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Lakshithe Wagalath. Modélisation mathématique du risque endogène dans les marchés financiers. Finance quantitative [q-fin.CP]. Université Pierre et Marie Curie - Paris VI, 2013. Français. NNT : . tel-00832234

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**THÈSE DE DOCTORAT DE L'UNIVERSITÉ
PIERRE ET MARIE CURIE**

Spécialité: **Mathématiques Appliquées**

présentée par

Lakshitha Wagalath

pour obtenir le grade de

Docteur de l'Université Pierre et Marie Curie

**Modélisation mathématique du risque
endogène dans les marchés financiers**

**Feedback effects and endogenous risk
in financial markets**

dirigée par RAMA CONT

Soutenue le 15 mars 2013, devant le jury composé de

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À ma famille

Remerciements

Je tiens tout d'abord à remercier Rama Cont de m'avoir accepté en thèse et de m'avoir offert un sujet lumineux, qui a été porté au premier plan par les récents bouleversements dans les marchés financiers. Je suis admiratif de ses immenses connaissances scientifiques et sa capacité à établir des ponts entre des domaines très divers et ma recherche a grandement bénéficié de son inventivité et son exigence scientifique. Rama a toujours eu une question à mes réponses, ce qui m'a permis d'explorer de la manière la plus complète possible ma problématique de recherche et d'éviter toute forme de paresse intellectuelle – c'est-à-dire d'apprendre le métier de chercheur. Je le remercie de m'avoir toujours encouragé à diffuser ma recherche dans de nombreux séminaires, groupes de travail et autres conférences, et à dialoguer avec les autres chercheurs. Je le remercie enfin et surtout de m'avoir transmis sa passion pour la recherche qui m'a permis de m'épanouir scientifiquement et humainement ces dernières années.

Jean-Pierre Fouque et Rüdiger Frey ont accepté d'être les rapporteurs de cette thèse. Je leur suis reconnaissant d'avoir pris le temps de lire et de commenter en détail mon travail. Merci également à Jean-Philippe Bouchaud, Nicole El Karoui, Damir Filipovic, Mathieu Rosenbaum et Peter Tankov d'avoir accepté de faire partie de mon jury de thèse. J'ai une pensée particulière pour Nicole qui m'a initié aux probabilités à l'X et a, depuis, soutenu mon désir de faire de la recherche.

Je remercie tous mes professeurs qui, chacun à leur manière, m'ont donné le goût pour les mathématiques et la finance. Je pense en particulier à Claude Deschamps, Nizar Touzi ou encore Jean Bertoin. Merci également à Charles-Albert Lehalle, Thierry Cantet et Raphaël Hasson de m'avoir guidé au cours de mes différents stages et appris à faire le lien entre mes connaissances théoriques et les exigences pratiques du monde de la banque. Enfin, je tenais à remercier Daniel Gabay et Yann Braouézec de m'avoir donné ma chance en tant que professeur de mathématiques et de finance.

J'ai une profonde gratitude envers mes amis, qui m'ont soutenu au cours de ces trois années de recherche et m'ont permis de ne pas toujours penser à la modélisation mathématique du risque endogène dans les marchés financiers; amis de lycée, de prépa, d'école ou de thèse ; amis que je ne croise que rarement ou que je vois toutes les semaines; amis adeptes d'un café à l'Inévitable, d'un cocktail au Pantalon, d'un burger au Petit Café ou d'une pizza-foot. Merci à tous !

Je remercie mes amis du laboratoire et de Jussieu, notamment Henri, Emmanuel, Sophie, Kilian, Pascal et Clément, pour m'avoir aidé et supporté, au jour le jour, au bureau. Un grand merci évidemment à Adrien, Amel, Andreea ainsi que toute la team Rama Cont à New-York, avec qui j'ai partagé d'excellents moments au cours de ces trois dernières années.

J'ai tant d'autres personnes à remercier – les membres du LPMA, François et Bilel pour leur aide en informatique, Jed Martin et Eckhart Tolle pour leurs apports philosophiques, Nicolas Théa pour son accueil à bras ouverts à chaque fois que je partais en échange à

New-York – qu’il me serait impossible de toutes les citer et je préfère dire un sincère merci à toutes ces personnes qui, de près ou de loin, ont contribué à cette thèse.

Je remercie enfin ma famille – mes deux soeurs, mes cousins, oncles, tantes et belle-famille – qui m’ont sans cesse entouré et soutenu dans mes projets. Je tiens à exprimer ma profonde gratitude pour mes parents qui ont placé l’épanouissement de leurs enfants avant toute autre chose et qui m’ont inspiré dans ces longues études et transmis leur soif d’apprendre, en me donnant un cadre affectif, intellectuel et matériel idéal. Je conclus naturellement par un grand merci à ma femme, qui m’a accompagné jour après jour dans cette aventure, en supportant les moments de folie et d’angoisse inhérents au travail de recherche et en se souciant, sans cesse, de mon équilibre personnel.

Résumé

Cette thèse propose un cadre mathématique pour la modélisation du risque endogène dans les marchés financiers. Le risque endogène désigne le risque généré, et amplifié, au sein du système financier lui-même, par les différents acteurs économiques et leurs interactions, par opposition au risque exogène, généré par des chocs extérieurs au système financier. Notre étude est motivée par l'observation des différentes crises financières passées, qui montre le rôle central du risque endogène dans les marchés financiers. Ainsi, les périodes de crises sont souvent associées à des phénomènes de liquidation/ventes éclair ('fire sales'), qui génèrent, de manière endogène, une importante volatilité pour les actifs financiers et des pics inattendus de corrélations entre les rendements de ces actifs, entraînant de fortes pertes pour les investisseurs. Alors que la structure de dépendance entre les rendements d'actifs financiers est traditionnellement modélisée de manière exogène, les faits décrits précédemment suggèrent qu'une telle modélisation exogène ne peut rendre compte du risque endogène observé dans les marchés financiers. L'idée principale de cette thèse est de distinguer entre deux origines pour la corrélation entre actifs. La première est exogène et reflète une corrélation fondamentale. La seconde est endogène et trouve son origine dans l'offre et la demande systématiques générées par les grandes institutions financières.

Nous modélisons la dynamique en temps discret des prix d'actifs financiers d'un marché multi-actifs par une chaîne de Markov dans lequel le rendement de chaque actif, à chaque période k , se décompose en un terme aléatoire – qui représente les fondamentaux de l'actif et est indépendant du passé – et un terme d'offre/demande systématique, généré de manière endogène. Dans chaque chapitre, nous caractérisons mathématiquement cette offre endogène et son impact sur les prix d'actifs financiers. Nous exhibons des conditions sous lesquelles la chaîne de Markov converge faiblement, lorsque le pas de temps du modèle discret tend vers zéro, vers la solution d'une équation différentielle stochastique dont nous donnons le drift et la volatilité multi-dimensionnels. L'étude du processus de covariation quadratique de la limite diffusive nous permet de quantifier l'impact de l'offre et demande endogènes systématiques sur la structure de dépendance entre actifs. Enfin, nous développons des outils statistiques et économétriques visant à résoudre le problème inverse d'identification et d'estimation des paramètres de notre modèle à partir de données de prix d'actifs financiers.

Le Chapitre 1 étudie le risque endogène généré par un fonds sujet à des ventes forcées en raison d'investisseurs qui sortent de leurs positions lorsque le fonds sous-performe et que sa valeur passe en-dessous d'un seuil. Nous modélisons l'offre et la demande provenant de telles ventes forcées en introduisant une 'fonction de liquidation' f qui mesure la vitesse à laquelle les investisseurs sortent de leurs positions dans le fonds. Nous supposons que l'offre en excès due au fonds sur chaque actif impacte le rendement de l'actif de manière linéaire et nous explicitons les conditions pour que la dynamique de prix en temps discret soit une chaîne de Markov dans $(\mathbb{R}_+^*)^n$, où n est le nombre d'actifs dans le marché. Nous exhibons des conditions sous lesquelles le modèle discret converge faiblement vers une diffusion en temps continu et calculons les drifts et volatilités multi-

dimensionnels de la dynamique de prix en temps continu. L'étude de la covariation quadratique de la limite diffusive permet d'expliciter l'impact des ventes forcées dans le fonds sur la structure de dépendance entre les actifs financiers. En particulier, nous montrons que la matrice de covariance réalisée s'écrit comme la somme d'une matrice de covariance fondamentale et une matrice de covariance en excès, qui dépend des positions du fonds, de la liquidité des actifs et de la trajectoire passée des prix et qui est nulle lorsqu'il n'y a pas de ventes forcées. Nous prouvons alors que cet impact endogène augmente la volatilité du fonds en question, exactement dans les scénarios où le fonds subit des pertes. Nous calculons également son impact sur la volatilité d'autres fonds investissant dans les mêmes actifs et prouvons l'existence d'une relation d'orthogonalité entre les positions du fonds de référence et d'un autre fonds telle que, si cette relation d'orthogonalité est vérifiée, des ventes forcées dans le fonds de référence n'affectent pas la volatilité de l'autre fonds.

Le Chapitre 2 étend les résultats du Chapitre 1 au cas de plusieurs fonds et d'un impact quelconque (pas nécessairement linéaire) de l'offre agrégée en excès provenant de ventes forcées dans ces fonds, sur les rendements de prix d'actifs. Nous exhibons des conditions sous lesquelles le modèle discret converge faiblement vers une diffusion en temps continu. La fonction de volatilité de la limite diffusive ne dépend de la fonction de d'impact qu'à travers sa dérivée première en zéro, montrant qu'un modèle de d'impact linéaire capture complètement l'impact des effets de rétroaction dus aux ventes forcées dans les différents fonds sur la structure de dépendance entre actifs. Nous calculons la matrice de covariance réalisée, en fonction des positions liquidées, en particulier dans un cas simple où les liquidations ont lieu à vitesse constante, dans un intervalle de temps fixé et nous donnons des conditions assurant que cette relation peut être inversée et les volumes de liquidations identifiés. Nous construisons alors un estimateur du volume de liquidation dans chaque actif, dont nous prouvons la consistance, et pour lequel nous dérivons un théorème central limite, qui nous permet de construire un test statistique testant si, pendant une période donnée, des liquidations ont eu lieu. Nous illustrons notre procédure d'estimation avec deux exemples empiriques: le 'quant event' d'août 2007 et les liquidations suivant la faillite de Lehman Brothers en Automne 2008.

Le Chapitre 3 étudie l'impact d'un investisseur institutionnel investissant une portion constante de sa richesse dans chaque actif (stratégie fixed-mix). Pour un vecteur d'allocations donné, nous prouvons l'existence d'une unique stratégie fixed-mix autofinancante. A chaque période, le prix des n actifs et la valeur du fonds fixed-mix sont obtenus comme la solution d'un problème de point fixe. Nous montrons que, sous certaines conditions que nous explicitons, le modèle discret converge vers une limite diffusive, pour laquelle nous calculons la covariance et la corrélation réalisée à l'ordre un en liquidité. Nos résultats montrent que la présence d'investisseurs institutionnels peut modifier les corrélations de façon significative. Nous calculons les vecteurs propres et valeurs propres de la matrice de corrélation réalisée (à l'ordre un en liquidité). L'étude des drifts de la limite continue nous permet de calculer les rendements espérés des actifs et montre qu'en raison de la présence de l'investisseur institutionnel, les rendements espérés des actifs avec grand (resp. faible) drift fondamental, comparés au rendement fondamen-

tal du fonds, diminuent (resp. augmentent). Nous calculons, dans un exemple simple, la stratégie efficiente pour un critère moyenne-variance et montrons qu'elle est différente de la stratégie optimale fondamentale (sans le fonds). L'étude de la frontière optimale dans cet exemple montre qu'un investisseur prenant en compte l'impact de l'investisseur institutionnel peut améliorer son rendement pour un niveau de risque donné.

Abstract

This thesis proposes a mathematical framework for studying feedback effects and endogenous risk in financial markets. We propose a multi-period model of a financial market with multiple assets, which takes into account the price impact generated by large shifts in supply and demand from financial institutions. Chapter 1 reviews the existing empirical and theoretical literature on feedback effects and motivates this thesis. Chapter 2 studies feedback effects from distressed selling in a large fund, in the case of a linear price impact model, while Chapter 3 extends such results to feedback effects from fire sales in multiple funds and allows for a general price impact function. Chapter 4 studies the impact of a large institutional investor keeping a fixed allocation/portion invested in each asset. We quantify the excess supply and demand generated in each case. The multi-period price dynamics in the presence of feedback effects is modeled as a Markov chain and we exhibit conditions under which it converges weakly, as the time step of the discrete-time model goes to zero, to the solution of a stochastic differential equation, for which we give the multi-dimensional drift and volatility explicitly. The study of the quadratic covariation process of the diffusion limit allows us to quantify the impact of feedback effects on the dependence structure of asset returns and the endogenous risk generated: under our model assumptions, we show that we can compute the impact of feedback effects on fund volatility and the spillover effects to other funds investing in the same assets. Finally, we give conditions for the identifiability of model parameters from time series of asset prices and build an estimator for the fund flows generated by systematic supply and demand by large financial institutions. We show that such estimator is consistent and derive a central limit theorem in Chapter 3. We illustrate our estimation procedure with two empirical examples.

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Notations

In this thesis, we use the following notations

- We denote the set of positive (resp. strictly positive) real numbers by:

$$\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\} \quad (\text{resp. } \mathbb{R}_+^* = \{x \in \mathbb{R} | x > 0\})$$

- For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the only integer such that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.
- For an integer $p \geq 1$, $\mathcal{C}^p(E)$ represents the set of mappings defined on E and p times continuously differentiable.
- For an integer $p \geq 1$, $\mathcal{C}_0^p(E)$ represents the subset of $\mathcal{C}^p(E)$ containing the mappings whose derivatives of order $1 \leq l \leq p$ have compact support (ie: are equal to zero outside of a compact set of E).
- $\mathcal{C}^\infty(E) = \{f | \forall p \geq 1, f \in \mathcal{C}^p(E)\}$ (resp. $\mathcal{C}_0^\infty(E) = \{f | \forall p \geq 1, f \in \mathcal{C}_0^p(E)\}$)
- For $(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n$, we denote the scalar product between X and Y and the norm of X as:

$$X.Y = \sum_{i=1}^n X_i Y_i \quad \text{and} \quad \|X\| = (X.X)^{\frac{1}{2}} = \left(\sum_{i=1}^n X_i^2 \right)^{\frac{1}{2}}$$

- For $X \in \mathbb{R}^n$ and $r > 0$, we denote $\mathcal{B}(X, r) = \{z \in \mathbb{R}^n | \|X - z\| \leq r\}$.

- For $X \in \mathbb{R}^n$, we write $\exp X = \begin{pmatrix} \exp X_1 \\ \vdots \\ \exp X_n \end{pmatrix}$.

- $\mathcal{M}_{p \times q}(\mathbb{R})$ (resp. $\mathcal{M}_p(\mathbb{R})$) denotes the set of real-valued matrices with p lines and q columns (resp. p columns).
- For $M \in \mathcal{M}_{p \times q}(\mathbb{R})$, $M^t \in \mathcal{M}_{q \times p}(\mathbb{R})$ denotes the transpose of M .
- The set of real-valued symmetric matrices is denoted:

$$\mathcal{S}_p(\mathbb{R}) = \{M \in \mathcal{M}_p(\mathbb{R}) | M^t = M\}$$

- The set of real valued symmetric positive semi-definite matrices is denoted:

$$\mathcal{S}_p^+(\mathbb{R}) = \{M \in \mathcal{S}_p(\mathbb{R}) | \forall X \in \mathbb{R}^n, X.MX \geq 0\}$$

- Convergence in law and probability: for a sequence $X^{(\tau)}$ of random variables, we denote the fact that $X^{(\tau)}$ converges in law (resp. in probability) to X when τ goes to zero by $X^{(\tau)} \xrightarrow[\tau \rightarrow 0]{} X$ (resp. $X^{(\tau)} \xrightarrow[\tau \rightarrow 0]{\mathbb{P}} X$)

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Chapter 1

Introduction

1.1 Feedback effects and endogenous risk in financial markets

The observation of financial markets in the past years shows that the global decline in asset prices during a financial crisis is associated with a surge in correlation between asset returns. Contrary to situations where liquidity 'dries up', those correlation spikes are not combined with a fall in traded volumes - on the contrary, traded volumes sometimes even increase during these episodes. Such unexpected correlation spikes generally occur during the liquidation of large positions and the failure of large financial institutions.

The recent financial crisis illustrates the surge in correlations associated to the collapse of a large financial institution. Figure 2.1 displays, on the left, the one-year exponentially-weighted moving average (EWMA) estimator of average pairwise correlations of daily returns in the main European equity index, the Eurostoxx 50 and on the right, the one year EWMA estimator of correlation between the energy sector and the technology sector of the S&P 500. In both graphs, we see that correlation increases significantly after the collapse of Lehman Brothers on September, 15th, 2008. For example, correlation between the energy sector and the utility sector of the S&P500 increases dramatically after September, 15th, 2008, from 5% to 85%.

Such correlation spikes were associated to the collapse of all equity markets around the world, following Lehman Brother's filing for chapter 11 bankruptcy protection. The top graph of Figure 1.2 shows the performance of the S&P500 index, which declined sharply in the Fall 2008. However, contrary to liquidity dry-up events, the global fall of stock markets and the correlation spikes observed between asset returns were not associated with a decrease in traded volumes. On the contrary, we see on the bottom graph of Figure 1.2, that traded volumes even increased in Fall 2008.

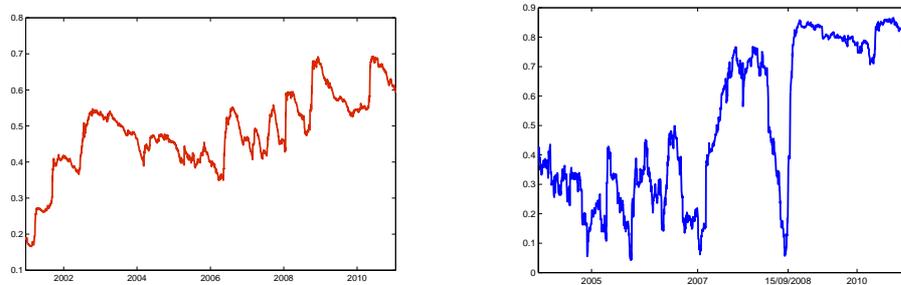


Figure 1.1: Left: One-year EWMA estimator of average pairwise correlations of daily returns in EuroStoxx 50 index. Right: One-year EWMA estimator of correlation between two sector indices of the S&P 500: SPDR XLE (energy) and SPDR XLK (technology).

The combination of falling markets, unexpected correlation spikes and stable (or larger-than-usual) trading volumes was originated by *the great deleveraging of Fall 2008*. The failure of a huge financial institution such as Lehman Brothers generated liquidations and deleveraging in all asset classes all over the world. It was such a shock to financial markets - major equity indices all lost around 10% on that day - that it triggered stop-

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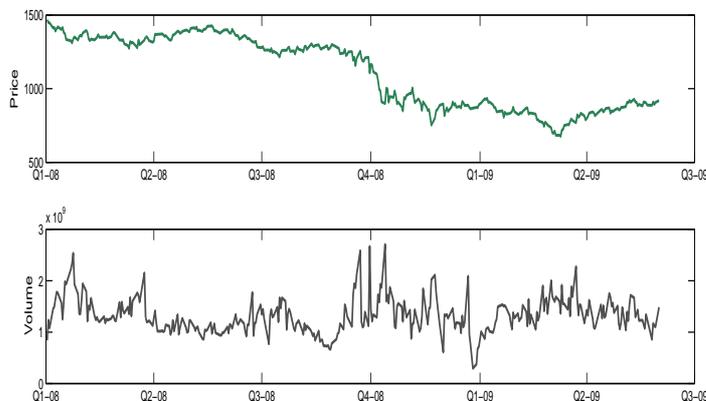


Figure 1.2: Level of the S&P500 and volume for the S&P futures

loss and deleveraging strategies among a remarkable number of financial institutions worldwide. Risk measures of portfolios, for example the value at risk, increased sharply, obliging financial institutions to hold more cash, which they got by deleveraging their portfolios, rather than by issuing debt which would have been very costly at such distressed times. This great deleveraging by financial institutions resulted in a generalized decline of asset prices, along with a rise in correlation between all asset classes, resulting in a high volatility environment and dramatic losses for investors.

The following examples also give a good intuition of how feedback effects from large shifts of supply and demand by financial institutions can lead to downward spirals of prices and unexpected correlation spikes and, ultimately, generate endogenous risk.

The 1987 Stock Market Crash

On October 19th 1987, the US stock market experienced its largest-ever one day drop: the Dow Industrial Average lost 22.6% which represented \$500 billion, leading all other markets of major economies to collapse in a similar way. The mechanisms which led to the "Black Monday" of 1987 were thoroughly investigated, by public authorities (Report of the Presidential Task Force on Market Mechanisms, 1988) and academics (Shiller, 1988; Rubinstein, 2000; Carlson, 2006) and show the key role played by rule-based strategies in the collapse of the stock market that day. During the years preceding the stock market crash, portfolio insurance strategies expanded dramatically and represented between \$60 and \$80 billion in 1987. Portfolio insurance consists in short selling stock index futures in order to hedge a portfolio of stocks and implies buying stocks when the market rises and selling stocks when it declines. Whereas the US stock market experienced a bull bubble from 1982 to early 1987, leading portfolio insurers to hold more and more stocks, the Fall of 1987 saw nervousness among investors: the second week

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before the crash, the S&P500 fell by 5.2% and the Dow Industrial Average lost 5% on October 16th. On the next trading day, October 19th, the decline resumed sharply at the opening of the stock markets, leading portfolio insurers to sell stocks in large blocks and automatically, as a result of their hedging strategy in a bear market. As portfolio insurers represented a significant portion of the traded volumes, their rule-based selling fed on itself, generating more selling and triggering stop-loss strategies and fire sales among other investors, leading ultimately to the largest fall in the history of the Dow Industrial Average. Contrary to the crisis of 1929, when markets took years to recover, stock markets recovered quickly from the Black Monday of 1987, which shows that, rather than reflecting a genuine economic decline, the stock crash was triggered by rule-based strategies, which led to self-reinforced fire sales and a decline in all stocks.

The collapse of Long Term Capital Management

LTCM was a hedge fund founded in 1994 by John Meriwether, the former vice-chairman and head of bond trading at Salomon Brothers and its board of director members included Scholes and Merton, who share the 1997 Nobel Prize in Economic Sciences. It followed quantitative strategies, mainly on government bonds (for example spreads between government securities, between swaps and US Treasuries but also Russian bonds) and built a portfolio of strategies which was supposedly well-diversified, among geographical zones (USA, Japan, European countries) and asset classes (it traded not only bonds but also equity and derivatives). Due to heavy losses in its investments in Russian bonds caused by the default of the Russian government in August 1998, LTCM was forced to liquidate part of its positions after a sudden increase in the correlations across its -previously uncorrelated- strategies. This led to a sharp increase in its volatility (Rosenfeld, 2010); unexpected spikes of correlation arose between asset classes that used to be uncorrelated (Russian bonds and US equity for instance), amplifying the fund's losses and leading to the collapse of LTCM.

The hedge fund losses of August 2007

From August 6th to August 9th 2007, long-short market-neutral equity funds experienced large losses: many funds lost around 10% per day during four days and experienced a rebound of around 15% on August 10th, 2007. During this week, as documented by Khandani and Lo (2011), market-neutral equity funds whose returns previously had a low historical volatility exhibited negative returns exceeding 20 standard deviations, while no major move was observed in major equity indices. Khandani and Lo (2011) simulate a contrarian strategy which implies buying (resp. selling) assets that decreased (resp. increased) the most one day before. They find that the profit and loss profile for this strategy is similar to that of market-neutral equity funds during the second week of August. They suggest that this event was due to a large market-neutral fund deleveraging its positions, which endogenously generated huge losses for other market-neutral funds, following the same rules of investments and having similar positions, while leaving

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index funds unaffected.

Such examples suggest that *feedback effects* from deleveraging and fire sales in large positions can lead to unexpected spikes in correlation between asset classes, which generate endogenous risk and can lead to the collapse of financial institutions. These examples illustrate that "asset correlations can be different during a liquidity crisis because price movements are caused by distressed selling and predatory trading rather than fundamental news" (Brunnermeier and Pedersen, 2005). We witness that one shock can be amplified within the financial system, due to the systematic supply and demand generated by financial institutions. This notion of endogenous risk has been studied by Danielsson and Shin (2003); Shin (2010) who define endogenous risk as "risks that are generated and amplified within the financial system, rather than risks from shocks that arrive from outside the financial system. The precondition for endogenous risk is the conjunction of circumstances where individual actors react to changes in their environment and when those individuals' actions affect their environment".

Such endogenous variations in volatility and correlations, generated by systematic patterns in supply and demand linked to fire sales, short-selling or rule-based trading strategies, have played an important role in past financial crises and have been the focus of several studies (Adrian and Shin, 2008, 2009; Brunnermeier, 2008; Brunnermeier and Pedersen, 2009; Carlson, 2006; Pedersen, 2009; Shin, 2010).

The key ingredient when feedback effects are concerned is *liquidity*. Because financial markets are not perfectly liquid, investors reacting to price movements in a systematic way and traders following specific rules of investments impact prices in a systematic way, which will make them trade more and so on. Due to the illiquidity of financial markets, feedback effects from fire sales and rule-based strategies can lead to the spirals of losses and correlation spikes described in the previous examples, ultimately causing systemic risk, which is the risk that the whole financial system collapses.

Systemic risk was particularly brought to light during the recent financial crisis. In the aftermath of the collapse of Lehman Brothers, public authorities injected hundreds of billions of dollars in financial markets, most of them dedicated to help financial institutions, so that the financial system and the whole economy do not collapse.

Whereas the observation of financial markets and, in particular, past financial crises, points to the key role played by feedback effects from large financial institutions, most models for asset prices do not take this feature into account and are purely statistical models, with exogenous parameters. Such models, which are more and more statistically sophisticated, allow to fit the returns of financial assets, thanks to an exogenous choice of parameters. However, they do not incorporate liquidity effects and feedback effects and hence cannot account for the empirical facts observed in past crisis times: mutually-exciting correlation spikes and fund losses.

Our aim is to propose a tractable model which takes those feedback effects into account and, in a sense, is more economically justified as it reflects the imbalances of supply and demand generated by investors.

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1.2 Empirical studies

Liquidity

Whereas financial markets are assumed to be infinitely liquid in numerous academic studies, and that investors are 'price-takers' and their strategies do not affect asset prices, the fact that financial markets are indeed not perfectly liquid is widely documented in the empirical literature. Empirical studies shed light on such market illiquidity at daily and intraday frequencies, for various asset classes and countries (Kraus and Stoll (1972); Holthausen et al. (1987); Kempf and Korn (1999); Bouchaud and Potters (2003); Almgren et al. (2005); Obizhaeva (2011); Cont et al. (2010)...). The fact that financial markets are not perfectly liquid allows for feedback effects from trading by investors, in particular large financial institutions, on asset prices. Depending on the strategy followed by large investors, the impact on asset prices will be different. In the next paragraphs, we review the empirical literature on feedback effects from different types of strategies.

Feedback effects from options trading

The observation of options markets reveals that stock prices and indices can behave abnormally near options expiration dates, suggesting that the action of options traders impacts the underlying stocks, especially on expiration dates. Numerous empirical studies (Stoll et al., 1986; Ni et al., 2005; Golez and Jackwerth, 2010) focus on the behavior of stock prices and indices near options expiration dates. They show a clustering effect for stock prices on the day when options expire, which is also known as stock pinning. The probability that a stock price or an index is equal (or very close) to an option's strike at the end of the day is significantly higher on options expiration dates. In addition, stocks and indices with a large options market have a higher probability of pinning at options maturity. Those studies suggest that stock pinning is caused by market-makers' rebalancing of delta hedges, which, on expiration dates, is strong enough to pull stocks and indices towards the nearest option strike. More precisely, they suggest that this clustering phenomenon is due to market makers who are long options and for whom delta hedging implies buying the underlying stock or index if it decreases below the strike price and selling the underlying if it exceeds the strike price.

Feedback effects from distressed selling

Evidence of distressed selling and its impact on market dynamics has been examined by several empirical studies. Ippolito (1992); Chevalier and Ellison (1997); Sirri and Tufano (1998) show that fund past performance and fund flows are strongly linked. Coval and Stafford (2007) study the relationship between ownership structure of open end mutual funds and the performance of those funds. They find that funds which underperformed experience outflows of capital leading to fire sales in existing positions. They show that the outflows of capital can be extreme and lead to a significant decrease of assets under management

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held by the funds. They explain such fire sales by the fact funds experiencing outflows of capital are often distressed funds which cannot borrow money easily. Combined to self-imposed constraints that prevent them from short-selling other securities, such outflows of capital result in fire sales in existing positions. This mechanism creates a price pressure on all assets held in common by distressed funds.

Fratzscher (2011) studies the impact of key events, such as the collapse of Lehman Brothers, on capital flows. He uses a dataset on portfolio capital flows and performance at the fund level containing daily, weekly and monthly flows for more than 16000 equity funds and 8000 bond funds, domiciled in 50 countries. He aggregates the net capital flows (ie net of valuation changes) for each country and finds that they are negative for all the countries of the study. This means that fund managers of such funds deleveraged their positions after the collapse of Lehman Brothers, sometimes in dramatic proportions: in some cases, the outflows can represent up to 30% of the assets under management by the funds.

Large financial institutions can also be forced to deleverage their positions for regulatory reasons. Berndt et al. (2005) shows how banks must sell their risky assets after large losses in the corporate debt market. Leland (2011) show how bank capital requirements, imposed by new regulation (Basel Committee on Banking Supervision, 2010), can generate fire sales.

Khandani and Lo (2011) simulate a market-neutral strategy and are able to reconstitute empirically the profile of losses of hedge funds during the second week of August 2007. They explain the hedge fund losses during this period by the deleveraging of a large market-neutral portfolio that impacted other market-neutral funds, which, in order to reduce risk exposure, were compelled to deleverage their market-neutral strategy, and reinforced the losses for all market-neutral funds, generating a series of unprecedented losses for such funds.

Short selling and predatory trading

As argued by the managers of LTCM in 1998, short selling and predatory trading can be associated to fund underperformance and distressed selling. Empirical studies by Comerton-Ford et al. (2010) show that short selling represents 40.2% of the total dollar volume on NYSE and 39.2% on the Nasdaq. Short selling is widely used by market participants in financial markets. Although it has been accused, among others by politicians, to be the key factor for market crashes and hence has periodically been banned (short selling on financial securities were banned in the US in Fall 2008; Germany banned short selling in 2010), several empirical studies, such as Haruvy and Noussair (2006), underline the fact that short selling allows prices in financial markets to reflect their fundamental values.

Price-mediated contagion

The empirical studies mentioned previously all focus on the impact of large fund flows on

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asset returns and do not study their impact on the dependence structure of assets. During the LTCM crisis, price-mediated contagion was described as "the rapid spread from one market to another of declining prices, declining liquidity, increased volatility, and increased correlation associated with the financial intermediaries' own effect on the markets in which they trade" (Kyle and Xiong, 2001). Bank for international settlements (1999); Kaminsky and Reinhart (2003) describe the mechanism which led to the market turmoil of Fall 1998 and the collapse of LTCM and show empirically that financial turbulence in one or more of the world's financial centers leads to contagion effects in the other markets. Hamao et al. (1990) study the interdependence of prices and volatility between New-York, London and Tokyo stock markets and find evidence of volatility spillover effects from New-York to Tokyo and London and from London to Tokyo. Lin et al. (1994) find that Tokyo New-York-daytime returns are correlated with the New-York Tokyo-overnight returns and explain this correlation by the flow of information from one market to another.

Jotikasthira et al. (2011) lead an empirical investigation on the effects of fund flows from developed countries to emerging markets. They show that underperformance of funds domiciled in developed countries lead to a deleveraging of their positions in emerging markets, hence affecting asset prices and correlations between emerging markets and developed countries and creating a new channel through which shocks are transmitted from developed markets to emerging markets. Anton and Polk (2008) find empirically that common active mutual fund ownership predicts cross-sectional variation in return realized covariance. They show that such fund-ownership-originated covariance is due to a contagion effect generated by trading from large fund managers. These studies show that the impact of large investors on a panel of assets can affect prices of other assets, through the price-mediated contagion mechanism described above, and hence modify the dependence structure between assets: losses in one asset class can lead to spiral of losses in all asset classes and a surge a correlation between those asset classes, which is exactly what happened in the examples described in Section 1.1.

1.3 Theoretical studies

Danielsson and Shin (2003) give an insightful description of feedback effects and endogenous risk. They first give the (non financial) example of the Millenium bridge in London, which, the day of its opening, wobbled dangerously due to the large numbers of pedestrians who reacted in a same systematic manner to an external shock (the wind, which made the bridge slightly oscillate) and amplified the bridge oscillation, which made them react again in the same way, and generate a self-reinforcing oscillation for the bridge. They focus in three financial examples which illustrate the feedback loop in financial markets. In particular, they show how feedback effects accelerated the collapse of LTCM: a distress in existing positions triggered margin calls, which were paid by deleveraging the fund's positions and hence generating adverse price moves and additional distress for the fund, opening the same loop again.

Theoretical models have been proposed in order to take into account the impact of

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large investors following rule-based strategies on asset prices. First, one has to model liquidity and characterize the impact of large trades on asset prices.

1.3.1 Price impact

Modeling price impact has become a controversial topic as many theoretical studies, backed with empirical data, flourished over the last years, yielding different, and sometimes contradictory, results. The most simple model for price impact is the linear model: a net demand/supply for an asset impacts the asset return in a linear way. This relationship is characterized by the market depth of the asset, which represents the number of assets one has to buy in order to make the asset price increase by one percent. Such linear relationship between excess demand/supply and asset returns is widely used in the theoretical literature and is documented by empirical results at daily (Obizhaeva, 2011) and intraday (Cont et al., 2010) frequencies. Other studies model price impact as logarithmic Bouchaud and Potters (2003), a square root (Zhang, 1999) or stochastic (Cetin et al., 2004). More complex models can be used in order to model market liquidity. Kyle (1985) introduces three parameters representing the tightness, depth and resilience of the market, to describe market liquidity. Kempf and Korn (1999) shows that the price impact function is concave and not symmetric: a net supply will have a larger impact than a net demand for the same volume of assets. Almgren and Chriss (2000) add a temporary impact to the permanent price impact function. The price impact model is then used to determine optimal liquidation strategies (Almgren and Chriss, 2000; Almgren and Lorenz, 2006; Almgren, 2009; Alfonsi et al., 2009), price options in illiquid markets (Cetin et al., 2006) or determine the impact of large institutional investors on asset prices (Pritsker, 2005; Allen et al., 2006).

Modeling price impact is not the purpose of this thesis. We take the price impact function as given and study the impact of feedback effects, from investors to asset prices. In order to study feedback effects from rule-based strategies and fire sales in particular, one has to characterize the supply and demand that they generate and propose a model for asset prices which takes this supply and demand pattern into account, in relation with liquidity. The existing literature can be divided into two types of models:

- equilibrium models: the supply and demand of each type of traders is specified ad hoc and, at each period, a (general or partial) equilibrium verifying a market clearing condition is attained (e.g. Frey and Stremme (1994, 1997), Platen and Schweizer (1998)).
- 'dynamic' models: at each time step, the movement of asset prices is decomposed into two components: a first 'exogenous' component which is modeled as independent from past moves in prices and classically modeled as a stochastic noise term, and a second, 'endogenous', component resulting from the price impact of the supply/demand generated by institutional investors who systematically rebalance their portfolios in reaction to price changes (e.g. Avellaneda and Lipkin (2003), Jeannin et al. (2007)).

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Both approaches have been proposed for modeling feedback effects from dynamic hedging and fire sales.

1.3.2 Modeling feedback effects from dynamic hedging

The impact of options hedging, portfolio insurance and other dynamic hedging strategies

A typical example where dynamic hedgers have to buy (resp. sell) stocks when the stock price has increased (resp. decreased) is the portfolio insurance strategy, which accelerated the market crash of 1987. Grossman (1988) focuses on feedback effects generated by portfolio insurance strategies, which can imply replicating a put option using index futures. Using an equilibrium model, Gennotte and Leland (1990) show that when hedgers follow strategies that generate an upward sloping demand, such as portfolio insurance, they tend to amplify small market shocks and increase market volatility. They allow for asymmetric information and show that, the better market participants understand that hedgers act in a systematic way and that their action do not carry information, the weaker the impact of dynamic hedging on asset prices. Platen and Schweizer (1994) develop a framework for studying the impact of portfolio insurance on asset volatility and smile and skewness. Frey and Stremme (1994, 1997) propose a temporary equilibrium model for quantifying the impact of dynamic hedging on volatility. They consider a multi-period market with one risky asset and two types of traders: 'reference traders', who are considered as the 'normal' traders, and 'program traders', who are running dynamic hedging strategies, such as delta hedging or portfolio insurance, in a systematic way. They characterize the supply and demand generated by each type of traders and show that at each period, there exists an equilibrium price verifying the market clearing equation. By studying the continuous-time limit of the discrete-time price dynamics, they are able to compute the asset volatility in the presence of feedback effects from program traders and quantify the increase of volatility generated by certain types of dynamic hedging. Platen and Schweizer (1998) decompose the aggregate demand for a stock into the sum of three terms: a noise term, a term originated by arbitrage-based agents and a term originated by dynamic hedging of options written on the stock. By characterizing the demand from options hedgers, they show the existence of a price equilibrium satisfying the market clearing condition and explain the smile and skewness effects for the asset's implied volatility, which are observed empirically. Their approach gives an endogenous explanation for asset volatility smile and skewness, contrary to the stochastic volatility approach (Heston, 1993; Renault and Touzi, 1996; Hobson and Rogers, 1998; Fouque et al., 2004; Christoffersen et al., 2009) or local volatility approach (Cox and Ross, 1976; Rubinstein, 1983; Dupire, 1996), which give an exogenous dynamics for asset volatility and generate, ad hoc, a smile and skewness structure for asset volatility. Schonbucher and Wilmott (2000) also propose a general equilibrium model in order to quantify the impact of replicating options by dynamic trading strategies and explain empirical observations such as volatility surges in the presence of feedback effects from delta-hedgers.

More recent studies proposed dynamic approaches in order to study feedback effects

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from dynamic hedgers. Avellaneda and Lipkin (2003) model the continuous-time dynamics of an asset as a Black Scholes dynamics with a perturbation due to dynamic hedgers. They consider the particular case where those dynamic hedgers are market makers who are long options and delta-hedge their positions. Using the first order approximation of the supply and demand generated by those hedgers, they show that the presence of delta hedgers who are long options generate stock-pinning: the probability that the stock price ends exactly at the strike price at maturity is strictly positive. As seen in Section 1.2, stock pinning has been documented in the empirical literature and results from the fact that short-option delta-hedgers have to sell (resp. buy) the stock if it goes above (resp. below) the option's strike price. Jeannin et al. (2007) give a complete representation of the impact of short options delta hedgers on price dynamics by taking into account the exact supply and demand that they generate and quantify the decrease of volatility that they generate.

Lions and Lasry (2006, 2007) develop a general framework for quantifying the impact of dynamic hedgers on asset volatility and asset prices. They show that, given a utility function and a price impact function - which is linear -for hedgers, there exists an optimal hedging strategy and they compute the impact of this strategy on asset dynamics. Surprisingly, they find that feedback effects in this framework result in a decrease of volatility. This result is counter-intuitive because some hedging strategies imply buying (resp. selling) when the price goes up (resp. down) and should yield higher volatility. It is due to the fact that Lasry and Lions only focus on feedback effects from hedgers following optimal strategies and hence not from all types of hedgers.

1.3.3 Fire sales and their impact on prices

The economic literature underlines the link between fire sales and market instability. Shleifer and Vishny (2011) describe the fire sale mechanism: when assets held by a financial institution experience a significant decrease, leading to losses for the institution, this institution is compelled, for various reasons that we describe in the sequel, to sell part of its positions. This forced sale is done at a discounted price to non specialist buyers as potential high valuation buyers are affected by the same shocks as the financial institution and may also be forced to sell. The authors underline the fact that in the presence of fire sales, losses by financial institutions with overlapping holdings become self-reinforcing, leading to downward spirals for asset prices and, ultimately, to systemic risk.

Fire sales can result from rule-based strategies, such as maintaining a fixed value at risk or a constant leverage for a portfolio of risky assets. Danielsson and Zigrand (2001); Danielsson et al. (2004) propose respectively a one-period and multi-period model of general equilibrium, where traders have value at risk constraints. They show that a decline in asset prices triggers a surge in perceived risk for the financial institution and hence a fire sale of the asset so as to maintain a constant value at risk, amplifying market shocks. Shin (2010) uses a one-period equilibrium model, where the equilibrium price of a risky asset results from the confrontation of noise traders and a large financial institution maintaining a constant leverage. He shows that the leverage of a financial institution

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increases due to losses in its portfolio and leads to fire sales so that the financial company can respect the leverage and capital constraint. As in the previous case, maintaining a portfolio with a constant leverage generates an upward sloping demand, which can lead to spirals of losses and endogenous risk.

Fire sales can also result from investors running for the exit. Investors holding risky assets can decide to sell those assets if they decrease below a threshold and hence stop their losses. Similarly, investors in mutual funds can redeem their positions when the mutual fund underperforms, the outflow of capital generating fire sales by the mutual fund, as documented by Coval and Stafford (2007). Pedersen (2009) gives a qualitative description of the impact of investors running for the exit on asset prices. He exhibits the mechanism through which they can generate spirals in prices and spillovers to other asset classes as well as a crowding effect. He illustrates such crowding effect with the example of the hedge fund losses of August 2007, that we presented in Section 1.1, and that he explains qualitatively by a self reinforcing deleveraging of quantitatively built market-neutral portfolios.

Fire sales are most commonly generated by creditors of distressed funds (Shleifer and Vishny, 2011), who can sell the assets of the distressed fund that they hold as collateral, precisely at the moment when those assets have lost value, generating fire sales and accelerating the decline of the fund's value. Hart and Moore (1994, 1995, 1998) model collateralized debt and assume that the lender has a the right to liquidate the collateral posted by the borrower in the case he defaults. They take the liquidation value of the collateral as exogenous. Shleifer and Vishny (1992) propose a model where this liquidation value is determined endogenously, by the confrontation of the demand by two types of potential investors: specialist buyers, who are bound to buy the collateral at its fair value and non-specialist buyer, who agree to buy the collateral but at a discounted price, hence triggering fire sales. The authors show that an exogenous shock which leads to the default of a company is bound to impact all other companies of the same sector and hence all other potential specialist buyers of collateral, resulting ultimately in fire sales and leading to a large negative impact on the value of collateral.

Brunnermeier and Pedersen (2005) studies a market with one risky asset (that can represent the value of a strategy on multiple assets) and models the impact on the asset price of a fund liquidating its positions as linear. The cause of the liquidation is not specified - it can be either of the causes described previously - and it leads, in the presence of other informed traders maximizing a mean-variance criteria, to short selling by those traders: the optimal strategy when a fund is liquidating its positions is to short sell the same positions and buy them back at the end of the liquidation. The supply and demand pattern generated by such predatory trading cannot be distinguished from that generated by the liquidation and amplifies the impact of fire sales on asset prices.

All the studies mentioned previously and most of the literature on feedback effects focus on a single asset. They model and quantify the impact of delta hedging or portfolio insurance or other types of dynamic hedging on the underlying asset's expected return and volatility (Section 1.3.2) or the mechanism leading to fire sales and distressed selling

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and short selling and their impact on the asset's performance (Section 1.3.3). However, as suggested by Section 1.1, one key aspect of feedback effects in financial markets, which led to numerous financial crisis, is their impact on the dependence structure of assets. The examples developed in Section 1.1 indeed strongly suggest that, due to the action of economic actors (who are selling in a period of distress, liquidating a portfolio or following rule-based strategies), losses in one asset class could spillover to other asset classes and generate spirals of losses and a surge in correlation between those asset classes. The empirical studies reviewed in the last part of Section 1.2 document the impact of feedback effects on the dependence structure of asset returns.

1.3.4 The dependence structure of assets

The theoretical literature on the impact of feedback effects on the dependence structure of asset returns is scarce. Brunnermeier (2008) explains the economic mechanisms through which losses in subprime mortgage-backed securities led to huge falls in equity markets, although the two asset classes used to be uncorrelated: investors suffering losses in subprime mortgage-backed securities deleveraged their portfolios by selling their positions in equity markets, leading to the fall of equity markets. Andrade et al. (2008) proposes a multi-asset equilibrium model in which liquidity providers hedge non-informational trading imbalances in one stock by buying or selling correlated stocks. Their model quantifies the price-mediated contagion from assets presenting trading imbalances to other assets and show that non-informational trading increases the volatility of stock returns. Greenwood and Thesmar (2011) propose a simple framework for modeling price dynamics which takes into account the ownership structure of financial assets, considered as given exogenously. They show that, in the presence of a concentrated ownership structure, assets are more likely to deviate from fundamentals and the correlation between assets is modified because liquidity shocks force investors to buy or sell at the same time, generating non fundamental trading.

Kyle and Xiong (2001) provides a framework which allows to quantify feedback effects in a market with two assets, where traders are divided into three categories: noise traders who trade randomly, long term investors whose supply and demand is characterized by a linear rule based on fundamentals and provide liquidity and convergence traders who trade optimally using a logarithmic utility function. The price in each market is the result of an equilibrium between the three types of traders and is determined endogenously by the aggregate action of traders. The price dynamics obtained shows the existence of contagion effects, when convergence traders suffer losses in one market and leverage their positions in both markets, spreading the loss from one market to another. Gromb and Vayanos (2002) study a market with two identical assets where arbitrageurs exploit the discrepancies between the two asset prices and show that the equilibrium attained is not socially optimal, in the sense that a change in the positions of the arbitrageurs can benefit to all other investors. In order to measure the connectedness between hedge funds, banks, broker/dealers and insurance companies, Billio et al. (2012) propose statistical tools based on the principal component analysis of the correlation matrix of returns of those four categories and shows that the four sectors have

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become highly interconnected, the banking sector playing a central role in the transmission of shocks between those four sectors. One of the main limits of such tools built on the study of returns is that studying returns does not take into account the size of each of the four sectors whereas, historically, the size of financial institutions played a central role in the propagation of shocks, as shown by the examples of Section 1.1.

Most of the studies described in this section either yield qualitative results or non tractable results. The few tractable studies that exist, such as Kyle and Xiong (2001), focus on a market with two assets. A quantitative framework is needed in order to account for the impact of feedback effects on the dependence structure of asset returns, in a tractable way. In particular, in such a framework, the realized correlation between asset returns in the presence of feedback effects, which is a key ingredient of the dependence structure of asset returns, should be tractable. Whereas the correlation structure of asset returns is often considered as constant (Markowitz, 1952, 1959; Pogue, 1970; Merton, 1972; Elton and Gruber, 1997) or varying stochastically (Engle and Kroner, 1995; Engle, 2002; Gouriéroux, 2006; Gouriéroux et al., 2009; Fonseca et al., 2007; Stelzer, 2010) with parameters that are specified exogenously, the evidence of feedback effects in financial markets and their impact on the dependence structure of asset returns call for modeling correlation in an endogenous way, in relation to the systematic supply and demand generated by large investors.

1.4 Summary of contributions

This thesis provides a quantitative analysis of feedback effects and endogenous risk in financial markets in a *multi-asset setting*. The empirical facts described in Section 1.1 strongly suggest that the flows of investments generated by large financial institutions have an endogenous impact on asset prices and the dependence structure of asset returns and can lead endogenously to huge losses for investors and systemic risk. Consequently, the main idea of this thesis is to consider that the returns of financial assets can be originated not only by:

- movements in fundamentals which are due to exogenous economic factors and reflect a fundamental structure between assets; but also by:
- the systematic supply and demand generated by large financial institutions which trade in reaction to fundamental price movements. In this thesis, we model the supply and demand generated by fire sales and short selling in distressed funds (Chapter 2 and 3) and by the rebalancing of a large institutional investor keeping a fixed allocation in each asset (Chapter 4).

As done in Föllmer and Schweizer (1993) for a market with one risky asset, we propose an intuitive discrete-time model which takes into account the price impact of large financial institutions in a multi-asset setting and we study the continuous-time limit of such multi-period model, which gives tractable results quantifying the impact of feedback effects on realized volatilities, correlations and expected returns.

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We formalize that idea as follows. We consider a financial market where n assets are traded at discrete dates $t_k = k\tau$, multiples of a time step τ (taken to be a trading day in the empirical examples). The value of asset i at date t_k is denoted S_k^i . It is useful, in the examples, to think of S^i as the value of an index or ETF representing a sector, asset class, or geographic zone or more generally the value of a strategy (long-short strategy for example). The impact of exogenous economic factors ('fundamentals') on prices is modeled through an IID sequence $(\xi_k)_{k \geq 1}$ of \mathbb{R}^n -valued centered random variables, such that in the absence of other effects than 'fundamentals', the return of asset i during period k is given by

$$\exp\left(\tau\left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\tau}\xi_{k+1}^i\right) - 1 \quad (1.1)$$

Here m_i represents the ('fundamental') expected return of asset i and the 'fundamental' covariance matrix Σ , defined by

$$\Sigma_{i,j} = \text{cov}(\xi_k^i, \xi_k^j)$$

represents the covariance structure of returns, in the absence of large systematic trades by institutional investors. Note that when ξ_k is normal: $\xi_k \sim \mathcal{N}(0, \Sigma)$, (1.1) is equal to the return of an asset in a discretized multi-variate Black-Scholes model.

The fundamental price movements described in (1.1) can trigger systematic trading by large financial institutions, which impacts prices further. Between t_k and t_{k+1} , the asset price dynamics can hence be summed up as follows:



We studied different origins for such systematic trading and allowed for general forms of price impact on asset returns. Our results allow to quantify the endogenous impact of large financial institutions on the dependence structure of asset returns, expected asset returns and the endogenous risk they generate: surge in fund volatility and spillover effects. We also develop econometric tools which allow to estimate the parameters of our model from time series of asset prices and investigate endogenous risk in a systematic way.

1.4.1 Chapter 2: Feedback effects from fire sales

Running for the exit: distressed selling and endogenous correlation in financial markets

Chapter 2 studies feedback effects generated by distressed selling and short selling in a distressed fund. We consider a large fund holding α_i units of asset i with $1 \leq i \leq n$. Thus, between t_k and t_{k+1} , exogenous economic factors described in (1.1) move the value

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of the fund from $V_k = \sum_{i=1}^n \alpha_i S_k^i$ to

$$V_{k+1}^* = \sum_{i=1}^n \alpha_i S_k^i \exp \left(\tau \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\tau} \xi_{k+1}^i \right)$$

Investors enter the fund at $t = 0$ when the fund is valued at $V_0 > 0$. Like most investors in mutual funds, investors in the fund adopt a passive, buy and hold behavior as long as the fund is performing well. If the fund value drops below a threshold $\beta_0 V_0 < V_0$, investors progressively may exit their positions, generating a negative demand across all assets held by the fund, proportionally to the positions held by the fund. Furthermore, short sellers can short the positions of the fund if it underperforms, as part of a predatory trading strategy (Brunnermeier and Pedersen, 2005). Short sellers generate the same supply and demand as distressed sellers and we will not make the difference between short selling and distressed selling in the sequel.

We model the supply/demand pattern generated by distressed selling by introducing a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which measures the rate at which investors in the fund exit their positions: when fund value drops from V_k to V_{k+1}^* , investors redeem a fraction $f(\frac{V_k}{V_0}) - f(\frac{V_{k+1}^*}{V_0})$ of their position in the fund. Thus, the net supply in asset i due to distressed selling (or short selling) is equal to

$$-\alpha_i \left(f\left(\frac{V_{k+1}^*}{V_0}\right) - f\left(\frac{V_k}{V_0}\right) \right)$$

The above assumptions on investor behavior imply that $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, constant on $[\beta_0, +\infty[$.

We furthermore assume that the fund is liquidated when the value reaches $\beta_{liq} V_0$ where $\beta_{liq} < \beta_0$ and distressed selling ceases. In practice, as the fund loses value and approaches liquidation, distressed selling becomes more intense: this feature is captured by choosing f to be *concave*. Figure 1.3 gives an example of such a function f : in this example, we see that the drop of fund value from V_k to V_{k+1}^* generates distressed selling of almost 40% of the fund's positions. Note that as long as the fund's value is above $\beta_0 V_0$, there is no distressed selling, as f is constant on $[\beta_0, +\infty[$.

As documented by empirical studies (Obizhaeva, 2011; Cont et al., 2010), we assume that the impact of the net supply generated by short sellers and distressed sellers on asset i 's return is linear and equal to

$$\frac{\alpha_i}{D_i} \left(f\left(\frac{V_{k+1}^*}{V_0}\right) - f\left(\frac{V_k}{V_0}\right) \right)$$

where D_i represents the *depth* of the market in asset i : a net demand of $\frac{D_i}{100}$ shares for security i moves the price of i by one percent.

The price dynamics can hence be summed as follows:

$$S_{k+1}^i = S_k^i \exp \left(\tau \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\tau} \xi_{k+1}^i \right) \left(1 + \frac{\alpha_i}{D_i} \left(f\left(\frac{V_{k+1}^*}{V_0}\right) - f\left(\frac{V_k}{V_0}\right) \right) \right)$$

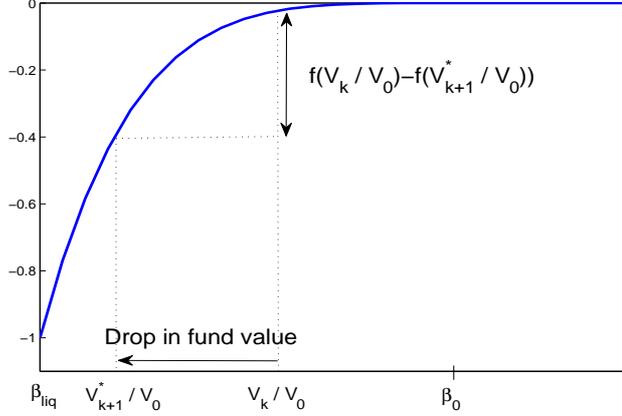


Figure 1.3: Net supply due to distressed selling and short selling is equal to $-\alpha_i(f(\frac{V_{k+1}^*}{V_0}) - f(\frac{V_k}{V_0}))$

where $V_k = \sum_{j=1}^n \alpha_j S_k^j$ and $V_{k+1}^* = \sum_{j=1}^n \alpha_j S_k^j \exp\left(\tau\left(m_j - \frac{\Sigma_{j,j}}{2}\right) + \sqrt{\tau}\xi_{k+1}^j\right)$ and we exhibit conditions under which S is a Markov chain in $(\mathbb{R}_+^*)^n$ (Proposition 2.2.1).

Simulations of the discrete-time model show that distressed selling by investors exiting the fund can generate significant realized correlation, even between assets with zero fundamental correlation, resulting in higher fund volatility. Realized correlation conditional on the fact that distressed selling took place is significantly higher than realized correlation in scenarios where there was no distressed selling, thus reducing the benefit of diversification for the fund, precisely in scenarios where such effects are needed.

We exhibit conditions (Assumption 2.4.1) under which the discrete-time price dynamics converges weakly to a diffusion limit and give the continuous-time limit of our discrete-time framework (Theorem 2.4.2). The study of this diffusion limit allows us to obtain analytical expressions for the realized covariance matrix of returns (Proposition 2.5.1): the realized covariance matrix between 0 and t is equal to $\frac{1}{t} \int_0^t c_s ds$ where the instantaneous covariance matrix c_s is given by

$$c_s = \Sigma + \frac{1}{V_0} f'(\frac{V_s}{V_0}) [\Lambda^t \pi_s \Sigma + \Sigma \pi_s^t \Lambda] + \frac{1}{V_0^2} (f')^2(\frac{V_s}{V_0}) (\Sigma \pi_s \cdot \pi_s) \Lambda^t \Lambda \quad \text{where}$$

- $\pi_t = \begin{pmatrix} \alpha_1 P_t^1 \\ \vdots \\ \alpha_n P_t^n \end{pmatrix}$ denotes the (dollar) holdings of the fund

- $\Lambda = \begin{pmatrix} \frac{\alpha_1}{D_1} \\ \vdots \\ \frac{\alpha_n}{D_n} \end{pmatrix}$ represents the positions of the fund in each market as a fraction of

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the respective market depth.

This result shows that realized covariance may be decomposed as the sum of a fundamental covariance and an 'excess' covariance which is liquidity-dependent and path-dependent. The excess covariance is exacerbated by illiquidity and is equal to zero when there is no distressed selling ($f' = 0$).

We compute the distressed fund's variance in the presence of fire sales (Proposition 2.6.1) and show that it is a sum of two regimes: a fundamental regime and an excess volatility regime, that is exacerbated with illiquidity (market depth D is small) or when the fund has large positions (α is large). Our results point to the limits of diversification, previously discussed by many authors, but also allow one to quantify these limits. We show that a fund manager investing in apparently uncorrelated strategies may experience significant realized correlation across his/her strategies in the case of distressed selling by investors facing losses, thus losing the benefit of diversification exactly when it is needed.

Finally, we study the spillover effect of a fund subject to distressed selling on other funds' variance (Proposition 2.6.3). We show that, in the presence of distressed selling in a reference fund, the variance of another fund with small positions μ_t^i on each asset i can also be decomposed into the sum of a fundamental variance and an additional variance. This additional variance is exacerbated when μ is collinear to the positions being liquidated α . On the contrary, if the allocations of the two funds verify the 'orthogonality' condition

$$\sum_{1 \leq i \leq n} \frac{\alpha_i}{D_i} \mu_t^i P_t^i = 0$$

distressed selling of investors in the reference fund does not affect the small fund's variance. This orthogonality condition allows for a quantitative explanation of the hedge fund losses of August 2007, when a large market-neutral fund deleveraged its portfolio, generating exacerbated volatility in other funds with similar (ie collinear) positions, while leaving index funds, which verified the orthogonality condition given above, unaffected.

1.4.2 Chapter 3: Inverse problem

Fire sales forensics: measuring endogenous risk

Chapter 3 develops econometric tools which allow for a quantitative investigation of fire sales from several distressed funds and the empirical reconstitution, a posteriori and using time series of asset prices, of the aggregate flow of investments during a period of distress.

It extends the framework used in Chapter 2 to a multi fund setting: consider J large funds, each fund j holding α_i^j units of asset i . As in Chapter 2, fundamental moves in asset values, described in (1.1), can generate fire sales in each fund, which are captured by a function f_j for each fund j . Such systematic net supply and demand generated by fire sales impact asset returns: whereas in the previous chapter, we assumed that price impact was linear, we allow here for a general price impact function: the impact on asset

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i 's return of a net demand of size v for asset i is equal to $\phi_i(v)$. As a consequence, the price dynamics is given by:

$$S_{k+1}^i = S_k^i \left(1 + \tau m_i + \sqrt{\tau} \xi_{k+1}^i + \phi_i \left(\sum_{1 \leq j \leq J} \alpha_i^j \left(f_j \left(\frac{(V_{k+1}^j)^*}{V_0^j} \right) - f_j \left(\frac{V_k^j}{V_0^j} \right) \right) \right) \right)$$

where $V_k^j = \sum_{1 \leq i \leq n} \alpha_i^j S_k^i$ and $(V_{k+1}^j)^* = \sum_{1 \leq i \leq n} \alpha_i^j S_k^i (1 + \tau m_i + \sqrt{\tau} \xi_{k+1}^i)$.

We exhibit conditions on the price impact functions $(\phi_i)_{1 \leq i \leq n}$, the deleveraging schedules $(f_j)_{1 \leq j \leq J}$ and the fundamental movements ξ (Assumption 3.2.3) such that the discrete-time price dynamics converges weakly, as the time step goes to zero and we give the expression of the diffusion limit (Theorem 3.2.4).

The limit diffusion process depends on the price impact functions only through their first and second derivatives in 0: $(\phi_i'(0))_{1 \leq i \leq n}$ and $(\phi_i''(0))_{1 \leq i \leq n}$. In particular, the local multi-dimensional volatility function depends only on $(\phi_i'(0))_{1 \leq i \leq n}$. In particular, a linear price impact model, with markets depths $D_i = \frac{1}{\phi_i'(0)}$ perfectly captures the impact of fire sales on the dependence structure of asset returns.

Corollary 3.2.6 gives the expression of the continuous-time limit in the case of linear price impact functions and shows that the magnitude of the impact of fire sales on price dynamics is captured by a $n \times n$ matrix Λ such that

$$\Lambda_{i,j} = \frac{\alpha_i^j}{D_i}$$

represents the size of fund j in asset i as a fraction of asset i 's market depth: $\Lambda_{i,j}$ is the liquidation impact of fund j 's position on asset i .

Our results allow to compute the realized covariance matrix in the presence of fire sales in multiple funds (Proposition 3.2.7). In particular, under the assumption that there are no fire sales between 0 and T and each fund j liquidates between T and $T + \tau_{liq}$ at a constant rate γ_j , we find (Corollary ??) that the realized covariance matrices between 0 and T and between T and $T + \tau_{liq}$ are respectively equal to:

$$C_{[0,T]} = \frac{1}{T} \int_0^T c_t dt = \Sigma$$

and

$$C_{[T, T+\tau_{liq}]} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t dt = \Sigma + LM_0\Pi\Sigma + \Sigma\Pi M_0L + O(\|\Lambda\|^2)$$

with

$$M_0 = \sum_{1 \leq j \leq J} \frac{\gamma_j}{V_0^j} \times \alpha^j (\alpha^j)^t$$

where $\alpha^j = \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_n^j \end{pmatrix}$ is the vector of positions of fund j and L, Π are diagonal matrices

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with i -th diagonal term equal respectively to $\frac{1}{D_i}$ and $\frac{1}{\tau_{iq}} \int_T^{T+\tau_{iq}} P_t^i dt$ and $\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2}$ is bounded almost surely when $\|\Lambda\| \rightarrow 0$.

We then consider the inverse problem of identifying the matrix M such that

$$C_{[T, T+\tau_{iq}]} = \Sigma + LM\Pi\Sigma + \Sigma\Pi ML$$

and hence investigate abnormal patterns of realized covariances.

We give conditions for the identifiability of M (Proposition 3.3.1) and show that the knowledge of M allows to estimate, up to an error term of order one in $\|\Lambda\|$, the volume of fire sales in asset class i between T and $T + \tau_{iq}$ (Corollary 3.3.2). We build an estimator of M (Section ??). We show that this estimator is consistent (Proposition 3.3.4) and we derive a central limit theorem (Proposition 3.3.5) which allows us to build a statistical test to determine whether the assumption of no fire sales during a period $[T, T + \tau_{iq}]$ can be rejected or not (Corollary 3.3.7). This econometric study allows us to estimate, thanks to the observation of price series, the magnitude of fire sales in each asset during a given period. We illustrate our estimation methodology with two empirical examples, described qualitatively in Section 1.1.

First, we study the Great Deleveraging of Fall 2008 (Section 3.4). We find that the aggregate liquidated portfolio was a long portfolio which was made essentially of financial stocks and energy stocks. In particular, for the Eurostoxx 50, the main European equity index, two financial stocks, Deutsche Bank and ING, account for more than a half of the fire sales during the aftermath of the collapse of Lehman Brothers. Our second example focuses on the hedge fund losses of August 2007 (Section 3.5). Our estimation procedure allows to reconstitute the long short structure of the positions liquidated during the second week of August 2007. We find that the aggregate portfolio liquidated is market-neutral: it is orthogonal to index funds, in the sense of the orthogonality condition given in Equation ?? and hence did not affect index funds, as predicted quantitatively by our model.

Our framework allows to explain large shifts in the realized covariance structure of asset returns in terms of supply and demand patterns across asset classes, which makes such events easier to analyze and understand. This estimation procedure may be useful for regulators in view of investigating unusual market events in a systematic way, moving a step in the direction proposed by Fielding et al. (2011), who underlined the importance of systematically investigating all 'systemic risk' events in financial markets, as done by the National Transportation Safety Board for major civil transportation accidents.

1.4.3 Chapter 4: Institutional investors and return correlations

Impact of large institutional investors on the dependence structure of asset returns

Chapter 4 studies the impact of a large institutional investor/fund following a fixed-mix strategy, ie: keeping a fixed allocation x_i in each asset i . Fixed-mix strategies are widely used by financial institutions as numerous studies Markowitz (1952); Evstigneev and Schenk-Hoppé (2002); Dempster et al. (2003); Mulvey and Kim (2008) show that this strategy can enhance the long-term growth rate of portfolios. In our model, at each period, the large

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fund has to rebalance its positions after exogenous price moves (1.1). Typically, when the value of an asset increased more than the other assets, its weight on the fund's portfolio increases and the fund has to sell some of its positions in this asset in order to maintain its target allocation. The fixed-mix strategy is an example of contrarian strategy, which implies 'buying low and selling high'.

We show that there exists a unique self-financing strategy which allows the large fund to follow a fixed-mix strategy (Proposition 4.2.1). At each date t_k , asset prices and fund value are given as the result of a fixed-point problem. Simulations of the multi-period model using realistic parameters, estimated from time series of the S&P500, show that the rebalancing by the fund at each period generates a significant increase in realized correlation between asset returns and modifies the principal component properties of the realized correlation matrix of returns. In particular, we find that starting with homogenous fundamental correlation between asset returns, equal to 15%, and a realistic size for the large institutional investor, the feedback mechanism that we propose leads to an average pairwise correlation of 21%, as observed for the S&P500 in 2006 (Section 4.2.2).

In order to confirm the phenomena observed in our numerical experiments, we exhibit conditions (Assumption 4.2.2) under which the multi-period model converges weakly to a diffusion limit and describe the diffusion limit (Theorem 4.2.3). As in the previous chapters, the expressions for the drift and the volatility of the diffusion limit show that the impact of the large fund is measured by a vector Λ such that

$$\Lambda_i = \frac{\phi_0^i}{D_i}$$

which represents the size of the fund's initial position as a fraction of asset market depth.

We give the expansion at order one in $\|\Lambda\|$ of the realized covariance matrix (Corollary 4.3.2) and the realized correlation matrix (Equation 4.18) in the presence of feedback effects from the large institutional investor. Starting with homoscedastic inputs, the economic mechanism of fund rebalancing naturally generates heteroscedasticity.

The realized correlation between assets i and j on $[0, T]$ is equal to:

$$\begin{aligned} R_{[0,T]}^{i,j} &= \frac{\Sigma_{i,j}}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} + \frac{\Lambda_i}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^i \right) \left(\sum_{1 \leq l \leq n} x_l \left(\Sigma_{j,l} - \frac{\Sigma_{i,j}\Sigma_{i,l}}{\Sigma_{i,i}} \right) \right) \\ &+ \frac{\Lambda_j}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^j \right) \left(\sum_{1 \leq l \leq n} x_l \left(\Sigma_{i,l} - \frac{\Sigma_{i,j}\Sigma_{j,l}}{\Sigma_{j,j}} \right) \right) + O(\|\Lambda\|^2) \end{aligned}$$

where $\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2}$ is bounded when Λ goes to zero. It is the sum of the fundamental correlation and an additional correlation which is liquidity-dependent and path-dependent. This formula exhibits the dependence of realized correlations to liquidity (Λ), the large fund's allocations ($(x_i)_{1 \leq i \leq n}$) and the volume of rebalancing that it generates for each asset ($d\Phi_t^i$ at date t for asset i). Our results show that when the fund invests in assets

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with zero fundamental correlation, feedback effects from the fixed-mix strategy decrease asset realized volatilities and generate positive realized correlation between asset returns, leading to a correlation matrix of returns which has the same features as those observed empirically: a first eigenvalue which is larger than the other eigenvalues and of the order of n , the number of assets in the market (Equation 4.20), and which is associated to an eigenvector with positive weights.

We give tractable formulas for the eigenvalues and the eigenvectors of the realized correlation matrix of returns (Proposition 4.4.1), which depend on the sizes and allocations of the large fund. We compute the formulas that we find in a simple example of homogenous fundamental correlation between assets and homogenous fundamental asset volatilities and show that the assets which generate large rebalancing volumes by the fund have large weights in the first principal component of the realized correlation matrix of returns.

Finally, we calculate asset expected returns (Proposition 4.5.1) and find that the presence of the large institutional investor increases (resp. decrease) the expected returns of assets i with fundamental expected m_i return lower (resp. large) than the benchmark return of the fund $\sum_{1 \leq j \leq n} x_j m_j$. Due to feedback effects, investors who use the large fund as a benchmark and overweigh (resp. underweigh) assets with large (resp. low) fundamental expected returns will experience lower-than-expected returns. We illustrate in a simple example how feedback effects modify optimal strategies associated to a mean-variance criteria (Proposition 4.5.3) and find that an investor who takes the presence of the fixed-mix fund into account can improve his risk/return trade-off (Figure 4.8).

Chapter 2

Running for the exit: distressed selling and endogenous correlation in financial markets

Abstract

We propose a simple multiperiod model of price impact from trading in a market with multiple assets, which illustrates how feedback effects due to distressed selling and short selling lead to endogenous correlations between asset classes. We show that distressed selling by investors exiting a fund and short selling of the fund's positions by traders may have non-negligible impact on the realized correlations between returns of assets held by the fund. These feedback effects may lead to positive realized correlations between fundamentally uncorrelated assets, as well as an increase in correlations across all asset classes and in the fund's volatility which is exacerbated in scenarios in which the fund undergoes large losses. By studying the diffusion limit of our discrete time model, we obtain analytical expressions for the realized covariance and show that the realized covariance may be decomposed as the sum of a fundamental covariance and a liquidity-dependent and path-dependent 'excess' covariance. Finally, we examine the impact of these feedback effects on the volatility of other funds. Our results provide insight into the nature of spikes in correlation associated with the failure or liquidation of large funds.

2.1 Introduction

Correlations in asset returns are a crucial ingredient for quantifying the risk of financial portfolios and a key input for asset allocation and trading. Correlations and covariances between returns of assets, indices and funds are routinely estimated from historical data and used by market participants as inputs for trading, portfolio optimization and risk management. Whereas sophisticated models –featuring stochastic volatility, conditional heteroskedasticity and jumps– have been proposed for univariate price dynamics, the dependence structure of returns is typically assumed to be stationary, either through a time-invariant correlation matrix or a copula, and estimated from historical time series of returns. For example, a popular method is to use (exponentially-weighted) moving average (EWMA) estimators of realized correlation.

On the other hand, empirical evidence points to high variability in *realized correlations* and model-based estimators of correlation (Engle, 2009): these estimators exhibit large spikes or dips associated to market events. Figure 2.1 shows examples of variability in time of (empirical) realized correlations in equity indices; we observe a sharp increase in realized correlations associated with the collapse of Lehman Brothers on September 15th, 2008. More generally, unexpected correlation spikes are often associated with the liquidation of large positions by market participants. For instance, in 1998, due to heavy losses in its investments in Russian bonds, Long Term Capital Management was forced to liquidate its positions after a sudden increase in the correlations across its –previously uncorrelated– positions which led to a sharp increase in its volatility (Rosenfeld, 2010). Unexpected spikes of correlation arose between asset classes that used to be uncorrelated (Russian bonds and US equity for instance), leading to the collapse of the fund. A more complex phenomenon occurred in August 2007: between August 7 and August 9 2007, all long-short equity market neutral hedge funds lost around 20% per day whereas major equity indices hardly moved. Khandani and Lo (2011) suggest that this ‘quant event’ of August 2007 was due to the unwinding of a large long-short market neutral hedge fund’s positions, that created extreme volatility on other funds with similar portfolios, while leaving index funds unaffected. These examples illustrate that “asset correlations can be different during a liquidity crisis because price movements are caused by distressed selling and predatory trading rather than fundamental news” (Brunnermeier and Pedersen, 2005).

The evidence for time-variation in the dependence structure of asset returns has motivated the development of new classes of stochastic models with time-dependent correlation structures (Engle, 2002; Da Fonseca et al., 2008; Gouriéroux et al., 2009; Stelzer, 2010) in which the conditional distribution of asset returns is given by a multivariate distribution with a randomly evolving covariance structure whose evolution is specified exogenously. However, such models where correlation is represented as an exogenous risk factor fail to explain the presence of spikes in correlations associated with market events such as the liquidation of large funds. The examples cited above suggest the existence of an *endogenous* component in asset correlations, which should be modeled by taking into account the impact of supply and demand generated by investors, in

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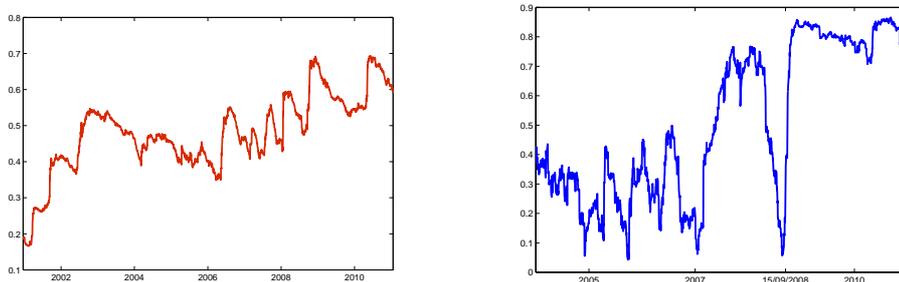


Figure 2.1: Left: One-year EWMA estimator of average pairwise correlations of daily returns in EuroStoxx 50 index. Right: One-year EWMA correlation between two sector indices of the S&P 500: SPDR XLE (energy) and SPDR XLK (technology).

particular in situations of market distress. Such endogenous variations in volatility and correlations, generated by systematic patterns in supply and demand linked to rule-based trading strategies, short selling or fire sales, have played an important role in past financial crises and have been the focus of several studies (Adrian and Shin, 2008, 2009; Brunnermeier, 2008; Brunnermeier and Pedersen, 2009; Carlson, 2006; Pedersen, 2009; Shin, 2010) which underline the link between liquidity and volatility in financial markets.

Our contribution is to show that the intuitive link between distressed selling and endogenous changes can be modeled quantitatively in a rather simple, analytically tractable framework which allows to quantify the endogenous risk generated by fire sales, when investors facing losses simultaneously try to exit a fund.

2.1.1 Summary

We consider a fund investing in various asset classes/ strategies whose returns are decomposed into random components that represent exogenous economic factors (fundamentals) and a term representing the price impact of sellers, which is a function of aggregate excess demand for each asset generated either by investors liquidating their positions or by speculators shorting the fund's position once the fund value drops below a threshold. Simulations of this discrete-time model reveal that, even in the case of assets with zero fundamental correlation, one observes a significant positive level of realized correlation resulting in higher than expected fund volatility. Furthermore, this realized correlation is observed to be path-dependent.

We confirm the generic nature of these simulation results by studying the continuous-time limit of our model. We exhibit conditions under which the discrete-time model exhibits a diffusion limit and provide explicit expressions for the realized covariance and correlation across assets and realized fund volatility for the limiting diffusion process. Our analytical results show that realized covariance is the sum of a fundamental covariance and an excess covariance term which is *path-dependent* and varies inversely with market liquidity. Furthermore, this excess covariance is *computable* in our model setting.

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Even in the absence of correlations between fundamentals, asset returns may exhibit significant positive realized correlation, resulting in higher realized fund volatility. We show that, even when market depth is constant, the liquidation of large fund positions can generate significant positive realized correlation between the fund's assets, and may also generate spillover effects, affecting the volatility of other funds holding similar assets. All these effects are shown to be analytical computable and expressions are given for their magnitude.

Our results point to the limits of diversification, previously discussed by many authors, but also allow one to quantify these limits. We show that a fund manager investing in apparently uncorrelated strategies may experience significant realized correlation across his/her strategies in the case of distressed selling by investors facing losses, thus losing the benefit of diversification exactly when it is needed. These results provide simple explanations for the sudden rise in correlations associated with the failure of LTCM in 1998 and the hedge fund losses of August 2007. Our study provides insight into the nature of spikes in correlation and fund volatility associated with the failure or liquidation of large funds and gives a quantitative framework to evaluate strategy crowding as a risk factor. In particular, the model explains how, in August 2007, the liquidation of a large long-short equity market neutral fund generated high volatility for funds with similar allocations while leaving index funds unaffected.

2.1.2 Related Literature

Empirical evidence of distressed selling and its impact on market dynamics has been documented by several previous studies. Funds experiencing large outflows sell their holdings, as documented by Coval and Stafford (2007). For regulatory reasons, after large losses, banks must sell risky assets, as discussed by Berndt et al. Berndt et al. (2005) for the corporate debt market. Khandani and Lo (2011) describe how the need to reduce risk exposure compelled market-neutral long-short equity hedge funds to liquidate large position in equity markets in the second week of August 2007, generating a series of huge losses which are explained quantitatively by our model. Comerton-Ford et al. (2010) show empirically the importance of short selling in financial markets (40.2% and 39.2% of total dollar volume on the NYSE and Nasdaq, respectively). Haruvy and Noussair (2006) examine empirically the effects of short selling restrictions finding that relaxing short selling constraints does not induce prices to track fundamentals. Our study provides a quantitative framework for analyzing these empirical observations.

Various theoretical models have been proposed for analyzing feedback effects resulting from fire sales in financial markets, mostly in a single-asset framework. Avellaneda and Lipkin (2009) show how short selling in a single asset generates price anomalies and higher price volatility and violation of Call-Put parity. Market losses in subprime mortgage-backed securities, largely seen as being uncorrelated with equity markets, led to huge falls in equity markets as explained by Brunnermeier (2008). Shin (2010) describes the mechanisms which amplified the recent financial crisis and the systemic risk they generate. Investors 'running for the exit' can generate spirals in prices and spillovers to other asset classes as well as a crowding effect, as discussed by Pedersen (2009). Andrade et al.

(2008) show how trading imbalances in one asset class can lead to deviation of prices from fundamental value in other asset classes. Short selling by predators is described in Brunnermeier and Pedersen (2005), where the authors show how shorting the portfolio of a fund approaching its liquidation value can lead to the collapse of the fund. Our detailed quantitative analysis confirms these predictions. Whereas these studies mainly focus on asset prices and fund value, our multi-asset framework allows for a computation of the impact of fund liquidation or short selling on realized correlation between assets and fund volatility.

2.1.3 Outline

The paper is organized as follows. Section 2.2 presents a multiperiod, multi-asset model of trading with price impact and introduces a simple model for distressed selling. Section 2.3 displays the results of the simulations of this model. In Section 2.4, we find the continuous-time limit of our discrete-time dynamics. Section 2.5 gives analytical expressions for the realized variance and covariance of asset returns in the continuous-time limit and uses these expressions to study the path-dependence of realized correlations and role of market depth. Using these analytical results, we show in Section 2.6 how feedback effects lead to endogenous volatility in a distressed fund and spillover effects across funds. Section 2.7 concludes.

2.2 A multi-asset model of price impact from distressed selling

Consider a market where n financial strategies/assets are traded at dates $t_k = k\tau$, where τ is the time step between two trading dates. The price of asset i at date t_k is denoted S_k^i and we denote $S_k = (S_k^1, \dots, S_k^n)$. It is useful, in the examples, to think of S^i as the value of an index or ETF representing a sector, asset class or geographic zone or more generally the value of a financial strategy (for example a long-short strategy).

At each period, the value of the assets is affected by exogenous economic factors, represented by an IID sequence $\xi_k = (\xi_k^1, \dots, \xi_k^n)_{1 \leq k \leq M}$ of centered random variables with covariance matrix Σ . In the absence of other effects, the return of asset i at period k would be

$$\exp\left(\tau\left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\tau}\xi_{k+1}^i\right) - 1$$

where m_i is the expected return of asset i in the absence of other effects than 'fundamentals'. We denote $(S_{k+1}^i)^* = S_k^i \exp\left(\tau\left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\tau}\xi_{k+1}^i\right)$.

We consider a large fund holding α_i units of asset i with $1 \leq i \leq n$. Thus, between t_k and t_{k+1} , exogenous economic factors move the (benchmark) value of the fund from

$$V_k = \sum_{i=1}^n \alpha_i S_k^i \text{ to}$$

$$V_{k+1}^* = \sum_{i=1}^n \alpha_i (S_{k+1}^i)^* = \sum_{i=1}^n \alpha_i S_k^i \exp\left(\tau\left(m_i - \frac{\Sigma_{i,i}}{2}\right) + \sqrt{\tau}\xi_{k+1}^i\right)$$

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Investors enter the fund at $t = 0$ when the fund is valued at $V_0 > 0$. Like most investors in mutual funds, investors in the fund adopt a passive, buy and hold behavior as long as the fund is performing well. If the fund value drops below a threshold $\beta_0 V_0 < V_0$, investors progressively may exit their positions, generating a negative demand across all assets held by the fund, proportionally to the positions held by the fund. Our purpose is to model the price impact of this *distressed selling* and investigate its effect on realized volatility and correlations of the assets held by the fund.

We model the supply/demand pattern generated by distressed selling by introducing a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which measures the rate at which investors in the fund exit their positions: when fund value drops from V_k to V_{k+1}^* , investors redeem a fraction $f(\frac{V_k}{V_0}) - f(\frac{V_{k+1}^*}{V_0})$ of their position in the fund. Thus, the net supply in asset i due to distressed selling (or short selling) is equal to

$$-\alpha_i(f(\frac{V_{k+1}^*}{V_0}) - f(\frac{V_k}{V_0}))$$

The above assumptions on investor behavior imply that $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing, constant on $[\beta_0, +\infty[$.

We furthermore assume that the fund is liquidated when the value reaches $\beta_{liq} V_0$ where $\beta_{liq} < \beta_0$. In practice, as the fund loses value and approaches liquidation, distressed selling becomes more intense: this feature is captured by choosing f to be *concave*. Figure 2.2 gives an example of such a function f .

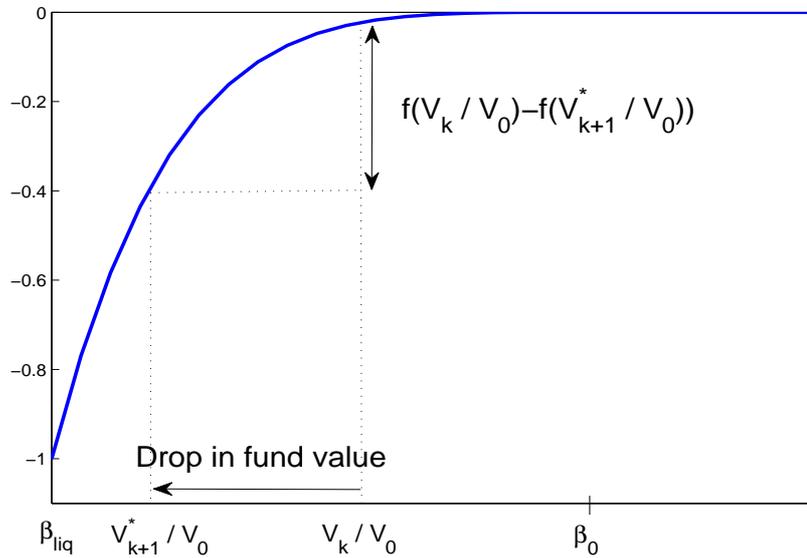


Figure 2.2: Net supply due to distressed selling and short selling is equal to $-\alpha_i(f(\frac{V_{k+1}^*}{V_0}) - f(\frac{V_k}{V_0}))$

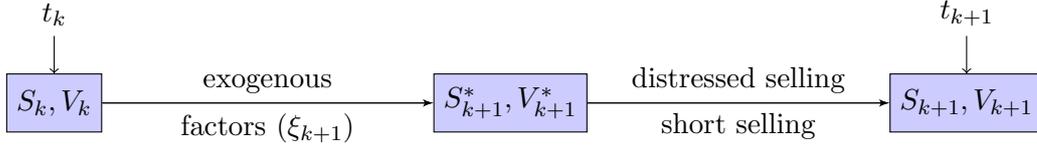
It is well documented that sale of large quantities of assets impacts prices. Empirical

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studies (Obizhaeva, 2011; Cont et al., 2010) provide evidence for approximate linearity of this price impact at daily and intraday frequencies. Between t_k and t_{k+1} , given the net supply generated by short sellers and distressed sellers, market impact on asset i 's return is equal to

$$\frac{\alpha_i}{D_i} \left(f\left(\frac{V_{k+1}^*}{V_0}\right) - f\left(\frac{V_k}{V_0}\right) \right)$$

where D_i represents the *depth* of the market in asset i : a net demand of $\frac{D_i}{100}$ shares for security i moves the price of i by one percent. Obizhaeva (2011) studies empirically the link between market depth and average daily volume (ADV) on NYSE and NASDAQ stocks, finding that $\frac{ADV_i \sqrt{250}}{D_i \sigma_i}$ is close to 1. We will use this relation to pick realistic values for the size of a large fund's positions α_i in terms of the market depth D_i , in the examples of Section 2.3. The supply/demand pattern generated by these distressed sellers exiting the fund may be amplified by short sellers or predatory traders: the presence of short sellers may result in scenarios where a fraction > 1 of the fund is exited/liquidated. From our perspective, their effect on price dynamics is similar and we will not distinguish between distressed (e.g. long) sellers and short sellers.



Summing up, the dynamics of asset prices is given by:

$$S_{k+1}^i = S_k^i \exp \left(\tau \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\tau} \xi_{k+1}^i \right) \left(1 + \frac{\alpha_i}{D_i} \left(f \left(\frac{V_{k+1}^*}{V_0} \right) - f \left(\frac{V_k}{V_0} \right) \right) \right) \quad (2.1)$$

where

$$V_k = \sum_{j=1}^n \alpha_j S_k^j \quad (2.2)$$

and

$$V_{k+1}^* = \sum_{j=1}^n \alpha_j S_k^j \exp \left(\tau \left(m_j - \frac{\Sigma_{j,j}}{2} \right) + \sqrt{\tau} \xi_{k+1}^j \right) \quad (2.3)$$

Proposition 2.2.1 *Under the assumption that $S_0 \in (\mathbb{R}_+^*)^n$ and $\|f\|_\infty < \frac{1}{2} \min_{1 \leq i \leq n} \frac{D_i}{|\alpha_i|}$, the price dynamics given by (2.1),(2.2) and (2.3) defines a Markov chain in $(\mathbb{R}_+^*)^n$.*

Proof (2.1), (2.2) and (2.3) show that S_{k+1} depends only on its value at t_k and on ξ_{k+1} , which is independent events previous to t_k . S is thus a Markov Chain. In addition, when $\|f\|_\infty < \frac{1}{2} \min_{1 \leq i \leq n} \frac{D_i}{|\alpha_i|}$, $1 + \frac{\alpha_i}{D_i} \left(f \left(\frac{V_{k+1}^*}{V_0} \right) - f \left(\frac{V_k}{V_0} \right) \right) > 0$, which ensures that, starting from $S_0 \in (\mathbb{R}_+^*)^n$, the Markov chain stays in $(\mathbb{R}_+^*)^n$.

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In the sequel, we work under the assumption of Proposition 2.2.1 which guarantees that S is a Markov chain in $(\mathbb{R}_+^*)^n$. Using the dimensionless variables $\tilde{S}_k = (\frac{S_k^1}{S_0^1}, \dots, \frac{S_k^n}{S_0^n})$ and $\tilde{V}_k = \frac{V_k}{V_0}$, we can rewrite (2.1)–(2.2)–(2.3) as

$$\tilde{S}_{k+1}^i = \tilde{S}_k^i \exp \left(\tau \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\tau} \xi_{k+1}^i \right) \left(1 + \frac{\alpha_i}{D_i} \left(f(\tilde{V}_{k+1}^*) - f(\tilde{V}_k) \right) \right)$$

where

$$\tilde{V}_k = \sum_{j=1}^n \frac{\alpha_j S_0^j}{V_0} \tilde{S}_k^j$$

and

$$\tilde{V}_{k+1}^* = \sum_{j=1}^n \frac{\alpha_j S_0^j}{V_0} \tilde{S}_k^j \exp \left(\tau \left(m_j - \frac{\Sigma_{j,j}}{2} \right) + \sqrt{\tau} \xi_{k+1}^j \right)$$

Hence the dynamics of \tilde{S}_k is entirely determined by

- the drift m
- the sequence (ξ_k) and its fundamental covariance matrix Σ
- the vector $(\frac{\alpha_1}{D_1}, \dots, \frac{\alpha_n}{D_n})$ which expresses the sizes of the fund's positions in each asset relative to the asset's market depth. This is a dimensionless measure of the size of positions, which is relevant for measuring market impact in case of liquidation.
- the dollar proportions $(\frac{\alpha_1 S_0^1}{V_0}, \dots, \frac{\alpha_n S_0^n}{V_0})$ initially invested in each asset by the fund
- the function f which describes the supply generated by distressed/short selling

2.3 Numerical experiments

2.3.1 Simulation procedure

We perform a Monte Carlo simulation (10^6 independent scenarios) of the multiperiod model described above for a fund investing in two strategies/asset classes with zero fundamental correlation and volatilities respectively given by 30% and 20%. We assume that the volume held by the fund on each asset is of the order of 20 times average daily volume for the asset. In comparison, LTCM's on-balance sheet assets totalled around \$125 billion, which represented 250 times average daily volume on the S&P 500 in 1998. We assume that the fund initially invests the same amount in both assets and that distressed sellers can trade once a day. We simulate the discrete-time model for a one-year period with the following parameters:

- $m=0$
- ξ is normal and $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$ with $\sigma_1 = 30\% \text{ year}^{-\frac{1}{2}}$ and $\sigma_2 = 20\% \text{ year}^{-\frac{1}{2}}$

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- $\frac{\alpha_1}{D_1} = \frac{\alpha_2}{D_2} = \frac{1}{10}$: the fund's position on asset 1 (resp. on asset 2) is equal to 10% of asset 1's (resp. asset 2's) market depth, or, using Obizhaeva (2011), around 15 times average daily volume for asset 1 (resp. around 20 times average daily volume for asset 2)
- $\frac{\alpha_1 S_0^1}{V_0} = \frac{\alpha_2 S_0^2}{V_0} = \frac{1}{2}$: the fund initially invests the same amount in 1 and 2
- We use the following choice for f : $f(x) = \frac{-1}{(\beta_{liq} - \beta_0)^4} (x - \beta_0)^4$ which satisfies the conditions described in Section 2.2, with $\beta_0 = 0.95$ and $\beta_{liq} = 0.55$.

2.3.2 Realized variance and realized correlations

In each simulated path, we compute the log-returns $r_k^i = \log\left(\frac{S_{k+1}^i}{S_k^i}\right)$ of asset i for $i = 1, 2$.

Let \bar{r}^i be the sample average of those returns: $\bar{r}^i = \frac{1}{M} \sum_{k=0}^{M-1} r_k^i$. For each sample path, we compute the realized covariance between assets i and j :

$$\widehat{C}^{i,j} = \frac{1}{T} \sum_{k=0}^{M-1} (r_k^i - \bar{r}^i)(r_k^j - \bar{r}^j)$$

and the realized correlation between i and j : $\frac{\widehat{C}^{i,j}}{(\widehat{C}^{i,i} \widehat{C}^{j,j})^{\frac{1}{2}}}$. The realized volatility for i is given by $(\widehat{C}^{i,i})^{\frac{1}{2}}$.

Figure 2.3 shows the distribution of the one-year realized correlation for the two strategies. In each scenario, we also computed realized correlation without feedback effects. Figure 2.4 is a scatter plot of the one-year realized correlation with and without feedback effects from distressed selling/short selling. Each point of the graph corresponds to one trajectory (for clarity, we choose to display only 1000 trajectories on scatter plots). For each point of the graph and hence each trajectory, realized correlation in the presence of feedback effects (resp. without feedback effects) can be read on the vertical axis (resp. the horizontal axis).

In the presence of distressed selling, the distribution of realized correlation is significantly modified. Our simulations show that *distressed selling by investors exiting funds with similar portfolios and short selling can generate significant realized correlation, even between assets with zero fundamental correlation*. In Figure 2.3, the distribution of realized correlation without feedback effects reflects the statistical error in the estimation of correlation. Hence, the aspect of the distribution of realized correlation with feedback effects due to distressed selling or short selling reflects the effects of such trading on correlation between assets: average correlation in the presence of feedback effects is higher than its fundamental value $\rho = 0$ and the profile of its distribution presents a thick upper tail. In Figure 2.4, all points are above the $Y=X$ axis, confirming the fact that distressed selling increases correlation between assets. In the presence of feedback effects, correlation becomes path dependant. It is interesting to examine the distribution

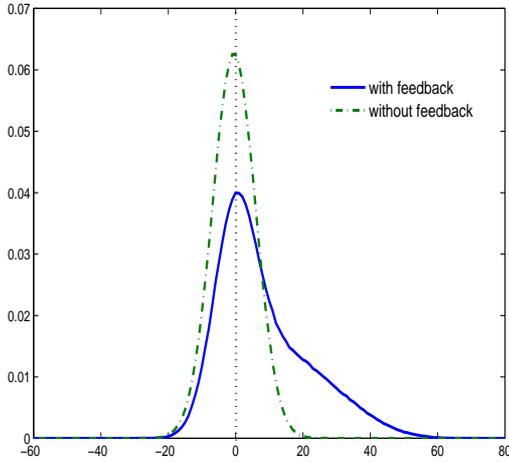


Figure 2.3: Distribution of realized correlation between the two securities (with $\rho = 0$) with and without feedback effects due to distressed selling

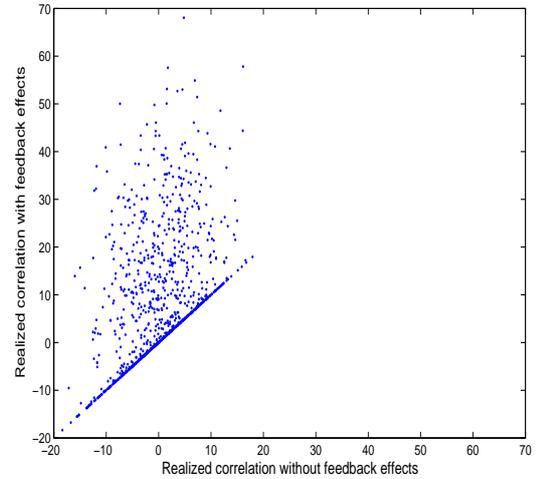


Figure 2.4: Scatter plot of realized correlation with and without feedback effects due to distressed selling (each data point represents one simulated scenario)

of realized correlation in scenarios where fund value reaches $\beta_0 V_0$, triggering distressed selling/short selling.

Conditional correlation: In Figure 2.5, we divide trajectories into two categories, whether fund value reaches $\beta_0 V_0$ or not and we display the distribution of realized correlation for those two categories: in plain line, the distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$, triggering distressed selling; in dotted line, the distribution of realized correlation in scenarios where fund value remains above $\beta_0 V_0$ and there is no distressed selling or short selling. *Realized correlation conditional on the fact that distressed selling took place is significantly higher than realized correlation in scenarios where there was no distressed selling.* In scenarios where distressed selling took place, price impact affects all assets of the fund in the same direction during the time the fund's market value is below the threshold $\beta_0 V_0$. This results in higher realized correlation in those scenarios: the average conditional correlation is equal to 18% whereas unconditional correlation is 9% and fundamental correlation is zero.

Asset volatility: *In the presence of feedback effects from distressed selling/short selling, asset volatility increases.* Figure 2.6 shows that the distribution of realized volatility of each asset, in scenarios where there was distressed selling, is centered around a higher value than the asset's fundamental volatility and presents a thick upper tail. In such scenarios, assets are more volatile than in scenarios without distressed selling. The action of distressed sellers (and short sellers) increases the amplitude of price moves and

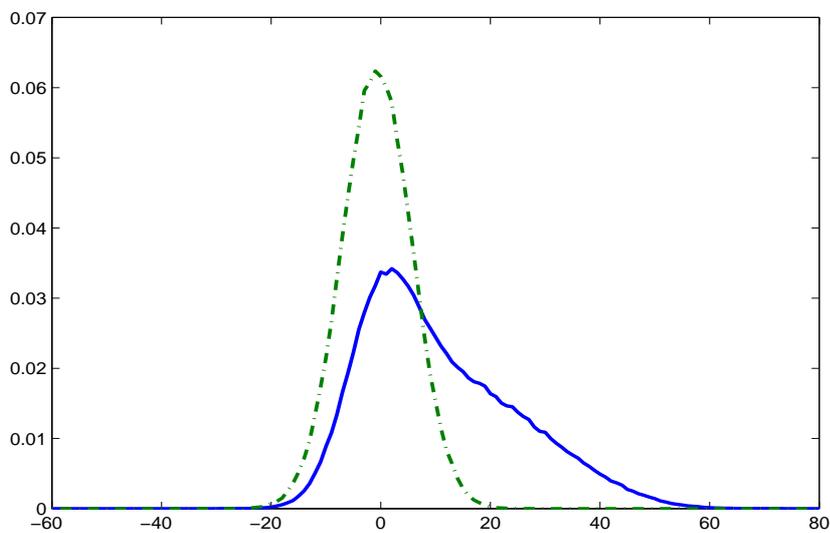


Figure 2.5: Distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$ (plain line) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted line)

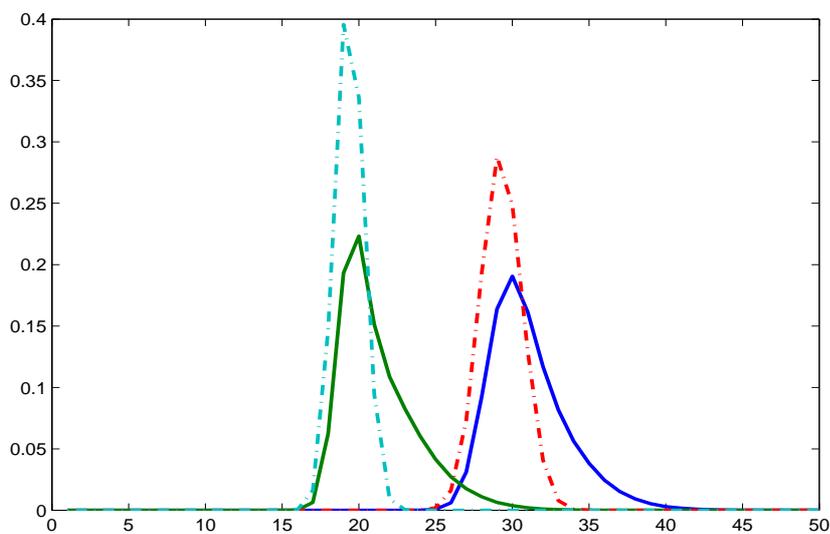


Figure 2.6: Distribution of realized volatilities for each security in scenarios where fund value reaches $\beta_0 V_0$ between 0 and T (plain lines) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted lines) (with $\sigma_1 = 30\%$ and $\sigma_2 = 20\%$)

generates higher asset volatility. This should result in higher fund volatility.

2.3.3 Fund volatility

Figure 2.7 is a scatter plot of fund volatility, with and without feedback effects from distressed selling/short selling. We also compare the distribution of fund volatility in scenarios where the fund reaches $\beta_0 V_0$ or not. Figure 2.8 displays the distributions of fund volatility in those two scenarios.

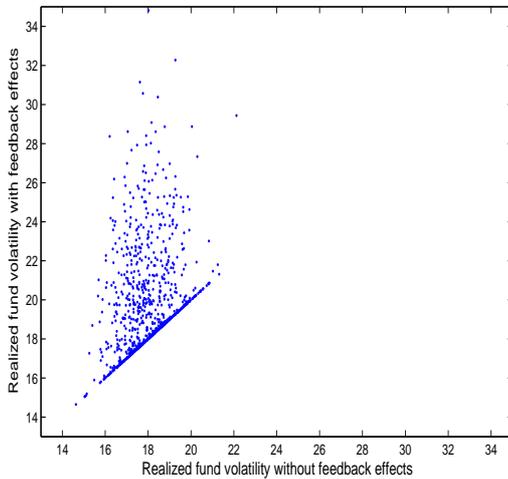


Figure 2.7: Scatter plot of realized volatility of the fund with and without feedback effects (each data point represents one simulated scenario)

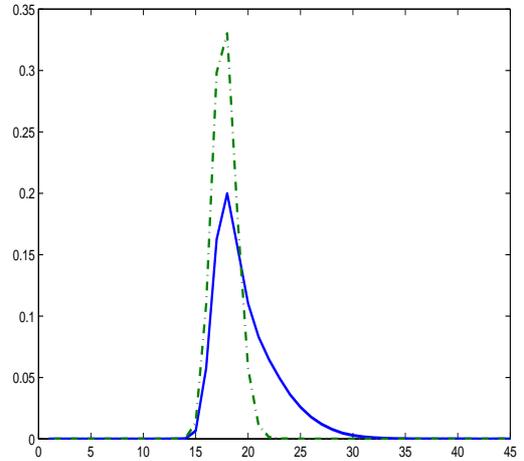


Figure 2.8: Distribution of realized fund volatility in scenarios where fund value reaches $\beta_0 V_0$ between 0 and T (plain line) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted line)

Distressed selling increases the fund's volatility: the distribution of realized fund volatility presents a thick upper tail when there are feedback effects from short sellers or distressed sellers. Figure 2.7 underlines the fact that feedback effects increase the fund's volatility. Figure 2.8 shows that when there is distressed selling, the fund is more volatile than when fund value remains above $\beta_0 V_0$ and there is no distressed selling.

Our simulations show that, even in the case of assets with zero fundamental correlation, one observes a significant positive level of realized correlation resulting in higher than expected fund volatility.

2.4 Diffusion limit

To confirm that the phenomena observed in the numerical experiments are not restricted to particular parameter choices or a particular choice of the function f , we will now analyze the continuous-time limit of our discrete-time model: the study of this limit allows one to obtain analytical formulas for realized correlation which confirm quantitatively the effects observed in the numerical experiments.

In order to study the continuous-time limit of the multi-period model, we work under the following assumption:

Assumption 2.4.1 *For all $1 \leq i \leq n$, $\alpha_i \geq 0$ and there exists $\eta > 0$ such that:*

$$\mathbb{E}(\|\exp(\eta\xi)\|) < \infty, \quad \mathbb{E}(\|\xi\|^{\eta+4}) < \infty \quad \text{and} \quad f \in \mathcal{C}_0^3(\mathbb{R})$$

where $\mathcal{C}_0^p(\mathbb{R})$ denotes the set of real-valued, p -times continuously differentiable maps whose derivatives of order $1 \leq l \leq p$ have compact support.

The assumption on f is natural if we assume that f is \mathcal{C}^3 and that f is constant on $]-\infty, \beta_{liq}]$ and $[\beta_0, +\infty[$.

Our main theoretical result is the following theorem which describes the diffusion limit of the price process.

Theorem 2.4.2 *Under Assumption 2.4.1, $(S_{\lfloor \frac{t}{\tau} \rfloor})_{t \geq 0}$ converges weakly to a diffusion $(P_t)_{t \geq 0}$ when τ goes to 0 where*

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \quad 1 \leq i \leq n$$

where μ (resp., σ) is a \mathbb{R}^n -valued (resp. matrix-valued) mapping defined by

$$\mu_i(P_t) = m_i + \frac{\alpha_i}{D_i} \frac{1}{2V_0^2} f''\left(\frac{V_t}{V_0}\right) \pi_t \cdot \Sigma \pi_t + \frac{\alpha_i}{D_i} \frac{1}{V_0} f'\left(\frac{V_t}{V_0}\right) (\pi_t \cdot \bar{m} + (\Sigma \pi_t)_i) \quad (2.4)$$

$$\sigma_{i,j}(P_t) = A_{i,j} + \frac{\alpha_i}{D_i} f'\left(\frac{V_t}{V_0}\right) \frac{(A^t \pi_t)_j}{V_0} \quad (2.5)$$

Here W_t is an n -dimensional Brownian motion, $\pi_t = \begin{pmatrix} \alpha_1 P_t^1 \\ \vdots \\ \alpha_n P_t^n \end{pmatrix}$ is the (dollar) allo-

cation of the fund, $V_t = \sum_{1 \leq k \leq n} \alpha_k P_t^k$ the value of the fund, $\bar{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$ and A is a

square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

- * When market depth is infinite (i.e. price impact is negligible) the continuous-time limit is a multivariate geometric Brownian motion and the covariance of the log-returns is given by the ‘fundamental’ covariance: $\text{cov}(\ln P_t^i, \ln P_t^j) = t \Sigma_{ij}$.
- * The expression of σ shows that distressed selling modifies correlation between assets, asset volatility and fund volatility. We will focus on this phenomenon in the next sections.

2.5 Realized correlations

2.5.1 Realized covariance

We define $C_{[t_1, t_2]}$ the realized covariance matrix of returns (Barndorff-Nielsen and Shephard, 2004; Andersen et al., 2003) between dates t_1 and t_2 as

$$C_{[t_1, t_2]}^{i,j} = \frac{1}{t_2 - t_1} ([\ln P^i, \ln P^j]_{t_2} - [\ln P^i, \ln P^j]_{t_1})$$

where $[\ln P^i, \ln P^j]_t$ is the quadratic covariation between $\ln P^i$ and $\ln P^j$ on $[0, t]$.

The following result follows by direct computation from Theorem 2.4.2:

Proposition 2.5.1 *The realized covariance matrix of returns between 0 and t is $\frac{1}{t} \int_0^t c_s ds$ where the instantaneous covariance c_s at date s is given by*

$$c_s = \Sigma + \frac{1}{V_0} f' \left(\frac{V_s}{V_0} \right) [\Lambda^t \pi_s \Sigma + \Sigma \pi_s^t \Lambda] + \frac{1}{V_0^2} (f')^2 \left(\frac{V_s}{V_0} \right) (\pi_s \cdot \Sigma \pi_s) \Lambda^t \Lambda \quad \text{where}$$

- $\pi_t = \begin{pmatrix} \alpha_1 P_t^1 \\ \vdots \\ \alpha_n P_t^n \end{pmatrix}$ denotes the (dollar) holdings of the reference fund
- $\Lambda = \begin{pmatrix} \frac{\alpha_1}{D_1} \\ \vdots \\ \frac{\alpha_n}{D_n} \end{pmatrix}$ represents the positions of the reference fund in each market as a fraction of the respective market depth.

The expression for the realized covariance of asset returns shows that realized covariance is the sum of the fundamental covariance and an *excess covariance term which is path-dependent and varies inversely with market depth*. Excess covariance depends on the ratio $\frac{\alpha}{D}$, which compares the positions of the fund to the market depth in each asset. When market depth is infinite, realized covariance reduces to fundamental covariance. Moreover, the expression of instantaneous covariance shows that it is a deterministic and continuous function of vector π_t , hence *the impact of distressed selling on realized covariance and correlation is computable* in this setting. Realized covariance and correlation between assets depend on the derivative of f , which represents the rate at which investors exit their positions when the fund underperforms.

In scenarios where the fund value stays above $\beta_0 V_0$ realized covariances converge to their fundamental value. However, as soon as the fund value falls below the threshold $\beta_0 V_0$ which triggers distressed selling, excess covariance appears: in such distress scenarios, realized correlation and realized variance differ from the values implied by the ‘fundamental covariance’ Σ . In the case where fundamental correlation is positive between all pairs of assets, distressed selling increases realized covariance. As shown by Eq. (2.5.1), the magnitude of this effect is determined by the size α_i of the positions being liquidated relative to the depth D_i of the market in these assets: this is further

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discussed in Section 2.5.4. It is also interesting to notice that when the fund invests significantly in an asset i compared to its market depth and when fund value drops below $\beta_0 V_0$, instantaneous covariance between i and any other asset j in the market is different from its fundamental value $\Sigma_{i,j}$ (as $\frac{\alpha_i}{D_i} f' \left(\frac{V_t}{V_0} \right) \frac{(\Sigma \pi_t)_j}{V_0} \neq 0$).

2.5.2 Case of zero fundamental correlations

We now focus on the case of a diagonal covariance matrix Σ ($\Sigma_{i,j} = 0$ for $i \neq j$): the n assets have zero fundamental correlation. We denote $\Sigma_{i,i} = \sigma_i^2$ (σ_i is asset i 's volatility).

Corollary 2.5.2 *If the fundamental covariance matrix Σ is diagonal, then, for all $1 \leq i, j \leq n$, the instantaneous covariances are given by*

$$c_t^{i,j} = \frac{\alpha_j}{D_j} f' \left(\frac{V_t}{V_0} \right) \frac{\alpha_i}{V_0} P_t^i \sigma_i^2 + \frac{\alpha_i}{D_i} f' \left(\frac{V_t}{V_0} \right) \frac{\alpha_j}{V_0} P_t^j \sigma_j^2 + \frac{\alpha_i \alpha_j}{D_i D_j} (f')^2 \left(\frac{V_t}{V_0} \right) \sum_{1 \leq l \leq n} \left(\frac{\alpha_l}{V_0} P_t^l \sigma_l \right)^2 \geq 0$$

and

$$c_t^{i,i} = \sigma_i^2 + 2 \frac{\alpha_i}{D_i} f' \left(\frac{V_t}{V_0} \right) \frac{\alpha_i}{V_0} P_t^i \sigma_i^2 + \left(\frac{\alpha_i}{D_i} \right)^2 (f')^2 \left(\frac{V_t}{V_0} \right) \sum_{1 \leq l \leq n} \left(\frac{\alpha_l}{V_0} P_t^l \sigma_l \right)^2 \geq \sigma_i^2$$

Realized correlation between i and j (resp realized variance for asset i) between 0 and T are equal to $\frac{\int_0^T c_t^{i,j} dt}{\left(\int_0^T c_t^{i,i} dt \int_0^T c_t^{j,j} dt \right)^{\frac{1}{2}}}$ (resp. $\left(\frac{1}{T} \int_0^T c_t^{i,i} dt \right)^{\frac{1}{2}}$).

Since f is increasing, realized correlations are positive and the realized volatility of asset i is greater than its fundamental volatility σ_i : *in the absence of fundamental correlation, distressed selling generates positive realized correlation across the fund's strategies and increase the volatility of all assets detained by the fund.* This is due to the fact that when $V_t < \beta_0 V_0$, all strategies owned by the fund face a net demand of the same sign. In particular a large fall in fund value generates a negative demand by investors across all positions held by the fund and increases the amplitude of price movements. These analytical results confirm the results of our simulation experiments.

Even if the fund invests in 'fundamentally' uncorrelated strategies, in scenarios where the fund experiences losses e.g. $V_t < \beta_0 V_0$ and approaches liquidation, distressed selling by investors leads to a positive realized correlation between the fund's strategies, reducing the benefit of diversification.

To check whether these asymptotic results are relevant in the case of daily rebalancing, we compare the theoretical formula for realized covariance in continuous time given in Corollary 2.5.2 and the realized covariance in a discrete-time market as calculated in 2.3.2. Figure 2.9 shows that the higher the trading frequency, the better the concordance between empirical realized covariance (calculated as in section 2.3.2) and the continuous-time result (given by Corollary 2.5.2). More precisely, a linear regression of the realized covariance with respect to the theoretical values computed using Corollary 2.5.2 shows good agreement between the empirical and theoretical values: the regression yields a slope of 0.95 ($R^2 = 0.63$) for $\tau = \frac{1}{250}$ and a slope of 0.99 ($R^2 = 0.96$) for $\tau = \frac{1}{2500}$.

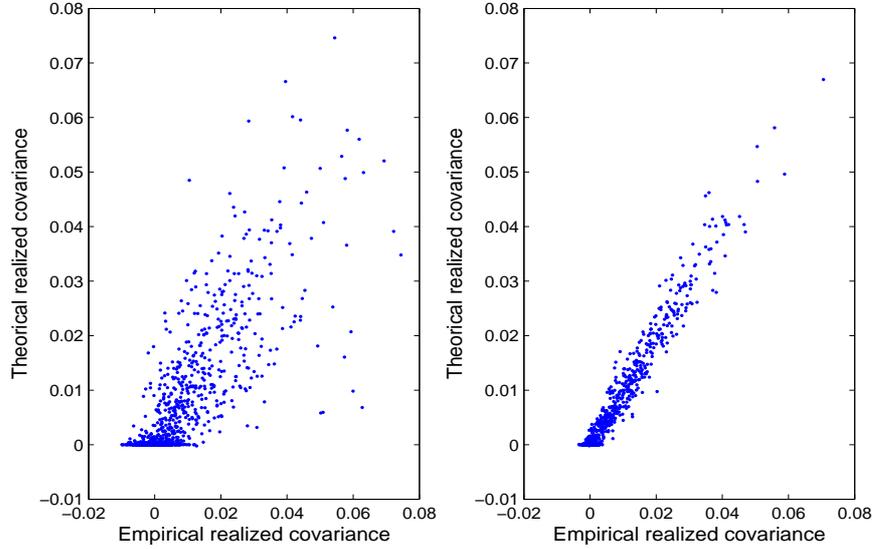


Figure 2.9: Scatter plot of theoretical realized covariance and empirical realized covariance for $\tau = \frac{1}{250}$ (left) and $\tau = \frac{1}{2500}$ (right)

2.5.3 The path-dependent nature of realized correlation

Proposition ?? shows that instantaneous covariance $c_t^{i,j}$ is a deterministic function of $\pi_t = (\alpha_1 P_t^1, \dots, \alpha_n P_t^n)^t$. Figure 2.10 shows an example of the evolution of correlation, given a trajectory π_t . We used the same parameters as in our simulations (section 2.3) and we display the evolution of $\frac{V_t}{V_0}$ on the graph at the left and the evolution of realized correlation $\frac{\frac{1}{t} \int_0^t c_s^{1,2}, ds}{\left(\left(\frac{1}{t} \int_0^t c_s^{1,1}, ds\right)\left(\frac{1}{t} \int_0^t c_s^{2,2}, ds\right)\right)^{\frac{1}{2}}}$, in the figure on the right. We see that as long as fund value stays above $\beta_0 V_0$, realized correlation is equal to 0. Losses greater than this threshold generate distressed selling and lead to a positive *endogenous* correlation between the asset returns. As shown in Figure 2.10, this excess correlation is path-dependent: it depends on the performance of the fund. Fund losses are amplified by investors exiting funds with similar allocations or by those trading against the fund. This not only drives down the fund value but increases the correlation between its two strategies to unexpected levels. As a result, realized correlation among strategies can be much higher than the 'fundamental' correlation, exactly when the fund is in dire need of the relief promised by diversification. The spiral can be triggered by a large loss in one of its strategies. This leads to investors exiting similar funds, others shorting its positions and thus generates a high correlation among all its positions.

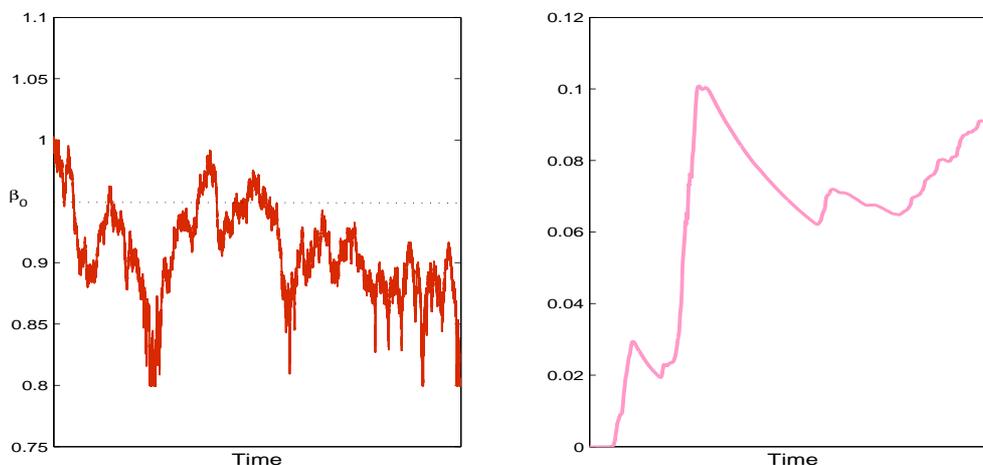


Figure 2.10: $\frac{V_t}{V_0}$ (left) and realized correlation on $[0, t]$ (right)

2.5.4 Liquidation impact

Theorem 2.4.2 and Proposition 2.5.1 show that the price dynamics and correlation between assets are functions of the positions of the fund relative to the market depth of each asset: $\Lambda = (\frac{\alpha_1}{D_1}, \dots, \frac{\alpha_n}{D_n})^t$. These results suggest that $\frac{\alpha_i}{D_i}$ may be used as an indicator of the impact on asset i of the liquidation of the fund's position. When $\frac{\alpha}{D} \rightarrow 0$, we find, as expected, a Black-Scholes model with constant correlation between assets. Proposition 2.5.1 shows that the excess covariance tends to 0 when market depth goes to infinity. Corollary 2.5.2 proves that in the case of assets with zero fundamental correlation, *the bigger the fund's positions compared to the market depth of each asset, the more correlated its strategies will be*, as can be seen on Figure 2.11.

It is interesting to underline the fact that when all assets, except one, denoted i_0 , have infinite market depths and when distressed selling/short selling occurred in the market ($\exists t_0, \frac{V_{t_0}}{V_0} \leq \beta_0$), all strategies are positively correlated with strategy i_0 (as $C_{i,i_0}^{t_0} > 0$). Figure 2.12 shows that when one asset (asset 2) has finite market depth, the underperformance of other assets (asset 1) leads to strictly positive realized correlation.

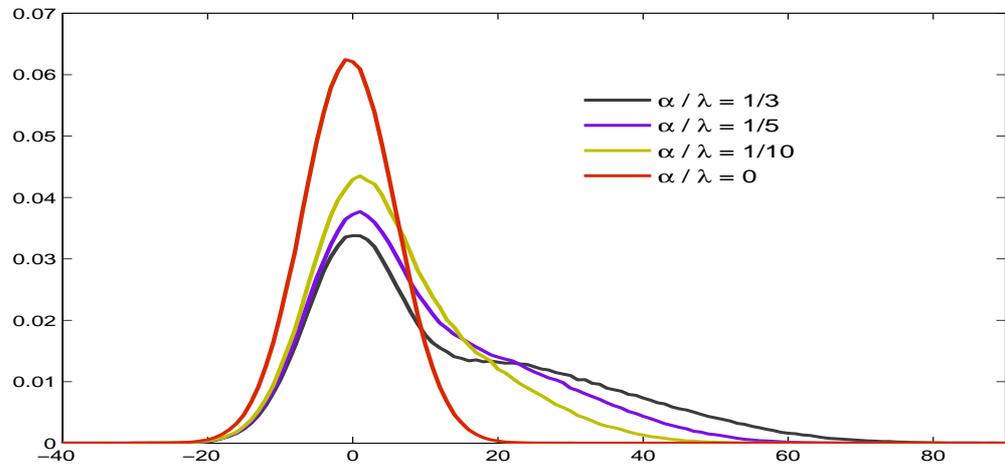


Figure 2.11: Distribution of realized correlation for different values of $\frac{\alpha}{D}$

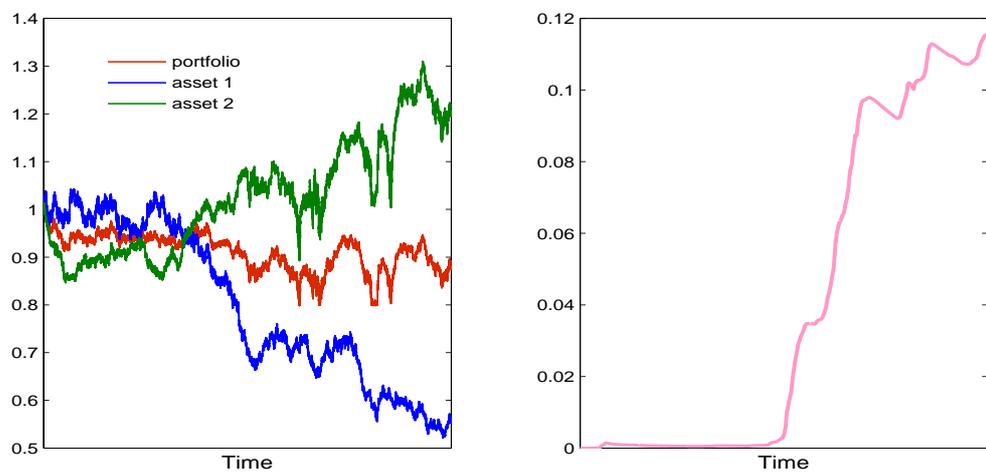


Figure 2.12: Fund value and asset value (left) and realized correlation (right) for $\frac{\alpha_1}{D_1} = 0$ and $\frac{\alpha_2}{D_2} = \frac{1}{5}$

2.6 Endogenous risk and spillover effects

The computation of realized correlations is relevant for the assessment of the (realized) volatility of portfolios: the explicit formulas obtained in Section 2.5 allow one to quantify the impact of distressed selling on the volatility of the fund being exited/shorted, and see how distressed selling on one fund affects other funds' volatility.

2.6.1 Realized variance of a fund subject to distressed selling

Proposition 2.6.1 *The fund's realized variance between 0 and t is equal to $\frac{1}{t} \int_0^t \Gamma_s ds$ where Γ_s , the instantaneous variance of the fund, is given by:*

$$\begin{aligned} \Gamma_s V_s^2 &= \pi_s \cdot \Sigma \pi_s + \frac{2}{V_0} f' \left(\frac{V_s}{V_0} \right) (\pi_s \cdot \Sigma \pi_s) \Lambda \cdot \pi_s \\ &\quad + \frac{1}{V_0^2} \left(f' \left(\frac{V_s}{V_0} \right) \right)^2 (\pi_s \cdot \Sigma \pi_s) (\Lambda \cdot \pi_s)^2 \end{aligned} \quad (2.6)$$

and π_s and Λ are defined in Proposition 2.5.1.

The proof is given in the appendix. We note that distressed selling increases the fund's volatility:

$$\Gamma_s V_s^2 \geq \pi_s \cdot \Sigma \pi_s$$

The fund's instantaneous variance is equal to its fundamental value $\frac{1}{V_s^2} \pi_s \cdot \Sigma \pi_s$ plus a term of order one in Λ and a term of order two in Λ . In a market with infinite market depth, Γ_s is equal to its fundamental value. As in the case of instantaneous covariance between assets, Γ_s is a continuous and deterministic function of π_s and is a superposition of two regimes: a fundamental regime and an excess volatility regime, that is exacerbated with illiquidity (D is small) or when the fund has big positions (α is large). Note that, even without liquidity drying up (D constant), feedback effects may significantly increase fund volatility when investors 'run for the exit', generating spikes in realized correlation, even in the absence of predatory trading by short sellers.

2.6.2 Fund volatility in the case of zero fundamental correlations

Corollary 2.6.2 *If the fundamental covariance matrix Σ is diagonal, the instantaneous variance of the fund value is given by*

$$\Gamma_t = \left(1 + \sum_{1 \leq i \leq n} \frac{\alpha_i P_t^i}{V_0} \frac{\alpha_i}{D_i} f' \left(\frac{V_t}{V_0} \right) \right)^2 \sum_{1 \leq j \leq n} \left(\frac{\alpha_j P_t^j}{V_t} \sigma_j \right)^2$$

Proof This is a consequence of Proposition 2.6.1 and the fact that Σ is a diagonal matrix.

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The fund's realized variance between 0 and T is equal to $\frac{1}{T} \int_0^T \Gamma_t dt$. Corollary 2.6.2 explains the observations of section 2.3 and confirms that, when the size of the fund's positions are non-negligible with respect to market depth, distressed selling leads to an increase in realized fund volatility, even when the fund invests in assets with zero fundamental correlation. Similarly, fund volatility increases when similar funds liquidate part of their positions.

These results point to the limits of diversification when price impact is not negligible: even if the fund manager invests in uncorrelated strategies, short selling and liquidation by investors facing losses will correlate them positively, exactly in scenarios where the fund experiences difficulty, increasing the volatility of the portfolio and reducing the benefit of diversification. This may arise either because the fund has large positions or because other large funds are following similar strategies (strategy crowding).

2.6.3 Spillover effects

We now examine the impact of distressed selling by investors in a large fund (called hereafter the reference fund) on the volatility of other funds.

Consider a (small) fund investing in the n securities and following a self-financing strategy. We denote by μ_t^i the number of units of i detained by the small fund at date t . Note that we allow for dynamic strategies. Its market value at t is $M_t = \sum_{1 \leq i \leq n} \mu_t^i P_t^i$.

As the small fund's strategy is self-financing, we have $dM_t = \sum_{1 \leq i \leq n} \mu_t^i dP_t^i$.

In our framework, the small fund's strategy should impact prices and its action should modify the dynamics of P given by Theorem 2.4.2. However, when its positions are very small compared to the size of the reference fund, the impact of its trading strategy is negligible compared to feedback effects due to distressed selling and short selling in the reference fund.

Under the assumption that the size of the small fund is small, its strategy does not impact prices and P still follows the dynamics given in Theorem 2.4.2 and we obtain

$$d[\ln M]_t = \pi_t^\mu \cdot c_t \pi_t^\mu dt$$

where $\pi_t^\mu = \begin{pmatrix} \mu_t^1 P_t^1 \\ \vdots \\ \mu_t^n P_t^n \end{pmatrix}$ is the dollar allocation of the small fund and c_t is the instantaneous covariance matrix of returns, given in Proposition 2.5.1.

Proposition 2.6.3 *Assume that the small fund's strategy does not impact prices. Then the quadratic variation of the fund value is given by*

$$[\ln M]_t = \int_0^t \gamma_s^M ds \quad \text{where}$$

$$\gamma_t^M = \pi_t^\mu \cdot \Sigma \pi_t^\mu + \frac{2f' \left(\frac{V_t}{V_0} \right)}{V_0} (\pi_t^\mu \cdot \Sigma \pi_t^\alpha) (\Lambda \cdot \pi_t^\mu) + \frac{f' \left(\frac{V_t}{V_0} \right)^2}{V_0^2} (\pi_t^\alpha \cdot \Sigma \pi_t^\alpha) (\Lambda \cdot \pi_t^\mu)^2$$

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where

- $M_t = \sum_{1 \leq i \leq n} \mu_t^i P_t^i$ is the value of the small fund
- $\pi_t^\alpha = (\alpha_1 P_t^1, \dots, \alpha_n P_t^n)^t$ denotes the (dollar) holdings of the reference fund,
- $\pi_t^\mu = (\mu_t^1 P_t^1, \dots, \mu_t^n P_t^n)^t$ denotes the (dollar) holdings of the small fund, and
- $\Lambda = (\frac{\alpha_1}{D_1}, \dots, \frac{\alpha_n}{D_n})^t$ represents the positions of the reference fund in each market as a fraction of the respective market depth.

This result shows how distressed selling in one fund affects the volatility of other funds. In the presence of feedback effects, the realized variance of the small fund is given by its fundamental value $\int_0^t \pi_s^\mu \cdot \Sigma \pi_s^\mu ds$ plus an additional variance term which correspond to the impact of distressed selling in the reference fund and depends on the 'liquidity vector' Λ .

It is interesting to note that this 'contagion' across portfolios depends on the similarity between the portfolio of the reference fund α and the small fund μ . In particular, when the portfolios α and μ are orthogonal for Σ ($\pi_t^\mu \cdot \Sigma \pi_t^\alpha = 0$), the term of order one in Λ in Proposition 2.6.3 is zero and the small fund's variance is equal to its fundamental value plus a term of order two in Λ , whose magnitude is much smaller.

More interestingly, if the allocations of the two funds verify the 'orthogonality' condition

$$\Lambda \cdot \pi_t^\mu = \sum_{i=1}^n \frac{\alpha_i}{D_i} \mu_t^i P_t^i = 0 \quad (2.7)$$

distressed selling of investors in the reference fund does not affect the small fund's variance:

$$[\ln M]_t = \int_0^t \pi_s^\mu \cdot \Sigma \pi_s^\mu ds$$

On the contrary, the excess volatility due to feedback is maximal when strategies μ and α are colinear (i.e. when the vectors π_t^μ and π_t^α are colinear). These results shed some light on the 'quant event' of August 2007. In August 2007, long-short equity market-neutral funds experienced extreme volatility and large losses during three days, whereas there was no tangible effect on major equity indices in the same period. An explanation which has been advanced is that a large position in such a market-neutral long-short fund, was liquidated by an investor in this three day period. Our model suggests that this rapid liquidation would then exacerbate the volatility of other long-short market-neutral funds following similar strategies (i.e. whose allocation vector has a positive projection on the allocation vector of the fund being exited). Since, by construction of market-neutral funds, the holdings of index funds are orthogonal to market-neutral funds in the sense of the orthogonality condition (2.7), our model predicts that index funds would not be subject to these feedback effects: indeed, they were insensitive to this event.

Alternative explanations advanced for the August 2007 events are sometimes based on a supposed drying up of liquidity in equity markets during that period (Khandani and Lo,

2011). However, there is no evidence such a dry-up in liquidity occurred: in fact, trading in equity indices occurred seamlessly during this period. By contrast, the mechanism underlying our model does not require any time-varying liquidity for these effects to occur: indeed, all these effects are present even when the market depth D_i is constant. Also, our explanation entails that the population of long-short market-neutral funds affected by this event had allocations with substantial 'colinearity' i.e. that "strategy crowding" was a major risk factor in this market. Our results show the relevance of *strategy crowding* as a risk factor and represent a first step in quantifying it. Our analysis points in particular to the necessity of using indicators based on the *size* of positions when quantifying crowding effects, via proxies such as the market capitalization of various strategies. Clearly, factors based on returns alone cannot capture such size effects.

2.7 Conclusion

We have presented a simple and analytically tractable model for investigating the impact of fire sales on volatility and correlations of assets held by a fund. Our model yields explicit results for the realized variance and realized correlations of assets held by the fund and shows that the realized covariance between returns of two assets may be decomposed into the sum of a 'fundamental' covariance and a liquidity-dependent 'excess covariance', which is found to be inversely proportional to the market depth of these assets.

We have shown that the presence of this excess covariance leads to endogenous risk for large portfolios –liquidating the positions of such a large portfolio entails a higher-than-expected volatility which may increase liquidation costs– as well as spillover effects: distressed selling of investors in a large fund may also exacerbate the volatility of funds with similar allocations, while leaving funds verifying an 'orthogonality' condition unaffected. This underlines the necessity of considering 'strategy crowding' as a risk factor and gives a quantitative framework to evaluate such risk.

More generally, our study shows that "liquidity risk" and "correlation risk", often treated as separate sources of risk, may be difficult to disentangle in practice: rather than being treated as an exogenous factor to be estimated using statistical methods, correlation risk needs to be modeled at its source, namely comovements in supply and demand across asset classes. Each of these observations raises a point which merits an independent, in-depth study. We plan to pursue some of these research directions in a forthcoming work.

2.8 Appendices

2.8.1 Proof of Theorem 2.4.2

We work under Assumption 2.4.1. We denote $Z_{k+1} = \tau\bar{m} + \sqrt{\tau}\xi_{k+1} \in \mathbb{R}^n$ where $\bar{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$. We can write the price dynamics (2.1) as follows:

$$S_{k+1}^i = S_k^i \exp(Z_{k+1}^i) \left[1 + \frac{\alpha_i}{D_i} \left(f \left(\sum_{j=1}^n \frac{\alpha_j S_k^j}{V_0} \exp(Z_{k+1}^j) \right) - f \left(\sum_{j=1}^n \frac{\alpha_j S_k^j}{V_0} \right) \right) \right]$$

As a consequence, we can write

$$S_{k+1} = \theta(S_k, Z_{k+1})$$

where $\theta : (\mathbb{R}_+^*)^n \times \mathbb{R}^n \mapsto (\mathbb{R}_+^*)^n$ is \mathcal{C}^3 as f is \mathcal{C}^3 .

As S is a Markov chain in $(\mathbb{R}_+^*)^n$, we study the convergence of the Markov chain on $(\mathbb{R}_+^*)^n$ and hence consider, without further precision, that $S \in (\mathbb{R}_+^*)^n$ in the sequel.

Define now a (resp., b) a $\mathcal{M}_n(\mathbb{R})$ -valued (resp. \mathbb{R}^n -valued) mapping such that

$$a_{i,j}(S) = \sum_{l=1}^n \frac{\partial \theta_i}{\partial z_l}(S, 0) \times A_{l,j} \quad (2.8)$$

$$b_i(S) = \sum_{j=1}^n \frac{\partial \theta_i}{\partial z_j}(S, 0) \bar{m}_j + \frac{1}{2} \sum_{j,l=1}^n \frac{\partial^2 \theta_i}{\partial z_j \partial z_l}(S, 0) \Sigma_{j,l} \quad (2.9)$$

In order to show Theorem 2.4.2, we first show the following lemma:

Lemma 2.8.1 *Under the assumptions of Theorem 2.4.2, for all $\epsilon > 0$ and $r > 0$:*

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \frac{1}{\tau} \mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S) = 0 \quad (2.10)$$

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E}(S_{k+1} - S_k | S_k = S) - b(S) \right\| = 0 \quad (2.11)$$

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E}((S_{k+1} - S_k)(S_{k+1} - S_k)^t | S_k = S) - aa^t(S) \right\| = 0 \quad (2.12)$$

where a and b are defined respectively in (2.8) and (2.9).

Proof Fix $\epsilon > 0$ and $r > 0$. As θ is \mathcal{C}^1 , there exists $C > 0$ such that, if $\|Z_{k+1}\| \leq \epsilon$ and $\|S_k\| \leq r$, then

$$\|S_{k+1} - S_k\| = \|\theta(S_k, Z_{k+1}) - \theta(S_k, 0)\| \leq C\|Z_{k+1}\| = C\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\|$$

We then have:

$$\mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S, \|S\| \leq r, \|Z_{k+1}\| \leq \epsilon)$$

$$\begin{aligned}
 &\leq \mathbb{P}(C\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon) \\
 &\leq \mathbb{P}\left(\|\xi_{k+1}\| \geq \frac{\epsilon - \|\bar{m}\|C\tau}{C\sqrt{\tau}}\right) \\
 &\leq \mathbb{E}\left[\left(\|\xi_{k+1}\| \frac{C(r)\sqrt{\tau}}{\epsilon - \|\bar{m}\|C(r)\tau}\right)^{2+\eta}\right] \\
 &\leq \tau^{1+\frac{\eta}{2}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{C(r)}{\epsilon - \|\bar{m}\|C(r)\tau}\right)^{2+\eta}
 \end{aligned}$$

which implies that

$$\begin{aligned}
 &\mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon, \|Z_{k+1}\| \leq \epsilon | S_k = S, \|S\| \leq r) \\
 &\leq \tau^{1+\frac{\eta}{2}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{C(r)}{\epsilon - \|\bar{m}\|C(r)\tau}\right)^{2+\eta}
 \end{aligned}$$

In addition, we have

$$\mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon, \|Z_{k+1}\| \geq \epsilon | S_k = S) \leq \mathbb{P}(\|Z_{k+1}\| \geq \epsilon)$$

and

$$\leq \tau^{1+\frac{\eta}{2}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{1}{\epsilon(r) - \|\bar{m}\|\tau}\right)^{2+\eta}$$

using the same methodology as we just did previously. Given that $\mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S)$ is equal to:

$$\mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon, \|Z_{k+1}\| \leq \epsilon | S_k = S) + \mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon, \|Z_{k+1}\| \geq \epsilon | S_k = S)$$

we find (2.10).

As θ is \mathcal{C}^2 we can write the Taylor expansion of θ_i in 0, for $1 \leq i \leq n$:

$$\begin{aligned}
 S_{k+1}^i - S_k^i &= \theta_i(S_k, Z_{k+1}) - \theta_i(S_k, 0) \\
 &= \frac{\partial\theta_i}{\partial z}(S_k, 0)Z_{k+1} + \frac{1}{2}Z_{k+1} \cdot \frac{\partial^2\theta_i}{\partial z\partial z'}(S_k, 0)Z_{k+1} + Z_{k+1} \cdot R_i(S_k, \tau, \xi_{k+1})Z_{k+1}
 \end{aligned}$$

where R_i converges uniformly to 0 when τ goes to 0, when $\|Z_{k+1}\| \leq \epsilon$ and $\|S_k\| \leq r$. We have:

$$\mathbb{E}\left(\frac{\partial\theta_i}{\partial z}(S, 0)Z_{k+1}\right) = \tau \sum_{j=1}^n \frac{\partial\theta_i}{\partial z_j}(S, 0)\bar{m}_j$$

and

$$\mathbb{E}\left(Z_{k+1} \cdot \frac{\partial^2\theta_i}{\partial z\partial z'}(S, 0)Z_{k+1}\right) = \tau \sum_{j,l=1}^n \frac{\partial^2\theta_i}{\partial z_l z_j} \Sigma_{j,l} + o(\tau)$$

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where $\frac{o(\tau)}{\tau}$ converges to zero when τ goes to zero. As a consequence, we find that:

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} [(S_{k+1} - S_k) | S_k = S, \|Z_{k+1}\| \leq \epsilon(r)] - b(S) \right\| = 0 \quad (2.13)$$

Remarking that

$$\begin{aligned} & \left\| \frac{1}{\tau} \mathbb{E} ((S_{k+1} - S_k) \mathbf{1}_{\|Z_{k+1}\| \leq \epsilon} | S_k = S) - b(S) \right\| \\ & \leq \left\| \left(\frac{1}{\tau} \mathbb{E} ((S_{k+1} - S_k) | S_k = S, \|Z_{k+1}\| \leq \epsilon) - b(S) \right) \right\| \mathbb{P}(\|Z_{k+1}\| \leq \epsilon(r)) + \|b(S)\| \mathbb{P}(\|Z_{k+1}\| \geq \epsilon) \end{aligned}$$

As we saw that $\mathbb{P}(\|Z_{k+1}\| \geq \epsilon) \leq \tau^{1+\frac{\eta}{2}} \mathbb{E} [(\|\xi_{k+1}\|)^{2+\eta}] \times \left(\frac{1}{\epsilon(r) - \|\bar{m}\| \tau} \right)^{2+\eta}$ and given (3.35) and the fact that b is continuous, we find that:

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} ((S_{k+1} - S_k) \mathbf{1}_{\|Z_{k+1}\| \leq \epsilon} | S_k = S) - b(S) \right\| = 0 \quad (2.14)$$

Similarly, we show that

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} ((S_{k+1} - S_k)(S_{k+1} - S_k)^t \mathbf{1}_{\|Z_{k+1}\| \leq \epsilon} | S_k = S) - aa^t(S) \right\| = 0 \quad (2.15)$$

Given (2.1), we have the following inequality for S_{k+1}^i :

$$S_{k+1}^i \leq S_k^i \exp(Z^i) \left(1 + \frac{2\alpha_i}{D_i} \|f\|_\infty \right) \leq S_k^i \exp(\bar{m}_i \tau + \sqrt{\tau} \xi_{k+1}^i) \left(1 + \frac{2\alpha_i}{D_i} \|f\|_\infty \right)$$

which implies that, conditional on $S_k = S$ and for $p > 0$ such that $p\sqrt{\tau} < \eta$, $S_{k+1} \in L^p$. Using this result for $p = 2$, we find that for $\sqrt{\tau} < \frac{\eta}{2}$, $S_{k+1} \in L^2$ and we can use Cauchy Schwarz inequality:

$$\begin{aligned} & |\mathbb{E} ((S_{k+1}^i - S_k^i) \mathbf{1}_{\|Z_{k+1}\| \geq \epsilon} | S_k = S)| \\ & \leq \sqrt{\mathbb{E} ((S_{k+1}^i - S_k^i)^2 | S_k = S) \mathbb{P}(\|Z_{k+1}\| \geq \epsilon(r))} \\ & \leq \sqrt{\mathbb{E} ((S_{k+1}^i - S_k^i)^2 | S_k = S)} \sqrt{\mathbb{E} \left(\frac{\|\xi_{k+1}\| \sqrt{\tau}}{\epsilon(r) - \|\mu\| \tau} \right)^{4+\eta}} \\ & \leq \tau^{1+\frac{\eta}{4}} \sqrt{\mathbb{E} ((S_{k+1}^i - S_k^i)^2 | S_k = S)} \sqrt{\mathbb{E} \left(\frac{\|\xi_{k+1}\|}{\epsilon - \|\mu\| \tau} \right)^{4+\eta}} \end{aligned}$$

As $\mathbb{E}(\|\xi_{k+1}\|^{4+\eta}) < \infty$, $S_{k+1} \in L^2$ and S_{k+1} stays L^2 bounded as $\|S_k\| \leq r$ and τ goes to 0, we obtain:

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} ((S_{k+1} - S_k) \mathbf{1}_{\|Z_{k+1}\| \geq \epsilon} | S_k = S) \right\| = 0 \quad (2.16)$$

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Using the same property with $p=4$, we show that

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} \left((S_{k+1} - S_k)(S_{k+1} - S_k)^t \mathbf{1}_{\|Z_{k+1}\| \geq \epsilon} | S_k = S \right) \right\| = 0 \quad (2.17)$$

(3.36) and (3.38) give (2.11). Similarly, (4.41) and (3.39) give (2.12).

The following lemma gives the explicit expressions of a and b defined in (2.8) and (2.9) respectively.

Lemma 2.8.2 (2.8) and (2.9) respectively can be written as

$$a_{i,j}(S) = S^i \left[A_{i,j} + \frac{\alpha_i}{D_i} \frac{1}{V_0} f' \left(\frac{V(S)}{V_0} \right) (A^t \pi(S))_j \right] \quad (2.18)$$

$$b_i(S) = S^i m_i + S^i \frac{\alpha_i}{D_i} \frac{1}{2V_0^2} f'' \left(\frac{V(S)}{V_0} \right) \pi(S) \cdot \Sigma \pi(S) \\ + S^i \frac{\alpha_i}{D_i} \frac{1}{V_0} f' \left(\frac{V(S)}{V_0} \right) (\pi(S) \cdot \bar{m} + (\Sigma \pi(S))_i) \quad (2.19)$$

where $\pi(S) = \begin{pmatrix} \alpha_1 S^1 \\ \vdots \\ \alpha_n S^n \end{pmatrix}$ and $V(S) = \sum_{j=1}^n \alpha_j S^j$.

Proof Given (2.1), θ is defined as:

$$\theta_i(S, z) = S^i \exp(z_i) \left(1 + \frac{\alpha_i}{D_i} \left(f \left(\sum_{j=1}^n \frac{\alpha_j S^j}{V_0} \exp(z_j) \right) - f \left(\frac{V(S)}{V_0} \right) \right) \right)$$

As a consequence, we find that for $l \neq i$:

$$\frac{\partial \theta_i}{\partial z_l}(S, 0) = S^i \frac{\alpha_i}{D_i} f' \left(\frac{V(S)}{V_0} \right) \frac{\alpha_l S^l}{V_0}$$

and

$$\frac{\partial \theta_i}{\partial z_i}(S, 0) = S^i + S^i \frac{\alpha_i}{D_i} f' \left(\frac{V(S)}{V_0} \right) \frac{\alpha_i S^i}{V_0}$$

In addition, for $l \neq i$ and $j \neq i$, we find that

$$\frac{\partial^2 \theta_i}{\partial z_j \partial z_l}(S, 0) = S^i \frac{\alpha_i}{D_i} f'' \left(\frac{V(S)}{V_0} \right) \frac{\alpha_l S^l}{V_0} \frac{\alpha_j S^j}{V_0}$$

$$\frac{\partial^2 \theta_i}{\partial z_i \partial z_l}(S, 0) = S^i \frac{\alpha_i}{D_i} \left(f'' \left(\frac{V(S)}{V_0} \right) \frac{\alpha_l S^l}{V_0} \frac{\alpha_i S^i}{V_0} + f' \left(\frac{V(S)}{V_0} \right) \frac{\alpha_l S^l}{V_0} \right)$$

and

$$\frac{\partial^2 \theta_i}{\partial z_i^2}(S, 0) = S^i + S^i \frac{\alpha_i}{D_i} \left(f'' \left(\frac{V(S)}{V_0} \right) \left(\frac{\alpha_i S^i}{V_0} \right)^2 + f' \left(\frac{V(S)}{V_0} \right) \frac{2\alpha_i S^i}{V_0} \right)$$

and by direct computation of (2.8) and (2.9) we find 3.40 and 3.41.

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Because f is \mathcal{C}^3 , a and b are \mathcal{C}^2 and \mathcal{C}^1 . Furthermore, because f' , and hence f'' and $f^{(3)}$, have a compact support and because we focus on $S \in (\mathbb{R}_+^*)^n$ and $\alpha_i \geq 0$, there exists $R > 0$ such that when $\|S\| \geq R$, $f' \left(\frac{V(S)}{V_0} \right) = f'' \left(\frac{V(S)}{V_0} \right) = f^{(3)} \left(\frac{V(S)}{V_0} \right) = 0$. As a consequence, there exists $K > 0$ such that for all $S \in (\mathbb{R}_+^*)^n$:

$$\|a(S)\| + \|b(S)\| \leq K\|S\| \quad (2.20)$$

Furthermore, as the first derivatives of a and b are bounded, a and b are Lipschitz.

Define the differential operator $G : C_0^\infty(\mathbb{R}_+^*)^n \mapsto C_0^1(\mathbb{R}_+^*)^n$ by

$$Gh(x) = \frac{1}{2} \sum_{1 \leq i, j \leq n} (aa^t)_{i,j}(x) \partial_i \partial_j h + \sum_{1 \leq i \leq n} b_i(x) \partial_i h$$

As a and b verify (2.20), (Ethier and Kurtz, 1986, Theorem 2.6, Ch.8) states that the martingale problem associated to (G, δ_{S_0}) is well-posed. As we have shown Lemma 2.8.1, by (Ethier and Kurtz, 1986, Theorem 4.2, Ch.7), when $\tau \rightarrow 0$, $\left(S_{\lfloor \frac{t}{\tau} \rfloor} \right)_{t \geq 0}$ converges in distribution to the solution $(\mathbb{P}, (P_t)_{t \geq 0})$ of the martingale problem associated to (G, δ_{S_0}) . Furthermore, as a and b are Lipschitz, the solution of this martingale problem is the unique strong solution of the stochastic differential equation:

$$dP_t = b(P_t)dt + a(P_t)dW_t \quad \text{and} \quad P_0 = S_0$$

which concludes the proof of Theorem 2.4.2.

2.8.2 Proof of Proposition 2.6.1

Theorem 2.4.2 gives us the dynamics of asset prices in the continuous-time limit:

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i$$

where W is an n dimensional Brownian motion and where the coefficients are Lipschitz. As a consequence, starting from $P_0 \in (\mathbb{R}_+^*)^n$, for all $1 \leq i \leq n$, P_t^i is strictly positive. We also have for all $1 \leq i \leq n$, $\alpha_i \geq 0$ and so: $V_t = \sum_{1 \leq k \leq n} \alpha_k P_t^k$ is strictly positive. Let's focus on the dynamics of the fund's position.

$$dV_t = \sum_{1 \leq i \leq n} \alpha_i dP_t^i = \sum_{1 \leq i \leq n} \alpha_i P_t^i (\mu_i(P_t)dt + (\sigma(P_t)dW_t)_i)$$

Dividing by V_t and denoting $x_t^i = \frac{\alpha_i P_t^i}{V_t}$, we obtain

$$\frac{dV_t}{V_t} = \sum_{1 \leq i \leq n} x_t^i (\mu_i(P_t)dt + (\sigma(P_t)dW_t)_i)$$

CHAPTER 2. DISTRESSED SELLING AND ENDOGENOUS CORRELATION

$$\begin{aligned}
 &= \sum_{1 \leq i \leq n} x_t^i \mu_i(P_t) dt + \sum_{1 \leq i \leq n} x_t^i (\sigma(P_t) dW_t)_i \\
 &= \sum_{1 \leq i \leq n} x_t^i \mu_i(P_t) dt + \sum_{1 \leq i \leq n} x_t^i \left(\sum_{1 \leq j \leq n} \sigma_{i,j}(P_t) dW_t^j \right) \\
 &= \sum_{1 \leq i \leq n} x_t^i \mu_i(P_t) dt + \sum_{1 \leq j \leq n} \sum_{1 \leq i \leq n} x_t^i \sigma_{i,j}(P_t) dW_t^j.
 \end{aligned}$$

As a consequence, the instantaneous variance of the fund is $\sum_{1 \leq j \leq n} \left(\sum_{1 \leq i \leq n} x_t^i \sigma_{i,j}(P_t) \right)^2$.

The statement follows.

Chapter 3

Fire sales forensics: measuring endogenous risk

Abstract

We propose a tractable framework for quantifying the impact of fire sales on the volatility and correlations of asset returns in a multi-asset setting. Our results enable to quantify the impact of fire sales on the covariance structure of asset returns and provide a quantitative explanation for spikes in volatility and correlations observed during liquidation of large portfolios. These results allow to estimate the impact and magnitude of fire sales from observation of market prices: we give conditions for the identifiability of model parameters from time series of asset prices, propose an estimator for the magnitude of fire sales in each asset class and study the consistency and large sample properties of the estimator. We illustrate our estimation methodology with two empirical examples: the hedge fund losses of August 2007 and the Great Deleveraging following the default of Lehman Brothers in Fall 2008.

3.1 Introduction

Fire sales or, more generally, the sudden deleveraging of large financial portfolios, have been recognized as a destabilizing factor in recent (and not-so-recent) financial crises, contributing to unexpected spikes in volatility and correlations of asset returns and resulting in spirals of losses for investors (Carlson, 2006; Brunnermeier, 2008; Khandani and Lo, 2011). In particular, unexpected increases in correlations across asset classes have frequently occurred during market downturns (Cont and Wagalath, 2012b; Bailey et al., 2012), leading to a loss of diversification benefits for investors, precisely when such benefits were desirable.

For instance, during the first week of August 2007, when a large fund manager deleveraged his/her positions in long-short market neutral equity strategies, other long-short market neutral equity funds experienced huge losses, while in the meantime, index funds were left unaffected (Khandani and Lo, 2011). On a larger scale, the Great Deleveraging of financial institutions' portfolios subsequent to the default of Lehman Brothers in fall 2008 led to an unprecedented peak in correlations across asset returns (Fratzscher, 2011).

The importance of fire sales as a factor of market instability is recognized in the economic literature. Shleifer and Vishny (1992, 2011) characterize an asset fire sale by a financial institution as a forced sale in which potential high valuation buyers are affected by the same shocks as the financial institution, resulting in a sale of the asset at a discounted price to non specialist buyers. They underline the fact that in the presence of fire sales, losses by financial institutions with overlapping holdings become self-reinforcing, leading to downward spirals for asset prices and, ultimately, to systemic risk. Pedersen (2009) describes qualitatively the effects of investors running for the exit and the spirals of losses and spillover effects they generate. Shin (2010) and ? propose equilibrium models which takes into account the supply and demand generated by investors reacting to a price move and show how feedback effects contribute to the amplification of volatility and market instability. ? emphasize the role of institutional investors in price-mediated contagion, suggesting that crisis spread through the asset holdings of international investors rather than through changes in fundamentals. Brunnermeier (2008) describes the channel through which losses in mortgage backed securities during the recent financial crisis led to huge losses in equity markets, although those two assets classes had been historically uncorrelated.

The empirical link between fire sales and increase in correlation across asset returns has been documented in several recent studies. Coval and Stafford (2007) give empirical evidence for fire sales by open-end mutual funds by studying the transactions caused by capital flows. They show that funds in distress experience outflows of capital by investors which result in fire sales in existing positions, creating a price pressure in the securities held in common by distressed funds. Jotikasthira et al. (2011) lead an empirical investigation on the effects of fund flows from developed countries to emerging markets. They show that such investment flows generate forced trading by fund managers, affecting asset prices and correlations between emerging markets and creating a

CHAPTER 3. FIRE SALES FORENSICS: MEASURING ENDOGENOUS RISK

new channel through which shocks are transmitted from developed markets to emerging markets. Anton and Polk (2008) find empirically that common active mutual fund ownership predicts cross-sectional variation in return realized covariance.

However, although the empirical examples cited above are related to liquidation of large *portfolios*, most theoretical studies focus for simplicity on fire sales in a single asset market and thus are not able to investigate the effect of fire sales on asset return correlations and the resulting limits to diversification alluded to above.

Kyle and Xiong (2001) propose an equilibrium model, which takes into account the supply and demand of three categories of traders: noise traders, long-term investors and convergence traders, in a market with two risky assets and find that convergence traders, who are assumed to trade using a logarithmic utility function, can react to a price shock in one asset by deleveraging their positions in both markets, leading to contagion effects. Greenwood and Thesmar (2011) propose a simple framework for modeling price dynamics which takes into account the ownership structure of financial assets, considered as given exogenously. Cont and Wagalath (2012b) model the systematic supply and demand generated by investors exiting a large distressed fund and quantify its impact on asset returns.

We propose here a tractable framework for modeling and estimating the impact of fire sales in multiple funds on the volatility and correlations of asset returns in a multi-asset setting. We explore the mathematical properties of the model in the continuous-time limit and derive analytical results relating the realized covariance of asset returns to the parameters describing the volume of fire sales. In particular, we show that, starting from homoscedastic inputs, feedback effects from fire sales naturally generate heteroscedasticity in the covariance structure of asset returns, thus providing an economic interpretation for various multivariate models of heteroscedasticity in the recent literature (Engle, 2002; Da Fonseca et al., 2008; Gouriéroux et al., 2009; Stelzer, 2010). Our results allow for a structural explanation for the variability observed in measures of cross sectional dependence in asset returns (Bailey et al., 2012), by linking such increases in cross-sectional correlation to the deleveraging of large portfolios.

The analytically tractable nature of these results allows to explore in detail the problem of *estimating* these parameters from empirical observations of price series; we explore the corresponding identification problem and propose a method for estimating the magnitude of distressed selling in each asset class, and study the consistency and large sample properties of the proposed estimator. These results provide a quantitative framework for the 'forensics analysis' of the impact of fire sales and distressed selling, which we illustrate with two empirical examples: the August 2007 hedge fund losses and the Great Deleveraging of bank portfolios following the default of Lehman Brothers in September 2008.

Our framework links large shifts in the realized covariance structure of asset returns with the liquidation of large portfolios, in a framework versatile enough to be amenable to empirical data. This provides a toolbox for risk managers and regulators in view of investigating unusual market events and their impact on the risk of portfolios in a systematic way, moving a step in the direction proposed by Fielding et al. (2011), who

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underlined the importance of systematically investigating all 'systemic risk' events in financial markets, as done by the National Transportation Safety Board for major civil transportation accidents.

Outline This paper is organized as follows. Section 3.2 presents a simple framework for modeling the impact of fire sales in various funds on asset returns. Section 3.3 resolves the question of the identification and estimation of the model parameters, characterizing the fire sales. Section 3.4 displays the results of our estimation procedure on liquidations occurring after the collapse of Lehman Brothers while Section 3.5 is focused on the study of the positions liquidated during the first week of August 2007.

3.2 Fire sales and endogenous risk

3.2.1 Impact of fire sales on price dynamics: a multiperiod model

Consider a financial market where n assets/financial strategies are traded at discrete dates $t_k = \frac{k}{N}$, multiples of a time step $\frac{1}{N}$ (taken to be a trading day in the empirical examples: $N = 250$). The value of asset/financial strategy i at date t_k is denoted S_k^i .

We consider J institutional investors trading in these assets: fund j initially holds α_i^j units of asset i . The value of this (benchmark) portfolio at date t_k is denoted

$$V_k^j = \sum_{i=1}^n \alpha_i^j S_k^i \quad (3.1)$$

The impact of (exogenous) economic factors ('fundamentals') on prices is modeled through an IID sequence $(\xi_k)_{k \geq 1}$ of \mathbb{R}^n -valued centered random variables such that, in the absence of fire sales, the return of asset i during period $[t_k, t_{k+1}]$ is given by

$$\exp \left(\frac{1}{N} \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\frac{1}{N}} \xi_{k+1}^i \right) - 1$$

Here m_i represents the expected return of asset i in the absence of fire sales and the 'fundamental' covariance matrix Σ , defined by

$$\Sigma_{i,j} = \text{cov}(\xi_k^i, \xi_k^j)$$

represents the covariance structure of returns in the absence of large systematic trades by institutional investors.

Typically, over short time horizons of a few days, institutional investors do not alter their portfolio allocations. However, the occurrence of large losses typically leads the fund to sell off part of its assets (Coval and Stafford, 2007; Jotikasthira et al., 2011; Shleifer and Vishny, 2011). Such *distressed selling* may be triggered endogenously by

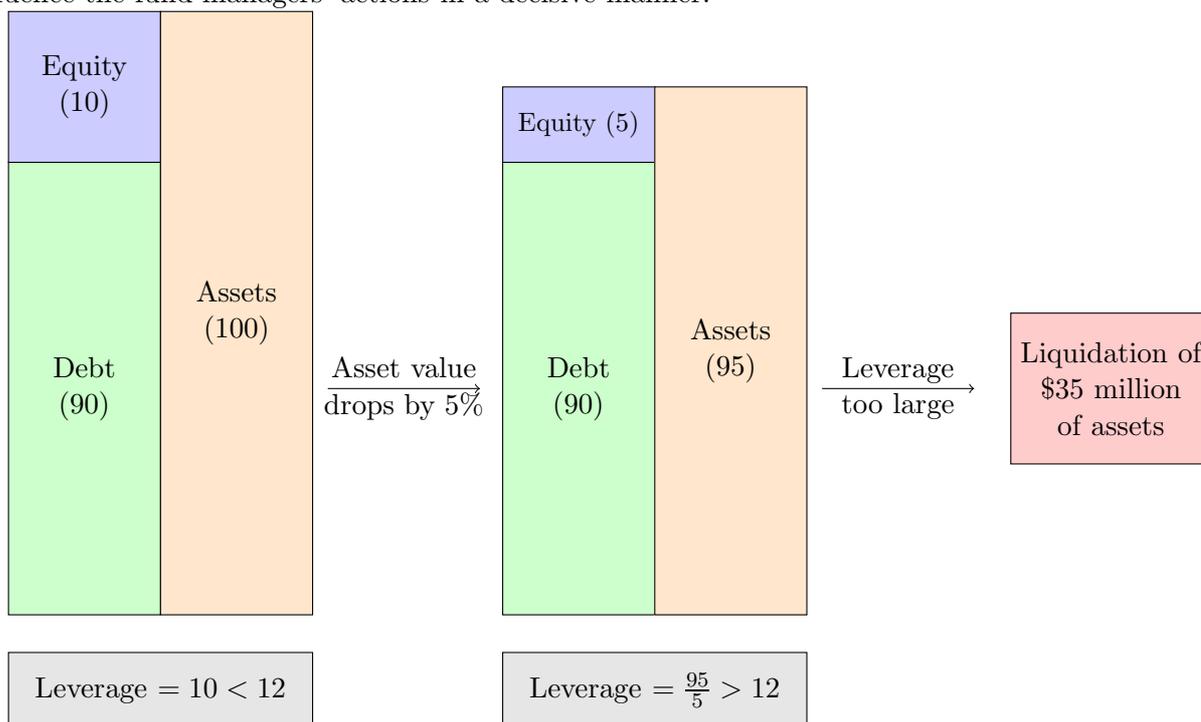
- capital requirements set by regulators or target leverage ratios set by fund managers, which lead financial institutions to deleverage their portfolios when faced

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with trading losses (Danielsson et al., 2004; Greenwood and Thesmar, 2011). Consider the simple example of a fund whose maximal leverage ratio is 12. Initially this fund possesses \$10 million of equity and borrows \$90 million to build a portfolio of assets worth \$100 million. The initial leverage of this fund is hence equal to $\frac{Assets}{Assets-Debt} = \frac{100}{100-90} = 10 < 12$.

A decline of d (expressed in percent) in the value of the assets held by the fund modifies the fund's leverage to a value of $\frac{100 \times (1-d)}{100 \times (1-d) - 90}$. As a consequence, a decline in asset value of more than 1.8% leads to a spike in the fund's leverage ratio above the maximum leverage ratio of 12. In order to maintain such maximum leverage ratio, the fund can either raise equity (which can be costly, especially at a time when its portfolio value is decreasing) or, most likely, engage in fire sales. The diagram below illustrates such endogenous mechanism for distressed selling when asset value drops by 5%, leading to liquidation of \$35 million of assets. On the contrary, as long as the drop in asset value is lower than 1.8%, the leverage of the fund remains below 12 and there is no distressed selling.

Note that this mechanism is asymmetric with respect to losses/gains: large losses trigger fire sales, but large gains do not necessarily result in massive buying. Once the capital requirement constraints or leverage constraints are not binding, they may cease to influence the fund managers' actions in a decisive manner.



Fire sales may be also due to:

- investors redeeming (or expanding) their positions depending on the performance of the funds, causing inflows and outflows of capital. This mechanism is described by

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Coval and Stafford (2007), who show empirically that funds in distress experience outflows of capital by investors and explain that, as the ability of borrowing is reduced for distressed funds and regulation and self-imposed constraints prevent them from short-selling other securities, such outflows of capital result in fire sales in existing positions.

- rule based strategies –such as portfolio insurance– which result in selling when a fund underperforms (Gennotte and Leland, 1990),
- sale of assets held as collateral by creditors of distressed funds (Shleifer and Vishny, 2011).

The impact of fire sales may also be exacerbated by short-selling and predatory trading: Brunnermeier and Pedersen (2005) show that, in the presence of fire sales in a distressed fund, the mean-variance optimal strategy for other investors is to short-sell the assets held by the distressed fund and buy them back after the period of distress. A common feature of these mechanisms is that they react to a (negative) change in fund value.

Here we do not attempt to model each of these mechanisms in detail but focus instead on their aggregate effect. This aggregate effect may be modeled in a parsimonious manner by introducing a *deleveraging schedule*, represented by a function f_j which measures the systematic supply/demand generated by the fund j as a function of the fund's return: when, due to market shocks, the value of the portfolio j moves over $[t_k, t_{k+1}]$ from V_k^j to

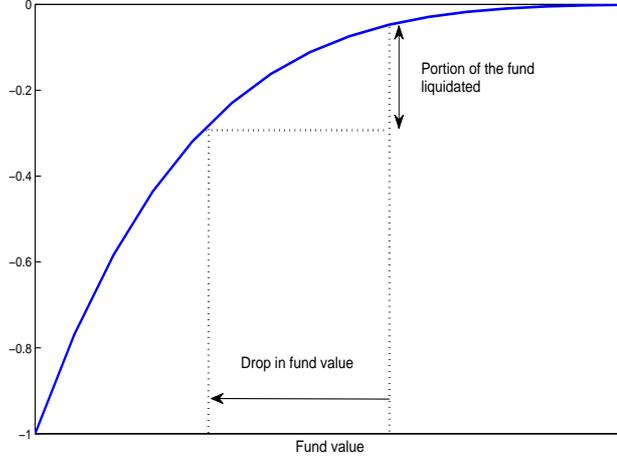
$$\sum_{l=1}^n \alpha_l^j S_k^l \exp\left(\frac{1}{N}(m_l - \frac{\Sigma_{l,l}}{2}) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)$$

a portion

$$f_j\left(\frac{V_k^j}{V_0^j}\right) - f_j\left(\frac{1}{V_0^j} \sum_{l=1}^n \alpha_l^j S_k^l \exp\left(\frac{1}{N}(m_l - \frac{\Sigma_{l,l}}{2}) + \sqrt{\frac{1}{N}}\xi_{k+1}^l\right)\right) \quad (3.2)$$

of fund j is liquidated between t_k and t_{k+1} , proportionally in each asset detained by the fund.

As shown in the previous example and by Jotikasthira et al. (2011), negative returns for a fund lead to outflows of capital from this fund: this implies that f_j is an increasing function. Fire sales occur when a fund underperforms significantly and its value goes below a threshold and it ends when the fund is entirely liquidated: as a consequence, we choose f_j to be constant for small and large values of its argument (i.e. constant outside an interval $[\beta_j^{liq}, \beta_j]$) with $\beta_j < 1$. Furthermore, we choose f_j to be concave, capturing the fact that fire sales accelerate as the fund exhibits larger losses. Figure 3.1 displays an example of such a deleveraging schedule f_j . As long as fund j 's value remains above $\beta_j V_0^j$, the portion liquidated, given in (3.2), is equal to zero, as f_j is constant on $[\beta_j, +\infty[$: there are no fire sales. A drop in fund value below that threshold generates fire sales of a portion of fund j , described in (3.2).


 Figure 3.1: Example of a deleveraging schedule f_j

When the trades are sizable with respect to the average trading volume, the supply/demand generated by this deleveraging strategy impacts asset prices. We introduce, for each asset i , a *price impact* function $\phi_i(\cdot)$ which captures this effect: the impact of buying v shares (where $v < 0$ represents a sale) on the return of asset i is $\phi_i(v)$. We assume that $\phi_i : \mathbb{R} \mapsto \mathbb{R}$ is increasing and $\phi_i(0) = 0$.

The impact of fire sales on the return of asset i is then equal to

$$\phi_i \left[\sum_{j=1}^J \alpha_i^j \left(f_j \left(\frac{1}{V_0^j} \sum_{l=1}^n \alpha_l^j S_k^l \exp \left(\frac{1}{N} \left(m_l - \frac{\Sigma_{l,l}}{2} \right) + \sqrt{\frac{1}{N}} \xi_{k+1}^l \right) \right) - f_j \left(\frac{V_k^j}{V_0^j} \right) \right) \right]$$

The price dynamics can be summed up as follows:

$$\boxed{S_k^i} \xrightarrow[\text{factors } (\xi_{k+1})]{\text{exogenous}} \boxed{S_k^i \exp \left(\frac{1}{N} \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\frac{1}{N}} \xi_{k+1}^i \right)} \xrightarrow{\text{fire sales}} \boxed{S_{k+1}^i}$$

$$S_{k+1}^i = S_k^i \exp \left(\frac{1}{N} \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\frac{1}{N}} \xi_{k+1}^i \right) \times$$

$$\left(1 + \phi_i \left[\sum_{j=1}^J \alpha_i^j \left(f_j \left(\frac{1}{V_0^j} \sum_{l=1}^n \alpha_l^j S_k^l \exp \left(\frac{1}{N} \left(m_l - \frac{\Sigma_{l,l}}{2} \right) + \sqrt{\frac{1}{N}} \xi_{k+1}^l \right) \right) - f_j \left(\frac{V_k^j}{V_0^j} \right) \right) \right] \right) \quad (3.3)$$

where V_k^j is the benchmark portfolio value of fund j at date t_k , defined in (3.1).

At each period, the return of asset i can be decomposed into a fundamental component, which is independent from the past, and an endogenous component due to the impact of fire sales. Note that when there are no fire sales, this endogenous term is equal to zero and the return of asset i is equal to its fundamental return.

Assumption 3.2.1 $S_0 \in (\mathbb{R}_+^*)^n$ and $\min_{1 \leq i \leq n} \phi_i \left(-2 \sum_{j=1}^n |\alpha_i^j| \times \|f_j\|_\infty \right) > -1$.

Proposition 3.2.2 *Under Assumption 3.2.1, (3.1)–(3.3) define a price dynamics S which is a discrete-time Markov process in $(\mathbb{R}_+^*)^n$.*

Proof Equations (3.1) and (3.3) show that S_{k+1} depends only on its value at t_k and on ξ_{k+1} , which is independent of events previous to t_k . The price vector S is thus a discrete-time Markov process. Furthermore, when $\min_{1 \leq i \leq n} \phi_i \left(-2 \sum_{j=1}^n |\alpha_i^j| \times \|f_j\|_\infty \right) > -1$, the endogenous price impact due to fire sales, is strictly larger than -1, which ensures that the Markov process stays in $(\mathbb{R}_+^*)^n$.

This multiperiod model exhibits interesting properties: in particular, as shown in (Cont and Wagalath, 2012b), the presence of distressed selling induces an endogenous, heteroscedastic component in the covariance structure of returns, which leads to path-dependent realized correlations, even in the absence of any heteroscedasticity in the fundamentals.

Figure 3.2 shows an example of such endogenous correlations: we simulated 10^6 price trajectories of this multiperiod model with the parameters used in (Cont and Wagalath, 2012b, Section 3) and for each trajectory, we computed the realized correlation between all pairs of assets. We find that even in the case where the exogenous shocks driving the asset values are independent (i.e. the 'fundamental' covariance matrix Σ is diagonal), the presence of distressed selling leads to significant realized correlations, thereby increasing the volatility experienced by investors holding the fund during episodes of fire sales. This phenomenon may substantially decrease the benefits of diversification.

Our goal is to explore such effects systematically and propose a method for estimating their impact on price dynamics.

3.2.2 Continuous-time limit

The multiperiod model described above is rather cumbersome to study directly; in the sequel we focus on its continuous-time limit, which is analytically tractable and more easily related to commonly used diffusion models for price dynamics. This will allow us to compute realized covariances between asset returns in the presence of feedback effects from distressed selling.

For two n -dimensional vectors x and y , we denote $x \cdot y = \sum_{1 \leq i \leq n} x_i y_i$ the scalar product between vectors x and y . For $M \in \mathcal{M}_n(\mathbb{R})$, M^t is the transpose of matrix M . $\mathcal{S}_n(\mathbb{R})$ (resp. $\mathcal{S}_n^+(\mathbb{R})$) denotes the set of real-valued symmetric matrices (resp. real-valued symmetric positive semi-definite matrices). For a sequence $X^{(N)}$ of random variables indexed by integers N , we denote the fact that $X^{(N)}$ converges in law (resp. in probability) to X

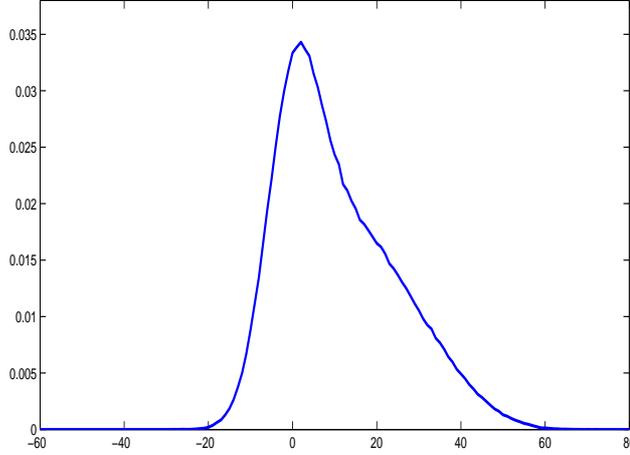


Figure 3.2: Distribution of realized correlation between two securities in the presence of distressed selling (case of zero fundamental correlation)

when N goes to infinity by $X^{(N)} \xrightarrow[N \rightarrow \infty]{\Rightarrow} X$ (resp. $X^{(N)} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} X$). For $(a, b) \in \mathbb{R}^2$, we denote $a \wedge b = \min(a, b)$.

In order to study the continuous-time limit of the multiperiod model described in the previous section, we make the following assumption.

Assumption 3.2.3 For $i = 1..n$, $j = 1..J$,

$$\phi_i \in \mathcal{C}^3(\mathbb{R}) , \quad f_j \in \mathcal{C}_0^3(\mathbb{R}) \quad \text{and} \quad \alpha_i^j \geq 0$$

$$\exists \eta > 0, \mathbb{E}(\|\exp(\eta\xi)\|) < \infty \quad \text{and} \quad \mathbb{E}(\|\xi\|^{\eta+4}) < \infty$$

where $\mathcal{C}_0^p(\mathbb{R})$ denotes the set of real-valued, p -times continuously differentiable maps whose first derivative has compact support.

Note that if $f_j \in \mathcal{C}_0^p(\mathbb{R})$, all its derivatives of order $1 \leq l \leq p$ have compact support. In particular f_j is constant for large values and very small values of its argument. This assumption has a natural interpretation in our context: fire sales occur when funds underperform, i.e. when the value of the fund relative to a benchmark falls below a threshold, and cease when the fund defaults, i.e. when the value of the fund relative to the benchmark decreases below a default threshold.

Theorem 3.2.4 Under Assumptions 3.2.1 and 3.2.3, the process $(S_{[Nt]})_{t \geq 0}$ converges weakly on the Skorokhod space $D([0, \infty[, \mathbb{R}^n)$, as $N \rightarrow \infty$, to a diffusion process $(P_t)_{t \geq 0}$ solution of the stochastic differential equation

$$\frac{dP_t^i}{P_t^i} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \quad 1 \leq i \leq n \quad (3.4)$$

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where μ (resp., σ) is a \mathbb{R}^n -valued (resp. matrix-valued) mapping defined by

$$\sigma_{i,k}(P_t) = A_{i,k} + \phi'_i(0) \sum_{j=1}^J \alpha_i^j f'_j\left(\frac{V_t^j}{V_0^j}\right) \frac{(A\pi_t^j)_k}{V_0^j} \quad (3.5)$$

$$\begin{aligned} \mu_i(P_t) = & m_i + \frac{\phi'_i(0)}{2} \sum_{j=1}^J \frac{\alpha_i^j}{(V_0^j)^2} f''_j\left(\frac{V_t^j}{V_0^j}\right) \pi_t^j \cdot \Sigma \pi_t^j \\ & + \sum_{j=1}^J \phi'_i(0) \frac{\alpha_i^j}{V_0^j} f'_j\left(\frac{V_t^j}{V_0^j}\right) \left(\pi_t^j \cdot \bar{m} + (\Sigma \pi_t^j)_i\right) + \frac{\phi''_i(0)}{2} \sum_{j,r=1}^J \frac{\alpha_i^j \alpha_i^r}{V_0^j V_0^r} f'_j\left(\frac{V_t^j}{V_0^j}\right) f'_r\left(\frac{V_t^r}{V_0^r}\right) \pi_t^j \cdot \Sigma \pi_t^r \end{aligned} \quad (3.6)$$

Here W_t is an n -dimensional Brownian motion, $\pi_t^j = \begin{pmatrix} \alpha_1^j P_t^1 \\ \vdots \\ \alpha_n^j P_t^n \end{pmatrix}$ is the (dollar) allocation of fund j , $V_t^j = \sum_{k=1}^n \alpha_k^j P_t^k$ is the value of fund j , $\bar{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$ and A is a square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

The proof of this Theorem is given in Appendix 3.6.1.

Remark 3.2.5 *The limit price process that we exhibit in Theorem 3.2.4 depends on the price impact functions only through their first and second derivatives in 0, $\phi'_i(0)$ and $\phi''_i(0)$. In particular, the expression of σ in (3.5) shows that realized volatilities and realized correlations of asset returns depend only on the slope $\phi'_i(0)$ of the price impact function. As a consequence, under our assumptions, a linear price impact function would lead to the same realized covariance structure for asset returns in the continuous-time limit.*

In the remainder of this paper, which is dedicated to the study of the impact of fire sales on the covariance structure of asset returns, we hence use the assumption of linear price impact: $D_i = \frac{1}{\phi'_i(0)}$ then corresponds to the market depth for asset i and is interpreted as the number of shares an investor has to buy in order to increase the price of asset i by 1%.

Corollary 3.2.6 (Case of linear price impact) *When $\phi_i(x) = \frac{x}{D_i}$, the drift and volatility of the stochastic differential equation (3.4) verified by the continuous-time price process are:*

$$\sigma_{i,k}(P_t) = A_{i,k} + \frac{1}{D_i} \sum_{1 \leq j \leq J} \alpha_i^j f'_j\left(\frac{V_t^j}{V_0^j}\right) \frac{(A\pi_t^j)_k}{V_0^j} \quad (3.7)$$

$$\mu_i(P_t) = m_i + \frac{1}{D_i} \sum_{j=1}^J \left(\frac{\alpha_i^j}{2(V_0^j)^2} f''_j\left(\frac{V_t^j}{V_0^j}\right) \pi_t^j \cdot \Sigma \pi_t^j + \frac{\alpha_i^j}{V_0^j} f'_j\left(\frac{V_t^j}{V_0^j}\right) \left(\pi_t^j \cdot \bar{m} + (\Sigma \pi_t^j)_i\right) \right) \quad (3.8)$$

where W_t , π_t^j , V_t^j , \bar{m} and A are defined in Theorem 3.2.4.

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When market depths are infinite, the price dynamics follows a multivariate exponential Brownian motion. In the presence of fire sales by distressed sellers, the fundamental dynamics of the assets is modified.

3.2.3 Realized covariance in the presence of fire sales

The *realized covariance* (Andersen et al., 2003; Barndorff-Nielsen and Shephard, 2004) between dates t_1 and t_2 computed on a time grid with step $\frac{1}{N}$ is defined as

$$\widehat{C}_{[t_1, t_2]}^{(N)} = \frac{1}{t_2 - t_1} ([X, X]_{t_2}^{(N)} - [X, X]_{t_1}^{(N)}) \quad (3.9)$$

where X is the log price process defined by $X_t^i = \ln P_t^i$ and $[X, X]_t^{(N)} = \left([X^i, X^k]_t^{(N)} \right)_{1 \leq i, k \leq n}$ with

$$[X^i, X^k]_t^{(N)} = \sum_{1 \leq l \leq \lfloor tN \rfloor} (X_{l/N}^i - X_{(l-1)/N}^i)(X_{l/N}^k - X_{(l-1)/N}^k) \quad (3.10)$$

As N goes to infinity, the process $\left([X, X]_t^{(N)} \right)_{t \geq 0}$ converges in probability on the Skorokhod space $D([0, \infty[, \mathbb{R}^n)$ to an increasing, $\mathcal{S}_n^+(\mathbb{R})$ -valued process $([X, X]_t)_{t \geq 0}$, the quadratic covariation of X (Jacod and Protter, 2012, Theorem 3.3.1). We define the $\mathcal{S}_n^+(\mathbb{R})$ -valued process $c = (c_t)_{t \geq 0}$, which corresponds intuitively to the 'instantaneous covariance' of returns, as the derivative of the quadratic covariation process. The realized covariance matrix of returns between t_1 and t_2 is denoted $C_{[t_1, t_2]}$.

$$[X, X]_t = \int_0^t c_s ds \quad C_{[t_1, t_2]} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} c_t dt \quad (3.11)$$

Theorem 3.2.6 allows to compute the realized covariance matrix for the n assets.

Proposition 3.2.7 *The instantaneous covariance matrix of returns, c_t , defined in (3.11), is given by:*

$$c_t = \Sigma + \sum_{j=1}^J \left[\frac{1}{V_0^j} f'_j \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] + \sum_{j,k=1}^J \frac{\pi_t^j \cdot \Sigma \pi_t^k}{V_0^j V_0^k} f'_j \left(\frac{V_t^j}{V_0^j} \right) f'_k \left(\frac{V_t^k}{V_0^k} \right) \Lambda_j \Lambda_k^t$$

where

$$\pi_t^j = \begin{pmatrix} \alpha_1^j P_t^1 \\ \vdots \\ \alpha_n^j P_t^n \end{pmatrix} \text{ denotes the (dollar) holdings of fund } j \text{ and } \Lambda_j = \begin{pmatrix} \frac{\alpha_1^j}{D_1} \\ \vdots \\ \frac{\alpha_n^j}{D_n} \end{pmatrix} \text{ rep-}$$

resents the positions of fund j in each market as a fraction of the respective market depth.

Fire sales impact realized covariances between assets. In the presence of fire sales, realized covariance is the sum of the fundamental covariance matrix Σ and an excess

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realized covariance which is liquidity-dependent and path-dependent. The magnitude of this endogenous impact is measured by the vectors Λ_j , which represent the positions of each fund as a fraction of asset market depths. The volume generated by fire sales in fund j on each asset i is equal to $\alpha_i^j \times f_j'$ and its impact on the return of asset i is equal to $\frac{\alpha_i^j}{D_i} \times f_j'$. This impact can be significant even if the asset is very liquid, when the positions liquidated are large enough compared to the asset's market depth. Thus, even starting with homoscedastic inputs, fire sales naturally lead to endogenous patterns of heteroscedasticity in the covariance structure of asset returns—in particular spikes or plateaux of high correlation during liquidation periods—similar to those observed in empirical data.

More precisely, we observe that the excess realized covariance terms due to fire sales contain a term of order one in $\|\Lambda\|$ plus higher order terms:

$$c_t = \Sigma + \sum_{j=1}^J \left[\frac{1}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] + O(\|\Lambda\|^2) \quad (3.12)$$

where

$$\Lambda = (\Lambda_1, \dots, \Lambda_J) \in \mathcal{M}_{n \times J}(\mathbb{R}) \quad (3.13)$$

where Λ_j is defined in Proposition 3.2.7 and $\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2}$ is bounded as $\|\Lambda\| \rightarrow 0$. This result is due to the fact that under Assumption 3.2.3, the second order terms $\frac{\pi_t^j \cdot \Sigma \pi_t^k}{V_0^j V_0^k} f_j' \left(\frac{V_t^j}{V_0^j} \right) f_k' \left(\frac{V_t^k}{V_0^k} \right)$ in the expression of c_t in Proposition 3.2.7 are bounded because for all $1 \leq j \leq n$, f_j' has a compact support.

In addition, if we denote γ_j the average rate of liquidation (for example $\gamma_j = \frac{f_j(\beta_j) - f_j(\beta_j^{liq})}{\beta_j - \beta_j^{liq}}$), we can approximate the terms of order one in $\|\Lambda\|$ in (3.12) as follows:

$$\sum_{j=1}^J \left[\frac{1}{V_0^j} f_j' \left(\frac{V_t^j}{V_0^j} \right) \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] = \sum_{j=1}^J \left[\frac{\gamma_j}{V_0^j} \left(\Lambda_j (\pi_t^j)^t \Sigma + \Sigma \pi_t^j \Lambda_j^t \right) \right] + O(\|f''\|)$$

where $\|f''\| = \sum_{j=1}^J \|f_j''\|_\infty$.

As a consequence, Proposition 3.2.7 may be interpreted as follows: if there are no fire sales between 0 and T , the realized covariance of returns between 0 and T is given by

$$C_{[0,T]} = \frac{1}{T} \int_0^T c_t dt = \Sigma$$

while the realized covariance between T and $T + \tau_{liq}$ (where liquidations could have occurred) contains an endogenous component, whose leading terms will be

$$C_{[T, T+\tau_{liq}]} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t dt = \Sigma + LM_0 \Pi \Sigma + \Sigma \Pi M_0 L + O(\|\Lambda\|^2, \|f''\|) \quad (3.14)$$

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where the remainder is composed of higher order corrections in $\|\Lambda\|^2$ and $\|f''\|$, and

$$M_0 = \sum_{j=1}^J \frac{\gamma_j}{V_0^j} \times \alpha^j (\alpha^j)^t \quad (3.15)$$

where $\alpha^j = \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_n^j \end{pmatrix}$ is the vector of positions of fund j and L and Π are diagonal ma-

trices with i -th diagonal term equal respectively to $\frac{1}{D_i}$ and $\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i dt$. In practice, as shown by simulation studies in (Cont and Wagalath, 2012b), this first order approximation is precise enough and we will focus on this approximation in the numerical examples.

In the absence of distressed selling between 0 and T , the realized covariances between asset returns during this period are equal to their fundamental value. Between T and $T + \tau_{liq}$, fire sales can affect the realized covariance between asset returns. The excess realized covariance is characterized by a matrix M_0 , defined in (3.15), which reflects the magnitude of the fire sales. Note that we do not assume that all the funds are liquidating between T and $T + \tau_{liq}$. A fund j which is not subject to fire sales during this period of time has a rate of liquidation γ_j equal to zero.

In (3.15), $\alpha^j (\alpha^j)^t$ is a $n \times n$ symmetric matrix representing an orthogonal projection on fund j 's positions and hence M_0 is a sum of projectors. The symmetric matrix M_0 captures the direction and intensity of liquidations in the J funds.

3.2.4 Spillover effects: price-mediated contagion

Consider now the situation where a reference fund with positions $(\alpha_1, \dots, \alpha_n)$ is subject to distressed selling. As argued above, this leads to endogenous volatility and correlations in asset prices, which then modifies the volatility experienced by any other fund holding the same assets.

Proposition 3.2.7 allows to compute the magnitude of this *volatility spillover* effect (Cont and Wagalath, 2012b). The following result shows that the realized variance of a (small) fund with positions $(\mu_t^i, i = 1..n)$ is the sum of the realized variance in the absence of distressed selling and an endogenous term which represents the impact of fire sales in the reference fund.

Corollary 3.2.8 (Spillover effects) *In the presence of fire sales in a reference fund with positions $(\alpha_1, \dots, \alpha_n)$, the realized variance for a small fund with positions $(\mu_t^i)_{1 \leq i \leq n}$ between t_1 and t_2 is equal to $\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \gamma_s ds$ where*

$$\gamma_s M_s^2 = \pi_s^\mu \cdot \Sigma \pi_s^\mu + \frac{2f'(\frac{V_s}{V_0})}{V_0} (\pi_s^\mu \cdot \Sigma \pi_s^\alpha) (\Lambda \cdot \pi_s^\mu) + \frac{f'(\frac{V_s}{V_0})^2}{V_0^2} (\pi_s^\alpha \cdot \Sigma \pi_s^\alpha) (\Lambda \cdot \pi_s^\mu)^2 \quad (3.16)$$

where $\pi_s^\alpha = \begin{pmatrix} \alpha_1 P_s^1 \\ \vdots \\ \alpha_n P_s^n \end{pmatrix}$ and $\pi_s^\mu = \begin{pmatrix} \mu_t^1 P_s^1 \\ \vdots \\ \mu_t^n P_s^n \end{pmatrix}$ denote the (dollar) holdings of the reference fund and the small fund respectively, $M_s = \sum_{i=1}^n \mu_s^i P_s^i$ is the small fund's value, and $\Lambda = (\frac{\alpha_1}{D_1}, \dots, \frac{\alpha_n}{D_n})^t$ represents the positions of the reference fund in each market as a fraction of the respective market depth.

The second and third term in (3.16), which represent the price-mediated contagion of endogenous risk from the distressed fund to other funds holding the same assets, are maximal for funds whose positions are colinear to those of the distressed fund. On the other hand, these endogenous terms are zero if the two portfolios verify an 'orthogonality condition':

$$\Lambda \cdot \pi_t^\mu = \sum_{i=1}^n \frac{\alpha_i}{D_i} \mu_t^i P_t^i = 0, \quad (3.17)$$

in which case the fund with positions μ_t is not affected by the fire sales of assets by the distressed fund.

3.3 Identification and estimation

Theorem 3.2.4 describes the convergence of the multiperiod model to its diffusion limit under the assumption that the funds liquidate long positions. However, the continuous-time model given in Theorem 3.2.4 makes sense in a more general setting where we relax the constraint on the sign of α_i^j i.e. when long-short portfolios are liquidated: in this case, the coefficients of the stochastic differential equation are still locally Lipschitz, so by (Ikeda and Watanabe, 1981, Theorem 3.1, Ch.4) the equation still has a unique strong solution on some interval $[0, \tau[$, where τ is a stopping time (possibly infinite).

In the sequel, we consider the continuous-time model given in Theorem 3.2.4 in this more general setting which allows for the liquidation of long-short portfolios. Note that the expressions for covariances and spillover effects are not modified.

3.3.1 Inverse problem and identifiability

Equation (3.14) describes the leading term in the impact of fire sales on the realized covariance matrix of returns. Conversely, given that realized covariances can be estimated from observation of prices series, one can use this relation to recover information about the volume of liquidation during a fire sales episode.

We now consider the inverse problem of explaining 'abnormal' patterns in realized covariance and volatility in the presence of fire sales and estimating the parameters of the liquidated portfolio from observations of prices. Mathematically, this boils down to answering the following question: for a given time period $[T, T + \tau_{liq}]$ where liquidations could have occurred, is it possible, given Σ , $C_{[T, T + \tau_{liq}]}$, L and Π , to find M such that

$$C_{[T, T + \tau_{liq}]} = \Sigma + LM\Pi\Sigma + \Sigma\Pi ML \quad (3.18)$$

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The following proposition gives conditions under which this inverse problem is well-posed i.e. the parameter M is identifiable:

Proposition 3.3.1 (Identifiability) *Let L and Π be diagonal matrices with*

$$L_{ii} = \frac{1}{D_i} \quad \Pi_{ii} = \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i dt$$

If $\Pi\Sigma L^{-1}$ is diagonalizable and there exists an invertible matrix Ω and ϕ_1, \dots, ϕ_n such that

$$\Omega^{-1}\Pi\Sigma L^{-1}\Omega = \begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_n \end{pmatrix}$$

and for all $1 \leq p, q \leq n$

$$\phi_p + \phi_q \neq 0$$

then there exists a unique symmetric $n \times n$ matrix M verifying (3.18) which is given by

$$M = \Phi(\Sigma, C_{[T, T+\tau_{liq}]}) \quad (3.19)$$

where $\Phi(\Sigma, C)$ is a $n \times n$ matrix defined by

$$[\Omega^t \Phi(\Sigma, C) \Omega]_{p,q} = \frac{1}{\phi_p + \phi_q} \times [\Omega^t L^{-1}(C - \Sigma)L^{-1}\Omega]_{p,q} \quad (3.20)$$

In this case, the unique solution M of (3.18) verifies

$$M = M_0 + O(\|\Lambda\|^2, \|f''\|) \quad (3.21)$$

where M_0 is defined in (3.15).

The proof of this proposition is given in Appendix 3.6.2. Thanks to (3.21), we deduce the following corollary:

Corollary 3.3.2 *The knowledge of M allows to estimate, up to an error term of order one in $\|\Lambda\|$ and zero in $\|f''\|$, the volume of fire sales in asset class i between T and $T + \tau_{liq}$:*

$$\begin{aligned} & \sum_{j=1}^J \frac{\alpha_i^j P_T^i}{V_T^j} \times \gamma_j \times \left(\frac{V_T^j - V_{T+\tau_{liq}}^j}{V_0^j} \right) \times V_T^j \\ &= (0, \dots, 0, P_T^i, 0, \dots, 0)M(P_T - P_{T+\tau_{liq}}) + O(\|\Lambda\|^2, \|f''\|) \end{aligned}$$

Note that the knowledge of M does not allow in general to reconstitute the detail of fire sales in each fund. Indeed, the decomposition of M given in (3.15) is not always unique. Nevertheless, when different funds engage in similar patterns of fire sales, the common component of these patterns may be recovered from the principal eigenvector of M . In the empirical examples, we find that M has one large eigenvalue, meaning that liquidations were concentrated in one direction.

3.3.2 Consistency and large sample properties

In the remainder of the paper, we make the following assumption, which guarantees that the identification problem is well-posed in the sense of Proposition 3.3.1:

Assumption 3.3.3 $\Pi\Sigma L^{-1}$ is diagonalisable with distinct eigenvalues ϕ_1, \dots, ϕ_n such that for all $1 \leq p, q \leq n$:

$$\phi_p + \phi_q \neq 0$$

As a consequence, (3.19) (3.20) (3.21) hold. We require that the eigenvalues of $\Pi\Sigma L^{-1}$ are distinct so that the set of matrices Σ verifying Assumption 3.3.3 is an open subset of $\mathcal{S}_n(\mathbb{R})$ which allows for the study of the differentiability of Φ defined in (3.20).

Proposition 3.3.1 states that if we know $L = \text{diag}(\frac{1}{D_i})$, $\Pi = \text{diag}(\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} P_t^i dt)$, the fundamental covariance matrix, Σ , and the realized covariance matrix between T and $T + \tau_{liq}$, $C_{[T, T+\tau_{liq}]}$, we can reconstitute M and hence the aggregate characteristics of the liquidation between T and $T + \tau_{liq}$, according to Corollary 3.3.2.

The market depth parameters (L) may be estimated using intraday data, following the methods outlined in Obizhaeva (2011); Cont et al. (2010). This is further discussed in Section 3.4. Π may be computed from time series of prices.

Σ and $C_{[T, T+\tau_{liq}]}$ are estimated using the realized covariance matrices computed on a time-grid with step $\frac{1}{N}$, defined in (3.9). In order to estimate Σ , we have to identify a period of time with no fire sales. Denote

$$\tau = \inf \{t \geq 0 \mid \exists 1 \leq j \leq J, V_t^j < \beta_j V_0^j\} \wedge T. \quad (3.22)$$

τ is the first time, prior to T , when fire sales occur. In our model, fire sales begin when the value of a fund j drops below a certain threshold $\beta_j V_0^j$, with $\beta_j < 1$. Given Corollary 3.2.6, asset prices and hence fund values are continuous, which implies that τ is a stopping time, bounded by T . Furthermore, as $\beta_j < 1$ for all $1 \leq j \leq J$, τ is strictly positive almost surely: $\mathbb{P}(\tau = 0) = 0$. As a consequence, we estimate the fundamental covariance matrix Σ using the sample realized covariance matrix on $[0, \tau]$, denoted $\widehat{\Sigma}^{(N)}$. In addition, a natural estimator for $C_{[T, T+\tau_{liq}]}$ is the sample realized covariance matrix between T and $T + \tau_{liq}$, denoted $\widehat{C}^{(N)}$. By (Jacod and Protter, 2012, Theorem 3.3.1), we find that the estimators of Σ and $C_{[T, T+\tau_{liq}]}$ are consistent:

$$\widehat{\Sigma}^{(N)} = \frac{1}{\tau} [X, X]_{\tau}^{(N)} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \Sigma \quad (3.23)$$

$$\widehat{C}^{(N)} = \frac{1}{\tau_{liq}} \left([X, X]_{T+\tau_{liq}}^{(N)} - [X, X]_T^{(N)} \right) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} C_{[T, T+\tau_{liq}]} \quad (3.24)$$

where the process $[X, X]^{(N)}$ is defined in (3.10) and τ is defined in (3.22). We can hence define an estimator $\widehat{M}^{(N)}$ of M by:

$$\widehat{M}^{(N)} = \Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}) \quad (3.25)$$

where Φ is defined in (3.20).

Proposition 3.3.4 (Consistency) $\widehat{M}^{(N)}$ defined in (3.25) is a consistent estimator of M :

$$\widehat{M}^{(N)} = \Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} M.$$

The proof of this proposition is given in Appendix 3.6.2. Proposition 3.3.4 shows that $\widehat{M}^{(N)}$ defined in 3.25 is a consistent estimator of M , which contains the information on liquidations between T and $T + \tau_{liq}$. The following proposition gives us the rate of this estimator $\widehat{M}^{(N)}$ and its asymptotic distribution.

Proposition 3.3.5 (Asymptotic distribution of estimator)

$$\sqrt{N} \left(\widehat{M}^{(N)} - M \right) \xrightarrow[N \rightarrow \infty]{\Rightarrow} \nabla \Phi \left(\Sigma, C_{[T, T + \tau_{liq}]} \right) \cdot \left(\frac{1}{\tau_{liq}} \left(\overline{Z}_{T + \tau_{liq}} - \overline{Z}_T \right) - \frac{1}{\tau} \overline{Z}_\tau \right) \quad (3.26)$$

where τ is defined in (3.22), $\nabla \Phi$ is the gradient of Φ , defined in (3.20), and

$$\overline{Z}_t^{ij} = \frac{1}{\sqrt{2}} \sum_{1 \leq k, l \leq n} \int_0^t \left(\tilde{V}_s^{ij, kl} + \tilde{V}_s^{ji, kl} \right) d\tilde{W}_s^{kl} \quad (3.27)$$

where \tilde{W} is a n^2 -dimensional Brownian motion independent from W and \tilde{V} is a $\mathcal{M}_{n^2 \times n^2}(\mathbb{R})$ -valued process verifying

$$(\tilde{V}_t \tilde{V}_t^t)^{ij, kl} = [\sigma \sigma^t(P_t)]_{i, k} [\sigma \sigma^t(P_t)]_{j, l} \quad (3.28)$$

where σ is defined in (3.7).

The proof of this proposition is given in Appendix 3.6.3. The Brownian motion \tilde{W} describes the estimation errors in (3.25): the fact that it is asymptotically independent from the randomness W driving the path of the price process allows to compute the asymptotic distribution of the estimator, conditioned on a given price path and derive confidence intervals, as explained below.

3.3.3 Testing for the presence of fire sales

Proposition 3.3.5 allows to test whether $M \neq 0$ i.e. if significant fire sales occurred between T and $T + \tau_{liq}$. Consider the null hypothesis

$$M = 0 \quad (H_0)$$

Under hypothesis (H_0) , there are no fire sales between T and $T + \tau_{liq}$. The central limit theorem given in Proposition 3.3.5 can be simplified as follows:

Proposition 3.3.6 Under the null hypothesis (H_0) , the estimator $\widehat{M}^{(N)}$ verifies the following central limit theorem:

$$\sqrt{N} \widehat{M}^{(N)} \xrightarrow[N \rightarrow \infty]{\Rightarrow} \Phi \left(\Sigma, \Sigma + \frac{1}{\tau_{liq}} \left(\overline{Z}_{T + \tau_{liq}} - \overline{Z}_T \right) - \frac{1}{\tau} \overline{Z}_\tau \right)$$

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where \bar{Z} is a n^2 -dimensional Brownian motion with covariance

$$\text{cov}(\bar{Z}^{i,j}, \bar{Z}^{k,l}) = \Sigma_{i,k}\Sigma_{j,l} + \Sigma_{i,l}\Sigma_{j,k}$$

and Φ and τ are defined in (3.20) and (3.22) respectively.

The proof of this proposition is given in Appendix 3.6.4. τ is given in (3.22) and can be simulated thanks to Corollary 3.2.6. This result allows to test whether the variability in the realized covariance of asset returns during $[T, T + \tau_{liq}]$ may be explained by the superposition of homoscedastic fundamental covariance structure and feedback effects from fire sales. To do this, we estimate the matrix M and test the nullity of the liquidation volumes derived in Corollary (3.3.2). In practice, it may be possible, for economic reasons, to identify a period $[0, T]$ with no fire sales and hence test the presence of fire sales during $[T, T + \tau_{liq}]$.

Corollary 3.3.7 *Under the null hypothesis (H_0) and if there are no fire sales between 0 and T ,*

$$\sqrt{N} \left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) \right) \xrightarrow[N \rightarrow \infty]{} \mathcal{N} \left(0, \left(\frac{1}{T} + \frac{1}{\tau_{liq}} \right) \sum_{i,j,k,l=1}^n m_{ij} m_{kl} (\Sigma_{ik}\Sigma_{jl} + \Sigma_{jk}\Sigma_{il}) \right)$$

with $m_{ij} = \sum_{p,q=1}^n \frac{[\Omega^{-1}P_T]_p [\Omega^{-1}(P_T - P_{T+\tau_{liq}})]_q}{\phi_p + \phi_q} \Omega_{ip}\Omega_{jq} D_i D_j$ where Ω and $(\phi_i)_{1 \leq i \leq n}$ are defined in Proposition 3.3.1, P_t is the vector of prices at date t and $(D_i)_{1 \leq i \leq n}$ are the asset market depths.

The proof of this corollary is given in Appendix 3.6.4. Corollary 3.3.7 gives the asymptotic law of $\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) \right)$, the estimated volume of liquidations, under the null hypothesis (H_0) and if there are no fire sales during $[0, T]$. We can then define a level l such that

$$\mathbb{P} \left(\left| P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) \right| > l \right) \leq 1 - p_l$$

where p_l is typically equal to 95% or 99%. If we find that $\left| P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) \right| > l$ and if we know that there were no fire sales during $[0, T]$, then the null hypothesis of no fire sales between T and $T + \tau_{liq}$ may be rejected at confidence level p_l .

3.3.4 Numerical experiments

To assess the accuracy of these estimators in samples of realistic size, we first apply this test to a simulated discrete-time market. We consider the case of one fund investing in $n = 20$ assets, with fundamental volatility 30% and zero fundamental correlation. Furthermore, we assume that all assets have the same market depth D and that the fund is initially equally weighted across these assets: $\frac{\alpha_i P_0^i}{V_0} = \frac{1}{n}$. The size of the fund can be captured by the vector Λ , defined in Proposition 3.2.7, which represents the size

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of the fund’s position in each asset as a fraction of the asset’s market depth. In our simulations, we choose this ratio equal to 20%.

We examine the results of our estimation method in the two following cases:

- the fund is not subject to distressed selling
- the fund is subject to distressed selling: when the fund value drops below $\beta_0 = 95\%$ of its initial value, the manager deleverages the fund portfolio.

Figure 3.3 displays a trajectory for the fund’s value, where the fund was subject to distressed selling between $T=116$ days and $T + \tau_{liq} = 127$ days.

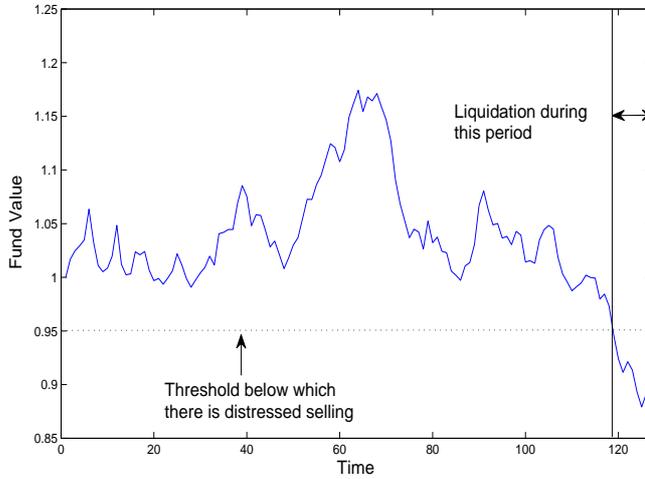


Figure 3.3: Fund value

We consider a market where trading is possible every day ($\frac{1}{N} = \frac{1}{250}$). We calculate $\widehat{\Sigma}^{(N)}$ and $\widehat{C}^{(N)}$ and we apply our estimation procedure and calculate in each case (no liquidation and liquidation cases) an estimate for the volume of liquidations. Using 3.3.7, we can determine, at confidence level 95%, for example, whether there has been a liquidation or not.

Under the assumption (H_0) that $M = 0$ and using Lemma 3.3.7 we find that

$$\mathbb{P} \left(\left| P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) \right| > 3.2 \times 10^3 \right) \leq 5\%$$

We find that

- when there are no fire sales, $P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) = 203 < 3.2 \times 10^3$ and we cannot reject assumption (H_0)
- when fire sales occur, $P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{liq}}) = 7 \times 10^3 > 3.2 \times 10^3$ and we reject (H_0) at a 95% confidence level.

Let us now focus on the results of our estimation procedure in the case where there were liquidations and check whether it allows for a proper reconstitution of the liquidated portfolio. We find that the estimates for the proportions liquidated $\frac{\alpha_i P_i^0}{V_0}$ are all positive and ranging from 2% to 10%, around the true value which is $\frac{1}{20} = 5\%$.

3.4 The Great Deleveraging of Fall 2008

Lehman Brothers was the fourth largest investment bank in the USA. During the year 2008, it experienced severe losses, caused mainly by the subprime mortgage crisis, and on September, 15th, 2008, it filed for chapter 11 bankruptcy protection, citing bank debt of \$613 billion, \$155 billion in bond debt, and assets worth \$639 billion, becoming the largest bankruptcy filing in the US history.

The failure of Lehman Brothers generated liquidations and deleveraging in all asset classes all over the world. The collapse of this huge institution was such a shock to financial markets - major equity indices all lost around 10% on that day - that it triggered stop loss and deleveraging strategies among a remarkable number of financial institutions worldwide. Risk measures of portfolios, for example the value at risk, increased sharply, obliging financial institutions to hold more cash, which they got by deleveraging their portfolios, rather than by issuing debt which would have been very costly at such distressed times.

This massive deleveraging has been documented in several empirical studies. Fratzscher (2011) studies the effect of key events, such as the collapse of Lehman Brothers, on capital flows. He uses a dataset on portfolio capital flows and performance at the fund level, from EPFR, and containing daily, weekly and monthly flows for more than 16000 equity funds and 8000 bond funds, domiciled in 50 countries. He aggregates the net capital flows (ie net of valuation changes) for each country and finds that they are negative for all the countries of the study. This means that fund managers of such funds deleveraged their positions after the collapse of Lehman Brothers, sometimes in dramatic proportions: in some cases, the outflows can represent up to 30% of the assets under management by the funds.

Our method allows to estimate the net effect of liquidations during this period. We report below the result of the estimation method described in Section 3.3 SPDRs and components of the Eurostoxx 50 index. Figure 3.4 shows that the increase of average correlation in these two equity baskets lasted for around three months after September, 15th, 2008. As a consequence, we examine liquidations that occurred between September, 15th, 2008 and December, 31st, 2008.

We calculate the realized covariance matrices respectively between 02/01/2008 and $T = 09/15/2008$ and between $T = 09/15/2008$ and $T + \tau_{liq} = 12/31/2008$ and apply the estimation procedure described in Section 3.3. We use a linear price impact model Obizhaeva (2011); Cont et al. (2010). To calibrate the market depth parameters D_i , we follow the approach proposed in Obizhaeva (2011): denoting by σ_i the average daily volatility of asset i and ADV_i the average daily trading volume, it was shown in Obizhaeva (2011) for a large panel of US stocks that the ratio $\frac{1}{D} \frac{ADV}{\sigma_r}$ does not vary

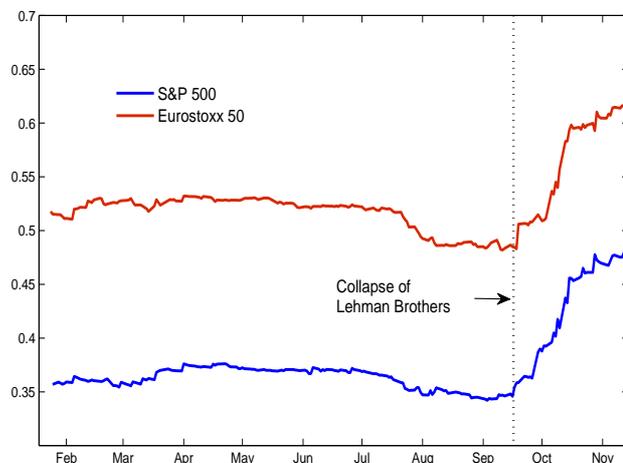


Figure 3.4: One-year EWMA estimator of average pairwise correlations of daily returns in S&P500 and EuroStoxx 50 index

significantly from one asset to another and

$$\frac{1}{D} \frac{ADV}{\sigma_r} \approx 0.33. \quad (3.29)$$

Obizhaeva (2011) also argues empirical evidence that the difference in price impact of buy-originated trades and sell-originated trades is not statistically significant. We use average daily volumes and average daily volatility to estimate the market depth of each asset, using (3.29). Alternatively one could use intraday data, following the methodology proposed in Cont et al. (2010).

3.4.1 Sector ETFs

We first study fire sales among sector SPDRs, which are sector sub indices of the S&P 500. There exist nine sector SPDRs: Financials (XLF), Consumer Discretionary (XLY), Consumer Staples (XLP), Energy (XLE), Health Care (XLV), Industrials (XLI), Materials (XLB), Technology (XLK) and Utilities (XLU) and our goal is to determine how economic actors investing in those SPDRs liquidated their portfolios following the collapse of Lehman Brothers.

In order to compute our estimation procedure, we need to know the market depth of each SPDR, which we can estimate as described in the previous section. Market depths are given in Table 3.1. We find that financials have the highest market depth and that other SPDRs have similar market depths.

We can then apply the estimation method described in Section 3.3 and find the magnitude of fire sales in each SPDR between September, 15th, 2008 and December, 31st, 2008.

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Sector SPDR	Estimated Market Depth $\times 10^8$ shares
Financials	34.8
Consumer Discretionary	4.4
Consumer Staples	6.2
Energy	8.8
Health Care	6.4
Industrials	8.1
Materials	7.0
Technology	7.9
Utilities	7.1

Table 3.1: Estimated market depth for SPDRs.

Our method yields an estimate of 86 billion dollars for fire sales affecting SPDRs between September, 15th, 2008 and December, 31st, 2008. Using Corollary 3.3.7, we can reject the hypothesis of no liquidation at a 95% confidence level for this period. The liquidation volume that we find is equivalent to a daily liquidation volume of 1.2 billion dollars per day. In comparison, the average volume on SPDRs before Lehman Brother's collapse was 5.1 billion dollars per day. This shows how massive the liquidations were after this market shock.

Corollary 3.3.2 allows us to determine the aggregate composition of liquidations between September 15th 2008 and December, 31st, 2008. The daily liquidated volumes and the proportions of each SPDR are given in Table 3.2. This shows that the aggregate portfolio liquidated after Lehman Brother's collapse was a long portfolio. This is consistent with the observation that many financial institutions liquidated equity holdings in order to meet capital requirements during this period, due to the increase of the risk associated with Lehman Brother's collapse. The highest volume of liquidations are associated with financial stocks, followed by the energy sector. Those two sectors represent 60% of the liquidations and more that 50 billion dollars liquidated before December, 31st, 2008.

As discussed in Section 3.3.1, the principal eigenvector of M reflects the common patterns of fire sales. Table 3.3 gives the proportions of fire sales associated to the principal eigenvector of M . We see that this portfolio is essentially made of financials, which have a weight of 78%. The large weight of XLF, the financial sector index, may be explained in terms of the loss of investor confidence in banks in the aftermath of the Lehman's collapse.

3.4.2 Eurostoxx 50

We now conduct our analysis on stocks belonging to the Eurostoxx 50 in order to determine the average composition of portfolios diversified among the components of the

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Sector SPDR	Daily amount liquidated $\times 10^6 \$$	Weight
Financials	320	28%
Consumer Discretionary	55	5%
Consumer Staples	38	3.5%
Energy	300	26%
Health Care	63	5.5%
Industrials	90	8%
Materials	110	9.5%
Technology	65	5.5%
Utilities	100	9%

Table 3.2: Daily volume and proportions of fire sales for SPDR between September 15th, 2008 and Dec 31,2008.

Sector SPDR	Weight
Financials	78%
Consumer Discretionary	0%
Consumer Staples	2.5%
Energy	4%
Health Care	0%
Industrials	0%
Materials	2.5%
Technology	10%
Utilities	3%

Table 3.3: Proportions of fire sales between September 15th, 2008 and December, 31st, 2008 associated to the principal eigenvector of M

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Eurostoxx 50 and that were liquidated after Lehman Brother’s filing for bankruptcy. The Eurostoxx 50 is an equity index regrouping the 50 largest capitalizations of the Euro zone. It is the most actively traded index in Europe and is used as a benchmark to measure the financial health of the euro zone.

We use the same methodology as in the previous section (choice of dates, estimation of Σ and market depths). Note that we restricted our study to 45 stocks of the index, for which we had clean data. The 5 stocks left correspond to the lowest capitalizations among the index components, with very low liquidity.

We find that 350 billion euros were liquidated on stocks belonging to the Eurostoxx 50 between September, 15th, 2008 and December, 31st, 2008. Our statistical test described in Corollary 3.3.7 allows us to reject the null hypothesis of no liquidation at a 99% confidence level. Our estimate for the liquidated volume is equivalent to a daily liquidation of 5 billion euros, which is equal to one third of the average daily volume of the index components before September, 15th, 2008.

Figure 3.5, where each bar represents the weight of a stock in the aggregate liquidated portfolio, shows that most of the liquidations following Lehman Brother’s collapse involved liquidation of long positions in stocks.

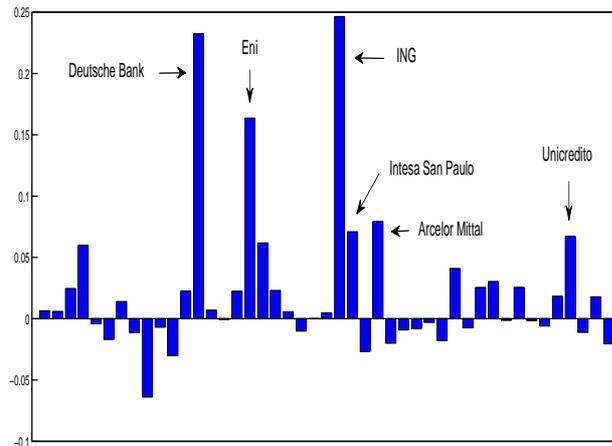


Figure 3.5: Fire sales in Eurostoxx 50 stocks in Fall 2008: each bar represents the weight of one stock in the aggregate liquidated portfolio

Figure 3.5 shows that fire sales are more intense for some stocks than others. Table 3.4 gives the detail of those stocks. As suggested by the previous section, we see that the fire sales in the Eurostoxx 50 index were concentrated in the financial and energy sectors. ING and Deutsche Bank account for almost half of the volume liquidated on the whole index.

Stock	Amount liquidated $\times 10^6 \text{ €}$	Weight
ING	1100	25%
Deutsche Bank	1000	23%
Eni	750	16%
Arcelor Mittal	350	8%
Intesa San Paolo	320	7%
Unicredito	300	6.5%

Table 3.4: Most liquidated stocks in the Eurostoxx 50 during the three months following September, 15th, 2008

3.5 The hedge fund losses of August 2007

From August 6th to August 9th 2007, long-short market-neutral equity funds experienced large losses: many funds lost around 10% per day and experienced a rebound of around 15% on August 10th, 2007. During this week, as documented by Khandani and Lo (2011), market-neutral equity funds whose returns previously had a low historical volatility exhibited negative returns exceeding 20 standard deviations, while no major move was observed in equity market indices.

Khandani and Lo (2011) suggested that this event was due to a large market-neutral fund deleveraging its positions. They simulate a contrarian long-short equity market neutral strategy implemented on all stocks in the CRSP Database and were able to reconstitute qualitatively the empirically observed profile of returns of quantitative hedge funds : low volatility before August 6th, huge losses during three days and a rebound on August 10th. We reconstituted empirically the returns for Khandani and Lo's equity market neutral strategy on the S&P500 for the first three quarters of 2007. Figure 3.6 shows that this strategy underperforms significantly during the second week of August 2007, while no major move occurred in the S&P 500. Such empirical results tend to confirm the hypothesis of the unwind of a large portfolio, which generated through price impact large losses across similar portfolios, as predicted by our model.

Using historical data on returns of 487 stocks from the S&P500 index, we have reconstituted the composition of the fund that deleveraged its positions during the second week of August 2007 using the estimation procedure described in Section 3.3 for the periods $[0, T] = [08/03/2006, 08/03/2007]$ and $[T, T + \tau_{liq}] = [08/06/2007, 08/09/2007]$.

Figure 3.7 displays the composition of the aggregate portfolio liquidated on the S&P500 during this period and found by our estimation method. The first and striking difference with the case of the deleveraging after Lehman Brother's collapse is that, during this quant event, the liquidated portfolio was a long-short portfolio. We clearly see in Figure 3.7 that for some stocks the liquidated position is significantly negative, meaning that a short position is being exited. More precisely, 250 stocks have positive weights in the liquidated portfolio, whereas 237 have negative weights. Furthermore,

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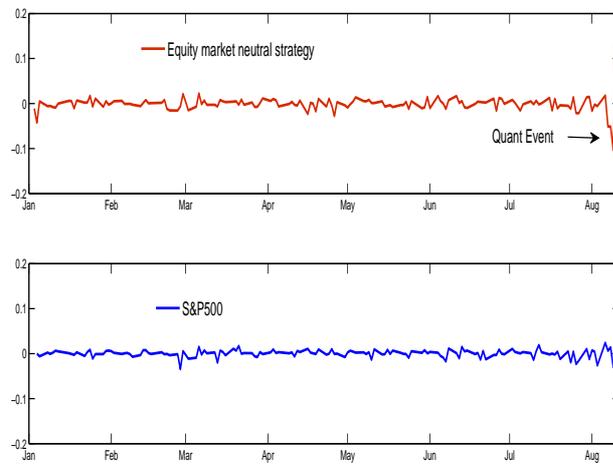


Figure 3.6: Returns of an market-neutral equity portfolio in 2007, compared with S&P500 returns.

we find that the liquidated portfolio was highly leveraged: for each dollar of capital, 15 dollars are invested in long positions and 14 dollars are invested in short positions.

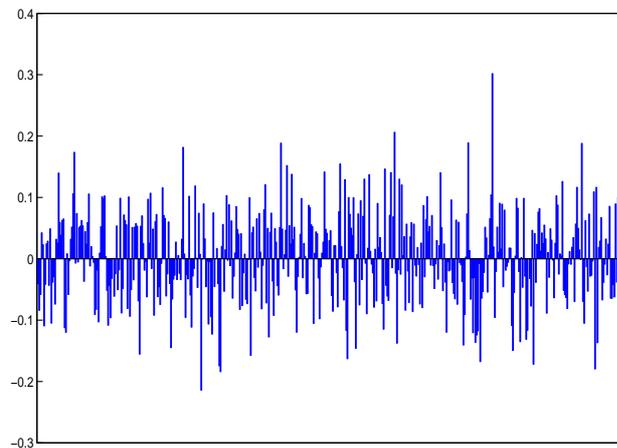


Figure 3.7: Equity positions liquidated during the 2nd week of August 2007.

Importantly, the estimated portfolio is market-neutral in the sense of Equation (3.16):

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using the notations of Section 3.2.4 we find

$$\frac{\hat{\Lambda} \cdot \pi_t^{\hat{\mu}}}{\|\hat{\Lambda}\| \|\pi_t^{\hat{\mu}}\|} = \frac{\sum_{i=1}^n \frac{\alpha_i}{D_i} \mu_t^i P_t^i}{\|\hat{\Lambda}\| \|\pi_t^{\hat{\mu}}\|} = 0.0958$$

which corresponds to an angle of 0.47π between the vectors $\hat{\Lambda}$ and $\pi_t^{\hat{\mu}}$, i.e. very close to orthogonality. This provides a quantitative explanation for the fact that, although massive liquidations occurred in the equity markets, index funds were not affected by this event. Note that, unlike other explanations proposed at the time, this explanation does not involve any assumption of liquidity drying up during the period of hedge fund turbulence.

3.6 Appendices

3.6.1 Proof of Theorem 3.2.4

We work under Assumptions 3.2.1 and 3.2.3. We denote $Z_{k+1} = \frac{1}{N}\bar{m} + \sqrt{\frac{1}{N}}\xi_{k+1} \in \mathbb{R}^n$ where $\bar{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$. We can write the price dynamics (3.3) as follows:

$$S_{k+1}^i = S_k^i \exp(Z_{k+1}^i) \left[1 + \phi_i \left(\sum_{j=1}^J \alpha_i^j \left(f_j \left(\sum_{l=1}^n \frac{\alpha_l^j S_k^l}{V_0^j} \exp(Z_{k+1}^l) \right) - f_j \left(\sum_{l=1}^n \frac{\alpha_l^j S_k^l}{V_0^j} \right) \right) \right) \right]$$

As a consequence, we have $S_{k+1} = \theta(S_k, Z_{k+1})$ where $\theta : (\mathbb{R}_+^*)^n \times \mathbb{R}^n \mapsto (\mathbb{R}_+^*)^n$ is $\mathcal{C}^3(\mathbb{R})$ as f_j and ϕ_i are $\mathcal{C}^3(\mathbb{R})$ for all $1 \leq j \leq J$ and $1 \leq i \leq n$.

Define now a (resp., b) a $\mathcal{M}_n(\mathbb{R})$ -valued (resp. \mathbb{R}^n -valued) mapping such that

$$a_{i,j}(S) = \sum_{l=1}^n \frac{\partial \theta_i}{\partial z_l}(S, 0) \times A_{l,j} \quad (3.30)$$

$$b_i(S) = \sum_{j=1}^n \frac{\partial \theta_i}{\partial z_j}(S, 0) \bar{m}_j + \frac{1}{2} \sum_{j,l=1}^n \frac{\partial^2 \theta_i}{\partial z_j \partial z_l}(S, 0) \Sigma_{j,l} \quad (3.31)$$

In order to show Theorem 3.2.4, we first show the following lemma:

Lemma 3.6.1 *Under Assumptions 3.2.1 and 3.2.3, for all $\epsilon > 0$ and $r > 0$:*

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} N \times \mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S) = 0 \quad (3.32)$$

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} \|N \times \mathbb{E}(S_{k+1} - S_k | S_k = S) - b(S)\| = 0 \quad (3.33)$$

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} \|N \times \mathbb{E}((S_{k+1} - S_k)(S_{k+1} - S_k)^t | S_k = S) - aa^t(S)\| = 0 \quad (3.34)$$

where a and b are defined respectively in (3.30) and (3.31).

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Proof Fix $\epsilon > 0$ and $r > 0$. As θ is \mathcal{C}^1 , for $\|S\| \leq r$, there exists $C > 0$ such that for all $Z \in \mathbb{R}^n$

$$\|\theta(S, Z) - \theta(S, 0)\| \leq C\|Z\|$$

As $S_{k+1} = \theta(S_k, Z_{k+1})$ and $S_k = \theta(S_k, 0)$, we find that:

$$\begin{aligned} & \mathbb{P}(\|S_{k+1} - S_k\| \geq \epsilon | S_k = S, \|S\| \leq r) \leq \mathbb{P}(C\|Z_{k+1}\| \geq \epsilon) \\ & \leq \mathbb{P}\left(C\frac{\bar{m}}{N} + \sqrt{\frac{1}{N}}\xi_{k+1} \geq \epsilon\right) \leq \mathbb{P}\left(\|\xi_{k+1}\| \geq \frac{\epsilon - \frac{\|\bar{m}\|C}{N}}{C\sqrt{\frac{1}{N}}}\right) \\ & \leq \mathbb{E}\left[\left(\|\xi_{k+1}\| \frac{C\sqrt{\frac{1}{N}}}{\epsilon - \frac{\|\bar{m}\|C}{N}}\right)^{2+\eta}\right] \leq \frac{1}{N^{1+\frac{\eta}{2}}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{C}{\epsilon - \frac{\|\bar{m}\|C}{N}}\right)^{2+\eta} \end{aligned}$$

which implies (3.32).

As θ is \mathcal{C}^2 , we can write the Taylor expansion of θ_i in 0, for $1 \leq i \leq n$:

$$\theta_i(S, Z) - \theta_i(S, 0) = \frac{\partial\theta_i}{\partial z}(S, 0)Z + \frac{1}{2}Z \cdot \frac{\partial^2\theta_i}{\partial z\partial z'}(S, 0)Z + Z \cdot R_i(S, Z)Z$$

where R_i converges uniformly to 0 when Z goes to 0, when $\|Z\| \leq \epsilon$ and $\|S\| \leq r$. We have:

$$\mathbb{E}\left(\frac{\partial\theta_i}{\partial z}(S, 0)Z_{k+1}\right) = \frac{1}{N}\sum_{j=1}^n \frac{\partial\theta_i}{\partial z_j}(S, 0)\bar{m}_j$$

and

$$\mathbb{E}\left(Z_{k+1} \cdot \frac{\partial^2\theta_i}{\partial z\partial z'}(S, 0)Z_{k+1}\right) = \frac{1}{N}\sum_{j,l=1}^n \frac{\partial^2\theta_i}{\partial z_l z_j}\Sigma_{j,l} + o\left(\frac{1}{N}\right)$$

Recalling that $S_{k+1}^i - S_k^i = \theta(S_k, Z_{k+1}) - \theta(S_k, 0)$, we find that:

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} \|N\mathbb{E}[(S_{k+1} - S_k) | S_k = S, \|Z_{k+1}\| \leq \epsilon] - b(S)\| = 0 \quad (3.35)$$

We remark that:

$$\|N\mathbb{E}((S_{k+1} - S_k)\mathbf{1}_{\|Z_{k+1}\| \leq \epsilon} | S_k = S) - b(S)\|$$

$$\leq \|(N\mathbb{E}((S_{k+1} - S_k) | S_k = S, \|Z_{k+1}\| \leq \epsilon) - b(S))\| \mathbb{P}(\|Z_{k+1}\| \leq \epsilon) + \|b(S)\| \mathbb{P}(\|Z_{k+1}\| \geq \epsilon)$$

As we saw that $\mathbb{P}(\|Z_{k+1}\| \geq \epsilon) \leq \frac{1}{N^{1+\frac{\eta}{2}}}\mathbb{E}\left[(\|\xi_{k+1}\|)^{2+\eta}\right] \times \left(\frac{1}{\epsilon - \frac{\|\bar{m}\|}{N}}\right)^{2+\eta}$ and given (3.35) and the fact that b is continuous, we find that:

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} \|N\mathbb{E}((S_{k+1} - S_k)\mathbf{1}_{\|Z_{k+1}\| \leq \epsilon} | S_k = S) - b(S)\| = 0 \quad (3.36)$$

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Similarly, we show that

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} \left\| N \mathbb{E} \left((S_{k+1} - S_k)(S_{k+1} - S_k)^t \mathbf{1}_{\|Z_{k+1}\| \leq \epsilon} | S_k = S \right) - aa^t(S) \right\| = 0 \quad (3.37)$$

Given (3.3), we have the following inequality:

$$S_{k+1}^i \leq S_k^i \exp \left(\frac{\bar{m}_i}{N} + \sqrt{\frac{1}{N}} \xi_{k+1}^i \right) \left(1 + \phi_i \left(2 \sum_{j=1}^J \frac{\alpha_i^j}{D_i} \|f_j\|_\infty \right) \right)$$

which implies that, conditional on $S_k = S$ and for $p > 0$ such that $p\sqrt{\frac{1}{N}} < \eta$, $S_{k+1} \in L^p$.

Using this result for $p = 2$, we find that for $\sqrt{\frac{1}{N}} < \frac{\eta}{2}$, $S_{k+1} \in L^2$ and we can use Cauchy Schwarz inequality:

$$\begin{aligned} & \left| \mathbb{E} \left((S_{k+1}^i - S_k^i) \mathbf{1}_{\|Z_{k+1}\| \geq \epsilon} | S_k = S \right) \right| \\ & \leq \sqrt{\mathbb{E} \left((S_{k+1}^i - S_k^i)^2 | S_k = S \right) \mathbb{P} \left(\|Z_{k+1}\| \geq \epsilon \right)} \\ & \leq \frac{1}{N^{1+\frac{\eta}{4}}} \sqrt{\mathbb{E} \left((S_{k+1}^i - S_k^i)^2 | S_k = S \right)} \sqrt{\mathbb{E} \left(\frac{\|\xi_{k+1}\|}{\epsilon - \frac{\|\bar{m}\|}{N}} \right)^{4+\eta}} \end{aligned}$$

As $\mathbb{E}(\|\xi_{k+1}\|^{4+\eta}) < \infty$, $S_{k+1} \in L^2$ and S_{k+1} stays L^2 bounded conditional on $S_k = S$ and $\|S\| \leq r$. As a consequence, we obtain:

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} \left\| N \mathbb{E} \left((S_{k+1} - S_k) \mathbf{1}_{\|Z_{k+1}\| \geq \epsilon} | S_k = S \right) \right\| = 0 \quad (3.38)$$

Using the same property with $p=4$, we show that

$$\lim_{N \rightarrow \infty} \sup_{\|S\| \leq r} \left\| N \mathbb{E} \left((S_{k+1} - S_k)(S_{k+1} - S_k)^t \mathbf{1}_{\|Z_{k+1}\| \geq \epsilon} | S_k = S \right) \right\| = 0 \quad (3.39)$$

(3.36) and (3.38) give (3.33). Similarly, (4.41) and (3.39) give (3.34).

The following lemma gives the expressions of a and b by direct computation of (3.30) – (3.31).

Lemma 3.6.2 (3.30) and (3.31) respectively can be written as

$$a_{i,k}(S) = S^i \left[A_{i,k} + \phi_i'(0) \sum_{j=1}^J \frac{\alpha_i^j}{V_0^j} f_j' \left(\frac{V_j(S)}{V_0^j} \right) (A^t \pi_j(S))_k \right] \quad (3.40)$$

$$b_i(S) = S^i m_i + S^i \frac{\phi_i'(0)}{2} \sum_{j=1}^J \frac{\alpha_i^j}{(V_0^j)^2} f_j'' \left(\frac{V_j(S)}{V_0^j} \right) \pi_j(S) \cdot \Sigma \pi_j(S) \quad (3.41)$$

$$\begin{aligned}
 & + S^i \phi_i'(0) \sum_{j=1}^J \frac{\alpha_i^j}{V_0^j} f_j' \left(\frac{V_j(S)}{V_0^j} \right) (\pi_j(S) \cdot \bar{m} + (\Sigma \pi_j(S))_i) \\
 & + S^i \frac{\phi_i''(0)}{2} \sum_{j,r=1}^J \frac{\alpha_i^j \alpha_i^r}{V_0^j V_0^r} f_j' \left(\frac{V_j(S)}{V_0^j} \right) f_r' \left(\frac{V_r(S)}{V_0^r} \right) \pi_j(S) \cdot \Sigma \pi_r(S)
 \end{aligned}$$

where $\pi_j(S) = \begin{pmatrix} \alpha_1^j S^1 \\ \vdots \\ \alpha_n^j S^n \end{pmatrix}$ and $V_j(S) = \sum_{l=1}^n \alpha_l^j S^l$.

Because f_j is \mathcal{C}^3 for $1 \leq j \leq J$, a and b are \mathcal{C}^2 and \mathcal{C}^1 respectively. Furthermore, because f_j' , and hence f_j'' and $f_j^{(3)}$, have a compact support, there exists $R > 0$ such that, for all $1 \leq j \leq J$, when $\|S\| \geq R$, $f_j' \left(\frac{V_j(S)}{V_0^j} \right) = f_j'' \left(\frac{V_j(S)}{V_0^j} \right) = f_j^{(3)} \left(\frac{V_j(S)}{V_0^j} \right) = 0$. As a consequence, there exists $K > 0$ such that for all $S \in (\mathbb{R}_+^*)^n$:

$$\|a(S)\| + \|b(S)\| \leq K \|S\| \quad (3.42)$$

Furthermore, as the first derivatives of a and b are bounded, a and b are Lipschitz.

Define the differential operator $G : C_0^\infty(\mathbb{R}_+^*)^n \mapsto C_0^1(\mathbb{R}_+^*)^n$ by

$$Gh(x) = \frac{1}{2} \sum_{1 \leq i, j \leq n} (aa^t)_{i,j}(x) \partial_i \partial_j h + \sum_{1 \leq i \leq n} b_i(x) \partial_i h$$

As a and b verify (3.42), one can apply (Ethier and Kurtz, 1986, Theorem 2.6, Ch.8) to conclude that the martingale problem associated to (G, δ_{S_0}) is well-posed. In fact, as a and b are Lipschitz, the solution of this martingale problem is given by the unique strong solution of the stochastic differential equation:

$$dP_t = b(P_t)dt + a(P_t)dW_t \quad \text{with } P_0 = S_0.$$

As we have shown Lemma 3.6.1, by (Ethier and Kurtz, 1986, Theorem 4.2, Ch.7), when $N \rightarrow \infty$, $(S_{\lfloor Nt \rfloor})_{t \geq 0}$ converges in distribution to the solution of the martingale problem associated to (G, δ_{S_0}) , which concludes the proof of Theorem 3.2.4.

3.6.2 Proofs of Propositions 3.3.1 and 3.3.4

Let us invert (3.18) under the assumptions of Proposition 3.3.1. Denote

$$\Omega^{(i)} = \begin{pmatrix} \Omega_{1,i} \\ \vdots \\ \Omega_{n,i} \end{pmatrix}$$

the i -th column of the matrix Ω . By definition, we know that $\Pi \Sigma L^{-1} \Omega^{(p)} = \phi_p \Omega^{(p)}$ which is equivalent to $(\Omega^{(p)})^t L^{-1} \Sigma \Pi = \phi_p (\Omega^{(p)})^t$. As (3.18) is equivalent to $M \Pi \Sigma L^{-1} +$

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$L^{-1}\Sigma\Pi M = L^{-1}(C_{[T, T+\tau_{iq}]} - \Sigma)L^{-1}$ and multiplying this equality on the left by $(\Omega^{(p)})^t$ and on the right by $\Omega^{(q)}$, we find that

$$(\phi_p + \phi_q)[\Omega^t M \Omega]_{p,q} = [\Omega^t L^{-1}(C_{[T, T+\tau_{iq}]} - \Sigma)L^{-1}\Omega]_{p,q}$$

which gives the matrix $\Omega^t M \Omega$ as a function of Σ and $C_{[T, T+\tau_{iq}]}$. As Ω is invertible, this characterizes the matrix M , as a function, denoted Φ of Σ and $C_{[T, T+\tau_{iq}]}$, proving (3.19) and (3.20).

Furthermore, notice that $M_0 = \Phi\left(\Sigma, C_{[T, T+\tau_{iq}]} + O(\|\Lambda\|^2, \|f''\|)\right)$. Given the expression for Φ in (3.20), (3.21) follows directly. This concludes the proof of Proposition 3.3.1.

Lemma 3.6.3 *The mapping Φ defined in (3.20) is C^∞ in a neighborhood of (Σ, C) .*

Proof The following map

$$F : \mathcal{S}_n^3(\mathbb{R}) \mapsto \mathcal{S}_n(\mathbb{R}), (S, C, N) \mapsto \text{LNIS} + \text{SIINL} + S - C \quad (3.43)$$

is infinitely differentiable, its gradient with respect to N given by

$$\frac{\partial F}{\partial N}(S, C, N) \cdot H_3 = LH_3\Pi S + S\Pi H_3L.$$

As Σ verifies Assumption 3.3.3, we showed that $\frac{\partial F}{\partial N}(\Sigma, C, N)$ is invertible for all C . As $\Phi(\Sigma, C)$ is defined as the only matrix verifying $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, the implicit function theorem states that Φ is C^∞ in a neighborhood of (Σ, C) .

As convergence in probability implies that a subsequence converges almost surely, we assume from now on that the estimators defined in (3.23) and (3.24) converge almost surely. As a consequence, for N large enough, $\widehat{\Sigma}^{(N)}$ also verifies Assumption 3.3.3. This is possible because the set of matrices Σ verifying this assumption is an open set and $\widehat{\Sigma}^{(N)}$ converges almost surely to Σ when N goes to infinity. We can hence define $\widehat{M}^{(N)}$ as in (3.25).

Lemma 3.6.3 implies in particular that Φ is continuous and hence that $\Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)})$ converges almost surely, and hence in probability, to $\Phi(\Sigma, C_{[T, T+\tau_{iq}]})$. As a consequence, $\Phi(\widehat{\Sigma}^{(N)}, \widehat{C}^{(N)})$ is a consistent estimator of $\Phi(\Sigma, C_{[T, T+\tau_{iq}]})$, meaning that $\widehat{M}^{(N)}$ is a consistent estimator of M . This shows Proposition 3.3.4.

3.6.3 Proof of Proposition 3.3.5

Using Theorem 3.2.6 and Ito's formula, we deduce that the log price $X_t^i = \ln(P_t^i)$ verifies the following stochastic differential equation:

$$dX_t^i = \left(\mu_i(e^{X_t}) - \frac{1}{2}(\sigma(e^{X_t})\sigma(e^{X_t})^t)_{i,i} \right) dt + (\sigma(e^{X_t})dW_t)_i$$

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where σ , μ and W are defined in Theorem 3.2.6 and e^{Xt} is a n -dimensional column vector with i -th term equal to $\exp X_t^i$. As a consequence, X is an Ito process which verifies, for $t \geq 0$,

$$\int_0^t \left(\sum_{1 \leq i \leq n} \left(\mu_i(P_t) - \frac{1}{2}(\sigma(P_t)\sigma_t(P_t)^t)_{i,i} \right)^2 + \|\sigma\sigma^t(P_t)\|^2 \right) ds < \infty$$

We are thus in the setting of (Jacod and Protter, 2012, Theorem 5.4.2, Ch.5) which describes the asymptotic distribution of the quadratic covariation of an Ito process with well-behaved coefficients. We need to extend $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ to a larger probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$. There exists \tilde{W} a n^2 -dimensional Brownian motion, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}})$ and independent from W , such that

$$\sqrt{N} \left([X, X]^{(N)} - [X, X] \right) \xrightarrow[N \rightarrow \infty]{s.l.} \bar{Z}$$

where the $n \times n$ dimensional process \bar{Z} is defined in (3.27) and s.l. means stable convergence in law (see (Jacod and Protter, 2012, Section 2.2.1)). The auxiliary Brownian motion \tilde{W} represents the estimation error. Furthermore, (Jacod and Protter, 2012, Equation 2.2.5) shows that

$$\left(\sqrt{N} \left([X, X]^{(N)} - [X, X] \right), \tau \right) \xrightarrow[N \rightarrow \infty]{\Rightarrow} (\bar{Z}, \tau)$$

This implies that the estimators $(\hat{\Sigma}^{(N)}, \hat{C}^{(N)})$ defined in (3.23) and (3.24) verify the following central limit theorem:

$$\sqrt{N} \left[\begin{pmatrix} \hat{\Sigma}^{(N)} \\ \hat{C}^{(N)} \end{pmatrix} - \begin{pmatrix} \Sigma \\ C_{[T, T+\tau_{liq}]} \end{pmatrix} \right] \xrightarrow[N \rightarrow \infty]{\Rightarrow} \begin{pmatrix} \frac{1}{\tau} \bar{Z}_\tau \\ \frac{1}{\tau_{liq}} (\bar{Z}_{T+\tau_{liq}} - \bar{Z}_T) \end{pmatrix} \quad (3.44)$$

Since $\Phi \in \mathcal{C}^1$, one can then apply the 'delta method' to $(\hat{\Sigma}^{(N)}, \hat{C}^{(N)})$ to derive the result in Proposition 3.3.5.

3.6.4 Proof of Proposition 3.3.6 and Corollary 3.3.7

Under the null hypothesis (H_0) , $\frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t dt = \Sigma$ and hence

$$\Phi \left(\Sigma, \frac{1}{\tau_{liq}} \int_T^{T+\tau_{liq}} c_t dt \right) = \Phi(\Sigma, \Sigma) = 0$$

Let us calculate now the first derivative of Φ on (Σ, Σ) . Recall that $\Phi(\Sigma, C)$ is defined as the only element of $\mathcal{S}_n(\mathbb{R})$ such that $F(\Sigma, C, \Phi(\Sigma, C)) = 0$, where F is defined in (4.13). F is affine in each component and as a consequence is \mathcal{C}^∞ and we can define its derivatives on (S, C, N) , $\frac{\partial F}{\partial S}(S, C, N)$, $\frac{\partial F}{\partial C}(S, C, N)$ and $\frac{\partial F}{\partial N}(S, C, N)$ which are linear mappings from $\mathcal{S}_n(\mathbb{R})$ to $\mathcal{S}_n(\mathbb{R})$ defined by:

$$\frac{\partial F}{\partial S}(S, C, N).H_1 = LN\Pi H_1 + H_1\Pi NL + H_1$$

$$\begin{aligned}\frac{\partial F}{\partial C}(S, C, N).H_2 &= -H_2 \\ \frac{\partial F}{\partial N}(S, C, N).H_3 &= LH_3\Pi S + S\Pi H_3 L\end{aligned}$$

As a consequence, we have

$$\nabla F(S, C, N).(H_1, H_2, H_3) = LN\Pi H_1 + H_1\Pi N L + H_1 - H_2 + LH_3\Pi S + S\Pi H_3 L$$

In the proof of Lemma 3.6.3, we showed that $\frac{\partial F}{\partial N}(\Sigma, C, N)$ is invertible. As a consequence we can apply the implicit function theorem in order to compute the gradient of Φ . As $F(\Sigma, C, \Phi(\Sigma, C)) = 0$ and $\Phi(\Sigma, \Sigma) = 0$, we find that $\frac{\partial F}{\partial S}(\Sigma, \Sigma, 0).H_1 = H_1$, $\frac{\partial F}{\partial C}(\Sigma, \Sigma, 0).H_2 = -H_2$ and $\frac{\partial F}{\partial N}(\Sigma, \Sigma, 0).H_3 = LH_3\Pi\Sigma + \Sigma\Pi H_3 L$ and hence the derivative of Φ on (Σ, Σ) is given by:

$$\nabla\Phi(\Sigma, \Sigma).(H_1, H_2) = \left(\frac{\partial F}{\partial N}(\Sigma, \Sigma, 0)\right)^{-1} (H_2 - H_1)$$

which is equivalent to

$$\nabla\Phi(\Sigma, \Sigma).(H_1, H_2) = \Phi(\Sigma, \Sigma + H_2 - H_1)$$

Using Proposition 3.3.5, we find that

$$\sqrt{N}\widehat{M}^{(N)} \xrightarrow{\mathcal{L}} \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{iq}}(\overline{Z}_{T+\tau_{iq}} - \overline{Z}_T) - \frac{1}{\tau}\overline{Z}_\tau\right)$$

which concludes the proof of Proposition 3.3.6.

If there are no fire sales between 0 and T , then $\tau = T$ almost surely. In addition, under (H_0) , we have $\sigma\sigma^t = \Sigma$ and the expression for the process \tilde{V}_t defined in (3.28) is simplified as

$$(\tilde{V}_t \tilde{V}_t^t)^{ij,kl} = \Sigma_{i,k} \Sigma_{j,l} \quad (3.45)$$

which implies that the process \overline{Z} defined in (3.27) is a Brownian motion.

Furthermore, given Proposition 3.3.6, under (H_0) , $\sqrt{N}\left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{iq}})\right)$ converges in law when N goes to infinity to the random variable

$$P_T^t \Phi\left(\Sigma, \Sigma + \frac{1}{\tau_{iq}}(\overline{Z}_{T+\tau_{iq}} - \overline{Z}_T) - \frac{1}{T}\overline{Z}_T\right)(P_T - P_{T+\tau_{iq}})$$

Given the expression for Φ given in (3.20), we find the expression for:

$$\begin{aligned}& P_T^t \Phi(\Sigma, C)(P_T - P_{T+\tau_{iq}}) \\ &= \sum_{1 \leq p, q \leq n} (\Omega^{-1} P_T)_p \frac{[\Omega^t L^{-1}(C - \Sigma)L^{-1}\Omega]_{p,q}}{\phi_p + \phi_q} (\Omega^{-1}(P_T - P_{T+\tau_{iq}}))_q\end{aligned}$$

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Given the fact that $L^{-1} = \text{diag}(D_i)$, we have $(\Omega^t L^{-1})_{p,i} = \Omega_{i,p} D_i$ and $(L^{-1} \Omega)_{j,q} = \Omega_{j,q} D_j$. As a consequence, denoting

$$m_{i,j} = \sum_{1 \leq p,q \leq n} \frac{[\Omega^{-1} P_T]_p [\Omega^{-1} (P_T - P_{T+\tau_{iq}})]_q}{\phi_p + \phi_q} \Omega_{ip} \Omega_{jq} D_i D_j$$

we can write $P_T^t \Phi(\Sigma, C)(P_T - P_{T+\tau_{iq}})$ as $\sum_{1 \leq i,j \leq n} m_{ij} (C_{i,j} - \Sigma_{i,j})$. Hence the limit of

$\sqrt{N} \left(P_T^t \widehat{M}^{(N)}(P_T - P_{T+\tau_{iq}}) \right)$ is equal to

$$\sum_{1 \leq i,j \leq n} m_{ij} \left(\frac{1}{\tau_{iq}} (\bar{Z}_{T+\tau_{iq}} - \bar{Z}_T) - \frac{1}{T} \bar{Z}_T \right)_{i,j}$$

Under the assumptions of Corollary 3.3.7, \bar{Z} is a Brownian motion on $[0, T + \tau_{iq}]$ (see (3.45)), so the limit process is a mean-zero Gaussian process. To compute its variance, we first compute the variance of $\sum_{1 \leq i,j \leq n} m_{ij} \bar{Z}_t^{i,j}$ which, given the expression of \bar{Z} in (3.27),

can be written as

$$\sum_{1 \leq k,l \leq n} \int_0^t \frac{1}{\sqrt{2}} \sum_{1 \leq i,j \leq n} m_{i,j} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) d\tilde{W}_s^{kl}.$$

Using the Ito isometry formula, its variance is thus equal to

$$\begin{aligned} & \sum_{1 \leq k,l \leq n} \int_0^t \left(\sum_{1 \leq i,j \leq n} \frac{1}{\sqrt{2}} m_{i,j} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \right)^2 ds \\ &= \frac{t}{2} \sum_{1 \leq k,l \leq n} \left(\sum_{1 \leq i,j,p,q \leq n} m_{i,j} m_{p,q} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl} \right) \right) \\ &= \frac{t}{2} \sum_{1 \leq i,j,p,q \leq n} m_{i,j} m_{p,q} \left(\sum_{1 \leq k,l \leq n} \left(\tilde{V}_s^{ij,kl} + \tilde{V}_s^{ji,kl} \right) \left(\tilde{V}_s^{pq,kl} + \tilde{V}_s^{qp,kl} \right) \right) \\ &= t \sum_{1 \leq i,j,p,q \leq n} m_{i,j} m_{p,q} (\Sigma_{i,p} \Sigma_{j,q} + \Sigma_{i,q} \Sigma_{j,p}) \end{aligned}$$

using the fact that $\sum_{1 \leq k,l \leq n} \tilde{V}_s^{ij,kl} \tilde{V}_s^{pq,kl} = \Sigma_{i,p} \Sigma_{j,q}$ as \tilde{V} verifies (3.45). Given the fact

that $\bar{Z}_{T+\tau_{iq}} - \bar{Z}_T$ and \bar{Z}_T are independent, we find that the variance of the limit $\sum_{1 \leq i,j \leq n} m_{ij} \left(\frac{1}{\tau_{iq}} (\bar{Z}_{T+\tau_{iq}} - \bar{Z}_T) - \frac{1}{T} \bar{Z}_T \right)_{i,j}$ is equal to

$$\left(\frac{1}{T} + \frac{1}{\tau_{iq}} \right) \sum_{1 \leq i,j,k,l \leq n} m_{ij} m_{kl} (\Sigma_{ik} \Sigma_{jl} + \Sigma_{jk} \Sigma_{il})$$

which concludes the proof of Corollary 3.3.7.

Chapter 4

Impact of institutional investors on the dependence structure of asset returns

Abstract

We propose a multi-period model of a financial market with multiple assets, which takes into account the impact of a large institutional investor rebalancing its positions at each period, so as to maintain a fixed allocation in each asset. We show that feedback effects can lead to significant excess realized correlation between asset returns and modify the principal component structure of the (realized) correlation matrix of returns. The continuous-time limit of the multi-period model allows to compute the realized correlation matrix of returns and its eigenvalues and eigenvectors in the presence of feedback effects. Our study naturally links the properties of the realized correlation matrix to the sizes and trading volumes of large institutional investors. Finally, we show that feedback effects flatten the differences between assets' expected returns and tend to align them with the returns of the institutional investor's portfolio. In particular, we show that an investor who tries to overperform the institutional investor by overweighting assets with large expected returns will experience lower-than-expected returns, due to feedback effects.

4.1 Introduction

International financial markets have become increasingly dominated by large institutional investors, who account for a large fraction of holdings and trades in financial assets. For instance, institutional investors in the US hold \$25000 billion in financial assets which represents 17.4% of total outstanding assets. Their positions in the US equity markets amount to \$12500 billion, which corresponds to holding 70% of the total equity assets in the US (Gonnard et al., 2008; Tonello and Rabimov, 2010). Two major features characterize large institutional investors over the last years. First, they build their portfolios with the use of indices and exchange traded funds (representing a sector, a geographical zone or an asset class for instance), which have become increasingly popular in the last years, and assets traded on large national exchanges (Gastineau, 2010; Fuhr, 2011; Boudreaux, 2012). Secondly, while such asset managers do not frequently modify their allocations, they do actively trade in the market: Carhart (2012) documents that the average turnover for US mutual funds is 75%.

Large institutional investors build and manage their portfolios comprising numerous assets taking into account the dependence structure between asset returns. In particular, the correlation between asset returns is a key ingredient for trading, portfolio optimization and risk management. It is very often considered as reflecting a structural correlation between fundamentals of asset returns and hence assumed not to vary a lot in time. Ever since Markowitz (1952), theoretical studies show that, under the assumption of a constant correlation structure between asset returns, optimal strategies are fixed-mix strategies: the investor maintains a fixed allocation in each asset of its portfolio. Typically, if the value of an asset increased, its weight in the portfolio increases and the investor following a fixed-mix strategy sells a part of its positions in this asset, so as to come back to the target allocation for this asset. The fixed-mix strategy implies 'buying low and selling high'. Numerous theoretical studies (Evstigneev and Schenk-Hoppé, 2002; Dempster et al., 2003; Mulvey and Kim, 2008) have shown that such strategies can enhance the long term growth rate of portfolios.

Whereas the price of financial assets is traditionally modeled as an exogenous stochastic process unaffected by investors' strategies, the presence of institutional investors, which have a large impact when trading, has implications for financial markets, in particular for the indices and ETFs that they trade, and hence for the components of those indices and ETFs.

In this paper, we examine how the market impact of a large institutional investor keeping a fixed allocation affects the realized correlation between asset returns and the expected asset returns. We derive tractable expressions for the principal components and the corresponding eigenvalues of the (realized) correlation matrix of returns in the presence of feedback effects. We describe and quantify how the economic mechanism of rebalancing positions by a large institutional investor can lead to the features commonly observed in empirical realized correlation matrices (Friedman and Weisberg, 1981; Bouchaud et al., 2000; Sandoval and Franca, 2012): positive average pairwise realized correlation for stock indices, largest eigenvalue of the realized correlation matrix

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which is significantly larger than the other eigenvalues (Figure 4.1) and associated to an eigenvector with positive weights (Figure 4.2).

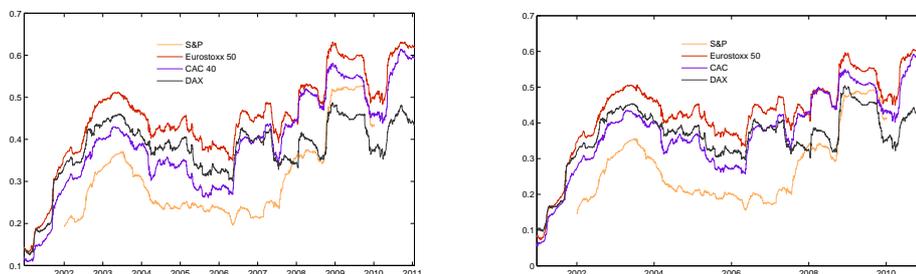


Figure 4.1: Left: One-year exponentially-weighted moving average estimator of average pairwise correlations of daily returns in equity indices/ Right: Difference between the first and second eigenvalue of the empirical correlation matrix normalized by the number of stocks in the index

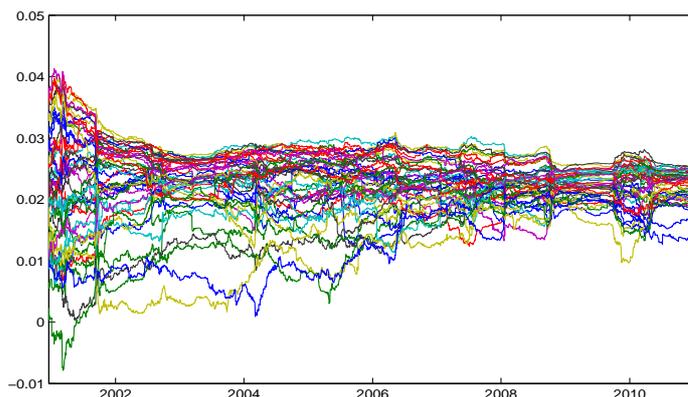


Figure 4.2: Components of the eigenvector associated to the largest eigenvalue of the empirical correlation matrix of returns for Eurostoxx 50

4.1.1 Summary

We propose a multi-period model of a financial market with multiple assets, in which a large institutional investor maintains a fixed allocation across assets. Simulations of this model, with realistic parameters estimated from time series of S&P500 stock returns, suggest that feedback effects from the fund's rebalancing lead to a significant increase in realized correlation between asset returns. We exhibit conditions under which the discrete-time model converges to a diffusion limit. By studying the multi-dimensional diffusion limit for the price dynamics, we show that the existence of a large institutional

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investor maintaining a constant allocation across asset classes may result in a significant and systematic impact on expected returns and the correlation of returns. In particular, such fixed-mix strategies dampen asset volatility but increase correlation across asset classes they invest in. This rebalancing effect leads to a systematic bias in the first principal component of the correlation matrix, overweighting assets with high turnover in the benchmark portfolio, as well as higher eigenvalue for this principal component. The impact of the large institutional investor biases asset expected returns and decreases the performance of funds who overweigh (resp. underweigh) assets with large (resp. low) expected returns. These findings have consequences for risk-management and asset allocation. We show that the impact of the large institutional investor modifies the risk/return trade off of portfolios composed from the same assets: an investor who factors these effects into his allocations can improve his risk/return trade off.

4.1.2 Related literature

Various empirical studies attest to the large market share of institutional investors. Gonnard et al. (2008) studies institutional investors of countries of the Organization for Economic Co-operation and Development (OECD) while Tonello and Rabimov (2010) focuses on the institutional investors in the US. Such investors comprise mutual funds, insurance companies and pension funds. Their investments amounted to \$40000 billion for funds in the OECD in 2005, which represents 150% of the gross domestic product of the OECD. US institutional investors account for more than half of those investments (\$25000 billion) and prefer investing in equity markets (50% of their positions).

The preferences of large institutional investors are examined in numerous empirical studies. Del Guercio (1996) finds empirically that banks, contrary to mutual funds, prefer investing in prudent stocks. Gompers and Metrick (2001) use a database with seventeen years of data on large institutional investors and show that they prefer holding liquid assets, while Ferreira and Matos (2008) find that institutional investors have a strong preference for the stocks of large firms and firms with good governance. Falkenstein (2012) shows that mutual funds prefer investing in liquid stocks with low transaction costs and are averse to stocks with low idiosyncratic volatility. Lakonishok et al. (1992) study the types of strategies followed by institutional investors and whether they follow trading practices which are potentially destabilizing for asset prices. The impact of institutional investors on asset returns is widely studied in the empirical literature. Aitken (1998) shows that the growth of capital invested by mutual funds and other institutional investors in emerging markets resulted in a sharp increase of autocorrelation for the assets in those markets. Sias and Starks (1997) also finds that the larger the institutional ownership of a stock in the NYSE, the larger its autocorrelations while Sias (1996) finds that an increase in ownership by institutional investors on a given stock results in a greater stock volatility.

Most theoretical studies model the impact of large institutional investors on a single asset's return and volatility. Almgren and Chriss (2000); Almgren and Lorenz (2006); Almgren (2009) model the permanent and temporary impact of a large investor liquidating a position on a single asset and derive an optimal liquidation strategy. Alfonsi et al.

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(2009) derive the optimal strategy to liquidate a large position on an asset by taking into account the order book of the asset. Gabaix et al. (2006) propose an equilibrium model which takes into account the supply and demand of a large institutional investor and show that their trades can lead to volatility spikes. They derive an optimal strategy for the institutional investor, in the presence of its own feedback effects. These theoretical studies explain quantitatively the facts described in the empirical literature cited previously, but mainly focus on a single asset and derive optimal strategies for institutional investors. In particular, they do not model the cross-asset impact of large institutional investors and the spillover effects that they can generate. Kyle and Xiong (2001) study a market with two risky assets and three types of traders: noise traders, convergence traders and long-term investors. They show how the strategies implemented by each type of traders can result in contagion effects and lead to endogenous correlation which can not be explained by assets fundamentals.

4.1.3 Outline

This paper is organized as follows: Section 4.2 presents a framework for modeling the impact of fixed-mix strategies on asset returns. Section 4.3 studies the realized covariances and correlations between asset returns in the presence of feedback effects from the large institutional investor while Section 4.4 gives analytical formulas for the eigenvalues and eigenvectors of the realized correlation matrix. Section 4.5 analyzes the impact of the large institutional investor on the asset expected returns.

4.2 Asset dynamics in the presence of feedback effects

4.2.1 Multi-period model

Consider a discrete-time market, where trading takes place at dates $t_k = k\tau$ and which comprises n financial assets. The value of asset i at t_k is S_k^i . Typically, one can consider that S^i is the value of an index or an ETF representing a sector, asset class or geographic zone. Between t_k and t_{k+1} , it is affected by economic moves, represented by an IID sequence $\xi_k = (\xi_k^1, \dots, \xi_k^n)_{1 \leq k \leq M}$ of centered random variables with covariance matrix Σ . In the absence of other effects, the value of asset i at t_{k+1} would be equal to

$$(S_{k+1}^i)^* = S_k^i \exp \left(\tau \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\tau} \xi_{k+1}^i \right) \quad (4.1)$$

where m_i is the 'fundamental' expected return of asset i and Σ is the 'fundamental' covariance matrix between asset returns and reflects the fundamental structure between the n assets.

Consider now a large institutional investor/fund investing in this market and following a (long) *fixed-mix strategy*, maintaining a fixed allocation in each asset. As discussed in Section 4.1, the fixed-mix strategy is widely used by institutional investors, in between two allocation dates, which correspond to a time frame of several months. At each date t_k , the fund holds a (constant, positive) proportion x_i of each asset i which means that

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the dollar amount invested by the fund in asset i at this date is equal to $x_i W_k$ where W_k is the fund value/wealth at t_k . Denoting by ϕ_k^i the number of units of asset i held at t_k , the fixed-mix strategy implies that at each date:

$$\phi_k^i S_k^i = x_i W_k \quad (4.2)$$

At each period, the asset values may change as described in (4.1) and the fund rebalances its positions to maintain the target proportion x_i in each asset i . Typically, if the value of an asset increased (resp. decreased) more than the others, the fund sells (resp. buys) units of this asset in order to maintain a fixed portion of this asset in its portfolio. The fixed-mix strategy is a typical example of a contrarian strategy, which implies 'buying low and selling high'. The rebalancing by the fund in order to maintain its target allocation generates a net demand of $\phi_{k+1}^i - \phi_k^i$ units of asset i between t_k and t_{k+1} , in a self-financing manner:

$$W_{k+1} = \sum_{i=1}^n \phi_{k+1}^i S_{k+1}^i = \sum_{i=1}^n \phi_k^i S_{k+1}^i \quad (4.3)$$

We assume that the impact of this net demand on the return of each asset i is linear (Obizhaeva, 2011; Cont et al., 2010) and is measured by the depth D_i of the market in asset i : a net demand of $\frac{D_i}{100}$ shares for security i moves the price of i by one percent.

As a result, the value of asset i at date t_{k+1} has to verify:

$$S_{k+1}^i = S_k^i \exp \left(\tau \left(m_i - \frac{\Sigma_{i,i}}{2} \right) + \sqrt{\tau} \xi_{k+1}^i \right) \left(1 + \frac{\phi_{k+1}^i - \phi_k^i}{D_i} \right) \quad (4.4)$$

However, as ϕ_{k+1}^i depends on S_{k+1}^i we have to prove that, at each period, the fund can rebalance its positions in a self-financing way so as to keep its fixed allocation ie: verify (4.4). This is done in Proposition 4.2.1. We remark that when market depths are infinite ($D_i = \infty$), the fund's rebalancing does not generate any feedback on asset returns and (4.4) is equivalent to (4.1), meaning that asset values move according to 'fundamentals' only, which are captured by the fundamental covariance matrix Σ and the fundamental expected returns m .

Proposition 4.2.1 *There exists a unique self-financing investment strategy which allows the fund to keep a constant proportion x_i invested in asset i . Furthermore, there exists*

$$\theta : (\mathbb{R}_+^*)^{n+1} \times \mathbb{R}^n \rightarrow (\mathbb{R}_+^*)^{n+1}$$

such that

$$\begin{pmatrix} S_{k+1} \\ W_{k+1} \end{pmatrix} = \begin{pmatrix} S_{k+1}^1 \\ \vdots \\ S_{k+1}^n \\ W_{k+1} \end{pmatrix} = \begin{pmatrix} \theta_1(S_k, W_k, \overline{m}\tau + \xi_{k+1}\sqrt{\tau}) \\ \vdots \\ \theta_n(S_k, W_k, \overline{m}\tau + \xi_{k+1}\sqrt{\tau}) \\ \theta_{n+1}(S_k, W_k, \overline{m}\tau + \xi_{k+1}\sqrt{\tau}) \end{pmatrix} \quad (4.5)$$

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and (S_k, W_k) verifies Equations 4.2, 4.3, and 4.4 for every k . Here, $\bar{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$ and, for every compact set $K \subset (\mathbb{R}_+^*)^{n+1}$, there exists $\epsilon_K > 0$ such that θ is C^∞ on $K \times \mathcal{B}(0, \epsilon_K)$.

In particular, (S_k, W_k) is a Markov chain in $(\mathbb{R}_+^*)^{n+1}$. The proof of this proposition is given in Appendix 4.6.1.

4.2.2 Simulation experiments

In this section, we make simulation experiments which illustrate the impact of feedback effects from a large institutional investor following a fixed-mix strategy on the realized correlation between assets and the principal component properties of the realized correlation matrix of returns.

We simulate the multi-period model described in the previous section in a very simple example of homogenous fundamental volatility, correlation and expected return. In order to compare our numerical results with empirical results on the S&P500 in 2006, we choose the following realistic parameters. The simulated market comprises $n = 500$ assets and trading is possible everyday ($\tau = \frac{1}{250}$). Each asset has a fundamental expected return $m_i = 11\%$ (equal to the return of the S&P500 in 2006) and a fundamental volatility $\sqrt{\Sigma_{i,i}} = 10\%$ (equal to the realized volatility of the S&P500 in 2006). We denote by $\rho = \frac{\Sigma_{i,j}}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}}$ the fundamental correlation between any pair of assets.

We consider an institutional investor maintaining a constant portion $x_i = \frac{1}{n} = 0.2\%$ invested in each asset i and the initial position of the fund in each asset is equal to one fifth of the respective asset market depth: for all i , $\frac{\phi_i^0}{D_i} = \frac{1}{5}$. This choice is legitimated by empirical studies: Tonello and Rabimov (2010) shows that the size of institutional investors over the last years is approximately \$25000 billion, among which 50%, ie: \$12500 billion are invested in US equity markets. As a proxy of market depth, we use, following Obizhaeva (2011),

$$D = \frac{\text{Average Daily Volume}}{0.33 \times \text{Daily Volatility}}$$

Given that in 2006 the average daily volume of the US equity market was \$80 billion and the realized volatility of the S&P500 was 10%, we find that the depth of the US equity markets is $\frac{80}{0.33 \times \frac{10\%}{\sqrt{250}}} \approx \38000 billions. Assuming, for example, that 60% of the institutional investors follow a fixed-mix strategy, this legitimates our choice of $\frac{\phi_i^0}{D_i} = 60\% \times \frac{12500}{38000} = \frac{1}{5}$.

Note that we choose to compare our numerical results to empirical results on the S&P500 in 2006 because the recent financial turmoil started in 2007 and, since then, the US equity market was subject to major fire sales and deleveraging phenomenon, which were the main source of feedback in those markets, as analysed in Cont and Wagalath (2012a).

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For each of the 10000 simulated scenarios, we compute the realized volatility of asset returns and the realized correlation between asset returns. The following figures display the distribution of those quantities, in the case of $\rho = 0$. We compare those distributions to the case without feedback effects (which corresponds to infinite market depth).

Realized correlation and realized variance Figure 4.3 shows that feedback effects increase realized correlation between assets. Whereas without feedback effects, the distribution of realized correlation between the two assets is centered around its fundamental value $\rho = 0$, we witness, in the presence of feedback effects, that the distribution of realized correlation is shifted towards strictly positive values, centered around an average value of 10% and with values over 20% with significant probability. Even starting with zero fundamental correlations, feedback effects generate, on average, a realized correlation of 10% between assets.

In Figure 4.4, we see that the price impact of fixed-mix strategies decreases asset volatility. This is due to the fact that the fund follows a 'contrarian' strategy: it buys (resp. sells) assets that decreased (resp. increased) the most, hence damping their decrease (resp. increase) and, overall, damping the amplitude of price moves.

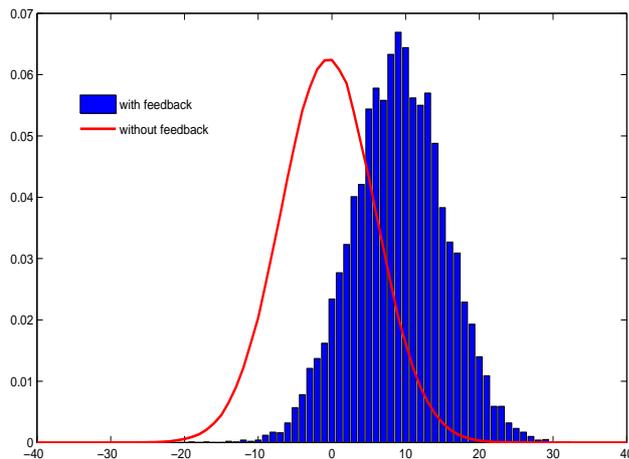


Figure 4.3: Distribution of realized correlation between assets 1 and 2 (with $\rho = 0$) with and without feedback effects

Average pairwise realized correlation and highest eigenvalue of the realized correlation matrix Table 4.1 displays the average for the average pairwise realized correlation over 10^4 simulations, for different values of fundamental correlation ρ and fund sizes $\frac{\phi_0^i}{D_i}$. It shows that, for each choice of parameters, the average pairwise correlation is higher than its fundamental value in the presence of feedback effects. Furthermore, we see that the larger the fund's positions as a fraction of asset depth (ie: the larger the

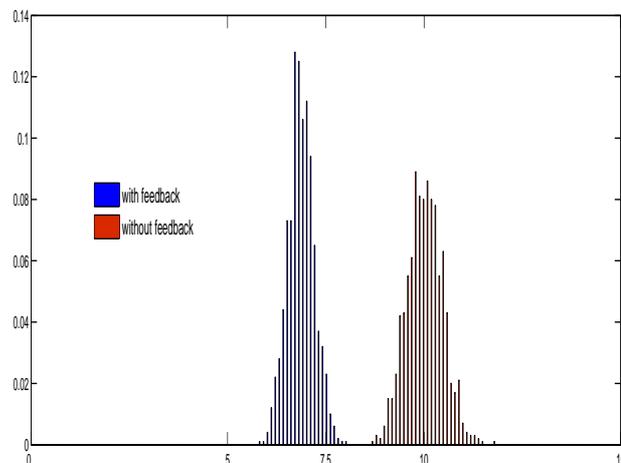


Figure 4.4: Distribution of realized volatility for asset 1 with and without feedback effects (fundamental volatility equal to 10%)

fund's positions or the lower the assets' depth or liquidity), the larger the impact on the average pairwise correlation. In 2006, the average pairwise one-year realized correlation in the S&P500 was 21%. Table 4.1 shows that *with a reasonable and realistic choice of parameters for the fund's size*, for example $\frac{\phi_0^i}{D_i} = \frac{1}{5}$ as discussed in the beginning of this section, *an homogenous fundamental correlation of only 15% combined to feedback effects generated by the rebalancing of the fund's positions generate a 22% average pairwise realized correlation, as observed empirically.*

Table 4.2 leads to the same conclusions: the presence of the large institutional investor increases the value of the largest eigenvalue of the realized correlation matrix, compared to its fundamental value which is given in the column $\frac{\phi_0^i}{D_i} = 0$. Furthermore, the larger the fund's positions as a fraction of market depth, the larger the eigenvalue of the realized correlation matrix. While the largest eigenvalue of the one-year realized correlation matrix of the S&P500 in 2006 was 110, we see that our model leads to this level of largest eigenvalue starting from a fundamental correlation of 15%, which corresponds to a fundamental largest eigenvalue of 76, combined to feedback effects, in the case where $\frac{\phi_0^i}{D_i} = \frac{1}{5}$.

First eigenvector of the realized correlation matrix It is also interesting to focus on the impact of feedback effects on the eigenvectors of the realized correlation matrix, and, in particular, on the first eigenvector of such matrix, associated to the largest eigenvalue. Given the set of reference parameters that we chose in our simulation experiments, the

first eigenvector of the fundamental correlation matrix is 'the market': $\frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. In

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Average pairwise realized correlation			
ρ	$\frac{\phi_0^i}{D_i} = \frac{1}{10}$	$\frac{\phi_0^i}{D_i} = \frac{1}{5}$	$\frac{\phi_0^i}{D_i} = \frac{1}{3}$
0	2%	5%	10%
10%	12%	15%	21%
15%	18%	22%	29%
25%	29%	35%	44%
50%	55%	61%	69%
75%	79%	82%	87%
90%	92%	93%	95%

Table 4.1: Average for the average pairwise realized correlation for different values of fundamental correlation ρ and fund sizes (as a fraction of market depth) $\frac{\phi_0^i}{D_i}$. In comparison, the average pairwise one-year realized correlation of the S&P500 in 2006 was 21%.

Largest eigenvalue of the realized correlation matrix				
ρ	$\frac{\phi_0^i}{D_i} = 0$	$\frac{\phi_0^i}{D_i} = \frac{1}{10}$	$\frac{\phi_0^i}{D_i} = \frac{1}{5}$	$\frac{\phi_0^i}{D_i} = \frac{1}{3}$
0	1	8	9	11
10%	51	64	79	105
15%	76	93	113	146
25%	126	149	175	216
50%	251	277	307	345
75%	375	394	412	434
90%	450	459	467	476

Table 4.2: Average for the largest eigenvalue of the realized correlation matrix for different values of fundamental correlation ρ and fund sizes (as a fraction of market depth) $\frac{\phi_0^i}{D_i}$. In comparison, the largest eigenvalue of the one-year realized correlation matrix of the S&P500 in 2006 was 110.

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Figure 4.5, we see that when there are no feedback effects, the angle between the first eigenvalue of the realized correlation matrix and the market is close to zero. However, in the presence of feedback effects from the fixed-mix strategy, we witness that this angle is larger, around two times larger than in the case without feedback effects: the first eigenvalue of the realized correlation matrix deviates from its fundamental direction. In Figure 4.6, we compare the angle between the first eigenvalue of the realized correlation matrix and the market for different choices of parameters: non homogenous liquidity ($\frac{\phi_0^{2i}}{D_{2i}} = \frac{2\phi_0^{2i+1}}{D_{2i+1}}$) and non homogenous fund allocations ($x_{2i} = 2x_{2i+1}$). *The eigenvectors of the realized correlation matrix depend on the size of the fund as a fraction of market depth (and hence are liquidity-dependent) and on the fund's allocations.* Feedback effects shape the first eigenvalue of the realized correlation matrix depending on the fund's strategy, positions and asset market depths.

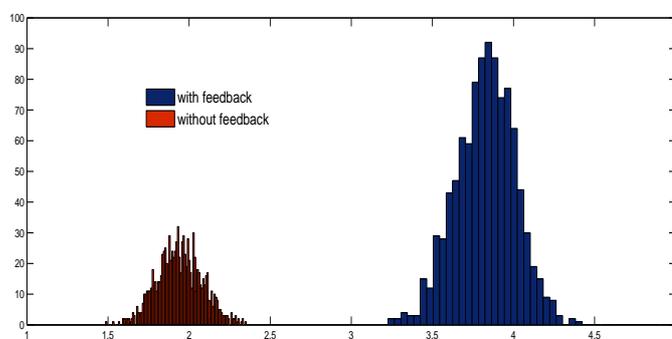


Figure 4.5: Histogram for the angle between the first eigenvector of the realized correlation matrix and the market with and without feedback effects

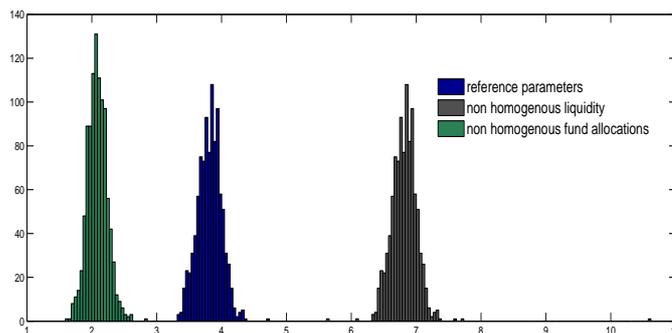


Figure 4.6: Histogram for the angle between the first eigenvector of the realized correlation matrix and the market for different sets of parameters

4.2.3 Continuous time limit

To confirm the phenomena observed in the simulation experiments, we now analyze the continuous-time limit of the multi-period model: the study of this limit allows one to obtain analytical formulas for realized correlation between asset returns, eigenvalues and eigenvectors of the realized correlation matrix and asset expected returns, which confirm quantitatively the effects observed in the numerical experiments.

Define a (resp., b) a $\mathcal{M}_{n+1 \times n}(\mathbb{R})$ -valued (resp. \mathbb{R}^{n+1} -valued) mapping such that

$$a_{i,j}(S, W) = \sum_{l=1}^n \frac{\partial \theta_i}{\partial z_l}(S, W, 0) \times A_{l,j} \quad (4.6)$$

$$b_i(S, W) = \sum_{j=1}^n \frac{\partial \theta_i}{\partial z_j}(S, W, 0) \bar{m}_j + \frac{1}{2} \sum_{j,l=1}^n \frac{\partial^2 \theta_i}{\partial z_j \partial z_l}(S, W, 0) \Sigma_{j,l} \quad (4.7)$$

where θ and \bar{m} are defined in Proposition 4.2.1 and A is a square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

In order to study the continuous-time limit of the multi-period model, we work under the following assumption:

Assumption 4.2.2 *There exists $\eta > 0$ such that:*

$$\mathbb{E}(\|\exp(\eta\xi)\|) < \infty \quad \text{and} \quad \mathbb{E}(\|\xi\|^{\eta+4}) < \infty$$

The following theorem describes the diffusion limit of the price process.

Theorem 4.2.3 *Under Assumption 4.2.2, $(S_{\lfloor \frac{t}{\tau} \rfloor}, W_{\lfloor \frac{t}{\tau} \rfloor})_{t \geq 0}$ converges weakly to a diffusion $(P_t, V_t)_{t \geq 0} = (P_t^1, \dots, P_t^n, V_t)_{t \geq 0}$ as τ goes to 0 with:*

$$\begin{aligned} dP_t^i &= b_i(P_t, V_t)dt + (a(P_t, V_t)dB_t)_i \quad 1 \leq i \leq n \\ dV_t &= b_{n+1}(P_t, V_t)dt + (a(P_t, V_t)dB_t)_{n+1} \end{aligned}$$

where a and b are defined in (4.6) and (4.7) respectively and B_t is an n -dimensional Brownian motion.

The proof of this theorem is given in Appendix 4.6.2. The expression for a and b can be deduced from Lemma 4.6.3 and Lemma 4.6.4 in Appendix 4.6.2.

In the case where market depths are infinite (for all i , $D_i = +\infty$), the expression for a and b simplifies to

$$\tilde{a}_{i,k}(\tilde{P}, \tilde{V}) = \tilde{P}^i A_{i,k} \quad \text{and} \quad \tilde{b}_i(\tilde{P}, \tilde{V}) = \tilde{P}^i m_i \quad 1 \leq i \leq n$$

and the price follows a multivariate Black Scholes dynamics with expected return m and covariance matrix Σ :

$$\tilde{P}_t^i = P_0^i \exp \left(\left(m_i - \frac{\Sigma_{i,i}}{2} \right) t + (AB_t)_i \right) \quad \text{and} \quad V_t = V_0 \exp \left(\left(X.m - \frac{X.\Sigma X}{2} \right) t + X.AB_t \right)$$

which implies that

$$\tilde{\Phi}_t^i = \frac{\tilde{\phi}_t^i}{\tilde{\phi}_0^i} = \exp \left(\left(X.m - m_i - \frac{X.\Sigma X - \Sigma_{i,i}}{2} \right) t + X.AB_t - (AB_t)_i \right) \quad (4.8)$$

In particular, we find that

$$\mathbb{E} \left(\tilde{\Phi}_t^i \right) = \exp \left(t \left(X.m - m_i + \Sigma_{i,i} - (\Sigma X)_i \right) \right) \quad (4.9)$$

When market depths are finite, feedback effects from the large fund modify this price dynamics.

In the continuous-time limit, at each date t , the fund allocates x_i to asset i . Its holdings in asset i are:

$$\phi_t^i = \frac{x_i V_t}{P_t^i} \quad (4.10)$$

4.3 Realized correlations in the presence of feedback effects

4.3.1 The covariance structure of asset returns

The realized covariance (Andersen et al., 2003; Barndorff-Nielsen and Shephard, 2004) matrix of asset returns between t_1 and t_2 , denoted $C_{[t_1, t_2]}$, is defined by

$$C_{[t_1, t_2]}^{i,j} = \frac{1}{t_2 - t_1} \left([\ln P^i, \ln P^j]_{t_2} - [\ln P^i, \ln P^j]_{t_1} \right) = \frac{1}{t_2 - t_1} \int_0^t c_s^{i,j} ds \quad (4.11)$$

where $[\ln P^i, \ln P^j]_t$ is the quadratic covariation between $\ln P^i$ and $\ln P^j$ on $[0, t]$ and c_s is the derivative of the quadratic covariation and corresponds to the instantaneous covariance matrix of asset returns. The following result follows by direct computation from Theorem 4.2.3 and Lemma 4.6.3.

Proposition 4.3.1 *The instantaneous covariance matrix at date t , c_t , defined in (4.11), is given by*

$$c_t = \Sigma + \Sigma(\Gamma_t^t - I_n)F_t + F_t(\Gamma_t - I_n)\Sigma + F_t(\Gamma_t - I_n)\Sigma(\Gamma_t^t - I_n)F_t \quad (4.12)$$

where F_t and Γ_t are $n \times n$ matrices such that F_t is diagonal with i -th term equal to

$$F_t^{i,i} = \frac{\Phi_t^i \Lambda_i}{1 + \Phi_t^i \Lambda_i} \quad \text{and} \quad \Gamma_t^{i,j} = \left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_t^p \Lambda_p} \right)^{-1} \frac{x_j}{1 + \Phi_t^j \Lambda_j} \quad (4.13)$$

where Λ is an n dimensional vector such that

$$\Lambda_i = \frac{\phi_0^i}{D_i} \quad (4.14)$$

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represents the initial holdings of the fund in asset i as a fraction of market depth and

$$\Phi_t^i = \frac{\phi_t^i}{\phi_0^i} \quad (4.15)$$

In the sequel, we study realized covariances and correlations on $[0, T]$ for clarity purpose. Using Proposition 4.3.1, we find that

$$\begin{aligned} C_{[0,T]} &= \Sigma + \frac{1}{T} \int_0^T (\Sigma(\Gamma_s^t - I_n)F_s + F_s(\Gamma_s - I_n)\Sigma) ds \\ &\quad + \frac{1}{T} \int_0^T (F_s(\Gamma_s - I_n)\Sigma(\Gamma_s^t - I_n)F_s) ds \end{aligned} \quad (4.16)$$

where F_s and Γ_s are defined in (4.13). In the presence of feedback effects, the realized covariance matrix is the sum of the fundamental covariance matrix Σ and an excess realized covariance matrix which is path-dependent and liquidity-dependent and is equal to zero when market depths are infinite. *Starting with homoscedastic inputs, feedback effects from institutional investors actions result in endogenous heteroscedasticity.* Our model shows that the economic mechanism of rebalancing naturally impacts the realized covariance between asset returns, this impact being quantified by Equation 4.16.

Corollary 4.3.2 *The realized covariance between assets i and j returns can be decomposed as follows:*

$$\begin{aligned} C_{[0,T]}^{i,j} &= \Sigma_{i,j} + \Lambda_i \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^i \right) \sum_{1 \leq l \leq n} x_l (\Sigma_{j,l} - \Sigma_{i,j}) \\ &\quad + \Lambda_j \left(1 + \int_0^T \left(1 - \frac{s}{T} d\Phi_s^j \right) \right) \sum_{1 \leq l \leq n} x_l (\Sigma_{i,l} - \Sigma_{i,j}) + O(\|\Lambda\|^2) \end{aligned} \quad (4.17)$$

where Λ and Φ are defined in (4.14) and (4.15) respectively, and $\mathbb{E} \left(\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2} \right)$ is bounded when Λ goes to zero.

The proof of this corollary is given in Appendix 4.6.3. We remark that when the fund invests significantly in one asset i_0 , such that $\Lambda_{i_0} > 0$, and even if its positions on the other assets are negligible ($\Lambda_i = 0$ for $i \neq i_0$), the realized covariance between those assets and i_0 is modified and different from the fundamental covariance.

4.3.2 Realized correlation between asset returns

The realized correlation between assets i and j between dates 0 and T is

$$R_{[0,T]}^{i,j} = \frac{C_{[0,T]}^{i,j}}{\left(C_{[0,T]}^{i,i} C_{[0,T]}^{j,j} \right)^{\frac{1}{2}}}$$

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where $C_{[0,T]}$, the realized covariance matrix between dates 0 and T , is defined in (4.16).

Using (4.17), the realized correlation between assets i and j can be decomposed as follows:

$$R_{[0,T]}^{i,j} = \frac{\Sigma_{i,j}}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} + \frac{\Lambda_i}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} \left(1 + \int_0^T \left(1 - \frac{s}{T}\right) d\Phi_s^i\right) \left(\sum_{1 \leq l \leq n} x_l \left(\Sigma_{j,l} - \frac{\Sigma_{i,j}}{\Sigma_{i,i}} \Sigma_{i,l}\right)\right) \\ + \frac{\Lambda_j}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} \left(1 + \int_0^T \left(1 - \frac{s}{T}\right) d\Phi_s^j\right) \left(\sum_{1 \leq l \leq n} x_l \left(\Sigma_{i,l} - \frac{\Sigma_{i,j}}{\Sigma_{j,j}} \Sigma_{j,l}\right)\right) + O(\|\Lambda\|^2) \quad (4.18)$$

where $\mathbb{E} \left(\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2} \right)$ is bounded when Λ goes to zero.

In the absence of other effects than exogenous economic moves in fundamentals, the realized correlation between assets i and j would be equal to the fundamental correlation between assets i and j : $\frac{\Sigma_{i,j}}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}}$. Feedback effects from the fund's rebalancing generate a realized correlation that is different from the fundamental correlation and is path-dependent and liquidity-dependent.

The magnitude of the fund/institutional investor's impact on asset returns is naturally measured by the quantities

$$\Lambda_i = \frac{\phi_0^i}{D_i}$$

which measure the size of the fund/institutional investor's positions in each asset, as a fraction of asset market depth. The larger the fund's positions, the greater the impact on realized correlation between assets. In Obizhaeva (2011), Obizhaeva shows empirically that D_i is proportional to $\frac{ADV_i}{\sqrt{\Sigma_{i,i}}}$, where ADV_i is the average daily volume for asset i .

This implies that the position of the fund in asset i can be measured by $\frac{\phi_0^i \sqrt{\Sigma_{i,i}}}{ADV_i}$.

Furthermore, at each date t , the fund buys a quantity $d\phi_t^i = \phi_0^i d\Phi_t^i$ of asset i in order to maintain a fixed proportion invested in asset i . Equation 4.18 shows that the larger the rebalancing of the fund's positions (more precisely, the larger the quantity $\int_0^T \left(1 - \frac{s}{T}\right) d\Phi_s^i$), the greater the impact of the fund on realized correlation between asset returns. *Feedback effects are exacerbated by the size of institutional investors and the volumes of rebalancing generated by their strategies.*

We define the average pairwise realized correlation between 0 and T , $\overline{R}_{[0,T]}$ by

$$\overline{R}_{[0,T]} = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} R_{[0,T]}^{i,j}$$

which can be decomposed, using (4.18), as the sum of the fundamental average pairwise correlation and excess pairwise realized correlation, which is also liquidity-dependent and path-dependent and exacerbated by the fund's positions and rebalancing volumes.

4.3.3 Case of zero fundamental correlations

In this section, we examine the case when the fund invests in assets with zero fundamental correlations: the fundamental covariance matrix Σ is diagonal and we write $\Sigma_{i,i} = \sigma_i^2$.

Using Corollary 4.3.2, we find that the realized variance of asset i returns is given by

$$C_{[0,T]}^{i,i} = \sigma_i^2 \left(1 - 2(1 - x_i) \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^i \right) \right) + O(\|\Lambda\|^2) < \sigma_i^2$$

Ito's formula gives: $1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^i = \frac{1}{T} \int_0^T \Phi_s^i ds > 0$, which implies that *feedback effects decrease the realized variance of asset returns*, as suggested by Figure 4.4. This is consistent with the fact that the fund buys (resp. sells) assets which decreased (resp. increased) the most, limiting the amplitude of asset movements.

Using (4.18), we can compute the realized correlation between assets i and j returns:

$$R_{[0,T]}^{i,j} = \Lambda_i \frac{\sigma_j x_j}{\sigma_i} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^i \right) + \Lambda_j \frac{\sigma_i x_i}{\sigma_j} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^j \right) + O(\|\Lambda\|^2) > 0 \quad (4.19)$$

Feedback effects from the fund's rebalancing generates *positive realized correlation* between assets with zero fundamental correlations. This stems from the fact that the systematic strategy used by the fund creates a similar pattern of behavior for all assets and shows that the presence of feedback effects reduces the benefits of diversification as they generate positive realized correlation between assets with zero fundamental correlations. This analytical result confirms quantitatively the numerical results of Figure 4.3.

(4.19) allows us to derive a lower bound for the realized correlation between assets i and j :

Corollary 4.3.3 *The large fund's rebalancing strategy generates a lower bound for realized correlation between asset returns, which depends on the size and rebalancing volumes of the institutional investor:*

$$R_{[0,T]}^{i,j} \geq 2\sqrt{x_i \Lambda_i x_j \Lambda_j} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^i \right)^{\frac{1}{2}} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\Phi_s^j \right)^{\frac{1}{2}} + O(\|\Lambda\|^2)$$

As the realized correlation matrix $R_{[0,T]}$ has strictly positive terms, the Perron Frobenius theorem states that it has an eigenvalue which is strictly higher than its other eigenvalues and which is associated to an eigenvector with strictly positive coordinates. Furthermore, this eigenvalue belongs to the interval:

$$\left[1 + (n-1)\overline{R_{[0,T]}} ; 1 + \max_i \sum_{j \neq i} R_{[0,T]}^{i,j} \right] \quad (4.20)$$

where $\overline{R_{[0,T]}}$ is the average pairwise realized correlation. This result is consistent with the observation of empirical realized correlation matrices, which have a largest

eigenvalue of the order of n , the number of assets in the basket. Our model naturally generates a highest eigenvalue of the order of n , as shown by (4.20), even starting with a fundamental correlation matrix equal to the identity matrix, which has all eigenvalues equal to 1. Furthermore, this result shows that this eigenvalue is associated to an eigenvector with strictly positive weights, as observed empirically (see Figure 4.2).

4.4 Eigenvalues and eigenvectors of the realized correlation matrix

In order to lead the principal component analysis of the realized correlation matrix, we derive analytical properties for the eigenvalues and eigenvectors of a proxy \tilde{R} for the realized correlation matrix $R_{[0,T]}$, which is equal to the first order expansion of the realized correlation matrix, i.e. the expression given in (4.18), up to the replacement of Φ by its fundamental value $\tilde{\Phi} = \Phi|_{\Lambda=0}$, whose explicit expression is given in (4.8).

We define the proxy for the realized correlation by:

$$\tilde{R} : \mathbb{R}^n \mapsto \mathcal{S}_n(\mathbb{R})$$

such that $[\tilde{R}(\Lambda)]_{i,i} = 1$ and, for $i \neq j$,

$$\begin{aligned} [\tilde{R}(\Lambda)]_{i,j} &= \frac{\Sigma_{i,j}}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} + \frac{\Lambda_i}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^i \right) \left(\sum_{1 \leq l \leq n} x_l \left(\Sigma_{j,l} - \frac{\Sigma_{i,j}}{\Sigma_{i,i}} \Sigma_{i,l} \right) \right) \\ &\quad + \frac{\Lambda_j}{\sqrt{\Sigma_{i,i}\Sigma_{j,j}}} \left(1 + \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^j \right) \left(\sum_{1 \leq l \leq n} x_l \left(\Sigma_{i,l} - \frac{\Sigma_{i,j}}{\Sigma_{j,j}} \Sigma_{j,l} \right) \right) \end{aligned}$$

For all $\Lambda \in \mathbb{R}^n$, $\tilde{R}(\Lambda)$ is a real-valued symmetric matrix and hence is diagonalisable in orthonormal basis: there exists n eigenvalues $v_1(\Lambda) \geq \dots \geq v_n(\Lambda)$ and n eigenvectors $\psi_1(\Lambda), \dots, \psi_n(\Lambda)$ such that

$$\tilde{R}(\Lambda)\psi_i(\Lambda) = v_i(\Lambda)\psi_i(\Lambda)$$

and

$$\psi_i(\Lambda) \cdot \psi_j(\Lambda) = \delta_{i,j}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

Notice that $\tilde{R}(\Lambda)$ is linear in Λ ; in particular, when Λ goes to zero, it converges almost surely to $\tilde{R}(0)$, which is the fundamental correlation matrix. Furthermore \tilde{R} is \mathcal{C}^∞ and

$$\nabla \tilde{R}(0) \cdot \Lambda = \tilde{R}(\Lambda) - \tilde{R}(0)$$

As the realized correlation matrix in the presence of feedback effects can be considered as a perturbation of the fundamental correlation matrix, one can expect that its eigenvalues and eigenvectors are also perturbations of the corresponding eigenvalues and eigenvectors of the fundamental correlation matrix. The following proposition quantifies this perturbation due to the impact of the large institutional investor.

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Proposition 4.4.1 *Under the assumption that $v_j(0)$, the j -th eigenvalue of the fundamental correlation matrix $\tilde{R}(0)$, is a simple eigenvalue, there exists an open set $U_j \subset \mathbb{R}^n$ containing 0 such that the restriction of v_j and ψ_j to U_j are C^∞ and for all $\Lambda \in U_j$, $v_j(\Lambda)$, the j -th eigenvalue of $\tilde{R}(\Lambda)$, is a simple eigenvalue and verifies*

$$v_j(\Lambda) = v_j(0) + \psi_j(0) \cdot \left(\tilde{R}(\Lambda) - \tilde{R}(0) \right) \psi_j(0) + o(\|\Lambda\|)$$

and is associated to the unit eigenvector $\psi_j(\Lambda)$, which is collinear to:

$$\psi_j(0) + \sum_{k \neq j} \frac{\psi_k(0) \cdot \left(\tilde{R}(\Lambda) - \tilde{R}(0) \right) \psi_j(0)}{v_j(0) - v_k(0)} \psi_k(0) + o(\|\Lambda\|)$$

where $\psi_k(0)$ is an eigenvector for $\tilde{R}(0)$, associated to the eigenvalue $v_k(0)$ and such that $\|\psi_k(0)\| = 1$ and $\frac{o(\|\Lambda\|)}{\|\Lambda\|}$ converges almost surely to zero when Λ goes to zero.

Proposition 4.4.1 gives a tractable formula for the eigenvalues and eigenvectors of the realized correlation matrix in the presence of feedback effects. We see that they become path-dependent and liquidity-dependent, as suggested by our numerical experiments. When market depths are infinite, those eigenvalues and eigenvectors are equal to the eigenvalues and eigenvectors of the fundamental correlation matrix. *In the presence of feedback effects from the fixed-mix strategy, the eigenvalues and eigenvectors of the realized correlation matrix depend on the sizes and allocations of institutional investors.*

Example and numerical tests In this paragraph, we illustrate how feedback effects impact the principal component properties of the realized correlation matrix of returns in a simple example where all parameters are homogenous: asset fundamental volatilities are equal to σ , fundamental correlation between any pair of assets is ρ , the large institutional investor's allocation in each asset i is $x_i = \frac{1}{n}$ and the size of its position in each asset as a fraction of market depth is $\Lambda_i = \bar{\Lambda}$.

Corollary 4.4.2 *Under the assumptions of our example, the largest eigenvalue of the realized correlation matrix of returns in the presence of feedback effects is equal to:*

$$v_1(\bar{\Lambda}) = 1 + (n-1)\rho + 2\bar{\Lambda} \left(1 - \frac{1}{n} \right) (1-\rho) \left(\rho + \frac{1-\rho}{n} \right) \sum_{1 \leq i \leq n} \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^i + o(\bar{\Lambda}) \quad (4.21)$$

and is associated to the first eigenvector, given by:

$$\psi_1(\bar{\Lambda}) = \frac{1}{\sqrt{n}} \left[1 + \bar{\Lambda} (1-\rho) \left(\rho + \frac{1-\rho}{n} \right) \left(n-1 + \frac{1}{n} \sum_{1 \leq i \leq n} \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^i \right) \right] \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$+ \frac{\bar{\Lambda}}{\sqrt{n}}(n-2)(1-\rho)\left(\rho + \frac{1-\rho}{n}\right) \begin{pmatrix} \int_0^T \left(1 - \frac{s}{T}\right) d\tilde{\Phi}_s^1 \\ \vdots \\ \int_0^T \left(1 - \frac{s}{T}\right) d\tilde{\Phi}_s^i \\ \vdots \\ \int_0^T \left(1 - \frac{s}{T}\right) d\tilde{\Phi}_s^n \end{pmatrix} + o(\bar{\Lambda}) \quad (4.22)$$

Proof In our example, the fundamental correlation matrix $\tilde{R}(0)$ is such that $[\tilde{R}(0)]_{i,i} = 1$ and $[\tilde{R}(0)]_{i,j} = \rho$ for $i \neq j$. As a consequence, it has a simple eigenvalue $v_1(0) = 1 + (n-1)\rho$, associated to the eigenvector $\psi_1(0) = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, and an eigenvalue of order $n-1$: $v_k(0) = 1 - \rho$ for $2 \leq k \leq n$, associated to eigenvectors $(\psi_k(0))_{2 \leq k \leq n}$, which form an orthonormal basis of the hyperplan of \mathbb{R}^n : $\mathcal{H} = \{z \in \mathbb{R}^n \text{ s.t. } \sum_{1 \leq i \leq n} z_i = 0\}$. As the largest eigenvalue of the fundamental correlation matrix, $v_1(0)$, is a simple eigenvalue, we can use Proposition 4.4.1 to compute the largest eigenvalue and the first eigenvector of the realized correlation matrix of returns in the presence of feedback effects.

Given the parameters of our example, we find that for $i \neq j$

$$[\tilde{R}(\Lambda) - \tilde{R}(0)]_{i,j} = \bar{\Lambda}(1-\rho)\left(\rho + \frac{1-\rho}{n}\right) \left(2 + \int_0^T \left(1 - \frac{s}{T}\right) d\tilde{\Phi}_s^i + \int_0^T \left(1 - \frac{s}{T}\right) d\tilde{\Phi}_s^j\right)$$

We hence find the expression for $v(\bar{\Lambda})$ given in Corollary 4.4.2 by direct computation of Proposition 4.4.1.

Proposition 4.4.1 states that the first eigenvalue of the realized correlation matrix of returns in the presence of feedback effects is given by:

$$\psi_1(\bar{\Lambda}) = \psi_1(0) + \frac{1}{n\rho} \sum_{k \geq 2} \left(\psi_k(0) \cdot \left(\tilde{R}(\Lambda) - \tilde{R}(0) \right) \psi_1(0) \right) \psi_k(0) + o(\bar{\Lambda})$$

as $v_1(0) - v_k(0) = n\rho$ for $k \geq 2$. Given the fact that $(\psi_k(0))_{2 \leq k \leq n}$ is an orthonormal basis of $\mathcal{H} = \{z \in \mathbb{R}^n \text{ s.t. } \sum_{1 \leq i \leq n} z_i = 0\}$, the term $\sum_{k \geq 2} \psi_k(0) \cdot \left(\tilde{R}(\Lambda) - \tilde{R}(0) \right) \psi_1(0) \psi_k(0)$ is the orthogonal projection of $\left(\tilde{R}(\Lambda) - \tilde{R}(0) \right) \psi_1(0)$ on \mathcal{H} . Given the equation defining \mathcal{H} , this orthogonal projection $p_{\mathcal{H}}$ on \mathcal{H} is expressed as

$$p_{\mathcal{H}}(z) = \begin{pmatrix} z_1 - \frac{1}{n} \sum_{1 \leq i \leq n} z_i \\ \vdots \\ z_n - \frac{1}{n} \sum_{1 \leq i \leq n} z_i \end{pmatrix}$$

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	Theoretical eigenvalue (4.21) vs Numerical eigenvalue	Fundamental eigenvalue vs Numerical eigenvalue
Average Error	10%	60%

Table 4.3: Average error for the largest eigenvector of the realized correlation matrix

$$\text{As } \left[\left(\tilde{R}(\Lambda) - \tilde{R}(0) \right) \psi_1(0) \right]_i = \frac{1}{\sqrt{n}} \bar{\Lambda} (1-\rho) \left(\rho + \frac{1-\rho}{n} \right) \sum_{j \neq i} \left(2 + \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^i + \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^j \right),$$

and given the expression for $p_{\mathcal{H}}$, we find the expression for $\psi_1(\bar{\Lambda})$ given in Corollary 4.4.2

Corollary 4.4.2 shows that the value of the largest eigenvalue of the realized correlation matrix increases in the presence of feedback effects, as suggested in the numerical results of Section 4.2.2. We simulated 10^4 price trajectories of our model and, for each trajectory, we calculated numerically the largest eigenvalue of the realized correlation matrix and computed the theoretical largest eigenvalue given by (4.21). Table 4.3 shows that the average error made by using the theoretical formula of (4.21) to estimate the largest eigenvalue of the realized correlation matrix is 10%, which is significantly lower than when using the fundamental largest eigenvalue (error of 60%, using fundamental largest eigenvalue, which is equal to $1 + (n - 1)\rho$).

Corollary 4.4.2 shows that the first eigenvector of the realized correlation matrix is path-dependent. On average, as we chose homogenous parameters, it is equal to

the fundamental eigenvector $\frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. However, *in scenarios where the rebalancing volume for one particular asset i is significantly larger than those for other assets, the weight of this asset i in the first eigenvector will be greater than for other assets,*

as the principal component of the realized correlation matrix is driven by the vector

$\begin{pmatrix} \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^1 \\ \vdots \\ \int_0^T \left(1 - \frac{s}{T} \right) d\tilde{\Phi}_s^n \end{pmatrix}$ which is equal to a weighted average of the rebalancing volume by the fund in each asset.

Corollary 4.4.2 shows that the weight in the first eigenvector of the realized correlation matrix of an asset which significantly overperforms or underperforms other assets should increase. Figure 4.7 shows that the weight of the energy sector increased from 2006 to 2007. This was associated to a significant overperformance of the energy sector (+44% in one year) over the other sectors, which generated large rebalancing volumes by large institutional investors and increased its weight in the first eigenvector, as predicted by Corollary 4.4.2.

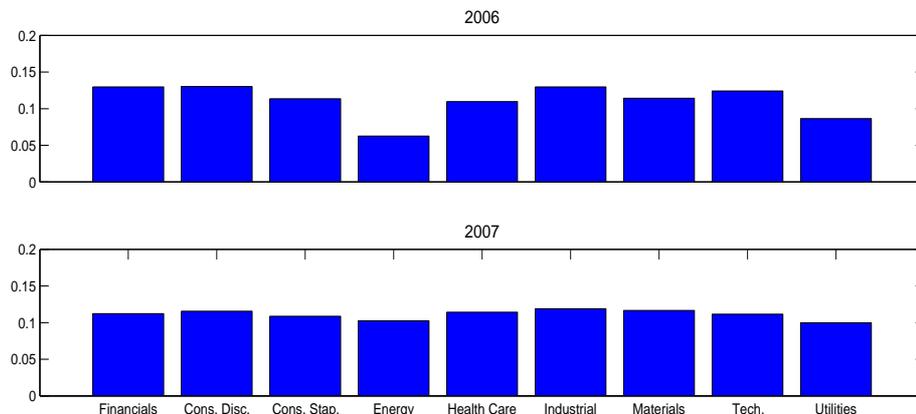


Figure 4.7: Weight of each sector in the first principal component of the empirical correlation matrix of US equity returns in 2006 and 2007.

4.5 Asset returns and fund performance in the presence of feedback effects

4.5.1 Impact on expected returns

Theorem (4.2.3) allows us to compute the expected return of each asset i in the presence of feedback effects. As P^i verifies the stochastic differential equation given in Theorem (4.2.3), the expected return of asset i at date t is $\frac{b_i(P_t, V_t)}{P_t^i}$ where b is defined in (4.7) and given explicitly by Lemmas 4.6.3 and 4.6.4. As we did for covariances and correlations between asset returns, we can study the expansion of the instantaneous expected return at order one in Λ , which gives an intuitive insight on the impact of feedback effects on the assets' expected return. Following the same method as in Corollary 4.3.2, we can prove the following Proposition.

Proposition 4.5.1 *The expected return of asset i at date t in the presence of feedback effects from the fixed-mix investor is:*

$$\frac{b_i(P_t, V_t)}{P_t^i} = m_i + \Phi_t^i \Lambda_i \left(\sum_{1 \leq l \leq n} x_l m_l - m_i \right) + O(\|\Lambda\|^2) \quad (4.23)$$

where Λ and Φ_t^i are defined respectively in (4.14) and (4.15) and $\mathbb{E} \left(\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2} \right)$ is bounded when Λ goes to zero.

When the positions of the fund are negligible compared to market depths ($\Lambda = 0$), the expected return of asset i at date t is equal to the 'fundamental' expected return m_i . In the presence of feedback effects, $\frac{b_i(P_t, V_t)}{P_t^i}$ is the sum of m_i and an additional expected

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return which is state-dependent and liquidity-dependent and is exacerbated by the size of the fund's positions in asset i as a fraction of the asset market depth.

The nature of the impact of feedback effects on the expected return of asset i depends on the difference between the fundamental expected return of asset i , m_i , and the fundamental expected return of the fund, $\sum_{1 \leq l \leq n} x_l m_l$. When $m_i > \sum_{1 \leq l \leq n} x_l m_l$, (4.23) shows

that $\frac{b_i(P_t, V_t)}{P_t^i} < m_i$ and *the impact of the large institutional investor decrease (resp. increase) the expected return of assets whose fundamental expected returns are larger (resp. lower) than the benchmark fund.* The action of the institutional investor endogenously dampens the difference between expected returns of assets with large fundamental expected returns (compared to the benchmark) and those with low fundamental expected returns.

Proposition 4.5.1 directly gives us the expected return for the large institutional investor and for other (small) investors in the presence of feedback effects.

Corollary 4.5.2 *At date t , the expected returns for the large institutional investor and for a (small) investor holding a portion y_t^i of each asset i are given respectively by*

$$\begin{aligned} & \sum_{1 \leq i \leq n} x_i m_i + \sum_{1 \leq i \leq n} x_i \Lambda_i \Phi_t^i \left(\sum_{1 \leq l \leq n} x_l m_l - m_i \right) + O(\|\Lambda\|^2) \\ & \sum_{1 \leq i \leq n} y_t^i m_i + \sum_{1 \leq i \leq n} y_t^i \Lambda_i \Phi_t^i \left(\sum_{1 \leq l \leq n} x_l m_l - m_i \right) + O(\|\Lambda\|^2) \end{aligned}$$

The expected return for the large institutional investor is lower (resp. higher) than its benchmark return $\sum_{1 \leq i \leq n} x_i m_i$ if it overweighs (resp. underweighs) assets with large fundamental expected returns. *When the large institutional investor allocates more to assets with low fundamental expected returns, he benefits from his own feedback effects and experience higher-than-expected returns.*

The expected return for a small fund with positions y_t is lower than the small fund's fundamental return $\sum_{1 \leq i \leq n} y_t^i m_i$ when it overweighs assets with large fundamental expected returns (ie when y_t^i is large for assets i which verify $m_i > \sum_{1 \leq l \leq n} x_l m_l$). *Whereas large institutional investors are considered as benchmarks by other investors, we show that, due to feedback effects, investors who try to beat the large fund by overweighting (resp. underweighting) assets with fundamental expected returns larger (resp. lower) than the large fund's benchmark will experience lower-than-expected returns.*

4.5.2 Optimal strategy and efficient frontier in a simple example

The following example illustrates how feedback effects from a large investor affects other funds. We consider the case of a market comprising $n = 2$ assets with zero fundamental

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correlation and identical fundamental volatility. We write:

$$\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

and we assume, for example, that $m_1 > m_2$. We consider that a large institutional investor starts investing in this market and keeps a constant proportion of each asset in its portfolio, equal to 50% for each asset:

$$X = \begin{pmatrix} 50\% \\ 50\% \end{pmatrix}$$

Consider now a small fund investing in those two assets and holding, at date t , a proportion y_t^1 of asset 1 and $y_t^2 = 1 - y_t^1$ of asset 2. We assume that this fund chooses its allocations by maximizing a mean-variance criteria. The small fund needs to estimate expected returns, variances and covariances and then calculates its allocation in each asset by solving:

$$y_t^1 = \arg \max\{U_t(y); y \in \mathbb{R}\} \quad (4.24)$$

where

$$U_t(y) = y\mathbb{E}(\text{return}_{1,t}) + (1 - y)\mathbb{E}(\text{return}_{2,t}) - \gamma \left(y^2\mathbb{E}(\text{variance}_{1,t}) + (1 - y)^2\mathbb{E}(\text{variance}_{2,t}) + 2y(1 - y)\mathbb{E}(\text{covariance}_{1,2,t}) \right)$$

where γ is a parameter of risk aversion for the small fund.

As long as the large institutional investor does not trade in the market:

$$\tilde{U}_t(y) = ym_1 + (1 - y)m_2 - \gamma\sigma^2 \left(y^2 + (1 - y)^2 \right) \quad (4.25)$$

However, in the presence of feedback effects from the rebalancing by the large institutional investor:

$$U_t(y) = \tilde{U}_t(y) + \left(\frac{y\Lambda_1 - (1 - y)\Lambda_2}{2}(m_2 - m_1) \right) \exp(t(m_1 - m_2)) \quad (4.26)$$

$$+ \gamma\sigma^2 \left(y^2\Lambda_1 + (1 - y)^2\Lambda_2 - y(1 - y)(\Lambda_1 + \Lambda_2) \right) \exp(t(m_1 - m_2)) + O(\|\Lambda\|^2)$$

Proof In our example $x_1 = x_2 = 50\%$, $\Sigma_{1,1} = \Sigma_{2,2} = \sigma^2$ and $\Sigma_{1,2} = \Sigma_{2,1} = 0$. Using 4.9, we find that $\mathbb{E}(\tilde{\Phi}_t^1) = \mathbb{E}(\tilde{\Phi}_t^2) = \exp(t(m_1 - m_2))$. Using (4.23) and (4.17), we find that

$$\mathbb{E}(\text{return}_{1,t}) = m_1 + \frac{\Lambda_1}{2}(m_2 - m_1) \exp(t(m_1 - m_2)) + O(\|\Lambda\|^2)$$

$$\mathbb{E}(\text{return}_{2,t}) = m_2 + \frac{\Lambda_2}{2}(m_1 - m_2) \exp(t(m_1 - m_2)) + O(\|\Lambda\|^2)$$

$$\mathbb{E}(\text{variance}_{1,t}) = \sigma^2 (1 - \Lambda_1 \exp(t(m_1 - m_2))) + O(\|\Lambda\|^2)$$

$$\mathbb{E}(\text{variance}_{2,t}) = \sigma^2 (1 - \Lambda_2 \exp(t(m_1 - m_2))) + O(\|\Lambda\|^2)$$

$$\mathbb{E}(\text{covariance}_{1,2,t}) = \frac{\sigma^2}{2} \left(\frac{\Lambda_1 + \Lambda_2}{2} \right) \exp(t(m_1 - m_2)) + O(\|\Lambda\|^2)$$

which gives (4.26).

The following Proposition follows immediately.

Proposition 4.5.3 *In the absence of large institutional investors, the optimal strategy is associated to (4.24) and (4.25) and is given by:*

$$\tilde{y}_1 = \frac{1}{2} + \frac{m_1 - m_2}{4\gamma\sigma^2}$$

When the large institutional investor trades in this market, the optimal strategy is associated to (4.24) and (4.26) and is given by:

$$y_t^1 = \tilde{y}_1 + (\Lambda_1 + \Lambda_2) \frac{m_1 - m_2}{8\gamma\sigma^2} + \frac{\Lambda_1 - \Lambda_2}{4} + O(\|\Lambda\|^2)$$

When there are no feedback effects, the mean variance criteria gives an optimal proportion in each asset which is constant and hence implies that the small fund's strategy will be a fixed-mix strategy. We see that the larger the difference between the fundamental expected return of asset 1 and 2, the greater the allocation in asset 1 (recall that $m_1 > m_2$). When risk aversion goes to infinity ($\gamma \rightarrow \infty$), the optimal allocations do not depend on the assets' fundamental expected returns.

We see that in the presence of feedback effects, this strategy (\tilde{y}_1) is no longer optimal, as $y_t^1 \neq \tilde{y}_1$. *The presence of large institutional investors generates a non-optimality for strategies which are optimal in the absence of feedback effects.* If the small fund does not take feedback effects into account, it will choose strategy \tilde{y}_1 which is not optimal. On the contrary, if the small fund estimates the fundamentals of the market (from price series when the large investor was not trading in the market) and knows the strategy of the large institutional investor (which is realistic, for example, for large mutual funds whose strategies are often public), the small fund will be able to follow the strategy y_t^1 which is optimal for the mean-variance criteria (4.24). Figure 4.8 shows that the efficient frontier is modified in the presence of feedback effects. We see that taking feedback effects into account allows the small investor to diminish the volatility of its portfolio for a given return. This stems from the fact that fixed-mix rebalancing induces a decrease of realized asset volatilities, as shown in Section 4.3.3, which can be used by an investor following the optimal strategy y_t^1 to build a less volatile portfolio. *Trading and risk-management decisions in financial markets need to take into account the allocations and sizes of large benchmark portfolios.*

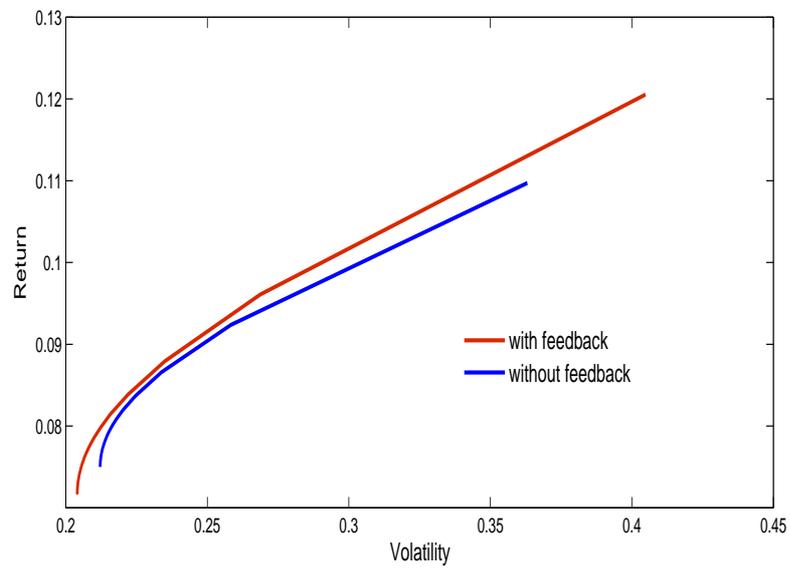


Figure 4.8: Efficient frontier with and without feedback effects

4.6 Appendices

We denote

$$M_k = \begin{pmatrix} S_k \\ W_k \end{pmatrix} \in \mathbb{R}^{n+1} \quad \text{and} \quad Z_{k+1} = \tau \bar{m} + \sqrt{\tau} \xi_{k+1} \in \mathbb{R}^n \quad (4.27)$$

where $\bar{m}_i = m_i - \frac{\Sigma_{i,i}}{2}$.

We obviously have $\sum_{i=1}^n x_i = 1$. In addition, as the large fund has long positions: $x_i \geq 0$ for all $1 \leq i \leq n$.

4.6.1 Proof of Proposition 4.2.1

Let $M_k \in (\mathbb{R}_+^*)^{n+1}$ and ξ_{k+1} be given, thus fixing the value of $\phi_k^i = \frac{x_i M_k^{n+1}}{M_k^i}$. We can write (4.4) as $M_{k+1}^i = A_i(M_k, Z_{k+1}) + \frac{1}{M_{k+1}^i} B_i(M_k, Z_{k+1}) M_{k+1}^{n+1}$ where M_k and Z_{k+1} are given in (4.27) and A_i and B_i are defined on $(\mathbb{R}_+^*)^{n+1} \times \mathbb{R}^n$ by:

$$A_i(M, Z) = M^i \exp(Z^i) \left(1 - \frac{x_i M^{n+1}}{M^i D_i} \right) \quad (4.28)$$

$$B_i(M, Z) = M^i \exp(Z^i) \frac{x_i}{D_i} \quad (4.29)$$

which implies that $M_{k+1}^i = \frac{1}{2} \left(A_i(M_k, Z_{k+1}) + \sqrt{A_i^2(M_k, Z_{k+1}) + 4B_i(M_k, Z_{k+1})M_{k+1}^{n+1}} \right)$. Reinjecting in Equation 4.3, we find that M_{k+1}^{n+1} is a fixed point of the function

$$v(x) = \frac{1}{2} \sum_{i=1}^n \frac{x_i M_k^{n+1}}{M_k^i} \left(A_i(M_k, Z_{k+1}) + \sqrt{A_i^2(M_k, Z_{k+1}) + 4B_i(M_k, Z_{k+1})x} \right)$$

As a consequence, proving Proposition 4.2.1 is equivalent to proving that v has a unique fixed point on \mathbb{R}_+^* .

Existence Given the expression of v , it is clear that for x large enough, $v(x) < x$. We then examine the three following possibilities:

- there exists i_0 such that $A_{i_0}(M_k, Z_{k+1}) > 0$, which implies that $v(0) > 0$ and, as v is a continuous function of x , that $v(x) > x$ for x small enough;
- there exists i_0 such that $A_{i_0}(M_k, Z_{k+1}) = 0$, which implies that

$$v(x) \geq \frac{1}{2} \frac{x_i M_k^{n+1}}{M_k^{i_0}} \sqrt{4B_{i_0}(M_k, Z_{k+1})x}$$

which is strictly larger than x for x small enough;

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- $\forall i A_i(M_k, Z_{k+1}) < 0$ which implies that $v(0) = 0$. Let us then calculate:

$$v'(0) = \sum_{i=1}^n \frac{x_i M_k^{n+1}}{M_k^i} \frac{B_i(M_k, Z_{k+1})}{|A_i(M_k, Z_{k+1})|} = \sum_{i=1}^n \frac{x_i M_k^{n+1}}{M_k^i} \frac{\frac{x_i}{D_i}}{\frac{x_i M_k^{n+1}}{M_k^i D_i} - 1}$$

This implies that:

$$v'(0) > \sum_{i=1}^n \frac{x_i M_k^{n+1}}{M_k^i} \frac{\frac{x_i}{D_i}}{\frac{x_i M_k^{n+1}}{M_k^i D_i}} = \sum_{i=1}^n x_i = 1$$

assuring that $v(x) > x$ for x small enough.

As v is a continuous function of x , that $v(x) > x$ for large x and $v(x) < x$ for small x , there exists at least one fixed point $x_0 > 0$ such that $v(x_0) = x_0$.

Unicity Suppose that there exist two fixed points of function v , denoted a and b with $0 < a < b$. As v is concave, for $0 < x < a$ we have $\frac{v(a)-v(x)}{a-x} \geq \frac{v(b)-v(a)}{b-a} = 1$, meaning that $x \geq v(x)$ which is in contradiction with the fact that $x < v(x)$ for x sufficiently small. As a consequence, v cannot have more than one fixed point. The unique fixed point of v is $M_{k+1}^{n+1} > 0$ and we can deduce, for $1 \leq i \leq n$:

$$M_{k+1}^i = \frac{1}{2} \left(A_i(M_k, Z_{k+1}) + \sqrt{A_i^2(M_k, Z_{k+1}) + 4B_i(M_k, Z_{k+1})M_{k+1}^{n+1}} \right) > 0$$

This proves the first part of Proposition 4.2.1.

We denote

$$\begin{aligned} \psi &: (\mathbb{R}_+^*)^{n+1} \times \mathbb{R}^n \times \mathbb{R}_+^* \rightarrow \mathbb{R} \\ \psi &: (M, Z, x) \rightarrow x - \frac{1}{2} \sum_{i=1}^n \frac{x_i M^{n+1}}{M^i} \left(A_i(M, Z) + \sqrt{A_i^2(M, Z) + 4B_i(M, Z)x} \right) \end{aligned} \quad (4.30)$$

where A_i and B_i are defined respectively in (4.28) and (4.29). ψ is C^∞ . Furthermore, we have $\psi(M, 0, M^{n+1}) = 0$ and $A_i(M, 0) = M^i \left(1 - \frac{x_i M^{n+1}}{M^i D_i} \right)$ and $B_i(M, 0) = M^i \frac{x_i}{D_i}$ according to (4.28) and (4.29), which implies that

$$A_i^2(M, 0) + 4B_i(M, 0)M^{n+1} = \left(M^i \left(1 + \frac{x_i M^{n+1}}{M^i D_i} \right) \right)^2 \quad (4.31)$$

Hence we find that:

$$\begin{aligned} \frac{\partial \psi}{\partial x}(M, 0, M^{n+1}) &= 1 - \sum_{i=1}^n \frac{x_i M^{n+1}}{M^i} \frac{B_i(M, 0)}{\sqrt{A_i^2(M, 0) + 4B_i(M, 0)M^{n+1}}} \\ &= 1 - \sum_{i=1}^n x_i \times \frac{\frac{x_i M^{n+1}}{M^i}}{D_i + \frac{x_i M^{n+1}}{M^i}} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^n x_i - \sum_{i=1}^n x_i \times \frac{\frac{x_i M^{n+1}}{M^i}}{D_i + \frac{x_i M^{n+1}}{M^i}} \\
 \frac{\partial \psi}{\partial x}(M, 0, M^{n+1}) &= \sum_{i=1}^n \frac{x_i}{1 + \frac{x_i M^{n+1}}{D_i M^i}} > 0
 \end{aligned} \tag{4.32}$$

As a consequence, if K is a compact set of $(\mathbb{R}_+^*)^{n+1}$, the implicit function theorem states that there exists $\epsilon_K > 0$ and θ_{n+1} which is \mathcal{C}^∞ on $K \times \mathcal{B}(0, \epsilon_K)$ such that:

$$\psi(M, Z, \theta_{n+1}(M, Z)) = 0 \tag{4.33}$$

and then, we deduce, for $1 \leq i \leq n$,

$$\theta_i(M, Z) = \frac{1}{2} \left(A_i(M, Z) + \sqrt{A_i^2(M, Z) + 4B_i(M, Z)\theta_{n+1}(M, Z)} \right)$$

and θ_i is \mathcal{C}^∞ on $K \times \mathcal{B}(0, \epsilon_K)$. This concludes the proof for the existence and smoothness of θ .

4.6.2 Proof of Theorem 4.2.3

Lemma 4.6.1 *Under the assumption that there exists $\eta > 0$ such that $\mathbb{E}(\|\xi\|^{\eta+2}) < \infty$, for all $\epsilon > 0$ and $r > 0$:*

$$\lim_{\tau \rightarrow 0} \sup_{\|M\| \leq r} \frac{1}{\tau} \mathbb{P}(\|M_{k+1} - M_k\| \geq \epsilon | M_k = M) = 0 \tag{4.34}$$

Proof Fix $\epsilon > 0$ and $r > 0$. Proposition 4.2.1 shows that there exists $\epsilon(r) > 0$ and $C(r) > 0$ such that, if $\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)$ and $\|M_k\| \leq r$, then

$$\|M_{k+1} - M_k\| = \|\theta(M_k, \bar{m}\tau + \sqrt{\tau}\xi_{k+1}) - \theta(M_k, 0)\| \leq C(r)\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\|$$

We then have:

$$\begin{aligned}
 &\mathbb{P}(\|M_{k+1} - M_k\| \geq \epsilon | M_k = M, \|M\| \leq r, \|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)) \\
 &\leq \mathbb{P}(C(r)\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon) \\
 &\leq \mathbb{P}\left(\|\xi_{k+1}\| \geq \frac{\epsilon - \|\bar{m}\|C(r)\tau}{C(r)\sqrt{\tau}}\right) \\
 &\leq \mathbb{E}\left[\left(\|\xi_{k+1}\| \frac{C(r)\sqrt{\tau}}{\epsilon - \|\bar{m}\|C(r)\tau}\right)^{2+\eta}\right] \\
 &\leq \tau^{1+\frac{\eta}{2}} \mathbb{E}\left[\left(\|\xi_{k+1}\|\right)^{2+\eta}\right] \times \left(\frac{C(r)}{\epsilon - \|\bar{m}\|C(r)\tau}\right)^{2+\eta}
 \end{aligned}$$

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which implies that

$$\begin{aligned} & \mathbb{P}(\|M_{k+1} - M_k\| \geq \epsilon, \|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r) | M_k = M, \|M\| \leq r) \\ & \leq \tau^{1+\frac{\eta}{2}} \mathbb{E} \left[(\|\xi_{k+1}\|)^{2+\eta} \right] \times \left(\frac{C(r)}{\epsilon - \|\bar{m}\|C(r)\tau} \right)^{2+\eta} \end{aligned}$$

Moreover, we have

$$\mathbb{P}(\|M_{k+1} - M_k\| \geq \epsilon, \|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r) | M_k = M) \leq \mathbb{P}(\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r))$$

and

$$\mathbb{P}(\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r)) \leq \tau^{1+\frac{\eta}{2}} \mathbb{E} \left[(\|\xi_{k+1}\|)^{2+\eta} \right] \times \left(\frac{1}{\epsilon(r) - \|\bar{m}\|\tau} \right)^{2+\eta} \quad (4.35)$$

Given that $\mathbb{P}(\|M_{k+1} - M_k\| \geq \epsilon | M_k = M)$ is equal to:

$$\begin{aligned} & \mathbb{P}(\|M_{k+1} - M_k\| \geq \epsilon, \|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r) | M_k = M) \\ & + \mathbb{P}(\|M_{k+1} - M_k\| \geq \epsilon, \|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r) | M_k = M) \end{aligned}$$

we find (4.34).

Lemma 4.6.2 *Under the assumption that there exists $\eta > 0$ such that $\mathbb{E}(\|\exp(\eta\xi)\|) < \infty$ and $\mathbb{E}(\|\xi\|^{\eta+4}) < \infty$, for all $\epsilon > 0$ and $r > 0$:*

$$\lim_{\tau \rightarrow 0} \sup_{\|M\| \leq r} \left\| \frac{1}{\tau} \mathbb{E}(M_{k+1} - M_k | M_k = M) - b(M) \right\| = 0 \quad (4.36)$$

$$\lim_{\tau \rightarrow 0} \sup_{\|S\| \leq r} \left\| \frac{1}{\tau} \mathbb{E}[(M_{k+1} - M_k)(M_{k+1} - M_k)^t | M_k = M] - aa^t(M) \right\| = 0 \quad (4.37)$$

where a and b are defined respectively in (4.6) and (4.7).

Proof Fix $r > 0$. We know by Proposition 4.2.1 that there exists $\epsilon(r) > 0$ such that θ is \mathcal{C}^∞ on $\mathcal{B}(0, r) \times \mathcal{B}(0, \epsilon(r))$. As a consequence, for $\|M_k\| \leq r$ and $\|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)$ we can right the Taylor expansion of θ_i ($1 \leq i \leq n+1$) in 0:

$$\begin{aligned} & M_{k+1}^i - M_k^i = \theta_i(M_k, m\tau + \sqrt{\tau}\xi_{k+1}) - \theta(M_k, 0) \\ & = \frac{\partial \theta_i}{\partial z}(M_k, 0)(\bar{m}\tau + \sqrt{\tau}\xi_{k+1}) + \frac{1}{2} (\bar{m}\tau + \sqrt{\tau}\xi_{k+1}) \cdot \frac{\partial^2 \theta_i}{\partial z \partial z'}(M_k, 0) (\bar{m}\tau + \sqrt{\tau}\xi_{k+1}) \\ & \quad + (\bar{m}\tau + \sqrt{\tau}\xi_{k+1}) \cdot R_i(M_k, \tau, \xi_{k+1}) (\bar{m}\tau + \sqrt{\tau}\xi_{k+1}) \end{aligned}$$

As θ_i is \mathcal{C}^∞ on $\mathcal{B}(0, r) \times \mathcal{B}(0, \epsilon(r))$, the rest $R_i(M_k, \tau, \xi_{k+1})$ converges uniformly to 0 when τ goes to 0. As a consequence, we obtain that:

$$\lim_{\tau \rightarrow 0} \sup_{\|M\| \leq r} \left\| \frac{1}{\tau} \mathbb{E}((M_{k+1} - M_k) | M_k = M, \|\bar{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)) - b(M) \right\| = 0 \quad (4.38)$$

Remarking that

$$\begin{aligned} & \left\| \frac{1}{\tau} \mathbb{E} \left((M_{k+1} - M_k) \mathbf{1}_{\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)} | M_k = M \right) - b(M) \right\| \\ & \leq \left\| \left(\frac{1}{\tau} \mathbb{E} \left((M_{k+1} - M_k) | M_k = M, \|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r) \right) - b(M) \right) \mathbb{P}(\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)) \right\| \\ & \quad + \|b(M)\mathbb{P}(\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r))\| \end{aligned}$$

and given (4.35) and (4.38) and the continuity of b (which implies that $b(M)$ is bounded for $\|M\| \leq r$), we find

$$\lim_{\tau \rightarrow 0} \sup_{\|M\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} \left((M_{k+1} - M_k) \mathbf{1}_{\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)} | M_k = M \right) - b(M) \right\| = 0 \quad (4.39)$$

Similarly, we show that:

$$\lim_{\tau \rightarrow 0} \sup_{\|M\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} \left((M_{k+1} - M_k)(M_{k+1} - M_k)^t \mathbf{1}_{\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \leq \epsilon(r)} | M_k = M \right) - aa^t(M) \right\| = 0 \quad (4.40)$$

By construction of M_{k+1} , we see that $M_{k+1} \leq g(M_k) \|\exp(\overline{m}\tau + \sqrt{\tau}\xi_{k+1})\|$. As a consequence, for $p > 0$, as soon as $p\sqrt{\tau} < \eta$ and given $M_k = M$, $M_{k+1} \in L^p$. Using this result for $p = 2$, we find that for $\sqrt{\tau} < \frac{\eta}{2}$, $M_{k+1} \in L^2$ and we can use Cauchy Schwarz inequality:

$$\begin{aligned} & \left| \mathbb{E} \left((M_{k+1}^i - M_k^i) \mathbf{1}_{\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r)} | M_k = M \right) \right| \\ & \leq \sqrt{\mathbb{E} \left((M_{k+1}^i - M_k^i)^2 | M_k = M \right) \mathbb{P} \left(\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r) \right)} \\ & \leq \sqrt{\mathbb{E} \left((M_{k+1}^i - M_k^i)^2 | M_k = M \right) \sqrt{\mathbb{E} \left(\frac{\|\xi_{k+1}\| \sqrt{\tau}}{\epsilon(r) - \|\mu\| \tau} \right)^{4+\eta}} \\ & \leq \tau^{1+\frac{\eta}{4}} \sqrt{\mathbb{E} \left((M_{k+1}^i - M_k^i)^2 | M_k = M \right) \sqrt{\mathbb{E} \left(\frac{\|\xi_{k+1}\|}{\epsilon(r) - \|\mu\| \tau} \right)^{4+\eta}} \end{aligned}$$

As $\mathbb{E}(\|\xi_{k+1}\|^{4+\eta}) < \infty$, $M_{k+1} \in L^2$ and M_{k+1} stays L^2 bounded as τ goes to 0 and we obtain:

$$\lim_{\tau \rightarrow 0} \sup_{\|M\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} \left((M_{k+1} - M_k) \mathbf{1}_{\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r)} | M_k = M \right) \right\| = 0 \quad (4.41)$$

Using the same property with $p=4$, we show that

$$\lim_{\tau \rightarrow 0} \sup_{\|M\| \leq r} \left\| \frac{1}{\tau} \mathbb{E} \left((M_{k+1} - M_k)(M_{k+1} - M_k)^t \mathbf{1}_{\|\overline{m}\tau + \sqrt{\tau}\xi_{k+1}\| \geq \epsilon(r)} | M_k = M \right) \right\| = 0 \quad (4.42)$$

(4.39) and (4.41) (resp. (4.40) and (4.42)) give (4.36) (resp. (4.37)).

The following lemma is a direct consequence of the implicit function theorem

Lemma 4.6.3 For $1 \leq l, i \leq n$:

$$\frac{\partial \theta_i}{\partial z_l}(M, 0) = \frac{M^i}{1 + \frac{x_i M^{n+1}}{M^i D_i}} \left(\delta_{i,l} + \frac{\frac{x_i M^{n+1}}{M^i D_i}}{\sum_{j=1}^n \frac{x_j}{1 + \frac{x_j M^{n+1}}{M^j D_j}}} \times \frac{x_l}{1 + \frac{x_l M^{n+1}}{M^l D_l}} \right) \quad (4.43)$$

$$\frac{\partial \theta_{n+1}}{\partial z_l}(M, 0) = \frac{M^{n+1}}{\sum_{j=1}^n \frac{x_j}{1 + \frac{x_j M^{n+1}}{M^j D_j}}} \frac{x_l}{1 + \frac{x_l M^{n+1}}{M^l D_l}} \quad (4.44)$$

Proof θ_{n+1} is defined implicitly by $\psi(M, Z, \theta_{n+1}(M, Z)) = 0$, where ψ is given in (4.30). The implicit function theorem gives, for $1 \leq l \leq n$,

$$\frac{\partial \theta_{n+1}}{\partial z_l}(M, Z) = \frac{-\frac{\partial \psi}{\partial z_l}(M, Z, \theta_{n+1}(M, Z))}{\frac{\partial \psi}{\partial x}(M, Z, \theta_{n+1}(M, Z))} \quad (4.45)$$

Given the expression for ψ given in (4.30), we find that

$$\frac{\partial \psi}{\partial z_l}(M, Z, \theta_{n+1}(M, Z)) = \frac{-1}{2} \frac{x_l M^{n+1}}{M^l} \left(A_l(M, Z) + \frac{A_l^2(M, Z) + 2B_l(M, Z)x}{\sqrt{A_l^2(M, Z) + 4B_l(M, Z)x}} \right) \quad (4.46)$$

and

$$\frac{\partial \psi}{\partial x}(M, Z, \theta_{n+1}(M, Z)) = 1 - \sum_{1 \leq i \leq n} \frac{x_i M^{n+1}}{M^i} \frac{B_i(M, Z)}{\sqrt{A_i^2(M, Z) + 4B_i(M, Z)x}} \quad (4.47)$$

Using (4.28) and (4.29), we find that

$$A_l^2(M, Z) + 2B_l(M, Z)M^{n+1} = (M^l)^2 \left(1 + \left(\frac{x_l M^{n+1}}{M^l D_l} \right)^2 \right) \quad (4.48)$$

Given (4.31) and the fact that $\theta_{n+1}(M, 0) = M^{n+1}$, we find that

$$\frac{\partial \psi}{\partial z_l}(M, 0, \theta_{n+1}(M, 0)) = \frac{-x_l M^{n+1}}{1 + \frac{x_l M^{n+1}}{M^l D_l}} \quad (4.49)$$

Using (4.45), (4.49) and (4.32), we find that

$$\frac{\partial \theta_{n+1}}{\partial z_l}(M, 0) = \frac{M^{n+1}}{\sum_{j=1}^n \frac{x_j}{1 + \frac{x_j M^{n+1}}{M^j D_j}}} \frac{x_l}{1 + \frac{x_l M^{n+1}}{M^l D_l}}$$

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Let us now calculate, for $1 \leq i \leq n$ and $1 \leq l \leq n$, $\frac{\partial \theta_i}{\partial z_l}(M, 0)$. Given the definition of θ_i , we remark that:

$$\frac{\partial \theta_i}{\partial z_l}(M, Z) = -\frac{\partial \psi}{\partial z_l}(M, Z, \theta_{n+1}(M, Z)) \frac{\delta_{i,l}}{\frac{x_i M^{n+1}}{M^i}} + \frac{B_i(M, Z) \frac{\partial \theta_{n+1}}{\partial z_l}(M, Z)}{\sqrt{A_l^2(M, Z) + 4B_l(M, Z)x}} \quad (4.50)$$

which, evaluated in 0 and given the expression for $\frac{\partial \theta_{n+1}}{\partial z_l}$ and (4.31) gives:

$$\frac{\partial \theta_i}{\partial z_l}(M, 0) = \frac{M^i}{1 + \frac{x_i M^{n+1}}{M^i D_i}} \left(\delta_{i,l} + \frac{\frac{x_i M^{n+1}}{M^i D_i}}{\sum_{j=1}^n \frac{x_j}{1 + \frac{x_j M^{n+1}}{M^j D_j}}} \times \frac{x_l}{1 + \frac{x_l M^{n+1}}{M^l D_l}} \right)$$

Lemma 4.6.4 For $1 \leq j, l \leq n$:

$$\frac{\partial^2 \theta_{n+1}}{\partial z_j \partial z_l}(M, 0) \frac{1}{M^{n+1}} \sum_{1 \leq p \leq n} \frac{x_p}{1 + \frac{x_p M^{n+1}}{D_p M^p}} = \frac{\delta_{j,l} x_l}{1 + \frac{x_l M^{n+1}}{D_l M^l}} \left(1 - \frac{2 \left(\frac{x_l M^{n+1}}{D_l M^l} \right)^2}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l} \right)^2} \right) \quad (4.51)$$

$$+ \frac{2x_j x_l}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l} \right) \left(1 + \frac{x_j M^{n+1}}{D_j M^j} \right)} \frac{1}{\sum_{1 \leq p \leq n} \frac{x_p}{1 + \frac{x_p M^{n+1}}{D_p M^p}}} \left(\frac{\left(\frac{x_l M^{n+1}}{D_l M^l} \right)^2}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l} \right)^2} + \frac{\left(\frac{x_j M^{n+1}}{D_j M^j} \right)^2}{\left(1 + \frac{x_j M^{n+1}}{D_j M^j} \right)^2} \right)$$

$$- \frac{2x_j x_l}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l} \right) \left(1 + \frac{x_j M^{n+1}}{D_j M^j} \right)} \frac{1}{\left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \frac{x_p M^{n+1}}{D_p M^p}} \right)^2} \sum_{1 \leq p \leq n} \frac{x_p \left(\frac{x_p M^{n+1}}{D_p M^p} \right)^2}{\left(1 + \frac{x_p M^{n+1}}{D_p M^p} \right)^3}$$

$$\frac{\partial^2 \theta_i}{\partial z_j \partial z_l}(M, 0, M^{n+1}) = \delta_{i,j} \delta_{j,l} \frac{M^i}{1 + \frac{x_i M^{n+1}}{D_i M^i}} \left(1 - \frac{2 \left(\frac{x_l M^{n+1}}{D_l M^l} \right)^2}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l} \right)^2} \right) \quad (4.52)$$

$$+ 2M^i \frac{\left(\frac{x_i M^{n+1}}{D_i M^i} \right)^2}{\left(1 + \frac{x_i M^{n+1}}{D_i M^i} \right)^3} \frac{1}{\sum_{1 \leq p \leq n} \frac{x_p}{1 + \frac{x_p M^{n+1}}{D_p M^p}}} \left(\delta_{i,l} \frac{x_j}{\left(1 + \frac{x_j M^{n+1}}{D_j M^j} \right)} + \delta_{i,j} \frac{x_l}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l} \right)} \right)$$

$$- 2M_i \frac{2 \left(\frac{x_l M^{n+1}}{D_l M^l} \right)^2}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l} \right)^3} \frac{\frac{x_j}{1 + \frac{x_j M^{n+1}}{D_j M^j}} \frac{x_l}{1 + \frac{x_l M^{n+1}}{D_l M^l}}}{\left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \frac{x_p M^{n+1}}{D_p M^p}} \right)^2} + \frac{\frac{x_i}{D_i}}{1 + \frac{x_i M^{n+1}}{D_i M^i}} \frac{\partial^2 \theta_{n+1}}{\partial z_j \partial z_l}(M, 0)$$

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Proof We first calculate for $1 \leq j, l \leq n$ $\frac{\partial^2 \theta_{n+1}}{\partial z_j \partial z_l}(M, 0)$. Deriving (4.46) with respect to z_l and z_j gives the following equation:

$$\frac{\partial^2 \psi}{\partial z_j \partial z_l}(M, Z, \theta_{n+1}(M, Z)) + \frac{\partial^2 \psi}{\partial x \partial z_l}(M, Z, \theta_{n+1}(M, Z)) \frac{\partial \theta_{n+1}}{\partial z_j}(M, Z) \quad (4.53)$$

$$\begin{aligned} &+ \frac{\partial^2 \psi}{\partial x \partial z_j}(M, Z, \theta_{n+1}(M, Z)) \frac{\partial \theta_{n+1}}{\partial z_l}(M, Z) + \frac{\partial^2 \psi}{\partial x^2}(M, Z, \theta_{n+1}(M, Z)) \frac{\theta_{n+1}}{\partial z_j}(M, Z) \frac{\theta_{n+1}}{\partial z_l}(M, Z) \\ &+ \frac{\partial \psi}{\partial x}(M, Z, \theta_{n+1}(M, Z)) \frac{\partial^2 \theta_{n+1}}{\partial z_j \partial z_l}(M, Z) = 0 \end{aligned}$$

Considering (4.46), we find that if $l \neq j$, then $\frac{\partial^2 \psi}{\partial z_j \partial z_l}(M, Z, \theta_{n+1}(M, Z)) = 0$. Deriving (4.46) with respect to z_l , we find that

$$\begin{aligned} \frac{\partial^2 \psi}{\partial z_l^2}(M, Z, x) &= -\frac{1}{2} \frac{x_l M^{n+1}}{M^l} \left(A_l(M, Z) + \frac{2A_l^2(M, Z) + 2B_l(M, Z)x}{\sqrt{A_l^2(M, Z) + 4B_l(M, Z)x}} - \frac{(A_l^2(M, Z) + 2B_l(M, Z)x)^2}{(A_l^2(M, Z) + 4B_l(M, Z)x)^{\frac{3}{2}}} \right) \\ &= \frac{\partial \psi}{\partial z_l}(M, Z, x) + \frac{2x_l M^{n+1}}{M^l} \frac{B_l^2(M, Z)x^2}{(A_l^2(M, Z) + 4B_l(M, Z)x)^{\frac{3}{2}}} \end{aligned}$$

Evaluating this equation in 0 and using (4.49) and (4.31), we find that

$$\frac{\partial^2 \psi}{\partial z_j \partial z_l}(M, 0, M^{n+1}) = \delta_{j,l} \frac{x_l M^{n+1}}{\left(1 + \frac{x_l M^{n+1}}{D_l M^l}\right)^3} \left(\left(1 - \frac{x_l M^{n+1}}{D_l M^l}\right)^2 - 1 \right) \quad (4.54)$$

Deriving (4.46) with respect to x , we find that

$$\frac{\partial^2 \psi}{\partial z_l \partial x}(M, Z, x) = -\frac{2x_l M^{n+1}(M, Z)}{M^l} \frac{B_l^2(M, Z)x}{(A_l^2(M, Z) + 4B_l(M, Z)x)^{\frac{3}{2}}}$$

which, evaluated in 0, gives

$$\frac{\partial^2 \psi}{\partial z_l \partial x}(M, 0, M^{n+1}) = -2x_l \left(\frac{M^{n+1}x_l}{M^l D_l} \right)^2 \frac{1}{\left(1 + \frac{M^{n+1}x_l}{M^l D_l}\right)^3} \quad (4.55)$$

Differentiating (4.47) with respect to x , we find that

$$\frac{\partial^2 \psi}{\partial x^2}(M, Z, x) = 2 \sum_{1 \leq p \leq n} \frac{x_p M^{n+1}}{M^p} \frac{B_p^2(M, Z)}{(A_p^2(M, Z) + 4B_p(M, Z)x)^{\frac{3}{2}}}$$

which implies that

$$\frac{\partial^2 \psi}{\partial x^2}(M, 0, M^{n+1}) = \frac{2}{M^{n+1}} \sum_{1 \leq p \leq n} \frac{x_p \left(\frac{x_p M^{n+1}}{D_p M^p} \right)^2}{\left(1 + \frac{x_p M^{n+1}}{D_p M^p} \right)^3} \quad (4.56)$$

Using (4.43), (4.44), (4.54), (4.55) and (4.56), the relationship given in (4.53) gives (4.51) of Lemma 4.6.4.

Using (4.50), we can calculate the second order derivative of θ_i for $1 \leq i \leq n$. For $1 \leq j, l \leq n$:

$$\begin{aligned} \frac{\partial^2 \theta_i}{\partial z_j \partial z_l}(M, Z, \theta_{n+1}(M, Z)) &= -\frac{\delta_{i,l}}{\frac{x_i M^{n+1}}{M^i}} \left(\frac{\partial^2 \psi}{\partial z_j \partial z_l}(M, Z, \theta_{n+1}(M, Z)) \right) \\ &\quad - \frac{\delta_{i,l}}{\frac{x_i M^{n+1}}{M^i}} \left(\frac{\partial^2 \psi}{\partial x \partial z_l}(M, Z, \theta_{n+1}(M, Z)) \frac{\partial \theta_{n+1}}{\partial z_j}(M, Z) \right) \\ &\quad + \delta_{i,j} \frac{\partial \theta_{n+1}}{\partial z_l}(M, Z) \frac{2B_i^2(M, Z)\theta_{n+1}(M, Z)}{(A_i^2(M, Z) + 4B_i(M, Z)\theta_{n+1}(M, Z))^{\frac{3}{2}}} \\ &\quad - 2 \frac{B_i^2(M, Z) \frac{\partial \theta_{n+1}}{\partial z_j}(M, Z) \frac{\partial \theta_{n+1}}{\partial z_l}(M, Z)}{(A_i^2(M, Z) + 4B_i(M, Z)\theta_{n+1}(M, Z))^{\frac{3}{2}}} + \frac{B_i(M, Z)}{\sqrt{A_i^2(M, Z) + 4B_i(M, Z)\theta_{n+1}(M, Z)}} \frac{\partial^2 \theta_{n+1}}{\partial z_j \partial z_l}(M, Z) \end{aligned}$$

which, for $Z = 0$, gives (4.52).

Define the differential operator $G : C_0^\infty(\mathbb{R}^{n+1}) \mapsto C_0^\infty(\mathbb{R}^{n+1})$ by

$$Gh(x) = \frac{1}{2} \sum_{1 \leq i, j \leq n} (aa^t)_{i,j}(x) \partial_i \partial_j h + \sum_{1 \leq i \leq n} b_i(x) \partial_i h$$

where a and b are defined in (4.6) and (4.7) respectively. a and b are continuous and Lemmas 4.6.3 and 4.6.4 show that for all $x \in \mathbb{R}^{n+1}$, $\|a(x)\| + \|b(x)\| \leq K\|x\|$. (Ethier and Kurtz, 1986, Theorem 2.6, Ch.8) states that the martingale problem for (G, δ_{S_0, W_0}) is well-posed. So, by (Ethier and Kurtz, 1986, Theorem 4.2, Ch.7), this implies, using Lemmas 4.6.1 and 4.6.2, that the process $(S_{\lfloor \frac{t}{\tau} \rfloor}, W_{\lfloor \frac{t}{\tau} \rfloor})$ converges in distribution to the solution $(\mathbb{P}, (P_t, V_t)_{t \geq 0})$ of the martingale problem for (G, δ_{S_0, W_0}) when $\tau \rightarrow 0$.

Furthermore, as a and b are C^∞ , they are locally Lipschitz and hence, by (Ikeda and Watanabe, 1981, Theorem 3.1, Ch.4), the solution of the martingale problem for (G, δ_{S_0, W_0}) is the unique strong solution of the stochastic differential equation given in Theorem 4.2.3.

4.6.3 Proof of Corollary 4.3.2

We study the expression of $C_{[0,T]}$ given in (4.16). Let us start with the term

$$\frac{1}{T} \int_0^T (F_s(\Gamma_s - I_n)\Sigma) ds$$

We have

$$[F_s(\Gamma_s - I_n)]_{i,k} = \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \left(\left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} \right)^{-1} \frac{x_k}{1 + \Phi_s^k \Lambda_k} - \delta_{i,k} \right)$$

which implies that

$$\begin{aligned} \int_0^T [F_s(\Gamma_s - I_n)\Sigma]_{i,j} ds &= \int_0^T \sum_{1 \leq k \leq n} [F_s(\Gamma_s - I_n)]_{i,k} \Sigma_{k,j} ds \\ &= - \int_0^T \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \Sigma_{i,j} ds + \int_0^T \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \sum_{1 \leq k \leq n} \left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} \right)^{-1} \frac{x_k}{1 + \Phi_s^k \Lambda_k} \Sigma_{k,j} ds \end{aligned}$$

We then remark that

$$\left| \int_0^T \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \Sigma_{i,j} ds - \int_0^T \Lambda_i \Phi_s^i \Sigma_{i,j} ds \right| \leq |\Sigma_{i,j}| \Lambda_i^2 \int_0^T \frac{(\Phi_s^i)^2}{1 + \Phi_s^i \Lambda_i} ds$$

and hence

$$\left| \int_0^T \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \Sigma_{i,j} ds - \int_0^T \Lambda_i \Phi_s^i \Sigma_{i,j} ds \right| \leq |\Sigma_{i,j}| \Lambda_i^2 \int_0^T (\Phi_s^i)^2 ds \quad (4.57)$$

Furthermore, we remark that

$$\begin{aligned} &\left| \int_0^T \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \sum_{1 \leq k \leq n} \left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} \right)^{-1} \frac{x_k}{1 + \Phi_s^k \Lambda_k} \Sigma_{k,j} ds - \int_0^T \Lambda_i \Phi_s^i \sum_{1 \leq k \leq n} x_k \Sigma_{k,j} ds \right| \\ &\leq \sum_{1 \leq k \leq n} x_k |\Sigma_{k,j}| \int_0^T \Phi_s^i \Lambda_i \left| \frac{1}{1 + \Phi_s^i \Lambda_i} \frac{1}{1 + \Phi_s^k \Lambda_k} \left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} \right)^{-1} - 1 \right| ds \\ &\leq \sum_{1 \leq k \leq n} x_k |\Sigma_{k,j}| \int_0^T \frac{\Phi_s^i \Lambda_i}{(1 + \Phi_s^k \Lambda_k)(1 + \Phi_s^i \Lambda_i)} \left| \left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} \right)^{-1} - (1 + \Phi_s^k \Lambda_k)(1 + \Phi_s^i \Lambda_i) \right| ds \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{1 \leq k \leq n} x_k |\Sigma_{k,j}| \int_0^T \Phi_s^i \Lambda_i \left| \left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} \right)^{-1} - (1 + \Phi_s^k \Lambda_k) (1 + \Phi_s^i \Lambda_i) \right| ds \\
 &\leq \sum_{1 \leq k \leq n} x_k |\Sigma_{k,j}| \int_0^T \Phi_s^i \Lambda_i \left| \Phi_s^i \Lambda_i + \Phi_s^k \Lambda_k + \Phi_s^i \Phi_s^k \Lambda_i \Lambda_k + \left(\frac{\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} - 1}{\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p}} \right) \right| ds \\
 &\leq \sum_{1 \leq k \leq n} x_k |\Sigma_{k,j}| \int_0^T \Phi_s^i \Lambda_i \left| \Phi_s^i \Lambda_i + \Phi_s^k \Lambda_k + \Phi_s^i \Phi_s^k \Lambda_i \Lambda_k + \left(\frac{\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} - \sum_{1 \leq p \leq n} x_p}{\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p}} \right) \right| ds \\
 &\leq \sum_{1 \leq k \leq n} x_k |\Sigma_{k,j}| \int_0^T \Phi_s^i \Lambda_i \left(\Phi_s^i \Lambda_i + \Phi_s^k \Lambda_k + \Phi_s^i \Phi_s^k \Lambda_i \Lambda_k - \left(\frac{\sum_{1 \leq p \leq n} \frac{x_p \Phi_s^p \Lambda_p}{1 + \Phi_s^p \Lambda_p}}{\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p}} \right) \right) ds \\
 &\leq \sum_{1 \leq k \leq n} x_k |\Sigma_{k,j}| \int_0^T \Phi_s^i \Lambda_i \left| \Phi_s^i \Lambda_i + \Phi_s^k \Lambda_k + \Phi_s^i \Phi_s^k \Lambda_i \Lambda_k + \sum_{1 \leq p \leq n} x_p \Phi_s^p \Lambda_p \sum_{1 \leq p \leq n} x_p (1 + \Phi_s^p \Lambda_p) \right| ds
 \end{aligned} \tag{4.58}$$

where we used that $\sum_{1 \leq p \leq n} x_p = 1$ and that for strictly positive real numbers $(y_i)_{1 \leq i \leq n}$,

$$\text{we have the convexity inequality } \left(\sum_{1 \leq p \leq n} \frac{x_p}{y_p} \right)^{-1} \leq \sum_{1 \leq p \leq n} x_p y_p.$$

Given Lemmas 4.6.3 and 4.6.4, we find that, for $1 \leq i, k \leq n$, $\frac{a_{i,k}(P_t, V_t)}{P_t^i}$, $\frac{a_{n+1,k}(P_t, V_t)}{V_t}$, $\frac{b_i(P_t, V_t)}{P_t^i}$ and $\frac{b_{n+1}(P_t, V_t)}{V_t}$, defined in (4.6) and (4.7), are bounded uniformly in Λ . As a consequence, by applying Itô's formula to $(\phi_t^i)^p (\phi_t^j)^q (\phi_t^k)^r$ for $p, q, r \geq 0$, we find that $\mathbb{E}((\phi_t^i)^p (\phi_t^j)^q (\phi_t^k)^r) \leq K \exp(Ct)$ where C does not depend on Λ .

Given that $\Phi_s^i = \frac{\phi_s^i}{\phi_0^i}$ and using (4.57), we find that

$$\int_0^T \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \Sigma_{i,j} ds = \int_0^T \Lambda_i \Phi_s^i \Sigma_{i,j} ds + O(\|\Lambda\|^2)$$

where $\mathbb{E} \left(\frac{O(\|\Lambda\|^2)}{\|\Lambda\|^2} \right)$ is bounded when Λ goes to zero.

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Similarly, using (4.58), we find that

$$\int_0^T \frac{\Lambda_i \Phi_s^i}{1 + \Phi_s^i \Lambda_i} \sum_{1 \leq k \leq n} \left(\sum_{1 \leq p \leq n} \frac{x_p}{1 + \Phi_s^p \Lambda_p} \right)^{-1} \frac{x_k}{1 + \phi_s^k \Lambda_k} \Sigma_{k,j} ds = \int_0^T \Lambda_i \Phi_s^i \sum_{1 \leq k \leq n} x_k \Sigma_{k,j} ds + O(\|\Lambda\|^2)$$

We then use the same methodology to study the other terms of $C_{[0,T]}$ given in (4.16).

We conclude this proof by using Ito's formula and the fact that $\Phi_0^i = \frac{\phi_0^i}{\phi_0^i} = 1$, which gives the relationship

$$1 + \int_0^T \left(1 - \frac{s}{T}\right) d\Phi_s^i = \frac{1}{T} \int_0^T \Phi_s^i ds$$

which leads to the decomposition of $C_{[0,T]}$ given in Corollary 4.3.2.

4.6.4 Proof of Proposition 4.4.1

Let $v_j(0)$ be a simple eigenvalue of the fundamental correlation matrix $\tilde{R}(0)$. Define $F : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, such that for $(\Lambda, v) \in \mathbb{R}^n \times \mathbb{R}$

$$F(\Lambda, v) = \det(vI_n - \tilde{R}(\Lambda))$$

F is polynomial in v and $(\Lambda_i)_{1 \leq i \leq n}$ so $F \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$. As $v_j(0)$ is a simple eigenvalue of $\tilde{R}(0)$

$$F(0, v_j(0)) = 0 \quad \text{and} \quad \frac{\partial F}{\partial v}(0, v_j(0)) \neq 0$$

The implicit function theorem ensures the existence of an open set \tilde{U}_j containing 0, and open set V_j containing $v_j(0)$ and a \mathcal{C}^∞ mapping $\tilde{v}_j : \tilde{U}_j \mapsto V_j$ such that $\forall (\Lambda, v) \in \tilde{U}_j \times V_j$:

- $F(\Lambda, v) = 0 \Leftrightarrow v = \tilde{v}_j(\Lambda)$
- $\frac{\partial F}{\partial v}(\Lambda, v) \neq 0$
- $\frac{\partial \tilde{v}_j}{\partial \Lambda_i}(\Lambda) = -\frac{\partial F}{\partial \Lambda_i}(\Lambda, \tilde{v}_j(\Lambda)) \times \frac{1}{\frac{\partial F}{\partial v}(\Lambda, \tilde{v}_j(\Lambda))}$

As a consequence, for all $\Lambda \in U$, $\tilde{v}_j(\Lambda)$ is a simple eigenvalue for the correlation matrix $\tilde{R}(\Lambda)$. Obviously $\tilde{v}_j(0) = v_j(0)$. As $v_j(0)$ is the j -th eigenvalue of $\tilde{R}(0)$ and is a simple eigenvalue: $v_{j-1}(0) < v_j(0) < v_{j+1}(0)$. We know that the eigenvalues of $\tilde{R}(\Lambda)$ are continuous functions of Λ , which implies that there exists an open set $U_j \subset \tilde{U}_j$ containing 0 such that: $\forall \Lambda \in U_j, \forall i \leq j-1$ and $k \geq j+1$, $v_i(\Lambda) < \tilde{v}_j(\Lambda) < v_k(\Lambda)$. As a consequence, for $\Lambda \in U_j$, $\tilde{v}_j(\Lambda)$ is the j -th eigenvalue of $\tilde{R}(\Lambda)$ and hence is equal to $v_j(\Lambda)$. As a consequence, the restriction of v_j to U_j is \mathcal{C}^∞ . Furthermore, the fact that $\frac{\partial F}{\partial v}(\Lambda, v) \neq 0$ implies that $v_j(\Lambda)$ is a simple eigenvalue for $\tilde{R}(\Lambda)$. Let us now calculate the derivative of v_j .

Denote $\tilde{\psi}_j(\Lambda)$ an eigenvector associated to the eigenvalue $v_j(\Lambda)$.

$$(v_j(\Lambda)I_n - \tilde{R}(\Lambda))\tilde{\psi}_j(\Lambda) = 0 \tag{4.59}$$

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As $v_j(\Lambda)$ is a simple eigenvalue, the rank of the matrix $v_j(\Lambda)I_n - \tilde{R}(\Lambda)$ is equal to $n - 1$ and it is possible to extract a $(n - 1) \times (n - 1)$ submatrix which is invertible. Without loss of generality, we can assume that $\left(\left[v_j(\Lambda)I_n - \tilde{R}(\Lambda) \right]_{i,k} \right)_{2 \leq i,k \leq n}$ is invertible. As a consequence, fixing the first component of the eigenvector $\tilde{\psi}_j(\Lambda)$ equal to one and remarking that $\Lambda \rightarrow \left(\left[v_j(\Lambda)I_n - \tilde{R}(\Lambda) \right]_{i,k} \right)_{2 \leq i,k \leq n}$ is also \mathcal{C}^∞ and invertible, which implies that its inverse is also a \mathcal{C}^∞ function of Λ , we deduce that the other components of $\tilde{\psi}_j(\Lambda)$ are \mathcal{C}^∞ functions of Λ , for $\Lambda \in U_j$. Hence, $\tilde{\psi}_j$ is a \mathcal{C}^∞ function of Λ on U_j and so is $\psi_j(\Lambda) = \frac{\tilde{\psi}_j(\Lambda)}{\|\tilde{\psi}_j(\Lambda)\|}$.

Differentiating (4.59) with respect to Λ_i , we obtain:

$$\left(\frac{\partial v_j}{\partial \Lambda_i}(0)I_n - \frac{\partial \tilde{R}}{\partial \Lambda_i}(0) \right) \psi_j(0) + (v_j(0)I_n - \tilde{R}(0)) \frac{\partial \psi_j}{\partial \Lambda_i}(0) = 0$$

Multiplying on the left by $\psi_j(0)^t$ and given the fact that

$$\psi_j(0)^t (v_j(0)I_n - \tilde{R}(0)) = 0$$

we obtain: $\|\psi_j(0) \cdot \psi_j(0)\|^2 \frac{\partial v_j}{\partial \Lambda_i}(0) = \psi_j(0)^t \frac{\partial \tilde{R}}{\partial \Lambda_i}(0) \psi_j(0)$. As a consequence, we have:

$$dv_j(0) \cdot \Lambda = \psi_j(0) \cdot \nabla \tilde{R}(0) \cdot \Lambda \psi_j(0) = \psi_j(0) \cdot \left(\tilde{R}(\Lambda) - \tilde{R}(0) \right) \psi_j(0)$$

which concludes the proof for the first part of Proposition 4.4.1.

We saw that the restriction of the unit eigenvector function (ψ_j) to U_j is also \mathcal{C}^∞ . We write

$$\frac{\partial \psi_j}{\partial \Lambda_i}(0) = \sum_{k=1}^n a_k^{i,j} \psi_k(0)$$

the decomposition of the differential of ψ_j in 0 with respect to Λ_i in the orthonormal basis $(\psi_k(0))_{1 \leq k \leq n}$.

Using this decomposition and the fact that

$$(v_j(0)I_n - \tilde{R}(0))\psi_k(0) = (v_j(0) - v_k(0))\psi_k(0)$$

and differentiating (4.59) with respect to Λ_i , we obtain:

$$\left(\frac{\partial v_j}{\partial \Lambda_i}(0)I_n - \frac{\partial \tilde{R}}{\partial \Lambda_i}(0) \right) \psi_j(0) + \sum_{k=1}^n (v_j(0) - v_k(0)) a_k^{i,j} \psi_k(0) = 0$$

Multiplying on the left by $\psi_k(0)^t$ for $k \neq j$ and using the fact that $\psi_k(0) \cdot \psi_j(0) = 0$ and $\|\psi_k(0)\| = 1$ we obtain:

$$(v_j(0) - v_k(0)) a_k^{i,j} = \psi_k(0) \cdot \frac{\partial \tilde{R}}{\partial \Lambda_i}(0) \psi_j(0)$$

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Differentiating the relation $\|\psi_j(\Lambda)\|^2 = 1$, we obtain that $\psi_j(0) \cdot \frac{\partial \psi_j}{\partial \Lambda_i}(0) = 0$ which implies that $a_j^{i,j} = 0$. Hence we have:

$$\frac{\partial \psi_j}{\partial \Lambda_i}(0) = \sum_{k \neq j} \frac{\psi_k(0) \cdot \frac{\partial \tilde{R}}{\partial \Lambda_i}(0) \psi_j(0)}{v_j(0) - v_k(0)} \psi_k(0)$$

By definition of the differential of $\psi_j(\Lambda)$ in 0 we can write:

$$\psi_j(\Lambda) = \psi_j(0) + \sum_{i=1}^n \frac{\partial \psi_j}{\partial \Lambda_i}(0) \Lambda_i + o(\|\Lambda\|)$$

which implies, up to a normalizing factor, that:

$$\psi_j(\Lambda) = \psi_j(0) + \sum_{k \neq j} \frac{\psi_k(0) \cdot \nabla \tilde{R}(0) \cdot \Lambda \psi_j(0)}{v_j(0) - v_k(0)} \psi_k(0) + o(\|\Lambda\|)$$

and hence the direction of $\psi_j(\Lambda)$ is given by the direction of:

$$\psi_j(0) + \sum_{k \neq j} \frac{\psi_k(0) \cdot (\tilde{R}(\Lambda) - \tilde{R}(0)) \psi_j(0)}{v_j(0) - v_k(0)} \psi_k(0) + o(\|\Lambda\|)$$

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Bibliography

- Adrian, T. and Shin, H. (2008). Financial intermediary leverage and value-at-risk. *Federal Reserve Bank of New York Staff Reports*, 338.
- Adrian, T. and Shin, H. (2009). Money, liquidity and monetary policy. *Federal Reserve Bank of New York Staff Reports*, 360.
- Aitken, B. (1998). Have institutional investors destabilized emerging markets? *Contemporary Economic Policy*, 16(2):173–184.
- Alfonsi, A., Fruth, A., and Schied, A. (2009). Optimal execution strategies in limit order books with general shape functions. *Quantitative Finance*, 10:143–157.
- Allen, F., Litov, L., and Mei, J. (2006). Large investors, price manipulation, and limits to arbitrage: an anatomy of market corners. *Review of Finance*, 10(4):645–693.
- Almgren, R. (2009). Optimal trading in a dynamic market. *Working paper*.
- Almgren, R. and Chriss, N. (2000). Optimal execution of portfolio transactions. *J. Risk*, 3:5–39.
- Almgren, R. and Lorenz, J. (2006). Adaptive arrival price. *Algorithmic Trading: Precision, Control, Execution*, 3:59–66.
- Almgren, R., Thum, C., Hauptmann, E., and Li, H. (2005). Direct estimation of equity market impact. *Risk*, 18(7):58–62.
- Andersen, T., Bollerslev, T., Diebold, F., and Labys, P. (2003). Modeling and forecasting realized volatility. *Econometrica*, 71(2):579–625.
- Andrade, S., Chang, C., and Seasholes, M. (2008). Trading imbalances, predictable reversals, and cross-stock price pressure. *Journal of Financial Economics*, 88(2):406–423.
- Anton, M. and Polk, C. (2008). Connected stocks. *Working paper, London Sch. Econ.*
- Avellaneda, M. and Lipkin, M. (2003). A market-induced mechanism for stock pinning. *Quantitative Finance*, 3:417–425.

BIBLIOGRAPHY

- Avellaneda, M. and Lipkin, M. (2009). A dynamic model for hard-to-borrow stocks. *Risk*, June:92–97.
- Bailey, N., Kapetanios, G., and Pesaran, M. (2012). Exponent of cross-sectional dependence: Estimation and inference. *working paper*.
- Bank for international settlements (1999). A review of financial market events in autumn 1998. *BIS working papers*.
- Barndorff-Nielsen, O. E. and Shephard, N. (2004). Econometric analysis of realised covariation: high frequency covariance, regression and correlation in financial economics. *Econometrica*, 72(2):885–925.
- Basel Committee on Banking Supervision (2010). Basel III: A global regulatory framework for more resilient banks and banking systems. *BIS working papers*.
- Berndt, A., Douglas, R., Duffie, D., Ferguson, M., and Schranz, D. (2005). Measuring default risk premia from default swap rates and EDFs. *BIS working papers*, (173).
- Billio, M., Getmansky, M., Lo, A., and Pelizzon, L. (2012). Econometric measures of connectedness and systemic risk in the finance and insurance sectors. *Journal of Financial Economics*.
- Bouchaud, J., Cizeau, P., Laloux, L., and Potters, M. (2000). Random matrix theory and financial correlations. *International Journal of Theoretical and Applied Finance*, 3(3):391–398.
- Bouchaud, J. and Potters, M. (2003). More statistical properties of order books and price impact. *Physica A: Statistical Mechanics and its Applications*, 324(1):133–140.
- Boudreaux, K. (2012). Discounts and premiums on closed-end mutual funds: A study in valuation. *The Journal of Finance*, 28(2):515–522.
- Brunnermeier, M. (2008). Deciphering the liquidity crunch 2007-2008. *Journal of Economic Perspectives*, 23:77–100.
- Brunnermeier, M. and Pedersen, L. (2005). Predatory trading. *Journal of Finance*, 60(4):1825–1863.
- Brunnermeier, M. and Pedersen, L. (2009). Market liquidity and funding liquidity. *Review of financial studies*, 22(6):2201–2238.
- Carhart, M. (2012). On persistence in mutual fund performance. *The Journal of finance*, 52(1):57–82.
- Carlson, M. (2006). A brief history of the 1987 stock market crash with a discussion of the federal reserve response. *FEDS working paper*, 13.

BIBLIOGRAPHY

- Cetin, U., Jarrow, R., and Protter, P. (2004). Liquidity risk and arbitrage pricing theory. *Finance and stochastics*, 8:311–341.
- Cetin, U., Jarrow, R., Protter, P., and Warachka, M. (2006). Pricing options in an extended black scholes economy with illiquidity: Theory and empirical evidence. *Review of Financial Studies*, 19(2):493–529.
- Chevalier, J. and Ellison, G. (1997). Risk taking by mutual funds as a response to incentives. *Journal of Political Economy*, 105:1167–1200.
- Christoffersen, P., Heston, S., and Jacobs, K. (2009). The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well.
- Comerton-Ford, C., Jones, C., and Putnins, T. (2010). Shorting at close range: a tale of two types. *Working paper*.
- Cont, R., Kukanov, A., and Stoikov, S. (2010). The price impact of order book events. Working paper.
- Cont, R. and Wagalath, L. (2012a). Fire sales forensics: measuring endogenous risk. *working paper*.
- Cont, R. and Wagalath, L. (2012b). Running for the exit: distressed selling and endogenous correlation in financial markets. *Mathematical Finance*, in press.
- Coval, J. and Stafford, E. (2007). Asset fire sales (and purchases) in equity markets. *Journal of Financial Economics*, 86(2):479–512.
- Cox, J. and Ross, S. (1976). The valuation of options for alternative stochastic processes. *Journal of financial economics*, 3(1):145–166.
- Da Fonseca, J., Grasselli, M., and Tebaldi, C. (2008). A multifactor volatility Heston model. *Quant. Finance*, 8(6):591–604.
- Danielsson, J. and Shin, H. (2003). *Endogenous risk*. Risk Books.
- Danielsson, J., Shin, H. S., and Zigrand, J.-P. (2004). The impact of risk regulation on price dynamics. *Journal of Banking and Finance*, 28(5):1069 – 1087.
- Danielsson, J. and Zigrand, J. (2001). What happens when you regulate risk? evidence from a simple equilibrium model. *FMG Discussion Papers*.
- Del Guercio, D. (1996). The distorting effect of the prudent-man laws on institutional equity investments. *Journal of Financial Economics*, 40(1):31–62.
- Dempster, M., Evstigneev, I., and Schenk-Hoppé, K. (2003). Exponential growth of fixed-mix strategies in stationary asset markets. *Finance and Stochastics*, 7(2):263–276.

BIBLIOGRAPHY

- Dupire, B. (1996). A unified theory of volatility. *Derivatives Pricing: The Classic Collection*, pages 185–196.
- Elton, E. and Gruber, M. (1997). Modern portfolio theory, 1950 to date. *Journal of Banking & Finance*, 21(11):1743–1759.
- Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate garch models. *Journal of Business and Economic Statistics*, 20:339–350.
- Engle, R. (2009). *Anticipating Correlations: A New Paradigm for Risk Management*. Princeton Univ Press.
- Engle, R. and Kroner, K. (1995). Multivariate simultaneous generalized arch. *Econometric theory*, 11(01):122–150.
- Ethier, S. and Kurtz, T. (1986). *Markov Processes: Characterization and Convergence*. Wiley.
- Evstigneev, I. and Schenk-Hoppé, K. (2002). From rags to riches: on constant proportions investment strategies. *International Journal of Theoretical and Applied Finance*, 5(06):563–573.
- Falkenstein, E. (2012). Preferences for stock characteristics as revealed by mutual fund portfolio holdings. *The Journal of Finance*, 51(1):111–135.
- Ferreira, M. and Matos, P. (2008). The colors of investors’ money: The role of institutional investors around the world. *Journal of Financial Economics*, 88(3):499–533.
- Fielding, E., Lo, A. W., and Yang, J. H. (2011). The National Transportation Safety Board: A model for systemic risk management. *Journal Of Investment Management.*, 9(1).
- Föllmer, H. and Schweizer, M. (1993). A microeconomic approach to diffusion models for stock prices. *Mathematical Finance*, 3(1):1–23.
- Fonseca, J., Grasselli, M., and Tebaldi, C. (2007). Option pricing when correlations are stochastic: an analytical framework. *Review of Derivatives Research*, 10(2):151–180.
- Fouque, J., Papanicolaou, G., Sircar, R., Solna, K., et al. (2004). Multiscale stochastic volatility asymptotics. *Multiscale Modeling and Simulation*, 2:22–42.
- Fratzscher, M. (2011). Capital flows, push versus pull factors and the global financial crisis. *European Central Bank working paper*.
- Frey, R. and Stremme, A. (1994). Portfolio insurance and volatility. *LSE Financial markets group discussion paper series*.
- Frey, R. and Stremme, A. (1997). Market volatility and feedback effects from dynamic hedging. *Mathematical Finance*, 7(4):351–374.

BIBLIOGRAPHY

- Friedman, S. and Weisberg, H. (1981). Interpreting the first eigenvalue of a correlation matrix. *Educational and Psychological Measurement*, 41(1):11.
- Fuhr, D. (2011). ETF Landscape. Industry Highlights. *Blackrock Presentation*.
- Gabaix, X., Gopikrishnan, P., Plerou, V., and Stanley, H. (2006). Institutional investors and stock market volatility. *The Quarterly Journal of Economics*, 121(2):461–504.
- Gastineau, G. (2010). *The exchange-traded funds manual*, volume 186. Wiley.
- Gennotte, G. and Leland, H. (1990). Market liquidity, hedging and crashes. *American Economic Review*, 80:999–1021.
- Golez, B. and Jackwerth, J. (2010). Pinning in the S&P 500 futures. *Working Paper Series of the Department of Economics, University of Konstanz*.
- Gompers, P. and Metrick, A. (2001). Institutional investors and equity prices. *Quarterly Journal of Economics*, 116:229–259.
- Gonnard, E., Kim, E., and Ynesta, I. (2008). Recent trends in institutional investors statistics. *Financial Market Trends*, 95:1–22.
- Gouriéroux, C. (2006). Continuous time wishart process for stochastic risk. *Econometric Reviews*, 25(2-3):177–217.
- Gouriéroux, C., Jasiak, J., and Sufana, R. (2009). The Wishart autoregressive process of multivariate stochastic volatility. *Journal of Econometrics*, 150(2):167–181.
- Greenwood, R. and Thesmar, D. (2011). Stock price fragility. *Journal of Financial Economics*, 102(3):471–490.
- Gromb, D. and Vayanos, D. (2002). Equilibrium and welfare in markets with financially constrained arbitrageurs. *Journal of financial Economics*, 66(2):361–407.
- Grossman, S. (1988). An analysis of the implications for stock and futures price volatility of program trading and dynamic hedging strategies. *Journal of Business*, 61:275–298.
- Hamao, Y., Masulis, R., and Ng, V. (1990). Correlations in price changes and volatility across international stock markets. *Review of Financial studies*, 3(2):281–307.
- Hart, O. and Moore, J. (1994). A theory of debt based on the inalienability of human capital. *Quarterly Journal of Economics*, 109(4):841–879.
- Hart, O. and Moore, J. (1995). Debt and seniority: an analysis of the role of hard claims in constraining management. *American Economic Review*, 85(3):567–585.
- Hart, O. and Moore, J. (1998). Default and renegotiation: a dynamic model of debt. *Quarterly Journal of Economics*, 113(1):1–41.

BIBLIOGRAPHY

- Haruvy, E. and Noussair, C. (2006). The effect of short-selling on bubbles and crashes in experimental spot asset markets. *Journal of finance*, 61(3):1119–1157.
- Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343.
- Hobson, D. and Rogers, L. (1998). Complete models with stochastic volatility. *Mathematical Finance*, 8(1):27–48.
- Holthausen, R., Leftwich, R., and Mayers, D. (1987). The effect of large block transactions on security prices: A cross-sectional analysis. *Journal of Financial Economics*, 19(2):237–267.
- Ikeda, N. and Watanabe, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. Elsevier.
- Ippolito, R. (1992). Consumer reaction to measures of poor quality: Evidence from the mutual fund industry. *Journal of law and Economics*, 35(1):45–70.
- Jacod, J. and Protter, P. (2012). *Discretization of processes*. Springer.
- Jeannin, M., Iori, G., and Samuel, D. (2007). Modeling stock pinning. *Physica A*, 376:467–479.
- Jotikasthira, P., Lundblad, C., and Ramadorai, T. (2011). Asset fire sales and purchases and the international transmission of funding shocks. *Journal of finance (forthcoming)*.
- Kaminsky, G. and Reinhart, C. (2003). The center and the periphery: the globalization of financial turmoil. Technical report, National Bureau of Economic Research.
- Kempf, A. and Korn, O. (1999). Market depth and order size. *Journal of Financial Markets*, 2:29–48.
- Khandani, A. and Lo, A. (2011). What happened to the quants in August 2007? Evidence from factors and transactions data. *Journal of Financial Markets*, 14:1–46.
- Kraus, A. and Stoll, H. (1972). Price impacts of block trading on the new york stock exchange. *The Journal of Finance*, 27(3):569–588.
- Kyle, A. (1985). Continuous auctions and insider trading. *Econometrica*, pages 1315–1335.
- Kyle, A. and Xiong, W. (2001). Contagion as a wealth effect. *The Journal of Finance*, 56(4):1401–1440.
- Lakonishok, J., Shleifer, A., and Vishny, R. (1992). The impact of institutional trading on stock prices. *Journal of financial economics*, 32(1):23–43.
- Leland, H. (2011). Leverage, Forced Asset Sales, and Market Stability: Lessons from Past Market Crises and the Flash Crash . *BIS working papers*.

BIBLIOGRAPHY

- Lin, W., Engle, R., and Ito, T. (1994). Do bulls and bears move across borders? international transmission of stock returns and volatility. *Review of Financial Studies*, 7(3):507–538.
- Lions, P. and Lasry, J. (2006). Towards a self-consistent theory of volatility. *Journal de Mathématiques Pures et Appliquées*, 86(6):541–551.
- Lions, P. and Lasry, J. (2007). Large investor trading impacts on volatility. In *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, volume 24, pages 311–323. Elsevier.
- Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1):77–91.
- Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley & Sons.
- Merton, R. (1972). An analytic derivation of the efficient portfolio frontier. *Journal of financial and quantitative analysis*, 7(4):1851–1872.
- Mulvey, J. and Kim, W. (2008). Constantly rebalanced portfolios-is mean-reverting necessary. Technical report, working paper Princeton University, Princeton, NJ, USA.
- Ni, S., N., P., and A.M., P. (2005). Stock price clustering on option expiration dates. *Journal of Financial Economics*, 78:49–87.
- Obizhaeva, A. (2011). Selection bias in liquidity estimates. *Working Paper*.
- Pedersen, L. (2009). When everyone runs for the exit. *International journal of central banking*, 5(4):177–199.
- Platen, E. and Schweizer, M. (1994). On smile and skewness. *Statistics research report*.
- Platen, E. and Schweizer, M. (1998). On feedback effects from hedging derivatives. *Mathematical Finance*, 8(1):67–84.
- Pogue, G. (1970). An extension of the markowitz portfolio selection model to include variable transactions' costs, short sales, leverage policies and taxes. *The Journal of Finance*, 25(5):1005–1027.
- Pritsker, M. (2005). Large investors: implications for equilibrium asset, returns, shock absorption, and liquidity. *FEDS Working Paper No. 2005-36*.
- Renault, E. and Touzi, N. (1996). Option hedging and implied volatilities in a stochastic volatility model. *Mathematical Finance*, 6(3):279–302.
- Rosenfeld, E. (2010). Long Term Capital Management. In Cont, R., editor, *Encyclopedia of Quantitative Finance*. Wiley.
- Rubinstein, M. (1983). Displaced diffusion option pricing. *Journal of Finance*, pages 213–217.

BIBLIOGRAPHY

- Rubinstein, M. (2000). Comments on the 1987 stock market crash: Eleven years later. *Risks in Investment Accumulation Products of Financial Institutions*.
- Sandoval, L. and Franca, I. (2012). Correlation of financial markets in times of crisis. *Physica A: Statistical Mechanics and its Applications*, 391(1):187–208.
- Schonbucher, P. and Wilmott, P. (2000). The feedback effect of hedging in illiquid markets. *SIAM Journal on Applied Mathematics*, 61(1):232–272.
- Shiller, R. J. (1988). Portfolio insurance and other investor fashions as factors in the 1987 stock market crash. *NBER Macroeconomics Annual 1988*.
- Shin, H. S. (2010). *Risk and Liquidity*. Oxford University Press.
- Shleifer, A. and Vishny, R. (1992). Liquidation values and debt capacity: A market equilibrium approach. *Journal of Finance*, 47(4):1343–1366.
- Shleifer, A. and Vishny, R. (2011). Fire sales in finance and macroeconomics. *Journal of Economic Perspectives*, 25(1):29–48.
- Sias, R. (1996). Volatility and the institutional investor. *Financial Analysts Journal*, pages 13–20.
- Sias, R. and Starks, L. (1997). Return autocorrelation and institutional investors. *Journal of Financial Economics*, 46(1):103–131.
- Sirri, E. and Tufano, P. (1998). Costly search and mutual fund flows. *Journal of Finance*, 53:1589–1622.
- Stelzer, R. (2010). Multivariate COGARCH(1, 1) processes. *Bernoulli*, 16(1):80–115.
- Stoll, H., Whaley, R., and for the Study of Financial Institutions, S. B. C. (1986). *Expiration day effects of index options and futures*. Salomon Brothers Center for the Study of Financial Institutions, Graduate School of Business Administration, New York University.
- Tonello, M. and Rabimov, S. (2010). The 2010 institutional investment report: trends in asset allocation and portfolio composition.
- Zhang, Y. (1999). Toward a theory of marginally efficient markets. *Physica A: Statistical Mechanics and its Applications*, 269(1):30–44.