_0(p,N) \in (0,2)$ such that

$$\int_{B_r(y)} \left(|x|^\alpha |u|^{p-1}\right)^{\frac{N}{2-\varepsilon_0}} dx \leq C r^{N - \frac{2N}{2-\varepsilon_0}}, \quad \forall\, B(y,2r) \subset \mathbb{R}^N \setminus \overline{B(0,R_0)}$$

where the constant C is independent of y and r. Let $|y| > 8R_0$ and $R = \frac{|y|}{8}$, so $B_{4R}(y) \subset \mathbb{R}^N \setminus \overline{B_{R_0}}$. Therefore

$$R^{\varepsilon_0} \|\beta_0 |x|^\alpha |u|^{p-1}\|_{L^{\frac{N}{2-\varepsilon_0}}(B_{2R}(y))} \leq \beta_0 R^{\varepsilon_0} \left(C R^{N-\frac{2N}{2-\varepsilon_0}}\right)^{\frac{2-\varepsilon_0}{N}} = C',$$

Applying (2.57) and Lemma 2.3.6,

$$\begin{aligned}
\omega(y) &\leq C R^{-\frac{N}{2}} \|\omega\|_{L^2(B_{2R}(y))} \\
&\leq C R^{-\frac{N}{2}} R^{-\frac{\alpha}{2}} \||x|^{\frac{\alpha}{2}}\omega\|_{L^2(B_{2R}(y))} = o\left(R^{-\frac{N+\alpha}{2}}\right) \quad \text{as} \quad |y| \to \infty.
\end{aligned}$$

Hence

$$|u(y)| = \omega(y)^{\frac{1}{\beta_0}} = o\left(R^{-\frac{N+\alpha}{2\beta_0}}\right) = o\left(R^{-\frac{2+\alpha}{p-1}}\right) \quad \text{as} \quad |y| \to \infty,$$

which shows the fast decay of u. To prove the second claim of (2.11), we observe that

$$-\Delta u(y) = |y|^\alpha |u|^{p-1}u = o\left(R^{-\frac{2+\alpha}{p-1}-2}\right) \quad \text{as} \quad |y| \to \infty.$$

The scaling argument and standard elliptic theory imply then

$$|\nabla u(y)| = o\left(R^{-\frac{2+\alpha}{p-1}-1}\right) \quad \text{as} \quad |y| \to \infty.$$

Now we consider the remaining case

$$\underline{p}(N,\alpha) < p < \frac{N+2+2\alpha}{N-2} \quad \text{and} \quad p < p(N,\alpha^-). \tag{2.58}$$

Here we use the Kelvin's transformation. Let

$$v(x) = |x|^{2-N} u\left(\frac{x}{|x|^2}\right), \quad \text{for } |x| > 0 \text{ small},$$

then near the origin, v verifies $-\Delta v = |x|^\beta |v|^{p-1}v$ with

$$\beta = (N-2)(p-1) - (4+\alpha).$$

With (2.58), as $\frac{N+\alpha}{N-2} < \underline{p}(N,\alpha) < p < p(N,\alpha)$, we have $\beta > -2$. Moreover, from the definition of $f(p)$ introduced in preliminaries, direct calculation yields

$$p\frac{(2+\beta)}{p-1}\left(N-2-\frac{2+\beta}{p-1}\right) = p\frac{(2+\alpha)}{p-1}\left(N-2-\frac{2+\alpha}{p-1}\right) > \frac{(N-2)^2}{4},$$

which means that $p < p(N,\beta)$.

Therefore we have the following properties :

- it is shown in [21] that, v is stable over $B_r \setminus \{0\}$ for $r > 0$ small since u is stable outside $B_{r^{-1}}$.
- $p < p(N,\beta^-)$ since $p < p(N,\beta)$ and $p < p(N,0)$ by (2.58).

Using Theorem 2.4.3 for v, we claim that v is a weak stable solution of $-\Delta v = |x|^\beta |v|^{p-1}v$ in B_r. Indeed, since $v(x) = O\left(|x|^{-\frac{2+\alpha}{p-1}}\right)$ and $\nabla v(x) = O\left(|x|^{-1-\frac{2+\alpha}{p-1}}\right)$, there holds

$$\int_{B_r} v^2 dx \leq C\int_0^r s^{N-1-\frac{2(2+\beta)}{p-1}} ds < \infty$$
$$\int_{B_r} |\nabla v|^2 dx \leq C\int_0^r s^{N-3-\frac{2(2+\beta)}{p-1}} ds < \infty,$$

In the above, we used the fact that $N-2-\frac{2(2+\beta)}{p-1} > 0$, since $p < \frac{N+2+2\alpha}{N-2}$ and $\beta = (N-2)(p-1)-(4+\alpha)$. Thus v is a weak solution in B_r. Since v is stable over

$B_r \setminus \{0\}$ and each singular point is of zero capacity in $\mathbb{R}^N (N \geq 2)$, by Lemma 2.4.4 we can find that v is stable in B_r therefore the desired claim is obtained.

Moreover, as $p < p(N, \beta^-)$ and $p < \frac{N+2+2\alpha}{N-2}$, v is continuous in B_r by Theorem 2.4.3, so we get

$$u(x) = |x|^{2-N} v\left(\frac{x}{|x|^2}\right) = O\left(|x|^{2-N}\right) = o\left(|x|^{-\frac{2+\alpha}{p-1}}\right) \quad \text{as } |x| \to \infty.$$

As $p < p(N, \beta^-)$, combining with (2.10) for v, we have

$$\nabla u(x) = O\left(|x|^{1-N}\right) + o\left(|x|^{\frac{2+\beta}{p-1}+1-N}\right) = o\left(|x|^{-1-\frac{2+\alpha}{p-1}}\right) \quad \text{as } |x| \to \infty.$$

The proof is completed. $\square$

2.4.4 Classification of finite Morse index solution in $\mathbb{R}^N$ for supercritical case

Now we are in position to prove Theorem 2.1.10 for p verifying

$$p > \frac{N+2+2\alpha}{N-2} \quad \text{and} \quad p < p(N, \alpha^-).$$

In fact, combining Theorems 2.1.13 and 2.1.14, the regularity result in Theorem 2.4.3, the supercritical exponent situation for Theorem 2.1.10 is a direct consequence of the following nonexistence result.

Proposition 2.4.7. *Let $\alpha > -2$ and $p > 1$ and $p \neq \frac{N+2+2\alpha}{2+\alpha}$. Then there does not exist any non trivial weak solution of* (2.2) *in $\mathbb{R}^N$ satisfying $u \in C^2(\mathbb{R}^N \setminus \{0\})$, the estimates* (2.11) *and*

$$\lim_{|x|\to 0} |x|^{\frac{2+\alpha}{p-1}} |u(x)| = \lim_{|x|\to 0} |x|^{1+\frac{2+\alpha}{p-1}} |\nabla u(x)| = 0. \tag{2.59}$$

Proof. As in [35, 64], we use the Emden-Fowler change of variable

$$u(r, \sigma) = r^{-\frac{2+\alpha}{p-1}} w(t, \sigma), \quad t = \ln r.$$

Therefore w satisfies

$$w_{tt} + A_1 w_t + \Delta_{S^{N-1}} w + A_2 w + |w|^{p-1} w = 0 \quad \text{in } \mathbb{R} \times S^{N-1},$$

where

$$A_1 = N - 2 - \frac{2+\alpha}{p-1} \neq 0, \quad A_2 = -\frac{2+\alpha}{p-1}\left(N - 2 - \frac{2+\alpha}{p-1}\right)$$

and $\Delta_{S^{N-1}}$ denotes the Laplace-Beltrami operator on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Set

$$E(w)(t) = \int_{S^{N-1}} \left(\frac{1}{2}|\nabla_{S^{N-1}} w|^2 - \frac{A_2}{2} w^2 - \frac{1}{p+1}|w|^{p+1} - \frac{1}{2} w_t^2\right) d\sigma.$$

Clearly,

$$\frac{d}{dt}E(w)(t) = A_1 \int_{S^{N-1}} w_t^2 d\sigma.$$

The estimates (2.11) and (2.59) yield

$$\lim_{t\to\pm\infty} w(t,\sigma) = \lim_{t\to\pm\infty} |w_t(t,\sigma)| = \lim_{t\to\pm\infty} |\nabla_{S^{N-1}} w(t,\sigma)| = 0,$$

hence $\lim_{t\to\pm\infty} E(w)(t) = 0$. On the other hand, integrating the equation of w, we get

$$0 = A_1 \int_{\mathbb{R}} \int_{S^{N-1}} w_t^2 d\sigma dt = A_1 \int_{\mathbb{R}^N} w_t^2 dx,$$

which means that $w_t = 0$ in $\mathbb{R}^N$. Therefore $w \equiv 0$ since $\lim_{t\to\pm\infty} w(t,\sigma) = 0$, so $u \equiv 0$. $\square$

Remark 2.4.8. We can see that the above proof works also for

$$\underline{p}(N,\alpha) < p < \min\left(\frac{N+2+2\alpha}{N-2},\ p(N,\alpha^-)\right).$$

2.4.5 Classification of finite Morse index solution in half Euclidian space

We end this chapter by proving Theorem 2.1.12. The proof is a simple adaptation of ideas for Theorem 9(b) in [35]. The main point is that we can consider v, the odd extension of u. Then v is a weak solution of (2.2) in $\mathbb{R}^N$. The crucial argument is that we can use test function as $\zeta_k(u)\psi$ with $\psi \in C_c^\infty(\mathbb{R}^N)$ as u verifies the Dirichlet boundary condition on $\partial\mathbb{R}^N_+$, we get all the corresponding estimate since all the boundary terms are zero when doing the integration by parts (see more details in [35]). For example, u is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^N_+$, so we can obtain the estimate (2.36) for $\psi \in C_c^\infty(\mathbb{R}^N \setminus \mathcal{K})$. By symmetry, the same estimate holds with v in $\mathbb{R}^N_-$ with $\psi \in C_c^\infty(\mathbb{R}^N \setminus \mathcal{K}')$ where $\mathcal{K}'$ is the mirror symmetry of $\mathcal{K}$. By taking the sum, we can claim that for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and $m \in \mathbb{N}$ large enough, there holds

$$\begin{aligned}&\int_\Omega \left(|\nabla(|v|^{\frac{\gamma-1}{2}} v)|^2 + |x|^\alpha |v|^{\gamma+p}\right) |\psi|^{2m} dx\\ &\leq C \int_\Omega |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}} dx,\end{aligned}$$

for any $\psi \in C_c^\infty(\mathbb{R}^N \setminus (\mathcal{K} \cup \mathcal{K}'))$ verifying $\|\psi\|_\infty \leq 1$.

In other words, we refind all the regularity results and the corresponding estimates for v as for weak solution in $\mathbb{R}^N$ with finite Morse index. For example, we can prove the corresponding result of Lemma 3.4 to solution of $-\Delta u = c(x)|u|^{p-1}u$ in B_1^+ verifying $u = 0$ on $B_1 \cap \partial\mathbb{R}^N_+$, where the contradiction comes from the classification result on half space, Theorem 9(b) in [35] instead of [35, Theorem 1].

Proceeding as for Theorem 2.1.10, we refine then $v \equiv 0$, so is u. $\square$

2.5 Further remarks and open questions

In general, the Hénon equation $-\Delta u = |x|^\alpha g(u)$ is more delicate to handle than the corresponding autonomous equation, i.e. when $\alpha = 0$. Many properties for the autonomous equation are no longer true or less understood for the nonautonomous situation.

For example, let $\alpha > 0$, we don't know if a *positive* and *continuous* solution to $-\Delta u = |x|^\alpha u^p$ in $\mathbb{R}^N$ could exist with a subcritical exponent

$$\frac{N + \max(2, \alpha)}{N - 2} < p < p_\alpha := \frac{N + 2 + 2\alpha}{N - 2}.$$

As mentioned yet, only very recently, Phan and Souplet proved the nonexistence of *bounded positive* solution in $\mathbb{R}^3$ for any $1 < p < p_\alpha$, see [49] and the references therein.

For the critical exponent case $p = p_\alpha$ with $\alpha > -2$, the existence of radial solutions in $\mathbb{R}^N$ is known via (2.9), and they are stable outside a compact set. When $\alpha = 0$ and $p = \frac{N+2}{N-2}$, Farina proved in [35, Theorem 2] a very interesting equivalence between the following assumptions :

(i) u is a finite Morse index solution of (2.2) in $\mathbb{R}^N$;

(ii) u is a solution of (2.2) in $\mathbb{R}^N$ and u is stable outside a compact set ;

(iii) u is a solution of (2.2) in $\mathbb{R}^N$ and $\nabla u \in L^2(\mathbb{R}^N)$.

Combining with [26], we get infinitely many (conformally non-equivalent) sign-changing solutions to $-\Delta u = |u|^{\frac{4}{n-2}} u$ in $\mathbb{R}^N$ with finite Morse index.

For weak solution to (2.2) with general $p \leq p_\alpha$ and $\alpha > -2$, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold true. For (ii) $\Rightarrow$ (iii), we can take just $\gamma = 1$ in the estimate (2.37). However it is difficult to give a general positive or negative answer for implication (ii) $\Rightarrow$ (i) or (iii) $\Rightarrow$ (ii), which seem to be completely open, even for $\alpha = 0$, except Farina's result.

An interesting work linked to that is [39] (see also the references therein), where the authors proved some relationships between the symmetry and low Morse index solutions for $-\Delta u = f(|x|, u)$. For example, we have

Theorem 2.5.1 (Gladiali-Pacella-Weth, [39])**.** *Let u be a stable solution of*

$$-\Delta u = f(|x|, u) \text{ in } \Omega = \mathbb{R}^N$$

or

$$-\Delta u = f(|x|, u) \text{ in } \Omega = \mathbb{R}^N \backslash B(0, R) \quad \text{and} \quad u = 0 \text{ on } \partial\Omega,$$

where $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,\alpha}$- function. If $|\nabla u| \in L^2(\Omega)$, then u is radial. In addition, if $\Omega = \mathbb{R}^N$ and f does not depend on $|x|$, then u is constant.

Therefore we can claim that for $\alpha > 1$, any stable solution to (2.2) in $\mathbb{R}^N$ verifying $\nabla u \in L^2(\mathbb{R}^N)$ must be radial.

When $\alpha \neq 0$ and $p = p_\alpha$, the existence of *non radial* or *sign-changing* entire solution to (2.2) with finite Morse index seems also unknown.

When $p \geq p(N, \alpha)$ and $\alpha > -2$, we have the following question :

Does there exist classical or weak solution u to (2.2) in $\mathbb{R}^N$ with $p \geq p(N, \alpha)$ and $0 < \mathrm{ind}(u) < \infty$?

As far as we are aware, the problem is open, even for $\alpha = 0$. All known solutions for $\alpha = 0$ and $p \geq p(N, 0)$, as radial solutions or lower dimensional solutions are either stable or have infinite Morse index in $\mathbb{R}^N$. Of course, we can ask the similar question for (2.1) with $N \geq 10 + 4\alpha$.

The understanding of finite Morse index but unstable solutions to (2.2) in $\mathbb{R}^N$ for $p \geq p(N, \alpha)$ is far from evident, even they have necessarily finite energy, i.e. $\nabla u \in L^2(\mathbb{R}^N)$. For example, when $\alpha = 0$, if a such solution u exists and if $u \to 0$ as $|x| \to \infty$, it must be *non radial* and *sign-changing*, since any positive solution is radial and all radial solutions are stable as $p \geq p(N, 0)$. Theorem 1.6 in [39] yields then $\mathrm{ind}(u) \geq N + 1 \geq 12$, which means that u must have a large Morse index.

The assumption $\lim_{|x|\to\infty} u(x) = 0$ seems to be *reasonable* for any finite Morse index solution of (2.2), but we don't know if it holds true in general for $p \geq p(N, 0)$.

Chapitre 3

Stable solutions for other Hénon type equation

ABSTRACT. In this chapter we investigate the non-autonomous elliptic equation $-\Delta u = |x|^\alpha u_+^p$ both in $\mathbb{R}^N$ and in $\mathbb{R}_+^N$ with the Dirichlet boundary condition, where $N \geq 2$, $p > 1$ and $\alpha > -2$. We consider the weak stable solutions and weak solutions with finite Morse index in both cases and give some classification results. In fact, the above equation is an example of the general Hénon equation $-\Delta u = |x|^\alpha f(u)$ in $\mathbb{R}^N$, where f is some general nonlinearity. We thus give a Liouville type result for stable solutions of the general equation in the low dimensional Euclidean spaces, under suitable conditions.

Solutions stables des autres équations de type Hénon

RÉSUMÉ. On considère les équations elliptiques non-linéaires $-\Delta u = |x|^\alpha u_+^p$ et $-\Delta u = |x|^\alpha f(u)$ dans $\Omega \subset \mathbb{R}^N$ avec $\alpha > -2$, $p > 1$ et $N \geq 2$ et f une fonction surlinéaire générale. On démontre quelques résultats de classification pour les solutions faibles stables en dehors d'un ensemble compact (en particulier, les solutions avec indice Morse fini) dans des espaces euclidiens ou dans des demi-espaces de petite dimension.

3.1 Introduction

We begin in this chapter with the general Hénon type elliptic equation

$$-\Delta u = |x|^\alpha f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{3.1}$$

with $N \geq 2$ and $\alpha > -2$. Here f is the function such that

$$f \in C^2(I, \mathbb{R}) \cap C^0(\overline{I}, \overline{\mathbb{R}}); f \geq 0, f' \geq 0 \text{ and } f \text{ is convex in } I, \tag{3.2}$$

where $I = (a, b) \subset \mathbb{R}$ is an open interval (bounded or not) and $\overline{I} = [a, b] \subset \overline{\mathbb{R}}$.

Remark 3.1.1. When $f(u) = u_+^p$, we have that u_+^p belongs to C^2 only if $p \geq 2$. Therefore, in the following when we study the particular case $f(u) = u_+^p$ with all $p > 1$, we do not need the condition $f \in C^2$.

Our goal is always to classify the stable solutions and finite Morse index solution of the equation (3.1). As before, we first give some useful definitions and notions.

Definition 3.1.2. We say that u is a weak solution of $-\Delta u = |x|^\alpha f(u)$ in a smooth domain $\Omega \subset \mathbb{R}^N$, if $u \in H^1_{loc}(\Omega)$ such that $|x|^\alpha f(u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega \Big[\nabla u \cdot \nabla \zeta - |x|^\alpha f(u)\zeta\Big] dx = 0 \quad \text{for all} \quad \zeta \in C^1_c(\Omega). \tag{3.3}$$

- Let u be a weak solution of $-\Delta u = |x|^\alpha f(u)$ in Ω, we say that u is stable if $|x|^\alpha f'(u) \in L^1_{loc}(\Omega)$ and

$$Q_u(\zeta) := \int_\Omega \Big[|\nabla \zeta|^2 - |x|^\alpha f'(u)\zeta^2\Big] dx \geq 0 \quad \text{for all} \quad \zeta \in C^1_c(\Omega). \tag{3.4}$$

- The Morse index of a solution u in Ω, $\mathrm{ind}_\Omega(u)$, is defined as the maximal dimension of all subspaces X of $C^1_c(\Omega)$ such that $Q_u(\zeta) < 0$ for any $\zeta \in X \backslash \{0\}$.

As before, with the above definition, we see that u is stable in Ω if and only if $\mathrm{ind}_\Omega(u) = 0$, and a finite Morse index solution u in Ω is stable outside a compact set $\mathcal{K} \subset \Omega$. Moreover, if u is stable in Ω, using a density argument, (3.4) holds for all $\zeta \in H^1_0(\Omega)$, and (3.3) holds for any compactly supported $\zeta \in H^1_0(\Omega) \cap L^\infty(\Omega)$.

Moreover, we define the zero set $Z(f)$ of the function $f(u)$:

$$Z(f) = \{u \in \overline{I} : f(u) = 0\}.$$

By assumption (3.2) on f, we observe that only four cases can occur :

1. $Z(f) = \emptyset$, therefore $f(u) \geq C > 0$.
2. $Z(f) = \{-\infty\}$. For example $f(u) = e^u$ with $I = \mathbb{R}$.
3. $Z(f) = \{c\}$ for some $c \in I$. For example $f(u) = u^p$ with $(p > 1)$ and $I = \mathbb{R}_+$.

4. $Z(f) = [c, d]$ for some $c, d \in I$. For example $f(u) = (u-d)_+^p$ with $(p > 2)$ and $I = \mathbb{R}$.

We don't consider the first case. Indeed, we may choose $f(u) \equiv 1$ since $f(u) \geq C > 0$, then equation (3.1) becomes to $-\Delta u = |x|^\alpha$, the existence of stable solution for this equation is trivial, therefore no Liouville type theorem hold in this case. From now on we assume that

$$Z(f) \neq \emptyset. \tag{3.5}$$

Before the study of the general equation (3.1), we firstly consider some of its examples - the non-autonomous elliptic equations

$$-\Delta u = |x|^\alpha u_+^p \quad \text{in } \mathbb{R}^N \tag{3.6}$$

and

$$-\Delta u = |x|^\alpha u_+^p \text{ in } \mathbb{R}_+^N := \mathbb{R}^{N-1} \times (0, \infty), \quad u = 0 \text{ on } \partial\mathbb{R}_+^N. \tag{3.7}$$

Here $p > 1$, $\alpha > -2$, $N \geq 2$ and $u_+ = \max(u, 0)$.

Recall that in the previous chapter, the classical Hénon type nonlinear elliptic equation

$$-\Delta u = |x|^\alpha |u|^{p-1} u \quad \text{in } \mathbb{R}^N \tag{3.8}$$

was considered. We have already proved some Liouville type results for both weak stable solutions and weak solutions with finite Morse index, see Theorem 2.1.10 and Theorem 2.1.12.

On the other hand, as mentioned in the previous chapter, for $N > 10 + 4\alpha$ and $p \geq p(N, \alpha)$, it is known by [21] that (3.8) admits a family of stable positive and radial solutions in $\mathbb{R}^N$. For $p = \frac{N+2+2\alpha}{N-2}$ and $\alpha > -2$, the radial positive function

$$V(x) = \lambda^{\frac{N-2}{2}} \left(\frac{\sqrt{(N+\alpha)(N-2)}}{1 + \lambda^{2+\alpha}|x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}} \quad \text{with} \quad \lambda > 0 \tag{3.9}$$

is a finite Morse index solutions of (3.8), see page 63.

We will investigate the solutions for the equation (3.6) and (3.7). In [54], the autonomous case $\alpha = 0$ was studied, Rehbi proved that

Theorem 3.1.3 (Rehbi, [54])**.** *Let u be a classical stable solution of*

$$-\Delta u = u_+^p \quad \textit{in } \mathbb{R}^N \tag{3.10}$$

or

$$-\Delta u = u_+^p \textit{ in } \mathbb{R}_+^N, \quad u = 0 \textit{ on } \partial\mathbb{R}_+^N. \tag{3.11}$$

Let $1 < p < p_{JL}$. Then u is a non-positive constant. Precisely, for (3.10)*, $u = c \leq 0$ and for* (3.11)*, $u = cx_N$ with $c \leq 0$.*

Here $p_{JL} := p(N,0)$ stands always the Joseph-Lundgren exponent, defined in page 59. Moreover he showed that the only classical solutions stable outside a compact set for $1 < p < p_{JL}$ and $p \neq \frac{N+2}{N-2}$ are non-positive constants.

In all what follows of this chapter, we denote also by $B_r(x)$ the ball of radius r centered at x, $B_r^+(x) = B_r(x) \cap \mathbb{R}^N_+$, and simply by B_r, B_r^+ when $x = 0$.

Our main results concerned with (3.6) and (3.7) are the following and we begin with the stable solutions.

Theorem 3.1.4. *Let $\alpha > -2$, $N \geq 2$ and let $1 < p < p(N,\alpha)$. Then any weak stable solution to* (3.6) *or* (3.7) *satisfies $u \leq 0$. More precisely, in this case, for* (3.6)*, u is a non-positive constant; and $u \equiv cx_N$ with $c \leq 0$ for* (3.7)*.*

Theorem 3.1.4 is sharp, indeed the positive stable solutions to (3.8) are also stable solutions to (3.6), when $p \geq p(N,\alpha)$ and $N > 10 + 4\alpha$.

Consider now the solutions with finite Morse index. In high dimensional situation, we have

Theorem 3.1.5. *Let $\alpha > -2$, $N \geq 3$ and let $1 < p < p(N,\alpha^-)$ with $p \neq \frac{N+2+2\alpha}{N-2}$. Let u be a weak solution of* (3.6) *or* (3.7) *which has finite Morse index, then $\mathrm{supp} u_+$ is compact. More precisely, for the equation* (3.6)*, we have either $u \equiv 0$ or*

$$u(x) = \frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{|y|^\alpha u_+^p(y)}{|x-y|^{N-2}} dy + c \quad \textit{with} \quad c < 0.$$

For the equation (3.7)*, we have either $u \equiv 0$ or*

$$u(x) = \int_{\mathbb{R}^N_+} |y|^\alpha u_+^p(y) G(x,y) dy + cx_N \quad \textit{with} \quad c < 0.$$

Here $G(x,y)$ denotes the Green function of $\mathbb{R}^N_+$ for $N \geq 3$ with the Dirichlet boundary condition :

$$\begin{cases} -\Delta_y G(x,y) = \delta_x(y) & \text{in} \quad \mathbb{R}^N_+, \\ G(x,y) = 0 & \text{if} \quad y \in \partial\mathbb{R}^N_+, \\ G(x,y) \to 0 & \text{when} \quad |y| \to \infty. \end{cases} \tag{3.12}$$

Theorems 3.1.5 is sharp for $\alpha \in (-2, 0]$, however, for $\alpha > 0$ we do not know if the classification holds true for $p(N,0) \leq p < p(N,\alpha)$ and $N > 10 + 4\alpha$.

The parallel classification result associated to finite Morse index solutions also holds in $\mathbb{R}^2$.

Theorem 3.1.6. *Let $\alpha > -2$, $p > 1$ and $N = 2$. Let u be a weak solution of* (3.6) *or* (3.7) *which has finite Morse index, then $\mathrm{supp}(u_+)$ is compact. More precisely, for the equation* (3.6)*, we have either u is a negative constant or*

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} |y|^\alpha u_+^p(y) \log|x-y| dy + c \quad \textit{with} \quad c \in \mathbb{R}.$$

For the equation (3.7), *we have either* $u \equiv 0$ *or*

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} |y|^\alpha u_+^p(y) \log \left(\frac{|\overline{x} - y|}{|x - y|} \right) dy + cx_2 \quad \textit{with} \quad c < 0,$$

where $\overline{x} = (x_1, -x_2)$.

Remark 3.1.7. For the solutions of (3.6) having finite Morse index, if $u_+ \not\equiv 0$, we have $\lim_{|x|\to\infty} u(x) = -\infty$ for $N = 2$. While for $N \geq 3$ we have $\lim_{|x|\to\infty} u(x) = c < 0$. On the other hand, in half space situation the result is similar for all $N \geq 2$.

Next we study the weak stable solution u in outside domains

$$-\Delta u = |x|^\alpha u_+^p \quad \text{in } B_R^c = \mathbb{R}^N \setminus B_R \tag{3.13}$$

and

$$-\Delta u = |x|^\alpha u_+^p \quad \text{in } B_R^c \cap \mathbb{R}^N_+ \quad \text{and} \quad u = 0 \quad \text{on } \partial(B_R^c) \cap \mathbb{R}^N_+, \tag{3.14}$$

We will not classify the solutions of (3.13) and (3.14), but we prove some fast decay estimate at infinity. More precisely, we have

Proposition 3.1.8. *Let* $\alpha > -2$, $N \geq 3$, $\frac{N+2+2\alpha}{N-2} \leq p < p(N, \alpha^-)$ *and* u *be a weak stable solution of* (3.13) *or* (3.14). *Then we have*

$$\lim_{|x|\to+\infty} |x|^{\frac{2+\alpha}{p-1}} u_+(x) = 0. \tag{3.15}$$

We now continue the study of the general equation (3.1). Set the supremum of $Z(F)$ be as

$$z = \sup Z(F).$$

With (3.5) in hand we can guarantee the existence of z. This z plays a crucial role in the following.

Working as in [30], we define some function $q(u)$ as

$$q(u) = \frac{f'^2}{ff''}(u) = \frac{(\ln f)'}{(\ln f')'}(u) \quad \text{for all } u \in \mathbb{R} \text{ and } u > z. \tag{3.16}$$

Moreover, we bring in the notations $\overline{q_0}$, $\underline{q_0}$ and $\overline{q_\infty}$, which are defined by

$$\overline{q_0} = \lim_{u\to z^+} \sup q(u), \quad \underline{q_0} = \lim_{u\to z^+} \inf q(u), \quad \overline{q_\infty} = \lim_{u\to b^-} \sup q(u). \tag{3.17}$$

With all the above preparing works we obtain our main result - a Liouville type theorem for weak stable solutions of (3.1).

Theorem 3.1.9. *Let $\alpha > -2$ and f satisfy the assumption* (3.2) *and* (3.5). *Let $u \in L^\infty_{loc}(\Omega)$ be a weak stable solution of* (3.1). *Then u is a constant if*

$$0 < \underline{q_0} \leq \overline{q_0} < +\infty, \quad 0 < \overline{q_\infty} < +\infty, \quad \textit{and} \quad N = 2 \tag{3.18}$$

or if

$$\overline{q_0} < +\infty, \quad \frac{2(2+\alpha)}{N-2}\left(1 + \frac{1}{\sqrt{\overline{q_0}}}\right) > \frac{1}{\underline{q_0}} \tag{3.19}$$

and

$$\overline{q_\infty} < +\infty, \quad \frac{2(2+\alpha)}{N-2}\left(1 + \frac{1}{\sqrt{\overline{q_\infty}}}\right) > \frac{1}{\overline{q_\infty}}, \quad \textit{and} \quad N \geq 3. \tag{3.20}$$

Mention that condition (3.20) is equivalent to

$$\begin{cases} 0 < \overline{q_\infty} < +\infty, \quad N < 10 + 4\alpha; \\ 1 < \overline{q_\infty} < +\infty, \quad N = 10 + 4\alpha; \\ q(N,\alpha) < \overline{q_\infty} < +\infty, \quad N > 10 + 4\alpha, \end{cases} \tag{3.21}$$

where $q(N,\alpha)$ is the conjugate exponent of $p(N,\alpha)$.

Theorem 3.1.9 tells us there is no non-trivial solution for (3.1) under conditions (3.18)-(3.20). It is sharp for the non-autonomous exponential-growth equation, the polynomial-growth case and also the polynomial-growth one with positive part. Indeed, as proved in the previous chapter, we know that the equation $-\Delta u = |x|^\alpha e^u$ has no stable solution for $N < 10 + 4\alpha$ (by defining $-\infty \in \overline{\mathbb{R}}$ be a constant we can say that for $N < 10 + 4\alpha$ the stable solution is $u \equiv -\infty$), moreover, we have the existence of radial stable solution in $\mathbb{R}^N$ when $N \geq 10 + 4\alpha$. On the other hand, it is shown by Dancer, Du and Guo in [21] that the equation $-\Delta u = |x|^\alpha |u|^{p-1}u$ has $u \equiv 0$ the only stable solution for $p < p(N,\alpha)$ and admits positive radial stable solution otherwise. In the first part of this chapter, the non existence of non-trivial stable solution for $-\Delta u = |x|^\alpha u_+^p$ when $p < p(N,\alpha)$ is investigated.

We consider now the finite Morse index solution of (3.1) and wish always to give some similar results. Unfortunately, even under the conditions of Theorem 3.1.9, we can not prove the Liouville type theorem for (3.1) associated to the finite Morse index solutions. We show a counter-example by using a result of Ni [48] and harmonic extension :

Let $\alpha > 0$, $1 \geq p < \frac{N+2+2\alpha}{N-2}$ and $N \geq 2$. Then the equation $-\Delta u = |x|^\alpha u_+^p$ admits a radial solution in $\mathbb{R}^N$ with $\mathrm{supp}(u_+)$ is support. Evidently, it's with finite Morse index in $\mathbb{R}^N$.

This chapter is organized as follows. We show some preliminaries in section 2, then in section 3, we prove Theorem 3.1.4 to get some Liouville type results for weak stable solutions of (3.6) and (3.7) in the first subsection ; the second subsection is devoted to show some classification results for finite Morse index solutions hence we give the demonstration of Theorem 3.1.5 and Theorem 3.1.6 ; in the last subsection we prove Proposition 3.1.8. In section 4, we consider the stable solutions of (3.1) in the first subsection and prove Theorem 3.1.9 ; finally we show the counter-example in the second subsection. In the following, c and c' denote some generic positive constants.

3.2 Preliminaries

In the previous chapter we have already introduced a well-known Pucci-Serrin's result (Lemma 2.3.6), notice that the proof of Proposition 3.1.8 for entire space problem (3.13) is based on Lemma 2.3.6. In order to prove the fast decay estimate in Proposition 3.1.8 for the half space problem (3.14), we need a half space version of Lemma 2.3.6.

Lemma 3.2.1. *Let $u \in H^1_{loc}(\mathbb{R}^N_+)$ be a weak solution of*

$$\begin{cases} -\Delta u \le a(x)u & in \quad \mathbb{R}^N_+ \\ u = 0 \quad on & \partial\mathbb{R}^{\mathbb{N}}_+ \end{cases} \tag{3.22}$$

Let $\theta = \frac{N}{2-\epsilon} > \frac{N}{2}$ with $\epsilon \in (0, 2)$ and $\lambda > 0$. Then for all $s > 1$ and $R > 0$, if $R^\epsilon \|a\|_{L^\theta(B^+_{2R})} \le \lambda$, we have

$$\sup_{B^+_R} u_+ \le cR^{-\frac{N}{s}} \|u_+\|_{L^s(B^+_{2R})},$$

where $c = c(N, \lambda, s, \theta)$ is a positive constant.

Proof of Lemma 3.2.1. This result could be well known, but we can not find the proof in the literature. For the sake of completeness, we here give a proof. The argument is similar to the one in [56] and we only prove the result for $s \ge 2$, (in fact for our purpose we need it only for $s = 2$). We assume initially that $R = 1$ (the general result follows by re-scaling). The idea is based on the study of auxiliary function v.

Let $M > 0$, $r \ge 1$ and define the function v such that :

$$v(t) = \begin{cases} t^r_+ & \text{if} \quad t \le M, \\ rM^{r-1}t - (r-1)M^r & \text{if} \quad t \ge M. \end{cases} \tag{3.23}$$

We choose $\eta \in C^\infty_c(B_2)$ a non negative function and take $\xi(t) = v(t)v'(t)$ (here B_2 denotes the ball in R^N centered at 0 and with radius 2). After a short calculation we can find,

$$\begin{aligned} &0 \le \xi(t) \le t\xi'(t), \quad tv'(t) \le rv(t), \\ \text{and} \quad &t^2\xi(t) \le 2r^2v^2(t), \quad |\nabla v(u)|^2 \le |\nabla u|^2\xi'(u). \end{aligned} \tag{3.24}$$

Since $u = 0$ on $\partial\mathbb{R}^N_+$ and $\eta = 0$ on ∂B_2^+, then the function $\varphi(x) = \eta^2(x)\xi(u(x))$ can serve as a test function for (3.22). That is,

$$\int_{B_2^+} \nabla u \nabla(\eta^2 \xi(u))dx \leq \int_{B_2^+} a(x) u \eta^2 \xi(u) dx,$$

which reads

$$\int_{B_2^+} |\nabla u|^2 \eta^2 \xi'(u) \leq \int_{B_2^+} a(x) u \eta^2 \xi(u) - 2 \int_{B_2^+} \eta \nabla(\eta) \nabla u \xi(u).$$

Since $0 \leq \xi(t) \leq t\xi'(t)$, we have

$$\int_{B_2^+} |\nabla u|^2 \eta^2 \xi'(u) \leq \int_{B_2^+} a(x) u \eta^2 \xi(u) + 2 \int_{B_2^+} \eta |\nabla(\eta)| |\nabla u| u \xi'(u). \tag{3.25}$$

Thanks to Young's inequality, we get that for all $\delta > 0$,

$$\int_{B_2^+} \eta |\nabla(\eta)| |\nabla u| u \xi'(u) \leq \delta \int_{B_2^+} \eta^2 |\nabla u|^2 \xi'(u) + \frac{1}{\delta} \int u^2 |\nabla \eta|^2 \xi'. \tag{3.26}$$

Therefore, combining (3.25) and (3.26), for δ small enough, there exits $c_\delta > 0$ such that

$$\int_{B_2^+} \eta^2 |\nabla u|^2 \xi'(u) \leq \int_{B_2^+} a(x) u \eta^2 \xi(u) + c_\delta \int u^2 |\nabla \eta|^2 \xi'. \tag{3.27}$$

Using (3.24), the equation (3.27) becomes

$$\int_{B_2^+} \eta^2 |\nabla v|^2 \leq r \int_{B_2^+} a(x) v^2 \eta^2 + c r^2 \int v^2 |\nabla \eta|^2. \tag{3.28}$$

Since $\nabla(v\eta) = \eta \nabla v + v \nabla \eta$, the Sobolev's inequality implies

$$\|\eta v\|_{2^*}^2 \leq c r \int_{B_2^+} a(x) v^2 \eta^2 + c(1 + r^2) \int v^2 |\nabla \eta|^2, \tag{3.29}$$

where $2^* = \frac{2N}{N-2}$. By Hölder's inequality, we get

$$\int_{B_2^+} a(x) v^2 \eta^2 \leq \|a\|_\theta \|v\eta\|_2^\epsilon \|v\eta\|_{2^*}^{2-\epsilon}.$$

Therefore applying again Young's inequality, we may find a constant $c > 0$ such that (3.29) becomes

$$\|\eta v\|_{2^*}^2 \leq c r^{\frac{2}{\epsilon}} \|a\|_\theta^{\frac{2}{\epsilon}} \int_{B_2^+} v^2 \eta^2 + c(1 + r^2) \int v^2 |\nabla \eta|^2, \tag{3.30}$$

then for any $\lambda > 0$, if $\|a\|_\theta \leq \lambda$, (3.30) implies

$$\|\eta v\|_{2^*}^2 \leq c r^\nu \left(\int_{B_2^+} v^2 \eta^2 + \int v^2 |\nabla \eta|^2 \right), \tag{3.31}$$

where $c = c(N, \lambda, s, \theta) > 0$ and $\nu = \max(2, \frac{2}{\epsilon})$. We now specify the function η more precisely. Let $1 < h' < h < 2$ such that $0 \le \eta \le 1$ and

$$\eta = \begin{cases} 1 & \text{on } B^+_{h'}, \\ 0 & \text{on } (B^+_h)^c, \\ |\nabla \eta| \le \frac{2}{h-h'} & \text{on } B^+_h \setminus B^+_{h'}. \end{cases}$$

Then from (3.31) there follows

$$\|v\|_{2^*,h'} \le c\frac{r^\nu}{h-h'}\|v\|_{2,h} \tag{3.32}$$

where $\|v\|_{p,h} = ||v||_{L^p(B_h)}$. At this stage let $M \to \infty$, then $v \to u_+^r$ and therefore by the monotone convergence theorem, (3.32) can be rewritten as

$$\|u_+^r\|_{2^*,h'} \le c\frac{r^\nu}{h-h'}\|u_+^r\|_{2,h}. \tag{3.33}$$

Denote by $k = \frac{2}{2^*}$, the last inequality reads

$$\|u_+\|_{2kr,h'} \le \left(c\frac{r^\nu}{h-h'}\right)^{\frac{1}{r}} \|u_+\|_{2r,h}. \tag{3.34}$$

As in [56], we iterate (3.34) by taking successively $r_j = k^j$, $h = h_j = 1 + \frac{1}{2^j}$ and $h' = h_{j+1}$. We obtain

$$\|u_+\|_{2k^{j+1},1} \le \|u_+\|_{2k^{j+1},h_{j+1}} \le c^{\sum_{i=1}^j \frac{1}{k^i}} K^{\sum_{i=1}^j \frac{i}{k^i}} \|u_+\|_{2,2} \text{ with } K = 2k^\nu.$$

Letting now $j \to \infty$ and since both series $\sum_1^j \frac{1}{k^i}$ and $\sum_1^j \frac{i}{k^i}$ converge, we derive that

$$\|u_+\|_{\infty,1} \le C\|u_+\|_{2,2} \le C\|u_+\|_{2,s}, \ \forall\, s \ge 2, \tag{3.35}$$

where the constant C depends only on N, λ, s and θ. Finally using a scaling argument, we conclude that

$$\sup_{B_R^+} u_+ \le cR^{-\frac{N}{s}}\|u_+\|_{L^s(B^+_{2R})}, \quad \forall\, s \ge 2.$$

So we are done. □

Remark 3.2.2. For $1 < s < 2$, we proceed as in [56] by changing the function v, the same estimates holds.

3.3 Classification results for $-\Delta u = |x|^\alpha u_+^p$

Here we study $-\Delta u = |x|^\alpha u_+^p$ to prove the classification results Theorem 3.1.4, Theorem 3.1.5 and Theorem 3.1.6 ; and some asymptotic behavior Proposition 3.1.8.

3.3.1 Stable solutions

This subsection is devoted to the study of the weak stable solutions of (3.6) and (3.7). The proof of Theorem 3.1.4 is based on the following proposition, which is similar to Proposition 1.7 in [21].

Proposition 3.3.1. *Let Ω be a domain of $\mathbb{R}^N$ with $N \geq 2$. Let u be a weak stable solution of $-\Delta u = |x|^\alpha u_+^p$ in Ω with $p > 1$ and $\alpha > -2$. Then for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geq \max\{\frac{p+\gamma}{p-1}, 2\}$, there exists a constant $C(p, m, \gamma, \alpha) > 0$ such that*

$$\int_\Omega \left(|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{p+\gamma} \right) \psi^{2m} \leq C \int_\Omega |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla \psi|^2 + |\psi||\Delta\psi| \right)^{\frac{p+\gamma}{p-1}} \quad (3.36)$$

for all test function $\psi \in C_c^2(\Omega)$ satisfying $|\psi| \leq 1$ in Ω.

Similarly, if we suppose that the weak solution u belongs to $H^1_{loc}(\Omega)$ such that $u = 0$ on $\partial\Omega$ and u is stable outside a compact set $\mathcal{K} \subset \Omega$, then the estimate (3.3.1) holds for all test function $\psi \in C_c^2(\mathbb{R}^N \backslash \mathcal{K})$ verifying $\|\psi\|_\infty \leq 1$.

Proof of Proposition 3.3.1. We follow the idea from Farina's work [35]. As for Proposition 2.2 in [54], we consider the cut-off function $\zeta_k(t) = \min\{t_+, k\}$ with $k \in \mathbb{N}$. Multiplying (3.6) by $\zeta_k(u)^\gamma \varphi^2$ with $\varphi \in C_c^2(\Omega)$ and $\varphi \in C_c^2(\mathbb{R}^N \backslash \mathcal{K})$ respectively, and then use $\zeta_k(u)^{\frac{\gamma+1}{2}} \varphi$ as a test function in (3.4). For the rest, we proceed in the same way as in the proof of Proposition 2.2 in [54] (see also [21] and [35]). So we omit the details. □

Proof of Theorem 3.1.4. We first consider the equation (3.6). Consider the function $\psi(x) = \varphi(\frac{|x|}{R})$, where $R > 0$, $\varphi \in C_c^\infty(-2, 2)$ satisfying

$$0 \leq \varphi \leq 1 \quad \text{and} \quad \varphi(t) = 1 \ \text{ if } \ |t| \leq 1.$$

Using Proposition 3.3.1, we get that for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geq \max\{\frac{p+\gamma}{p-1}, 2\}$,

$$\begin{aligned} \int_{B_R} \left(|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{p+\gamma} \right) dx &\leq C \int_{\mathbb{R}^N} |x|^{-\frac{(\gamma+1)\alpha}{p-1}} \left(|\nabla\psi|^2 + |\psi||\Delta\psi| \right)^{\frac{p+\gamma}{p-1}} dx \\ &\leq C R^{N - \frac{2(p+\gamma)+(\gamma+1)\alpha}{p-1}}, \end{aligned} \quad (3.37)$$

where $C = C(p, m, \gamma, \alpha, N, \varphi) > 0$ is independent of R.

Applying Lemma 2.2.1, fix such a γ so that (2.13) holds and let $R \to \infty$ in (3.37), we conclude that

$$\int_{\mathbb{R}^N} \left(|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{p+\gamma} \right) dx = 0$$

which implies that $u_+ \equiv 0$, hence $u \leq 0$ and then $\Delta u = 0$ in $\mathbb{R}^N$, by Liouville theorem we derive that $u = c$ with $c \leq 0$. This completes the proof for the entire space case.

For the equation (3.7), thanks to the Dirichlet boundary condition, using Proposition 3.3.1 with $\Omega = \mathbb{R}^N_+$ we have that (3.37) holds always true by replacing B_R and $\mathbb{R}^N$ by B^+_R and $\mathbb{R}^N_+$, thus we also obtain $u_+ = 0$ in $\mathbb{R}^N_+$. So we get a non positive harmonic function in $\mathbb{R}^N_+$, i.e.

$$\Delta u = 0 \text{ in } \mathbb{R}^N_+, \quad u = 0 \text{ on } \partial\mathbb{R}^N_+ \quad \text{and} \quad u \le 0 \text{ in } \mathbb{R}^N_+. \tag{3.38}$$

Now we claim

Lemma 3.3.2. *Let u satisfy* (3.38)*, then there exists $c \le 0$ such that $u(x) = cx_N$.*

Proof of Lemma 3.3.2. We use some estimation on the Green's function. Let $x = (x', x_N)$ and introduce the Green function of the half unit ball B_1^+ with the Dirichlet boundary condition. For any $x, y \in B_1^+$ and $x \ne y$,

$$G_{B_1^+}(x,y) = -\frac{1}{2\pi}\left(\log\frac{|x-y|}{|\overline{x}-y|} - \log\left|\frac{x}{|x|^2} - y\right| + \log\left|\frac{\overline{x}}{|x|^2} - y\right|\right) \quad \text{if} \quad N = 2$$

and if $N \ge 3$, $G_{B_1^+}(x,y) =$

$$C_N\left[|x-y|^{2-N} - |\overline{x}-y|^{2-N} + |x|^{2-N}\left(\left|\frac{x}{|x|^2} - y\right|^{2-N} - \left|\frac{\overline{x}}{|x|^2} - y\right|^{2-N}\right)\right],$$

where $C_N = \frac{1}{N(N-2)w_N}$ and $\overline{x} = (x', -x_N)$. We have then

Lemma 3.3.3. *Let $N \ge 2$. There exists $\varepsilon > 0$ depending on N such that for all $x \in B_1^+$ with $|x| \le \varepsilon$, there holds*

$$\frac{\partial}{\partial x_N}\frac{\partial}{\partial \nu_y} G_{B_1^+}(x,y) \le 0 \quad \textit{for all} \quad y \in S_1^+, \tag{3.39}$$

where $S_1^+ = \partial B_1 \cap \mathbb{R}^N_+$.

Proof. For $y \in S_1^+$, since $|x-y| = |x|\left|\frac{x}{|x|^2} - y\right|$ for any $x \in \mathbb{R}^N$, a direct computation shows that when $N \ge 3$

$$\frac{\partial G_{B_1^+}}{\partial \nu_y}(x,y) = \frac{|x|^2 - 1}{Nw_N}\left(\frac{1}{|x-y|^N} - \frac{1}{|\overline{x}-y|^N}\right) \quad \text{for} \quad x \in B_1^+.$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial x_N}\frac{\partial}{\partial \nu_y} G_{B_1^+}(x,y) =& \frac{1-|x|^2}{w_N}\left(\frac{x_N - y_N}{|x-y|^{N+2}} - \frac{x_N + y_N}{|\overline{x}-y|^{N+2}}\right) \\ &+ \frac{2x_N}{Nw_N}\left(\frac{1}{|x-y|^N} - \frac{1}{|\overline{x}-y|^N}\right).\end{aligned} \tag{3.40}$$

Rewriting $\dfrac{\partial}{\partial x_N}\dfrac{\partial}{\partial \nu_y}G_{B_1^+}(x,y) := I_1 + I_2 + I_3$ with

$$I_1 = \frac{x_N(1-|x|^2)}{w_N}\left(\frac{1}{|x-y|^{N+2}} - \frac{1}{|\overline{x}-y|^{N+2}}\right),$$
$$I_2 = \frac{2x_N}{Nw_N}\left(\frac{1}{|x-y|^{N}} - \frac{1}{|\overline{x}-y|^{N}}\right),$$
$$I_3 = \frac{y_N(|x|^2-1)}{w_N}\left(\frac{1}{|x-y|^{N+2}} + \frac{1}{|\overline{x}-y|^{N+2}}\right).$$

Direct computation yields that there exist two positive constants c_1 and c_2 such that for $x \in B_1^+$, $|x| < 1/2$ and $y \in S_1^+$, we have

$$|I_1 + I_2| \le c_1 x_N^2 y_N \quad \text{and} \quad I_3 \le -c_2 y_N.$$

We derive that for $|x| \le \varepsilon$ with ε small enough, $I_1 + I_2 + I_3 \le 0$, thus (3.39) holds for $N \ge 3$.

When $N = 2$, we have

$$\begin{aligned}\frac{\partial}{\partial x_2}\frac{\partial}{\partial \nu_y}G_{B_1^+}(x,y) =& \frac{1-|x|^2}{\pi}\left(\frac{x_2-y_2}{|x-y|^4} - \frac{x_2+y_2}{|\overline{x}-y|^4}\right)\\ &+ \frac{x_2}{\pi}\left(\frac{1}{|x-y|^2} - \frac{1}{|\overline{x}-y|^2}\right) \quad \text{for } x \in B_1^+\end{aligned}$$

Repeat again the above proof, we obtain the desired result. This completes the proof of Lemma 3.3.3. □

Back to the problem (3.38) and carry on the proof of Lemma 3.3.2. For $R > 0$ and $x \in B_R^+$ we have the representation formula

$$u(x) = -\int_{S_R^+} \frac{\partial G_{B_R^+}}{\partial \nu_y}(x,y)u(y)dS(y)$$

where $S_R^+ = \partial B_R \cap \mathbb{R}_+^N$ and $G_{B_R^+}$ is the Green function on B_R^+ with the Dirichlet boundary condition. As $G_{B_R^+}(x,y) = R^{2-N}G_{B_1^+}(\frac{x}{R}, \frac{y}{R})$ and applying Lemma 3.3.3, we get that $\partial_{x_N} u \le 0$ for $x \in B_R^+$ with $|x| \le \varepsilon R$. Let $R \to +\infty$ we obtain then

$$\frac{\partial u}{\partial x_N} \le 0 \quad \text{for all } x \in \mathbb{R}_+^N. \tag{3.41}$$

We extend oddly u in (3.38) to get a harmonic function in $\mathbb{R}^N$, still denoted by u. Hence (3.41) implies that $\partial_{x_N} u \le 0$ in $\mathbb{R}^N$. Since u is a harmonic function in $\mathbb{R}^N$, we get

$$\Delta \frac{\partial u}{\partial x_N} = 0 \quad \text{in} \quad \mathbb{R}^N \tag{3.42}$$

It means that $\partial_{x_N} u$ is a non-positive harmonic function in $\mathbb{R}^N$, so it is a non-positive constant c. This implies that $u = cx_N$ with $c \le 0$ and completes the proof of Lemma 3.3.2 and therefore the proof of Theorem 3.1.4. □

3.3.2 Finite Morse index solutions

In this subsection, we consider the finite Morse index solutions and move Theorem 3.1.5 and 3.1.6.

3.3.2.1 Slow decay estimate for stable solutions

We prove first the slow decay estimate of weak stable solutions in exterior domains.

Lemma 3.3.4. *Let $\alpha > -2$, $N \geq 2$, $1 < p < p(N, 0)$ and u be weak stable solution of*

$$-\Delta u = |x|^\alpha u_+^p \quad \text{in } \mathbb{R}^N \setminus B_r.$$

Then there holds

$$u_+(x) = O\left(|x|^{-\frac{2+\alpha}{p-1}}\right), \quad \text{for } |x| \to \infty. \tag{3.43}$$

Similarly, the estimate (3.43) holds for any weak stable solution of

$$-\Delta u = |x|^\alpha u_+^p \quad \text{in } \mathbb{R}_+^N \setminus B_r \quad \text{and} \quad u = 0 \quad \text{on } \partial\mathbb{R}_+^N \setminus B_r.$$

The key argument to prove (3.43) is a uniform estimate inspired by the interesting work of Phan & Souplet [49]. We have

Lemma 3.3.5. *Let $N \geq 2$, $1 < p < p(N, 0)$ and $0 < \beta < 1$. Assume that $c(x) \in C^{0,\beta}(\overline{B_1})$ verifies $\|c(x)\|_{C^{0,\beta}(\overline{B_1})} \leq C_1$ and $c(x) \geq C_2 > 0$ in $\overline{B_1}$. Then there exists a constant $C > 0$ depending only on β, C_1, C_2, p and N, such that any stable solution u to $-\Delta u = c(x)u_+^p$ in B_1 satisfies*

$$u_+^{\frac{p-1}{2}}(x) \leq \frac{C}{1 - |x|}, \quad \forall\, x \in B_1. \tag{3.44}$$

Similarly, assume that $c(x) \in C^{0,\beta}(\overline{B_1^+})$ verifies $\|c(x)\|_{C^{0,\beta}(\overline{B_1^+})} \leq C_1$ and $c(x) \geq C_2 > 0$ in $\overline{B_1^+}$. Then there exists a constant $C > 0$ depending only on β, C_1, C_2, p and N, such that any stable solution u to $-\Delta u = c(x)u_+^p$ in B_1^+ satisfying $u = 0$ on ∂B_1^+ also verifies the estimate (3.44) for all $x \in B_1^+$.

Proof. For the unit ball case, we argue by contradiction and repeat similarly the proof of Lemma 2.1 in [49] (except that a upper-bound estimate for $u_+^{\frac{p-1}{2}}(x)$ is sufficient here) then we arrive at a nontrivial stable solution verifying $-\Delta v = C_0 v_+^p$ in $\mathbb{R}^N$ with the constant $C_0 > 0$ and $v_+(0) = 1$. This is impossible by the classification result Theorem 2.1 in [54] as $1 < p < p(N, 0)$, so we are done.

For the half unit ball case, the Doubling Lemma and the classical blow-up argument also work and two cases may occur after exploding, due to the Dirichlet boundary condition. We give briefly the discussion. Proceed by contradiction and consider the sequences c_k, u_k verifying and points $y_k \in B_1^+$ such that the defined functions $M_k := u_{k,+}^{\frac{p-1}{2}}$ satisfy

$$M_k(y_k) > \frac{2k}{1-|y_k|}.$$

Applying Doubling Lemma 2.2.3 with $D = B_1^+$ and $\Gamma = \partial B_1^+$, we get that for $k \in \mathbb{N}^*$ there exists $x_k \in B_1^+$ such that

$$M_k(x_k) \geq M_k(y_k), \qquad M_k(x_k) > \frac{2k}{1-|x_k|}$$

and

$$M_k(z) \leq 2M_k(x_k) \quad \text{for all } z \in \{z \in B_1^+ \,:\, |z-x_k| \leq k\lambda_k\} \tag{3.45}$$

where $\lambda_k := M_k^{-1}(x_k) \to 0$ as $k \to \infty$ since $M_k(x_k) \geq M_k(y_k) > 2k$.

Consider now

$$v_k(y) = \lambda_k^{\frac{p-1}{2}} u_k(x_k + \lambda_k y) \quad \text{and} \quad \overline{c}_k(y) = c_k(x_k + \lambda_k y).$$

Note that $v_{k,+}(0) = 1$ and $v_{k,+}(y) \leq 2^{\frac{2}{p-1}}$ for $|y| \leq k$ due to (3.45). Moreover, we observe that v_k satisfies in B_k the following equation

$$-\Delta v_k = \overline{c}_k(y) v_{k,+}^p \quad \text{in } B_k \quad \text{and} \quad v_k = 0 \quad \text{on } B_k \cap \partial\mathbb{R}_+^N. \tag{3.46}$$

Up to a subsequence, $\dfrac{x_{k,N}}{\lambda_k} \to l$ as $k \to \infty$, with $l \in [0, +\infty]$.

If $l = \infty$, after extracting a subsequence, $\overline{c}_k \to C_0$ in $C_{loc}(\mathbb{R}^N)$ with $C_0 > 0$ a constant and we may also assume that $v_k \to v$ in $C^2_{loc}(\mathbb{R}^N)$, and v is a stable solution of

$$-\Delta v = C_0 v_+^p \quad \text{in } \mathbb{R}^N \qquad \text{and} \qquad v_+(0) = 1.$$

If $\dfrac{x_{k,N}}{\lambda_k} \to l < \infty$, we can prove that $l > 0$ (see Theorem 5 in [1]) thus we get a stable solution of

$$-\Delta v = C_0 v_+^p \quad \text{in } \mathbb{R}_+^N, \qquad v = 0 \quad \text{on } \partial\mathbb{R}_+^N \qquad \text{and} \qquad v_+(l) = 1.$$

As $1 < p < p(N, 0)$, both the above two cases contradict Theorem 2.1 in [54]. Therefore we obtain that

$$u_+^{\frac{p-1}{2}}(x) \leq \frac{C}{1-|x|} \quad \text{for } x \in B_1^+,$$

which completes the proof of Lemma 3.3.5. □

Proof of Lemma 3.3.4. With Lemma 3.3.5 in our hand, we will use now the scaling argument. More precisely, for the entire space case, let $\rho > 0$, $\Omega = \{x \in$

$\mathbb{R}^N : |x| > \rho\}$, $|x_0| > 2\rho$ and $R = \frac{1}{2}|x_0|$. Define $U(y) = R^{(2+\alpha)/(p-1)}u(x_0 + Ry)$ with $y \in B_1$, we observe that $x_0 + Ry \in \Omega$ for all $y \in B_1$. Therefore $U(y)$ is a solution of

$$-\Delta U(y) = c(y)U_+^p(y) \quad \text{in } B_1, \quad \text{with} \quad c(y) = \left|\frac{x_0}{R} + y\right|^\alpha. \tag{3.47}$$

Since $\left|\frac{x_0}{R} + y\right| \in [1, 3]$ in $\overline{B_1}$ and $\|c\|_{C^{1,\alpha}(\overline{B_1})} \leq C_\alpha$ which is a constant independent of x_0, applying Lemma 2.4.5, we obtain $U_+(0) \leq C$, which is equivalent to say

$$u_+(x_0) \leq CR^{-\frac{2+\alpha}{p-1}} \quad \text{for } |x_0| \text{ large enough},$$

which means that $u_+(x) = O\left(|x|^{-\frac{2+\alpha}{p-1}}\right)$ when $|x| \to \infty$.

For the half space case, we need to distinguish two situation. For $x = (x', x_N) \in \mathbb{R}^N_+$, if $|x'|$ is large compared with x_N such that $x_0 + Ry \in \Omega$ for all $y \in B_1$, then we can use Lemma 2.4.5 with B_1 and proceed as for entire space case. If x_N is large compared with $|x'|$, we have $x_0 + Ry \in \Omega$ for all $y \in B_1^+$, so the estimate corresponding to B_1^+ in Lemma 2.4.5 works and the proof is similar except that now we use $\mathbb{R}^N_+$ and B_1^+ instead of $\mathbb{R}^N$ and B_1. So we are done. □

3.3.2.2 Higher dimensional case : $N \geq 3$

To prove Theorem 3.1.5, we need the following representation formula.

Lemma 3.3.6. *Let $1 < p < p(N, 0)$ and $N \geq 3$, let u be a solution to (3.6) which is stable outside a compact set. Then*

$$v(x) = \frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{|y|^\alpha u_+^p(y)}{|x-y|^{N-2}} dy. \tag{3.48}$$

is well defined, $v \geq 0$ and $v(x) \to 0$ as $|x| \to \infty$.

Proof. Define $f(x) = |x|^\alpha u_+^p$ and $g(x) = |x|^{2-N}$. Clearly, $g \in L^1(B_1)$ and $g \in L^q(B_1^c)$ for any $q > \frac{N}{N-2}$. Moreover, $f \in L^1(B_1)$ by using the definition of weak solution, and $f \in L^q(B_1^c)$ by using (3.43) for all $q > \frac{N}{2}$, since

$$\alpha - \frac{2+\alpha}{p-1}p = -2 - \frac{2+\alpha}{p-1} < -2.$$

Therefore $N(N-2)\omega_N v = f * g$ is well defined by Young's inequality.

Obviously $v \geq 0$. Now we consider the behavior of $v(x)$ as $|x| \to \infty$. Let $A_1 = B(x, \frac{|x|}{2})$, $A_2 = B^c(x, 3|x|)$ and $A_3 = \mathbb{R}^N \setminus (A_1 \cup A_2)$. Rewrite v as follows

$$N(N-2)\omega_N v(x) = \sum_{i=1}^{3} \int_{A_i} \frac{|y|^\alpha u_+^p(y)}{|x-y|^{N-2}} dy := I_1 + I_2 + I_3.$$

For $|x|$ large, using Lemma 3.3.4 we see that

$$I_1 \le C|x|^{\alpha - \frac{\alpha+2}{p-1}p} \int_{A_1} \frac{dy}{|x-y|^{N-2}} = C'|x|^{-\frac{\alpha+2}{p-1}} \to 0 \quad \text{as} \quad |x| \to \infty.$$

For $y \in A_2$, one has $|y-x| \ge 3|x|$, hence $|y| \ge 2|x|$ and $|y-x| \ge \frac{|y|}{2}$. This together with Lemma 3.3.4 imply that for large $|x|$,

$$I_2 \le C \int_{A_2} |y|^{\alpha - \frac{\alpha+2}{p-1}p} |y|^{2-N} dy = C'|x|^{-\frac{\alpha+2}{p-1}} \to 0 \quad \text{as} \quad |x| \to \infty.$$

For $y \in A_3$, we have

$$I_3 \le C|x|^{2-N} \int_{A_3 \bigcap B^c(0,R_0)} |y|^\alpha u_+^p(y) dy + \int_{B(0,R_0)} \frac{|y|^\alpha u_+^p(y)}{|x-y|^{N-2}} dy := I_{31} + I_{32}$$

Fix $R_0 > 0$ larger enough such that we can use the decay estimate given by Lemma 3.3.4 for $|y| \ge R_0$, then

$$I_{31} \le C|x|^{2-N} \left(|x|^{\alpha+N-\frac{\alpha+2}{p-1}p} - R_0^{\alpha+N-\frac{\alpha+2}{p-1}p} \right) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Since $|y|^\alpha u_+^p \in L^1(B(0,R_0))$, the dominated convergence theorem yields that for $\lim_{|x|\to\infty} I_{32} = 0$. This completes the proof of Lemma 3.3.6. $\square$

Similarly, we have the half space version.

Lemma 3.3.7. *Let $\alpha > -2$, $N \ge 3$, $1 < p < p(N,0)$ and u be a solution to* (3.7) *which is stable outside a compact set and*

$$v(x) = \int_{\mathbb{R}_+^N} |y|^\alpha u_+^p(y) G(x,y) dy, \tag{3.49}$$

where $G(x,y)$ is the Green function given by (3.12). *Then v is well defined in $\mathbb{R}_+^N$, $-\Delta v = |x|^\alpha u_+^p$, $v \ge 0$ in $\mathbb{R}_+^N$, $v = 0$ on $\partial\mathbb{R}_+^N$ and $v(x) \to 0$ as $|x| \to \infty$.*

Proof . Indeed, for any $x, y \in \mathbb{R}_+^N$ and $x \ne y$, we have

$$G(x,y) = \frac{1}{N(N-2)w_N} \left(\frac{1}{|x-y|^{N-2}} - \frac{1}{|\overline{x}-y|^{N-2}} \right)$$

with $\overline{x} = (x_1, ..., -x_N)$. Thanks to the fact that $|\overline{x}-y| > |x-y|$ for any $x, y \in \mathbb{R}_+^N$ and $x \ne y$, the proof follows the same lines of proof for Lemma 3.3.6. $\square$

We are now ready to prove the desired Theorem 3.1.5.

Proof of Theorem 3.1.5. For the equation (3.6), we clearly have $\Delta(u-v) = 0$ in $\mathbb{R}^N$ with v defined by (3.48). Moreover, as $p(N,\alpha^-) \le p(N,0)$, $u - v \le u \le u_+$ and we have $\lim_{|x|\to\infty} u_+ = 0$ by Lemma 3.3.4. So $(u-v)$ is upper bounded in $\mathbb{R}^N$. Thus by Liouville theorem we derive that $u = v + c$ with $c \in \mathbb{R}$.

If $c \geq 0$, then $u = v + c \geq 0$. By Theorem 2.1.11 in Chapiter 2, we get $u \equiv 0$. If $c < 0$, as $\lim_{|x|\to\infty} v(x) = 0$, we derive that the $\mathrm{supp}(u_+)$ is compact.

For the equation (3.7), we also have $\Delta(u-v) = 0$ in $\mathbb{R}^N_+$ where $v \geq 0$ is defined by (3.49). As $p(N, \alpha^-) \leq p(N, 0)$, by Lemma 3.3.4 we have $\lim_{|x|\to\infty} u_+ = 0$, so $u - v \leq C$ with $C > 0$ and $\lim_{|x|\to\infty}(u - v)_+ = 0$ in $\mathbb{R}^N_+$.

Let $w = u - v$, then w verifies

$$\begin{cases} \Delta w = 0, w \leq C & \text{in} \quad \mathbb{R}^N_+ \\ w = 0 & \text{on} \quad \partial\mathbb{R}^N_+ \\ \lim_{|x|\to\infty} w_+(x) = 0 & \text{for} \quad x \in \mathbb{R}^N_+. \end{cases} \tag{3.50}$$

We have the following result.

Lemma 3.3.8. *Let w be a solution of the problem* (3.50)*, then there exists $c \leq 0$ such that $w = cx_N$.*

Proof. Thanks to Lemma 3.3.2, we only need to prove that $w \leq 0$ in $\mathbb{R}^N_+$. For that we proceed by contradiction and suppose the contrary, then $M = \sup_{\mathbb{R}^N_+} w > 0$. As $w = 0$ on $\partial\mathbb{R}^N_+$ and $\lim_{|x|\to\infty} w_+(x) = 0$, so M must be reached by $x_0 \in \mathbb{R}^N_+$, as w is continuous. However, w is a harmonic function, the strong maximum principle implies that w is constant in $\mathbb{R}^N_+$, which is impossible. So we are done. □

We carry on the proof of Theorem 3.1.5. By Lemma 3.3.8, we have then $u = v + cx_N$ with $c \leq 0$ in $\mathbb{R}^N_+$. If $c = 0$, $u = v \geq 0$, by Theorem 2.1.12 in Chapiter 2, we get $u \equiv 0$. To complete the proof of Theorem 3.1.5, it remains only to prove that the support of u_+ is compact when $c < 0$. This is the content of the following lemma.

Lemma 3.3.9. *Under the assumptions of Theorem 3.1.5, let u be a weak solution with finite Morse index of the equation* (3.7)*, then supp*(u_+) *is compact.*

Proof. We borrow the idea from the proof of Lemma 2.8 in [40], where the authors dealt with the autonomous case for $1 < p < \frac{N+2}{N-2}$.

Firstly, we claim that $u_+(x) = 0$ for $x_N > 0$ large enough. Suppose $u_+ \not\equiv 0$, as $u(x) = v(x) + cx_N$ with $c < 0$ and $v(x) \to 0$ as $|x| \to \infty$, we have immediately

$$\lim_{x_N \to +\infty} u(x) = -\infty.$$

Therefore there exists $A > 0$ such that $u(x) < 0$ for $x_N > A$.

We claim then $u_+(x) = 0$ for $|x'|$ large enough. Indeed, we have

$$\frac{\partial v}{\partial x_N}(x) = \int_{\mathbb{R}^N_+} |y|^\alpha u_+^p(y) \frac{\partial G}{\partial x_N}(x, y) dy$$

$$\text{with} \quad \frac{\partial G}{\partial x_N}(x, y) = -\frac{1}{N w_N} \left(\frac{x_N - y_N}{|x - y|^N} - \frac{x_N + y_N}{|\overline{x} - y|^N} \right).$$

Since $|x_N - y_N| \leq |x-y|$, $|x_N + y_N| \leq |\overline{x} - y|$ and $|\overline{x} - y| > |x-y|$ for all $x \neq y \in \mathbb{R}^N_+$, we deduce that

$$-C\delta(x) \leq \frac{\partial v}{\partial x_N}(x) \leq C\delta(x) \quad \textit{with} \quad \delta(x) = \int_{\mathbb{R}^N_+} \frac{|y|^\alpha u_+^p(y)}{|x-y|^{N-1}} dy, \tag{3.51}$$

where $C = 2(Nw_N)^{-1} > 0$ is a universal constant.

Following the proof of Lemma 3.3.6, we can prove that $\delta(x)$ is well defined, $\delta(x) \geq 0$ in $\mathbb{R}^N_+$, and $\delta(x) \to 0$ as $|x| \to \infty$. Therefore

$$\frac{\partial v}{\partial x_N}(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Moreover, it follows from $u = v + cx_N$ that $\dfrac{\partial u}{\partial x_N}(x) = \dfrac{\partial v}{\partial x_N} + c$ with $c < 0$. Therefore we have

$$\frac{\partial u}{\partial x_N}(x) < 0 \quad \text{for } |x'| \text{ large enough.}$$

Since $u = 0$ when $x_N = 0$, we derive that $u < 0$ for $|x'|$ large enough. Hence $u_+(x) = 0$ in exterior domain of the half space. □

3.3.2.3 Two dimensional case : $N = 2$

Before proving Theorem 3.1.6, we show first some basic properties.

Lemma 3.3.10. *Let $N = 2$, $p > 1$ and u be a weak solution of* (3.6) *with finite Morse index. Then $u_+ \in L^\infty_{loc}(\mathbb{R}^2)$ and $|y|^\alpha u_+^p \in L^1(\mathbb{R}^2)$.*

Proof. We prove first that $u_+ \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Since the Morse index is finite, by argument of capacity, we have that for all $x_0 \in \mathbb{R}^2$ there exists $r = r(x_0) > 0$ such that u is stable in $B_r(x_0)$.

For $x_0 \neq 0$, using (3.37) in $B_R(x_0)$ with $R = \min\left(\frac{r}{2}, \frac{|x_0|}{2}\right)$ we have

$$\int_{B_R(x_0)} |x|^\alpha u_+^{p+\gamma} < \infty \quad \text{for all } \gamma \in \left[1, 2p + 2\sqrt{p(p-1)} - 1\right). \tag{3.52}$$

We get then $u_+^{p+\gamma} \in L^1(B_R(x_0))$ which implies also $|x|^\alpha u_+^p \in L^{1+\frac{\gamma}{p}}(B_R(x_0))$. By the equation (3.6), we have $u \in W^{2,1+\frac{\gamma}{p}}(B_R(x_0))$. As $N = 2$, we see that u is a Hölder continuous function in $B_R(x_0)$.

For $x_0 = 0$, using (3.37) we still have (3.52) with $R = \frac{r}{2}$. By Hölder inequality and since $\alpha > -2$, we can choose $\varepsilon > 0$ small enough such that $|x|^\alpha u_+^p \in L^{1+\varepsilon}(B_R)$. As $N = 2$, we get again that u is a Hölder continuous function in B_R.

Now we prove that $|y|^\alpha u_+^p \in L^1(\mathbb{R}^2)$. Indeed, as $u_+^p \in L^\infty_{loc}(\mathbb{R}^2)$ and $\alpha > -2$ we have $|y|^\alpha u_+^p \in L^1_{\text{loc}}(\mathbb{R}^2)$. On the other hand, by Lemma 3.3.4 we see that $|y|^\alpha u_+^p = O(|y|^q)$ with $q = \alpha - \frac{2+\alpha}{p-1}p < -2$. So we are done. □

For the problem (3.6), we have the following.

Lemma 3.3.11. *Let $N = 2$, $\alpha > -2$, $p > 1$ and u be a solution to* (3.6) *with finite Morse index. Then*

$$v(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} |y|^\alpha u_+^p(y)\log|x-y|dy \quad \text{in } \mathbb{R}^2$$

is well defined and satisfies $-\Delta v(x) = |x|^\alpha u_+^p(x)$ in $\mathbb{R}^2$. Moreover, we have

$$\frac{v(x)}{\log|x|} \to -\frac{1}{2\pi}\int_{\mathbb{R}^2} |y|^\alpha u_+^p(y)dy \quad \text{as} \quad |x| \to \infty. \tag{3.53}$$

Proof. We prove that $v(x)$ is well defined. Indeed, fix $R > 0$ larger enough and write $-2\pi v = I_1 + I_2$, where

$$I_1 = \int_{B_R} |y|^\alpha u_+^p(y)\log|x-y|dy \quad \text{and} \quad I_2 = \int_{B_R^c} |y|^\alpha u_+^p(y)\log|x-y|dy.$$

Since $u_+ \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ and the function $|y|^\alpha \log|x-y| \in L^1_{\text{loc}}(\mathbb{R}^2)$ as $\alpha > -2$, so I_1 exists. Then, as $|y|^\alpha u_+^p = O(|y|^q)$ if $|y| \to \infty$ with $q = \alpha - \frac{2+\alpha}{p-1}p < -2$, I_2 is also finite, so v is well defined. Moreover, it is obvious that $-\Delta v(x) = |x|^\alpha u_+^p(x)$ in $\mathbb{R}^2$.

Notice first that the r.h.s. in (3.53) is well defined since $|x|^\alpha u_+^p(x) \in L^1(\mathbb{R}^2)$. In order to prove (3.53), we use a Chen-Li's technic in [16] and we only need to verify that

$$I := \int_{\mathbb{R}^2} \frac{\log|x-y| - \log|x|}{\log|x|}|y|^\alpha u_+^p(y)dy \to 0 \quad \text{as} \quad |x| \to \infty.$$

Write now $I = I_1 + I_2$, where I_1, I_2 are the integrals on the two regions : $\Omega_1 = B_1(x)$ and $\Omega_2 = B_1^c(x)$. Without loss of generality, we may assume that $|x| \geq 3$. For I_1, since

$$I_1 \leq C\int_{\Omega_1} |y|^\alpha u_+^p(y)dy + \frac{1}{\log|x|}\int_{\Omega_1} \log|x-y||y|^\alpha u_+^p(y)dy,$$

by Lemma 3.3.4, as $|x| \to \infty$, we have $I_1 \to 0$.

For I_2, we use the fact that for $|x-y| > 1$,

$$\left|\frac{\log|x-y| - \log|x|}{\log|x|}\right| \leq C,$$

therefore $I_2 \to 0$ as $|x| \to \infty$ by using again Lemma 3.3.4. So we are done. □

For the problem (3.7), we have a similar result.

Lemma 3.3.12. *Let $N = 2$, $\alpha > -2$, $p > 1$ and u be a weak solution of* (3.7), *then*

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2_+} |y|^\alpha u_+^p(y)\log\left(\frac{|\overline{x}-y|}{|x-y|}\right)dy \quad \text{in } \mathbb{R}^2_+$$

is well defined, $v(x) \geq 0$ but now $v(x) \to 0$ as $|x| \to \infty$.

Proof. As for Lemma 3.3.11, v is well defined and $|y|^\alpha u_+^p \in L^1(\mathbb{R}^2_+)$. Clearly $v \geq 0$ since $|\overline{x} - y| > |x - y|$ for any $x \neq y \in \mathbb{R}^2_+$. Now we consider the behavior of v as $|x| \to \infty$. Let $A_1 = B_R^+$ (with R a constant to be determined), $A_2 = B_{\frac{|x|}{2}}(x) \bigcap \mathbb{R}^2_+$ and $A_3 = \mathbb{R}^2_+ \setminus (A_1 \bigcup A_2)$, and rewrite v as follows

$$2\pi v(x) = \sum_{i=1}^{3} \int_{A_i} |y|^\alpha u_+^p(y) \log\left(\frac{|\overline{x} - y|}{|x - y|}\right) dy := I_1 + I_2 + I_3.$$

For $y \in A_3$, we can verify that $1 \leq \frac{|\overline{x}-y|}{|x-y|} \leq 10$ which reads $0 \leq \log \frac{|\overline{x}-y|}{|x-y|} \leq \log 10$. Together with the fact $|y|^\alpha u_+^p \in L^1\left(\mathbb{R}^2_+\right)$, we get for any $\varepsilon > 0$,

$$I_3 \leq C \int_{\mathbb{R}^2_+ \setminus B_R^+} |y|^\alpha u_+^p dy < \varepsilon,$$

provided that R is larger enough. Once R is fixed, we see that for $y \in A_1$,

$$0 \leq \log \frac{|\overline{x} - y|}{|x - y|} \leq \log\left(1 + \frac{2R}{|x| - R}\right).$$

Since $|y|^\alpha u_+^p \in L^1\left(\mathbb{R}^2_+\right)$, the dominated convergence theorem implies that $I_1 \to 0$ as $|x| \to \infty$.

For $y \in A_2$, obviously $|y| > \frac{|x|}{2}$ and $\log \frac{|\overline{x}-y|}{|x-y|} \leq \log(\frac{3}{2}|x|) + \Big|\log|x - y|\Big|$. Therefore, direct computation yields

$$I_2 \leq c|x|^{\alpha - \frac{2+\alpha}{p-1}p} \left(|x|^2 \log|x| + |x|^2\right).$$

As $\alpha - \frac{2+\alpha}{p-1}p < -2$ we derive again $I_2 \to 0$ as $|x| \to \infty$. The proof is completed. □

Now we are ready to prove the classification result in dimension $N = 2$.

Proof of Theorem 3.1.6. For the equation (3.6), set $w = u - v$ with v in Lemma 3.3.11, then $\Delta w \equiv 0$ in $\mathbb{R}^2$ and we have $w = u - v \leq u_+ - v$. Lemma 3.3.4 implies that $u_+ \to 0$ as $|x| \to \infty$ and (3.53) implies that $v \leq C\log(|x| + 1)$, so w is bounded above in $\mathbb{R}^2$. Therefore, w is a constant by Liouville's theorem, so $u = v + c$ in $\mathbb{R}^2$. Moreover, using again (3.53) we have either $u_+ \equiv 0$ or $v \to -\infty$ as $|x| \to \infty$, thus $\text{supp}(u_+)$ is compact in each case.

For the half space case (3.7), the proof follows the main lines as for $N \geq 3$. Let $w = u - v$ with v in Lemma 3.3.12, we arrive at the problem (3.50) with $N = 2$. By Lemma 3.3.8, we get that $w = cx_N$ with $c \leq 0$ thus $u = v + cx_N$. If $c = 0$, applying again Theorem 2.1.12 in Chapter 2 we get $u \equiv 0$. It rests to show that $\text{supp}(u_+)$ is compact when $c < 0$. The arguments used in the proof of Lemma 3.3.9 still work for $N = 2$ so we are done. □

3.3.3 Fast decay estimate for solutions stable at infinity

We prove here Proposition 3.1.8. Consider firstly the equation (3.13). We need the two following lemmas.

Lemma 3.3.13. *Let $\alpha > -2$, $p > 1$ and u be a weak stable solution of* (3.13). *Then there exists $R_0 > 0$ such that :*

– *for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and $r > R_0 + 3$,*

$$\int_{\{R_0+2<|x|<r\}} \left[|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{\gamma+p} \right] dx \leq A + Br^{N - \frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \quad (3.54)$$

where A and B are non negative constants independent of r.

– *for any $\gamma \in [1, 2p+2\sqrt{p(p-1)}-1)$ and any ball $B_r(y)$ such that $B_{2r}(y) \subset \mathbb{R}^N \setminus \overline{B_{R_0}}$,*

$$\int_{B_r(y)} \left[|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{\gamma+p} \right] dx \leq Cr^{N - \frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \quad (3.55)$$

where C is a non-negative constant independent of both r and y.

Lemma 3.3.14. *Let $\alpha > -2$, $p > 1$ and u be a weak stable solution of* (3.14). *Then there exists $R_0 > 0$ such that :*

– *for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and $r > R_0 + 3$,*

$$\int_{\{R_0+2<|x|<r\}\cap\mathbb{R}^N_+} \left[|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{\gamma+p} \right] dx \leq A + Br^{N - \frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \quad (3.56)$$

hbox where A and B are non negative constants independent of r.

– *for any $\gamma \in [1, 2p+2\sqrt{p(p-1)}-1)$ and any ball $B_r(y)$ such that $B^+_{2r}(y) \subset \mathbb{R}^N_+ \setminus \overline{B_{R_0}}$,*

$$\int_{B^+_r(y)} \left[|\nabla(u_+^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u_+^{\gamma+p} \right] dx \leq Cr^{N - \frac{(2+\alpha)\gamma+2p+\alpha}{p-1}}, \quad (3.57)$$

where C is a non-negative constant independent of both r and y.

Proof . By using Proposition 3.3.1, the demonstrations of Lemma 3.3.13 and Lemma 3.3.14 follow the same line as in [35]. Indeed, for Lemma 3.3.13, we consider two functions $\eta_0(x) = \eta(|x| - R_0)$ and $\phi_R(x) = \phi(R^{-1}|x|)$, where $\eta \in C^\infty(\mathbb{R})$ is

$$0 \leq \eta \leq 1 \text{ in } \mathbb{R}, \quad \eta(t) = 0 \text{ if } t \leq 1 \quad \text{and} \quad \eta(t) = 1 \text{ if } t \geq 2;$$

and $\phi \in C_0^\infty(\mathbb{R})$ satisfies

$$0 \leq \phi \leq 1 \text{ in } \mathbb{R}, \quad \phi(t) = 1 \text{ if } |t| \leq 1 \quad \text{and} \quad \phi(t) = 0 \text{ if } |t| \geq 2.$$

For $R > R_0 + 3$, define ψ_R as follows :

$$\psi_R(x) = \begin{cases} \eta_0(x) & \text{if } |x| \le R_0 + 3 \\ \phi_R(|x|) & \text{if } |x| \ge R_0 + 3. \end{cases}$$

Notice that $\psi_R \in C_0^\infty(\mathbb{R}^N \backslash \overline{B(0,R_0)})$ and $0 \le \psi_R \le 1$ in $\mathbb{R}^N$. Therefore Proposition 3.3.1 together with $m = 5$ and the test function ψ_R imply the claim (3.54). Using Proposition 3.3.1 again with the test function $\phi_R(x-y)$, we obtain the estimate (3.55). The proof of Lemma 3.3.14 is similar so we omit the details here. $\square$

Proof of Proposition 3.1.8. We are ready to give the proof of Proposition 3.1.8 for equation (3.13). By Kato's inequality (see [44]) we have

$$-\Delta u_+ \le |x|^\alpha u_+^p \quad \text{in } \mathbb{R}^N. \tag{3.58}$$

Since $\frac{N+2+2\alpha}{N-2} \le p < p(N,\alpha^-)$, one can find $\gamma_1 \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ (see Lemma 2.2.1 and the proof of Theorem 2.1.14) such that

$$N(p-1) - (2+\alpha)\gamma_1 - 2p - \alpha = 0. \tag{3.59}$$

Set now

$$\beta = \frac{\gamma_1 + p}{2} > 1 \quad \text{and} \quad \omega(x) = (u_+(x))^\beta \ge 0, \tag{3.60}$$

using (3.54) with $\gamma = \gamma_1$ and letting $r \to +\infty$, we have

$$\int_{\mathbb{R}^N \backslash \overline{B_{R_0+2}}} |x|^\alpha w^2 < \infty. \tag{3.61}$$

Next (3.58) implies that

$$\Delta w + \beta |x|^\alpha (u_+)^{p-1} w \ge 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.62}$$

Moreover, like in the proof of the estimate (2.40), by (3.55) and since $p < p(N,\alpha^-)$, we may find $0 < \varepsilon_0 < 2$ such that for any $B_{2R}(y) \subset \mathbb{R}^N \setminus \overline{B_{R_0}}$, there holds

$$\int_{B_R(y)} \left(|x|^\alpha u_+^{p-1}\right)^{\frac{N}{2-\varepsilon_0}} \le C R^{N - \frac{2N}{2-\varepsilon_0}}. \tag{3.63}$$

This implies

$$R^{\varepsilon_0} \|\beta |x|^\alpha (u_+)^{p-1}\|_{L^{\frac{N}{2-\varepsilon_0}}(B_{2R}(y))} \le c\beta R^{\varepsilon_0} (R^{N - \frac{2N}{2-\varepsilon_0}})^{\frac{2-\varepsilon_0}{N}} = C \tag{3.64}$$

with C a constant independent of y and R. Then the Pucci-Serrin's argument (Lemma 2.3.6) with $s = 2$ gives

$$w(y) \le cR^{-\frac{N}{2}} \|w\|_{L^2(B_{2R}(y))}.$$

By (3.61) this reads,

$$w(y) \leq cR^{-\frac{N+\alpha}{2}} \|x^{\frac{\alpha}{2}} w\|_{L^2(B_{2R}(y))} = o(R^{-\frac{N+\alpha}{2}}) = o(|y|^{-\frac{N+\alpha}{2}}), \text{ as } |y| \to \infty$$

which is equivalent to

$$u_+(y) = w^{\frac{1}{\beta}}(y) = o(|y|^{-\frac{N+\alpha}{2\beta}}) = o(|y|^{-\frac{2+\alpha}{p-1}}), \text{ as } |y| \to \infty.$$

We now prove the fast decay estimate for problem (3.14). By Kato's inequality we also have $-\Delta u_+ \leq |x|^\alpha u_+^p$ in $\mathbb{R}^N_+$. There always exists $\gamma_1 \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ such that (3.59) holds true. Set β, ω as in (3.60), using (3.56) with $\gamma = \gamma_1$ and letting $r \to +\infty$, we have (3.61) with $\mathbb{R}^N \backslash \overline{B_{R_0+2}}$ replaced by $\mathbb{R}^N_+ \backslash \overline{B_{R_0+2}}$. Clearly, we always have $\Delta w + \beta |x|^\alpha (u_+)^{p-1} w \geq 0$ in $\mathcal{D}'(\mathbb{R}^N_+)$.

Moreover, by (3.57) and since $p < p(N, \alpha^-)$, there exists $0 < \varepsilon_0 < 2$ such that for any $B^+_{2r}(y) \subset \mathbb{R}^N_+ \setminus \overline{B_{R_0}}$, we have

$$\int_{B^+_r(y)} \left(|x|^\alpha u_+^{p-1}\right)^{\frac{N}{2-\varepsilon_0}} \leq Cr^{N - \frac{2N}{2-\varepsilon_0}}, \tag{3.65}$$

We will distinguish two cases.

Case 1. $y = (y', y_N) \in R^N_+$, $|y| > 4R_0$ and $y_N \geq \frac{|y'|}{4}$, where $y' = (y_1, ..., y_{N-1})$.

Let $R = \frac{|y|}{10}$, thus we easily verify that $B_{2R}(y) \subset \mathbb{R}^N_+ \setminus \overline{B_{R_0}}$. Applying (3.63) we obtain

$$\int_{B_{2R}(y)} \left(|x|^\alpha u_+^{p-1}\right)^{\frac{N}{2-\varepsilon_0}} \leq CR^{N - \frac{2N}{2-\varepsilon_0}},$$

which implies

$$R^{\varepsilon_0} \|\beta |x|^\alpha (u_+)^{p-1}\|_{L^{\frac{N}{2-\varepsilon_0}}(B_{2R}(y))} \leq c\beta R^{\varepsilon_0} (R^{N - \frac{2N}{2-\varepsilon_0}})^{\frac{2-\varepsilon_0}{N}} = c$$

with c a constant. Using now Lemma 2.3.6 with $s = 2$ and following exactly the same arguments for the equation (3.13), we can get in this case

$$u_+(y) = w^{\frac{1}{\beta}}(y) = o(|y|^{-\frac{N+\alpha}{2\beta}}) = o(|y|^{-\frac{2+\alpha}{p-1}}), \text{ as } |y| \to \infty.$$

Case 2. $y = (y', y_N) \in R^N_+$, $|y| > 4R_0$ and $y_N < \frac{|y'|}{4}$.

In this case we take $R = \frac{|y|}{4}$ and we verify that

$$B^+_{2R}(y) \subset \mathbb{R}^N_+ \backslash \overline{B_{R_0}} \quad \text{and} \quad y \in B^+_R(y).$$

Applying again (3.57) with $B^+_{2R}(y)$, we have

$$\int_{B^+_{2R}(y)} \left(|x|^\alpha u_+^{p-1}\right)^{\frac{N}{2-\varepsilon_0}} \leq CR^{N - \frac{2N}{2-\varepsilon_0}},$$

which implies :

$$R^{\varepsilon_0}\|\beta|x|^\alpha(u_+)^{p-1}\|_{L^{\frac{N}{2-\varepsilon_0}}(B^+_{2R}(y))} \leq c\beta R^{\varepsilon_0}(R^{N-\frac{2N}{2-\varepsilon_0}})^{\frac{2-\varepsilon_0}{N}} = c$$

with c a constant. Since w satisfies $-\Delta w \leq \beta|x|^\alpha(u_+)^{p-1}w$, applying Lemma 3.2.1 to the function w with $a(x) = \beta|x|^\alpha(u_+)^{p-1}$ and $s = 2$ we derive

$$\sup_{B^+_R(y)} w \leq cR^{-\frac{N}{2}}\|w\|_{L^2(B^+_{2R}(y))} \leq cR^{-\frac{N+\alpha}{2}}\|x^{\frac{\alpha}{2}}w\|_{L^2(B^+_{2R}(y))}.$$

As $y \in B^+_R(y)$ we conclude that $w(y) = o(|y|^{-\frac{N+\alpha}{2}})$ which in turn reads

$$u_+(y) = w^{\frac{1}{\beta}}(y) = o(|y|^{-\frac{N+\alpha}{2\beta}}) = o(|y|^{-\frac{2+\alpha}{p-1}}), \text{ as } |y| \to \infty.$$

The proof of Proposition 3.1.8 is therefore completed. □

3.4 Classification results for general non-linearity

We investigate now the general equation $-\Delta u = |x|^\alpha f(u)$ where f satisfies (3.2), and give the demonstrations of Theorem 3.1.9.

3.4.1 Classification of stable solutions

This first subsection is devoted to prove Theorem 3.1.9. We borrow some ideas presented by Crandall-Rabinowitz in [17] (see also [65], [30]) and state first an important integral estimate.

Proposition 3.4.1. *Let $\alpha > -2$ and $\Omega \subset \mathbb{R}^N$ (bounded or not) with $N \geq 2$. Let f satisfy* (3.2) *and $u \in L^\infty_{loc}(\Omega)$ be a weak stable solution of*

$$-\Delta u \leq |x|^\alpha f(u) \quad in \quad \Omega \tag{3.66}$$

Moreover, let $\phi(u) \in W^{1,\infty}_{loc}(I,\mathbb{R})$ be a convex function, $\psi(u)$ and $\xi(u)$ belong to $W^{1,\infty}_{loc}(I,\mathbb{R})$ and denote two non-negative functions such that

$$\psi' = \phi'^2, \quad \xi' = \psi.$$

Then we have

$$\begin{aligned}&\int_\Omega |x|^\alpha \left[(f'\phi^2 - f\psi)\circ u\right]\eta^2 dx\\ &\leq \int_\Omega [\xi\circ u]\Delta(\eta^2)dx - \int_\Omega [\phi^2\circ u]\eta\Delta\eta dx, \quad \textit{for all test function } \eta \in C^2_c(\Omega).\end{aligned} \tag{3.67}$$

Proof of Proposition 3.4.1. Firstly, multiplying (3.66) by $\psi(u)\eta^2$ and integrating by parts, we get

$$\int_\Omega \phi'(u)^2|\nabla u|^2\eta^2 dx \leq \int_\Omega \xi(u)\Delta(\eta^2)dx + \int_\Omega |x|^\alpha f(u)\psi(u)\eta^2 dx. \tag{3.68}$$

Then, using the stability property (3.4) with the test function $\zeta = \phi(u)\eta$, by directly computation we obtain

$$\begin{aligned}\int_\Omega |x|^\alpha f'(u)\phi^2(u)\eta^2 dx &\leq \int_\Omega |\nabla(\phi(u)\eta)|^2 dx = \int_\Omega |\phi'(u)\eta\nabla u + \phi(u)\nabla\eta|^2 dx \\ &\leq \int_\Omega \phi'(u)^2|\nabla u|^2\eta^2 dx - \int_\Omega \phi^2(u)\eta\Delta\eta dx. \end{aligned} \tag{3.69}$$

Combine (3.68) and (3.69) we have the desired estimation (3.67). □

We are ready to prove Theorem 3.1.9.

Proof of Theorem 3.1.9. As $z = \sup Z(F)$ does exist, let now

$$\tilde{u} = z + (u - z)_+.$$

Observe that by Kato's lemma, $\tilde{u}$ solves (3.66) and satisfies also the stability condition (3.4). If $\tilde{u} \equiv z$ then $(u - z)_+ \equiv 0$, which implies that $u \leq z$, therefore $f(u) \equiv 0$, i.e. $F(x, u) \equiv 0$. u is harmonic and bounded above so u is a constant.

The rest work is to prove that

$$\tilde{u} \equiv z,$$

for $\tilde{u}$ verifying (3.66), (3.4) and $\tilde{u} \geq z$.

We firstly need the following estimate.

Lemma 3.4.2. *There exists a convex function $\phi \in W^{1,\infty}_{loc}((z, b), \mathbb{R})$ such that*

$$f'\phi^2 - f\psi \geq cf'\phi^2 \quad in \quad (z, b), \tag{3.70}$$

where $c > 0$ is a constant and the function $\psi \geq 0$ such that $\psi \in W^{1,\infty}_{loc}((z, b), \mathbb{R})$ and $\psi' = \phi'^2$.

Proof. Claim that the desired function ϕ is defined as

$$\phi(s) = \begin{cases} f(s)^\gamma & \text{if } z < s \leq \tilde{z} \\ f(\tilde{z})^{\gamma-1} f(s) exp\left(\int_{\tilde{z}}^s \sqrt{\frac{f''}{f}} ds\right) & \text{if } \tilde{z} \leq s \leq M \\ f(s)^\beta + A & \text{if } M \leq s < b, \end{cases} \tag{3.71}$$

where $\gamma \in [1, 1 + \frac{1}{\sqrt{q_0}})$, $\beta \in [1, 1 + \frac{1}{\sqrt{q_\infty}})$, $\tilde{z} > z$, $M \in (\tilde{z}, b)$ and A is such that ϕ belongs to $H^1_{loc}((z, b), \mathbb{R})$. We borrow the main technics showed in [30] to give briefly the reason.

Taking $\phi(s) := f^\gamma(s)$ with $\gamma \geq 1$, we notice firstly that there exists a function $\psi(s)$ such that $\psi' = \phi'^2$ and $\lim_{s\to z^+}\psi(s) = 0$ so we may take $\psi(s) = \int_z^s \phi'^2 dt$. Indeed, since f is convex and non-decreasing, we have

$$\begin{aligned}\int_{z_n}^{t_0} \phi'^2 dt &= \gamma^2 \int_{z_n}^{t_0} f^{2\gamma-2} f'^2 dt \\ &\leq \gamma^2 f'(t_0) \int_{z_n}^{t_0} f^{2\gamma-2} f' dt \\ &\leq \frac{\gamma^2}{2\gamma-1} f'(t_0) f^{2\gamma-1}(t_0) \\ &< +\infty \quad \text{for a given } t_0 \in I \text{ and a sequence } z_n \to z.\end{aligned}$$

Let now (s_n) be any sequence along which $\frac{f'\phi^2}{f\psi}$ converges, by Cauchy's mean value theorem, there exists $t_n \in (z, s_n)$ such that

$$\frac{f'\phi^2}{f\psi}(s_n) = \frac{f' f^{2\gamma-1}}{\psi}(s_n) = \left.\frac{f'' f^{2\gamma-1} + (2\gamma-1) f^{2\gamma-2} f'^2}{\gamma^2 f^{2\gamma-2} f'^2}\right|_{s=t_n}$$

Passing to the limit we obtain

$$\liminf_{s\to z^+} \frac{f'\phi^2}{f\psi}(s) \geq \frac{1}{\gamma^2}\left(\frac{1}{\overline{q_0}} + 2\gamma - 1\right) > 1 \quad \text{for all } \gamma \in [1, 1 + \tfrac{1}{\sqrt{\overline{q_0}}}).$$

We note that $\overline{q_0} < +\infty$ ensures the non-emptiness of the above interval $[1, 1 + 1/\sqrt{\overline{q_0}})$. Therefore there exists $c > 0$ such that $f'\phi^2 - f\psi \geq cf'\phi^2$ in the neighborhood $(z, \tilde{z}]$.

Similar as above, we can check without difficulty that $\liminf_{s\to b^-} \frac{f'\phi^2}{f\psi}(s) > 1$ for all $\beta \in [1, 1 + 1/\sqrt{\overline{q_\infty}})$. So (3.70) holds in $[M, b)$ with some $M \in (\tilde{z}, b)$.

The rest work is to prove (3.70) is true for $s \in [\tilde{z}, M]$. Since

$$\left(\frac{f'}{f}\phi^2 - \psi\right)' = \phi^2\left[\frac{f''}{f} - \left(\frac{f'}{f} - \frac{\phi'}{\phi}\right)^2\right],$$

by (3.71), we have $\left(\frac{f'}{f}\phi^2 - \psi\right)'(s) = 0$ for all $s \in [\tilde{z}, M]$ so $\left[\frac{f'}{f}\phi^2 - \psi\right](s)$ is a constant over $[\tilde{z}, M]$. Hence for $s \in [\tilde{z}, M]$

$$\begin{aligned}f'\phi^2 - f\psi = f\left(\frac{f'}{f}\phi^2 - \psi\right) &= f(s)\left(\frac{f'(\tilde{z})}{f(\tilde{z})}\phi^2(\tilde{z}) - \psi(\tilde{z})\right) \\ &\geq f'(\tilde{z})\phi^2(\tilde{z}) - f(\tilde{z})\psi(\tilde{z}) > 0 \quad \text{here we use (3.70) for } s = \tilde{z}.\end{aligned}$$

$f'\phi^2$ is bounded in $[\tilde{z}, M]$ by the definitions of f and ϕ, so (3.70) holds in $[\tilde{z}, M]$ and Lemma 3.4.2 is proved. □

Go back to the proof of Theorem 3.1.9. Recall the definition of $\tilde{u}$:

$$\tilde{u} = z + (u - z)_+.$$

With (3.70) in hand, by direct calculation, for all $\zeta \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \le \zeta \le 1$, we use (3.67) with $\eta = \zeta^m$, $m > 1$, $m' = \frac{m}{m-1} > 1$ and $\Omega = \mathbb{R}^N$ to get

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^\alpha [f'\phi^2 \circ \tilde{u}]\zeta^{2m} dx \le & C\left(\int_{\mathbb{R}^N} |x|^\alpha [(\phi^2+\xi)^{m'} \circ \tilde{u}]\zeta^{2m} dx\right)^{\frac{1}{m'}} \\ & \times \left(\int_{\mathbb{R}^N} |x|^{-\frac{\alpha m}{m'}} (|\nabla\zeta|^2 + |\zeta||\Delta\zeta|)^m dx\right)^{\frac{1}{m}}.\end{aligned} \tag{3.72}$$

Along the analysis in [30], we claim, when $2\gamma m' \ge \frac{1}{q_1} + 2\gamma$ and $2\beta m' \ge \frac{1}{q_2} + 2\beta$,

$$f'\phi^2 \circ \tilde{u} \ge c(\phi^2+\xi)^{m'} \circ \tilde{u} \quad \text{for all } \tilde{u} \in (z, b), \tag{3.73}$$

where $c > 0$ is a constant, $0 < q_1 < \underline{q_0}$ and $0 < q_2 < \overline{q_\infty}$ are arbitrary.

Indeed, fixing q_1 such that $0 < q_1 < \underline{q_0}$, by the definition of $\underline{q_0}$, we observe that

$$\frac{ff''}{(f')^2} \le \frac{1}{q_1} \quad \text{for } \tilde{u} \in (z, \tilde{z}],$$

which yields $\left(\frac{f'}{f^{1/q_1}}\right)' \le 0$, so $\frac{f'}{f^{1/q_1}}$ is non-increasing, this together with the fact $\frac{f'}{f^{1/q_1}} \ge 0$ imply that $f' \ge cf^{1/q_1}$ in $(z, \tilde{z}]$. Therefore, with the expression of ϕ (3.71), we have

$$[f'\phi^2](\tilde{u}) \ge c[f'f^{2\gamma}](\tilde{u}) \ge c' f^{2\gamma+1/q_1}(\tilde{u}) \quad \text{for } \tilde{u} \in (z, \tilde{z}]. \tag{3.74}$$

Moveover, it is easy to verify that

$$\left(\phi^2+\xi\right)^{m'}(\tilde{u}) \le cf^{2\gamma m'} \quad \text{for } \tilde{u} \in (z, \tilde{z}]. \tag{3.75}$$

Combining (3.74) and (3.75), we conclude (3.73) in $(z, \tilde{z}]$ if $2\gamma m' \ge \frac{1}{q_1} + 2\gamma$. Similarly, we can prove that, for a given $0 < q_2 < \overline{q_\infty}$, it follows that (3.73) holds true in $[\tilde{u}, b)$ if $2\beta m' \ge \frac{1}{q_2} + 2\beta$. So we get (3.73).

Since $\alpha > -2$, we have $\frac{N+\alpha}{N-2} > 1$, therefore m' can be chosen close to $\frac{N+\alpha}{N-2} > 1$. Moreover by choosing γ close to $1 + \frac{1}{\sqrt{\underline{q_0}}}$, β close to $1 + \frac{1}{\sqrt{\overline{q_\infty}}}$, q_1 close to $\underline{q_0}$ and q_2 close to $\overline{q_\infty}$, the conditions (3.18) and (3.19) can hold true. Combining (3.72) and (3.73), we get that for some $m > \frac{N+\alpha}{2+\alpha}$,

$$\int_{\mathbb{R}^N} |x|^\alpha [(\phi^2+\xi)^{m'} \circ \tilde{u}]\zeta^{2m} dx \le C\int_{\mathbb{R}^N} |x|^{-\frac{\alpha m'}{m}} (|\nabla\zeta|^2 + |\zeta||\Delta\zeta|)^m dx. \tag{3.76}$$

Now for every $R > 0$, consider the function $\varphi_R(x) = \varphi(R^{-1}|x|)$, where $\varphi \in C_0^\infty(\mathbb{R})$ satisfies $0 \le \varphi \le 1$ in $\mathbb{R}$, $\varphi(t) = 1$ if $|t| \le 1$ and $\varphi(t) = 0$ if $|t| \ge 2$. Applying (3.76) with $\zeta = \varphi_R$, we obtain

$$\begin{aligned}\int_{B_R} |x|^\alpha [(\phi^2+\xi)^{m'} \circ \tilde{u}] dx &\le \int_{\mathbb{R}^N} |x|^\alpha \left[(\phi^2+\xi)^{m'} \circ \tilde{u}\right] \varphi_R^{2m} dx \\ &\le CR^{N-(2+\alpha)m+\alpha}, \quad \forall\, R > 0.\end{aligned}$$

Note that $N - (2+\alpha)m + \alpha < 0$ since $m > \frac{N+\alpha}{2+\alpha}$. Letting $R \to \infty$,

$$\int_{\mathbb{R}^N} |x|^\alpha [(\phi^2+\xi)^{m'} \circ \tilde{u}] dx = 0,$$

which shows that $\tilde{u} \equiv z$ and completes the proof. □

3.4.2 A counter-example associated to the finite Morse index solutions

At the end, we show a simple counter-example to explain that in general, we cannot expect to get some Liouville type result for finite Morse index solutions as in Theorem 3.1.9, that is we cannot hope to get only constant solutions. For that we consider the Hénon equation with Dirichlet boundary condition, i.e.

$$\begin{cases} -\Delta v = |x|^\alpha v^p & \text{in} \quad B_1 \\ v = 0 & \text{on} \quad \partial B_1 \\ v > 0 & \text{in} \quad B_1, \end{cases} \tag{3.77}$$

with $\alpha > 0$, $1 \geq p < \frac{N+2+2\alpha}{N-2}$ and $N \geq 2$. It is shown by Ni in [48] that (3.77) processes a non-trivial positive radial solution v. Now we extend v by harmonic extension to get a radially symmetric and decreasing function, still denoted by v such that $v \in C^2(\mathbb{R}^N \backslash \{0\})$ and solves the equation

$$-\Delta v_1 = |x|^\alpha v_+^p \quad \text{in} \quad \mathbb{R}^N.$$

Since $\text{supp}(v_+)$ is compact, of course v has finite Morse index. This completes our proof.

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