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MECANIQUE PAR HOMOGENEISATION

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MATHEMATICAL MODELING OF SOME MECHANICS PROBLEMS BY  
HOMOGENIZATION.

NUTTAWAT SONTICHAJ

Ph.D. (MECHANICS)

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ABSTRACT

We proposed, by homogenization, some models for effective behaviors of structures made of two materials with a high contrast of stiffness (or permittivity) and of size in the occupied domains. Because each steady-state problem considered here may be formulated in terms of minimization, we used the method of variational convergence. The first two parts dealt with fibered structures: first in a non-linear scalar case and the next in a linear vector case. Whilst aiming to generalize the results of Bellieud–Bouchitté and Gruais, we proposed a different strategy. The third study concerned 2-D modeling of some thin flat masonry where the mortar occupied a rather thin domain and is far softer than the bricks. A variant devoted to 3-D linearly elastic stratified media was the final part of this work.

KEY WORDS : HOMOGENIZATION, VARIATIONAL CONVERGENCE,  
ELASTIC MATERIALS

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## RÉSUMÉ DE LA THÈSE

Dans cette étude nous avons proposé par homogénéisation quelques modèles mathématiques de comportement effectif de structures faites de deux phases avec un fort contraste de rigidité (ou conductivité) et de taille des domaines occupés. Parce que chaque problème (de statique) considéré dans ce travail peut être écrit comme un problème de minimisation, la méthode utilisée est celle de la convergence variationnelle (propriété de compacité pour les suites d'énergie uniformément bornées, bornes inférieures et supérieures de l'énergie de ces suites).

D'abord, nous étudions une structure renforcée par des fibres dans le cas scalaire (électricité ou thermique stationnaire). Le domaine occupé par la structure est  $\Omega := \omega \times (0, L)$  avec  $L > 0$  et  $\omega$  un domaine de  $\mathbb{R}^2$ . Pour chaque  $\varepsilon > 0$ , nous considérons une distribution périodique de cellules  $(Y_\varepsilon^i)_{i \in I_\varepsilon}$  telle que  $Y_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$ , et  $I_\varepsilon := \{i \in \mathbb{Z}^2 \mid Y_\varepsilon^i \subset \omega\}$ . Soient  $(D_{r_\varepsilon}^i)_{i \in I_\varepsilon}$  la famille de disques de  $\mathbb{R}^2$  centrés en  $\hat{x}_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2)$  de rayon  $r_\varepsilon \ll \varepsilon$ ,  $T_\varepsilon^i := D_{r_\varepsilon}^i \times (0, L)$  et  $T_\varepsilon := \cup_{i \in I_\varepsilon} T_\varepsilon^i$ . L'ensemble de minces cylindres parallèles  $T_\varepsilon$  représente les fibres. Le coefficient de conductivité  $a_\varepsilon$  est

$$a_\varepsilon(x) = \begin{cases} 1, & \text{si } x \in \Omega \setminus T_\varepsilon, \\ \lambda_\varepsilon, & \text{sinon.} \end{cases}$$

Nous supposons

$$r_\varepsilon \rightarrow 0, \quad \frac{r_\varepsilon}{\varepsilon} \rightarrow 0, \quad \lambda_\varepsilon \rightarrow +\infty, \quad k_\varepsilon := \lambda_\varepsilon \frac{r_\varepsilon^2}{\varepsilon^2} \rightarrow k, \quad k \geq 0 \quad \text{quand } \varepsilon \rightarrow 0.$$

La donnée au bord  $\Gamma_0$   $u_0$  est Lipschitz, pendant que le chargement  $(f, g) \in L^{p'}(\Omega) \times L^{p'}(\Gamma_1)$ ,  $p' = p/(p-1)$ ,  $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ .

Ce problème peut être écrit sous forme de problème de minimisation :

$$(\mathcal{P}_\varepsilon) \quad \min \{ F_\varepsilon(w) - L(w) \mid w \in W_{\Gamma_0}^{1,p}(\Omega) \},$$

où

$$\begin{aligned} W_{\Gamma_0}^{1,p}(\Omega) &:= \{ w \in W^{1,p}(\Omega) \mid w = u_0 \text{ sur } \Gamma_0 \}, \\ F_\varepsilon(w) &:= \int_{\Omega} a_\varepsilon \phi_p(\nabla w) \, dx, \\ \phi_p(\xi) &:= \frac{1}{p} |\xi|^p, \quad \forall \xi \in \mathbb{R}^n, \quad n = 1, 2, 3, \\ L(w) &:= \int_{\Omega} f w \, dx + \int_{\Gamma_1} g w \, dH^2. \end{aligned}$$

L' étude d' homogénéisation consiste en l'examen du comportement de la suite de solutions  $(u_\varepsilon)$  quand  $\varepsilon$  tend vers zéro. Dans [4], il a été montré que la limite de  $(\mathcal{P}_\varepsilon)$  est :

$$\min \{ \Phi(u, v) - L(u) \mid (u, v) \in (L^p(\Omega))^2 \},$$

où

$$\Phi(u, v) = \begin{cases} \int_{\Omega} \phi_p(\nabla u) \, dx + \frac{k\pi}{p} \int_{\Omega} \left| \frac{\partial v}{\partial x_3} \right|^p \, dx + \frac{2\pi\gamma}{p} \int_{\Omega} |v - u|^p \, dx, \\ \text{si } \begin{cases} (u, v) \in W_{\Gamma_0}^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L)), \\ v = u_0 \text{ sur } \Gamma_0 \cap (\omega_0 \cup \omega_L), \end{cases} \\ +\infty \quad \text{sinon,} \end{cases}$$

et

$$[0, +\infty] \ni \gamma = \begin{cases} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} |\log r_\varepsilon|^{-1}, & \text{si } p = 2, \\ \lim_{\varepsilon \rightarrow 0} \left| \frac{2-p}{p-1} \right|^{p-1} r_\varepsilon^{2-p} \varepsilon^{-2}, & \text{si } p \neq 2. \end{cases}$$

Nous sommes concernés par l'extension de ce résultat pour des sections de fibres plus générales et des densités d'énergie plus générales que  $\phi_p$ . Notre but est de fournir une autre preuve que nous espérons plus apte à traiter de tels cas plus réalistes. Les étapes de la preuve de [4] consistent à établir successivement :

- (i) une propriété de compacité pour les suites  $(u_\varepsilon)$  telles que  $F_\varepsilon(u_\varepsilon) < C$ ,
- (ii) une inégalité de borne inférieure pour les suites  $(F_\varepsilon(u_\varepsilon))$ ,
- (iii) une inégalité de borne supérieure pour les suites  $(F_\varepsilon(u_\varepsilon))$ .

Ici, nous remplaçons les étapes (ii) et (iii) par

- (ii') une égalité de borne supérieure de la suite  $(F_\varepsilon(u_\varepsilon))$ .



(iii') une inégalité de borne inférieure de la suite  $(F_\varepsilon(u_\varepsilon))$ , qui essentiellement utilise une inégalité sous-différentielle.

La précédente analyse peut être aisément étendue au cas où  $\phi_p$  est remplacée par n'importe quelle fonction convexe qui satisfait :

$$\exists M > 0, \exists r \in (1, p); \quad |W(\xi) - \phi_p(\xi)| \leq M|\xi|^r \quad \forall \xi \in \mathbb{R}^3,$$

la densité associée à  $\Phi(u, v)$  devient :

$$W(\nabla u) + 2\pi\gamma|v - u|^p + W\left(\frac{\partial v}{\partial x_3}\right).$$

En fait, les arguments clés de notre analyse sont l'identification de  $\gamma$  en termes de solutions de problèmes capacitaires et d'utiliser la  $p$ -positive homogénéité et la convexité de  $\phi_p$  et le fait que  $\phi_p(\xi) \geq \phi_p(\xi_3)$ ,  $\xi = (\xi_1, \xi_2, \xi_3)$ . Donc, il est facile de deviner ce que pourrait être  $\phi(u, v)$ , quand  $\phi_p$  est remplacée par n'importe quelle fonction strictement convexe et quand les sections des fibres sont des domaines de  $\mathbb{R}^2$  étoilés réguliers. Nous espérons que notre stratégie sera capable de réduire et de surmonter les difficultés techniques mises en jeu.

Ensuite, nous étudions le comportement effectif d'une structure cylindrique micro-fibrée comme au chapitre précédent, faite d'un matériau isotrope linéairement élastique entourant une distribution périodique de fibres isotropes linéairement élastiques très fines et de très grande rigidité.

Les coefficients de Lamé  $\lambda_\varepsilon, \mu_\varepsilon$  de la structure sont tels que

$$\lambda_\varepsilon(x) = \begin{cases} \lambda_0 > 0, & \text{si } x \in \Omega \setminus T_\varepsilon \\ \lambda_{\varepsilon 1}, & \text{si } x \in T_\varepsilon \end{cases}, \quad \mu_\varepsilon(x) = \begin{cases} \mu_0 > 0, & \text{si } x \in \Omega \setminus T_\varepsilon \\ \mu_{\varepsilon 1}, & \text{si } x \in T_\varepsilon. \end{cases}$$

La structure est fixée sur la partie  $\Gamma_0 := \omega \times \{0, L\}$  de la frontière  $\partial\Omega$  de  $\Omega$ , soumise à des forces volumiques de densité  $f$  et surfaciques de densité  $g$  sur  $\Gamma_1 := \partial\Omega \setminus \Gamma_0$ . Il est bien connu que si  $\lambda_{\varepsilon 1}, \mu_{\varepsilon 1} > 0$ ,  $f \in L^2(\Omega; \mathbb{R}^3)$ ,  $g \in L^2(\Gamma_1; \mathbb{R}^3)$ , la détermination d'un équilibre conduit au problème de minimisation

$$(\mathcal{P}_\varepsilon) \quad \min \{ F_\varepsilon(w) - L(w) \mid w \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \},$$

où

$$F_\varepsilon(w) := \int_{\Omega} W_\varepsilon(e(w)) dx, \quad L(w) := \int_{\Omega} f \cdot w dx - \int_{\Gamma_1} g \cdot w ds,$$

et

$$W_\varepsilon(e) := \frac{1}{2} \lambda_\varepsilon \operatorname{tr}^2 e + \mu_\varepsilon |e|^2, \quad \forall e \in \mathbb{S}^3 \text{ l'espace des matrices symétriques } 3 \times 3,$$

$$H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) = \{ v \in H^1(\Omega; \mathbb{R}^3) \mid v = 0 \text{ sur } \Gamma_0 \},$$

qui a une solution unique  $\bar{u}_\varepsilon$ .

Pour déterminer le comportement effectif de la structure microfibrée, nous étudions le comportement asymptotique de  $\bar{u}_\varepsilon$  quand  $\varepsilon$  tend vers zéro. Soit

$$k_\varepsilon := \mu_{\varepsilon 1} \frac{|T_\varepsilon|}{|\Omega|}, \quad l_\varepsilon := \frac{\lambda_{\varepsilon 1}}{\mu_{\varepsilon 1}},$$

et supposons que, quand  $\varepsilon \rightarrow 0$ ,

$$r_\varepsilon \rightarrow 0, \quad \frac{r_\varepsilon}{\varepsilon} \rightarrow 0, \quad \lambda_{\varepsilon 1} \rightarrow +\infty, \quad \mu_{\varepsilon 1} \rightarrow +\infty,$$

$$k_\varepsilon \rightarrow k \in [0, +\infty], \quad r_\varepsilon^2 k_\varepsilon \rightarrow \kappa \in [0, +\infty], \quad l_\varepsilon \rightarrow l \in [0, +\infty),$$

$$(\varepsilon^2 |\ln r_\varepsilon|)^{-1} \rightarrow \gamma \in [0, +\infty).$$

Soit  $1_{T_\varepsilon}$  la fonction caractéristique de  $T_\varepsilon$  et  $\mathcal{M}_b(\Omega; \mathbb{R}^3)$  l'espace des mesures sur  $\Omega$  bornées à valeurs dans  $\mathbb{R}^3$ . Il a été prouvé dans [5] que, quand  $\varepsilon$  tend vers zéro,  $\bar{u}_\varepsilon$  converge faiblement dans  $H^1(\Omega; \mathbb{R}^3)$  vers  $\bar{u}$  et que  $\bar{v}_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} \bar{u}_\varepsilon 1_{T_\varepsilon}$  converge \* faible dans  $\mathcal{M}_b(\Omega; \mathbb{R}^3)$  vers un élément  $\bar{v}$  de  $L^2(\Omega; \mathbb{R}^3)$  solution de

$$(\mathcal{P}^{\text{eff}}) \quad \min \{ \Phi(u, v) - L(u) \mid (u, v) \in L^2(\Omega; \mathbb{R}^3)^2 \},$$

avec

$$\Phi(u, v) = \begin{cases} \int_{\Omega} W_0(e(u)) dx \\ + \mu_0 \pi \gamma \int_{\Omega} (v - u)^T \begin{bmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{bmatrix} (v - u) dx & \text{si } (u, v) \in \mathcal{D}, \\ + \frac{1}{2} \frac{3l+2}{2(l+1)} k \int_{\Omega} \left| \frac{\partial v_3}{\partial x_3} \right|^2 dx \\ + \frac{1}{2} \frac{3l+2}{2(l+1)} \frac{\kappa}{4} \int_{\Omega} \left| \frac{\partial^2 v_1}{\partial x_3^2} \right|^2 + \left| \frac{\partial^2 v_2}{\partial x_3^2} \right|^2 dx, & \\ +\infty, & \text{sinon,} \end{cases}$$

et

$$W_0(e(u)) := \frac{1}{2} \lambda_0 \operatorname{tr}^2 e(u) + \mu_0 |e(u)|^2, \quad \chi := \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0},$$

$$\mathcal{D} := H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times \left\{ v \in L^2(\omega, H_0^2(0, L; \mathbb{R}^3)) \mid \frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = 0 \text{ sur } \Gamma_0 \right\}.$$

Donc, le comportement effectif de la structure microfibrée est celle d'un milieu continu élastique généralisé mettant en jeu une variable d'état supplémentaire et ses deux premières dérivées. Cette variable d'état supplémentaire rend compte de la microstructure dans la mesure où elle décrit le comportement asymptotique du déplacement dans les fibres judicieusement mis à l'échelle. Notre objectif était de comprendre ce résultat plus profondément dans un cadre plus général, quant à la géométrie des sections des fibres et au comportement de la matrice et des fibres. Néanmoins, nous ne sommes, à nouveau, que parvenus à donner une autre preuve du résultat de [5] en étudiant directement la convergence variationnelle (comme dans le cas scalaire) de  $F_\varepsilon$  et avons divisé notre preuve en trois étapes :

1. une propriété de compacité pour toute suite  $(u_\varepsilon)$  telle que  $F_\varepsilon(u_\varepsilon)$  soit uniformément bornée,
2. une borne supérieure atteinte pour une suite  $(F_\varepsilon(u_\varepsilon))$ ,
3. une inégalité de borne inférieure pour une suite  $(F_\varepsilon(u_\varepsilon))$ .

L'ingrédient essentiel est la construction de champs oscillants appropriés fournissant la "meilleure" borne supérieure. Donc, il semble possible avec des sections de fibres plus générales ( $r_\varepsilon\Delta$  avec  $\partial\Delta$  suffisamment régulière) et une densité d'énergie de déformation quadratique plus générale  $W_M$  pour la matrice, dans la mesure où les solutions  $\theta_\varepsilon^\alpha$  des problèmes capacitaires

$$\min \left\{ \int_{(-\varepsilon, \varepsilon)^2} W_M(e(\varphi)) d\hat{x} \left| \begin{array}{l} \varphi \in H^1((-\varepsilon, \varepsilon)^2; \mathbb{R}^3), \\ \varphi(\hat{x}) = e^\alpha \text{ sur } r_\varepsilon\Delta, \{e^\alpha\}_{\alpha=1,2,3} \text{ base de } \mathbb{R}^3, \\ \varphi(\hat{x}) = 0 \text{ sur } (-\varepsilon, \varepsilon)^2 \setminus D(0, R_\varepsilon). \end{array} \right. \right\}$$

sont telles que :

- i)  $\exists w_M^{\text{cap}} \in \mathbb{S}^3$  telle que  $(w_M^{\text{cap}})_{\alpha\beta} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{(-\varepsilon, \varepsilon)^2} w_M(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) d\hat{x}$ ,
- ii)  $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon i + r_\varepsilon \partial\Delta} W'_M(e(\theta_\varepsilon^\alpha)) \cdot (u_\varepsilon - \bar{u}_\varepsilon) dl$   
 $= \lim_{\varepsilon \rightarrow 0} \int_{\partial D_{R_\varepsilon}^i} W'_M(e(\theta_\varepsilon^\alpha)) \cdot (u_\varepsilon - \bar{u}_\varepsilon) dl = 0.$

Dans l'étude suivante, nous appliquons la théorie de l'homogénéisation au comportement statique de maçonneries linéairement élastiques minces et plates.

Soit  $a$  un nombre réel positif,  $H \in a\mathbb{N}$ , et  $h$  un nombre réel positif petit. Le domaine occupé par la maçonnerie mince considérée ici est  $\Omega_h := \omega \times (-h, h)$  où  $\omega := (0, L) \times (0, H)$ . Ce mur est constitué d'une distribution périodique de briques reliées l'une à l'autre par un mortier adhésif occupant un domaine très mince. Plus précisément si  $\hat{Y} := (0, 1) \times (0, a)$ ,

$$\begin{aligned} S &:= \{ \hat{y} \in \hat{Y} \mid y_2 = a/4 \text{ ou } 3a/4; \\ &\quad y_1 = 1/4, \text{ et } 0 < y_2 < a/4 \text{ ou } 3a/4 < y_2 < a; \\ &\quad y_1 = b \text{ et } a/4 < y_2 < 3a/4, \text{ où } b \in (0, 3/4] \}, \end{aligned}$$

$$\hat{M}_l := \{ \hat{x} \in \hat{Y} \mid \text{dist}(\hat{x}, S) < l \}, \quad l < \min\{b, a/4\},$$

$$\hat{B}_l := \hat{Y} \setminus \hat{M}_l,$$

$$\varepsilon = L/n, \quad n \in \mathbb{N},$$

$$\hat{Y}_\varepsilon^i := \varepsilon i + \varepsilon \hat{Y}, \quad i \in I_\varepsilon := \{ i \in \mathbb{Z}^2 \mid \hat{Y}_\varepsilon^i \subset \omega \},$$

$$\hat{M}_{l\varepsilon}^i = \varepsilon i + \varepsilon \hat{M}_l, \quad \hat{M}_{l\varepsilon} = \cup_{i \in I_\varepsilon} \hat{M}_{l\varepsilon}^i,$$

$$\hat{B}_{l\varepsilon}^i = \varepsilon i + \varepsilon \hat{B}_l, \quad \hat{B}_{l\varepsilon} = \cup_{i \in I_\varepsilon} \hat{B}_{l\varepsilon}^i,$$

alors  $B_{l\varepsilon h} := \hat{B}_{l\varepsilon} \times (-h, h)$  et  $M_{l\varepsilon h} := \hat{M}_{l\varepsilon} \times (-h, h)$  sont les domaines respectivement occupés par les briques et le mortier. Les briques sont faites d'un matériau homogène linéairement élastique dont la densité d'énergie  $W$  est une fonction quadratique strictement convexe vérifiant

$$\exists \alpha, \beta > 0; \quad \alpha |e|^2 \leq W(e) \leq \beta |e|^2 \quad \forall e \in \mathbb{S}^3,$$

l'espace des matrices  $3 \times 3$  symétriques. Le mortier est supposé constitué d'un matériau isotrope linéairement élastique de densité d'énergie  $W_{\lambda\mu}$  dont les coefficients de Lamé  $\lambda$  et  $\mu$  sont supposés être bien plus faibles que  $\alpha$  et  $\beta$ . Enfin, le mur est encasté le long de  $\Gamma_{0h} = \gamma_0 \times (-h, h)$ ,  $\partial\Omega_h$ ,  $\gamma_0$  étant une partie de  $\partial\omega$  de longueur positive, et est soumis à des forces volumiques de densité  $f^h$ .

Pour trouver les configurations d'équilibre du mur, on est conduit au problème

$$(\mathcal{P}) \quad \min \left\{ \int_{B_{l\varepsilon h}} W(e(u)(x)) dx + \int_{M_{l\varepsilon h}} W_{\lambda\mu}(e(u)(x)) dx - \int_{\Omega_h} f^h(x) \cdot u(x) dx \right. \\ \left. \mid u \in H_{\Gamma_{0h}}^1(\Omega_h; \mathbb{R}^3) \right\}$$

où

$$\begin{cases} H_{\Gamma_{0h}}^1(\Omega_h; \mathbb{R}^3) := \{v \in H^1(\Omega_h; \mathbb{R}^3) \mid v = 0 \text{ sur } \Gamma_{0h} \text{ au sens des traces}\}, \\ e(v) := \frac{1}{2}(\nabla v + \nabla v^T). \end{cases}$$

Supposant  $f^h$  dans  $L^2(\Omega_h; \mathbb{R}^3)$ , ce problème clairement a une solution unique, mais, à cause des faibles valeurs de  $(\varepsilon, l, \lambda, \mu, h)$ , l'obtention d'une approximation numérique peut être difficile. Donc, il est intéressant de proposer un modèle simplifié mais suffisamment précis. Une première tentative [6] est de remplacer  $(\mathcal{P})$  par un problème bidimensionnel approprié posé sur la surface moyenne du mur qui peut s'écrire :

$$(\hat{\mathcal{P}}_{\hat{s}}) \quad \min \left\{ \int_{\hat{B}_{l\varepsilon}} \hat{W}(e(u)(\hat{x})) d\hat{x} + \int_{\hat{M}_{l\varepsilon}} \hat{W}_{\lambda'\mu}(e(u)(\hat{x})) d\hat{x} - \int_{\omega} \hat{f}(\hat{x}) \cdot u(\hat{x}) d\hat{x} \right. \\ \left. \mid u \in H_{\gamma_0}^1(\omega; \mathbb{R}^2) \right\},$$

où

- $\hat{W}$ , déduite de  $W$ , est une fonction strictement convexe vérifiant

$$\exists \hat{\alpha}, \hat{\beta} > 0; \quad \hat{\alpha}|e|^2 \leq \hat{W}(e) \leq \hat{\beta}|e|^2 \quad \forall e \in \mathbb{S}^2,$$

- $\hat{W}_{\lambda'\mu}(e) = \frac{\lambda'}{2}(e_{11} + e_{22})^2 + \mu|e|^2$ ,  $\lambda' = \frac{2\lambda\mu}{\lambda+2\mu}$ , qui correspond à  $W_{\lambda\mu}$  sous l'hypothèse des contraintes planes,
- $\hat{f}$  est déduite de  $f^h$ ,
- $\hat{s} := (\varepsilon, l, \lambda', \mu)$ .

Supposant encore que  $\hat{f} \in L^2(\omega; \mathbb{R}^2)$ , le problème  $(\hat{\mathcal{P}}_{\hat{s}})$  a une unique solution  $\hat{u}_{\hat{s}}$ , mais, à cause des faibles valeurs de  $\hat{s}$ , obtenir une approximation numérique reste difficile. Aussi, prenant en compte ces faibles valeurs, nous proposerons un modèle simplifié mais suffisamment précis en étudiant le comportement asymptotique de  $(\hat{\mathcal{P}}_{\hat{s}})$  quand  $\hat{s}$  tend vers zéro.

En fait,  $(\hat{\mathcal{P}}_{\hat{s}})$  ressemble à un problème d'homogénéisation périodique puisque la géométrie et les propriétés mécaniques de la microstructure sont  $\varepsilon\hat{Y}$ -périodiques, mais la géométrie met en jeu un paramètre supplémentaire  $l$  tandis

que la densité de déformation du mortier met en jeu deux autres  $\lambda', \mu$ . Si  $(l, \lambda', \mu)$  étaient fixes, la densité effective de la structure hétérogène serait

$$W_{l\lambda'\mu}^{\text{eff}}(E) := \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{B}_l} \hat{W}(E + e(v)(y)) dy + \int_{\hat{M}_l} \hat{W}_{\lambda'\mu}(E + e(v)(y)) dy \right) \right. \\ \left. \mid v \in H_{\text{per}}^1(\hat{Y}; \mathbb{R}^2) \right\}$$

où

$$H_{\text{per}}^1(\hat{Y}; \mathbb{R}^2) := \{ v \in H^1(\hat{Y}; \mathbb{R}^2) \mid \text{les traces de } v \text{ sur les côtés opposés de } \hat{Y} \text{ sont égales} \}.$$

Le comportement asymptotique de  $W_{l\lambda'\mu}^{\text{eff}}$  quand  $(l, \lambda', \mu) \rightarrow 0$  est un problème de modélisation de jonctions élastiques souples. Avec l'hypothèse supplémentaire

$$\exists \bar{\lambda}', \bar{\mu} \in (0, +\infty); \quad \lambda' \sim 2\bar{\lambda}'l, \quad \mu \sim 2\bar{\mu}l,$$

qui correspond au cas le plus intéressant du point de vue mécanique, les arguments de [7] et [1] montrent que

$$\lim_{(l, \lambda', \mu) \rightarrow 0} W_{l\lambda'\mu}^{\text{eff}}(E) = \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v](y)) dl \right) \right. \\ \left. \mid v \in H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) \right\},$$

où

$$H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) := \{ v \in H^1(\hat{Y} \setminus S; \mathbb{R}^2) \mid \text{les traces de } v \\ \text{sur les côtés opposés de } \hat{Y} \text{ sont égales} \},$$

$$\bar{W}_{\bar{\lambda}'\bar{\mu}}(v) := \hat{W}_{\bar{\lambda}'\bar{\mu}}(v \otimes_s n) \quad \forall v \in \mathbb{R}^2, \quad a \otimes_s b := \frac{1}{2}(a \otimes b + b \otimes a),$$

$n$  est une normale unitaire à  $S$ ,  $[v]$  est la différence, prise dans le sens de  $n$ , des traces de  $v$  sur  $S$ .

Dans la suite, nous prouvons que cette limite est *la densité effective d'énergie de déformation* du mur en déterminant le comportement asymptotique de la fonctionnelle énergie de déformation totale  $F_{\hat{s}}$  mise en jeu par  $(\hat{P}_{\hat{s}})$  quand  $\hat{s} \rightarrow 0$  avec la condition supplémentaire précédente.

On établit la convergence variationnelle, pour la convergence faible \* dans  $BD(\omega)$ , de  $F_{\hat{s}}$  vers une fonctionnelle de densité  $W^{\text{eff}}$  donnée par la précédente limite de  $W_{l, \lambda', \mu}^{\text{eff}}$ .

En fait, nous n'avons pas réussi à prouver que les points d'accumulation des suites d'énergies uniformément bornées appartiennent à  $H^1(\omega; \mathbb{R}^2)$  et vérifient une condition homogène de Dirichlet sur  $\gamma_0$ . Donc, on ne peut ([2], [3]) affirmer que  $\hat{u}_s$  converge (\* faible dans  $BD(\omega)$ ) vers l'unique solution de

$$(\bar{\mathcal{P}}) \quad \min \left\{ \int_{\omega} W^{\text{eff}}(e(v)) dx - \int_{\omega} \hat{f} \cdot v dx \mid v \in H_{\gamma_0}^1(\omega; \mathbb{R}^2) \right\}.$$

Le problème  $(\bar{\mathcal{P}})$  décrit l'équilibre d'un corps élastique bidimensionnel homogène occupant  $\omega$  comme configuration de référence, de densité d'énergie  $W^{\text{eff}}$ , encastré le long de  $\gamma_0$  et soumis à des forces de densité  $\hat{f}$ . Nous affirmons seulement qu'un candidat raisonnable comme densité d'énergie de déformation de l'assemblage briques mortier est  $W^{\text{eff}}$ . Ainsi, notre modèle est plus simple que le modèle de départ puisqu'il met en jeu un corps homogène et aussi suffisamment précis à cause du résultat de convergence. Ceci fut a été obtenu en [6] par des arguments plutôt heuristiques.

Le matériau homogène équivalent est plus faible que le matériau constituant les briques. Ceci est le prix à payer dû à la difficulté à faire un mur homogène. La résistance effective du mur serait plus grande que celle des briques si du mortier très rigide était utilisé. Il serait intéressant de considérer les cas où  $\lambda', \mu$  étaient d'ordre de grandeur  $1/l$ .

Enfin, nous avons procédé à une extension bidimensionnelle en considérant des milieux stratifiés constitués à partir de deux constituants linéairement élastiques, le constituant le plus faible occupant des couches beaucoup plus minces que l'autre. Il est à noter que dans ce cas les domaines occupés par chaque phase ne sont pas connexes.

Comme précédemment, nous ne sommes parvenus qu'à prouver la convergence variationnelle (vis à vis de la convergence \* faible dans  $BD$ ) des énergies de déformations totales.

Nous conjecturons qu'il y a aussi convergence des minimiseurs pour certaines géométries. C'est ce que nous avons observé sur un exemple unidimensionnel où l'on peut faire des calculs analytiques : la \*faible limite des minimiseurs est aussi une limite uniforme et satisfait aux conditions aux limites de Dirichlet.

## REFERENCES

- [1] A. Aitmoussa, *Modélisation et étude des Singularités de Contraintes d'un Joint Collé très mince*, Ph.D. Thesis, University Montpellier 2, France, 1989.
- [2] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, 1985.
- [3] H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces : Application to PDEs and Optimization*, MPS-SIAM Book Series on Optimization, 2005.
- [4] M. Bellieud, G. Bouchitté, *Homogenization of Elliptic Problems in a Fiber Reinforced Structure. Non local effects*, Ann. Scuola Norm. Sup. CI Sci. (4) 26 (1998) p.407–432.
- [5] M. Bellieud, I. Gruais, *Homogenization of an Elastic Material Reinforced by Very Stiff or Heavy Fibers. Non Local Effects. Memory Effects*, J. Math. Pures Appl. 84 (2005), 55–96.
- [6] A. Cecchi, K. Sab, *A Multi-Parameter Homogenization Study for Modeling Elastic Masonry*, European Journal of Mechanics A/Solids, 21 (2002), p.249–268.
- [7] C. Licht, G. Michaille, *A Modelling of Elastic Adhesive Bounded Joints*, Advances in Mathematical Sciences and Applications, 7 (1997), no 2, p.711–740.



## CHAPTER I

### INTRODUCTION

Mechanics of materials is the study of behavior of forces inside and outside the solid bodies. Formulas developed in mechanics of materials relate stresses to internal forces and moments. The study of mechanics materials is very useful for helping to choose the kind of materials with appropriated using in product, industrial and electrical designs.

In this thesis, we consider some composite materials consisting of short or long, continuous or discontinuous and one or in multiple directions fibers embedded in a matrix. Such materials offer advantages over conventional isotropic structural materials such as steel, aluminum, and other types of metal. These advantages include high strength, low weight as well as good fatigue and corrosion resistance. In addition, by changing the arrangements of the fibers, the properties of the material can be tailored to meet the requirements of a specific design.

The excellent properties of composites are achieved by the favorable characteristics of the two major constituents, namely the fiber and the matrix. In low-performance composites provide some stiffening but very little strengthening. They usually are in the form of particles, short or chopped fiber. The load is mainly carried by the matrix. In high-performance composites, continuous fibers, provide the desirable stiffness and strength, whereas the matrix provides protection and support for the fibers as well as helps redistribute the load from broken to adjacent intact fibers.

Composite materials are heterogeneous materials obtained by mixing several phases or constituent materials on a very fine (microscopic) scale. However, one is usually interested only in the large scale (macroscopic) properties of such a composite. Therefore, the main problem with composite materials is to determine their effective properties without determining their fine scale structure.

Composite materials can be studied from a number of different viewpoints each of which requires a different kind of expertise. On a scale that is large with respect to the fiber diameter, the fiber and matrix properties may be averaged, and the material may be treated as homogeneous. This assumption, commonly employed in macromechanical analysis of composites, the design steps from micromechanics (which takes into account the fiber and matrix properties) through macromechanics (which treats the properties of the composite) to structural analysis. Hence, the material is considered to be quasi-homogeneous, which implies that the properties are taken to be the same at every point. These properties are not the same as the properties of either the fiber or the matrix but are a combination of the properties of the constituents.

Mathematicians have been interested in composite materials since the 1970's. Their first main contribution in this field was to give a theoretical basis for the notion of effective properties of a composite material. Indeed, homogenization theory permits one to properly define a composite material as a limit, in the sense of homogenization, of a sequence of increasingly finer mixtures of the constituent phases. Effective properties are now defined as homogenized coefficients. The application of homogenization to the modeling of composite materials has become a popular subject in applied mathematics. The physical ideas of homogenization have a very long history going back at least to Maxwell [1], Poisson [2] and Rayleigh [3]. The homogenization method in optimal design was initiated by Murat and Tartar in the late 1970's. The first relevant references are the works of Murat and Tartar [4], [5], [6] and [7].

Our primary technique, in this work, is the homogenization of composite materials [8]. The procedures are mainly based on the definition of a local surrounding of a macroscopic material point with volume and the boundary. This volume represents a characteristic part of the material which is sufficient to describe its structure and behavior, respectively. This implies that the size of heterogeneities at the microscopic level has to be one scale smaller than the size of the volume of the macroscopic point. Homogenization has many potential applications, but we consider it only as a tool for deriving macroscopic or

effective properties of microscopically heterogeneous media. As such it provides a firm basis to the notion of composite material obtained by mixing, on a very fine scale, several phase components.

Although homogenization is not restricted to periodic problems, it has been used often for the asymptotic analysis of periodic structures. Indeed, in many fields of science and mechanics one has to solve boundary value problems in periodic media. Quite often the size of the period is small compared to the size of a sample of the medium, and, is often denoted by  $\varepsilon$ . An asymptotic analysis, as  $\varepsilon$  goes to zero, allows one to reduce the complexity of the problem. Starting from a microscopic description of a problem, one seeks a macroscopic, or effective model. This process of making an asymptotic analysis in order to derive an averaged formulation is precisely the goal of homogenization.

In this thesis we focus our attention on minimization problems and apply the convergence theory for sequences of functions, variational convergence. This convergence may be regarded as the weak convergence, which guarantees that an approach to the limit existence in the corresponding minimization problems. Therefore this concept of convergence has natural applications in all branches of optimization theory.

In the framework of this thesis, the second chapter focuses on the study of fiber reinforced structures by considering the scalar case. Here we consider the homogenization of the quasi-linear elliptic problem of degree  $p$ ,  $1 < p < +\infty$ . Bellieud and Bouchitté [9] already succeed in this problem. Here we present another proof that we expect to be more suitable to treat more general geometrical and physical cases. Because the energy density is assumed to be positively homogeneous of degree  $p$ , the suitable Sobolev space that we use here is the  $W^{1,p}$ . The main theorem is the variational convergence problem. The steps of our proof are the compactness properties, upper bound equality and lower bound inequality, respectively. Here the upper bound equality is to be proved before the lower bound inequality. Because we have to use the results in the upper bound for complete the proof in the lower bound by subdifferential inequality.

The third chapter is nearly devoted like in the previous chapter, but it focuses in vector case and is considered in a homogenization of a linearly elliptic boundary value problem of elasticity. The isotropic elastic material is reinforced by a periodic distribution of very thin parallel fibers in which the Lamé coefficients are assumed to have high values. Bellieud and Gruais [10] proved that the macroscopic behavior is the one of a generalized continuum medium involving an additional state variable accounting for the microstructure. Here we propose a proof of this result by studying the variational convergence of the energy functional. The processes of the proof are likely in the second chapter, but this one is more complicated. We have to choose the suitable capacity problem for furnishing the result.

The fourth chapter is concerned with the homogenization method for some elastic masonries. Elastic masonry is a heterogeneous medium which shows an anisotropic and inhomogeneous nature. In particular, the inhomogeneity is due to its composite materials, mortar and bricks, which have very different mechanical properties. The anisotropy is due to the different masonry patterns since the mechanical response is affected by the geometrical arrangement of the components. The unquestionable importance of a lot of real masonry estate requires researcher particular attention for this kind of structures. Therefore, in order to design an efficient response for repairing existing masonry structures, a large number of theoretical studies, experimental laboratory activities and computational procedures have been proposed in scientific literature [11], [12] and [13]; moreover due to the low values of parameters, obtaining numerical approximations may be difficult. Thus, it is of interest to propose a simplified but accurate enough model. This work is illustrated by results of 2-dimensional model. A first attempt [11] is to replace 3-dimensional problem by a suitable 2-dimensional problem set in the cross section of the wall. The techniques of bonding [12], [13] and homogenization are used for deriving the effective behavior.

Finally, the fifth chapter is devoted to stratified medium models, it extends the problem of the fourth chapter but we consider a 3-dimensional case. We mostly focus on a stratified medium made from two homogeneous linearly

constituents, one being far weaker than the other. Moreover, as in the case of mortar in the fourth chapter, the weak constituent occupies layers far thinner than the thin layers occupied by the stronger material. This chapter also states the example in 1-dimensional case, where closed-form formulas are available, for studying the results and comparing with the result in 3-dimensional case that corresponding or not.

In the sequel, we do not point out open problems, but it is clear that the range of applications covered by this thesis, although very important. Of course, there are many other types of homogenization problems that have not yet been attacked. It is our hope that this thesis can serve as a basis for further developments in new directions.

## REFERENCES

- [1] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd Ed., Clarendon Press, Oxford, 1881.
- [2] S. Poisson, *Second Mémoire sur la Théorie du Magnétisme*, Mem. Acad, France, 5 (1822).
- [3] J. W. Rayleigh, *On the Influence of Obstacles Arranged in Rectangular Order Upon the Properties of a Medium*, Phil. Mag., 32 (1892), p.481–491.
- [4] F. Murat, L. Tartar, *Calcul des Variations et Homogénéisation*, Les Méthodes de l'Homogénéisation Théorie et Applications en Physique, Coll. Dir. Etudes et Recherches EDF, 57, Eyrolles, Paris, 1985, p.319–369.
- [5] F. Murat, L. Tartar, *Optimality Conditions and Homogenization*, in *Non-linear Variational Problems*, A. Marino et al. eds., Pitman, Boston, 1985, p.1–8.
- [6] L. Tartar, *Problèmes de Contrôle des Coefficients dans des équations aux Dérivées Partielles*, Control Theory, Numrical Methods and Computer Systems Modelling, International Symposium, Rocquencourt, June 1974, A. Bensoussan, J. L. Lions eds., Lecture Notes in Economy and Mathematical Systems, Springer Verlag, 107 (1975), p.420–426.
- [7] L. Tartar, *Estimation de Coefficients Homogénéisés*, Computing methods in applied sciences and engineering, Third International Symposium, December 1977, R. Glowinski, J. L. Lions eds., Lecture Notes in Math, Springer Verlag, 704 (1979), p.364–373.
- [8] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, 1985.
- [9] M. Bellieud, G. Bouchitté, *Homogenization of Elliptic Problems in a Fiber Reinforced Structure. Non local effects*, Ann. Scuola Norm. Sup. CI Sci. (4) 26 (1998), p.407–432.

- [10] M. Bellieud, I. Gruais, *Homogenization of an Elastic Material Reinforced by Very Stiff or Heavy Fibers. Non Local Effects. Memory Effects*, J. Math. Pures Appl. 84 (2005), 55–96.
- [11] A. Cecchi, K. Sab, *A Multi-Parameter Homogenization Study for Modeling Elastic Masonry*, European Journal of Mechanics A/Solids, 21 (2002), p.249–268.
- [12] C. Licht, G. Michaille, *A Modelling of Elastic Adhesive Bounded Joints*, Advances in Mathematical Sciences and Applications, 7 (1997), no 2, p.711–740.
- [13] A. Aitmoussa, *Modélisation et étude des Singularités de Contraintes d'un Joint Collé très mince*, Ph.D. Thesis, University Montpellier 2, France, 1989.

## CHAPTER II

### MATHEMATICAL MODELING OF FIBER REINFORCED STRUCTURES BY HOMOGENIZATION

#### 2.1 Setting the Problem

Let  $p \in (1, +\infty)$ , we consider the homogenization of the elliptic problem

$$\begin{cases} -\operatorname{div} \sigma_\varepsilon = f & \text{on } \Omega, \\ \sigma_\varepsilon = a_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon, & \\ u_\varepsilon = u_0 & \text{on } \Gamma_0, \\ \sigma_\varepsilon \cdot n = g & \text{on } \Gamma_1, \end{cases} \quad (2.1)$$

where  $\Omega := \omega \times (0, L)$  with  $L > 0$  and  $\omega$  is a bounded domain of  $\mathbb{R}^2$  with smooth boundary and containing the origin of coordinates. The homogenization study of (2.1) consists in examining the behavior of the sequence of the solution  $(u_\varepsilon)$  as  $\varepsilon$  tends to zero. The permittivity coefficient  $a_\varepsilon$  is  $\varepsilon$ -periodic and satisfies a uniform lower bound,  $\Gamma_0$  is an open subset of  $\partial\Omega$  with Hausdorff measure  $H^2(\Gamma_0)$  strictly positive,  $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ , and  $n$  is the unit exterior normal on  $\partial\Omega$ . The boundary data  $u_0$  is Lipschitz, while  $(f, g) \in L^{p'}(\Omega) \times L^{p'}(\Gamma_1)$ ,  $p' = p/(p-1)$ .

The problem (2.1) is related to the minimization problem

$$(\mathcal{P}_\varepsilon) \quad \min \{ F_\varepsilon(w) - L(w) \mid w \in W_{\Gamma_0}^{1,p}(\Omega) \},$$

where

$$\begin{aligned} W_{\Gamma_0}^{1,p}(\Omega) &:= \{ w \in W^{1,p}(\Omega) \mid w = u_0 \text{ on } \Gamma_0 \}, \\ F_\varepsilon(w) &:= \int_{\Omega} a_\varepsilon \phi_p(\nabla w) \, dx, \\ \phi_p(\xi) &:= \frac{1}{p} |\xi|^p, \quad \forall \xi \in \mathbb{R}^n, \quad n = 1, 2, 3, \\ L(w) &:= \int_{\Omega} f w \, dx + \int_{\Gamma_1} g w \, dH^2. \end{aligned} \quad (2.2)$$



We are interested in the asymptotic behavior of  $(\mathcal{P}_\varepsilon)$  as  $\varepsilon \rightarrow 0$ . We present another proof of a study of Bellieud and Bouchitté [1] that we expect to be more suitable to treat more general geometrical and physical cases.

The bases of the cylindrical domain  $\Omega$  are denoted by  $\omega_0 := \omega \times \{0\}$  and  $\omega_L := \omega \times \{L\}$ . For each  $\varepsilon$ , we consider a periodic distribution of cells  $(Y_\varepsilon^i)_{i \in I_\varepsilon}$  such that  $Y_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$ , and  $I_\varepsilon := \{i \in \mathbb{Z}^2 \mid Y_\varepsilon^i \subset \omega\}$ . Let  $(D_{r_\varepsilon}^i)_{i \in I_\varepsilon}$  be the family of disks of  $\mathbb{R}^2$  centered at  $\hat{x}_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2)$  of radius  $r_\varepsilon \ll \varepsilon$ ,  $T_\varepsilon^i := D_{r_\varepsilon}^i \times (0, L)$  and  $T_\varepsilon := \cup_{i \in I_\varepsilon} T_\varepsilon^i$ . The set of thin parallel cylinders  $T_\varepsilon$  represents the fibers (see Figure 2.1 and Figure 2.2). The conductivity coefficient  $a_\varepsilon$  is

$$a_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in \Omega \setminus T_\varepsilon, \\ \lambda_\varepsilon, & \text{otherwise.} \end{cases}$$

We make the assumptions

$$r_\varepsilon \rightarrow 0, \quad \frac{r_\varepsilon}{\varepsilon} \rightarrow 0, \quad \lambda_\varepsilon \rightarrow +\infty, \quad k_\varepsilon := \lambda_\varepsilon \frac{r_\varepsilon^2}{\varepsilon^2} \rightarrow k, \quad k \geq 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (2.3)$$

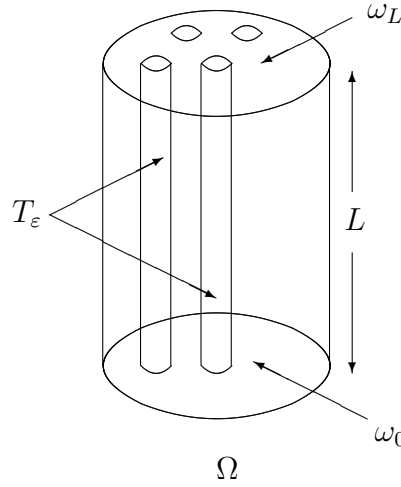


Figure 2.1: The domain  $\Omega = \omega \times (0, L)$  occupied by a composite material

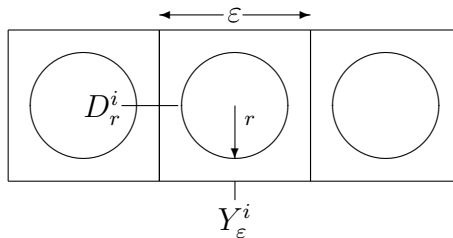


Figure 2.2: The circular cross section of the fiber  $Y_\varepsilon^i \subset \omega$

In [1], it was shown that the asymptotic limit of  $(\mathcal{P}_\varepsilon)$  is

$$(\mathcal{P}^{\text{eff}}) \quad \min \{ \Phi(u, v) - L(u) \mid (u, v) \in (L^p(\Omega))^2 \},$$

where

$$\Phi(u, v) = \begin{cases} \int_{\Omega} \phi_p(\nabla u) dx + \frac{k\pi}{p} \int_{\Omega} \left| \frac{\partial v}{\partial x_3} \right|^p dx + \frac{2\pi\gamma}{p} \int_{\Omega} |v - u|^p dx, \\ \text{if } \begin{cases} (u, v) \in W_{\Gamma_0}^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L)), \\ v = u_0 \text{ on } \Gamma_0 \cap (\omega_0 \cup \omega_L), \end{cases} \\ +\infty \quad \text{otherwise,} \end{cases} \quad (2.4)$$

and  $\gamma \in [0, +\infty]$ ,

$$\gamma = \begin{cases} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} |\log r_\varepsilon|^{-1}, & \text{if } p = 2, \\ \lim_{\varepsilon \rightarrow 0} \left| \frac{2-p}{p-1} \right|^{p-1} r_\varepsilon^{2-p} \varepsilon^{-2}, & \text{if } p \neq 2. \end{cases} \quad (2.5)$$

Here, the boundary data  $u_0$  is assumed to be Lipschitz in order to ensure that the infimum value of problem  $(\mathcal{P}_\varepsilon)$  remains finite as  $\varepsilon \rightarrow 0$ . In case  $k = +\infty$ , we add further assumption

$$k_\varepsilon r_\varepsilon \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \quad (2.6)$$

The conditions

$$k > 0 \quad \text{and} \quad \{ \gamma > 0 \text{ or } \omega_0 \subset \Gamma_0 \text{ or } \omega_L \subset \Gamma_0 \} \quad (2.7)$$

guarantee that the functional  $\Phi$  is coercive in  $W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L))$ .

We are concerned with the extension of this result to more general cross sections of the fibers and more general energy density than  $\phi_p$ . The aim of this paper is therefore to provide another proof that we expect to be more suitable to treat such general cases. The steps of the proof in [1] are to successively establish :

- (i) a compactness property of the sequence  $(u_\varepsilon)$  such that  $F_\varepsilon(u_\varepsilon) < C$ ,
- (ii) a lower bound inequality of the sequence  $(F_\varepsilon(u_\varepsilon))$ ,
- (iii) an upper bound inequality of the sequence  $(F_\varepsilon(u_\varepsilon))$ .

Here we replace the steps (ii) and (iii) by

- (ii') an upper bound equality of the sequence  $(F_\varepsilon(u_\varepsilon))$ ,
- (iii') a lower bound inequality of the sequence  $(F_\varepsilon(u_\varepsilon))$  which essentially uses a subdifferential inequality.

## 2.2 An Alternative Strategy

It consists, under (2.3), (2.5), (2.6) and (2.7), in proving the following three propositions. In the sequel, the symbols  $\rightarrow$ ,  $\rightharpoonup$  and  $\overset{*}{\rightharpoonup}$  stand for the strong convergence, the weak convergence and the weak star convergence, respectively. As usual, the letter  $C$  denotes various constants and for all  $\xi = (\xi_1, \xi_2, \xi_3)$  in  $\mathbb{R}^3$ ,  $\hat{\xi}$  stands for  $(\xi_1, \xi_2)$ .

**Proposition 2.1 (compactness property)** *Let  $(u_\varepsilon)$  be a sequence such that  $\sup F_\varepsilon(u_\varepsilon)$  is finite. Then  $(u_\varepsilon)$  is strongly relatively compact in  $L^p(\Omega)$  and  $(v_\varepsilon)$ , given by  $v_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u_\varepsilon$ , is bounded in  $L^1(\Omega)$  and, up to a subsequence,  $(v_\varepsilon)$  weakly\* converges in the space of bounded measures  $\mathcal{M}_b(\Omega)$  to an element  $v$  of  $L^p(\Omega)$ .*

**Proposition 2.2 (upper bound equality)** *For all  $(u, v)$  in  $(L^p(\Omega))^2$ , such that  $\Phi(u, v) < +\infty$ , there exists a sequence  $(u_\varepsilon)$  such that  $u_\varepsilon \rightarrow u$  in  $L^p(\Omega)$ ,  $v_\varepsilon \overset{*}{\rightharpoonup} v$  in  $\mathcal{M}_b(\Omega)$  and*

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \Phi(u, v).$$

**Proposition 2.3 (lower bound inequality)** *For all  $u$  in  $L^p(\Omega)$  and for all sequences  $(u_\varepsilon)$  such that  $u_\varepsilon \rightarrow u$  in  $L^p(\Omega)$ ,  $v_\varepsilon \overset{*}{\rightharpoonup} v$  in  $\mathcal{M}_b(\Omega)$ , one has :*

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

The proofs of these propositions are presented in the following sections.

### Proof of Proposition 2.1 :

Compactness property was already proved in [1].

□

### Proof of Proposition 2.2 :

Our sole contribution is to prove that we can replace inequality by equality, for that we use the same approximation  $u'_\varepsilon$  of  $u$  as in [1]

$$u'_\varepsilon = (1 - \theta_\varepsilon)u + \theta_\varepsilon w_\varepsilon.$$

The function  $\theta_\varepsilon$  is first defined on the closure of  $\omega_\varepsilon := \cup_{i \in I_\varepsilon} Y_\varepsilon^i$  as a  $(-\varepsilon/2, \varepsilon/2)^2$ -periodic continuous function which satisfies  $0 \leq \theta_\varepsilon \leq 1$ ,  $\theta_\varepsilon = 1$  on  $D_{r_\varepsilon} := \cup_{i \in I_\varepsilon} D_{r_\varepsilon}^i$ ,  $\theta_\varepsilon = 0$  on  $\bar{\omega}_\varepsilon \setminus \cup_{i \in I_\varepsilon} D_{R_\varepsilon}^i$ , where  $D_{R_\varepsilon}^i$  is the disk of  $\mathbb{R}^2$  centered at  $\hat{x}_\varepsilon^i$  of radius  $R_\varepsilon$  such that  $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$ . Next  $\theta_\varepsilon$  is assumed to vanish on  $\bar{\omega} \setminus \omega_\varepsilon$  and

$$w_\varepsilon(\hat{x}, x_3) = \sum_{i \in I_\varepsilon} \left( \frac{1}{|D_{r_\varepsilon}^i|} \int_{D_{r_\varepsilon}^i} v(\hat{y}, x_3) d\hat{y} \right) 1_{Y_\varepsilon^i}(\hat{x}).$$

The approximation  $u'_\varepsilon$  does not satisfy the boundary condition on  $\Gamma_0 \cap (\omega_0 \cup \omega_L)$  so that, as in [1], we introduce a sharper approximation

$$u_\varepsilon^\# := u\varphi_\varepsilon + u'_\varepsilon(1 - \varphi_\varepsilon).$$

Here  $\varphi_\varepsilon$  is a  $C^\infty(\bar{\Omega})$  function which satisfies  $\varphi_\varepsilon = 1$  on  $\Gamma_0$ ,  $\varphi_\varepsilon = 0$  on  $\bar{\Omega} \setminus \Sigma_\varepsilon$ ,  $|\nabla \varphi_\varepsilon| \leq C/r_\varepsilon$  on  $\bar{\Omega}$  where  $\Sigma_\varepsilon := \{x \in \Omega \mid \text{dist}(x, \Gamma_0) < r_\varepsilon\}$ . We assume that  $u$  and  $v$  are Lipschitz on  $\bar{\Omega}$  and there exists  $L > 0$  such that

$$\left| \frac{\partial v}{\partial x_3}(\hat{x}', x_3) - \frac{\partial v}{\partial x_3}(\hat{x}'', x_3) \right| < L|\hat{x}' - \hat{x}''| \quad \forall (\hat{x}', x_3), (\hat{x}'', x_3) \in \bar{\Omega}. \quad (2.8)$$

Letting  $\Psi$  be any continuous function on  $\bar{\Omega}$  such that  $0 \leq \Psi \leq 1$ , we introduce  $F_\varepsilon^\Psi$ ,  $\Phi^\Psi$  defined by similar formulae as the ones of  $F_\varepsilon$  and  $\Phi$  but with  $\Psi dx$  in place of  $dx$ . We will prove the lemma :

#### Lemma 2.4

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^\Psi(u'_\varepsilon) = \Phi^\Psi(u, v).$$

If Lemma 2.4 is proved, then, by a classical approximation process, we can deduce

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u'_\varepsilon) = \Phi(u, v). \quad (2.9)$$

Finally, we complete the proof of (2.9) for any  $(u, v)$  such that  $\Phi(u, v) < +\infty$  by approximation and diagonalization arguments.

**Proof.** We split  $F_\varepsilon^\Psi(u'_\varepsilon)$  in three parts

$$F_\varepsilon^\Psi(u'_\varepsilon) = F_\varepsilon^{\Psi 1}(u'_\varepsilon) + F_\varepsilon^{\Psi 2}(u'_\varepsilon) + F_\varepsilon^{\Psi 3}(u'_\varepsilon). \quad (2.10)$$

First, we consider

$$F_\varepsilon^{\Psi 1}(u'_\varepsilon) := \int_{\Omega \setminus B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u'_\varepsilon) \Psi dx = \int_{\Omega \setminus B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u) \Psi dx,$$

where  $B_\varepsilon := \cup_{i \in I_\varepsilon} D_{R_\varepsilon}^i \setminus \bar{D}_{r_\varepsilon}^i \times (0, L)$ . Hence, the assumption  $R_\varepsilon \ll \varepsilon$  yields  $\lim_{\varepsilon \rightarrow 0} |B_\varepsilon \cup T_\varepsilon| = 0$  and, consequently,

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi 1}(u'_\varepsilon) = \int_{\Omega} \phi_p(\nabla u) \Psi \, dx.$$

Next, we pay attention to

$$F_\varepsilon^{\Psi 2}(u'_\varepsilon) := \int_{B_\varepsilon} \phi_p(\nabla u'_\varepsilon) \Psi \, dx.$$

Writing

$$z_\varepsilon := (v - u) \widehat{\nabla} \theta_\varepsilon, \quad (2.11)$$

we obtain

$$\nabla u'_\varepsilon = z_\varepsilon + (w_\varepsilon - v) \nabla \theta_\varepsilon + (1 - \theta_\varepsilon) \nabla u + \theta_\varepsilon \nabla w_\varepsilon.$$

Let us show

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon)) \Psi \, dx = 0. \quad (2.12)$$

The function  $\phi_p$ , being convex and positively homogeneous of degree  $p$ , satisfies

$$\forall \xi, \eta \in \mathbb{R}^n, \quad n = 1, 2, 3, \quad |\phi_p(\xi) - \phi_p(\eta)| \leq C |\xi - \eta| (|\xi|^{p-1} + |\eta|^{p-1}). \quad (2.13)$$

Therefore, Hölder inequality yields

$$\begin{aligned} & \left| \int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon)) \Psi \, dx \right| \\ & \leq C \left( \int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p \, dx \right)^{\frac{1}{p}} \left( \int_{B_\varepsilon} |\nabla u'_\varepsilon|^p \, dx + \int_{B_\varepsilon} |z_\varepsilon|^p \, dx \right)^{\frac{1}{p'}}. \end{aligned}$$

The smoothness of  $(u, v)$  implies

$$\left. \begin{aligned} u'_\varepsilon &= u \text{ on } \Omega \setminus (B_\varepsilon \cup T_\varepsilon), \quad |u'_\varepsilon| \leq C \text{ on } \Omega, \quad |\nabla w_\varepsilon| \leq C \text{ on } B_\varepsilon, \\ u'_\varepsilon &= w_\varepsilon \text{ on } T_\varepsilon, \quad |w_\varepsilon - v| \leq CR_\varepsilon \text{ on } B_\varepsilon, \end{aligned} \right\} \quad (2.14)$$

so that

$$\begin{aligned} & \int_{B_\varepsilon} |\nabla u'_\varepsilon|^p \, dx + \int_{B_\varepsilon} |z_\varepsilon|^p \, dx \leq C \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \theta_\varepsilon) \, d\hat{x}, \\ & \int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p \, dx \leq CR_\varepsilon^p \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \theta_\varepsilon) \, d\hat{x}, \end{aligned}$$

where  $D(r_\varepsilon, R_\varepsilon) := \{ \hat{x} \in \mathbb{R}^2 \mid r_\varepsilon < |\hat{x}| < R_\varepsilon \}$ . Hence, if we choose  $\theta_\varepsilon$  such that

$$\exists M > 0; \quad \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \theta_\varepsilon) \, d\hat{x} \leq M \quad \forall \varepsilon > 0, \quad (2.15)$$

then (2.12) is true. We finally have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi^2}(u'_\varepsilon) \\
&= \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi_p(z_\varepsilon) \Psi \, dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} |v - u|^p \phi_p(\widehat{\nabla} \theta_\varepsilon) \Psi \, dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \theta_\varepsilon) \, d\hat{x} \int_0^L \sum_{i \in I_\varepsilon} |v - u|^p(\hat{y}_\varepsilon^i, x_3) \Psi(\hat{y}_\varepsilon^i, x_3) \, dx_3 \quad (\text{with } \hat{y}_\varepsilon^i \in Y_\varepsilon^i) \\
&= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \theta_\varepsilon) \, d\hat{x} \int_0^L \sum_{i \in I_\varepsilon} |Y_\varepsilon^i| |v - u|^p(\hat{y}_\varepsilon^i, x_3) \Psi(\hat{y}_\varepsilon^i, x_3) \, dx_3.
\end{aligned}$$

Observe that  $\lim_{\varepsilon \rightarrow 0} \int_0^L \sum_{i \in I_\varepsilon} |Y_\varepsilon^i| |v - u|^p(\hat{y}_\varepsilon^i, x_3) \Psi(\hat{y}_\varepsilon^i, x_3) \, dx_3 = \int_\Omega |v - u|^p \Psi \, dx$ .

To get the lowest upper bound in Proposition 2.2, it is clear that  $\theta_\varepsilon$  has to be the solution of the capacity problem

$$(\mathcal{P}_\varepsilon^{\text{cap}}) \quad \min \left\{ \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \varphi) \, d\hat{x} \left| \begin{array}{l} \varphi \in W^{1,p}(D(r_\varepsilon, R_\varepsilon)), \\ \varphi(\hat{x}) = 1 \text{ on } |\hat{x}| = r_\varepsilon, \\ \varphi(\hat{x}) = 0 \text{ on } |\hat{x}| = R_\varepsilon. \end{array} \right. \right\}$$

As observed in [1], we have

$$\theta_\varepsilon = \begin{cases} \frac{\log R_\varepsilon - \log |\hat{x}|}{\log R_\varepsilon - \log r_\varepsilon}, & \text{if } p = 2, \\ \frac{R_\varepsilon^s - |\hat{x}|^s}{R_\varepsilon^s - r_\varepsilon^s}, & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}) \end{cases}$$

and

$$\int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \theta_\varepsilon) \, d\hat{x} = \frac{2\pi}{p} \Gamma_p(r_\varepsilon, R_\varepsilon),$$

where

$$\Gamma_p(r_\varepsilon, R_\varepsilon) := \begin{cases} \frac{1}{\log R_\varepsilon - \log r_\varepsilon}, & \text{if } p = 2, \\ \left( \frac{s}{R_\varepsilon^s - r_\varepsilon^s} \right)^{p-1}, & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}). \end{cases}$$

Note that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \Gamma_p(r_\varepsilon, R_\varepsilon) = \gamma.$$

If  $\gamma < +\infty$ , then (2.15) is satisfied and

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi^2}(u'_\varepsilon) = \frac{2\pi\gamma}{p} \int_\Omega |v - u|^p \Psi \, dx.$$

When  $\gamma = +\infty$ , it suffices to prove that  $\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi^2}(u'_\varepsilon) = 0$ . Due to (2.14), the result is a consequence of  $F_\varepsilon^{\Psi^2}(u'_\varepsilon) \leq CR_\varepsilon^p \varepsilon^{-2} \Gamma_p(r_\varepsilon, R_\varepsilon)$ , which tends to zero.

Now, we consider the remaining part

$$F_\varepsilon^{\Psi 3}(u'_\varepsilon) := \int_{T_\varepsilon} \lambda_\varepsilon \phi_p(\nabla w_\varepsilon) \Psi \, dx.$$

Recalling the assumption (2.8) on  $v$  and using the local Lipschitz property (2.13), we deduce

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi 3}(u'_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|T_\varepsilon|} \int_{T_\varepsilon} \lambda_\varepsilon \phi_p \left( \frac{\partial v}{\partial x_3} \right) \Psi \, dx \\ &= \frac{k\pi}{p} \int_{\Omega} \left| \frac{\partial v}{\partial x_3} \right|^p \Psi \, dx, \end{aligned}$$

as shown in [1].

□

Now, we will prove the upper bound equality by using the sharper approximation ( $u_\varepsilon^\#$ ). We start with

$$F_\varepsilon(u_\varepsilon^\#) = \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) \, dx + \int_{\Omega \setminus \Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) \, dx. \quad (2.16)$$

Conditions (2.14) imply  $|u'_\varepsilon - u| \leq C(r_\varepsilon 1_{T_\varepsilon} + R_\varepsilon 1_{B_\varepsilon})$ . Hence

$$\int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) \, dx \leq C \left( |\Sigma_\varepsilon| + \int_{\Sigma_\varepsilon} a_\varepsilon(x) |\nabla u'_\varepsilon|^p \, dx + \lambda_\varepsilon |T_\varepsilon \cap \Sigma_\varepsilon| + \left( \frac{R_\varepsilon}{r_\varepsilon} \right)^p |\Sigma_\varepsilon| \right).$$

Lemma 2.4 implies that for every  $\Psi \in C_0(\bar{\Omega}, [0, 1])$ , such that  $\Psi = 1$  on a small neighborhood of  $\Gamma_0 \cap (\omega_0 \cup \omega_L)$ ,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) \, dx &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} a_\varepsilon \phi_p(\nabla u'_\varepsilon) \Psi \, dx \\ &= \int_{\Omega} \left( \phi_p(\nabla u) + \frac{k\pi}{p} \left| \frac{\partial v}{\partial x_3} \right|^p + \frac{2\pi\gamma}{p} |v - u|^p \right) \Psi \, dx. \end{aligned}$$

Thus, by letting  $\Psi$  tend to zero, we deduce

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) \, dx = 0,$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) \, dx &= \lim_{\varepsilon \rightarrow 0} \left( \int_{\Omega} a_\varepsilon \phi_p(\nabla u'_\varepsilon) \, dx - \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) \, dx \right) \\ &= \int_{\Omega} \phi_p(\nabla u) + \frac{k\pi}{p} \left| \frac{\partial v}{\partial x_3} \right|^p + \frac{2\pi\gamma}{p} |v - u|^p \, dx, \end{aligned}$$

which proves the result for  $(u, v)$  smooth. We complete the proof by a standard approximation of  $(u, v)$  and a diagonalization argument [2].

□

### Proof of Proposition 2.3

It is enough to consider  $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) < +\infty$ . Due to the compactness property,  $(u, v)$  is in  $(L^p(\Omega))^2$ . We first consider the term  $F_\varepsilon^2(u_\varepsilon)$ . Let  $(u_\eta, v_\eta)$  be Lipschitz on  $\bar{\Omega}$  such that  $\lim_{\eta \rightarrow 0} \|u_\eta - u\|_{L^p(\Omega)} + \|v_\eta - v\|_{L^p(\Omega)} = 0$ . We define  $(v_\eta - u_\eta)_\varepsilon := \sum_{i \in I_\varepsilon} (v_\eta - u_\eta)(\hat{x}_\varepsilon^i, x_3) 1_{Y_\varepsilon^i}$  and  $z_{\eta\varepsilon} := (v_\eta - u_\eta)_\varepsilon \widehat{\nabla} \theta_\varepsilon$ . Because of the local Lipschitz property (2.13) of  $\phi_p$  and  $(u, v) \in (L^p(\Omega))^2$ , Hölder inequality implies

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} (\phi_p(z_{\eta\varepsilon}) - \phi_p(z_\varepsilon)) dx = 0.$$

The proof of the upper bound equality allows us to write

$$\lim_{\varepsilon \rightarrow 0} \phi_p(z_{\eta\varepsilon}) = \frac{2\pi\gamma}{p} \int_{\Omega} |v_\eta - u_\eta|^p dx.$$

The convexity of  $\phi_p$  and the fact that  $\phi_p(\nabla u_\varepsilon) \geq \phi_p(\widehat{\nabla} u_\varepsilon)$  yield

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^2(u_\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi_p(\widehat{\nabla} u_\varepsilon) dx \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi_p(z_{\eta\varepsilon}) dx \\ &\quad + \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot (\widehat{\nabla} u_\varepsilon - z_{\eta\varepsilon}) dx. \end{aligned} \quad (2.17)$$

The very definition of  $\phi_p$  implies

$$\begin{aligned} \phi'_p(\xi) &= |\xi|^{p-2} \xi && \forall \xi \in \mathbb{R}^n, \quad n = 1, 2, 3, \\ \phi'_p(t\xi) &= \phi'_p(t) \phi'_p(\xi) && \forall (t, \xi) \in \mathbb{R} \times \mathbb{R}^n, \quad n = 1, 2, 3, \\ \phi'_p(\xi) \cdot \xi &= p\phi_p(\xi) && \forall \xi \in \mathbb{R}^n, \quad n = 1, 2, 3. \end{aligned}$$

Hence

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot z_{\eta\varepsilon} dx = 2\pi\gamma \int_{\Omega} |v_\eta - u_\eta|^p dx. \quad (2.18)$$

For the other term of (2.17), we have

$$\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot \widehat{\nabla} u_\varepsilon dx = \sum_{i \in I_\varepsilon} \int_0^L \phi'_p(v_\eta - u_\eta)(\hat{x}_\varepsilon^i, x_3) \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \widehat{\nabla} u_\varepsilon d\hat{x} dx_3,$$



where  $D^i(r_\varepsilon, R_\varepsilon) = D_{R_\varepsilon}^i \setminus \bar{D}_{r_\varepsilon}^i$ . Let  $\nu$  be the outer normal on  $\partial D^i(r_\varepsilon, R_\varepsilon)$ , the very definition of  $\theta_\varepsilon$  as a solution of  $(\mathcal{P}_\varepsilon^{\text{cap}})$  yields

$$\begin{aligned} \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \widehat{\nabla}u_\varepsilon d\hat{x} &= \int_{\partial D^i(r_\varepsilon, R_\varepsilon)} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) u_\varepsilon dl \\ &= \int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) u_\varepsilon dl + \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) u_\varepsilon dl \\ &= -\tilde{u}_\varepsilon^i \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) dl + \tilde{v}_\varepsilon^i \int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) u_\varepsilon dl \end{aligned}$$

where  $\tilde{u}_\varepsilon^i := \frac{\int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) u_\varepsilon dl}{\int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) dl} = \frac{1}{2\pi R_\varepsilon} \int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) u_\varepsilon dl$ ,  $\tilde{u}_\varepsilon := \sum_{i \in I_\varepsilon} \tilde{u}_\varepsilon^i 1_{Y_\varepsilon^i}$ ,  $\tilde{v}_\varepsilon^i := \frac{\int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) v_\varepsilon dl}{\int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) dl} = \frac{1}{2\pi r_\varepsilon} \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) v_\varepsilon dl$ ,  $\tilde{v}_\varepsilon := \sum_{i \in I_\varepsilon} \tilde{v}_\varepsilon^i 1_{Y_\varepsilon^i}$ . Thus,

$$\begin{aligned} \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \widehat{\nabla}u_\varepsilon d\hat{x} &= (\tilde{v}_\varepsilon^i - \tilde{u}_\varepsilon^i) \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \nu) dl \\ &= (\tilde{v}_\varepsilon^i - \tilde{u}_\varepsilon^i) \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla}\theta_\varepsilon) \cdot \widehat{\nabla}\theta_\varepsilon d\hat{x} \\ &= 2\pi\Gamma_p(r_\varepsilon, R_\varepsilon)(\tilde{v}_\varepsilon^i - \tilde{u}_\varepsilon^i), \end{aligned}$$

and

$$\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot \widehat{\nabla}u_\varepsilon dx = 2\pi\Gamma_p(r_\varepsilon, R_\varepsilon) \int_{\Omega} \phi'_p((v_\eta - u_\eta)_\varepsilon)(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon) dx.$$

It was shown in [1] that  $(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon) \rightharpoonup (v - u)$  in  $L^p(\Omega)$ . On the other hand,  $(v_\eta - u_\eta)$  being smooth and  $\phi'_p$  being continuous from  $L^p(\Omega)$  to  $L^{p'}(\Omega)$ , we have  $\phi'_p((v_\eta - u_\eta)_\varepsilon) \rightarrow \phi'_p(v_\eta - u_\eta)$  in  $L^{p'}(\Omega)$ . Hence,

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot \widehat{\nabla}u_\varepsilon dx = 2\pi\gamma \int_{\Omega} \phi'_p(v_\eta - u_\eta)(v - u) dx. \quad (2.19)$$

Therefore, (2.17), (2.18) and (2.19) imply

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^2(u_\varepsilon) &\geq \frac{2\pi\gamma}{p} \int_{\Omega} |v_\eta - u_\eta|^p dx \\ &\quad + 2\pi\gamma \left[ \int_{\Omega} |v_\eta - u_\eta|^p dx - \int_{\Omega} \phi'_p(v_\eta - u_\eta)(v - u) dx \right]. \end{aligned}$$

The expected lower bound for  $F_\varepsilon^2(u_\varepsilon)$  is obtained by letting  $\eta$  tend to zero.

To complete the proof it suffices to use the arguments of [1] concerning the lower bounds for  $F_\varepsilon^1(u_\varepsilon)$ ,  $F_\varepsilon^3(u_\varepsilon)$  and the fact that  $v$  belongs to  $L^p(\omega, W^{1,p}(0, L))$ .

□

### The Final Result

The following theorem, a convergence result for the minimizer of  $(\mathcal{P}_\varepsilon)$ , is a standard consequence of the previous three propositions.

**Theorem 2.5** *Let the assumptions (2.3) and (2.5) hold with  $(k, \gamma) \in (0, +\infty)^2$ . Then the unique solution  $\bar{u}_\varepsilon$  of  $(\mathcal{P}_\varepsilon)$  converges weakly in  $W^{1,p}(\Omega)$  to the unique solution  $\bar{u}$  of the problem*

$$\min \left\{ \min \{ \Phi(u, v) - L(u) \mid v \in L^p(\Omega) \} \mid u \in L^p(\Omega) \right\},$$

where  $\Phi$  and  $L$  are defined by (2.4) and (2.2) respectively.

**Proof.** The proof of this theorem is the same as that in [1].

□

### 2.3 Conclusions and Remarks

The previous analysis can be easily extended to the case when  $\phi_p$  is replaced by any strictly convex function which satisfies

$$\exists M > 0, \exists r \in (1, p); \quad |W(\xi) - \phi_p(\xi)| \leq M|\xi|^r \quad \forall \xi \in \mathbb{R}^3, \quad (2.20)$$

the density function associated with  $\Phi(u, v)$  becomes

$$W(\nabla u) + 2\pi\gamma|v - u|^p + W\left(\frac{\partial v}{\partial x_3}\right).$$

Indeed, (2.20) and Hölder inequality imply

$$\left| \int_{B_\varepsilon} W(\nabla u_\varepsilon) dx - \int_{B_\varepsilon} \phi_p(\nabla u_\varepsilon) dx \right| \leq M|B_\varepsilon|^{1-\frac{r}{p}} \int_{\Omega} |\nabla u_\varepsilon|^p dx,$$

while our arguments and those of [1] to derive the upper bound and lower bound respectively are valid when  $\phi_p$  is replaced by any convex function satisfying a growth condition like

$$\exists \alpha, \beta > 0; \quad \alpha(|\xi|^p - 1) \leq W(\xi) \leq \beta(1 + |\xi|^p) \quad \forall \xi \in \mathbb{R}^3,$$

which is an obvious consequence of (2.20).

Eventually, the key arguments of our analysis are the identification of  $\gamma, \theta_\varepsilon$  in terms of the solution of capacitary problems and the use of the  $p$ -positive homogeneity and convexity of  $\phi_p$  and of the fact that  $\phi_p(\xi) \geq \phi_p(\hat{\xi})$ . Thus, it is easy to guess what could be  $\Phi(u, v)$ , when  $\phi_p$  is replaced by any strictly convex function and when the cross sections of the fibers are smooth star-shaped domains of  $\mathbb{R}^2$ . We hope that our proposed strategy will be able to reduce and overcome the involved technical difficulties.

*Note that this chapter was the subject of the following article :*

S. Orankitjaroen, N. Sontichai, C. Licht, A. Kananthai, *Mathematical Modeling of Fiber Reinforced Structures by Homogenization*, Thai Journal of Mathematics, Special Issue Annual Meeting in Mathematics (2008), p.103–115.

## REFERENCES

- [1] M. Bellieud, G. Bouchitté, *Homogenization of Elliptic Problems in a Fiber Reinforced Structure. Non local effects*, Ann. Scuola Norm. Sup. CI Sci. (4) 26 (1998), p.407-432.
- [2] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, 1985.

## CHAPTER III

### A REMARK ON THE HOMOGENIZATION OF A MICROFIBERED LINEARLY ELASTIC MATERIAL

#### 3.1 Setting the Problem

We intend to study the macroscopic behavior of a cylindrical micro-fibered structure made of a linearly isotropic elastic matrix surrounding a periodic distribution of very thin linearly isotropic elastic fibers of very high stiffness. As usual, we make no difference between the real physical space and  $\mathbb{R}^3$  whose orthonormal basis is denoted by  $\{e^\alpha\}$  and, for all  $\xi = (\xi_1, \xi_2, \xi_3)$  of  $\mathbb{R}^3$ ,  $\hat{\xi}$  stands for  $(\xi_1, \xi_2)$ . Let  $\omega$  a bounded domain of  $\mathbb{R}^2$ , containing the origin, with a Lipschitz continuous boundary  $\partial\omega$  and  $L$  a positive number so that  $\Omega := \omega \times (0, L)$  is a reference configuration of the fibered structure which can be described as follows (see Figure 3.1).

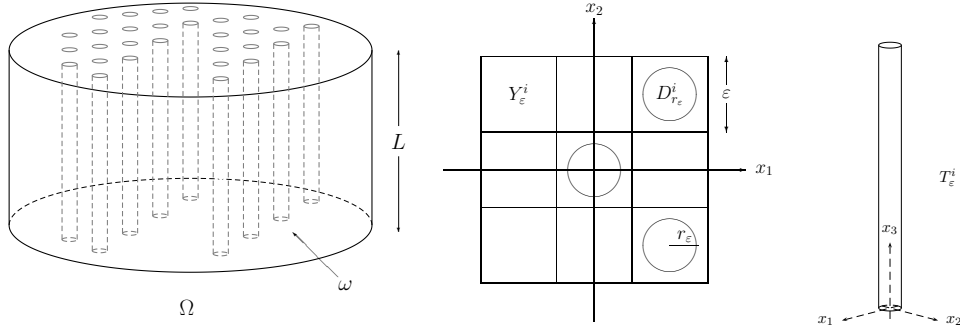


Figure 3.1: The fibered structure

For each  $\varepsilon > 0$ ,  $(Y_\varepsilon^i)_{i \in I_\varepsilon}$ , where  $Y_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$  and  $I_\varepsilon := \{i \in \mathbb{Z}^2 \mid Y_\varepsilon^i \subset \omega\}$ , denotes a periodic distribution of cells. Let  $(D_{r_\varepsilon}^i)_{i \in I_\varepsilon}$  the family of disk of  $\mathbb{R}^2$  centered at  $\hat{x}_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2)$  of radius  $r_\varepsilon \ll \varepsilon$ , and  $T_\varepsilon^i := D_{r_\varepsilon}^i \times (0, L)$ . The set  $T_\varepsilon := \cup_{i \in I_\varepsilon} T_\varepsilon^i$  of thin parallel cylinders is the domain occupied by the fibers.

The Lamé coefficients  $\lambda_\varepsilon$  and  $\mu_\varepsilon$  of the structure are such that

$$\lambda_\varepsilon(x) = \begin{cases} \lambda_0 > 0 & \text{if } x \in \Omega \setminus T_\varepsilon, \\ \lambda_{\varepsilon 1} & \text{otherwise,} \end{cases} \quad \mu_\varepsilon(x) = \begin{cases} \mu_0 > 0 & \text{if } x \in \Omega \setminus T_\varepsilon, \\ \mu_{\varepsilon 1} & \text{otherwise.} \end{cases}$$

The structure is clamped on the part  $\Gamma_0 := \omega \times \{0, L\}$  of the boundary  $\partial\Omega$  of  $\Omega$ , subjected to body forces of density  $f$  and to surface forces of density  $g$  on  $\Gamma_1 := \partial\Omega \setminus \Gamma_0$ .

The problem of finding the equilibrium configuration of the structure reads as

$$\begin{cases} -\operatorname{div} \sigma_\varepsilon = f & \text{in } \Omega, \\ \sigma_\varepsilon := \lambda_\varepsilon \operatorname{tr} e(u_\varepsilon) I + 2\mu_\varepsilon e(u_\varepsilon), & e(u_\varepsilon) = (\nabla u_\varepsilon)_s := \frac{1}{2}(\nabla u_\varepsilon + \nabla u_\varepsilon^T), \\ u_\varepsilon = 0 & \text{on } \Gamma_0, \\ \sigma_\varepsilon n = g & \text{on } \Gamma_1. \end{cases} \quad (3.1)$$

where  $u_\varepsilon, \sigma_\varepsilon$  denote the displacement and stress fields and  $n$  is the unit outward normal. It is well-known that if  $\lambda_{\varepsilon_1}, \mu_{\varepsilon_1} > 0$ ,  $f \in L^2(\Omega; \mathbb{R}^3)$ ,  $g \in L^2(\Gamma_1; \mathbb{R}^3)$ , then the problem, which can also be written

$$(\mathcal{P}_\varepsilon) \quad \min \{ F_\varepsilon(w) - L(w) \mid w \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \}, \quad (3.2)$$

where

$$F_\varepsilon(w) := \int_\Omega W_\varepsilon(e(w)) dx, \quad L(w) := \int_\Omega f \cdot w dx + \int_{\Gamma_1} g \cdot w ds,$$

and

$$W_\varepsilon(e) := \frac{1}{2} \lambda_\varepsilon \operatorname{tr}^2 e + \mu_\varepsilon |e|^2 \quad \forall e \in \mathbb{S}^3 \text{ the space of symmetric } 3 \times 3 \text{ matrices,}$$

$$H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) = \{ v \in H^1(\Omega; \mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_0 \}$$

has a unique solution  $\bar{u}_\varepsilon$ .

To determine the macroscopic (or effective) behavior of the microfibered structure, we aim to study the asymptotic behavior of  $\bar{u}_\varepsilon$  when  $\varepsilon$  goes to zero. Let

$$k_\varepsilon := \mu_{\varepsilon_1} \frac{|T_\varepsilon|}{|\Omega|}, \quad l_\varepsilon := \frac{\lambda_{\varepsilon_1}}{\mu_{\varepsilon_1}},$$

and assume that, as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} r_\varepsilon \rightarrow 0, \quad \frac{r_\varepsilon}{\varepsilon} \rightarrow 0, \quad \lambda_{\varepsilon_1} \rightarrow +\infty, \quad \mu_{\varepsilon_1} \rightarrow +\infty, \\ k_\varepsilon \rightarrow k \in [0, +\infty], \quad r_\varepsilon^2 k_\varepsilon \rightarrow \kappa \in [0, +\infty], \quad l_\varepsilon \rightarrow l \in [0, +\infty), \\ (\varepsilon^2 |\ln r_\varepsilon|)^{-1} \rightarrow \gamma \in [0, +\infty). \end{aligned} \quad (3.3)$$

Let  $1_{T_\varepsilon}$  the characteristic function of  $T_\varepsilon$  and  $\mathcal{M}_b(\Omega; \mathbb{R}^3)$  the space of bounded  $\mathbb{R}^3$ -valued measures in  $\Omega$ , it was proven in [1] that, as  $\varepsilon$  tends to zero,  $\bar{u}_\varepsilon$  weakly converges in  $H^1(\Omega; \mathbb{R}^3)$  toward  $\bar{u}$  and  $\bar{v}_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} \bar{u}_\varepsilon 1_{T_\varepsilon}$  weakly\* converges in  $\mathcal{M}_b(\Omega; \mathbb{R}^3)$  toward an element  $\bar{v}$  of  $L^2(\Omega; \mathbb{R}^3)$  solving

$$(\mathcal{P}^{\text{eff}}) \quad \min \{ \Phi(u, v) - L(u) \mid (u, v) \in L^2(\Omega; \mathbb{R}^3)^2 \},$$

with

$$\Phi(u, v) = \begin{cases} \int_{\Omega} W_0(e(u)) dx \\ + \mu_0 \pi \gamma \int_{\Omega} (v - u)^T \begin{bmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{bmatrix} (v - u) dx & \text{if } (u, v) \in \mathcal{D}, \\ + \frac{1}{2} \frac{3l+2}{2(l+1)} k \int_{\Omega} \left| \frac{\partial v_3}{\partial x_3} \right|^2 dx \\ + \frac{1}{2} \frac{3l+2}{2(l+1)} \frac{\kappa}{4} \int_{\Omega} \left| \frac{\partial^2 v_1}{\partial x_3^2} \right|^2 + \left| \frac{\partial^2 v_2}{\partial x_3^2} \right|^2 dx, & \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$W_0(e(u)) := \frac{1}{2} \lambda_0 \text{tr}^2 e(u) + \mu_0 |e(u)|^2, \quad \chi := \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0},$$

$$\mathcal{D} := H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \times \left\{ v \in L^2(\omega, H_0^2(0, L; \mathbb{R}^3)) \mid \frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = 0 \text{ on } \Gamma_0 \right\}.$$

Thus, the macroscopic behavior of the micro-fibered structure is the one of a so-called generalized elastic continuum medium involving an additional state variable and its first two derivatives. This additional state variable accounts for the microstructure in the extent where it describes the asymptotic behavior of a suitable scaling of the displacement field in the fibers. Our main concern is to understand this result more deeply and in a more general setting e.g., a different cross-section of the fibers, a more general behavior of the matrix or the fibers. Nevertheless, here, we confine to give another proof of the result of [1] by directly studying the variational convergence (as in the scalar case [2]) of  $F_\varepsilon$  and shall divide our proof into three steps :

1. a compactness property for each sequence  $(u_\varepsilon)$  such that  $F_\varepsilon(u_\varepsilon)$  is bounded,

2. an upper bound equality for the sequence  $(F_\varepsilon(u_\varepsilon))$ ,
3. a lower bound inequality for the sequence  $(F_\varepsilon(u_\varepsilon))$ .

### 3.2 A Different Approach

Actually, the result of [1] is a standard consequence [3] of the following three propositions :

**Proposition 3.1 (compactness property)** *Let  $(u_\varepsilon)$  be a sequence such that  $\sup_\varepsilon F_\varepsilon(u_\varepsilon)$  is finite. Then  $(u_\varepsilon)$  is strongly relatively compact in  $L^2(\Omega; \mathbb{R}^3)$  and  $(v_\varepsilon)$  is bounded in  $L^1(\Omega; \mathbb{R}^3)$  and, up to a subsequence,  $(v_\varepsilon)$  weakly\* converges in  $\mathcal{M}_b(\Omega; \mathbb{R}^3)$  to an element  $v$  of  $L^2(\Omega; \mathbb{R}^3)$ .*

**Proposition 3.2 (upper bound equality)** *For all  $(u, v)$  in  $L^2(\Omega; \mathbb{R}^3)^2$  with  $\Phi(u, v) < +\infty$ , there exists a sequence  $(u_\varepsilon)$  such that  $u_\varepsilon \rightarrow u$  in  $L^2(\Omega; \mathbb{R}^3)$ ,  $v_\varepsilon \xrightarrow{*} v$  in  $\mathcal{M}_b(\Omega; \mathbb{R}^3)$  and*

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \Phi(u, v).$$

**Proposition 3.3 (lower bound inequality)** *For all  $u$  in  $L^2(\Omega; \mathbb{R}^3)$  and for all sequence  $(u_\varepsilon)$  such that  $u_\varepsilon \rightarrow u$  in  $L^2(\Omega; \mathbb{R}^3)$ ,  $v_\varepsilon \xrightarrow{*} v$  in  $\mathcal{M}_b(\Omega; \mathbb{R}^3)$ , one has :*

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

The convergence symbols  $\rightarrow$ ,  $\rightharpoonup$  and  $\xrightarrow{*}$  are used for the strong convergence, the weak convergence and the weak\* convergence, respectively. The proof of these propositions are presented in the following subsections, where, as a common practice,  $C$  denote various constants which may vary from line to line.

#### Proof of Proposition 3.1

A proof of this proposition can be found in [1].

□



**Proof of Proposition 3.2**

We split  $F_\varepsilon$  into three parts :

$$F_\varepsilon^1(w) := \int_{\Omega \setminus (B_\varepsilon \cup T_\varepsilon)} W_\varepsilon(e(w)) dx,$$

$$F_\varepsilon^2(w) := \int_{B_\varepsilon} W_\varepsilon(e(w)) dx, \quad F_\varepsilon^3(w) := \int_{T_\varepsilon} W_\varepsilon(e(w)) dx,$$

where  $B_\varepsilon := (D_{R_\varepsilon} \setminus \overline{D_{r_\varepsilon}}) \times (0, L)$ ,  $D_{r_\varepsilon} := \cup_{i \in I_\varepsilon} D_{r_\varepsilon}^i$ ,  $D_{R_\varepsilon} := \cup_{i \in I_\varepsilon} D_{R_\varepsilon}^i$ ,  $D_{R_\varepsilon}^i$  is the disk of  $\mathbb{R}^2$  centered at  $\hat{x}_\varepsilon^i$  of radius  $R_\varepsilon$  such that  $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$ . We point out that our proof is in the same spirit as that of [2], where the main ingredient stems from [1] and essentially confine to the convergence of  $(F_\varepsilon^2)$ . We first assume  $u$  and  $v$  to be smooth on  $\overline{\Omega}$  and construct an  $L^2$ -approximation  $u_\varepsilon$  of  $u$  by :

$$u_\varepsilon = \sum_{\alpha=1}^3 (u_\alpha(e^\alpha - \theta_\varepsilon^\alpha) + \bar{w}_{\varepsilon\alpha} \theta_\varepsilon^\alpha). \quad (3.4)$$

Here, for each  $\alpha = 1, 2, 3$ , the vector field  $\theta_\varepsilon^\alpha$  is first defined on the closure of  $\omega_\varepsilon := \cup_{i \in I_\varepsilon} Y_\varepsilon^i$  as a  $(-\varepsilon/2, \varepsilon/2)^2$ -periodic element of  $H^1(\Omega; \mathbb{R}^3)$  which does not depend on  $x_3$  and satisfies  $\theta_\varepsilon^\alpha = e^\alpha$  on  $D_{r_\varepsilon}$ ,  $\theta_\varepsilon^\alpha = 0$  on  $\bar{\omega}_\varepsilon \setminus D_{R_\varepsilon}$ . Next  $\theta_\varepsilon^\alpha$  is assumed to vanish on  $\bar{\omega} \setminus \omega_\varepsilon$  and

$$\bar{w}_\varepsilon(x) = w_\varepsilon(x) + V_\varepsilon(x) \quad (3.5)$$

where

$$w_\varepsilon(\hat{x}, x_3) = \sum_{i \in I_\varepsilon} \left( \frac{1}{|D_{r_\varepsilon}^i|} \int_{D_{r_\varepsilon}^i} v(\hat{y}, x_3) d\hat{y} \right) 1_{Y_\varepsilon^i}(\hat{x}),$$

and  $V_\varepsilon$  stems from  $w_\varepsilon$  in such a way that  $F_\varepsilon^3(u_\varepsilon)$  converges. The true expressions of  $V_\varepsilon$  can be found in [1] (formula (5.16) and (5.52) with  $\psi$  and  $\varphi$  in place of  $v$  and  $u$ ).

As  $R_\varepsilon \ll \varepsilon$  implies  $\lim_{\varepsilon \rightarrow 0} |B_\varepsilon \cup T_\varepsilon| = 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^1(u_\varepsilon) = \int_{\Omega} W_0(e(u)) dx.$$

To find the limit of  $F_\varepsilon^2(u_\varepsilon)$ , we introduce

$$z_\varepsilon := \sum_{\alpha=1}^3 (v - u)_\alpha e(\theta_\varepsilon^\alpha), \quad (3.6)$$

and compute the linearized strain of  $u_\varepsilon$  from (3.4) :

$$e(u_\varepsilon) = z_\varepsilon + e(u) + \sum_{\alpha=1}^3 [(\bar{w}_{\varepsilon\alpha} - v_\alpha) e(\theta_\varepsilon^\alpha) + (\theta_\varepsilon^\alpha \otimes_s \nabla(\bar{w}_\varepsilon - u)_\alpha)].$$

We claim that a good choice of  $\theta_\varepsilon^\alpha$  yields

$$\lim_{\varepsilon \rightarrow 0} \left( \int_{B_\varepsilon} W_0(e(u_\varepsilon)) dx - \int_{B_\varepsilon} W_0(z_\varepsilon) dx \right) = 0.$$

Note that  $W_0$ , being convex and positively homogeneous of degree 2, satisfies (see [4]) :

$$\forall \xi, \eta \in \mathbb{S}^3, \quad |W_0(\xi) - W_0(\eta)| \leq C|\xi - \eta|(|\xi| + |\eta|) \quad (3.7)$$

so that Cauchy-Schwarz inequality implies

$$\begin{aligned} & \left| \int_{B_\varepsilon} W_0(e(u_\varepsilon)) - W_0(z_\varepsilon) dx \right| \\ & \leq C \left( \int_{B_\varepsilon} |e(u_\varepsilon) - z_\varepsilon|^2 dx \right)^{1/2} \left( \int_{B_\varepsilon} |e(u_\varepsilon)|^2 dx + \int_{B_\varepsilon} |z_\varepsilon|^2 dx \right)^{1/2}. \end{aligned}$$

Because  $u$  and  $v$  are smooth, we have

$$|\nabla \bar{w}_\varepsilon| \leq C \text{ on } B_\varepsilon, \quad |\bar{w}_\varepsilon - v| \leq CR_\varepsilon \text{ on } B_\varepsilon,$$

consequently,

$$\int_{B_\varepsilon} |z_\varepsilon|^2 dx \leq C\varepsilon^{-2} \sum_{\alpha=1}^3 \int_{D(r_\varepsilon, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 d\hat{x}$$

with  $D(r_\varepsilon, R_\varepsilon) = D(0, R_\varepsilon) \setminus \overline{D(0, r_\varepsilon)}$ , where  $D(0, R) := \{\hat{x} \in \mathbb{R}^2 \mid |\hat{x}| < R\}$  for all  $R > 0$  and

$$\begin{aligned} & \int_{B_\varepsilon} |e(u_\varepsilon) - z_\varepsilon|^2 dx \\ & \leq CR_\varepsilon^2 \varepsilon^{-2} \left( 1 + \sum_{\alpha=1}^3 R_\varepsilon^2 \int_{D(0, R_\varepsilon)} |\theta_\varepsilon^\alpha|^2 d\hat{x} + \int_{D(0, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 d\hat{x} \right) \\ & \leq CR_\varepsilon^2 \varepsilon^{-2} \left( 1 + \sum_{\alpha=1}^3 \int_{D(0, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 d\hat{x} \right) \end{aligned}$$

by due account of the Korn's inequality in  $D(0, R_\varepsilon)$ . Therefore,

$$\begin{aligned} & \left| \int_{B_\varepsilon} W_0(e(u_\varepsilon)) dx - \int_{B_\varepsilon} W_0(z_\varepsilon) dx \right| \\ & \leq CR_\varepsilon \varepsilon^{-1} \left( 1 + \sum_{\alpha=1}^3 \int_{D(0, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 d\hat{x} \right)^{1/2} \cdot \varepsilon^{-1} \left( \sum_{\alpha=1}^3 \int_{D(0, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 d\hat{x} \right)^{1/2}, \end{aligned}$$

thus, assuming that  $\theta_\varepsilon^\alpha$  satisfies

$$\int_{D(0, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 d\hat{x} \leq \frac{C}{|\ln r_\varepsilon|}, \quad \forall \alpha = 1, 2, 3, \quad (3.8)$$

it suffices to study the asymptotic behavior of  $\int_{B_\varepsilon} W_0(z_\varepsilon) dx$ . Let us denote the bilinear form associated with the quadratic form  $W_0$  by  $w_0$  :

$$w_0(e, e') = \frac{1}{2} \lambda_0 (\text{tr } e)(\text{tr } e') + \mu_0 e \cdot e', \quad \forall e, e' \in \mathbb{S}^3.$$

Note that

$$\begin{aligned} \int_{B_\varepsilon} W_0(z_\varepsilon) dx &= \sum_{\alpha, \beta=1}^3 \int_{B_\varepsilon} (v-u)_\alpha (v-u)_\beta w_0(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) dx \\ &= \varepsilon^{-2} \sum_{\alpha, \beta=1}^3 \left( \int_{D(r_\varepsilon, R_\varepsilon)} w_0(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) d\hat{x} \right. \\ &\quad \left. \int_0^L \sum_{i \in I_\varepsilon} |Y_\varepsilon^i| (v-u)_\alpha(\hat{x}_\varepsilon^i, x_3) (v-u)_\beta(\hat{x}_\varepsilon^i, x_3) dx_3 \right) + O(\varepsilon). \end{aligned}$$

Here, it clearly appears that in order to get the lowest upper bound for  $F_\varepsilon^2$ ,  $\theta_\varepsilon^\alpha$  has to be the unique solution of the capacitary problem

$$(\mathcal{P}_\varepsilon^{\text{cap}, \alpha}) \quad \min \left\{ \int_{D(r_\varepsilon, R_\varepsilon)} W_\varepsilon(e(\varphi)) d\hat{x} \left| \begin{array}{l} \varphi \in H^1((-\varepsilon, \varepsilon)^2; \mathbb{R}^3), \\ \varphi(\hat{x}) = e^\alpha \text{ on } D(0, r_\varepsilon) = \{|\hat{x}| < r_\varepsilon\}, \\ \varphi(\hat{x}) = 0 \text{ on } (-\varepsilon, \varepsilon)^2 \setminus D(0, R_\varepsilon). \end{array} \right. \right\}$$

It is shown in [1] (see Appendix) that

$$\left. \begin{array}{l} \text{i) inequality (3.8) is true,} \\ \text{ii) } \exists w_0^{\text{cap}} \in \mathbb{S}^3 \text{ such that} \\ \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} w_0(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) d\hat{x} = (w_0^{\text{cap}})_{\alpha\beta} \quad \forall \alpha, \beta = 1, 2, 3. \end{array} \right\} \quad (3.9)$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} W_0(z_\varepsilon) dx = \int_\Omega w_0^{\text{cap}}(v-u) \cdot (v-u) dx,$$

with (see Appendix)

$$w_0^{\text{cap}} = \pi \gamma \mu_0 \begin{bmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We complete the proof of the convergence of  $(F_\varepsilon^1(u_\varepsilon))$  and  $(F_\varepsilon^2(u_\varepsilon))$  for any  $(u, v)$  such that  $\Phi(u, v) < +\infty$  by approximation and diagonalization arguments. Eventually, as mentioned earlier,  $V_\varepsilon$  is chosen in such a way that a tedious computation shows that  $F_\varepsilon^3(u_\varepsilon)$  has the expected limit.

□

### Proof of Proposition 3.3

We assume here that  $\liminf_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ . Compactness property yields that  $(u, v)$  belongs to  $L^2(\Omega; \mathbb{R}^3)^2$ .

We begin with the term  $F_\varepsilon^2(u_\varepsilon)$ . Let  $(u_\eta, v_\eta)$  be Lipschitz on  $\bar{\Omega}$  with the property  $\lim_{\eta \rightarrow 0} \|u_\eta - u\|_{L^2(\Omega; \mathbb{R}^3)} + \|v_\eta - v\|_{L^2(\Omega; \mathbb{R}^3)} = 0$ . Next we define an approximation  $(v_\eta - u_\eta)_\varepsilon := \sum_{i \in I_\varepsilon} (v_\eta - u_\eta)(\hat{x}_\varepsilon^i, x_3) 1_{Y_\varepsilon^i}$  of  $(v_\eta - u_\eta)$ , and associate  $z_{\eta\varepsilon}$  to  $(u_\eta, v_\eta)$  by (3.6). Let  $\tilde{z}_{\eta\varepsilon} := \sum_{\alpha=1}^3 (v_\eta - u_\eta)_{\varepsilon\alpha} e(\theta_\varepsilon^\alpha)$ . Because of local Lipschitz property (3.7) of  $W_0$  and  $(u, v) \in L^2(\Omega; \mathbb{R}^3)^2$ , Cauchy-Schwarz inequality implies

$$\lim_{\varepsilon \rightarrow 0} \left( \int_{B_\varepsilon} W_0(\tilde{z}_{\eta\varepsilon}) dx - \int_{B_\varepsilon} W_0(z_{\eta\varepsilon}) dx \right) = 0.$$

The proof of upper bound equality shows

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} W_0(z_{\eta\varepsilon}) dx = \int_{\Omega} w_0^{\text{cap}}(v_\eta - u_\eta) \cdot (v_\eta - u_\eta) dx.$$

Therefore,  $W_0$ , being convex and 2-positively homogeneous, the subdifferential inequality gives :

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} W_0(e(u_\varepsilon)) dx \\ & \geq \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} W_0(\tilde{z}_{\eta\varepsilon}) dx + \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon}) \cdot (e(u_\varepsilon) - \tilde{z}_{\eta\varepsilon}) dx \quad (3.10) \\ & = - \int_{\Omega} w_0^{\text{cap}}(v_\eta - u_\eta) \cdot (v_\eta - u_\eta) dx + \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon}) \cdot e(u_\varepsilon) dx \end{aligned}$$

Letting  $D^i(r_\varepsilon, R_\varepsilon) := D_{R_\varepsilon}^i \setminus \overline{D_{r_\varepsilon}^i}$ , we have :

$$\begin{aligned} & \int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon}) \cdot e(u_\varepsilon) dx \\ & = \sum_{i \in I_\varepsilon} \sum_{\alpha=1}^3 \int_0^L (v_\eta - u_\eta)_\alpha(\hat{x}_\varepsilon^i, x_3) \left( \int_{D^i(r_\varepsilon, R_\varepsilon)} W_0'(e(\theta_\varepsilon^\alpha)) \cdot e(u_\varepsilon) d\hat{x} \right) dx_3. \end{aligned}$$

If  $\nu$  denotes the outer normal along both  $\partial D_{r_\varepsilon}^i$  and  $\partial D_{R_\varepsilon}^i$ , the very definition of  $\theta_\varepsilon^\alpha$  as a solution of  $(\mathcal{P}_\varepsilon^{\text{cap},\alpha})$  and Green's formula implies :

$$\begin{aligned}
 & \int_{D^i(r_\varepsilon, R_\varepsilon)} W'_0(e(\theta_\varepsilon^\alpha)) \cdot e(u_\varepsilon) \, d\hat{x} \\
 &= - \int_{\partial D_{r_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot u_\varepsilon \, dl + \int_{\partial D_{R_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot u_\varepsilon \, dl \\
 &= - \int_{\partial D_{r_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot (u_\varepsilon - \bar{u}_\varepsilon) \, dl + \int_{\partial D_{R_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot (u_\varepsilon - \bar{u}_\varepsilon) \, dl \\
 &\quad + \int_{\partial D_{r_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot (\bar{u}_\varepsilon - \bar{u}_\varepsilon) \, dl \\
 &= - \int_{\partial D_{r_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot (u_\varepsilon - \bar{u}_\varepsilon) \, dl + \int_{\partial D_{R_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot (u_\varepsilon - \bar{u}_\varepsilon) \, dl \\
 &\quad + 2 \sum_{\beta=1}^3 (\bar{u}_\varepsilon^\beta - \bar{u}_\varepsilon^\beta) \int_{D(r_\varepsilon, R_\varepsilon)} w_0(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) \, d\hat{x},
 \end{aligned}$$

where

$$\begin{aligned}
 \bar{u}_\varepsilon^i(x_3) &= \frac{1}{|\partial D_{r_\varepsilon}^i|} \int_{\partial D_{r_\varepsilon}^i} u_\varepsilon(\hat{x}, x_3) \, dl, & \bar{u}_\varepsilon(x) &= \sum_{i \in I_\varepsilon} \bar{u}_\varepsilon^i(x_3) 1_{Y_\varepsilon^i}(\hat{x}), \\
 \bar{u}_\varepsilon^i(x_3) &= \frac{1}{|\partial D_{R_\varepsilon}^i|} \int_{\partial D_{R_\varepsilon}^i} u_\varepsilon(\hat{x}, x_3) \, dl, & \bar{u}_\varepsilon(x) &= \sum_{i \in I_\varepsilon} \bar{u}_\varepsilon^i(x_3) 1_{Y_\varepsilon^i}(\hat{x}),
 \end{aligned}$$

$u_\varepsilon(\cdot, x_3)$  being, by Fubini's theorem, well defined in  $H^1(\omega; \mathbb{R}^3)$  for  $x_3 \in (0, L)$  a.e..

In fact, the standard estimates

$$\begin{aligned}
 \int_0^L \int_{\partial D_{r_\varepsilon}^i} |u_\varepsilon - \bar{u}_\varepsilon^i| \, dl \, dx_3 &\leq r_\varepsilon \left( \int_{T_\varepsilon^i} |\nabla u_\varepsilon|^2 \, dx \right)^{1/2} \\
 \int_0^L \int_{\partial D_{R_\varepsilon}^i} |u_\varepsilon - \bar{u}_\varepsilon^i| \, dl \, dx_3 &\leq R_\varepsilon \left( \int_{D_{R_\varepsilon}^i \times (0, L)} |\nabla u_\varepsilon|^2 \, dx \right)^{1/2}
 \end{aligned}$$

and the estimates (see Appendix)

$$|W'_0(e(\theta_\varepsilon^\alpha)) \nu|_{L^\infty(\partial D(0, r_\varepsilon))} \leq \frac{C}{r_\varepsilon |\ln r_\varepsilon|}, \quad |W'_0(e(\theta_\varepsilon^\alpha)) \nu|_{L^\infty(\partial D(0, R_\varepsilon))} \leq \frac{C}{R_\varepsilon |\ln r_\varepsilon|}$$

for  $\alpha = 1, 2, 3$  imply that

$$\begin{aligned}
 & \left| \sum_{i \in I_\varepsilon} \sum_{\alpha=1}^3 \int_0^L (v_\eta - u_\eta)_\alpha(\hat{x}_\varepsilon^i, x_3) \left( \int_{\partial D_{r_\varepsilon}^i} W'_0(e(\theta_\varepsilon^\alpha)) \nu \cdot (u_\varepsilon - \bar{u}_\varepsilon) \, dl \right) \, dx_3 \right| \\
 & \leq \frac{C}{|\ln r_\varepsilon|} \sum_{i \in I_\varepsilon} \left( \int_{T_\varepsilon^i} |\nabla u_\varepsilon|^2 \, dx \right)^{1/2} \leq \frac{C\varepsilon}{\varepsilon^2 |\ln r_\varepsilon|} \left( \int_\Omega |\nabla u_\varepsilon|^2 \, dx \right)^{1/2} \leq C\varepsilon
 \end{aligned}$$

and

$$\begin{aligned} & \left| \sum_{i \in I_\varepsilon} \sum_{\alpha=1}^3 \int_0^L (v_\eta - u_\eta)_\alpha(\hat{x}_\varepsilon^i, x_3) \left( \int_{\partial D_{R_\varepsilon}^i} W_0'(e(\theta_\varepsilon^\alpha)) \nu \cdot (u_\varepsilon - \bar{u}_\varepsilon) dl \right) dx_3 \right| \\ & \leq \frac{C}{|\ln r_\varepsilon|} \sum_{i \in I_\varepsilon} \left( \int_{T_\varepsilon^i} |\nabla u_\varepsilon|^2 dx \right)^{1/2} \leq \frac{C\varepsilon}{\varepsilon^2 |\ln r_\varepsilon|} \left( \int_\Omega |\nabla u_\varepsilon|^2 dx \right)^{1/2} \leq C\varepsilon. \end{aligned}$$

Thus,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon}) \cdot e(u_\varepsilon) dx \\ & = 2 \sum_{i \in I_\varepsilon} \sum_{\alpha, \beta=1}^3 \int_0^L (v_\eta - u_\eta)_\alpha(\hat{x}_\varepsilon^i, x_3) \left( \int_{D(r_\varepsilon, R_\varepsilon)} w_0(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) d\hat{x} \right) (\bar{u}_\varepsilon - \bar{u}_\varepsilon)_\beta^i(x_3) dx_3 \\ & = 2 \int_\Omega w_0^{\text{cap}}(v_\eta - u_\eta) \cdot (v - u) dx, \end{aligned}$$

by due account of (3.9) and of the weak convergence in  $L^2(\Omega; \mathbb{R}^3)$  of  $(\bar{u}_\varepsilon, \bar{u})$  toward  $(v, u)$  (see [1] p.68). Hence, letting  $\eta$  tends to zero yields

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon^2(u_\varepsilon) \geq \int_\Omega w_0^{\text{cap}}(v - u) \cdot (v - u) dx.$$

Because  $|B_\varepsilon \cup T_\varepsilon|$  tends to zero, a classical semi-continuity argument taking into account the convexity of  $W_0$  yields

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon^1(u_\varepsilon) \geq \int_\Omega W_0(e(u)) dx.$$

For the third term  $F_\varepsilon^3(u_\varepsilon)$ , we may extend the strategy of [5] to all cases of relative behaviors of the parameters by due account of the function  $V_\varepsilon$  introduced by [1].

□

### 3.3 Conclusions and Remarks

Here was presented another proof of a result of [1] concerning the homogenization of a cylindrical fibered structure. Instead of passing to the limit on a formulation of the problem in terms of variational equality through an appropriate sequence of oscillating test fields, we study the variational convergence of the energy functional. Hence, the ingredients in the construction of the appropriate oscillating test fields are clearly justified as providing the “best” upper

bound. Thus, it seems possible to consider a more general cross section for the fibers (say  $r_\varepsilon\Delta$  with  $\partial\Delta$  smooth enough) and a more general quadratic bulk energy density  $W_M$  for the matrix in the extent where the solutions  $\theta_\varepsilon^\alpha$  of the involved capacity problems

$$\min \left\{ \int_{(-\varepsilon, \varepsilon)^2} W_M(e(\varphi)) d\hat{x} \left| \begin{array}{l} \varphi \in H^1((-\varepsilon, \varepsilon)^2; \mathbb{R}^3), \\ \varphi(\hat{x}) = e^\alpha \text{ on } r_\varepsilon\Delta, \\ \varphi(\hat{x}) = 0 \text{ on } (-\varepsilon, \varepsilon)^2 \setminus D(0, R_\varepsilon). \end{array} \right. \right\}$$

are such that

- i)  $\exists w_M^{\text{cap}} \in \mathbb{S}^3$  such that  $(w_M^{\text{cap}})_{\alpha\beta} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{(-\varepsilon, \varepsilon)^2} w_M(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) d\hat{x}$ ,
- ii)  $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon i + r_\varepsilon \partial\Delta} W'_M(e(\theta_\varepsilon^\alpha)) \cdot (u_\varepsilon - \bar{u}_\varepsilon) dl$   
 $= \lim_{\varepsilon \rightarrow 0} \int_{\partial D_{R_\varepsilon}^i} W'_M(e(\theta_\varepsilon^\alpha)) \cdot (u_\varepsilon - \bar{u}_\varepsilon) dl = 0$ .

### 3.4 Appendix : The Vector Capacity Problem

Taking advantage of the cylindrical geometry, Bellieud and Gruais [1] showed that  $\theta_\varepsilon^\alpha$  and  $\sigma_\varepsilon^\alpha := W'_0(e(\theta_\varepsilon^\alpha))$  are such that

$$\begin{aligned} \theta_\varepsilon^1(x_1, x_2) &= \theta_\varepsilon^2(x_2, x_1), \quad \theta_{\varepsilon 3}^1 = \theta_{\varepsilon 3}^2 = 0, \\ \theta_\varepsilon^3 &= \frac{\ln(R_\varepsilon/r)}{\ln(R_\varepsilon/r_\varepsilon)} e^3 \text{ in } D(r_\varepsilon, R_\varepsilon), \quad r = |\hat{x}|, \end{aligned}$$

$$\begin{aligned} (\theta_{\varepsilon 1}^1 + i\theta_{\varepsilon 2}^1)(\hat{x}) &= \frac{A}{2\mu_0} \left( \chi(\ln z + \overline{\ln z}) + \frac{z^2}{r_\varepsilon^2 + R_\varepsilon^2} - \frac{z}{\bar{z}} - \frac{2z\bar{z}}{\chi(r_\varepsilon^2 + R_\varepsilon^2)} \right. \\ &\quad \left. + \frac{2\chi(r_\varepsilon^2 \ln R_\varepsilon - R_\varepsilon^2 \ln r_\varepsilon)}{R_\varepsilon^2 - r_\varepsilon^2} + \frac{r_\varepsilon^2 R_\varepsilon^2}{(r_\varepsilon^2 + R_\varepsilon^2)\bar{z}^2} \right) + \frac{R_\varepsilon^2}{R_\varepsilon^2 - r_\varepsilon^2}, \end{aligned}$$

with

$$\chi := \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0}, \quad A := \frac{\mu_0}{\frac{R_\varepsilon^2 - r_\varepsilon^2}{\chi(r_\varepsilon^2 + R_\varepsilon^2)} - \chi \ln \frac{R_\varepsilon}{r_\varepsilon}}, \quad z \text{ the complex number } x_1 + ix_2,$$

and

$$\begin{aligned} \sigma_\varepsilon^1 \nu &= \frac{\mu_0(\chi + 1)(1 + o(1))}{\chi r_\varepsilon |\ln r_\varepsilon|} e^1 \quad \text{on } \partial D(0, r_\varepsilon), \\ \sigma_\varepsilon^1 \nu &= \frac{\mu_0(1 + o(1))}{\chi R_\varepsilon |\ln r_\varepsilon|} \left[ 4\left(1 + \frac{1}{\chi} \cos^2 \theta - \left(1 + \chi + \frac{2}{\chi}\right)\right) \right] e^1 + 2\left(1 + \frac{1}{\chi}\right) \sin 2\theta e^2 \quad \text{on } \partial D(0, R_\varepsilon), \\ \sigma_\varepsilon^3 \nu &= \frac{-\mu}{r \ln R_\varepsilon / r_\varepsilon} e^3 \quad \text{on } \partial D(0, r). \end{aligned}$$

Thus, for each  $\alpha, \beta = 1, 2, 3$ ,

i)

$$|\sigma_\varepsilon^\alpha \nu|_{L^\infty(\partial D(0, r_\varepsilon))} \leq \frac{C}{r_\varepsilon |\ln r_\varepsilon|}, \quad |\sigma_\varepsilon^\alpha \nu|_{L^\infty(\partial D(0, R_\varepsilon))} \leq \frac{C}{R_\varepsilon |\ln r_\varepsilon|},$$

ii)

$$\int_{D(0, R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 d\hat{x} \leq C \int_{D(0, R_\varepsilon)} W_0(e(\theta_\varepsilon^\alpha)) d\hat{x} \leq \frac{C}{2} \int_{\partial D(0, R_\varepsilon)} (\sigma_\varepsilon^\alpha \nu)_\alpha dl \leq \frac{C}{|\ln r_\varepsilon|},$$

iii)

$$\begin{aligned} (w_0^{\text{cap}})_{\alpha\beta} &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} w_0(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)) d\hat{x} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-2}}{2} \int_{\partial D(0, r_\varepsilon)} (\sigma_\varepsilon^\alpha \nu)_\beta dl, \end{aligned}$$

satisfies

$$w_0^{\text{cap}} = \pi\gamma\mu_0 \begin{bmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

*Note that this chapter was the subject of the following article :*

N. Sontichai, S. Orankitjaroen, C. Licht, A. Kananthai, *A Remark on the Homogenization of a Microfibered Linearly Elastic Material*, East-West J. of Mathematics, a special volume, 2010, p.255–267.



## REFERENCES

- [1] M. Bellieud, I. Gruais, *Homogenization of an Elastic Material Reinforced by Very Stiff or Heavy Fibers. Non Local Effects. Memory Effects*, J. Math. Pures Appl. 84 (2005), 55–96.
- [2] S. Orankitjaroen, N. Sontichai, C. Licht, A. Kananthai, *Mathematical Modeling of Fiber Reinforced Structures by Homogenization*, Thai Journal of Mathematics, Special Issue Annual Meeting in Mathematics (2008), p.103–115.
- [3] H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces : Application to PDEs and Optimization*, MPS-SIAM Book Series on Optimization, 2005.
- [4] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag, 1989.
- [5] C. Pideri, P. Seppecher, *A Second Gradient Material Resulting from the Homogenization of an Heterogeneous Linear Elastic Medium*, Continuum Mech. Thermodyn. **9** (1997), p.241–257.

## CHAPTER IV

### A SIMPLIFIED 2-DIMENSIONAL MODEL FOR SOME ELASTIC MASONRIES

#### 4.1 Setting the Problem

We do not make any difference between the physical Euclidean space of  $\mathbb{R}^3$ , if  $x := (x_1, x_2, x_3) \in \mathbb{R}^3$  then  $\hat{x}$  stands for  $(x_1, x_2)$ . A model of static behavior of a masonry can be described as follows. Let  $a$  a positive real number,  $H \in a\mathbb{N}$ , and  $h$  a small positive number. The domain occupied by the thin masonry under considerations here is  $\Omega_h := \omega \times (-h, h)$  where  $\omega := (0, L) \times (0, H)$ . This masonry is made of a periodic distribution of bricks linked to each other by adhesive mortar occupying a very thin domain (see Figure 4.1). More precisely, if  $\hat{Y} := (0, 1) \times (0, a)$ ,

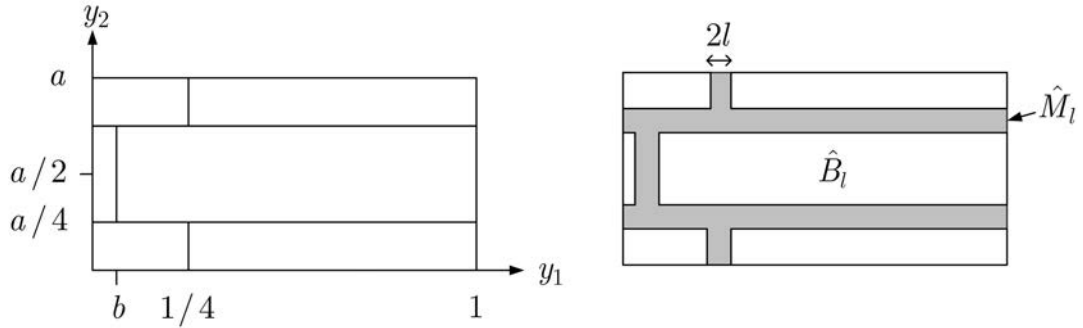


Figure 4.1: The unit cell  $\hat{Y}$ ,  $S$ ,  $\hat{M}_l$  and  $\hat{B}_l$

$$\begin{aligned}
 S &:= \{ \hat{y} \in \hat{Y} \mid y_2 = a/4 \text{ or } 3a/4; \\
 &\quad y_1 = 1/4, \text{ and } 0 < y_2 < a/4 \text{ or } 3a/4 < y_2 < a; \\
 &\quad y_1 = b \text{ and } a/4 < y_2 < 3a/4, \text{ where } b \in (0, 3/4] \}, \\
 \hat{M}_l &:= \{ \hat{x} \in \hat{Y} \mid \text{dist}(\hat{x}, S) < l \}, \quad l < \min\{b, a/4\}, \\
 \hat{B}_l &:= \hat{Y} \setminus \hat{M}_l, \\
 \varepsilon &= L/n, \quad n \in \mathbb{N}, \\
 \hat{Y}_\varepsilon^i &:= \varepsilon i + \varepsilon \hat{Y}, \quad i \in I_\varepsilon := \{ i \in \mathbb{Z}^2 \mid \hat{Y}_\varepsilon^i \subset \omega \}, \\
 \hat{M}_{l\varepsilon}^i &= \varepsilon i + \varepsilon \hat{M}_l, \quad \hat{M}_{l\varepsilon} = \cup_{i \in I_\varepsilon} \hat{M}_{l\varepsilon}^i, \\
 \hat{B}_{l\varepsilon}^i &= \varepsilon i + \varepsilon \hat{B}_l, \quad \hat{B}_{l\varepsilon} = \cup_{i \in I_\varepsilon} \hat{B}_{l\varepsilon}^i,
 \end{aligned}$$

then  $B_{l\varepsilon h} := \hat{B}_{l\varepsilon} \times (-h, h)$  and  $M_{l\varepsilon h} := \hat{M}_{l\varepsilon} \times (-h, h)$  are the domains occupied by the bricks and the mortar, respectively. The bricks are made of a homogeneous linearly elastic material whose bulk energy density denoted by  $W$  is a strictly convex quadratic function satisfying

$$\exists \alpha, \beta > 0; \quad \alpha |e|^2 \leq W(e) \leq \beta |e|^2 \quad \forall e \in \mathbb{S}^3,$$

the space of symmetric  $3 \times 3$  matrices. The mortar is assumed to be made of a homogeneous, isotropic linearly elastic material of density  $W_{\lambda\mu}$  whose Lamé coefficients  $\lambda$  and  $\mu$  are supposed to be far smaller than  $\alpha$  and  $\beta$ . Eventually, the wall is assumed to be clamped on a part  $\Gamma_{0h} = \gamma_0 \times (-h, h)$  of  $\partial\Omega_h$ ,  $\gamma_0$  being a part of  $\partial\omega$  with positive length, and subjected to body forces of density  $f^h$ .

To find the equilibrium configurations of the wall, we are led to the problem :

$$(\mathcal{P}) \quad \min \left\{ \int_{B_{l\varepsilon h}} W(e(u)(x)) dx + \int_{M_{l\varepsilon h}} W_{\lambda\mu}(e(u)(x)) dx - \int_{\Omega_h} f^h(x) \cdot u(x) dx \right. \\
 \left. \mid u \in H_{\Gamma_{0h}}^1(\Omega_h; \mathbb{R}^3) \right\}$$

where

$$\begin{cases} H_{\Gamma_{0h}}^1(\Omega_h; \mathbb{R}^3) := \{ v \in H^1(\Omega_h; \mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_{0h} \text{ in the sense of the traces } \}, \\ e(v) := \frac{1}{2}(\nabla v + \nabla v^T). \end{cases}$$

Assuming  $f^h$  in  $L^2(\Omega_h; \mathbb{R}^3)$ , this problem clearly has a unique solution, but, due to the low values of  $(\varepsilon, l, \lambda, \mu, h)$ , obtaining numerical approximation may

be difficult. Thus, it is of interest to propose a simplified but accurate enough model. A first attempt [1] is to replace  $(\mathcal{P})$  by a suitable 2-dimensional problem set in the cross section  $\omega$  of the wall which may read as :

$$(\hat{\mathcal{P}}_{\hat{s}}) \quad \min \left\{ \int_{\hat{B}_{l\varepsilon}} \hat{W}(e(u)(\hat{x})) d\hat{x} + \int_{\hat{M}_{l\varepsilon}} \hat{W}_{\lambda'\mu}(e(u)(\hat{x})) d\hat{x} - \int_{\omega} \hat{f}(\hat{x}) \cdot u(\hat{x}) d\hat{x} \right. \\ \left. \mid u \in H_{\gamma_0}^1(\omega; \mathbb{R}^2) \right\},$$

where

- we will still denote the symmetric part of the gradient in the sense of distribution of  $\hat{x} \in \omega \mapsto u(\hat{x}) \in \mathbb{R}^2$  by  $e(u)$  which is then a distribution on  $\omega$  with value in  $\mathbb{S}^2$  the space of  $2 \times 2$  symmetric matrices,
- $\hat{W}$ , deduced from  $W$ , is a strictly convex quadratic function satisfying

$$\exists \hat{\alpha}, \hat{\beta} > 0; \quad \hat{\alpha}|e|^2 \leq \hat{W}(e) \leq \hat{\beta}|e|^2 \quad \forall e \in \mathbb{S}^2,$$

- $\hat{W}_{\lambda'\mu}(e) = \frac{\lambda'}{2}(e_{11} + e_{22})^2 + \mu|e|^2$ ,  $\lambda' = \frac{2\lambda\mu}{\lambda+2\mu}$ , which corresponds to  $W_{\lambda\mu}$  in the plane stress assumption.
- $\hat{f}$  is deduced from  $f^h$ ,
- $\hat{s} := (\varepsilon, l, \lambda', \mu)$ .

We will use the following notations :

$$\hat{W}_{\hat{s}}(\hat{y}, e) := \begin{cases} \hat{W}(e) & \text{if } \hat{y} \in \hat{B}_l \\ \hat{W}_{\lambda'\mu}(e) & \text{if } \hat{y} \in \hat{M}_l \end{cases} \quad (4.1)$$

which is extended into  $\mathbb{R}^2 \times \mathbb{S}^2$  by  $\hat{Y}$ -periodicity, and

$$F_{\hat{s}}(u) = \int_{\omega} \hat{W}_{\hat{s}}(\hat{x}/\varepsilon, e(u)(\hat{x})) d\hat{x}.$$

Assuming again that  $\hat{f} \in L^2(\omega; \mathbb{R}^2)$ , problem  $(\hat{\mathcal{P}}_{\hat{s}})$  has a unique solution  $\hat{u}_{\hat{s}}$  but, due to the low values of  $\hat{s}$ , getting numerical approximations remains difficult. Thus, taking into account this low values, we will propose a simplified but accurate enough model by studying the asymptotic behavior of  $(\hat{\mathcal{P}}_{\hat{s}})$  when

$\hat{s}$  goes to zero. Since we will only consider 2-dimensional problems in the sequel, to shorten notation we will denote a current point of  $\mathbb{R}^2$  by  $x = (x_1, x_2)$  or  $y = (y_1, y_2)$  and  $s$  not by  $\hat{x}$ ,  $\hat{y}$  and  $\hat{s}$ , respectively.

Actually,  $(\hat{\mathcal{P}}_s)$  looks like a problem of periodic homogenization since the geometry and the mechanical properties of the microstructure are  $\varepsilon\hat{Y}$ -periodic, but the geometry involves an additional parameter  $l$  while the bulk energy of the mortar involves two other ones  $\lambda'$ ,  $\mu$ . If  $(l, \lambda', \mu)$  were fixed, the asymptotic effective bulk energy of the heterogeneous structure will be given by

$$W_{l\lambda'\mu}^{\text{eff}}(E) := \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{B}_l} \hat{W}(E + e(v)(y)) dy + \int_{\hat{M}_l} \hat{W}_{\lambda'\mu}(E + e(v)(y)) dy \right) \right. \\ \left. \mid v \in H_{\text{per}}^1(\hat{Y}; \mathbb{R}^2) \right\}$$

where

$$H_{\text{per}}^1(\hat{Y}; \mathbb{R}^2) := \{ v \in H^1(\hat{Y}; \mathbb{R}^2) \mid \text{the traces of } v \text{ on the opposite sides of } \hat{Y} \text{ are equal} \}.$$

The asymptotic behavior of  $W_{l\lambda'\mu}^{\text{eff}}$  when  $(l, \lambda', \mu) \rightarrow 0$  is a problem of modelling of soft elastic junctions. By using the arguments of [2] in a very general setting or of [3] in a setting close to the present one, it may be shown that

$$\lim_{(l, \lambda', \mu) \rightarrow 0} W_{l\lambda'\mu}^{\text{eff}}(E) = \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v](y)) dl \right) \right. \\ \left. \mid v \in H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) \right\},$$

where

$$H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) := \{ v \in H^1(\hat{Y} \setminus S; \mathbb{R}^2) \mid \text{the traces of } v \\ \text{on the opposite sides of } \hat{Y} \text{ are equal} \},$$

we made the additional assumption

$$\exists \bar{\lambda}', \bar{\mu} \in (0, +\infty); \quad \lambda' \sim 2\bar{\lambda}'l, \quad \mu \sim 2\bar{\mu}l, \quad (4.2)$$

which leads to the most interesting case from the mechanical point of view,

$$\begin{cases} \bar{W}_{\bar{\lambda}'\bar{\mu}}(v) := \hat{W}_{\bar{\lambda}'\bar{\mu}}(v \otimes_s n) & \forall v \in \mathbb{R}^2 \\ a \otimes_s b := \frac{1}{2}(a \otimes b + b \otimes a), \end{cases}$$

$n$  is a chosen unit normal to  $S$ ,  $[v]$  is the difference, taken in the direction of  $n$ , of the traces of  $v$  on  $S$ .

In fact, in the sequel we prove that this previous limit is *the effective bulk energy of the wall* by studying the asymptotic behavior of the strain energy functional  $F_s$  involved by  $(\hat{P}_s)$  when  $s$  goes to zero with the sole condition (4.2). This will be done in the setting of variational convergence by establishing that the strain energy functional converges in some sense to an integral functional on  $\omega$  whose density is :

$$W^{\text{eff}}(E) := \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v](y)) dl \right) \right. \quad (4.3)$$

$$\left. \left| v \in H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) \right. \right\}.$$

## 4.2 The Asymptotic Model

### 4.2.1 An Auxiliary Problem

We have just emphasized the role played, for all  $E$  in  $\mathbb{S}^2$ , by the problem

$$(\hat{P}_E) \quad \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v](y)) dl \right) \right.$$

$$\left. \left| v \in H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) \right. \right\}$$

which has a unique (up to a constant element of  $\mathbb{R}^2$ ) solution  $v_E$  and we have

$$W^{\text{eff}}(E) = \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v_E)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v_E](y)) dl \right) \quad (4.4)$$

and

$$\exists \gamma, \beta > 0; \quad \gamma|E|^2 \leq W^{\text{eff}}(E) \leq \hat{W}(E) \leq \beta|E|^2 \quad \forall E \in \mathbb{S}^2. \quad (4.5)$$

Let us still denote the extension of  $v_E$  into  $\mathbb{R}^2$  by  $\hat{Y}$ -periodicity by  $v_E$  and let  $\sigma_E = \hat{W}'(E + e(v_E))$ . It is easy to check that  $\text{div } \sigma_E = 0$  in the sense of the distributions on  $\mathbb{R}^2$ . Thus,  $v_E$ , being solution of an elastostatic problem in quadrants  $Q_i$  with boundary conditions like  $\sigma_E n \in H^{1/2}(\partial Q_i; \mathbb{R}^2)$ , belongs to  $H^2(\hat{Y} \setminus S; \mathbb{R}^2)$  [4], [5]. The Sobolev embedding implies that  $v_E$  on each connected components of  $\hat{Y} \setminus S$  is the restriction of Hölder continuous functions on  $\mathbb{R}^2$  and that  $e(v_E)$  and

consequently  $\sigma_E$  belongs to  $L^r(\hat{Y}; \mathbb{S}^2) \forall r \in [1, +\infty)$ . This regularity property of  $\sigma_E$  is crucial to prove the lower bound : the function  $x \in \omega \mapsto \sigma_E(x/\varepsilon) \in \mathbb{S}^2$  weakly converges in  $L^r(\omega; \mathbb{S}^2)$  toward  $\int_{\hat{Y}} \sigma_E(y) dy \forall r \in [1, +\infty)$ !

The field  $v_E$  does not belong to  $H^1(\hat{Y}; \mathbb{R}^2)$ , thus in order to use this field for building suitable test functions we proceed to a regularization as in [1] and [2]. For all  $v$  in  $H^1_{\text{per}}(\hat{Y} \setminus S; \mathbb{R}^2)$  we define  $R_i v$ ,  $i = 1, 2, 3, 4$ , by :

$$\begin{aligned} R_1 v(y) &:= \frac{1}{2} \left[ \min \left\{ 1, \frac{1}{l} \left| y_1 - \frac{1}{4} \right| \right\} \left( v(y) - v \left( \frac{1}{2} - y_1, y_2 \right) \right) + v(y) + v \left( \frac{1}{2} - y_1, y_2 \right) \right] \\ R_2 v(y) &:= \frac{1}{2} [\min\{1, |y_1 - b|/l\} (v(y) - v(2b - y_1, y_2)) + v(y) + v(2b - y_1, y_2)] \\ R_3 v(y) &:= \frac{1}{2} \left[ \min \left\{ 1, \frac{1}{l} \left| y_2 - \frac{a}{4} \right| \right\} \left( v(y) - v \left( y_1, \frac{a}{2} - y_2 \right) \right) + v(y) + v \left( y_1, \frac{a}{2} - y_2 \right) \right] \\ R_4 v(y) &:= \frac{1}{2} \left[ \min \left\{ 1, \frac{1}{l} \left| y_2 - \frac{3a}{4} \right| \right\} \left( v(y) - v \left( y_1, \frac{3a}{2} - y_2 \right) \right) + v(y) + v \left( y_1, \frac{3a}{2} - y_2 \right) \right] \end{aligned}$$

and let  $R_l := R_4 \circ R_3 \circ R_2 \circ R_1$ .

Clearly for all  $v$  in  $H^1_{\text{per}}(\hat{Y} \setminus S; \mathbb{R}^2)$ ,  $R_l v$  belongs to  $H^1_{\text{per}}(\hat{Y}; \mathbb{R}^2)$  and  $R_l v(y) = v(y) \forall y \in \hat{B}_l$ . It is straight forward to check that

$$\lim_{l \rightarrow 0} |R_l v_E - v_E|_{L^r(\hat{Y}; \mathbb{R}^2)} = 0 \quad \forall r \in [1, +\infty) \quad (4.6)$$

$$\lim_{s \rightarrow 0} |\hat{W}'_{\lambda, \mu}(e(R_l v_E))|_{L^r(\hat{M}_l; \mathbb{S}^2)} = 0 \quad \forall r \in [1, +\infty) \quad (4.7)$$

$$\lim_{l \rightarrow 0} |e(v_E)|_{L^r(\hat{M}_l; \mathbb{S}^2)} = 0 \quad \forall r \in [1, +\infty) \quad (4.8)$$

As for  $v_E$  we still denote the extension of  $R_l v_E$  into  $\mathbb{R}^2$  by  $\hat{Y}$ -periodicity by  $R_l v_E$ .

### 4.2.2 The Convergence Result

We start by establishing a compactness property for sequences with bounded energy. Let  $\mathcal{M}_b(\omega; \mathbb{S}^2)$  the space of bounded  $\mathbb{S}^2$ -valued measures on  $\omega$  and  $BD(\omega) := \{ u \in L^1(\omega; \mathbb{R}^2) \mid e(u) \in \mathcal{M}_b(\omega; \mathbb{S}^2) \}$ .

**Proposition 4.1 (compactness property)** *Let  $(u_s)$  a sequence in  $H^1_{\gamma_0}(\omega; \mathbb{R}^2)$  such that  $F_s(u_s) \leq C$ , then there exist  $u$  in  $BD(\omega)$  and a not relabelled subsequence such that  $u_s$  weak\* converges in  $BD(\omega)$  toward  $u$  and consequently strongly in  $L^q(\omega; \mathbb{R}^2)$ ,  $q$  arbitrary in  $[1, 2)$  and weakly in  $L^2(\omega; \mathbb{R}^2)$ .*

**Proof of Proposition 4.1 :**

We have

$$\begin{aligned}
 C &\geq \hat{\alpha} \int_{\hat{B}_{l\varepsilon}} |e(u_s)|^2 dx + \mu \int_{\hat{M}_{l\varepsilon}} |e(u_s)|^2 dx \\
 &\geq \frac{\hat{\alpha}}{a} \left( \int_{\hat{B}_{l\varepsilon}} |e(u_s)| dx \right)^2 + \frac{\mu}{|\hat{M}_{l\varepsilon}|} \left( \int_{\hat{M}_{l\varepsilon}} |e(u_s)| dx \right)^2 \\
 &\geq \frac{\hat{\alpha}}{a} \left( \int_{\hat{B}_{l\varepsilon}} |e(u_s)| dx \right)^2 + \frac{C\mu}{l} \left( \int_{\hat{M}_{l\varepsilon}} |e(u_s)| dx \right)^2
 \end{aligned}$$

Hence (4.2) and the boundary condition  $u_s = 0$  on  $\gamma_0$  imply that  $u_s$  is bounded in  $LD(\omega; \mathbb{R}^2) := \{u \in L^1(\omega; \mathbb{R}^2) \mid e(u) \in L^1(\omega; \mathbb{S}^2)\}$  which gives the desired assertion [6].

□

Now, we are in a position to state our main convergence result.

**Theorem 4.2 (upper and lower bound)**

- *Upper bound : for all  $u$  in  $H^1(\omega; \mathbb{R}^2)$  there exists a sequence  $(u_s)$  in  $H^1(\omega; \mathbb{R}^2)$  such that  $u_s$  weak\* converges in  $BD(\omega)$  toward  $u$  and*

$$F^{\text{eff}}(u) := \int_{\omega} W^{\text{eff}}(e(u)) dx = \lim_{s \rightarrow 0} F_s(u_s).$$

- *Lower bound : for all  $u$  in  $H^1(\omega; \mathbb{R}^2)$  and all sequences  $(u_s)$  in  $H^1(\omega; \mathbb{R}^2)$  which weak\* converges in  $BD(\omega)$  toward  $u$ , we have :*

$$F^{\text{eff}}(u) \leq \liminf_{s \rightarrow 0} F_s(u_s).$$

**Proof of the upper bound :**

We add some ingredients of the mathematical theory of bonding joints [2] and [3] to the classical proof by [7] for homogenization of elliptic operators.

First we assume that  $u$  is affine  $u(x) = Ex + d$ ,  $E \in \mathbb{S}^2$ ,  $d \in \mathbb{R}^2$ . Let  $w_{E_s}$  such that  $w_{E_s}(x) = \varepsilon(R_l v_E)(x/\varepsilon)$ , (4.6) and (4.8) imply :

$$\begin{aligned}
 \lim_{s \rightarrow 0} \int_{\omega} |w_{E_s}(x) - \varepsilon v_E(x/\varepsilon)|^r dx &= 0 \quad \forall r \in [1, 2] \\
 \lim_{s \rightarrow 0} \int_{\omega} |\varepsilon v_E(x/\varepsilon)|^r dx &= 0
 \end{aligned}$$



so that the field  $u_s = u + w_{E_s}$  belongs to  $H^1(\omega; \mathbb{R}^2)$  and strongly converges in  $L^q(\omega; \mathbb{R}^2)$ , and in  $L^2(\omega; \mathbb{R}^2)$ , toward  $u$ . Moreover

$$\begin{aligned} F_s(u_s) &= \int_{\hat{B}_{l\varepsilon}} \hat{W}(E + e(v_E)(x/\varepsilon)) dx + \int_{\hat{M}_{l\varepsilon}} \hat{W}_{\lambda'\mu}(E + e(R_l v_E)(x/\varepsilon)) dx \\ &= \frac{|\omega|}{|\hat{Y}|} \left( \int_{\hat{B}_l} \hat{W}(E + e(v_E)(y)) dy + \int_{\hat{M}_l} \hat{W}_{\lambda'\mu}(E + e(R_l v_E)(y)) dy \right). \end{aligned}$$

Hence, a simple computation (see [2] and [3]) gives

$$\begin{aligned} \lim_{s \rightarrow 0} F_s(u_s) &= \frac{|\omega|}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v_E)(y)) dy + \int_S \bar{W}_{\lambda'\mu}([v_E](y)) dl \right) \quad (4.9) \\ &= \int_{\omega} W^{\text{eff}}(e(u)(x)) dx. \end{aligned}$$

Next, we take  $u$  as a piecewise continuous affine function :  $u(x) = E^i x + d^i$  on  $\omega^i$ ,  $i \in I$  finite, where the  $\omega^i$  form a partition by polyhedral sets. Like in first step, we define  $u_s$  by  $u_s^i(x) = u(x) + w_{E^i_s}$  on each  $\omega^i$ . But by due account to the possible discontinuities on the interface  $\Sigma_{jk}$  between  $\omega^j$  and  $\omega^k$  we need to introduce  $\phi_\delta$  in  $W^{1,\infty}(\omega)$ ,  $0 \leq \phi_\delta \leq 1$ ,  $\phi_\delta = 1$  on  $\Sigma_{jk\delta} = \{x \in \omega \mid \text{dist}(x, \Sigma_{jk\delta}) < \delta\}$ ,  $\delta > 0$ ,  $\phi_\delta = 0$  on  $\omega \setminus \Sigma_{jk2\delta}$  and

$$u_{\delta s} = \phi_\delta u + (1 - \phi_\delta) u_s^i \quad \text{on } \omega^i.$$

Hence, we can repeat the end of the proof by [7] p.47–48 because  $u_s^i$  converges strongly toward  $u$  not only in  $L^q(\omega^i; \mathbb{R}^2)$  but also in  $L^2(\omega^i; \mathbb{R}^2)$  while  $\hat{W}_{\lambda'\mu}$  is convex and satisfies  $\hat{W}_{\lambda'\mu}(e) \leq C|e|^2$ .

Eventually, the proof is complete by a diagonalization and density argument. □

### Proof of the lower bound :

Once more we proceed by introducing a continuous piecewise affine function  $v(x) = E^i x + d^i$  as *approximation in  $H^1$*  of  $u$  on  $\omega^i$ . For each  $\omega^i$ , let us introduce  $\phi^i \in \mathcal{D}(\omega^i)$  such that  $0 \leq \phi^i \leq 1$ . The subdifferential inequality yields

$$\begin{aligned} F_s(u_s) &\geq \sum_{i \in I} \left( \int_{\omega^i} \phi^i(x) \hat{W}_s(x/\varepsilon, E^i + e(w_{E^i_s})(x)) dx \right. \\ &\quad \left. + \int_{\omega^i} \phi^i(x) \hat{W}'_s(x/\varepsilon, E^i + e(w_{E^i_s})(x)) \cdot e(u_s - v - w_{E^i_s}) dx \right). \end{aligned}$$

A slight and obvious modification of the argument used in establishing the first step of the proof of the upper bound gives :

$$\lim_{s \rightarrow 0} \int_{\omega^i} \phi^i(x) \hat{W}_s(x/\varepsilon, E^i + e(w_{E^i_s})(x)) dx = \int_{\omega^i} \phi^i(x) W^{\text{eff}}(e(v))(x) dx.$$

Moreover, (4.6), (4.7) and (4.8) imply :

$$\begin{aligned} & \lim_{s \rightarrow 0} \int_{\omega^i} \phi^i(x) \hat{W}'_s(x/\varepsilon, E^i + e(w_{E^i_s})(x)) \cdot e(u_s - v - w_{E^i_s}) dx \\ &= \lim_{s \rightarrow 0} \int_{\omega^i} \phi^i(x) \hat{W}'_s(x/\varepsilon, E^i + e(v_{E^i})(x/\varepsilon)) \cdot e(u_s - v - w_{E^i_s}) dx \\ &= \lim_{s \rightarrow 0} \int_{\omega^i} \sigma_{E^i}(x/\varepsilon) \cdot e(\phi^i(u_s - v - w_{E^i_s})) dx \\ &\quad - \int_{\omega^i} \sigma_{E^i}(x/\varepsilon) \cdot \nabla \phi^i \otimes_s (u_s - v - w_{E^i_s}) dx \\ &= - \int_{\omega^i} \sigma_{E^i}(x/\varepsilon) \cdot \nabla \phi_i \otimes_s (u_s - v - w_{E^i_s}) dx \quad (\text{div } \sigma_{E^i} = 0) \\ &= - \int_{\omega^i} \left( \int_Y \sigma_{E^i}(y) dy \right) \cdot \nabla \phi^i \otimes_s (u - v) dx \end{aligned}$$

because  $\sigma_{E^i}(\cdot/\varepsilon)$  weakly converges in  $L^{q'}(\omega; \mathbb{S}^2)$  toward  $\int_Y \sigma_{E^i}(y) dy$  and  $(u_s - v - w_{E^i_s})$  converges strongly in  $L^{q'}(\omega; \mathbb{R}^2)$  to  $u - v$ . Hence,

$$\liminf_{s \rightarrow 0} F_s(u_s) \geq \sum_i \int_{\omega^i} \phi_i W^{\text{eff}}(e(v)) dx + \sum_i \int_{\omega^i} \phi_i (W^{\text{eff}})'(e(v)) \cdot e(u - v) dx$$

And we conclude as in [7] by letting  $\phi_i$  converge increasingly to one on  $\omega_i$  for the first term and using (4.5) and the density of the piecewise affine continuous functions in  $H^1(\omega; \mathbb{R}^2)$  for the second term. □

### 4.2.3 Mechanical Interpretation, a Proposal of Model

Indeed, we did not succeed in proving that cluster points of sequence with uniformly bounded finite strain energy belongs to  $H^1(\omega; \mathbb{R}^2)$  and satisfies an homogeneous Dirichlet boundary condition on  $\gamma_0$ . Thus we cannot (see [7], [8]) assert that the unique solution  $\hat{u}_s$  of  $(\hat{\mathcal{P}}_s)$  weak\* converges in  $BD(\omega)$  toward the obviously unique solution  $\bar{u}$  of  $(\mathcal{P}^{\text{eff}})$

$$(\mathcal{P}^{\text{eff}}) \quad \min \left\{ \int_{\omega} W^{\text{eff}}(e(v)) dx - \int_{\omega} \hat{f} \cdot v dx \mid v \in H^1_{\gamma_0}(\omega; \mathbb{R}^2) \right\}.$$

Problem ( $\mathcal{P}^{\text{eff}}$ ) describes the equilibrium of a homogeneous elastic flat body occupying  $\omega$  as reference configuration with bulk energy  $W^{\text{eff}}$ , clamped on  $\gamma_0$  and subjected to body forces of density  $\hat{f}$ . Hence, our model is simpler than the genuine one because it involves a homogeneous body and accurate enough due to our convergence result. We only can claim that a reasonable candidate for effective energy bulk energy density of the assembly bricks mortar is  $W^{\text{eff}}$ . This was also derived in [1] through rather heuristical arguments.

Due to (4.5) the effective homogeneous material is weaker than the genuine material of the bricks. This is the price to pay due to the difficulty of making a homogeneous wall. The effective strength of the wall should be greater than the one of the bricks when stiff mortar is used. It should be interesting to consider the case when the magnitudes of  $\lambda', \mu$  is of order  $1/l$ . Hence our model is simpler than the genuine one because it involves a homogeneous body and is accurate enough due to our convergence result.

## REFERENCES

- [1] A. Cecchi, K. Sab, *A Multi-Parameter Homogenization Study for Modeling Elastic Masonry*, European Journal of Mechanics A/Solids, 21 (2002), p.249–268.
- [2] C. Licht, G. Michaille, *A Modelling of Elastic Adhesive Bounded Joints*, Advances in Mathematical Sciences and Applications, 7 (1997), no 2, p.711–740.
- [3] A. Aitmousa, *Modélisation et étude des Singularités de Contraintes d'un Joint Collé très mince*, Ph.D. Thesis, University Montpellier 2, France, 1989.
- [4] P. Grisvard, *Behavior of the Solution of an Elliptic Boundary Value Problem in a Polygonal or Polyhedral domain*, Symposium on Numerical Solutions of Partial Differential Equations III, B. Hubbard Editor, 1975, p.207–274.
- [5] P. Grisvard, *Boundary Value Problems in Plane Polygone. Instructions for Use*, EDF Bulletin de la Direction des Etudes et Recherches Série C' Mathématiques, Informatique, 1 (1986), p.21–59.
- [6] R. Temam, *Problèmes Mathématiques en Plasticité*, Gauthier-Villars, Paris, 1983.
- [7] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, 1985.
- [8] H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces Application to PDEs and Optimization*, MPS-SIAM Book Series on Optimization, Philadelphia, 2006.

## CHAPTER V

### MODELING OF SOME STRATIFIED MEDIA

#### 5.1 Setting the Problem

We are concerned here by a stratified medium made from two homogeneous linearly constituents, one being far weaker than the other. Moreover, as in the case of mortar in the previous chapter, the weak constituent occupies layers far thinner than the thin layers occupied by the stronger material. This may model plywood for instance. Except the fact that both the hard phase and the soft phase fill out not connected sets of  $\mathbb{R}^3$ , the situation is mathematically similar to that of masonries and we will confine to explicit the slight differences.

As previously, we do not make any difference between the physical Euclidean space and  $\mathbb{R}^3$ , as well as if  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ ,  $\hat{x}$  stands for  $(x_1, x_2)$ .

Let  $Y := (0, 1)^3$ ,  $S := \{y \in Y \mid y_3 = 1/2\}$ ,  $M_l = \{y \in Y \mid \text{dist}(y, S) < l\}$  for  $0 < l < 1/2$ ,  $B_l = Y \setminus M_l$ ,  $M_{l_\varepsilon}^i := \varepsilon i + \varepsilon M_l$ ,  $i = (i_1, i_2, i_3) \in \mathbb{Z}^3$ ,  $B_{l_\varepsilon}^i := \varepsilon i + \varepsilon B_l$ ,  $i \in \mathbb{Z}^3$ .

Let  $\Omega$  a domain of  $\mathbb{R}^3$ , with a Lipschitz-continuous boundary, which corresponds to the domain occupied by the stratified media under consideration. The two sets  $M_{l_\varepsilon} = \Omega \cap (\cup_{i \in \mathbb{Z}^3} M_{l_\varepsilon}^i)$  and  $B_{l_\varepsilon} = \Omega \cap (\cup_{i \in \mathbb{Z}^3} B_{l_\varepsilon}^i)$  correspond to the domain occupied by each component. Hence they appear layers included in

$$\{ \varepsilon(i_3 + (1/2) - l) < x_3 < \varepsilon(i_3 + (1/2) + l) \}$$

and

$$\{ \varepsilon(i_3 + (1/2) + l) < x_3 < \varepsilon(i_3 + (3/2) - l) \}$$

of thickness  $2l\varepsilon$  and  $\varepsilon - 2l\varepsilon$ . We assume that  $B_{l_\varepsilon}$  contains a homogeneous linearly elastic material whose bulk energy density denoted by  $W$  is a strictly convex quadratic function satisfying

$$\exists \alpha, \beta > 0; \quad \alpha|e|^2 \leq W(e) \leq \beta|e|^2 \quad \forall e \in \mathbb{S}^3$$

$\mathbb{S}^3$  being the space of symmetric  $3 \times 3$  matrices, while  $M_{l_\varepsilon}$  contains a homogeneous isotropic linearly elastic material of density  $W_{\lambda\mu}$ , where the Lamé coefficients  $\lambda$  and  $\mu$  are supposed to be far smaller than  $\alpha$  and  $\beta$ .

The layers are assumed to be perfectly stuck together and the body made of them is clamped on a part  $\Gamma_0$  of  $\partial\Omega$  with positive surface measure and subjected to body forces of density  $f$ .

To find equilibrium configurations of the stratified body, we are led to the problem

$$(\mathcal{P}_s) \quad \min \left\{ \int_{B_{l_\varepsilon}} W(e(u)) \, dx + \int_{M_{l_\varepsilon}} W_{\lambda\mu}(e(u)) \, dx - \int_{\Omega} f \cdot u \, dx \mid u \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \right\}$$

involving the quadruple  $s = (\varepsilon, l, \lambda, \mu)$  of parameters and which has a unique solution  $\bar{u}_s$ . These parameters being small, it is again worthwhile for numerical reasons to derive a simplified but accurate enough model by studying the asymptotic behavior of  $(\mathcal{P}_s)$  when  $s$  tends to 0 with the assumption

$$\exists \bar{\lambda}, \bar{\mu} \in (0, \infty); \quad \lambda \sim 2\bar{\lambda}l, \quad \mu \sim 2\bar{\mu}l.$$

## 5.2 Asymptotic Modeling

As in the case of masonries, the effective bulk energy density will be obtained through a minimization problem set on the unit cell  $Y$  which reads as :

$$(\mathcal{P}_E) \quad \min \left\{ \frac{1}{|Y|} \int_{Y \setminus S} W(E + e(v)) \, dy + \int_S \bar{W}_{\bar{\lambda}\bar{\mu}}([v]) \, dy \mid v \in H_{\text{per}}^1(Y \setminus S; \mathbb{R}^3) \right\}$$

where we used the notations of the previous chapter. This problem has a unique (up to a constant element of  $\mathbb{R}^3$ ) solution  $v_E$  and

$$W^{\text{eff}}(E) = \frac{1}{|Y|} \int_{Y \setminus S} W(E + e(v_E)) \, dy + \int_S \bar{W}_{\bar{\lambda}\bar{\mu}}([v_E]) \, dy$$

$$\exists \gamma, \beta > 0; \quad \gamma|E|^2 \leq W^{\text{eff}}(E) \leq W(E) \leq \beta|E|^2 \quad \forall E \in \mathbb{S}^3.$$

If, one more time,  $v_E$  still denotes the extension of  $v_E$  into  $\mathbb{R}^3$  by  $Y$ -periodicity, it is obvious, with this simpler geometry of  $Y \setminus S$  that  $e(v_E)$  and  $\sigma_E$  belongs to  $H^1(Y \setminus S; \mathbb{R}^3)$  so that  $\sigma_E \in L^6(Y; \mathbb{R}^3)$ . This regularity property is crucial to prove the lower bound in the following variational convergence theorem for the strain energy functionals : the function  $x \in \Omega \mapsto \sigma_E(x/\varepsilon)$  weakly

converges in  $L^6(Y; \mathbb{S}^3)$  toward  $\int_Y \sigma_E dy$ . The necessary smoothing operator from  $H_{\text{per}}^1(Y \setminus S; \mathbb{R}^3)$  toward  $H_{\text{per}}^1(Y; \mathbb{R}^3)$  is defined by :

$$R_l v(y) := \frac{1}{2} \left[ \min \left( 1, \frac{1}{l} \left| y_3 - \frac{1}{2} \right| \right) (v(\hat{y}, y_3) - v(\hat{y}, 1 - y_3)) + v(\hat{y}, y_3) + v(\hat{y}, 1 - y_3) \right]$$

so that :

$$\lim_{l \rightarrow 0} \|R_l v_E - v_E\|_{L^r(Y; \mathbb{R}^3)} = 0 \quad \forall r \in [1, +\infty)$$

$$\lim_{s \rightarrow 0} \|W'_{\lambda\mu}(e(R_l v_E))\|_{L^6(M_l; \mathbb{S}^3)} = 0$$

$$\lim_{l \rightarrow 0} \|e(v_E)\|_{L^6(M_l; \mathbb{S}^3)} = 0.$$

### 5.2.1 A Convergence Result

Because  $|M_{l\varepsilon}| = O(l)$ , the inequalities

$$\mu \int_{M_{l\varepsilon}} |e(u)|^2 dx \geq \frac{\mu}{|M_{l\varepsilon}|} \left( \int_{M_{l\varepsilon}} |e(u)| dx \right)^2 \geq C \frac{\mu}{l} \left( \int_{M_{l\varepsilon}} |e(u)| dx \right)^2$$

yield that for all sequences in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  with uniformly bounded bulk energy

$$F_s(u_s) = \int_{B_{l\varepsilon}} W(e(u_s)) dx + \int_{M_{l\varepsilon}} W_{\lambda\mu}(e(u_s)) dx,$$

there exist  $u$  in  $BD(\Omega; \mathbb{R}^3)$  and a relabelled subsequence such that  $u_s$  weak\* converges in  $BD(\Omega; \mathbb{R}^3)$  toward  $u$  and consequently strongly in  $L^q(\Omega; \mathbb{R}^3)$ , for all  $q$  in  $[1, 3/2)$ , and weakly in  $L^{3/2}(\Omega; \mathbb{R}^3)$ .

Hence, all the ingredients are gathered for proving, as in the case of masonries, the following convergence result :

#### Theorem 5.1 (upper and lower bound)

- *Upper bound* : for all  $u$  in  $H^1(\Omega; \mathbb{R}^3)$  there exists a sequence  $(u_s)$  in  $H^1(\Omega; \mathbb{R}^3)$  such that  $u_s$  weak\* converges in  $BD(\Omega; \mathbb{R}^3)$  towards  $u$  and

$$F^{\text{eff}}(u) := \int_{\omega} W^{\text{eff}}(e(u)) dx = \lim_{s \rightarrow 0} F_s(u_s).$$

- *Lower bound* : for all  $u$  in  $H^1(\Omega; \mathbb{R}^3)$  and all sequences  $(u_s)$  in  $H^1(\Omega; \mathbb{R}^3)$  which weak\* converges in  $BD(\Omega; \mathbb{R}^3)$  toward  $u$ , we have :

$$F^{\text{eff}}(u) \leq \liminf_{s \rightarrow 0} F_s(u_s).$$

### 5.2.2 Mechanical Interpretation, Proposal of Model

As previously, we did not succeed in proving that the cluster points of sequences in  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  with uniformly bounded finite strain energy do belong to  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$ . Thus, we can only claim that a reasonable proposal for effective bulk energy density is  $W^{\text{eff}}$ . We make the conjecture that with special geometries for  $\Omega$  these cluster points belong to  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  so that  $\bar{u}_s$  unique solution of  $(\mathcal{P}_s)$  does weak\* converge in  $BD(\Omega; \mathbb{R}^3)$  toward the unique solution of

$$(\mathcal{P}) \quad \min \left\{ F^{\text{eff}}(u) - \int_{\Omega} f \cdot u \, dx \mid u \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^3) \right\}$$

which describes the equilibrium of a homogeneous body with  $\Omega$  as reference configuration, clamped on  $\Gamma_0$ , subjected to body forces  $f$  and made of a linearly elastic material with bulk energy density  $W^{\text{eff}}$ .

The next section is devoted to a 1-dimensional case where this is true.

### 5.3 A 1-Dimensional Case

Let  $\Omega = (0, 1)$  and  $U, k$  two fixed positive real numbers. Let  $n \in \mathbb{N}^*$  intended to go to  $+\infty$  and  $l, \mu$  two small positive numbers intended to tend to 0, we set  $s = (1/n, l, \mu)$ . We are interested in the asymptotic behavior when  $s \rightarrow 0$ , with  $\exists \bar{\mu} \in (0, \infty)$  such that  $\mu \sim 2\bar{\mu}l$ , of the unique solution  $u_s$  of :

$$\sigma'_s = 0 \quad \text{in } \Omega \tag{5.1}$$

$$\sigma_s = 2k_s u'_s \tag{5.2}$$

with

$$k_s = \begin{cases} \mu & \text{in } M_{ln} = \cup_{i=1}^n \left\{ \left| x - \frac{1}{n}(i - \frac{1}{2}) \right| < \frac{l}{n} \right\} \\ k & \text{in } \Omega \setminus \bar{M}_{ln} \end{cases} \tag{5.3}$$

$$u_s(0) = 0, \quad u_s(1) = U. \tag{5.4}$$

where, of course,  $\sigma'_s$  and  $u'_s$  denote the derivatives in the distribution sense. Obviously (5.1)–(5.4) corresponds to the problem

$$(\mathcal{P}_s) \quad \min \left\{ \int_{\Omega} k_s(x) (u'(x))^2 \, dx \mid u \in H^1(\Omega), u(0) = 0, u(1) = U \right\}.$$



Clearly (5.1), (5.2) imply that there exists some constant  $C_s$  such that  $u'_s = C_s/2k_s$ , so that :

$$U = \int_0^1 u'_s(x) dx = C_s \left( \frac{1-2l}{2k} + \frac{2l}{2\mu} \right)$$

i.e :

$$C_s = 2U / \left( \frac{1-2l}{k} + \frac{2l}{\mu} \right).$$

Consequently  $(\sigma_s, u_s)$  are unique and are given by :

$$\begin{aligned} u_s(x) &= \left( U / \left( \frac{1-2l}{k} + \frac{2l}{\mu} \right) \right) \int_0^x \frac{dt}{k_s(t)} \quad \forall x \in [0, 1] \\ \sigma_s(x) &= 2U / \left( \frac{1-2l}{k} + \frac{2l}{\mu} \right) \quad \forall x \in [0, 1]. \end{aligned}$$

Note that  $u'_s$  is uniformly bounded in  $L^1(\Omega)$  because

$$\int_0^1 |u'_s(x)| dx = \int_0^1 u'_s(x) dx = U$$

whereas  $u'_s$  is not uniformly bounded in all  $L^p(\Omega)$ ,  $1 < p \leq \infty$ , because

$$\int_0^1 |u'_s(x)|^p dx = C_s^p \left( (1-2l) \frac{1}{(2k)^p} + \frac{2l}{2\mu} \frac{1}{(2\mu)^{p-1}} \right).$$

Clearly  $\frac{C_s}{2k} \rightarrow \frac{U\bar{\mu}}{k+\bar{\mu}}$ ,  $\frac{C_s}{2\mu} \sim \frac{U}{2l} \frac{k}{k+\bar{\mu}}$ , hence, by using the theorem of the mean, we deduce that for all functions  $\phi$  continuous on  $\bar{\Omega}$  and vanishing at 0 and 1,

$$\lim_{s \rightarrow 0} \int_0^1 \phi(x) u'_s(x) dx = \int_0^1 \phi(x) \frac{U\bar{\mu}}{k+\bar{\mu}} dx + \int_0^1 \phi(x) \frac{Uk}{k+\bar{\mu}} dx = \int_0^1 \phi(x) U dx.$$

Hence, the measure  $u'_s dx$  weak\* converges in  $\mathcal{M}^1(\Omega)$  towards  $U dx$ , thus  $u_s$  **weak\* converges in  $BV(\Omega)$  toward  $\mathcal{U}$**  where

$$\mathcal{U}(x) = Ux \quad \forall x \in \bar{\Omega},$$

while  $\sigma_s = C_s$  uniformly converges on  $\bar{\Omega}$  toward  $\frac{2k\bar{\mu}}{k+\bar{\mu}}U$ .

Actually, we have a little bit more. Because  $u'_s$  is  $1/n$  periodic and consequently  $u_s(1/n) = \frac{1}{n} \int_0^1 u'_s(t) dt = \frac{U}{n} = \mathcal{U}(1/n)$ , we have

$$\max \{ |u_s(x) - \mathcal{U}(x)| \mid x \in \bar{\Omega} \} = |u_s((1-4l)/2n) - \mathcal{U}((1-4l)/2n)| \leq C \frac{1-4l}{2n}$$

so that  $u_s$  **converges uniformly on  $\bar{\Omega}$  towards  $\mathcal{U}$** .

Note that  $u'_s$  does not converge strongly to  $U$  in  $L^1(\Omega)$  because

$$\int_0^1 |u'_s(x) - U| dx = (1 - 2l) \left| \frac{C_s}{2k} - U \right| + 2l \left| \frac{C_s}{2\mu} - U \right|$$

Anyway, the limits  $(u_0, \sigma_0)$  of  $(u_s, \sigma_s)$  satisfy :

$$\sigma_0 = \frac{1}{1/k + 1/\bar{\mu}} u'_0. \quad (5.5)$$

Eventually, the unique solution up to a constant  $v_E$  of

$$(\mathcal{P}_E) \quad \min \left\{ \int_0^{1/2} k(u'(y))^2 dy + \int_{1/2}^1 k(u'(y))^2 dy + \bar{\mu} |u(1/2^+) - u(1/2^-)| \right. \\ \left. \left| u \in H^1_{\text{per}}(\Omega \setminus \{1/2\}) \right. \right\}$$

allows us to compute  $W^{\text{eff}}(E)$  as the infimum for  $(\mathcal{P}_E)$ . Necessarily  $v_E$  is affine on  $[0, 1/2)$  and  $(1/2, 1]$ . Because of the periodicity condition and the uniqueness up to a constant for  $v_E$ , symmetry considerations imply that the slopes of  $v_E$  are the same on both intervals. Hence  $u(x) = (-E + p)x$  on  $[0, 1/2)$ ,  $u(x) = (-E + p)(x - 1)$  on  $(1/2, 1]$ , so that  $u(1/2^+) - u(1/2^-) = E - p$ . Thus  $p$  is such that  $J(p) := kp^2 + \bar{\mu}(E - p)^2$  be minimal. Clearly  $p_{\text{opt}} = \frac{\bar{\mu}}{k + \bar{\mu}} E$  and consequently

$$W^{\text{eff}}(E) = J(p_{\text{opt}}) = \frac{1}{1/k + 1/\bar{\mu}} E^2$$

which is confirm to (5.5)!

In this 1-dimensional example we have shown the uniform convergence on  $\bar{\Omega}$  and the weak\* convergence in  $BV(\Omega)$  of the unique solution  $u_s$  of  $(\mathcal{P}_s)$  toward the unique solution of

$$(\mathcal{P}^{\text{eff}}) \quad \min \left\{ \int_{\Omega} W^{\text{eff}}(u'(x)) dx \left| u \in H^1(\Omega), u(0) = 0, u(1) = U \right. \right\}.$$

In this case the limit  $\bar{u}$  of  $u_s$  belongs to  $H^1(\Omega)$  and satisfies the Dirichlet boundary conditions!

A last worth noting point is that if  $l$  and  $\mu$  were fixed the classical effective stiffness will be classically such that

$$\frac{1}{K^{\text{eff}}} = \frac{1 - 2l}{k} + \frac{2l}{\mu}.$$

So that the effective stiffness given by (5.5) is obtained by letting  $l$  and  $\mu$  going to zero with  $\frac{l}{2l}$  tending to  $\bar{\mu}$ !

## CHAPTER VI CONCLUSION AND DISCUSSION

*At the end of this chapter are the general references of all this thesis and the [ ] in the following text refer to these general references.* In this

thesis, we have presented Mathematical modeling of some mechanical problems by homogenization. We consider many kind of problems spitted in four chapters.

The first kind problem is shown in Chapter II. It concerns the scalar case in a fibered medium with energy densities proportional to  $\phi_p = \frac{1}{p} |\cdot|^p$ ,  $1 < p < +\infty$ . By using variational convergence, we got the properties of effective energy density as in [4]. But our work can be easily extended to the case when  $\phi_p$  is replaced by any strictly convex function which satisfies

$$\exists M > 0, \exists r \in (1, p) ; \quad |W(\xi) - \phi_p(\xi)| \leq M|\xi|^r \quad \forall \xi \in \mathbb{R}^3.$$

Then, the effective energy will involve an additional state variable which accounts for the fibered microstructure. The upper bound and lower bound of this effective energy are derived by our arguments (which essentially use the subdifferential inequality after proving an upper bound equality) and those of [4]. The key arguments of our analysis are the identification of some coefficients in terms of the solutions of capacitary problems and the use of the  $p$ -positive homogeneity and convexity of  $\phi_p$  and of the fact that  $\phi_p(\xi) \geq \phi_p(\hat{\xi})$ . Thus, it is easy to guess what could be effective energy when  $\phi_p$  is replaced by any strictly convex function and when the cross sections of the fibers are smooth star-shaped domains of  $\mathbb{R}^2$ . We hope that our proposed strategy will be able to reduce and overcome the involved technical difficulties.

Chapter III presented another proof of a result of [5] concerning the homogenization of a cylindrical fibered linearly elastic structure. Instead of passing to the limit on a formulation of the problem in terms of variational equality through an appropriate sequence of oscillating test fields, we study the variational

convergence of the energy functional. Hence, the ingredients in the construction of the appropriate oscillating test fields are clearly justified as providing the “best” upper bound.

Another mechanical problem concerns thin, flat and linearly elastic masonries (as shown in Chapter IV). We have considered a 2-dimensional heterogeneous model set in a domain  $\omega$  of  $\mathbb{R}^2$  (middle section of the wall). Here we did not succeed in proving that cluster points of sequence with uniformly bounded finite strain energy belongs to  $H^1(\omega; \mathbb{R}^2)$  and satisfies an homogeneous Dirichlet boundary condition. Thus we cannot (see [2], [3]) assert that the unique solution of minimization problem weak\* converges in  $BD(\omega)$  toward the obviously unique solution of the limit minimization problem. We only can claim that a reasonable candidate for effective bulk energy density of the assembly bricks mortar is the one of the variational limit of the total strain energy of the wall. This was also derived in [6] through rather heuristical arguments. Our model is simpler than the genuine one because it involves a homogeneous body and accurate enough due to our convergence result. The effective homogeneous material is weaker than the genuine material of the bricks. This is the price to pay due to the difficulty of making a homogeneous wall. The effective strength of the wall should be greater than the one of the bricks when stiff mortar is used. It should be interesting to consider the case when the magnitudes of  $\lambda', \mu$  is of order  $1/l$ .

In final chapter, we extend the results of the fourth chapter but in a 3-dimensional case and with both phases occupying unconnected domain. The results that we got are similar to the previous chapter. Because we did not succeed in proving that the cluster points of sequences  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$  with uniformly bounded finite strain energy do belong to  $H_{\Gamma_0}^1(\Omega; \mathbb{R}^3)$ , we can only claim that a reasonable proposal for effective bulk energy density  $W^{\text{eff}}$  is obtained by through the variational limit of the total energy.

In a 1-dimensional example we have shown the uniform convergence on  $\bar{\Omega}$  and the weak\* convergence in  $BV(\Omega)$  of the unique solution  $u_s$  of  $(\mathcal{P}_s)$  toward the unique solution of

$$(\mathcal{P}^{\text{eff}}) \quad \min \left\{ \int_{\Omega} W^{\text{eff}}(u'(x)) dx \mid u \in H^1(\Omega), u(0) = 0, u(1) = U \right\}.$$

In this case, the limit  $\bar{u}$  of  $u_s$  belongs to  $H^1(\Omega)$  and satisfies the Dirichlet boundary conditions! A last worth noting point is that if  $l$  and  $\mu$  (the stiffness of the weak and strong materials respectively) were fixed the classical effective stiffness would be such that

$$\frac{1}{K^{\text{eff}}} = \frac{1 - 2l}{k} + \frac{2l}{\mu},$$

$2l$  being the total length of the domain occupied by the weak phase. So that the effective stiffness is obtained by letting  $l$  and  $\mu$  going to zero with  $\frac{\mu}{2l}$  tending to  $\bar{\mu}$ !

## REFERENCES

- [1] A. Aitmousa, *Modélisation et étude des Singularités de Contraintes d'un Joint Collé très mince*, Ph.D. Thesis, University Montpellier 2, France, 1989.
- [2] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman Advanced Publishing Program, Boston, 1985.
- [3] H. Attouch, G. Buttazzo, G. Michaille, *Variational Analysis in Sobolev and BV Spaces : Application to PDEs and Optimization*, MPS-SIAM Book Series on Optimization, 2005.
- [4] M. Bellieud, G. Bouchitté, *Homogenization of Elliptic Problems in a Fiber Reinforced Structure. Non local effects*, Ann. Scuola Norm. Sup. CI Sci. (4) 26 (1998) p.407–432.
- [5] M. Bellieud, I. Gruais, *Homogenization of an Elastic Material Reinforced by Very Stiff or Heavy Fibers. Non Local Effects. Memory Effects*, J. Math. Pures Appl. 84 (2005), 55–96.
- [6] A. Cecchi, K. Sab, *A Multi-Parameter Homogenization Study for Modeling Elastic Masonry*, European Journal of Mechanics A/Solids, 21 (2002), p.249–268.
- [7] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer Verlag, 1989.
- [8] P. Grisvard, *Behavior of the Solution of an Elliptic Boundary Value Problem in a Polygonal or Polyhedral domain*, Symposium on Numerical Solutions of Partial Differential Equations III, B. Hubbard Editor, 1975, p.207–274.

- [9] P. Grisvard, *Boundary Value Problems in Plane Polygone. Instructions for Use*, EDF Bulletin de la Direction des Etudes et Recherches Série C' Mathématiques, Informatique, 1 (1986), p.21–59.
- [10] C. Licht, G. Michaille, *A Modelling of Elastic Adhesive Bounded Joints*, Advances in Mathematical Sciences and Applications, 7 (1997), no 2, p.711–740.
- [11] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd Ed., Clarendon Press, Oxford, 1881.
- [12] F. Murat, L. Tartar, *Calcul des Variations et Homogénéisation*, Les Méthodes de l'Homogénéisation Théorie et Applications en Physique, Coll. Dir. Etudes et Recherches EDF, 57, Eyrolles, Paris, 1985, p.319–369.
- [13] F. Murat, L. Tartar, *Optimality Conditions and Homogenization*, in *Non-linear Variational Problems*, A. Marino et al. eds., Pitman, Boston, 1985, p.1–8.
- [14] S. Orankitjaroen, N. Sontichai, C. Licht, A. Kananthai, *Mathematical Modeling of Fiber Reinforced Structures by Homogenization*, Thai Journal of Mathematics, Special Issue Annual Meeting in Mathematics (2008), p.103–115.
- [15] C. Pideri, P. Seppecher, *A Second Gradient Material Resulting from the Homogenization of an Heterogeneous Linear Elastic Medium*, Continuum Mech. Thermodyn. **9** (1997), p.241–257.
- [16] S. Poisson, *Second Mémoire sur la Théorie du Magnétisme*, Mem. Acad, France, 5 (1822).
- [17] J. W. Rayleigh, *On the Influence of Obstacles Arranged in Rectangular Order Upon the Properties of a Medium*, Phil. Mag., 32 (1892), p.481–491.

- [18] L. Tartar, *Problèmes de Contrôle des Coefficients dans des équations aux Dérivées Partielles*, Control Theory, Numrical Methods and Computer Systems Modelling, International Symposium, Rocquencourt, June 1974, A. Bensoussan, J. L. Lions eds., Lecture Notes in Economy and Mathematical Systems, Springer Verlag, 107 (1975), p.420–426.
- [19] L. Tartar, *Estimaiton de Coefficients Homogénéisés*, Computing methods in applied sciences and engineering, Third International Symposium, December 1977, R. Glowinski, J. L. Lions eds., Lecture Notes in Math, Springer Verlag, 704 (1979), p.364–373.
- [20] R. Temam, *Problèmes Mathématiques en Plasticité*, Gauthier-Villars, Paris, 1983.



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**RÉSUMÉ en français**

On étudie quelques problèmes singuliers d'homogénéisation en élasticité linéarisée ou électricité non linéaire où en plus de la période de distributions des hétérogénéités apparaissent d'autres paramètres comme la très forte (ou faible) rigidité (ou permittivité) d'une des phases et la taille relative de celle-ci. Sont visés des milieux fibrés ou stratifiés et des maçonneries planes et minces

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**TITRE en anglais**

**MATHEMATICAL MODELING OF SOME MECHANICS PROBLEMS BY HOMOGENIZATION**

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**RÉSUMÉ en anglais**

We study some singular problems of homogenization in linearized elasticity or nonlinear electricity where in addition to the period of the distribution of heterogeneities other parameters do appear as the very high (or low) stiffness (or permittivity) of one phase and its relative size. Fibered media, stratified media and flat and thin masonries are considered

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**DISCIPLINE**

MECANIQUE

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**MOTS-CLES**

Homogénéisation, Gamma convergence, Milieux élastiques

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