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▶ **To cite this version:**

Yang Shi. Q-discrete Painlevé equation and associated linear problem. High Energy Physics - Theory [hep-th]. Université Pierre et Marie Curie - Paris VI, 2010. English. NNT : 2010PA066526 . tel-00815010

HAL Id: tel-00815010
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Submitted on 18 Apr 2013

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q-Discrete Painlevé equation and the Associated Linear Problems

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A thesis submitted for the degree of
Doctor of Philosophy

Under the Cotutelle scheme of

Applied Mathematics
University of Sydney

and

Physique Théorique
Université Pierre et Marie Curie

December 2009

Acknowledgements

First and foremost I wish to thank my two supervisors Nalini Joshi (University of Sydney) and Claude Viallet (Université Pierre et Marie Curie (Paris VI)). To Nalini, for choosing a good problem for me to work on and supporting me in all the possible ways throughout my PhD, right to the end, meticulously proof reading for me. I admire her enthusiasm and optimism for overcoming difficulties in Mathematics and difficulties of life in general, but most of all I am grateful for her unfaltering confidence in me during the very long period of time during which I tried and tried to prove the main results of my thesis. My French supervisor Claude Viallet has accepted me as a student upon the request of Nalini on my behalf, hence uncomplainingly taking on the responsibility of battling the impossibilities of French-Australian administrations for my Cotutelle arrangement. Only then could I have split my PhD time in between two of the most wonderful cities in the world during which I have learnt and enjoyed myself a great deal.

I would like to thank Embassy of France in Australia from whom I was awarded the Australian Cotutelle grant in 2006 and the "French Lessons for Scientists" scholarships for three consecutive years, 2006, 2007 and 2008. I would also like to give my sincere thanks to Fiona Grisard and Frédéric Vanhove from the Science and Technology Section of Embassy of France in Australia. They have alway been very efficient and often went out of their way to help.

I wish to thank the friends who did the proof reading for me: John Bourke Chris Cornet, Chris Ormerod, Lewis Mitchell and Thomas Fung for his help with LaTeX. However, it is Stewart Wilcox who has got the worst of it. He was the first person to read my LaTeX file. It was just a sequence of calculations with minimal explanations in between which was so rough that it could not be called it a draft. I was worried about what it might do to our friendship for such a big ask. I am glad that I asked him now, because it gave us something to hang out, talk and laugh about during his two months in Sydney, even if it was my thesis that we were laughing about.

I would like to thank my office mates: John Bourke for the non-obstructive distractions; Neil Saunders for the humour, the cherry ripes, the Lindt bunny, the

Ferrero Rochers; Lewis Mitchell who has been my constant companion for the last three months of my PhD, having to stand my conversations which got increasingly boring as time wore on and eventually reduced to just the chapter numbers of this thesis, and finally to Ben Wilson and Tegan Morrison for having started our room 530 posse and the beginning of the era of our fraternité under which we have passed the good and sometimes not so good times as PhD students.

I want to thank my friends from Paris: Jiali Huang, Léonce Ki, Namrata Khandekar and Christian Scimiterna with whom I have shared good company, good food, and the beautiful Paris together. My girl friends Julia Zhao, Naiwen Duan, Jessie Yao and Sofia Lim, for although they did not understand what is it that I do, they supported, tolerated and were always willing to accommodate whatever our plans around this PhD thing. I specially thank my best friend Jacky Siu and all members of his family, who took me in as one of their own thirteen years ago when I first came to Australia from China, they from Hong Kong.

I want to thank my aunt and my two cousins with whom I lived for the past two years. This is going to get embarrassing. I swear all Chinese aunties are like this. My aunt has cooked and washed for me; picked me up from uni when it was getting too dark, too wet or too cold. She has even brushed my hair for me, when, the second last week before I submitted I hurt my back from all that typing and could not lift my arms for a day or two. In her ever enthusiasm she even offered to plait my hair to which I gently but firmly declined. To my two riotous cousins, who have also given me support in their own ways and still manage to shoo whatever lowly organism of living creatures in the forms of spider, cockroach, moths, lizard . . . they could find in my direction. Thank goodness there has not been a rat yet.

To the rest of my family in China, my mum, grandma . . . to whom I dedicate this thesis and to whom I can finally go back to on finishing this.

The content of this thesis is original work of which I am the sole author. The use of existing works is explicitly and duly acknowledged in the text.

Yang Shi
Sydney, December 16, 2009.

CONTENTS

CHAPTER 1

Introduction

The objective of this thesis is to study the q-**discrete Painlevé Equations**. Two of the remarkable aspects of the Painlevé equations will be our main interests, namely the determinant form of their special solutions and the associated linear problems.Using results from the continuous Painlevé equations as an analogue, we study the q-discrete Painlevé equations, in particular their special solutions via their associated linear problems in a q-discrete setting. We introduce the Painlevé Equations and the discrete analogue of the Painlevé equations; give the motivation for this work and state the main results. The structure of the thesis is outlined at the end of this chapter.

1.1. Why the Painlevé equations?

The six Painlevé equations (denoted PI-PVI) were first discovered a little over one hundred years ago by Painlevé [**61**], Gambier [**23**] and Fuchs [**20**] in the quest of finding and classifying all second-order ordinary differential equations (ODEs) whose solutions can be globally continued in the complex plane.

Second-order differential equations are particularly relevant in the study of physics. Classical Special Functions which are defined as the solutions of second-order differential equations (DEs) appear throughout the history of physics. From the governing equations of classical mechanics: Newton's laws of motion, fluid dynamics, optics; to special relativity, quantum mechanics and statistical mechanics. As the models in physics were getting more and more sophisticated, the governing equations one had to solve at the end became more and more complicated. The question became rather: can these second-order DEs be solved at all, hence the model is considered any "good", therefore an Integrable model? Ultimately, we need to be able

to make predictions from the theory to compare with observations from experiments.

Painlevé and the mathematicians of his time looked to classify second-order ODEs by their singularity structure. An ODE is said to have the Painlevé property if none of its solutions have any movable singularities around which the solutions are multi-valued, in which case we say it is integrable. In finding all second-order ODEs of the type

$$y'' = F(t; y, y'), \qquad ' = \frac{d}{dt},$$

that have this property, fifty canonical types (up to a Möbius transformation) of equations were found, six of which had never been seen before. These six equations **define** new special functions, in the same sense that equation[1]

$$y'' = -ty \tag{1.1}$$

known as the Airy equation, together with an initial condition defines the Airy function $Ai(t)$.

The solutions of the six Painlevé equations are called the **Painlevé transcendents**. They have been found to naturally arise in many domains of physics: statistical mechanics models [**54**], random matrix theory [**19**], quantum gravity [**31**], quantum field theory [**16**], non-linear optics [**26**] and general relativity [**53**] to name just a few.

The Painlevé equations are also known to relate to important partial differential equations (PDEs) [**3**]. In fact, a primary reason for the intense interest and activity in the the area over the last 30 years is due to the fact that Painlevé equations can be obtained by similarity reduction from important, completely integrable PDEs such as the nonlinear Schrödinger equation [**11**], Boussinesq equation [**12**], Kadomstev-Petviashvilli (KP) equation, sine-Gordon equation [**50**], Korteweg-de Vries (KdV) equation [**10**] and the modified Korteweg-de Vries (mKdV) equation [**57**]. These integrable PDEs are the governing equations of numerous important physical systems and can all be solved by the inverse scattering transform (IST). IST is a technique developed by Gardner et al. [**24**], Zakharov and Shabat

[1]While Airy equation does not usually have the minus sign on the right, we chose to use this convention in our work for convenience.

[**77**] and Ablowitz et al. [**2**] in the 1960s for solving integrable non-linear PDEs via their associated linear problems. For a recent review on the IST technique, see [**1**].

Having the Painlevé property, or equivalently being integrable, means that the equation possesses many beautiful properties which can be used to study its solutions. In this work, we concern ourselves with two of these characteristics, namely the special solutions and the associated linear problems of the Painlevé equations.

1.2. Special solutions of the Painlevé equations

As stated before, the general solutions (i.e. the Painlevé transcendents) of the Painlevé equations cannot be expressed in terms of any of the classical special functions. However, for special values of the parameters, the Painlevé equations can admit special solutions of rational and hypergeometric type [**6, 60, 30**]. In particular, PII-PVI admit hypergeometric type special solutions of Airy, Bessel, Parabolic cylinder, Whittaker and Gauss hypergeometric type respectively. The rational type special solutions are related to the so called "special polynomials", in particular, Yablonskii-Vorobiev polynomials for PII [**75, 76**], Okamoto polynomials for PIV [**60**] and Umemura polynomials for PIII, PV and PVI [**22**]. PI does not have any special solutions as it does not contain any parameters.

More recently, these special polynomials were found to be expressible as determinants whose matrix entries are classical orthogonal polynomials such as Laguerre and Hermite polynomials [**59, 15, 45, 43, 46**].

In this thesis we will use the second Painlevé equation (PII) to illustrate the various characteristics of the Painlevé equations [2]. PII has one parameter in its equation,

$$f'' = 2f^3 + tf - a. \tag{1.2}$$

Let us denote the solution of equation (1.2) with parameter a as

$$f(t) = f_a(t).$$

[2] See Appendix A for the list of all the six Painlevé equations.

Example 1.2.1. The simplest rational type special solution for PII occurs when $a = 0$. It is easy to check that

$$f_0(t) = 0. \tag{1.3}$$

Example 1.2.2. The simplest hypergeometric type special solution occurs when $a = \frac{1}{2}$. In this case PII reduces to a Riccati equation

$$f'_{\frac{1}{2}} = -f^2_{\frac{1}{2}} - \frac{t}{2}, \tag{1.4}$$

which can be linearized using the standard transformation $f_{\frac{1}{2}} = \frac{u'}{u}$. In this case $u(t)$ satisfies the Airy type equation

$$u'' = -\frac{t}{2}u,$$

hence

$$f_{\frac{1}{2}}(t) = \frac{A(2^{-\frac{1}{3}}t)'}{A(2^{-\frac{1}{3}}t)}, \tag{1.5}$$

where $A(2^{-\frac{1}{3}}t)$ is a solution of the Airy equation (1.1) with respect to the variable $2^{-\frac{1}{3}}t$.

1.2.1. Bäcklund transformation , special solution hierarchies and determinantal forms. The Painlevé equations possess what are called the Bäcklund transformations, which is another one of the nice properties of the Painlevé equations. The Bäcklund transformation relates a solution of the Painlevé equation for a particular value of the parameter to another solution for a different value of the parameter.
The Bäcklund transformation for PII (1.2), is

$$f_{a+1}(t) = -f_a - \frac{(a + 1/2)}{f'_a - f^2_a - t/2}. \tag{1.6}$$

Equation (1.6) relates the solution $f_{a+1}(t)$ of PII equation for when the parameter is $a + 1$, to $f_a(t)$ and its derivative. A hierarchy of rational type special solutions $f_k(t)$, for when the parameter $a = k$, k is an integer, can be generated from the simplest rational type special solution

$$f_0(t) = 0$$

by successively applying the Bäcklund transformation (1.6. These special solutions were first obtained by Airault [**6**]. Here is a few examples of some

rational hierarchy of special solutions of PII are:

$$\begin{aligned} f_0(t) &= 0, \\ f_1(t) &= \frac{1}{t}, \\ f_2(t) &= \frac{2(-2+t^3)}{t(4+t^3)}, \\ f_3(t) &= \frac{3t^2(160+8t^3+t^6)}{(4+t^3)(-80+20t^3+t^6)}. \end{aligned}$$

Notice that the numerator and denominator of $f_k(t)$ can be factorized in terms of some special polynomial. These polynomials are called the Yablonskii-Vorobiev polynomials [**75, 76**]. However, there is a more concise way of writing these special polynomials hence expressing $f_k(t)$ in the form of determinants whose entries are well-known orthogonal polynomials. Determinantal expressions of the special solutions of the Painlevé equations were first derived by Flaschka and Newell for the special solutions of PII equation. In the case of PII, the polynomials are of Laguerre type [**18**].

Theorem 1.2.3. [**18**]
PII admits rational type special solutions $f_k(t)$ when $a = k$, $k = 0, 1, 2, ...$

$$f_k(t) = \frac{d}{dt} \ln \frac{\tau_k(t)}{\tau_{k-1}(t)},$$

where $\tau_k(t)$ is a polynomial of degree $k(k+1)/2$ in t and can be represented as the determinant of a $k \times k$ matrix

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-k+2} & T_{-k+3} & \dots & T_1 \end{vmatrix}.$$

Here $T_k(t)$ denotes the Laguerre type polynomial of degree k in t, with the generating function

$$\sum_{k=0}^{\infty} T_k(t) x^k = \exp\left(i\frac{4}{3}x^3 + itx \right), \tag{1.7}$$

and recurrence type relations

$$\begin{aligned} kT_k &= 4iT_{k-3} + itT_{k-1}, \\ \frac{dT_k}{dt} &= iT_{k-1}, \end{aligned}$$

and $T_k = 0, k < 0$. A few examples of T_k,

$$T_0 = 1 \ , \ T_1 = it \ , \ T_2 = -\frac{t^2}{2} \ , \ T_3 = \frac{4i}{3} - \frac{it^3}{6}.$$

Similarly, a hierarchy of hypergeometric type solutions for when a is a half integer, $a = \frac{2k+1}{2}$, $k = 0, 1, 2, ...$, can be generated from the simplest hypergeometric type special solution for when $k = 0$

$$f_{\frac{1}{2}}(t) = \frac{A(2^{-\frac{1}{3}}t)'}{A(2^{-\frac{1}{3}}t)}$$

by successively applying the Bäcklund transformation . The hypergeometric type solutions $f_{k+\frac{1}{2}}(t)$ can also be expressed in the determinantal form.

Theorem 1.2.4. [**18**]
PII admits hypergeometric type solutions $f_{k+\frac{1}{2}}(t)$ for $a = \frac{2k+1}{2}$, $k = 0, 1, 2, ...$ with

$$f_{k+\frac{1}{2}}(t) = \frac{d}{dt} \ln \frac{\tau_k(t)}{\tau_{k-1}(t)},$$

where $\tau_k(t)$ is expressed as the determinant of a matrix of Airy type function and derivatives of Airy type function $A(2^{-\frac{1}{3}}t)$,

$$\tau_k(t) = \begin{vmatrix} A & \frac{d}{dt}A & \dots & \frac{d^{k-1}}{dt^{k-1}}A \\ \frac{d}{dt}A & \frac{d^2}{dt^2}A & \dots & \frac{d^k}{dt^k}A \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{k-1}}{dt^{k-1}}A & \frac{d^k}{dt^k}A & \dots & \frac{d^{2k-2}}{dt^{2k-2}}A \end{vmatrix},$$

where

$$\frac{d^2A(2^{-\frac{1}{3}}t)}{dt^2} = -\frac{t}{2}A(2^{-\frac{1}{3}}t).$$

Determinantal expressions for the special solutions of the Painlevé equations have since been derived from different perspectives (algebraical or geometrical) by Okamoto and Kajiwara et al. see [**60**] and [**45, 43, 46**] for example.

1.3. The 2×2 associated linear problems of the Painlevé equations

Associated linear problems (also known as the **Lax pair** or the **iso-monodromy deformation problem**) of the Painlevé equations can be obtained in two ways. The first is by deformation of a 2×2 linear differential system also is referred to as the spectral problem that preserves the monodromy data of the original linear system, termed the iso-monodromy deformation. Fuchs [**21**], Jimbo and Miwa [**36**] have found that the Painlevé equations arise as the compatibility condition of the spectral linear system and the deformation equation when the deformation is monodromy preserving.

As we have stated earlier that the Painlevé equations are related to some completely integrable PDEs by similarity reduction. These PDEs each come with a Lax pair which is seen as an indication of their integrability. Another way to obtain a Lax pair for the Painlevé equation is by performing similarity reduction on the Lax pair of the Integrable PDE, to which the Painlevé equation in question is related. For example, Ablowitz and Segur [**4**] have found that PII can be obtained from the mKdV equation by similarity reduction. The Lax pair for PII was then derived by similarity reduction of the Lax pair of mKdV equation. The special case of PIII and its Lax pair studied in [**18**] was derived by Newell [**56**] from similarity reduction of a integrable PDE and its Lax pair.

In their 1980 paper [**18**], Flaschka and Newell have developed an analytical technique for studying the Painlevé equations via their associated linear problems. The technique is based on the 2×2 associated linear systems of PII and a special case of PIII.

Flaschka and Newell's approach led to many classes of special solutions of PII and PIII, some of which were previously known, found by investigating the non-linear equations themselves rather than the associated linear problems. What is appealing about their technique is that by exploring the connection between the Painlevé equation and its associated linear problems, the determinantal structure of the hierarchies of special solutions of the Painlevé equation emerge naturally out of this approach. This is true for both rational and hypergeometric type special solutions. Studying the Painlevé equations via their associated linear problems approach was also

employed by Dubrovin and Mazzoco [**17**], who have studied the special solutions of PVI via its associated linear problems from a geometrical perspective.

1.4. Discrete Painlevé equations

Due to recent interests in discrete problems in mathematics and physics, discrete or difference equations have taken a prominent place in the mathematical physics community. Although the general analytical theory for discrete equations has been stated long ago along with that for differential equations by the Birkhoff school [**14, 9**], the development of discrete equations has lagged behind that of differential equations. With the progress in discrete analogues of the classical special functions such as discrete orthogonal polynomials [**51**], and discrete/basic hypergeometric type functions [**25**], it was clear that the discrete Painlevé equations should follow. In fact, the first appearance of the discrete Painlevé equations arose as a discrete analogue of PI (d-PI) from the study of orthogonal polynomials [**70**]. More recently, examples of the discrete analogues of the Painlevé equations have been found in the context of physics. Another discrete analogue of PI [3] was discovered in the study of the partition function in a 2D model of quantum gravity [**13, 35**]. Soon after, a discrete analogue of PII (d-PII) was derived both in a physics context [**62**], and from a similarity reduction of discrete analogue of the mKdV equation [**58**].

Discrete analogues of PIII-PV were found using the singularity confinement technique [**64**]. The form of the discrete analogue of PVI remained elusive until Jimbo and Sakai (1996) [**37**] derived the q-discrete analogue of PVI (q-PVI) by formulating the iso-monodromy deformation problem in a q-discrete setting. Since then, a variety of different forms of the discrete analogues of the Painlevé equations have been derived, and their properties have been studied via different perspectives [**29**]. In 2001, Sakai [**68**] classified all possible types of the discrete Painlevé equations by means of geometry of rational surfaces.

[3] Discrete equations are named according to what differential equations they become in the continuum limit, however many distinct discrete equations can have the same differential equation in the continuum limit, hence are referred to by the same name unfortunately.

1.5. Special solutions of the discrete Painlevé equations

Although more fundamental than the differential equations, the discrete Painlevé equations are found to possess many of the beautiful properties of their differential counterparts, such as admitting special solutions of rational and hypergeometric type. Like the Painlevé equations, apart from discrete analogue of PI (d-PI), discrete PII-PVI admit basic hypergeometric type special solutions which are the discrete analogues of Airy [**63**], Bessel [**27**], parabolic cylinder [**71**], confluent hypergeometric [**72**], and the Gauss hypergeometric functions [**37**], respectively. We demonstrate this by using a q-discrete analogue of PII (q-PII)

$$g(x/\lambda)g(\lambda x) = \frac{\alpha x^2(g(x)+x^2)}{g(x)(g(x)-1)} \tag{1.8}$$

as an example. This equation has one parameter α. Let $\alpha = \frac{1}{\lambda^{4k}}$ and denote the solution of equation (1.8) as $g_k(x)$.

Example 1.5.1. Equation (1.8) admits the simplest rational type special solution for $k = 0$, $\alpha = 1$

$$g_0(x) = -ix. \tag{1.9}$$

Example 1.5.2. Equation (1.8) reduces to a q-discrete analogue of the Riccati equation (q-Riccati) for $k = \frac{1}{2}$, $\alpha = \frac{1}{\lambda^2}$,

$$g_{\frac{1}{2}}(\lambda x) = \frac{x^2 + g_{\frac{1}{2}}(x)}{g_{\frac{1}{2}}(x)}. \tag{1.10}$$

q-Riccati equation (1.10) can be linearized by letting

$$g_{\frac{1}{2}}(x) = -i\frac{x}{\lambda}\frac{ai(x/\lambda)}{ai(x/\lambda^2)} \tag{1.11}$$

, and we find

$$ai(x/\lambda^2) - \frac{i}{\lambda x}ai(x/\lambda) + \frac{ai(x)}{\lambda} = 0, \tag{1.12}$$

which we define to be a q-discrete analogue of the Airy equation (q-Airy). Equations (1.8), (1.10) and (1.12) are called q-PII, q-Riccati and q-Airy equations because in the continuum limit $\frac{1}{\lambda} \to 1$, they tends to these equations respectively. See Appendix C on the continuum limits of equations (1.8), (1.10) and (1.12).

1.5.1. Bäcklund transformation and the hierarchies of special solutions of the discrete Painlevé equations. A simple and systematic procedure for deriving the Bäcklund transformation of the discrete Painlevé equations was developed by Joshi et al. [**39**]. A hierarchy of special solutions can be calculated by repeatedly applying the Bäcklund transformation on the simplest solution. The Bäcklund transformation of q-PII equation (1.8) [**39**] is given by

$$g_{k+1}(x) = \frac{(\alpha x^2 - g_k(x/\lambda)g_k(x)\lambda^2 + \alpha g_k(x))\, x^2}{\lambda^2 g_k(x)\left(g_k(x)(g_k(x/\lambda) - 1) - x^2\right)}. \tag{1.13}$$

Equation (1.13) relates solution of equation (1.8) at $\alpha = \frac{1}{\lambda^{4(k+1)}}$ to that at $\alpha = \frac{1}{\lambda^{4k}}$. When we apply the Bäcklund transformation (1.13) on the rational solution (1.9) we obtain a hierarchy of rational type special solutions of q-PII for $\alpha = \frac{1}{\lambda^{4k}}$, k is an integer. If we apply the Bäcklund transformation (1.13) on the q-Airy type special solution (1.11) we obtain a hierarchy of q-hypergeometric type special solutions of q-PII for $\alpha = \frac{1}{\lambda^{4k}}$, k an half integer.

Example 1.5.3. Let us look at few examples of some rational special solutions. For $k = 0, 1, 2, 3$ respectively we have:

$$g_0(x) = -ix,$$

$$g_1(x) = -i\frac{x\left(-i + x(1+\lambda)\right)}{\lambda^2\left(-i + \frac{x}{\lambda}(1+\lambda)\right)} = -i\frac{xP_1(n)}{\lambda^2 P_1(n+1)}, \tag{1.14}$$

$$g_2(x) = -i\frac{x}{\lambda^4}\frac{P_3(n)P_1(n+1)}{P_1(n)P_3(n+1)}, \tag{1.15}$$

$$g_3(x) = -i\frac{x}{\lambda^6}\frac{P_6(n)P_3(n+1)}{P_3(n)P_6(n+1)}, \tag{1.16}$$

where $x = \frac{1}{\lambda^n}$,

$$P_1(n) := -i + x(1+\lambda), \tag{1.17}$$

$$\begin{aligned} P_3(n) := {} & ix^3\lambda^3(1+\lambda)^3(1-\lambda+\lambda^2) \\ & +x^2\lambda^2(1+\lambda)^2(1-\lambda+\lambda^2)(1+\lambda+\lambda^2) \\ & -ix\lambda^2(1+\lambda)(1-\lambda+\lambda^2)(1+\lambda+\lambda^2) - \lambda^4, \end{aligned} \tag{1.18}$$

$$
\begin{aligned}
& P_6(n): \\
= \; & -x^6\lambda^7(1+\lambda)^6(1-\lambda+\lambda^2)^2(1-\lambda+\lambda^2-\lambda^3+\lambda^4) \\
& +ix^5\lambda^5(1+\lambda)^5(1+\lambda^2)(1-\lambda+\lambda^2)^2(1+\lambda+\lambda^2) \\
& \qquad \times(1-\lambda+\lambda^2-\lambda^3+\lambda^4) \\
& +x^4\lambda^4(1+\lambda)^4(1-\lambda+\lambda^2)^2(1+\lambda+\lambda^2)(1+\lambda+\lambda^2+\lambda^3+\lambda^4) \\
& \qquad \times(1-\lambda+\lambda^2-\lambda^3+\lambda^4) \qquad (1.19) \\
& -ix^3\lambda^4(1+\lambda)^5(1-\lambda+\lambda^2)^2(1+\lambda+\lambda^2+\lambda^3+\lambda^4) \\
& \qquad \times(1-\lambda+\lambda^2-\lambda^3+\lambda^4) \\
& -x^2\lambda^5(1+\lambda)^2(1-\lambda+\lambda^2)(1+\lambda+\lambda^2)(1+\lambda+\lambda^2+\lambda^3+\lambda^4) \\
& \qquad \times(1-\lambda+\lambda^2-\lambda^3+\lambda^4) \\
& +ix\lambda^7(1+\lambda)(1-\lambda+\lambda^2)(1+\lambda+\lambda^2)(1+\lambda^4)+\lambda^{11}.
\end{aligned}
$$

In calculating above rational type solutions we have noticed that as in the case of rational special solutions of PII earlier (section 1.2.1) the numerator and the denominator of $g_k(x)$ factorize into products of some "special polynomial" in x (1, $P_1(n)$, $P_3(n)$, $P_6(n)$ for example) just as in the continuous case. However, it does not reveal the determinant structure of these polynomials and hence the rational type special solutions of q-PII.

1.5.2. Determinant form of special solutions of discrete PII. Using Hirota's bilinear formalism, Kajiwara et. al. showed that the hierarchies of special solutions of both rational and hypergeometric type for some discrete analogues of PII-PV [**48, 49, 47, 42, 44, 34, 32, 40, 41**] can be put into the determinantal form. The determinantal form of the q-hypergeometric type special solutions of q-PVI was found by Sakai [**67**]. More recently Tsuda and Masuda [**74**] have obtained the determinantal form of algebraic special solutions of q-PVI, again via geometrical means. The determinantal structures of the two types of special solutions of q-PII (1.8) example considered before have not been found, which will be the principal concern of this thesis. Here we present some known results on the determinantal forms of some special solutions of another discrete analogue of PII. It is known that the special solutions of discrete analogues of the Painlevé equations tend to the special solutions of the Painlevé equations in the continuum limit (one

of the many evidences of parallel behaviors of the discrete and differential integrable systems), we can compare these discrete results with the formulae from our examples of PII in section 1.2.1. These results will also give us an idea of what the determinantal solutions of our q-PII (1.8) might look like.

Theorem 1.5.4. [**49**] Equation

$$w_{n+1} + w_{n-1} = \frac{2}{x}\frac{(n+1)w_n - (N+1)}{1 - w_n^2}, \tag{1.20}$$

where x and N are parameters, is a discrete analogue of PII. Equation (1.20) admits a hierarchy of rational type special solutions indexed by $N = 0, 1, 2, ...$, expressible in terms of determinants involving the discrete analogue of the Laguerre polynomials given by

$$w_n = \frac{\tau_{N+1}^{n+1}\tau_N^n}{\tau_{N+1}^n\tau_N^{n+1}} - 1 \tag{1.21}$$

where τ_N^n is a determinant of size $N \times N$ matrix:

$$\tau_N^n = \begin{vmatrix} L_N^n & L_{N+1}^n & \dots & L_{2N-1}^n \\ L_{N-2}^n & L_{N-1}^n & \dots & L_{2N-3}^n \\ \vdots & \vdots & \ddots & \vdots \\ L_{-N+2}^n & L_{-N+3}^n & \dots & L_1^n \end{vmatrix},$$

and $L_k^n(\xi)$ is a discrete analogue of the Laguerre polynomial of kth degree in n, defined by

$$\sum_{k=0}^{\infty} L_k^n(\xi)\, \lambda^k = (1-\lambda^2)^{-\frac{\xi}{2}}(1-\lambda)^{-n-\xi} \exp\left(-\frac{\xi\lambda}{1-\lambda}\right), \tag{1.22}$$
$$L_k^n(\xi) = 0, \quad (k < 0).$$

Solutions (1.21) correspond to the hierarchy of the rational special solutions of PII in Theorem 1.2.3 in the continuum limit. Equation (1.22) is equation (1.7) when $\xi = -\frac{1}{2\epsilon^3},\ n = \frac{t}{\epsilon},\ \lambda = \epsilon x$, as $\epsilon \to 0$.

Theorem 1.5.5. [**48**] Equation

$$w_{n+1} + w_{n-1} = \frac{(\alpha n + \beta)w_n + \gamma}{1 - w_n^2} \tag{1.23}$$

is the PII equation (1.2) with $a = -(2N+1)$, if $\alpha = 2p, \beta = (2N-1)p + 2q, \gamma = -(2N+1)p$, $p = -\epsilon^3,\ q = 1, w_n = \epsilon f$, and $n = -\frac{t}{2\epsilon}$, in the limit $\epsilon \to 0$. Equation (1.23) admits a hierarchy of special solutions of d-hypergeometric type. The hierarchy is indexed by $N = 0, 1, 2, ...$,

$$w_n = \frac{\tau_{N+1}^{n+1}\tau_N^n}{\tau_{N+1}^n\tau_N^{n+1}} - 1, \tag{1.24}$$

where τ_N^n is a determinant of a size $N \times N$ matrix,

$$\tau_N^n = \begin{vmatrix} A_n & A_{n+2} & \dots & A_{n+2N-2} \\ A_{n+1} & A_{n+3} & \dots & A_{n+2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+N-1} & A_{n+N+1} & \dots & A_{n+3N-3} \end{vmatrix},$$

where A_n satisfies a discrete analogue of Airy (d-Airy) equation

$$A_{n+2} = 2A_{n+1} - (pn+q)A_n. \tag{1.25}$$

Solutions (1.24) are the q-discrete analogue of the hierarchy of hypergeometric type special solutions of PII in Theorem 1.2.4. The simplest special solution of this type is when $N = 0$, $\tau_0^n = A_n$,

$$w_n = \frac{A_{n+1}}{A_n} - 1,$$

using the determinantal formula.
This can be checked that this is the solution by looking at d-PII equation (1.23) itself. When $N = 0$, $\alpha = 2p$, $\gamma = -p$, $\gamma = -\frac{\alpha}{2}$ and d-PII (1.23) reduces to the discrete analogue of the Riccati equation

$$w_{n+1} = \frac{w_n - (pn+q-1)}{1+w_n},$$

which can be linearized using

$$w_n = \frac{G_{n+1}}{G_n} - 1,$$

where G_n satisfies

$$G_{n+2} = 2G_{n+1} - (pn+q)G_n,$$

which is d-Airy equation (1.25).

Results such as theorems 1.5.4, 1.5.5 and the preceding works of Kajiwara et al. on the determinantal forms of some special solutions of discrete

analogues of the other Painlevé equations [**47, 42, 44, 34, 32, 67, 40, 41**] let us to believe that the determinantal forms of the rational and the q-hypergeometric type special solutions of q-PII equation (1.8) exist.

On the other hand, we know that there is this other approach of Flaschka and Newell [**18**] from which the determinantal forms of some special solutions of the Painlevé equations can be obtained via the study of the associated linear problems.

1.6. The 2×2 associated linear problems of the discrete Painlevé equations

As in the continuous case, the 2×2 associated linear problems of discrete Painlevé equations can be found in two ways. One by iso-monodromy deformation of a 2×2 discrete linear system. The Lax pair for q-PVI was derived in this way by Jimbo and Sakai [**37**]. Murata [**55**] has found the Lax pairs for q-PV-q-PI by a coalescence procedure on Sakai's Lax pair for q-PVI. The other way to obtain Lax pairs for discrete Painlevé equations is by similarity reduction of the Lax pairs of the integrable lattice equations (discrete analogues of the integrable PDEs) to which the discrete Painlevé equations are related. A 2×2 Lax pair for the q-PII equation (1.8) was found by Hay et al. [**33**] by performing similarity reduction on the Lax pair of a q-discrete analogue of the mKdV (q-mKdV) equation. A 2×2 Lax pair of a special case of q-PIII was derived by Nijhoff and reported in [**38**] from similarity reduction of the Lax pair of an integrable lattice equation.

Our aim is to follow Flaschka and Newell's work in the q-discrete setting and develop the technique for studying q-Painlevé equations via their associated linear problems. To the author's knowledge, this has not been done before. Our study is based on the 2×2 Lax pair for q-PII equation (1.8) [**33**].

There are of course Lax pairs for discrete Painlevé equations of size other than 2×2, for example there is a 4×4 Lax pair [**28**] for a discrete analogue of PIII (d-PIII). For simplicity we will only deal with 2×2 Lax pairs in this thesis. There are also of course discrete Painlevé equations with 2×2 Lax pairs other than q-PII equation (1.8). We chose to investigate this particular analogue of PII equation because like the Lax pair used by Flaschka and Newell in [**18**] which was obtained by similarity reduction of

the Lax pair of mKdV equation, the Lax pair of equation (1.8) was obtained by similarity reduction of the Lax pair of a q-mKdV equation.

1.6.1. The Main Results. By following a close analogy with Flaschka and Newell's work on the associated linear problems of PII, we have found the determinantal forms of two types of hierarchies of special solutions of q-PII equation (1.8). These results are consistent with the special solutions of equation (1.8) calculated using the Bäcklund transformation in section 1.5.1.

Theorem 1.6.1. q-PII equation (1.8) with parameter $\alpha = \frac{1}{\lambda^{4k}}$ where k are integers admits a hierarchy of rational type special solutions $g_k(x)$, given by

$$g_k(x) = -\frac{ix}{\lambda^{2k}} \frac{\tau_k(x/\lambda)\tau_{k-1}(x/\lambda^2)}{\tau_k(x/\lambda^2)\tau_{k-1}(x/\lambda)}$$

$$\tau_k(x) = \begin{vmatrix} T_k & T_{k+1} & \cdots & & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \cdots & & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ T_2 & T_3 & \cdots & & T_k & T_{k+1} \\ & & \ddots & & \vdots & \vdots \\ & & & \cdots & T_2 & T_3 \\ 0 & \cdots & & \cdots & T_0 & T_1 \end{vmatrix}.$$

$T_k(x)$ is the q discrete polynomial of degree k in x where

$$T_j(\frac{1}{\lambda^{2j}} - 1) = -\frac{i(-i + \lambda x + \lambda^2 x)}{\lambda} T_{j-1} - \frac{x(-i - i\lambda + x\lambda^2)}{\lambda} T_{j-2} + x^2 T_{j-3}$$

$$T_{j-1}(x) = \frac{1}{ix\lambda}(T_j(x/\lambda) - T_j(x))$$

and $T_0(x) = 1$, $T_k = 0, k < 0$. $T_j(x)$ has the generating function $u(\nu, x)$,

$$u(\nu, x) = \frac{1}{(\nu/\lambda; \lambda^{-2})_\infty (ix\nu; \lambda^{-2})_\infty (ix\lambda\nu; \lambda^{-2})_\infty} = \sum_{j=0}^{\infty} T_j(x)\nu^j.$$

Theorem 1.6.2. q-PII equation (1.8) with parameter $\alpha = \frac{1}{\lambda^{4k}}$, where k are half integers, that is $k = n + \frac{1}{2}$, $n = 0, 1, 2, ...$ admits a hierarchy of

hypergeometric type special solutions $g_{n+\frac{1}{2}}(x)$, given by

$$g_{n+\frac{1}{2}}(x)\begin{cases} = -\frac{ix}{\lambda^{2n+1}}\frac{\tau_{n+1}(x/\lambda)\tau_n(x/\lambda^2)}{\tau_{n+1}(x/\lambda^2)\tau_n(x/\lambda)} \text{ for } n \text{ even}, \\ = -\frac{ix}{\lambda^{2n+2}}\frac{\tau_{n+1}(x/\lambda)\tau_n(x/\lambda^2)}{\tau_{n+1}(x/\lambda^2)\tau_n(x/\lambda)} \text{ for } n \text{ odd}, \end{cases}$$

where $\tau_n(x)$ is the determinant of a $n \times n$ matrix

$$\tau_n(x) = \begin{vmatrix} \frac{a_n^{(\frac{1}{2})}}{\lambda^n} & \frac{\overline{a}_n^{(\frac{1}{2})}}{\lambda^n} & & \cdots & \frac{a_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} & \frac{\overline{a}_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} \\ \frac{a_{n-2}^{(\frac{1}{2})}}{\lambda^{n-2}} & \frac{\overline{a}_{n-2}^{(\frac{1}{2})}}{\lambda^{n-2}} & & \cdots & \frac{a_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} & \frac{\overline{a}_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} & \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ 0 & 0 & \cdots & 0 & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix} \text{ for } n \text{ even},$$

$$\tau_n(x) = \begin{vmatrix} \frac{a_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \frac{a_{n+1}^{(\frac{1}{2})}}{\lambda^{n+1}} & \frac{\overline{a}_{n+1}^{(\frac{1}{2})}}{\lambda^{n+1}} & \cdots & \frac{a_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} & \frac{\overline{a}_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} \\ \frac{a_{n-3}^{(\frac{1}{2})}}{\lambda^{n-3}} & \frac{a_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \frac{\overline{a}_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \cdots & \frac{a_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} & \frac{\overline{a}_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} & \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ 0 & 0 & 0 & \cdots & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix} \text{ for } n \text{ odd},$$

where

$$\begin{aligned} \overline{a}_{2j}^{(\frac{1}{2})} &= ix\lambda^2 a_{2j}^{(\frac{1}{2})} - \frac{i}{x\lambda^2}\left(\frac{1}{\lambda^{4j+4}} - 1\right)a_{2j+2}^{(\frac{1}{2})}, \\ \overline{\overline{a}}_{2j}^{(\frac{1}{2})} &= -\frac{1}{\lambda}a_{2j}^{(\frac{1}{2})} - x^2\lambda^3 a_{2j-2}^{(\frac{1}{2})} + \frac{i}{x\lambda}\overline{a}_{2j}^{(\frac{1}{2})}. \end{aligned}$$

In particular $a_0^{(\frac{1}{2})}$ satisfies the q-Airy equation (1.12)

$$\overline{\overline{a}}_0^{(\frac{1}{2})} - \frac{i}{x\lambda}\overline{a}_0^{(\frac{1}{2})} + \frac{a_0^{(\frac{1}{2})}}{\lambda} = 0.$$

1.7. Outline of the thesis

The rest of thesis is organized as follows: in **chapter 2**, we focus on the continuous Painlevé equations using PII as an example. Two important aspects of the Painlevé equations are discussed in detail: the special solutions and the associated linear problems. We describe in detail the method of Flaschka and Newell [**18**] of how special solutions of the non-linear problem, that is the Painlevé equations, can be obtained through the analysis of their associated linear problems.

To study the associated linear problems of q-Painlevé equations, one need the theory of linear analysis in the q-discrete setting. In **chapter 3** we review the theorems of the Birkhoff school [**14, 9, 52**] on the analysis of q-linear systems, and the application on two examples. In **chapter 4**, we apply the theorems to the 2×2 Lax pair [**33**] of q-PII equation (1.8). We show that one can find:

- the special values of the parameters for which q-PII admits special solutions,
- the simplest special solutions of q-PII for both q-rational and q-hypergeometric type,
- the solutions of the linear problems corresponding to the two simplest cases where q-PII admit special solutions,
- the Bäcklund transformation of the solution of the q-PII equation,
- the Schlesinger transformation of the solutions of the linear systems of q-PII.

Finally we deduce the determinantal forms of the hierarchies of special solutions of both rational and hypergeometric type. In **chapter 5** we present a summary and conclusions of this work.

We have included several appendices in this thesis. Appendix A has the list of the six Painlevé equations. Appendix B gives a summary and all the formulae of hypergeometric and q-hypergeometric type functions we have used. In appendix C, we show how to take the continuum limit of q-discrete equations.

CHAPTER 2

The second Painlevé equation

This Chapter is a review on Flaschka and Newell's work in [**18**]. First, we show that PII comes as the compatibility equation of a pair of linear differential systems of equations, that is the Lax pair of PII. The **Spectral** half of the Lax pair is studied analytically around its two singularities, a regular singularity at $x = 0$, and an irregular singularity of rank 3 at $x = \infty$. Formal solutions in the form of series expansions around the singularities are then established. We study **the simplest two** of the special cases when the linear systems can be solved exactly, namely for $a = 0$ and $\frac{1}{2}$. These two cases correspond in turn to PII admitting some simple special solutions of rational and hypergeometric type, respectively. Furthermore, we show that the solutions of the linear system for all of the special values of the parameter of PII (that is a are integers, or half integers) can be built on the first two exact solutions. Determinantal forms of the hierarchies of some special solutions of PII for both the rational type and the hypergeometric type can then be obtained using this fact.

We have made the derivation of some expressions more explicit and provided the complete derivation of the determinantal forms of the solutions; the latter were not included in Flaschka and Newell's paper [**18**]. In particular, we have found the Schlesinger transformation of the associated linear problem of PII

$$\Phi_1^{(a+1)}(x,t) = L_a(x,t)\Phi_1^{(a)}(x,t),$$

which is used to obtain the closed form of the $\Phi_1^{(a+1)}(x,t)$ in terms of the two simplest exact solutions of the linear problem: $\Phi_1^{(0)}(x,t)$ and $\Phi_1^{(\frac{1}{2})}(x,t)$, which proves the form of the solution $\Phi_1^{(a+1)}(x,t)$ (proposition 2.5.4 in this thesis; equation (3.53) in [**18**]) used by Flaschka and Newell to obtain the determinantal form of the hypergeometric type special solutions for when a are half integers. We have also showed in proposition 2.5.5 that the determinantal form that results from Flaschka and Newell's approach is the same

as the determinantal form, equation (3) in Kajiwara's paper, derived using the bilinear forms of PII.

2.1. The iso-monodromy deformation problem of PII

The iso-monodromy deformation problem :
When analyzing a linear differential equation we look its singularity structure. For example, where are the singularities and of which kind they are. Formal solutions of certain forms can be constructed around the singularities according to their types. Deformation of the linear system with respect to a parameter (a constant) of the linear system is monodromy preserving if:

a) the Stokes multipliers associated with formal solutions about an irregular singular point,
b) the monodromy matrix about a regular singular point, and
c) the matrix connecting the formal solutions around different singularities

remain unchanged.
The iso-monodromy deformation problem of PII, consists of **the spectral problem**

$$\begin{aligned}\Psi_x &= M(x,t)\Psi \qquad (2.1)\\ &= \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \frac{1}{x} + \begin{pmatrix} -(2f(t)^2+t) & 2if'(t) \\ -2if'(t) & (2f(t)^2+t) \end{pmatrix} + \begin{pmatrix} 0 & 4f(t) \\ 4f(t) & 0 \end{pmatrix} x \right.\\ &\qquad \left. + \begin{pmatrix} -i4 & 0 \\ 0 & i4 \end{pmatrix} x^2 \right\} \Psi, \quad {}_x = \frac{d}{dx},\end{aligned}$$

and **the deformation problem**

$$\Psi_t = N(x,t)\Psi = \begin{pmatrix} -ix & f(t) \\ f(t) & ix \end{pmatrix} \Psi, \quad {}_t = \frac{d}{dt}, \qquad (2.2)$$

which describes the deformation of $\Psi(x,t)$ in t.
From the form of (2.1) we see that the spectral problem (2.1) has two singular points, $x = \infty$ (irregular of rank three) and $x = 0$ (regular). Monodromy data are calculated around these two points. The deformation, or differentiation equation (2.2) with respect to t, preserves the monodromy found in

equation (2.1) [**18**]. The compatibility of the two linear systems

$$\Psi_{tx} = \Psi_{xt} \quad \Rightarrow \quad M_t - N_x = NM - MN$$

forces $f(t)$ to satisfy a second-order non-linear ODE. In this case

$$f''(t) = 2f(t)^3 + tf(t) - a \qquad ' = \frac{d}{dt}, \tag{2.3}$$

which is the second Painlevé equation (PII). Hence t, the deformation parameter is referred to as the Painlevé variable, x as the spectral variable. The pair of linear systems (2.1, 2.2) is also called the Lax pair or the associated linear problems of PII.

We will concentrate our linear analysis on equation (2.1), the spectral half of the Lax pair . This is because the coefficient matrix $M(x,t)$ of the spectral problem (2.1) has polynomial dependence on the spectral variable x whereas the deformation equation depends transcendentally on its variable t (via $f(t)$ the solution of PII). Therefore, to analyze $\Psi(x,t)$ in the t plane we would have to know how $f(t)$ behaves which is precisely what we are trying to find out in the first place!

2.2. Linear analysis of the spectral problem

Solutions of a $n \times n$ linear differential system of equations have prescribed form of expansions around the singularities. Near a regular singular point $x = 0$, the solutions are given by

$$x^{\rho_i} \sum_{j=0}^{\infty} a_{ij} x^j \tag{2.4}$$

where ρ_i ($i = 1, ..., n$) are the eigenvalues of the coefficient matrix at $x = 0$. For an irregular singular point at infinity, the solutions have the form

$$e^{\sum_{k=1}^{n} \omega_k x^k} x^l \sum_{j=0}^{\infty} \alpha_{ij} \frac{1}{x^j} \tag{2.5}$$

where n is the rank of the irregularity at $x = \infty$, ω_k, l are constants.

2.2.1. Asymptotic expansions at $x = 0$. Let the 2×2 fundamental solution matrix of the Lax pair (2.1, 2.2) at $x = 0$ be $\Phi(x,t) = \{\phi_1, \phi_2\}$.

Proposition 2.2.1. The two linearly independent vector solutions have Frobenius series expansions at $x = 0$ of the form

$$\begin{aligned}\phi_1(x,t) &= x^a \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j} x^{2j} \\ c_{2j+1} x^{2j+1} \end{pmatrix} \qquad (2.6)\\ &\sim e^{f(t)} x^a \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i(2f' - 2f^2 - t)}{(1+2a)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} x + \ldots \right\},\end{aligned}$$

$$\begin{aligned}\phi_2(x,t) &= x^{-a} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} b_{2j+1} x^{2j+1} \\ d_{2j} x^{2j} \end{pmatrix} \qquad (2.7)\\ &\sim e^{-f(t)} x^{-a} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{i(2f' + 2f^2 + t)}{(1-2a)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \ldots \right\},\end{aligned}$$

where the expansion coefficients satisfy the recurrence relations

$$ja_j = -i(2f(t)^2 + t + 2f'(t))c_{j-1} + 4f(t)a_{j-2} - 4ic_{j-3} \qquad (2.8)$$

$$(j+2a)c_j = i(2f'(t) - 2f(t)^2 - t)a_{j-1} - 4f(t)c_{j-2} - 4ia_{j-3} \qquad (2.9)$$

$$(j-2a)b_j = -i(2f'(t) + 2f(t)^2 + t)d_{j-1} + 4f(t)b_{j-2} - 4id_{j-3} \qquad (2.10)$$

$$jd_j = i(2f(t)^2 - t - 2f'(t))b_{j-1} - 4f(t)d_{j-2} - 4ib_{j-3}. \qquad (2.11)$$

and

$$a_{\text{odd}} = d_{\text{odd}} = 0 \qquad \text{and} \qquad c_{\text{even}} = b_{\text{even}} = 0. \qquad (2.12)$$

In particular

$$\frac{a_0'(t)}{a_0(t)} = f(t). \qquad (2.13)$$

Proof. Linear problem (2.1) has a regular singularity at $x = 0$, hence the solutions take the form of expansion (2.4). First we find ρ_1 and ρ_2, the two eigenvalues of the coefficient matrix $M(x,t)$ of equation (2.1) near $x = 0$. Since $M(x,t)$ is off-diagonal at $x = 0$, we diagonalize $M(x,t)$ by conjugation with $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Let

$$\Phi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Phi_1, \qquad (2.14)$$

then the Lax pair becomes

$$\begin{aligned}\Phi_{1x} &= \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \frac{1}{x} + \begin{pmatrix} 0 & -i(2f(t)^2 + t + 2f'(t)) \\ i(2f'(t) - 2f(t)^2 - t) & 0 \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} 4f(t) & 0 \\ 0 & -4f(t) \end{pmatrix} x + \begin{pmatrix} 0 & -i4 \\ -i4 & 0 \end{pmatrix} x^2 \right\} \Phi_1 \qquad (2.15)\end{aligned}$$

$$\Phi_{1t} = \begin{pmatrix} f(t) & -ix \\ -ix & -f(t) \end{pmatrix} \Phi_1. \qquad (2.16)$$

We found $\rho_{1,2} = a, -a$. Therefore the solution matrix $\Phi_1 = \{\phi_{11}, \phi_{12}\}$ has the Frobenius series expansion of the form

$$\Phi_1(x,t) = \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}x^{2j} & b_{2j+1}x^{2j+1} \\ c_{2j+1}x^{2j+1} & d_{2j}x^{2j} \end{pmatrix} \begin{pmatrix} x^a & 0 \\ 0 & x^{-a} \end{pmatrix} \qquad (2.17)$$

The two vector solutions of the spectral equation (2.15) have the leading behaviour x^a and x^{-a}, respectively. Note that a is also the parameter of PII equation (2.3). The coefficients a_j, b_j, c_j, d_j are in general functions of the Painlevé variable t and satisfy recurrence relations which are found by substituting solution (2.17) into the spectral equation (2.15) and equating powers of x around $x = 0$, we have

$$\begin{aligned} ja_j &= -i(2f(t)^2 + t + 2f'(t))c_{j-1} + 4f(t)a_{j-2} - 4ic_{j-3} \\ (j+2a)c_j &= i(2f'(t) - 2f(t)^2 - t)a_{j-1} - 4f(t)c_{j-2} - 4ia_{j-3} \\ (j-2a)b_j &= -i(2f'(t) + 2f(t)^2 + t)d_{j-1} + 4f(t)b_{j-2} - 4id_{j-3} \\ jd_j &= +i(2f(t)^2 - t - 2f'(t))b_{j-1} - 4f(t)d_{j-2} - 4ib_{j-3}, \end{aligned}$$

For $a \neq 0$, the recurrence relations at $j = 0$ are

$$\begin{aligned} 0 \times a_0 &= 0 \\ 2a \times c_0 &= 0 \\ -2a \times b_0 &= 0 \\ 0 \times d_0 &= 0. \end{aligned}$$

That is $b_0 = c_0 = 0$, while a_0 and d_0 are arbitrary constants (constant with respect to x, and in general are functions of t). Furthermore,

$$a_{\text{odd}} = d_{\text{odd}} = 0 \qquad \text{and} \qquad c_{\text{even}} = b_{\text{even}} = 0,$$

that is

$$\phi_{11}(x,t) \sim a_0 x^a \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_1/a_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} x + a_2/a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} x^2 + ... \right\}, \tag{2.18}$$

$$\phi_{12}(x,t) \sim d_0 x^{-a} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_1/d_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + d_2/d_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} x^2 + ... \right\}. \tag{2.19}$$

The coefficients c_1/a_0 and b_1/d_0 can be calculated from the recurrence relations (2.9) and (2.10) respectively for $j = 1$,

$$\begin{aligned} \frac{c_1}{a_0} &= \frac{i(2f' - 2f^2 - t)}{(1+2a)}, \\ \frac{b_1}{d_0} &= \frac{i(2f' + 2f^2 + t)}{(1-2a)}. \end{aligned}$$

The rest of the coefficients of the Frobenius expansion (2.17) can be calculated in a similar manner. As stated earlier the coefficients a_j, b_j, c_j, d_j in general are functions of t. To find how they depend on t we need to make use of the other half of the Lax pair, **deformation equation** (2.16). We substitute solution (2.17) into equation (2.16) and equating the leading behaviour at $x = 0$, to leading order yields

$$\frac{a_0'}{a_0} = \frac{d}{dt} \ln a_0(t) = f(t) \quad \Rightarrow \quad a_0(t) = e^{f(t)}, \tag{2.20}$$

$$\frac{d_0'}{d_0} = \frac{d}{dt} \ln d_0(t) = -f(t) \quad \Rightarrow \quad d_0(t) = e^{-f(t)}. \tag{2.21}$$

Equations (2.20, 2.21) relate $f(t)$ (the solution of PII) to the leading behaviours $a_0(t)$ and $d_0(t)$ of the solutions of the corresponding associated linear problems. This fact will be used to obtain the determinantal forms of the hierarchies of special solutions of PII in the later section. □

2.2.2. Expansions at $x = \infty$.

Proposition 2.2.2. Let $\Psi(x,t) = \{\psi_1, \psi_2\}$ be the fundamental solution matrix of the Lax pair (2.1, 2.2) at $x = \infty$. The two vector solutions at

$x=\infty$ are of the form:

$$\psi_1(x,t)=e^{-i(\frac{4}{3}x^3+tx)}\sum_{k=0}^{\infty}\begin{pmatrix}\alpha_k(t)\\ \gamma_k(t)\end{pmatrix}\frac{1}{x^k}, \tag{2.22}$$

$$\psi_2(x,t)=e^{i(\frac{4}{3}x^3+tx)}\sum_{k=0}^{\infty}\begin{pmatrix}\beta_k(t)\\ \delta_k(t)\end{pmatrix}\frac{1}{x^k}, \tag{2.23}$$

where the coefficients $\alpha_k,\beta_k,\gamma_k,\delta_k$ satisfy the recurrence relations:

$$4f\gamma_{k-1}=-2if^2\alpha_{k-2}-2if'\gamma_{k-2}-(k-3)\alpha_{k-3}-a\gamma_{k-3}, \tag{2.24}$$

$$\begin{aligned}8i\gamma_k \;=\; &-4f\alpha_{k-1}+2if'\alpha_{k-2}-(2if^2+2it)\gamma_{k-2} \\ &-(k-3)\gamma_{k-3}-a\alpha_{k-3},\end{aligned} \tag{2.25}$$

$$\begin{aligned}-8i\beta_k=&-4f\delta_{k-1}-2if'\delta_{k-2}+(2if^2+2it)\beta_{k-2} \\ &-(k-3)\beta_{k-3}-a\delta_{k-3},\end{aligned} \tag{2.26}$$

$$4f\beta_{k-1}=-2if^2\delta_{k-2}+2if'\beta_{k-2}-(k-3)\delta_{k-3}-a\beta_{k-3}, \tag{2.27}$$

in particular

$$\gamma_1 \;=\; \frac{if(t)}{2}, \qquad \beta_1=-\frac{if(t)}{2}, \tag{2.28}$$

$$\alpha_{1t} \;=\; -\frac{if(t)^2}{2}, \qquad \delta_{1t}=\frac{if(t)^2}{2}. \tag{2.29}$$

Proof. The spectral equation (2.1) has an irregular singular point of rank three at $x=\infty$. Therefore solutions are of the forms given by (2.5). On substituting expansion (2.5) into equation (2.1) we found that $\omega_3=\pm i\frac{4}{3}$, $\omega_2=0$, $\omega_1=\pm it$, and $l=0$. That is the solution matrix $\Psi(x,t)$ near $x=\infty$ has the formal expansion

$$\Psi(x,t)=\sum_{k=0}^{\infty}\begin{pmatrix}\alpha_k(t) & \beta_k(t)\\ \gamma_k(t) & \delta_k(t)\end{pmatrix}\frac{1}{x^k}\begin{pmatrix}e^{-i(\frac{4}{3}x^3+tx)} & 0\\ 0 & e^{i(\frac{4}{3}x^3+tx)}\end{pmatrix}. \tag{2.30}$$

To find the recurrence relations for the coefficients $\alpha_k,\beta_k,\gamma_k,\delta_k$ we substitute solution (2.30) into the spectral equation (2.1) and equating powers of

$\frac{1}{x}$ at $x = \infty$, we have

$$\begin{aligned}
4f\gamma_{k-1} &= -2if^2\alpha_{k-2} - 2if'\gamma_{k-2} - (k-3)\alpha_{k-3} - a\gamma_{k-3},\\
8i\gamma_k &= -4f\alpha_{k-1} + 2if'\alpha_{k-2} - (2if^2 + 2it)\gamma_{k-2}\\
&\quad -(k-3)\gamma_{k-3} - a\alpha_{k-3},\\
-8i\beta_k &= -4f\delta_{k-1} - 2if'\delta_{k-2} + (2if^2 + 2it)\beta_{k-2}\\
&\quad -(k-3)\beta_{k-3} - a\delta_{k-3},\\
4f\beta_{k-1} &= -2if^2\delta_{k-2} + 2if'\beta_{k-2} - (k-3)\delta_{k-3} - a\beta_{k-3}.
\end{aligned}$$

Equation (2.24) and (2.27) at $k = 1$ tells us that $\gamma_0 = 0$ and $\beta_0 = 0$ respectively. In general $\alpha_k, \beta_k, \gamma_k, \delta_k$ are functions of t. To find the t dependence we substitute solution (2.30) into the deformation equation (2.2) and equate powers of $\frac{1}{x}$ at $x = \infty$. We found that α_0 and δ_0 are constants, that is they are not functions of t, and are normalized to be 1. Equations (2.25) and (2.26) at $k = 1$ gives us

$$\gamma_1 = \frac{if(t)}{2}, \qquad \beta_1 = -\frac{if(t)}{2}.$$

We have also found

$$\alpha_{1t} = -\frac{if(t)^2}{2}, \qquad \delta_{1t} = \frac{if(t)^2}{2},$$

from the deformation equation. The rest of the coefficients $\alpha_k, \beta_k, \gamma_k$ and δ_k can be calculated similarly. □

2.3. Special solutions

To find the general solution of the associated linear problems of PII, that is to solve the Riemann-Hilbert (RH) problem of the associated linear problems of PII is not trivial. In fact the associated linear problems are no easier to solve than the non-linear ODE which they are related to. This makes sense, since otherwise the associated linear problems could not possibly be a representation of the transcendental Painlevé equation. However there are some special cases when the associated linear problems can be solved. We found that these cases correspond to PII admitting some special solutions. We know from the analytic theory of linear differential equations (see [**8**] for example), that solutions of the form (2.6, 2.7) at $x = 0$ are not valid in

general when the difference of the leading powers $2a$ is an integer. Inconsistencies can arise in either of the recurrence relations (2.9) or (2.10) when $2a$ is an integer, leaving c_{2a} or b_{2a} undefined corresponding to whether $2a$ a negative or a positive integer respectively. When this happens one of the solutions (2.6) or (2.7) is no longer valid. Since PII possesses the symmetry, $f(t) \to -f(t) \Rightarrow a \to -a$, we only need to consider $a > 0$ here.

The case $2a$ is an integer separates into two types: (1) $2a$ even, (2) $2a$ odd. Let us consider type 1 first. For $2a$ even, we see that when the LHS of equation (2.10) vanishes at $j = 2a$, the RHS also vanishes since $d_{\text{odd}} = 0$ and $b_{\text{even}} = 0$. Hence for $2a$ even, solutions (2.6) and (2.7) are still the valid forms of the expansions of the vector solutions at $x = 0$. The simplest example of this type is when $a = 0$.

2.3.1. The simplest rational type special solution.

Proposition 2.3.1. A vector solution of the corresponding associated linear problem when $a = 0$ and PII is solved by $f(t) = 0$ is given by

$$\psi_1(x,t) = \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} \\ e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} (-1)^j T_j(t) x^j \\ T_j(t) x^j \end{pmatrix}. \tag{2.31}$$

Proof. It is easy to see that when $a = 0$, $f(t) = 0$ is a solution of PII. The Lax pair (2.1, 2.2) in this case reduces to:

$$\Psi_x = \begin{pmatrix} -i(4x^2+t) & 0 \\ 0 & i(4x^2+t) \end{pmatrix} \Psi, \tag{2.32}$$

$$\Psi_t = \begin{pmatrix} -ix & 0 \\ 0 & ix \end{pmatrix} \Psi. \tag{2.33}$$

They can be solved easily, since now the second-order system has decoupled into two first order equations. A solution of Lax pair (2.32, 2.33) is

$$\psi_1(x,t) = \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} \\ e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix}.$$

Proposition 2.3.2. The function $e^{i(\frac{4}{3}x^3+tx)}$ has series expression near $x = 0$. The coefficients of the expansion are found to be orthogonal polynomials

of Laguerre type

$$\sum_{j=0}^{\infty} T_j(t)x^j = \exp\left(i\frac{4}{3}x^3 + itx\right)$$

where

$$jT_j = 4iT_{j-3} + itT_{j-1}, \tag{2.34}$$

$$\frac{dT_j}{dt} = iT_{j-1}, \tag{2.35}$$

$T_0 = 1$, and $T_j = 0, j < 0$.

Proof. Relations (2.34, 2.35) can be obtained using the definition of $T_j(t)$ as follows. Let

$$\exp\left(i\frac{4}{3}x^3 + itx\right) = e^{\theta(x,t)},$$

differentiate with respect to x gives

$$\begin{aligned} \frac{d}{dx}e^{\theta(x,t)} &= i(4x^2 + t)e^{\theta(x,t)} \\ \Rightarrow jT_j &= 4iT_{j-3} + itT_{j-1}, \end{aligned}$$

and differentiate with respect to t gives

$$\begin{aligned} \frac{d}{dt}e^{\theta(x,t)} &= ixe^{\theta(x,t)} \\ \Rightarrow \frac{dT_j}{dt} &= iT_{j-1}. \end{aligned}$$

A few examples of T_j are

$$T_0 = 1\ ,\ T_1 = it\ ,\ T_2 = -\frac{t^2}{2}\ ,\ T_3 = \frac{4i}{3} - \frac{it^3}{6}.$$

□

Hence

$$\psi_1(x,t) = \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} \\ e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} (-1)^j T_j(t)x^j \\ T_j(t)x^j \end{pmatrix}.$$

□

Type (2): for $2a$ odd, while the LHS of equation (2.10) is zero at $j = 2a$, the RHS is in general not zero. Unlike the $2a$ even case, here $d_{\text{even}} \neq 0$ and $b_{\text{odd}} \neq 0$. Hence when $j = 2a$ is odd, extra constraints are needed on the coefficients which are expressions involving $f(t)$ for the RHS of equation

(2.10) to vanish, to ensure that no inconsistency arises and solutions $\phi_1(x,t)$ and $\phi_2(x,t)$ of the form (2.6) and (2.7) remain valid. Let us consider the simplest example of this type.

2.3.2. The simplest hypergeometric type special solution. The simplest example for $2a$ odd is $2a = 1$. In this case equation (2.10) at $j = 1$ gives

$$0 \times b_1 \quad = \quad -i(2f(t)^2 + t + 2f'(t))d_0.$$

Since $d_0 \neq 0$, it follows that in order for the RHS to vanish $f(t)$ has to satisfy a Riccati type equation

$$f'(t) = -f^2 - \frac{t}{2}. \tag{2.36}$$

In other words PII (2.3) has reduced to a first order non-linear ODE when $a = \frac{1}{2}$. What about the solution of the corresponding linear system? Firstly, we know from equation (2.13) that $f(t) = \frac{a_0'}{a_0}$. Substituting this into the Riccati equation (2.36) we obtain an equation for $a_0(t)$,

$$a_0'' = -\frac{t}{2}a_0. \tag{2.37}$$

That is $a_0(t)$, the coefficient of the leading behaviour of vector solution (2.6) of the linear problems near $x = 0$, is a Airy type function with respect to t. That is

$$a_0(t) = A(2^{-\frac{1}{3}}t),$$

where $A(2^{-\frac{1}{3}}t)$ is a solution of the Airy equation (1.1) with respect to the variable $2^{-\frac{1}{3}}t$. Hence a solution of PII equation (2.3) when $a = \frac{1}{2}$ denoted $f_{\frac{1}{2}}(t)$ is

$$f_{\frac{1}{2}}(t) = \frac{A(2^{-\frac{1}{3}}t)'}{A(2^{-\frac{1}{3}}t)}. \tag{2.38}$$

Proposition 2.3.3. A solution of the linear problems (2.1, 2.2) when $a = \frac{1}{2}$ and $f(t)$ defined by (2.38) satisfies Riccati equation (2.36) is given by:

$$\begin{aligned} \phi_1(x,t) &= x^{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-if(t)}{x} & \frac{i}{x} \end{pmatrix} \begin{pmatrix} A(z) \\ A_t(z) \end{pmatrix} \\ &= x^{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-if(t)}{x} & \frac{i}{x} \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(t)x^{2j} \\ a_{2j}(t)'x^{2j} \end{pmatrix} \end{aligned} \tag{2.39}$$

where $z(x,t) = 2^{\frac{2}{3}}x^2 + 2^{-\frac{1}{3}}t$ and $A(z)$ is the solution of the Airy equation, that is

$$\frac{d^2A(z)}{dz^2} = -zA(z).$$

Function $A(z)$ has the series expansion around $x = 0$

$$A(z) = \sum_{j=0}^{\infty} a_{2j}(t)x^{2j}, \tag{2.40}$$

where a_{2j} are defined by two recurrence type relations:

$$a_{2j} = \frac{2}{j}\frac{da_{2j-2}}{dt}, \tag{2.41}$$

$$\frac{d^2a_{2j}}{dt^2} = -\frac{t}{2}a_{2j} - a_{2j-2}, \tag{2.42}$$

in particular,

$$\frac{d^2a_0}{dt^2} = -\frac{t}{2}a_0. \tag{2.43}$$

Proof. We use the asymptotic behaviour of $\phi_1(x,t)$ at $x = 0$ as a guide as to how to find the solution in the closed form. Recall in order to obtain the leading behavior of the solution of the linear problem (2.1) we need to diagonalize the coefficient matrix $M(x,t)$ at $x = 0$.This gives us the first transformation:

$$\Phi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Phi_1 \tag{2.44}$$

where

$$\begin{aligned} \Phi_{1x} = &\left\{ \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \frac{1}{x} + \begin{pmatrix} 0 & 0 \\ i4f'(t) & 0 \end{pmatrix} \right. \\ &\left. + \begin{pmatrix} 4f(t) & 0 \\ 0 & -4f(t) \end{pmatrix} x + \begin{pmatrix} 0 & -i4 \\ -i4 & 0 \end{pmatrix} x^2 \right\} \Phi_1, \end{aligned} \tag{2.45}$$

$$\Phi_{1t} = \begin{pmatrix} f(t) & -ix \\ -ix & -f(t) \end{pmatrix} \Phi_1. \tag{2.46}$$

We know from equation (2.17) that a vector solution of the Lax pair (2.45, 2.46) with asymptotic behaviour $x^{\frac{1}{2}}$, that is the first column of the solution

matrix $\Phi_1 = \{\phi_{11}, \phi_{12}\}$ is given by :

$$\phi_{11}(x,t) = x^{\frac{1}{2}} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(t)x^{2j} \\ c_{2j+1}(t)x^{2j+1} \end{pmatrix}$$

where $a_0(t)$ satisfies the Airy type equation (2.37). This indicates the next transformation

$$\Phi_1 = x^{\frac{1}{2}}\Phi_2, \tag{2.47}$$

where the Lax pair is now

$$\begin{aligned} \Phi_{2x} = & \left\{ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{x} + \begin{pmatrix} 0 & 0 \\ i4f'(t) & 0 \end{pmatrix} \right. \\ & \left. + \begin{pmatrix} 4f(t) & 0 \\ 0 & -4f(t) \end{pmatrix} x + \begin{pmatrix} 0 & -i4 \\ -i4 & 0 \end{pmatrix} x^2 \right\} \Phi_2, \end{aligned} \tag{2.48}$$

$$\Phi_{2t} = \begin{pmatrix} f(t) & -ix \\ -ix & -f(t) \end{pmatrix} \Phi_2. \tag{2.49}$$

Let a solution of the Lax pair (2.48, 2.49) be

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}x^{2j} \\ c_{2j+1}x^{2j+1} \end{pmatrix},$$

then $\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$ satisfies the Lax pair of equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 4fx & -4ix^2 \\ 4if' - 4ix^2 & -\frac{1}{x} - 4fx \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.50}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} f(t) & -ix \\ -ix & -f(t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.51}$$

We observe that the top entry of vector equation (2.50) is

$$v = \frac{u_x - 4f(t)xu}{-i4x^2}.$$

This tells us how $v(x,t)$ is related to $u(x,t)$. We want to transform to a new "$v(x,t)$", which relates to $u(x,t)$ more simply. Since we know

$$u(x,t) = \sum_{j=0}^{\infty} a_{2j}x^{2j},$$

and

$$\frac{u_x}{x} = \sum_{j=0}^{\infty} (2j+2) a_{2j+2} x^{2j},$$

we choose v_1 to be $\frac{u_x}{4x}$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-if(t)}{x} & \frac{i}{x} \end{pmatrix} \begin{pmatrix} u \\ v_1 \end{pmatrix}, \tag{2.52}$$

where $\begin{pmatrix} u \\ v_1 \end{pmatrix}$ satisfies the Lax pair

$$\Phi_{3x} = \begin{pmatrix} 0 & 4x \\ -x(2t+4x^2) & 0 \end{pmatrix} \Phi_3 \tag{2.53}$$

$$\Phi_{3t} = \begin{pmatrix} 0 & 1 \\ -(t/2+x^2) & 0 \end{pmatrix} \Psi_3. \tag{2.54}$$

We see that transform (2.52) has significantly simplified the Lax pair. For instance, equations (2.53, 2.54) no longer have $f(t)$ (that is the solution of PII) in their coefficient matrices. The Lax pair (2.53, 2.54) give us two equations for $u(x,t)$:

$$\frac{du}{dt}(x,t) = \frac{1}{4x}\frac{du}{dx}(x,t) \tag{2.55}$$

and

$$u_{tt} = -(t/2+x^2)u. \tag{2.56}$$

Equation (2.55) says a solution of the Lax pair (2.53) and (2.54) is

$$\begin{pmatrix} u \\ v_1 \end{pmatrix} = \begin{pmatrix} u \\ \frac{u_x}{4x} \end{pmatrix} = \begin{pmatrix} u \\ \frac{du}{dt} \end{pmatrix}.$$

In particular it suggests the change of variable

$$u(x,t) = g\left(z(\alpha x^2 + \beta t)\right), \quad \text{where } \alpha, \beta \text{ are constants.} \tag{2.57}$$

Since

$$\begin{aligned} u_x &= g_z 2\alpha x, \\ u_t &= g_z \beta, \end{aligned}$$

satisfy equation (2.55) for some choice of α and β. Now, substitute solution (2.57) into equation (2.56)

$$\begin{aligned} u_{tt} &= -(t/2 + x^2)u \\ \beta^2 g_{zz} &= -(t/2 + x^2)g \\ g_{zz} &= -\left(\frac{1}{\beta^2}x^2 + \frac{t}{2\beta^2}\right)g \\ &= -(\alpha x^2 + \beta t)g \\ &= -zg. \end{aligned}$$

For the last line to be true

$$\begin{aligned} \alpha &= \frac{1}{\beta^2} \qquad \text{and} \qquad \beta = \frac{1}{2\beta^2}, \\ \Rightarrow \qquad \alpha &= 2^{\frac{2}{3}} \qquad \text{and} \qquad \beta = 2^{-\frac{1}{3}}. \end{aligned}$$

This says that $u(x,t) = g\left(z(x,t)\right)$ satisfy the Airy equation (1.1) with respect to z, if the new variable is

$$z(x,t) = \alpha x^2 + \beta t = 2^{\frac{2}{3}}x^2 + 2^{-\frac{1}{3}}t,$$

that is

$$u(x,t) = A(z),$$

for

$$z(x,t) = 2^{\frac{2}{3}}x^2 + 2^{-\frac{1}{3}}t.$$

A vector solution of equations (2.53, 2.54) in closed form is then :

$$\begin{pmatrix} u \\ v_1 \end{pmatrix} = \begin{pmatrix} u \\ \frac{du}{dt} \end{pmatrix} = \begin{pmatrix} A(z) \\ A_t(z) \end{pmatrix} = \begin{pmatrix} A(z) \\ 2^{-\frac{1}{3}}A_z(z) \end{pmatrix}.$$

Finally we use transformations (2.44, 2.47, 2.52) to go back to the solution of the original Lax pair (2.1, 2.2)

$$\begin{aligned}
\phi_1 &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \phi_{11} \\
&= x^{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= x^{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-if(t)}{x} & \frac{i}{x} \end{pmatrix} \begin{pmatrix} u \\ v_1 \end{pmatrix} \\
&= x^{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-if(t)}{x} & \frac{i}{x} \end{pmatrix} \begin{pmatrix} u \\ \frac{du}{dt} \end{pmatrix}
\end{aligned}$$

where $u(x,t) = A(z)$. Now

$$u(x,t) = A(z) = \sum_{j=0}^{\infty} a_{2j}(t) x^{2j},$$

so that equations (2.55) and (2.56) for $u(x,t)$:

$$\begin{aligned}
\frac{du}{dt} &= \frac{1}{4x} \frac{du}{dx} \\
\frac{d^2u}{dt^2} &= -(t/2 + x^2)u
\end{aligned}$$

give us two recurrence type relations for a_{2j}:

$$\begin{aligned}
a_{2j} &= \frac{2}{j} \frac{da_{2j-2}}{dt}, \\
\frac{d^2 a_{2j}}{dt^2} &= -\frac{t}{2} a_{2j} - a_{2j-2},
\end{aligned}$$

in particular,

$$\frac{d^2 a_0}{dt^2} = -\frac{t}{2} a_0.$$

□

2.4. Schlesinger transformations of the linear problem

Recall the solution of the original Lax pair (2.1, 2.2) is $\Phi(x,t)$ and to analyze the linear system (2.1) around $x=0$ we let

$$\Phi(x,t)=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\Phi_1(x,t),$$

where $\Phi_1(x,t)$ satisfies the Lax pair (2.15, 2.16). Let us denote the solution of PII (2.3) with parameter a as $f_a(t)$ and the corresponding linear problems by the superscript $^{(a)}$

$$\begin{aligned}\Phi^{(a)}_{1x} &= \left\{\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}\frac{1}{x}+\begin{pmatrix}0 & -i(2f_a^2+t+2f_a')\\ i(2f_a'-2f_a^2-t) & 0\end{pmatrix}\right.\\ &\quad\left.+\begin{pmatrix}4f_a & 0\\ 0 & -4f_a\end{pmatrix}x+\begin{pmatrix}0 & -i4\\ -i4 & 0\end{pmatrix}x^2\right\}\Phi_1^{(a)} \qquad (2.58)\\ \Phi^{(a)}_{1t} &= \begin{pmatrix}f_a & -ix\\ -ix & -f_a\end{pmatrix}\Phi_1^{(a)}. \qquad (2.59)\end{aligned}$$

We would like to find $L_a(x,t)$ such that

$$\Phi_1^{(a+1)}(x,t)=L_a(x,t)\Phi_1^{(a)}(x,t).$$

This is referred to as the Schlesinger transformation of the linear system (2.58). Let

$$\begin{pmatrix}u^{(a)}(x,t)\\ v^{(a)}(x,t)\end{pmatrix}=x^a\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}^{(a)}(t)x^{2j}\\ c_{2j+1}^{(a)}(t)x^{2j+1}\end{pmatrix} \qquad (2.60)$$

be the vector solution of the Lax pair (2.58, 2.59) with the leading order x^a near $x=0$, that is the first column of solution matrix (2.17). It has the asymptotic behavior

$$\begin{pmatrix}u^{(a)}(x,t)\\ v^{(a)}(x,t)\end{pmatrix}\sim x^a\begin{pmatrix}a_0^{(a)}(t)\\ c_1^{(a)}(t)x\end{pmatrix},\qquad x\to 0. \qquad (2.61)$$

Then the solution of the linear system with parameter $a+1$ and $-(a+1)$, where the corresponding PII equation is solved by $f_{a+1}(t)$ is

$$\begin{pmatrix}u^{(a+1)}(x,t)\\ v^{(a+1)}(x,t)\end{pmatrix}=x^{a+1}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}^{(a+1)}(t)x^{2j}\\ c_{2j+1}^{(a+1)}(t)x^{2j+1}\end{pmatrix}, \qquad (2.62)$$

with the asymptotic behaviour

$$\begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix} \sim x^{a+1} \begin{pmatrix} a_0^{(a+1)}(t) \\ c_1^{(a+1)}(t)x \end{pmatrix}. \tag{2.63}$$

Note that $v^{(a)}(x,t)$ has the same leading order in x as $u^{(a+1)}(x,t)$, namely x^{a+1}. This observation has led us to investigate whether $v^{(a)}(x,t)$ and $u^{(a+1)}(x,t)$ are simply related. In fact, we will show that $v^{(a)}(x,t) = u^{(a+1)}(x,t)$, but first we prove that their respective leading coefficients $c_1^{(a)}(t)$ and $a_0^{(a+1)}(t)$ are the same function of t.

Proposition 2.4.1.

$$c_1^{(a)}(t) = a_0^{(a+1)}(t)$$

Proof. Recall equation (2.13) relates the solution of PII to the leading coefficient of the expansion of the solution of the linear problems at $x = 0$ $a_0(t)$, that is

$$f_a(t) = \frac{a_0^{(a)\,\prime}}{a_0^{(a)}}, \qquad ' = \frac{d}{dt}. \tag{2.64}$$

First we will use the recurrence relation for $c_1^{(a)}(t)$ and Bäcklund transformation of PII to show

$$\frac{a_0^{(a+1)\,\prime}}{a_0^{(a+1)}} = f_{a+1}(t) = \frac{c_1^{(a)\,\prime}}{c_1^{(a)}},$$

that is $a_0^{(a+1)}(t)$ is proportional to $c_1^{(a)}(t)$. The recurrence relation (2.9) for $c_1^{(a)}(t)$ gives

$$c_1^{(a)} = \frac{i(2f_a' - 2f_a^2 - t)}{(1+2a)} a_0^{(a)},$$

then

$$\begin{aligned} \frac{c_1^{(a)\,\prime}}{c_1^{(a)}} &= \frac{\left((f_a' - f_a^2 - \frac{t}{2})a_0^{(a)}\right)'}{(f_a' - f_a^2 - \frac{t}{2})a_0^{(a)}} \\ &= \frac{(f_a' - f_a^2 - \frac{t}{2})' a_0^{(a)}}{(f_a' - f_a^2 - \frac{t}{2})a_0^{(a)}} + \frac{(f_a' - f_a^2 - \frac{t}{2})a_0^{(a)\,\prime}}{(f_a' - f_a^2 - \frac{t}{2})a_0^{(a)}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(f_a' - f_a^2 - \frac{t}{2})'}{(f_a' - f_a^2 - \frac{t}{2})} + \frac{a_0^{(a)\prime}}{a_0^{(a)}} \\
&= -2f_a - \frac{a + \frac{1}{2}}{(f_a' - f_a^2 - \frac{t}{2})} + f_a \\
&= -f_a - \frac{a + \frac{1}{2}}{(f_a' - f_a^2 - \frac{t}{2})} \quad ,
\end{aligned}
$$

where we have used the PII equation (2.3) to replace f_a'' in the expression. The last line is the well-known Bäcklund transformation of PII. That is

$$f_{a+1}(t) = -f_a - \frac{a + \frac{1}{2}}{(f_a' - f_a^2 - \frac{t}{2})}, \tag{2.65}$$

hence we have shown

$$\frac{c_1^{(a)\prime}}{c_1^{(a)}} = f_{a+1}(t) = \frac{a_0^{(a+1)\prime}}{a_0^{(a+1)}} \quad ,$$

which says that $a_0^{(a+1)}(t)$ is proportional to $c_1^{(a)}(t)$. The constant of proportion can be set to be 1 without loss of generality. □

We have shown that $v^{(a)}(x,t)$ and $u^{(a+1)}(x,t)$ have the same asymptotic behavior $x^{a+1}c_1^{(a)}(t)$ near $x = 0$. Now we will show that

Proposition 2.4.2.

$$v^{(a)}(x,t) = u^{(a+1)}(x,t).$$

Proof. Recall the spectral equation (2.58) is of the form

$$\begin{pmatrix} u^{(a)} \\ v^{(a)} \end{pmatrix}_x = M^{(a)}(x,t) \begin{pmatrix} u^{(a)} \\ v^{(a)} \end{pmatrix} = \begin{pmatrix} M_{11}^{(a)} & M_{12}^{(a)} \\ M_{21}^{(a)} & M_{22}^{(a)} \end{pmatrix} \begin{pmatrix} u^{(a)} \\ v^{(a)} \end{pmatrix}.$$

This 2×2 linear system of coupled first order differential equations (DEs) can be written as two second order scalar DEs:

$$u_{xx}^{(a)} = \frac{M_{12_x}^{(a)}}{M_{12}^{(a)}} u_x^{(a)} + \left(M_{11_x}^{(a)} - \frac{M_{12_x}^{(a)}}{M_{12}^{(a)}} M_{11}^{(a)} - \det M^{(a)} \right) u^{(a)} \tag{2.66}$$

$$v_{xx}^{(a)} = \frac{M_{21_x}^{(a)}}{M_{21}^{(a)}} v_x^{(a)} + \left(M_{22_x}^{(a)} - \frac{M_{21_x}^{(a)}}{M_{21}^{(a)}} M_{22}^{(a)} - \det M^{(a)} \right) v^{(a)} \tag{2.67}$$

where

$$\begin{aligned} M_{11}^{(a)}(x,t) &= \frac{a}{x} + 4f_a x \\ M_{12}^{(a)}(x,t) &= -i(2f_a^2 + t + 2f_a') - 4\,i\,x^2 \\ M_{21}^{(a)}(x,t) &= i(-2f_a^2 - t + 2f_a') - 4\,i\,x^2 \\ M_{22}^{(a)}(x,t) &= -\frac{a}{x} - 4f_a x. \end{aligned} \tag{2.68}$$

Then $u^{(a+1)}(x,t)$ solves equation

$$\begin{aligned} & u_{xx}^{(a+1)} \\ = \; & \frac{M_{12_x}^{(a+1)}}{M_{12}^{(a+1)}}\, u_x^{(a+1)} + \left(M_{11_x}^{(a+1)} - \frac{M_{12_x}^{(a+1)}}{M_{12}^{(a+1)}} M_{11}^{(a+1)} - \det M^{(a+1)} \right) u^{(a+1)}. \end{aligned} \tag{2.69}$$

We can calculate $M_{11}^{(a+1)}(x,t)$, $M_{12}^{(a+1)}(x,t)$, $M_{21}^{(a+1)}(x,t)$ and $M_{22}^{(a+1)}(x,t)$ by letting $a \to a+1$ and using Bäcklund transformation (2.65) for f_{a+1}. It can be easily checked that equation (2.67) is the same as equation (2.69) provided f_a satisfies the PII equation (2.3). Together with the fact that $v^{(a)}(x,t)$ and $u^{(a+1)}(x,t)$ have the same leading behaviour $x^{a+1}c_1^{(a)}(t)$ (or $x^{a+1}a_0^{(a+1)}(t)$) at $x=0$, we have proved the proposition. □

Now we can relate $\begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix}$ with $\begin{pmatrix} u^{(a)}(x,t) \\ v^{(a)}(x,t) \end{pmatrix}$, that is to find the Schlesinger transformation $L_a(x,t)$.

Proposition 2.4.3. The solutions of Lax pair (2.58, 2.59) for parameter $a+1$ and a are related by $L_a(x,t)$, that is

$$\begin{aligned} \begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix} &= L_a(x,t) \begin{pmatrix} u^{(a)}(x,t) \\ v^{(a)}(x,t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & -\frac{i(1+2a)}{x(t+2f_a^2-2f_a')} \end{pmatrix} \begin{pmatrix} u^{(a)}(x,t) \\ v^{(a)}(x,t) \end{pmatrix}. \end{aligned} \tag{2.70}$$

Proof. We already know that

$$u^{(a+1)}(x,t) = v^{(a)}(x,t).$$

It is only left to find how $v^{(a+1)}(x,t)$ can be written in terms of $u^{(a)}(x,t)$ and $v^{(a)}(x,t)$. Recall $\begin{pmatrix} u^{(a+1)} \\ v^{(a+1)} \end{pmatrix}$ satisfies the linear system

$$\begin{pmatrix} u^{(a+1)} \\ v^{(a+1)} \end{pmatrix}_x = \begin{pmatrix} M_{11}^{(a+1)} & M_{12}^{(a+1)} \\ M_{21}^{(a+1)} & M_{22}^{(a+1)} \end{pmatrix} \begin{pmatrix} u^{(a+1)} \\ v^{(a+1)} \end{pmatrix}, \tag{2.71}$$

where $M_{11}^{(a)}, M_{12}^{(a)}, M_{21}^{(a)}$ and $M_{11}^{(a)}$ are defined in equation (2.68). The top entry of this vector equation is:

$$\begin{aligned} & u_x^{(a+1)} &=& M_{11}^{(a+1)} u^{(a+1)} + M_{12}^{(a+1)} v^{(a+1)}, \\ \text{or} \qquad & v^{(a+1)} &=& \frac{u_x^{(a+1)} - M_{11}^{(a+1)} u^{(a+1)}}{M_{12}^{(a+1)}} \\ & &=& \frac{v_x^{(a)} - M_{11}^{(a+1)} v^{(a)}}{M_{12}^{(a+1)}} \\ & &=& \frac{M_{21}^{(a)} u^{(a)} + M_{22}^{(a)} v^{(a)}}{M_{12}^{(a+1)}} - \frac{M_{11}^{(a+1)}}{M_{12}^{(a+1)}} v^{(a)}, \end{aligned}$$

where we have used

$$v_x^{(a)} = M_{21}^{(a)} u^{(a)} + M_{22}^{(a)} v^{(a)}.$$

Hence

$$\begin{aligned} \begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{M_{21}^{(a)}}{M_{12}^{(a+1)}} & \frac{M_{22}^{(a)}}{M_{12}^{(a+1)}} - \frac{M_{11}^{(a+1)}}{M_{12}^{(a+1)}} \end{pmatrix} \begin{pmatrix} u^{(a)}(x,t) \\ v^{(a)}(x,t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & -\frac{i(1+2a)}{x(t+2f_a^2-2f_a')} \end{pmatrix} \begin{pmatrix} u^{(a)}(x,t) \\ v^{(a)}(x,t) \end{pmatrix}. \end{aligned}$$

□

We have found the Schlesinger transformation of the linear system (2.58)

$$\begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix} = L_a(x,t) \begin{pmatrix} u^{(a)}(x,t) \\ v^{(a)}(x,t) \end{pmatrix}$$

where $L_a(x,t)$ has the form

$$L_a(x,t) = \begin{pmatrix} 0 & 1 \\ 1 & \frac{l_a(t)}{x} \end{pmatrix}$$

and

$$l_a(t) = -\frac{i(1+2a)}{x(t+2f_a^2-2f_a')}.$$

Successively applying $L_a(x,t)$ we can eventually relate $\begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix}$ to $\begin{pmatrix} u^{(0)}(x,t) \\ v^{(0)}(x,t) \end{pmatrix}$ or $\begin{pmatrix} u^{(\frac{1}{2})}(x,t) \\ v^{(\frac{1}{2})}(x,t) \end{pmatrix}$ depending on whether a is an integer or a half integer, respectively. That is

$$\begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix} = L_a L_{a-1} \dots L_0 \begin{pmatrix} u^{(0)}(x,t) \\ v^{(0)}(x,t) \end{pmatrix},$$

and

$$\begin{pmatrix} u^{(a+1)}(x,t) \\ v^{(a+1)}(x,t) \end{pmatrix} = L_a L_{a-1} \dots L_{\frac{1}{2}} \begin{pmatrix} u^{(\frac{1}{2})}(x,t) \\ v^{(\frac{1}{2})}(x,t) \end{pmatrix}.$$

The reason we want to do this is because we have solved the associated linear problems in closed form when the parameter takes value $a = 0$, corresponding to PII admitting the simplest rational special solution $f_0(t) = 0$,

$$\begin{pmatrix} u^{(0)}(x,t) \\ v^{(0)}(x,t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-i(\frac{4}{3}x^3+tx)} \\ e^{i(\frac{4}{3}x^3+tx)} \end{pmatrix}, \tag{2.72}$$

and $a = \frac{1}{2}$, corresponding to PII admitting the simplest hypergeometric special solution $f_{\frac{1}{2}}(t) = \frac{A(2^{-\frac{1}{3}}t)'}{A(2^{-\frac{1}{3}}t)}$,

$$\begin{pmatrix} u^{(\frac{1}{2})}(x,t) \\ v^{(\frac{1}{2})}(x,t) \end{pmatrix} = x^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ -i\frac{f(t)}{x} & \frac{i}{x} \end{pmatrix} \begin{pmatrix} A(z) \\ A_t(z) \end{pmatrix}. \tag{2.73}$$

in subsection 2.3.1 and 2.3.2 respectively. We also know that the solution of PII is related to the leading coefficient of the solution of the associated linear problems by

$$f_{a+1}(t) = \frac{a_0^{(a+1)\prime}}{a_0^{(a+1)}}.$$

That is we can find $f_{a+1}(t)$ via the expression of $a_0^{(a+1)}(t)$ in terms of the coefficients $a_j^{(0)}$ (that is T_j, $(j = 0, 1, \dots)$ defined by equations (2.34, 2.35)) when the parameter a is an integer, or in terms of the coefficients $a_{2j}^{(\frac{1}{2})}$ (that

is a_{2j}, $(j = 0, 1, \ldots)$ defined by equations (2.41, 2.42)) when the parameter a is a half integer.

2.5. Determinantal representations of special solutions

2.5.1. Determinantal form of rational special solution hierarchy.

Theorem 2.5.1. [**18**] PII equation (2.3) when a are integers $a = k$, $k = 0, 1, 2...$ admits rational type special solutions given by

$$f_k(t) = \frac{d}{dt} \ln \frac{\tau_k(t)}{\tau_{k-1}(t)}$$

where the function $\tau_k(t)$ is defined by

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \ldots & T_{2k-1} \\ T_{k-2} & T_{k-1} & \ldots & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-k+2} & T_{-k+3} & \ldots & T_1 \end{vmatrix}. \tag{2.74}$$

The $\tau_k(t)$ function is a polynomial of degree $k(k+1)/2$ in t, presented as the determinant of a $k \times k$ matrix, where $T_k(t)$ denotes the Laguerre polynomial of degree k in t,

$$\begin{aligned} jT_j &= 4iT_{j-3} + itT_{j-1}, \\ \frac{dT_j}{dt} &= iT_{j-1}, \end{aligned}$$

$T_0 = 1$, and $T_j = 0, j < 0$.

Proof. In [**18**], Flaschka and Newell did not use the Schlesinger transformation to obtain the determinantal form of these rational special solutions of PII. Instead, they have proved this result by considering a special case of the Riemann-Hilbert problem of the linear problem (2.1) of PII for when k are integers, see [**18**] for the details.
When $a = k$, k is an integer, the two vector solutions of the the Lax pair (2.1, 2.2) have infinite series expansions near $x = 0$ is given by

$$\phi_1^{(k)}(x,t) \begin{pmatrix} u^{(k)}(x,t) \\ v^{(k)}(x,t) \end{pmatrix} = x^k \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}^{(k)}(t) x^{2j} \\ c_{2j+1}^{(k)}(t) x^{2j+1} \end{pmatrix},$$

$$\phi_2^{(k)}(x,t)\begin{pmatrix}u^{(k)}(x,t)\\ v^{(k)}(x,t)\end{pmatrix} = x^k\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}b_{2j+1}^{(k)}(t)x^{2j+1}\\ d_{2j}^{(k)}(t)x^{2j}\end{pmatrix},$$

and the two vector solutions near $x=\infty$ is given by

$$\psi_1^{(k)}(x,t) = e^{-i(\frac{4}{3}x^3+tx)}\left\{\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{x}\begin{pmatrix}\alpha_1^{(k)}(t)\\ \gamma_1^{(k)}(t)\end{pmatrix}+...+\frac{1}{x^k}\begin{pmatrix}\alpha_k^{(k)}(t)\\ \gamma_k^{(k)}(t)\end{pmatrix}\right\},$$

$$\psi_2^{(k)}(x,t) = e^{i(\frac{4}{3}x^3+tx)}\left\{\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{x}\begin{pmatrix}\beta_1^{(k)}(t)\\ \delta_1^{(k)}(t)\end{pmatrix}+...+\frac{1}{x^k}\begin{pmatrix}\beta_k^{(k)}(t)\\ \delta_k^{(k)}(t)\end{pmatrix}\right\},$$

where $\beta_1^{(k)} = -i\frac{f_k(t)}{2}$. Note the series expansions of $\psi_1^{(k)}$ and $\psi_2^{(k)}$ terminate at the power $\frac{1}{x^k}$ in this case. Since $\phi_1^{(k)}(x,t)$, $\phi_2^{(k)}(x,t)$, $\psi_1^{(k)}(x,t)$ and $\psi_2^{(k)}(x,t)$ are solutions of the same 2×2 first-order linear systems, $\psi_2^{(k)}(x,t)$ must be a linear combination of $\phi_1^{(k)}(x,t)$ and $\phi_2^{(k)}(x,t)$, that is

$$\psi_2^{(k)}(x,t) = A\phi_1^{(k)}(x,t) + B\phi_2^{(k)}(x,t),$$

where A and B are constants and recall $e^{\left(i\frac{4}{3}x^3+itx\right)} = \sum_{k=0}^{\infty}T_k(t)x^k$, and $T_k(t)$ satisfies relations (2.34, 2.35). Hence we have

$$\begin{aligned}\sum_{j=0}^{\infty}T_j(t)x^j&\left\{\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{x}\begin{pmatrix}\beta_1^{(k)}\\ \delta_1^{(k)}\end{pmatrix}+...+\frac{1}{x^k}\begin{pmatrix}\beta_k^{(k)}\\ \delta_k^{(k)}\end{pmatrix}\right\}\\ &= Ax^k\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left\{\begin{pmatrix}a_0^{(k)}\\0\end{pmatrix}+x\begin{pmatrix}0\\c_1^{(k)}\end{pmatrix}+...\right\}\\ &\quad +Bx^{-k}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left\{\begin{pmatrix}0\\d_0^{(k)}\end{pmatrix}+x\begin{pmatrix}b_1^{(k)}\\0\end{pmatrix}+...\right\}.\end{aligned}$$

Equating powers of x near $x=0$, we see that on the RHS the series summation of vectors with leading power of x^{-k} is going to be dominant. We take $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ matrix factor inside of the series expansion. Since the vectors of the expansion alternate between $b_{2j+1}^{(k)}\begin{pmatrix}1\\0\end{pmatrix}$ and $d_{2j}^{(k)}\begin{pmatrix}0\\1\end{pmatrix}$, after being pre-multiplied by $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ the vectors alternate between $b_{2j+1}^{(k)}\begin{pmatrix}1\\1\end{pmatrix}$ and

$$d_{2j}^{(k)}\begin{pmatrix}1\\-1\end{pmatrix},$$

$$\begin{aligned}\sum_{k=0}^{\infty}T_k(t)x^k&\left\{\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{x}\begin{pmatrix}\beta_1^{(k)}\\\delta_1^{(k)}\end{pmatrix}+...+\frac{1}{x^k}\begin{pmatrix}\beta_k^{(k)}\\\delta_k^{(k)}\end{pmatrix}\right\}\\
&= Ax^k\begin{pmatrix}1&1\\1&-1\end{pmatrix}\left\{\begin{pmatrix}a_0^{(k)}\\0\end{pmatrix}+x\begin{pmatrix}0\\c_1^{(k)}\end{pmatrix}+...\right\}\\
&\quad +Bx^{-k}\Bigg\{d_0^{(k)}\begin{pmatrix}1\\-1\end{pmatrix}+b_1^{(k)}x\begin{pmatrix}1\\1\end{pmatrix}+d_2^{(k)}x^2\begin{pmatrix}1\\-1\end{pmatrix}\\
&\quad +b_3^{(k)}x^3\begin{pmatrix}1\\1\end{pmatrix}+...+d_{k-2}^{(k)}x^{k-2}\begin{pmatrix}1\\-1\end{pmatrix}+b_{k-1}^{(k)}x^{k-1}\begin{pmatrix}1\\1\end{pmatrix}\\
&\quad +...+d_{2k-2}^{(k)}x^{2k-2}\begin{pmatrix}1\\-1\end{pmatrix}+b_{2k-1}^{(k)}x^{2k-1}\begin{pmatrix}1\\1\end{pmatrix}+...\Bigg\}.\end{aligned}$$

Note the last line is when k is an even integer, the case for k are odd integers can be proved similarly. On equating the first $2k$ powers of x near $x=0$, that is from x^{-k} to x^{k-1}, we obtain $2k$ equations. Let $\xi_j=\beta_j^{(k)}+\delta_j^{(k)}$, $\eta_j=\beta_j^{(k)}-\delta_j^{(k)}$. Then

$$\begin{aligned}\frac{1}{x^k}:\quad & T_0\begin{pmatrix}\beta_k^{(k)}\\\delta_k^{(k)}\end{pmatrix}=Bd_0^{(k)}\begin{pmatrix}1\\-1\end{pmatrix}\\
&\Rightarrow T_0\xi_k=0,\\
\frac{1}{x^{k-1}}:\quad & T_1\begin{pmatrix}\beta_k^{(k)}\\\delta_k^{(k)}\end{pmatrix}+T_0\begin{pmatrix}\beta_{k-1}^{(k)}\\\delta_{k-1}^{(k)}\end{pmatrix}=Bb_1^{(k)}\begin{pmatrix}1\\1\end{pmatrix}\\
&\Rightarrow T_1\eta_k+T_0\eta_{k-1}=0,\\
&\vdots\qquad\qquad\vdots\\
\frac{1}{x^2}:\quad & T_{k-2}\xi_k+T_{k-3}\xi_{k-1}+...+T_0\xi_2=0\\
\frac{1}{x}:\quad & T_{k-1}\eta_k+T_{k-2}\eta_{k-1}+...+T_0\eta_1=0\\
&\vdots\qquad\qquad\vdots\end{aligned}$$

$$x^{k-2}: \quad T_{2k-2}\xi_k + T_{2k-3}\xi_{k-1} + ... + T_{k-1}\xi_1 = -T_{k-2}\xi_0$$
$$x^{k-1}: \quad T_{2k-1}\eta_k + T_{2k-2}\eta_{k-1} + ... + T_k\eta_1 = -T_{k-1}\eta_0.$$

Recall $\beta_0^{(k)} = 0$ and $\delta_0^{(k)} = 1$, so $\eta_0 = \beta_0^{(k)} - \delta_0^{(k)} = -1$, $\xi_0 = \beta_0^{(k)} + \delta_0^{(k)} = 1$. We have k equations for $\xi_k, ...\xi_1$ which can be rewritten in a matrix form

$$\begin{pmatrix} T_{k-1} & T_k & \cdots & T_{2k-4} & T_{2k-3} & T_{2k-2} \\ T_{k-3} & T_{k-2} & \cdots & T_{2k-6} & T_{2k-5} & T_{2k-4} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ T_1 & T_2 & \cdots & T_{k-2} & T_{k-1} & T_k \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & T_0 & T_1 & T_2 \\ 0 & \cdots & & \cdots & 0 & T_0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \\ \\ \vdots \\ \\ \xi_k \end{pmatrix} = - \begin{pmatrix} T_{k-2} \\ T_{k-4} \\ \vdots \\ T_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (2.75)$$

and k equations for $\eta_k, ...\eta_1$ which can be rewritten in a matrix form

$$\begin{pmatrix} T_k & T_{k+1} & \cdots & & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \cdots & & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ T_2 & T_3 & \cdots & & T_k & T_{k+1} \\ & & \ddots & & \vdots & \vdots \\ & & & \cdots & T_2 & T_3 \\ 0 & \cdots & & \cdots & T_0 & T_1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \\ \\ \vdots \\ \\ \eta_k \end{pmatrix} = \begin{pmatrix} T_{k-1} \\ T_{k-3} \\ \vdots \\ T_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (2.76)$$

Now we can evaluate ξ_1 using Cramer's rule. We recall Cramer's rule here, since we will use it repeatedly below.

Cramer's rule:

$$\sum_{j=1}^{n} \mathbf{C}_j x_j = \mathbf{B},$$

where

$$\mathbf{B} = (b_1, b_2, ..., b_n)^{\mathbf{T}} \qquad \mathbf{T}, \text{transpose}$$
$$A = |\mathbf{C_1 C_2 ... C_j ... C_n}|,$$

then x_j can be calculated using

$$x_j = \frac{1}{A}|\mathbf{C_1 C_2 ... B ... C_n}|.$$

In equation (2.75), $\mathbf{B} = (T_{k-2}, T_{k-4}, ..., T_0, 0, ..., 0)^{\mathbf{T}}$, $(\mathbf{C_1}...\mathbf{C_j}...\mathbf{C_n})$ is the matrix in equation (2.75), and A is the determinant of this matrix. So for ξ_1, applying Cramer's rule

$$\xi_1 = \frac{1}{A}|\mathbf{B}\ \mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}|.$$

Using the definitions of $\mathbf{B}$ and $\mathbf{C_j}$ and relation (2.35) for the T_js, we have

$$\frac{d\mathbf{C_j}}{dt} = i\mathbf{C_{j-1}}, \tag{2.77}$$

and we see that $\mathbf{B} = i\frac{d\mathbf{C_1}}{dt}$. That is

$$\xi_1 = \frac{i}{A}|\frac{d\mathbf{C_1}}{dt}\ \mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}|.$$

Next we will show that

Proposition 2.5.2.

$$|\frac{d\mathbf{C_1}}{dt}\ \mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}| = \frac{dA}{dt}.$$

Proof. Since

$$\begin{aligned}
A &= |\mathbf{C_1}\mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}| \\
\frac{dA}{dt} &= \sum_{j=1}^{n} |\mathbf{C_1}\mathbf{C_2}...\frac{d\mathbf{C_j}}{dt}...\mathbf{C_n}| \\
&= |\frac{d\mathbf{C_1}}{dt}\mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}| + ... + |\mathbf{C_1}\mathbf{C_2}...\frac{\mathbf{C_j}}{dt}...\mathbf{C_n}| + ... \\
&\quad +|\mathbf{C_1}\mathbf{C_2}...\mathbf{C_j}...\frac{d\mathbf{C_n}}{dt}| \\
&= |\frac{d\mathbf{C_1}}{dt}\mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}| + ...i|\mathbf{C_1}\mathbf{C_2}...\mathbf{C_{j-1}}...\mathbf{C_n}| + ... \\
&\quad +i|\mathbf{C_1}\mathbf{C_2}...\mathbf{C_j}...\mathbf{C_{n-1}}| \\
&= |\frac{d\mathbf{C_1}}{dt}\mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}|.
\end{aligned}$$

Again we have used relation (2.77) and the fact that a determinant is zero if two of its columns are the same. □

So now we have:

$$\xi_1 = \frac{i}{A}|\frac{d\mathbf{C_1}}{dt}\ \mathbf{C_2}...\mathbf{C_j}...\mathbf{C_n}| = i\frac{1}{A}\frac{dA}{dt} = i\frac{d}{dt}\ln A.$$

Let $\tau_k(t)$ to be the determinant

$$\tau_k(t) = \begin{vmatrix} T_k & T_{k+1} & \dots & T_{2k-1} \\ T_{k-2} & T_{k-1} & \dots & T_{2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-k+2} & T_{-k+3} & \dots & T_1 \end{vmatrix}. \tag{2.78}$$

The $\tau_k(t)$ function (2.78) is the determinant of the $k \times k$ matrix in equation (2.76) since T_k is 0 for $k < 0$. We observe that A, the determinant of the matrix in equation (2.75) is $T_0\tau_{k-1}(t)$. Finally we have

$$\xi_1 = i\frac{1}{A}\frac{dA}{dt} = i\frac{d}{dt}\ln\tau_{k-1}(t),$$

as $T_0 = 1$ so that $\frac{dT_0}{dt} = 0$.
Similarly it can be shown that

$$\eta_1 = -i\frac{d}{dt}\ln\tau_k(t). \tag{2.79}$$

Now recall, $\xi_1 = \beta_1^{(k)} + \delta_1^{(k)}$, $\eta_1 = \beta_1^{(k)} - \delta_1^{(k)}$ where we know that $\beta_1^{(k)} = -i\frac{f_k(t)}{2}$, hence

$$\frac{\xi_1 + \eta_1}{2} = \beta_1^{(k)} \quad = \quad -i\frac{f_k(t)}{2} = \frac{i}{2}\frac{d}{dt}\left(\ln\tau_{k-1} - \ln\tau_k\right).$$

We have finally

$$f_k(t) = \frac{d}{dt}\ln\left(\frac{\tau_k}{\tau_{k-1}}\right).$$

□

2.5.2. Determinantal form of hypergeometric type special solution hierarchy.

Theorem 2.5.3. PII equation (2.3) when a are half integers $a = k + \frac{1}{2}$, $k = 0, 1, 2, ...$ admits hypergeometric type special solutions given by

$$f_{k+\frac{1}{2}}(t) = \frac{d}{dt}\ln\frac{\tau_{k+1}(t)}{\tau_k(t)},$$

$$\tau_k(t) \quad = \quad \begin{vmatrix} A & \frac{d}{dt}A & \dots & \frac{d^{k-1}}{dt^{k-1}}A \\ \frac{d}{dt}A & \frac{d^2}{dt^2}A & \dots & \frac{d^k}{dt^k}A \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{k-1}}{dt^{k-1}}A & \frac{d^k}{dt^k}A & \dots & \frac{d^{2k-2}}{dt^{2k-2}}A \end{vmatrix},$$

where

$$\frac{d^2}{dt^2}A(t) = -\frac{t}{2}A(t).$$

Proof. A vector solution $\phi_1^{(k+\frac{1}{2})}(x,t)$ of the Lax pair (2.1, 2.2) at $x=0$ has the asymptotic behaviour given by equation (2.6)

$$\phi_1^{(k+\frac{1}{2})}(x,t) \sim x^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_0^{(k+\frac{1}{2})}(t)x^k + ...\\ c_1^{(k+\frac{1}{2})}(t)x^{k+1} + ...\end{pmatrix}. \quad (2.80)$$

The leading coefficient of the solution of the linear problem is related to the solution of PII by

$$\frac{a_0^{(k+\frac{1}{2})\prime}}{a_0^{(k+\frac{1}{2})}} = \frac{d}{dt}\ln a_0^{(k+\frac{1}{2})}(t) = f_{k+\frac{1}{2}}(t).$$

Therefore if we knew what $a_0^{(k+\frac{1}{2})}(t)$ is, then we have found $f_{k+\frac{1}{2}}(t)$, some special solution of PII for when $a = k+\frac{1}{2}$. First we derive $\phi_1^{(k+\frac{1}{2})}(x,t)$ in closed form using the Schlesinger transformation.

Proposition 2.5.4. A solution of Lax pair (2.1, 2.2) when $a = k+\frac{1}{2}$, $k = 0,1,2,...$ with leading behaviour $x^{k+\frac{1}{2}}$ is:

$$\phi_1^{(k+\frac{1}{2})}(x,t) = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}u^{(k+\frac{1}{2})}(x,t)\\ v^{(k+\frac{1}{2})}(x,t)\end{pmatrix} = \quad (2.81)$$

k even : $x^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1+\frac{s_2}{x^2}+...+\frac{s_k}{x^k} & \frac{t_2}{x^2}+...+\frac{t_k}{x^k}\\ \frac{s_1}{x}+\frac{s_3}{x^3}+...+\frac{s_{k+1}}{x^{k+1}} & \frac{i}{x}+...+\frac{t_{k+1}}{x^{k+1}}\end{pmatrix}\begin{pmatrix}A(z)\\ A_t(z)\end{pmatrix}$

k odd : $x^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\frac{s_1}{x}+\frac{s_3}{x^3}+...+\frac{s_k}{x^k} & \frac{i}{x}+...+\frac{t_{k+1}}{x^{k+1}}\\ 1+\frac{s_2}{x^2}+...+\frac{s_{k+1}}{x^{k+1}} & \frac{t_2}{x^2}+...+\frac{t_{k+1}}{x^{k+1}}\end{pmatrix}\begin{pmatrix}A(z)\\ A_t(z)\end{pmatrix}$,

where

$$A(z) = \sum_{j=0}^{\infty} a_{2j}(t)x^{2j},$$

and

$$\begin{aligned} a_{2j} &= \frac{2}{j}\frac{da_{2j-2}}{dt},\\ \frac{d^2a_{2j}}{dt^2} &= -\frac{t}{2}a_{2j} - a_{2j-2}, \end{aligned}$$

in particular $\frac{d^2a_0}{dt^2} = -\frac{t}{2}a_0,\ a_0(t) = A(t)$.

Proof. The solution matrix of Lax pair (2.1, 2.2) is $\Phi(x,t)$ and let

$$\Phi(x,t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Phi_1(x,t),$$

where $\Phi_1(x,t)$ satisfies Lax pair (2.58, 2.59). Let $\begin{pmatrix} u^{(k+\frac{1}{2})}(x,t) \\ v^{(k+\frac{1}{2})}(x,t) \end{pmatrix}$ be the first column of the matrix solution $\Phi_1(x,t)$.

For $a = k + \frac{1}{2}$, $k = 0, 1, 2, ...$ Apply proposition 2.4.3 repetitively to obtain

$$\begin{pmatrix} u^{(k+\frac{1}{2})}(x,t) \\ v^{(k+\frac{1}{2})}(x,t) \end{pmatrix} = L_{(k-1)+\frac{1}{2}} \cdot L_{(k-2)+\frac{1}{2}} \cdots L_{\frac{1}{2}} \begin{pmatrix} u^{(\frac{1}{2})}(x,t) \\ v^{(\frac{1}{2})}(x,t) \end{pmatrix}.$$

Recall that $\begin{pmatrix} u^{(\frac{1}{2})}(x,t) \\ v^{(\frac{1}{2})}(x,t) \end{pmatrix}$ is given by

$$\begin{pmatrix} u^{(\frac{1}{2})}(x,t) \\ v^{(\frac{1}{2})}(x,t) \end{pmatrix} = x^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{-if_{\frac{1}{2}}(t)}{x} & \frac{i}{x} \end{pmatrix} \begin{pmatrix} A(z) \\ A_t(z) \end{pmatrix}.$$

Since $L_a(x,t)$ has simple dependence with respect to x

$$L_a(x,t) = \begin{pmatrix} 0 & 1 \\ 1 & \frac{l_a(t)}{x} \end{pmatrix},$$

hence $L_{(k-1)+\frac{1}{2}} \cdot L_{(k-2)+\frac{1}{2}} \cdots L_{\frac{1}{2}} \cdot \begin{pmatrix} 1 & 0 \\ \frac{-if_{\frac{1}{2}}(t)}{x} & \frac{i}{x} \end{pmatrix}$ has the form:

$$k \text{ even}: \quad \nu^{\frac{1}{2}} \begin{pmatrix} 1 + \frac{s_2}{x^2} + ... + \frac{s_k}{x^k} & \frac{t_2}{x^2} + ... + \frac{t_k}{x^k} \\ \frac{s_1}{x} + \frac{s_3}{x^3} + ... + \frac{s_{k+1}}{x^{k+1}} & \frac{i}{x} + ... + \frac{t_{k+1}}{x^{k+1}} \end{pmatrix},$$

$$k \text{ odd}: \quad x^{\frac{1}{2}} \begin{pmatrix} \frac{s_1}{x} + \frac{s_3}{x^3} + ... + \frac{s_k}{x^k} & \frac{i}{x} + ... + \frac{t_k}{x^k} \\ 1 + \frac{s_2}{x^2} + ... + \frac{s_{k+1}}{x^{k+1}} & \frac{t_2}{x^2} + ... + \frac{t_{k+1}}{x^{k+1}} \end{pmatrix}$$

where $s_0 = 1,\ t_0 = 0, t_1 = i$, $s_1, ..., s_{k-1}$ and $t_2, ..., t_{k-1}$ are functions of t. Hence we have

$$\begin{pmatrix} u^{(k+\frac{1}{2})}(x,t) \\ v^{(k+\frac{1}{2})}(x,t) \end{pmatrix} = \tag{2.82}$$

$$k \text{ even}: \quad x^{\frac{1}{2}} \begin{pmatrix} 1 + \frac{s_2}{x^2} + ... + \frac{s_k}{x^k} & \frac{t_2}{x^2} + ... + \frac{t_k}{x^k} \\ \frac{s_1}{x} + \frac{s_3}{x^3} + ... + \frac{s_{k+1}}{x^{k+1}} & \frac{i}{x} + ... + \frac{t_{k+1}}{x^{k+1}} \end{pmatrix} \begin{pmatrix} A(z) \\ A_t(z) \end{pmatrix}$$

$$k \text{ odd}: \quad x^{\frac{1}{2}} \begin{pmatrix} \frac{s_1}{x} + \frac{s_3}{x^3} + ... + \frac{s_{k+1}}{x^{k+1}} & \frac{i}{x} + ... + \frac{t_{k+1}}{x^{k+1}} \\ 1 + \frac{s_2}{x^2} + ... + \frac{s_k}{x^k} & \frac{t_2}{x^2} + ... + \frac{t_k}{x^k} \end{pmatrix} \begin{pmatrix} A(z) \\ A_t(z) \end{pmatrix}.$$

Let $\phi_1^{(k+\frac{1}{2})}(x,t)$ be the first column of $\Phi(x,t)$, that is

$$\phi_1^{(k+\frac{1}{2})}(x,t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u^{(k+\frac{1}{2})}(x,t) \\ v^{(k+\frac{1}{2})}(x,t) \end{pmatrix}$$

we have proved proposition 2.5.4.

For the case $k = 0$, $a = \frac{1}{2}$, $s_1 = -if_{\frac{1}{2}}(t)$ where $f_{\frac{1}{2}}(t)$ solves Riccati equation (2.36), the formula for $\phi_1^{(\frac{1}{2})}(x,t)$ given by proposition 2.5.4 is consistent with equation (2.39) from earlier section 2.3.2:

$$\phi_1^{(\frac{1}{2})}(x,t) = x^{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-if_{\frac{1}{2}}(t)}{x} & \frac{i}{x} \end{pmatrix} \begin{pmatrix} A(z) \\ A_t(z) \end{pmatrix},$$

where $z(x,t) = 2^{\frac{2}{3}}x^2 + 2^{-\frac{1}{3}}t$ and $A(z)$ is the Airy function, where

$$\frac{d^2A}{dz^2} = -zA.$$

where

$$A(z) = \sum_{j=0}^{\infty} a_{2j}(t)x^{2j},$$

and

$$\begin{aligned} a_{2j} &= \frac{2}{j}\frac{da_{2j-2}}{dt}, \\ \frac{d^2a_{2j}}{dt^2} &= -\frac{t}{2}a_{2j} - a_{2j-2}, \end{aligned}$$

in particular

$$\frac{d^2a_0}{dt^2} = -\frac{t}{2}a_0,\ a_0(t) = A(t).$$

We have omitted the superscript $^{(\frac{1}{2})}$ on the coefficients a_{2j} from $\phi_1^{(\frac{1}{2})}(x,t)$ for simplicity. □

Now we are ready to use solution $\phi_1^{(k+\frac{1}{2})}(x,t)$ to find $f_{k+\frac{1}{2}}(t)$. Let us look at the case where k is even (case for k odd can be proved similarly). Consider

$$\phi_1^{(k+\frac{1}{2})}(x,t) = \tag{2.83}$$
$$x^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1+\frac{s_2}{x^2}+...+\frac{s_k}{x^k} & \frac{t_2}{x^2}+...+\frac{t_k}{x^k}\\ \frac{s_1}{x}+\frac{s_3}{x^3}+\ldots+\frac{s_{k+1}}{x^{k+1}} & \frac{i}{x}+...+\frac{t_{k+1}}{x^{k+1}}\end{pmatrix}\begin{pmatrix}A(z)\\ A_t(z)\end{pmatrix}.$$

Recall at $x=0$

$$\begin{aligned} A(z) &= A(2^{\frac{2}{3}}x^2+2^{-\frac{1}{3}}t) = \sum_{j=0}^{\infty} a_{2j}x^{2j} \\ A_t(z) &= A_t(2^{\frac{2}{3}}x^2+2^{-\frac{1}{3}}t) = \sum_{j=0}^{\infty}\frac{da_{2j}}{dt}x^{2j}. \end{aligned}$$

Moreover, we know the asymptotic behaviour of $\phi_1^{(k+\frac{1}{2})}(x,t)$ is given by equation (2.80)

$$\sim\quad x^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_0^{(k+\frac{1}{2})}(t)x^k+...\\ c_1^{(k+\frac{1}{2})}(t)x^{k+1}+...\end{pmatrix}. \tag{2.84}$$

Equating the two expressions (2.83) and (2.84) of $\phi_1^{(k+\frac{1}{2})}(x,t)$ in the neighbourhood of $x=0$,

$$\begin{aligned} &\phi_1^{(k+\frac{1}{2})}(x,t) \\ =\quad & x^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1+\frac{s_2}{x^2}+...+\frac{s_k}{x^k} & \frac{t_2}{x^2}+...+\frac{t_k}{x^k}\\ \frac{s_1}{x}+\frac{s_3}{x^3}+...+\frac{s_{k+1}}{x^{k+1}} & \frac{i}{x}+...+\frac{t_{k+1}}{x^{k+1}}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}x^{2j}\\ \frac{a_{2j}}{dt}x^{2j}\end{pmatrix} \\ &\sim\quad x^{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_0^{(k+\frac{1}{2})}(t)x^k+...\\ c_1^{(k+\frac{1}{2})}(t)x^{k+1}+...\end{pmatrix}, \qquad x\to 0. \end{aligned}$$

For solution (2.83) to have the correct asymptotic behaviour, s_j and t_j need to satisfy systems of equations. From the top entry, equating powers in x,

from $x^{-k}, x^{-k+2}, ...,$ to x^k gives us $k+1$ equations for the $k+1$ variables $s_k, s_{k-2}, ...s_0$ and $t_k, t_{k-2}, ...t_2$:

$$\begin{aligned}
\frac{1}{x^k} &: && s_k a_0 + t_k a_0' = 0 \\
\frac{1}{x^{k-2}} &: && s_k a_2 + t_k a_2' + s_{k-2} a_0 + t_{k-2} a_0' = 0 \\
\vdots &: && \vdots \\
x^{k-2} &: && s_k a_{2k-2} + t_k a_{2k-2}' + \cdots + t_2 a_k' + a_{k-2} = 0 \\
x^k &: && s_k a_{2k} + t_k a_{2k}' + \cdots + t_2 a_{k-2}' + a_k = a_0^{(k+\frac{1}{2})}.
\end{aligned}$$

Rewriting these in the form of a $(k+1) \times (k+1)$ matrix equation:

$$\begin{pmatrix}
a_0 & a_0' & 0 & 0 & \cdots & & 0 \\
a_2 & a_2' & a_0 & a_0' & 0 & \cdots & 0 \\
 & & & & & & \\
\vdots & & & \ddots & & & \vdots \\
 & & & & & & \\
a_{2k-2} & a_{2k-2}' & & \cdots & a_{k-4} & a_{k-4}' & a_{k-2} \\
a_{2k} & a_{2k}' & & \cdots & a_{k-2} & a_{k-2}' & a_k
\end{pmatrix}
\begin{pmatrix} s_k \\ t_k \\ s_{k-2} \\ \vdots \\ s_2 \\ t_2 \\ 1 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \\ 0 \\ a_0^{(k+\frac{1}{2})} \end{pmatrix} \quad (2.85)$$

Let us denote the matrix equation (2.85) as

$$\sum_{j=1}^{k+1} \mathbf{C}_j x_j \;=\; \mathbf{B},$$

that is let

$$(\mathbf{C_1 C_2 ... C_j ... C_{k+1}})$$

be the matrix in equation (2.85) and A is the determinant of this matrix

$$\begin{aligned}
A &= |\mathbf{C_1 C_2 ... C_j ... C_{k+1}}|, \\
\mathbf{B} &= (0, ..., 0, a_0^{(k+\frac{1}{2})})^{\mathbf{T}}, \\
(x_1, x_2 ..., x_{k+1}) &= (s_k, t_k, ..., 1).
\end{aligned}$$

Let $\tau_{k+1}(t)$ be the determinant of the $k+1 \times k+1$ matrix

$$\tau_{k+1}(t) = \begin{vmatrix} a_0 & a_0' & 0 & 0 & \cdots & & 0 \\ a_2 & a_2' & a_0 & a_0' & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ a_{2k-2} & a_{2k-2}' & a_{2k-4} & \cdots & a_{k-4} & a_{k-4}' & a_{k-2} \\ a_{2k} & a_{2k}' & a_{2k-2} & \cdots & a_{k-2} & a_{k-2}' & a_k \end{vmatrix}.$$

That is

$$\tau_{k+1}(t) = |\mathbf{C_1 C_2 ... C_j ... C_{k+1}}| = A.$$

Applying Cramer's rule:

$$x_j = \frac{1}{A}|\mathbf{C_1 C_2 ... B ... C_{k+1}}|$$

to evaluate x_{k+1}, we obtain

$$\begin{aligned} x_{k+1} = 1 &= \frac{a_0^{(k+\frac{1}{2})} \begin{vmatrix} a_0 & a_0' & 0 & 0 & \cdots & 0 \\ a_2 & a_2' & a_0 & a_0' & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ a_{2k-2} & a_{2k-2}' & & \cdots & a_{k-4} & a_{k-4}' \end{vmatrix}}{A} \\ &= \frac{a_0^{(k+\frac{1}{2})} \tau_k(t)}{\tau_{k+1}(t)}. \end{aligned}$$

Hence we have shown that

$$a_0^{(k+\frac{1}{2})}(t) = \frac{\tau_{k+1}(t)}{\tau_k(t)}.$$

To prove theorem 2.5.3 we still need to show

Proposition 2.5.5.

$$\tau_{k+1}(t) = \begin{vmatrix} a_0 & a_0' & 0 & 0 & \cdots & & 0 \\ a_2 & a_2' & a_0 & a_0' & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ a_{2k-2} & a_{2k-2}' & a_{2k-4} & \cdots & a_{k-4} & a_{k-4}' & a_{k-2} \\ a_{2k} & a_{2k}' & a_{2k-2} & \cdots & a_{k-2} & a_{k-2}' & a_k \end{vmatrix} \tag{2.86}$$

$$= \mu \begin{vmatrix} a_0 & a_0' & \cdots & & \frac{d^k a_0}{dt^k} \\ a_0' & a_0'' & \cdots & & \frac{d^{k+1} a_0}{dt^{k+1}} \\ \vdots & & \ddots & & \vdots \\ \frac{d^{k-1} a_0}{dt^{k-1}} & \frac{d^k a_0}{dt^k} & \cdots & \frac{d^{2k-1} a_0}{dt^{2k-1}} & \frac{d^{2k} a_0}{dt^{2k}} \\ \frac{d^k a_0}{dt^k} & \frac{d^{k+1} a_0}{dt^{k+1}} & \cdots & \frac{d^{2k} a_0}{dt^{2k}} & \frac{d^{2k+1} a_0}{dt^{2k+1}} \end{vmatrix} \tag{2.87}$$

where μ is a constant.

Proof. Let

$$(\mathbf{C_1}\ \mathbf{C_2}\ \ldots\ \mathbf{C_{k+1}}) = \begin{pmatrix} a_0 & a_0' & 0 & 0 & \cdots & & 0 \\ a_2 & a_2' & a_0 & a_0' & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ a_{2k-2} & a_{2k-2}' & a_{2k-4} & \cdots & a_{k-4} & a_{k-4}' & a_{k-2} \\ a_{2k} & a_{2k}' & a_{2k-2} & \cdots & a_{k-2} & a_{k-2}' & a_k \end{pmatrix} \tag{2.88}$$

and

$$(\mathbf{D_1}\ \mathbf{D_2}\ \ldots\ \mathbf{D_{k+1}}) = \begin{pmatrix} a_0 & a_0' & \cdots & & \frac{d^k a_0}{dt^k} \\ a_0' & a_0'' & \cdots & & \frac{d^{k+1} a_0}{dt^{k+1}} \\ \vdots & & \ddots & & \vdots \\ \frac{d^{k-1} a_0}{dt^{k-1}} & \frac{d^k a_0}{dt^k} & \cdots & \frac{d^{2k-1} a_0}{dt^{2k-1}} & \frac{d^{2k} a_0}{dt^{2k}} \\ \frac{d^k a_0}{dt^k} & \frac{d^{k+1} a_0}{dt^{k+1}} & \cdots & \frac{d^{2k} a_0}{dt^{2k}} & \frac{d^{2k+1} a_0}{dt^{2k+1}} \end{pmatrix}. \tag{2.89}$$

We notice that by definition

$$(\mathbf{D_1}\ \mathbf{D_2}\ ...\ \mathbf{D_{k+1}}) = \left(\mathbf{D_1}\ \mathbf{D_1^{(1)}}\ ...\ \mathbf{D_1^{(k)}}\right) \tag{2.90}$$

where $^{(n)} = \frac{d^n}{dt^n}$. We are going to prove proposition 2.5.5 in several steps. First we show

Proposition 2.5.6.

$$(\mathbf{C_1}\ \mathbf{C_2}\ ...\ \mathbf{C_{k+1}}) \ = \ (\mathbf{C_1}\ \mathbf{C_1}'\ ...\ \mathbf{C_1^{(k)}}) \begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & -1 & \\ & \mathbf{0} & & \ddots \\ & & & \end{pmatrix},$$

where matrix $\begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & -1 & \\ & \mathbf{0} & & \ddots \\ & & & \end{pmatrix}$ is an upper triangular matrix. The $*$ denotes the fact that the upper triangular entries are in general functions of t. The bold faced **0** denotes the fact that the lower triangular entries are all zero. The diagonal entries are 1 or -1 only.

Proof. Let us look at the $1 \times (k+1)$ column vectors $\mathbf{C_n}$ (n=1,..., k+1) of the matrix (2.88) first. We describe the even columns $\mathbf{C_{2j}}$ and the odd columns $\mathbf{C_{2j-1}}$ separately.
The even column vector $\mathbf{C_{2j}}$, is defined to have 0 entries until the jth entry ($j = 1, 2, ...$) which is a_0', then followed by a_2', a_4'..., up to $a'_{2(k-j+1)}$,

$$\mathbf{C_{2j}} \ = \ (0, ..., a_0', a_2', ..., a'_{2(k-j+1)})^{\mathbf{T}}.$$

The odd column vectors $\mathbf{C_{2j-1}}$ have 0 entries until the jth entry which is a_0, then followed by a_2, a_4..., up to $a_{2(k-j+1)}$,

$$\mathbf{C_{2j-1}} \ = \ (0, ..., a_0, a_2, ..., a_{2(k-j+1)})^{\mathbf{T}}.$$

Immediately from the definition we see that

$$\frac{d\mathbf{C_{2j-1}}}{dt} = \mathbf{C_{2j}}. \tag{2.91}$$

Another relation for the column vectors can be obtained by differentiating $\mathbf{C_{2j}}$,

$$\begin{aligned}\frac{d\mathbf{C_{2j}}}{dt} &= (0,...,a_0'',a_2'',...,a_{2(k-j+1)}'')^{\mathbf{T}} \\ &= -\frac{t}{2}(0,...,a_0,a_2,...,a_{2(k-j+1)})^{\mathbf{T}} \\ &- (0,...,0,a_0,...,a_{2(k-j)})^{\mathbf{T}} \\ &= -\frac{t}{2}\mathbf{C_{2j-1}} - \mathbf{C_{2j+1}} \end{aligned} \tag{2.92}$$

where we have used relation (2.42) to rewrite a_{2j}''. Proposition 2.5.6 is equivalent to the statement:
for $n = 1, ..., k+1$,

$$\mathbf{C_n} = \begin{cases} \mathbf{C_{2j+1}} = (-1)^j \mathbf{C_1}^{(n-1)} + \sum_{k<n} \mu_k(t)\mathbf{C_k} & \text{for} \quad n \text{ odd}, \\ \mathbf{C_{2j+2}} = (-1)^j \mathbf{C_1}^{(n-1)} + \sum_{k<n} \mu_k(t)\mathbf{C_k} & \text{for} \quad n \text{ even} \end{cases} \tag{2.93}$$

where $\mu_k(t)$ denote functions of t. We will prove this by induction.
For the n odd case, $n = 1$, $j = 0$,

$$\mathbf{C_1} = \mathbf{C_1}.$$

For the n even case, $n = 2$, $j = 0$, by the definitions of $\mathbf{C_2}$ in equation (2.88)

$$\mathbf{C_2} = \mathbf{C_1}^{(1)}.$$

Now we want to show that in general when $n+1$ is odd

$$\mathbf{C_{n+1}} = \mathbf{C_{2j+3}} = \mathbf{C_{2(j+1)+1}} = (-1)^{j+1}\mathbf{C_1}^{(n)} + \sum_{k<n+1} \mu_k(t)\mathbf{C_k}$$

and when $n+1$ is even

$$\mathbf{C_{n+1}} = \mathbf{C_{2j+2}} = \mathbf{C_{2j+2}} = (-1)^j \mathbf{C_1}^{(n)} + \sum_{k<n+1} \mu_k(t)\mathbf{C_k}.$$

When $n+1$ is odd, n is even. Rewrite $\mathbf{C_{n+1}}$ using relation (2.92),

$$\mathbf{C_{n+1}} = -\frac{t}{2}\mathbf{C_{n-1}} - \mathbf{C_n}'.$$

Now use equation (2.93) for the expression of $\mathbf{C_n}$ for n is even

$$\mathbf{C_{n+1}} = -\frac{t}{2}\mathbf{C_{n-1}} - \left((-1)^j \mathbf{C_1}^{(n-1)} + \sum_{k<n} \mu_k(t)\mathbf{C_k}\right)' \tag{2.94}$$

$$= -\frac{t}{2}\mathbf{C_{n-1}} + (-1)^{j+1}\mathbf{C_1}^{(n)} + \sum_{k<n} \mu_k(t)\mathbf{C_k}' + \sum_{k<n} \mu_k'(t)\mathbf{C_k}. \tag{2.95}$$

Let us look at the first summation on line (2.95). Since

$$\mathbf{C_{2j}}' = -\frac{t}{2}\mathbf{C_{2j-1}} - \mathbf{C_{2j+1}},$$

the derivatives of $\mathbf{C_k}$ (k even) can all be expressed in terms of $\mathbf{C_{k-1}}$ and $\mathbf{C_{k+1}}$. The derivatives of $\mathbf{C_k}$ (k odd) can be rewritten in terms of $\mathbf{C_{k+1}}$ with equation (2.91). As k can be at most $n-1$, then the derivatives of $\mathbf{C_k}$ in this sum can be rewritten in terms of $\mathbf{C_k}$ with k at most n. The first and second summation in line (2.95) can be combined as one summation

$$\sum_{k<n+1} \tilde{\mu}_k(t)\mathbf{C_k}$$

where $\tilde{\mu}_k(t)$ are functions of t and

$$\mathbf{C_{n+1}} = (-1)^{j+1}\mathbf{C_1}^{(n)} + \sum_{k<n+1} \tilde{\mu}_k(t)\mathbf{C_k}.$$

For $n+1$ even, n is odd, use equation (2.91), then (2.93) for $\mathbf{C_n}$ odd

$$\begin{aligned}
\mathbf{C_{n+1}} &= \frac{d\mathbf{C_n}}{dt} \\
&= \left((-1)^j \mathbf{C_1}^{(n-1)} + \sum_{k<n} \mu_k(t)\mathbf{C_k}\right)' \\
&= (-1)^j \mathbf{C_1}^{(n)} + \sum_{k<n} \mu_k(t)\mathbf{C_k}' + \sum_{k<n} \mu_k'(t)\mathbf{C_k} \\
&= (-1)^j \mathbf{C_1}^{(n)} + \sum_{k<n+1} \hat{\mu}_k(t)\mathbf{C_k},
\end{aligned}$$

where $\hat{\mu}_k(t)$ are functions of t.

Equation (2.93) means that each $\mathbf{C_n}$ can be written in terms of $\mathbf{C_1^{(n-1)}}$ and $\mathbf{C_1}, \ldots, \mathbf{C_{n-1}}$, which really means that each $\mathbf{C_n}$ can be written in terms of

$\mathbf{C_1}, \ldots, \mathbf{C_1^{(n-1)}}$. In matrix notation

$$(\mathbf{C_1}\ \mathbf{C_2}\ \ldots\ \mathbf{C_{k+1}}) \;=\; (\mathbf{C_1}\ \mathbf{C_1}'\ \ldots\ \mathbf{C_1^{(k)}}) \begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & -1 & \\ & \mathbf{0} & & \ddots \end{pmatrix},$$

where $*$ denotes upper triangular entries which are in general functions of t, $\mathbf{0}$ denotes that all the lower triangular entries are zero. The diagonal entries are 1 or -1 only. We have proved proposition 2.5.6. □

Next we show that

Proposition 2.5.7.

$$(\mathbf{C_1}\ \mathbf{C_1}'\ \ldots\ \mathbf{C_1^{(k)}}) = \begin{pmatrix} 1 & & & \\ & 2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \frac{2^k}{k!} \end{pmatrix} (\mathbf{D_1}\ \mathbf{D_2}\ \ldots\ \mathbf{D_{k+1}}) \qquad (2.96)$$

where $\begin{pmatrix} 1 & & & \\ & 2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \frac{2^k}{k!} \end{pmatrix}$ is a constant diagonal matrix.

Proof. Recall

$$\mathbf{C_1} = (a_0, a_2, \ldots, a_{2k})^{\mathbf{T}}$$

and

$$\mathbf{D_1} = (a_0,\ \frac{da_0}{dt},\ \frac{d^2a_0}{dt^2},\ \ldots\ \frac{d^ka_0}{dt^k})^{\mathbf{T}}.$$

From equation (2.55) we have

$$\begin{aligned} a_{2j} &= \frac{2}{j}\frac{da_{2j-2}}{dt} \\ &= \frac{2}{j}\frac{2}{j-1}\frac{d^2a_{2j-4}}{dt^2} \\ &\ \ \vdots \\ &= \frac{2^j}{j!}\frac{d^ja_0}{dt^j}, \end{aligned}$$

therefore

$$\mathbf{C_1} = (a_0,\ 2\frac{da_0}{dt},\ 2\frac{d^2a_0}{dt^2},\ \ldots\ \frac{2^k}{k!}\frac{d^ka_0}{dt^k})^{\mathbf{T}} = \begin{pmatrix} 1 & & & \\ & 2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \frac{2^k}{k!} \end{pmatrix} \mathbf{D_1} \quad (2.97)$$

where the matrix multuplying $\mathbf{D_1}$ is a constant diagonal matrix. Differentiate equation (2.97) with respect to t n times we have

$$\mathbf{C_1^{(n)}} = \frac{d^n\mathbf{C_1}}{dt^n} = \begin{pmatrix} 1 & & & \\ & 2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \frac{2^k}{k!} \end{pmatrix} \mathbf{D_1^{(n)}},$$

then the matrix is

$$\begin{aligned} & (\mathbf{C_1}\ \mathbf{C_1}'\ \ldots\ \mathbf{C_1^{(k)}}) \\ = \ & (\mathbf{C_1}\ \frac{d\mathbf{C_1}}{dt}\ \ldots\ \frac{d^k\mathbf{C_1}}{dt^k}) = \begin{pmatrix} 1 & & & \\ & 2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \frac{2^k}{k!} \end{pmatrix} (\mathbf{D_1}\ \frac{d\mathbf{D_1}}{dt}\ \ldots\ \frac{d^k\mathbf{D_1}}{dt^k}) \\ = \ & \begin{pmatrix} 1 & & & \\ & 2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \frac{2^k}{k!} \end{pmatrix} (\mathbf{D_1}\ \mathbf{D_2}\ \ldots\ \mathbf{D_{k+1}}). \end{aligned}$$

We have proved proposition 2.5.7. □

Proposition 2.5.6 and 2.5.7 together give us

$$\begin{aligned}
&(\mathbf{C_1}\ \mathbf{C_2}\ ...\ \mathbf{C_{k+1}}) \\
=\ &(\mathbf{C_1}\ \mathbf{C_1}'\ ...\ \mathbf{C_1^{(k)}}) \begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & -1 & \\ & \mathbf{0} & & \ddots \\ & & & \end{pmatrix} \\
=\ &\begin{pmatrix} 1 & & & \\ & 2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \frac{2^k}{k!} \end{pmatrix} (\mathbf{D_1}\ \mathbf{D_2}\ ...\ \mathbf{D_{k+1}}) \begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & -1 & \\ & \mathbf{0} & & \ddots \\ & & & \end{pmatrix}.
\end{aligned}$$

Take the determinant on both sides, note the determinants of the first and third matrix on the RHS are both just products of their diagonal entries which are all constants. We have finally

$$|\mathbf{C_1}\ \mathbf{C_2}\ \mathbf{C_3}\ ...\ \mathbf{C_{k+1}}| = \mu \left| \mathbf{D_1}\ \frac{d\mathbf{D_1}}{dt}\ ...\ \frac{d^k\mathbf{D_1}}{dt^k} \right|$$

where μ is a constant. We have proved proposition 2.5.5. □

Thus, we have shown that

$$f_{k+\frac{1}{2}}(t) = \frac{d}{dt} \ln a_0^{(k+\frac{1}{2})}(t) = \frac{d}{dt} \ln \frac{\tau_{k+1}}{\tau_k}$$

where

$$\tau_{k+1}(t) = \begin{vmatrix} a_0 & a_0' & \cdots & \frac{d^k a_0}{dt^k} \\ a_0' & a_0'' & \cdots & \frac{d^{k+1} a_0}{dt^{k+1}} \\ \vdots & & \ddots & \vdots \\ \frac{d^{k-1} a_0}{dt^{k-1}} & \frac{d^k a_0}{dt^k} & \cdots & \frac{d^{2k-1} a_0}{dt^{2k-1}} \quad \frac{d^{2k} a_0}{dt^{2k}} \\ \frac{d^k a_0}{dt^k} & \frac{d^{k+1} a_0}{dt^{k+1}} & \cdots & \frac{d^{2k} a_0}{dt^{2k}} \quad \frac{d^{2k+1} a_0}{dt^{2k+1}} \end{vmatrix}.$$

We have thus proved theorem 2.5.3 □

In this chapter, we have described only two of the cases which Flaschka and Newell have studied in [**18**]. This is because we have only investigated

the q-analogues of these two cases for the q-discrete Painlevé equation considered in our work. They are the two most obvious special cases for which the associated linear problems simplify and indicate special solutions for the Painlevé equations. However, there are many aspects of the Painlevé equations which may be understood via their associated linear problems, such as the asymptotic behaviour, the connection formula, the Bäcklund transformation , the integral representation etc. see [**18**] for more details.

CHAPTER 3

q-linear analysis

The aim of this thesis is to develop the technique for studying the q-Painlevé equations via their associated linear problems, for which we will need results on the analytic theory of the q-linear system developed a hundred years ago by Birkhoff and his students. We review theorems of the Birkhoff school [**14, 9, 52**] on q-linear equations, and see how they work for the two simplest cases.

3.1. Analytic theory of the q-discrete linear system

Carmichael [**14**] and Birkhoff [**9**] studied the $n \times n$ q-linear system of the form

$$Y(qx) = \left(A_0 + A_1 x + A_2 x^2 + ... + A_\mu x^\mu\right) Y(x)$$

where where A_0 has eigenvalues q^{θ_j}, A_μ has eigenvalues q^{ρ_j}, $j = 1, ...n$, and for $i \neq j$, $\theta_i - \theta_j$, $\rho_i - \rho_j \neq$ integer. This is referred as the "non-resonant" condition for the coefficient matrix of the q-linear system. Carmichael has shown under the non-resonant condition the existence of two sets of fundamental solutions, one in a neighbourhood of $x = 0$ and the other in a neighbourhood of $x = \infty$. He has further shown that these two fundamental solution matrix can be related by a q-constant connection matrix $P(x)$, that is $P(qx) = P(x)$. Hence giving the ingredients of a theory that can be regarded as the answer to the q-discrete analogue of the Riemann-Hilbert problem. Birkhoff later has provided a explicit formula for the connection matrix $P(x)$, and demonstrated by applying on the simplest example. That is the 1×1 linear system with μ being 1. He has shown that the solution of this problem is the q-discrete analogue of the Gamma function. Le Caine [**52**], another student of Birjhoff investigated the next case from the point of view of simplicity. That is the 2×2 linear system with μ being 1 case. She

has found that the solutions are the q-discrete analogue of the Gauss hypergeometric functions. We reproduce these two examples in the next section. First we state the theorems of Carmichael and Birkhoff.

Theorem 3.1.1. [**14**] Consider the $n \times n$ q-discrete linear system

$$Y(qx) = \left(A_0 + A_1 x + A_2 x^2 + ... + A_\mu x^\mu\right) Y(x), \tag{3.1}$$

where A_0 has eigenvalues q^{θ_j}, A_μ has eigenvalues q^{ρ_j}, $j = 1, ...n$. For $i \neq j$, $\theta_i - \theta_j$, $\rho_i - \rho_j \neq$ integer, the q-linear system (3.1) has fundamental matrix solutions $Y_0(x)$, $Y_\infty(x)$ given by

$$\begin{cases} Y_0(x) = \left(x^{\theta_j}\epsilon_{ij}(x)\right)_{1\leq i,j\leq n}, \\ Y_\infty(x) = q^{\frac{\mu}{2}(t^2-t)}\left(x^{\rho_j}\delta_{ij}(x)\right)_{1\leq i,j\leq n}, \end{cases} \tag{3.2}$$

where $t = \frac{\ln x}{\ln q}$, $(\epsilon_{ij})_{1\leq i,j\leq n}$ and $(\delta_{ij})_{1\leq i,j\leq n}$ are $n \times n$ matrices of analytic functions which can be expanded as a power-series in x or $\frac{1}{x}$ around $x = 0$ and ∞, respectively.

Birkhoff [**9**] has completed the study of the non-resonant case of the $n \times n$ linear q-discrete system by solving its generalized Riemann problem, giving an explicit formula for the connection matrix $P(x)$.

Theorem 3.1.2. [**9**] The two fundamental matrix solutions $Y_0(x)$ and $Y_\infty(x)$ are related by the connection matrix $P(x)$,

$$Y_0(x) = Y_\infty(x)P(x)$$

where $P(x)$ is a $n \times n$ matrix of functions, defined in terms of the Weierstrass sigma function $\sigma(t)$. It is "q constant", that is $P(qx) = P(x)$. More explicitly $P(x) = p_{ij}$, for $i, j = 1, 2, ...n$

$$p_{ij} = c_{ij}e^{-\frac{\eta}{2}\mu t^2 + \left(\eta_1(\theta_i-\rho_j) - \frac{\eta_2\mu}{2}\right)t}\sigma(t-a_1^{(i,j)})\sigma(t-a_2^{(i,j)})...\sigma(t-a_\mu^{(i,j)}) \tag{3.3}$$

where $\omega_1 = 1$ and $\omega_2 = \frac{2\pi i}{\ln q}$ are the two periods of the $\sigma(t)$ function, which satisfies the relations

$$\begin{cases} \sigma(t+\omega_1) = -\exp\left(\eta_1(t+\frac{1}{2})\right)\sigma(t), \\ \sigma(t+\omega_2) = -\exp\left(\eta_2(t+\frac{\omega_2}{2})\right)\sigma(t), \quad \text{with} \quad \eta_1\omega_2 - \eta_2 = 2\pi i. \end{cases}$$

The μ constants, $a_\lambda^{(i,j)}$, satisfy the constraint

$$\sum_{\lambda=1}^{\mu} a_\lambda^{(i,j)} = \theta_i + \rho_j - \frac{\mu\pi i}{\ln q}. \tag{3.4}$$

In the same paper [**14**] Carmichael had an inverse of theorem 3.1.1, stated as follows.

Theorem 3.1.3. [**14**] Let $Y_0(x)$ and $Y_\infty(x)$ be two sets of $n \times n$ single-valued functions which, away from zero and infinity, are analytic, defined by

$$\begin{cases} Y_0(x) = \left(x^{\theta_j}\epsilon_{ij}(x)\right)_{1\le i,j\le n}, & |\left(\epsilon_{ij}(x)\right)_{1\le i,j\le n}| \neq 0 \\ Y_\infty(x) = q^{\frac{\mu}{2}(t^2-t)}\left(x^{\rho_j}\delta_{ij}(x)\right)_{1\le i,j\le n}, & |\left(\delta_{ij}(x)\right)_{1\le i,j\le n}| \neq 0, \end{cases} \tag{3.5}$$

where $t = \frac{\ln x}{\ln q}$ and for $i \neq j$, $\theta_i - \theta_j$, $\rho_i - \rho_j \neq$ integer. The two setes of functions being connected by the relation

$$Y_0(x) = Y_\infty(x)P(x), \qquad P(qx) = P(x).$$

Then the two sets of functions $Y_0(x)$ and $Y_\infty(x)$ are the two simple fundamental solutions of a system of q-discrete equations

$$Y(qx) = A(x)Y(x),$$

in which the $n \times n$ coefficient matrix $A(x)$ is a polynomial of degree μ in x.

Above theorem is not simply a repetition of theorem 3.1.1. Rather, it has a significant implication on the iso-monodromy deformation problem of q-Painlevé equations and especially relevant in this thesis. It says that given the forms of the solutions of the linear problem and how they are related to each other, one can reconstruct the equation they satisfy. In the context of the associated linear problems for the q-Painlevé equations, it means that once we have solved the spectral linear problem of the Lax pair we can reconstruct information of the solution of the q-Painlevé equation of which the coefficients of the linear problem are defined in terms of.

Note: Ramis [**65**] has introduced analytic functions which can be used in place of the "$t = \frac{\ln x}{\ln q}$" function used by the Birkhoff school, so that there is no logarithmic singularity at $x = 0$ or ∞. However, in this thesis, we follow Birkhoff school, due to its simplicity of expansion.

3.2. Case 1×1, degree 1

From constraint (3.4) we can see that in general, the only cases where $a_\lambda^{(i,j)}$ can be evaluated explicitly are when $\mu = 1$. Birkhoff [**9**] has solved the simplest case, the 1×1, $\mu = 1$ case.
Consider the q-discrete equation

$$y(qx) = (1 - x)y(x), \tag{3.6}$$

where $\theta_1 = 0$, $\rho_1 = -\frac{i\pi}{\ln q}$, and $\mu = 1$.
The solutions at $x = 0$ and $x = \infty$ are given by the infinite products

$$\begin{cases} y_0(x) = \left(1 - \frac{x}{q}\right)\left(1 - \frac{x}{q^2}\right)\ldots, \\ y_\infty(x) = q^{\frac{1}{2}(t^2-t)}e^{-\pi i t}\frac{1}{1-\frac{1}{x}}\frac{1}{1-\frac{1}{qx}}\ \ldots, \end{cases} \tag{3.7}$$

which can be verified by substitution. The connection matrix $P(x)$ for the 1×1 case is $p(x)$ where

$$y_0(x) = y_\infty(x)p(x). \tag{3.8}$$

The constant a_1 in the definition (3.3) of $p(x)$ can be calculated from equation (3.4)

$$\sum_{\lambda=1}^{1} a_\lambda = \theta_1 - \rho_1 - \frac{\pi i}{\ln q} = 0 + \frac{\pi i}{\ln q} - \frac{\pi i}{\ln q} = 0.$$

Now we are ready to evaluate the function $p(x)$ using formula (3.3), we have:

$$\begin{aligned} p(x) &= ce^{-\frac{1}{2}t^2+[(\eta_1(\theta_1-\rho_1)-\frac{\eta_2}{2}]t}\sigma(t-a_1) \\ &= ce^{-\frac{1}{2}t^2+[(\eta_1(\frac{i\pi}{\ln q})-\frac{\eta_2}{2}]t}\sigma(t) \\ &= ce^{-\frac{1}{2}t^2+i\pi t}\sigma(t), \end{aligned} \tag{3.9}$$

where we have used the relation

$$\eta_1\frac{i\pi}{\ln q} - \eta_2 = 2i\pi.$$

The constant c can be found by evaluating equation (3.8) at $x = 1$ using expressions (3.7) and (3.9), we have

$$y_0(x) = (x-1)y_\infty(x)\frac{p(x)}{x-1} \tag{3.10}$$

$$\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)\ldots = c\,\frac{1}{\left(1-\frac{1}{q}\right)}\frac{1}{\left(1-\frac{1}{q^2}\right)}\ldots \tag{3.11}$$

$$\Rightarrow \qquad c = \left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{q^2}\right)^2\ldots \tag{3.12}$$

where we have used the fact $\sigma(0) = 0$, $\sigma'(0) = 1$.

Note: The solution of equation (3.6) is denoted as $\Gamma_q(1-x)$, and is the q-discrete analogue of the Gamma function. In Birkhoff's notation, the solutions at $x = 0$ and $x = \infty$ are denoted as

$$\begin{cases}\Gamma_q^0(1-x),\\ \Gamma_q^\infty(1-x),\end{cases}$$

and the two solutions are related by

$$\Gamma_q^0(1-x) = \Gamma_q^\infty(1-x)p(x).$$

$\Gamma_q(1-x)$ is a special case of the so called basic or q-hypergeometric function defined in Appendix B as

$$\begin{aligned}&{}_m\Phi_n\left(a_1,\ldots a_m;b_1,\ldots b_n;q,z\right) \\ &= \sum_{k\geq 0}\frac{(a_1;q)_k\ldots(a_m;q)_k}{(b_1;q)_k\ldots(b_n;q)_k(q;q)_k}\left[(-1)^k q^{\binom{n}{2}}\right]^{1+n-m} z^k\end{aligned} \tag{3.13}$$

where

$$(a;q)_k = \begin{cases}1, & k=0,\\ (1-a)(1-aq)\ldots(1-aq^{k-1}), & k=1,2,\ldots.\end{cases}$$

Proposition 3.2.1. Solution of equation (3.6) is given by

$$\Gamma_q(1-x) = {}_1\Phi_0(0,-;q,x) = \sum_{j=0}^{\infty}\frac{x^j}{(q;q)_j}.$$

Note in this case the n in ${}_m\Phi_n$ is 0, which means there is no b_j, this is denoted by "$-$ " in the notation above.

Proof. Let

$$y(x) = \sum_{j=0}^{\infty} \frac{x^j}{(q;q)_j},$$

then

$$\begin{aligned} y(qx) - y(x) &= \sum_{j=0}^{\infty} \frac{x^j(q^j - 1)}{(q;q)_j} \\ &= -x \sum_{j=1}^{\infty} \frac{x^{j-1}}{(q;q)_{j-1}} \\ &= -xy(x) \\ y(qx) &= (1-x)y(x). \end{aligned}$$

□

Proposition 3.2.2. $\Gamma_q(1-x)$ also has the infinite product expression

$$\Gamma_q(1-x) = \frac{1}{(x;q)_\infty}$$

where

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\dots.$$

Proof. Let

$$y(x) = \frac{1}{(1-x)(1-qx)(1-q^2x)...},$$

then

$$\begin{aligned} y(qx) &= \frac{1}{(1-qx)(1-q^2x)(1-q^3x)...} \\ &= \frac{(1-x)}{(1-x)(1-qx)(1-q^2x)(1-q^3x)...} \\ &= (1-x)y(x). \end{aligned}$$

We have showed that the two different definitions for $\Gamma_q(1-x)$ in Proposition 3.2.2 and 3.2.1 satisfy the same equation, and since they have the same asymptotic behavior at $x = 0$, they must be the same function. □

To summarize, the solution of equation (3.6) is

$$\begin{aligned}\Gamma_q(1-x) = \frac{1}{(x;q)_\infty} &= {}_1\Phi_0(0,-;q,x) \\ &= \sum_{j=0}^{\infty} \frac{x^j}{(q;q)_j},\end{aligned} \tag{3.14}$$

where we have the $\Gamma_q(1-x)$ function in both the products form and the summation form.

3.3. Case 2×2, degree 1

The next case of the q-linear system (3.1) to consider from the point of view of simplicity is the 2×2, $\mu = 1$ case. Le Caine [**52**] considered the q-linear system

$$Y(qx) = (Ax + B)Y(x) \tag{3.15}$$

where A has eigenvalues a_1, a_2, and B has eigenvalues b_1, b_2. The coefficient matrix has zeros at α_1, α_2, that is

$$|Ax + B| = 0$$

when $x = \alpha_1, \alpha_2$. The six constants $a_1, a_2, b_1, b_2, \alpha_1$ and α_2 are called the characteristic constants of system (3.15), they satisfy the constraint $b_1 b_2 = a_1 a_2 \alpha_1 \alpha_2$. Le Caine proved that any 2×2 system $Y(qx) = (Ax + B)Y(x)$ with these six characteristic constants can be transformed into what she called the "normal form":

$$Y(qx) = \begin{pmatrix} 0 & a_1(x-\alpha_1) \\ -a_2(x-\alpha_2) & (a_1+a_2)x + b_1 + b_2 \end{pmatrix} Y(x), \tag{3.16}$$

hence we will consider equation (3.16) from now on.

Theorem 3.3.1. [**52**] The q-linear system (3.16) has fundamental solution matrices $Y^0(x)$,

$$\begin{aligned} Y^0(x) &= \Gamma_q^0\left(1 - \frac{x}{\alpha_1}\right) \times \\ &\begin{pmatrix} {}_2\Phi_1(a,b;c;q,\frac{x}{\alpha_2}) & {}_2\Phi_1(aq/c,bq/c,q^2/c;q,\frac{x}{\alpha_2}) \\ -\frac{b_1}{a_1\alpha_1}{}_2\Phi_1(a,b;c;q,\frac{qx}{\alpha_2}) & -\frac{b_2}{a_1\alpha_1}{}_2\Phi_1(aq/c,bq/c,q^2/c;q,\frac{qx}{\alpha_2}) \end{pmatrix} \begin{pmatrix} b_1^t & 0 \\ 0 & b_2^t \end{pmatrix}, \end{aligned}$$

and $Y^{\infty}(x)$,

$$\begin{aligned} Y^{\infty}(x) &= e^{\pi i t}\Gamma_q^{\infty}\left(1-\frac{\alpha_1}{x}\right)\times \\ &\begin{pmatrix} {}_2\Phi_1(a,\frac{aq}{c},\frac{aq}{b};q,\frac{q^2\alpha_1}{x}) & {}_2\Phi_1(b,\frac{bq}{c},\frac{bq}{a};q,\frac{q^2\alpha_1}{x}) \\ {}_2\Phi_1(a,\frac{aq}{c},\frac{aq}{b};q,\frac{q\alpha_1}{x}) & {}_2\Phi_1(b,\frac{bq}{c},\frac{bq}{a};q,\frac{q\alpha_1}{x}) \end{pmatrix}\times\begin{pmatrix} a_1^t & 0\\ 0 & a_2^t\end{pmatrix}, \end{aligned}$$

where ${}_2\Phi_1(a,b;c;q,z)$ is the q-discrete analogue of Gauss's hypergeometric function, and is discussed in detail in Appendix B.
The solutions are related by

$$Y^0(x) = Y^{\infty}(x)P(x),$$

$$P(x) = p_{ij}, \qquad p_{ij} = p(-a_i x/b_j)c_{ij}, \qquad (i,j=1,2)$$

where

$$p(x) = \exp\left((-\eta/2)t^2 - \pi i t\right)\sigma(t),$$

$$\begin{aligned} c_{11} &= \prod_{\nu=0}^{\infty}\frac{(1+a_1\alpha_1 q^{\nu}/b_2)\,(1+b_1q^{\nu+1}/(a_2\alpha_2))}{(1-b_1q^{\nu+1}/b_2)(1-a_1q^{\nu}/a_2)}\\ c_{21} &= \prod_{\nu=0}^{\infty}\frac{(1+a_2\alpha_2 q^{\nu}/b_2)\,(1+b_1q^{\nu+1}/(a_1\alpha_2))}{(1-b_1q^{\nu+1}/b_2)(1-a_2q^{\nu}/a_2)}\\ c_{12} &= \prod_{\nu=0}^{\infty}\frac{(1+a_1\alpha_2 q^{\nu}/b_1)\,(1+b_2q^{\nu+1}/(a_2\alpha_2))}{(1-b_2q^{\nu+1}/b_1)(1-a_1q^{\nu}/a_2)}\\ c_{22} &= \prod_{\nu=0}^{\infty}\frac{(1+a_2\alpha_2 q^{\nu}/(b_1q^{\nu}))\,(1+b_2q^{\nu+1}/(a_1\alpha_2))}{(1-b_2q^{\nu+1}/b_1)(1-a_2q^{\nu}/a_1)}. \end{aligned}$$

Proof. By Theorem 3.1.1, we know the matrix solutions of equation (3.16) at $x=0$ and $x=\infty$ have the form

$$Y^0(x) = (E^{(1)}(x), E^{(2)}(x))\begin{pmatrix} b_1^t & 0\\ 0 & b_2^t\end{pmatrix}, \tag{3.17}$$

and

$$Y^{\infty}(x) = (F^{(1)}(x), F^{(2)}(x))\begin{pmatrix} a_1^t & 0\\ 0 & a_2^t\end{pmatrix}, \tag{3.18}$$

where $E^{(1)}(x), E^{(2)}(x)$ and $F^{(1)}(x), F^{(2)}(x)$ are 1×2 vectors of functions, analytic at $x=0$ and $x=\infty$ respectively.

To find $E^{(1)}(x)$, substitute the vector solution $b_1^t E^{(1)}(x)$ into equation (3.16), we have

$$E^{(1)}(qx) = \begin{pmatrix} 0 & \frac{a_1}{b_1}(x-\alpha_1) \\ -\frac{a_2}{b_1}(x-\alpha_2) & \frac{(a_1+a_2)x+b_1+b_2}{b_1} \end{pmatrix} E^{(1)}(x).$$

Now try to get this into a form where we can compare it with the 2×2 system for the q-hypergeometric equation (B.15), solved in terms of ${}_2\Phi_1(a,b;c;q,z)$. Let

$$E^{(1)}(x) = g_1(x)E_1^{(1)}(x),$$

where

$$g_1(qx) = \left(1-\frac{x}{\alpha_1}\right) g_1(x),$$

that is $g_1(x) = \Gamma_q^0\left(1-\frac{x}{\alpha_1}\right)$.
Let

$$E_1^{(1)}(x) = \begin{pmatrix} u \\ v_1 \end{pmatrix},$$

now

$$E_1^{(1)}(qx) = \begin{pmatrix} u(qx) \\ v_1(qx) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{a_1\alpha_1}{b_1} \\ -\frac{a_2(x-\alpha_2)}{b_1\left(1-\frac{x}{\alpha_1}\right)} & \frac{(a_1+a_2)x+b_1+b_2}{b_1\left(1-\frac{x}{\alpha_1}\right)} \end{pmatrix} \begin{pmatrix} u(x) \\ v_1(x) \end{pmatrix}.$$

A further transformation is required so that the $(1,2)$ entry is normalized to be 1. Let

$$v(x) = -\frac{a_1\alpha_1}{b_1}v_1(x) = u(qx),$$

then

$$\begin{pmatrix} u(qx) \\ v(qx) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b_2\left(\frac{x}{\alpha_2}-1\right)}{b_1\left(1-\frac{x}{\alpha_1}\right)} & \frac{(a_1+a_2)x+b_1+b_2}{b_1\left(1-\frac{x}{\alpha_1}\right)} \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

where we have used the relation $b_1b_2 = a_1a_2\alpha_1\alpha_2$ to simplify the expression.

Compare this with the q-hypergeometric system (B.15) to obtain

$$\begin{aligned} z &= \frac{x}{\alpha_2}, \\ abz - \frac{c}{q} &= \frac{b_1}{b_2}\left(\frac{x}{\alpha_1} - 1\right), \\ -\frac{(a_1+a_2)x + b_1 + b_2}{b_2} &= (a+b)z - \left(1 + \frac{c}{q}\right), \end{aligned}$$

which can be solved to get

$$\begin{aligned} z &= \frac{x}{\alpha_2}, \\ \frac{c}{q} &= \frac{b_1}{b_2}, \\ a &= -\frac{a_2\alpha_2}{b_2}, \\ b &= -\frac{a_1\alpha_2}{b_2}, \end{aligned}$$

and

$$u(x) = {}_2\Phi_1\left(a, b; c; q, \frac{x}{\alpha_2}\right).$$

Hence, one of the solutions of the 2×2, $\mu = 1$ linear system (3.15) is

$$b_1^t \Gamma_q^0 \left(1 - \frac{x}{\alpha_1}\right) \begin{pmatrix} {}_2\Phi_1(a, b; c; q, \frac{x}{\alpha_2}) \\ -\frac{a_1\alpha_1}{b_1}\,{}_2\Phi_1(a, b; c; q, \frac{qx}{\alpha_2}) \end{pmatrix}.$$

The other vector solution can be found similarly. It is also possible to find $c_{11,12,21,22}$ in the connection matrix $P(x)$ since the exact form of Y_0 and Y_∞ are known to be expressed in terms of q-hypergeometric functions. For the details see [**52**]. □

All the theorems and examples in this Chapter apply to q-linear systems of non-resonant type. Adams [**5**] has dropped the non-resonant condition on the coefficient matrix and studied the solutions of the general q-linear system. The Riemann problem for the general q-linear system was looked at by Trjitzinsky [**73**], and more recently by Ramis, Sauloy and Zhang [**66, 69**]. However, since in the study of the associated linear problems of q-Painlevé equations, we have only looked at q-linear systems of non-resonant type , therefore we will not go into details of the theorems for the general q-linear system here.

CHAPTER 4

A q-discrete analogue of the second Painlevé equation

We have seen that the 2×2 q-linear system (3.1.1) with μ being 1 is solved by the q-hypergeometric function ${}_2\Phi_1(a, b; c; q, z)$. Its continuous analogue is Gauss hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; t)$. However the associated linear problems of the q-Painlevé equations are 2×2 q-linear systems, whose coefficient matrices are polynomials of degree μ greater than 1. In this chapter, we will show that these linear systems can be solved in terms of the classical q-special function for the special cases when their associated non-linear equation, that is the q-Painlevé equations, admit special solutions for particular values of the parameters. This fact is then inverted and used to find the determinantal forms of some special solutions of the q-Painlevé equations which is the main result of this work.

4.1. A q-discrete analogue of PII and its associated linear problems

The associated linear problems consist of **the spectral problem**

$$\widehat{\Psi}(\nu, x) = \Psi(\nu/\lambda^2, x) = A(\nu, x)\Psi(\nu, x) \tag{4.1}$$

and **the deformation equation**

$$\overline{\Psi}(\nu, x) = \Psi(\nu, x/\lambda) = B(\nu, x)\Psi(\nu, x) \tag{4.2}$$

where

$$\begin{aligned} A(\nu, x) &= A_0(x) + A_1(x)\nu + A_2(x)\nu^2 + A_3(x)\nu^3 \\ &= \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} 0 & m_1(x) \\ m_2(x) & 0 \end{pmatrix} \nu + \begin{pmatrix} n_1(x) & 0 \\ 0 & n_2(x) \end{pmatrix} \nu^2 \\ &\quad + \begin{pmatrix} 0 & f_1(x) \\ f_2(x) & 0 \end{pmatrix} \nu^3, \end{aligned}$$

$$B(\nu,x)=\begin{pmatrix} -i\sqrt{\alpha}\frac{\lambda x}{g(\lambda x)} & i\frac{\lambda^2 x}{\sqrt{\alpha}}\nu \\ i\frac{x}{\sqrt{\alpha}}\nu & i\frac{g(\lambda x)}{\sqrt{\alpha}x\lambda}\end{pmatrix}.$$

Note, ν is the q-discrete analogue of the spectral variable, e_1, e_2 are constants, $m_1(x)$, $m_2(x)$, $n_1(x)$, $n_2(x)$, $f_1(x)$ and $f_2(x)$ are functions of $g(x)$ and x the q-discrete Painlevé variable

$$m_1(x)=\frac{e_2 g(x\lambda)\left(-e_1x^2\lambda^2-e_2g(x)+e_2g(x)g(x\lambda)\right)}{e_1x^2\lambda}, \tag{4.3}$$

$$m_2(x)=-\frac{x^2\left(e_1x^2\lambda^2-e_2g(x)g(x\lambda)\lambda^2+e_1g(x\lambda)\right)}{\lambda g(x)g(x\lambda)^2}, \tag{4.4}$$

$$n_1(x)=\frac{e_2\left(e_1x^2+e_2g(x)g(x\lambda)^2-e_2g(x)g(x\lambda)\right)}{e_1g(x\lambda)}, \tag{4.5}$$

$$n_2(x)=-\frac{e_2\left(-e_1x^4\lambda^4-e_1x^2g(x\lambda)\lambda^2+e_2g(x)g(x\lambda)^2\right)}{e_1\lambda^2g(x)g(x\lambda)}, \tag{4.6}$$

$$f_1(x)=-\frac{e_2^2x^2\lambda}{e_1}, \tag{4.7}$$

$$f_2(x)=-\frac{e_2^2x^2}{e_1\lambda}. \tag{4.8}$$

The compatibility of q-linear systems (4.1, 4.2), $\overline{\widehat{\Psi}}=\widehat{\overline{\Psi}}$ is :

$$A(\nu,x/\lambda)B(\nu,x)-B(\nu/\lambda^2,x)A(\nu,x)=0$$

which forces $g(x)$ to satisfy a second-order non-linear q-discrete equation

$$g(x/\lambda)g(\lambda x)=\frac{\alpha x^2(g(x)+x^2)}{g(x)(g(x)-1)}. \tag{4.9}$$

This is a q-discrete analogue of PII, denoted q-PII. Equations (4.1, 4.2) together are called the Lax pair of equation (4.9).

4.2. q-Linear analysis of the spectral problem

We apply Carmichael's Theorem (3.1.1) on the formal solutions of the linear q-discrete system to the spectral problem (4.1) of q-PII which is a 2×2 q-linear system with polynomial coefficient matrix of degree $\mu=3$. We concentrate our analysis on the spectral half of the Lax pair for the reasons similar to those of the continuous case. We see that the coefficient matrix of the spectral problem has polynomial dependence on its variable

ν, whereas the dependence of the deformation equation on its variable x is transcendental, via the solution of q-PII $g(x)$.

4.2.1. Series expansion near $\nu = 0$.

Proposition 4.2.1. There exists a fundamental solution matrix Φ of the q-linear systems (4.1, 4.2) at $\nu = 0$, for $\frac{e_1}{e_2} \neq \frac{1}{\lambda^{4k}}$, k is an integer or half integer,

$$\Phi(\nu, x) = \{\phi_1(\nu, x), \phi_2(\nu, x)\},$$

$$\phi_1(\nu, x) = e_1^t \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(x)\nu^{2j} \\ c_{2j+1}(x)\nu^{2j+1} \end{pmatrix}, \tag{4.10}$$

$$\phi_2(\nu, x) = e_2^t \sum_{j=0}^{\infty} \begin{pmatrix} b_{2j+1}(x)\nu^{2j+1} \\ d_{2j}(x)\nu^{2j} \end{pmatrix} \tag{4.11}$$

where $t = -\frac{1}{2}\frac{\ln \nu}{\ln \lambda}$, and

$$e_1 a_j(1/\lambda^{2j} - 1) = m_1 c_{j-1} + n_1 a_{j-2} + f_1 c_{j-3} \tag{4.12}$$

$$c_j(e_1/\lambda^{2j} - e_2) = m_2 a_{j-1} + n_2 c_{j-2} + f_2 a_{j-3} \tag{4.13}$$

$$e_2 d_j(1/\lambda^{2j} - 1) = m_2 b_{j-1} + n_2 d_{j-2} + f_2 b_{j-3} \tag{4.14}$$

$$b_j(e_2/\lambda^{2j} - e_1) = m_1 d_{j-1} + n_1 b_{j-2} + f_1 d_{j-3}, \tag{4.15}$$

where $m_1, m_2, n_1, n_2, f_1, f_2$ are defined earlier by equations (4.3)-(4.8).

In particular we will show that

$$g(x) = -ix\sqrt{\alpha}\frac{a_0(x/\lambda)}{a_0(x/\lambda^2)}, \tag{4.16}$$

$$g(x) = -ix\sqrt{\alpha}\frac{d_0(x/\lambda^2)}{d_0(x/\lambda)}. \tag{4.17}$$

Proof. By Carmichael's theorem, we know near $\nu = 0$ the solution matrix has the form

$$\Phi(\nu, x) = \sum_{j=0}^{\infty} \begin{pmatrix} a_j(x) & b_j(x) \\ c_j(x) & d_j(x) \end{pmatrix} \nu^j \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix}. \tag{4.18}$$

To obtain the recurrence relations (4.12-4.15) for the coefficients a_j, b_j, c_j, d_j, substitute solution (4.18) into equation (4.1) and equate powers of ν near

$\nu = 0$. We see for $e_1 \neq e_2$, $b_0 = c_0 = 0$, while a_0, d_0 are arbitrary. The fact that $b_0 = c_0 = 0$ further implies

$$a_{\text{odd}} = c_{\text{even}} = d_{\text{odd}} = b_{\text{even}} = 0. \tag{4.19}$$

Hence the fundamental matrix solution has the form:

$$\begin{aligned}\Phi(\nu, x) &= \left\{ \begin{pmatrix} a_0(x) & 0 \\ 0 & d_0(x) \end{pmatrix} + \begin{pmatrix} 0 & b_1(x) \\ c_1(x) & 0 \end{pmatrix} \nu + \begin{pmatrix} a_2(x) & 0 \\ 0 & d_2(x) \end{pmatrix} \nu^2 \right. \\ &\quad \left. + \begin{pmatrix} 0 & b_3(x) \\ c_3(x) & 0 \end{pmatrix} \nu^3 + \ldots \right\} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix}.\end{aligned}$$

Let $\Phi = \{\phi_1, \phi_2\}$, then

$$\begin{aligned}\phi_1(\nu, x) &= e_1^t \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(x)\nu^{2j} \\ c_{2j+1}(x)\nu^{2j+1} \end{pmatrix}, \\ \phi_2(\nu, x) &= e_2^t \sum_{j=0}^{\infty} \begin{pmatrix} b_{2j+1}(x)\nu^{2j+1} \\ d_{2j}(x)\nu^{2j} \end{pmatrix}.\end{aligned}$$

Note that a_j, b_j, c_j and d_j are in general functions of x as m_1, m_2, n_1, n_2, f_1 and f_2 are functions of x. To find the x dependence of $a_j(x), c_j(x)$, for example, we substitute $\phi_1(\nu, x)$ solution (4.10) into the other half of the Lax pair, the deformation equation (4.2),

$$\begin{aligned}\phi_1(\nu, x/\lambda) &= e_1^t \begin{pmatrix} a_0(x/\lambda) + \ldots \\ c_1(x/\lambda)\nu + \ldots \end{pmatrix} \\ = e_1^t \begin{pmatrix} -i\sqrt{\alpha}\frac{\lambda x}{g(\lambda x)} & i\frac{\lambda^2 x}{\sqrt{\alpha}}\nu \\ i\frac{x}{\sqrt{\alpha}}\nu & i\frac{g(\lambda x)}{\sqrt{\alpha} x \lambda} \end{pmatrix} &\begin{pmatrix} a_0(x) + \ldots \\ c_1(x)\nu + \ldots \end{pmatrix}.\end{aligned}$$

Equating powers in ν:

$$\begin{aligned}\nu^0 : \qquad a_0(x/\lambda) &= -i\sqrt{\alpha}\frac{\lambda x}{g(\lambda x)} a_0(x) \\ \Rightarrow \qquad g(x) &= -ix\sqrt{\alpha}\frac{a_0(x/\lambda)}{a_0(x/\lambda^2)}\end{aligned} \tag{4.20}$$

This says that the solution of q-PII is given by the ratio of the leading coefficient of the solution of its associated linear problems around $\nu = 0$, with different shifts in the x direction. This difference in shift corresponds to differentiation in the continuous setting. Equation (4.17) can be found similarly, so are the rest of the coefficients in the expansion (4.18). □

4.2.2. Series expansion near $\nu = \infty$.

Proposition 4.2.2. The fundamental solution matrix to the q-linear systems (4.1, 4.2) at $\nu = \infty$ is

$$\Psi = \{\psi_1, \psi_2\},$$

$$\psi_1(\nu, x) = I(\nu)J(x)u(\nu, x) \begin{pmatrix} \lambda & \lambda \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} \alpha_j(x) \\ \gamma_j(x) \end{pmatrix} \frac{1}{\nu^j}, \quad (4.21)$$

$$\psi_2(\nu, x) = I(\nu)J(x)v(\nu, x) \begin{pmatrix} \lambda & \lambda \\ 1 & -1 \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} \beta_j(x) \\ \delta_j(x) \end{pmatrix} \frac{1}{\nu^j} \quad (4.22)$$

where

$$I(\nu) = \left(\frac{e_2^2}{e_1}\right)^t, \qquad t = -\frac{1}{2}\frac{\ln \nu}{\ln \lambda},$$
$$J(x) = \left(\frac{e_2}{e_1}\right)^{\frac{s}{2}}, \qquad s = -\frac{\ln x}{\ln \lambda},$$

$$u(\nu, x) = \Gamma_{\lambda^{-2}}(1 + \nu/\lambda)\Gamma_{\lambda^{-2}}(1 + ix\nu)\Gamma_{\lambda^{-2}}(1 + ix\lambda\nu),$$
$$v(\nu, x) = \Gamma_{\lambda^{-2}}(1 - \nu/\lambda)\Gamma_{\lambda^{-2}}(1 - ix\nu)\Gamma_{\lambda^{-2}}(1 - ix\lambda\nu).$$

Coefficients $\alpha_j, \beta_j, \gamma_j$ and δ_j satisfy the recurrence relations

$$\begin{aligned} & -x^2\frac{e_2^2}{e_1}\alpha_j(\lambda^{2j} - 1) \qquad (4.23) \\ = & \left(\frac{(n_1 + n_2)}{2} + \frac{e_2^2}{e_1}\frac{x(-i - i\lambda + x\lambda^2)\lambda^{2j-2}}{\lambda}\right)\alpha_{j-1} + \left(\frac{(n_1 - n_2)}{2}\right)\gamma_{j-1} \\ & \left(\frac{(m_1 + m_2\lambda^2)}{2\lambda} - \frac{e_2^2}{e_1}\frac{i(-i + \lambda x + \lambda^2 x)\lambda^{2j-4}}{\lambda}\right)\alpha_{j-2} + \frac{(-m_1 + m_2\lambda^2)}{2\lambda}\gamma_{j-2} \\ & + \left(\frac{(e_1 + e_2)}{2} - \frac{e_2^2\lambda^{2j-6}}{e_1}\right)\alpha_{j-3} + \frac{(e_1 - e_2)}{2}\gamma_{j-3}, \end{aligned}$$

$$
\begin{aligned}
& -x^2\frac{e_2^2}{e_1}\gamma_j(\lambda^{2j}+1) \qquad (4.24)\\
= \;& \left(\frac{(n_1+n_2)}{2}+\frac{e_2^2}{e_1}\frac{x(-i-i\lambda+x\lambda^2)\lambda^{2j-2}}{\lambda}\right)\gamma_{j-1}+\left(\frac{(n_1-n_2)}{2}\right)\alpha_{j-1}\\
& \left(\frac{-(m_1+m_2\lambda^2)}{2\lambda}-\frac{e_2^2}{e_1}\frac{i(-i+\lambda x+\lambda^2 x)\lambda^{2j-4}}{\lambda}\right)\gamma_{j-2}+\frac{(m_1-m_2\lambda^2)}{2\lambda}\alpha_{j-2}\\
& +\left(\frac{(e_1+e_2)}{2}-\frac{e_2^2\lambda^{2j-6}}{e_1}\right)\gamma_{j-3}+\frac{(e_1-e_2)}{2}\alpha_{j-3},
\end{aligned}
$$

$$
\begin{aligned}
& x^2\frac{e_2^2}{e_1}\beta_j(\lambda^{2j}+1) \qquad (4.25)\\
= \;& \left(\frac{(n_1+n_2)}{2}+\frac{e_2^2}{e_1}\frac{x(-i-i\lambda+x\lambda^2)\lambda^{2j-2}}{\lambda}\right)\beta_{j-1}+\left(\frac{(n_1-n_2)}{2}\right)\delta_{j-1}\\
& \left(\frac{(m_1+m_2\lambda^2)}{2\lambda}+\frac{e_2^2}{e_1}\frac{i(-i+\lambda x+\lambda^2 x)\lambda^{2j-4}}{\lambda}\right)\beta_{j-2}+\frac{(-m_1+m_2\lambda^2)}{2\lambda}\delta_{j-2}\\
& +\left(\frac{(e_1+e_2)}{2}-\frac{e_2^2\lambda^{2j-6}}{e_1}\right)\beta_{j-3}+\frac{(e_1-e_2)}{2}\delta_{j-3},
\end{aligned}
$$

$$
\begin{aligned}
& x^2\frac{e_2^2}{e_1}\delta_j(\lambda^{2j}-1) \qquad (4.26)\\
= \;& \left(\frac{(n_1+n_2)}{2}+\frac{e_2^2}{e_1}\frac{x(-i-i\lambda+x\lambda^2)\lambda^{2j-2}}{\lambda}\right)\delta_{j-1}+\left(\frac{(n_1-n_2)}{2}\right)\beta_{j-1}\\
& \left(\frac{-(m_1+m_2\lambda^2)}{2\lambda}+\frac{e_2^2}{e_1}\frac{i(-i+\lambda x+\lambda^2 x)\lambda^{2j-4}}{\lambda}\right)\delta_{j-2}+\frac{(m_1-m_2\lambda^2)}{2\lambda}\beta_{j-2}\\
& +\left(\frac{(e_1+e_2)}{2}-\frac{e_2^2\lambda^{2j-6}}{e_1}\right)\delta_{j-3}+\frac{(e_1-e_2)}{2}\beta_{j-3},
\end{aligned}
$$

with $\beta_0=\gamma_0=0$, and α_0, δ_0 are arbitrary constants set to be 1 without loss of generality.

Proof. First we need to diagonalize

$$
A_3(x)=\begin{pmatrix}0 & f_1(x)\\ f_2(x) & 0\end{pmatrix}=\begin{pmatrix}0 & -\frac{e_2^2x^2\lambda}{e_1}\\ -\frac{e_2^2x^2}{e_1\lambda} & 0\end{pmatrix}
$$

for the leading behavior of the series expansion at $\nu=\infty$. Since $\frac{f_1(x)}{f_2(x)}=\lambda^2$, this can be done by conjugation with the constant matrix $C=\frac{1}{2}\begin{pmatrix}\lambda & \lambda\\ 1 & -1\end{pmatrix}$.

Let $\Psi(\nu,x)=C\Psi_1(\nu,x)$,

$$\widehat{\Psi}_1 = C^{-1}A(\nu,x)C\Psi_1, \tag{4.27}$$
$$\overline{\Psi}_1 = C^{-1}B(\nu,x)C\Psi_1 \tag{4.28}$$

where

$$\begin{aligned} & C^{-1}A(\nu,x)C \\ = & \begin{pmatrix} -\frac{e_2^2}{e_1}x^2 & 0 \\ 0 & +\frac{e_2^2}{e_1}x^2 \end{pmatrix}\nu^3+\frac{1}{2}\begin{pmatrix} n_1+n_2 & n_1-n_2 \\ n_1-n_2 & n_1+n_2 \end{pmatrix}\nu^2 \\ & +\frac{1}{2\lambda}\begin{pmatrix} m_1+m_2\lambda^2 & -m_1+m_2\lambda^2 \\ m_1-m_2\lambda^2 & -(m_1+m_2\lambda^2) \end{pmatrix}\nu+\frac{1}{2}\begin{pmatrix} e_1+e_2 & e_1-e_2 \\ e_1-e_2 & e_1+e_2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} & C^{-1}B(\nu,x)C \\ = & \begin{pmatrix} i\sqrt{\frac{e_2}{e_1}}x\lambda & 0 \\ 0 & -i\sqrt{\frac{e_2}{e_1}}x\lambda \end{pmatrix}\nu+\begin{pmatrix} -\frac{i(e_1x^2\lambda^2-e_2g(x\lambda)^2)}{x\lambda g(x\lambda)} & -\frac{i(e_1x^2\lambda^2+e_2g(x\lambda)^2)}{x\lambda g(x\lambda)} \\ -\frac{i(e_1x^2\lambda^2+e_2g(x\lambda)^2)}{x\lambda g(x\lambda)} & -\frac{i(e_1x^2\lambda^2-e_2g(x\lambda)^2)}{x\lambda g(x\lambda)} \end{pmatrix}. \end{aligned}$$

The matrix solution of the q-linear systems (4.27) at $\nu=\infty$ therefore has the form:

$$\Psi_1(\nu,x)=I(\nu)J(x)\sum_{j=0}^{\infty}\begin{pmatrix} \alpha_j(x) & \beta_j(x) \\ \gamma_j(x) & \delta_j(x) \end{pmatrix}\frac{1}{\nu^j}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \tag{4.29}$$

where on substituting into the spectral linear system (4.27) we have conditions:

$$I(\nu/\lambda^2) = \frac{e_2^2}{e_1}I(\nu)$$

and

$$\begin{aligned} u(\nu/\lambda^2,x) &= \left(1+\nu\frac{i}{\lambda}(-i+\lambda x+\lambda^2x)-\nu^2\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}-x^2\nu^3\right)u(\nu,x) \\ &= (1+\nu/\lambda)(1+ix\nu)(1+ix\lambda\nu)u(\nu,x), \end{aligned}$$

$$\begin{aligned} v(\nu/\lambda^2,x) &= \left(1-\nu\frac{i}{\lambda}(-i+\lambda x+\lambda^2x)-\nu^2\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}+x^2\nu^3\right)v(\nu,x) \\ &= (1-\nu/\lambda)(1-ix\nu)(1-ix\lambda\nu)v(\nu,x). \end{aligned}$$

For (4.29) also to be a solution of the deformation equation (4.28), the following equations with respect to the Painlevé variable x need to be satisfied

$$J(x/\lambda) = \sqrt{\frac{e_2}{e_1}}J(x),$$

and

$$\begin{aligned} u(\nu, x/\lambda) &= (1+ix\lambda\nu)u(\nu,x), \\ v(\nu, x/\lambda) &= (1-ix\lambda\nu)v(\nu,x). \end{aligned}$$

Equations for $I(\nu)$ and $J(x)$ are solved by

$$\begin{aligned} I(\nu) &= \left(\frac{e_2^2}{e_1}\right)^t, \qquad t = -\frac{1}{2}\frac{\ln\nu}{\ln\lambda}, \\ J(x) &= \left(\frac{e_2}{e_1}\right)^{\frac{s}{2}}, \qquad s = -\frac{\ln x}{\ln\lambda}, \end{aligned}$$

whereas

$$\begin{aligned} u(\nu,x) &= \Gamma_{\lambda^{-2}}(1+\nu/\lambda)\Gamma_{\lambda^{-2}}(1+ix\nu)\Gamma_{\lambda^{-2}}(1+ix\lambda\nu), \\ v(\nu,x) &= \Gamma_{\lambda^{-2}}(1-\nu/\lambda)\Gamma_{\lambda^{-2}}(1-ix\nu)\Gamma_{\lambda^{-2}}(1-ix\lambda\nu) \end{aligned}$$

satisfy the equations for $u(\nu,x)$ and $v(\nu,x)$. $\Gamma_q(1-z)$ is the q-discrete analogue of the Gamma function. The recurrence relations for the coefficients α_j, β_j, γ_j, δ_j can be found by substituting solution (4.29) into (4.27) and equate powers of ν at $\nu=\infty$, for example

$$\nu^3: \qquad -x^2\frac{e_2^2}{e_1}\begin{pmatrix}\alpha_0\\ \gamma_0\end{pmatrix} = x^2\frac{e_2^2}{e_1}\begin{pmatrix}-\alpha_0\\ \gamma_0\end{pmatrix}$$

implies that α_0 is arbitrary and $\gamma_0 = 0$,

$$\begin{aligned} \nu^2: \qquad & -x^2\frac{e_2^2}{e_1}\begin{pmatrix}\alpha_1\\ \gamma_1\end{pmatrix}\lambda^2 - \frac{e_2^2}{e_1}\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}\begin{pmatrix}\alpha_0\\ 0\end{pmatrix} \\ & = x^2\frac{e_2^2}{e_1}\begin{pmatrix}-\alpha_1\\ \gamma_1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}(n_1+n_2)\alpha_0\\ (n_1-n_2)\alpha_0\end{pmatrix}, \end{aligned}$$

implies

$$\begin{aligned}\alpha_1 &= -\frac{\alpha_0\left(2x(-i-i\lambda+x\lambda^2)+\frac{e_1}{e_2^2}\lambda(n_1+n_2)\right)}{2x^2\lambda(\lambda^2-1)},\\ \gamma_1 &= \frac{e_1}{e_2^2}\frac{\alpha_0(-n_1+n_2)}{2x^2(\lambda^2+1)}.\end{aligned} \tag{4.30}$$

In general

$$\begin{aligned}\nu^{3+j}:\quad & \frac{e_2^2}{e_1}\left(-x^2\begin{pmatrix}\alpha_j\\ \gamma_j\end{pmatrix}\lambda^{2j}-\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}\begin{pmatrix}\alpha_{j-1}\\ \gamma_{j-1}\end{pmatrix}\lambda^{2j-2}\right.\\ &\left.+\frac{i}{\lambda}(-i+\lambda x+\lambda^2 x)\begin{pmatrix}\alpha_{j-2}\\ \gamma_{j-2}\end{pmatrix}\lambda^{2j-4}+\begin{pmatrix}\alpha_{j-3}\\ \gamma_{j-3}\end{pmatrix}\lambda^{2j-6}\right)\\ &= x^2\frac{e_2^2}{e_1}\begin{pmatrix}-\alpha_j\\ \gamma_j\end{pmatrix}+\frac{1}{2}\begin{pmatrix}(n_1+n_2)\alpha_{j-1}+(n_1-n_2)\gamma_{j-1}\\ (n_1-n_2)\alpha_{j-1}+(n_1+n_2)\gamma_{j-1}\end{pmatrix}\\ &+\frac{1}{2\lambda}\begin{pmatrix}(m_1+m_2\lambda^2)\alpha_{j-2}+(-m_1+m_2\lambda^2)\gamma_{j-2}\\ (m_1-m_2\lambda^2)\alpha_{j-2}-(m_1+m_2\lambda^2)\gamma_{j-2}\end{pmatrix}\\ &+\frac{1}{2}\begin{pmatrix}(e_1+e_2)\alpha_{j-3}+(e_1-e_2)\gamma_{j-3}\\ (e_1-e_2)\alpha_{j-3}+(e_1+e_2)\gamma_{j-3}\end{pmatrix},\end{aligned}$$

gives the recurrence relations (4.23) for α_j and (4.24) for γ_j, and

$$\begin{aligned}\nu^{3+j}:\quad & \frac{e_2^2}{e_1}\left(x^2\begin{pmatrix}\beta_j\\ \delta_j\end{pmatrix}\lambda^{2j}-\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}\begin{pmatrix}\beta_{j-1}\\ \delta_{j-1}\end{pmatrix}\lambda^{2j-2}\right.\\ &\left.-\frac{i}{\lambda}(-i+\lambda x+\lambda^2 x)\begin{pmatrix}\beta_{j-2}\\ \delta_{j-2}\end{pmatrix}\lambda^{2j-4}+\begin{pmatrix}\beta_{j-3}\\ \delta_{j-3}\end{pmatrix}\lambda^{2j-6}\right)\\ &= x^2\frac{e_2^2}{e_1}\begin{pmatrix}-\beta_j\\ \delta_j\end{pmatrix}+\frac{1}{2}\begin{pmatrix}(n_1+n_2)\beta_{j-1}+(n_1-n_2)\delta_{j-1}\\ (n_1-n_2)\beta_{j-1}+(n_1+n_2)\delta_{j-1}\end{pmatrix}\\ &+\frac{1}{2\lambda}\begin{pmatrix}(m_1+m_2\lambda^2)\beta_{j-2}+(-m_1+m_2\lambda^2)\delta_{j-2}\\ (m_1-m_2\lambda^2)\beta_{j-2}-(m_1+m_2\lambda^2)\delta_{j-2}\end{pmatrix}\\ &+\frac{1}{2}\begin{pmatrix}(e_1+e_2)\beta_{j-3}+(e_1-e_2)\delta_{j-3}\\ (e_1-e_2)\beta_{j-3}+(e_1+e_2)\delta_{j-3}\end{pmatrix},\end{aligned}$$

gives the recurrence relations (4.25) for β_j and (4.26) for δ_j. Similarly it can be shown that δ_0 is arbitrary, $\beta_0 = 0$. The rest of the coefficients can be calculated using the recurrence relations. For example

$$\begin{aligned} \delta_1 &= \frac{\alpha_0\left(2x(-i-i\lambda+x\lambda^2)+\frac{e_1}{e_2^2}\lambda(n_1+n_2)\right)}{2x^2\lambda(\lambda^2-1)}, \\ \beta_1 &= \frac{e_1}{e_2^2}\frac{\alpha_0(n_1-n_2)}{2x^2(\lambda^2+1)}. \end{aligned} \tag{4.31}$$

The coefficients α_j, β_j, γ_j and δ_j are in general functions of x since n_1, n_2, m_1, m_2, f_1 and f_2 are all functions of x. Their dependence on x can be found by substituting solution (4.29) into (4.28). We found that α_0 and δ_0 does not depend on x, that is $\alpha_0(x/\lambda) = \alpha_0(x)$ and $\delta_0(x/\lambda) = \delta_0(x)$(the function $J(x)$ is used to take the x dependence of the leading coefficients). Since they are constants with respect to both variable ν and x, therefore can be set to be 1 without loss of generality. □

4.3. Special solutions

Solutions of the form (4.10, 4.11) are in general not valid when $\frac{e_1}{e_2}$ is an integer power of $\frac{1}{\lambda^2}$. Inconsistency can arise in either recurrence relations (4.13) or (4.15) when this is the case. The case $\frac{e_1}{e_2} = \alpha = 1/\lambda^{4k}$ separates into two types: (1) k is an integer, (2) k is a half integer. Since q-PII (4.9) has the symmetry,

$$g(x) \to -x^2/g(x) \quad \Rightarrow \quad \alpha \to 1/\alpha,$$

we only need to consider $k > 0$.

Definition 4.3.1. Type (1): $\frac{e_1}{e_2} = \alpha = \lambda^{-4k}$, k is a positive integer.
For the linear system (4.1) we find that no inconsistencies arise in this case, and solutions at $\nu = 0$ of the form (4.10, 4.11) are still valid. This can be easily checked as follows, when $\frac{e_1}{e_2} = \frac{1}{\lambda^{4k}}$, the j in equation (4.15) is even,

$$b_{\text{even}} \times 0 = m_1(x)d_{\text{odd}} + n_1(x)b_{\text{even}} + f_1(x)d_{\text{odd}}.$$

Since we found in the expansion of solution of the Lax pair (4.1, 4.2) that $d_{\text{odd}} = b_{\text{even}} = 0$ (equation (4.19)), the RHS is also zero. Hence no inconsistency arises in this case.

Definition 4.3.2. Type (2): $\frac{e_1}{e_2} = \alpha = \lambda^{-4k}$, k is a positive half integer.
In this case inconsistency can occur for the recurrence relations. For example when $\frac{e_1}{e_2} = \frac{1}{\lambda^{4k}}$, the j in equation (4.15) is odd

$$b_{\text{odd}} \times 0 = m_1(x)d_{\text{even}} + n_1(x)b_{\text{odd}} + f_1(x)d_{\text{even}}$$

but b_{odd}, d_{even} are not necessarily 0, so the RHS is not zero in general . However, there is still a possibility for avoiding inconsistency, if special conditions are imposed on the coefficients $m_1(x)$, $n_1(x)$ and $f_1(x)$.

We will look at the simplest case of type (1) and (2) respectively, which is $k = 0$, $\frac{e_1}{e_2} = 1$ and $k = \frac{1}{2}$, $\frac{e_1}{e_2} = \frac{1}{\lambda^2}$.

Note that the linear problem (4.1) is not easy to solve in general, being the 2×2 system of (3.1) with $\mu = 3$. However, the problem simplifies when $A_0(x)$, $A_1(x)$, $A_2(x)$ and $A_3(x)$ in $A(\nu, x)$ commute, which means $A(\nu, x)$ can be diagonalized by conjugation with a constant matrix, and the second-order linear problem reduces to two first-order ones.

4.3.1. The simplest rational type special solution. We look at type (1) first, as it includes the case $\frac{e_1}{e_2} = 1$, when the linear system can be diagonalized by conjugation with a constant matrix and is exactly solved in terms of the q-Gamma functions.

Proposition 4.3.3. A solution of the Lax pair (4.1, 4.2) when $\alpha = \frac{e_1}{e_2} = 1$ and the corresponding q-PII equation (4.9) admits solution $g(x) = -ix$, is given by

$$\begin{aligned} & \phi_1^{(0)}(\nu, x) \qquad (4.32)\\ = \; & \frac{1}{2}\begin{pmatrix} \lambda & \lambda \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \Gamma_{\lambda^{-2}}(1+\nu/\lambda)\Gamma_{\lambda^{-2}}(1+ix\nu)\Gamma_{\lambda^{-2}}(1+ix\lambda\nu) \\ \Gamma_{\lambda^{-2}}(1-\nu/\lambda)\Gamma_{\lambda^{-2}}(1-ix\nu)\Gamma_{\lambda^{-2}}(1-ix\lambda\nu) \end{pmatrix} \\ = \; & \sum_{j=0}\begin{pmatrix} \lambda T_{2j}\nu^{2j} \\ T_{2j+1}\nu^{2j+1} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} T_j(\frac{1}{\lambda^{2j}} - 1) = & -\frac{i(-i+\lambda x+\lambda^2 x)}{\lambda}T_{j-1} - \frac{x(-i-i\lambda+x\lambda^2)}{\lambda}T_{j-2} \qquad (4.33)\\ & +x^2 T_{j-3}, \end{aligned}$$

$$T_{j-1}(x) = \frac{1}{ix\lambda}\left(T_j(x/\lambda) - T_j(x)\right). \qquad (4.34)$$

Proof. On asking the four coefficient matrices $A_0(x)$, $A_1(x)$, $A_2(x)$ and $A_3(x)$ in equation (4.1) to commute with each other, we arrive at the conditions

$$e_1 = e_2, \;\; n_1(x) = n_2(x), \;\; m_1(x) = \lambda^2 m_2(x), \;\; f_1(x) = \lambda^2 f_2(x) \quad (4.35)$$

and

$$g(x) = -ix. \quad (4.36)$$

Substituting the special solution (4.36) into the linear problems (4.1, 4.2) gives

$$\Psi(\nu/\lambda^2, x) = A_0(\nu, x)\Psi(\nu, x), \quad (4.37)$$
$$\Psi(\nu, x/\lambda) = B_0(\nu, x)\Psi(\nu, x) \quad (4.38)$$

where

$$A_0(\nu, x) = \begin{pmatrix} 1 + \nu^2 \frac{x(-i - i\lambda + x\lambda^2)}{\lambda} & \nu i(-i + \lambda x + \lambda^2 x) - x^2\lambda\nu^3 \\ \nu \frac{i(-i+\lambda x+\lambda^2 x)}{\lambda^2} - \frac{x^2}{\lambda}\nu^3 & 1 - \nu^2 \frac{x(-i-i\lambda+x\lambda^2)}{\lambda} \end{pmatrix},$$
$$B_0(\nu, x) = \begin{pmatrix} 1 & i\lambda^2 x\nu \\ ix\nu & 1 \end{pmatrix}.$$

We have denoted the reduced $A(\nu, x), B(\nu, x)$ as $A_0(\nu, x), B_0(\nu, x)$ in the case of $\frac{e_1}{e_2} = 1$ and the other conditions (4.35, 4.36). The solution matrix in this case is denoted as $\Phi = \{\phi_1^{(0)}, \phi_2^{(0)}\}$. The constant matrix which diagonalizes $A_0(\nu, x)$ is $C = \frac{1}{2}\begin{pmatrix} \lambda & \lambda \\ 1 & -1 \end{pmatrix}$. Let $\Psi(\nu, x) = C\Psi_1(\nu, x)$. Then

$$\widehat{\Psi}_1 = C^{-1}A_0(\nu, x)C\Psi_1, \quad (4.39)$$
$$\overline{\Psi}_1 = C^{-1}B_0(\nu, x)C\Psi_1 \quad (4.40)$$

where

$$
\begin{aligned}
&C^{-1}A_0(\nu,x)C \\
&\quad = \begin{pmatrix} 1+\nu\frac{i}{\lambda}(-i+\lambda x+\lambda^2 x)-\nu^2\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}-x^2\nu^3 & 0 \\ 0 & 1-\nu\frac{i}{\lambda}(-i+\lambda x+\lambda^2 x)-\nu^2\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}+x^2\nu^3 \end{pmatrix}\Psi_1 \\
&\quad = \begin{pmatrix} (1+\frac{\nu}{\lambda})(1+ix\nu)(1+ix\lambda\nu) & 0 \\ 0 & (1-\frac{\nu}{\lambda})(1-ix\nu)(1-ix\lambda\nu) \end{pmatrix}\Psi_1, \\
&C^{-1}B_0(\nu,x)C \;=\; \begin{pmatrix} (1+ix\lambda\nu) & 0 \\ 0 & (1-ix\lambda\nu) \end{pmatrix}\Psi_1.
\end{aligned}
$$

Let the first column of the solution matrix of equations (4.39, 4.40) be $\begin{pmatrix} u(\nu,x) \\ v(\nu,x) \end{pmatrix}$, that is

$$\phi_1^{(0)}(\nu,x) = \frac{1}{2}\begin{pmatrix} \lambda & \lambda \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u(\nu,x) \\ v(\nu,x) \end{pmatrix}$$

where

$$
\begin{aligned}
u(\nu/\lambda^2,x) &= (1+\nu/\lambda)(1+ix\nu)(1+ix\lambda\nu)u(\nu,x), && (4.41)\\
u(\nu,x/\lambda) &= (1+ix\lambda\nu)u(\nu,x), && (4.42)\\
v(\nu/\lambda^2,x) &= (1-\nu/\lambda)(1-ix\nu)(1-ix\lambda\nu)v(\nu,x), && (4.43)\\
v(\nu,x/\lambda) &= (1-ix\lambda\nu)v(\nu,x). && (4.44)
\end{aligned}
$$

We see that equation (4.41) has solution

$$u(\nu,x) = u_1(\nu,x)u_2(\nu,x)u_3(\nu,x),$$

where

$$
\begin{aligned}
u_1(\nu/\lambda^2,x) &= (1+\nu/\lambda)u_1(\nu,x),\\
u_2(\nu/\lambda^2,x) &= (1+ix\nu)u_2(\nu,x),\\
u_3(\nu/\lambda^2,x) &= (1+ix\lambda\nu)u_3(\nu,x),
\end{aligned}
$$

each of which can be solved in terms of the q-Gamma function $\Gamma_q(1-z)$. We have

$$\begin{aligned} u_1(\nu,x) &= \Gamma_{\lambda^{-2}}(1+\nu/\lambda), \\ u_2(\nu,x) &= \Gamma_{\lambda^{-2}}(1+ix\nu), \\ u_3(\nu,x) &= \Gamma_{\lambda^{-2}}(1+ix\lambda\nu). \end{aligned}$$

Hence

$$u(\nu,x) = \Gamma_{\lambda^{-2}}(1+\nu/\lambda)\Gamma_{\lambda^{-2}}(1+ix\nu)\Gamma_{\lambda^{-2}}(1+ix\lambda\nu). \quad (4.45)$$

It is easy to check that solution (4.45) also satisfies equation (4.42). To see this we use the infinite product formula of $\Gamma_q(1-z)$:

$$\Gamma_q(1-z) = \frac{1}{(z;q)_\infty},$$

where $(z;q)_\infty = (1-z)(1-qz)(1-q^2z)\ldots$.
Now

$$\begin{aligned} u(\nu,x) &= \frac{1}{(-\frac{\nu}{\lambda};\lambda^{-2})_\infty}\frac{1}{(-ix\nu;\lambda^{-2})_\infty}\frac{1}{(-ix\lambda\nu;\lambda^{-2})_\infty} \\ &= \frac{1}{(1+\frac{\nu}{\lambda})(1+\frac{\nu}{\lambda^3})\ldots}\frac{1}{(1+ix\nu)(1+\frac{ix\nu}{\lambda^2})\ldots}\frac{1}{(1+ix\lambda\nu)(1+\frac{ix\nu}{\lambda})\ldots}, \end{aligned}$$

and

$$\begin{aligned} u(\nu,x/\lambda) &= \frac{1}{(1+\frac{\nu}{\lambda})(1+\frac{\nu}{\lambda^3})\ldots}\frac{1}{(1+\frac{ix\nu}{\lambda})(1+\frac{ix\nu}{\lambda^3})\ldots}\frac{1}{(1+ix\nu)(1+\frac{ix\nu}{\lambda^2})\ldots} \\ &= (1+ix\lambda\nu)u(\nu,x). \end{aligned}$$

Similarly it can be shown that

$$v(\nu,x) = \Gamma_{\lambda^{-2}}(1-\nu/\lambda)\Gamma_{\lambda^{-2}}(1-ix\nu)\Gamma_{\lambda^{-2}}(1-ix\lambda\nu) \quad (4.46)$$

satisfies equations (4.43) and (4.44). Hence we have

$$\phi_1^{(0)}(\nu,x) = \frac{1}{2}\begin{pmatrix}\lambda & \lambda\\ 1 & -1\end{pmatrix}\begin{pmatrix}\Gamma_{\lambda^{-2}}(1+\nu/\lambda)\Gamma_{\lambda^{-2}}(1+ix\nu)\Gamma_{\lambda^{-2}}(1+ix\lambda\nu)\\ \Gamma_{\lambda^{-2}}(1-\nu/\lambda)\Gamma_{\lambda^{-2}}(1-ix\nu)\Gamma_{\lambda^{-2}}(1-ix\lambda\nu)\end{pmatrix}.$$

To show the series summation expression of $\phi_1^{(0)}(\nu,x)$ in equation (4.32) recall that $\Gamma_q(1-z)$ (see equation (3.14)) has a series summation expression

at $z = 0$:

$$\begin{aligned}\Gamma_q(1-z) &= {}_1\Phi_0(0,-;q,z)\\ &= \sum_{j=0}^{\infty}\frac{z^j}{(q;q)_j}.\end{aligned}$$

Thus

$$\begin{aligned}&u(\nu,x)\\ =\ & {}_1\Phi_0(0,-;\lambda^{-2},-\nu/\lambda)\ \times {}_1\Phi_0(0,-;\lambda^{-2},-ix\nu)\ \times {}_1\Phi_0(0,-;\lambda^{-2},-ix\nu\lambda)\\ =\ & \sum_h \frac{(-\frac{\nu}{\lambda})^h}{(\lambda^{-2};\lambda^{-2})_h}\sum_j\frac{(-ix\nu)^j}{(\lambda^{-2};\lambda^{-2})_j}\sum_k\frac{(-ix\nu\lambda)^k}{(\lambda^{-2};\lambda^{-2})_k}\\ =\ & 1+\frac{i\nu\lambda}{(1-\lambda^{-2})}\big(-i+x\lambda(1+\lambda)\big)+\ldots\\ =\ & \sum_{j=0}^{\infty}T_j(x)\nu^j\end{aligned}$$

where $T_0(x) = 1$. From the definition of T_j and equations (4.41, 4.42) for $u(\nu,x)$ we have the relations which define $T_j(x)$, $j = 1,2,\ldots$

$$\begin{aligned}T_j(\frac{1}{\lambda^{2j}}-1) = &-\frac{i(-i+\lambda x+\lambda^2 x)}{\lambda}T_{j-1}-\frac{x(-i-i\lambda+x\lambda^2)}{\lambda}T_{j-2}\\ &+x^2T_{j-3}\\ T_{j-1}(x) = &\frac{1}{ix\lambda}\left(T_j(x/\lambda)-T_j(x)\right).\end{aligned}$$

Similarly for $v(\nu,x)$

$$\begin{aligned}&v(\nu,x)\\ =\ & {}_1\Phi_0(0,-;\lambda^{-2},\nu/\lambda)\times {}_1\Phi_0(0,-;\lambda^{-2},ix\nu)\times {}_1\Phi_0(0,-;\lambda^{-2},ix\nu\lambda)\\ =\ & \sum_h \frac{(\frac{\nu}{\lambda})^h}{(\lambda^{-2};\lambda^{-2})_h}\sum_j\frac{(ix\nu)^j}{(\lambda^{-2};\lambda^{-2})_j}\sum_k\frac{(ix\nu\lambda)^k}{(\lambda^{-2};\lambda^{-2})_k}\\ =\ & 1-\frac{i\lambda\nu}{(1-\lambda^{2})}\big(-i+x\lambda(1+\lambda)\big)+\ldots\\ =\ & \sum_{j=0}^{\infty}(-1)^jT_j(x)\nu^j.\end{aligned}$$

Finally we have the solution of equations (4.37, 4.38)

$$\begin{aligned}
\phi_1^{(0)}(\nu,x) &= \frac{1}{2}\begin{pmatrix}\lambda & \lambda\\ 1 & -1\end{pmatrix}\begin{pmatrix}u(\nu,x)\\ v(\nu,x)\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}\lambda & \lambda\\ 1 & -1\end{pmatrix}\begin{pmatrix}1+T_1\nu+T_2\nu^2+T_3\nu^3+...\\ 1-T_1\nu+T_2\nu^2-T_3\nu^3+...\end{pmatrix}\\
&= \begin{pmatrix}\lambda(1+T_2\nu^2+...)\\ T_1\nu+T_3\nu^3+...\end{pmatrix}\\
&= \sum_{j=0}\begin{pmatrix}\lambda T_{2j}\nu^{2j}\\ T_{2j+1}\nu^{2j+1}\end{pmatrix}.
\end{aligned}$$

□

4.3.2. The simplest q-hypergeometric type special solution. Type (2): $k=\frac{1}{2}$, $\frac{e_1}{e_2}=\alpha=\frac{1}{\lambda^2}$. In this case the recurrence relation (4.15) at $j=1$ is:

$$b_1\times 0 = m_1(x)d_0.$$

Since $d_0\neq 0$, $m_1(x)$ needs to be zero for the equation above to be consistent. Recall m_1 is defined by equation (4.3),

$$\begin{aligned}
m_1(x) &= \frac{\lambda^2 g(x\lambda)\left(-x^2-g(x)+g(x)g(x\lambda)\right)}{x^2}=0\\
& & g(x/\lambda) &= \tfrac{x^2}{\lambda^2(g(x)-1)} \qquad (4.47)\\
\Rightarrow & & \text{or} & \\
& & g(x\lambda) &= \tfrac{x^2+g(x)}{g(x)}. \qquad (4.48)
\end{aligned}$$

Hence like its continuous counter-part, the q-PII equation (4.9) can be reduced to a q-discrete analogue of the Riccati equation. In Appendix C we show that the contiuum limit of equation (4.48) is a differential Riccati equation.

Consider now the linear systems in this special case. We know from equation (4.20) that

$$\begin{aligned}
g(x) &= -ix\sqrt{\alpha}\frac{a_0(x/\lambda)}{a_0(x/\lambda^2)}\\
&= -i\frac{x}{\lambda}\frac{a_0(x/\lambda)}{a_0(x/\lambda^2)}
\end{aligned}$$

where $a_0(x)$ is the coefficient of the leading behaviour of one of the vector solution of the linear problems at $\nu = 0$. q-Riccati equation (4.48) implies

$$\left(-ix\frac{a_0(x)}{a_0(x/\lambda)}\right)\left(\frac{-ix}{\lambda}\frac{a_0(x/\lambda)}{a_0(x/\lambda^2)}\right) = x^2 - \frac{ix}{\lambda}\frac{a_0(x/\lambda)}{a_0(x/\lambda^2)}$$

$$a_0(x/\lambda^2) - \frac{i}{x\lambda}a_0(x/\lambda) + \frac{a_0(x)}{\lambda} = 0. \tag{4.49}$$

This is a q-discrete analogue of the Airy equation. In Appendix C we show that the contiuum limit of equation (4.49) tends to the differential Airy equation as $\frac{1}{\lambda} \to 1$. So we have found that when $\frac{e_1}{e_2} = \alpha = \frac{1}{\lambda^2}$, the q-PII equation (4.9) reduces to the q-Riccati equation (4.48, or 4.47) and the leading coefficient $a_0(x)$ of the solution of the linear problems at $\nu = 0$ satisfies the q-discrete Airy equation with respect to the q-discrete Painlevé variable x. Using q-Riccati equation (4.48) to replace the $g(x\lambda)$ in the Lax pair (4.1, 4.2), recall in this case we have $m_1(x) = 0$,

$$\Psi(\nu/\lambda^2, x) = A_{\frac{1}{2}}(\nu, x)\Psi(\nu, x), \tag{4.50}$$

$$\Psi(\nu, x/\lambda) = B_{\frac{1}{2}}(\nu, x)\Psi(\nu, x) \tag{4.51}$$

where

$$A_{\frac{1}{2}}(\nu, x) = \begin{pmatrix} \frac{1}{\lambda} + n_1(x)\nu^2 & f_1(x)\nu^3 \\ m_2(x)\nu + f_2(x)\nu^3 & \lambda + n_2(x)\nu^3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\lambda} + \left(\frac{x^4\lambda^3 + x^2\lambda g(x) + x^2\lambda^3 g(x)}{x^2 + g(x)}\right)\nu^2 & -x^2\lambda^4\nu^3 \\ \frac{x^2(-x^2 - g(x) - x^2\lambda^2 g(x) + x^2\lambda^4 g(x) + \lambda^4 g(x)^2)}{\lambda^2(x^2+g(x))^2}\nu - x^2\lambda^2\nu^2 & \lambda + \frac{\lambda(-x^2 + x^4\lambda^2 - g(x))}{x^2+g(x)}\nu^3 \end{pmatrix},$$

$$B_{\frac{1}{2}}(\nu, x) = \begin{pmatrix} -\frac{ix}{1+\frac{x^2}{g(x)}} & i\lambda^3 x\nu \\ ix\lambda\nu & \frac{ix}{1+\frac{x^2}{g(x)}} \end{pmatrix}.$$

We have denoted $A(\nu, x), B(\nu, x)$ for the case $\alpha = \frac{e_1}{e_2} = \frac{1}{\lambda^{4k}} = \frac{1}{\lambda^2}$, as $A_{\frac{1}{2}}(\nu, x), B_{\frac{1}{2}}(\nu, x)$.

Proposition 4.3.4. A solution of the Lax pair (4.1, 4.2) for the case $\alpha = \frac{e_1}{e_2} = \frac{1}{\lambda^2}$, and the corresponding q-PII (4.9) is solved by $g(x) = -i\frac{x}{\lambda}\frac{a_0(x/\lambda)}{a_0(x/\lambda^2)}$

(where $a_0(x)$ satisfies a q-Airy equation (4.49)) is given by

$$\begin{aligned}\phi_1^{\left(\frac{1}{2}\right)}(\nu,x) &= \nu^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ -\frac{n_1(x)}{f_1(x)\nu} & \frac{1}{\lambda f_1(x)\nu}\end{pmatrix}\begin{pmatrix}1 & 0\\ x^2\lambda^4 & ix\lambda^2\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}(x)\nu^{2j}\\ a_{2j}(x/\lambda)\nu^{2j}\end{pmatrix}\\ &= \nu^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ \frac{g(x)}{\lambda^3(x^2+g(x))\nu} & -\frac{i}{x\lambda^3\nu}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}(x)\nu^{2j}\\ a_{2j}(x/\lambda)\nu^{2j}\end{pmatrix} \qquad (4.52)\end{aligned}$$

where

$$\overline{a}_{2j} = ix\lambda^2 a_{2j} - \frac{i}{x\lambda^2}(1/\lambda^{4j+4}-1)a_{2j+2}, \qquad (4.53)$$

$$\overline{\overline{a}}_{2j} = -\frac{1}{\lambda}a_{2j} - x^2\lambda^3 a_{2j-2} + \frac{i}{x\lambda}\overline{a}_{2j}, \qquad (4.54)$$

in particular,

$$\overline{\overline{a}}_0 - \frac{i}{x\lambda}\overline{a}_0 + \frac{a_0}{\lambda} = 0,$$

recall $\overline{y}(x) = y(\frac{x}{\lambda})$.

Proof. We will use the asymptotic behaviour (4.10) of $\phi_1(\nu,t)$ near $\nu = 0$ as a guide as to how to transform the Lax pair (4.50, 4.51) to a simpler form, for when $\alpha = \frac{e_1}{e_2} = \frac{1}{\lambda^2}$ and $g(x)$ satisfy equation (4.48).
Since

$$\begin{aligned}e_1^t &= (1/\lambda)^t = \exp\left(-\ln\lambda\frac{\ln\nu}{\ln\lambda^{-2}}\right) = \exp\left(\frac{1}{2}\ln\nu\right) = \nu^{\frac{1}{2}},\\ e_2^t &= (\lambda)^t = \exp\left(\ln\lambda\frac{\ln\nu}{\ln\lambda^{-2}}\right) = \exp\left(-\frac{1}{2}\ln\nu\right) = \nu^{-\frac{1}{2}},\end{aligned}$$

we know from equation (4.10), there is a solution of the Lax pair (4.50, 4.51) with asymptotic behaviour $\nu^{\frac{1}{2}}$ of the form

$$\phi_1(\nu,x) = \nu^{\frac{1}{2}}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}\nu^{2j}\\ c_{2j+1}\nu^{2j+1}\end{pmatrix}$$

where $a_0(x)$ satisfies equation (4.49). This indicates the first transformation:

$$\Psi(\nu,x) = \nu^{\frac{1}{2}}\Psi_1(\nu,x). \qquad (4.55)$$

Then,

$$\begin{aligned}\widehat{\Psi}_1(\nu,x) &= \Psi_1(\nu/\lambda^2,x) = \lambda A_{\frac{1}{2}}(\nu,x)\Psi_1(\nu,x) \qquad (4.56)\\ &= \begin{pmatrix} 1+\lambda n_1(x)\nu^2 & \lambda f_1(x)\nu^3 \\ \lambda m_2(x)\nu + \lambda f_2(x)\nu^3 & \lambda^2 + \lambda n_2(x)\nu^3 \end{pmatrix} \Psi_1(\nu,x).\end{aligned}$$

This transformation does not change equation (4.51) that is

$$\Psi_1(\nu,x/\lambda) = B_{\frac{1}{2}}(\nu,x)\Psi_2(\nu,x) = \begin{pmatrix} -\frac{ix}{1+\frac{x^2}{g(x)}} & i\lambda^3 x\nu \\ ix\lambda\nu & \frac{ix}{1+\frac{x^2}{g(x)}} \end{pmatrix} \Psi_1(\nu,x). \quad (4.57)$$

Let $\begin{pmatrix} u(\nu,x) \\ v(\nu,x) \end{pmatrix}$ be a vector solution of the Lax pair (4.56, 4.57), then

$$\begin{pmatrix} u(\nu/\lambda^2,x) \\ v(\nu/\lambda^2,x) \end{pmatrix} = \begin{pmatrix} 1+\lambda n_1(x)\nu^2 & \lambda f_1(x)\nu^3 \\ \lambda m_2(x)\nu + \lambda f_2(x)\nu^3 & \lambda^2 + \lambda n_2(x)\nu^3 \end{pmatrix} \begin{pmatrix} u(\nu,x) \\ v(\nu,x) \end{pmatrix} \quad (4.58)$$

and

$$\begin{pmatrix} u(\nu,x/\lambda) \\ v(\nu,x/\lambda) \end{pmatrix} = \begin{pmatrix} -\frac{ix}{1+\frac{x^2}{g(x)}} & i\lambda^3 x\nu \\ ix\lambda\nu & \frac{ix}{1+\frac{x^2}{g(x)}} \end{pmatrix} \begin{pmatrix} u(\nu,x) \\ v(\nu,x) \end{pmatrix}. \quad (4.59)$$

The top entry of equation (4.58) gives us

$$\frac{\widehat{u}-u}{\nu^2} = \lambda n_1(x)u + \nu\lambda f_1(x)v. \quad (4.60)$$

Equation (4.60) shows how $v(\nu,x)$ is related to $u(\nu,x)$. We would like to transform to a new "v"which relates to u more simply. Let

$$v_1 = \frac{\widehat{u}-u}{\nu^2} = \lambda n_1(x)u + \nu\lambda f_1(x)v,$$

then

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -\frac{n_1(x)}{f_1(x)\nu} & \frac{1}{\lambda f_1(x)\nu} \end{pmatrix} \begin{pmatrix} u \\ v_1 \end{pmatrix} = Q(\nu,x) \begin{pmatrix} u \\ v_1 \end{pmatrix} \qquad (4.61)\\ &= \begin{pmatrix} 1 & 0 \\ \frac{x^2\lambda^2+g(x)+\lambda^2 g(x)}{\lambda^3(x^2+g(x))\nu} & -\frac{1}{x^2\lambda^5\nu} \end{pmatrix} \begin{pmatrix} u \\ v_1 \end{pmatrix}.\end{aligned}$$

The reason for transformation (4.61) is that we know in q-discrete calculus:

$$\frac{\overline{y}(x)-y(x)}{x^2(q-1)} = \frac{y(qx)-y(x)}{x^2(q-1)} \to \frac{1}{x}\frac{dy}{dx}, \qquad \text{in the limit } q \to 1,$$

which is similar to what we have in the case of PII for the special case $a = \frac{1}{2}$.
The Lax pair is now

$$\begin{pmatrix} \widehat{u} \\ \widehat{v}_1 \end{pmatrix} = \widehat{Q}^{-1} \left(\lambda A_{\frac{1}{2}}(\nu, x) \right) Q \begin{pmatrix} u \\ v_1 \end{pmatrix}, \tag{4.62}$$

$$\begin{pmatrix} \overline{u} \\ \overline{v}_1 \end{pmatrix} = \overline{Q}^{-1} B_{\frac{1}{2}}(\nu, x) Q \begin{pmatrix} u \\ v_1 \end{pmatrix} \tag{4.63}$$

where

$$\widehat{Q}^{-1} \left(\lambda A_{\frac{1}{2}}(\nu, x) \right) Q$$
$$= \begin{pmatrix} 1 & \nu^2 \\ -x^2\lambda^2(-1-\lambda^2+x^2\lambda^4)\nu^2 + x^4\lambda^6\nu^4 & 1 + (-1+x^2\lambda^2+x^2\lambda^4)\nu^2 \end{pmatrix},$$

$$\overline{Q}^{-1} B_{\frac{1}{2}}(\nu, x) Q = \begin{pmatrix} ix\lambda^2 & -\frac{i}{x\lambda^2} \\ ix(-1-\lambda^2+x^2\lambda^4) - ix^3\lambda^4\nu^2 & -\frac{i(x^2\lambda^2-1)}{x\lambda^2} \end{pmatrix}.$$

Now, we see that the transformation equation (4.61) simplifies the Lax pair significantly. For instance, equations (4.62, 4.63) no longer have $g(x)$ the solution of q-PII in their coefficient matrices. The top entry of the vector equation (4.63) gives us

$$\overline{u} = ix\lambda^2 u - \frac{i}{x\lambda^2} \left(\frac{\widehat{u} - u}{\nu^2} \right) \tag{4.64}$$

which relates the operation in x to the operation in ν.
Finally, let

$$v_2 = \overline{u} = ix\lambda^2 u - \frac{i}{x\lambda^2} \left(\frac{\widehat{u} - u}{\nu^2} \right),$$

so

$$v_2 = \sum_{j=0}^{\infty} a_{2j}(x/\lambda)\nu^{2j}.$$

That is

$$\begin{pmatrix} u \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^2\lambda^4 & ix\lambda^2 \end{pmatrix} \begin{pmatrix} u \\ v_2 \end{pmatrix}, \tag{4.65}$$

then

$$\begin{pmatrix} \widehat{u} \\ \widehat{v}_2 \end{pmatrix} = \begin{pmatrix} 1+x^2\lambda^4\nu^2 & ix\lambda^2\nu^2 \\ -ix\nu^2 - ix^3\lambda^4\nu^4 & 1+(x^2\lambda^2-1)\nu^2 \end{pmatrix} \begin{pmatrix} u \\ v_2 \end{pmatrix}, \quad (4.66)$$

$$\begin{pmatrix} \overline{u} \\ \overline{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\lambda} - x^2\lambda^3\nu^2 & \frac{i}{x\lambda} \end{pmatrix} \begin{pmatrix} u \\ v_2 \end{pmatrix}. \quad (4.67)$$

Recall $u(\nu, x)$ at the neighborhood of $\nu = 0$ has the series expansion:

$$u(\nu, x) = \sum_{j=0}^{\infty} a_{2j}(x)\nu^{2j} = a_0(x) + a_2(x)\nu^2 + a_4(x)\nu^4 + ...$$

then equation (4.64) gives us a relation (4.53) for a_{2j}:

$$\overline{a}_{2j} = ix\lambda^2 a_{2j} - \frac{i}{x\lambda^2}(1/\lambda^{4j+4} - 1)a_{2j+2}.$$

The bottom entry of equation (4.67) is:

$$\overline{v}_2 = \overline{\overline{u}} = \left(-\frac{1}{\lambda} - x^2\lambda^3\nu^2\right) u + \frac{i}{x\lambda}\overline{u} \quad (4.68)$$

which gives us the other relation (4.54) for a_{2j}:

$$\overline{\overline{a}}_{2j} = -\frac{1}{\lambda}a_{2j} - x^2\lambda^3\nu^2 a_{2j-2} + \frac{i}{x\lambda}\overline{a}_{2j}.$$

Transformations (4.55), (4.61), and (4.65) means we went:

$$\nu^{\frac{1}{2}} \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u \\ \frac{\widehat{u}-(1+\lambda n_1\nu^2)u}{\lambda f_1\nu^3} \end{pmatrix} \to \begin{pmatrix} u \\ \frac{\widehat{u}-u}{\nu^2} \end{pmatrix} \to \begin{pmatrix} u \\ \overline{u} \end{pmatrix}.$$

A solution of the Lax pair (4.66, 4.67) is

$$\begin{pmatrix} u \\ v_2 \end{pmatrix} = \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}(x)\nu^{2j} \\ a_{2j}(x/\lambda)\nu^{2j} \end{pmatrix}$$

and a_{2j} satisfy the relations (4.53) and (4.54). In particular, $a_0(x)$ satisfies the q-Airy equation (4.49). The solution of the original Lax pair (4.50, 4.51)

can be constructed back using the transformations (4.55), (4.61), and (4.65),

$$\begin{aligned}
\phi_1^{\left(\frac{1}{2}\right)}(\nu,x) &= \nu^{\frac{1}{2}}\begin{pmatrix}u\\ v\end{pmatrix} = \nu^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ -\frac{n_1(x)}{f_1(x)\nu} & \frac{1}{\lambda f_1(x)\nu}\end{pmatrix}\begin{pmatrix}u\\ v_1\end{pmatrix}\\
&= \nu^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ -\frac{n_1(x)}{f_1(x)\nu} & \frac{1}{\lambda f_1(x)\nu}\end{pmatrix}\begin{pmatrix}1 & 0\\ x^2\lambda^4 & ix\lambda^2\end{pmatrix}\begin{pmatrix}u\\ v_2\end{pmatrix}\\
&= \nu^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ \frac{g(x)}{\lambda^3(x^2+g(x))\nu} & -\frac{i}{x\lambda^3\nu}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}(x)\nu^{2j}\\ a_{2j}(x/\lambda)\nu^{2j}\end{pmatrix}.
\end{aligned}$$

□

4.4. Schlesinger transformation L_k

Let the Lax pair for when $e_1 = \frac{1}{\lambda^{2k}}, e_2 = \lambda^{2k}$, $g_k(x)$ is a solution of the q-PII equation

$$g(x/\lambda)g(\lambda x) = \frac{\alpha x^2(g(x)+x^2)}{g(x)(g(x)-1)}, \qquad \text{for} \quad \alpha = \frac{e_1}{e_2} = \frac{1}{\lambda^{4k}},$$

be denoted as:

$$\widehat{\Psi}^{(k)}(\nu,x) = A_k(\nu,x)\Psi^{(k)}(\nu,x), \tag{4.69}$$

$$\overline{\Psi}^{(k)}(\nu,x) = B_k(\nu,x)\Psi^{(k)}(\nu,x) \tag{4.70}$$

where $A_k(\nu,x)$ and $B_k(\nu,x)$ denote the coefficient matrices of the spectral (4.1) and deformation (4.2) equations.

Let $\begin{pmatrix}u^{(k)}(\nu,x)\\ v^{(k)}(\nu,x)\end{pmatrix}$ be the first column $\phi_1^{(k)}(\nu,x)$ of the fundamental matrix solution of the Lax pair (4.69, 4.70) given by proposition 4.2.1. Therefore,

$$\begin{pmatrix}u^{(k)}(\nu,x)\\ v^{(k)}(\nu,x)\end{pmatrix} \sim \nu^k\begin{pmatrix}a_0^{(k)}(x)\\ c_1^{(k)}(x)\nu\end{pmatrix} \qquad \text{as } \nu\to 0,$$

where the superscript $^{(k)}$ denotes the fact $e_1 = \frac{1}{\lambda^{2k}}$, $e_2 = \lambda^{2k}$ and the corresponding solution of the the q-PII equation is $g_k(x)$. Then, the solution of the linear system when the parameter $\alpha = \frac{e_1}{e_2} = \frac{1}{\lambda^{4(k+1)}}$ is

$$\begin{pmatrix}u^{(k+1)}(\nu,x)\\ v^{(k+1)}(\nu,x)\end{pmatrix} \sim \nu^{k+1}\begin{pmatrix}a_0^{(k+1)}(x)\\ c_1^{(k+1)}(x)\nu\end{pmatrix} \qquad \text{as } \nu\to 0.$$

The Schlesinger transformation $L_k(\nu, x)$ is such that

$$\begin{pmatrix} u^{(k+1)}(\nu,x) \\ v^{(k+1)}(\nu,x) \end{pmatrix} = L_k(\nu,x) \begin{pmatrix} u^{(k)}(\nu,x) \\ v^{(k)}(\nu,x) \end{pmatrix}.$$

Proposition 4.4.1. The Schlesinger transformation of the associated linear problem (4.69) of q-PII is given by

$$\Psi^{(k+1)}(\nu,x) = L_k(\nu,x)\Psi^{(k)}(\nu,x), \tag{4.71}$$

where

$$L_k(\nu,x) = \begin{pmatrix} \frac{-A_{21}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} & \frac{A_{11}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} \\ \frac{A_{11}^{(k+1)}A_{21}^{(k)}(\lambda^2\nu)}{A_{12}^{(k+1)}|A_k(\lambda^2\nu)|} & \frac{1}{A_{12}^{(k+1)}}\left(1 - \frac{A_{11}^{(k+1)}}{|A_k(\lambda^2\nu)|}A_{11}^{(k)}(\lambda^2\nu)\right) \end{pmatrix}. \tag{4.72}$$

Proof. We first relate $v^{(k)}(\nu,x)$ with $u^{(k+1)}(\nu,x)$, making use of the Bäcklund transformation of $g_k(x)$. We are motivated by an observation that $v^{(k)}(\nu,x)$ has the same order of leading behavior in ν with $u^{(k+1)}(\nu,x)$ in the neighbourhood of $\nu = 0$, namely,

$$\begin{aligned} v^{(k)}(\nu,x) &\sim c_1^{(k)}(x)\nu^{k+1}, \\ u^{(k+1)}(\nu,x) &\sim a_0^{(k+1)}(x)\nu^{k+1}. \end{aligned}$$

Proposition 4.4.2. There exists a constant μ such that

$$c_1^{(k)}(x) = \mu a_0^{(k+1)}(x).$$

Proof. Recall equation (4.20) relates the solution of q-PII to the leading behaviour of the solution of its associated linear problems

$$g_k(x) = -i\frac{x}{\lambda^{2k}}\frac{a_0^{(k)}(x/\lambda)}{a_0^{(k)}(x/\lambda^2)}. \tag{4.73}$$

Then proposition 4.4.2 is true if we can show

$$g_{k+1}(x) = -i\frac{x}{\lambda^{2k+2}}\frac{a_0^{(k+1)}(x/\lambda)}{a_0^{(k+1)}(x/\lambda^2)} = -i\frac{x}{\lambda^{2k+2}}\frac{c_1^{(k)}(x/\lambda)}{c_1^{(k)}(x/\lambda^2)}.$$

From the recurrence relation (4.13) for $c_1^{(k)}(x)$ we have

$$\begin{aligned} c_1^{(k)}(x)\left(e_1/\lambda^2 - e_2\right) &= m_2(x)a_0^{(k)}(x) \\ &= -\frac{x^2\left(e_1x^2\lambda^2 - e_2g_k(x)g_k(x\lambda)\lambda^2 + e_1g_k(x\lambda)\right)}{\lambda g_k(x)g_k(x\lambda)^2}a_0^{(k)}(x), \end{aligned}$$

then,

$$\frac{c_1^{(k)}(x/\lambda)}{c_1^{(k)}(x/\lambda^2)} = \frac{m_2(x/\lambda)a_0^{(k)}(x/\lambda)}{m_2(x/\lambda^2)a_0^{(k)}(x/\lambda^2)}$$
$$= \frac{\lambda^4 g_k(x/\lambda^2)g_k(x/\lambda)^2\left(e_1x^2 - e_2g_k(x/\lambda)g_k(x)\lambda^2 + e_1g_k(x)\right)a_0^{(k)}(x/\lambda)}{\lambda^2 g_k(x/\lambda)g_k(x)^2\left(e_1\frac{x^2}{\lambda^2} - e_2g_k(x/\lambda^2)g_k(x/\lambda)\lambda^2 + e_1g_k(x/\lambda)\right)a_0^{(k)}(x/\lambda^2)}.$$

Use the q-PII equation

$$g_k(x/\lambda^2)g_k(x) = \frac{\frac{e_1}{e_2}\frac{x^2}{\lambda^2}\left(g_k(x/\lambda) + \frac{x^2}{\lambda^2}\right)}{g_k(x/\lambda)\left(g_k(x/\lambda) - 1\right)}$$

to eliminate $g_k(x/\lambda^2)$, and rewrite $\frac{a_0^{(k)}(x/\lambda)}{a_0^{(k)}(x/\lambda^2)}$ in terms of $g_k(x)$, we have

$$\frac{c_1^{(k)}(x/\lambda)}{c_1^{(k)}(x/\lambda^2)} = \frac{\left(\alpha x^2 - g_k(x/\lambda)g_k(x)\lambda^2 + \alpha g_k(x)\right)x^2}{g_k(x)^2\left(g_k(x)(g_k(x/\lambda) - 1) - x^2\right)}\frac{g_k(x)\lambda^{2k}i}{x}$$
$$= i\lambda^{2k}\frac{\left(\alpha x^2 - g_k(x/\lambda)g_k(x)\lambda^2 + \alpha g_k(x)\right)x}{g_k(x)\left(g_k(x)(g_k(x/\lambda) - 1) - x^2\right)}.$$

Then

$$-i\frac{x}{\lambda^{2k+2}}\frac{c_1^{(k)}(x/\lambda)}{c_1^{(k)}(x/\lambda^2)} = \frac{\left(\alpha x^2 - g_k(x/\lambda)g_k(x)\lambda^2 + \alpha g_k(x)\right)x^2}{\lambda^2 g_k(x)\left(g_k(x)(g_k(x/\lambda) - 1) - x^2\right)}.$$

The last line is the Bäcklund transformation of our q-PII equation (4.9), found by Joshi et al. [**39**]. That is

$$g_{k+1}(x) = \frac{\left(\alpha x^2 - g_k(x/\lambda)g_k(x)\lambda^2 + \alpha g_k(x)\right)x^2}{\lambda^2 g_k(x)\left(g_k(x)(g_k(x/\lambda) - 1) - x^2\right)}, \tag{4.74}$$

and

$$g_{k+1}(x) = -i\frac{x}{\lambda^{2k+2}}\frac{a_0^{(k+1)}(x/\lambda)}{a_0^{(k+1)}(x/\lambda^2)} = -i\frac{x}{\lambda^{2k+2}}\frac{c_1^{(k)}(x/\lambda)}{c_1^{(k)}(x/\lambda^2)}.$$

Therefore $a_0^{(k+1)}(x)$ is proportional to $c_1^{(k)}(x)$. □

This motivates the following statement

Proposition 4.4.3.

$$v^{(k)}(\lambda^2\nu, x) = u^{(k+1)}(\nu, x).$$

Proof. Recall the spectral equation (4.69) is of the form

$$\begin{pmatrix} \widehat{u}^{(k)} \\ \widehat{v}^{(k)} \end{pmatrix} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{pmatrix} \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} \tag{4.75}$$

where

$$\begin{aligned} A_{11}^{(k)} &= \frac{1}{\lambda^{2k}} + \frac{\lambda^{6k}\left(\lambda^{-4k}x^2 + g_k(x)g_k(x\lambda)^2 - g_k(x)g_k(x\lambda)\right)\nu^2}{g_k(x\lambda)} \qquad (4.76)\\ A_{12}^{(k)} &= \frac{\lambda^{6k}g_k(x\lambda)\left(-\lambda^{-4k}x^2\lambda^2 - g_k(x) + g_k(x)g_k(x\lambda)\right)\nu}{x^2\lambda} - x^2\lambda^{6k}\lambda\nu^3 \\ A_{21}^{(k)} &= -\frac{\lambda^{6k}x^2\nu^3}{\lambda} - \frac{x^2\lambda^{-2k}\left(x^2\lambda^2 - \lambda^{4k}g_k(x)g_k(x\lambda)\lambda^2 + g_k(x\lambda)\right)\nu}{\lambda g_k(x)g_k(x\lambda)^2} \\ A_{22}^{(k)} &= \lambda^{2k} - \frac{\lambda^{2k}\nu^2\left(-x^4\lambda^4 - x^2g_k(x\lambda)\lambda^2 + \lambda^{4k}g_k(x)g_k(x\lambda)^2\right)}{\lambda^2 g_k(x)g_k(x\lambda)}. \end{aligned}$$

It is useful to note that

$$|A_k(\nu,x)| = (1 - \frac{\lambda^{4k}}{\lambda^2}\nu^2)(1+\lambda^{4k}x^2\nu^2)(1+\lambda^{4k}x^2\lambda^2\nu^2). \tag{4.77}$$

This 2×2 system of coupled first order q-discrete equations (q-$\triangle$Es) can be written as two second order q-$\triangle$Es:

$$\widehat{\widehat{u}}^{(k)} = (A_{22}^{(k)}\frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}} + \widehat{A}_{11}^{(k)})\widehat{u}_k + \frac{\widehat{A}_{12}^{(k)}}{A_{12}^{(k)}}|A_k|u^{(k)} \tag{4.78}$$

$$\widehat{\widehat{v}}^{(k)} = (A_{11}^{(k)}\frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}} + \widehat{A}_{22}^{(k)})\widehat{v}^{(k)} + \frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}}|A_k|v^{(k)} \tag{4.79}$$

and the equation for $u^{(k+1)}$ is then

$$\widehat{\widehat{u}}^{(k+1)} = (A_{22}^{(k+1)}\frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}} + \widehat{A}_{11}^{(k+1)})\widehat{u}^{(k+1)} + \frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}}|A_{k+1}|u^{(k+1)}. \tag{4.80}$$

We observe from equation (4.77) that

$$|A_k(\nu,x)| = |A_{k+1}(\nu/\lambda^2,x)| = |\widehat{A}_{k+1}| \tag{4.81}$$

and using the q-PII equation (4.9) for $g_k(x)$ and its Bäcklund transformation (4.74) we can easily show

$$\frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}}(\nu,x) = \frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}}(\nu/\lambda^2,x) \tag{4.82}$$

and

$$(A_{11}^{(k)}\frac{\widehat{A}_{21}^{(k)}}{A_{21}^{(k)}}+\widehat{A}_{22}^{(k)})(\nu,x) \quad = \quad (A_{22}^{(k+1)}\frac{\widehat{A}_{12}^{(k+1)}}{A_{12}^{(k+1)}}+\widehat{A}_{11}^{(k+1)})(\nu/\lambda^2,x).$$

That is $v^{(k)}(\nu,x)$ and $u^{(k+1)}(\nu/\lambda^2,x)$ satisfy the same equation. The asymptotic behaviors of $v^{(k)}(\nu,x)$ and $u^{(k+1)}(\nu/\lambda^2,x)$ are

$$c_1^{(k)}(x)\nu^{k+1}=\mu a_0^{(k+1)}(x)\nu^{k+1}$$

and

$$a_0^{(k+1)}(x)\frac{\nu^{k+1}}{\lambda^{2k+2}}$$

respectively. If we chose μ to be $\frac{1}{\lambda^{2k+2}}$, that is

$$c_1^{(k)}(x)=\frac{a_0^{(k+1)}(x)}{\lambda^{2k+2}}, \tag{4.83}$$

then we have

$$u^{(k+1)}(\nu/\lambda^2,x)=v^{(k)}(\nu,x), \qquad \text{or} \qquad u^{(k+1)}(\nu,x)=v^{(k)}(\nu\lambda^2,x).$$

□

So now the problem of writing $u^{(k+1)}(\nu,x)$ in terms of $u^{(k)}(\nu,x)$ and $v^{(k)}(\nu,x)$ is reduced to writing $v^{(k)}(\nu\lambda^2,x)$ in terms of $u^{(k)}(\nu,x)$ and $v^{(k)}(\nu,x)$. Pre-multiply the linear system (4.75) by the inverse of $A_k(\nu,x)$,

$$\frac{1}{|A_k|}\begin{pmatrix}A_{22}^{(k)} & -A_{12}^{(k)}\\ -A_{21}^{(k)} & A_{11}^{(k)}\end{pmatrix}\begin{pmatrix}u^{(k)}\\ v^{(k)}\end{pmatrix}(\nu/\lambda^2,x) \quad = \quad \begin{pmatrix}u^{(k)}\\ v^{(k)}\end{pmatrix}(\nu,x)$$

$$\Rightarrow v^{(k)}=\frac{1}{|A_k|}(A_{11}^{(k)}\widehat{v}^{(k)}-A_{21}^{(k)}\widehat{u}^{(k)}),$$

or

$$v^{(k)}(\nu\lambda^2)=\frac{1}{|A_k(\nu\lambda^2)|}(A_{11}^{(k)}(\nu\lambda^2)v^{(k)}-A_{21}^{(k)}(\nu\lambda^2)u^{(k)}).$$

Note we have made the x dependence implicit, since all the operations are on the variable ν and x does not change.
Hence $u^{(k+1)}$ in terms of $u^{(k)}$ and $v^{(k)}$ is

$$u^{(k+1)}(\nu)=v^{(k)}(\lambda^2\nu) \quad = \quad \frac{1}{|A_k(\lambda^2\nu)|}(A_{11}^{(k)}(\lambda^2\nu)v^{(k)}-A_{21}^{(k)}(\lambda^2\nu)u^{(k)}).$$

Now it is only left for us to write $v^{(k+1)}$ in terms of $u^{(k)}$ and $v^{(k)}$ to obtain $L_k(\nu, x)$.
From the spectral equation for $v^{(k+1)}(\nu)$, we have

$$\begin{aligned} v^{(k+1)}(\nu) &= \frac{1}{A_{12}^{(k+1)}}(\widehat{u}^{(k+1)} - A_{11}^{(k+1)}u^{(k+1)}) \\ &= \frac{1}{A_{12}^{(k+1)}}\left(v^{(k)} - \frac{A_{11}^{(k+1)}}{|A_k(\lambda^2\nu)|}\left(A_{11}^{(k)}(\lambda^2\nu)v^{(k)} - A_{21}^{(k)}(\lambda^2\nu)u^{(k)}\right)\right) \\ &= \frac{1}{A_{12}^{(k+1)}}\left\{\left(1 - \frac{A_{11}^{(k+1)}}{|A_k(\lambda^2\nu)|}A_{11}^{(k)}(\lambda^2\nu)\right)v^{(k)} + \frac{A_{11}^{(k+1)}A_{21}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|}u^{(k)}\right\}. \end{aligned}$$

We are now ready to relate $\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix}$ to $\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix}$.

$$\begin{aligned} &\begin{pmatrix} u^{(k+1)}(\nu, x) \\ v^{(k+1)}(\nu, x) \end{pmatrix} \\ = \; & L_k(\nu, x)\begin{pmatrix} u^{(k)}(\nu, x) \\ v^{(k)}(\nu, x) \end{pmatrix} \\ = \; & \begin{pmatrix} \frac{-A_{21}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} & \frac{A_{11}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} \\ \frac{A_{11}^{(k+1)}A_{21}^{(k)}(\lambda^2\nu)}{A_{12}^{(k+1)}|A_k(\lambda^2\nu)|} & \frac{1}{A_{12}^{(k+1)}}\left(1 - \frac{A_{11}^{(k+1)}}{|A_k(\lambda^2\nu)|}A_{11}^{(k)}(\lambda^2\nu)\right) \end{pmatrix}\begin{pmatrix} u^{(k)}(\nu, x) \\ v^{(k)}(\nu, x) \end{pmatrix}. \end{aligned}$$

□

Recall that the special solutions of PII form an infinite sequence which can be expressed in a determinantal form [**18**]. In section 1.5.1 we saw that those rational and q-hypergeometric type special solutions of q-PII also form infinite sequences. We prove here that these also have a determinantal representation and relate special solutions of q-PII to the special solutions of the associated linear problem.

We know that the solution of q-PII $g_k(x)$ is related to the leading coefficient $a_0^{(k)}(x)$ of the solution of the associated linear problems $\phi_1^{(k)}(\nu, x)$ by

$$g_k(x) = -i\frac{x}{\lambda^{2k}}\frac{a_0^{(k)}(x/\lambda)}{a_0^{(k)}(x/\lambda^2)},$$

that is one can find $g_k(x)$ via the expression of $a_0^{(k)}(x)$. We also have the Schlesinger transformation of the associated linear problem of q-PII, which allows us to write down the solution of the linear problem when $e_1/e_2 = \frac{1}{\lambda^{4k}}$, for k is any integer or half integer.
For k is an integer,

$$\begin{aligned}\phi_1^{(k)}(\nu,x) &= \begin{pmatrix} u^{(k)}(\nu,x) \\ v^{(k)}(\nu,x)\end{pmatrix} = L_{k-1}L_{k-2}\dots L_0\ \phi_1^{(0)}(\nu,x) \quad (4.84)\\ &= L_{k-1}L_{k-2}\dots L_0 \begin{pmatrix} u^{(0)}(\nu,x) \\ v^{(0)}(\nu,x)\end{pmatrix},\end{aligned}$$

and from proposition 4.3.3 we know

$$\begin{aligned}&\phi_1^{(0)}\\ =\ & \begin{pmatrix} u^{(0)}(\nu,x) \\ v^{(0)}(\nu,x)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\lambda & \lambda \\ 1 & -1\end{pmatrix}\begin{pmatrix}\Gamma_{\lambda^{-2}}(1+\frac{\nu}{\lambda})\Gamma_{\lambda^{-2}}(1+ix\nu)\Gamma_{\lambda^{-2}}(1+ix\lambda\nu) \\ \Gamma_{\lambda^{-2}}(1-\frac{\nu}{\lambda})\Gamma_{\lambda^{-2}}(1-ix\nu)\Gamma_{\lambda^{-2}}(1-ix\lambda\nu)\end{pmatrix}\\ =\ & \sum_{j=0}\begin{pmatrix} a_{2j}^{(0)}\nu^{2j} \\ c_{2j+1}^{(0)}\nu^{2j+1}\end{pmatrix} = \sum_{j=0}\begin{pmatrix}\lambda T_{2j}\nu^{2j} \\ T_{2j+1}\nu^{2j+1}\end{pmatrix} \quad (4.85)\end{aligned}$$

where

$$\begin{aligned}T_j(\frac{1}{\lambda^{2j}}-1) =& -\frac{i(-i+\lambda x+\lambda^2 x)}{\lambda}T_{j-1} - \frac{x(-i-i\lambda+x\lambda^2)}{\lambda}T_{j-2},\\ &+x^2T_{j-3},\\ T_{j-1}(x) =& \frac{1}{ix\lambda}\left(T_j(x/\lambda)-T_j(x)\right).\end{aligned}$$

For k is a half integer,

$$\begin{aligned}\phi_1^{(k)}(\nu,x) &= \begin{pmatrix} u^{(k)}(\nu,x) \\ v^{(k)}(\nu,x)\end{pmatrix} = L_{k-1}L_{k-2}\dots L_{\frac{1}{2}}\ \phi_1^{(\frac{1}{2})}(\nu,x) \quad (4.86)\\ &= L_{k-1}L_{k-2}\dots L_{\frac{1}{2}} \begin{pmatrix} u^{(\frac{1}{2})}(\nu,x) \\ v^{(\frac{1}{2})}(\nu,x)\end{pmatrix},\end{aligned}$$

and from proposition 4.3.4 we know

$$\begin{aligned}\phi_1^{(\frac{1}{2})}(\nu,x) &= \begin{pmatrix} u^{(\frac{1}{2})}(\nu,x) \\ v^{(\frac{1}{2})}(\nu,x)\end{pmatrix} \\ &= \nu^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{\lambda^3(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^3\nu}\end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}^{(\frac{1}{2})}(x)\nu^{2j} \\ a_{2j}^{(\frac{1}{2})}(x/\lambda)\nu^{2j}\end{pmatrix}, \quad (4.87)\end{aligned}$$

where

$$\begin{aligned}\overline{a}_{2j}^{(\frac{1}{2})} &= ix\lambda^2 a_{2j}^{(\frac{1}{2})} - \frac{i}{x\lambda^2}(1/\lambda^{4j+4}-1)a_{2j+2}^{(\frac{1}{2})}, \\ \overline{\overline{a}}_{2j}^{(\frac{1}{2})} &= -\frac{1}{\lambda}a_{2j}^{(\frac{1}{2})} - x^2\lambda^3 a_{2j-2}^{(\frac{1}{2})} + \frac{i}{x\lambda}\overline{a}_{2j}^{(\frac{1}{2})}.\end{aligned}$$

Hence it is possible to find $g_k(x)$ via the expression of $a_0^{(k)}(x)$ in terms of T_j, $(j=0,1,\ldots)$ when k is an integer, or in terms of $a_{2j}^{(\frac{1}{2})}(x)$ and $a_{2j}^{(\frac{1}{2})}(x/\lambda)$, $(j=0,1,\ldots)$ when k is a half integer.

4.4.1. Using Schlesinger transformation L_k for the evaluation of $a_0^{(k)}$. Recall the Schlesinger transformation for the solutions of the associated linear problem of q-PII is

$$L_k(\nu,x) = \begin{pmatrix} \frac{-A_{21}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} & \frac{A_{11}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} \\ \frac{A_{11}^{(k+1)}A_{21}^{(k)}(\lambda^2\nu)}{A_{12}^{(k+1)}|A_k(\lambda^2\nu)|} & \frac{1}{A_{12}^{(k+1)}}\left(1-\frac{A_{11}^{(k+1)}}{|A_k(\lambda^2\nu)|}A_{11}^{(k)}(\lambda^2\nu)\right)\end{pmatrix},$$

where $A_{11}^{(k)}$, $A_{12}^{(k)}$, $A_{21}^{(k)}$ and $A_{22}^{(k)}$ are defined by equation (4.76). We see that as $\nu \to 0$,

$$\begin{aligned}A_{11}^{(k)}(\nu,x) &\sim O(1), \\ A_{12}^{(k)}(\nu,x) &\sim O(\nu), \\ A_{21}^{(k)}(\nu,x) &\sim O(\nu), \\ A_{22}^{(k)}(\nu,x) &\sim O(1), \\ |A_k(\nu,x)| &\sim O(1).\end{aligned}$$

Hence

$$L_k(\nu,x) \sim \begin{pmatrix} O(\nu) & O(1) \\ O(1) & O(1/\nu)\end{pmatrix},$$

that is $L_k(\nu, x)$ is a matrix function with infinite series expansion in the spectral variable ν in the neighbourhood of $\nu = 0$, unlike the Schlesinger transformation $L_a(x, t)$ of the linear problem of PII which is a matrix function of simple dependence of its spectral variable x :

$$L_a(x,t) \;=\; \begin{pmatrix} 0 & 1 \\ 1 & \frac{l_a(t)}{x} \end{pmatrix},$$

where

$$l_a(t) = -\frac{i(1+2a)}{x(t+2f_a^2-2f_a')}.$$

We have found that Schlesinger transformation $L_k(\nu, x)$ is not particularly helpful for finding determinantal expressions of $a_0^{(k)}$. Let us look at a few examples from equation (4.84):

$$\phi_1^{(k)}(\nu,x) = L_{k-1}L_{k-2}\dots L_0\ \phi_1^{(0)}(\nu,x),$$

for k are integers as to see why this is the case.
For $k = 1$,

$$\phi_1^{(1)}(\nu,x) = L_0(\nu,x)\phi_1^{(0)}(\nu,x), \tag{4.88}$$

with

$$\phi_1^{(0)}(\nu,x) = \sum_{j=0} \begin{pmatrix} a_{2j}\nu^{2j} \\ c_{2j+1}\nu^{2j+1} \end{pmatrix},$$

where we have omitted the superscript $^{(0)}$ on the coefficients of expansion here for simplicity, and

$$\begin{aligned} L_0(\nu,x) \;&=\; \begin{pmatrix} \frac{-A_{21}^{(0)}(\lambda^2\nu)}{|A_0(\lambda^2\nu)|} & \frac{A_{11}^{(0)}(\lambda^2\nu)}{|A_0(\lambda^2\nu)|} \\ \frac{A_{11}^{(01)}A_{21}^{(0)}(\lambda^2\nu)}{A_{12}^{(1)}|A_0(\lambda^2\nu)|} & \frac{1}{A_{12}^{(1)}}\left(1 - \frac{A_{11}^{(1)}}{|A_0(\lambda^2\nu)|}A_{11}^{(0)}(\lambda^2\nu)\right) \end{pmatrix} \\ &\sim\; \begin{pmatrix} s_1\nu + s_3\nu^3 + \dots & \frac{1}{\nu}(t_1\nu + t_3\nu^3 + \dots) \\ s_0 + s_2\nu^2 + \dots & \frac{1}{\nu}(t_0 + t_2\nu^2 + \dots) \end{pmatrix}, \end{aligned}$$

where we have named the coefficients of ν^j in the $(1,1)$ and $(2,1)$ entries to be s_j, the coefficients of ν^{j-1} in the $(1,2)$ and $(2,2)$ entries to be t_j, $(j = 0, 1, \dots)$. Note s_j and t_j are in general functions of x, which can be

calculated from the definitions (4.76) of $A^{(k)}_{mn}(\nu, x)$, $(m, n = 1, 2)$.
The asymptotic behavior of $\phi^{(1)}_1(\nu, x)$ near $\nu = 0$ is given by

$$\phi^{(1)}_1(\nu, x) \sim \begin{pmatrix} a^{(1)}_0 \nu \\ c^{(1)}_1 \nu^2 \end{pmatrix}, \qquad \text{as } \nu \to 0$$

but from equation (4.88) $\phi^{(1)}_1(\nu, x)$ is given by

$$\phi^{(1)}_1(\nu, x) = \begin{pmatrix} s_1\nu + s_3\nu^3 + \ldots & \frac{1}{\nu}(t_1\nu + t_3\nu^3 + \ldots) \\ s_0 + s_2\nu^2 + \ldots & \frac{1}{\nu}(t_0 + t_2\nu^2 + \ldots) \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}\nu^{2j} \\ c_{2j+1}\nu^{2j+1} \end{pmatrix}.$$

Equating powers in ν of the two expressions of $\phi^{(1)}_1(\nu, x)$:
-the top equation gives

$$\nu \;\; : \;\; s_1 a_0 + t_1 c_1 = a^{(1)}_0,$$

-the bottom equation gives

$$\begin{aligned} \nu^0 \;\; &: \;\; s_0 a_0 + t_0 c_1 = 0 \\ \nu^2 \;\; &: \;\; s_0 a_2 + t_0 c_3 + s_2 a_0 + t_2 c_1 = c^{(1)}_1, \end{aligned}$$

which we can rewrite in the form of a matrix equation, recall from equation (4.83) that $c^{(1)}_1 = \frac{a^{(2)}_0}{\lambda^4}$,

$$\begin{pmatrix} a_0 & c_1 & a_2 & c_3 \\ 0 & 0 & a_0 & c_1 \end{pmatrix} \begin{pmatrix} s_2 \\ t_2 \\ s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} \frac{a^{(2)}_0}{\lambda^4} \\ 0 \end{pmatrix}. \tag{4.89}$$

Equation (4.84) for $k = 2$ is,

$$\phi^{(2)}_1(\nu, x) = L_1 L_0 \, \phi^{(0)}_1(\nu, x), \tag{4.90}$$

where

$$L_1 L_0 \;\; \sim \;\; \begin{pmatrix} s_0 + s_2\nu^2 + \ldots & \frac{1}{\nu}(t_0 + t_2\nu^2 + \ldots) \\ \frac{s_{-1}}{\nu} + s_1\nu s_3\nu^3 + \ldots & \frac{1}{\nu}\left(\frac{t_{-1}}{\nu} + t_1\nu + t_3\nu^3 + \ldots\right) \end{pmatrix},$$

where we have renamed the coefficients of ν^j in the $(1, 1)$ and $(2, 1)$ entries to be s_j, the coefficients of ν^{j-1} in the $(1, 2)$ and $(2, 2)$ entries to be t_j, $(j = -1, 0, 1, \ldots)$. Note the s_j and t_j for the case $k = 2$ are obviously

different functions from the s_j and t_j for the case $k = 1$.
We know $\phi_1^{(2)}(\nu, x)$'s asymptotic behaviour is given by

$$\phi_1^{(2)}(\nu, x) \sim \nu^2 \begin{pmatrix} a_0^{(2)} \\ c_1^{(2)}\nu \end{pmatrix}, \qquad \text{as } \nu \to 0$$

but from equation (4.90) $\phi_1^{(2)}(\nu, x)$ is given by

$$\begin{aligned} &\phi_1^{(2)}(\nu, x) \\ &= \begin{pmatrix} s_0 + s_2\nu^2 + \ldots & \frac{1}{\nu}(t_0 + t_2\nu^2 + \ldots) \\ \frac{s_{-1}}{\nu} + s_1\nu + s_3\nu^3 + \ldots & \frac{1}{\nu}\left(\frac{t_{-1}}{\nu} + t_1\nu + t_3\nu^3 + \ldots\right) \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}\nu^{2j} \\ c_{2j+1}\nu^{2j+1} \end{pmatrix}. \end{aligned}$$

Equating powers in ν of the two expressions of $\phi_1^{(2)}(\nu, x)$:
-the top equation gives

$$\begin{aligned} \nu^0 &: \quad s_0a_0 + t_0c_1 = 0 \\ \nu^2 &: \quad s_0a_2 + t_0c_3 + s_2a_0 + t_2c_1 = a_0^{(2)} \end{aligned}$$

which we have already expressed in the form of a matrix equation (4.89),
-the bottom equation gives

$$\begin{aligned} \frac{1}{\nu} &: \quad s_{-1}a_0 + t_{-1}c_1 = 0 \\ \nu &: \quad s_{-1}a_2 + t_{-1}c_3 + s_1a_0 + t_1c_1 = 0 \\ \nu^3 &: \quad s_{-1}a_4 + t_{-1}c_5 + s_1a_2 + t_1c_3 + s_3a_0 + t_3c_1 = c_1^{(2)} \end{aligned}$$

which we can rewrite in the form of a matrix equation, and recall from equation (4.83) that $c_1^{(2)} = \frac{a_0^{(3)}}{\lambda^6}$,

$$\begin{pmatrix} a_0 & c_1 & a_2 & c_3 & a_4 & c_5 \\ 0 & 0 & a_0 & c_1 & a_2 & c_3 \\ 0 & 0 & 0 & 0 & a_0 & c_1 \end{pmatrix} \begin{pmatrix} s_3 \\ t_3 \\ s_1 \\ t_1 \\ s_{-1} \\ t_{-1} \end{pmatrix} = \begin{pmatrix} \frac{a_0^{(3)}}{\lambda^6} \\ 0 \\ 0 \end{pmatrix}. \tag{4.91}$$

Equation (4.84) for $k = 3$ is,

$$\phi_1^{(3)}(\nu, x) = L_2L_1L_0\ \phi_1^{(0)}(\nu, x), \tag{4.92}$$

where

$$L_2L_1L_0 \sim \begin{pmatrix} \frac{s_{-1}}{\nu}+s_1\nu+s_3\nu^3+\dots & \frac{1}{\nu}\left(\frac{t_{-1}}{\nu}+t_1\nu+t_3\nu^3+\dots\right) \\ \frac{s_{-2}}{\nu^2}+s_0+s_2\nu^2+\dots & \frac{1}{\nu}\left(\frac{t_{-2}}{\nu^2}+t_0+t_2\nu^2+\dots\right) \end{pmatrix}.$$

We know

$$\phi_1^{(3)}(\nu,x) \sim \nu^3 \begin{pmatrix} a_0^{(3)} \\ c_1^{(3)}\nu \end{pmatrix}, \qquad \text{as } \nu \to 0$$

but from equation (4.92) $\phi_1^{(3)}(\nu,x)$ is given by

$$\begin{aligned} &\phi_1^{(3)}(\nu,x) \\ &= \begin{pmatrix} \frac{s_{-1}}{\nu}+s_1\nu+s_3\nu^3+\dots & \frac{1}{\nu}\left(\frac{t_{-1}}{\nu}+t_1\nu+t_3\nu^3+\dots\right) \\ \frac{s_{-2}}{\nu^2}+s_0+s_2\nu^2+\dots & \frac{1}{\nu}\left(\frac{t_{-2}}{\nu^2}+t_0+t_2\nu^2+\dots\right) \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}\nu^{2j} \\ c_{2j+1}\nu^{2j+1} \end{pmatrix}. \end{aligned}$$

Equating powers in ν of the two expressions of $\phi_1^{(3)}(\nu,x)$:
-the top equation gives

$$\begin{aligned} \frac{1}{\nu} &: \quad s_{-1}a_0+t_{-1}c_1=0 \\ \nu &: \quad s_{-1}a_2+t_{-1}c_3+s_1a_0+t_1c_1=0 \\ \nu^3 &: \quad s_{-1}a_4+t_{-1}c_5+s_1a_2+t_1c_3+s_3a_0+t_3c_1=a_0^{(3)} \end{aligned}$$

which we have already expressed in the form of a matrix equation (4.91),
-the bottom equation gives

$$\begin{aligned} \frac{1}{\nu^2} &: \quad s_{-2}a_0+t_{-2}c_1=0 \\ \nu^0 &: \quad s_{-2}a_2+t_{-2}c_3+s_0a_0+t_0c_1=0 \\ \nu^2 &: \quad s_{-2}a_4+t_{-2}c_5+s_0a_2+t_0c_3+s_2a_0+t_2c_1=0 \\ \nu^4 &: \quad s_{-2}a_6+t_{-2}c_7+s_0a_4+t_0c_5+s_2a_2+t_2c_3+s_4a_0+t_4c_1=c_1^{(3)}, \end{aligned}$$

which we can rewrite in the form of a matrix equation, recall from equation (4.83) that $c_1^{(3)} = \frac{a_0^{(4)}}{\lambda^8}$,

$$\begin{pmatrix} a_0 & c_1 & a_2 & c_3 & a_4 & c_5 & a_6 & c_7 \\ 0 & 0 & a_0 & c_1 & a_2 & c_3 & a_4 & c_5 \\ 0 & 0 & 0 & 0 & a_0 & c_1 & a_2 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_0 & c_1 \end{pmatrix} \begin{pmatrix} s_4 \\ t_4 \\ s_2 \\ t_2 \\ s_0 \\ t_0 \\ s_{-2} \\ t_{-2} \end{pmatrix} = \begin{pmatrix} \frac{a_0^{(4)}}{\lambda^8} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the above examples we see that although we can calculate all the s_j, t_j and the coefficients $a_{2j}^{(0)}$, $c_{2j+1}^{(0)}$ $(j = 0, 1, \ldots)$ of the expansion of $\phi_1^{(0)}(\nu, x)$ and hence calculate $a_0^{(k)}$, it does not however give us $a_0^{(k)}(x)$ in the desired determinantal form. This is due to the fact that $L_k(\nu, x)$ in the neighbourhood of $\nu = 0$ has infinite series expansion in the spectral variable ν. In the next subsection we will show that there is another Schlesinger transformation which reveals the determinantal structure of $a_0^{(k)}(x)$.

4.5. A simpler Schlesinger transformation Λ_k

While relation (4.84) is correct, its complexity does not help us to find $a_0^{(k)}(x)$ in a determinantal form. We will show a simpler Schlesinger transformation of the associated linear problem from which the determinantal form of $a_0^{(k)}(x)$ and hence $g_k(x)$ can be obtained. Recall the associated linear problem (4.69)

$$\begin{aligned} \widehat{\Psi}^{(k)}(\nu, x) &= A_k(\nu, x)\Psi^{(k)}(\nu, x) \\ &= \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{pmatrix} \Psi^{(k)}(\nu, x), \end{aligned}$$

where $A_{mn}^{(k)}$ $(m, n = 1, 2)$, are defined in equation (4.76). We also have the Schlesinger transformation (4.71) that relates $\Psi^{(k+1)}(\nu, x)$ to $\Psi^{(k)}(\nu, x)$

$$\Psi^{(k+1)}(\nu, x) = L_k(\nu, x)\Psi^{(k)}(\nu, x),$$

where $L_k(\nu, x)$ is defined by equation (4.72)

$$L_k(\nu,x) = \begin{pmatrix} \frac{-A_{21}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} & \frac{A_{11}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} \\ \frac{A_{11}^{(k+1)}A_{21}^{(k)}(\lambda^2\nu)}{A_{12}^{(k+1)}|A_k(\lambda^2\nu)|} & \frac{1}{A_{12}^{(k+1)}}\left(1 - \frac{A_{11}^{(k+1)}}{|A_k(\lambda^2\nu)|}A_{11}^{(k)}(\lambda^2\nu)\right) \end{pmatrix}.$$

Definition 4.5.1. Let us define a new system of vector functions $F^{(k)}(\nu, x)$, which are related to $\phi_1^{(k)}(\nu, x)$ by

$$\begin{aligned} F^{(k)}(\nu,x) &= \phi_1^{(k)}(\nu/\lambda^{2k}, x) \\ &= \frac{\nu^k}{\lambda^{2k^2}} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}^{(k)}(x)\frac{\nu^{2j}}{\lambda^{4kj}} \\ c_{2j+1}^{(k)}(x)\frac{\nu^{2j+1}}{\lambda^{2k(2j+1)}} \end{pmatrix}. \end{aligned} \tag{4.93}$$

Proposition 4.5.2. Now we show that vector function $F^{(k)}(\nu, x)$ have Schlesinger transformation defined by

$$F^{(k+1)}(\nu,x) = \Lambda_k(\nu,x)F^{(k)}(\nu,x) \tag{4.94}$$

$$\Lambda_k(\nu,x) = A_{k+1}(\nu/\lambda^{2k},x)L_k(\nu/\lambda^{2k},x) = \begin{pmatrix} 0 & 1 \\ \frac{1}{\lambda^2} & \frac{\rho_k(x)}{\nu} \end{pmatrix}, \tag{4.95}$$

where

$$\rho_k(x) = \frac{\lambda^{2k}\left(e_1 - e_2\lambda^2\right) g_k(x) g_k(x\lambda)^2}{x^2\lambda^3\left(e_1x^2\lambda^2 + \left(e_1 - e_2\lambda^2 g_k(x)\right) g_k(x\lambda)\right)}, \tag{4.96}$$

recall $e_1 = \frac{1}{\lambda^{2k}},\ e_2 = \lambda^{2k}$.

Proof. We know

$$\begin{aligned} \widehat{\phi_1}^{(k+1)}(\nu,x) &= A_{k+1}(\nu,x)\phi_1{}^{(k+1)}(\nu,x) \\ &= A_{k+1}(\nu,x)L_k(\nu,x)\phi_1^{(k)}(\nu,x), \end{aligned}$$

now let $\nu \to \frac{\nu}{\lambda^{2k}}$,

$$\begin{aligned} \widehat{\phi_1}^{(k+1)}(\nu/\lambda^{2k},x) &= A_{k+1}(\nu/\lambda^{2k},x)L_k(\nu/\lambda^{2k},x)\phi_1^{(k)}(\nu/\lambda^{2k},x) \\ \phi_1{}^{(k+1)}(\nu/\lambda^{2k+2},x) &= A_{k+1}(\nu/\lambda^{2k},x)L_k(\nu/\lambda^{2k},x)\phi_1^{(k)}(\nu/\lambda^{2k},x) \\ F^{(k+1)}(\nu,x) &= A_{k+1}(\nu/\lambda^{2k},x)L_k(\nu/\lambda^{2k},x)F^{(k)}(\nu,x). \end{aligned}$$

Use the definition of $A_{k+1}(\nu,x)$ and $L_k(\nu,x)$ we have

$$\begin{aligned}
& A_{k+1}(\nu,x)L_k(\nu,x) \\
= & \begin{pmatrix} A_{11}^{(k+1)}(\nu,x) & A_{12}^{(k+1)}(\nu,x) \\ A_{21}^{(k+1)}(\nu,x) & A_{22}^{(k+1)}(\nu,x) \end{pmatrix} \\
& \qquad \times \begin{pmatrix} \frac{-A_{21}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} & \frac{A_{11}^{(k)}(\lambda^2\nu)}{|A_k(\lambda^2\nu)|} \\ \frac{A_{11}^{(k+1)}A_{21}^{(k)}(\lambda^2\nu)}{A_{12}^{(k+1)}|A_k(\lambda^2\nu)|} & \frac{1}{A_{12}^{(k+1)}}\left(1-\frac{A_{11}^{(k+1)}}{|A_k(\lambda^2\nu)|}A_{11}^{(k)}(\lambda^2\nu)\right) \end{pmatrix} \\
= & \begin{pmatrix} 0 & 1 \\ \frac{A_{21}^{(k)}(\lambda^2\nu)|A_{k+1}(\nu)|}{A_{12}^{(k+1)}(\nu)|A_k(\lambda^2\nu)|} & \frac{-A_{11}^{(k)}(\lambda^2\nu)|A_{k+1}(\nu)|+A_{22}^{(k+1)}(\nu,x)|A_k(\lambda^2\nu)|}{A_{12}^{(k+1)}(\nu)|A_k(\lambda^2\nu)|} \end{pmatrix}.
\end{aligned}$$

An observation from equation (4.77) tells us that

$$|A_{k+1}(\nu,x)| = |A_k(\lambda^2\nu,x)|.$$

Using Bäcklund transformation (4.74) of $g_k(x)$ and q-PII equation (4.9) it can be easily checked that

$$\frac{A_{21}^{(k)}(\lambda^2\nu)}{A_{12}^{(k+1)}(\nu)} = \frac{1}{\lambda^2},$$

and

$$\frac{-A_{11}^{(k)}(\lambda^2\nu)+A_{22}^{(k+1)}(\nu,x)}{A_{12}^{(k+1)}(\nu)} = \frac{(e_1-e_2\lambda^2)\,g_k(x)g_k(x\lambda)^2}{x^2\lambda^3\left(e_1x^2\lambda^2+(e_1-e_2\lambda^2g_k(x))g_k(x\lambda)\right)\nu}.$$

Hence

$$\begin{aligned}
\Lambda_k(\nu,x) &= A_{k+1}(\nu/\lambda^{2k},x)L_k(\nu/\lambda^{2k},x) \\
&= \begin{pmatrix} 0 & 1 \\ \frac{1}{\lambda^2} & \frac{\lambda^{2k}\left(e_1-e_2\lambda^2\right)g_k(x)g_k(x\lambda)^2}{x^2\lambda^3(e_1x^2\lambda^2+(e_1-e_2\lambda^2g_k(x))g_k(x\lambda))\nu} \end{pmatrix}.
\end{aligned}$$

□

Proposition (4.5.2) gives a much simpler Schlesinger transformation $\Lambda_k(\nu,x)$ then $L_k(\nu,x)$ which is rational in ν. This observation is crucial in allowing us to obtain the determinantal form of $a_0^{(k)}(x)$ and therefore that of $g_k(x)$. We now have,

for k is an integer

$$F^{(k+1)}(\nu,x)=\Lambda_k\Lambda_{k-1}\dots\begin{pmatrix}0 & 1\\ \frac{1}{\lambda^2} & \frac{\rho_0(x)}{\nu}\end{pmatrix}F^{(0)}(\nu,x) \tag{4.97}$$

and for k is a half integer

$$F^{(k+1)}(\nu,x)=\Lambda_k\Lambda_{k-1}\dots\begin{pmatrix}0 & 1\\ \frac{1}{\lambda^2} & \frac{\rho_{\frac{1}{2}}(x)}{\nu}\end{pmatrix}F^{(\frac{1}{2})}(\nu,x). \tag{4.98}$$

More over from the definition of $F^{(k)}(\nu,x)$ (4.93) we have

$$F^{(0)}(\nu,x)=\phi_1^{(0)}(\nu,x)=\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}\nu^{2j}\\ c_{2j+1}\nu^{2j+1}\end{pmatrix}, \tag{4.99}$$

where we have omitted the superscript $^{(0)}$ on the coefficients of expansion for simplicity, and

$$\begin{aligned}F^{(\frac{1}{2})}(\nu,x) &= \phi_1^{\left(\frac{1}{2}\right)}(\nu/\lambda,x) \qquad (4.100)\\ &= \frac{\nu^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\begin{pmatrix}1 & 0\\ \frac{g_{\frac{1}{2}}(x)}{\lambda^2(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^2\nu}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}^{(\frac{1}{2})}(x)\frac{\nu^{2j}}{\lambda^{2j}}\\ a_{2j}^{(\frac{1}{2})}(x/\lambda)\frac{\nu^{2j}}{\lambda^{2j}}\end{pmatrix}.\end{aligned}$$

4.5.1. $F^{(k+1)}(\nu,x)$ **when** k **is an integer.** Let us first consider equation (4.97 of the case where k is an integer

$$\Lambda_k(\nu,x)=\begin{pmatrix}0 & 1\\ \frac{1}{\lambda^2} & \frac{\rho_k(x)}{\nu}\end{pmatrix},$$

where $\rho_k(x)$ are functions of $g_k(x)$ and x.
Some examples:

$$\Lambda_1\Lambda_0=\begin{pmatrix}s_0 & \frac{t_0}{\nu}\\ \frac{s_1}{\nu} & \frac{1}{\nu}\left(\frac{t_1}{\nu}+t_{-1}\nu\right)\end{pmatrix},$$

$$\Lambda_2\Lambda_1\Lambda_0=\begin{pmatrix}\frac{s_1}{\nu} & \frac{1}{\nu}\left(\frac{t_1}{\nu}+t_{-1}\nu\right)\\ \frac{s_2}{\nu^2}+s_0 & \frac{1}{\nu}\left(\frac{t_2}{\nu^2}+t_0\right)\end{pmatrix},$$

where we have renamed the coefficients of ν^{-j} in the $(1,1)$ and $(2,1)$ entries in each case to be s_j and the coefficients of ν^{-j-1} in the $(1,2)$ and $(2,2)$ entries to be t_j ($j=0,\dots,l$). Note the particular form of $\Lambda_k(\nu,x)$ implies

that the s_0 and t_{-1} in each case are constants whereas s_j and t_j in general are functions of $g_k(x)$ and x. In general for k, an odd integer

$$\Lambda_k \dots \Lambda_1 \Lambda_0 = \begin{pmatrix} \frac{s_{k-1}}{\nu^{k-1}} + \dots + \frac{s_2}{\nu^2} + s_0 & \frac{1}{\nu}\left(\frac{t_{k-1}}{\nu^{k-1}} + \dots + \frac{t_2}{\nu^2} + t_0\right) \\ \frac{s_k}{\nu^k} + \dots + \frac{s_3}{\nu^3} + \frac{s_1}{\nu} & \frac{1}{\nu}\left(\frac{t_k}{\nu^k} + \dots + \frac{t_3}{\nu^3} + \frac{t_1}{\nu} + t_{-1}\nu\right) \end{pmatrix}.$$

For k, an even integer

$$\Lambda_k \dots \Lambda_1 \Lambda_0 = \begin{pmatrix} \frac{s_{k-1}}{\nu^{k-1}} + \dots + \frac{s_3}{\nu^3} + \frac{s_1}{\nu} & \frac{1}{\nu}\left(\frac{t_{k-1}}{\nu^{k-1}} + \dots + \frac{t_3}{\nu^3} + \frac{t_1}{\nu} + t_{-1}\nu\right) \\ \frac{s_k}{\nu^k} + \dots + \frac{s_2}{\nu^2} + s_0 & \frac{1}{\nu}\left(\frac{t_k}{\nu^k} + \dots + \frac{t_2}{\nu^2} + t_0\right) \end{pmatrix}.$$

4.5.2. $F^{(k+1)}(\nu, x)$ when k is a half integer. For k is a half integer

$$F^{(k+1)}(\nu, x) = \Lambda_k \Lambda_{k-1} \dots \begin{pmatrix} 0 & 1 \\ \frac{1}{\lambda^2} & \frac{\rho_{\frac{1}{2}}(x)}{\nu} \end{pmatrix} F^{(\frac{1}{2})}(\nu, x),$$

where

$$\begin{aligned} F^{(\frac{1}{2})}(\nu, x) &= \phi_1^{\left(\frac{1}{2}\right)}(\nu/\lambda, x) \\ &= \frac{\nu^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{\lambda^2(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^2\nu} \end{pmatrix} \sum_{j=0}^{\infty} \begin{pmatrix} a_{2j}^{(\frac{1}{2})}(x)\frac{\nu^{2j}}{\lambda^{2j}} \\ a_{2j}^{(\frac{1}{2})}(x/\lambda)\frac{\nu^{2j}}{\lambda^{2j}} \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} \overline{a}_{2j}^{(\frac{1}{2})} &= ix\lambda^2 a_{2j}^{(\frac{1}{2})} - \frac{i}{x\lambda^2}(1/\lambda^{4j+4} - 1)a_{2j+2}^{(\frac{1}{2})}, \\ \overline{\overline{a}}_{2j}^{(\frac{1}{2})} &= -\frac{1}{\lambda}a_{2j}^{(\frac{1}{2})} - x^2\lambda^3 a_{2j-2}^{(\frac{1}{2})} + \frac{i}{x\lambda}\overline{a}_{2j}^{(\frac{1}{2})}. \end{aligned}$$

We write $k = n - \frac{1}{2}$, $(n = 1, 2, \dots)$, and consider a few examples:

$$n = 1: \qquad \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{\lambda^2(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^2\nu} \end{pmatrix} = \begin{pmatrix} \frac{s_1}{\nu} & \frac{t_1}{\nu} \\ s_0 + \frac{s_2}{\nu^2} & \frac{t_2}{\nu^2} \end{pmatrix},$$

$$n = 2: \qquad \Lambda_{\frac{3}{2}}\Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{\lambda^2(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^2\nu} \end{pmatrix} = \begin{pmatrix} s_0 + \frac{s_2}{\nu^2} & \frac{t_2}{\nu^2} \\ \frac{s_1}{\nu} + \frac{s_3}{\nu^3} & \frac{t_1}{\nu} + \frac{t_3}{\nu^3} \end{pmatrix},$$

$$n = 3: \qquad \Lambda_{\frac{5}{2}}\Lambda_{\frac{3}{2}}\Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{\lambda^2(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^2\nu} \end{pmatrix} = \begin{pmatrix} \frac{s_1}{\nu} + \frac{s_3}{\nu^3} & \frac{t_1}{\nu} + \frac{t_3}{\nu^3} \\ s_0 + \frac{s_2}{\nu^2} + \frac{s_4}{\nu^4} & \frac{t_2}{\nu^2} + \frac{t_4}{\nu^4} \end{pmatrix},$$

where we have named the coefficients of ν^{-j} in the $(1,1)$ and $(2,1)$ entries on the RHS of the equations in each case to be s_j, the coefficients of ν^{-j} in the $(1,2)$ and $(2,2)$ entries to be t_j, $(j = 0, 1, \ldots, n+1)$. From the particular form of the $\Lambda_{n-\frac{1}{2}}$ and matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{g_{\frac{1}{2}}(x)}{\lambda^2(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^2\nu} \end{pmatrix},$$

we found that s_0 is a different constant and $t_1 = -\frac{i}{x\lambda^2}$ for all the cases. In general for n is an odd integer, $n = 1, 3, \ldots$

$$\Lambda_{n-\frac{1}{2}} \ldots \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g(x)}{\lambda^3(x^2+g(x))\nu} & -\frac{i}{x\lambda^3\nu} \end{pmatrix} = \begin{pmatrix} \frac{s_1}{\nu} + \frac{s_3}{\nu^3} + \ldots + \frac{s_n}{\nu^n} & \frac{t_1}{\nu} + \ldots + \frac{t_n}{\nu^n} \\ s_0 + \frac{s_2}{\nu^2} + \ldots + \frac{s_{n+1}}{\nu^{n+1}} & \frac{t_2}{\nu^2} + \ldots + \frac{t_{n+1}}{\nu^{n+1}} \end{pmatrix}$$

and for n is an even integer, $n = 2, 4, \ldots$

$$\Lambda_{n-\frac{1}{2}} \ldots \Lambda_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{g(x)}{\lambda^3(x^2+g(x))\nu} & -\frac{i}{x\lambda^3\nu} \end{pmatrix} = \begin{pmatrix} s_0 + \frac{s_2}{\nu^2} + \ldots + \frac{s_n}{\nu^n} & \frac{t_2}{\nu^2} + \ldots + \frac{t_n}{\nu^n} \\ \frac{s_1}{\nu} + \frac{s_3}{\nu^3} + \ldots + \frac{s_{n+1}}{\nu^{n+1}} & \frac{t_1}{\nu} + \ldots + \frac{t_{n+1}}{\nu^{n+1}} \end{pmatrix}.$$

4.6. Determinantal representations of special solutions

4.6.1. Determinantal form of rational type special solutions of q-PII.

Theorem 4.6.1. q-PII equation (4.9) with parameter $\alpha = \frac{1}{\lambda^{4k}}$ where k are integers admits a hierarchy of rational type special solutions $g_k(x)$, given by

$$g_k(x) = -\frac{ix}{\lambda^{2k}} \frac{\tau_k(x/\lambda)\tau_{k-1}(x/\lambda^2)}{\tau_k(x/\lambda^2)\tau_{k-1}(x/\lambda)}$$

$$\tau_k(x) = \begin{vmatrix} T_k & T_{k+1} & \cdots & & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \cdots & & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ T_2 & T_3 & \cdots & & T_k & T_{k+1} \\ & & \ddots & & \vdots & \vdots \\ & & & \cdots & T_2 & T_3 \\ 0 & \cdots & & \cdots & T_0 & T_1 \end{vmatrix}.$$

$T_k(x)$ is the q discrete polynomial of degree k in x where

$$T_j(x)(\frac{1}{\lambda^{2j}}-1) = -\frac{i(-i+\lambda x+\lambda^2 x)}{\lambda}T_{j-1}(x) - \frac{x(-i-i\lambda+x\lambda^2)}{\lambda}T_{j-2}(x) + x^2 T_{j-3}(x)$$
$$T_{j-1}(x) \;=\; \frac{1}{ix\lambda}(T_j(x/\lambda) - T_j(x))$$

and $T_0(x)=1$, $T_j = 0, j<0$.

Proof. We consider the case when k is an odd integer, will show later that the case for k is even can be proved along the way. Recall the Schlesinger transformation relating $F^{(k+1)}$ to $F^{(0)}$,

$$F^{(k+1)}(\nu,x) = \Lambda_k\Lambda_{k-1}\dots\Lambda_0 F^{(0)}(\nu,x)$$
$$= \begin{pmatrix} \frac{s_{k-1}}{\nu^{k-1}}+\ldots+\frac{s_2}{\nu^2}+s_0 & \frac{1}{\nu}\left(\frac{t_{k-1}}{\nu^{k-1}}+\ldots+\frac{t_2}{\nu^2}+t_0\right) \\ \frac{s_k}{\nu^k}+\ldots+\frac{s_3}{\nu^3}+\frac{s_1}{\nu} & \frac{1}{\nu}\left(\frac{t_k}{\nu^k}+\ldots+\frac{t_3}{\nu^3}+\frac{t_1}{\nu}+t_{-1}\nu\right)\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix} a_{2j}\nu^{2j} \\ c_{2j+1}\nu^{2j+1}\end{pmatrix},$$

and from the definition (4.93) of $F^{(k)}$ we know

$$F^{(k+1)}(\nu,x) = \phi_1^{(k+1)}(\nu/\lambda^{2(k+1)},x) \sim \frac{\nu^{k+1}}{\lambda^{2(k+1)^2}}\begin{pmatrix} a_0^{(k+1)} \\ c_1^{(k+1)}\frac{\nu}{\lambda^{2(k+1)}}\end{pmatrix}.$$

Equating the two expressions of $F^{(k+1)}$ in powers of ν near $\nu=0$:
-the top entry gives

$$\begin{aligned}
\frac{1}{\nu^{k-1}} &: s_{k-1}a_0 + t_{k-1}c_1 = 0 \\
\frac{1}{\nu^{k-3}} &: s_{k-1}a_2 + t_{k-1}c_3 + + s_{k-3}a_0 + t_{k-3}c_1 = 0 \\
\vdots &: \vdots \\
\nu^{k-1} &: s_{k-1}a_{2k-2} + t_{k-1}c_{2k-1} + \ldots + s_0 a_{k-1} + t_0 c_k = 0 \\
\nu^{k+1} &: s_{k-1}a_{2k} + t_{k-1}c_{2k+1} + \ldots + s_0 a_{k+1} + t_0 c_{k+2} = \frac{a_0^{(k+1)}}{\lambda^{2(k+1)^2}},
\end{aligned}$$

-the bottom entry gives

$$\begin{aligned}
\frac{1}{\nu^k} &: \quad s_k a_0 + t_k c_1 = 0 \\
\frac{1}{\nu^{k-2}} &: \quad s_k a_2 + t_k c_3 + + s_{k-2} a_0 + t_{k-2} c_1 = 0 \\
\vdots &: \quad \vdots \\
\nu^k &: \quad s_k a_{2k} + t_k c_{2k+1} + \ldots + s_1 a_{k+1} + t_1 c_{k+2} + t_{-1} c_k = 0 \\
\nu^{k+2} &: \quad s_k a_{2k+2} + t_k c_{2k+3} + \ldots + s_1 a_{k+3} + t_1 c_{k+4} + t_{-1} c_{k+2} = \frac{c_1^{(k+1)}}{\lambda^{2(k+1)(k+2)}}.
\end{aligned}$$

We can write the equations we have from the top entry in a $(k+1)\times(k+1)$ matrix form

$$\begin{pmatrix} a_{k+1} & c_{k+2} & \ldots & a_{2k} & c_{2k+1} \\ a_{k-1} & c_k & \ldots & a_{2k-2} & c_{2k-1} \\ & & \ddots & & \\ & & & & \\ 0 & & \ldots & a_0 & c_1 \end{pmatrix} \begin{pmatrix} s_0 \\ t_0 \\ s_2 \\ \vdots \\ s_{k-1} \\ t_{k-1} \end{pmatrix} = \begin{pmatrix} \frac{a_0^{(k+1)}}{\lambda^{2(k+1)^2}} \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix} \tag{4.101}$$

and the equations from the bottom entry in a $(k+2)\times(k+2)$ matrix form

$$\begin{pmatrix} c_{k+2} & a_{k+3} & \ldots & a_{2k+2} & c_{2k+3} \\ c_k & a_{k+1} & \ldots & a_{2k} & c_{2k+1} \\ & & \ddots & & \\ & & & & \\ 0 & & \ldots & a_0 & c_1 \end{pmatrix} \begin{pmatrix} t_{-1} \\ s_1 \\ t_1 \\ \vdots \\ s_k \\ t_k \end{pmatrix} = \begin{pmatrix} \frac{c_1^{(k+1)}}{\lambda^{2(k+1)(k+2)}} \\ 0 \\ \\ \vdots \\ \\ 0 \end{pmatrix}. \tag{4.102}$$

Definition 4.6.2. Recall k is an odd integer, therefore $k+1$ is even, let $\sigma_{\text{even}}(x)$ be

$$\sigma_{k+1} = \begin{vmatrix} a_{k+1} & c_{k+2} & \ldots & a_{2k} & c_{2k+1} \\ a_{k-1} & c_k & \ldots & a_{2k-2} & c_{2k-1} \\ & & \ddots & & \\ & & & & \\ 0 & & \ldots & a_0 & c_1 \end{vmatrix}$$

and $\sigma_{\rm odd}(x)$ be

$$\sigma_{k+2} = \begin{vmatrix} c_{k+2} & a_{k+3} & \dots & a_{2k+2} & c_{2k+3} \\ c_k & a_{k+1} & \dots & a_{2k} & c_{2k+1} \\ & & \ddots & & \\ 0 & & \dots & a_0 & c_1 \end{vmatrix}.$$

Apply Cramer's rule on matrix equation (4.101) to evaluate s_0, recall s_0 is a constant,

$$\begin{aligned} s_0 &= \frac{a_0^{(k+1)}(x)\sigma_k(x)}{\sigma_{k+1}(x)} \\ \Rightarrow a_0^{(k+1)}(x) &= \frac{s_0\sigma_{k+1}(x)}{\sigma_k(x)}, \end{aligned} \tag{4.103}$$

where $\sigma_k = \sigma_{\rm odd}$, $\sigma_{k+1} = \sigma_{\rm even}$ are defined by definition 4.6.2. We have $a_0^{(\rm even)}$ in terms of determinants.

Apply Cramer's rule on matrix equation (4.102) for t_{-1}, recall t_{-1} is also a constant,

$$\begin{aligned} t_{-1} &= \frac{c_1^{(k+1)}(x)\sigma_{k+1}(x)}{\sigma_{k+2}(x)} \\ c_1^{(k+1)}(x) &= \frac{t_{-1}\sigma_{k+2}(x)}{\sigma_{k+1}(x)}, \end{aligned}$$

however from equation (4.83) we know that $c_1^{(k+1)} = \frac{a_0^{(k+2)}}{\lambda^{2(k+2)}}$, hence

$$\frac{a_0^{(k+2)}(x)}{\lambda^{2(k+2)}} = \frac{t_{-1}\sigma_{k+2}(x)}{\sigma_{k+1}(x)}. \tag{4.104}$$

Now we also have $a_0^{(\rm odd)}$ in terms of determinants. Recall

$$F^{(0)}(\nu, x) = \sum_{j=0} \begin{pmatrix} a_{2j}\nu^{2j} \\ c_{2j+1}\nu^{2j+1} \end{pmatrix} = \phi_1^{(0)}(\nu, x) = \sum_{j=0} \begin{pmatrix} \lambda T_{2j}\nu^{2j} \\ T_{2j+1}\nu^{2j+1} \end{pmatrix},$$

that is

$$a_{2j} = \lambda T_{2j}, \qquad c_{2j+1} = T_{2j+1}, \qquad j = 0, 1, \dots.$$

Then for k is any integer, let

$$\tau_k(x) = \begin{vmatrix} T_k & T_{k+1} & \cdots & & T_{2k-2} & T_{2k-1} \\ T_{k-2} & T_{k-1} & \cdots & & T_{2k-4} & T_{2k-3} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ T_2 & T_3 & \cdots & & T_k & T_{k+1} \\ & & \ddots & & \vdots & \vdots \\ & & & \cdots & T_2 & T_3 \\ 0 & \cdots & & \cdots & T_0 & T_1 \end{vmatrix}. \tag{4.105}$$

The τ_k function defined by equation (4.105) is proportional to the σ_k function defined in definition 4.6.2 and

$$a_0^{(k)}(x) = \mu_k \frac{\tau_k(x)}{\tau_{k-1}(x)},$$

where μ_k is a constant. Finally recall (4.73) we have

$$\begin{aligned} g_k(x) &= -\frac{ix}{\lambda^{2k}} \frac{a_0^{(k)}(x/\lambda)}{a_0^{(k)}(x/\lambda^2)} \\ &= -\frac{ix}{\lambda^{2k}} \frac{\tau_k(x/\lambda)\tau_{k-1}(x/\lambda^2)}{\tau_k(x/\lambda^2)\tau_{k-1}(x/\lambda)}. \end{aligned}$$

□

4.6.2. Determinantal form of q-hypergeometric type special solutions of q-PII.

Theorem 4.6.3. q-PII equation (4.9) with parameter $\alpha = \frac{1}{\lambda^{4k}}$, where k are half integers, that is $k = n + \frac{1}{2}$, $n = 0, 1, 2, ...$ admits a hierarchy of hypergeometric type special solutions $g_{n+\frac{1}{2}}(x)$, given by

$$g_{n+\frac{1}{2}}(x) \begin{cases} = -\frac{ix}{\lambda^{2n+1}} \frac{\tau_{n+1}(x/\lambda)\tau_n(x/\lambda^2)}{\tau_{n+1}(x/\lambda^2)\tau_n(x/\lambda)} \text{ for } n \text{ even}, \\ = -\frac{ix}{\lambda^{2n+2}} \frac{\tau_{n+1}(x/\lambda)\tau_n(x/\lambda^2)}{\tau_{n+1}(x/\lambda^2)\tau_n(x/\lambda)} \text{ for } n \text{ odd}, \end{cases}$$

where $\tau_n(x)$ is the determinant of a $n \times n$ matrix

$$\tau_n(x) = \begin{vmatrix} \frac{a_n^{(\frac{1}{2})}}{\lambda^n} & \frac{\overline{a}_n^{(\frac{1}{2})}}{\lambda^n} & & \cdots & \frac{a_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} & \frac{\overline{a}_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} \\ \frac{a_{n-2}^{(\frac{1}{2})}}{\lambda^{n-2}} & \frac{\overline{a}_{n-2}^{(\frac{1}{2})}}{\lambda^{n-2}} & & \cdots & \frac{a_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} & \frac{\overline{a}_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} & \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ 0 & 0 & \cdots & 0 & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix} \text{ for } n \text{ even},$$

$$\tau_n(x) = \begin{vmatrix} \frac{a_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \frac{a_{n+1}^{(\frac{1}{2})}}{\lambda^{n+1}} & \frac{\overline{a}_{n+1}^{(\frac{1}{2})}}{\lambda^{n+1}} & \cdots & \frac{a_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} & \frac{\overline{a}_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} \\ \frac{a_{n-3}^{(\frac{1}{2})}}{\lambda^{n-3}} & \frac{a_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \frac{\overline{a}_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \cdots & \frac{a_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} & \frac{\overline{a}_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} & \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ 0 & 0 & 0 & \cdots & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix} \text{ for } n \text{ odd},$$

where

$$\begin{aligned} \overline{a}_{2j}^{(\frac{1}{2})} &= ix\lambda^2 a_{2j}^{(\frac{1}{2})} - \frac{i}{x\lambda^2}\left(\frac{1}{\lambda^{4j+4}} - 1\right) a_{2j+2}^{(\frac{1}{2})}, \\ \overline{\overline{a}}_{2j}^{(\frac{1}{2})} &= -\frac{1}{\lambda} a_{2j}^{(\frac{1}{2})} - x^2\lambda^3 a_{2j-2}^{(\frac{1}{2})} + \frac{i}{x\lambda}\overline{a}_{2j}^{(\frac{1}{2})}. \end{aligned}$$

Proof. Let us consider $F^{(k+1)}(\nu, x)$, k is a half integer and we write $k = n - \frac{1}{2}$, $n = 1, 2, \ldots$. We study the case when $n = 2, 4, \ldots$, the case for n is

odd can also be proved here as we will see later.

$$
\begin{aligned}
& F^{(n+\frac{1}{2})}(\nu,x) \qquad (4.106)\\
&= \Lambda_{n-\frac{1}{2}}\Lambda_{n-\frac{3}{2}}\dots\Lambda_{\frac{1}{2}}F^{(\frac{1}{2})}(\nu,x)\\
&= \Lambda_{n-\frac{1}{2}}\Lambda_{n-\frac{3}{2}}\dots\Lambda_{\frac{1}{2}}\frac{\nu^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\begin{pmatrix}1 & 0\\ \frac{g_{\frac{1}{2}}(x)}{\lambda^2(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^2\nu}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}^{(\frac{1}{2})}(x)\frac{\nu^{2j}}{\lambda^{2j}}\\ a_{2j}^{(\frac{1}{2})}(x/\lambda)\frac{\nu^{2j}}{\lambda^{2j}}\end{pmatrix}\\
&= \frac{\nu^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\begin{pmatrix}s_0+\frac{s_2}{\nu^2}+\dots+\frac{s_n}{\nu^n} & \frac{t_2}{\nu^2}+\dots+\frac{t_n}{\nu^n}\\ \frac{s_1}{\nu}+\frac{s_3}{\nu^3}+\dots+\frac{s_{n+1}}{\nu^{n+1}} & \frac{t_1}{\nu}+\dots+\frac{t_{n+1}}{\nu^{n+1}}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}^{(\frac{1}{2})}(x)\frac{\nu^{2j}}{\lambda^{2j}}\\ a_{2j}^{(\frac{1}{2})}(x/\lambda)\frac{\nu^{2j}}{\lambda^{2j}}\end{pmatrix},
\end{aligned}
$$

where $a_{2j}^{(\frac{1}{2})}$ is defined by equations (4.53, 4.54). From the definition of $F^{\left(n+\frac{1}{2}\right)}$ we also know that as $\nu\to 0$

$$
\begin{aligned}
F^{\left(n+\frac{1}{2}\right)}(\nu,x) &= \phi_1^{\left(n+\frac{1}{2}\right)}(\nu/\lambda^{2n+1},x)\\
&\sim \frac{\nu^{n+\frac{1}{2}}}{\lambda^{(2n+1)(n+1/2)}}\begin{pmatrix}a_0^{\left(n+\frac{1}{2}\right)}\\ c_1^{\left(n+\frac{1}{2}\right)}\frac{\nu}{\lambda^{2n+1}}\end{pmatrix}. \qquad (4.107)
\end{aligned}
$$

For $F^{(n+\frac{1}{2})}(\nu,x)$ given by equation (4.106) to have the correct leading behavior (4.107) in the neighbourhood of $\nu=0$, s_j and t_j ($j=0,\dots,n+1$) need to satisfy systems of equations. Equating the two expressions of $F^{\left(n+\frac{1}{2}\right)}(\nu,x)$, from the top entry we get $n+1$ equations $(\nu^{-n},\nu^{-n+2},...,\nu^n)$:

$$
\begin{aligned}
\frac{1}{\nu^n} &: \quad s_n a_0^{(\frac{1}{2})}+t_n\overline{a}_0^{(\frac{1}{2})}=0\\
\frac{1}{\nu^{n-2}} &: \quad s_n\frac{a_2^{(\frac{1}{2})}}{\lambda^2}+t_n\frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2}+s_{n-2}a_0^{(\frac{1}{2})}+t_{n-2}\overline{a}_0^{(\frac{1}{2})}=0\\
\vdots &: \quad \vdots\\
\nu^{n-2} &: \quad s_n\frac{a_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}}+t_n\frac{\overline{a}_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}}+\dots+t_2\frac{\overline{a}_n^{(\frac{1}{2})}}{\lambda^n}+s_0\frac{a_{n-2}^{(\frac{1}{2})}}{\lambda^{n-2}}=0\\
\nu^n &: \quad s_n\frac{a_{2n}^{(\frac{1}{2})}}{\lambda^{2n}}+t_n\frac{\overline{a}_{2n}^{(\frac{1}{2})}}{\lambda^{2n}}+\dots+t_2\frac{\overline{a}_{n+2}^{(\frac{1}{2})}}{\lambda^{n+2}}+s_0\frac{a_n^{(\frac{1}{2})}}{\lambda^n}=\frac{a_0^{(n+\frac{1}{2})}}{\lambda^{(2n+1)(n+1/2)}}.
\end{aligned}
$$

Rewriting this in the form of a $(n+1)\times(n+1)$ matrix equation

$$\begin{pmatrix} \frac{a_n^{(\frac12)}}{\lambda^n} & \frac{a_{n+2}^{(\frac12)}}{\lambda^{n+2}} & \frac{\overline{a}_{n+2}^{(\frac12)}}{\lambda^{n+2}} & & \cdots & \frac{a_{2n}^{(\frac12)}}{\lambda^{2n}} & \frac{\overline{a}_{2n}^{(\frac12)}}{\lambda^{2n}} \\ \frac{a_{n-2}^{(\frac12)}}{\lambda^{n-2}} & \frac{a_n^{(\frac12)}}{\lambda^n} & \frac{\overline{a}_n^{(\frac12)}}{\lambda^n} & & \cdots & \frac{a_{2n-2}^{(\frac12)}}{\lambda^{2n-2}} & \frac{\overline{a}_{2n-2}^{(\frac12)}}{\lambda^{2n-2}} \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & a_0^{(\frac12)} & \overline{a}_0^{(\frac12)} & \frac{a_2^{(\frac12)}}{\lambda^2} & \frac{\overline{a}_2^{(\frac12)}}{\lambda^2} \\ 0 & 0 & 0 & \cdots & & a_0^{(\frac12)} & \overline{a}_0^{(\frac12)} \end{pmatrix} \begin{pmatrix} s_0 \\ s_2 \\ \vdots \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} \frac{a_0^{(n+\frac12)}}{\lambda^{(2n+1)(n+1/2)}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.108}$$

For the bottom entry, we get $n+2$ equations $(\nu^{-n-1}, \nu^{-n+1}, ..., \nu^{n+1})$:

$$\begin{aligned} \frac{1}{\nu^{n+1}} \quad &: \quad s_{n+1}a_0^{(\frac12)} + t_{n+1}\overline{a}_0^{(\frac12)} = 0 \\ \frac{1}{\nu^{n-1}} \quad &: \quad s_{n+1}\frac{a_2^{(\frac12)}}{\lambda^2} + t_{n+1}\frac{\overline{a}_2^{(\frac12)}}{\lambda^2} + s_{n-1}a_0^{(\frac12)} + t_{n-1}\overline{a}_0^{(\frac12)} = 0 \\ \vdots \quad &: \quad \vdots \\ \nu^{n-1} \quad &: \quad s_{n+1}\frac{a_{2n}^{(\frac12)}}{\lambda^{2n}} + t_{n+1}\frac{\overline{a}_{2n}^{(\frac12)}}{\lambda^{2n}} + \cdots + s_1\frac{a_n^{(\frac12)}}{\lambda^n} + t_1\frac{\overline{a}_n^{(\frac12)}}{\lambda^n} = 0 \\ \nu^{n+1} \quad &: \quad s_{n+1}\frac{a_{2n+2}^{(\frac12)}}{\lambda^{2n+2}} + t_{n+1}\frac{\overline{a}_{2n+2}^{(\frac12)}}{\lambda^{2n+2}} + \cdots + s_1\frac{a_{n+2}^{(\frac12)}}{\lambda^{n+2}} + t_1\frac{\overline{a}_{n+2}^{(\frac12)}}{\lambda^{n+2}} = \frac{c_1^{\left(n+\frac12\right)}}{\lambda^{(2n+1)(n+3/2)}}. \end{aligned}$$

Rewriting this in the form of a $(n+2)\times(n+2)$ matrix equation

$$\begin{pmatrix} \frac{a_{n+2}^{(\frac12)}}{\lambda^{n+2}} & \frac{\overline{a}_{n+2}^{(\frac12)}}{\lambda^{n+2}} & & \cdots & \frac{a_{2n+2}^{(\frac12)}}{\lambda^{2n+2}} & \frac{\overline{a}_{2n+2}^{(\frac12)}}{\lambda^{2n+2}} \\ \frac{a_n^{(\frac12)}}{\lambda^n} & \frac{\overline{a}_n^{(\frac12)}}{\lambda^n} & & \cdots & \frac{a_{2n}^{(\frac12)}}{\lambda^{2n}} & \frac{\overline{a}_{2n}^{(\frac12)}}{\lambda^{2n}} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & a_0^{(\frac12)} & \overline{a}_0^{(\frac12)} & \frac{a_2^{(\frac12)}}{\lambda^2} & \frac{\overline{a}_2^{(\frac12)}}{\lambda^2} \\ 0 & 0 & \cdots & 0 & a_0^{(\frac12)} & \overline{a}_0^{(\frac12)} \end{pmatrix} \begin{pmatrix} s_1 \\ t_1 \\ \vdots \\ s_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{c_1^{\left(n+\frac12\right)}}{\lambda^{(2n+1)(n+3/2)}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.109}$$

Let $\tau_n(x)$ be the determinant of the $n \times n$ matrix

$$\tau_n(x) = \begin{vmatrix} \frac{a_n^{(\frac{1}{2})}}{\lambda^n} & \frac{\overline{a}_n^{(\frac{1}{2})}}{\lambda^n} & & \cdots & \frac{a_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} & \frac{\overline{a}_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} \\ \frac{a_{n-2}^{(\frac{1}{2})}}{\lambda^{n-2}} & \frac{\overline{a}_{n-2}^{(\frac{1}{2})}}{\lambda^{n-2}} & & \cdots & \frac{a_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} & \frac{\overline{a}_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} & \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ 0 & 0 & \cdots & 0 & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix}, \text{ for } n \text{ even, } (4.110)$$

$$\tau_n(x) = \begin{vmatrix} \frac{a_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \frac{a_{n+1}^{(\frac{1}{2})}}{\lambda^{n+1}} & \frac{\overline{a}_{n+1}^{(\frac{1}{2})}}{\lambda^{n+1}} & \cdots & \frac{a_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} & \frac{\overline{a}_{2n-2}^{(\frac{1}{2})}}{\lambda^{2n-2}} \\ \frac{a_{n-3}^{(\frac{1}{2})}}{\lambda^{n-3}} & \frac{a_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \frac{\overline{a}_{n-1}^{(\frac{1}{2})}}{\lambda^{n-1}} & \cdots & \frac{a_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} & \frac{\overline{a}_{2n-4}^{(\frac{1}{2})}}{\lambda^{2n-4}} \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} & \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ 0 & 0 & 0 & \cdots & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix}, \text{ for } n \text{ odd. } (4.111)$$

Then by Cramer's rule on equation (4.108) for s_0, recall s_0 is a constant and we assumed n is an even integer

$$a_0^{(n+\frac{1}{2})}(x) = \mu_n \frac{\tau_{n+1}(x)}{\tau_n(x)}, \qquad \mu_n \text{ is a constant,}$$

we have a formula for $a_0^{(n+\frac{1}{2})}(x)$, n is even. Applying Cramer's rule on equation (4.109) for t_1, recall $t_1 = -\frac{i}{x\lambda^2}$,

$$c_1^{(n+\frac{1}{2})}(x) = \eta_n a_0^{(n+1+\frac{1}{2})}(x) = \frac{\varsigma_n'}{x} \frac{\tau_{n+2}(x)}{\tau_{n+1}(x)}, \qquad \eta_n, \varsigma_n' \text{ are constants.}$$

Since $n+1$ is an odd integer when n is an even integer we now also have the formula for $a_0^{(n+\frac{1}{2})}(x)$, n is odd

$$a_0^{(n+\frac{1}{2})}(x) = \frac{\varsigma_n}{x} \frac{\tau_{n+1}(x)}{\tau_n(x)}, \qquad \varsigma_n \text{ is a constant.}$$

Recall

$$g_n(x) = -\frac{ix}{\lambda^{2n}} \frac{a_0^{(n)}(x/\lambda)}{a_0^{(n)}(x/\lambda^2)},$$

$a_0^{(n+\frac{1}{2})}(x)$ for n is an even integer is

$$a_0^{(n+\frac{1}{2})}(x) = \mu_n \frac{\tau_{n+1}(x)}{\tau_n(x)},$$

and when n is an odd integer is

$$a_0^{(n+\frac{1}{2})}(x) = \frac{\varsigma_n}{x} \frac{\tau_{n+1}(x)}{\tau_n(x)},$$

where $\tau_n(x)$ is defined by equations (4.110, 4.111). Recall μ_n and ς_n are constants, and

$$\frac{x/\lambda}{x/\lambda^2} = \frac{1}{\lambda}.$$

Hence for n is an even integer,

$$\begin{aligned} g_{n+\frac{1}{2}}(x) &= -\frac{ix}{\lambda^{2n+1}} \frac{a_0^{(n+\frac{1}{2})}(x/\lambda)}{a_0^{(n+\frac{1}{2})}(x/\lambda^2)} \\ &= -\frac{ix}{\lambda^{2n+1}} \frac{\tau_{n+1}(x/\lambda)\tau_n(x/\lambda^2)}{\tau_{n+1}(x/\lambda^2)\tau_n(x/\lambda)} \end{aligned}$$

and for n is an odd integer,

$$\begin{aligned} g_{n+\frac{1}{2}}(x) &= -\frac{ix}{\lambda^{2n+1}} \frac{a_0^{(n+\frac{1}{2})}(x/\lambda)}{a_0^{(n+\frac{1}{2})}(x/\lambda^2)} \\ &= -\frac{ix}{\lambda^{2n+2}} \frac{\tau_{n+1}(x/\lambda)\tau_n(x/\lambda^2)}{\tau_{n+1}(x/\lambda^2)\tau_n(x/\lambda)}. \end{aligned}$$

□

Let us look at a few examples of the determinantal formula of the q-hypergeometric type special solutions of q-PII. For $n = 1$,

$$g_{\frac{1}{2}} = -\frac{ix}{\lambda} \frac{\tau_1(x/\lambda)}{\tau_1(x/\lambda^2)},$$

for $n = 2$,

$$g_{\frac{3}{2}} = -\frac{ix}{\lambda} \frac{\tau_2(x/\lambda)\tau_1(x/\lambda^2)}{\tau_2(x/\lambda^2)\tau_1(x/\lambda)},$$

for $n = 3$,

$$g_{\frac{5}{2}} = -\frac{ix}{\lambda}\frac{\tau_3(x/\lambda)\tau_2(x/\lambda^2)}{\tau_3(x/\lambda^2)\tau_2(x/\lambda)},$$

where

$$\begin{aligned}
\tau_1(x) &= a_0^{(\frac{1}{2})}, \\
\tau_2(x) &= \begin{vmatrix} \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix}, \\
\tau_3(x) &= \begin{vmatrix} \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{a_4^{(\frac{1}{2})}}{\lambda^4} & \frac{\overline{a}_4^{(\frac{1}{2})}}{\lambda^4} \\ a_0^{(\frac{1}{2})} & \frac{a_2^{(\frac{1}{2})}}{\lambda^2} & \frac{\overline{a}_2^{(\frac{1}{2})}}{\lambda^2} \\ 0 & a_0^{(\frac{1}{2})} & \overline{a}_0^{(\frac{1}{2})} \end{vmatrix}.
\end{aligned}$$

In this chapter, a general relation between the solution of the linear problem and the solution of q-PII equation was established. We have found exact solutions for two special cases of the associated linear problem of q-PII, corresponding to q-PII admitting a simple rational and q-hypergeometric type special solution respectively. Schlesinger transformation of the linear problem was found and used to found all the solutions of the linear problem where the corresponding q-PII equation admits either rational or q-hypergeometric type special solutions. This fact was then used to find the determinantal forms of the hierarchies of the rational and the q-hypergeometric type special solutions with these two simple special solutions as the first member respectively.

CHAPTER 5

Conclusions

Motivated by the results of Kajiwara et al. on the determinantal forms of some special solutions of the discrete analogues of the Painlevé equations, we aimed to follow Flaschka and Newell's approach of obtaining determinantal forms of some special solutions of the Painlevé equations via the associated linear problem in the q-discrete setting. We have found that the relationship between q-Painlevé equations and their associated q-linear problems is very similar to that of continuous Painlevé equations and their associated linear problems. Hence the method developed by Flaschka and Newell [**18**] can be followed analogously and used to study q-Painlevé equations in the q-discrete setting. The idea is to use the properties of the linear problem. In particular since it is the special solutions of the q-Painlevé equations (that is for special values of its parameters the q-Painlevé equation admits simpler solutions of rational and hypergeometric type) which we want to study, therefore we have considered the special cases of the associated linear problem when it admits simpler solutions for special values of the parameters.

In differential linear analysis, the fundamental solutions take the forms of simple Frobenius series expansions when the equation satisfies the "non-resonant condition", whereas for the "resonant"case the solutions in general involves the logarithm function. Flaschka and Newell have considered two types of the special cases of the associated linear problem under the "resonant"condition and found the conditions on the equation for which the fundamental solutions still take the forms of simple Frobenius series without logarithm functions. They have found that the Painlevé equation admits rational and hypergeometric type special solutions corresponding to these two types of the special cases of the linear problem. Further more, the closed form of all of the special solutions of the linear problem of these two

type can be found by using the Schlesinger transform of the linear problem, iterating from the simplest special solutions of each type respectively. Determinantal expression of rational and hypergeometric type special solutions of the Painlevé equations can obtained through the special solutions of the linear problem.

For the q-discrete linear problem there is the q-discrete analogue of the "non-resonant condition"and fundamental solutions of the forms of simple Frobenius series expansions and the "resonant"case where the solutions in general involves q-discrete analogue of the logarithm function. We have considered two types of the special cases of the associated linear problem of q-PII of the "resonant"case and asked for the conditions for the fundamental solutions to be without the logarithm function. We have found the conditions are precisely where q-PII admits rational and q-hypergeometric type special solutions. The two simplest cases of the two type of the q-linear problem was solved:
equation (4.32),

$$\begin{aligned}\phi_1^{(0)}(\nu,x) &= \frac{1}{2}\begin{pmatrix}\lambda & \lambda\\ 1 & -1\end{pmatrix}\begin{pmatrix}\Gamma_{\lambda^{-2}}(1+\frac{\nu}{\lambda})\Gamma_{\lambda^{-2}}(1+ix\nu)\Gamma_{\lambda^{-2}}(1+ix\lambda\nu)\\ \Gamma_{\lambda^{-2}}(1-\frac{\nu}{\lambda})\Gamma_{\lambda^{-2}}(1-ix\nu)\Gamma_{\lambda^{-2}}(1-ix\lambda\nu)\end{pmatrix}\\ &= \sum_{j=0}\begin{pmatrix}\lambda T_{2j}\nu^{2j}\\ T_{2j+1}\nu^{2j+1}\end{pmatrix},\end{aligned}$$

and equation (4.52),

$$\phi_1^{\left(\frac{1}{2}\right)}(\nu,x) = \nu^{\frac{1}{2}}\begin{pmatrix}1 & 0\\ \frac{g_{\frac{1}{2}}(x)}{\lambda^3(x^2+g_{\frac{1}{2}}(x))\nu} & -\frac{i}{x\lambda^3\nu}\end{pmatrix}\sum_{j=0}^{\infty}\begin{pmatrix}a_{2j}^{(\frac{1}{2})}(x)\nu^{2j}\\ a_{2j}^{(\frac{1}{2})}(x/\lambda)\nu^{2j}\end{pmatrix}.$$

The Schlesinger transformation (4.94) $\Lambda(\nu,x)$ of a related q-linear system

$$F^{(k)}(\nu,x) = \phi_1^{(k)}(\nu/\lambda^{2k},x)$$

was found:

$$F^{(k+1)}(\nu,x) = \Lambda_k(\nu,x)F^{(k)}(\nu,x)$$

and used to express all the special solutions $F^{(k)}(\nu,x)$ of the q-linear problem in the closed form in terms of $F^{(0)}(\nu,x)$ and $F^{(\frac{1}{2})}(\nu,x)$. That is for k

is an integer

$$F^{(k+1)}(\nu,x)=\Lambda_k\Lambda_{k-1}\ldots\begin{pmatrix}0 & 1\\ \frac{1}{\lambda^2} & \frac{\rho_0(x)}{\nu}\end{pmatrix}F^{(0)}(\nu,x)$$

and for k is a half integer

$$F^{(k+1)}(\nu,x)=\Lambda_k\Lambda_{k-1}\ldots\begin{pmatrix}0 & 1\\ \frac{1}{\lambda^2} & \frac{\rho_{\frac{1}{2}}(x)}{\nu}\end{pmatrix}F^{(\frac{1}{2})}(\nu,x).$$

We worked with the set of solutions $F^{(k)}(\nu,x)$ instead of $\phi_1^{(k)}(\nu,x)$ because the Schlesinger transformation for $F^{(k)}(\nu,x)$ are much simpler and that allowed us to obtain the determinantal expression of the special solutions of the q-PII equation of both rational theorem 4.6.1 and q-hypergeometric type theorem 4.6.3 where we have found that the entries of the matrix, which the determinants are defined involve only the coefficients of the the solution of the linear problem for the simplest case of its type, equation (4.105) for rational type special solutions

$$\tau_k(x)=\begin{vmatrix} T_k & T_{k+1} & \cdots & & T_{2k-2} & T_{2k-1}\\ T_{k-2} & T_{k-1} & \cdots & & T_{2k-4} & T_{2k-3}\\ \vdots & \vdots & \ddots & & \vdots & \vdots\\ T_2 & T_3 & \cdots & & T_k & T_{k+1}\\ & & \ddots & & \vdots & \vdots\\ & & & \cdots & T_2 & T_3\\ 0 & \cdots & & \cdots & T_0 & T_1 \end{vmatrix},$$

and equations (4.110 and 4.111) for q-hypergeometric type special solutions of q-PII. In other words, the solutions of the linear problem for the two simplest cases $\phi_1^{(0)}(\nu,x)$ and $\phi_1^{(\frac{1}{2})}(\nu,x)$ are the generating functions of the entries of the matrices on which these determinants are defined. This shed some light on the peculiar asymmetric structure of the determinant forms of special solutions for the discrete Painlevé equations first found by Kajiwara et al. [**48, 47**].

In this thesis, we have considered the associated q-linear problems of a q-analogue of PII. The method we have developed to study the q-discrete Painlevé equation via its associated q-linear problem should work for other discrete analogues of the Painlevé equations which are in possession of a

2×2 Lax pair. Such examples are the 2×2 Lax pair of Jimbo and Sakai's q-PVI [**37**] and the Lax pairs for q-PV-q-PI found by a coalescence procedure on the Lax pair of this q-PVI. The reason of why we have not performed our analysis on these Lax pairs already is because the corresponding q-Painlevé equations are represented in the coupled form, for example the q-PVI in [**37**] is

$$\begin{aligned} y\bar{y} &= \frac{(\bar{z}-tb_1)(\bar{z}-tq/b_1)}{(\bar{z}-b_3)(\bar{z}-1/b_3)} \\ z\bar{z} &= \frac{(y-ta_1)(y-t/a_1)}{(y-a_3)(y-1/a_3)} \end{aligned}$$

where $y=y(t), z=z(t)$, $\bar{f}=f(qt), \underline{f}=f(t/q)$ and a_1, a_3, b_1, b_3 are constants. Whereas the example of q-PII considered in this thesis is in the scalar form.

One might ask why we should study the Painlevé equations of both differential and discrete type via the associated linear problems when many of the results may be obtained by studying the non-linear equations directly. We have in fact shown that the associated linear problems readily reveals the beautiful determinantal structure of the special solutions of the Painlevé equations. The technique is entirely analytical; no algebraic or geometrical knowledge is needed. All of the operations used require only elementary linear q-discrete analysis, which is very similar to linear differential analysis. Furthermore, when a non-linear equation has an Lax pair it implies integrability, in the sense that the non-linear problem is "reduced" to a linear problem. Therefore it is advantageous to make use of the linearity of these very special non-linear equations, the Painlevé and the discrete Painlevé equations.

APPENDIX A

List of the six Painlevé equations

$$\frac{d^2y}{dz^2} = 6y^2 + z \tag{A.1}$$

$$\frac{d^2y}{dz^2} = 2y^3 + zy + \alpha \tag{A.2}$$

$$\frac{d^2y}{dz^2} = \frac{1}{y}\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{\alpha y^2 + \beta}{z} + \gamma y^3 + \frac{\delta}{y} \tag{A.3}$$

$$\frac{d^2y}{dz^2} = \frac{1}{2y}\left(\frac{dy}{dz}\right)^2 + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y} \tag{A.4}$$

$$\frac{d^2y}{dz^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{z} + \frac{\delta y(y+1)}{y-1} \tag{A.5}$$

$$\begin{aligned}\frac{d^2y}{dz^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)\left(\frac{dy}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)\frac{dy}{dz} \\ &\quad + \frac{y(y-1)y-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{y^2} + \frac{\gamma(z-1)}{(y-1)^2} + \frac{\delta z(z-1)}{(y-z)^2}\right)\end{aligned} \tag{A.6}$$

APPENDIX B

Differential and discrete hypergeometric functions

B1. Differential hypergeometric functions

A good reference on the special functions is a book [7] by Andrews, Askey and Roy, whose conventions on the notations of the special functions have been followed in this thesis.

Definition B1.1. The general hypergeometric function

$$ {}_mF_n\left(a_1,...,a_m;b_1,...,b_n;z\right)=\sum_{k\geq 0}\frac{(a_1)_k...(a_m)_k}{(b_1)_k...(b_n)_k k!}z^k, $$

where

$$ (\alpha)_n=\alpha(\alpha+1)...(\alpha+n-1), $$

solves the general differential hypergeometric equation

$$ \{\delta(\delta+b_1-1)...(\delta+b_n-1)-z(\delta+a_1)...(\delta+a_m)\}\,{}_mF_n\left(a_1,...,a_m;b_1,...,b_n;z\right)=0 $$

where $\delta=z\frac{d}{dz}$.

For $m=2,n=1$, $a_1=\alpha$, $a_2=\beta$ and $b_1=\gamma$, it is the Gauss's hypergeometric function

$$ {}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n\geq 0}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\ n!}z^n \tag{B.1} $$

which solves the hypergeometric equation

$$ (1-z)\delta^2 y-\{(\alpha+\beta)z+(1-\gamma)\}\delta y-\alpha\beta z y=0. \tag{B.2} $$

Equation (B.2) has fundamental solutions:

$$ \begin{cases} {}_2F_1(\alpha,\beta;\gamma;z), \\ z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z), \end{cases} $$

at $z = 0$, and

$$\begin{cases} (z)^{-\alpha}{}_2F_1(\alpha, \alpha-\gamma+1; \alpha-\beta+1; 1/z) \\ (-z)^{-\beta}{}_2F_1(\beta, \beta-\gamma+1; \beta-\alpha+1; 1/z) \end{cases}$$

at $z = \infty$. These are related by

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-z)^{-\alpha}{}_2F_1(\alpha, \alpha-\gamma+1; \alpha-\beta+1; 1/z) \\ &\quad + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)}(-z)^{-\beta}{}_2F_1(\beta, \beta-\gamma+1; \beta-\alpha+1; 1/z) \end{aligned}$$

where $\Gamma(a+1) = a\Gamma(a)$ is the Gamma function.

B2. The q-hypergeometric function

A good reference on the q-discrete (or basic) special functions is a book [**25**] by Gasper and Rahman, whose conventions on the notations of the q-special functions have been followed in this thesis.

Definition B2.1. The general q-hypergeometric function (also referred to as the **basic hypergeometric function**) is:

$${}_m\Phi_n\left(a_1, ..., a_m; b_1, ..., b_n; q, z\right) = \sum_{k\geq 0} \frac{(a_1; q)_k ... (a_m; q)_k}{(b_1; q)_k ... (b_n; q)_k (q; q)_k} \left[(-1)^k q^{\binom{n}{2}}\right]^{1+n-m} z^k$$

where

$$(a; q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad |q| < 1.$$

For $m = 2, n = 1$, $a_1 = a$, $a_2 = b$ and $b_1 = c$, it is the q-discrete analogue of Gauss's hypergeometric function

$${}_2\Phi_1(a, b; c; q, z) = \sum_{n\geq 0} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n. \tag{B.3}$$

Proposition B2.2. Equation

$$(abz - c/q)\sigma^2\Phi - \left((a+b)z - (1+c/q)\right)\sigma\Phi + (z-1)\Phi = 0 \tag{B.4}$$

is solved by the q-hypergeometric function (B.3), where

$$\sigma y(z) = y(qx),$$

Proof. Using the definition for ${}_2\Phi_1(a,b;c;q,z)$,

$$\begin{aligned} {}_2\Phi_1(a,b;c;q,z) &= \sum_{n\geq 0}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n, \\ \sigma_2\Phi_1(a,b;c;q,z) &= \sum_{n\geq 0}\frac{(a;q)_n(b;q)_nq^n}{(c;q)_n(q;q)_n}z^n, \end{aligned}$$

and

$$\begin{aligned} (1-\sigma)_2\Phi_1(a,b;c;q,z) &= \sum_{n\geq 0}\frac{(a;q)_n(b;q)_n(1-q^n)}{(c;q)_n(q;q)_n}z^n \\ &= \sum_{n\geq 0}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_{n-1}}z^n \\ &= \frac{(1-a)(1-b)z}{1-c}\sum_{n\geq 1}\frac{(aq;q)_{n-1}(bq;q)_{n-1}}{(cq;q)_{n-1}(q;q)_{n-1}}z^{n-1} \\ &= \frac{(1-a)(1-b)z}{1-c}{}_2\Phi_1(aq,bq;cq;q,z), \end{aligned}$$

hence

$$\begin{aligned} (1-\sigma)_2\Phi_1(a,b;c/q;q,z) &= \frac{(1-a)(1-b)z}{1-c/q}\sum_{n\geq 1}\frac{(aq;q)_{n-1}(bq;q)_{n-1}}{(c;q)_{n-1}(q;q)_{n-1}}z^{n-1} \\ &= \frac{(1-a)(1-b)z}{1-c/q}{}_2\Phi_1(aq,bq,c;q,z), \end{aligned}$$

and

$$\begin{aligned}
& {}_2\Phi_1(aq, bq, c; q, z) \\
= & \sum_{n\geq 0} \frac{(aq;q)_n (bq;q)_n}{(c;q)_n (q;q)_n} z^n \\
= & \frac{1}{(1-a)(1-b)} \sum_{n\geq 0} \frac{(a;q)_n (bq;q)_n (1-aq^n)(1-bq^n)}{(c;q)_n (q;q)_n} z^n \\
= & \frac{1}{(1-a)(1-b)} \sum_{n\geq 0} \frac{(a;q)_n (bq;q)_n (1-(a+b)q^n + abq^{2n})}{(c;q)_n (q;q)_n} z^n \\
= & \frac{1}{(1-a)(1-b)} \big\{ {}_2\Phi_1(a,b;c;q,z) - (a+b)\,{}_2\Phi_1(a,b;c;q,qz) \\
& \qquad\qquad + ab\,{}_2\Phi_1(a,b;c;q,q^2z) \big\} \\
= & \frac{1}{(1-a)(1-b)} \left\{ 1 - (a+b)\sigma + ab\sigma^2 \right\} {}_2\Phi_1(a,b;c;q,z),
\end{aligned}$$

but

$$\begin{aligned}
& {}_2\Phi_1(a, b; c/q; q, z) \\
= & \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n}{(c/q;q)_n (q;q)_n} z^n \\
= & \frac{1}{1-c/q} \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n (1-cq^{k-1})}{(c;q)_n (q;q)_n} z^n \\
= & \frac{(1-c\sigma/q)}{(1-c/q)} {}_2\Phi_1(a,b;c;q,z),
\end{aligned}$$

therefore

$$\begin{aligned}
& (1-\sigma)\,{}_2\Phi_1(a, b; c/q; q, z) \\
= & (1-\sigma)\frac{(1-c\sigma/q)}{(1-c/q)} {}_2\Phi_1(a,b;c;q,z) \\
= & \frac{z}{(1-c/q)} \left\{ 1 - (a+b)\sigma + ab\sigma^2 \right\} {}_2\Phi_1(a,b;c;q,z).
\end{aligned}$$

Rearrange, taking terms to one side, we get

$$\left\{ (abz - c/q)\sigma^2 - \big((a+b)z - (1+c/q)\big)\sigma + z - 1 \right\} {}_2\Phi_1(a,b;c;q,z) = 0$$

□

Proposition B2.3. The q-hypergeometric equation (B.4) has fundamental solutions

$$\begin{cases} {}_2\Phi_1(a,b;c;q,z), \\ e_{\frac{q}{c}}(z){}_2\Phi_1(a',b';c';q,z) = e_{\frac{q}{c}}(z){}_2\Phi_1(qa/c,qb/c;q^2/c;q,z), \end{cases} \tag{B.5}$$

at $z=0$ and

$$\begin{cases} e_a(cq/abz){}_2\Phi_1(a,aq/c;\frac{aq}{b};q,cq/abz), \\ e_b(cq/abz){}_2\Phi_1(b,bq/c;\frac{bq}{a};q,cq/abz), \end{cases} \tag{B.6}$$

at $z=\infty$, where

$$\sigma e_a(z) = e_a(qz) = ae_a(z).$$

Proof. First, we find and solve the q-discrete analogue of the indicial equation for equation (B.4). That is, $\phi(z)$ is a solution of equation (B.4) with leading behaviour:

$$\sigma\phi(z) = \rho\phi(z)$$

then, let

$$\phi(z) = e_\rho(z)\phi_1(z)$$

where

$$\sigma e_\rho(z) = \rho e_\rho(z), \text{ and } \sigma\phi_1(z) \sim \phi_1(z), \text{ to leading order as } z \to 0.$$

Substitute $\phi(x)$ into equation (B.4), we get

$$(abz - c/q)\rho^2 - \big((a+b)z - (1+c/q)\big)\rho + z - 1 = 0. \tag{B.7}$$

At $z=0$, equation (B.7) is

$$\begin{aligned} \frac{c}{q}\rho^2 - (1+c/q)\rho + 1 &= 0 \\ (c\rho/q - 1)(\rho - 1) &= 0 \end{aligned}$$

which gives

$$\rho = \frac{q}{c},\ \ 1.$$

At $z=\infty$ equation (B.7) goes to

$$\begin{aligned} ab\rho^2 - (a+b)\rho + 1 &= 0 \\ (a\rho - 1)(b\rho - 1) &= 0 \end{aligned}$$

which gives

$$\rho = \frac{1}{a}, \quad \frac{1}{b}.$$

We have already found one of the solution at $z = 0$ with leading behaviour $\rho = 1$, which is ${}_2\Phi_1(a, b; c; q, z)$. Let $\phi(z)$ be the other solution with leading behaviour $\rho = \frac{q}{c}$, that is

$$\phi(z) = e_{\frac{q}{c}}(z)\phi_1(z), \tag{B.8}$$

where

$$\sigma e_{\frac{q}{c}}(z) = \frac{q}{c} e_{\frac{q}{c}}(z). \tag{B.9}$$

Then

$$\begin{aligned} \sigma\phi &= \sigma(e_{\frac{q}{c}}\phi_1) \\ &= \frac{q}{c} e_{\frac{q}{c}} \sigma\phi_1 \\ \sigma^2\phi &= \frac{q^2}{c^2} e_{\frac{q}{c}} \sigma^2\phi_1 \end{aligned}$$

Substitute above into equation (B.4),

$$\frac{q^2}{c^2} e_{\frac{q}{c}} \sigma^2\phi_1 - \lambda \frac{q}{c} e_{\frac{q}{c}} \sigma\phi_1 + \mu e_{\frac{q}{c}} \phi_1 = 0 \tag{B.10}$$

which simplifies to

$$\sigma^2\phi_1 - \lambda'\sigma\phi_1 + \mu'\phi_1 = 0 \tag{B.11}$$

with

$$\begin{cases} \lambda' = \frac{\lambda}{\frac{q}{c}} = \frac{\frac{q}{c}(a+b)z - (1+q/c)}{\frac{q^2abz}{c^2} - \frac{q}{c}}, \\ \mu' = \frac{\mu}{\frac{q^2}{c^2}} = \frac{z-1}{\frac{q^2}{c^2}(abz - c/q)} = \frac{z-1}{(q^2abz/c^2 - q/c)}. \end{cases} \tag{B.12}$$

Equation (B.11) is solved by

$$\phi_1(z) = {}_2\Phi_1(a', b', c'; q, z),$$

where by comparing (B.11) with (B.4) we find

$$\begin{aligned} a' &= \frac{qa}{c}, \\ b' &= \frac{qb}{c}, \\ c' &= \frac{q^2}{c}. \end{aligned}$$

Hence equation (B.4) have solutions

$$\begin{cases} {}_2\Phi_1(a,b;c;q,z), \\ e_{\frac{q}{c}}(z){}_2\Phi_1(a',b';c';q,z) = e_{\frac{q}{c}}(z){}_2\Phi_1(qa/c,qb/c;q^2/c;q,z), \end{cases}$$

at $z=0$.

To find the two fundamental solutions of (B.4) at $z=\infty$ with leading behavior $\rho = \frac{1}{a}, \frac{1}{b}$, we first rewrite equation (B.4) let

$$\begin{aligned} z &= \frac{1}{t}, \\ qz &= \frac{q}{t}, \\ q^2z &= \frac{q^2}{t}, \end{aligned}$$

then

$$\begin{aligned} & z \to qz, \\ \Rightarrow \quad & t \to \frac{t}{q} \\ & z \to q^2z, \\ \Rightarrow \quad & t \to \frac{t}{q^2}. \end{aligned}$$

Rewrite equation (B.4) in terms of the variable t,

$$\left(\frac{ab}{t}-\frac{c}{q}\right){}_2\Phi_1(t/q^2) - \left(\frac{a+b}{t}-(1+c/q)\right){}_2\Phi_1(t/q) + \left(\frac{1}{t}-1\right){}_2\Phi_1(t) = 0.$$

On letting

$$t \to q^2t,$$

we have

$$\begin{aligned} & \left(\frac{ab}{q^2t}-\frac{c}{q}\right){}_2\Phi_1(t) - \left(\frac{a+b}{q^2t}-(1+c/q)\right){}_2\Phi_1(tq) + \left(\frac{1}{q^2t}-1\right){}_2\Phi_1(q^2t) \\ = \; & (ab-cqt){}_2\Phi_1(t) - \left((a+b)-(1+c/q)q^2t\right){}_2\Phi_1(qt) + (1-q^2t){}_2\Phi_1(q^2t) \\ = \; & -ab\left(\frac{cqt}{ab}-1\right){}_2\Phi_1(t) - ab\left(\frac{a+b}{ab}-(1+c/q)\frac{q^2t}{ab}\right){}_2\Phi_1(qt) \\ & -ab\left(\frac{q^2t-1}{ab}\right){}_2\Phi_1(q^2t), \end{aligned}$$

and

$$\left(\frac{q^2t-1}{ab}\right){}_2\Phi_1(q^2t)+\left(\frac{a+b}{ab}-(1+c/q)\frac{q^2t}{ab}\right){}_2\Phi_1(qt) \qquad \text{(B.13)}$$
$$+\left(\frac{cqt}{ab}-1\right){}_2\Phi_1(t)=0.$$

Compare this with equation (B.4), we see the new variable is $\frac{cqt}{ab}=\frac{cq}{abz}$. We use the hint that at $z=\infty$ the q-hypergeometric function has leading behavior $\rho=\frac{1}{a},\frac{1}{b}$. To find the solution with $\rho=\frac{1}{a}$ behavior let

$$\begin{aligned}
{}_2\Phi_1(t) &= e_a(t)Y(t),\\
{}_2\Phi_1(qt) &= e_a(qt)Y(qt)\\
&= ae_a(t)Y(qt),\\
{}_2\Phi_1(q^2t) &= ae_a(qt)Y(q^2t)\\
&= a^2e_a(t)Y(q^2t).
\end{aligned}$$

Substitute these into (B.13)

$$\left(\frac{q^2t-1}{ab}\right)a^2Y(q^2t)+\left(\frac{a+b}{ab}-(1+c/q)\frac{q^2t}{ab}\right)aY(qt)+\left(\frac{cqt}{ab}-1\right)Y(t)=0.$$

Rearrange in terms of the new variable then compare with equation (B.4) shows

$$Y(t)={}_2\Phi_1(a',b';c';q,cqt/ab)={}_2\Phi_1(a,aq/c;qa/b;q,cq/abz).$$

Solution with $\rho=\frac{1}{a}$ behavior at $z=\infty$ can be found similarly by letting

$${}_2\Phi_1(t) = e_b(t)Y(t).$$

In this case we found

$$Y(t)={}_2\Phi_1(a',b';c';q,cqt/ab)={}_2\Phi_1(b,bq/c;qb/a;q,cq/abz).$$

Hence the solutions at ∞ are

$$\begin{cases} e_a(cq/abz){}_2\Phi_1(a,aq/c;qa/b;q,cq/abz)\\ e_b(cq/abz){}_2\Phi_1(b,bq/c;qb/a;q,cq/abz). \end{cases} \qquad \text{(B.14)}$$

□

Fundamental solutions of equation (B.4) at $z = 0, \infty$ are related by the formula derived by Watson (1910) using a q-analogue of the contour integral representation for ${}_2\Phi_1(a, b; c; q, z)$. For details, see section "Analytic continuation of ${}_2\Phi_1(a, b; c; q, z)$" in Gasper and Rahman's book [**25**],

$$\begin{aligned} {}_2\Phi_1(a, b; c; q, z) &= \frac{(b, c/a)_\infty (az, q/az)_\infty}{(c, b/a)_\infty (z, q/z)_\infty} {}_2\Phi_1(a, aq/c; aq/b; q, cq/abz) \\ &\quad + \frac{(a, c/b)_\infty (bz, q/bz)_\infty}{(c, a/b)_\infty (z, q/z)_\infty} {}_2\Phi_1(b, bq/c; bq/b; q, cq/abz). \end{aligned}$$

The q-hypergeometric equation (B.4)

$$(abz - c/q)\sigma^2\Phi - ((a + b)z - (1 + c/q))\,\sigma\Phi + (z - 1)\Phi = 0$$

can be put in form of a 2×2 matrix eqaution:

$$\sigma \begin{pmatrix} u \\ \sigma u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{(z-1)}{(abz-c/q)} & \frac{(a+b)z-(1+c/q)}{(abz-c/q)} \end{pmatrix} \begin{pmatrix} u \\ \sigma u \end{pmatrix}, \tag{B.15}$$

where u is a solution of equation (B.4).

B3. Continuum limits

Proposition B3.1. The q-hypergeometric equation (B.4)

$$\left\{(abz - c/q)\sigma^2 - \big((a + b)z - (1 + c/q)\big)\sigma + z - 1\right\} {}_2\Phi_1(a, b; c; q, z) = 0$$

becomes the hypergeometric equation equation (B.2)

$$(1 - z)\delta^2 y - \{(\alpha + \beta)z + (1 - \gamma)\}\delta y - \alpha\beta z y = 0$$

in the limit $q \to 1$

Proof. We know

$$\frac{\sigma - 1}{q - 1} = z\frac{d}{dz} = \delta$$

$$\frac{(\sigma - 1)^2}{(q - 1)^2} = \frac{(\sigma^2 - 2\sigma + 1)}{(q - 1)^2} = \delta^2$$

in the limit $q \to 1$.
Rewriting (B.4) in terms of δ and δ^2:

$$\begin{aligned}
& \sigma^2\Phi - \lambda\sigma\Phi + \mu\Phi \\
= \;& \frac{(\sigma^2 - 2\sigma + 1)}{(q-1)^2}\Phi + \frac{(2\sigma - 1)}{(q-1)^2}\Phi - \frac{\lambda\sigma\Phi}{(q-1)^2} + \frac{\mu\Phi}{(q-1)^2} \\
= \;& \frac{(\sigma-1)^2}{(q-1)^2}\Phi + \frac{(2-\lambda)(\sigma-1)}{(q-1)^2}\Phi + \frac{(2-\lambda)}{(q-1)^2}\Phi - \frac{(1-\mu)}{(q-1)^2}\Phi \\
= \;& \delta^2\Phi + \frac{2-\lambda}{q-1}\delta\Phi + \frac{1-\lambda+\mu}{(q-1)^2}\Phi \qquad \text{(B.16)} \\
= \;& 0
\end{aligned}$$

what is left is to show

$$\frac{2-\lambda}{q-1} \sim -\frac{(\alpha+\beta)z + (1-\gamma)}{1-z}$$

and

$$\frac{1-\lambda+\mu}{(q-1)^2} \sim \frac{z\alpha\beta}{1-z}$$

as $q \to 1$
we have

$$\begin{aligned}
\frac{2-\lambda}{q-1} &= \frac{2abz - 2c/q - (a+b)z + (1+c/q)}{(abz - c/q)(q-1)} \\
&= \frac{-(a(1-b)z + b(1-a)z + (c/q - 1)}{(abz - c/q)(q-1)} \\
&= \frac{-(a(1-b)z + b(1-a)z + (c/q - 1)}{(c/q - abz)(1-q)} \\
&= \frac{-(q^\alpha(1-q^\beta)z + q^\beta(1-q^\alpha)z + (q^{\gamma-1} - 1)}{(q^{\gamma-1} - q^{\alpha\beta}z)(1-q)} \\
&\sim -\frac{(\beta+\alpha)z + (1-\gamma)}{1-z}
\end{aligned}$$

and

$$\begin{aligned}\frac{1-\lambda+\mu}{(q-1)^2} &= \frac{abz-c/q-(a+b)z+(1+c/q)+z-1}{(abz-c/q)(q-1)^2}\\ &= \frac{z(ab-(a+b)+1)}{(abz-c/q)(q-1)^2}\\ &= \frac{z(1-a)(1-b))}{(abz-c/q)(1-q)^2}\\ &= \frac{z(1-q^{\alpha})(1-q^{\beta})}{(q^{\alpha\beta}z-q^{\gamma-1})(1-q)^2}\\ &\sim -\frac{z\alpha\beta}{1-z}\end{aligned}$$

we have used $\frac{1-q^k}{1-q} \to k$, as $q \to 1$. □

Proposition B3.2. The q-hypergeometric series equation (B.3)

$$ {}_2\Phi_1(a,b;c;q,z) = \sum_{n\geq 0} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n$$

becomes the hypergeometric series equation (B.1)

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n\geq 0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n$$

as $q \to 1$

Proof. If we let $a = q^{\alpha}, b = q^{\beta}, c = q^{\gamma}$, then it is easily shown using the fact $\frac{1-q^k}{1-q} \to k$, as $q \to 1$. □

APPENDIX C

Taking the continuum limit

We show how to take the continuum limit as $q \to 1$,using the example of q-PII. We show how q-PII reduces to q-Riccati, and q-Riccati to q-Airy, and how they each tends to their differential analogues. First we recapitulate what happens in the continuous case.

C1. PII, Riccati, Airy

PII equation:

$$y'' = 2y^3 + ty - a, \qquad ' = \frac{d}{dt} \tag{C.1}$$

when $a = \frac{1}{2}$ reduces to the Riccati equation:

$$y' = -y^2 - \frac{t}{2}. \tag{C.2}$$

Riccati equation (C.2) can be linearized into an Airy type equation by letting $y(t) = \frac{Ai'}{Ai}$, that is

$$Ai'' = -\frac{t}{2}Ai. \tag{C.3}$$

C2. q-PII, q-Riccati, q-Airy

A q-PII equation:

$$g(x/\lambda)g(\lambda x) = \frac{\alpha x^2(g(x) + x^2)}{g(x)(g(x) - 1)}, \quad \text{note here } q = \tfrac{1}{\lambda} \tag{C.4}$$

when $\alpha = \frac{1}{\lambda^2}$ reduces to a q-Riccati equation:

$$g(\lambda x) = \frac{x^2 + g(x)}{g(x)}. \tag{C.5}$$

Equation (C.5) can be linearized into a q-Airy equation by letting $g(x) = -i\frac{x}{\lambda}\frac{ai(x/\lambda)}{ai(x/\lambda^2)}$, which is

$$ai(x/\lambda^2) - \frac{i}{\lambda x}ai(x/\lambda) + \frac{ai(x)}{\lambda} = 0. \tag{C.6}$$

C3. Continuum limits

Now we show why we call equations (C.4), (C.5), and (C.6): q-PII, q-Riccati, and q-Airy respectively.

Proposition C3.1. The q-discrete equation:

$$g(x/\lambda)g(\lambda x) = \frac{\alpha x^2(g(x)+x^2)}{g(x)(g(x)-1)}$$

tends to PII

$$y'' = 2y^3 + ty - a$$

in the continuum limit as $\frac{1}{\lambda} \to 1$ where

$$\begin{aligned} x^2 &= -\frac{1}{4}\frac{1}{\lambda^{2n}} \\ g(x) &= \frac{1}{2}\left(1 - y(t)\epsilon\right) \\ \alpha &= (1 + a\epsilon^3) \\ \frac{1}{\lambda^2} &= (1 + \epsilon^3/2) \\ t &= n\epsilon \end{aligned}$$

as $\epsilon \to 0$.

Proof. The q-discrete variable x increments by multiples of $\frac{1}{\lambda}$,

$$x^2 = -\frac{1}{4}\frac{1}{\lambda^{2n}},$$

where the additive discrete variable n increments by the integers.

The continuous independent variable t is related to the q-difference variable x through the additive difference variable n:

$$t = n\epsilon, \qquad .$$

Hence, iterating the q-discrete variable x implies:

$$\begin{aligned} x &\to x/\lambda \\ \Rightarrow n &\to n+1 \\ \Rightarrow t &\to t+\epsilon. \end{aligned}$$

The continuum limit is taken when $\epsilon \to 0$, $\frac{1}{\lambda} \to 1$,

$$\begin{aligned} x^2 &= -\frac{1}{4}\left(\frac{1}{\lambda^2}\right)^n, \\ &= -\frac{1}{4}\left(\frac{1}{\lambda^2}\right)^{\frac{t}{\epsilon}}, \\ &= -\frac{1}{4}(1+\epsilon^3/2)^{\frac{t}{\epsilon}}, \\ &= -\frac{1}{4}e^{\frac{t}{\epsilon}\ln(1+\epsilon^3/2)}, \\ &\sim -\frac{1}{4}e^{\frac{t}{\epsilon}(\epsilon^3/2+O(\epsilon^6))}, \\ &\sim -\frac{1}{4}e^{\frac{t\epsilon^2}{2}}, \\ &\sim -\frac{1}{4}(1+t\epsilon^2/2), \end{aligned}$$

as $\epsilon \to 0$.

We can expand the dependent variable $g(x) = g(t)$ when ϵ is small

$$\begin{aligned} g(x) = g(t) &= g_0(t) + g_1(t)\epsilon + g_2(t)\epsilon^2 + ... \\ &= \sum_{m=0} g_m(t)\epsilon^m, \end{aligned} \tag{C.7}$$

where

$$\begin{aligned} g_k(x/\lambda) = g_k(t+\epsilon) &= g_k(t) + \epsilon g_k' + \epsilon^2 \frac{g_k''}{2!} + ... \\ &= \sum_n \epsilon^n \frac{g_k^{(n)}}{n!}, \end{aligned}$$

and

$$\begin{aligned} g(x/\lambda) = g(t+\epsilon) &= g_0(t+\epsilon) + g_1(t+\epsilon)\epsilon + g_2(t+\epsilon)\epsilon^2 + ... \\ &= \sum_{m=0} g_m(t+\epsilon)\epsilon^m \\ &= \sum_{m+n=0} \epsilon^{m+n}\frac{g_m^{(n)}}{n!}, \qquad f^{(n)} = \frac{d^n f}{dt^n}. \end{aligned} \tag{C.8}$$

Similarly,

$$\begin{aligned} g(x\lambda) = g(t-\epsilon) &= g_0(t-\epsilon) + g_1(t-\epsilon)\epsilon + g_2(t-\epsilon)\epsilon^2 + ... \\ &= \sum_{m=0} g_m(t-\epsilon)\epsilon^m \\ &= \sum_{m+n=0} \epsilon^{m+n}(-1)^n \frac{g_m^{(n)}}{n!}. \end{aligned} \tag{C.9}$$

Now substitute expressions (C.7), (C.8) and (C.9) into q-PII equation (C.4), and equate powers in ϵ:

$$\begin{aligned} \epsilon^0 : \qquad g_0^2 g_0(g_0-1) &= -\frac{1}{4}\left(g_0 - \frac{1}{4}\right) \\ \Rightarrow g_0 &= \frac{1}{2}, \text{ or } -\frac{1}{2}, \\ \epsilon : \qquad g_0^3(g_0-1) &= -\frac{1}{4}\left(g_0 - \frac{1}{4}\right) \\ \Rightarrow g_0 &= \frac{1}{2}. \end{aligned}$$

Equating terms of order ϵ^2 shows that the equation is consistent but does not give any new information.

$$\epsilon^3 : \qquad -\frac{g_1''}{8} + g_1^3 = -\frac{t}{8}g_1 - a/16,$$

recall that we have chosen

$$\alpha = (1 + a\epsilon^3).$$

Let $g_1 \to -g_1/2$

$$g_1'' = 2g_1^3 + tg_1 - a.$$

□

Proposition C3.2. The q-discrete equation

$$g(\lambda x) = \frac{x^2 + g(x)}{g(x)}$$

has Riccati equation

$$y' = -y^2 - \frac{t}{2}$$

as its continuum limit as $\frac{1}{\lambda} \to 1$.

Proof. Substitute the expressions (C.7), (C.8) and (C.9)for $g(x), g(x/\lambda), g(x\lambda), x^2$ in terms of $y(t)$ and t, equating powers in ϵ will show that

$$y' = -y^2 - \frac{t}{2}$$

and $\alpha = \frac{1}{\lambda^2} = (1 + \epsilon^3/2) = (1 + a\epsilon^3)$ implies that $a = \frac{1}{2}$. □

Proposition C3.3. The q-discrete equation

$$ai(x/\lambda^2) - \frac{i}{\lambda x} ai(x/\lambda) + \frac{ai(x)}{\lambda} = 0$$

has Airy type equation

$$w'' = -\frac{t}{2} w$$

as continuum limit as $\frac{1}{\lambda} \to 1$ for $ai(x) = w(t)$.

Proof. Since

$$\begin{aligned} x^2 &\sim -\frac{1}{4}\left(1 + \epsilon^2 t/2\right), \\ \frac{1}{x} &\sim \left(-\frac{1}{4}(1 + \epsilon^2/t2)\right)^{-\frac{1}{2}} \\ &\sim \frac{2i}{\sqrt{1 + \epsilon^2 t/2}} \\ &\sim 2i(1 - \epsilon^2 t/4). \end{aligned}$$

Substitute the expressions for $ai(x)$ and $\frac{1}{x}$ in terms of $w(t)$ and t, equating powers in ϵ, we obtain

$$w'' = -\frac{t}{2} w.$$

□

To summarize, we have shown in this appendix:

$$\begin{array}{rcl} q\text{-PII eqn.}(C.4) & \to & \text{PII eqn.}(2.3) \\ \downarrow & & \downarrow \\ q\text{-Riccati eqn.}(C.5) & \to & \text{Riccati eqn.}(C.2) \\ \downarrow & & \downarrow \\ q\text{-Airy eqn.}(C.6) & \to & \text{Airy eqn.}(C.3). \end{array}$$

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