Etude de la théorie de Yang-Mills N=4

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THÈSE DE DOCTORAT
Spécialité :
Physique théorique

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Topics in $\mathcal{N} = 4$ Yang-Mills Theory

Thèse dirigée par David Kosower
Soutenue le : 19 Octobre 2012

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Abstract

This thesis describes some of the developments in calculational techniques for scattering amplitudes in super-symmetric gauge field theory. The focus is on on-shell recursion relations and on the use of unitarity methods for loop calculations. In particular, on-shell recursion is related to the BCFW rules for computing tree-level gauge amplitudes. Combinations of unitarity cut techniques and recursion are used to compute the loop level amplitude. And finally start from amplitudes to obtain energy-energy correlation function in Super-Yang-Mills $\mathcal{N} = 4$ Theory with the aid of Mellin–Barnes representation. In the last chapter, we were trying to find a convergent contour for the Mellin–Barnes integrals in multi-dimension obtained by looking for some approximation stationary-phase contour.

Keywords: scattering amplitudes, phase space integral, Mellin–Barnes representation, stationary-phase contour, supersymmetric Yang-Mills theory, decay of Higgs Boson

Contents

Chapter 1

Introduction to the model and to the notations

This thesis is devoted in part to the computation of the next-to-leading order (NLO) corrections to the energy-energy correlation (EEC in [30] due to Richards, Stirling, S.D. Ellis, [31] due to the SLD Collaboration, [32] due to Kramer and Spiesberger , [34] due to Basham, Brown, Ellis and Love also in [9] due to Hofman and Maldacena, etc) in the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. The EEC measures the correlation between energy emitted in two different directions from a collision or decay. It is a classic observable in electron–positron collisions, and can usefully be adapted to the $\mathcal{N} = 4$ super-Yang–Mills theory. Its interest here is in offering a finite observable that can be studied in both the weak- and strong-coupling regimes.

The energy-energy correlation function is defined by:

$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j} \int dPS_n |A_n|^2 \frac{E_i E_j}{E_{total}^2} \delta(\cos\theta_{ij} - \cos\chi). \tag{1.1}$$

where χ is the relative angle between two fluxes of energy, E_i and E_j and the self-correlation $i = j$, $\chi = 0$ are included. In weak coupling regime, the EEC [34] was first computed in QCD by Basham, Brown, Ellis and Love. This correlation function was introduced in order to offer a precise direct test of perturbative QCD in high energy e^+e^- annihilation because it is calculable and also accessible to experiment. The initial state was chosen to be high energy e^+e^- because there is no hadron in the initial state and the coupling constant vanishes asymptotically so that the EEC can be calculated. Richards, Stirling and Ellis [30] studied the $\mathcal{O}(\alpha_s^2)$ QCD correction to the EEC for the same process. One of the applications of this observable is measuring the strong coupling α_s, as was done by the SLD collaboration [31].

To connect weak and strong coupling, a correspondence between a gauge theory and string theory was introduced. It was suggested by 't Hooft that theory might simplify when the number of colors N_c is large [38]. The diagrammatic expansion of the field theory suggests that the large N_c theory is a free string theory and that the string coupling constant is $1/N_c$. In this way the large N_c limit connects gauge theories with string theories. The $\mathcal{N} = 4$ supersymmetric Yang–Mills theory is a conformal gauge theory where the coupling constant does not run as a function of the energy scale. It is the maximal possible supersymmetric field theory in four dimensions. It contains one gluon, four fermions and six scalars. There is a global $SU(4)$ R-symmetry that rotates the six scalar fields and the four fermions. The conformal group in four dimensions is $SO(4, 2)$. If we want to look for a string theory to which $\mathcal{N} = 4$ supersymmetric Yang–Mills theory corresponds, we would demand that the string theory should reflect the symmetry, for example R-symmetry in $\mathcal{N} = 4$. One would look for a five dimensional geometry which has these symmetries. Locally there is only one space with $SO(4, 2)$ isometries: five dimensional Anti-de-Sitter space, or AdS_5 . Anti-de Sitter space is the maximally symmetric solution of Einstein's equations with a negative cosmological constant. In this supersymmetric case we expect the strings to also be supersymmetric. The superstrings move in ten dimensions. Now that we have added one more dimension it is not surprising any more to add

five more to get to a ten dimensional space. Since the gauge theory has an $SU(4) \cong SO(6)$ global symmetry it is rather natural that the extra five dimensional space should be a five sphere, S^5. This suggests that $\mathcal{N} = 4$ $U(N_c)$ Yang-Mills theory could be the same as ten dimensional superstring theory on $AdS_5 \times S^5$. This is called AdS/CFT correspondence ([46], [47], [48], [49] due to Maldacena, etc).

Hofman and Maldacena [9] studied the general properties of energy and charge correlations in conformal field theory. The energy correlation functions is defined in terms of correlation functions of local gauge invariant operators. They explained that the integrated energy flux per unit angle at a large sphere of radius r can be expressed in term of energy-momentum tensor,

$$E(\theta) = \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} dt\, n^i\, T_i^0(t, r\overrightarrow{n}) \tag{1.2}$$

where $\overrightarrow{n}$ is a unit vector in R^3 and it specifies the point on the S^2 at infinity.

In order to compare the EEC in non-perturbative conformal theory and the EEC in perturbative quantum field theory, it is interesting to calculate EEC in the $\mathcal{N} = 4$ Yang-Mills theory. I will use the decay of a scalar coupled to the $\mathcal{N} = 4$ Lagrangian in order to compute the EEC.

This thesis describes some of the recent progress on calculating the amplitude using only on-shell states. It utilize these on-shell method to produce the tree-level and one-loop level three particle amplitudes and tree-level four particle amplitudes which are needed to calculate the EEC to NLO (next-to-leading order). When integrating the differential cross-section obtained from the amplitudes in phase space, the thesis develops the use of Mellin–Barnes techniques for computing phase-space integrals. It applies them to the computation of the integrals needed for the EEC. It also develops techniques for use of Mellin–Barnes representations numerically where prior methods lead to numerically unstable or nonconvergent results.

The first chapter introduces some basic notation and background material. In the second chapter, I compute the tree amplitudes required for the leading order and real-emission corrections. This includes all decays with up to four particles in the final state. And at the end of the second chapter, I also obtain the LO (leading order) EEC. In the third chapter, I compute the one-loop corrections to the scalar decay amplitudes using unitarity, and obtain the virtual corrections to the EEC. In the fourth chapter, I apply Mellin–Barnes techniques to calculate all the basic integrals needed for the real-emission corrections to the EEC In the fifth and final chapter, I study techniques for improving the numerical convergence of higher-dimensional Minkowski-domain Mellin–Barnes integrals, using approximations to integration contours of stationary phase.

1.1 Introduction to the Model and the Motivation

One of main focus of my thesis is the perturbative calculation, through NLO, of the energy-energy correlation function in supersymmetric $\mathcal{N} = 4$ Yang–Mills. I have calculated it using the decay of a scalar coupled to the $\mathcal{N} = 4$ Lagrangian. The effective Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \lambda H \mathcal{L}_{\mathcal{N}=4}. \tag{1.3}$$

I will only calculate the contributions from each pair $i \neq j$, $\theta \neq 0$ in eq. (1.1). Then EEC can be written as,

$$\frac{d\Sigma}{dc} = \sum_{i<j} \int dPS_n |A_n|^2 \frac{E_i E_j}{E_{total}^2} \delta(\cos\theta_{ij} - c). \tag{1.4}$$

where $c = \cos\chi$. Expand this EEC to series in coupling g,

$$\frac{d\Sigma}{dc} = g^2 EEC_{LO} + g^4 EEC_{NLO} \tag{1.5}$$

For calculating the EEC_{LO}, I did not take the two particle decays state into account, because the final two particle are outgoing back-to-back in center-mass system, so the angle between the momenta of the two final particles is a constant π. To obtain the leading order (LO) contribution, I calculate the tree level amplitudes of H to three-particle decays. In my calculation, I will only calculate different

In order to obtain a finite result at NLO, EEC_{NLO}, we have to combine the one loop virtual contribution EEC_V and the real emission EEC_{real}. The virtual contributions arise from integrating the interference term of one-loop three-particle amplitudes and their tree-level counterparts over D-dimensional($D = 4-2\epsilon$) three-particle phase space. The one-loop amplitudes were obtained using generalized unitarity. The real-emission contributions arise from integrating the squared tree-level four-particle amplitudes over the D-dimensional four-particle phase space (see fig. 1.1). The divergence in virtual contribution comes from the one-loop amplitude caused by singularities at small loop momenta. The divergence in real emission contribution comes from the integration of soft and collinear singularities in the phase space. In both of the events, we use D-dimensional regularization to resolve the singularities.

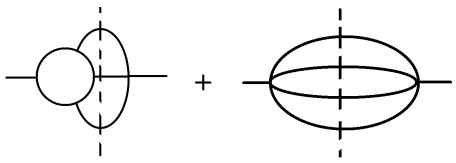

Figure 1.1: A diagrammatic representation of the infra-red cancellation between the virtual contributions and the real emission contributions.

For computing and expressing the amplitudes of gauge theory in a sensible way, as already well explained in articles [36] due to Mangano and Parke, [2] due to Dixon and [1] due to Parke and Taylor, we'd better to track all possible informations (color and helicity) about the external particles. The price to pay is to calculate all different color-ordering amplitudes with different helicity configurations. In the following sections, we will review the techniques: color management, spinor helicity notations, super-symmetry.

1.2 Color management

In gauge theory, we track the informations, for example color and helicity, to decompose an amplitude into sum of partial amplitude. A tree level gluon amplitude with definite helicity for each particle can be written as a the sum of product of a single trace and a color-ordered partial amplitude (in review article [2] due to Dixon):

$$\mathcal{A}_n^{\text{tree}}(k_i, \lambda_i, a_i) = \sum_{\sigma=S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} ... T^{a_{\sigma(n)}}) A_n^{tree}(\sigma(1^{\lambda_1}), \sigma(2^{\lambda_2}), ..., \sigma(n^{\lambda_n})), \tag{1.6}$$

where k_i, λ_i are respectively the momentum and the helicity of the i-th parton which is simply denoted as i^{λ_i} and the matrices T are the generators of the gauge group in the adjoin representation, $a = 1, \cdots, N_c$. The one-loop level amplitude can be decomposed as,

$$\begin{aligned} \mathcal{A}_n^{1-\text{loop}}(k_i, \lambda_i, a_i) = & \sum_{\sigma=S_n/Z_n} (N_c \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}) A_{n:1}^{1-\text{loop}}(\sigma(1^{\lambda_1}), \sigma(2^{\lambda_2}) \dots) \\ & + \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma=S_n/S_{n:c}} G_{n:c} A_{n:c}^{1-\text{loop}}(\sigma(1^{\lambda_1}), \sigma(2^{\lambda_2}) \dots)), \end{aligned} \tag{1.7}$$

where

$$G_{n:c} = \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}}) \text{Tr}(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}}) \,. \tag{1.8}$$

Each partial amplitude can be computed by Feynman digram with color-order vertex. In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, all the amplitude even involving fermion and scalar particle can be decomposed in the above way, since all those particles are in adjoin representation of gauge group. Finally, when we calculate the differential cross-section, we need to sum all the different color contributions. The following formulas are very useful to manipulate T^a matrices. The generators T are normalized in the following way:

$$\text{Tr}\left(T^a T^b\right) = \delta^{ab} \tag{1.9}$$

and

$$(T^a)^{\bar{j}}_i (T^a)^{\bar{l}}_k = \delta^{\bar{l}}_i \delta^{\bar{j}}_k - \frac{1}{N_c} \delta^{\bar{j}}_i \delta^{\bar{l}}_k. \tag{1.10}$$

Using the above two properties, we can simplify the trace of product of matrices T to a polynomial of N_c (We neglect the $1/N_c$ term in eq. (1.10) for we consider the large N_c limit):

$$\mathrm{Tr}(T^a M_1)\mathrm{Tr}(T^a M_2) = \mathrm{Tr}(M_1 M_2) \tag{1.11}$$

and

$$\mathrm{Tr}(T^a M_1 T^a M_2) = \mathrm{Tr}(M_1)\mathrm{Tr}(M_2) \Rightarrow \mathrm{Tr}(T^a T^a M_1) = N_c \mathrm{Tr}(M_1), \tag{1.12}$$

where the symbols M_1 and M_2 represent two products of matrices. Since we are only interested in the large N_c limit, we will try to calculate all $\mathcal{A}_4^{tree}$ eq. (1.6) and the leading color-order contributions of $\mathcal{A}_3^{1-loop}$ (the single trace term in eq. (1.7)) for all kinds of external particles.

1.3 Spinor Helicity

Spinor variables, spinor products, and the spinor-helicity method allow us to represent amplitudes in a very compact form. We track the helicities of all external particles and write all the different helicity amplitudes using spinor products. Spinors originate in solutions of the Weyl equation,

$$p_\mu \sigma^\mu u(p) = 0. \tag{1.13}$$

From the on-shell condition:

$$p^2 = 0 \Leftrightarrow \mathrm{Det}(p_\mu \sigma^\mu) = 0, \tag{1.14}$$

the rank of $p_\mu \sigma^\mu$ drops to one so we may decompose $p_\mu \sigma^\mu$ into a product of a complex two-dimensional vector and its complex conjugate:

$$p^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \tag{1.15}$$

where the elements of λ^α and of $\tilde{\lambda}^{\dot{\alpha}}$ are functions of momentum p and α, $\dot{\alpha} = 0, 1$. If the momentum p is real, then $\tilde{\lambda}^{\dot{\alpha}}$ is complex conjugate of λ^α. We define spinor product:

$$\begin{aligned} \langle ij \rangle &= \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta, \\ [ij] &= \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}}. \end{aligned} \tag{1.16}$$

where the spinors λ_i^α and $\tilde{\lambda}_i^{\dot{\alpha}}$ correspond to momentum p_i. Because $\epsilon_{\alpha\beta}$ is anti-symmetric under exchange of labels:

$$\begin{aligned} \langle ij \rangle &= -\langle ji \rangle, \\ \langle ii \rangle &= 0. \end{aligned} \tag{1.17}$$

The two-dimensional spinor can be also considered as the eigenvector or projection of four-dimensional spinor of helicity operator.

$$\begin{aligned} u^+(p) &= \lambda_\alpha(p) \equiv |p\rangle \\ u^-(p) &= \tilde{\lambda}^{\dot{\alpha}}(p) \equiv |p], \end{aligned} \tag{1.18}$$

whose conjugate spinors are

$$\begin{aligned} \bar{u}^-(p) &= \lambda^\alpha(p) \equiv \langle p|, \\ \bar{u}^+(p) &= \tilde{\lambda}_{\dot{\alpha}}(p) \equiv [p|. \end{aligned} \tag{1.19}$$

We use a concrete representation of γ matrices to find following properties in [2] due to Dixon,

$$\langle p|\gamma^\mu|q] = [q|\gamma^\mu|p\rangle, \tag{1.20}$$

$$\langle q|\gamma^\mu|q] = 2q^\mu, \tag{1.21}$$

$$\langle p|\gamma^\mu|q]\langle r|\gamma_\mu|s] = -2\langle pr\rangle[qs] \qquad \text{(Fierz rearrangement)}, \tag{1.22}$$

$$\langle p|\gamma^\mu\gamma^\nu|q\rangle = -\langle q|\gamma^\nu\gamma^\mu|p\rangle, \tag{1.23}$$

$$\langle pq\rangle\langle r| + \langle qr\rangle\langle p| + \langle rp\rangle\langle q| = 0 \qquad \text{(Schouten Identity)}. \tag{1.24}$$

We can introduce a spinor representation for the polarization vector for a massless gauge boson of definite helicity ± 1 in ref. [39] due to Xu, Zhang and Chang,

$$\varepsilon_\mu^\pm(k,q) \;=\; \pm\frac{\langle q^\mp|\gamma_\mu|k^\mp\rangle}{\sqrt{2}\langle q^\mp|k^\pm\rangle}\;, \tag{1.25}$$

where k is the vector-boson momentum and q is an auxiliary massless vector, called the reference momentum, reflecting the freedom of performing on-shell gauge transformations. We can verify that the introduced polarization vector satisfy all the properties of polarization vectors,

$$\varepsilon^\pm(p)\cdot p = 0,\;\; \varepsilon_\mu^+(p) = (\varepsilon_\mu^-(p))^*, \tag{1.26}$$

$$\varepsilon^+(p)\cdot\varepsilon^-(p) = -1,\;\; \varepsilon^+(p)\cdot\varepsilon^-(p) = 0. \tag{1.27}$$

$$\sum_{\lambda=\pm}\varepsilon_\mu^\lambda(k,q)\,(\varepsilon_\nu^\lambda(k,q))^* \;=\; -\eta_{\mu\nu} + \frac{k_\mu q_\nu + k_\nu q_\mu}{k\cdot q}\;. \tag{1.28}$$

A *Mathematica* package, called S@M [25] due to Maître and Mastrolia, was used extensively in this thesis to perform computations with spinors. All the tree-level and one-loop amplitudes I computed were obtained analytically using programs based on this package.

1.4 Supersymmetry

In super-Yang–Mills $\mathcal{N}=4$, there is an extended SUSY algebra:

$$\left\{Q_\alpha^a, \bar{Q}_{\dot\alpha b}\right\} = 2\delta_b^a \mathbf{P}_{\alpha\dot\alpha}, \tag{1.29}$$

where $a, b = 1,\ldots,4$. We define another supercharge:

$$Q^a(q,\eta) = \eta q^\alpha Q_\alpha^a, \tag{1.30}$$

where q is a reference momentum like the one we used to define polarization vectors in last section, and η is a Grassmann variable. If there is no spontaneous breaking of supersymmetry, then $Q^a(q,\eta)$ should annihilate the vacuum. So:

$$\langle 0|\left[Q^a(q,\eta), \Phi_1\Phi_2\cdots\Phi_n\right]|0\rangle = \sum_{i=1}^{n}\langle 0|\Phi_1\cdots[Q^a(q,\eta),\Phi_i]\cdots\Phi_n|0\rangle \tag{1.31}$$

where Φ_i are operators creating the helicity eigenstates. In $\mathcal{N}=4$, Φ_i are $g^\pm$, $f^\pm$ and s. We have one gluon $g^\pm$, four fermions $f_a^\pm$, six real scalars $s_{ab} = -s_{ba}$, along with the algebra

$$\begin{aligned}
[Q_a(q,\eta), g^+(k)] &= \eta[kq]f_a^+,\\
[Q_a(q,\eta), f_b^+(k)] &= \eta\delta_{ab}\langle kq\rangle g^+ + \eta[kq]s_{ab},\\
[Q_a(q,\eta), s_{bc}(k)] &= \eta\delta_{ab}\langle kq\rangle f_c^+ - \eta\delta_{ac}\langle kq\rangle f_b^+ + \eta[qk]\epsilon_{abcd}f_d^-,\\
[Q_a(q,\eta), f_b^-(k)] &= \eta\delta_{ab}[qk]g^- + \frac{1}{2}\eta\langle qk\rangle\epsilon_{abcd}s_{cd}\\
[Q_a(q,\eta), g^-(k)] &= \eta\langle qk\rangle f_a^-.
\end{aligned} \tag{1.32}$$

Equipped with these relations, in $\mathcal{N} = 4$ Yang–Mills, we can deduce all tree-level MHV amplitudes from $A^{\text{MHV}}(\cdots, g_i^-, \cdots, g_j^- \cdots)$ where $\cdots$ stands for an arbitrary number of positive-helicity gluons g^+. The generalization to N^{K}MHV amplitude is in [35] due to Elvang, Freedman, Kiermaier. In this thesis, we consider the decay of a scalar particle H in $\mathcal{N} = 4$ Yang–Mills. We may impose a trivial action of the supercharge operator on H,

$$[Q, H] = 0 \tag{1.33}$$

as was done by Badger, Glover, Khoze [12].

1.5 Mellin–Barnes Integrals

The Mellin–Barnes approach has been extensively used to evaluate Feynman (loop) integrals analytically, and numerically for integrals with massless propagators. A pedagogical introduction to this approach can be found in Smirnov's book [28]. I use the Mellin–Barnes representation to compute the phase-space integrals. Several implementations, for example `MB.m` and `MBasymptotics.m`(due to Czakon [22]), `MBresolve.m` (due to Smirnov) and `barnesroutines.m`(due to Kosower) in *Mathematica* were used to evaluate the Mellin–Barnes representations analytically and numerically. In the last chapter of this thesis, we try to find a convergent contour to evaluate Mellin–Barnes representation numerically. We apply the idea of finding the steepest descent contour to the Mellin–Barnes integral. The introduction to the steepest descent contour can be found in many textbooks, for example, Wong's book [29]. In one-complex dimension, the contour of steepest descent is the contour of stationary phase passing one or more stationary points (zeros of derivative of integrand). In higher dimension, the contour of steepest descent is not defined, so we try to find a contour avoiding all the singularities in the space of stationary phase as a possible candidate for the numerical evaluation of Mellin–Barnes integrals.

Chapter 2

Calculations of all tree level amplitudes

In this chapter, I give a brief introduction in section 2.1 to our model. I calculate all tree-level amplitudes with up to four particles in the final state using the BCFW on-shell recursion relations due to Britto, Cachazo and Feng [4] in section 2.2, and using a supersymmetric form of the BCFW recursion in section 2.3 as a more rapid way and also as a cross-check. And finally, from the amplitudes I obtain the differential cross section in section 2.5. Using H plus three-particle differential cross-section, in section 2.6, I obtained the leading order EEC function.

2.1 Set up

We consider the scalar particle H coupled to all terms $\mathcal{N} = 4$ Lagrangian. To simplify the calculation, I rewrite the $\mathcal{N} = 4$ Lagrangian slightly:

$$\partial_\mu s \partial^\mu s \longrightarrow s \Box s, \tag{2.1}$$

where s represents a scalar particle. This replacement changes nothing in the $\mathcal{N} = 4$ Lagrangian. With the second form, the amplitude $H \to s\ s$ and $H \to f\ f$ vanish if the final particles are on-shell, so such decay channels do not arise at the tree level. The only two-particle final state $g\ g$. The Lagrangian term for this decay is:

$$\mathcal{L}_H^{int} = \frac{1}{2} H \mathrm{Tr} G_{\mu\nu} G^{\mu\nu} \tag{2.2}$$

which is also a term in the QCD effective Lagrangian in the large top-mass limit. Accordingly we can apply the decomposition of Badger, Glover, and Khoze [12]. The Higgs boson is considered to be the real part of a complex field, $\phi = \frac{1}{2}(H + iA)$. We can write down the interaction term $H\mathrm{Tr} G_{\mu\nu} G^{\mu\nu}$ as follows,

$$\mathcal{L}_{H,A}^{int} = \phi \mathrm{Tr} G_{SD\mu\nu} G_{SD}^{\mu\nu} + \phi^\dagger \mathrm{Tr} G_{ASD\mu\nu} G_{ASD}^{\mu\nu} \tag{2.3}$$

where ${}^*G^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}, G_{SD\mu\nu} = \frac{1}{2}(G_{\mu\nu} + {}^*G_{\mu\nu})$, $G_{ASD\mu\nu} = \frac{1}{2}(G_{\mu\nu} - {}^*G_{\mu\nu})$. To obtain the Higgs amplitude $A_n(H, \dots)$, we need to calculate both the ϕ and $\phi^\dagger$ amplitudes $A_n(\phi, \dots)$ and $A_n(\phi^\dagger, \dots)$ and sum them,

$$A_n(H, \dots) = A_n(\phi, \dots) + A_n(\phi^\dagger, \dots). \tag{2.4}$$

We do this because the ϕ amplitudes $A_n(\phi, \dots)$ are relatively simple to calculate; and we can obtain $A_n(\phi^\dagger, \dots)$ from $A_n(\phi, \dots)$ by parity. The two-particle amplitudes were computed using Feynman diagrams in ref. [40] (due to Dixon and Khoze),

$$\begin{aligned} A^{\mathrm{tree}}(\phi,\, 1_g^-, 2_g^-) &= -\langle 12 \rangle^2 \\ A^{\mathrm{tree}}(\phi^\dagger,\, 1_g^+, 2_g^+) &= -[12]^2 \end{aligned} \tag{2.5}$$

where the symbol 1_g^- represents a gluon with momentum p_1 and negative helicity. The amplitudes in QCD with multi-final-particle states were calculated in ref. [12] (due to Badger, Glover, and Khoze) using MHV rules [13] due to Cachazo, Svrček and Witten.

The MHV rules were introduced in ref. [13] due to Cachazo, Svrček and Witten. The authors provide a method to build amplitude by connecting the off-shell continuations of the maximally helicity-violating (MHV) amplitude by scalar propagators. The MHV amplitude in pure gluon gauge theory contains two negative helicity gluons,

$$A^{\rm MHV}(1^+,\cdots,j^-,\cdots,k^-,\cdots,n^+) = \frac{i\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1n\rangle\langle n1\rangle} \tag{2.6}$$

where 1^+, n^+ and $\cdots$ represent positive helicity gluons. The amplitude in pure gluon gauge theory contains three negative gluons is called NMHV amplitude. By gluing the MHV amplitude together, one obtains the amplitude with more negative helicities.

2.2 BCFW On-Shell Recursion Relations

The standard technique for computing the amplitude is that of Feynman diagrams. Even with spinor representation and with the color-ordered vertices (as explained, for example in Dixon's lecture notes [2]), for each amplitude, there are a lot of diagrams to calculate. In 2004, two on-shell methods were born. Unlike Feynman rules, these methods — due to Britto, Cachazo and Feng [4] (BCF) and Cachazo, Svrček and Witten [13](CSW). BCF— use only on-shell states to reproduce all amplitudes. We may take the BCF recursion relations to be a special case of BCFW in [5] due to Britto, Cachazo, Feng and Witten. If one shift the spinors of two adjacent external legs in BCFW, one can obtain the BCF relation. The operation "shift" will be explained in the following text. Here, we use the BCFW on-shell recursion relations to calculate all the required tree amplitudes. The BCFW recursion effectively allows one to reconstruct the full form of amplitude by using Cauchys Theorem. To use it, define a shift of two external momenta p_j and p_l using a complex variable z. The $[j,l\rangle$ shift is defined by

$$\tilde{\lambda}_j \to \tilde{\lambda_j} - z\tilde{\lambda_l},\ \lambda_l \to \lambda_l + z\lambda_j, \tag{2.7}$$

where λ_j and $\tilde{\lambda}_j$ are the spinors corresponding to p_j, as introduced in eq. (1.15) . The conservation of momentum holds for the shifted momenta, because

$$p_j(z) + p_l(z) = \lambda_j(\tilde{\lambda_j} - z\tilde{\lambda_l}) + (\lambda_l + z\lambda_j)\tilde{\lambda_l} = \lambda_j\tilde{\lambda_j} + \lambda_l\tilde{\lambda_l} = p_j + p_l\,. \tag{2.8}$$

After a $[j,l\rangle$ shift, an amplitude $\mathcal{A}$ becomes $\mathcal{A}(z)$, a function of z; the physical amplitude is given by $\mathcal{A}(0)$. Let us consider a certain contour integral of $\mathcal{A}$, taken around the circle at infinity,

$$\frac{1}{2\pi i}\oint dz\frac{\mathcal{A}(z)}{z} = 0 \Rightarrow \mathcal{A} = \mathcal{A}(0) = -\sum_{z_{\rm poles}} {\rm Res}\frac{\mathcal{A}(z)}{z} \tag{2.9}$$

As shown in the above equation eq. (2.9), $\mathcal{A}(0)$ can be obtained by taking the residue of z at non zero poles and removing the contribution when $z \to \infty$. If we calculate a tree level amplitude by the color-ordered Feynman diagrams, we find that the denominators of the amplitude could be a products of propagators, so we notice the singularities of tree level amplitudes is the zeros of the multi-particle propagators. For example, let's consider

$$K^2_{r\ldots j\ldots s} = (p_r + \cdots + p_j + \cdots + p_s)^2 = 0 \tag{2.10}$$

which is one of the singularity of a tree level amplitude, where $p_r + \cdots + p_j + \cdots + p_s$ is a sum of adjacent momenta and the momentum p_l is not in the sum. If we perform the shift $[j,l\rangle$, then

$$\hat{K}^2_{r\ldots j\ldots s} = (p_r + \cdots + p_j(z) + \cdots + p_s)^2 = (p_r + \cdots + \lambda_j(\tilde{\lambda_j} - z\tilde{\lambda_l}) + \cdots + p_s)^2 \tag{2.11}$$

The shifted propagator $\hat{K}^2_{r\ldots j\ldots s} = 0$ is one of poles of the shifted amplitude $\mathcal{A}(z)$. So we can use the on-shell condition solving for z,

$$\hat{K}^2(z_{r\ldots j\ldots s}) = 0, z_{rs} = \frac{K^2_{r\ldots j\ldots s}}{\langle j|K_{r\ldots s}|l]}. \tag{2.12}$$

Then the BCFW recursion tells us, the residue of $\mathcal{A}(z)/z$ at non-zero z_{rs} (in eq. (2.12)) can be obtained by sub-amplitudes, (see fig. 2.1),

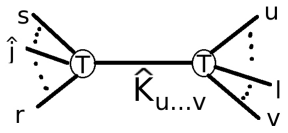

Figure 2.1: The digram representation of $D^h_{r\ldots s}$

$$D^h_{r\ldots s} = A^{\text{right}}(r, \ \ldots \ , \hat{\jmath}, \ \ldots \ , s, \ -\hat{K}^h_{r\ldots j\ldots s}) \frac{i}{K^2_{r\ldots j\ldots s}} A^{\text{left}}(u, \ \ldots \ , \hat{l}, \ \ldots \ , v, \ \hat{K}^{-h}_{r\ldots j\ldots s}) \tag{2.13}$$

where A^{right} and A^{left} are two sub-amplitudes. One can replace the momentum $-\hat{K} \to \hat{K}$ and multiply the formula (2.13) by a minus sign. The denominator is $K^2_{r\ldots j\ldots s}$ is the sum of unshifted momenta. We next replace all shifted momentum spinors by the unshifted spinors, using the suitably fixed value of the complex variable z_{rs},

$$\langle \hat{\jmath} a\rangle = \langle ja\rangle, \quad [\hat{\jmath} a] = [ja] - z_{rs}[la], \tag{2.14}$$

$$[\hat{l} a] = [la], \quad \langle \hat{l} a\rangle = \langle la\rangle + z_{rs}\langle ja\rangle, \tag{2.15}$$

$$[\hat{\jmath}\hat{K}] = \frac{\langle jKj] - z_{rs}\langle jKl]}{\langle \hat{K} j\rangle}, \tag{2.16}$$

$$[a\hat{K}] = \frac{\langle j|K|a]}{\langle \hat{K} j\rangle}, \tag{2.17}$$

$$\langle \hat{K}\hat{l}\rangle = -\frac{\langle l|K|l] - z_{rs}\langle j|K|l]}{[\hat{K} l]}, \tag{2.18}$$

$$\langle \hat{K} a\rangle = -\frac{\langle a|K|l]}{[\hat{K} l]}. \tag{2.19}$$

$$\tag{2.20}$$

All spinor products corresponding to the internal momentum $\hat{K}$ must match between angle and bracket products, so match them up, $\hat{K}$ spinors, $\langle a\hat{K}\rangle[\hat{K} b] = \langle a|\hat{K}|b]$ and then replace,

$$\langle a|\hat{K}|b] = \langle a|K|b] + z_{rs}\langle aj\rangle[lb]. \tag{2.21}$$

Performing the replacements eq. (2.14)-(2.20) and eq. (2.21) in eq. (2.13), we obtain a formula which is a function of z_{rs} with all hatted variables replaced. Substitute the solution for z_{rs} in eq. (2.12) into $D^h_{r\ldots s}$ to obtain contribution in this channel. To generate the total amplitude, we sum all contributions of all different channels in order to obtain the full form of the amplitude $\mathcal{A}(0)$,

$$A = \mathcal{A}(0) = \sum_{r,s,h} A^h_L \frac{i}{K^2_{r\ldots s}} A^{-h}_R \tag{2.22}$$

Before apply BCFW to compute the tree level amplitude, we introduce three point MHV and $\overline{\text{MHV}}$ as building blocks. For three point amplitudes $A(1,2,3)$, if the three momenta are real and on-shell, then one obtain

$$s_{12} = (p_1 + p_2)^2 = (-p_3)^2 = 0, \tag{2.23}$$

$$s_{12} = 0 \Rightarrow s_{12} = \langle 12\rangle[21] \Rightarrow \langle 12\rangle = 0,\ [12] = 0. \tag{2.24}$$

where the spinor product [21] is the complex conjugate of $\langle 12\rangle$ if p_1 and p_2 are real. All the spinors' products vanish. For solving this problem, we need to take the momenta to be complex. It is possible to choose the spinors which obey ,

$$\tilde{\lambda_1} \sim \tilde{\lambda_2} \sim \tilde{\lambda_3} \text{ and } c_1\,\lambda_1 + c_2\,\lambda_2 + c_3\,\lambda_3 = 0 \tag{2.25}$$

or

$$\lambda_1 \sim \lambda_2 \sim \lambda_3 \text{ and } c_1\,\tilde{\lambda_1} + c_2\tilde{\lambda_2} + c_3\tilde{\lambda_3} = 0, \tag{2.26}$$

as explained in review [3] due to Bern, Dixon and Kosower. Each of the above two classes of spinors permits us to define the three point amplitudes,

$$A^{\rm MHV}(1^-, 2^-, 3^+) = \frac{i\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \text{ and } A^{\overline{\rm MHV}}(1^+, 2^+, 3^-) = -\frac{i[12]^4}{[12][23][31]}. \tag{2.27}$$

Let us compute two examples here explicitly in order to demonstrate this procedure. First, we want to calculate the NMHV amplitude $A(\phi, 1_g^-, 2_g^-, 3_g^-, 4_g^+)$. Using the shift $[1^-, 2^-\rangle$, there are four diagrams which need to be calculated. They are:

$$A(\phi, 1_g^-, 2_g^-, 3_g^-, 4_g^+) = D^{(a)} + D^{(b)} + D^{(c)} + D^{(d)}, \tag{2.28}$$

where the diagrams corresponding the contributions $D^{(a)},\ D^{(b)},\ D^{(c)},\ D^{(d)}$ were shown in fig. 2.2. The

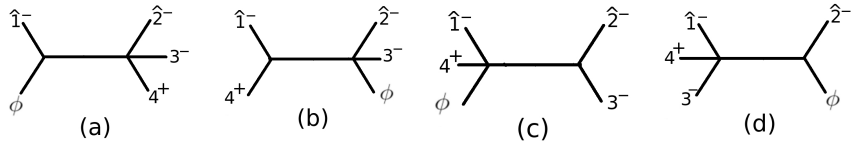

Figure 2.2: The four on-shell recursive diagrams obtained from the $[1, 2\rangle$ shift in the computation of $A(\phi, 1_g^-, 2_g^-, 3_g^-, 4_g^+)$. The contribution of diagram (c) vanishes, as explained in the text.

contributions of each diagram in fig. 2.2 are,

$$D^{(a)} = A_2^{\rm tree}(\phi,\ \hat{1}^-,\ \hat{K}_{234}^-)\frac{i}{s_{234}}A_4^{\rm tree}(\hat{2}^-,\ 3^-,\ 4^+,\ -\hat{K}_{234}^+), \tag{2.29}$$

$$D^{(b)} = A_3^{\rm tree}(\hat{1}^-,\ 4^+,\ -\hat{K}_{14}^-)\frac{i}{s_{14}}A_3^{tree}(\phi,\ \hat{2}^-,\ 3^-,\ \hat{K}_{14}^+), \tag{2.30}$$

$$D^{(d)} = A_4^{\rm tree}(\hat{1}^-,\ -\hat{K}_{134}^+,\ 3^-,\ 4^+)\frac{i}{s_{134}}A_2^{\rm tree}(\phi,\ \hat{2}^-,\ \hat{K}_{134}^-). \tag{2.31}$$

The contribution of (c) (see fig. 2.2) vanishes because in three-particle-amplitude, the complex variable z should be calculated by

$$s_{\hat{23}} = \langle\hat{23}\rangle[3\hat{2}] = \langle\hat{23}\rangle[32] = 0 \Rightarrow \hat{\lambda_2} \propto \lambda_3 \tag{2.32}$$

The three particle amplitude $A_3^{\rm tree}(\hat{2}^-,\ 3^-,\ \hat{K}_{14}^+)$ vanishes when $\hat{\lambda_2} \propto \lambda_3$, so the contribution of $D^{(c)}$ vanishes. For the other contributions,

$$\begin{aligned} D^{(a)} &= \langle\hat{1}\hat{K}_{234}\rangle^2\frac{i}{s_{234}}\frac{i[4\hat{K}_{234}]^4}{[\hat{23}][34][\hat{K}_{234}\hat{2}]} \\ &= \langle 1\hat{K}_{234}\rangle^2\frac{i}{s_{234}}\frac{i[4\hat{K}_{234}]^4}{[23][34][\hat{K}_{234}2]} \\ &= \frac{\langle 1|\hat{K}_{234}|4]^3}{s_{234}\langle 1|\hat{K}_{234}|2][32][43]} \\ &= \frac{\langle 1|2+3|4]^3}{s_{234}\langle 1|3+4|2][32][43]}. \end{aligned} \tag{2.33}$$

We also calculate $D^{(b)}$ and $D^{(d)}$ in the same way. We obtain,

$$A(\phi,1_g^-,2_g^-,3_g^-,4_g^+)=-\frac{m_H^4\langle 13\rangle^4}{s_{134}\langle 14\rangle\langle 34\rangle\langle 1|3+4|2]\langle 3|1+4|2]}-\frac{\langle 3|1+2|4]^3}{s_{124}\langle 3|1+4|2][21][41]}+\frac{\langle 1|2+3|4]^3}{s_{234}\langle 1|3+4|2][32][43]}. \tag{2.34}$$

where m_H is the mass of the scalar H. If we apply the color-ordered Feynman rules to calculate this amplitude, the possible denominators are $s_{i\ i+1}$ or $s_{i\ i+1\ i+2}$, where i and $i+1$ represent two neighbor momenta. We notice that in the expression of $A(\phi,1_g^-,2_g^-,3_g^-,4_g^+)$, eq. (2.34), there are denominators like $\langle 1|3+4|2]$ and $\langle 3|1+4|2]$. It might appear that the amplitude will have poles when $p_1+p_4\propto p_2$ or $p_1+p_4\propto p_3$. Such poles in the amplitude are not physical because they are not corresponding to soft or collinear singularities, and they are called spurious poles. In fact, they cancel between different terms, and we can use spinor formulas to remove them. After all the spurious poles have been removed, $A(\phi,1_g^-,2_g^-,3_g^-,4_g^+)$ becomes:

$$\begin{aligned}A(\phi,1_g^-,2_g^-,3_g^-,4_g^+) &= \frac{\langle 23\rangle^2(\langle 1|2|4]+\langle 1|3|4])^2}{s_{23}s_{34}s_{234}}+\frac{\langle 13\rangle^2(\langle 2|1|4]+\langle 2|3|4])^2}{s_{14}s_{34}s_{134}}+\frac{\langle 12\rangle^2(\langle 3|1|4]]+\langle 3|2|4])^2}{s_{12}s_{14}s_{124}}\\ &+\frac{s_{123}\langle 13\rangle^2}{\langle 4|1|2]\langle 4|3|2]}-\frac{s_{23}\langle 13\rangle(\langle 3|1|4]+\langle 3|2|4])}{s_{14}[12]\langle 4|3|2]}+\frac{s_{12}(\langle 1|2|4]+\langle 1|3|4])\langle 13\rangle}{s_{34}\langle 4|1|2][32]}\end{aligned} \tag{2.35}$$

For calculating the squared amplitude $|A|^2$, it is not necessary to remove spurious poles at the amplitude level. We could replace all the spinor products by the invariants s_{ij} and s_{ijk} after squaring to calculate the differential cross section, and then factorize the result. The spurious poles will cancel automatically.

The other example is the amplitude $A(\phi,1_g^-,2_g^-,3_g^-)$ which contains only negative helicity gluons. Still using the shift $[1,2\rangle$, we notice there are only two contributions,

$$D^{(a)}=A(\hat{1}_g^-,-\hat{K}_a^+,3_g^-)\frac{i}{K_a^2}A(\phi,\hat{2}_g^-,\hat{K}_a^-) \tag{2.36}$$

where the internal propagator $K_a=p_1+p_3$. and

$$D^{(b)}=A(\phi,\hat{1}_g^-,-\hat{K}_b^-)\frac{i}{K_b^2}A(\hat{2}_g^-,3_g^-,\hat{K}_b^+) \tag{2.37}$$

where the internal propagator $K_b=p_2+p_3$. The contribution $D^{(b)}$ vanishes because of eq. (2.32), as explained in the first example. Then

$$A(\phi,1_g^-,2_g^-,3_g^-)=-\frac{m_H^4}{[12][23][31]}. \tag{2.38}$$

In fact, for the amplitude $A(\phi,1_g^-,\cdots,n_g^-)$ which contains only negative helicity gluons, if we compute it using the $[1,2\rangle$, there is only one non-vanishing contribution,

$$A(\phi,1_g^-,\cdots,n_g^-)=A(\hat{1}_g^-,-\hat{K}^+,n_g^-)\frac{i}{K^2}A(\phi,\hat{2}_g^-,\cdots,(n-1)_g^-,\hat{K}^-) \tag{2.39}$$

where $K=p_2+p_3+\cdots+p_{n-1}$. Then we obtain,

$$A(\phi,1_g^-,\cdots,n_g^-)=\frac{(-1)^n m_H^4}{[12][23]\cdots[n-1n][n1]}, \tag{2.40}$$

as calculated in ref. [40] due to Dixon and Khoze.

2.3 Super-BCFW

Brandhuber, Heslop and Travaglini [10] used Grassmann variables to construct superamplitudes combining all different particle and helicity states. Shifting both spinors and Grassman variables, they could reproduce

all $\mathcal{N}=4$ super-Yang–Mills amplitudes. The MHV generating function is defined as follows [11] due to Nair,

$$A_n^{\rm MHV}(1,2,...,n) = \frac{i}{\prod_{j=1}^n \langle jj+1\rangle}\delta^{(8)}(\sum_{j=1}^n \lambda_j^\alpha \eta_j^a)\,. \tag{2.41}$$

We suppress the momentum-conserving delta function here. The remaining delta function, $\delta^{(8)}(\sum_{j=1}^n \lambda_j^\alpha \eta_j^a)$ enforces supercharge conservation. The Grassmann variable η_j^a are anti-commutating numbers,

$$\eta_1\,\eta_2 = -\eta_2\,\eta_1. \tag{2.42}$$

The upper index of η_j^a varies from 1 to 4 corresponding to R-charge index. The following properties will be very useful to manipulate the Grassmann variable η, if $f = a + b\eta$

$$\int d\eta f(\eta) = \frac{d}{d\eta}f(\eta) = a,\ \delta(\eta) = \eta \tag{2.43}$$

We also need some differential operators which will act on the generating function to extract the component MHV amplitudes with different helicity configurations. Here are the rules of extraction:

$$g^+(i) \leftrightarrow 1, f^+(i) \leftrightarrow f^a(i) \leftrightarrow \frac{\partial}{\partial\eta_{ia}}, S \leftrightarrow S^{ab} \leftrightarrow \frac{\partial}{\partial\eta_{ia}\partial\eta_{ib}}, \tag{2.44}$$

$$f^-(i) \leftrightarrow f^{abc} \leftrightarrow \frac{\partial}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}}, g^-(i) \leftrightarrow g^{abcd} \leftrightarrow \frac{\partial}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}}. \tag{2.45}$$

For example,

$$\begin{aligned}\left\langle g^-(1)f^+(2)S(3)g^+(4)f^+(5)\right\rangle &= g^{abcd}(1)f^a(2)S^{bc}g^+(4)f^d(5)A_5^{\rm MHV}(1,2,3,4,5)\\ &= i\frac{\langle 12\rangle\langle 13\rangle^2\langle 15\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}.\end{aligned} \tag{2.46}$$

Similarly, we may define the anti-MHV or $\overline{\rm MHV}$ generating function as follows,

$$A_n^{\overline{\rm MHV}}(1,2,...,n) = \frac{i(-1)^n}{\prod_{j=1}^n [jj+1]}\delta^{(8)}(\sum_{j=1}^n \tilde\lambda_j^{\dot\alpha}\tilde\eta_j^a). \tag{2.47}$$

We can relate these two superspaces via a Grassmann Fourier transform defined as

$$\hat F = \int [\prod_{i,a} d\tilde\eta_{ia}]\exp(\sum_{b,j}\eta_j^b\tilde\eta_{jb}). \tag{2.48}$$

Evaluation of the Fourier transform

$$\hat F A_n^{\overline{\rm MHV}} = \frac{i(-1)^n}{\prod_{i=1}^n [i(i+1)]}\prod_{a=1}^4\sum_{i<j}^n [ij]\frac{\partial}{\partial\eta_{ia}}\frac{\partial}{\partial\eta_{ja}}\eta_1^a\eta_2^a...\eta_n^a \tag{2.49}$$

is then equivalent to the following the rules:

$$[ij]\tilde\eta_{ia}\tilde\eta_{ja} \to \eta_1^a...\eta_{i-1}^a[i|\eta_{i+1}^a...\eta_{j-1}^a|j]\eta_{j+1}^a...\eta_n^a\,. \tag{2.50}$$

So we can write down $\hat F A_3^{\overline{\rm MHV}}$ and $\hat F A_4^{\overline{\rm MHV}}$ explicitly:

$$\hat F A_3^{\overline{\rm MHV}} = \frac{-i}{[12][23][31]}\prod_{a=1}^4([12]\eta_3^a + [23]\eta_1^a + [31]\eta_2^a) \tag{2.51}$$

and

$$\begin{aligned}\hat{F}A_4^{\overline{\rm MHV}} &= \frac{i}{[12][23][34][41]}\prod_{a=1}^{4}([12]\eta_3^a\eta_4^a+[14]\eta_2^a\eta_3^a+[31]\eta_2^a\eta_4^a \\ &\quad +[23]\eta_1^a\eta_4^a+[42]\eta_1^a\eta_3^a+[34]\eta_1^a\eta_2^a)\end{aligned} \tag{2.52}$$

Inspired by this calculation in the pure $\mathcal{N}=4$ Super-Yang-Mills theory, I introduce two new superamplitudes using Grassmann numbers η in our calculations,

$$\begin{aligned}A^{\rm tree}(\phi,\,1_g^-,2_g^-) &= -\langle 12\rangle^2\times\prod_{A=1}^{4}\eta_1^A\eta_2^A\,, \\ A^{\rm tree}(\phi^\dagger,\,1_g^+,2_g^+) &= -[12]^2\,,\end{aligned} \tag{2.53}$$

generalizing the amplitudes seen earlier eq. (2.5). With these, the tree-level amplitudes for H plus four particles can also be generated via supersymmetric recursion relations. Their collinear limits can be checked using splitting amplitudes [14] due to Bern, Dixon, Dunbar and Kosower. This computation yields results agreeing with the results I obtained above using BCFW. For the super-BCFW recursion relation, we shift not only the spinors but also the Grassmann numbers η. The reason is that we need to conserve not only momentum but also the supercharge. The conservation of momentum is still hold for the shifted momentum as the same case in BCFW eq. (2.8) . The conservation of supercharge in this case means that we must require,

$$\hat{\eta}_i^A\lambda_j+\eta_j^A\hat{\lambda}_l=\eta_i^A\lambda_j+\eta_j^A\lambda_l \tag{2.54}$$

We can achieve this by shifting the Grassmann numbers $\hat{\eta}_i^A$:

$$\hat{\eta}_i^A=\eta_i^A+z\eta_j^A \tag{2.55}$$

Then we can repeat the same procedure to solve for, and substitute for, values of the complex variable z in the BCFW case, and find the total amplitude:

$$A=\sum_{r,s,h}\int d^4\eta_{\hat{K}_{r\dots s}}A_L^h\,\frac{i}{K_{r\dots s}^2}\,A_R^{-h} \tag{2.56}$$

where $\eta_{K_{r\dots s}}$ is the grassmann variable associated to the shifted internal propagator $\hat{K}_{r\dots s}$. Using the property of grassmann variable in eq. (2.43), we can calculate the integration over grassmann variable directly as calculate the derivative of the integrand. For each channel, we can also perform the replacement eqs. (2.14)-(2.20) and (2.21), (2.12) so that all the hatted spinor are replaced. We again must sum contributions from all channels and helicity configurations to obtain the total amplitude.

For the ϕ plus three-particle MHV case, we only need to calculate one diagram:

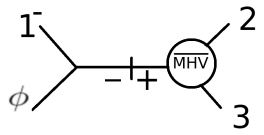

Figure 2.3: The one diagram required to compute $A_3^{\rm MHV}(\phi,1,2,3)$.

$$A_3^{\rm MHV}(\phi,1,2,3)=\int d^4\eta_K A^{\rm tree}(\phi,\,\hat{1},\,\hat{K}_{23})\frac{i}{s_{23}}A_3^{\overline{\rm MHV}}(\hat{2},3,\hat{K}_{23}) \tag{2.57}$$

The result is simple:

$$A_3^{\rm MHV}(\phi,1,2,3)=\frac{1}{i}A_3^{\rm MHV}(1,2,3)=\frac{\delta^{(8)}(\lambda_1^\alpha\eta_1^a+\lambda_2^\alpha\eta_2^a+\lambda_3^\alpha\eta_3^a)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \tag{2.58}$$

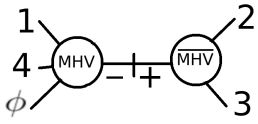

Figure 2.4: The one diagram required to compute $A_4^{\text{MHV}}(\phi, 1, 2, 3, 4)$.

For MHV amplitudes with four final-state particles, we also need to calculate only one diagram:

$$A_4^{\text{MHV}}(\phi, 1, 2, 3, 4) = \int d^4\eta_K A^{\text{MHV}}(\phi,\ \hat{1},\ K_{23}, 4)\frac{i}{s_{23}} A_3^{\overline{\text{MHV}}}(\hat{2}, 3, K_{23}) \tag{2.59}$$

This result is also simple:

$$A_4^{\text{MHV}}(\phi, 1, 2, 3, 4) = \frac{1}{i} A_4^{\text{MHV}}(1, 2, 3, 4) = \frac{\delta^{(8)}(\lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a + \lambda_3^\alpha \eta_3^a + \lambda_4^\alpha \eta_4^a)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{2.60}$$

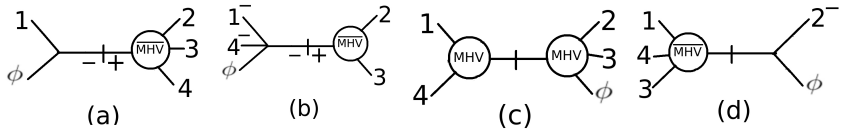

Figure 2.5: The four diagrams require for the calculation of $A_4^{\text{NMHV}}(\phi, 1, 2, 3, 4)$

For NMHV amplitudes with four final-state particles, we need to calculate four diagrams,

$$A_4^{\text{NMHV}}(\phi, 1, 2, 3, 4) = D^{(a)} + D^{(b)} + D^{(c)} + D^{(d)}\,, \tag{2.61}$$

shown in fig. 2.5. These give rise to the contributions,

$$D^{(a)} = \int d^4\eta_K A^{\text{tree}}(\phi,\ \hat{1},\ K_{234})\frac{i}{s_{234}} A_3^{\text{MHV}}(\hat{2}, 3, K_{234}), \tag{2.62}$$

$$D^{(b)} = \int d^4\eta_K A^{\text{tree}}(\phi,\ \hat{1}^-,\ K_{23}^-,\ 4^-)\frac{i}{s_{23}} A_3^{\overline{\text{MHV}}}(\hat{2}, 3, K_{23}), \tag{2.63}$$

$$D^{(c)} = \int d^4\eta_K A_3^{\text{MHV}}(\hat{1}, -K_{14}, 4)\frac{i}{s_{14}} A_3^{\text{MHV}}(\phi, \hat{2}, 3, K_{14}), \tag{2.64}$$

$$D^{(d)} = \int d^4\eta_K A_4^{\overline{\text{MHV}}}(\hat{1}, 4, -K_{134}, 3)\frac{i}{s_{134}} A^{\text{tree}}(\phi,\ \hat{2},\ K_{134}). \tag{2.65}$$

Here, the contribution of $D^{(c)}$ is nonvanishing, because in $A_3^{\text{MHV}}(\phi, 1, 2, 3)$ the momenta p_1, p_2 and p_3 can be real and on-shell at the same time. Using the super-BCFW method, we can reproduce all the amplitudes we need in our model.

2.4 Some Results for tree level amplitudes

In this section, I show some results for different amplitudes. The results can be obtained via BCFW and also via Super-BCFW. I denote the scalars in $\mathcal{N} = 4$ super-Yang–Mills by S_{12}, S_{13}, S_{14}, S_{34}, S_{42}, S_{23} in the index the fermions by $f_1^+, f_2^+, f_3^+, f_4^+, f_{234}^-, f_{143}^-, f_{124}^-, f_{132}^-$. The indices correspond to R-charge indices, and accordingly also represent the Grassmann derivatives needed to project onto the state. The order of the Grassmann derivative yields the proper sign when we use our generating function eqs. (2.58), (2.60) and (2.61) to obtain the amplitudes. All of the examples showed in this section are generated by the super-amplitude obtained in previous section.

For ϕ plus three-particle case:

The example of NMHV amplitude (calculate in section 2.2) is

$$A(\phi, 1_g^-, 2_g^-, 3_g^-) = -\frac{m_H^4}{[12][23][31]}, \tag{2.66}$$

which is also the only NMHV amplitude in four final particles state.

The example of MHV amplitude is,

$$A(\phi, 1_g^+, 2_g^-, 3_g^-) = \frac{\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \tag{2.67}$$

For ϕ plus four-particle case:

The example of NNMHV amplitude is

$$A(\phi, 1_g^-, 2_g^-, 3_g^-, 4_g^-) = -\frac{m_H^4}{[12][23][34][41]}, \tag{2.68}$$

which is also the only NNMHV amplitude in ϕ plus four-particle case.

The examples of NMHV amplitude are

$$\begin{aligned}
A(\phi, 1_g^-, 2_g^-, 3_g^-, 4_g^+) &= -\frac{m_H^4\langle 13\rangle^4}{s_{134}\langle 14\rangle\langle 34\rangle\langle 1|3+4|2]\langle 3|1+4|2]} && (2.69)\\
&\quad -\frac{\langle 3|1+2|4]^3}{s_{124}\langle 3|1+4|2][21][41]} + \frac{\langle 1|2+3|4]^3}{s_{234}\langle 1|3+4|2][32][43]}, && (2.70)\\
A(\phi, 1_g^-, 2_g^-, 3_{f_1^+}, 4_{f_{234}^-}) &= \frac{m_H^4\langle 24\rangle^3}{s_{234}\langle 34\rangle\langle 2|3+4|1]\langle 4|2+3|1]} + \frac{\langle 4|1+2|3]^2}{\langle 4|2+3|1][21][32]} && (2.71)\\
&\quad +\frac{\langle 2|1+3|4]\langle 2|1+4|3]^2}{s_{134}\langle 2|3+4|1][41][43]}, && (2.72)\\
A(\phi, 1_g^-, 2_g^-, 3_{S_{12}}, 4_{S_{34}}) &= -\frac{m_H^4\langle 13\rangle^2\langle 14\rangle}{s_{134}\langle 34\rangle\langle 1|3+4|2]\langle 3|1+4|2]} - \frac{s_{124}\langle 3|1+2|4]}{\langle 3|1+4|2][21][41]} && (2.73)\\
&\quad +\frac{\langle 1|2+3|4]\langle 1|2+4|3]^2}{s_{234}\langle 1|3+4|2][32][43]}, && (2.74)\\
A(\phi, 1_g^-, 2_{S_{12}}, 3_{f_{234}^-}, 4_{f_{143}^-}) &= \frac{\langle 3|1+4|2]}{[21][41]} + \frac{\langle 1|3+4|2]\langle 1|2+4|3]}{s_{234}[32][43]}. && (2.75)
\end{aligned}$$

Then the examples of MHV amplitude are

$$\begin{aligned}
A(\phi, 1_g^-, 2_g^-, 3_g^+, 4_g^+) &= -\frac{\langle 12\rangle^3}{\langle 14\rangle\langle 23\rangle\langle 34\rangle}, && (2.76)\\
A(\phi, 1_g^-, 2_g^+, 3_{S_{12}}, 4_{S_{34}}) &= -\frac{\langle 13\rangle^2\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}, && (2.77)\\
A(\phi, 1_{S_{12}}, 2_{S_{34}}, 3_{S_{12}}, 4_{S_{34}}) &= -\frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 14\rangle\langle 23\rangle\langle 34\rangle}, && (2.78)\\
A(\phi, 1_g^-, 2_g^+, 3_{f_1^+}, 4_{f_{234}^-}) &= -\frac{\langle 13\rangle\langle 14\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}, && (2.79)\\
A(\phi, 1_g^-, 2_{S_{12}}, 3_{f_3^+}, 4_{f_4^+}) &= -\frac{\langle 12\rangle\langle 13\rangle}{\langle 23\rangle\langle 34\rangle}, && (2.80)\\
A(\phi, 1_{S_{12}}, 2_{S_{34}}, 3_{f_1^+}, 4_{f_{234}^-}) &= \frac{\langle 24\rangle}{\langle 23\rangle} - \frac{\langle 14\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle} && (2.81)\\
A(\phi, 1_{f_{132}^-}, 2_{f_{234}^-}, 3_{f_4^+}, 4_{f_1^+}) &= -\frac{\langle 12\rangle}{\langle 34\rangle} && (2.82)\\
& && (2.83)
\end{aligned}$$

All these amplitudes have been checked by collinear limits in ref. [14] (due to Bern, Dixon, Dunbar and Kosower). As we also have another superamplitude,

$$A^{\rm tree}(\phi^\dagger,\, 1_g^+, 2_g^+) = -[1\,2]^2, \tag{2.84}$$

we can reproduce all $\phi^\dagger$ amplitudes via super-BCFW recursion in a similar way to the ϕ amplitudes. The $\phi^\dagger$ amplitudes can also be obtained by applying parity to the ϕ amplitudes. As explained by Badger, Glover, and Khoze [12], for practical purposes this means that we compute the amplitudes with ϕ, and reverse the helicities of every particle. Then we let $\langle ij\rangle \leftrightarrow [ji]$ to get the desired $\phi^\dagger$ amplitude . We finally obtain the H amplitudes by simply add ϕ and $\phi^\dagger$ amplitudes together,

$$\begin{aligned}
A^{\rm tree}(H,1^-,2^-,3^-) &= A^{\rm tree}(\phi,1^-,2^-,3^-)\,, && (2.85)\\
A^{\rm tree\ MHV}(H,1,2,3) &= A^{\rm tree\ MHV}(\phi,1,2,3)\,, && (2.86)\\
A^{\rm tree\ \overline{MHV}}(H,1^-,2^-,3^-) &= A^{\rm tree\ \overline{MHV}}(\phi^\dagger,1,2,3)\,. && (2.87)\\
A^{\rm tree}(H,1^+,2^+,3^+) &= A^{\rm tree}(\phi^\dagger,1^+,2^+,3^+) && (2.88)\\
A^{\rm tree}(H,1^-,2^-,3^-,4^-) &= A^{\rm tree}(\phi,1^-,2^-,3^-,4^-)\,, && (2.89)\\
A^{\rm tree\ NMHV}(H,1,2,3,4) &= A^{\rm tree\ NMHV}(\phi,1,2,3,4)\,, && (2.90)\\
& && (2.91)
\end{aligned}$$

and

$$\begin{aligned}
A^{\rm tree\ MHV}(H,1,2,3,4) &= A^{\rm tree\ MHV}(\phi,1,2,3,4) + A^{\rm tree\ MHV}(\phi^\dagger,1,2,3,4)\,, && (2.92)\\
A^{\rm tree}(H,1^+,2^+,3^+,4^-) &= A^{\rm tree}(\phi^\dagger,1^+,2^+,3^+,4^-)\,, && (2.93)\\
A^{\rm tree}(H,1^+,2^+,3^+,4^+) &= A^{\rm tree}(\phi^\dagger,1^+,2^+,3^+,4^+)\,. && (2.94)
\end{aligned}$$

2.5 From Tree-level Amplitudes to the Differential Cross-section

In previous sections, I obtained all the tree level partial amplitudes up to four-particle final states. In order to obtain the differential cross section, we need first calculate the following term for each particle type,

$$\sum_{Helicities}\sum_{colors} |\mathcal{A}|^2 = \sum_{Helicities}\sum_{colors} \overline{\mathcal{A}}\mathcal{A} \tag{2.95}$$

where $\mathcal{A}$ in eq. (1.6) is the sum of product of partial amplitude and trace of color matrices. For example, for ϕ plus three-particle case

$$\mathcal{A}(H,1,2,3) = {\rm Tr}(T^{a_1}T^{a_2}T^{a_3})A(H,1,2,3) + {\rm Tr}(T^{a_2}T^{a_1}T^{a_3})A(H,2,1,3) \tag{2.96}$$

and

$$\overline{\mathcal{A}}(H,1,2,3) = \overline{{\rm Tr}(T^{a_1}T^{a_2}T^{a_3})}\,\overline{A}(H,1,2,3) + \overline{{\rm Tr}(T^{a_2}T^{a_1}T^{a_3})}\,\overline{A}(H,2,1,3). \tag{2.97}$$

Because T are generator of $SU(N_c)$ group, so T are hermitian matrix, $T^\dagger = T$, so

$$\overline{{\rm Tr}(T^{a_1}T^{a_2}T^{a_3})} = {\rm Tr}(T^{a_3}T^{a_2}T^{a_1}) \tag{2.98}$$

then using the properties (1.11) and (1.12), we calculate

$$\overline{{\rm Tr}(T^{a_1}T^{a_2}T^{a_3})}{\rm Tr}(T^{a_1}T^{a_2}T^{a_3}) \sim N_c^3 \tag{2.99}$$

and

$$\overline{{\rm Tr}(T^2T^1T^3)}{\rm Tr}(T^1T^2T^3) \sim N_c \tag{2.100}$$

Then for the leading color contribution of tree level amplitude,

$$\sum_{colors} |\mathcal{A}_3|^2 \sim N_c^3\left(|A(H,1,2,3)|^2 + |A(H,2,1,3)|^2\right) \tag{2.101}$$

for ϕ plus four-particle case, we obtain

$$\sum_{colors} |\mathcal{A}_4|^2 \sim N_c^4 \sum_{\sigma\in S_3} |A(H,\sigma(1),\sigma(2),\sigma(3),4)|^2 \tag{2.102}$$

Then we need to sum all different helicity contributions and we obtain the contribution to differential cross-section of a specific particle type,

$$\sum_{Helicities}\sum_{colors} |\mathcal{A}_3|^2 \sim N_c^3 \sum_{Helicities} \left(|A(H,1,2,3)|^2 + |A(H,2,1,3)|^2\right) \tag{2.103}$$

$$\sum_{Helicities}\sum_{colors} |\mathcal{A}_4|^2 \sim N_c^4 \sum_{Helicities}\sum_{\sigma\in S_3} |A(\sigma(1),\sigma(2),\sigma(3),4)|^2 \tag{2.104}$$

For example, let's consider the contribution of the four-particle final state $1_f,\ 2_f,\ 3_g,\ 4_g$ to the differential cross-section,

$$|A|^2(H,1_f,\ 2_f,\ 3_g,\ 4_g) = \sum_{Helicities}\sum_{colors} |\mathcal{A}_4(H,1_f,\ 2_f,\ 3_g,\ 4_g)|^2. \tag{2.105}$$

We want to replace this form by a fully-symmetrized one,

$$\begin{aligned} &|A|^2(H,1_f,\ 2_f,\ 3_g,\ 4_g) + |A|^2(H,1_f,\ 2_g,\ 3_f,\ 4_g) + |A|^2(H,1_f,\ 2_g,\ 3_g,\ 4_f) \\ &+|A|^2(H,1_g,\ 2_f,\ 3_f,\ 4_g) + |A|^2(H,1_g,\ 2_f,\ 3_g,\ 4_f) + |A|^2(H,1_g,\ 2_g,\ 3_f,\ 4_f). \end{aligned} \tag{2.106}$$

With the above expression, we would be counting the same point six times over. So for the integrand in the phase space, we should multiply the above formula by $1/6 \times 1/4$, where $1/6$ comes from the over-counting and $1/4$ comes from the symmetry factor of two pairs of identical particles. For the total amplitude, we symmetrize as above, also summing over all different particle types. We may expect that after we've summed all amplitudes together, the final result will be simpler than each of the contributions separately. I consider each of the classes MHV, NMHV, NNMHV separately, but summing over all helicities and particle type within each class. The sum over color will be performed at last.

For H plus three-particles case: Squaring the NMHV amplitude yields,

$$|A|^2(H,\ 1_g^-,\ 2_g^-,\ 3_g^-) \quad = \quad \frac{m_H^8}{s_{12}s_{23}s_{13}}. \tag{2.107}$$

Squaring the MHV amplitude yields,

$$\textstyle\sum_{\rm MHV} |A|^2(H) \quad = \quad \frac{m_H^8}{s_{12}s_{23}s_{13}}. \tag{2.108}$$

Squaring the $\overline{\rm MHV}$ amplitude yields,

$$\textstyle\sum_{\overline{\rm MHV}} |A|^2(H) \quad = \quad \frac{m_H^8}{s_{12}s_{23}s_{13}}. \tag{2.109}$$

Squaring the all-plus helicity amplitude yields,

$$|A|^2(H,\ 1_g^+,\ 2_g^+,\ 3_g^+) \quad = \quad \frac{m_H^8}{s_{12}s_{23}s_{13}}. \tag{2.110}$$

We denote the sum of all different class MHV, NMHV, $\overline{\rm MHV}$, all-plus by $|A|^2(1,2,3)$, then,

$$\begin{aligned} |A|^2(1,2,3) = &\ \textstyle\sum_{\rm NMHV} |A|^2 + \sum_{\rm MHV} |A|^2 + \sum_{\overline{\rm MHV}} |A|^2 + \sum_{\text{All plus helicity}} |A|^2 \\ = &\ \frac{4m_H^8}{s_{12}s_{23}s_{13}} \end{aligned} \tag{2.111}$$

To sum over all color contribution, we need only calculate,

$$\begin{aligned} \sum_{color} |A|^2(1,2,3) = & \sum_{\sigma_2 \in S_2} |A|^2(\sigma(1),\sigma(2),3) \\ = & 2|A|^2(1,2,3) \end{aligned} \tag{2.112}$$

So the fully-symmetrized differential cross section for H plus three-particles case, denoted as $|A(1,2,3)|^2$ is

$$\begin{aligned} |A(1,2,3)|^2 &= \frac{1}{6}\sum_{color} |A|^2(1,2,3) \\ &= \frac{4m_H^8}{3s_{12}s_{23}s_{13}} \end{aligned} \tag{2.113}$$

where the prefactor 1/6 avoid the problems of overcompensating and problems of over-counting.

For H plus four-particles case:

Squaring the NNMHV amplitude yields,

$$\begin{aligned} |A|^2(H,\, 1_g^-,\, 2_g^-,\, 3_g^-,\, 4_g^-,\,) &= |A|^2(\phi,\, 1_g^-,\, 2_g^-,\, 3_g^-,\, 4_g^-,\,) \\ &= \frac{m_H^8}{s_{12}s_{23}s_{34}s_{41}}. \end{aligned} \tag{2.114}$$

In order to sum the NMHV amplitudes, note that

$$\sum_{\text{NMHV}} |A|^2(H) = \sum_{\text{NMHV}} |A|^2(\phi). \tag{2.115}$$

Performing the sum, we obtain,

$$\begin{aligned} \textstyle\sum_{\text{NMHV}} |A|^2(H) &= \frac{2m_H^4}{s_{14}s_{23}s_{123}s_{124}} + \frac{2m_H^8}{s_{12}s_{14}s_{123}s_{134}} + \frac{2m_H^4}{s_{14}s_{23}s_{123}s_{134}} + \frac{2m_H^8}{s_{12}s_{34}s_{123}s_{134}} \\ &+ \frac{2m_H^4}{s_{23}s_{34}s_{123}s_{134}} + \frac{2m_H^8}{s_{12}s_{34}s_{124}s_{134}} + \frac{2m_H^4}{s_{12}s_{34}s_{124}s_{234}} + \frac{2m_H^8}{s_{14}s_{23}s_{124}s_{234}} \\ &+ \frac{2m_H^4}{s_{12}s_{34}s_{124}s_{234}} + \frac{2m_H^8}{s_{14}s_{34}s_{124}s_{234}} + \frac{2m_H^4}{s_{14}s_{23}s_{134}s_{234}}. \end{aligned} \tag{2.116}$$

To sum the MHV amplitudes, use

$$\sum_{\text{MHV}} |A|^2(H) = \sum_{\text{MHV}} |A(\phi) + A(\phi^\dagger)|^2. \tag{2.117}$$

Performing the MHV sum, we obtain,

$$|A|^2(H) = \frac{2m_H^8}{s_{12}s_{23}s_{34}s_{41}}. \tag{2.118}$$

We know that the all plus-helicity gluon configuration yields the same contribution as $|A^{\text{NNMHV}}|^2$ and that the contribution of all $\overline{\text{NMHV}} = (+,+,+,-)$ configuration yields the same contribution as $\sum_{\text{NMHV}} |A|^2$. So the total contribution of this color ordering is,

$$|A|^2(1,\,2,\,3,\,4) = (2\sum_{\text{NNMHV}} |A|^2 + 2\sum_{\text{NMHV}} |A|^2 + \sum_{\text{MHV}} |A|^2). \tag{2.119}$$

Adding all the color orderings, we find for the leading-color contribution,

$$\begin{aligned}\sum_{\sigma\in S_3}|A|^2(\sigma(1),\,\sigma(2),\,\sigma(3),\,4) \;=\; & \frac{8m_H^8}{s_{13}s_{14}s_{23}s_{24}}+\frac{8m_H^8}{s_{12}s_{14}s_{23}s_{34}}+\frac{8m_H^8}{s_{12}s_{13}s_{24}s_{34}}+\frac{8m_H^8}{s_{13}s_{14}s_{123}s_{124}}\\ &+\frac{16m_H^8}{s_{14}s_{23}s_{123}s_{124}}+\frac{16m_H^8}{s_{13}s_{24}s_{123}s_{124}}+\frac{8m_H^8}{s_{23}s_{24}s_{123}s_{124}}+\frac{8m_H^8}{s_{12}s_{14}s_{123}s_{134}}\\ &+\frac{16m_H^8}{s_{14}s_{23}s_{123}s_{134}}+\frac{16m_H^8}{s_{12}s_{34}s_{123}s_{134}}+\frac{8m_H^8}{s_{23}s_{34}s_{123}s_{134}}+\frac{8m_H^8}{s_{12}s_{13}s_{124}s_{134}}\\ &+\frac{16m_H^8}{s_{13}s_{24}s_{124}s_{134}}+\frac{16m_H^8}{s_{12}s_{34}s_{124}s_{134}}+\frac{8m_H^8}{s_{24}s_{34}s_{124}s_{134}}+\frac{8m_H^8}{s_{12}s_{24}s_{123}s_{234}}\\ &+\frac{16m_H^8}{s_{13}s_{24}s_{123}s_{234}}+\frac{16m_H^8}{s_{12}s_{34}s_{123}s_{234}}+\frac{8m_H^8}{s_{13}s_{34}s_{123}s_{234}}+\frac{8m_H^8}{s_{12}s_{23}s_{124}s_{234}}\\ &+\frac{16m_H^8}{s_{14}s_{23}s_{124}s_{234}}+\frac{16m_H^8}{s_{12}s_{34}s_{124}s_{234}}+\frac{8m_H^8}{s_{14}s_{34}s_{124}s_{234}}+\frac{8m_H^8}{s_{13}s_{23}s_{134}s_{234}}\\ &+\frac{16m_H^8}{s_{14}s_{23}s_{134}s_{234}}+\frac{16m_H^8}{s_{13}s_{24}s_{134}s_{234}}+\frac{8m_H^8}{s_{14}s_{24}s_{134}s_{234}}\end{aligned} \tag{2.120}$$

The the fully-symmetrized differential cross section for H plus four-particles case, denoted as $|A(1,2,3,4)|^2$ is

$$|A(1,2,3,4)|^2=\frac{1}{24}\sum_{\sigma\in S_3}|A|^2(\sigma(1),\,\sigma(2),\,\sigma(3),\,4) \tag{2.121}$$

where the prefactor 1/24 avoid the problems of overcompensating and problems of over-counting.

2.6 The Leading-Order Energy-Energy Correlation Function

In previous section, we calculate the fully-symmetrized differential cross section for H plus three-particles case (2.113). Now we can calculate the leading order energy-energy function, defined in (1.4). For H plus three-particles case, the EEC becomes,

$$EEC=\int dPS_3|A(1,2,3)|^2\left(\frac{E_1E_2\delta(\cos\theta_{12}-c)}{E_{total}^2}+\frac{E_2E_3\delta(\cos\theta_{23}-c)}{E_{total}^2}+\frac{E_1E_3\delta(\cos\theta_{13}-c)}{E_{total}^2}\right) \tag{2.122}$$

where $E_{total}=m_H$. Since $|A(1,2,3)|^2$ is fully-symmetrized differential cross section (2.113),

$$EEC_{LO}=\int dPS_3\,3\,|A(1,2,3)|^2\frac{E_2E_3\delta(\cos\theta_{23}-c)}{E_{total}^2} \tag{2.123}$$

We write down the measure of three-particle phase space($D=4-2\epsilon$ and $s_{ij}=s_{ij}/m_H^2$),

$$dPS_3=\Pi_{i=1}^3\frac{dp_i^D\delta(p_i^2)}{(2\pi)^3}(2\pi)^4\delta^D(p_H-\sum_i p_i)=\frac{dp_2^{D-1}}{2E_2(2\pi)^3}\frac{dp_3^{D-1}}{2E_3(2\pi)^3}(2\pi)^4. \tag{2.124}$$

In practice, it is very convenient to write the three-particle phase-space measure in term of invariants. Then we can obtain,

$$dPS_3=m_H^{2-4\epsilon}\frac{4^{3\epsilon-4}\pi^{2\epsilon-\frac{5}{2}}s_{12}^{-\epsilon}s_{23}^{-\epsilon}(1-s_{12}-s_{23})^{-\epsilon}}{\Gamma(1-\epsilon)\Gamma\left(3/2-\epsilon\right)},\qquad s_{12}+s_{23}<1 \tag{2.125}$$

Because there is a δ function $\delta(\cos\theta_{23}-c)$ in the integrand of eq. (2.123), it is reasonable to change variables so that the integral can be reduced to an one-dimensional integral. The function $\cos\theta_{23}$ can be expressed as,

$$\cos\theta_{23}=1-\frac{s_{23}}{2E_2E_3} \tag{2.126}$$

In the center-mass system, $p_H = -p_1 - p_2 - p_3 = (-m_H, \overrightarrow{0})$,

$$\cos\theta_{23} = 1 - \frac{s_{23}}{2E_2E_3} = 1 - \frac{2s_{23}}{(s_{12}+s_{23})(s_{13}+s_{23})} \tag{2.127}$$

Then we notice we should change variables:

$$u_1 = \frac{s_{23}}{(s_{12}+s_{23})(1-s_{12})}, \quad t = s_{12}+s_{23} \tag{2.128}$$

so that $\cos\theta_{23} = 1 - 2u_1$. The three-particle phase-space measure become,

$$dPS_3 = m_H^{2-4\epsilon}\frac{2^{4\epsilon-7}\pi^{2\epsilon-3}(1-t)^{1-2\epsilon}t^{1-2\epsilon}(1-u_1)^{-\epsilon}u_1^{-\epsilon}(1-tu_1)^{2\epsilon-2}}{\Gamma(2-2\epsilon)}, \quad (0<u_1<1,\ 0<t<1) \tag{2.129}$$

We define

$$u = \frac{1-c}{2} \tag{2.130}$$

So we write the measure for the EEC function in the following way,

$$\begin{aligned} EEC_{LO} &= \int_0^1 dt \frac{2^{4\epsilon-8}\pi^{2\epsilon-3}(1-t)^{-2\epsilon}t^{-2\epsilon}(1-u)^{-\epsilon-1}u^{-\epsilon-1}(1-tu)^{2\epsilon-1}}{\Gamma(2-2\epsilon)} \\ &= \frac{2^{8\epsilon-9}\pi^{2\epsilon-\frac{5}{2}}(1-u)^{-\epsilon-1}u^{-\epsilon-1}\,{}_2F_1(1-2\epsilon,1-2\epsilon;2-4\epsilon;u)}{(1-2\epsilon)\Gamma(3/2-2\epsilon)} \\ &= -m_H^4\frac{\ln(1-u)}{256\pi^3(1-u)u^2}, \text{ where } \epsilon \text{ is set to 0.} \\ &= m_H^4\frac{(\omega+1)^3}{256\pi^3\omega}\ln\left(1+\frac{1}{\omega}\right) \end{aligned} \tag{2.131}$$

where $\omega = \cot^2(\chi/2)$. We notice immediately that $EEC_{LO}(u)$ diverges in the limits $u \to 0$ and $u \to 1$. We can compare this result with the leading order EEC of QCD in [30] due to Richards, Stirling and Ellis. The leading order EEC of QCD is proportional to $g^{(1)}$,

$$\begin{aligned} g^{(1)} &= \frac{1}{8}C_F(1+\omega)^3\frac{1+3\omega}{\omega}\left[(2-6\omega^2)\ln(1+\frac{1}{\omega})+6\omega-3\right] \\ &= \frac{1}{8}C_F\left[\frac{(2u-3)\left(6u-9u^2\right)}{(u-1)u^5}+\frac{(2u-3)\left(4u^2-12u+6\right)\ln(1-u)}{(u-1)u^5}\right]. \end{aligned} \tag{2.132}$$

We find that in the limit $u \to 0$ or $\chi \to 0^\circ$, the leading singularities of EEC_{LO} and of $g^{(1)}$ is the same,

$$EEC_{LO} \sim g^{(1)} \sim \frac{1}{u}. \tag{2.133}$$

In the limit $u \to 1$ or $\chi \to 180^\circ$,

$$EEC_{LO} \sim g^{(1)} \sim \frac{\ln u}{u}. \tag{2.134}$$

Chapter 3

Calculation of One-loop Amplitudes

3.1 Calculation of One-loop Amplitudes

In previous chapter, I calculated all tree-level amplitudes with up to four particles in final state. These include those which we need to compute the one-loop amplitude. Beyond those amplitudes, we also need amplitudes not containing the external scalar H, which can be obtained from [10] due to Brandhuber, Heslop, Travaglini. In this chapter, I will use generalized unitarity to compute the cut-constructible part of loop amplitudes. I review the unitarity [14] due to Bern, Dixon, Dunbar and Kosower (also in [15] due to Berger, Bern, Dixon, Forde and Kosower, [16], [17] due to Bern, Dixon and Kosower) and generalized unitarity methods section 3.1.2. The one-loop cut-constructible part contain box, triangle and bubble integrals. The method I will use for computing box, triangle, and bubble coefficients were introduced respectively by Britto, Cachazo and Feng [18], by Forde [6], and by Mastrolia [7]. Using these amplitudes, I will compute the virtual contributions to the energy-energy correlation in the last section of this chapter.

3.1.1 Structure of One-loop Amplitude in Four Dimension

The standard technique for computing the one-loop amplitude is still that of Feynman diagrams. We may use the color-ordered vertices to compute it directly. For n-point amplitude in gauge theory, we would obtain a sum of n-point integrals,

$$\int d^D l \frac{\prod_{i=1}^{n} L_i^{\mu_i}}{(l-k_1)^2(l-k_1-k_2)^2\cdots(l+k_n)^2} \tag{3.1}$$

where l is loop momentum, k_i is outgoing momentum and $L_i^{\mu_i}$ is some vector constructed by l and k_i, and also lower point integral (The rank of the tensor in numerator is less than n and the number of loop propagator is also less than n). Using Passarino–Veltman reduction method [44] or many other reduction methods in [58] and [59] due to Pittau, [60] due to Weinzierl, we may be able to reduce the above integrals to four–point, three–point and two–point integrals. Or using the OPP method [63] due to Ossola, Papadopoulos and Pittau (also in [64] due to Mastrolia, Ossola, Papadopoulos, Pittau) as an alternative approach, we rewrite the integrand in terms of a standard set in which the term corresponds to vanishing integral or to one of n–point scalar integral ($n \leq 4$). In general, an one-loop amplitude can be expanded as follows:

$$A^{\text{one-loop}} = \sum_i C_{4,i}\mathcal{I}_{4,i} + \sum_i C_{3,i}\mathcal{I}_{3,i} + \sum_i C_{2,i}\mathcal{I}_{2,i} + R \tag{3.2}$$

where the integral $\mathcal{I}_{n,i}$ which are basis integrals are n-point scaler loop integrals and R which does not contain any loop integral is a rational function of spinor products. We define the sum of the terms which contain loop integrals as

$$A^{\text{one-loop, }CC} = \sum_i C_{4,i}\mathcal{I}_{4,i} + \sum_i C_{3,i}\mathcal{I}_{3,i} + \sum_i C_{2,i}\mathcal{I}_{2,i}. \tag{3.3}$$

In the following sections, I will calculate $A^{\text{one-loop, }CC}$ for each particle type using unitarity methods and generalized unitarity methods.

The basis integral $\mathcal{I}$ contains scalar box integrals $\mathcal{I}_4$, triangle integrals $\mathcal{I}_3$ and bubble integrals $\mathcal{I}_2$. The box integrals $\mathcal{I}_4$ is defined as,

$$\mathcal{I}_4(k_1,k_2,k_3) = -i\mu^{2\epsilon}\int \frac{d^Dl}{(2\pi)^D}\frac{1}{l^2(l+k_1)^2(l+k_1+k_2)^2(l+k_1+k_2+k_3)^2}, \tag{3.4}$$

where k_i, $i=1,2,3,4$ are outgoing momenta, $k_1+k_2+k_3+k_4=0$ so $l+k_1+k_2+k_3 = l-k_4$. The triangle integral $\mathcal{I}_{3,i}$ is defined as,

$$\mathcal{I}_3(k_1,k_2) = i\mu^{2\epsilon}\int \frac{d^Dl}{(2\pi)^D}\frac{1}{l^2(l+k_1)^2(l+k_1+k_2)^2}, \tag{3.5}$$

and the bubble integrals $\mathcal{I}_2$ is defined as,

$$\mathcal{I}_2(k) = -\mu^{2\epsilon}\int \frac{d^Dl}{(2\pi)^D}\frac{1}{l^2(l+k)^2}. \tag{3.6}$$

We also define a dimensionless function denoted by F_4 from the basis integral $\mathcal{I}_4$

$$\mathcal{I}_4(\ldots) = \frac{c_\Gamma}{\mathcal{G}}F_4(\ldots) \tag{3.7}$$

where

$$c_\Gamma = \frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon+1)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)} \tag{3.8}$$

and $\mathcal{G}$ which is Gram determinants are given by

$$\begin{aligned}\mathcal{G}^{0m} &= \mathcal{G}^{1m} = \mathcal{G}^{2mh} = -\frac{1}{2}s\,t\\ \mathcal{G}^{2me} &= \mathcal{G}^{3m} = -\frac{1}{2}(s\,t\, - k_2^2\,k_4^2),\end{aligned} \tag{3.9}$$

where $s=(k_1+k_2)^2$, $t=(k_2+k_3)^2$. The index $0m$ means all outgoing momenta k_i, $i=1,2,3,4$ are massless. The index $1m$ means there is only one outgoing massive momentum. The index $2mh$ means there are two adjacent outgoing massive momenta, for example $k_i^2\neq 0$ and $k_{i+1}^2\neq 0$. The index $2me$ means there are two non-adjacent outgoing massive momenta, for example $k_i^2\neq 0$ and $k_{i+2}^2\neq 0$. The index $3m$ means there are three outgoing massive momenta. We also define the dimensionless function F_3 from the basis integral $\mathcal{I}_3$,

$$\mathcal{I}_3(\ldots) = \frac{c_\Gamma}{\mathcal{G}}F_3(\ldots) \tag{3.10}$$

where

$$\begin{aligned}\mathcal{G}^{1m} &= -k_1^2 \text{ where } k_1^2\neq 0\\ \mathcal{G}^{2m} &= -k_1^2+k_2^2 \text{ where } k_1^2\neq 0,\ k_2^2\neq 0.\end{aligned} \tag{3.11}$$

And at last we define function F_2 from the basis integral $\mathcal{I}_2$,

$$\mathcal{I}_2(k) = \frac{c_\Gamma}{\epsilon(1-2\epsilon)}\left(\frac{\mu^2}{-k^2}\right) = c_\Gamma F_2(k) \tag{3.12}$$

The analytic forms of these integrals $\mathcal{I}$ and F are listed in ref. [14] due to Bern, Dixon, Dunbar and Kosower. The problem of calculating the cut–constructible part of one–loop amplitude is reduced to finding the coefficients $C_{4,i}$, $C_{3,i}$, $C_{2,i}$ in eq. (3.3). In the following section, we will use unitarity and generalized unitarity to calculate these coefficients.

3.1.2 Calculation of Cut–constructible Contributions via Unitarity

In this section, I review the unitarity method in [14] due to Bern, Dixon, Dunbar and Kosower. The conservation of probability requires the unitarity of the scattering S-matrix.

$$S\ S^{\dagger} = 1 \tag{3.13}$$

We can decompose the S-matrix into two parts,

$$S = 1 + iT, \tag{3.14}$$

where only T part describes the non-forward scattering. The unitarity constraint implies

$$2\Im T = TT^{\dagger}. \tag{3.15}$$

In perturbation theory, we can expand both sides of the equality in a series in the coupling. At one-loop case, the left-hand side corresponds to a discontinuity of a one-loop amplitude and the right-hand side which may be obtained from loop amplitude by cutting it corresponds to a product of two tree amplitudes with all possible internal states inserted. The unitarity method are sewing tree amplitude to construct the imaginary part of loop amplitude then by replacing $\delta(l^2)$ by $1/l^2$, obtain the full contribution of unitarity cut to loop amplitude. In practice, we sew tree amplitudes together as

$$\int d^4 l_1 \delta(l_1^2)\delta(l_2^2) A_L^{\text{tree}}(-l_2, ..., l_1) A_R^{\text{tree}}(l_2, ..., -l_1). \tag{3.16}$$

Then we replace the $\delta(l^2)$ by $1/l^2$ in the integrand

$$\int d^4 l_1 \frac{1}{l_1^2 l_2^2} A_L^{\text{tree}}(-l_2, ..., l_1) A_R^{\text{tree}}(l_2, ..., -l_1) \tag{3.17}$$

where $l_2 = l_1 - K$ and l_1, l_2 are still regarded as on-shell momenta, replace the spinor product in the denominator of the integrand by:

$$\frac{1}{\langle a l_1 \rangle} = \frac{\pm[l_1 a]}{(l_1 \pm a)^2} \tag{3.18}$$

then the numerator a priory can be written as:

$$\text{Tr}_{\pm}(ab...l_1...l_2...) \tag{3.19}$$

We simplify such expressions in terms of the normal vector product, for example:

$$\text{Tr}_{\pm}(abcd) = 2((a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)) \pm \epsilon^{\mu\nu\rho\gamma} a_\mu b_\nu c_\rho d_\gamma \tag{3.20}$$

In our case, there are just three independent external momenta so the contributions from $\epsilon^{\mu\nu\rho\gamma} a_\mu b_\nu c_\rho d_\gamma$ will vanish. The numerator of the integrand becomes a function of $l_i \cdot k_i$. Now we can simplify the integrand to:

$$\sum_{ij} c_{4,ij} \frac{1}{l_1^2 l_2^2} \frac{1}{D_i D_j} + \sum_i c_{3,i} \frac{1}{l_1^2 l_2^2} \frac{1}{D_i} + c_2 \frac{1}{l_1^2 l_2^2} \tag{3.21}$$

In this calculation, we select only terms whose denominators contains the factor $l_1^2 l_2^2$. In this integrand, the term,

$$\frac{1}{l_1^2 l_2^2} \frac{1}{D_i D_j} \tag{3.22}$$

correspond to a box integral; the term,

$$\frac{1}{l_1^2 l_2^2} \frac{1}{D_i} \tag{3.23}$$

to a triangle integral, and the term

$$\frac{1}{l_1^2 l_2^2}\,, \tag{3.24}$$

to a bubble integral. We need to add all internal states and all different K-channel contributions and suppress the overlapped integral from different unitarity cut, then we obtain the cut-constructible part of a one-loop amplitude.

Let us consider two examples to demonstrate how we use the unitarity method to obtain the cut-constructible part of a one–loop amplitude.

The first example is to calculate the cut-constructible part of the amplitude $A^{\text{one-loop}}(\phi, 1_g^-, 2_g^-, 3_g^-)$. For this amplitude, there are six unitarity cuts which contribute. Two of them are shown in fig. 3.1. Since the scalar particle ϕ carries no color, for a color ordered diagram, ϕ can be inserted between any two out going legs. The other four contributions can be obtained from moving the external legs 1,2 and 3 clockwisely. We apply the unitarity method to the graph (a) in fig. 3.1.

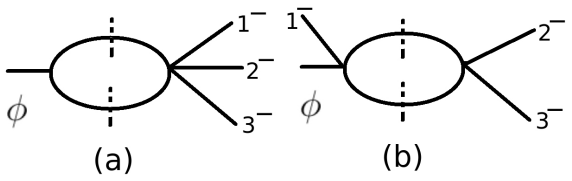

Figure 3.1: Two of six unitarity cuts contributing to $A^{\text{one-loop MHV}}(\phi, 1_g^-, 2_g^-, 3_g^-)$

$$\begin{aligned}
A_{(a)}^{\text{one loop}} \to & \int d^4l \delta(l_1^2)\delta(l_2^2) A(\phi, l_1^-, -l_2^-) A(-l_1^+, l_2^+, 1_g^-, 2_g^-, 3_g^-) \mid_{\delta(l^2)\to 1/l^2} \\
\to & \int \frac{d^4l}{l_1^2 l_2^2} A(\phi, l_1^-, -l_2^-) A(-l_1^+, l_2^+, 1_g^-, 2_g^-, 3_g^-) \\
\to & \int \frac{d^4l}{l_1^2 l_2^2} \frac{m_H^4}{[l_1 l_2][l_2 l_1]} \frac{[l_1 l_2]^4}{[l_1 l_2][l_2 1][12][23][3 l_1]} \\
\to & A(\phi, 1_g^-, 2_g^-, 3_g^-) \int \frac{d^4l}{l_1^2 l_2^2} \frac{[l_1 l_2][31]}{[l_2 1][3 l_1]} \\
\to & A(\phi, 1_g^-, 2_g^-, 3_g^-) \int \frac{d^4l}{l_1^2 l_2^2} \frac{\text{Tr}_+(l_1 l_2 k_1 k_3)}{(l_2+k_1)^2 (l_1-k_3)^2}
\end{aligned} \tag{3.25}$$

Using eq. (3.20) and suppressing the $\epsilon^{\mu\nu\rho\gamma}$ term, we obtain,

$$\frac{A_{(a)}^{\text{one loop}}}{A(\phi, 1_g^-, 2_g^-, 3_g^-)} \to \int \frac{d^4l}{l_1^2 l_2^2} \frac{\text{Tr}_+(l_1 l_2 k_1 k_3)}{(l_2+k_1)^2 (l_1+k_3)^2} \to \int d^4l \, 2 \, \frac{l_1 \cdot l_2 \, k_1 \cdot k_3 - l_1 \cdot k_1 \, l_2 \cdot k_3 + l_1 \cdot k_3 \, l_2 \cdot k_1}{l_1^2 l_2^2 (l_2+k_1)^2 (l_1+k_3)^2} \tag{3.26}$$

Then we simplify the above integral to (We suppress the terms which are proportional to l_1^2 or to l_2^2 in the numerator of the integrand.),

$$\begin{aligned}
\frac{A_{(a)}^{\text{one loop}}}{A(\phi, 1_g^-, 2_g^-, 3_g^-)} \to & -\frac{1}{2} s_{23} s_{12} \int \frac{d^4l}{l_1^2 l_2^2 (l_2+k_1)^2 (l_1+k_3)^2} \\
& -\frac{1}{2}(m_H^2 - s_{12}) \int \frac{d^4l}{l_1^2 l_2^2 (l_1+k_3)^2} - \frac{1}{2}(m_H^2 - s_{23}) \int \frac{d^4l}{l_1^2 l_2^2 (l_2+k_1)^2},
\end{aligned} \tag{3.27}$$

where we notice that the coefficients of the loop integral in the above formula are proportional to the Gram determinants defined in eqs. (3.9) and (3.11). We obtained one box integral and two triangle integrals which are shown in fig. 3.2. Then we apply the unitarity method to the graph (b) in fig. 3.1.

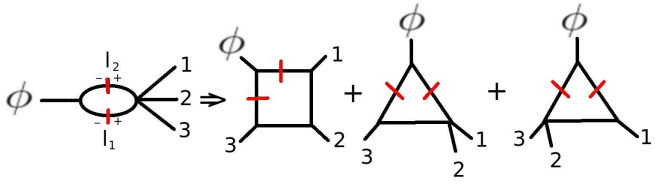

Figure 3.2: A graphic representation of contribution $A^{\text{one loop}}_{(a)}$

$$
\begin{aligned}
A^{\text{one loop}}_{(b)} \to & \int d^4l\, \delta(l_1^2)\delta(l_2^2) A(\phi,1^-,l_1^-,-l_2^-) A^{\text{MHV}}(2^-,3^-,-l_1^+,l_2^+)\mid_{\delta(l^2)\to 1/l^2} \\
\to & \int \frac{d^4l}{l_1^2 l_2^2} A(\phi,1^-,l_1^-,l_2^-) A^{MHV}(2^-,3^-,l_1^+,l_2^+) \\
\to & \int \frac{d^4l}{l_1^2 l_2^2} \frac{m_H^4}{[1l_2][l_2l_1][l_11]} \frac{[l_1l_2]^4}{[l_1l_2][l_22][23][3l_1]} \\
\to & A(\phi,1^-,2^-,3^-) \int \frac{d^4l}{l_1^2 l_2^2} \frac{\text{Tr}_-(k_1l_1l_2k_1k_2k_3)}{(l_2-k_1)^2(l_1+k_1)^2(l_2+k_2)^2} \\
\to & A(\phi,1^-,2^-,3^-) \int \frac{d^4l}{l_1^2 l_2^2} \frac{(l_1+k_1)^2\text{Tr}_-(l_2k_1k_2k_3)+(l_2-k_1)^2\text{Tr}_-(l_1k_1k_2k_3)}{(l_2-k_1)^2(l_1+k_1)^2(l_2+k_2)^2} \\
\to & A(\phi,1^-,2^-,3^-) \int \frac{d^4l}{l_1^2 l_2^2} \left(\frac{\text{Tr}_-(l_2k_1k_2k_3)}{(l_2-k_1)^2(l_2+k_2)^2} + \frac{\text{Tr}_-(l_1k_1k_2k_3)}{(l_1+k_1)^2(l_2+k_2)^2} \right)
\end{aligned}
\tag{3.28}
$$

where the property of γ matrix

$$
\{\gamma^\mu,\ \gamma^\nu\} = 2g^{\mu\nu} \tag{3.29}
$$

was use to simplify the trace in the above integrand. Then we can simplify the above integral to (We suppress the terms which are proportional to l_1^2 or to l_2^2 in the numerator of the above integrand.),

$$
\begin{aligned}
\frac{A^{\text{one loop}}_{(b)}}{A(\phi,1^-,2^-,3^-)} \to & -\frac{1}{2}s_{12}s_{23}\int \frac{d^4l}{l_1^2l_2^2(l_2-k_1)^2(l_2+k_2)^2} - \frac{1}{2}(m_H^2-s_{23})\int \frac{d^4l}{l_1^2l_2^2(l_2-k_1)^2} \\
& -\frac{1}{2}s_{13}s_{23}\int \frac{d^4l}{l_1^2l_2^2(l_1+k_1)^2(l_2+k_2)^2} - \frac{1}{2}(m_H^2-s_{23})\int \frac{d^4l}{l_1^2l_2^2(l_1+k_1)^2}
\end{aligned}
\tag{3.30}
$$

We obtained two box integrals and two triangle integrals. Finally for calculating the cut-constructible part of the amplitude $A^{\text{one-loop}}(\phi,1_g^-,2_g^-,3_g^-)$, we need to sum over all six diagrams and suppress the overlapped loop integrals, we obtain

$$
A^{\text{one-loop, }CC}(\phi,1_g^-,2_g^-,3_g^-) = c_\Gamma A^{\text{tree}}(\phi,1_g^-,2_g^-,3_g^-)V_3 \tag{3.31}
$$

$$
\begin{aligned}
V_3 = & \ F_4^{1m}(k_1,k_2,k_3,k_\phi) + F_4^{1m}(k_2,k_3,k_1,k_\phi) + F_4^{1m}(k_3,k_1,k_2,k_\phi) \\
& +3\frac{1}{\epsilon^2}(-\frac{\mu^2}{s_{123}})^\epsilon - \frac{1}{\epsilon^2}(-\frac{\mu^2}{s_{12}})^\epsilon - \frac{1}{\epsilon^2}(-\frac{\mu^2}{s_{13}})^\epsilon - \frac{1}{\epsilon^2}(-\frac{\mu^2}{s_{23}})^\epsilon,
\end{aligned}
\tag{3.32}
$$

where the dimensionless function F_4^{1m} is explicitly given below,

$$
\begin{aligned}
F_4^{1m}(k_1,k_2,k_3,K_4) = & -\frac{1}{\epsilon^2}\left[\left(\frac{\mu^2}{-s}\right)^\epsilon + \left(\frac{\mu^2}{-t}\right)^\epsilon - \left(\frac{\mu^2}{-K_4^2}\right)^\epsilon\right] \\
& +\text{Li}\left(1-\frac{K_4^2}{s}\right) + \text{Li}\left(1-\frac{K_4^2}{t}\right) + \frac{1}{2}\ln^2\left(\frac{-s}{-t}\right) + \frac{\pi^2}{6}.
\end{aligned}
\tag{3.33}
$$

Since all different unitarity cuts do not contribute any bubble integrals, we notice that $A^{\text{one-loop, }CC}(\phi,1_g^-,2_g^-,3_g^-)$ is free of bubble integrals.

Now let us consider another example. We would like to compute the cut-constructible part of the amplitude $A^{\text{one-loop MHV}}(\phi, 1, 2, 3)$. Thanks to the super-amplitude representation, we can calculate all different particle types together. There are still six unitarity cuts which contribute. Two of them are shown in fig. 3.3. The other contributions can be obtained from moving the external legs 1,2 and 3 clockwisely.

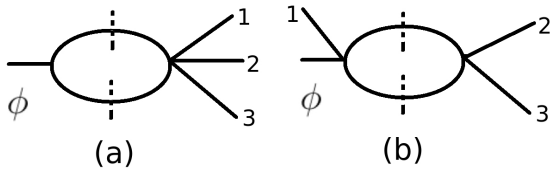

Figure 3.3: Two of six unitarity cuts contributing to $A^{\text{one-loop MHV}}(\phi, 1, 2, 3)$

We can use the unitarity method to calculate the contribution (a) in fig. 3.3,

$$\begin{aligned} A^{\text{one-loop MHV}}_{(a)}(\phi,1,2,3) \to & \int \frac{d^4l}{l_1^2 l_2^2} A(\phi, l_1^-, -l_2^-) A^{\text{MHV}}(-l_1, l_2, 1, 2, 3) \\ \to & \int \frac{d^4l}{l_1^2 l_2^2} \langle l_1 l_2\rangle^2 A^{\text{MHV}}(1,2,3) \frac{\langle 31\rangle}{\langle l_1 l_2\rangle\langle l_2 1\rangle\langle 3 l_1\rangle} \\ \to & A^{\text{MHV}}(1,2,3) \int \frac{d^4l}{l_1^2 l_2^2} \frac{\langle 31\rangle\langle l_1 l_2\rangle}{\langle l_2 1\rangle\langle 3 l_1\rangle} \\ \to & A^{\text{MHV}}(1,2,3) \int d^4l \frac{\text{Tr}_-(l_1 l_2 k_1 k_3)}{l_1^2 l_2^2 (l_2+k_1)^2 (l_1-k_3)^2} \end{aligned} \tag{3.34}$$

We notice that this integral is almost same integral in eq. (3.25) since in our case $\text{Tr}_-(abcd) = \text{Tr}_+(abcd)$ so we do not need to simplify the integrand again. Then we use the unitarity method to calculate the contribution (b) in fig. 3.3. We obtained the tree level MHV amplitude for ϕ plus tree-particle in eq. (2.58) and we have the tree level MHV amplitude in eq. (2.41), so we can use super cut in [8] due to Brandhuber, Spence and Travaglini. The contribution of the graph (b) in fig. 3.3 is

$$A^{\text{one-loop MHV}}_{(b)}(\phi,1,2,3) \to \int \frac{d^4 l_1}{l_1^2 l_2^2} \iint d^4\eta_{l_1}^A d^4\eta_{l_2}^A A^{\text{MHV}}(\phi, 1, l_2, -l_1) A^{\text{MHV}}(l_1, -l_2, 2, 3) \tag{3.35}$$

We will first integrate out of the fermionic loop variables, as calculated in [8] due to Brandhuber, Spence and Travaglini.

$$\begin{aligned} & \iint d^4\eta_{l_1}^A d^4\eta_{l_2}^A \delta^{(8)}(\lambda_1^\alpha \eta_1^A - \lambda_{l_2}^\alpha \eta_{l_2}^A - \lambda_{l_1}^\alpha \eta_{l_1}^A) \delta^{(8)}(\lambda_2^\alpha \eta_2^A + \lambda_3^\alpha \eta_3^A + \lambda_{l_2}^\alpha \eta_{l_2}^A - \lambda_{l_1}^\alpha \eta_{l_1}^A) \\ = & \ \delta^{(8)}(\lambda_1^\alpha \eta_1^A + \lambda_2^\alpha \eta_2^A + \lambda_3^\alpha \eta_3^A) \langle l_1 l_2\rangle^4. \end{aligned} \tag{3.36}$$

Then

$$\begin{aligned} A^{\text{one-loop MHV}}_{(b)} \to & \ A_3^{\text{MHV}}(\phi,1,2,3) \int \frac{dl_1^4}{l_1^2 l_2^2} \frac{\langle l_1 l_2\rangle^2 \langle 12\rangle\langle 31\rangle}{\langle 1 l_2\rangle\langle l_1 1\rangle\langle l_2 2\rangle\langle 23\rangle\langle 3 l_1\rangle} \\ \to & \ A_3^{\text{MHV}}(\phi,1,2,3) \int \frac{dl_1^4}{l_1^2 l_2^2} \frac{\text{Tr}_-(k_1 l_1 l_2 k_1 k_2 k_3)}{(l_2-k_1)^2 (l_1+k_1)^2 (l_1-k_3)^2}. \end{aligned} \tag{3.37}$$

We can simply obtain the above formula by replace the overall factor $A(\phi, 1^-, 2^-, 3^-)$ by $A_3^{\text{MHV}}(\phi, 1, 2, 3)$ in eq. (3.28). So we do not have to simplify this integrand again. For calculating the cut-constructible part of the amplitude $A^{\text{one-loop MHV}}(1, 2, 3)$, we should also sum over six unitarity cuts and suppress the overlapped loop integrals, we obtain the cut-constructible part of the amplitude $A^{\text{one-loop MHV}}(1, 2, 3)$,

$$A^{\text{one-loop MHV, } CC}(1,2,3) = c_\Gamma A^{\text{tree MHV}}(\phi,1,2,3) V_3 \tag{3.38}$$

where V_3 is defined in eq. (3.32). We conclude that for one-loop MHV and all-negative helicity ϕ plus three-particle amplitude, the cut-constructible parts are

$$A^{\text{one-loop, }CC}(\phi,1,2,3) = c_\Gamma A^{\text{tree}}(\phi,1,2,3)V_3 \tag{3.39}$$

where V_3 was defined in eq. (3.32).

3.1.3 Box Integral coefficient

In previous section, we used unitarity to build the cut-constructible parts of the one–loop amplitudes $A^{\text{one-loop, }CC}(\phi,1,2,3)$. In this section and the following two sections, we will try to cut more than two internal propagators to obtain directly the coefficients of the basic integrals. In order to extract the coefficient of box, a quadruple cut expression was suggested by Britto, Cachazo and Feng in ref. [18]. For a specific box defined by eq. (3.4), its coefficient is obtained from

$$\begin{aligned} & \int d^4l \, \frac{A_1\, A_2\, A_3\, A_4}{l^2(l+k_1)^2(l+k_1+k_2)^2(l-k_4)^2} \, |_{1/(l^2+i\epsilon)\to\delta^+(l^2)} \\ \to \quad & \int d^4l \delta^+(l^2)\delta^+((l+k_1)^2)\delta^+((l+k_1+k_2)^2)\delta^+((l-k_4)^2) A_1\, A_2\, A_3\, A_4 \\ \to \quad & \frac{1}{2} \sum_{solutions\ of\ l} A_1\, A_2\, A_3\, A_4 = C_4 \end{aligned} \tag{3.40}$$

where the tree amplitudes $A_1, \cdots, A_4$ are the amplitudes in the four corners if one cut four internal propagators. The four on-shell conditions can totally fix the internal loop momentum l. If we perform a quadruple cut to an one-loop amplitude, then we need only to write down a product of four tree amplitude and substitute the solution l in the product. The solutions for l can be found in ref. [42] due to Berger, Bern, Dixon, Febres Cordero, Forde, Ita, Kosower and Maitre or in ref. [43] due to Risager,

$$l^\mu_\pm = \frac{\langle 1^\mp | \not{k}_2 \not{k}_3 \not{k}_4 \gamma^\mu | 1^\pm \rangle}{2\langle 1^\mp | \not{k}_2 \not{k}_4 | 1^\pm \rangle} \tag{3.41}$$

where k_1 is a massless momentum.

In our one-loop amplitude calculation, there are 3 boxes contributing to an one-loop amplitude. One contribution from the quadruple cut is shown in fig. 3.4. We can obtain the other contributions by moving the outgoing legs 1,2,3 clockwisely. We can see from fig. 3.4, that there are only one massive outgoing leg

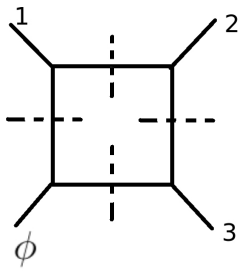

Figure 3.4: One of three contributions of quadruple cut

and three massless outgoing legs. In each massless leg's corner, the three point vertex could be MHV and $\overline{\text{MHV}}$. When a box graph contains a sequence of three point vertexes, a non-vanishing solution is only found when the vertexes alternate between MHV and $\overline{\text{MHV}}$-types [18]. For three-point vertexes $A(1,2,3)$, as explained in section 2.2, we can choose that

$$\tilde{\lambda_1} \sim \tilde{\lambda_2} \sim \tilde{\lambda_3} \text{ or } \lambda_1 \sim \lambda_2 \sim \lambda_3. \tag{3.42}$$

If two three point vertexes are neighbors, let l be the common propagator, then λ_l and $\tilde{\lambda}_l$ must be proportional to the spinors of different outgoing legs. There are two consequences form above argument. First consequence is that l must be complex momentum so that λ_l is not complex conjugate of $\tilde{\lambda}_l$. Second consequence is that for two adjacent three-point vertexes, one of them is MHV then the other must be $\overline{\text{MHV}}$.

Knowing the above properties and the solutions for l in [43] due to Risager, we can start to calculate the contribution of the quadruple cut of $A^{\text{one loop}}(1_g^-, 2_g^-, 3_g^-)$ in fig. 3.4. The only particle which can circulate in the loop is gluon. And there is only one possible helicity configuration: the vertex attached to leg 1 must be MHV. The coefficient of the basic integral shown in fig. 3.4 is,

$$\begin{aligned} C_{4,\phi,1,2,3} &= \frac{1}{2} A^{\text{tree}}(\phi, -l_1^-, l_2^-) A^{\text{tree,MHV}}(1_g^-, l_3^-, -l_2^+) A^{\text{tree},\overline{\text{MHV}}}(2_g^-, l_4^+, -l_3^+) A^{\text{tree,MHV}}(3_g^-, l_1^+, -l_4^-) \\ &= \frac{1}{2} \frac{m_H^2}{[l_1 l_2]^2} \frac{\langle l_3 1\rangle^4}{\langle l_3 1\rangle\langle l_3 l_2\rangle\langle l_2 1\rangle} \frac{[l_4 l_3]^4}{[l_4 l_3][l_3 2][2 l_4]} \frac{\langle l_4 3\rangle^4}{\langle l_4 3\rangle\langle 3 l_1\rangle\langle l_1 l_4\rangle} \end{aligned} \tag{3.43}$$

We substitute the solution,

$$\begin{aligned} l_1 &= \frac{|(-p_{123})1][3|}{[31]} \quad & l_2 &= \frac{|(-p_{123})3][1|}{[31]} \\ l_3 &= \langle 2|\frac{[23]}{[13]}[1| & l_4 &= \langle 2|\frac{[23]}{[13]}[3| \end{aligned} \tag{3.44}$$

in the eq. (3.43) and we obtain,

$$C_{4,\phi,1,2,3} = -\frac{1}{2} s_{12} s_{23} A^{\text{tree}}(1_g^-, 2_g^-, 3_g^-) \tag{3.45}$$

We notice that the coefficient $C_{4,\phi,1,2,3}$ is a product of the Gram determinant $\mathcal{G}^{1m} = -s\,t/2$ and tree level amplitude $A^{\text{tree}}(1_g^-, 2_g^-, 3_g^-)$. We can calculate the other two box coefficients $C_{4,\phi,2,3,1}$ and $C_{4,\phi,3,1,2}$ in a similar way:

$$\begin{aligned} C_{4,\phi,2,3,1} &= -\frac{1}{2} s_{23} s_{13} A^{\text{tree}}(1_g^-, 2_g^-, 3_g^-) \\ C_{4,\phi,3,1,2} &= -\frac{1}{2} s_{13} s_{12} A^{\text{tree}}(1_g^-, 2_g^-, 3_g^-) \end{aligned} \tag{3.46}$$

Sum all above three box contribution together. One can obtain the quadruple cut contribution to $A^{\text{one-loop, }CC}$,

$$\begin{aligned} A^{\text{one-loop}}_{\text{cut box}}(\phi, 1, 2, 3) &= c_\Gamma A^{\text{tree}}(\phi, 1, 2, 3)(F_4^{1m}(k_1, k_2, k_3, k_\phi) \\ &\quad + F_4^{1m}(k_2, k_3, k_1, k_\phi) + F_4^{1m}(k_3, k_1, k_2, k_\phi)). \end{aligned} \tag{3.47}$$

3.1.4 Triangle Integral coefficient

In this section, we would like to extract the coefficient of triangle integrals directly using the method introduced in [6] due to Forde. To extract this triangle integral coefficient, for example as shown in fig. 3.5, the three cut propagators are

$$l_0^2 = 0,\ l_1^2 = (l - K_1)^2 = 0,\ l_2^2 = (l - K_2)^2 = 0. \tag{3.48}$$

For solving these three on-shell conditions, we introduce two on-shell momenta

$$\begin{aligned} K_1^{\flat,\mu} &= \frac{K_1^\mu - (S_1/\gamma)K_2^\mu}{1 - (S_1 S_2/\gamma^2)} \\ K_2^{\flat,\mu} &= \frac{K_2^\mu - (S_1/\gamma)K_1^\mu}{1 - (S_1 S_2/\gamma^2)} \end{aligned} \tag{3.49}$$

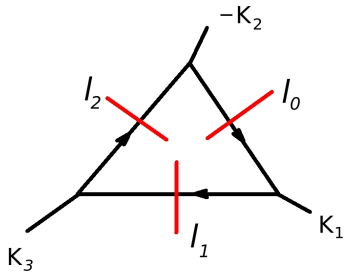

Figure 3.5: The triple cut used to compute the scalar triangle coefficient

where $S_1 = K_1^2$, $S_2 = K_2^2$ and γ can be expressed in terms of K_1^μ and K_2^μ:

$$\gamma_\pm = (K_1 \cdot K_2) \pm \sqrt{\Delta}, \quad \Delta = (K_1 \cdot K_2)^2 - K_1^2 K_2^2. \tag{3.50}$$

There are two solutions for γ so that at the end we should calculate the average of the above two solutions. Then we express the loop momentum l_0, l_1, l_2 as:

$$\langle l_i^- | = t\langle K_1^{\flat,-}| + \alpha_{i1}\langle K_2^{\flat,-}|, \ \langle l_i^+| = \frac{\alpha_{i2}}{t}\langle K_1^{\flat,+}| + \langle K_2^{\flat,+}| \tag{3.51}$$

where t is a parameter because we only have three constraints and the values of α_{ij} are

$$\begin{aligned} \alpha_{01} &= \frac{S_1(\gamma - S_2)}{(\gamma^2 - S_1 S_2)}, & \alpha_{02} &= \frac{S_2(\gamma - S_1)}{(\gamma^2 - S_1 S_2)}, \\ \alpha_{11} &= \alpha_{01} - \frac{S_1}{\gamma}, & \alpha_{12} &= \alpha_{02} - 1, \\ \alpha_{21} &= \alpha_{01} - 1, & \alpha_{22} &= \alpha_{02} - \frac{S_2}{\gamma}. \end{aligned} \tag{3.52}$$

The loop momentum can not be totally determined. With the above parametrization, the loop integration over l^μ becomes,

$$\int d^4 l \prod_{i=0}^{2} \delta(l_i^2) A_1 A_2 A_3 \to \int dt \, J_t \, A_1(l(t)) A_2(l(t)) A_3(l(t)) \tag{3.53}$$

where J_t is the Jacobian determinant. Since we only want to extract the coefficient of a triangle integral, we must exclude the influences of boxes which contains the chosen triple cut. The box integral must contains the fourth propagator, for example $1/(l - k_4)^2$. We substitute the parametrization l (3.51) in $1/(l - k_4)^2$ and we obtain,

$$1/(l - k_4)^2 \propto \frac{t}{(t - t_+)(t - t_-)} \tag{3.54}$$

In order to remove the above influences, we expand the product of the three tree amplitudes around $t = \infty$, then it yields,

$$\int dt \, J_t \, \mathrm{Inf}_t A_1(l(t)) A_2(l(t)) A_3(l(t)) \to \int dt \, J_t \sum c_i t^i \tag{3.55}$$

Then we use the following properties

$$\begin{aligned} &\langle K_1^\flat | K_1 | K_2^\flat] = 0 \\ &\langle K_1^\flat | K_2 | K_2^\flat] = 0 \\ &\langle K_1^\flat | \gamma^\mu | K_2^\flat] \langle K_1^\flat | \gamma^\mu | K_2^\flat] = 0 \end{aligned} \tag{3.56}$$

we obtain,

$$\begin{aligned}\int \frac{\langle K_1^\flat | l | K_2^\flat]}{l^2(l-K_1)^2(l-K_2)^2} = 0 \quad &\rightarrow \quad \int dt J_t \frac{1}{t^n} = 0 \; n \geq 1 \\ \int \frac{\langle K_2^\flat | l | K_1^\flat]}{l^2(l-K_1)^2(l-K_2)^2} = 0 \quad &\rightarrow \quad \int dt J_t t^n = 0 \; n \geq 1\end{aligned} \tag{3.57}$$

Because the above integrations vanish, there is only one term, $\int dt\, J_t\, c_0$ in eq, (3.55) non-vanishing. So the integral coefficient for a triple cut is the constant term in the series

$$\int dt\, J_t\, \mathrm{Inf}_t(A_1(l(t))A_2(l(t))A_3(l(t))) \rightarrow \int dt\, J_t\, c_0. \tag{3.58}$$

The integration over t is associated with the scalar triangle integral so that the coefficient of the triangle integral is c_0.

Now, we can apply the above procedures to our model and try to extract the coefficients of triangle integral. For the triangle integrals in our model, there are 9 contributions, which can be obtained from moving legs 1,2,3 clockwisely of that shown in fig. 3.6.

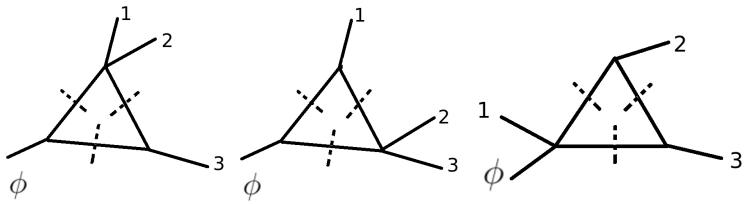

Figure 3.6: Three of nine contributions of triple cut of the amplitude $A^{\text{one loop}}(1_g^-, 2_g^-, 3_g^-)$

For example, we want to calculate the coefficient of the first left graph in fig. 3.6 of the amplitude $A^{\text{one loop}}(1_g^-, 2_g^-, 3_g^-)$. We take $K_1 = k_1 + k_2$ and $K_2 = k_1 + k_2 + k_3$ in eq. (3.48) . We can write down the product of the three tree amplitudes (see fig. 3.7):

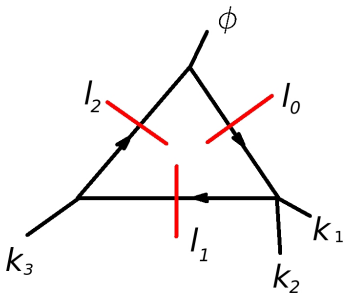

Figure 3.7: The triple cut used to compute the scalar triangle coefficient $C_{3,\phi,12,3}$

$$\begin{aligned}A_1 A_2 A_3 &= A^{\text{tree}}(\phi, l_0^-, -l_2^-) A^{\text{tree, MHV}}(1_g^-, 2_g^-, l_1, -l_0) A^{\text{tree}}(l_2, 3, -l_1) \\ &= -\frac{m_H^2}{[l_0 l_2]^2} \frac{i\langle 12\rangle^4}{\langle 12\rangle\langle 2 l_1\rangle\langle l_1 l_0\rangle\langle l_0 1\rangle} \frac{i\langle l_1 3\rangle^4}{\langle l_1 3\rangle\langle 3 l_2\rangle\langle l_2 l_1\rangle}\end{aligned} \tag{3.59}$$

We obtain two solutions for γ from eq. (3.50):

$$\gamma_+ = s_{123}, \;\; \gamma_- = s_{12} \tag{3.60}$$

For the solution $\gamma = \gamma_- = s_{12} = S_1$, if we substitute γ_- in eq. (3.51), the spinor product $\langle l_1 l_2\rangle$ will be replaced as,

$$\langle l_1 l_2\rangle \to -t(1 - \frac{S_1}{\gamma_-})\langle K_1^\flat K_2^\flat\rangle = 0 \tag{3.61}$$

then the product of three tree amplitudes,

$$A_1 A_2 A_3 \to \infty. \tag{3.62}$$

We should throw away the solution $\gamma = \gamma_-$ and only keep the solution $\gamma = \gamma_+$. We substitute γ_+ in eq. (3.51), then we have the solution for l_i, $i = 0, 1, 2$, so we can replace all the spinor product involving l_i, $i = 0, 1, 2$ in eq. (3.59) and expand the result around $t = \infty$. The coefficient of t^0 is

$$\begin{aligned} C_{3,\phi,12,3} = \frac{1}{2}\mathrm{Inf}_{t\to\infty} A_1 A_2 A_3 &= = \frac{1}{2}\frac{m_H^4 \langle K_1^\flat 3\rangle^2 \langle 12\rangle^2}{s_{12}(s_{12} - s_{123})\langle K_1^\flat 1\rangle\langle K_1^\flat 2\rangle} \\ &= \frac{1}{2}\frac{m_H^4 \langle 3K_1^\flat 2]^2 \langle 12\rangle^2}{s_{12}(s_{12} - s_{123})\langle 1K_1^\flat 2]\langle 2K_1^\flat 2]} \end{aligned} \tag{3.63}$$

where $K_1^\flat$ can be replaced by K_1 and K_2 in eq. (3.49). The we have

$$C_{3,\phi,12,3} = \frac{1}{2}A^{\text{tree}}(1_g^-, 2_g^-, 3_g^-)(s_{12} - s_{123}). \tag{3.64}$$

At last, we can extract all the triangle-integral coefficients using the above method. After we extract the coefficients of triangle integrals, we notice for different particle types that our theory is very similar to $\mathcal{N} = 4$ Yang-Mills itself and that the leading singularities are the same for different particle type:

$$A^{\text{one-loop}}(\phi, 1, 2, 3) = -\frac{c_\Gamma A^{\text{tree}}(\phi, 1, 2, 3)}{\epsilon^2}((\frac{\mu^2}{s_{12}})^\epsilon + (\frac{\mu^2}{s_{23}})^\epsilon + (\frac{\mu^2}{s_{13}})^\epsilon) + \mathcal{O}[1], \tag{3.65}$$

which can be also obtained from eq. (3.39). The sum of the triangle diagrams contributions and the box contributions yields above leading singularities.

3.1.5 Bubble Integral coefficient

To extract the coefficients of bubble integral, we employ the spinor integration as presented by Mastrolia [7]. He used the spinorial variables to parametrize the Lorentz invariant phase-space (LIPS) (introduced in [13] due to Cachazo, Svrcek and Witten) in the K^2-channel,

$$\int d^4 l_1 \delta^{(+)}(l_1^2)\delta^{(+)}((l_1 - K)^2) = \int\int \frac{\langle l\, dl\rangle[dl\, l]}{\langle l|K|l]} \int t dt \delta^{(+)}(t - \frac{K^2}{\langle l|K|l]}). \tag{3.66}$$

The above formula is obtained by rescaling the original loop-variable l_1^μ as,

$$\ell_1^\mu = \frac{\langle l_1|\gamma^\mu|l_1]}{2} \equiv t l^\mu = t\frac{\langle l|\gamma^\mu|l]}{2} \tag{3.67}$$

with $l_1^2 = l^2 = 0$. In terms of spinor variables, the rescaling reads,

$$|l_1\rangle = \sqrt{t}\ |l\rangle\ , l_1] = \sqrt{t}\ |l]\ , \tag{3.68}$$

where the integration of the rescaling parameter t is directly can be done as a consequence of the on-shell conditions $\delta^{(+)}((l_1 - K)^2)$, and l^μ becomes new loop integration variable.

In ref. [7] Mastrolia takes two massless momenta, p_μ and q_μ fulfilling the conditions,

$$p_\mu + q_\mu = K_\mu,\ p^2 = q^2 = 0\ ,\ 2p\cdot q = 2p\cdot K = 2q\cdot K \equiv K^2 \tag{3.69}$$

and decompose l_μ in a basis of four massless momenta constructed out of them,

$$l_\mu = p_\mu + z\,\bar{z}\,q_\mu + \frac{z}{2}\langle q|\gamma_\mu|p] + \frac{\bar{z}}{2}\langle p|\gamma_\mu|q] \tag{3.70}$$

Notice that the vectors $\langle q|\gamma_\mu|p]$ and $\langle p|\gamma_\mu|q]$ are orthogonal to both p_μ and q_μ. The above decomposition can be realized starting from the definition of l_μ in terms of spinor variables, $l^\mu = \langle l|\gamma_\mu|l]/2$, and performing the following spinor decomposition,

$$|l\rangle \equiv |p\rangle + z|q\rangle \ , \quad |l] \equiv |p] + \bar{z}|q] \ . \tag{3.71}$$

In practice, if the number of the K-channel outgoing particles is 2, we can assign p and q to be the two outgoing momenta. If the number of the K-channel outgoing particles is more than 2, we can introduce a reference spinor χ_μ, which can be a outgoing momentum , then

$$K^\mu = K^{\flat,\mu} + \frac{K^2}{\langle\chi|K|\chi]}\chi^\mu \tag{3.72}$$

where $(K^\flat)^2$ vanishes. So we can set p and q to $K^\flat$ and $\dfrac{K^2}{\langle\chi|K|\chi]}\chi$.

With the above parametrization, the integration over the loop momenta l^μ becomes,

$$\begin{aligned} &\int d^4l\,\delta^{(+)}(l_1^2)\delta^{(+)}((l_1-K)^2)\,A_L^{\rm tree}(l_1)A_R^{\rm tree}(l_1) \\ = \ &\oint dz \int d\bar{z} \int t^2\ dt\ \delta(t - \frac{1}{(1+z\bar{z})})t^{\alpha_L+\alpha_R}A_L^{\rm tree}(t,l)A_R^{\rm tree}(t,l) \end{aligned} \tag{3.73}$$

where $\alpha_{L,R}$ parametrizes the scaling behavior of $A_{L,R}^{\rm tree}$. The t-integration can be performed, because of the presence of the δ-function.

$$\int d^4l\,\delta^{(+)}(l_1^2)\delta^{(+)}((l_1-K)^2)A_L^{\rm tree}(l_1)A_R^{\rm tree}(l_1) = \oint dz \int d\bar{z}\ f(z,\bar{z}) \tag{3.74}$$

To begin with the integration, we find a primitive of f with respect to $\bar{z}$ by keeping z as independent variable,

$$F(z,\bar{z}) = \int d\bar{z}\ f(z,\bar{z}) \ , \tag{3.75}$$

$$F(z,\bar{z}) = F^{\rm rat}(z,\bar{z}) + F^{\log}(z,\bar{z}) \ . \tag{3.76}$$

Then the z-integration will be performed by applying Cauchy's Residue Theorem. We also know a priory that the coefficients of bubble integral are rational functions. So we only calculate the integration of $F^{\rm rat}$ in z,

$$C_{2;K} \equiv -\frac{1}{2\pi i}\oint dz\ F^{\rm rat}(z,\bar{z}). \tag{3.77}$$

Then we can apply the above method to our model. There are 6 contributions, which can be obtained from cyclic permutations (without changing the order of the outgoing legs 1,2,3) of that shown in fig. 3.8. We denote $C_{2;\phi,123}$ as the coefficient of the bubble integral in graph (a) and $C_{2;\phi 1}$ as the coefficient of the bubble integral in graph (b) (see fig. 3.8). The other four coefficients are $C_{2;\phi,312}$, $C_{2;\phi,231}$, $C_{2;\phi 2}$ and $C_{2;\phi 3}$.

Each bubble integral contributes a $1/\epsilon$ pole:

$$\mathcal{I}_2 \propto \frac{1}{(1-2\epsilon)\epsilon}\left(\frac{\mu^2}{-s}\right)^\epsilon = \frac{1}{\epsilon} + \left(\ln\left(-\frac{\mu}{s}\right) + 2\right) + \mathcal{O}[\epsilon] \tag{3.78}$$

But the leading singularities in ϵ of the cut–constructible part obtained in eq. (3.65) which are already reproduced by box and triangle contribution. The sum over all $1/\epsilon$ poles of the bubble contribution should vanish. so the sum of the bubble of different channels vanishes:

$$C_{2;\phi,123} + C_{2;\phi,231} + C_{2;\phi,312} + C_{2;\phi 1} + C_{2;\phi 2} + C_{2;\phi 3} = 0 \tag{3.79}$$

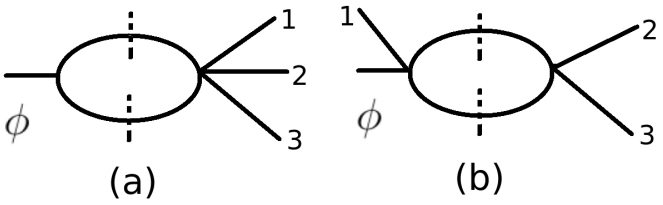

Figure 3.8: The coefficient of (a) is $C_{2;\phi,123}$. The coefficient of (b) is $C_{2;\phi 1}$

Using the above extraction method, we found that not only the sum of $C_{2;\phi i}$ vanishes, but also each $C_{2;\phi i}$ vanishes. And this is expected by the result (3.39) obtained via unitarity method.

As ending this section, we give an example that we apply this method to calculate $C_{2;\phi 1}$ which is the coefficient of bubble integral in K_{23} channel in $A^{\text{one-loop}}(\phi, 1_g^-, 2_{f_1}^+, 3_{f_{234}}^-)$.

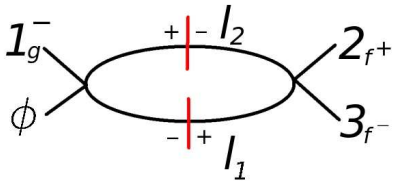

Figure 3.9: The double cut to compute the bubble coefficient $C_{2;\phi 1,g}$ when calculating $A^{\text{one-loop}}(\phi, 1_g^-, 2_{f_1}^+, 3_{f_{234}}^-)$.

Let us compute the coefficient of a bubble integral shown in fig. 3.9. We calculate the case that it is a gluon which circulating in the loop. The product of two tree amplitudes is,

$$\begin{aligned} A^R A^L &= A^{\text{MHV}}(\phi, 1_g^-, -l_2^+, l_1^-) A^{\text{MHV}}(-l_1^+, l_2^-, 2_{f_1}, 3_{f_{234}}) \\ &= \frac{\langle l_1 1\rangle^3 \langle l_2 3\rangle^3}{\langle l_1 l_2\rangle^2 \langle l_1|3\rangle\langle l_2|1\rangle\langle 2|3\rangle} \\ &= \frac{\langle l_1 1\rangle^3 \langle 3|l_2|l_1]^3}{\langle l_1 3\rangle\langle 23\rangle\langle l_1|l_2|l_1]^2\langle 1|l_2|l_1]} \end{aligned} \tag{3.80}$$

Then we can replace the momentum l_2 by $l_2 = l_1 - k_2 - k_3$. The above formula becomes,

$$A^R A^L = \frac{\langle l_1 1\rangle^3 \langle 3|2|l_1]^3}{\langle l_1 3\rangle\langle 23\rangle\langle l_1|2+3|l_1]^2\langle 1|2+3|l_1]} \tag{3.81}$$

Now, we can replace the loop momentum l_1 by $l_1 = t\ l$, and for K–channel, $K = k_2 + k_3$, we assign $p = k_2$ and $q = k_3$ in eq. (3.69) so that we can replace the loop momentum spinor l by spinor k_2, k_3, z and $\bar{z}$ using the formula (3.71). We obtain,

$$\begin{aligned} A^R A^L &= \frac{\langle l1\rangle^3 \langle 3|2|l]^3}{\langle l3\rangle\langle 23\rangle\langle l|2+3|l]^2\langle 1|2+3|l]} \\ &= \frac{\bar{z}^3(-\langle 12\rangle - z\langle 13\rangle)^3\langle 23\rangle[32]^3}{(-\bar{z}\langle 12\rangle[32] + \langle 13\rangle[32])(\langle 23\rangle[32] + z\bar{z}\langle 23\rangle[32])^2} \end{aligned} \tag{3.82}$$

The bubble integral coefficient $C_{2;\phi 1}$ is,

$$\begin{aligned} C_{2;\phi 1,g} &= -\frac{1}{2\pi i}\oint dz \int d\bar{z} \int dt\, t^2\, \delta(t - \frac{1}{(1+z\bar{z})}) A^R A^L \\ &= -\frac{1}{2\pi i}\oint dz \left(\frac{-\left(2+6z\bar{z}+6z^2\bar{z}^2\right)\langle 1|2\rangle^2 - z\left(7+21z\bar{z}+18z^2\bar{z}^2\right)\langle 1|2\rangle\langle 1|3\rangle}{6z^3(1+z\bar{z})^3\langle 2|3\rangle} \right. \\ &\quad \left. -\frac{\left(11+27z\bar{z}+18z^2\bar{z}^2\right)\langle 1|3\rangle^2}{6z(1+z\bar{z})^3\langle 2|3\rangle} \right)\Bigg|_{\ln\to 0} \end{aligned} \tag{3.83}$$

We apply Cauchys Residue Theorem to the above formula. We only need to calculate the residue of the integrand at $z = 0$. We finally obtain

$$C_{2;\phi 1,g} = -\frac{11}{6} A^{\text{tree}}(\phi, 1_g^-, 2_{f_1}^+, 3_{f_{234}}^-). \tag{3.84}$$

Following above procedures, we calculate the contributions from all different helicity configuration and from different particle type (circulating in the loop). The results are shown below. I remove a overall factor of $A^{\text{tree}}(\phi, 1_g^-, 2_{f_1}^+, 3_{f_{234}}^-)$, so that the above calculation will just yield a number. We obtain the different contributions from the different particle circulating around the loop. Let $C_{2;\phi 1,g}$ be the contribution of gluon loop to $C_{2;\phi 1}$. There are two helicity-configurations for $C_{2;\phi 1,g}$. The final results are:

$$\begin{aligned} C_{2;\phi 1,g} &= -\frac{1}{3} - \frac{11}{6} \\ C_{2;\phi 1,f_1} &= \frac{1}{3} + \frac{11}{6} \\ C_{2;\phi 1,f_i} &= \frac{1}{3} + \frac{1}{6},\ i = 2,3,4 \\ C_{2;\phi 1,s} &= -\frac{1}{3} - \frac{1}{6} \end{aligned} \tag{3.85}$$

We add all these contributions together. We find that $C_{2;\phi 1}$ vanishes. We can repeat the above calculation for MHV and all-negative helicity amplitude $A^{\text{one-loop, }CC}(\phi, 1, 2, 3)$. Just as we expected, $A^{\text{one-loop, }CC}(\phi, 1, 2, 3)$ does not contain any bubble integrals.

At last, we can directly reproduce the cut-constructible part of all–negative helicity and MHV amplitudes $A^{\text{one-loop, }CC}(\phi, 1, 2, 3)$, in eq. (3.39) by summing the products of the basic integral and their coefficients obtained via generalized unitarity method. We can also obtain the cut-constructible part of all–positive helicity and $\overline{\text{MHV}}$ amplitudes $A^{\text{one-loop, }CC}(\phi^\dagger, 1, 2, 3)$,

$$A^{\text{one-loop, }CC}(\phi^\dagger, 1, 2, 3) = A^{\text{one-loop, }CC}(\phi^\dagger, 1, 2, 3) = c_\Gamma A^{\text{tree}}(\phi^\dagger, 1, 2, 3) V_3 \tag{3.86}$$

where V_3 was defined in eq. (3.32). The rational part of the above amplitudes, the amplitudes $A^{\text{one-loop}}(\phi, 1, 2, 3)$ of all–positive helicity and of $\overline{\text{MHV}}$ and the amplitudes $A^{\text{one-loop}}(\phi^\dagger, 1, 2, 3)$ of all–negative helicity and of MHV will be calculated in the next section because these functions which all rational functions of spinor products can be computed by loop level recursion relations.

3.2 Rational part

Until this section, we only calculated the cut-constructible part of one-loop amplitudes via both unitarity and generalized unitarity method. In this section, we will reviews the method of loop-level recursion relations [45] due to Berger, Bern, Dixon, Forde and Kosower and compute the cut–unpredictable part of an one-loop amplitude.

For loop level amplitude, we still consider the shift $[j, l\rangle$ defined in eq. (2.7),

$$A^{\text{one loop}} = A^{\text{one loop, }CC}(z) + R(z) \tag{3.87}$$

At tree amplitude level, the residue of the shifted amplitude can be found at its physical singularities(collinear and soft singularities). At one–loop level, the shifted cut-constructible part $A^{\text{one loop, }CC}(z)$ often contains unphysical poles, for example it may contains,

$$\frac{\ln(s/t)}{(s-t)^n} \text{ or } \frac{1}{(s-t)^n}. \tag{3.88}$$

where $s = t$ which corresponds to no physical singularities is called spurious singularity. We may write down a new decomposition of one–loop amplitude,

$$\begin{aligned} A^{\text{one loop}} &= A^{\text{one loop, }CC}(z) + \widehat{CR}(z) + R(z) - \widehat{CR}(z) \\ &= A^{\text{one loop, }CC}(z) + \widehat{CR}(z) + \widehat{R}(z). \end{aligned} \tag{3.89}$$

where $\widehat{CR}(z)$ are the rational functions added in order to cancel the spurious singularities in z of $A^{\text{one loop, }CC}(z)$. Since we can find the cut-constructible part $A^{\text{one loop, }CC}$ via unitarity or generalized unitarity, in general, we can write down the counter term $\widehat{CR}(z)$ from the observation of $A^{\text{one loop, }CC}$. Then we expect that $\widehat{R}(z)$ only contains physical singularities so that it can be calculated by on-shell recursion relations,

$$\widehat{R}(0) = \sum_{r,s,h} \left(R_L \frac{i}{K^2_{r\cdots s}} A^{\text{tree}}_R + A^{\text{tree}}_L \frac{i}{K^2_{r\cdots s}} R_R + A^{\text{tree}}_L \frac{i\mathcal{F}(r\cdots s)}{K^2_{r\cdots s}} A^{\text{tree}}_R \right). \tag{3.90}$$

where the sum is over all affected multi-particle momentum $K_{r\cdots s}$ by the shift $[j, l\rangle$, R_L and R_R are the rational part of sub-amplitudes and $\mathcal{F}(r\cdots s)$, called factorization function may be found in ref. [45]. Remember that

$$\widehat{R}(z) = R(z) - \widehat{CR}(z) \tag{3.91}$$

The counter term $\widehat{CR}(z)$ may also contain physical poles. We denote O as the sum of the residues of the shifted counter term $\widehat{CR}(z)$ at the physical poles,

$$O = \sum_{poles\ \alpha} \text{Res}_{z=z_\alpha} \frac{\widehat{CR}(z)}{z} \tag{3.92}$$

Finally the rational part of one–loop amplitude defined in eq. (3.87) can be computed as,

$$R(0) = \widehat{R}(0) + \widehat{CR} + O \tag{3.93}$$

In the rest part of this section, we will use the above method to calculate the rational part of the amplitudes $A^{\text{one-loop}}(\phi, 1, 2, 3)$ with all–negative helicity or MHV and also calculate the amplitudes $A^{\text{one-loop}}(\phi, 1, 2, 3)$ with all–positive helicity or $\overline{\text{MHV}}$ which does not contain cut–constructible part. The sub-amplitudes we need can be found in ref. [24] by Berger, Vittorio Del Duca, and Dixon,

$$\begin{aligned} A^{\text{one loop}}(\phi, 1^+, 2^+) &= -2\, c_\Gamma A^{\text{tree}}_2(\phi^\dagger, 1^+, 2^+), \\ A^{\text{one loop}}(\phi, 1^\pm, 2^\mp) &= 0, \\ A^{\text{one loop}}(\phi, 1^-, 2^-) &= c_\Gamma \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon \left[-\frac{2}{\epsilon^2} + 2 \right] A^{\text{tree}}_2(\phi, 1^-, 2^-). \end{aligned} \tag{3.94}$$

If we want to use the loop level recursion relation method introduced in the beginning of this section to construct the one–loop rational part in our model, we notice that method can be easily applied. Because there is no spurious poles in the obtained cut–constructible part eq. (3.39), so the counter term $\widehat{CR}(z)$ vanishes and the rational parts of one loop amplitudes in pure $\mathcal{N} = 4$ supersymmetric theory vanish. So that the cut-unpredictable rational part of ϕ can be directly calculated via BCFW.

The rational parts of all–negative helicity and MHV amplitudes $A^{\text{one-loop}}(\phi,1,2,3)$ are,

$$R(\phi,1,2,3) = 2c_\Gamma A^{\text{tree}}(\phi,1,2,3) \tag{3.95}$$

The all–positive helicity and $\overline{\text{MHV}}$ amplitudes $A^{\text{one-loop}}(\phi,1,2,3)$ are

$$A^{\text{one-loop}}(\phi,1,2,3) = -2\,c_\Gamma A_2^{\text{tree}}(\phi^\dagger,1,2,3) \tag{3.96}$$

And we can obtain the amplitude $A^{\text{one-loop}}(\phi^\dagger,1,2,3)$ by parity. We will need to calculate the amplitude with $A(H) = A(\phi) + A(\phi^\dagger)$, for example

$$\begin{aligned} A^{\text{one-loop}}(H,1^-,2^-,3^-) &= A^{\text{one-loop}}(\phi,1^-,2^-,3^-) + A^{\text{one-loop}}(\phi^\dagger,1^-,2^-,3^-) \\ &= c_\Gamma A^{\text{tree}}(\phi,1^-,2^-,3^-)(V_3+2) - 2\,c_\Gamma A_2^{\text{tree}}(\phi,1^-,2^-,3^-) \\ &= c_\Gamma A^{\text{tree}}(\phi,1^-,2^-,3^-)V_3 \end{aligned} \tag{3.97}$$

where V_3 is defined in eq. (3.32). At last, we notice that for any particle types with any helicity configurations, the rational parts of $A^{\text{one-loop}}(H,1,2,3)$ vanish. The one loop amplitudes $A^{\text{one-loop}}(H,1,2,3)$ are

$$A^{\text{one-loop}}(H,1,2,3) = c_\Gamma A^{\text{one-loop}}(H,1,2,3)V_3. \tag{3.98}$$

3.3 Calculations of the Virtual Contribution to the EEC Function

In previous sections, we obtained the one loop amplitudes for H plus three–particle cases in eq. (3.98). It is simple to obtain the fully-symmetrized differential cross-section from these amplitudes by repeating the calculations we have done in section 2.5. The sum over the all final states (of the interference between one-loop amplitudes and tree amplitude) are also can be obtained simply:

$$\sum_{\text{all states}} A^{\text{tree}}(1,2,3)A^{\text{one-loop}}(H,1,2,3) = \frac{4c_\Gamma s_{123}^4}{3s_{12}s_{13}s_{23}}V_3 \tag{3.99}$$

We will calculate the virtual contribution to EEC function. The Energy-Energy correlation function of the one-loop contribution is:

$$\begin{aligned} EEC_V &= 2\int dPS_3 \frac{3E_2E_3\delta(\cos\theta_{23}-c)}{E_{tot}^2} \sum_{\text{all states}} A^{\text{tree}}(1,2,3)A^{\text{one-loop}}(1,2,3) \\ &= 8c_\Gamma \int dPS_3 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E_{tot}^2} \frac{m_H^8}{s_{12}s_{13}s_{23}}V_3 \end{aligned} \tag{3.100}$$

where the overall factor 2 before the integral comes from the fact that $\sum A^{\text{tree}}(1,2,3)A^{\text{one-loop}}(1,2,3)$ and $\sum A^{\text{one-loop}}(1,2,3)A^{\text{tree}}(1,2,3)$ will yield the same contribution. In the following calculation, we will calculate dimensionless EEC'_V instead of EEC_V,

$$EEC'_V = EEC_V/(m_H^{4-4\epsilon}S_\Gamma), \tag{3.101}$$

where

$$S_\Gamma = \frac{2^{6\epsilon-11}\pi^{3\epsilon-5}}{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)} \tag{3.102}$$

The function EEC'_V will be more easily compared with the real emission contribution to EEC function in the future. The parametrization of the three-particle phase-space was already introduced in eq. (2.128) in the section 2.6. We defined $V_{3,LS}$

$$V_{3,LS} = \frac{1}{\epsilon^2}(-(\frac{-\mu^2}{s_{12}})^\epsilon - (\frac{-\mu^2}{s_{13}})^\epsilon - (\frac{-\mu^2}{s_{23}})^\epsilon). \tag{3.103}$$

Using parametrization (2.128), we obtain the virtual contribution EEC'_V,

$$EEC'_V = \int_0^1 dt\, J(u,t) V_{3,LS} + \int_0^1 dt\, J(u,t)(V_3 - V_{3,LS}) \tag{3.104}$$

where

$$J(u,t) = \frac{(1-t)^{-2\epsilon} t^{-2\epsilon} (1-u)^{-\epsilon-1} u^{-\epsilon-1} \Gamma^3(1-\epsilon)\Gamma(\epsilon+1)(1-tu)^{2\epsilon-1}}{\Gamma(1-2\epsilon)}. \tag{3.105}$$

We perform the integration of $\int_0^1 dt\, J(u,t) V_{3,LS}$ directly in *Mathematica*. We obtain,

$$\begin{aligned}
\int_0^1 dt\, J(u,t) V_{3,LS} = \; & \frac{3\pi^2 (1-u)^{-\epsilon} u^{-\epsilon-1} \csc(\pi\epsilon)\csc(3\pi\epsilon)\Gamma(1-\epsilon)^2\, {}_2F_1(1-2\epsilon, 1-2\epsilon; 2-5\epsilon; u)}{(u-1)\Gamma(2-5\epsilon)\Gamma(3\epsilon+1)} \\
& + \frac{3\pi^2 (1-u)^{-2\epsilon} u^{-\epsilon-1} \csc(\pi\epsilon)\csc(3\pi\epsilon)\Gamma(1-\epsilon)^2\, {}_2F_1(1-3\epsilon, 1-3\epsilon; 2-5\epsilon; u)}{(u-1)\Gamma(2-5\epsilon)\Gamma(3\epsilon+1)} \\
& - \frac{\pi^3 2^{8\epsilon-1} (1-u)^{-\epsilon} u^{-2\epsilon-1} \csc(\pi\epsilon)\csc(3\pi\epsilon)\Gamma(-\epsilon)\, {}_2F_1(1-3\epsilon, 1-3\epsilon; 2-6\epsilon; u)}{(u-1)\Gamma\left(\frac{3}{2}-3\epsilon\right)\Gamma\left(\frac{1}{2}-\epsilon\right)\Gamma(3\epsilon)}
\end{aligned} \tag{3.106}$$

We can expand the above formula in ϵ up to ϵ^0 order, obtaining,

$$\begin{aligned}
\int_0^1 dt\, J(u,t) V_{3,LS} \;=\; & \frac{3\ln(1-u)}{\epsilon^2(1-u)u^2} + \frac{16\mathrm{Li}_2(u) + 4\ln(1-u)\ln((1-u)u)}{\epsilon(u-1)u^2} \\
& + \frac{86\mathrm{Li}_2(u)}{(u-1)u^2} - \frac{21\mathrm{Li}_2(u)\ln(1-u)}{(u-1)u^2} - \frac{22\mathrm{Li}_2(u)\ln(u)}{(u-1)u^2} - \frac{5\ln^3(1-u)}{2(u-1)u^2} \\
& - \frac{5\ln(u)\ln^2(1-u)}{(u-1)u^2} + \frac{32\ln^2(2)\ln(1-u)}{(u-1)u^2} - \frac{8\ln^2(4)\ln(1-u)}{(u-1)u^2} \\
& - \frac{3\ln^2(u)\ln(1-u)}{(u-1)u^2} + \frac{7\pi^2\ln(1-u)}{2(u-1)u^2} + \mathcal{O}[\epsilon]
\end{aligned} \tag{3.107}$$

Then we need to calculate the integration $\int_0^1 dt\, J(u,t)(V_3 - V_{3,LS})$. For calculating this integration, we can simply set the regulator ϵ to 0, then perform the integration directly in *Mathematica*. There is one term in $V_3 - V_{3,LS}$ which is not easy to integrate. It is:

$$CL_3(u) = \int_0^1 dt \frac{2\mathrm{Li}_2\left(1 - \frac{tu-1}{(t-1)tu}\right)}{(1-u)u(1-tu)} \tag{3.108}$$

We will calculate $CL_3(u)$ via Mellin–Barnes method in the fourth chapter. At last, we obtain the virtual contribution to the EEC function,

$$\begin{aligned}
EEC'_V = \; & \frac{3\ln(1-u)}{\epsilon^2(1-u)u^2} + \frac{16\mathrm{Li}_2(u) + 4\ln(1-u)\ln((1-u)u)}{\epsilon(u-1)u^2} \\
& + \frac{86\mathrm{Li}_2(u)}{(u-1)u^2} + \frac{4\mathrm{Li}_3(1-u)}{(u-1)u^2} - \frac{2\mathrm{Li}_3(u)}{(u-1)u^2} - \frac{20\mathrm{Li}_2(u)\ln(-(u-1)u)}{(u-1)u^2} - \frac{4\zeta(3)}{(u-1)u^2} \\
& - \frac{8\ln^3(1-u)}{3(u-1)u^2} - \frac{4\ln(u)\ln^2(1-u)}{(u-1)u^2} - \frac{2\ln^2(u)\ln(1-u)}{(u-1)u^2} + \frac{17\pi^2\ln(1-u)}{6(u-1)u^2} \\
& + CL_3(u)
\end{aligned} \tag{3.109}$$

The one-loop energy-energy correlation function EEC'_V was calculated in this chapter.

Chapter 4

Calculation of the Real-Emission Contribution to the Energy-Energy Correlation

4.1 The Real-Emission Contribution to the Energy-Energy Correlation

In previous chapters, I computed the virtual contribution to the energy-energy correlation, as well as the integrands we need to calculate the real-emission contribution. In this chapter, I calculate the latter contribution. I proceed as follows: use a Mellin–Barnes representation of the integrand; perform the phase-space integrals. I obtain analytic representations for some of the remaining Mellin–Barnes integrals by comparing the series expansion of the integral with the series expansion of a sum of chosen polylogarithms with arbitrary coefficients. There are ten basic integrals, $C_{8,t}$, $t = a,\ b, \ldots j$. Of these, I obtain compact results for four — $C_{8,a}$ in section 4.4.3, $C_{8,b}$ in section 4.4.4, $C_{8,d}$ in section 4.4.5 and $C_{8,e}$ in section 4.4.6. For the other six integrals, I obtain compact results for the divergent parts in ϵ, and series expansions of the finite part in ϵ. We calculate the basic integral $C_{8,c}$ in section 4.4.8, the integral $C_{8,f}$ in section 4.4.9, the integral $C_{8,g}$ in section 4.4.10, the integral $C_{8,h}$ in section 4.4.11, the integral $C_{8,i}$ in section 4.4.12 and the integral $C_{8,j}$ in section 4.4.13. In section 4.5, I evaluate all the basic integrals numerically and compare the results with the analytic forms obtained using Mellin–Barnes representations. In the section 4.7, I assemble all the different parts of the real-emission contribution to the energy-energy correlation, and show that the divergent terms cancel against the virtual contribution (3.109). This provides a strong cross-check of the correctness of the calculation.

The real-emission contribution at this order in the coupling arises from the four-particle decay of H,

$$\frac{d\Sigma}{dc} = \sum_{i<j} \int dPS_4 |A(H \to p_1\, p_2\, p_3\, p_4)|^2 \frac{E_i E_j}{E_{total}^2} \delta(\cos\theta_{ij} - c) \tag{4.1}$$

where $|A|^2$ is the differential cross-section as a function of kinematic invariants $s_{kl} = 2\, p_k \cdot p_l$ and θ_{ij} is the angle between the two outgoing particles i and j.

4.2 Four-Particle Phase Space

The four-particle phase space measure in the decay $q \to p_1\, p_2\, p_3\, p_4$ is,

$$dPS_4 = \Pi_{i=1}^4 \frac{dp_i^D \delta(p_i^2)}{(2\pi)^{D-1}} (2\pi)^d \delta^D(q - \sum_i p_i) \tag{4.2}$$

in dimensional regularization with $D = 4 - 2\epsilon$ space-time dimensions. In a practical application, this expression in terms of particle momenta is not convenient. Let us rewrite it in terms of kinematic invariants s_{ij} and angular volumes. The measure becomes:

$$\begin{aligned} dPS_4 &= (2\pi)^{4-3D}(q^2)^{3-\frac{D}{2}}2^{1-2D}(-\Delta_4)^{\frac{D-5}{2}}\,\Theta(-\Delta_4)\,\delta(q^2 - s_{12} - s_{13} - s_{14} - s_{23} - s_{24} - s_{34}) \\ &\quad \mathrm{d}\Omega_{d-1}\,\mathrm{d}\Omega_{d-2}\,\mathrm{d}\Omega_{d-3}\mathrm{d}s_{12}\,\mathrm{d}s_{13}\mathrm{d}s_{14}\mathrm{d}s_{23}\mathrm{d}s_{24}\mathrm{d}s_{34}, \end{aligned} \tag{4.3}$$

where the Gram determinant Δ_4 is given by

$$\Delta_4 = \lambda(s_{12}s_{34}, s_{13}s_{24}, s_{14}s_{23})\ ,\ \lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)\ . \tag{4.4}$$

Noticing that there is a delta function $\delta(\cos\theta_{ij} - c)$ in the integrand of the energy-energy correlation function, we want a parameterization expressing $\cos\theta_{ij}$ as simply as possible, so that the delta function directly removes one integration.

We will use the parameterization similar to that used by Gehrmann-De Ridder, Gehrmann and Heinrich [21],

$$\begin{aligned} &s_{12} \to (1-z)(-t(1-z_1) - z_1 + 1), && s_{23} \to (1-z)z_1, \\ &s_{24} \to t(1-z)(1-z_1), && s_{34} \to -vz(1-z_1) - s_{13} + z, \\ &s_{13} \to s_{13}, && s_{14} \to vz(1-z_1) \end{aligned} \tag{4.5}$$

We can rewrite $\cos\theta_{ij}$ as follows,

$$\cos\theta_{ij} = 1 - \frac{s_{ij}}{2E_iE_j}\,. \tag{4.6}$$

We want a representation of the fraction in this equation to be as simple as possible, and furthermore that it avoid posing any further constraints on the boundaries of integration variables. Without loss of generality, we may choose $i = 2$ and $j = 3$

$$\frac{s_{23}}{2E_2E_3} \propto \frac{s_{23}}{p_2 \cdot p_{1234} p_3 \cdot p_{1234}} \propto \frac{s_{23}}{(s_{12} + s_{23} + s_{24})(s_{13} + s_{23} + s_{34})} \tag{4.7}$$

We change variables so that the above form can be replaced by a single integration variable u_2.

Introduce the following parameterization:

$$\begin{aligned} &s_{12} \to (t-1)(u_1-1)w, && s_{23} \to \frac{(u_1-1)u_1u_2w}{u_1u_2 - 1}, \\ &s_{24} \to -t(u_1-1)w, && s_{34} \to -\frac{(u_1-1)u_1u_2w}{u_1u_2-1} + u_1 - x, \\ &s_{13} \to s_{13}, && s_{14} \to (u_1-1)(w-1) \end{aligned} \tag{4.8}$$

where

$$u_2 = \frac{1 - \cos\theta_{23}}{2} \tag{4.9}$$

then

$$\cos\theta_{23} = 2u_2 - 1. \tag{4.10}$$

So the δ function $\delta(\cos\theta_{23} - c)$ in eq. (4.1) becomes,

$$\delta(\cos\theta_{23} - c) = \delta(2u_2 - 1 - c) = \delta(u_2 - u)/2 \tag{4.11}$$

if we define $u = (1-c)/2$.

Using the above parameterization, the physical constraint

$$\Delta_4 = \lambda(s_{12}s_{34}, s_{13}s_{24}, s_{14}s_{23}) < 0 \tag{4.12}$$

becomes:

$$\begin{aligned}&\frac{(u_1-1)^2u_1^2w^2(t(u_1-1)u_2w+t(-u_1)u_2+t+u_2-1)^2}{(u_1u_2-1)^2}\\&+\frac{2(u_1-1)^2u_1w^2(t(u_2((u_1-1)w-u_1+2)-1)-u_2+1)}{u_1u_2-1}s_{13}\\&+(u_1-1)^2w^2s_{13}^2<0\end{aligned}\tag{4.13}$$

To find the boundaries for s_{13}, we need only solve for s_{13} in the above expression. The roots are:

$$s_{13\pm}=\frac{u_1}{1-u_1u_2}(\sqrt{(1-t)(1-u_2)}\pm\sqrt{tu_2(1-u_1)(1-w)})^2\tag{4.14}$$

To perform the integrations, start by integrating s_{13} from s_{13-} to s_{13+}, where s_{13-} and s_{13+} are functions of u_2, t, v and u_1. Then integrate u_2, t, w, u_1 each from 0 to 1. The variable u_2 is fixed by the δ function (4.11).

4.3 Basic Integrals for the Energy-Energy Correlation

In order to obtain a finite result for the next-to-leading order correction, we have to add together the one-loop virtual contribution and the real emission. I obtained the one-loop virtual contribution (3.109) by integrating the interference of the tree and one-loop three-parton amplitudes over three-particle phase space in $D=4-2\epsilon$ dimensions in last chapter. For the real-emission contribution, there are basis integrals in terms of which we can express the result. Gehrmann-De Ridder, Gehrmann, and Heinrich [21] introduced four master integrals (the integration by parts technique was used to reduce the number of independent integrals.) for the cross-section in massless QCD,

$$\begin{aligned}R_4&=\int dPS_4\,,\\R_6&=\int dPS_4\frac{1}{s_{124}s_{134}}\,,\\R_{8,a}&=\int dPS_4\frac{1}{s_{13}s_{23}s_{14}s_{24}}\,,\\R_{8,b}&=\int dPS_4\frac{1}{s_{13}s_{23}s_{123}s_{234}}\,,\end{aligned}\tag{4.15}$$

along with another integral which is not linearly independent of this basis,

$$R_{8,r}=\int dPS_4\frac{1}{s_{13}s_{123}s_{24}s_{124}}\tag{4.16}$$

In our case, the presence of the delta function makes it tricky to use integration by parts identities (IBP) to reduce the basis. The basic integrals are not found by using any IBP relations.

We find our basic integrals in the following way, The total fully-symmetrized differential cross-section denoted as $|A(1,2,3,4)|^2$ was obtained in eq. (2.121). The real emission contribution to the energy-energy correlation function is defined by,

$$\begin{aligned}EEC_{real}&=\sum_{i<j}\int dPS_n|A(1,2,3,4)|^2\frac{E_iE_j}{E_{total}^2}\delta(\cos\theta_{ij}-c)\\&=\int dPS_n\,6\,|A(1,2,3,4)|^2\frac{E_2E_3}{E_{total}^2}\delta(\cos\theta_{23}-c)\end{aligned}\tag{4.17}$$

In the integrand of the above formula, we can freely exchange momenta 1 and 4, or 2 and 3 and finally reduce the number of different denominators to ten. Using these denominators, we define ten basic integrals

separately. For example, the integral $C_{8,a}$ is defined as,

$$C_{8,a} = \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{13} s_{23} s_{14} s_{24}} \tag{4.18}$$

which is one of the basic integrals. Its dimension is the same as $m_H^{-4-6\epsilon}$. In the following text, I will calculate the dimensionless integral $C'_{8,a}$ instead of $C_{8,a}$, where $C'_{8,a}$ is defined as,

$$C'_{8,a} = 8 f_{dim} C_{8,a} \tag{4.19}$$

where the overall trivial factor 8 is just for simplicity and f_{dim} was defined as follow,

$$f_{dim} = m_H^{4+6\epsilon} / S_\Gamma \tag{4.20}$$

where S_Γ is defined in eq. (3.102) which is the same factor we extracted from the virtual contribution to EEC function in eq. (3.101), coming from the measure of four-particle phase space.

The ten basic dimensionless integrals are shown below,

$$\begin{aligned}
C'_{8,a} &= 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{13} s_{23} s_{14} s_{24}}, \quad C'_{8,b} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{12} s_{24} s_{13} s_{34}} \\
C'_{8,c} &= 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{s_{12} s_{123} s_{24} s_{234}}, \quad C'_{8,d} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{12} s_{124} s_{13} s_{134}} \\
C'_{8,e} &= 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{23} s_{123} s_{24} s_{124}}, \quad C'_{8,f} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{24} s_{234} s_{14} s_{134}} \\
C'_{8,g} &= 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{13} s_{134} s_{24} s_{124}}, \quad C'_{8,h} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{23} s_{123} s_{14} s_{124}} \\
C'_{8,i} &= 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{13} s_{123} s_{24} s_{234}}, \quad C'_{8,j} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{13} s_{123} s_{24} s_{124}}
\end{aligned} \tag{4.21}$$

I will focus on calculating these basic integrals in following sections.

4.4 Mellin–Barnes representation for the Calculation of the Basic Integrals

Before I perform the calculation of the integrals $C_{8,t}$, $t = a, \cdots, j$, I would like to introduce the powerful Mellin–Barnes representation which was used to obtain the result. In general, the integrals $C_{8,t}$, $t = a, \cdots, j$ contain singularities in ϵ, which are not easy to resolve when the integration variable u_2 is fixed by δ function (4.11). In order to proceed with an analytic calculation, I used a Mellin–Barnes representation. This representation allows us to trade the initial variables for a set of Mellin–Barnes integrations. We can integrate over our initial variables, and are left with several integrations over contours in the complex plane. This approach also offers us a way of resolving the singularities in ϵ. In order to use this representation, replace $(A+B)^{-\nu}$ and hypergeometric functions ${}_2F_1$ and ${}_pF_q$ as follows,

$$\frac{1}{(A+B)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{\mathcal{C}} dz \frac{A^z}{B^{\nu+z}} \Gamma(-z)\Gamma(\nu+z), \tag{4.22}$$

where $|\arg A - \arg B| < \pi$.

$$ {}_2F_1(\alpha,\beta,\gamma,u) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma(\alpha+z)\Gamma(\beta+z)\Gamma(-z)}{\Gamma(\gamma+z)}(-u)^z dz \tag{4.23} $$

and

$$ {}_pF_q\left(a_1,\cdots,a_p;b_1,\cdots,b_q;u\right) = \frac{\prod_{k=1}^q\Gamma\left(b_k\right)}{2\pi i\prod_{k=1}^p\Gamma\left(a_k\right)}\int_{\mathcal{C}}dz\frac{\Gamma(z)\prod_{k=1}^p\Gamma\left(a_k-z\right)}{(-u)^z\prod_{k=1}^q\Gamma\left(b_k-z\right)}. \tag{4.24} $$

The contour of integration is chosen in such a way that the poles of the functions $\Gamma(\cdots+z)$ lie to its left, and the poles of the functions $\Gamma(\cdots-z)$ lie to its right.

After applying the above representation, we can integrate the original phase-space variables t, w, u_1. In general, we will be left a product of following type:

$$ (-1)^{P_1(z_1,\cdots,z_n)}u^{P_2(z_1,\cdots,z_n)}\frac{\prod_{k=3}^m\Gamma(P_k(z_1,\cdots,z_n))}{\prod_{k=m+1}^n\Gamma(P_k(z_1,\cdots,z_n))} \tag{4.25} $$

where $P_k(z_1,\cdots,z_n)$, $k=1,2,3,\cdots,m$ are linear function of $z_1,\cdots,z_n$. The coefficients of $z_1,\cdots,z_n$ are typically 1 or 2. The "constant" term $P_k(0,\cdots,0)$ are typically $a+b\epsilon$ where a and b are two integers.

We can then use the publicly-available *Mathematica* packages `MB.m` (due to Czakon [22]), `MBresolve.m` (due to Smirnov [23]), `barnesroutines.m` (due to Kosower) to continue evaluate the product (4.25). We also used `pslq.m` (due to Straub [33]). We mainly use the following routines in these packages:

1. the routine `MBresolve` of the package `MBresolve.m`;
2. the routines `MBexpand`, `MBapplyBarnes` and `MBintegrate` of the package `MB.m`;
3. the routine `DoAllBarnes` of the package `barnesroutines.m`
4. the routine `PSLQ` of the package `pslq.m`

First I apply the routine `MBresolve` of the package `MBresolve.m` to the product (4.25) in order to resolve singularities in ϵ. The singularities in ϵ can be resolved in the following way: in a integrand, pick numerical values for the real parts of the z_i contours, along with a value of ϵ that gives positive values to the arguments of all gamma functions. When we try to take $\epsilon\to 0$, we will notice that in some gamma functions, the poles to the left and to the right of the contour will merge, and the contour will be pinched in between them. This signals the presence of an singularity in ϵ. In order to extract it, shift the contour past the overlapping poles. This will yield a contribution corresponding to the residue at the pole. Repeating this procedure makes all singularities in ϵ explicit.

After resolving the ϵ singularities of a Mellin–Barnes representation, we used the function `MBexpand` to perform a series expansion in ϵ. The routine `MBexpand` expands the integrand directly to a series in ϵ. The coefficients of $1/\epsilon^2$, $1/\epsilon$, and ϵ^0 are three sums of several multi-dimensional Mellin–Barnes integrals. In general, we will be left with a sums of several multi-dimensional Mellin–Barnes integrals. We simplify the notation as follows to represent a multi-dimensional Mellin–Barnes integral,

$$ \frac{1}{(2\pi i)^m}\int_{C_1-i\infty}^{C_1+i\infty}dz_1\cdots\int_{C_m-i\infty}^{C_m+i\infty}dz_m G(z_1,\cdots,z_m)\to\frac{1}{(2\pi i)^m}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^m dz_j G(z_1,\cdots,z_m) \tag{4.26} $$

and also indicate the intersection of the contour $\mathcal{C}$ and the real slice:

$$ \text{where } C_1 = N_1,\ \cdots,\ C_m = N_m \tag{4.27} $$

$N_1,\ \cdots,N_m$ are real numbers, normally chosen by `MBresolve.m`.

Then we will apply Barnes' lemmas to each multi-dimensional Mellin–Barnes integral in the summation to reduce the dimension of these integrals. There are two Barnes' lemmas. If the contour separate the poles

of $\Gamma(\cdots + t)$ and the poles of $\Gamma(\cdots - t)$, then the first Barnes' lemma states,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+t)\Gamma(b+t)\Gamma(c-t)\Gamma(d-t)\,dt\\&=\frac{\Gamma(a+t)\Gamma(b+c)\Gamma(a+d)\Gamma(b+d)}{\Gamma(a+b+c+d)}\end{aligned}\tag{4.28}$$

and the second Barnes lemma states

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dt\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(1-d-t)\Gamma(-t)}{\Gamma(e+t)}\\&=\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1-d+a)\Gamma(1-d+b)\Gamma(1-d+c)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}.\end{aligned}\tag{4.29}$$

provided that $d+e=a+b+c$. The introduction to Barnes lemma and its generalization (integrand with polygamma function $\psi^{(k)}(\cdots \pm t)$) can be found in Smirnov's textbook [28]. We will use the routine `MBapplyBarnes` in the package `MB.m` and an other routine, `DoAllBarnes` , in the package `barnesroutines.m` to apply the Barnes lemmas to the multi-dimensional Mellin–Barnes integrals to reduce their dimension or to calculate them directly.

Finally, we apply the routine `MBasymptotics` in `MBasymptotics.m` (also written by Czakon) to the coefficients $1/\epsilon^2$, $1/\epsilon$, and ϵ^0. We will be left with three series expansions in u. To obtain a series expansion in u from Mellin–Barnes integral, we can do this by shifting the contour of the variable that appears in the exponent of u. Each time we cross a pole, we pick up a residue, and the coefficient of a new term in the series expansion. Shifting across higher-order poles can give rise to powers of $\ln u$. This procedure is accomplished by `MBasymptotics`.

The coefficients of u^k obtained by applying `MBasymptotics` to multi-dimensional Mellin–Barnes integrals could be still a (sum of) multi-dimensional Mellin–Barnes integrals. To calculate these multi-dimensional Mellin–Barnes integrals analytically, we can apply Barnes' lemmas to them again or sum the residues of one variable z to reduce the dimension of Mellin–Barnes integral. Finally we may be able to reduce the dimensions of these integrals to one or calculate the result for these integrals. For many one-dimensional Mellin–Barnes integrals, we cannot apply Barnes lemmas (for example, integrands with six or more gamma functions), or the application of these lemmas is very slow. In this case, it is more efficient to evaluate the integrals numerically, and to deduce the exact form of the coefficient, in terms of rational numbers, π, and ζ constants, from the numerical evaluation. To do so, we can make use of the PSLQ algorithm. The PSLQ algorithm was formalized in the routine `PSLQ` of the package `pslq.m` (due to Straub [33]) `PSLQ` is an algorithm for finding integer relations. Given n real numbers $\mathbf{x}=(x_1,x_2,\cdots,x_n)$, `PSLQ` tries to find integers $\mathbf{m}=(m_1,m_2,\cdots,m_n)$, not all zero, such that

$$\mathbf{x}\cdot\mathbf{m}=\sum_{k=1}^{n}=0\tag{4.30}$$

In the package, the routine `MBintegrate` can evaluate a one-dimensional Mellin–Barnes integral to any precision. In our calculation, for example, using `MBintegrate`, we evaluate the one-dimensional Mellin–Barnes integral numerically to an accuracy which is high enough to apply the PSLQ algorithm to then deduce an exact form for the result.

Finally, we obtain the coefficients of u^k as real number. Then we will try to obtain analytic formulas by matching the coefficients of u^k in such series expansions to an ansatz built using a set of basis functions. Before showing the possible basis function, we need to introduce the definition of degree of transcendentality. Physicists computing multi-loop Feynman diagrams have introduced conjectures that involve the notion of degree of transcendentality of a number (DoT). One defines

1. $\mathrm{DoT}(r)=0$, r is rational;
2. $\mathrm{DoT}(\pi^k)=k$, $k\in\mathbb{N}$;

3. $\mathrm{DoT}(\zeta_k) = k$;

4. $\mathrm{DoT}(a \times b) = \mathrm{DoT}(a) + \mathrm{DoT}(b)$.

Then we will try to define degree of transcendentality of a function. We first define algebraic functions. An algebraic function is a function $f(x)$ which satisfies $P(x, f(x)) = 0$, where $P(x, y)$ is a polynomial in x and y with integer coefficients. Functions that can be constructed using only a finite number of elementary operations are examples of algebraic functions. A function which is not algebraic is, by definition, a transcendental function. The trigonometric functions, the exponential function, and their inverses. Then we define the Degree of Transcendentality of a function (DoT) in the following way:

1. $\mathrm{DoT}(P(x)) = 0$, $P(x)$ is an algebraic function;

2. $\mathrm{DoT}(\pi^k P(x)) = k$ and $DoT(\zeta_k P(x)) = k$

3. $\mathrm{DoT}(\ln^k(P(x))) = k$, $k \in \mathbb{N}$;

4. $\mathrm{DoT}(\mathrm{Li}_k(P(x))) = k$;

5. $\mathrm{DoT}(f(x) + g(x)) = \mathrm{Max}(\mathrm{DoT}(f(x)), \mathrm{DoT}(g(x)))$.

6. $\mathrm{DoT}(f(x) \times g(x)) = \mathrm{DoT}(f(x)) + \mathrm{DoT}(g(x))$.

The functions arcsin or arccos are also taken to be logarithmic, with complex arguments. We suppose that the coefficients of $1/\epsilon^2$, $1/\epsilon$, and ϵ^0 meet the following constraints,

1. The coefficient of $1/\epsilon^2$ can be function whose DoT $\leq$1, for example: $\ln P(u)$

2. The coefficient of $1/\epsilon$ can be function whose DoT $\leq$1 DoT $\leq$2, for example: $\ln \times \ln$, π^2, Li_2.

3. The coefficient of ϵ^0 can be function whose DoT $\leq$1 DoT $\leq$2 DoT $\leq$3, for example, product of lower transcendental degree functions, ζ_3, Li_3,

4. The arguments of the above functions are algebraic functions of u

5. Compute the discontinuity from MB representation, by replacing u^z by $\dfrac{u^z \pi}{\Gamma(z)\Gamma(1-z)}$, to help us to find the argument of basis functions.

In the first place, we assume that the arguments of these transcendental function are u, $1 - u$.

4.4.1 A Shortcut

In the calculation of several of the basic integrals, I will make use of a "non-rigorous" shortcut. After we've integrated the variable s_{13} with the parameterization (4.8) and the boundary (4.14) or using the parameterization (4.5), we obtain a hypergeometric function of the form,

$$\frac{1}{(A-B)^2}\,{}_2F_1\left(1, \frac{1}{2} - \epsilon, 1 - 2\epsilon, -\frac{4AB}{(A-B)^2}\right). \tag{4.31}$$

One can directly perform the replacement eqs. (4.22) and (4.23) in this hypergeometric function to obtain its Mellin–Barnes representation and one can integrate the variables u_1, v and t. In this way, we obtain a rigorous representation of eq. (4.31). The following "non-rigorous" shortcut offers a another way to find a Mellin–Barnes representation of the hypergeometric function in eq. (4.31) involving fewer complex variables than its completely rigorous Mellin–Barnes representation. In the "non-rigorous" shortcut, we continue to use a hypergeometric transformation, we can rewrite it as follows,

$$\begin{aligned}
&\frac{1}{(A-B)^2}\,{}_2F_1\left(1, \frac{1}{2} - \epsilon, 1 - 2\epsilon, -\frac{4AB}{(A-B)^2}\right) = \frac{1}{A^2}\,{}_2F_1\left(1, 1+\epsilon; 1-\epsilon; \frac{B^2}{A^2}\right) \text{ where } B < A, \\
&\frac{1}{(A-B)^2}\,{}_2F_1\left(1, \frac{1}{2} - \epsilon, 1 - 2\epsilon, -\frac{4AB}{(A-B)^2}\right) = \frac{1}{B^2}\,{}_2F_1\left(1, 1+\epsilon; 1-\epsilon; \frac{A^2}{B^2}\right) \text{ where } A < B.
\end{aligned} \tag{4.32}$$

The aim of this hypergeometric transformation is to factorize the argument of hypergeometric function. For example, if we integrate the variable s_{13} with the parameterization (4.5), the arguments of the hypergeometric function are

$$-\frac{4\sqrt{(1-t)t(1-v)vz_1}}{(\sqrt{(1-t)(1-v)}-\sqrt{t\,v\,z_1})^2}=-\frac{4AB}{(A-B)^2} \tag{4.33}$$

if we set

$$A=\sqrt{(1-t)(1-v)},\ B=\sqrt{t\,v\,z_1}. \tag{4.34}$$

After we've performed the hypergeometric transformation (4.32), the arguments of of the hypergeometric function are

$$\frac{A^2}{B^2}=\frac{(1-t)(1-v)}{t\,v\,z_1}\ \text{or}\ \frac{B^2}{A^2}=\frac{t\,v\,z_1}{(1-t)(1-v)} \tag{4.35}$$

These two argument are totally factorized.

However, this splits the original integral into two integrals with either parameterization, eq. (4.8) or (4.5). For the former, the integration region for the variables u_1, v and t is dependent on $u_2=u$. Until now, the calculations we performed are still rigorous. In stead of finding a Mellin–Barnes representation in those two splitted regions, I find a representation of eq. (4.31) differently. To clarify the issues, I give two explicit examples of such calculation in this section. The shortcut has two steps:

1. The "non-rigorous" step: we perform the following replacement in the integrand,

$$\frac{1}{(A-B)^2}\,{}_2F_1\left(1,\frac{1}{2}-\epsilon,1-2\epsilon,-\frac{4AB}{(A-B)^2}\right)\to\frac{1}{A^2}\,{}_2F_1\left(1,1+\epsilon;1-\epsilon;\frac{B^2}{A^2}\right) \tag{4.36}$$

   which is in fact valid only when $B<A$. Then we obtain the Mellin–Barnes representation of the integrand. Finally, we integrate the variables in their original integration region ignoring the constraint $B<A$.

2. The correction step: in the Mellin–Barnes transform, $(-1)^{(-w-1)}$ is replaced by $\dfrac{\Gamma(1+\epsilon)\Gamma(-\epsilon)}{\Gamma(\epsilon-w)\Gamma(1-\epsilon+w)}$

With these two steps, we can obtain a Euclidean Mellin–Barnes representation of the integrand which contains the hypergeometric function (4.31).

I give two examples below. We can calculate the results for both using a rigorous calculation, and the shortcut above, and see that we obtain the same result. More generally, we can always check the result numerically.

The first example is straightforward, and applies to the integral

$$R_0=\int dPS_4\frac{1}{s_{13}}\,. \tag{4.37}$$

The phase-space is symmetric for the final four particles, so that we can rewrite this integral as,

$$R_0=\int dPS_4\frac{1}{s_{14}}\,. \tag{4.38}$$

Using the parameterization of ref. [21], we find by direct (and rigorous) calculation,

$$\begin{aligned}R_0=&\int dPS_4\frac{1}{s_{14}}\\ =&\ S_\Gamma(m_H)^{2-6\epsilon}\iiiint dtdvdzdz_1(1-t)^{-\epsilon}t^{-\epsilon}(1-v)^{-\epsilon}v^{-1-\epsilon}(1-z)^{1-2\epsilon}z^{-2\epsilon}(1-z_1)^{-2\epsilon}z_1^{-\epsilon}\\ =&\ S_\Gamma(m_H)^{2-6\epsilon}\frac{\Gamma(1-2\epsilon)\Gamma^4(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-4\epsilon)\Gamma(2-3\epsilon)}.\end{aligned} \tag{4.39}$$

We can apply the shortcut to the original form,

$$\begin{aligned} R_0 = & \int dPS_4 \frac{1}{s_{13}} \\ = & S_\Gamma(m_H)^{2-6\epsilon} \iiiint dtdvdzdz_1 (1-t)^{-\epsilon} t^{-\epsilon} (1-v)^{-\epsilon} v^{-\epsilon} (1-z)^{1-2\epsilon} z^{1-2\epsilon} (1-z_1)^{1-2\epsilon} z_1^{-\epsilon} \\ & \times \frac{1}{S_{13-}} {}_2F_1(1, \frac{1}{2} - \epsilon, 1-2\epsilon, -\frac{\Delta S}{S_{13-}}) \end{aligned} \tag{4.40}$$

where $\Delta S = 4\sqrt{(1-t)t(1-v)vz^2 z_1}$, $S_{13-} = z(\sqrt{(1-v)(1-t)} - \sqrt{t\,v\,z_1})^2$. We first apply the shortcut step, given by the first equation in eq. (4.32), with $A = \sqrt{(1-v)(1-t)}$, $B = \sqrt{t\,v\,z_1}$. We should in fact consider the two regions $A > B$ and $A < B$ separately. Although the transformation is only valid when $A > B$, we still apply it to our integrand ignoring this constraint. We obtain,

$$\begin{aligned} R_0 = & \int dPS_4 \frac{1}{s_{13}} \\ = & S_\Gamma(m_H)^{2-6\epsilon} \iiiint_0^1 dtdvdzdz_1 (1-t)^{-\epsilon} t^{-\epsilon} (1-v)^{-\epsilon} v^{-\epsilon} (1-z)^{1-2\epsilon} z^{1-2\epsilon} (1-z_1)^{1-2\epsilon} z_1^{-\epsilon} \frac{1}{z} \\ & \times {}_2F_1\left(1, 1+\epsilon; 1-\epsilon; \frac{tvz_1}{(1-t)(1-v)}\right) \end{aligned} \tag{4.41}$$

We integrate the variables t, v, z z_1 from 0 to 1 ignoring the constraint $A > B$ Then the Mellin–Barnes representation to replace the hypergeometric function:

$$\begin{aligned} R_0 = & \int dPS_4 \frac{1}{s_{13}} \\ = & S_\Gamma(m_H)^{2-6\epsilon} \iiiint_0^1 dtdvdzdz_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dw \frac{(-1)^{-w-1}(1-z)^{1-2\epsilon} z^{-2\epsilon} (1-z_1)^{1-2\epsilon} \Gamma(1-\epsilon)}{\Gamma(\epsilon+1)\Gamma(-\epsilon-w)} \\ & \times \Gamma(-w)\Gamma(w+1)(1-t)^{-\epsilon+w+1} t^{-\epsilon-w-1} (1-v)^{-\epsilon+w+1} v^{-\epsilon-w-1} z_1^{-\epsilon-w-1} \Gamma(\epsilon-w) \end{aligned}$$

Integrate the variables t, v, z, $z_1$from 0 to 1 ignoring the constraint $A > B$

$$\begin{aligned} & \to S_\Gamma(m_H)^{2-6\epsilon} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dw \frac{(-1)^{-w-1}\Gamma(1-2\epsilon)\Gamma(1-\epsilon)\Gamma(-w)\Gamma(w+1)\Gamma^2(-\epsilon-w)\Gamma(\epsilon-w)}{\Gamma(3-4\epsilon)\Gamma(\epsilon+1)\Gamma(-3\epsilon-w+2)} \\ & \times \Gamma^2(-\epsilon+w+2) \end{aligned} \tag{4.42}$$

If we simply perform this integral by summing residues, we will obtain the wrong answer. This is no surprise, because the transformation we used above is not valid everywhere. We may notice, however, that this representation has multiple series of residues on the left-hand side of the contour; and that if we exclude one of these series, we will obtain the correct answer. We can implement this exclusion as follows, by replacing

$$(-1)^{-w-1} \longrightarrow \frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)}{\Gamma(\epsilon-w)\Gamma(1-\epsilon+w)}\,. \tag{4.43}$$

The two forms are equivalent for integer w, that is at poles we want to keep, but eliminates the poles at $w = -1+\epsilon-k$, $k = 0,\ 1\ldots$, which we need to discard. This replacement step is the correction step of the shortcut procedure. Applying it to R_0 in eq. (4.42), we obtain

$$R_0 = S_\Gamma m_H^{2-6\epsilon} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dw \frac{\Gamma^2(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-w)\Gamma(w+1)\Gamma^2(-\epsilon-w)\Gamma(-\epsilon+w+1)}{\Gamma(3-4\epsilon)\Gamma(1-2\epsilon)\Gamma(-3\epsilon-w+2)} \tag{4.44}$$

When $\epsilon \to 0$, the right-hand residues at $w = 0,\ ,1\ 2,\ldots$ and at $w = -\epsilon,\ ,1-\epsilon\ 2-\epsilon,\ldots$ will always remain separated from the left-hand residues at $w = -1,\ ,-2\ -3,\ldots$ and at $w = -1+\epsilon,\ ,-2+\epsilon\ -3+\epsilon,\ldots$,

so no additional poles will be generated from gamma functions which involve w. The intersection of the contour and the real axis can be chosen to be $c = -1/2$. The second Barnes lemma (4.29) permits us to obtain the result directly,

$$R_0 = S_\Gamma m_H^{2-6\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma^4(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-4\epsilon)\Gamma(2-3\epsilon)} \tag{4.45}$$

This result is in agreement with eq. (4.39). The shortcut permits us to use fewer complex variables to represent the integrands we encounter, and yields a Mellin–Barnes representation in the Euclidean region which is simpler to evaluate numerically than the one in the Minkowski region.

The second example is the dimensionless integral

$$R_{8,a} = \int dPS_4 \frac{m_H^{4+6\epsilon}}{s_{13}s_{14}s_{23}s_{24}}. \tag{4.46}$$

After integrating the variable s_{13}, again use the first equation in eq. (4.32), where again $A = \sqrt{(1-v)(1-t)}$, $B = \sqrt{t\,v\,z_1}$. From the shortcut step, we obtain,

$$\begin{aligned} R_{8,a} = & \int dPS_4 \frac{1}{s_{13}s_{14}s_{23}s_{24}} \\ = & S_\Gamma \frac{\Gamma(-2\epsilon)^2}{\Gamma(-4\epsilon)} \iiint_0^1 dv\,dt\,dz_1 (t(1-t))^{-\epsilon-1}(v(1-v))^{-\epsilon-1}(1-z_1)^{-2\epsilon-1} z_1^{-\epsilon-1} \\ & \times {}_2F_1\left(1, 1+\epsilon; 1-\epsilon; \frac{tvz_1}{(1-t)(1-v)}\right) \end{aligned} \tag{4.47}$$

We insert a Mellin–Barnes representation for the hypergeometric function, and integrate the remaining variables from 0 to 1 ignoring the constraint $A > B$, to obtain,

$$S_\Gamma \frac{1}{2\pi i} \int_{\mathcal{C}} dw \frac{(-1)^{-w-1}\Gamma(1-\epsilon)\Gamma(-2\epsilon)\Gamma(-w)\Gamma(w+1)\Gamma^3(-\epsilon-w-1)\Gamma(\epsilon-w)\Gamma^2(-\epsilon+w+1)}{\Gamma(-4\epsilon)\Gamma(\epsilon+1)\Gamma(-3\epsilon-w-1)\Gamma(-\epsilon-w)} \tag{4.48}$$

We can now apply the second step of the shortcut procedure. To correct this Mellin–Barnes integrand, we perform the replacement (4.43) then we obtain,

$$R_{8,a} = S_\Gamma \frac{1}{2\pi i} \int_{\mathcal{C}} dw \frac{\Gamma(1-\epsilon)\Gamma(-3\epsilon)\Gamma(-w)\Gamma(w+1)\Gamma^3(-\epsilon-w-1)\Gamma(-\epsilon+w+1)}{\Gamma(-2\epsilon)\Gamma^2(-\epsilon)\Gamma(-3\epsilon-w-1)\Gamma(-\epsilon-w)} \tag{4.49}$$

where $-1 < c < 0$. This gives us the right result. For this one-dimensional MB representation, we can add all the residues on the left side of the contour in the complex plane to obtain the analytical result:

$$\begin{aligned} R_{8,a} = & S_\Gamma \frac{\Gamma(-2\epsilon)\Gamma(-\epsilon)^5\,{}_4F_3(1,-\epsilon,-\epsilon,-\epsilon;1-\epsilon,-3\epsilon,\epsilon+1;1)}{\Gamma(-4\epsilon)\Gamma(-3\epsilon)} \\ & + S_\Gamma \frac{\Gamma^2(1-\epsilon)\Gamma^4(-2\epsilon)\Gamma(\epsilon)\Gamma(-\epsilon)\,{}_3F_2(-2\epsilon,-2\epsilon,-2\epsilon;1-2\epsilon,-4\epsilon;1)}{\Gamma(1-2\epsilon)\Gamma^2(-4\epsilon)}. \end{aligned} \tag{4.50}$$

This agrees with the result given by Gehrmann-De Ridder, Gehrmann and Heinrich [21].

4.4.2 Another Method to Compute the Coefficients of $1/\epsilon^2$ and $1/\epsilon$

In this section, I present a method (different from the method in section 4.4) for calculating the coefficient of $1/\epsilon^2$ and of $1/\epsilon$. Normally, the Mellin–Barnes representation of these coefficient are just one-fold or two-fold integrals whose integrands contain only gamma functions and powers. As an example, consider the dimensionless integral

$$C_{EX} = \frac{8m_H^{-4+6\epsilon}}{S_\Gamma} \int dPS_4 E_2 E_3 \delta(\cos\theta_{23} - c) \frac{s_{14}s_{23}}{E_{tot}^2 s_{12}s_{13}}\,. \tag{4.51}$$

Using the shortcut step, we obtain the following Mellin–Barnes representation of this integral:

$$\begin{aligned} C_{EX} = \;& 8m_H^{-4+6\epsilon}/S_\Gamma \int dPS_4 E_2 E_3 \delta(\cos\theta_{23} - c) \frac{s_{14}s_{23}}{E_{tot}^2 s_{12} s_{13}} \\ = \;& \frac{1}{(2\pi i)^2} \iint_{\mathcal{C}} dz_1 dz_2 \frac{\Gamma(3-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_1)\Gamma(-\epsilon-z_2+1)\Gamma(z_2-\epsilon)}{\Gamma(-2\epsilon)\Gamma(-3\epsilon-z_2+4)\Gamma(-\epsilon+z_2+1)\Gamma(-3\epsilon-z_2-z_1+4)} \\ & \times\Gamma(-2\epsilon+z_1+3)(1-u)^{-\epsilon+z_2+z_1} u^{\epsilon-z_2-z_1-3}\Gamma(-\epsilon-z_2-z_1+1) \end{aligned} \tag{4.52}$$

In this Mellin–Barnes integrand, normally, we need to apply eq. (4.22) again to the factor $(1-u)^{-\epsilon+z_1+z_2}$ and replace it with

$$(1-u)^{-\epsilon+z_1+z_2} \to \frac{1}{2\pi i}\int_{\mathcal{C}} dz_3 \frac{\Gamma(-z_3)(-1)^{-e+z_1+z_2-z_3} u^{-e+z_1+z_2-z_3}\Gamma(e-z_1-z_2+z_3)}{\Gamma(e-z_1-z_2)} \tag{4.53}$$

in order to be able to calculate the series expansion in u. The variable u is a positive real number, so when we apply this representation, we implicitly replace:

$$(1-u)^{-\epsilon+w+z_2} \to (1+(-1+\delta i)u)^{-\epsilon+w+z_2} \tag{4.54}$$

where $0 < \delta \ll 1$ so that $|-\arg((-1+\delta i)u)| < \pi$.

For this one and for all the Mellin–Barnes representation of the coefficient of $1/\epsilon^2$ and of $1/\epsilon$ of the basic integrals, we did not have to calculate the series expansion of u but we perform the expansion over ϵ directly of its Mellin–Barnes representation. First, we apply `MBresolve` to Mellin–Barnes representation (4.52) to resolve its singularities in ϵ. Then we can apply `MBexpand` to expand the integrand to a series in ϵ. The series expansion in ϵ is the summation of integrals,

$$C_{EX} = \sum_{k=1}^{8} J_k + \mathcal{O}[\epsilon] \tag{4.55}$$

where the integrals J_k are given in eqs. (4.56)-(4.63):

$$J_1 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{-2(1-u)^{z_1}u^{-z_1-3}\Gamma(1-z_1)\Gamma(-z_1)\Gamma(z_1+3)}{3\epsilon\Gamma(4-z_1)}, \text{where } C_1 = -1.19327 \tag{4.56}$$

$$J_2 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{-5(1-u)^{z_1}u^{-z_1-3}\Gamma(1-z_1)\Gamma(-z_1)\Gamma(z_1+3)}{3\Gamma(4-z_1)}, \text{where } C_1 = -1.19327 \tag{4.57}$$

$$J_3 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{2(1-u)^{z_1}u^{-z_1-3}\ln(1-u)\Gamma(1-z_1)\Gamma(-z_1)\Gamma(z_1+3)}{3\Gamma(4-z_1)}, \text{where } C_1 = -1.19327 \tag{4.58}$$

$$J_4 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{-2(1-u)^{z_1}u^{-z_1-3}\ln u\Gamma(1-z_1)\Gamma(-z_1)\Gamma(z_1+3)}{3\Gamma(4-z_1)}, \text{where } C_1 = -1.19327 \tag{4.59}$$

$$J_5 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{2(1-u)^{z_1}u^{-z_1-3}\Gamma(1-z_1)\Gamma(-z_1)\Gamma(z_1+3)\psi^{(0)}(1-z_1)}{3\Gamma(4-z_1)}, \text{where } C_1 = -1.19327 \tag{4.60}$$

$$J_6 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{-2(1-u)^{z_1}u^{-z_1-3}\Gamma(1-z_1)\Gamma(-z_1)\Gamma(z_1+3)\psi^{(0)}(4-z_1)}{\Gamma(4-z_1)}, \text{where } C_1 = -1.19327 \tag{4.61}$$

$$J_7 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{4(1-u)^{z_1}u^{-z_1-3}\Gamma(1-z_1)\Gamma(-z_1)\Gamma(z_1+3)\psi^{(0)}(z_1+3)}{3\Gamma(4-z_1)}, \text{where } C_1 = -1.19327 \tag{4.62}$$

$$\begin{aligned} J_8 = & \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}} dz_1 dz_2 \frac{4\Gamma(1-z_2)\Gamma(-z_2)\Gamma(z_2)\Gamma(-z_1)\Gamma(z_1+3)}{\Gamma(4-z_2)\Gamma(-z_2-z_1+4)} \\ & \times(1-u)^{z_2+z_1}u^{-z_2-z_1-3}\Gamma(-z_2-z_1+1). \end{aligned} \tag{4.63}$$

where $C_1 = -1.1932,\ C_2 = 0.1581$

Where J_1 contains a factor $1/\epsilon$. From the above expression, we can calculate all the proper residues and add them analytically to obtain the final result:

$$\begin{aligned} C_{EX} = & \frac{6(u-1)((u-8)u+10)\ln(1-u) - 2u(u(10u-39)+30)}{9\epsilon u^6} \\ & + \frac{4(u-1)((u-8)u+10)\mathrm{Li}_2\left(\frac{u}{u-1}\right)}{u^6} \\ & + \frac{4(u-1)((u-8)u+10)\ln^2(1-u)}{3u^6} \\ & - \frac{2(u(u(8u-15)-3)+6(u-1)((u-8)u+10)\ln u+9)\ln(1-u)}{9u^6} \\ & + \frac{u(495-127u)+4(u(10u-39)+30)\ln u - 378}{9u^5} + \mathcal{O}[\epsilon]. \end{aligned} \tag{4.64}$$

In this way, we can obtain directly the compact analytic form of the expansion in ϵ . But for some more complicated Mellin–Barnes representations, we will perform the replacement in eq. (4.53) to calculate the series expansion in u.

4.4.3 Calculation of the Integral $C'_{8,a}$

The integral $C'_{8,a}$ is given by,

$$C'_{8,a} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{13} s_{23} s_{14} s_{24}} \tag{4.65}$$

First, integrate the variable s_{13}, then use the MB representation and integrate all the original phase-space variables. We obtain,

$$\begin{aligned} IntA_0 = & \frac{1}{(2\pi i)^3} \iiint_{\mathcal{C}} dz_1 dz_2 dz_3 \frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)u^{-z_3-3}\Gamma(-z_2)\Gamma(-z_3)\Gamma(-2\epsilon+z_2+1)}{\Gamma(-2\epsilon)\Gamma(-3\epsilon-z_1-3z_2)\Gamma(\epsilon-z_1-3z_2)\Gamma(-3\epsilon-z_1-2z_2)} \\ & \times \frac{\Gamma(z_1+2z_2+1)(-1)^{-\epsilon+z_1+3z_2-z_3}\Gamma(-\epsilon-z_1-3z_2-1)\Gamma(-\epsilon-z_1-2z_2-1)^2}{\Gamma(-\epsilon-z_1-2z_2)} \\ & \times \Gamma(\epsilon-z_1-3z_2+z_3)\Gamma(-z_1-2z_2) \end{aligned} \tag{4.66}$$

where the z_i are Mellin–Barnes variables integrated along contours chosen as described earlier.

Amongst the MB tools, I first apply `MBresolve` to the above expression in order to resolve the singularities in ϵ. We obtain a sum of two terms, $\sum_{k=1}^{2} IntA_k$,

$$\begin{aligned} IntA_1 = & \frac{1}{(2\pi i)^2} \iint_{\mathcal{C}} dz_1 dz_2 \frac{-u^{-z_1-3}\Gamma(-z_1)\Gamma(-z_2)\Gamma(1-2\epsilon)\Gamma(-\epsilon)^3(-1)^{-z_1+z_2-\epsilon}}{\Gamma(1-3\epsilon)\Gamma(-2\epsilon)\Gamma(-z_2-3\epsilon+1)\Gamma(-z_2+\epsilon+1)} \\ & \times \Gamma(z_2-2\epsilon+1)\Gamma(-z_2-\epsilon)\Gamma(z_1-z_2+\epsilon+1) \\ & \text{where } C_1 = -0.503,\ C_2 = -0.997, \end{aligned} \tag{4.67}$$

and

$$\begin{aligned} IntA_2 = & \frac{1}{(2\pi i)^3} \iiint_{\mathcal{C}} \prod_{j=1}^{3} dz_j \frac{u^{-z_1-3}\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_1)\Gamma(-z_2)\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)\Gamma(-z_3-3\epsilon)\Gamma(-z_3-\epsilon)\Gamma(-z_3-z_2-3\epsilon)} \\ & \times \frac{\Gamma(-\epsilon)\Gamma(-z_3-\epsilon-1)^2\Gamma(z_2-2\epsilon+1)(-1)^{z_3-z_1+z_2-\epsilon}\Gamma(-z_3-z_2-\epsilon-1)}{\Gamma(-z_3-z_2+\epsilon)} \\ & \times \Gamma(-z_3+z_1-z_2+\epsilon), \\ & \text{where } C_1 = -0.815,\ C_2 = -0.6026,\ C_3 = -1.041. \end{aligned} \tag{4.68}$$

At this point, we need only to expand the integrands in a series in ϵ (which we could do, for example, by applying `MBexpand` to these integrals). In the present integral, the highest Mellin–Barnes integration dimension is 3. For the integral $C_{8,a}$, however, we can get the coefficients analytically because the remaining integrals are simple. For the coefficient of ϵ^0, denoted $C_{8,a,0}$, we obtain the following series expansion about $u = 0$,

$$\begin{aligned} C'_{8,a,0} = \quad & \frac{307}{4} - \frac{9\pi^2}{2} + \frac{50}{u} - \frac{3\pi^2}{u} + \frac{10415u}{108} - \frac{11\pi^2 u}{2} + \frac{97201u^2}{864} - \frac{25\pi^2 u^2}{4} + \frac{545633u^3}{4320} - \frac{137\pi^2 u^3}{20} \\ & + \frac{2992259u^4}{21600} - \frac{147\pi^2 u^4}{20} + \frac{1108226033u^5}{7408800} - \frac{1089\pi^2 u^5}{140} + \frac{9466497871u^6}{59270400} - \frac{2283\pi^2 u^6}{280} \\ & - \frac{31\ln u}{2} - \frac{10\ln u}{u} - \frac{353}{18}u\ln u - \frac{1655}{72}u^2\ln u - \frac{9319}{360}u^3\ln u - \frac{10241}{360}u^4\ln u \\ & - \frac{542453u^5\ln u}{17640} - \frac{2318051u^6\ln u}{70560} + \frac{3\ln^2 u}{2} + \frac{\ln^2 u}{u} + \frac{11}{6}u\ln^2 u + \frac{25}{12}u^2\ln^2 u \\ & + \frac{137}{60}u^3\ln^2 u + \frac{49}{20}u^4\ln^2 u + \frac{363}{140}u^5\ln^2 u + \frac{761}{280}u^6\ln^2 u \cdots. \end{aligned} \tag{4.69}$$

The coefficients of $1/\epsilon^2$ and $1/\epsilon$ can be obtained via the method in section 4.4.2:

$$\frac{2\ln(1-u)}{(u-1)u^2\epsilon^2} + \frac{3\ln(1-u)^2 + 2\ln(1-u)\ln u + 10\mathrm{Li}_2(u)}{(1-u)u^2\epsilon}. \tag{4.70}$$

Let us guess that the denominator in the analytic form of $C_{8,a,0}$ is again $(u-1)u^2$. Noticing that the DoT (degree of transcendentality) of the coefficient of $1/\epsilon^2$ is 1, and that of the coefficient of $1/\epsilon$ is 2, we expect the degree of $C_{8,a,0}$ to be 3. I will take the simplest ansatz for the basis functions, in which the possible arguments of transcendental function are u or $1-u$.

First, we extract the terms which are proportional to $\ln^2 u$. It is:

$$\left(\frac{3}{2} + \frac{1}{u} + \frac{11u}{6} + \frac{25u^2}{12} + \frac{137u^3}{60} + \frac{49u^4}{20} + \frac{363u^5}{140} + \frac{761u^6}{280} + \cdots\right)\ln^2 u \tag{4.71}$$

The coefficient of $\ln^2 u$ is a function of at most DoT = 1. So we can compare the series coefficients with those of the function,

$$\frac{a\ln(1-u)}{(1-u)u^2}, \tag{4.72}$$

and find agreement with $a = -1$. Then, we extract the terms which are proportional to $\ln u$. It is:

$$\left(-\frac{31}{2} - \frac{10}{u} - \frac{353u}{18} - \frac{1655u^2}{72} - \frac{9319u^3}{360} - \frac{10241u^4}{360} - \frac{542453u^5}{17640} - \frac{2318051u^6}{70560} + \cdots\right)\ln u + \cdots \tag{4.73}$$

We compare this series coefficients with an ansatz whose DoT is 2:

$$\frac{b\ln^2(1-u) + c\mathrm{Li}_2(u)}{(1-u)u^2} \tag{4.74}$$

and find agreement with $b = -3$ and $c = -10$. We also extract the terms which are proportional to π^2,

$$\pi^2\left(-\frac{9}{2} - \frac{3}{u} - \frac{11u}{2} - \frac{25u^2}{4} - \frac{137u^3}{20} - \frac{147u^4}{20} - \frac{1089u^5}{140} - \frac{2283u^6}{280} + \cdots\right) \tag{4.75}$$

Using the ansatz whose DoT is 1:

$$\frac{d\ln(1-u)}{(1-u)u^2}, \tag{4.76}$$

we find agreement with $d = 3$. Finally, this leaves us with a series which is free of $\ln u$ and π:

$$\frac{307}{4} + \frac{50}{u} + \frac{10415u}{108} + \frac{97201u^2}{864} + \frac{545633u^3}{4320} + \frac{2992259u^4}{21600} + \frac{1108226033u^5}{7408800} + \frac{9466497871u^6}{59270400} + \cdots \quad (4.77)$$

The ansatz whose DoT is 3 for this series expansion is:

$$\frac{e\mathrm{Li}_3(u)}{(1-u)u^2} + \frac{f\mathrm{Li}_2(u)\ln(1-u)}{(1-u)u^2} + \frac{g\ln^3(1-u)}{(1-u)u^2} + \frac{h\left(\mathrm{Li}_3(1-u) + \frac{1}{2}\ln u \ln^2(1-u) - \frac{1}{6}\pi^2\ln(1-u) - \zeta_3\right)}{(1-u)u^2} \quad (4.78)$$

where the above basis function is chosen because it is the only combination involving $\mathrm{Li}_3(1-u)$ that is free of ζ_3, $\ln u$ and π. By comparing series expansions, we find that the ansatz does indeed match if $e = -2$, $f = -4$, $g = 50$, and $h = 22$. This finally gives us an analytic form for $C'_{8,a}$,

$$\begin{aligned} C'_{8,a} &= \int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)}{s_{13}s_{23}s_{14}s_{24}} \\ &= \frac{2\ln(1-u)}{(u-1)u^2\epsilon^2} + \frac{3\ln(1-u)^2 + 2\ln(1-u)\ln u + 10\mathrm{Li}_2(u)}{(1-u)u^2\epsilon} \\ &+ \frac{\left(\ln(1-u)\left(2\pi^2 + 6\ln(1-u)^2 - 24\ln(1-u)\ln u + 3\ln u^2\right)\right)}{3(u-1)u^2} \\ &+ \frac{6(2\ln(1-u) + 5\ln u)\mathrm{Li}_2(u) - 66\mathrm{Li}_3(1-u)}{3(u-1)u^2} \\ &+ \frac{-150\mathrm{Li}_3(u) + 66\zeta_3}{3(u-1)u^2} + \mathcal{O}[\epsilon] \end{aligned} \quad (4.79)$$

4.4.4 Calculation of the Integral $C'_{8,b}$

The integral $C'_{8,b}$ is given by

$$C'_{8,b} = 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)}{E^2_{total}s_{12}s_{24}s_{13}s_{34}} \quad (4.80)$$

In the parametrization (4.8), we need first to integrate the variable s_{13}. The factors in denominator of the integrand of $C_{8,b}$, s_{13} and s_{34} are both not free of s_{13}. But the sum

$$s_{13} + s_{34} = s_{134} - s_{14} \quad (4.81)$$

is in fact free of s_{13}. We can use the symmetry in the phase-space to rewrite the denominator of $C_{8,b}$ so that there is only one factor in the denominator of the integrand not free of s_{13} in order to make the integration over the s_{13} simpler. We rewrite the integral $C_{8,b}$ as:

$$\begin{aligned} C'_{8,b} = &\ 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)}{E^2_{total}s_{12}s_{24}s_{13}s_{34}} \\ = &\ 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)(s_{13} + s_{34})}{E^2_{total}s_{12}s_{24}s_{13}s_{34}(s_{13} + s_{34})} \\ = &\ 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)}{E^2_{total}s_{12}s_{24}s_{13}(s_{13} + s_{34})} + 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)}{E^2_{total}s_{12}s_{24}s_{34}(s_{13} + s_{34})} \end{aligned} \quad (4.82)$$

We can freely exchange the index of momentum 1 and 4 and we obtain,

$$\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)}{E^2_{total}s_{12}s_{24}s_{34}(s_{13} + s_{34})} = \int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23} - c)}{E^2_{total}s_{12}s_{24}s_{13}(s_{13} + s_{34})}. \quad (4.83)$$

Then

$$C'_{8,b} = 8f_{dim}\int dPS_4 \frac{2E_2E_3\delta(\cos\theta_{23} - c)}{E^2_{total}s_{12}s_{24}s_{13}(s_{13} + s_{34})}. \quad (4.84)$$

For any integrand whose denominator contains only one s_{13}, the non-rigorous short calculation (in section 4.4.1) is applicable. We obtain the Mellin–Barnes representation of the integral $C_{8,b}$:

$$\begin{aligned} IntB_0 = \quad & IntB_0 \iiiint_{\mathcal{C}} \prod_{j=1}^{4} dz_j \frac{2\Gamma(1-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)u^{-z_1-3}\Gamma(-z_4)\Gamma(z_4+1)\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)}{\Gamma(-2\epsilon-1)\Gamma(-\epsilon-z_4)\Gamma(-\epsilon+z_4+1)\Gamma(-3\epsilon-z_4-z_3)} \\ & \times \frac{\Gamma(-\epsilon-z_4-z_2-z_3-1)\Gamma(\epsilon-z_4+z_1-z_2-z_3)\Gamma(-\epsilon-z_4-1)\Gamma(z_4-\epsilon)\Gamma(-2\epsilon+z_2+1)}{\Gamma(-3\epsilon-z_4-z_2-z_3)\Gamma(\epsilon-z_4-z_2-z_3)} \\ & \times \Gamma(z_3+1)\Gamma(-\epsilon-z_4-z_3-1)(-1)^{-\epsilon+z_4-z_1+z_2+z_3} \end{aligned} \tag{4.85}$$

It is not complicated to repeat all the steps that have been done for $C_{8,a}$ in last subsection. First, applying `MBresolve` to eq. (4.85), we ca resolve its singularities in ϵ. Then we obtain a sum of four integrals, $\sum_{k=1}^{4} IntB_k$, where $IntB_k$ are shown below, eqs. (4.86)-(4.89),

$$\begin{aligned} IntB_1 = \quad & \frac{1}{(2\pi i)^3} \iiint_{\mathcal{C}} \prod_{j=1}^{3} dz_j (-1) \frac{2u^{-z_1-3}\Gamma(z_3+1)\Gamma(1-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)^2\Gamma(\epsilon+1)\Gamma(-z_2-z_3)}{\Gamma(-2\epsilon)\Gamma(-z_3-2\epsilon+1)\Gamma(-z_2-z_3-2\epsilon+1)\Gamma(-z_2-z_3+2\epsilon+1)} \\ & \times \Gamma(z_2-2\epsilon+1)(-1)^{-z_1+z_2+z_3-2\epsilon}\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)^2\Gamma(z_1-z_2-z_3+2\epsilon+1), \\ & \text{where } C_1 = -1.278,\ C_2 = -0.426,\ C_3 = -0.351, \end{aligned} \tag{4.86}$$

$$\begin{aligned} IntB_2 = \quad & IntB_2 \frac{1}{(2\pi i)^2} \iint_{\mathcal{C}} dz_1 dz_2 (-1) \frac{2u^{-z_1-3}(-1)^{z_2-z_1}\Gamma(-z_1)\Gamma(-z_2)\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-z_2)\Gamma(1-4\epsilon)\Gamma(-z_2-4\epsilon+1)} \\ & \times \Gamma(-\epsilon)^2\Gamma(\epsilon+1)\Gamma(z_1-z_2+1)\Gamma(-z_2-2\epsilon)\Gamma(z_2-2\epsilon+1), \\ & \text{where }, C_1 = -0.5037,\ C_2 = -0.9974, \end{aligned} \tag{4.87}$$

$$\begin{aligned} IntB_3 = \quad & \frac{1}{(2\pi i)^3} \iiint_{\mathcal{C}} \prod_{j=1}^{3} dz_j \frac{2u^{-z_1-3}\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(1-2\epsilon)}{\Gamma(-2\epsilon)\Gamma(-z_2-z_3)\Gamma(-z_3-4\epsilon)\Gamma(-z_2-z_3-4\epsilon)} \\ & \times \Gamma(1-\epsilon)\Gamma(-\epsilon)^2\Gamma(\epsilon+1)(-1)^{-z_1+z_2+z_3}\Gamma(z_2-2\epsilon+1)\Gamma(-z_3-2\epsilon-1)\Gamma(z_1-z_2-z_3) \\ & \times \Gamma(-z_2-z_3-2\epsilon-1), \\ & \text{where } C_1 = -0.6406,\ C_2 = -0.4143,\ C_3 = -1.37048, \end{aligned} \tag{4.88}$$

$$\begin{aligned} IntB_4 = \quad & \frac{1}{(2\pi i)^4} \iiiint_{\mathcal{C}} \prod_{j=1}^{4} z_j \frac{2u^{-z_1-3}\Gamma(-z_4)\Gamma(z_4+1)\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)}{\Gamma(-2\epsilon-1)\Gamma(-z_4-\epsilon)\Gamma(z_4-\epsilon+1)\Gamma(-z_4-z_3-3\epsilon)} \\ & \times \frac{(-1)^{z_4-z_1+z_2+z_3-\epsilon}\Gamma(z_4-\epsilon)\Gamma(z_2-2\epsilon+1)\Gamma(-z_4-z_3-\epsilon-1)\Gamma(-z_4-z_2-z_3-\epsilon-1)}{\Gamma(-z_4-z_2-z_3-3\epsilon)\Gamma(-z_4-z_2-z_3+\epsilon)} \\ & \times \Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_4-\epsilon-1)\Gamma(-z_4+z_1-z_2-z_3+\epsilon)\Gamma(z_3+1)\Gamma(1-2\epsilon), \\ & \text{where } C_1 = -1.12224,\ C_2 = -0.598878,\ C_3 = -0.519235,\ C_4 = -0.836055 \end{aligned} \tag{4.89}$$

We apply the routine `MBexpand` to all these integrals $IntB_k$ in order to series expand each integral up to ϵ^0 order, then we apply the routine `MBasymptotics` to the coefficient of ϵ^0 in order to find its power expansion in u. The coefficients of $1/\epsilon^2$ and $1/\epsilon$ can be obtained via the method of section 4.4.2. The coefficients of

u^k are already some real numbers. We can use the same ansatz in the calculation of the integral $C_{8,a}$, so we can fit the unknown coefficients in our ansatz. We find that:

$$\begin{aligned} C'_{8,b} = & -\frac{2\ln(1-u)}{\epsilon^2(1-u)u^2} + \frac{2\ln^2(1-u) + 4\ln(1-u)\ln u + 12\mathrm{Li}_2(u)}{\epsilon(1-u)u^2} \\ & + \frac{2}{(1-u)u^2}(6\ln^2(1-u)\ln u - \ln(1-u)\left(\pi^2 + 2\ln^2 u\right) + 2(\ln(1-u) - 6\ln u)\mathrm{Li}_2(u) \\ & + 18\mathrm{Li}_3(1-u) + 36\mathrm{Li}_3(u) - 18\zeta_3) + \mathcal{O}[\epsilon] \end{aligned} \tag{4.90}$$

4.4.5 Calculations of the Integral $C'_{8,d}$

The integral $C'_{8,d}$is given by

$$C'_{8,d} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{12} s_{124} s_{13} s_{134}} \tag{4.91}$$

We can apply the non-rigorous shortcut calculation tricks to it and obtain the Mellin–Barnes representation of the above integral $C_{8,d}$:

$$\begin{aligned} IntD_0 = & \frac{1}{(2\pi i)^4} \iiiint_{\mathcal{C}} dz_1 dz_2 dz_3 dz_4 \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_4)\Gamma(z_4+1)\Gamma(-z_1)\Gamma(z_1+1)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-\epsilon+z_4+1)\Gamma(-3\epsilon-z_4-z_1+1)} \\ & \times \frac{\Gamma(-2\epsilon+z_1+2)\Gamma(-2\epsilon+z_2+1)u^{z_1-z_3-2}\Gamma(-\epsilon-z_4-z_1-1)(-1)^{-\epsilon+z_4+z_1-z_3+z_2}\Gamma(-z_3)}{\Gamma(-3\epsilon-z_4-z_2+1)\Gamma(\epsilon-z_4-z_1-z_2)} \\ & \times \Gamma(-z_2)\Gamma(z_4-\epsilon)\Gamma(-\epsilon-z_4-z_1-z_2-1)\Gamma(\epsilon-z_4-z_1+z_3-z_2) \end{aligned} \tag{4.92}$$

Using the routine `MBresolve`, we resolve the singularities in ϵ of the above Mellin–Barnes representation and we obtain a summation of the three integrals, $\sum_{k=1}^{3} IntD_k$. The integrals $IntD_k$ are shown below eqs. (4.93)-(4.95):

$$\begin{aligned} IntD_1 = & \frac{1}{(2\pi i)^2} \iint_{\mathcal{C}} dz_2 dz_3 - \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)^2\Gamma(\epsilon+1)u^{-z_3-3}(-1)^{z_2-z_3}}{\Gamma(2-4\epsilon)\Gamma(1-z_2)\Gamma(-4\epsilon-z_2+1)} \\ & \times \Gamma(-2\epsilon+z_2+1)\Gamma(z_3-z_2+1)\Gamma(-z_3)\Gamma(-z_2)\Gamma(-2\epsilon-z_2) \\ & \text{where } C_2 = -0.997497, C_3 = -0.503702 \end{aligned} \tag{4.93}$$

$$\begin{aligned} IntD_2 = & \frac{1}{(2\pi i)^3} \iiint_{\mathcal{C}} dz_1 dz_2 dz_3 \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)^2\Gamma(\epsilon+1)\Gamma(-z_1)\Gamma(z_1+1)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-4\epsilon-z_1+1)\Gamma(-4\epsilon-z_2+1)\Gamma(-z_1-z_2)} \\ & \times \Gamma(-2\epsilon+z_1+2)\Gamma(-2\epsilon+z_2+1)u^{z_1-z_3-2}(-1)^{z_1-z_3+z_2}\Gamma(-2\epsilon-z_1-z_2-1)\Gamma(-z_1+z_3-z_2) \\ & \times \Gamma(-z_3)\Gamma(-z_2)\Gamma(-2\epsilon-z_1-1), \\ & \text{where } C_1 = -1.04134, C_2 = -0.602695, C_3 = -0.815423, \end{aligned} \tag{4.94}$$

$$
\begin{aligned}
IntD_3 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}} dz_1 dz_2 dz_3 dz_4 \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_4)\Gamma(z_4+1)\Gamma(-z_1)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-\epsilon+z_4+1)\Gamma(-3\epsilon-z_4-z_1+1)} \\
& \times \frac{\Gamma(z_4-\epsilon)\Gamma(-2\epsilon+z_1+2)\Gamma(-2\epsilon+z_2+1)u^{z_1-z_3-2}\Gamma(-\epsilon-z_4-z_1-1)(-1)^{-\epsilon+z_4+z_1-z_3+z_2}}{\Gamma(-3\epsilon-z_4-z_2+1)\Gamma(\epsilon-z_4-z_1-z_2)} \\
& \times \Gamma(-\epsilon-z_4-z_1-z_2-1)\Gamma(\epsilon-z_4-z_1+z_3-z_2)\Gamma(z_1+1)\Gamma(-z_3)\Gamma(-z_2) \\
& \text{where } C_1 = -0.495544, C_2 = -0.546209, C_3 = -0.760409, C_4 = -0.812599
\end{aligned}
\tag{4.95}
$$

We repeat the same process: we first apply `MBexpand` to obtain the series expansions in ϵ up to ϵ^0 order of every integral $IntD_k$. The coefficients of $1/\epsilon^2$ and $1/\epsilon$ can be obtained via the method of section 4.4.2. Then we apply `MBasymptotics` to the coefficient of ϵ^0 in order to obtain the series coefficients of u. The coefficients of u^k are sums of some one dimension integrals. We may calculate the each coefficient of u^k numerically, then we resort to `PSLQ` to find their analytic value. Or we may use the following tricks to rewrite an one-dimensional integral to a summation of several one-dimensional integrals, so that we can apply the first Barnes lemma to every in that summation. Here is an example:

$$
\begin{aligned}
& MB_1 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \frac{\Gamma(-z_1-4)\Gamma(-z_1-3)\Gamma(-z_1-1)\Gamma(z_1+4)\Gamma(z_1+5)\psi^{(0)}(-z_1-2)}{\Gamma(-z_1-2)} \\
& \text{where } C_1 = -0.8154.
\end{aligned}
\tag{4.96}
$$

We want to rewrite the integral MB_1 to a sum of the integrals like:

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \Gamma(-z_1+a)\Gamma(z_1+b)\Gamma(z_1+c)\Gamma(-z_1-2)\psi^{(0)}(-z_1-2) \\
& \text{where } C_1 = -0.8154.
\end{aligned}
\tag{4.97}
$$

Using identities of the form,

$$
\Gamma(-z_1-4)\Gamma(-z_1-3)\Gamma(z_1+4)\Gamma(z_1+5) = -\Gamma(-z_1-2)^2\Gamma(z_1+3)^2 \tag{4.98}
$$

The integral MB_1 becomes:

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{\mathcal{C}} dz_1 \Gamma(-z_1-1)\Gamma(z_1+3)^2\Gamma(-z_1-2)\psi^{(0)}(-z_1-2) \\
& \text{where } C_1 = -0.8154.
\end{aligned}
\tag{4.99}
$$

to which we can apply the generalization of the first Barnes lemma.

In general, in order to further simplify the integral, replace $\Gamma(a-z)\Gamma(b+z)$ in the numerator, where $a+b \geq 1$, by Polynomial$(z)\Gamma(c-z)\Gamma(d+z)$ where $\Gamma(c-z)\Gamma(d+z)$ cancels a denominator factor or where $c-z$ or $d+z$ are the arguments of the polygamma function $\psi^{(0)}$. We can rewrite any product Polynomial$(z)\Gamma(c-z)$ as a weighted sum of gamma functions,

$$
\text{Polynomial}(z)\Gamma(c-z) = \sum_{i=0} c_i \Gamma(c+i-z) \tag{4.100}
$$

or

$$
\text{Polynomial}(z)\Gamma(d+z) = \sum_{i=0} c_i \Gamma(d+i+z) \tag{4.101}
$$

and thereby reduce the integral to sum of integrals to which we can apply Barnes' lemmas. This allows us to compute all coefficients in the series expansion in u analytically. We can obtain the series in u to a

sufficiently high order to compare with a known ansatz and fix coefficients uniquely. This gives us the final analytic form of $C_{8,d}$,

$$\begin{aligned} C'_{8,d} = & -\frac{\ln(1-u)}{\epsilon^2(1-u)u^2} + \frac{\frac{\pi^2 u}{3} + (1+u)\ln(1-u)^2 + 2\ln(1-u)\ln u + 2(3+u)\mathrm{Li}_2(u)}{\epsilon(1-u)u^2} \\ & + \frac{-12(7u+4)\mathrm{Li}_3(1-u) - 48u\mathrm{Li}_3(u) - 108\mathrm{Li}_3(u) - 6(5u+1)\mathrm{Li}_2(u)\ln(1-u) + 5u\ln^3(1-u)}{3(u-1)u^2} \\ & , + \frac{\ln^3(1-u) + 2\pi^2\ln(1-u) + 15\pi^2 u\ln(1-u) + 48\zeta_3}{3(u-1)u^2} \\ & + \frac{\ln u\left(12(u+3)\mathrm{Li}_2(u) + 2\pi^2 u - 36u\ln^2(1-u) - 18\ln^2(1-u)\right)}{3(u-1)u^2} \\ & + \frac{2\ln(1-u)\ln^2 u}{(u-1)u^2} + \mathcal{O}[\epsilon] \end{aligned} \tag{4.102}$$

4.4.6 Calculation of the Integral $C'_{8,e}$

The integral $C'_{8,e}$ is given by

$$C'_{8,e} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{23} s_{123} s_{24} s_{124}} \tag{4.103}$$

Unfortunately, for this integral, we cannot apply the shortcut used above; but we can always write down its Mellin–Barnes representation. The dimension of the Mellin–Barnes representation is 6. For this integral, we find its Mellin–Barnes representation is,

$$\begin{aligned} IntE_0 = & \frac{1}{(2\pi i)^6}\int\cdots\int\prod_{j=1}^{6} dz_j \frac{\Gamma(-z_1)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5)\Gamma(1-\epsilon)u^{-z_1-z_3-3}}{\Gamma(-z_6)\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-z_6-z_3-4\epsilon)} \\ & \times \frac{\Gamma(z_5-2\epsilon+1)\Gamma(z_6+z_3-z_5+1)(-1)^{-z_1+z_2+z_5-\epsilon}\Gamma(z_2+z_3+z_4+1)\Gamma(z_3+z_4-2\epsilon+1)}{\Gamma(-z_6+z_5-4\epsilon+1)\Gamma(-z_6+z_2+z_5-2\epsilon+1)} \\ & \times\Gamma(-2z_6+z_2+2z_5-2\epsilon)\Gamma(z_6-z_2-z_5+\epsilon)\Gamma(-z_6-z_3+z_5-2\epsilon-1) \\ & \times\Gamma(-z_3-2\epsilon+1)\Gamma(z_1-z_6)\Gamma(-z_6-2\epsilon-1)\Gamma(-z_6-z_3-z_4+z_5-2\epsilon) \end{aligned} \tag{4.104}$$

Applying the routine `MBresolve` to the above representation, we resolve its singularities in ϵ. Then we obtain a sum of four integrals, $\sum_{k=1}^{4} IntE_k$. The integrals $IntE_k$ are shown below, eqs. (4.105)-(4.108):

$$\begin{aligned} IntE_1 = & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} dz_1 dz_3 dz_5 \frac{-\Gamma(-z_1)\Gamma(-z_3)\Gamma(-z_5)\Gamma(1-\epsilon)\Gamma(-\epsilon)u^{-z_1-z_3-3}}{\Gamma(1-2\epsilon)\Gamma(z_3-2\epsilon+2)\Gamma(-z_5-2\epsilon+1)\Gamma(z_3-z_5+2\epsilon+1)} \\ & \times\Gamma(z_3-2\epsilon+1)\Gamma(z_3+2\epsilon+1)\Gamma(z_5-2\epsilon+1)(-1)^{-z_1-z_3+z_5-\epsilon} \\ & \times\Gamma(z_3-z_5)\Gamma(-z_3-2\epsilon+1)\Gamma(z_1+z_3-z_5+2\epsilon+1), \\ & \text{where } C_1 = -0.755281, C_3 = -0.211418, C_5 = -0.35509. \end{aligned} \tag{4.105}$$

$$
\begin{aligned}
IntE_2 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=1,j\neq 4}^{5} dz_j \frac{\Gamma(-z_1)\Gamma(-z_3)\Gamma(-z_5)\Gamma(1-\epsilon)u^{-z_1-z_3-3}\Gamma(z_2+z_3+1)\Gamma(z_3-z_5)}{\Gamma(1-2\epsilon)\Gamma(z_2+z_3+2)\Gamma(z_3-2\epsilon+2)} \\
& \times\frac{\Gamma(z_3-2\epsilon+1)\Gamma(z_5-2\epsilon+1)(-1)^{-z_1+z_2+z_5-\epsilon}\Gamma(-z_2-z_3-\epsilon-1)\Gamma(z_2+2z_3+2\epsilon+2)}{\Gamma(-z_5-2\epsilon+1)\Gamma(z_3-z_5+2\epsilon+1)} \\
& \times\Gamma(z_1+z_3-z_5+2\epsilon+1)\Gamma(-z_3-2\epsilon+1), \\
& \text{where } C_1=-0.796243, C_2=-1.01894, C_3=-0.0754048, C_5=-0.684528.
\end{aligned}
\tag{4.106}
$$

$$
\begin{aligned}
IntE_3 = \quad & \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^{5} dz_j \frac{\Gamma(-z_1)\Gamma(-z_3)\Gamma(1-z_4)\Gamma(-z_4)\Gamma(-z_5)\Gamma(1-\epsilon)u^{-z_1-z_3-3}\Gamma(z_3-z_5)}{\Gamma(1-2\epsilon)\Gamma(z_2+z_3+2)\Gamma(z_3-2\epsilon+2)} \\
& \times\frac{\Gamma(z_5-2\epsilon+1)(-1)^{-z_1+z_2+z_5-\epsilon}\Gamma(z_2+z_3+z_4+1)\Gamma(-z_2-z_3-\epsilon-1)\Gamma(z_2+2z_3+2\epsilon+2)}{\Gamma(-z_5-2\epsilon+1)\Gamma(z_3-z_5+2\epsilon+1)} \\
& \times\Gamma(z_3+z_4-2\epsilon+1)\Gamma(z_1+z_3-z_5+2\epsilon+1)\Gamma(-z_3-2\epsilon+1), \\
& \text{where } C_1=-1.03638, C_2=-0.824306, C_3=-0.39701, C_4=0.274106, C_5=-0.711066,
\end{aligned}
\tag{4.107}
$$

$$
\begin{aligned}
IntE_4 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^{6} dz_j \frac{\Gamma(-z_1)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5)\Gamma(1-\epsilon)u^{-z_1-z_3-3}\Gamma(z_1-z_6)}{\Gamma(-z_6)\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-z_6-z_3-4\epsilon)} \\
& \times\frac{\Gamma(z_5-2\epsilon+1)\Gamma(z_6+z_3-z_5+1)(-1)^{-z_1+z_2+z_5-\epsilon}\Gamma(z_2+z_3+z_4+1)\Gamma(z_3+z_4-2\epsilon+1)}{\Gamma(-z_6+z_5-4\epsilon+1)\Gamma(-z_6+z_2+z_5-2\epsilon+1)} \\
& \times\Gamma(-2z_6+z_2+2z_5-2\epsilon)\Gamma(z_6-z_2-z_5+\epsilon)\Gamma(-z_6-z_3+z_5-2\epsilon-1) \\
& \times\Gamma(-z_3-2\epsilon+1)\Gamma(-z_6-2\epsilon-1)\Gamma(-z_6-z_3-z_4+z_5-2\epsilon) \\
& \text{where } C_1=-0.600371, C_2=-0.408977, C_3=-0.103174, \\
& C_4=-0.016356, C_5=-0.892485, C_6=-1.1896.
\end{aligned}
\tag{4.108}
$$

To the integral $IntE_k$, we apply `MBexpand` in order to expand them to a series in ϵ up to ϵ^0 order. The coefficients of $1/\epsilon^2$ and $1/\epsilon$ can be obtained via the method of section 4.4.2. Then we apply `MBasymptotics` to the coefficient of ϵ^0 in order to obtain its series coefficients of u^k. The series coefficients of u^k can be reduced to one-dimensional Mellin–Barnes integrals to which we can apply the Barnes lemmas. We calculate series coefficients of u^k to a order to compare with the ansatz to obtain the analytic form of the integral $C_{8,e}$,

$$
\begin{aligned}
C'_{8,e} = \quad & \frac{-\pi^2 u/6+(1/2-u/2)\ln(1-u)^2+(1-u)\mathrm{Li}_2(u)}{(1-u)u^2\epsilon} \\
& -\frac{1}{6(1-u)u^2}(\ln(1-u)\left(\pi^2(7-11u)+\ln(1-u)(-7(-1+u)\ln(1-u)+3(-7+9u)\ln u)\right)) \\
& +\frac{1}{(1-u)u^2}(2u\ln(1-u)\mathrm{Li}_2(u)+(-7+9u)\mathrm{Li}_3(1-u)+(-7+5u)\mathrm{Li}_3(u)+7\zeta_3)+\mathcal{O}[\epsilon]
\end{aligned}
\tag{4.109}
$$

4.4.7 Calculations of the Remaining Basic Integrals

In following sections, we will calculate the rest of the basic integrals. We can again obtain series expansions in u, but here the basis of functions for the ansatz is not large enough, and so I could not find a compact

analytic form for them. For each $C_{8,*}$, let us define:

$$C'_{8,t} = \frac{C'_{8,t,-2}}{\epsilon^2} + \frac{C'_{8,t,-1}}{\epsilon} + C'_{8,t,0} + \mathcal{O}[\epsilon], \tag{4.110}$$

where $t = c, f, g, h, i, j, k$ and

$$C'_{8,t,0} = C'_{8,t,0,\ln u} + C'_{8,t,0,\not\ln u} \tag{4.111}$$

where $C'_{8,t,0}$ is the coefficient of ϵ^0 and $C'_{8,t,0,\ln u}$ are the terms which contain $\ln u$, and $C'_{8,t,0,\not\ln u}$ are the terms of order ϵ^0 which are free of $\ln u$. Similarly we can obtain a series expansion about $u = 1$ by replacing $u = 1 - u'$ in $C'_{8,*}$, and again computing the series expansion in ϵ,

$$C'_{8,t}(u') = \frac{C'_{8,t,-2}(u')}{\epsilon^2} + \frac{C'_{8,t,-1}(u')}{\epsilon} + C'_{8,t,0}(u') + \mathcal{O}[\epsilon], \tag{4.112}$$

$$C'_{8,t,0}(u') = C'_{8,t,0,\ln u'} + C_{8,t,0,\not\ln u'} \tag{4.113}$$

In practice, we will need to do this only for the finite terms. The coefficients of $1/\epsilon^2$ and $1/\epsilon$, $C_{8,t,-2}$ and $C_{8,t,-1}$ can always be obtained via the method of section 4.4.2.

4.4.8 Calculations of the Integral $C'_{8,c}$

The integral $C'_{8,c}$ is given by

$$C'_{8,c} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E^2_{total} s_{12} s_{123} s_{24} s_{234}}. \tag{4.114}$$

Since in our parametrization (4.8), the factors s_{123} and s_{234} in the denominator of the integrand are not free of s_{13}. The sum $s_{123} + s_{234}$ is free of s_{13}. We use the symmetry in the four-particle phase space to rewrite the integrand in order to perform the integration over s_{13} simple.

$$\begin{aligned} C'_{8,c} = &\ 8 f_{dim} \int dPS_4 \frac{(s_{123} + s_{234}) E_2 E_3 \delta(\cos\theta_{23} - c)}{E^2_{total} s_{12} s_{123} s_{24} s_{234} (s_{123} + s_{234})} \\ = &\ 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E^2_{total} s_{12} s_{123} s_{24} (s_{123} + s_{234})} + 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E^2_{total} s_{12} s_{24} s_{234} (s_{123} + s_{234})} \end{aligned} \tag{4.115}$$

We can freely exchange the momentum indices 1 and 4 in the second term of the above formula,

$$\int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E^2_{total} s_{12} s_{24} s_{234} (s_{123} + s_{234})} = \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E^2_{total} s_{12} s_{123} s_{24} (s_{123} + s_{234})} \tag{4.116}$$

Then we obtain,

$$C'_{8,c} = 8 f_{dim} \int dPS_4 \frac{2 E_2 E_3 \delta(\cos\theta_{23} - c)}{E^2_{total} s_{12} s_{123} s_{24} (s_{234} + s_{123})}. \tag{4.117}$$

The Mellin–Barnes representation of the integral $C_{8,c}$ is:

$$\begin{aligned} IntC_0 = &\ \frac{1}{(2\pi i)^7} \int \cdots \int \prod_{j=1}^{7} dz_j \frac{2\Gamma(1-\epsilon)\Gamma(-z_1)\Gamma(z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(-\epsilon+z_5-1)}{\Gamma(-2\epsilon-1)\Gamma(e+z_5)\Gamma(-\epsilon-z_2+1)\Gamma(-2\epsilon-z_6+1)} \\ &\times \frac{\Gamma(z_5-z_7)\Gamma(z_7-z_2)(-1)^{-\epsilon-z_5-z_1-z_2}\Gamma(\epsilon+z_5+z_1)\Gamma(-2\epsilon-z_3+z_6+2)\Gamma(-\epsilon+z_3-z_7)}{\Gamma(-3\epsilon-z_6+z_7)\Gamma(-3\epsilon+z_5-z_3+z_6+1)} \\ &\times \Gamma(-\epsilon-z_3+z_7-1) u^{-z_1-z_3+z_6-2} \Gamma(-2\epsilon-z_5-z_6+z_7+1)\Gamma(-2\epsilon+z_3+z_4-z_6) \\ &\times \Gamma(-\epsilon-z_3-z_4+z_7)\Gamma(-z_2+z_3+z_4-z_7+1)) \end{aligned} \tag{4.118}$$

We can again apply the routine `MBresolve` to eq. (4.118) in order to resolve its singularities in ϵ. Then we obtain a sum of seven integrals $\sum_{k=1}^{7} IntC_k$ in eqs. (4.119)-(4.125):

$$\begin{aligned} IntC_1 = \quad & \frac{1}{(2\pi i)^4}\int\cdots\int_{\mathcal{C}}\prod_{j=1,4,5,6} dz_j \Big(\frac{2(-1)^{-\epsilon-z_5-z_1}u^{-z_1+z_6-2}\Gamma(-\epsilon)\Gamma(-\epsilon+z_5-1)\Gamma(-z_1)}{\Gamma(-4\epsilon-z_6)\Gamma(-2\epsilon-z_6+1)\Gamma(-3\epsilon+z_5+z_6+1)} \\ & \times\Gamma(-2\epsilon-z_4)\Gamma(-z_4)\Gamma(\epsilon+z_4+1)\Gamma(-3\epsilon-z_5-z_6+1)\Gamma(-2\epsilon+z_4-z_6)\Gamma(-z_6) \\ & \times\Gamma(z_6+1)\Gamma(-2\epsilon+z_6+2)\Gamma(\epsilon+z_5+z_1), \\ & \text{where } C_1=-1.42428, C_4=-0.391587, C_5=1.77712, C_6=-0.929342 \end{aligned} \tag{4.119}$$

$$\begin{aligned} IntC_2 = \quad & \frac{1}{(2\pi i)^4}\int\cdots\int_{\mathcal{C}}\prod_{j=1,5,6,7} dz_j \frac{2(-1)^{-z_5-z_1}u^{\epsilon-z_1+z_6-z_7-1}\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-\epsilon+z_5-1)\Gamma(-z_1)}{(\Gamma(\epsilon+z_5)\Gamma(-2\epsilon-z_6+1)\Gamma(-3\epsilon-z_6+z_7))} \\ & \times\frac{\Gamma(\epsilon+z_5+z_1)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(\epsilon-z_7+1)\Gamma(z_5-z_7)\Gamma(-\epsilon+z_6-z_7+3)\Gamma(\epsilon+z_7)}{\Gamma(-2\epsilon+z_5+z_6-z_7+2)} \\ & \times\Gamma(-3\epsilon-z_6+z_7-1)\Gamma(-2\epsilon-z_5-z_6+z_7+1)), \\ & \text{where } C_1=-0.815526, C_5=1.77872, C_6=-0.709401, C_7=0.8297450. \end{aligned} \tag{4.120}$$

$$\begin{aligned} IntC_3 = \quad & \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 3}^{6} dz_j \frac{2(-1)^{-\epsilon-z_5-z_1-z_2}u^{-z_1+z_6-2}\Gamma(1-\epsilon)\Gamma(-\epsilon+z_5-1)\Gamma(-z_1)}{\Gamma(-\epsilon-z_2+1)\Gamma(-4\epsilon-z_6)\Gamma(-2\epsilon-z_6+1)\Gamma(-3\epsilon+z_5+z_6+1)} \\ & \times\Gamma(\epsilon+z_5+z_1)\Gamma(-\epsilon-z_2)\Gamma(z_2)\Gamma(-2\epsilon-z_4)\Gamma(-z_4)\Gamma(\epsilon-z_2+z_4+1)\Gamma(-3\epsilon-z_5-z_6+1) \\ & \times\Gamma(-2\epsilon+z_4-z_6)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(-2\epsilon+z_6+2) \\ & \text{where } C_1=-0.787005, C_2=-0.461073, C_4=-0.205599, C_5=1.51091, C_6=-0.623463. \end{aligned} \tag{4.121}$$

$$\begin{aligned} IntC_4 = \quad & \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 3 \text{ and } 4}^{7} dz_j \frac{2(-1)^{-\epsilon-z_5-z_1-z_2}u^{\epsilon-z_1+z_6-z_7-1}\Gamma(1-\epsilon)\Gamma(-z_1)}{\Gamma(\epsilon+z_5)\Gamma(-\epsilon-z_2+1)\Gamma(-2\epsilon-z_6+1)} \\ & \times\frac{\Gamma(\epsilon+z_5+z_1)\Gamma(-\epsilon-z_2)\Gamma(z_2)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(\epsilon-z_7+1)\Gamma(z_5-z_7)\Gamma(-\epsilon+z_6-z_7+3)}{\Gamma(-2\epsilon+z_5+z_6-z_7+2)\Gamma(-3\epsilon-z_6+z_7)} \\ & \times\Gamma(z_7-z_2)\Gamma(-3\epsilon-z_6+z_7-1)\Gamma(-2\epsilon-z_5-z_6+z_7+1)\Gamma(-\epsilon+z_5-1) \\ & \text{where } C_1=-1.15658, C_2=0.494661, C_5=1.76871, C_6=-0.805709, C_7=0.900758, \end{aligned} \tag{4.122}$$

$$
\begin{aligned}
IntC_5 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 3}^{7} dz_j \frac{2(-1)^{-\epsilon-z_5-z_1-z_2}u^{-\epsilon-z_1+z_6-z_7-2}\Gamma(1-\epsilon)}{\Gamma(\epsilon+z_5)\Gamma(-\epsilon-z_2+1)\Gamma(-2\epsilon-z_6+1)} \\
& \times\frac{\Gamma(-2\epsilon-z_4)\Gamma(-z_4)\Gamma(\epsilon-z_2+z_4+1)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(-\epsilon-z_7)\Gamma(z_5-z_7)}{\Gamma(-4\epsilon+z_5+z_6-z_7+1)\Gamma(-3\epsilon-z_6+z_7)} \\
& \times\Gamma(z_7-z_2)\Gamma(-2\epsilon-z_5-z_6+z_7+1)\Gamma(-\epsilon+z_4-z_6+z_7)\Gamma(\epsilon+z_5+z_1)\Gamma(z_2) \\
& \Gamma(-\epsilon+z_5-1)\Gamma(-z_1)\Gamma(-3\epsilon+z_6-z_7+2) \\
& \text{where } C_1=-0.906632, C_2=0.17358, C_4=-0.375981, \\
& C_5=1.21199, C_6=-0.598146, C_7=0.721839,
\end{aligned}
\tag{4.123}
$$

$$
\begin{aligned}
IntC_6 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 3}^{7} dz_j \frac{2(-1)^{-\epsilon-z_5-z_1-z_2}u^{\epsilon-z_1+z_6-z_7-1}\Gamma(1-\epsilon)\Gamma(-\epsilon+z_5-1)}{\Gamma(\epsilon+z_5)\Gamma(-\epsilon-z_2+1)\Gamma(-2\epsilon-z_6+1)} \\
& \frac{\times\Gamma(z_2)\Gamma(1-z_4)\Gamma(-z_4)\Gamma(-\epsilon-z_2+z_4)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(\epsilon-z_7+1)}{\Gamma(-2\epsilon+z_5+z_6-z_7+2)\Gamma(-3\epsilon-z_6+z_7)} \\
& \times\Gamma(z_7-z_2)\Gamma(-2\epsilon-z_5-z_6+z_7+1)\Gamma(-3\epsilon+z_4-z_6+z_7-1)\Gamma(-z_1)\Gamma(\epsilon+z_5+z_1), \\
& \times\Gamma(z_5-z_7)\Gamma(-\epsilon+z_6-z_7+3) \\
& \text{where } C_1=-1.41909, C_2=0.00569086, C_4=0.543388, \\
& C_5=1.60584, C_6=-0.707894, C_7=0.934493,
\end{aligned}
\tag{4.124}
$$

$$
\begin{aligned}
IntC_7 = \quad & \frac{1}{(2\pi i)^7}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^{7} dz_j \frac{2(-1)^{-\epsilon-z_5-z_1-z_2}u^{-z_1-z_3+z_6-2}\Gamma(1-\epsilon)\Gamma(-\epsilon+z_5-1)}{\Gamma(-2\epsilon-1)\Gamma(\epsilon+z_5)\Gamma(-\epsilon-z_2+1)\Gamma(-2\epsilon-z_6+1)} \\
& \times\frac{\Gamma(z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-2\epsilon+z_3+z_4-z_6)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(-2\epsilon-z_3+z_6+2)}{\Gamma(-3\epsilon-z_6+z_7)\Gamma(-3\epsilon+z_5-z_3+z_6+1)}
\end{aligned}
\tag{4.125}
$$

$$
\begin{aligned}
& \times\Gamma(-z_2+z_3+z_4-z_7+1)\Gamma(z_7-z_2)\Gamma(-\epsilon-z_3+z_7-1)\Gamma(-\epsilon-z_3-z_4+z_7) \\
& \times\Gamma(z_5-z_7)\Gamma(-\epsilon+z_3-z_7)\Gamma(-2\epsilon-z_5-z_6+z_7+1)\Gamma(-z_1)\Gamma(\epsilon+z_5+z_1) \\
& \text{where } C_1=-0.208252, C_2=0.0118178, C_3=-0.264238, \\
& C_4=-0.102125, C_5=1.59921, C_6=-0.650907, C_7=0.472031.
\end{aligned}
$$

We then apply the routine `MBexpand` to expand every integral in the $\sum_{k=1}^{7} IntC_k$ to a series in ϵ up to ϵ^0 order. The coefficients of $1/\epsilon^2$ and $1/\epsilon$ are simple to be obtained via the method in section 4.4.2:

$$
C'_{8,c,-2}=0 \tag{4.126}
$$

$$
C'_{8,c,-1}=\frac{(2-2u)\mathrm{Li}_2(u)-\pi^2 u/3+(1-u)\ln^2(1-u)}{(1-u)u^2} \tag{4.127}
$$

Now we focus on the coefficient of ϵ^0 of this function. We apply the routine `MBasymptotics` to the coefficient of ϵ^0 in order to obtain a series expansion in u. Even We can compute the series coefficients to a high order of u. But it seems that our base function set is not large enough to fit the series. So we try

to calculate $C_{8,c,\epsilon^0,\ln u}$ directly. We can extract a part of the coefficient of $\ln u$ as a sum of one-dimensional Mellin–Barnes integrals which can not be fitted into our ansatz. After a simplification, it is:

$$C'_{8,c,sp} = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_7 \sum_{k=-1}^{\infty}(-1)\frac{4u^k \ln u\Gamma(-z_7)\Gamma(z_7)^2\Gamma(k-z_7+3)\Gamma(k+z_7+2)}{\Gamma(k+2)\Gamma(k+3)\Gamma(z_7+1)},\ \text{where } C_1 = 0.8674 \tag{4.128}$$

We can try to apply Barnes' lemma to the above integral for each k and sum the series. Finally, it yields:

$$\begin{aligned} C'_{8,c,sp} = \ & \frac{2\ln u\left(-3u\mathrm{Li}_2(u) + u\left(\left(\pi^2-6\right)u+6\right) + 3(u+1)\ln^2\left(1-\sqrt{u}\right) + 3(u+1)\ln^2\left(\sqrt{u}+1\right)\right)}{3(u-1)u^2} \\ & + \ln u\frac{-(u+1)\ln^2(1-u) + 4\ln\left(\sqrt{u}+1\right)\ln\left(1-\sqrt{u}\right)}{(u-1)u^2} \end{aligned} \tag{4.129}$$

So we can obtain $C'_{8,c,0,\ln u}$ as a compact function of u:

$$\begin{aligned} C'_{8,c,0,\ln u} = \ & \frac{2\ln u\left(-3u\mathrm{Li}_2(u) + 6u(1-u) + 3(u+1)\ln^2\left(1-\sqrt{u}\right) + 3(u+1)\ln^2\left(\sqrt{u}+1\right)\right)}{3(u-1)u^2} \\ & + \ln u\frac{-(u+1)\ln^2(1-u) + 4\ln\left(\sqrt{u}+1\right)\ln\left(1-\sqrt{u}\right)}{(u-1)u^2} \end{aligned} \tag{4.130}$$

Then I compute the $C'_{8,c,0,\ln u}$ as a series expansion of u. It yields:

$$\begin{aligned} C'_{8,c,0,\ln u} = \ & -\frac{2\left(-39u\zeta_3 + \pi^2\ln(1-u)\right)}{3(u-1)u^2} \\ & + \frac{919}{36} + \frac{26}{u} + \frac{67553u}{2700} + \frac{25979117u^2}{1058400} + \frac{382997833u^3}{15876000} + \frac{5475628693u^4}{230519520} + \frac{10647403347619u^5}{454507653600} \\ & + \frac{84123675413329u^6}{3636061228800} + \frac{21635338069741247 9u^7}{9457395256108800} + \frac{3865377949315103804 63u^8}{17070598437276384000} \\ & + \frac{42123035432820191232 97u^9}{187776582810040224000} + \frac{22092400280692601568 92923u^{10}}{9933381230651127849 6000} \\ & + \frac{10958987039388856759508453u^{11}}{4966690615325563924800 00} + \frac{14142482618157427678318 1893u^{12}}{6456697799923233102240000} \\ & + \frac{36162068403392936530675909063u^{13}}{16622702601230935833624000 00} + \frac{61388906715518646271204935718 13u^{14}}{28398963910725207708644736 0000} + \cdots \end{aligned} \tag{4.131}$$

I tried to fit the series expansion of u to the enlarged the base function of ansatz with the arguments $1-\sqrt{u}$ and $1+\sqrt{u}$, but it seems it still lacks basis functions, so I replace u by $1-u'$ in the Mellin–Barnes representation of the integral $C_{8,c}$ in eq. (4.118) and try to expand $C'_{8,c}$ at $u'=0$. In this way, I can obtain the series expansion at $u=1$ and see if we can extract more information about $C'_{8,c}$.

The coefficients of u'^k in series expansion of $C'_{8,c,0}(u)$ is more complicated to calculate then the coefficients of u^k in the series expansion of $C'_{8,c,0}(u)$. In the calculation, we found that the coefficient of u'^k is a sum of several genuine divergent Mellin–Barnes integrals. For example, here is a one-dimensional genuinely divergent integral MB_2:

$$MB_2 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz\,(-1)^z\,\Gamma(z)\Gamma(-z),\ \text{where } C = -0.5 \tag{4.132}$$

The sum of the residues on the left of the contour is

$$MB_2 \sim \sum_{k=0}^{\infty}\frac{1}{1+k} \tag{4.133}$$

This is so-called harmonic series and there is no regularization method for summing this series. So this Mellin–Barnes integral is genuinely divergent. The contribution of the integral MB_2 is calculated in the

following way. We multiply the integrand by a factor v^z, so the integral MB_2 becomes

$$MB_2 \to MB_2'(v) = \frac{1}{2\pi i}\int_{\mathcal{C}} dz\,(-1)^z v^z \Gamma(z)\Gamma(-z),\ \text{where } C = -0.5 \tag{4.134}$$

Then we compute the contribution of the integral MB_2',

$$MB_2'(v) = -\ln\left(\frac{-1+v}{v}\right) \tag{4.135}$$

We take the function $MB_2'(v)$ to be the contribution of the integral MB_2. The function $MB_2'(v)$ goes to infinity when $v \to 1$. For 2-dimensional Mellin–Barnes integral, the genuinely divergent integrals also appear. For example:

$$\begin{aligned} MB_3 &= \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}} dz_2 dz_6 \frac{2(-1)^{-z_2}\Gamma(-z_2-1)\Gamma^2(-z_2)\Gamma(z_2+1)\Gamma^2(-z_6)\Gamma(z_6)\Gamma(z_6+1)}{\Gamma(-z_2-z_6)\Gamma(z_6-z_2)}, \\ &\text{where } C_2 = -\tfrac{3}{2}, C_6 = -\tfrac{20}{19}. \end{aligned} \tag{4.136}$$

To regularize this Mellin–Barnes integral, we multiply the integrand with v^{-z_2} and calculate the sum of the residues of z_2 on the left side of the contour. It yields:

$$\begin{aligned} MB_3 \to MB_3' = &\ \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}} dz_2 dz_6 v^{-z_2} \frac{2(-1)^{-z_2}\Gamma(-z_2-1)\Gamma^2(-z_2)\Gamma(z_2+1)\Gamma^2(-z_6)\Gamma(z_6)\Gamma(z_6+1)}{\Gamma(-z_2-z_6)\Gamma(z_6-z_2)}, \\ &\text{where } C_2 = -\frac{3}{2}, C_6 = -\frac{20}{19} \\ = &\ \frac{1}{2\pi i}\int_{\mathcal{C}} dz_6 (-1)\frac{2v^2\Gamma^2(-z_6)\Gamma(z_6)\Gamma(z_6+1)\,{}_3F_2(1,1,2;2-z_6,z_6+2;v)}{\Gamma(2-z_6)\Gamma(z_6+2)}, \\ &\text{where } C_6 = -\frac{20}{19} \end{aligned} \tag{4.137}$$

Then we expand the integrand in the above formula up to 0 order near $v = 1$. We obtain a new integral:

$$\begin{aligned} MB_3'' = &\ \frac{1}{2\pi i}\int_{\mathcal{C}} dz_6\ 2z_6^2\Gamma(z_6)\Gamma(z_6+1)\Gamma^2(-z_6)\,{}_4F_3(1,1,1-z_6,z_6+1;2,2,2;1) \\ &+\frac{2\ln(1-v)\Gamma(z_6)\Gamma(z_6+1)\Gamma^2(-z_6)}{v}, \\ &\text{where } C_6 = -\frac{20}{19}. \end{aligned} \tag{4.138}$$

We take the integral MB_3'' to be the contribution of the integral MB_3. The integral MB_3'' goes to infinity when $v \to 1$. Finally, we should add the contributions from different genuine divergent integrals to cancel to singularity at $v = 1$, so that we can obtain a finite result for the series coefficient of u'^k. There is no need to calculate the coefficient of $1/\epsilon$ again. The part which are not free of $\ln u'$ in $C_{8,c,0}(u')$ is

$$\begin{aligned} C'_{8,c,0,\ln u'} = &\ \frac{\ln(u')\left((4-2u')\mathrm{Li}_2(u') + (-4u'-3)\ln^2(1-u')\right)}{(1-u')^2u'} - \frac{7\ln^3(u')}{3(1-u')^2} \\ &+\frac{(5u'+2)\ln(1-u')\ln^2(u')}{(1-u')^2u'} + \frac{\pi^2(5u'-4)\ln(u')}{3(u'-1)^2}. \end{aligned} \tag{4.139}$$

Then the part which are free of $\ln u'$ in $C_{8,c,0}(u')$ is

$$\begin{aligned} C'_{8,c,0,\not\ln u'}(u') = & \frac{(12u'-13)\zeta_3}{(u'-1)^2u'} + \frac{2\pi^2\ln(2)}{(1-u')^2} - 4 - 2\pi^2 + 11u' - \frac{35\pi^2u'}{6} + \frac{2035u'^2}{54} - \frac{171\pi^2u'^2}{16} \\ & + \frac{581u'^3}{8} - \frac{2341\pi^2u'^3}{144} + \frac{6150647u'^4}{54000} - \frac{515591\pi^2u'^4}{23040} + \frac{2164013u'^5}{13500} - \frac{18527\pi^2u'^5}{640} \\ & + \frac{976937939u'^6}{4630500} - \frac{7719433\pi^2u'^6}{215040} + \frac{19658396729u'^7}{74088000} - \frac{3094743\pi^2u'^7}{71680} \\ & + \frac{1291903885331u'^8}{4000752000} - \frac{837968669\pi^2u'^8}{16515072} + \cdots \end{aligned} \tag{4.140}$$

4.4.9 Calculations of the integral $C'_{8,f}$

The integral $C'_{8,f}$ is given by

$$C'_{8,f} = 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{24}s_{234}s_{14}s_{134}}. \tag{4.141}$$

We exchange the index of momentum 1 and 4 in the integrand of $C_{8,f}$,

$$C'_{8,f} = 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{12}s_{123}s_{14}s_{134}} \tag{4.142}$$

The Mellin–Barnes representation of the integral $C_{8,f}$ is:

$$\begin{aligned} IntF_0 = & \frac{1}{(2\pi i)^7}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^{7}dz_j \frac{\Gamma(1-\epsilon)\Gamma(-z_1)\Gamma(z_1+1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-4\epsilon+z_6-z_3+1)} \\ & \times\frac{\Gamma(-2\epsilon+z_6-z_1-1)(-1)^{\epsilon+z_1-z_2-z_5-2z_6-z_7}\Gamma(-2\epsilon+z_1-z_3+2)\Gamma(-2\epsilon+z_3+z_4+2)}{\Gamma(-z_6-z_2-z_5+1)\Gamma(-4\epsilon+z_6-z_1+z_5)\Gamma(z_6-z_1)} \\ & \times\Gamma(-z_6+z_3-z_5)\Gamma(-\epsilon+z_6+z_2+z_5)\Gamma(-2\epsilon+z_6-z_3+z_5) \\ & \times\Gamma(2\epsilon-2z_6-z_2+z_3+z_4-2z_5+1)\Gamma(-z_7)\Gamma(-z_1+z_6+z_7)u^{-z_3-2+z_1-z_7} \\ & \times\Gamma(-2\epsilon+z_5+1)\Gamma(-2\epsilon+z_6-z_1-z_3-z_4+z_5-2) \end{aligned} \tag{4.143}$$

As usual, we can use the routine `MBresolve` in order to resolve the singularity in ϵ. We obtain a sum of eleven integrals, $\sum_{k=1}^{11} IntF_k$. The integrals $IntF_k$ $k = 1, \cdots, 11$ eqs. (4.144)-(4.154)

$$\begin{aligned} IntF_1 = & \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}} dz_5dz_7 \frac{(-1)^{z_7}u^{z_7-z_5-2}\Gamma(-\epsilon)^2\Gamma(2\epsilon+1)\Gamma(-z_7)\Gamma(-2\epsilon-z_5)}{\Gamma(1-z_5)\Gamma(-4\epsilon-z_5+1)} \\ & \times\Gamma(z_7-z_5+1)\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1), \\ & \text{where } C_5 = -0.997497, C_7 = -0.503702. \end{aligned} \tag{4.144}$$

$$\begin{aligned} IntF_2 = & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} dz_4dz_5dz_7 \frac{(-1)^{z_7}u^{z_7-z_5-2}\Gamma(-\epsilon)\Gamma(-z_7)\Gamma(-2\epsilon-z_4-1)\Gamma(-z_4)}{\Gamma(1-4\epsilon)\Gamma(1-z_5)\Gamma(-4\epsilon-z_5+1)} \\ & \times\Gamma(-2\epsilon-z_5)\Gamma(z_7-z_5+1)\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1)\Gamma(-2\epsilon+z_4+2)\Gamma(\epsilon+z_4+1), \\ & \text{where } C_4 = -0.814507, C_5 = -0.615896, C_7 = -0.960356. \end{aligned} \tag{4.145}$$

$$IntF_3 = \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=1,4,5,7} dz_j \frac{(-1)^{z_7}u^{z_7-z_5-2}\Gamma(-\epsilon)\Gamma(-z_1)\Gamma(z_1+1)\Gamma(-2\epsilon+z_1+2)\Gamma(-z_7)}{\Gamma(1-2\epsilon)\Gamma(-4\epsilon-z_1)\Gamma(-4\epsilon-z_5+1)\Gamma(-z_1-z_5)}$$

$$\times\Gamma(-2\epsilon-z_1-z_4-2)\Gamma(-z_4)\Gamma(-2\epsilon+z_4+2)\Gamma(\epsilon+z_4+1)\Gamma(-2\epsilon-z_1-z_5-1)$$

$$\times\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1)\Gamma(-z_1+z_7-z_5),$$

where $C_1 = -1.64659, C_4 = -0.941542, C_5 = -0.294187, C_7 = -0.178151.$

(4.146)

$$IntF_4 = \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} dz_3 dz_5 dz_7 \frac{(-1)^{z_7}u^{z_7-z_5-2}\Gamma(-\epsilon)\Gamma(\epsilon+1)\Gamma(-z_7)\Gamma(-2\epsilon-z_3+1)\Gamma(-z_3)}{\Gamma(1-2\epsilon)\Gamma(-4\epsilon+z_3+1)\Gamma(-4\epsilon-z_5+1)\Gamma(z_3-z_5+1)},$$

$$\times\Gamma(z_3-\epsilon)\Gamma(-2\epsilon+z_3-z_5)\Gamma(z_7+z_3-z_5+1)\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1)\Gamma(-3\epsilon+z_3+1),$$

where $C_3 = 0.295632, C_5 = -0.490526, C_7 = -1.36711.$

(4.147)

$$IntF_5 = \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=1,3,5,7} dz_j \frac{(-1)^{z_7}u^{z_7-z_5-2}\Gamma(\epsilon+1)\Gamma(-\epsilon-z_1-1)\Gamma(-z_1)\Gamma(z_1+1)\Gamma(-z_7)}{(\Gamma(1-2\epsilon)\Gamma(-4\epsilon-z_1+z_3)\Gamma(-4\epsilon-z_5+1)\Gamma(-z_1+z_3-z_5))}$$

$$\times\Gamma(-z_3)\Gamma(-3\epsilon+z_3+1)\Gamma(z_3-\epsilon)\Gamma(-2\epsilon-z_1+z_3-z_5-1)\Gamma(-z_1+z_7+z_3-z_5)\Gamma(-z_5)$$

$$\times\Gamma(-2\epsilon+z_5+1)\Gamma(-2\epsilon+z_1-z_3+2),$$

where $C_1 = -1.23415, C_3 = 0.654241, C_5 = -0.4631, C_7 = -0.936405.$

(4.148)

$$IntF_6 = \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}}\prod_{j=1,3,4,5,7} dz_j \frac{(-1)^{z_7}u^{z_7-z_5-2}\Gamma(-z_1)\Gamma(z_1+1)\Gamma(-z_7)\Gamma(-2\epsilon+z_1-z_3+2)}{(\Gamma(1-2\epsilon)\Gamma(-4\epsilon-z_1+z_3)\Gamma(-4\epsilon-z_5+1)\Gamma(-z_1+z_3-z_5))}$$

$$\times\Gamma(-2\epsilon-z_1-z_4-2)\Gamma(-z_4)\Gamma(\epsilon+z_4+1)\Gamma(-2\epsilon+z_3+z_4+2)\Gamma(-2\epsilon-z_1+z_3-z_5-1)$$

$$\times\Gamma(-z_1+z_7+z_3-z_5)\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1)\Gamma(-z_3)\Gamma(z_3-\epsilon),$$

where $C_1 = -0.933515, C_3 = 0.799302, C_4 = -1.51591, C_5 = -0.654726, C_7 = -0.917014.$

(4.149)

$$IntF_7 = \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=3,4,5,7} dz_j \frac{(-1)^{z_7}u^{\epsilon+z_7+z_4-z_5-1}\Gamma(-\epsilon)\Gamma(\epsilon+1)\Gamma(-z_7)\Gamma(-3\epsilon-z_3+1)\Gamma(-z_3)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-3\epsilon+z_4-z_5+2)\Gamma(2\epsilon+z_3+z_4-z_5+2)}$$

$$\times\Gamma(-\epsilon-z_4-1)\Gamma(-z_4)\Gamma(-\epsilon+z_4+1)\Gamma(z_3+z_4+1)\Gamma(z_3+z_4-z_5+1)$$

$$\times\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1)\Gamma(2\epsilon+z_7+z_3+z_4-z_5+2),$$

where $C_3 = -0.475555, C_4 = -0.047522, C_5 = -0.772042, C_7 = -0.221211.$

(4.150)

$$
\begin{aligned}
IntF_8 = \quad & \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}}\prod_{j=1,3,4,5,7} dz_j \frac{(-1)^{z_7}u^{\epsilon+z_7+z_4-z_5-1}\Gamma(-\epsilon-z_1-1)\Gamma(-z_1)\Gamma(z_1+1)\Gamma(-z_7)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-3\epsilon-z_1+z_3+z_4+1)\Gamma(-3\epsilon+z_4-z_5+2)} \\
& \times\frac{\Gamma(-2\epsilon+z_1-z_3+2)\Gamma(-z_3)\Gamma(-\epsilon-z_4-1)\Gamma(-z_4)\Gamma(-\epsilon+z_4+1)}{\Gamma(\epsilon-z_1+z_3+z_4-z_5+1)} \\
& \times\Gamma(-\epsilon-z_1+z_3+z_4-z_5)\Gamma(\epsilon-z_1+z_7+z_3+z_4-z_5+1)\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1) \\
& \times\Gamma(z_3+z_4+1)\Gamma(-2\epsilon+z_3+z_4+2), \\
& \text{where } C_1=-0.659871, C_3=-0.457658, C_4=-0.444918, \\
& C_5=-0.566508, C_7=-1.23697.
\end{aligned}
\tag{4.151}
$$

$$
\begin{aligned}
IntF_9 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 6}^{7} dz_j \frac{(-1)^{z_7}u^{\epsilon+z_7-z_2-z_3-z_5-2}\Gamma(-z_1)\Gamma(z_1+1)\Gamma(-z_7)\Gamma(-z_2)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-3\epsilon-z_1-z_2)} \\
& \times\frac{\Gamma(-z_3)\Gamma(-\epsilon+z_2+z_3)\Gamma(-\epsilon-z_1-z_2-z_3-z_4-2)\Gamma(-z_4)\Gamma(-2\epsilon+z_3+z_4+2)}{\Gamma(\epsilon-z_1-z_2-z_5)\Gamma(-3\epsilon-z_2-z_3-z_5+1)} \\
& \times\Gamma(-\epsilon-z_1-z_2-z_5-1)\Gamma(\epsilon-z_1+z_7-z_2-z_5)\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1) \\
& \times\Gamma(-2\epsilon+z_1-z_3+2)\Gamma(-\epsilon-z_2-z_3)\Gamma(z_2+z_3+z_4+1), \\
& \text{where } C_1=-0.362762, C_2=-0.484523, C_3=-0.271797, C_4=-0.991377, \\
& C_5=-0.654658, C_7=-0.356614.
\end{aligned}
\tag{4.152}
$$

$$
\begin{aligned}
IntF_{10} = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 6}^{7} dz_j \frac{(-1)^{-\epsilon+z_7-z_2-z_3}u^{2\epsilon+z_7-z_5-2}\Gamma(1-\epsilon)\Gamma(-z_1)\Gamma(z_1+1)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-z_2-z_3+1)\Gamma(-2\epsilon-z_1+z_3)} \\
& \times\frac{\Gamma(-z_3)\Gamma(\epsilon+z_2+z_3)\Gamma(-z_1-z_4-2)\Gamma(-z_4)\Gamma(-2\epsilon-z_2-z_3+z_4+1)}{\Gamma(-2\epsilon-z_5+1)\Gamma(2\epsilon-z_1+z_3-z_5)} \\
& \times\Gamma(-z_1+z_3-z_5-1)\Gamma(2\epsilon-z_1+z_7+z_3-z_5)\Gamma(-2\epsilon+z_5+1)\Gamma(-2\epsilon+z_1-z_3+2) \\
& \times\Gamma(-z_7)\Gamma(-z_2)\Gamma(-2\epsilon+z_3+z_4+2)\Gamma(-z_5), \\
& \text{where } C_1=-0.677624, C_2=-0.485681, C_3=-0.408594, C_4=-1.34095, \\
& C_5=-0.750213, C_7=-0.468925.
\end{aligned}
\tag{4.153}
$$

$$
\begin{aligned}
IntF_{11} = \quad & \frac{1}{(2\pi i)^7}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^{7}dz_j\frac{(-1)^{\epsilon-z_6+z_7-z_2-z_5}u^{z_6+z_7-z_3-2}\Gamma(1-\epsilon)\Gamma(-2\epsilon+z_6-z_1-1)\Gamma(-z_1)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(z_6-z_1)\Gamma(-4\epsilon+z_6-z_3+1)} \\
& \times\frac{\Gamma(z_6-z_1+z_7)\Gamma(-z_2)\Gamma(-2\epsilon+z_1-z_3+2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-2\epsilon+z_3+z_4+2)}{\Gamma(-z_6-z_2-z_5+1)\Gamma(-4\epsilon+z_6-z_1+z_5)} \\
& \times\Gamma(-z_5)\Gamma(-2\epsilon+z_5+1)\Gamma(-\epsilon+z_6+z_2+z_5)\Gamma(-2\epsilon+z_6-z_3+z_5) \\
& \times\Gamma(z_1+1)\Gamma(-z_7)\Gamma(-2\epsilon+z_6-z_1-z_3-z_4+z_5-2)\Gamma(-z_6+z_3-z_5), \\
& \text{where } C_1=-0.809246, C_2=-0.559843, C_3=-0.0963445, \\
& C_4=-1.77612, C_5=-0.89669, C_6=0.72256, C_7=-0.621523.
\end{aligned}
\tag{4.154}
$$

We then apply the routine `MBexpand` to expand every integral in the $\sum_{k=1}^{11} IntF_k$ to a series in ϵ up to ϵ^0 order. The coefficients of $1/\epsilon^2$ and $1/\epsilon$ are simple to be obtained via the method in section 4.4.2:

$$
C'_{8,f,-2} = -\frac{\ln(1-u)}{(1-u)u^2},
\tag{4.155}
$$

$$
C'_{8,f,-1} = -\frac{6(u+7)\mathrm{Li}_2(u)+\pi^2 u+3\ln(1-u)\left((u+7)\ln(1-u)-8\tanh^{-1}(1-2u)\right)}{6(u-1)u^2},
\tag{4.156}
$$

Then we apply the routine `MBasymptotics` to the coefficient of ϵ^0 in order to expand it to a expansion in u. During the evaluation of the coefficient of u, we will confront some 3-D Mellin–Barnes integral like:

$$
\begin{aligned}
MB_4 = \quad & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}}dz_1dz_2dz_3\frac{u^5(-1)^{-z_2}\Gamma(-z_1)\Gamma(z_1+1)\Gamma(z_2)\Gamma(-z_3)\Gamma(z_3+2)\Gamma(-z_1-z_2-1)}{5040\Gamma(-z_1-z_2+z_3+1)} \\
& \times\Gamma(z_1-z_3+2)\Gamma(-z_1+z_3+6)\Gamma(z_3-z_2) \\
& \text{where } C_1=-0.677624, C_2=-0.894274, C_3=-0.408594.
\end{aligned}
\tag{4.157}
$$

It seems there is no way to reduce the dimension of this integral by Barnes' lemma. We can multiply the integrand of MB_4 by v^{-z_2}. The integrand MB_4 becomes MB'_4,

$$
\begin{aligned}
& MB'_4 = \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}}dz_1dz_2dz_3v^{-z_2}\frac{u^5(-1)^{-z_2}\Gamma(-z_1)\Gamma(z_1+1)\Gamma(z_2)\Gamma(-z_3)\Gamma(z_3+2)\Gamma(-z_1-z_2-1)}{5040\Gamma(-z_1-z_2+z_3+1)} \\
& \times\Gamma(z_1-z_3+2)\Gamma(-z_1+z_3+6)\Gamma(z_3-z_2) \\
& \text{where } C_1=-0.677624, C_2=-0.894274, C_3=-0.408594.
\end{aligned}
\tag{4.158}
$$

We reduce the dimension of the integral MB'_4 by calculating the sum of the residues of variable z_2 on the left side of contour. The sum of the residues is a two-dimensional integral whose integrand is a function of v; we then expand the integrand to a series in v near $v=1$ The integral MB_4 in eq. (4.157) is reduced to MB''_4 in eq. (4.159):

$$
\begin{aligned}
& MB''_4 = \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}}dz_1dz_3\frac{u^5\Gamma(-z_1)^2\Gamma(z_1+1)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(z_3+2)\Gamma(z_1-z_3+2)\Gamma(-z_1+z_3+6)}{5040(z_1+1)z_3\Gamma(-z_1+z_3+2)} \\
& \times\left(\frac{z_1\Gamma(-z_1+z_3+1)}{\Gamma(1-z_1)\Gamma(z_3+2)}-\frac{z_3\Gamma(-z_1+z_3+1)}{\Gamma(1-z_1)\Gamma(z_3+2)}-\frac{\Gamma(-z_1+z_3+1)}{\Gamma(1-z_1)\Gamma(z_3+2)}-z_1+z_3+1\right) \\
& \text{where } C_1=-0.677624, C_3=-0.408594.
\end{aligned}
\tag{4.159}
$$

Again use,

$$\text{Polynomial}(z)\Gamma(c \pm z) = \sum_k \Gamma(k + c \pm z), \tag{4.160}$$

and then we can change variables, use the relations between Γ functions and apply Barnes lemmas on MB_4'' to reduce the dimension from 2 to 1. So finally, we can evaluate the integral MB_4.

During the evaluation of the integral $C_{8,f}$, we notice again that our ansatz of the transcendental function is not large enough for the coefficient of ϵ^0. We can calculate the series coefficients of u up to a high order of the coefficient of $\ln u$ in $C_{8,f,0,\ln u}$. But I still did not find the compact analytical form of the coefficient of $\ln u$. We will extract the part of the series expansion which can not be fit into base function,

$$\begin{aligned} MB_5 = \frac{1}{2\pi i}\int_{\mathcal{C}} dz_3 \sum_{m=2}^{k+2}\sum_{n=1}^{m-1}\sum_{i=0}^{k-m+2} & \frac{2(-1)^{m-n+1}u^k\Gamma(1-z_3)\Gamma(-z_3)\Gamma(k-m+3)}{\Gamma(-i+k-m+3)\Gamma(m+1)\Gamma(n+1)\Gamma(k-m+3)} \\ & \times\Gamma(n+z_3)\Gamma(-i+k+z_3+2)\ln u, \end{aligned} \tag{4.161}$$

where $C_3 = -0.475$

We try to apply Barnes' lemmas to the sum of integrals, MB_5,

$$MB_5 = \sum_{m=2}^{k+2}\sum_{n=1}^{m-1}\sum_{i=0}^{k-m+2} \frac{2u^k(-1)^{m-n-1}\Gamma(n)\Gamma(n+1)\ln u\Gamma(-i+k+2)\Gamma(-i+k+3)\Gamma(k-m+3)}{m!n!(k-m+2)!\Gamma(-i+k-m+3)\Gamma(-i+k+n+3)} \tag{4.162}$$

For obtaining the analytical compact result, we change summation variables in MB_5, then:

$$MB_5 = \sum_{n=1}^{\infty}\sum_{i=n-1}^{\infty}\sum_{m=n+1}^{i+2}\sum_{k=i}^{\infty} -\frac{2\Gamma(i+2)\Gamma(i+3)u^k(-1)^{m-n}\Gamma(n)\ln u}{\Gamma(m+1)\Gamma(i-m+3)\Gamma(i+n+3)} \tag{4.163}$$

So that the sum over k and m can be performed,

$$MB_5 = \sum_{n=1}^{\infty}\left(\sum_{n=1}^{i+1} -\frac{2u^i\Gamma(i+2)^2\ln u}{n(u-1)\Gamma(i-n+2)\Gamma(i+n+3)}\right) \tag{4.164}$$

The sums over i and n are not easily performed. So we use this identity

$$\frac{\Gamma(i+2)\Gamma(n+1)}{\Gamma(i+n+3)} \to \int_0^1 x^{i+1}(1-x)^n\,dx \tag{4.165}$$

Then we change the order of integration and summations. The summation over i and n can be also performed. It yields:

$$MB_5 = \int_0^1 dx\,\frac{\ln\left(\frac{-ux^2+2ux-1}{ux-1}\right)}{u(1-ux)(1-u)} \tag{4.166}$$

This integration can finally be performed in *Mathematica*. After some simplification, the coefficient of $\ln u$ in $C'_{8,f,\ln u}$ is:

$$\frac{2\text{Li}_2\left(\left(-i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u}\right)+2\text{Li}_2\left(\left(i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u}\right)+\text{Li}_2(u)+2\ln^2\left(\sqrt{1-u}\right)-4\sin^{-1}\left(\sqrt{u}\right)^2}{u^2(1-u)} \tag{4.167}$$

We notice that the arguments of the transcendental functions can be both real and complex:

$$u,\ 1-u,\ \sqrt{u},\ 1-\sqrt{u}\cdots \tag{4.168}$$

or

$$\left(i\sqrt{1-u}+\sqrt{u}\right),\ \left(-i\sqrt{1-u}+\sqrt{u}\right),\ \left(-i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u},\ \left(i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u}\cdots. \tag{4.169}$$

But even with these new arguments, I still fail to find the compact form of $C_{8,f,0}$. I show our results here:

$$\begin{aligned}
C'_{8,f,0,\ln u} = \quad & \ln u \frac{2u\mathrm{Li}_2(u) + 14\mathrm{Li}_2(u) + \frac{\pi^2 u}{3} + u\ln^2(1-u) + 3\ln^2(1-u)}{(u-1)u^2} \\
& + \ln u \frac{2\mathrm{Li}_2\left(\left(-i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u}\right) + 2\mathrm{Li}_2\left(\left(i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u}\right)}{(1-u)u^2} \\
& + \ln u \frac{\mathrm{Li}_2(u) + 2\ln^2\left(\sqrt{1-u}\right) - 4\sin^{-1}\left(\sqrt{u}\right)^2}{(1-u)u^2} \\
& - \frac{2\ln(1-u)\ln^2 u}{(1-u)u^2},
\end{aligned} \tag{4.170}$$

$$\begin{aligned}
C'_{8,f,0,\not\ln u} = \quad & \frac{2\pi^2\ln(1-u)}{(1-u)u^2} + \frac{13\zeta_3}{(1-u)u} \\
& + \frac{49}{u} + \frac{5759}{72} + \frac{68941u}{675} + \frac{253947397u^2}{2116800} + \frac{306543431u^3}{2268000} + \frac{107002425869u^4}{720373500} \\
& + \frac{22341346725889u^5}{139134996000} + \frac{18714980259530311u^6}{109081836864000} + \frac{85933268242990271 21u^7}{47286976280544000} \\
& + \frac{32638299124989380397 59u^8}{17070598437276384000} + \frac{37570986230088650735479u^9}{187776582810040224000} \\
& + \frac{25885497512443101878488 67u^{10}}{12416726538313909812000} + \frac{13974198097028900828718 41687u^{11}}{64566977999232331022400 00} \\
& + \frac{14463392180416116956153 76049u^{12}}{64566977999232331022400 00} + \frac{11079538477124079473059 48655893u^{13}}{47912495732959756226328 00000} \\
& + \frac{29829874753576916897595744 7129189481u^{14}}{12523943084629816599512 32857600000} \\
& + \frac{26064964194608270879447602 22183549801u^{15}}{10645351621935344109585479289600000} \\
& + \frac{80244836503654683031220710 36791475423u^{16}}{31936054865806032328756437868800000} \\
& + \frac{21387008662638442782177831924073067 1756673u^{17}}{83068872311448070690328370540535 6800000} \\
& + \frac{21884404011922853103799840041315950 4200851u^{18}}{83068872311448070690328370540535 6800000} \\
& + \frac{37597292878696041057495591081495048 9364227181u^{19}}{13963877435554420683044199087864047 80800000} \\
& + \frac{70969044699602613189733460669298875 9706198559369u^{20}}{25819209378340123842948724113460624396992 00000} \\
& + \frac{16647088528990414339419148739208259664960673074267u^{21}}{59384181570182284838782065460959436113081600000} \\
& + \frac{46837132242598809415319331092509265284749753835383 31u^{22}}{16397457136066583401108697825407424296724656800000} \\
& + \frac{46039991009062747181331588155120291802310439837189261u^{23}}{15832027579650494318311846176255444148561737 6000000} \\
& + \frac{30461338936432885399434884577099056126934654558744 7u^{24}}{10296676380575562575264488430397126716938560 00000} \\
& + \frac{11636094666165731787018444537318642136357443452924332136971u^{25}}{38690783959977267529920039040918342073579631199008000000} + \cdots,
\end{aligned} \tag{4.171}$$

We can also replace $u \to 1-u'$ in eq. (4.143) to repeat all these procedures and obtains (here is the part not free of $\ln u'$)

$$
\begin{aligned}
C'_{8,f,0,\ln u'} = \;& \frac{\left(\frac{5u'}{6}-2\right)\ln^3(u')}{(1-u')^2u'} + \frac{\left(9-\frac{3u'}{2}\right)\ln(1-u')\ln^2(u')}{(1-u')^2u'} + \frac{\left(\pi^2\left(\frac{3u'}{2}-3\right)\right)\ln(u')}{(1-u')^2u'} \\
& -3\ln u' - \frac{199}{12}u'\ln u' - \frac{613}{15}u'^2\ln u' - \frac{378619u'^3\ln u'}{5040} - \frac{82979}{700}u'^4\ln u' \\
& - \frac{141694673u'^5\ln u'}{831600} - \frac{2175913073u'^6\ln u'}{9459450} - \frac{92178179u'^7\ln u'}{310464} \\
& - \frac{43337182367u'^8\ln u'}{116953200} - \frac{18879389019341u'^9\ln u'}{41902660800} - \frac{865588495026817u'^{10}\ln u'}{1613252440800} \\
& - \frac{1553992667930937 1u'^{11}\ln u'}{24736537425600} - \frac{34984572077231 6029u'^{12}\ln u'}{482362479799200} \\
& - \frac{72570822935524493u'^{13}\ln u'}{87702269054400} - \frac{860077529412359197u'^{14}\ln u'}{920296836459000} \\
& - \frac{4653819554303528436 53u'^{15}\ln u'}{444762942912288000} - \frac{1224439419747084757 2917u'^{16}\ln u'}{10531351112530248000} \\
& - \frac{1135247785436065492608479u'^{17}\ln u'}{884633493452540832000} - \frac{8756989761133180608012 2851u'^{18}\ln u'}{62189734589713620489600} \\
& - \frac{1874164907882251493068 7881u'^{19}\ln u'}{12194065605826200096000} + \cdots
\end{aligned}
\tag{4.172}
$$

The part which is free of $\ln u'$ is:

$$
\begin{aligned}
C'_{8,f,0,\not\ln u'} = \;& \frac{(37-9u')\zeta_3}{(1-u')^2u'} + \frac{\pi^2\left(6-\frac{5u'}{6}\right)\ln(1-u')}{(1-u')^2} \\
& +29 + \frac{5275u'}{72} + \frac{711997u'^2}{5400} + \frac{142680259u'^3}{705600} + \frac{8966052119u'^4}{31752000} \\
& + \frac{854983022489u'^5}{2305195200} + \frac{6363012934411109u'^6}{13635229608000} + \frac{1590973423946021u'^7}{2796970176000} \\
& + \frac{5817980335299046147u'^8}{8597632051008000} + \frac{4279752042402833 14847u'^9}{541923759913536000} \\
& + \frac{3408053799567798603 44087u'^{10}}{375553165620080448000} + \cdots
\end{aligned}
\tag{4.173}
$$

4.4.10 Calculations of the Integral $C'_{8,g}$

The integral $C'_{8,g}$ is given by

$$
C'_{8,g} = 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}s_{134}s_{24}s_{124}} \tag{4.174}
$$

The non-rigorous shortcut calculation (in section 4.4.1) can be applied to this integral. The MB representation of the integral $C'_{8,g}$ is

$$
\begin{aligned}
IntG_0 = \;& \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=1}^{4}dz_j \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_1)\Gamma(-z_4)\Gamma(-z_2)\Gamma(z_2+1)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-\epsilon-z_3)\Gamma(-3\epsilon-z_3-z_1+1)} \\
& \times\frac{\Gamma(-\epsilon-z_3-1)\Gamma(-2\epsilon+z_1+1)\Gamma(-2\epsilon+z_2+2)u^{-z_4+z_2-2}\Gamma(-\epsilon-z_3-z_2-1)}{\Gamma(-3\epsilon-z_3-z_2+1)\Gamma(\epsilon-z_3-z_1-z_2)} \\
& \times(-1)^{-\epsilon+z_3+z_1-z_4+z_2}\Gamma(-\epsilon-z_3-z_1-z_2-1)\Gamma(\epsilon-z_3-z_1+z_4-z_2))
\end{aligned}
\tag{4.175}
$$

We use the routine `MBresolve` to $IntG_0$ in order to resolve the singularities in ϵ. Then we obtain a sum of integrals, $\sum_{k=1}^{2} IntG_k$,

$$\begin{aligned} IntG_1 = \quad & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} \prod_{j=1,j\neq 3}^{4} dz_j - \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)^2\Gamma(\epsilon+1)\Gamma(-z_1)\Gamma(-z_4)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-2\epsilon-z_1+2)} \\ & \frac{\Gamma(-2\epsilon+z_1+1)\Gamma(-2\epsilon+z_2+2)u^{-z_4+z_2-2}\Gamma(-z_1-z_2)(-1)^{-2\epsilon+z_1-z_4+z_2}}{\Gamma(-2\epsilon-z_2+2)\Gamma(2\epsilon-z_1-z_2+1)} \\ & \times\Gamma(-z_2)^2\Gamma(z_2+1)\Gamma(2\epsilon-z_1+z_4-z_2+1), \\ & \text{where } C_1=-0.55884, C_2=-0.217643, C_4=-1.49097. \end{aligned} \tag{4.176}$$

and

$$\begin{aligned} IntG_2 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}} \prod_{j=1}^{4} dz_j \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_1)\Gamma(-z_4)\Gamma(-z_2)\Gamma(z_2+1)}{(\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(-\epsilon-z_3)\Gamma(-3\epsilon-z_3-z_1+1)} \\ & \times\frac{\Gamma(-\epsilon-z_3-1)\Gamma(-2\epsilon+z_1+1)\Gamma(-2\epsilon+z_2+2)u^{-z_4+z_2-2}\Gamma(-\epsilon-z_3-z_2-1)}{\Gamma(-3\epsilon-z_3-z_2+1)\Gamma(\epsilon-z_3-z_1-z_2)} \\ & \times\Gamma(-\epsilon-z_3-z_1-z_2-1)\Gamma(\epsilon-z_3-z_1+z_4-z_2)(-1)^{-\epsilon+z_3+z_1-z_4+z_2} \\ & \text{where } C_1=-0.887538, C_2=-0.417936, C_3=-0.927911, C_4=-0.954985. \end{aligned} \tag{4.177}$$

We then apply the routine `MBexpand` to expand every integral in the $\sum_{k=1}^{2} IntG_k$ to a series in ϵ up to ϵ^0 order. The coefficients of $1/\epsilon^2$ and of $1/\epsilon$ are always straight forwardly obtained via the method introduced in section 4.4.2,

$$C'_{8,g,-2} = 0, \tag{4.178}$$

$$C'_{8,g,-1} = \frac{1}{3(u-1)u^2}(6u^2\mathrm{Li}_2(u) + 3u^2\ln^2(1-u) + \left(\pi^2 u + 6\right)u + 6(u+1)\ln(1-u)) \tag{4.179}$$

Then we apply the routine `MBasymptotics` to the coefficient of ϵ^0 in order to expand it to a expansion in u. We compute the part is not free of $\ln u$ in $C_{8,g,0}$:

$$C'_{8,g,0,\ln u} = -\frac{\ln u\left(2u^2\mathrm{Li}_2(u) + u^2\ln^2(1-u) + 2u + 2(u+1)\ln(1-u)\right)}{(u-1)u^2} + \frac{\pi^2\ln u}{3(1-u)} \tag{4.180}$$

But at the end, even with enlarged ansatz for fitting the series, I still fail to compute the compact form of

the function $C_{8,g,0,\not{n}\,u}$. Here is the series expansion of $C_{8,g,0,\not{n}\,u}$:

$$
\begin{aligned}
C'_{8,g,0,\not{n}\,u} \;=\;& -\frac{22\zeta_3}{1-u}+\frac{\pi^2\ln(1-u)}{3(1-u)} \\
& \frac{49}{2}+\frac{229u}{9}+\frac{197u^2}{8}+\frac{2509u^3}{108}+\frac{466061u^4}{21600}+\frac{104722441u^5}{5292000}+\frac{5271577u^6}{294000}+\frac{254605363u^7}{15876000} \\
& +\frac{598184233u^8}{42336000}+\frac{1690236021371u^9}{138311712000}+\frac{285435379279u^{10}}{27662342400}+\frac{433459506471197u^{11}}{51424294521600} \\
& +\frac{67423591066453u^{12}}{10284858904320}+\frac{3142343537836277u^{13}}{668515828780800}+\frac{1915388192774831u^{14}}{668515828780800} \\
& +\frac{10147696126291005 13u^{15}}{966005372588256000}-\frac{5741964880768600291u^{16}}{7728042980706048000}-\frac{17040663566544767858141u^{17}}{677528568179900 2368000} \\
& -\frac{67439289583171411560569u^{18}}{15808999924197672192000}-\frac{18008539145601122672988 11u^{19}}{3003709985597557716480 00} \\
& -\frac{33058118478920235041 7773u^{20}}{42910142651393681664000}-\frac{10446488123000653303242109933u^{21}}{11122738076667755622412544000} \\
& -\frac{12301459962808130212621333933u^{22}}{11122738076667755622412544000}-\frac{32509039508910586597262709 8027u^{23}}{25582297576335839315488512000} \\
& -\frac{18337532750677211365110571501 491u^{24}}{12791148788167919657744256000 0}-\frac{15297211238615748044817839 4906857u^{25}}{95933615911259397433081920000 00} \\
& -\frac{62309267489414545146039661548 91u^{26}}{35530968856021999049289600000 0}-\frac{41627493646861547810314914972269 71499u^{27}}{21783646164969671375129911574400 0000} \\
& -\frac{45015440888152751920717573709 18538299u^{28}}{21783646164969671375129911574400 0000} \\
& -\frac{13479097148521482429454955410976 1819685231u^{29}}{60708843497153977155349550566695360 00000} \\
& -\frac{14402657074170714528017719046410 5867681451u^{30}}{60708843497153977155349550566695360 00000} \\
& -\frac{47480487229237626910936215373602 71082670081u^{31}}{18819741484117732918158360675675561 6000000} \\
& -\frac{40226173633002339520845754541576226080445173u^{32}}{15055793187294186334526688540540449280 00000} \\
& -\frac{42444536237016947318737126297419957904998773u^{33}}{15055793187294186334526688540540449280 00000} \\
& -\frac{44640055391975908485830483998033289951425973u^{34}}{15055793187294186334526688540540449280 00000}+\cdots
\end{aligned}
\tag{4.181}
$$

As usual, u can be replaced by $1-u'$ in eq. (4.175). And the series expansion in u' near $u'=1$ can be also calculated. Here is the series expansion of $C'_{8,g,0}$:

$$
\begin{aligned}
C'_{8,g,0} =\;& -\frac{\pi^2\left(7u'+5(u'-1)^2\ln(1-u')-12\right)}{3(u'-1)^2u'} \\
& -\frac{14\zeta_3+8}{u'}-56-\frac{249u'}{4}-\frac{4013u'^2}{54}-\frac{25349u'^3}{288}-\frac{922199u'^4}{9000}-\frac{5867u'^5}{50} \\
& -\frac{16359479u'^6}{123480}-\frac{417220813u'^7}{2822400}-\frac{4666707299u'^8}{28576800}-\frac{568018103u'^9}{3175200} \\
& -\frac{14950462061u'^{10}}{76839840}-\frac{16160296469u'^{11}}{76839840}-\frac{7634410393343u'^{12}}{33763425696}-\frac{21995222370059u'^{13}}{90901530720} \\
& -\frac{27904887668159u'^{14}}{108216108000}-\frac{1137751607418677u'^{15}}{4155498547200}-\frac{73946089043283439u'^{16}}{255199554529920} \\
& -\frac{22949180922857129u'^{17}}{75058692508800}-\frac{16565442882693742969 9u'^{18}}{514827571917859200} \\
& -\frac{18306525524207215993u'^{19}}{54192375991353600}+\cdots
\end{aligned}
\tag{4.182}
$$

4.4.11 Calculations of the Integral $C'_{8,h}$

The integral $C'_{8,h}$ is given by

$$C'_{8,h} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{23} s_{123} s_{14} s_{124}} \tag{4.183}$$

The Mellin–Barnes representation of the integral $C'_{8,h}$ is

$$\begin{aligned} IntH_0 = \; & \frac{1}{(2\pi i)^6} \int \cdots \int_{\mathcal{C}} \prod_{j=1}^{6} dz_j \frac{\Gamma(1-\epsilon)\Gamma(z_6)\Gamma(-z_1)\Gamma(-z_5)\Gamma(-z_3)\Gamma(-z_4)}{\Gamma(1-2\epsilon)^2\Gamma(-\epsilon - z_6 + 1)\Gamma(-3\epsilon - z_2 + 1)\Gamma(\epsilon - z_1 - z_2)} \\ & \times \frac{u^{-z_5-z_3-3}\Gamma(-z_6 - z_2)\Gamma(-\epsilon - z_1 - z_2 - 1)\Gamma(-\epsilon - z_2 - z_3)\Gamma(-\epsilon + z_2 + z_3 + 1)}{\Gamma(-3\epsilon - z_1 - z_2 - z_3)} \\ & \times (-1)^{-\epsilon - z_6 + z_1 - z_5 + z_2} \Gamma(\epsilon - z_1 + z_5 - z_2)\Gamma(-\epsilon - z_2 - z_3 - z_4 - 1)\Gamma(-z_6 + z_2 + z_3 + z_4 + 1)) \\ & \times \Gamma(-2\epsilon + z_1 + 1)\Gamma(-2\epsilon - z_3 + 1)\Gamma(-2\epsilon + z_3 + z_4 + 2). \end{aligned} \tag{4.184}$$

We again apply the routine `MBresolve` to the representation $IntH_0$ in order to resolve the singularities in ϵ, then we obtain a sum of three integrals, $\sum_{k=1}^{3} IntH_k$. The integrals $IntH_k$ are shown below in ea. (4.185)-(4.187),

$$\begin{aligned} IntH_1 = \; & \frac{1}{(2\pi i)^4} \iiiint_{\mathcal{C}} \prod_{j=1, j\neq 2}^{5} dz_j \frac{-\Gamma(-\epsilon)\Gamma(-z_1)\Gamma(-z_5)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-2\epsilon + z_1 + 1)}{\Gamma(1-2\epsilon)^2\Gamma(-3\epsilon - z_1 + z_4 + 1)\Gamma(-3\epsilon + z_3 + z_4 + 2)} \\ & \times \frac{\Gamma(-\epsilon - z_4)\Gamma(-\epsilon + z_4 + 1)u^{-z_5-z_3-3}\Gamma(z_3 + z_4 + 1)\Gamma(-2\epsilon + z_3 + z_4 + 2)(-1)^{-\epsilon + z_1 - z_5 - z_3 - z_4}}{\Gamma(\epsilon - z_1 + z_3 + z_4 + 1)} \\ & \times \Gamma(-\epsilon - z_1 + z_3 + z_4)\Gamma(\epsilon - z_1 + z_5 + z_3 + z_4 + 1)\Gamma(-2\epsilon - z_3 + 1), \\ & \text{where } C_1 = -0.536594, C_3 = -0.263892, C_4 = -0.213652, C_5 = -0.3408, \end{aligned} \tag{4.185}$$

$$\begin{aligned} IntH_2 = \; & \frac{1}{(2\pi i)^5} \int \cdots \int_{\mathcal{C}} \prod_{j=1}^{5} dz_j \frac{(\Gamma(-z_1)\Gamma(-z_5)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-2\epsilon + z_1 + 1)}{\Gamma(1-2\epsilon)^2\Gamma(-3\epsilon - z_2 + 1)\Gamma(\epsilon - z_1 - z_2)\Gamma(-3\epsilon - z_1 - z_2 - z_3)} \\ & \times (-1)^{-\epsilon + z_1 - z_5 + z_2} \Gamma(-\epsilon - z_1 - z_2 - 1)\Gamma(-\epsilon - z_2 - z_3)\Gamma(-\epsilon + z_2 + z_3 + 1) \\ & \times \Gamma(z_2 + z_3 + z_4 + 1)\Gamma(\epsilon - z_1 + z_5 - z_2)\Gamma(-\epsilon - z_2 - z_3 - z_4 - 1)\Gamma(\epsilon - z_1 - z_2) \\ & \times \Gamma(-2\epsilon - z_3 + 1)u^{-z_5-z_3-3}\Gamma(-3\epsilon - z_1 - z_2 - z_3)\Gamma(-2\epsilon + z_3 + z_4 + 2), \\ & \text{where } C_1 = -0.996162, C_5 = -0.220155, C_2 = -0.111428, C_3 = -0.581723, C_4 = -1.14501 \end{aligned} \tag{4.186}$$

and

$$
\begin{aligned}
IntH_3 = \quad & \frac{1}{(2\pi i)^4}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^{6}dz_j\frac{\Gamma(1-\epsilon)\Gamma(z_6)\Gamma(-z_1)\Gamma(-z_5)\Gamma(-z_3)\Gamma(-z_4)}{\Gamma(1-2\epsilon)^2\Gamma(-\epsilon-z_6+1)\Gamma(-3\epsilon-z_2+1)\Gamma(\epsilon-z_1-z_2)} \\
& \times\frac{u^{-z_5-z_3-3}\Gamma(-z_6-z_2)\Gamma(-\epsilon-z_1-z_2-1)\Gamma(-\epsilon-z_2-z_3)\Gamma(-\epsilon+z_2+z_3+1)}{\Gamma(-3\epsilon-z_1-z_2-z_3)} \\
& \times\Gamma(-2\epsilon+z_3+z_4+2)(-1)^{-\epsilon-z_6+z_1-z_5+z_2}\Gamma(\epsilon-z_1+z_5-z_2) \\
& \times\Gamma(-\epsilon-z_2-z_3-z_4-1)\Gamma(-z_6+z_2+z_3+z_4+1)\Gamma(-2\epsilon+z_1+1)\Gamma(-2\epsilon-z_3+1), \\
& \text{where } C_1=-0.795606, C_2=-0.467461, C_3=-0.410091, C_4=-0.259369, \\
& C_5=-0.97089, C_6=-0.524884
\end{aligned}
\tag{4.187}
$$

Then we apply the routine `MBexpand` to each $IntH_k$ in order to expand the integrands to a series in ϵ up to ϵ^0 order. The coefficients of $1/\epsilon^2$ and $1/\epsilon$ are still simple to be obtained using the method introduced in section 4.4.2. The coefficients of $1/\epsilon^2$ and $1/\epsilon$ are,

$$C'_{8,h,-2}=0, \tag{4.188}$$

and

$$C'_{8,h,-1}=\frac{2\mathrm{Li}_2(u)+\ln(1-u)\ln u}{2(u-1)u^2}. \tag{4.189}$$

Then we focus on the coefficient of ϵ^0. We use the routine `MBasymptotics` to expand it to a series in u near $u=0$. Then we evaluate the series coefficients. During this calculation, we will confront some two-dimensional Mellin–Barnes integrals:

$$
\begin{aligned}
MB_6 = & \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}}dz_3dz_6\frac{(-1)^{-z_6}\Gamma(-z_6-6)\Gamma(-z_6)\Gamma(z_6)\Gamma(1-z_3)\Gamma(-z_3)}{5040\Gamma(1-z_6)\Gamma(-z_6+z_3+2)} \\
& \times\Gamma(z_3+2)\Gamma(z_3+7)\Gamma(-z_6+z_3+7) \text{ where } C_3=-0.410091, C_6=-0.524884
\end{aligned}
\tag{4.190}
$$

Let G be the integrand. we can reduce the dimension of this integral MB_6 to an one-dimensional integral by calculating the sum of residues of z_6 on the left side of contour.

$$
\begin{aligned}
& MB'_6=\frac{1}{2\pi i}\int_{\mathcal{C}}dz_3\sum\mathrm{Residue}_{z_6=-1-k}G\times v^{-z_6}, \\
& \text{where } C_3=-0.410091
\end{aligned}
\tag{4.191}
$$

We need to expand the integrand $\sum\mathrm{Residue}_{z_6=-1-k}G\times v^{-z_6}$ to a series in v near $v=1$. But the sum of the residue involve higher order hypergeometric function:

$$
\begin{aligned}
\sum\mathrm{Residue}_{z_6=-1-k}G\times v^{-z_6} = \quad & \frac{\Gamma(1-z_3)\Gamma(-z_3)\Gamma(z_3+2)\Gamma(z_3+7)\Gamma(z_3+14)}{177811200\Gamma(z_3+9)} \\
& \times{}_4F_3(1,1,7,z_3+14;8,8,z_3+9;v)
\end{aligned}
\tag{4.192}
$$

which seems to be difficult to expand as a series expansion of v in *Mathematica* due to the hypergeometric function ${}_4F_3$. So we will simplify the initial integral MB_6's integrand as follows:

$$\frac{\Gamma(-z_6+z_3+7)}{\Gamma(-z_6+z_3+2)}\to\sum_{ij}a_{ij}z_6^iz_3^j \tag{4.193}$$

allows us to cancel a factor in the denominator of MB_6's integrand and then use the relation,

$$z_6^i\Gamma(a+z_6)=\sum_j b_j\Gamma(a+j+z_6). \tag{4.194}$$

Finally, we can add the sum of the residues of z_6 without involving higher-order hypergeometric functions but at most involving ${}_2F_1$s. Then we can expand the $\sum \text{Residue}_{z_6=-1-k}G\times v^{-z_6}$ at $v=1$. Following this procedures, all the higher dimensional Mellin–Barnes integrals can be reduced to one-dimensional ones.

I show the result of the integral $C_{8,h}$ here:

$$C'_{8,h,0,\ln u}=\frac{\ln u\left(\left(u+\frac{7}{2}\right)\text{Li}_2(u)+\left(\frac{u}{2}+\frac{1}{4}\right)\ln^2(1-u)\right)}{(1-u)u^2}-\frac{3\ln(1-u)\ln^2 u}{4(1-u)u^2}+\frac{\frac{1}{6}\pi^2\ln u}{(1-u)u}, \tag{4.195}$$

$$\begin{aligned}C'_{8,h,0,\ln u}&=\frac{-9}{u}-\frac{119}{8}-\frac{437}{24}u-\frac{11831}{576}u^2-\frac{267989}{12000}u^3\\&-\frac{5138111}{216000}u^4-\frac{68631869}{2744000}u^5-\frac{5150106413}{197568000}u^6\\&-\frac{750017983}{27783000}u^7-\frac{74209121117}{2667168000}u^8-\frac{101422904493181}{3550000608000}u^9\\&-\frac{207673649064787}{7100001216000}u^{10}-\frac{465995053348439339}{15598702671552000}u^{11}\\&-\frac{67855921985658227}{2228386095936000}u^{12}-\frac{161117282726954011}{5199567557184000}u^{13}\\&-\frac{3929276705428702609}{124789621372416000}u^{14}-\frac{4898091019793567222063}{153272852450669952000}u^{15}\\&\frac{7448757830666749178737}{229909278676004928000}u^{16}-\frac{8624915908135197833588273}{26282462373978630019200 0}u^{17}+\cdots\\&+\frac{\pi^2\left(-\frac{u}{6}-\frac{1}{4}\right)\ln(1-u)}{(1-u)u^2}+\frac{-3\zeta_3}{(1-u)u}.\end{aligned} \tag{4.196}$$

I can also calculate the series expansion of $u'=1-u$.

For the part which are not free of $\ln u'$

$$C'_{8,h,0,\ln u'}=\frac{\ln u'\left(\left(\frac{3}{2}-u'\right)\text{Li}_2(u')+\left(\frac{9}{4}-\frac{u'}{2}\right)\ln^2(1-u')\right)}{(1-u')^2u'}+\frac{\pi^2\left(\frac{1}{3}-\frac{u'}{6}\right)\ln u'}{(1-u')^2u'}-\frac{3\ln(1-u')\ln^2 u'}{4(1-u')^2u'} \tag{4.197}$$

For $C'_{8,h,0,\ln u'}$, we can reduce all high dimensional Mellin–Barnes integrals to one-dimensional ones except:

$$\begin{aligned}&MB_7=\frac{1}{(2\pi i)^2}\iint_{\mathcal{C}}dz_2dz_3\Gamma(-z_2)\Gamma(z_2+1)^2\Gamma(-z_3)\psi^{(0)}(z_3+2)\Gamma(z_2-z_3+1)\Gamma(z_3-z_2)^2,\\&\text{where } C_2=-\tfrac{1}{3}, C_3=-\tfrac{1}{5}.\end{aligned} \tag{4.198}$$

This integral was evaluated numerically in *Mathematica* with `NIntegrate` routine. But no matter how high we set the option: WorkingPrecision, PrecisionGoal and AccuracyGoal in *Mathematica*, the order of error estimated is still 10^{-10}. Fortunately, it is enough to use the routine `PSLQ` to reproduce the analytic result.

The value of the integral MB_7 (4.198) is

$$\frac{1}{6}\left(-27\zeta_3-\gamma_E\pi^2+6\pi^2\ln(2)\right) \tag{4.199}$$

So at last, we can generate the series of u'

$$
\begin{aligned}
C'_{8,h,0,1,n\,u'} = \quad & \frac{\left(-2u'^3+4u'^2+u'-9\right)\zeta_3}{(1-u')^2} \\
& -\frac{19}{2}-\frac{57u'}{4}-\frac{445u'^2}{18}-\frac{3559u'^3}{96}-\frac{3673081u'^4}{72000}-\frac{4775609u'^5}{72000} \\
& \frac{7\pi^2}{4}+\frac{61\pi^2u'}{24}+\frac{77\pi^2u'^2}{18}+\frac{299\pi^2u'^3}{48}+\frac{376\pi^2u'^4}{45}+\frac{2551\pi^2u'^5}{240}+\cdots
\end{aligned}
\tag{4.200}
$$

4.4.12 Calculations of the integral $C'_{8,i}$

The integral $C'_{8,i}$ is given by

$$
C'_{8,i} = 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}s_{123}s_{24}s_{234}} \tag{4.201}
$$

This integral is the most complicated integral among the basic integrals. In our parametrization (4.8), the factors s_{13}, s_{123} and s_{234} in the denominator of the integrand are not free of s_{13}. But $s_{123}-s_{13}$ and $s_{123}+s_{234}$ are free of s_{13}. We use the symmetry in the four-particle phase space to rewrite the integrand in order to perform the integration over s_{13} simple. So We rewrite the integrand of $C_{8,i}$ in the following way, so that the integration over the variable s_{13} can be directly performed:

$$
\begin{aligned}
C'_{8,i} &= 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}s_{123}s_{24}s_{234}} \\
&= 8f_{dim}\int dPS_4 \frac{2E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}s_{123}s_{24}(s_{123}+s_{234})} \\
&= 8f_{dim}\left(\int dPS_4 \frac{2E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}(s_{123}-s_{13})s_{24}(s_{123}+s_{234})} - \frac{2E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{123}(s_{123}-s_{13})s_{24}(s_{123}+s_{234})}\right) \\
&= C_{8,k}-C_{8,l}
\end{aligned}
\tag{4.202}
$$

First, we compute the integral $C_{8,k}$ using non-rigorous shortcut calculation (in section 4.4.1):

$$
C'_{8,k} = 8f_{dim}\int dPS_4 \frac{2E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}(s_{123}-s_{13})s_{24}(s_{123}+s_{234})} \tag{4.203}
$$

The Mellin–Barnes representation of $C'_{8,k}$ is

$$
\begin{aligned}
IntK_0 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=2}^{5}dz_j \frac{2\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_4)\Gamma(z_4+1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_3)\Gamma(z_3+1)}{\Gamma(-2\epsilon-1)\Gamma(-\epsilon+z_4+1)\Gamma(-3\epsilon-z_4-z_3)} \\
& \times\frac{(-1)^{-\epsilon+z_4+z_5-z_1}\Gamma(-\epsilon-z_4-z_2-2)\Gamma(-\epsilon+z_4+z_2+1)\Gamma(-\epsilon-z_4-z_5)\Gamma(-z_5)}{\Gamma(\epsilon-z_4-z_5)\Gamma(-3\epsilon-z_4-z_2+z_3-z_5+1)} \\
& \times\Gamma(-2\epsilon-z_3+z_5)u^{-z_1-z_2+z_3-2}\Gamma(-z_1)\Gamma(\epsilon-z_4-z_5+z_1)\Gamma(-2\epsilon-z_2+z_3+1)
\end{aligned}
\tag{4.204}
$$

Using the routine `MBresolve`, we can resolve the singularity in ϵ of the representation $IntK_0$ and obtain

a sum of four integrals, $\sum_{k=1}^{4} IntK_k$, shown in eqs. (4.205)-(4.208):

$$\begin{aligned} IntK_1 = \quad & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} dz_1 dz_3 dz_5 \frac{4e^{-i\pi(z_1-z_5)}\Gamma(-z_1)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(z_3+2)\Gamma(-z_5)u^{-z_1+z_3-1}}{\epsilon\Gamma(1-z_3)\Gamma(z_3-z_5+3)} \\ & \times\Gamma(z_1-z_5+1)\Gamma(z_5-z_3)(2\epsilon\psi^{(0)}(z_1-z_5+1)+2\epsilon\psi^{(0)}(z_3-z_5+3)-2\epsilon\psi^{(0)}(z_5-z_3) \\ & +2\epsilon\psi^{(0)}(1-z_3)-2\epsilon\psi^{(0)}(z_3+2)-2\epsilon\psi^{(0)}(1-z_5)-2i\pi\epsilon+1), \\ & \text{where } C_1=-0.723612, C_3=-0.795342, C_5=-0.00644131, \end{aligned} \tag{4.205}$$

$$\begin{aligned} IntK_2 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}} \prod_{j=1,j\neq 4}^{5} dz_j \frac{2\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)^2\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_5)e^{-i\pi(z_1+z_2-z_5)}}{\epsilon\Gamma(z_2-z_3+1)\Gamma(z_3-z_5+2)} \\ & \times\Gamma(-z_2+z_3+1)\Gamma(z_5-z_3)u^{-z_1-z_2+z_3-2}\Gamma(z_1+z_2-z_5+1)(2\epsilon\psi^{(0)}(z_1+z_2-z_5+1) \\ & +2\epsilon\psi^{(0)}(z_2-z_3+1)-2\epsilon\psi^{(0)}(-z_2+z_3+1)-2\epsilon\psi^{(0)}(z_2-z_5+1)+\epsilon\psi^{(0)}(-z_2) \\ & +\epsilon\psi^{(0)}(z_2+1)+2\epsilon\psi^{(0)}(z_3-z_5+2)-2\epsilon\psi^{(0)}(z_5-z_3)-2i\pi\epsilon+2\gamma\epsilon+2\epsilon+1), \\ & \text{where } C_1=-0.550581, C_2=-0.0566196, C_3=-0.537339, C_5=-0.111268, \end{aligned} \tag{4.206}$$

$$\begin{aligned} IntK_3 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}} \prod_{j=1,j\neq 4}^{5} dz_j - \frac{2\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(z_2+2)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_5)}{\epsilon\Gamma(z_2-z_3+2)\Gamma(z_3-z_5+3)} \\ & \times e^{-i\pi(z_1+z_2-z_5)}\Gamma(-z_2+z_3+1)\Gamma(z_5-z_3)u^{-z_1-z_2+z_3-2}\Gamma(z_1+z_2-z_5+2) \\ & \times(2\epsilon\psi^{(0)}(z_1+z_2-z_5+2)+2\epsilon\psi^{(0)}(z_2-z_3+2)-2\epsilon\psi^{(0)}(-z_2+z_3+1)-2\epsilon\psi^{(0)}(z_2-z_5+2) \\ & +\epsilon\psi^{(0)}(-z_2-1)+\epsilon\psi^{(0)}(z_2+2)+2\epsilon\psi^{(0)}(z_3-z_5+3)-2\epsilon\psi^{(0)}(z_5-z_3)-2i\pi\epsilon+2\gamma\epsilon+1), \\ & \text{where } C_1=-0.183608, C_2=-1.16656, C_3=-0.873415, C_5=-0.207961, \end{aligned} \tag{4.207}$$

and

$$\begin{aligned} IntK_4 = \quad & \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}} \prod_{j=1}^{5} dz_j \frac{-4\Gamma(-z_4)\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_5)}{\Gamma(-z_4-z_3)\Gamma(-z_4-z_2+z_3-z_5+1)} \\ & \times e^{i\pi(z_4-z_1+z_5)}\Gamma(-z_4-z_2-2)\Gamma(z_4+z_2+1)\Gamma(-z_2+z_3+1)\Gamma(z_5-z_3) \\ & \times u^{-z_1-z_2+z_3-2}\Gamma(-z_4+z_1-z_5), \\ & \text{where } C_1=-0.664508, C_2=-0.176261, C_3=-0.706882, C_4=-0.81131, C_5=-0.493648. \end{aligned} \tag{4.208}$$

Here we focus on the coefficient of ϵ^0 obtained by applying the routine `MBexpand` to the sum $\sum_{k=1}^{4} IntK_k$.

We can calculate the series coefficient of u. we show the result of the coefficient of ϵ^0 directly

$$\begin{aligned}
C'_{8,k,0} = \quad & \frac{-\frac{2}{3}\left(-12+\pi^2\right)\ln u+\frac{2}{3}\left(-57+\pi^2+33\zeta_3\right)}{u} \\
& -\frac{89}{9}+\frac{5\ln u}{3}+u\left(-\frac{8807}{720}+\frac{53\ln u}{30}\right)+u^2\left(-\frac{182381}{13230}+\frac{1147\ln u}{630}\right) \\
& +u^3\left(-\frac{189835363}{12700800}+\frac{4673\ln u}{2520}\right)+u^4\left(-\frac{30490461667}{1920996000}+\frac{65053\ln u}{34650}\right) \\
& +u^5\left(-\frac{6482635635643}{389577988800}+\frac{1023889\ln u}{540540}\right)+u^6\left(-\frac{491307008551}{28406728350}+\frac{481027\ln u}{252252}\right) \\
& +u^7\left(-\frac{10728163667601811}{600469540070400}+\frac{9394675\ln u}{4900896}\right) \\
& +u^8\left(-\frac{4480186534338826747}{243865691961091200}+\frac{1008277793\ln u}{523783260}\right) \\
& +u^9\left(-\frac{5100690686464370563}{270961879956768000}+\frac{1124170193\ln u}{581981400}\right) \\
& +u^{10}\left(\frac{28523329189\ln u}{14724129420}-\frac{3032750638501442144947}{157672717946843299200}\right) \\
& +\frac{u^{11}(33402042418087091548 8\ln u+14116206544833208289 50882069)}{172006601396556326400} \\
& +\frac{u^{12}(22844578076908804760544\ln u+24532194474724074670867414 4177)}{11739450545314969276800}+\cdots
\end{aligned} \tag{4.209}$$

As for the integral $C'_{8,l}$ defined in eq. (4.202),

$$C'_{8,l} = 8 f_{dim}\int dPS_4 \frac{2E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{123}(s_{123}-s_{13})s_{24}(s_{123}+s_{234})} \tag{4.210}$$

The Mellin–Barnes representation of $C'_{8,l}$ is:

$$\begin{aligned}
IntL_0 = \quad & \frac{1}{(2\pi i)^8}\int\cdots\int_{\mathcal{C}}\prod_{j=1}^{8}dz_j \frac{2\Gamma(1-\epsilon)\Gamma(z_8)\Gamma(-z_1)\Gamma(-z_4)\Gamma(-z_5)\Gamma(z_5+1)}{\Gamma(-2\epsilon-1)\Gamma(-\epsilon-z_8+1)\Gamma(-2\epsilon-z_6)\Gamma(-3\epsilon-z_2-z_6)} \\
& \times\frac{\Gamma(-z_7)\Gamma(-2\epsilon-z_3+1)u^{-z_1-z_3-2}\Gamma(-z_8-z_2)\Gamma(-\epsilon-z_2-z_7)\Gamma(-2\epsilon-z_6+z_7)}{\Gamma(-3\epsilon-z_2-z_3-z_7+1)\Gamma(\epsilon-z_2-z_7)} \\
& \times\Gamma(-z_3+z_5-z_6)(-1)^{-\epsilon-z_8-z_1+z_2+z_7}\Gamma(\epsilon+z_1-z_2-z_7)\Gamma(-z_6)\Gamma(z_6+1) \\
& \times\Gamma(-\epsilon-z_2-z_3-z_6-2)\Gamma(-\epsilon+z_2+z_3+z_6+1)\Gamma(-2\epsilon+z_3+z_4-z_5) \\
& \times\Gamma(-\epsilon-z_2-z_3-z_4+z_5-z_6)\Gamma(-z_8+z_2+z_3+z_4-z_5+z_6+1)
\end{aligned} \tag{4.211}$$

Applying the routine `MBexpand` to the Mellin–Barnes representation, we obtain a sum of integrals, $\sum_{k=1}^{5} IntL_k$, shown in eqs. (4.212)-(4.216):

$$
\begin{aligned}
IntL_1 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}} dz_1 dz_3 dz_6 dz_7 - \frac{2e^{-i\pi(z_1+z_3+z_6-z_7)}u^{-z_1-z_3-2}\Gamma(-z_1)\Gamma(1-z_3)\Gamma(z_3+1)}{\epsilon\Gamma(z_3+2)\Gamma(z_6-z_7+3)} \\
& \times\Gamma(-z_3-z_6-1)\Gamma(z_6+1)\Gamma(z_3+z_6+2)\Gamma(z_1+z_3+z_6-z_7+2)\Gamma(-z_7)\Gamma(z_7-z_6) \\
& \times(-2\psi^{(0)}(1-z_3)\epsilon - \psi^{(0)}(z_3+1)\epsilon + 2\psi^{(0)}(z_3+2)\epsilon - \psi^{(0)}(-z_3-z_6-1)\epsilon + 2\psi^{(0)}(-z_6)\epsilon \\
& +\psi^{(0)}(z_3+z_6+2)\epsilon + 2\psi^{(0)}(z_6-z_7+3)\epsilon - 2\psi^{(0)}(z_3+z_6-z_7+2)\epsilon \\
& +2\psi^{(0)}(z_1+z_3+z_6-z_7+2)\epsilon - 2\psi^{(0)}(z_7-z_6)\epsilon - 2i\pi\epsilon + \gamma\epsilon + 1), \\
& \text{where } C_1 = -0.0429503, C_3 = -0.560317, C_6 = -0.906813, C_7 = -0.597238.
\end{aligned} \tag{4.212}
$$

$$
\begin{aligned}
IntL_2 = \quad & \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}} \prod_{j=1,3,5,6,7} dz_j \frac{2e^{-i\pi(z_1+z_3+z_6-z_7)}u^{-z_1-z_3-2}\Gamma(-z_1)\Gamma(1-z_3)}{\Gamma(z_3+2)\Gamma(z_6-z_7+3)} \\
& \times\Gamma(-z_3+z_5-z_6)\Gamma(z_6+1)\Gamma(z_3+z_6+2)\Gamma(z_1+z_3+z_6-z_7+2)\Gamma(-z_7)\Gamma(z_7-z_6) \\
& \times\Gamma(-z_5)\Gamma(z_5+1)\Gamma(z_5+2)\Gamma(-z_5-1)\Gamma(z_3-z_5), \\
& \text{where } C_1 = -0.136446, C_3 = -0.291374, C_5 = -0.912513, C_6 = -0.952179, C_7 = -0.429662.
\end{aligned} \tag{4.213}
$$

and

$$
\begin{aligned}
IntL_3 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}} \prod_{j=1,j\neq 2,4}^{8} dz_j \frac{2e^{-i\pi(z_8+z_1+z_3+z_6-z_7)}u^{-z_1-z_3-2}\Gamma(z_8)\Gamma(-z_1)\Gamma(1-z_3)}{\Gamma(1-z_8)\Gamma(z_3+2)\Gamma(z_6-z_7+3)} \\
& \times\Gamma(-z_8-z_5-1)\Gamma(-z_3+z_5-z_6)\Gamma(z_6+1)\Gamma(-z_8+z_3+z_6+2)\Gamma(z_1+z_3+z_6-z_7+2) \\
& \times\Gamma(-z_5)\Gamma(z_5+1)\Gamma(z_5+2)\Gamma(-z_7)\Gamma(z_7-z_6)\Gamma(z_3-z_5), \\
& \text{where } C_1 = -1.16316, C_3 = -0.109891, C_5 = -0.646538, C_6 = -0.761777, \\
& C_7 = -0.616226, C_8 = -0.372296.
\end{aligned} \tag{4.214}
$$

$$
\begin{aligned}
IntL_4 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}} \prod_{j=1,j\neq 2,4}^{8} dz_j \frac{2e^{-i\pi(z_8+z_1+z_3+z_6-z_7)}u^{-z_1-z_3-2}\Gamma(z_8)\Gamma(-z_1)\Gamma(1-z_3)}{\Gamma(1-z_8)\Gamma(z_3+2)\Gamma(-z_8+z_3-z_5+1)\Gamma(z_6-z_7+3)} \\
& \times\Gamma(-z_8-z_5-1)\Gamma(z_3-z_5)\Gamma(-z_5)\Gamma(z_5+1) \\
& \times\Gamma(-z_3+z_5-z_6)\Gamma(z_6+1)\Gamma(-z_8+z_3+z_6+2)\Gamma(z_1+z_3+z_6-z_7+2)\Gamma(-z_7)\Gamma(z_7-z_6), \\
& \times(\Gamma(1-z_8)\Gamma(z_3+2) - \Gamma(-z_8+z_3-z_5+1)\Gamma(z_5+2)) \\
& \text{where } C_1 = -0.36198, C_3 = -0.35813, C_5 = -0.785229, C_6 = -0.653286, \\
& C_7 = -0.0715289, C_8 = 0.0419341.
\end{aligned} \tag{4.215}
$$

$$
\begin{aligned}
IntL_5 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 2,4}^{8} dz_j \frac{-2e^{-i\pi(z_8+z_1+z_3+z_6-z_7)}u^{-z_1-z_3-2}\Gamma(z_8)\Gamma(-z_1)}{\Gamma(-z_8+z_3-z_5+1)\Gamma(z_6-z_7+2)} \\
& \times\Gamma(-z_5)\Gamma(z_5+1)\Gamma(-z_3+z_5-z_6)\Gamma(z_6+1)\Gamma(-z_8+z_3+z_6+1)\Gamma(z_1+z_3+z_6-z_7+1) \\
& \times\Gamma(-z_7)\Gamma(z_7-z_6)\Gamma(z_3-z_5)\Gamma(1-z_3)\Gamma(-z_8-z_5), \\
& \text{where } C_1=-0.217128, C_3=0.0802147, C_5=-0.492807, C_6=-0.739881, \\
& C_7=-0.281083, C_8=0.144034.
\end{aligned}
\tag{4.216}
$$

We apply the routine `MBexpand` to each integral in the sum $\sum_{k=1}^{5} IntL_k$ in order to expand the integrands to series in ϵ up to ϵ^0 order. We can extract the coefficient of ϵ^0 and we apply the routine `MBasymptotics` to it in order to obtain the series coefficients of u. Finally we obtain,

$$
C'_{8,l,0,\ln u} = 0, \tag{4.217}
$$

$$
\begin{aligned}
C'_{8,l,0} = & \frac{6-4\zeta_3}{u} - \frac{29}{18} - \frac{8731u}{3600} - \frac{27057u^2}{9800} - \frac{36957947u^3}{12700800} - \frac{1141251883u^4}{384199200} - \frac{166106486657u^5}{55653998400} \\
& - \frac{1351212763391u^6}{454507653600} - \frac{1769177780855371u^7}{600469540070400} - \frac{236640076333522157u^8}{81288563987030400} - \frac{22226635892993141u^9}{7741767998764800} \\
& - \frac{6370622752753409 7409u^{10}}{22524673992406185600} - \frac{17104838347531015392 97u^{11}}{614309290701986880000} - \frac{11489004763475648632 7111u^{12}}{4192660909041060456 0000} \\
& - \frac{1023918325069285511466900 71u^{13}}{37972607340807266006880000} - \frac{1037423044147512803037355203 11u^{14}}{39098223915552624249226800000} \\
& - \frac{4356034514719793544059146058 99u^{15}}{16681908870635786346336768000 0} - \frac{4555516003545138295614790087 19u^{16}}{17724528175050522992982816000 0} \\
& - \frac{6500867961105826640081747171607 11u^{17}}{25692224899387940446651914816000 0} - \frac{6757110346949237024355358889969 51u^{18}}{27119570727131714915910354528000 0} \\
& - \frac{6198260624258540138161484804337398 9u^{19}}{25256508804602998766562496377600000} - \frac{2419539495607350491634091135981924349u^{20}}{10006989595652345297008726243324800 00} \\
& - \frac{1224000949832107546288219008498947469 41u^{21}}{51369213257682039191311461382400640000} + \cdots
\end{aligned}
\tag{4.218}
$$

To obtain the basic integral defined at the beginning of this section $C_{8,i}$, we use:

$$
C'_{8,i} = C'_{8,k} - C'_{8,l} \tag{4.219}
$$

As usual, $C'_{8,i,-2}$ and $C'_{8,i,-1}$ can be expressed as a compact form due to the method in section 4.4.2:

$$
C'_{8,i,-2} = 0, \tag{4.220}
$$

$$
C'_{8,i,-1} = \frac{\left(2u-2u^2\right)\mathrm{Li}_2(u) - \frac{1}{3}\pi^2u^2 + \left(u-u^2\right)\ln^2(1-u) + \left(\frac{\pi^2}{3}-2\right)u + (2-2u)\ln(1-u)}{(1-u)u^2}. \tag{4.221}
$$

But We are not able to find the compact form of $C'_{8,i,0}$.

4.4.13 Calculations of the Integral $C'_{8,j}$

The integral $C'_{8,j}$ is defined as:

$$
C'_{8,j} = 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}s_{123}s_{24}s_{124}}. \tag{4.222}
$$

In our parametrization (4.8), the factors s_{13}, s_{123} in the denominator of the integrand are not free of s_{13}. But $s_{123}-s_{13}$ is free of s_{13}. We use the symmetry in the four-particle phase space to rewrite the integrand in order to perform the integration over s_{13} simple. So We rewrite the integrand of $C_{8,i}$ in the following way,

$$\begin{aligned} C'_{8,j} &= 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}s_{123}s_{24}s_{124}} \\ &= 8f_{dim}\int dPS_4 \frac{E_2E_3}{E^2_{total}}\delta(\cos\theta_{23}-c)(\frac{1}{s_{13}(s_{123}-s_{13})s_{24}s_{124}}-\frac{1}{s_{123}(s_{123}-s_{13})s_{24}s_{124}}) \\ &= C'_{8,m}-C'_{8,n} \end{aligned} \tag{4.223}$$

We calculate these two integrals $C'_{8,m}$ and $C'_{8,n}$ separately. The integral $C'_{8,m}$ is given by

$$C'_{8,m} = 8f_{dim}\int dPS_4 \frac{E_2E_3\delta(\cos\theta_{23}-c)}{E^2_{total}s_{13}(s_{123}-s_{13})s_{24}s_{124}} \tag{4.224}$$

We can apply non-rigorous calculation (in section 4.4.1) to the integral $C_{8,m}$ and obtain its Mellin–Barnes representation:

$$\begin{aligned} IntM_0 = \; & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}}\prod_{j=2}^{4}dz_j \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_4)\Gamma(z_4+1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_3)\Gamma(-2\epsilon-z_2+1)}{\Gamma(-2\epsilon-1)\Gamma(-3\epsilon-z_4+1)\Gamma(-\epsilon+z_4+1)\Gamma(-5\epsilon-z_4-z_2+z_3+1)} \\ & \times\Gamma(-2\epsilon+z_3+1)(1-u)^{\epsilon+z_4-z_3-1}\Gamma(-\epsilon-z_4-z_2-2)\Gamma(-\epsilon+z_4+z_2+1) \\ & \times\Gamma(-3\epsilon-z_4+z_3)u^{-\epsilon-z_4-z_2+z_3-2} \end{aligned} \tag{4.225}$$

we can resolve the singularities in ϵ by using the routine `MBresolve`. We obtain a sum of four integrals, $\sum_{k=1}^{4} IntM_k$, shown in eqs. (4.226)-(4.229):

$$\begin{aligned} IntM_1 = \; & \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}} dz_1dz_3 \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon+1)\Gamma(-z_1)\Gamma(-z_3)u^{\epsilon-z_1-2}(-1)^{\epsilon-z_1-z_3}}{\Gamma(2-3\epsilon)\Gamma(-4\epsilon+z_3+3)\Gamma(-\epsilon+z_3+2)}, \\ & \times\Gamma(-3\epsilon+z_3+1)\Gamma(-2\epsilon+z_3+1)\Gamma(-\epsilon+z_1+z_3+2) \\ & \text{where } C_1=-0.52912, C_3=-0.0786174. \end{aligned} \tag{4.226}$$

$$\begin{aligned} IntM_2 = \; & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}}\prod_{j=1}^{3}dz_j \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_1)\Gamma(z_2+1)\Gamma(-z_3)\Gamma(-2\epsilon-z_2+1)\Gamma(\epsilon-z_2)}{\Gamma(-4\epsilon+z_2+2)\Gamma(-6\epsilon+z_3+2)\Gamma(-2\epsilon+z_2+z_3+2)} \\ & \times\Gamma(-\epsilon+z_2+1)\Gamma(-2\epsilon+z_3+1)u^{-z_1-z_2-3}(-1)^{2\epsilon-z_1-z_2-z_3}\Gamma(-4\epsilon+z_2+z_3+1) \\ & \times\Gamma(-2\epsilon+z_1+z_2+z_3+2), \\ & \text{where , }\{C_1=-0.580475, C_2=-0.393907, C_3=-0.146543. \end{aligned} \tag{4.227}$$

$$\begin{aligned} IntM_3 = \; & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}}\prod_{j=1}^{3}dz_j(-1) \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_3)\Gamma(-2\epsilon-z_2+1)}{\Gamma(-2\epsilon-z_2-1)\Gamma(-2\epsilon+z_2+3)\Gamma(-4\epsilon+z_3+3)\Gamma(z_2+z_3+3)} \\ & \times\Gamma(-\epsilon-z_2-1)\Gamma(\epsilon+z_2+2)\Gamma(-2\epsilon+z_3+1)u^{-z_1-z_2-3}(-1)^{-z_1-z_2-z_3}\Gamma(-2\epsilon+z_2+z_3+2) \\ & \times\Gamma(z_1+z_2+z_3+3), \\ & \text{where } C_1=-0.106881, C_2=-0.58034, C_3=-0.649423. \end{aligned} \tag{4.228}$$

and

$$IntM_4 = \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=1}^{4}dz_j\frac{\Gamma(-z_3)\Gamma(-2\epsilon-z_2+1)\Gamma(-2\epsilon+z_3+1)u^{-z_1-z_2-3}(-1)^{\epsilon+z_4-z_1-z_3}}{\Gamma(-5\epsilon-z_4-z_2+z_3+1)}$$

$$\times\Gamma(-\epsilon-z_4-z_2-2)\Gamma(-\epsilon+z_4+z_2+1)\Gamma(-3\epsilon-z_4+z_3)\Gamma(-\epsilon-z_4+z_1+z_3+1))$$

$$\text{where } C_1=-0.491396, C_2=-0.129013, C_3=-0.140706, C_4=-0.959049 \tag{4.229}$$

We apply the routine `MBexpand` to the sum $\sum_{k=1}^{4} IntM_k$ in order to obtain the series coefficient of ϵ^0 order. Applying the routine `MBasymptotics` to the coefficient of ϵ^0, we obtain its series expansion in u. The tricks to reduce the dimension of the integral are always Barnes' lemmas and summing residues. Using these

tricks, we obtain series coefficient of u, shown below, The coefficient of ϵ^0 in $C_{8,m}$ is

$$
\begin{aligned}
C'_{8,m,0} &= \frac{\frac{1}{3}\pi^2 \ln u - 2\ln u - 11\zeta_3 + 8}{u} + 2\ln u - \frac{25}{2} + u\left(\frac{5\ln u}{2} - \frac{643}{36}\right) + u^2\left(\frac{26\ln u}{9} - \frac{4807}{216}\right) \\
&+u^3\left(\frac{77\ln u}{24} - \frac{187417}{7200}\right) + u^4\left(\frac{87\ln u}{25} - \frac{33037}{1125}\right) + u^5\left(\frac{223\ln u}{60} - \frac{214124}{6615}\right) \\
&+u^6\left(\frac{962\ln u}{245} - \frac{28902313}{823200}\right) + u^7\left(\frac{4609\ln u}{1120} - \frac{956007457}{25401600}\right) + u^8\left(\frac{4861\ln u}{1134} - \frac{1142589613}{28576800}\right) \\
&+u^9\left(\frac{55991\ln u}{12600} - \frac{32410183139}{768398400}\right) + u^{10}\left(\frac{58301\ln u}{12705} - \frac{13355963813}{301870800}\right) \\
&+u^{11}\left(\frac{785633\ln u}{166320} - \frac{4499148011627}{97394497200}\right) + u^{12}\left(\frac{811373\ln u}{167310} - \frac{10138685972431}{211021410600}\right) \\
&+u^{13}\left(\frac{835397\ln u}{168168} - \frac{2057974969621}{41318877600}\right) + u^{14}\left(\frac{3431678\ln u}{675675} - \frac{28655404433533}{556539984000}\right) \\
&+u^{15}\left(\frac{29889983\ln u}{5765760} - \frac{63767083804710961}{1200939080140800}\right) u^{16}\left(\frac{30570663\ln u}{5785780} - \frac{69722313786879683}{1275997772649600}\right) \\
&+u^{17}\left(\frac{197698279\ln u}{36756720} - \frac{5474826027829547861}{97546276784436480}\right) \\
&+u^{18}\left(\frac{201578155\ln u}{36858822} - \frac{658461374262928777}{11440612709285760}\right) \\
&+u^{19}\left(\frac{8210931\ln u}{1478048} - \frac{6387514094646026599}{108384751982707200}\right) \\
&+u^{20}\left(\frac{13920029\ln u}{2469012} - \frac{527579233070405501}{8754153044757120}\right) \\
&+u^{21}\left(\frac{325333835\ln u}{56904848} - \frac{76122575911199089973}{1236648768210535680}\right) \\
&+u^{22}\left(\frac{990874363\ln u}{171037867} - \frac{295792185337692234011}{4709704562048566080}\right) \\
&+u^{23}\left(\frac{25128807667\ln u}{4283383104} - \frac{34410322798784177872453}{537520629364238520000}\right) \\
&+\frac{u^{24}(6819153254798232348000\ln u - 7487814111987837625043)}{114854835334239000000} \\
&+\frac{u^{25}(1258916299567905001023 0\ln u - 1390662320864140395540 01)}{2096330454520530228000} \\
&+\frac{u^{26}(1321668448062458267772 00\ln u - 1468388308258415694228811)}{21769585489251660060000} \\
&+\frac{u^{27}(2117785574883335542387 0200\ln u - 2365894283732282526281 40319)}{34520552128006605460800 00} \\
&+\frac{u^{28}(1218508610498688580941 87000\ln u - 1368502138563239180064 234163)}{19664385944346619896420000} \\
&+\frac{u^{29}(1063523833464404572676 2758600\ln u - 1200556620934084143353 22920011)}{16999227789370706195316000 00} + \cdots
\end{aligned}
\tag{4.230}
$$

The series expansion of $C'_{8,m,0}$ in $u' = 1 - u$ is

$$
\begin{aligned}
C_{8,m,0}(u') &= 22 + \frac{8}{u'} + 19u' + \frac{165u'^2}{8} + \frac{5129u'^3}{216} + \frac{1194391u'^4}{43200} + \frac{6920507u'^5}{216000} + \frac{389145223u'^6}{10584000} \\
&+ \frac{3092551381u'^7}{74088000} + \frac{9266782631u'^8}{197568000} + \frac{835609133671u'^9}{16003008000} + \frac{111627118076167u'^{10}}{1936363968000} \\
&+ \frac{122338524059767u'^{11}}{1936363968000} + \frac{22512854885482123u'^{12}}{327245510592000} + \frac{3168602942900023 99u'^{13}}{4254191637696000} \\
&+ \frac{3413155165708221 19u'^{14}}{4254191637696000} + \frac{3659990723702359 03u'^{15}}{4254191637696000} + \frac{9037223306209936 59371u'^{16}}{9835691066353152000} \\
&+ \frac{1634832181648479 3433147u'^{17}}{167206748128003584000} + \frac{6259595607747633354606947u'^{18}}{6036163607420929382400 0} \\
&+ \frac{1257698873774777 68141937273u'^{19}}{1146871085409976582656000} + \frac{1326421116234085 58368163201u'^{20}}{1146871085409976582656000} \\
&- 14\ln u' - 24u'\ln u' - \frac{415}{12}u'^2\ln u' - \frac{553}{12}u'^3\ln u' - \frac{2803}{48}u'^4\ln u' - \frac{85691u'^5\ln u'}{1200} \\
&- \frac{2142811u'^6\ln u'}{25200} - \frac{17497537u'^7\ln u'}{176400} - \frac{80319593u'^8\ln u'}{705600} - \frac{272860859u'^9\ln u'}{2116800} \\
&- \frac{3361531081u'^{10}\ln u'}{23284800} - \frac{41029623491u'^{11}\ln u'}{256132800} - \frac{587175551323u'^{12}\ln u'}{3329726400} \\
&- \frac{8345873776159u'^{13}\ln u'}{43286443200} - \frac{9070878422239u'^{14}\ln u'}{43286443200} - \frac{9807505441367u'^{15}\ln u'}{43286443200} \\
&- \frac{717743870305531u'^{16}\ln u'}{2943478137600} - \frac{13077726819892907u'^{17}\ln u'}{50039128339200} \\
&- \frac{7960094391747345 79u'^{18}\ln u'}{2852230315334400} - \frac{1609676103693945 3961u'^{19}\ln u'}{54192375991353600} \\
&- \frac{3416071628801202389u'^{20}\ln u'}{10838475198270720} + \cdots \\
&+ \frac{\left(\left(\frac{3u'^2}{2} - \frac{3u'}{2}\right)\ln(1-u') + \frac{5u'}{2} - 4\right)\ln^2(u')}{(1-u')^2 u'} + \frac{5\ln^3(u')}{6(1-u')} + \frac{4\pi^2\ln(u')}{3(1-u')}
\end{aligned}
\tag{4.231}
$$

The integral $C'_{8,n}$ is given by:

$$
C'_{8,n} = 8 f_{dim} \int dPS_4 \frac{E_2 E_3 \delta(\cos\theta_{23} - c)}{E_{total}^2 s_{123}(s_{123} - s_{13}) s_{24} s_{124}} \tag{4.232}
$$

It is not difficult to obtain its Mellin–Barnes representation. The Mellin–Barnes representation of the integral $C_{8,n}$ is

$$
\begin{aligned}
IntN_0 = \quad & \frac{1}{(2\pi i)^7} \int \cdots \int_{\mathcal{C}} \prod_{j=1}^{7} dz_j \frac{\Gamma(1-\epsilon)\Gamma(-z_1)\Gamma(-z_4)\Gamma(-z_5)\Gamma(z_5+1)\Gamma(-z_6)\Gamma(z_7)\Gamma(-2\epsilon - z_3 + 1)}{\Gamma(-2\epsilon-1)\Gamma(1-2\epsilon)\Gamma(-3\epsilon - z_2 + 1)\Gamma(-\epsilon - z_7 + 1)} \\
& \times \frac{\Gamma(-2\epsilon + z_6 + 1) u^{-z_1 - z_3 - 3} \Gamma(-z_2 - z_7)\Gamma(z_5 - z_3)\Gamma(-\epsilon - z_2 - z_3 - 2)\Gamma(-\epsilon + z_2 + z_3 + 1)}{\Gamma(-\epsilon - z_2 + z_6 + 1)\Gamma(-5\epsilon - z_2 - z_3 + z_6 + 1)} \\
& \times \Gamma(-3\epsilon - z_2 + z_6)(-1)^{\epsilon - z_1 + z_2 - z_6 - z_7 - 1} \Gamma(-\epsilon + z_1 - z_2 + z_6 + 1)\Gamma(-2\epsilon + z_3 + z_4 - z_5 + 1). \\
& \times \Gamma(-\epsilon - z_2 - z_3 - z_4 + z_5)\Gamma(z_2 + z_3 + z_4 - z_5 - z_7 + 1)
\end{aligned}
\tag{4.233}
$$

As usual, the ϵ singularities can be resolved by the routine `MBresolve`. It yields a sum of integrals

$\sum_{k=1}^{5} IntN_k$ eqs, (4.234)-(4.238):

$$
\begin{aligned}
IntN_1 = \quad & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}}\prod_{j=1,3,6} dz_j(-1)\frac{\Gamma(-z_1)\Gamma(-z_3-1)\Gamma(1-z_3)\Gamma(z_3+2)^2\Gamma(-z_6)\Gamma(z_6+1)u^{-z_1-z_3-3}}{\epsilon\Gamma(z_3+3)\Gamma(z_6+3)\Gamma(z_3+z_6+3)} \\
& \times(-1)^{-z_1-z_3-z_6}\Gamma(z_3+z_6+2)(2\epsilon\psi^{(0)}(z_3+z_6+2)+\epsilon\psi^{(0)}(-z_3-1)+2\epsilon\psi^{(0)}(1-z_3) \\
& -2\epsilon\psi^{(0)}(z_3+3)+2\epsilon\psi^{(0)}(z_6+1)-4\epsilon\psi^{(0)}(z_6+3)+\gamma\epsilon-1)\Gamma(z_1+z_3+z_6+3) \\
& \text{where } C_1=-1.27429, C_3=-1.05823, C_6=-0.108367,
\end{aligned}
\tag{4.234}
$$

$$
\begin{aligned}
IntN_2 = \quad & \frac{1}{(2\pi i)^4}\iiiint_{\mathcal{C}}\prod_{j=1,3,5,6} dz_j(-1)\frac{\Gamma(-z_1)\Gamma(1-z_3)\Gamma(z_3+2)\Gamma(-z_5-1)\Gamma(-z_5)}{\Gamma(z_3+3)\Gamma(z_6+3)\Gamma(z_3+z_6+3)} \\
& \times\Gamma(-z_6)\Gamma(z_6+1)u^{-z_1-z_3-3}(-1)^{-z_1-z_3-z_6}\Gamma(z_3-z_5+1)\Gamma(z_5-z_3)\Gamma(z_3+z_6+2) \\
& \times\Gamma(z_1+z_3+z_6+3)\Gamma(z_5+1)\Gamma(z_5+2) \\
& \text{where } C_1=-0.648314, C_3=-1.04316, C_5=-0.759735, C_6=-0.630036,
\end{aligned}
\tag{4.235}
$$

$$
\begin{aligned}
IntN_3 = \quad & \frac{1}{(2\pi i)^5}\int\cdots\int_{\mathcal{C}}\prod_{j=1,3,5,6,7} dz_j(-1)\frac{\Gamma(-z_1)\Gamma(1-z_3)\Gamma(-z_5)\Gamma(z_5+1)\Gamma(z_5+2)\Gamma(-z_6)\Gamma(z_6+1)}{\Gamma(z_3+3)\Gamma(z_6+3)\Gamma(1-z_7)\Gamma(z_3+z_6+3)} \\
& \times\Gamma(z_7)u^{-z_1-z_3-3}\Gamma(z_3-z_5+1)\Gamma(z_5-z_3)\Gamma(z_3+z_6+2)\Gamma(z_3-z_7+2)\Gamma(-z_5-z_7-1) \\
& \times(-1)^{-z_1-z_3-z_6-z_7}\Gamma(z_1+z_3+z_6+3) \\
& \text{where } C_1=-0.529905, C_3=-0.847164, C_5=-0.222106, C_6=-0.400865, C_7=-0.936406,
\end{aligned}
\tag{4.236}
$$

$$
\begin{aligned}
IntN_4 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 2}^{7} dz_j(-1)\frac{\Gamma(-z_1)\Gamma(1-z_3)\Gamma(-z_4)\Gamma(-z_5)\Gamma(z_5+1)\Gamma(-z_6)\Gamma(z_6+1)}{\Gamma(z_3+3)\Gamma(z_6+3)\Gamma(1-z_7)\Gamma(z_3+z_6+3)} \\
& \times\Gamma(z_7)u^{-z_1-z_3-3}\Gamma(z_5-z_3)\Gamma(z_3+z_6+2)\Gamma(z_3-z_7+2)\Gamma(-z_4+z_5+2)(-1)^{-z_1-z_3-z_6-z_7} \\
& \times\Gamma(z_1+z_3+z_6+3)\Gamma(z_3+z_4-z_5+1)\Gamma(z_4-z_5-z_7-1) \\
& \text{where } C_1=-0.327002, C_3=-1.30572, C_4=0.806859, C_5=-0.922141, C_6=-0.240195, \\
& C_7=0.0640556,
\end{aligned}
\tag{4.237}
$$

$$
\begin{aligned}
IntN_5 = \quad & \frac{1}{(2\pi i)^6}\int\cdots\int_{\mathcal{C}}\prod_{j=1,j\neq 2}^{7} dz_j \frac{\Gamma(-z_1)\Gamma(1-z_3)\Gamma(-z_4)\Gamma(-z_5)\Gamma(z_5+1)\Gamma(-z_6)\Gamma(z_6+1)\Gamma(z_7)}{\Gamma(z_3+2)\Gamma(z_6+2)\Gamma(1-z_7)\Gamma(z_3+z_6+2)} \\
& \times\Gamma(z_5-z_3)\Gamma(z_3+z_6+1)\Gamma(z_3-z_7+1)\Gamma(-z_4+z_5+1)(-1)^{-z_1-z_3-z_6-z_7} \\
& \times\Gamma(z_3+z_4-z_5+1)\Gamma(z_4-z_5-z_7)u^{-z_1-z_3-3}\Gamma(z_1+z_3+z_6+2) \\
& \text{where } C_1=-0.184662, C_3=-0.581526, C_4=-0.0305276, C_5=-0.370808, \\
& C_6=-0.238721, C_7=0.118729.
\end{aligned}
\tag{4.238}
$$

We apply the routine `MBexpand` to the sum $\sum_{k=1}^{5} IntN_k$ in order to extract the series coefficient of ϵ^0. Then we apply the routine `MBasymptotics` to the coefficient of ϵ^0 and reduce the dimension of the integral to 0 or 1 so that we can evaluate them. Finally, we obtain

$$
C_{8,n,0,\ln u} = 0, \tag{4.239}
$$

$$
\begin{aligned}
C'_{8,n,0} = \quad & \frac{2\zeta_3-1}{u}+\frac{73}{8}+u\left(-\frac{\pi^2}{3}+\frac{563}{54}\right)+u^2\left(-\frac{\pi^2}{2}+\frac{18299}{1728}\right)+u^3\left(-\frac{11\pi^2}{18}+\frac{1134803}{108000}\right)\\
&+u^4\left(-\frac{25\pi^2}{36}+\frac{139769}{13500}\right)+u^5\left(-\frac{137\pi^2}{180}+\frac{53870867}{5292000}\right)+u^6\left(-\frac{49\pi^2}{60}+\frac{5930352569}{592704000}\right)\\
&+u^7\left(-\frac{121\pi^2}{140}+\frac{78720340229}{8001504000}\right)+u^8\left(-\frac{761\pi^2}{840}+\frac{15491048003}{1600300800}\right)\\
&+u^9\left(-\frac{7129\pi^2}{7560}+\frac{20302960466023}{2130000364800}\right)+u^{10}\left(-\frac{7381\pi^2}{7560}+\frac{12505024970521}{1331250228000}\right)\\
&+u^{11}\left(-\frac{83711\pi^2}{83160}+\frac{216762917586113291}{23398054007328000}\right)+u^{12}\left(-\frac{86021\pi^2}{83160}+\frac{213935726791516661}{23398054007328000}\right)\\
&+u^{13}\left(-\frac{1145993\pi^2}{1081080}+\frac{30184313206442393}{3342579143904000}\right)\\
&+u^{14}\left(-\frac{1171733\pi^2}{1081080}+\frac{3340967047099684493}{374368864117248000}\right)\\
&+u^{15}\left(-\frac{1195757\pi^2}{1081080}+\frac{8115559088713485059087}{919637114704019712000}\right)\\
&+u^{16}\left(-\frac{2436559\pi^2}{2162160}+\frac{8029444016449441376227}{919637114704019712000}\right)\\
&+u^{17}\left(-\frac{42142223\pi^2}{36756720}+\frac{5451715509153842448862 7813}{6307790969754871204608000}\right)\\
&+u^{18}\left(-\frac{14274301\pi^2}{12252240}+\frac{5399129519150163401878 7123}{6307790969754871204608000}\right)\\
&+u^{19}\left(-\frac{275295799\pi^2}{232792560}+\frac{5349376214756224807997 2793}{6307790969754871204608000}\right)\\
&+u^{20}\left(-\frac{55835135\pi^2}{46558512}+\frac{5302221599015047754558 4953}{6307790969754871204608000}\right)\\
&+u^{21}\left(-\frac{18858053\pi^2}{15519504}+\frac{6396746869368733352528 89720051}{7674689272900751794646553 6000}\right)\\
&+u^{22}\left(-\frac{6364399\pi^2}{5173168}+\frac{7210180661763510569761268377}{8721237810114490675734720 00}\right)\\
&+u^{23}\left(-\frac{444316699\pi^2}{356948592}+\frac{7154140935392002124412382564381}{8721237810114490675734720 00000}\right)\\
&+u^{24}\left(-\frac{1347822955\pi^2}{1070845776}+\frac{4201598437702326748232 3015581}{5160495745629876139488000000}\right)+\cdots\\
&-\frac{9\zeta_3}{1-u}
\end{aligned}
\tag{4.240}
$$

The series expansion of $C'_{8,n,0}$ in $u' = 1 - u$ are also obtained,

$$\begin{aligned}
C'_{8,n,0}(u') &= \frac{7(u'-2)\ln^3(u')}{6(u'-1)^2} + \frac{\pi^2\left(10u'^2-19u'+2\right)\ln u'}{6(u'-1)^2u'} \\
&-\frac{7\ln^2(u')}{2} + 4\ln u' + u'\left(-\frac{1}{2}25\ln^2(u') - \frac{7\ln u'}{2}\right) + u'^2\left(-\frac{1}{6}131\ln^2(u') - \frac{667\ln u'}{36}\right) \\
&+u'^3\left(-\frac{1}{6}191\ln^2(u') - \frac{2783\ln u'}{72}\right) + u'^4\left(-\frac{1}{20}849\ln^2(u') - \frac{226987\ln u'}{3600}\right) \\
&+u'^5\left(-\frac{1}{120}6433\ln^2(u') - \frac{36481\ln u'}{400}\right) + u'^6\left(-\frac{1}{70}4567\ln^2(u') - \frac{21647321\ln u'}{176400}\right) \\
&+u'^7\left(-\frac{1}{10}773\ln^2(u') - \frac{55492427\ln u'}{352800}\right) + u'^8\left(-\frac{1}{72}6461\ln^2(u') - \frac{1236277051\ln u'}{6350400}\right) \\
&+u'^9\left(-\frac{1}{720}73811\ln^2(u') - \frac{1490230009\ln u'}{6350400}\right) \\
&+u'^{10}\left(-\frac{1}{660}76301\ln^2(u') - \frac{42582505607\ln u'}{153679680}\right) \\
&+u'^{11}\left(-\frac{1021583\ln^2(u')}{7920} - \frac{247257473761\ln u'}{768398400}\right) \\
&\frac{-19\zeta_3 - 2\pi^2\ln(2)}{4u'} + \frac{1}{3}\left(3\zeta_3 - 4\pi^2 - 6\right) + \cdots
\end{aligned} \tag{4.241}$$

So the basic integral defined in the beginning of this section $C_{8,j}$ is

$$C'_{8,j} = C'_{8,m} - C'_{8,n} \tag{4.242}$$

As usual, $C'_{8,j,-2}$ and $C'_{8,j,-1}$ can be expressed as a compact form due to the method in section 4.4.2:

$$C'_{8,j,-2} = 0, \tag{4.243}$$

$$C'_{8,j,-1} = \frac{\left(2u^2-u-1\right)\mathrm{Li}_2(u) + \frac{\pi^2u^2}{3} + \left(u^2-\frac{u}{2}-\frac{1}{2}\right)\ln^2(1-u) + \left(2-\frac{\pi^2}{6}\right)u + 2u\ln(1-u)}{(1-u)u^2}. \tag{4.244}$$

We still fail to obtain the compact form of $C'_{8,j,0}$.

4.5 Numerical Evaluations of all Base Integrals

In this section, we would like to check numerically the results of the basic integrals obtained in last several sections.

Using the parametrization (4.8), we can obtain a definite integral representation for 10 basic integrals $C_{8,t}$, $t = a, b, \cdots, j$. We can also obtain a Mellin–Barnes representation from a definite integral representation. In this section, we will compare the numerical evaluation result of the definite integral representation of each basic integral by the routine `NIntegrate` in *Mathematica* and the numerical evaluation result of Mellin–Barnes representation of each basic integral by the routine `MBintegrate` in the package `MB.m` at the regulator $\epsilon = -9/10$ in order to check if the Mellin–Barnes representations of each basic integral are right. For these basic integrals, in the following tables, we show the values of its definite integral representation in the second column and the values of its Mellin–Barnes representation in the third column. At this step, we do not resolve the singularities in ϵ in both representations. To obtain a finite value for both

representations, we set $\epsilon = -9/10$,

$C'_{8,a}(\epsilon, u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`
$\epsilon = -\frac{9}{10}, u = \frac{1}{3}$	0.022911	0.022911
$\epsilon = -\frac{9}{10}, u = \frac{1}{2}$	0.0296757	0.0296756
$\epsilon = -\frac{9}{10}, u = \frac{2}{3}$	0.0418084	0.0418082

(4.245)

$C'_{8,b}(\epsilon, u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`
$\epsilon = -\frac{9}{10}, u = \frac{1}{3}$	0.0100865	0.0100864
$\epsilon = -\frac{9}{10}, u = \frac{1}{2}$	0.0191328	0.0191322
$\epsilon = -\frac{9}{10}, u = \frac{2}{3}$	0.0369737	0.0369738

(4.246)

$C'_{8,c}(\epsilon, u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`
$\epsilon = -\frac{9}{10}, u = \frac{1}{3}$	0.0016019	0.001627
$\epsilon = -\frac{9}{10}, u = \frac{1}{2}$	0.0025476	0.0026152
$\epsilon = -\frac{9}{10}, u = \frac{2}{3}$	0.004022	0.004076

(4.247)

$C'_{8,d}(\epsilon, u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`
$\epsilon = -\frac{9}{10}, u = \frac{1}{3}$	0.00170543	0.00170542
$\epsilon = -\frac{9}{10}, u = \frac{1}{2}$	0.00317033	0.00317032
$\epsilon = -\frac{9}{10}, u = \frac{2}{3}$	0.00600573	0.0060057

(4.248)

$C'_{8,e}(\epsilon, u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`
$\epsilon = -\frac{9}{10}, u = \frac{1}{3}$	0.00239834	0.00228992
$\epsilon = -\frac{9}{10}, u = \frac{1}{2}$	0.00273212	0.00276919
$\epsilon = -\frac{9}{10}, u = \frac{2}{3}$	0.0032645	0.00325049

(4.249)

$C'_{8,f}(\epsilon,u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`	
$\epsilon=-\frac{9}{10},u=\frac{1}{3}$	0.00245795	0.00244877	(4.250)
$\epsilon=-\frac{9}{10},u=\frac{1}{2}$	0.00417472	0.00421465	
$\epsilon=-\frac{9}{10},u=\frac{2}{3}$	0.00718465	0.0070891	

$C'_{8,g}(\epsilon,u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`	
$\epsilon=-\frac{9}{10},u=\frac{1}{3}$	0.00051421	0.000514202	(4.251)
$\epsilon=-\frac{9}{10},u=\frac{1}{2}$	0.00117996	0.00117998	
$\epsilon=-\frac{9}{10},u=\frac{2}{3}$	0.00276185	0.00276183	

$C'_{8,h}(\epsilon,u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`	
$\epsilon=-\frac{9}{10},u=\frac{1}{3}$	0.00277253	0.00281535	(4.252)
$\epsilon=-\frac{9}{10},u=\frac{1}{2}$	0.00298851	0.00301179	
$\epsilon=-\frac{9}{10},u=\frac{2}{3}$	0.00336561	0.0033495	

$C'_{8,i}(\epsilon,u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`	
$\epsilon=-\frac{9}{10},u=\frac{1}{3}$	0.00120544	0.00119843	(4.253)
$\epsilon=-\frac{9}{10},u=\frac{1}{2}$	0.001941	0.00193831	
$\epsilon=-\frac{9}{10},u=\frac{2}{3}$	0.00303227	0.00303423	

$C'_{8,j}(\epsilon,u)$	Definite Integral Representation evaluated by `NIntegrate`	Mellin–Barnes representation evaluated by `MBintegrate`	
$\epsilon=-\frac{9}{10},u=\frac{1}{3}$	0.000956393	0.000394748	(4.254)
$\epsilon=-\frac{9}{10},u=\frac{1}{2}$	0.00173016	0.00181475	
$\epsilon=-\frac{9}{10},u=\frac{2}{3}$	0.00317329	0.00321134	

We checked that the Mellin–Barnes representation of the basic integrals are numerically right in the above tables. Then we apply the routines `MBresolve` and `MBexpand` to the Mellin–Barnes representations of these basic integral in order to obtain its series expansion in ϵ up to ϵ^0 order. The coefficients of $1/\epsilon^2$, $1/\epsilon$ and ϵ^0 are still high dimensional Mellin–Barnes integrals. We will not expand the coefficients of ϵ into series in u but evaluate these coefficients with fixed u numerically by applying the routine `MBintegrate`. we obtain,

$$C'_{8,t} = \frac{C'_{N,8,t,-2}}{\epsilon^2} + \frac{C'_{N,8,t,-1}}{\epsilon} + C'_{N,8,t,0} + \mathcal{O}[\epsilon] \tag{4.255}$$

We show the values of $C'_{N,8,t,-2}$, $C'_{N,8,t,-1}$ and $C'_{N,8,t,0}$ in the second column of the following table.

On the other hand, we defined functions $C'_{8,t,-2}$, $C'_{8,t,-1}$, $C'_{8,t,0}$ for each basic integral.

$$C'_{8,t} = \frac{C'_{8,t,-2}}{\epsilon^2} + \frac{C'_{8,t,-1}}{\epsilon} + C'_{8,t,0} + \mathcal{O}[\epsilon] \tag{4.256}$$

I obtain compact function results of $C'_{8,t,-2}$, $C'_{8,t,-1}$, $C'_{8,t,0}$ for four integrals $C'_{8,a}$ in section 4.4.3, $C'_{8,b}$ in section 4.4.4, $C'_{8,d}$ in section 4.4.5 and $C'_{8,e}$ in section 4.4.6. For the other six integrals, I obtain their compact functions results of $C'_{8,t,-2}$, $C'_{8,t,-1}$. But we were only able to obtain the series expansion results of $C'_{8,t,0}$ in variable u. The results are shown $C'_{8,c}$ in section 4.4.8, $C'_{8,f}$ in section 4.4.9, $C'_{8,g}$ in section 4.4.10, $C'_{8,h}$ in section 4.4.11, $C'_{8,i}$ in section 4.4.12, $C'_{8,j}$ in section 4.4.13. We can simply set $u = 1/3$, $1/2$, $2/3$ to obtain the values of the functions $C'_{8,t,-2}$, $C'_{8,t,-1}$, $C'_{8,t,0}$ at these points. The results are shown in the third column of the following tables. We compare the results of eq. (4.255) with the results of eq. (4.256) to check if the function $C'_{8,t,-2}$, $C'_{8,t,-1}$, $C'_{8,t,0}$, $t = a, b, \cdots, i$ are right. For the basic integral $C'_{8,a}$,

$$\begin{array}{lll}
C'_{8,a}(\epsilon,u) & C'_{N,8,a,-2}/\epsilon^2 + C'_{N,8,a,-1}/\epsilon + C'_{N,8,a,0} & C'_{8,a,-2}/\epsilon^2 + C'_{8,a,-1}/\epsilon + C'_{8,a,0} \\
u = \dfrac{1}{3} & \dfrac{10.94755}{\epsilon^2} + \dfrac{68.12418}{\epsilon} + 184.9 & \dfrac{10.9476}{\epsilon^2} + \dfrac{68.1242}{\epsilon} + 184.9 \\
u = \dfrac{1}{2} & \dfrac{11.090354}{\epsilon^2} + \dfrac{65.79736}{\epsilon} + 166.238 & \dfrac{11.0904}{\epsilon^2} + \dfrac{65.7974}{\epsilon} + 166.238 \\
u = \dfrac{2}{3} & \dfrac{14.83126}{\epsilon^2} + \dfrac{86.7001}{\epsilon} + 211.4 & \dfrac{14.8313}{\epsilon^2} + \dfrac{86.7001}{\epsilon} + 211.4
\end{array} \tag{4.257}$$

For the basic integral $C'_{8,b}$,

$$\begin{array}{lll}
C'_{8,b}(\epsilon,u) & C'_{N,8,b,-2}/\epsilon^2 + C'_{N,8,b,-1}/\epsilon + C'_{N,8,b,0} & C'_{8,b,-2}/\epsilon^2 + C'_{8,b,-1}/\epsilon + C'_{8,b,0} \\
u = \dfrac{1}{3} & \dfrac{10.94755}{\epsilon^2} + \dfrac{87.8196}{\epsilon} + 341.109 & \dfrac{10.9476}{\epsilon^2} + \dfrac{87.8196}{\epsilon} + 341.109 \\
u = \dfrac{1}{2} & \dfrac{11.09035}{\epsilon^2} + \dfrac{78.9568}{\epsilon} + 270.676 & \dfrac{11.0904}{\epsilon^2} + \dfrac{78.9568}{\epsilon} + 270.676 \\
u = \dfrac{2}{3} & \dfrac{14.83126}{\epsilon^2} + \dfrac{95.816}{\epsilon} + 292.996 & \dfrac{14.8313}{\epsilon^2} + \dfrac{95.816}{\epsilon} + 292.995
\end{array} \tag{4.258}$$

For the basic integral $C'_{8,c}$,

$$\begin{array}{lll}
C'_{8,c}(\epsilon,u) & C'_{N,8,c,-2}/\epsilon^2 + C'_{N,8,c,-1}/\epsilon + C'_{N,8,c,0} & C'_{8,c,-2}/\epsilon^2 + C'_{8,c,-1}/\epsilon + C'_{8,c,0} \\
u = \dfrac{1}{3} & -\dfrac{6.73299}{\epsilon} - 74.11 & -\dfrac{6.73295}{\epsilon} - 74.1157 \\
u = \dfrac{1}{2} & -\dfrac{6.57976}{\epsilon} - 67.12 & -\dfrac{6.57974}{\epsilon} - 67.1073 \\
u = \dfrac{2}{3} & -\dfrac{8.33911}{\epsilon} - 81.484 & -\dfrac{8.33905}{\epsilon} - 81.4832
\end{array} \tag{4.259}$$

For the basic integral $C'_{8,d}$,

$$\begin{array}{lll} C'_{8,d}(\epsilon,u) & C'_{N,8,d,-2}/\epsilon^2+C'_{N,8,d,-1}/\epsilon+C'_{N,8,d,0} & C'_{8,d,-2}/\epsilon^2+C'_{8,d,-1}/\epsilon+C'_{8,d,0} \\ u=\dfrac{1}{3} & \dfrac{5.473778959461754}{\epsilon^2}+\dfrac{62.75}{\epsilon}+397.361 & \dfrac{5.47378}{\epsilon^2}+\dfrac{62.75}{\epsilon}+397.361 \\ u=\dfrac{1}{2} & \dfrac{5.545177444480486}{\epsilon^2}+\dfrac{59.2179}{\epsilon}+354.784 & \dfrac{5.54518}{\epsilon^2}+\dfrac{59.2176}{\epsilon}+354.784 \\ u=\dfrac{2}{3} & \dfrac{7.415632948510536}{\epsilon^2}+\dfrac{75.6431}{\epsilon}+438.071 & \dfrac{7.41563}{\epsilon^2}+\dfrac{75.6431}{\epsilon}+438.071 \end{array} \tag{4.260}$$

For the basic integral $C'_{8,e}$,

$$\begin{array}{lll} C'_{8,e}(\epsilon,u) & C'_{N,8,e,-2}/\epsilon^2+C'_{N,8,e,-1}/\epsilon+C'_{N,8,e,0} & C'_{8,e,-2}/\epsilon^2+C'_{8,e,-1}/\epsilon+C'_{8,e,0} \\ u=\dfrac{1}{3} & -\dfrac{3.36648}{\epsilon}-21.4505 & -\dfrac{3.36648}{\epsilon}-21.4504 \\ u=\dfrac{1}{2} & -\dfrac{3.28989}{\epsilon}-22.1022 & -\dfrac{3.28987}{\epsilon}-22.0966 \\ u=\dfrac{2}{3} & -\dfrac{4.16956}{\epsilon}-29.694 & -\dfrac{4.16952}{\epsilon}-30.1737 \end{array} \tag{4.261}$$

When we evaluate $C'_{N,8,e,0}$, we notice there is a Mellin–Barnes integral MB_7 which can not be directly evaluated by `MBintegrate`. We will not try to find a better contour of integration for it. We will perform the numerical evaluation in the following way.

$$\begin{aligned} MB_7 = \; & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}}\prod_{j=2,3,5}dz_j-\frac{(-1)^{z_2}(1-u)^{-z_3-1}u^{-z_5-2}\Gamma(-z_2-1)\Gamma(z_2+1)\Gamma(z_3)}{\Gamma(1-z_5)\Gamma(z_2+z_3+z_5+3)} \\ & \times\Gamma(-z_3-z_5)\Gamma(-z_3-z_5+1)\Gamma(z_3+z_5+1)\Gamma(z_2+z_3+z_5+2)\Gamma(-z_5)\Gamma(z_5+1), \\ & \text{where } C_2=-0.811713, C_3=0.24245, C_5=-0.350811 \end{aligned} \tag{4.262}$$

We sum all the residues of z_2 on the right side of the contour and obtain a two-dimensional integral:

$$\begin{aligned} MB'_7 = \; & \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}}\prod_{j=3,5}dz_j\frac{(1-u)^{-z_3}u^{-z_5-2}(z_3+z_5+2)\Gamma(z_3)\Gamma(-z_5)\Gamma(z_5+1)}{(u-1)(z_3+z_5+1)\Gamma(1-z_5)\Gamma(z_3+z_5+3)} \\ & \times\Gamma(z_3+z_5+1)\Gamma(z_3+z_5+2)(\psi^{(0)}(z_3+z_5+2)+\gamma_E)\Gamma(-z_3-z_5)\Gamma(-z_3-z_5+1), \\ & \text{where } C_3=0.24245, C_5=-0.350811 \end{aligned} \tag{4.263}$$

The integral MB'_7 then can be evaluated by MB.m already.

For the basic integral $C'_{8,f}$,

$$\begin{array}{lll} C'_{8,f}(\epsilon,u) & C'_{N,8,f,-2}/\epsilon^2+C'_{N,8,f,-1}/\epsilon+C'_{N,8,f,0} & C'_{8,f,-2}/\epsilon^2+C'_{8,f,-1}/\epsilon+C'_{8,f,0} \\ u=\dfrac{1}{3} & \dfrac{5.47377}{\epsilon^2}+\dfrac{59.3836}{\epsilon}+360.923 & \dfrac{5.47378}{\epsilon^2}+\dfrac{59.3835}{\epsilon}+360.917 \\ u=\dfrac{1}{2} & \dfrac{5.54517}{\epsilon^2}+\dfrac{55.9279}{\epsilon}+320.213 & \dfrac{5.54518}{\epsilon^2}+\dfrac{55.9278}{\epsilon}+320.207 \\ u=\dfrac{2}{3} & \dfrac{7.415632}{\epsilon^2}+\dfrac{71.4736}{\epsilon}+393.65 & \dfrac{7.41563}{\epsilon^2}+\dfrac{71.4736}{\epsilon}+393.621 \end{array} \tag{4.264}$$

For the basic integral $C'_{8,g}$,

$$\begin{array}{lll} C'_{8,g}(\epsilon,u) & C'_{N,8,g,-2}/\epsilon^2 + C'_{N,8,g,-1}/\epsilon + C'_{N,8,g,0} & C'_{8,g,-2}/\epsilon^2 + C'_{8,g,-1}/\epsilon + C'_{8,g,0} \\ u = \frac{1}{3} & -\frac{0.683302}{\epsilon} - 5.45868 & -\frac{0.683301}{\epsilon} - 5.45867 \\ u = \frac{1}{2} & -\frac{1.23407}{\epsilon} - 9.55625 & -\frac{1.23407}{\epsilon} - 9.55623 \\ u = \frac{2}{3} & -\frac{2.77131}{\epsilon} - 21.4174 & -\frac{2.77131}{\epsilon} - 21.4098 \end{array} \tag{4.265}$$

For the basic integral $C'_{8,h}$,

$$\begin{array}{lll} C'_{8,h}(\epsilon,u) & C'_{N,8,h,-2}/\epsilon^2 + C'_{N,8,h,-1}/\epsilon + C'_{N,8,h,0} & C'_{8,h,-2}/\epsilon^2 + C'_{8,h,-1}/\epsilon + C'_{8,h,0} \\ u = \frac{1}{3} & -86.1516 - \frac{7.95075}{\epsilon} & -86.1488 - \frac{7.95066}{\epsilon} \\ u = \frac{1}{2} & -69.9998 - \frac{6.5798}{\epsilon} & -70.0021 - \frac{7.95066}{\epsilon} \\ u = \frac{2}{3} & -76.6599 - \frac{7.128}{\epsilon} & -76.6334 - \frac{7.12798}{\epsilon} \end{array} \tag{4.266}$$

For the basic integral $C'_{8,i}$,

$$\begin{array}{lll} C'_{8,i}(\epsilon,u) & C'_{N,8,i,-2}/\epsilon^2 + C'_{N,8,i,-1}/\epsilon + C'_{N,8,i,0} & C'_{8,i,-2}/\epsilon^2 + C'_{8,i,-1}/\epsilon + C'_{8,i,0} \\ u = \frac{1}{3} & -\frac{3.73829}{\epsilon} - 39.4743 & -\frac{3.73828}{\epsilon} - 39.4581 \\ u = \frac{1}{2} & -\frac{3.67564}{\epsilon} - 35.8596 & -\frac{3.67557}{\epsilon} - 35.8289 \\ u = \frac{2}{3} & -\frac{4.69883}{\epsilon} - 43.9556 & -\frac{4.69871}{\epsilon} - 43.2167 \end{array} \tag{4.267}$$

For the basic integral $C'_{8,j}$,

$$\begin{array}{lll} C'_{8,j}(\epsilon,u) & C'_{N,8,j,-2}/\epsilon^2 + C'_{N,8,j,-1}/\epsilon + C'_{N,8,j,0} & C'_{8,j,-2}/\epsilon^2 + C'_{8,j,-1}/\epsilon + C'_{8,j,0} \\ u = \frac{1}{3} & -46.3947 - \frac{3.8428}{\epsilon} & -46.3914 - \frac{3.8428}{\epsilon} \\ u = \frac{1}{2} & -46.0311 - \frac{4.12494}{\epsilon} & -46.1216 - \frac{4.12491}{\epsilon} \\ u = \frac{2}{3} & -63.1001 - \frac{5.96306}{\epsilon} & -64.2366 - \frac{5.96303}{\epsilon} \end{array} \tag{4.268}$$

In the numerical evaluation of $C'_{N,8,j,0}$, we found a Mellin–Barnes integral MB_8 which can not be evaluated by `MBintegrate`:

$$\begin{aligned} MB_8 = \; & \frac{1}{(2\pi i)^4} \iiiint_{\mathcal{C}} \prod_{j=2,3,5,6} dz_j \frac{e^{-i\pi z_2}(1-u)^{-z_3-3}u^{z_6}\Gamma(z_2+2)\Gamma(z_3+2)\Gamma(-z_5)\Gamma(z_5+1)\Gamma(z_5+2)}{\Gamma(-z_2-1)\Gamma(z_6+3)\Gamma(z_3-z_6+3)} \\ & \times\Gamma(-z_6)\Gamma(z_6+1)\Gamma(-z_2-z_5-3)\Gamma(-z_3+z_6+1)\Gamma(-z_2+z_3-z_6) \\ & \Gamma(z_3-z_5-z_6+1)\Gamma(-z_3+z_5+z_6), \\ & \text{where } C_2 = -1.67048, C_3 = -1.31036, C_5 = -0.586244, C_6 = -0.495201. \end{aligned} \tag{4.269}$$

We will sum the residues of z_2 on the left-side of the contours. The residues are at $z_2 = -2, -3, \cdots$, and at $z_2 = -3 - z_5$. We obtain $MB_8 = MB_{8,1} + MB_{8,2}$:

$$\begin{aligned} MB_{8,1} = & \frac{1}{(2\pi i)^3} \iiint_{\mathcal{C}} \prod_{j=3,5,6} dz_j - \frac{e^{i\pi z_5}(1-u)^{-z_3} u^{z_6} \Gamma(z_3+2)\Gamma(-z_5-1)\Gamma(-z_5)\Gamma(z_5+1)}{(u-1)^3 \Gamma(z_6+3)\Gamma(z_3-z_6+3)} \\ & \times \Gamma(-z_3+z_6+1)\Gamma(z_3-z_5-z_6+1)\Gamma(z_3+z_5-z_6+3)\Gamma(-z_3+z_5+z_6)\Gamma(-z_6)\Gamma(z_6+1), \\ & \text{where } C_3 = -1.31036, C_5 = -0.586244, C_6 = -0.495201 \end{aligned}$$

$$\begin{aligned} MB_{8,2} = & \frac{1}{(2\pi i)^3} \iiint_{\mathcal{C}} \prod_{j=3,5,6} dz_j \frac{(1-u)^{-z_3-3} u^{z_6} \Gamma(z_3+2)\Gamma(-z_5-1)\Gamma(-z_5)\Gamma(z_5+1)}{\Gamma(z_6+3)\Gamma(z_3-z_6+3)\Gamma(-z_3+z_6-1)} \\ & \times \Gamma(z_3-z_6+2)\Gamma(-z_3+z_6+1)\Gamma(z_3-z_5-z_6+1)\Gamma^2(-z_3+z_5+z_6)\Gamma(-z_6)\Gamma(z_6+1), \\ & \text{where } C_3 = -1.31036, C_5 = -0.586244, C_6 = -0.495201 \end{aligned} \tag{4.270}$$

Then there two integrals $MB_{8,1}$ and $MB_{8,2}$ can be evaluated by the routine `MBintegrate`.

4.6 The Calculation of the Definite Integral $CL_3(u)$

The definite integral $CL_3(u)$ which was defined in eq. (3.108) is a part of the virtual contribution EEC_V to EEC function, (see section 3.3). We denote

$$CL_3'(u) = \frac{1}{2}(1-u)uCL_3(u) = \int_0^1 dt \mathrm{Li}_2(1 - \frac{1-tu}{(1-t)tu})/(1-t\,u) \tag{4.271}$$

Using the Mellin–Barnes representation of the polylog function $\mathrm{Li}_2(z)$

$$\mathrm{Li}_2(z) = -\frac{1}{2\pi i} \int ds \frac{\Gamma(s+1)\Gamma(-s)^3}{(-z)^s \Gamma(1-s)^2} \tag{4.272}$$

and the replacement (4.22), we can obtain the Mellin–Barnes representation of $CL_3(u)$.

$$CL_3'(u) = -\frac{1}{2\pi i} \int dz_1 \frac{\Gamma(z_1+1)\Gamma(-z_1)^3}{\Gamma(1-z_1)^2(1-t\,u)} \left(\frac{1-tu}{(1-t)tu} - 1\right)^{-z_1} \tag{4.273}$$

where the contour for z_1 must be chosen to be $C_1 \in (-1, 0)$. If we want replace this factor in the above formula,

$$\left(\frac{1-tu}{(1-t)tu} - 1\right)^{-z_1} \to \frac{1}{2\pi i} \int_{\mathcal{C}} dz_2 (-1)^{-z_2} \left(\frac{1-tu}{(1-t)tu} - 1\right)^{-z_1-z_2} \frac{\Gamma(-z_2)\Gamma(z_1+z_2)}{\Gamma(z_1)} \tag{4.274}$$

We see it is impossible to find any contour C_2 so that $-C_2 > 0$ and $C_1 + C_2 > 0$ where $-1 < C_1 < 0$. So we use a regulator to solve this problem. We perform the replacement (4.22) to the following term,

$$\left(\frac{1-tu}{(1-t)tu} - 1\right)^{-z_1+\epsilon} \to \frac{1}{2\pi i} \int_{\mathcal{C}} dz_2 (-1)^{-z_2} \left(\frac{1-tu}{(1-t)tu} - 1\right)^{-z_1-z_2+\epsilon} \frac{\Gamma(-z_2)\Gamma(z_1+z_2-\epsilon)}{\Gamma(z_1-\epsilon)}. \tag{4.275}$$

where ϵ is a regulator. Then at least for the above Mellin–Barnes integral, we can find the contour C_2 for z_2: $-C_2 > 0$ and $C_1 + C_2 - \epsilon > 0$ for the ϵ which satisfies $\epsilon - C_1 < 0$. Then we can perform again the replacement (4.22) and till we can integrate the variable t. We obtain

$$
\begin{aligned}
CL_3'(u) = & -\frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} dz_1 dz_2 dz_3 \frac{\Gamma^3(-z_1)\Gamma(z_1+1)u^{-z_2-1}\Gamma(-z_2)\Gamma^2(-z_3)(-1)^{\epsilon-z_1-z_3-1}}{\Gamma(1-z_1)\Gamma(z_1-\epsilon)\Gamma(-\epsilon+z_1+z_2-z_3+1)} \\
& \times\Gamma(-\epsilon+z_1+z_2)\Gamma(-\epsilon+z_1+z_2+z_3+1)
\end{aligned} \tag{4.276}
$$

Now, we can apply the routine `MBresolve` to the above formula to resolve its singularities and apply `MBexpand` to expand it as a series in ϵ. We find $CL_3'(u) = \sum_{k=1}^{3} IntCL_k$. The three Mellin–Barnes integrals are free of ϵ singularities, shown below,

$$
\begin{aligned}
IntCL_1 = & \frac{1}{2\pi i}\int_{\mathcal{C}} dz_2(-1)\frac{e^{-i\pi z_3}u^{-z_3-1}\Gamma^2(-z_3)\Gamma(z_3+1)}{6\Gamma(1-z_3)}\Big(-3(\psi^{(0)}(1-z_3))^2 \\
& +6(\psi^{(0)}(z_3+1)-i\pi)\psi^{(0)}(1-z_3)-3\psi^{(0)}(z_3+1)^2+6i\pi\psi^{(0)}(z_3+1)+3\psi^{(1)}(1-z_3) \\
& +2\pi^2-3\psi^{(1)}(z_3+1)\Big), \\
& \text{where } C_2=-0.895075,
\end{aligned} \tag{4.277}
$$

$$
\begin{aligned}
IntCL_2 = & \frac{1}{(2\pi i)^2}\iint_{\mathcal{C}} dz_2 dz_3 \frac{e^{-i\pi z_3}u^{-z_3-1}\Gamma(-z_2)\Gamma(z_2)\Gamma^2(-z_3)\Gamma(z_2+z_3+1)}{\Gamma(z_2-z_3+1)}\Big(-\psi^{(0)}(z_2-z_3+1) \\
& +\psi^{(0)}(z_2+z_3+1)+\psi^{(0)}(z_2)+\gamma-i\pi\Big), \\
& \text{where } C_2=0.318695, C_3=-0.682014,
\end{aligned} \tag{4.278}
$$

and

$$
\begin{aligned}
IntCL_3 = & \frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} dz_1 dz_2 dz_3 \frac{e^{-i\pi(z_1+z_3)}\Gamma^3(-z_1)\Gamma(z_1+1)u^{-z_3-1}\Gamma(-z_2)\Gamma^2(-z_3)}{\Gamma^2(1-z_1)\Gamma(z_1)\Gamma(z_1+z_2-z_3+1)} \\
& \times\Gamma(z_1+z_2)\Gamma(z_1+z_2+z_3+1), \\
& \text{where } C_1=0.684888, C_2=-0.410392, C_3=-0.323002.
\end{aligned} \tag{4.279}
$$

The series expansion in u can be calculated by `MBasymptotics` directly. Then we can use some ansatz to fit the series expansion in u. Finally, we obtain,

$$
\begin{aligned}
CL_3'(u) = & -4-\frac{5u}{9}-\frac{457u^2}{2700}-\frac{2059u^3}{29400}-\frac{134503u^4}{3969000}-\frac{4124627u^5}{230519520}-\frac{14976107u^6}{1515025512}-\frac{102499427u^7}{18551332800} \\
& -\frac{36261929293u^8}{12063004153200}-\frac{319795204259u^9}{215049111076800}-\frac{192750775411u^{10}}{354831033276720}+\cdots \\
& -\frac{1}{2}\ln(1-u)\ln^2(u)+\frac{\mathrm{Li}_2\left(\left(-i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u}\right)+\mathrm{Li}_2\left(\left(i\sqrt{1-u}+\sqrt{u}\right)\sqrt{u}\right)}{u}\ln u \\
& +\frac{2\mathrm{Li}_2(u)-\ln^2(1-u)}{4u}\ln u
\end{aligned} \tag{4.280}
$$

We can check numerically that the series expansion calculated by Mellin–Barnes representation is actually the series expansion of $CL_3'(u)$ near $u=0$.

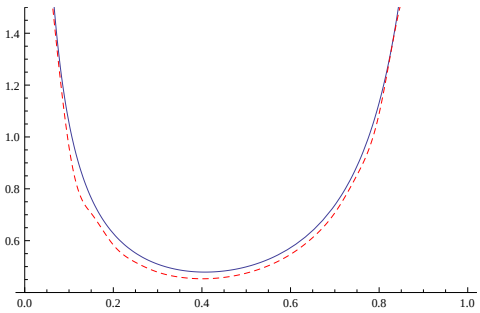

Figure 4.1: In this plot, the blue line represents the leading order EEC $g^2\, PS_3\, EEC_{LO}(u)$. The red dashed line represents the result of $g^2\, PS_3\, EEC_{LO}(u) + g^4\, PS_3\, EEC_{NLO}(u)$.

4.7 Combining All Contributions

In this section, we will try to combine some of our results to finally obtain EEC_{real} defines in (4.17). As explained in section 4.3, we can calculate EEC_{real} by calculating 10 basic integral. We obtain,

$$f_{dim} EEC_{real} = \frac{1}{2} C'_{8,a} + \frac{1}{4} C'_{8,b} + \frac{1}{2} C'_{8,c} + \frac{1}{2} C'_{8,d} + C'_{8,e} + C'_{8,f} + C'_{8,g} + 2C'_{8,h} + C_{8,i} + 2C'_{8,j} \tag{4.281}$$

We substitute our results obtained in this chapter in the contribution EEC_{real}. Then we can obtain the coefficient of $1/\epsilon^2$ and $1/\epsilon$ in the contribution $f_{dim} EEC_{real}$. We obtain

1. the coefficient of $1/\epsilon^2$:
$$-\frac{3\ln(1-u)}{(1-u)u^2} \tag{4.282}$$
2. the coefficient of $1/\epsilon$:
$$\frac{16\mathrm{Li}_2(u) + 4\ln(1-u)\ln((1-u)u)}{(1-u)u^2} \tag{4.283}$$

We sum the real emission contribution EEC_{real} and the virtual contribution $EEC_{one-loop}$, eq. (3.109). The coefficients of $1/\epsilon^2$ and $1/\epsilon$ of the sum of the two contribution are exactly canceled so we can have a finite result for the energy-energy correlation function which is a strong cross-check of the correctness of energy-energy correlation. The plots of

$$g^2\, EEC_{LO}(u)/PS_3 \text{ and } g^2\, EEC_{LO}(u)/PS_3 + g^4\, EEC_{NLO}(u)/PS_3 \tag{4.284}$$

, where the coupling g is set to 0.3 and $PS_3 = 1/(256\pi^3)$ is the three-particle phase space volume , are given in the fig. 4.1.

Chapter 5

The Numerical Evaluation of Mellin–Barnes Integrals

5.1 The Numerical Evaluation of Mellin–Barnes Integrals

In this thesis, I have used the Mellin–Barnes approach to calculate several phase-space integrals analytically (in [50] due to Anastasiou, Tausk and Tejeda-Yeomans, in [51] and [52] due to V. A. Smirnov). The Mellin–Barnes approach has been widely used to evaluate Feynman (loop) integrals analytically, and numerically for integrals with massless propagators. We can use a Mellin–Barnes representation of a massive propagator,

$$(m^2 - p^2)^{-\lambda} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{(m^2)^z \Gamma(-z)(-p^2)^{-\lambda+z}\Gamma(\lambda+z)}{\Gamma(\lambda)}, \tag{5.1}$$

to put integrals with such propagators into the same form as those with massless ones. The contour, which must separate poles of $\Gamma(-z)$ and $\Gamma(\lambda+z)$ is chosen to be parallel to the imaginary axis. From the following example, we can see how Mellin–Barnes representation arises in loop amplitude (This example comes from the pedagogical introduction in Smirnov's book [28]):

$$F_\Gamma(q^2, m^2; a_1, a_2, D) = \int \frac{d^D p}{(p^2 - m^2)^{a_1}[(q-p)^2]^{a_2}}. \tag{5.2}$$

Apply eq. (5.1) to this loop integral to replace a massive propagator by a massless one and also use the following equation,

$$\int \frac{d^D p}{(-p^2)^{\lambda_1}[-(q-p)^2]^{\lambda_2}} = i\pi^{D/2} \frac{H(\lambda_1, \lambda_2)}{(-q^2)^{\lambda_1+\lambda_2+\epsilon-2}} \tag{5.3}$$

where

$$H(\lambda_1, \lambda_2) = \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)\Gamma(2-\epsilon-\lambda_1)\Gamma(2-\epsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(4-\lambda_1-\lambda_2-2\epsilon)}. \tag{5.4}$$

We obtain the result:

$$\begin{aligned} F_\Gamma(q^2, m^2; a_1, a_2, D) = \quad & \frac{i\pi^{D/2}(-1)^{a_1+a_2}\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \\ & \times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(a_1+a_2+\epsilon-2+z)\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-a_1-a_2-2\epsilon-z)} \end{aligned} \tag{5.5}$$

The rules for choosing an integration contour that goes from $-i\infty$ to $+i\infty$ are the same as before: the poles of $\Gamma(\cdots - z)$ and the poles of $\Gamma(\cdots + z)$ should be separated by the contour.

For analytic use, that we choose the contour is parallel to the imaginary axis raises no issues. However, the integrand may not be absolutely convergent for invariants in the Minkowski region, and the oscillations

of the integrand lead to problems with convergence for numerical use. In this chapter, I explore a method to resolve such problems in higher-dimensional Mellin–Barnes integrations, by deforming integration contours to a stationary phase contour. This generalizes work of Gluza, Kosower, and Yudin [26] in the one-dimensional cases. The approach is based on looking for a hyper-surface of stationary phase, obtained by gluing an approximation around a local stationary point to a suitable asymptotic form.

I will first review the case of one-dimensional Mellin–Barnes integrals where the integrand has one or two stationary points in the fundamental region. When the integrand is real, a lone stationary point will be a stationary point on the interval containing the original contour. I will then show how to generalize the one-dimensional ansatz to two-dimensional Mellin–Barnes integrals. I investigate Mellin–Barnes integrals where the integrand has one, two, or even more stationary points in a region surrounding the original contour, obtaining an ansatz for a modified contour which can be applied to each of these points. This ansatz should be applicable both to two-dimensional and higher-dimensional Mellin–Barnes integrals.

5.2 One-Dimensional Mellin–Barnes Integrals

In this section, I review the issues surrounding numerical evaluation of one-dimensional Mellin–Barnes integrals by contour integration. In general, a one-dimensional Mellin–Barnes integral arising from Feynman integrals can be written in the following way:

$$\mathcal{I}(u) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dz\, G(z) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dz \frac{u^z \prod_i \Gamma(a_i z + b_i)}{\prod_j \Gamma(c_j z + d_j)} \left\{1, \prod_k \psi^{(f_k)}(p_k z + r_k)\right\}, \tag{5.6}$$

where individual terms may contain polygamma functions ($f_k = 0$) or its derivatives ($f_k > 0$). We will be interested in the case that the difference $\Delta = \sum_i a_i - \sum_j c_j = 0$. In this case, the Mellin–Barnes integral will typically define an analytic function. The function could be obtained by summing residues: summing the integrand's residues closing the contour to the left yields an analytic continuation of the function obtained by summing the residues closing the contour to the right.

In this chapter, it will be useful to introduce the notion of a 'fundamental region', denoting a region on the real slice of z about the original value of C, in which we can vary continuously where the contour intersects the real slice without crossing any poles. Any value of this intersection in the fundamental region will then yield the same value for the integral. In the one-dimensional case, the fundamental region is just an interval on the real axis. If we find that a stationary point of the integrand to be complex, being in the fundamental region will be understood to mean that the real parts of the locations of such points are in the fundamental region.

The integral (5.6) is called a Euclidean integral when the parameter $u > 0$, and a Minkowski integral when $u < 0$. For Euclidean Mellin–Barnes integrals, we can choose the contour parallel to the imaginary axis; I will call this choice the 'naive' contour. In the Minkowski region for the parameter u, the choice of the naive contour may lead to convergence problems. Gluza, Kosower, and Yudin [26] suggested solving these problems by deforming the contour to an approximation to a contour of stationary phase. In one dimension, the contour of stationary phase is also the contour of steepest descent. In one-dimensional integrals, the stationary phase constraint completely fixes the form of the contour, as in one complex dimension (that is, two real dimensions) we need only one constraint to fix a curve. We can find the introduction to the method of steepest descent in a complex-analysis textbook, for example in Wong's book [29]. For a contour integral:

$$I(\lambda) = \int_{\mathcal{C}} g(z) e^{\lambda f(z)}, \tag{5.7}$$

where $f(z)$ and $g(z)$ are analytic functions and λ is a large positive parameter. The basic idea is to deform the contour $\mathcal{C}$ into a new path of integration $\mathcal{C}'$ so that the following conditions are met:

1. $\mathcal{C}'$ passes through one or more zeros of $f'(z)$;
2. The imaginary part of $f(z)$ is constant.

With the above two constraints, we try to find the contour of steepest decent for Mellin–Barnes integrals. The one-dimensional Mellin–Barnes integral (5.6) can be written in the following way:

$$\mathcal{I}(u) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dz G(z) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dz \exp\ln G(z), \tag{5.8}$$

where $\ln G(z)$ is still a analytic function, so that The method of steepest descent can be applied to the one-dimensional Mellin–Barnes integral (5.6):

1. The deformed contour $\mathcal{C}'$ passes through one or more zeros of

$$\frac{dG(z)}{dz}; \tag{5.9}$$

2. $\Im \ln G(z) = \arg G(z)$ is constant on $\mathcal{C}'$.

The first step in this procedure is to find a stationary point of the integrand. The contour of stationary phase will pass through that stationary point. The next step is to find a Padé approximation to the exact contour. In one dimension, the singularities of the integrand are all located on the real axis so they can be avoided simply. We can use the function `ContourPlot` in the symbolic computer package *Mathematica* to draw the exact stationary phase contour, and contrast it with the approximation.

In this section, I review the idea of finding stationary phase contour for an example:

$$\mathcal{I}(u) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dz G(z) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dz \frac{u^z \Gamma^3(-z)\Gamma(z+1)}{\Gamma(-2z)}, \tag{5.10}$$

similar to the example given by Czakon [22]. The intersection of the contour and the real axis is $C = \Re(z) = -1/2$ so that the fundamental region is the interval $-1 < z < 0$ between two poles $z = -1,\ 0$ of the integrand. Let us focus on this interval. For this example, we can perform the integration analytically, by summing over all the residues of the integrand to the left of $\Re(z) = -1/2$,

$$\mathcal{I}(u) = \frac{4\mathrm{csch}^{-1}\left(2\sqrt{u}\right)}{\sqrt{4u+1}}, \tag{5.11}$$

where

$$\mathrm{csch}^{-1}(x) = \begin{cases} \ln\left(\dfrac{1-\sqrt{x^2+1}}{x}\right) & x < 0, \\ \ln\left(\dfrac{1+\sqrt{x^2+1}}{x}\right) & x > 0. \end{cases} \tag{5.12}$$

$\mathcal{I}(u)$ has a branch cut running from $u = -1/4$ to $u = 0$. For $u > 0$, the integrand is real so that the value of $\mathcal{I}(u)$ is real. For $u < -1/4$, even though the integrand is complex, the value of $\mathcal{I}(u)$ is still real, so $u = -1/4$ is called the threshold of the integrand. For this integrand, the difference $\Delta = -1 \times 3 + 1 - (-2) = 0$, so it is within the class of interest.

5.2.1 A Euclidean integral with one stationary point in the fundamental region

For $u > 0$, the integral (5.10) is in the Euclidean region, and could be evaluated numerically using the naive contour parallel to the imaginary axis. Let us see how a stationary-phase contour would compare. To find the stationary phase contour of the integrand, first find a stationary point. If we look at the plot of $G(z)$, shown in fig. 5.1 for $u = 1/2$, we can see that it indeed has a minimum on the real axis, in the fundamental region $-1 < z < 0$; furthermore, the integrand does not vanish in $-1 < z < 0$. We can find the point numerically by calculating the solution of

$$\frac{dG(z)}{dz} = 0 \tag{5.13}$$

or by finding the minimum of

$$\left|\frac{1}{G(z)}\frac{dG(z)}{dz}\right|, \tag{5.14}$$

where z is real in the fundamental region is $-1 < z < 0$. The factor $1/G(z)$ in eq. (5.14) is helpful to stabilizing a numerical solution for the location of the minimum. For example, when $u = 1/2$, one finds,

$$z_0 = -0.657527. \tag{5.15}$$

This fixes the base point of the stationary phase contour.

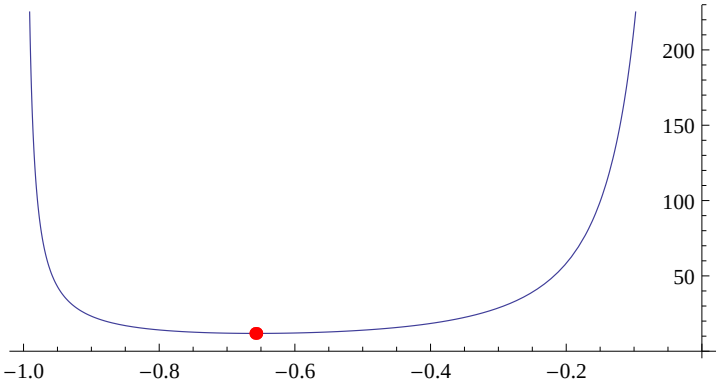

Figure 5.1: The plot of the integrand $G(z) = (1/2)^z\Gamma(-z)^3\Gamma(z+1)/\Gamma(-2z)$ in the region $-1 < \Re z < 0$. There is a minimum when $-1 < z < 0$.

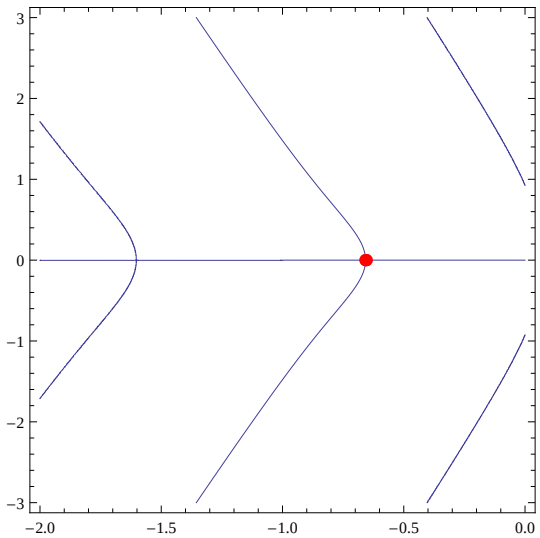

Figure 5.2: The red large point represents the local minimum point which is also the stationary point of the integrand $G(z)$ and the blue line is the exact stationary phase contour.

We can see the exact stationary phase contour in the plot shown in fig. 5.2. We choose the contour which passes the stationary point going from $-i\infty$ to $i\infty$. We will not try to find the exact functional form for the stationary phase contour. The idea is to find a Padé approximation for it. When the contour is close to the local minimum, $|z(t) - z_0| \lesssim 1$ the ansatz is,

$$z(t) = z_0 + it + dt + at^2. \tag{5.16}$$

We can substitute the ansatz into the integrand. The integrand evaluated at the base point, $G(z_0)$, is real. On the stationary phase contour, the integrand is always real for u in the Euclidean region. Let us expand $G(z(t))$ around $t = 0$ to third order,

$$G(z_0 + it + at^2) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \mathcal{O}(t^4)\,, \tag{5.17}$$

and require the imaginary part of this series expansion to vanish. $c_0 = G(z_0)$ is real.

$$\Im(c_1 t + c_2 t^2 + c_3 t^3) = t^3 \left(a G''(z_0) + \frac{1}{2} d^2 G^{(3)}(z_0) - \frac{1}{6} G^{(3)}(z_0) \right) + d\, t^2\, G''(z_0) + t\, G'(z_0) = 0\,. \tag{5.18}$$

The coefficient of t, $G'(z_0)$ vanishes because we are at an extremum. The vanishing of the coefficient of t^2 requires that $d = 0$ if $G''(z_0)$ does not vanish; $d = 0$ means that the tangent of the stationary phase contour at z_0 is orthogonal to the real axis. The condition that the coefficient of t^3 vanishes permits us to solve for a,

$$a = \frac{G^{(3)}(z_0)}{6G''(z_0)}. \tag{5.19}$$

For the example we are considering,

$$a = -0.425854. \tag{5.20}$$

We have thus found a quadratic approximation to the stationary-phase contour.

In the other limit, when $|z| \to \infty$ or $|z| \gg 1$, perform the replacements:

$$\begin{aligned} u^z \quad &\to \quad \exp(z \ln u), \\ \Gamma(z) \quad &\to \quad \exp\left(-z + z \ln z - \frac{1}{2} \ln z \right), \end{aligned} \tag{5.21}$$

the latter being the asymptotic expansion of the Γ function. The integrand becomes:

$$G(z \to \infty) \to \exp\left(z \ln u + (2z + \frac{1}{2}) \ln 2 - (z+1) \ln(-z) + (z + \frac{1}{2}) \ln(z+1) \right). \tag{5.22}$$

Take the imaginary part of the exponent, which is $\arg G(z \to \infty)$, as the approximation of the asymptotic phase of the integrand. For the given example, $\arg G(z \to \infty)$ is then,

$$\begin{aligned} \arg G(z \to \infty) \quad &= \quad \Im(z \ln u + (2z + \frac{1}{2}) \ln 2 - (z+1) \ln(-z) \\ &\qquad + (z + \frac{1}{2}) \ln(z+1)) \\ &= \quad \frac{\arg z}{2} - \arg(-z) + \pi \Re z\, \mathrm{sign} \Im z + \Im z\, \ln(4u), \end{aligned} \tag{5.23}$$

which is a linear equation as $z \to \infty$. The stationary phase constraint reads:

$$\arg G(z \to \infty) \cong 0 \mod 2\pi. \tag{5.24}$$

The `ContourPlot` result and the fact $\Delta = 0$ lead us to notice (ref. [26] due to Gluza, Kosower and Yudin) that the asymptotic stationary phase contour is a straight line. Let us then take as an ansatz for the asymptotic region ($|z| \gg 1$) a linear form,

$$z(t) = z_\infty + it + bt. \tag{5.25}$$

Now substitute this ansatz into eq. (5.23) and expand it in a series expansion about $t = \infty$. We can solve for b by requiring that the coefficient of t^1 vanish and solve for z_∞ by requiring that the coefficient of t^0 vanish. We obtain,

$$\begin{aligned} b &= -0.220636, \\ z_\infty &= -0.715438. \end{aligned} \tag{5.26}$$

for $t > 0$. Finally, we can write down a Padé ansatz combining both regions of the approximated contour. In our example,

$$z(t) = z_0 \frac{1 + |t| z_\infty / z_0}{1 + |t|} + it + \frac{at^2 + bt^4}{1 + |t|^3}. \tag{5.27}$$

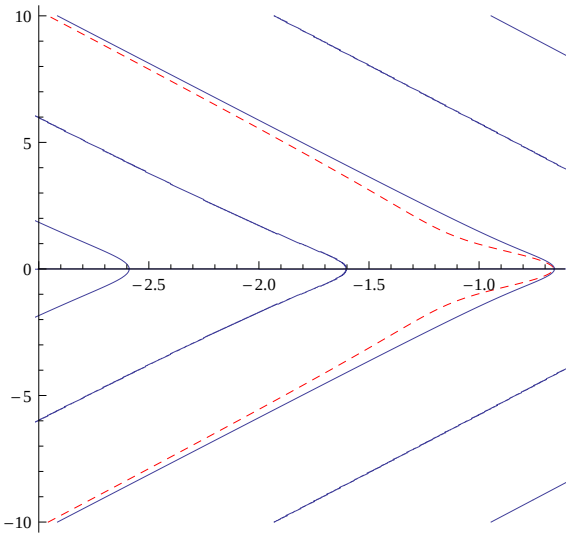

Figure 5.3: The blue and dashed red lines represent the exact stationary-phase contour and the Padé approximation to it, respectively, for the Euclidean integral.

The exact stationary-phase contour and the approximate contour using the ansatz in eq. (5.27) are shown in fig. 5.3. In principle, we could improve the approximation further by adding more parameters to the ansatz [26] (due to Gluza, Kosower and Yudin), but for stability and convergence purposes this isn't necessary.

For this integrand we are considering here, we would not be able to find a stationary point on the real axis if we were to choose the intersection of the contour to be somewhere in the region $\Re z > 0$. We would instead find two complex stationary points which are complex conjugates of each other. I postpone the discussion of this case to the third subsection.

5.2.2 A Minkowski integral with a lone stationary point in the fundamental region

When $u < 0$, the integral (5.10) is in the Minkowski domain. In this region, we cannot always use the naive contour to evaluate the integral numerically. To look instead for a stationary-phase contour, proceed as in the Euclidean case. First find the stationary point by finding the minimum of eq. (5.14) where z is complex and $-1 < \Re z < 0$ to make sure that the real part of the stationary point is in the fundamental region. In general, this point has a complex value. When $u = -1 + \delta i$, with δ a small positive number,

$$z_0 = -0.789320 - 0.174532i. \tag{5.28}$$

Following the same approach as in Euclidean case, when $|z(t) - z_0| \lesssim 1$, we substitute an ansatz for $z(t)$ into the integrand and expand it in a power series expansion in t. We require that each coefficient have the same phase as $G(z_0)$. The tangent of the stationary phase contour at z_0 is no longer orthogonal to the real axis (see fig. 5.4). In this case, we do need to add a linear term in t to the real part of $z(t)$. The simplest way to do this is to take the following form for our ansatz,

$$z(t) = z_0 + i\, t\, e^{i\phi_0} + a\, t^2, \tag{5.29}$$

where a may be real or complex. Expand $G(z(t))$ to fourth order in t,

$$\begin{aligned} G(z(t)) = \quad & G(z_0) - \frac{1}{2}t^2 e^{2i\phi_0} G''(z_0) \\ & -\frac{1}{6}it^3 e^{i\phi_0}\left(-6aG''(z_0) + e^{2i\phi_0}G^{(3)}(z_0)\right) \\ & +\frac{1}{24}t^4\left(12a^2G''(z_0) - 12ae^{2i\phi_0}G^{(3)}(z_0) + e^{4i\phi_0}G^{(4)}(z_0)\right) + \mathcal{O}(t^5). \end{aligned} \tag{5.30}$$

Then the stationary-phase condition requires that every term in this series expansion have the same phase:

$$\begin{aligned} &\arg(G(z_0)) = \arg\left(-e^{2i\phi_0}G''(z_0)\right) = \arg\left(-ie^{i\phi_0}\left(-6aG''(z_0) + e^{2i\phi_0}G^{(3)}(z_0)\right)\right) \\ &= \arg\left(12a^2G''(z_0) - 12ae^{2i\phi_0}G^{(3)}(z_0) + e^{4i\phi_0}G^{(4)}(z_0)\right) \end{aligned} \tag{5.31}$$

Rewrite last equations as follows,

$$\begin{aligned} &\Im\left(\frac{e^{2i\phi_0}G''(z_0)}{G(z_0)}\right) = \Re\left(\frac{e^{i\phi_0}\left(e^{2i\phi_0}G^{(3)}(z_0) - 6aG''(z_0)\right)}{G(z_0)}\right) \\ &= \Im\left(\frac{-12e^{2i\phi_0}aG^{(3)}(z_0) + 12a^2G''(z_0) + e^{4i\phi_0}G^{(4)}(z_0)}{G(z_0)}\right) = 0 \end{aligned} \tag{5.32}$$

We can use these two constraints to solve for ϕ_0 and a. If we take a to be real, then at best we can make the coefficients through that of t^3 have the same phase as $G(z_0)$. If we take a to be complex, then we can make both the coefficients of t^3 and of t^4 have the same phase as $G(z_0)$. For the integral (5.10) with $u = -1$, we obtain,

$$\begin{aligned} &\phi_0 = -0.528899, \\ &a = -1.26804,\ a \text{ is real} \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} &\phi_0 = -0.528899, \\ &a = -0.931716 + 0.575462i,\ a \text{ is complex.} \end{aligned} \tag{5.34}$$

See fig. 5.4 to compare the approximated contours with the exact stationary contour.

However, if we choose $a = -0.931716 + 0.575462i$, the contour will ultimately cross the real axis a second time. It will then head in a bad direction asymptotically, and will fail to enclose all the residues to the left of $\Re z = -1/2$. In one-dimensional integrals, it is still straightforward to correct this bad behavior of the contour using the asymptotic behavior. In order to simplify the ansatz in higher-dimensional integrals, though, I will choose the counterpart of a there to be a real function.

In order to analyze the behavior of the contour at the asymptotic region, $|z| \gg 1$, let us again use the asymptotic approximation for the Γ function (5.21), and then take the imaginary part of the exponent as the approximation to the asymptotic phase of the integrand. The expression for $\arg G(z \to \infty)$ is once again,

$$\begin{aligned} \arg G(z \to \infty) = \quad & \Im(z \ln u + (2z + \frac{1}{2})\ln 2 - (z+1)\ln(-z) \\ & +(z + \frac{1}{2})\ln(z+1)) \\ = \quad & \frac{\arg z}{2} - \arg(-z) + \Re z\,(\pi \text{sign}\Im z + \arg u) \\ & +\Im z\,\ln(4|u|) \end{aligned} \tag{5.35}$$

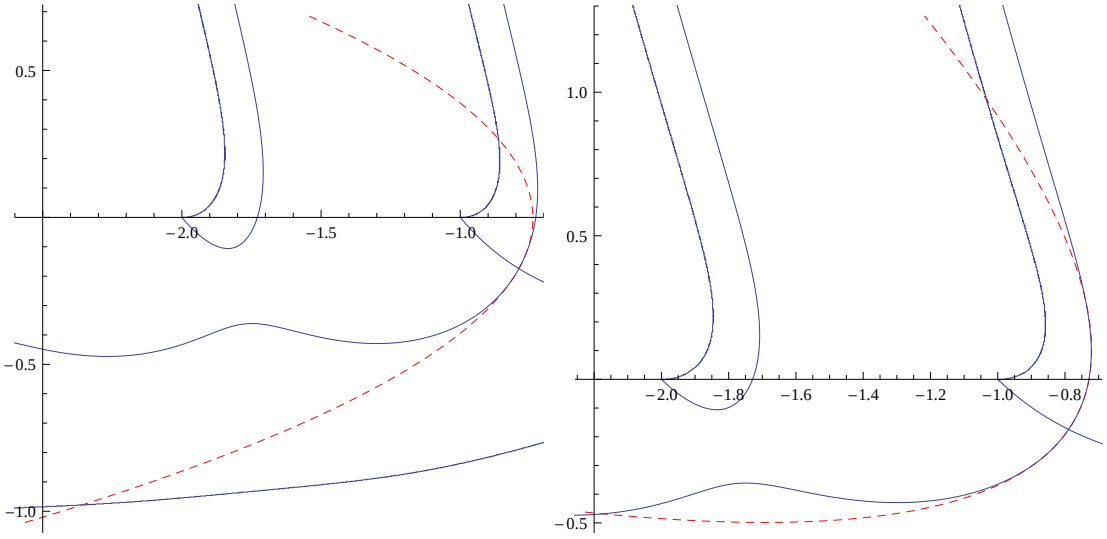

Figure 5.4: Here are the two plots in the region where z is close to the stationary phase point. The red dashed line in the first plot represents the approximated contour when we consider a is a real number $a = -1.26804$. In the second plot, we consider a is a complex number $a = -0.931716 + 0.575462i$.

In this case, looking at the plot of exact stationary phase contours for our integrand, shown in fig. 5.5, we notice that the contour becomes asymptotically parallel to the real axis (see fig. 5.5) when $\Im z(t) < 0$. This suggests the ansatz,

$$z(t) = z_{\pm\infty} + ite^{i\phi_\infty}. \tag{5.36}$$

As in the Euclidean case, we substitute this ansatz into eq. (5.35) and then expand G in a series expansion about $t = \infty$. We can solve for ϕ_∞ by requiring that the coefficient of t^1 vanish, and solve for z_∞ by requiring that the coefficient of t^0 have the same phase as $G(z_0)$. We obtain,

$$\begin{aligned} z_{+\infty} &= -0.722182, \phi_{+\infty} = 0.217156 \text{ for } \Im z(t) > 0\,, \\ z_{-\infty} &= -0.5187953i, \phi_{-\infty} = -\frac{\pi}{2} \text{ for } \Im z(t) < 0\,. \end{aligned} \tag{5.37}$$

The value $\phi_{-\infty} = \pi/2$ means that the asymptotic stationary-phase contour is parallel to the real axis. Finally, we glue the two parts of contour together to obtain the full approximation to the stationary-phase contour,

$$z(t) = \frac{z_0 + (z_{+\infty}\Theta(t) + z_{-\infty}\Theta(-t))|t|^3}{1+|t|^3} + it\frac{e^{i\phi_0} + (e^{i\phi_\infty}\Theta(t) + e^{i\phi_\infty}\Theta(-t))|t|^3}{1+|t|^3} + \frac{at^2}{1+|t|^3}, \tag{5.38}$$

where $\Theta(x)$ is step function defined by:

$$\begin{aligned} \Theta(x) &= 1, \text{ If } x > 0, \\ \Theta(x) &= 0, \text{ If } x < 0, \end{aligned} \tag{5.39}$$

as shown in fig. 5.5.

In these last two subsections, I reviewed the general idea of finding the stationary-phase contour based on a lone stationary point in one-dimensional Mellin–Barnes integrals. In the next subsection, I consider integrands which have two or more stationary points in the fundamental region.

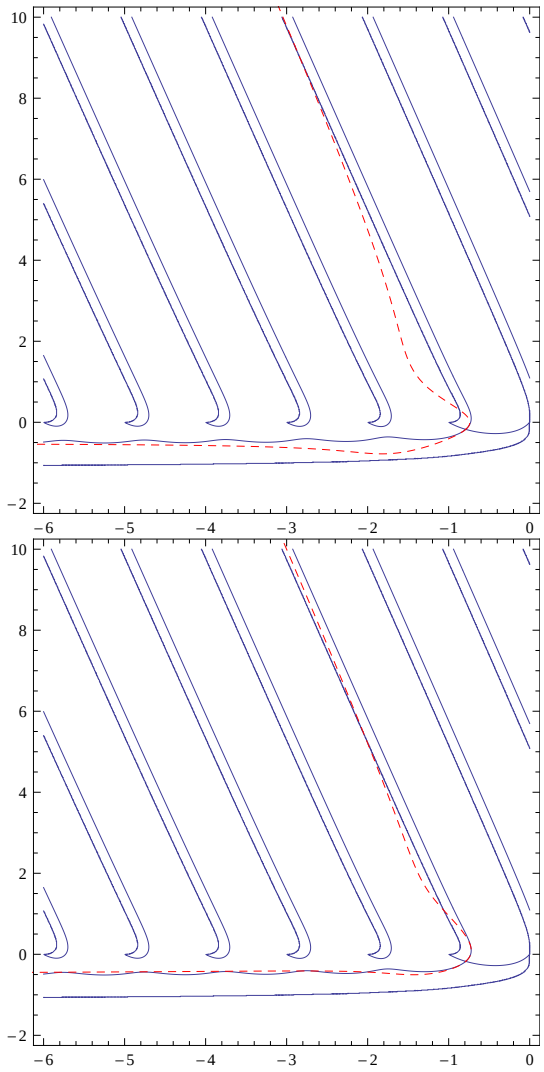

Figure 5.5: From up to down, the red dashed lines represent contours with $a = -1.26804$ contour, with $a = -0.931716 + 0.575462i$ (matched to the correct asymptotic form), and the blue line represents the exact contour.

5.2.3 Integrands with two stationary points in the fundamental region

In this section, I consider the integrand of eq. (5.10), multiplied by a polygamma function $\psi(z)$, or the same integrand shifted to a different fundamental region, in order to see how the stationary-phase contour can pass two stationary points of the integrand.

An example is given by the following integrand,

$$G(z) = \frac{u^z \Gamma^3(-z)\Gamma(z+1)\psi^{(0)}(z)}{\Gamma(-2z)}. \tag{5.40}$$

Take $u = 1$ so the integral is in the Euclidean domain. If we want to find a stationary point on the real axis, we must choose the fundamental region to be $\Re z \in (1, 2)$. We would find one local minimum on the real axis, at $z = 1.48088$. It lies between two zeros of the integrand, at $z = 1.46163$ and at $z = \frac{3}{2}$. However, the stationary-phase contour which passes through $z = 1.48088$ ends at the pole of the integrand at $z = 2$ (see fig. 5.6), so this would be a bad choice of base point. Instead we must seek stationary points off the

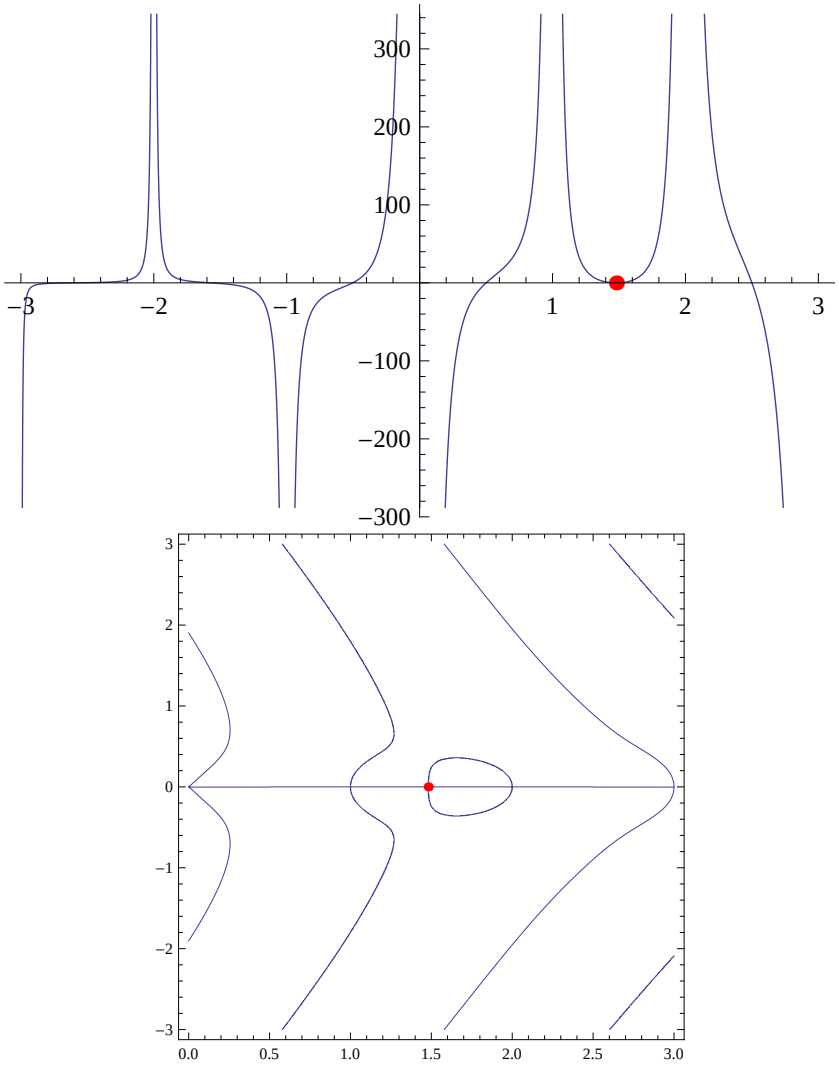

Figure 5.6: Up: the plot of $G(z) = \Gamma^3(-z)\Gamma(z+1)\psi^{(0)}(z)/\Gamma(-2z)$, where $-3 < z < 3$. Down: the plot of the stationary phase contour passing $z = 1.48088$ ends at the pole $z = 2$ of $G(z)$. The red points represent the stationary point point $z = 1.48088$ which is the only stationary point point on the real axis.

real axis. We could find a pair of complex-conjugate points in this fundamental region,

$$z_{0\pm} = 1.42804 \pm 0.43725i\,, \tag{5.41}$$

but we may as well use the complex conjugate pair we would have found in the original fundamental region, $-1 < \Re z < 0$. In this region, there is no stationary point on the real axis (see fig. 5.6). Looking for a minimum of eq. (5.14), we can however find two stationary points (see fig. 5.7),

$$z_{0\pm} = -0.587017 \pm 0.227805i, \quad \arg G(z_0) = \pm 2.177. \tag{5.42}$$

If we draw the exact stationary-phase contour, shown in fig. 5.7, we see that the contour has two segments, with different phases. These two segments intersect at,

$$z_{\text{join}} = -0.504083. \tag{5.43}$$

We notice that $G(z_{\text{join}})$ vanishes due to the polygamma function; two stationary-phase contours can only

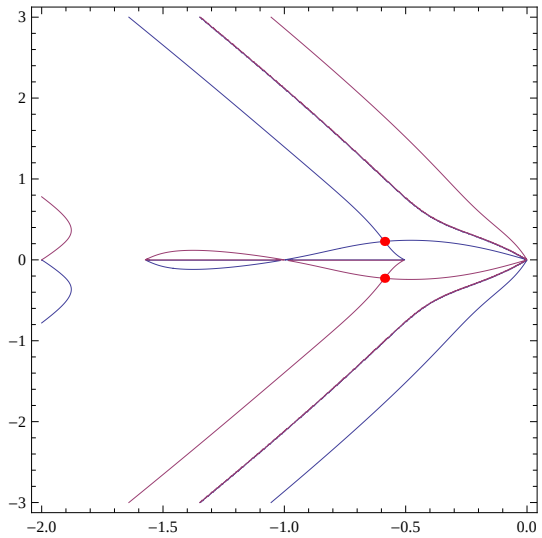

Figure 5.7: The red points are the local minimum which we found $z_0 = -0.587017 \pm 0.227805i$ for the integrand: $\Gamma^3(-z)\Gamma(z+1)\psi^{(0)}(z)/\Gamma(-2z)$

intersect with each other at zeros or singularities of the integrand. We can use the split ansatz:

$$\begin{aligned} z(t) &= z_{0+} + ite^{i\phi_{0+}} + at^2 + bt^3 \text{ when } \Im z(t) > 0, \\ z(t) &= z_{0-} + ite^{i\phi_{0-}} + at^2 + bt^3 \text{ when } \Im z(t) < 0 \end{aligned} \tag{5.44}$$

in the two halves of the complex plane. As usual, we can solve for ϕ_0 and a by expanding $G(z_0 + ite^{i\phi_0} + at^2)$ in series in t and solve the eqs. (5.32). The presence of the cubic term bt^3 allows us to ensure that the contour passes through the zero of the integrand. We can solve for b using the following equation,

$$\begin{aligned} &\Im z(t) = \Im z_{0\pm} t \cos\phi_{0\pm} = 0, \; t = -\frac{\Im z_{0\pm}}{\cos\phi_{0\pm}}, \\ &z(t = -\frac{\Im z_{0\pm}}{\cos\phi_{0\pm}}) = z_{\text{join}} = -0.504083 \end{aligned} \tag{5.45}$$

When $t \to \infty$, we can ensure the right asymptotic behavior of the contour by using the ansatz,

$$\begin{aligned} z(t) &= z_{+\infty} + i\, te^{i\phi_{\infty+}} \text{ when } \Im z(t) > 0, \\ z(t) &= z_{-\infty} + ite^{i\phi_{\infty-}} \text{ when } \Im z(t) < 0, \end{aligned} \tag{5.46}$$

and requiring that the phase match that at the corresponding base point. Indeed for Euclidean integrals, if the integrand vanishes at points (one or several) in the fundamental region, even if we are able to find a minimum of the integrand on the real axis, it may be better to look for the stationary points away from the real axis to serve as base points for the stationary-phase contour. Alternatively, we could shift the contour to a region where the integrand does not vanish so we can find a stationary point on the real axis. (In this case there would be additional contributions from crossing poles.) We could then apply the ansatz in eq. (5.27) given in an earlier subsection to compute an approximate contour.

Let us now examine the Minskowski region ($u < 0$), taking for example $u = -1$. The integrand is then,

$$G(z) = \frac{(-1)^z \Gamma^3(-z)\Gamma(z+1)\psi^{(0)}(z)}{\Gamma(-2z)}. \tag{5.47}$$

The stationary points in the fundamental region $-1 < \Re z < 0$, found by minimizing eq. (5.14) are at (see fig. 5.8),

$$\begin{aligned} z_{0+} &= -0.549808 + 0.153654i\,, \quad \arg G(z_{0+}) = 0.399108\,, \\ z_{0-} &= -0.709606 - 0.354869i\,, \quad \arg G(z_{0-}) = 2.11785\,. \end{aligned} \tag{5.48}$$

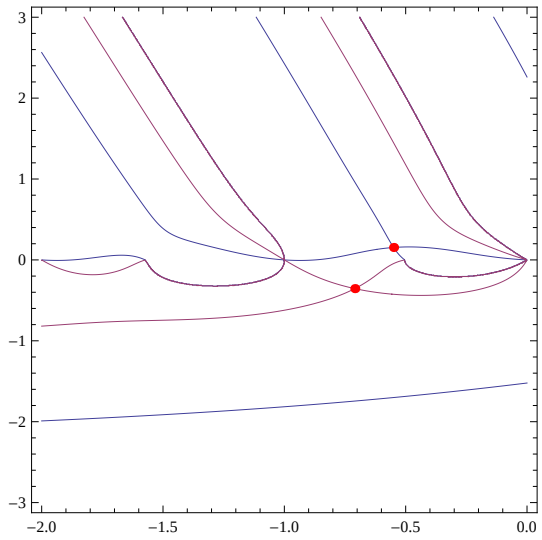

Figure 5.8: The red points are the local minimum which we found at z_{+0} and z_{-0} of $G(z) = (-1)^z\Gamma(-z)^3\Gamma(z+1)\psi^{(0)}(z)/\Gamma(-2z)$

We can use the ansatz given in eqs. (5.44) and (5.46) to parametrize the contour, just as in the Euclidean case with complex stationary points discussed in the beginning of this subsection.

The contour of stationary phase passes through two stationary points in this case. We could further generalize this for integrand whose contour of stationary phase passes stationary points from different fundamental region, if desired.

5.3 Two-Dimensional Integrals

In this section, I seek to extend the one-dimensional approach reviewed in the previous section to finding stationary-phase contours for higher-dimensional Mellin–Barnes integrals. In one-dimensional Mellin–Barnes integrals, the constraint of stationarity of the phase completely fixes the form of the contour. In addition, the singularities of the integrand are located only on the real axis. In n-dimensional Mellin–Barnes integrals, the subspace that satisfies the stationary-phase condition (5.9) is a $(2n-1)$-real-dimensional volume. The surface of integration is an n-real-dimensional subspace; our aim is to find a n-dimensional surface bypassing all the singularities within the stationary-phase volume. I will add an additional $n-1$ constraints in a very simple way in order to obtain a n-dimensional surface of stationary phase. The hyper-surface we will obtain in this chapter is then just one of infinitely many possibilities.

Let us focus on the two-dimensional case. We will be interested in Mellin–Barnes integrands $G(z_1,\ z_2)$ which are products of gamma functions whose arguments are linear combinations of the two integration variables z_1 and z_2,

$$\Gamma(a+bz_1),\ \Gamma(a+bz_2),\ \Gamma(a+bz_1+cz_2)\,, \tag{5.49}$$

along with polygamma functions with the same types of arguments. For example,

$$G(z_1,\ z_2) = \frac{u_1^{z_1}u_2^{z_2}\prod_i\Gamma(a_i+b_iz_1+c_iz_2)}{\prod_j\Gamma(d_j+e_jz_1+f_jz_2)}\left\{1, \prod_k \psi^{(f_k)}(p_kz_1+q_kz_2+r_k)\right\}. \tag{5.50}$$

The singularities of the integrand are no longer located only on the real slice $\Im z_1 = 0 = \Im z_2$, but also on the hyperplanes in (z_1, z_2) where the imaginary part of the arguments of gamma functions vanish, for example, $\Gamma(z_1 - z_2)$ has singularities on the hyperplane where $\Im(z_1 - z_2) = 0$, on these lines where $\Re(z_1 - z_2) = -k$, with k an integer. There are, roughly speaking, two kinds of singularities:

1. $z_i = -\frac{a}{b} - \frac{1}{b}k$;
2. $z_1 + \frac{c}{b}z_2 = -\frac{a}{b} - \frac{1}{b}k$ or $bz_1 + cz_2 = -a - k$.

where k is a non-negative integer.

We can now define two differences, Δ_1 and Δ_2, as sums of the coefficients of z_1 and of z_2 in the arguments of the gamma functions. For the integrand (5.50),

$$\Delta_1 = \sum_i b_i - \sum_j e_j, \quad \Delta_2 = \sum_i c_i - \sum_j f_j. \tag{5.51}$$

We are interested in the case that both Δ_1 and Δ_2 vanish. The examples I consider satisfy both of these conditions. To find contours for two or higher-dimensional Mellin–Barnes integrals, we will apply the following procedures:

1. the deformed contour $\mathcal{C}'$ should pass through one or more zeros (z_{10}, z_{20}) of

$$\frac{\partial G(z_1, z_2)}{\partial z_1} = 0 = \frac{\partial G(z_1, z_2)}{\partial z_2}; \tag{5.52}$$

2. $\mathcal{C}'$ should be inside the subspace where $\arg G(z_1, z_2)$ is constant.

5.3.1 Parametrization of the 2-D Surface in Euclidean Case with one single stationary point in the fundamental region

First, we take a look at the Euclidean case, with $u_{1,2} > 0$, which is simpler than the Minkowski one. We could evaluate such integrals by integrating along the naive contour, and examine a stationary-phase choice for contrast. Consider the following explicit example,

$$\begin{aligned} G(z_1,\ z_2) = \quad & \frac{u_1^{z_1} u_2^{z_2} \Gamma^2(-z_1)\Gamma(-z_2)\Gamma^3(z_2+1)\Gamma^2(-z_1-z_2-1)}{\Gamma(2z_2+2)} \\ & \times\Gamma(z_1+z_2+2)\Gamma(2z_1+2z_2+2)\Gamma(z_1-z_2). \end{aligned} \tag{5.53}$$

The fundamental region surrounds $\Re z_1 = -1/3$, $\Re z_1 = -1/7$. The integrand never vanishes in this region, so the stationary point is located on the real slice and can be found by solving eq. (5.52) or, equivalently by finding the minimum of the function:

$$\left| \frac{1}{G(z_1,\ z_2)} \frac{\partial G(z_1,\ z_2)}{\partial z_1} \right|^2 + \left| \frac{1}{G(z_1,\ z_2)} \frac{\partial G(z_1,\ z_2)}{\partial z_2} \right|^2 \tag{5.54}$$

For the given example, there is a lone stationary point which is also the minimum of the integrand in the fundamental region so that we do not need to shift the contour and calculate additional residues. Were we to fail to find a stationary point in that region, we would loosen this condition, and perhaps shift the contour to a new region.

For the above example, when $u_1 = 20$, $u_2 = 1/3$, we find the following stationary point $z_1 = z_{10}$ and $z_2 = z_{20}$,

$$z_{10} = -0.577373,\ \ z_{20} = -0.0981137\,. \tag{5.55}$$

For further convenience, since we will need to integrate in the region $-\infty < \Im z_1 < \infty$ and $-\infty < \Im z_2 < \infty$, let us use a polar coordinate system to represent $\Im z_1$ and $\Im z_2$. We may parametrize,

$$\begin{aligned} \Im z_1 &= t \sin\theta, \\ \Im z_2 &= t \cos\theta, \end{aligned} \tag{5.56}$$

where $t > 0$ and $0 < \theta < 2\pi$. We use the same group of parameters for all regions of $\Im z_1$ and $\Im z_2$, which permit us to glue an approximation near the stationary point with an asymptotic form in a simple manner, as stated in the introduction. This simple change of variables applies only to this simple Euclidean case. In the remaining examples of two-dimensional integrals, the variables $(\Im z_1, \Im z_2)$ and (θ, t) are not simply related to each other. We take $\Re z_1$ and $\Re z_2$ to be general real functions of t and θ. Our ansatz is then,

$$\begin{aligned} z_1(\theta, t) &= z_{10} + it \sin\theta + f_1(\theta, t), \\ z_2(\theta, t) &= z_{20} + it \cos\theta + f_2(\theta, t) \end{aligned} \tag{5.57}$$

where $f_1(\theta, t)$ and $f_2(\theta, t)$, as inspired by the one-dimensional case, have the form $P(m,t)/Q(n,t)$ where the coefficients of t in the $P(m,t)$ are functions of θ, and m and n are the orders of polynomials P and Q. Our aim is to require the surface to avoid all the singularities. In the one-dimensional case, if the imaginary part of variable z does not vanish, then every imaginary part of argument of gamma function does not vanish either, so the contour can avoid all the singularities. In higher-dimension case, it is not easy to make every imaginary part of every argument of a gamma function to be nonvanishing on the whole surface. In this ansatz, we require that wherever the imaginary part of arguments of gamma functions vanish, $f_1(\theta, t)$, $f_2(\theta, t)$ also vanish. This imposes the following constraints to the functions $f_1(\theta, t)$ and $f_2(\theta, t)$, no matter what the value of t is.

1. $f_1(\theta, t)$ and $f_2(\theta, t)$ are real periodic functions of θ. The period is π.
2. If the integrand contains $\Gamma(bz_1 + d)$, then for $\theta = 0,$ and π, $\Im z_1 = t \sin(k\pi) = 0 = f_1(0, t) = f_1(\pi, t)$;
3. If the integrand contains $\Gamma(cz_2 + d)$, then for $\theta = \pi/2,$ and $3\pi/2$,$\Im z_2 = t \cos(k\pi/2) = 0 = f_2(\pi/2, t) = f_2(3\pi/2, t)$;
4. If the integrand contains $\Gamma(z_1 - z_2 + c)$, then for $\theta = \pi/4,$ and $5\pi/4$, $\Im(z_1 - z_2) = 0 = f_1(\theta, t) - f_2(\theta, t)$;
5. If the integrand contains $\Gamma(z_1 + z_2 + c)$, then for $\theta = 3\pi/4,$ and $7\pi/4$, $\Im(z_1 + z_2) = 0 = f_1(\theta, t) + f_2(\theta, t)$;
6. In general, if the integrand contains $\Gamma(bz_1 + cz_2 + d)$, then $\forall$ b and c, If $\Im(bz_1 + cz_2) = 0$, then $bf_1(\theta, t) + cf_2(\theta, t) = 0$.

As concerns the first constraint, we want the period of $f_1(\theta, t)$ and $f_2(\theta, t)$ to be π instead of 2π because the Euclidean integral is real, so that we expect that the value of the integrand has the following property on whatever contour we find,

$$G(z_1, z_2) = \overline{G}(z_1, z_2) = G(\overline{z_1}, \overline{z_2}) \tag{5.58}$$

so that

$$z_1 = \overline{z_1},\ z_2 = \overline{z_2}. \tag{5.59}$$

For our ansatz, the property (5.59) reads,

$$\begin{aligned} \overline{z_1(\theta, t)} = &\quad z_{10} - it \sin\theta + f_1(\theta, t) = z_{10} + it \sin(\theta + \pi) + f_1(\theta, t), \\ = &\quad z_1(\theta + \pi, t) = z_{10} + it \sin(\theta + \pi) + f_1(\theta + \pi, t) \\ \overline{z_2(\theta, t)} = &\quad z_{20} - it \cos\theta + f_2(\theta, t) = z_{10} + it \cos(\theta + \pi) + f_2(\theta, t) \\ = &\quad z_2(\theta + \pi, t) = z_{20} + it \cos(\theta + \pi) + f_2(\theta + \pi, t)\,; \end{aligned} \tag{5.60}$$

accordingly,

$$f_1(\theta+\pi,t)=f_1(\theta,t),\ f_2(\theta+\pi,t)=f_2(\theta,t). \tag{5.61}$$

Let's see an example to understand how the ansatz (5.57) with the above constraints can avoid the singularities of a Γ function. The singularities of $\Gamma(bz_1+cz_2+d)$ are the surfaces:

$$\Im(bz_1+cz_2+d)=\Im(bz_1+cz_2)=0,\ \Re(bz_1+cz_2+d)=-k \tag{5.62}$$

where k is a non-negative integer. Substitute the ansatz (5.57) in $\Im(bz_1+cz_2+d)$:

$$\begin{aligned}&\Im(bz_1+cz_2)=t\,(b\sin\theta+c\cos\theta)=0\\ &\Downarrow \text{ (with the constraints imposed on the functions } f_1(\theta,t) \text{ and } f_2(\theta,t))\\ &\Re(bz_1+cz_2+d)=b\,z_{10}+c\,z_{20}+d+(bf_1(\theta,t)+cf_2(\theta,t))=b\,z_{10}+c\,z_{20}+d\neq -k,\end{aligned} \tag{5.63}$$

so long as the integrand is not singular at our stationary point (z_{10},z_{20}). We can indeed verify that with these constraints, the ansatz surface can avoid all singularities of the integrand. Next we will expand $f_1(\theta,t)$, $f_2(\theta,t)$ as a series at $t=0$ and $t=+\infty$, we demand that every coefficient, which is a function of θ, should also meet the requirement for the functions $f_1(\theta,t)$ and $f_2(\theta,t)$.

For $|z_1-z_{10}|\lesssim 1$ and $|z_2-z_{20}|\lesssim 1$ or $t\lesssim 1$, use a quadratic ansatz:

$$z_1(\theta,t)=z_{10}+i\,t\,\sin\theta+F_s(\theta)t^2 \tag{5.64}$$

where $0<t$, $0<\theta<2\pi$,

$$f_1(\theta,t)=F_s(\theta)t^2+\mathcal{O}[t^3] \text{ near } t=0. \tag{5.65}$$

$f_1(\theta,t)$ is real function so that $F_s(\theta)$ is a real function. The constraint on $f_1(\theta,t)$ must also be applied to $F_s(\theta)$.

The imaginary part of z_1 vanishes when:

$$\begin{aligned}&t=0,\ z_1(\theta,0)=z_{10},\\ &\forall t,\ \text{for } \theta=0,\text{ or }\pi,\ z_1(\theta,t)=z_{10}+F_s(\theta)t^2=z_{10}\end{aligned} \tag{5.66}$$

So we must require $F_s(0)=F_s(\pi)=0$. The ansatz for z_2 can be chosen as:

$$z_2=z_{20}+i\,t\,\cos\theta+F_s(\theta)\cot\theta t^2 \tag{5.67}$$

We also require that $F_s(\pi/2)$ and $F_s(3\pi/2)$ be finite so that $F_s(\pi/2)\cot(\pi/2)=F_s(3\pi/2)\cot(3\pi/2)=0$; and we also require that $F_s(\theta)\cot\theta$ be finite when $0\leq\theta<2\pi$. Now, we can verify for the argument of $\Gamma(bz_1+cz_2+d)$,

$$bz_1(\theta,t)+cz_2(\theta,t)+d=bz_{10}+cz_{20}+d+i\,t\,(b\sin\theta+c\cos\theta)+(bF_s(\theta)+cF_s(\theta)\cot\theta)t^2. \tag{5.68}$$

We can see how this surface avoid the poles involving z_1 and z_2: if for some $\theta=\theta'$ the imaginary part of $bz_1(\theta,t)+cz_2(\theta,t)+d$ vanishes,

$$\Im(bz_1(\theta',t)+cz_2(\theta',t)+d)=t\,(b\sin\theta'+c\cos\theta')=0 \tag{5.69}$$

then

$$(bF_s(\theta')+cF_s(\theta')\cot\theta')t^2=\frac{F_s(\theta')}{\sin\theta'}(b\sin\theta'+c\cos\theta')=0, \tag{5.70}$$

as $F_s(\theta)\cot\theta$ is finite so the prefactor $F_s(\theta)/\sin\theta$ is finite when $0\leq\theta<2\pi$. So when the imaginary part of $bz_1(\theta,t)+cz_2(\theta,t)+d$ vanishes, the real part of $bz_1(\theta,t)+cz_2(\theta,t)+d$ is constant:

$$\Re(bz_1(\theta',t)+cz_2(\theta',t)+d)=bz_{10}+cz_{20}+d+(bF_s(\theta)+cF_s(\theta')\cot\theta')t^2=bz_{10}+cz_{20}+d. \tag{5.71}$$

The choice (5.67) is not unique. There are infinitely many choices like that of $F_s(\theta)\cot\theta$ in z_2 which also satisfy the conditions I impose here.

The choice made here permits us to solve for the unknown function $F_s(\theta)$ in the ansatz analytically, when we have a single stationary point located on the real slice in a fundamental region where the integral doesn't generically vanish.

The basic idea is to solve for the unknown functions in the ansatz; this generalizes the idea of solving for the unknown parameters in one-dimensional integrals. We substitute the ansatz into the integrand and expand it in power series in t up to the third order. We require that every term in this series expansion have the same phase, which is simply 0 in the Euclidean case:

$$\begin{aligned} G(z_1(\theta,t),z_2(\theta,t)) &= G(z_{10},z_{20}) + \frac{dG(z_{10},z_{20})}{dt}t + \frac{d^2G(z_{10},z_{20})}{dt^2}t^2 \\ &\quad + \frac{d^3G(z_{10},z_{20})}{dt^3}t^3 + \mathcal{O}[t^4] \\ &= c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \mathcal{O}[t^4]. \end{aligned} \tag{5.72}$$

We want the imaginary part of this series expansion to vanish. Let's look term by term. The first term, $c_0 = G(z_{10},z_{20})$, is real. The next term is,

$$\Im c_1 = \Im\frac{dG(z_{10},z_{20})}{dt} = \sin\theta G^{(1,0)}(z_{10},z_{20}) + \cos\theta G^{(0,1)}(z_{10},z_{20}) = 0. \tag{5.73}$$

This term vanishes trivially because (z_{10},z_{20}) is a stationary point of the integrand (5.52). The following term,

$$\begin{aligned} c_2 = &\quad \frac{d^2G(z_{10},z_{20})}{dt^2} \\ = &\quad F_s(\theta)\cot\theta\ G^{(0,1)}(z_{10},z_{20}) + F_s(\theta)G^{(1,0)}(z_{10},z_{20}) - \tfrac{1}{2}\sin^2\theta\, G^{(2,0)}(z_{10},z_{20}) \\ &\quad -\tfrac{1}{2}\cos^2\theta\, G^{(0,2)}(z_{10},z_{20}) - \sin\theta\ \cos\theta\, G^{(1,1)}(z_{10},z_{20}), \end{aligned} \tag{5.74}$$

is a real function so its imaginary part vanishes.

If we demand that the following term vanish,

$$\begin{aligned} \Im c_3 = \Im\frac{d^3G(z_{10},z_{20})}{dt^3} = 0 \quad &= \frac{1}{6\sin\theta}(-6F_s(\theta)\sin^2\theta\, G^{(2,0)}(z_{10},z_{20}) - 6F_s(\theta)\cos^2\theta\, G^{(0,2)}(z_{10},z_{20}) \\ &\quad -12F_s(\theta)\sin\theta\ \cos\theta\, G^{(1,1)}(z_{10},z_{20}) + \sin^4\theta\, G^{(3,0)}(z_{10},z_{20}) \\ &\quad + \sin\theta\ \cos^3\theta\, G^{(0,3)}(z_{10},z_{20}) + 3\sin^2\theta\ \cos^2\theta\, G^{(1,2)}(z_{10},z_{20}) \\ &\quad +3\sin^3\theta\ \cos\theta\, G^{(2,1)}(z_{10},z_{20})), \end{aligned} \tag{5.75}$$

we have a linear equation in $F_s(\theta)$, and we can solve for it,

$$\begin{aligned} F_s(\theta) = &\quad \frac{\sin^4\theta G^{(3,0)}(z_{10},z_{20}) + \cos^3\theta\sin\theta G^{(0,3)}(z_{10},z_{20})}{6\left(\sin^2\theta G^{(2,0)}(z_{10},z_{20}) + 2\cos\theta\sin\theta G^{(1,1)}(z_{10},z_{20}) + \cos^2\theta G^{(0,2)}(z_{10},z_{20})\right)} \\ &+\frac{3\sin^2\theta\cos^2\theta G^{(1,2)}(z_{10},z_{20}) + 3\sin^3\theta\cos\theta G^{(2,1)}(z_{10},z_{20})}{6\left(\sin^2\theta G^{(2,0)}(z_{10},z_{20}) + 2\cos\theta\sin\theta G^{(1,1)}(z_{10},z_{20}) + \cos^2\theta G^{(0,2)}(z_{10},z_{20})\right)} \end{aligned} \tag{5.76}$$

At the local maximum of the integrand, we have:

$$\begin{aligned} &G^{(2,0)}(z_{10},\ z_{20})G^{(0,2)}(z_{10},\ z_{20}) - (G^{(1,1)}(z_{10},\ z_{20}))^2 > 0\,, \\ &G^{(2,0)}(z_{10},\ z_{20}) < 0,\ G^{(0,2)}(z_{10},\ z_{20}) < 0\,. \end{aligned} \tag{5.77}$$

The denominator of $F_s(\theta)$ does not vanish, thanks to eq. (5.77). Also, $F_s(\theta) \propto \sin\theta$ so $F_s(\theta)\cot\theta \propto \cos\theta$ satisfies the condition given earlier. For the given example, $F_s(\theta)$ can be calculated:

$$F_s(\theta) = \frac{105.8\sin^3\theta - 1450\cos^3\theta + 367\sin\theta\cos^2\theta + 381\sin^2\theta\cos\theta}{-248\sin\theta - 267\cos\theta - 696\cos\theta\cot\theta} \tag{5.78}$$

where the coefficients are all numerical (not exact)

In the region where the integration variables $|z_1| \gg 1$ and $|z_2| \gg 1$, just as the in one-dimensional case, we perform the asymptotic replacements (5.21). The integrand $G(z_1,\ z_2)$ becomes

$$R_{G(z_1\to\infty, z_2\to\infty)} \exp(i\arg(G(z_1\to\infty, z_2\to\infty))), \tag{5.79}$$

where

$$R_{G(z_1\to\infty, z_2\to\infty)} \sim |G(z_1\to\infty, z_2\to\infty)|. \tag{5.80}$$

We require the asymptotic value of the integrand on the surface we are looking to be a real number, so that the phase of the integrand over the whole contour is $0 \mod 2\pi$,

$$\arg(G(z_1\to\infty, z_2\to\infty) = 0 \mod 2\pi. \tag{5.81}$$

The expression for $\arg(G(z_1\to\infty, z_2\to\infty))$ is shown below,

$$\begin{aligned}\arg(G(z_1\to\infty, z_2\to\infty))) = \quad & \Im(z_1)\ln\left(\frac{80|z_1-z_2||z_1+z_2|}{|z_1|}\right) + \Im(z_2)\ln\left(\frac{|z_1+z_2|}{3|z_1-z_2|}\right) \\ & +\Re(z_1)(2\pi\text{sign}\Im(z_1+z_2) + \arg(z_1-z_2) + \arg(z_1+z_2) - 2\arg(-z_1)) \\ & -\Re(z_2)(2\pi\text{sign}\Im(z_1+z_2) + \arg(z_1-z_2) - \arg(z_1+z_2) \\ & + \arg(-z_2) - \arg(z_2)) \\ & +\frac{1}{2}(6\pi\text{sign}\Im(z_1+z_2) - \arg(z_1-z_2) - 2\arg(-z_1) - \arg(-z_2)).\end{aligned} \tag{5.82}$$

This is almost a linear equation in $\Re(z_1)$, $\Im(z_1)$, $\Re(z_2)$ and $\Im(z_2)$.

When $t \gg 1$, use the ansatz

$$\begin{aligned} z_1 &= z_{10\infty}(\theta) + i\,t\,\sin\theta + F_l(\theta)t, \\ z_2 &= z_{20\infty}(\theta) + i\,t\,\cos\theta + F_l(\theta)\cot\theta\ t \end{aligned} \tag{5.83}$$

when $t\to\infty$, $z_1/z_2 = \tan\theta$. The coefficients of $\Im(z_1)$ and of $\Im(z_2)$ in eq. (5.82) are then free of the scale parameter t. Then we expand $\arg(G(z_1\to\infty, z_2\to\infty))$ as a series about $t=\infty$. We extract the coefficient of t in this series and we require it to vanish so that $\arg(G(z_1\to\infty, z_2\to\infty))$ is not proportional to t. This ansatz permits us to write this constraint as a linear equation in $F_l(\theta)$ and also permits us to solve for $F_l(\theta)$ explicitly as a function of θ. We obtain:

$$F_l(\theta) = \frac{F_{l,N}}{F_{l,D}}, \tag{5.84}$$

$$F_{l,N} = \cos\theta\ln\left(\frac{\sin(2\theta)+1}{9(\cos\theta-\sin\theta)^2}\right) + \sin\theta\ln\left(6400\cos^2(2\theta)\csc^4\theta\right). \tag{5.85}$$

And

$$\begin{aligned}
F_{l,D} = \quad & 4\tan^{-1}(-F_l(\theta), -\sin\theta) + 2\cot\theta\tan^{-1}(-F_l(\theta)\cot\theta, -\cos\theta) \\
& -2\cot\theta\tan^{-1}(F_l(\theta)\cot\theta, \cos\theta) + 4\tan^{-1}(-F_l(\theta)(\cot\theta+1), -\sin\theta-\cos\theta) \\
& -6\tan^{-1}(F_l(\theta)\cot\theta + F_l(\theta), \sin\theta+\cos\theta) - 2\tan^{-1}(F_l(\theta) - F_l(\theta)\cot\theta, \sin\theta-\cos\theta) \\
& +4\cot\theta\tan^{-1}(-F_l(\theta)(\cot\theta+1), -\sin\theta-\cos\theta) \\
& -6\cot\theta\tan^{-1}(F_l(\theta)\cot\theta + F_l(\theta), \sin\theta+\cos\theta) \\
& +2\cot\theta\tan^{-1}(F_l(\theta) - F_l(\theta)\cot\theta, \sin\theta-\cos\theta), \\
= \quad & \pi\cot(\theta)\left[2\text{sign}(\cos\theta) + \text{sign}(\cos\theta-\sin\theta) + 5\text{sign}(\sin\theta+\cos\theta)\right] \\
& +\pi\left[2\text{sign}(\sin\theta) + \text{sign}(\sin\theta-\cos\theta) + 5\text{sign}(\sin\theta+\cos\theta)\right]
\end{aligned} \tag{5.86}$$

where $\tan^{-1}(x,y)$ gives the arc tangent of y/x, taking into account which quadrant the point (x,y) is in. At first glance, this solution is useless, because $F_{l,D}$ appears to depend on $F_l(\theta)$. If we first simplify the equation in different intervals of θ, however, we will find that in fact $F_{l,D}$ is independent of $F_l(\theta)$, and we could just set $F_l(\theta) = 1$ in the above expression to obtain the solution. Equivalently, we can use

$$\tan^{-1}(ax, b) = \tan^{-1}\left(x, \frac{b}{a}\right) - \pi\Theta(-a)\text{sign}\left(\frac{b}{a}\right), \tag{5.87}$$

where $\Theta(-a)$ is the step function defined in eq. (5.39); the dependence on $F_l(\theta)$ in $F_{l,D}$ will cancel out. We can check this result by using `FindRoot` and `Plot` in *Mathematica* to draw the solution of eq. (5.84) numerically and compare the two plots to see that we obtain the right solution for $F_l(\theta)$. Setting $F_l(\theta) = 1$ in $F_{l,D}$, we obtain

$$\begin{aligned}
F_{l,D} = \quad & 4\tan^{-1}(-1, -\sin\theta) + 2\cot\theta\tan^{-1}(-\cot\theta, -\cos\theta) - 2\cot\theta\tan^{-1}(\cot\theta, \cos\theta) \\
& +4\tan^{-1}(-\cot\theta-1, -\sin\theta-\cos\theta) - 2\tan^{-1}(1-\cot\theta, \sin\theta-\cos\theta) \\
& -6\tan^{-1}(\cot\theta+1, \sin\theta+\cos\theta) + 4\cot\theta\tan^{-1}(-\cot\theta-1, -\sin\theta-\cos\theta) \\
& +2\cot\theta\tan^{-1}(1-\cot\theta, \sin\theta-\cos\theta) - 6\cot\theta\tan^{-1}(\cot\theta+1, \sin\theta+\cos\theta)
\end{aligned} \tag{5.88}$$

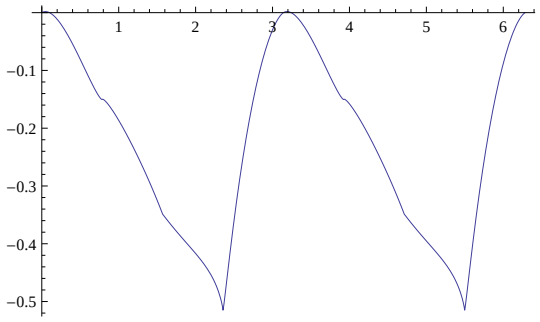

Figure 5.9: Plot of $F_l(\theta)$

The functions $F_s(\theta)$ and $F_l(\theta)$ are well defined and finite everywhere in the interval $\theta \in (0, 2\pi)$ (see fig. 5.9).

Next we write down a very simple Padé approximation,

$$\begin{aligned} z_1 &= z_{10} + i\,t\,\sin\theta + \frac{F_s(\theta)t^2 + F_l(\theta)t^p}{1+t^{p-1}} \\ z_2 &= z_{20} + i\,t\,\cos\theta + \cot\theta\frac{F_s(\theta)t^2 + F_l(\theta)t^p}{1+t^{p-1}}, \end{aligned} \tag{5.89}$$

where $p \geq 4$.

The numerical evaluation on the stationary phase contour agrees with the result obtained by integration over the naive contour. For a Mellin–Barnes integrand of which the stationary point is on the real slice $\Im z_1 = \Im z_2 = 0$, the ansatz gives an approximation to a stationary-phase contour. The figs. 5.10, 5.11 show the value of the integrand on the surface we found. The way we define the functions F_s and F_l will be generalized for yet-higher dimensions in the next section.

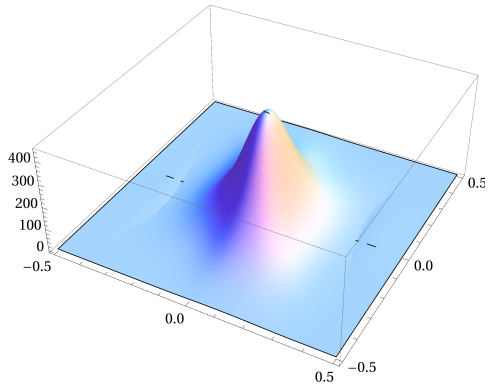

Figure 5.10: The value of logarithm of the the integrand $|\Re G(z_1, z_2)|$ on the surface we found when $-1/2 < \Im z_1 < 1/2$, $-1/2 < \Im z_2 < 1/2$

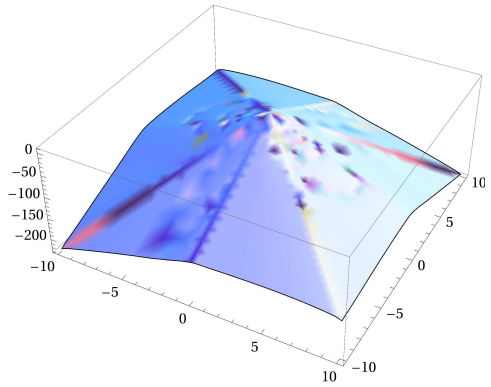

Figure 5.11: The value of the integrand $\ln|\Re G(z_1, z_2)|$ on the surface we found when $-10 < \Im z_1 < 10$, $-10 < \Im z_2 < 10$

In the final part of this section, let us discuss how to solve for $z_{10\infty}(\theta)$ and $z_{20\infty}(\theta)$ in the ansatz (5.83). We again expand $\arg(G(z_1 \to \infty, z_2 \to \infty))$ as a series at $t = \infty$. We take the imaginary part of this series expansion. In addition to the term proportional to t, we have a "constant" term which is still a function

of θ. It is a linear function of $z_{10\infty}(\theta)$ and $z_{20\infty}(\theta)$. For the given example, it is

$$\begin{aligned}
&(\pi\text{sign}(\sin\theta) + \frac{\pi}{2}\text{sign}(\sin\theta - \cos\theta) + \frac{5\pi}{2}\text{sign}(\sin\theta + \cos\theta))z_{10\infty}(\theta) \\
&+(\pi\text{sign}(\cos\theta) - \frac{\pi}{2}\text{sign}(\sin\theta - \cos\theta) + \frac{5\pi}{2}\text{sign}(\sin\theta + \cos\theta))z_{20\infty}(\theta) \\
&+\frac{3}{2}\pi\text{sign}(\sin\theta) + \frac{1}{4}\pi\text{sign}(\cos\theta) - \frac{1}{4}\pi\text{sign}(\sin\theta - \cos\theta) + 3\pi\text{sign}(\sin\theta + \cos\theta) \\
&-2\tan^{-1}(F_l(\theta), \sin\theta) = 0 \mod 2\pi.
\end{aligned} \tag{5.90}$$

We have two unknown functions with only one equation. In general, we cannot solve for $z_{10\infty}(\theta)$ and $z_{20\infty}(\theta)$. I propose an ansatz for solving this equation, but will not offer a generalization to higher dimensions. We can plot the coefficients of $z_{10\infty}(\theta)$ and $z_{20\infty}(\theta)$, and notice that these two coefficients are not continuous functions of θ (see fig. 5.12). Numerical investigations show that the coefficients would be

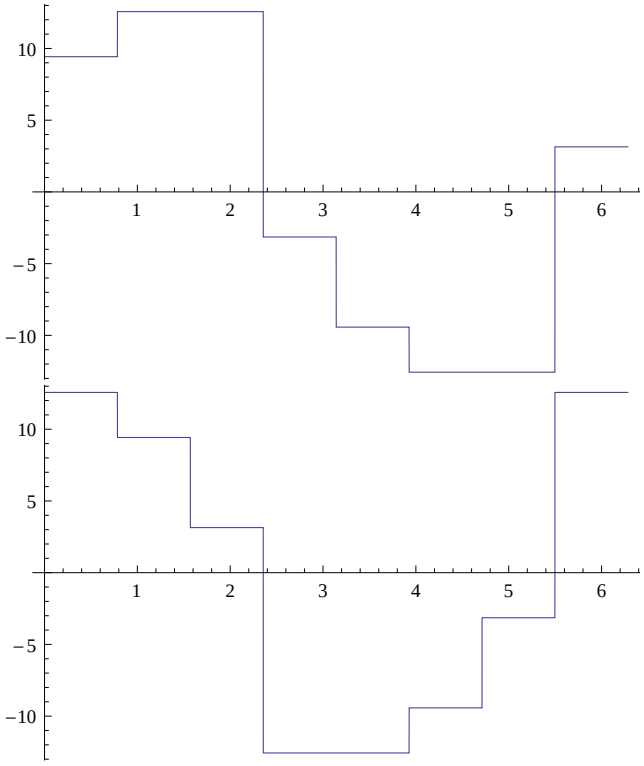

Figure 5.12: From up to down: Plots of the coefficient of $z_{10\infty}(\theta)$ and the one of $z_{20\infty}(\theta)$

discontinuous at values of $\theta = \theta'$, where, given a factor $\Gamma(bz_1 + cz_2 + d)$ in the integrand, θ' satisfies

$$\Im(bz_1 + cz_2 + d) = b\sin\theta' + c\cos\theta' = 0. \tag{5.91}$$

For the given example, θ' makes one of $\sin\theta' \pm \cos\theta'$, $\sin\theta'$, or $\cos\theta'$ vanish. That is, θ' can be one of 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$, $\frac{7\pi}{4}$, 2π.

We would like the asymptotic points $z_{10\infty}(\theta)$ and $z_{20\infty}(\theta)$ to be continuous periodic functions of θ. The period of$z_{10\infty}(\theta)$ and $z_{20\infty}(\theta)$ is π as we require the property (5.59). Because the coefficient of $z_{20\infty}(\theta)$ never vanishes, we can solve for $z_{20\infty}(\theta)$ as a function of $z_{10\infty}(\theta)$ (choosing $0 \mod 2\pi = 0$ in eq. (5.90); we

could also choose $0 \mod 2\pi = 2\pi k$ where k is integer),

$$\begin{aligned} z_{20\infty}(\theta) = & \frac{8\tan^{-1}(F_l(\theta), \sin\theta) - 6\pi \text{sign}(\sin\theta) - \pi \text{sign}(\cos\theta)}{2\pi(2\text{sign}(\cos\theta) - \text{sign}(\sin\theta - \cos\theta) + 5\text{sign}(\sin\theta + \cos\theta))} \\ & + \frac{\pi \text{sign}(\sin\theta - \cos\theta) - 12\pi \text{sign}(\sin\theta + \cos\theta)}{2\pi(2\text{sign}(\cos\theta) - \text{sign}(\sin\theta - \cos\theta) + 5\text{sign}(\sin\theta + \cos\theta))} \\ & + \frac{z_{10\infty}(\theta)(-4\pi \text{sign}(\sin\theta) - 2\pi \text{sign}(\sin\theta - \cos\theta) - 10\pi \text{sign}(\sin\theta + \cos\theta))}{2\pi(2\text{sign}(\cos\theta) - \text{sign}(\sin\theta - \cos\theta) + 5\text{sign}(\sin\theta + \cos\theta))}. \end{aligned} \tag{5.92}$$

The requirement of continuity when $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$ gives us numerical values for $z_{10\infty}$ at various angles,

$$\begin{aligned} & z_{20\infty}(2\pi - \delta) = z_{20\infty}(\delta) \Rightarrow z_{10\infty}(0) = -0.499985, z_{20\infty}(0) = -0.638886, \\ & z_{20\infty}(\frac{\pi}{4} - \delta) = z_{20\infty}(\frac{\pi}{4} + \delta) \Rightarrow z_{10\infty}(\frac{\pi}{4}) = -0.26669, z_{20\infty}(\frac{\pi}{4}) = -0.76668, \\ & z_{20\infty}(\frac{\pi}{2} - \delta) = z_{20\infty}(\frac{\pi}{2} + \delta) \Rightarrow z_{10\infty}(\frac{\pi}{2}) = -0.634099, z_{20\infty}(\frac{\pi}{2}) = -0.2500, \\ & z_{20\infty}(\frac{3\pi}{4} - \delta) = z_{20\infty}(\frac{3\pi}{4} + \delta) \Rightarrow z_{10\infty}(\frac{\pi}{2}) = -0.46638, z_{20\infty}(\frac{\pi}{2}) = -0.7336, \\ & z_{20\infty}(\pi - \delta) = z_{20\infty}(\pi + \delta) \Rightarrow z_{10\infty}(\pi) = -0.499985, z_{20\infty}(\pi) = -0.638886, \end{aligned} \tag{5.93}$$

where $\delta = 10^{-5}$. That is, we have five constraints for $z_{10\infty}(\theta)$. As explained above, we want $z_{10\infty}(\theta)$ to be a periodic function with the period π. We can thus take it to have the following form,

$$\begin{aligned} & z_{10\infty}(\theta) = a + b\sin\theta + c\cos\theta + d\sin(2\theta) + e\cos(2\theta),\ \theta \in (0,\ \pi), \\ & z_{10\infty}(\theta) = a + b\sin(\theta - \pi) + c\cos(\theta - \pi) + d\sin(2\theta - 2\pi) + e\cos(2\theta - 2\pi),\ \theta \in (\pi,\ 2\pi), \end{aligned} \tag{5.94}$$

Using the five constraints (5.93), we can solve for the unknown coefficients in last formula

$$a = -1.0511,\ b = 0.968123,\ c = 0.,\ d = 0.099847,\ e = 0.551118. \tag{5.95}$$

We plot the function $z_{10\infty}(\theta)$ (see fig. 5.13). While $z_{10\infty}(\theta)$ is continuous, its derivative is not. We can

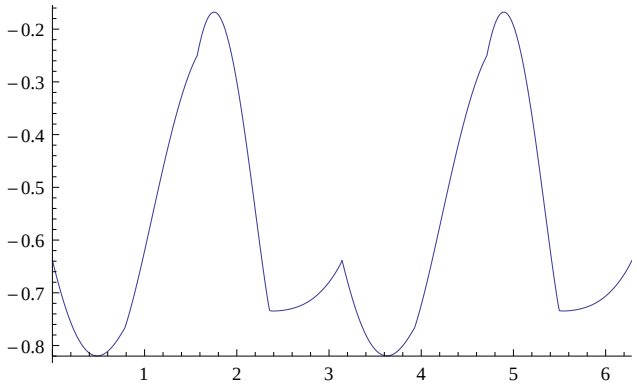

Figure 5.13: The plot of $z_{10\infty}(\theta)$

finally perform the following replacement in the ansatz (5.89)

$$\begin{aligned} z_{10} & \to z_{10}\frac{1 + t\, z_{10\infty}(\theta)/z_{10}}{1 + t} \\ z_{20} & \to z_{20}\frac{1 + t z_{20\infty}(\theta)/z_{20}}{1 + t} \end{aligned} \tag{5.96}$$

to obtain an approximation to a contour of stationary phase. This does not provide a universal solution for the asymptotic region, however, because the values of $z_{10\infty}(\theta)$ and $z_{20\infty}(\theta)$ are determined solely by the continuity requirement (5.93) (with θ drawn from solutions to eq. (5.91)). These requirements do not prevent their values, in particular of the imaginary parts of $z_1 \pm z_2$, from vanishing, and hence landing on a pole of a gamma function.

5.4 Higher Dimension Euclidean Case with the extremum on the real slice

For n-dimensional Euclidean integrals, if we find only one stationary point on the real slice in a fundamental region, for example by minimizing

$$\sum_{k=1}^{n} \left| \frac{1}{G(z_1, \cdots, z_n)} \frac{\partial G(z_1, \cdots, z_n)}{\partial z_k} \right|, \tag{5.97}$$

where $G(z_1, \cdots, z_n)$ is the integrand of the integral, and if the integrand has no zero in that region, we can simply generalize the ansatz (5.89) to the n-dimensional case. The ansatz for the j-th variable z_j is:

$$z_j(\phi_1, \cdots, \phi_{n-1}) = z_{j0} + i x_j t + \frac{x_j}{x_1} \frac{F_s t^2 + F_l t^p}{1 + t^{p-1}}, \; p \geq 4 \tag{5.98}$$

where

$$\begin{aligned} x_j &= \prod_{k=1}^{j-1} \sin\theta_k \, \cos\theta_j, \text{ where } j < n \\ x_n &= \prod_{k=1}^{n-1} \sin\theta_k, \end{aligned} \tag{5.99}$$

where $0 \leq \theta_j \leq \pi$ when $j \neq n$, $0 \leq \theta_n < 2\pi$, and F_s and F_l are functions of θ_j, $j = 1, \cdots, n-1$. We can solve for F_s as follows. We substitute the ansatz (5.98) into the integrand and expand it in a power series in t near 0, up to the third order. We require that every term in this series expansion have the same phase, which is simply 0 in the Euclidean region. We obtain,

$$G(z_1, \cdots, z_n) = G(z_{10}, \cdots, z_{n0}) + \frac{dG(z_{10}, \cdots, z_{n0})}{dt} t + \frac{d^2 G(z_{10}, \cdots, z_{n0})}{dt^2} t^2 + \frac{d^3 G(z_{10}, \cdots, z_{n0})}{dt^3} t^3 + \mathcal{O}[t^4]. \tag{5.100}$$

Here, $G(z_{10}, \cdots, z_{n0})$ is real because the stationary point is in the real slice. Every term in $\dfrac{dG(z_{10}, \cdots, z_{20})}{dt}$ is proportional to a partial derivative $\dfrac{\partial G(z_1, \cdots, z_n)}{\partial z_k}$, so it vanishes at the stationary point. The second derivative $\dfrac{d^2 G(z_{10}, \cdots, z_{20})}{dt^2}$ is a real function so its imaginary part vanishes automatically. We must require

$$\Im \frac{d^3 G(z_{10}, \cdots, z_{20})}{dt^3} = 0\,. \tag{5.101}$$

We can solve the above equation for F_s because it is a linear equation.

In the region where $t \gg 1$, we perform the asymptotic replacements (5.21). The integrand $G(z_1, \cdots, z_n)$ becomes

$$R_{G(z_1 \to \infty, \cdots, z_n \to \infty)} \exp(i \arg(G(z_1 \to \infty, \cdots, z_n \to \infty))), \tag{5.102}$$

where

$$R_{G(z_1 \to \infty, \cdots, z_n \to \infty)} \sim |G(z_1 \to \infty, \cdots, z_n \to \infty)|. \tag{5.103}$$

Then we expand $\arg(G(z_1 \to \infty, \cdots, z_n \to \infty))$ as a series about $t = \infty$. We extract the coefficient of t in this series and we require it to vanish so that $\arg(G(z_1 \to \infty, \cdots, z_n \to \infty))$ is not proportional to t. This

ansatz (5.98) permits us to write this constraint as a linear equation in $F_l(\theta)$ and also permits us to solve for $F_l(\theta)$ explicitly as a function of $\theta_1, \cdots, \theta_{n-1}$. Here we can give a brief example:

$$\frac{1}{(2\pi i)^3}\iiint_{\mathcal{C}} u_1^{z_1}u_2^{z_2}u_3^{z_3}\frac{\Gamma^2(-z_3-1)\Gamma(z_3+1)\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_3-z_2-1)\Gamma(-z_3+z_1-z_2)}{\Gamma(-z_3)\Gamma(-z_3-z_2)^2} \tag{5.104}$$

where u_1, u_2 and u_3 are set to 1. We find a stationary points

$$z_{10} = -0,95310,\ z_{20} = -0.22476,\ z_{30} = -0.15258. \tag{5.105}$$

We just give the plots of $F_s(\theta_1\ \theta_2)$, $F_l(\theta_1\ \theta_2)$ (see fig. 5.14). The numerical evaluation of the integral on

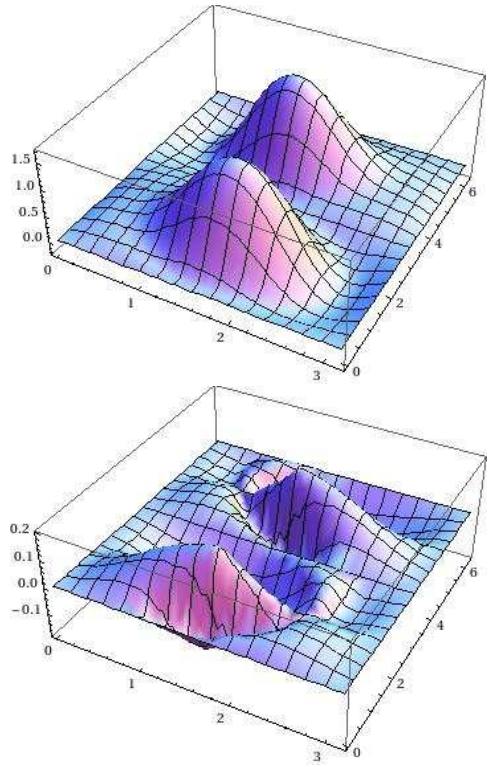

Figure 5.14: From up to down: the plots of $F_s(\theta_1\ \theta_2)$, $F_l(\theta_1,\ \theta_2)$ of the integrand.

the surface obtained by substituting the solution $F_s(\theta_1,\ \theta_2)$, $F_l(\theta_1,\ \theta_2)$ into the ansatz (5.98) agrees with that on the naive contour.

5.5 A First Minkowski Integral

In this section, I return to the two-dimensional case, considering an example in the Minkowski region, both $u_1 < 0$ and $u_2 < 0$. With these arguments, the particular integral I will consider can be integrated along the naive contour; in the next section, I will consider values of the arguments for which this is no longer true.

One example can be given here is

$$\begin{aligned} G(z_1,z_2) = \quad & u_1^{z_1}u_2^{z_2}\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma^2(z_2+1)\Gamma^2(-z_1-z_2-1)}{\Gamma(2z_2+2)} \\ & \times\Gamma(z_1+z_2+2)\Gamma(2z_1+2z_2+2), \end{aligned} \tag{5.106}$$

where u_1 and u_2 are set to $-1/5+i\delta$ and $-7/5+i\delta$, $0<\delta\ll 1$. As discussed earlier in the one-dimensional case, we should first find the stationary points, that is the points at where the first derivatives of integrand vanish. Here, z_{10} and z_{20} can be found numerically by looking for the minimum of eq. (5.54). We choose the fundamental region to be the one surrounding $\Re z_1=-1/3$, $\Re z_1=-1/7$. The stationary point is

$$z_{10}=-0.279124-0.0525383i,\ z_{20}=-0.260158-0.0419805i. \tag{5.107}$$

In general, the stationary point is now located out at complex values, not on the real slice.

Consider the following hybrid of one-dimensional and two-dimensional Euclidean ansatz, in the region where z_1 and z_2 are close to the stationary point,

$$\begin{aligned} z_1(\theta,t) &= z_{10}+i\,t\,\sin\theta\, e^{i\phi_0(\theta)}+F_s(\theta)t^2,\\ z_2(\theta,t) &= z_{20}+i\,t\,\cos\theta\, e^{i\phi_0(\theta)}+\cot\theta\, F_s(\theta)t^2 \end{aligned} \tag{5.108}$$

We want to keep the imaginary part of $z_1(\theta,t)$ and $z_2(\theta,t)$ as simple as possible. So we take $F_s(\theta)$ to be a real function. Denoting the integrand by $G(z_1(\theta,t),z_2(\theta,t))$, we expand it as a power series in t. We can solve for $\phi_0(\theta)$ and $F_s(\theta)$ as follows. As before, we substitute the above ansatz into the integrand and expand it in a power series in t near 0 up to the third order. We require that the every term in this series expansion have the same phase, which is $\arg G(z_{10},z_{20})$. We still obtain eq. (5.72).

$$c_0=G(z_{10},z_{20}). \tag{5.109}$$

The coefficient c_1 of t^1 vanishes because each term in c_1 is proportional to

$$\frac{\partial G(z_{10},z_{20})}{\partial z_{10}}\ \text{or}\ \frac{\partial G(z_{10},z_{20})}{\partial z_{20}}. \tag{5.110}$$

Explicitly, the coefficient c_2 of t^2 is

$$c_2=-\frac{1}{2}e^{2i\phi_0}\left(\sin^2\theta\, G^{(2,0)}(z_{10},z_{20})+\sin(2\theta)G^{(1,1)}(z_{10},z_{20})+\cos^2\theta\, G^{(0,2)}(z_{10},z_{20})\right), \tag{5.111}$$

while the coefficient c_3 of t^3 is,

$$\begin{aligned} c_3 = \quad & iF_s(\theta)e^{i\phi_0}\sin\theta G^{(2,0)}(z_{10},z_{20})+2iF_s(\theta)e^{i\phi_0}\cos\theta G^{(1,1)}(z_{10},z_{20})\\ & +iF_s(\theta)e^{i\phi_0}\cos\theta\cot\theta G^{(0,2)}(z_{10},z_{20})-\frac{1}{6}ie^{3i\phi_0}\sin^3\theta\, G^{(3,0)}(z_{10},z_{20})\\ & -\frac{1}{6}ie^{3i\phi_0}\cos^3\theta\, G^{(0,3)}(z_{10},z_{20})-\frac{1}{2}ie^{3i\phi_0}\sin\theta\cos^2\theta\, G^{(1,2)}(z_{10},z_{20})\\ & -\frac{1}{2}ie^{3i\phi_0}\sin^2\theta\,\cos\theta G^{(2,1)}(z_{10},z_{20}). \end{aligned} \tag{5.112}$$

We require that c_2 and c_0 have the same phase, that is,

$$\Im(-\frac{c_2}{c_0})=0. \tag{5.113}$$

We can solve for ϕ_0 as a function of θ:

$$\phi_0(\theta)=-\frac{1}{2}\arg\left(\frac{\sin^2\theta\, G^{(2,0)}(z_{10},z_{20})+\cos^2\theta\, G^{(0,2)}(z_{10},z_{20})+2\sin\theta\,\cos\theta\, G^{(1,1)}(z_{10},z_{20})}{G(z_{10},z_{20})}\right). \tag{5.114}$$

We also require that,

$$\Im\frac{c_3}{c_0}=0, \tag{5.115}$$

The last formula is a linear equation in F_s, so that we can find F_s as a function of θ:

$$\begin{aligned}
F_s(\theta) = \quad & \frac{F_{s,N}}{F_{s,D}}, \\
F_{s,N}(\theta) = \quad & \Im(ie^{3i\phi_0}(\sin^3\theta G^{(3,0)}(z_{10},z_{20}) + \cos^3\theta\, G^{(0,3)}(z_{10},z_{20}) + 3\sin\theta\,\cos^2\theta\, G^{(1,2)}(z_{10},z_{20}) \\
& +3\sin^2\theta\,\cos\theta\, G^{(2,1)}(z_{10},z_{20}))/G(z_{10},z_{20})), \\
F_{s,D}(\theta) = \quad & 6\frac{1}{\sin\theta}\Im(ie^{i\phi_0}(\sin^2\theta\, G^{(2,0)}(z_{10},z_{20}) + 2\cos\theta\,\sin\theta\, G^{(1,1)}(z_{10},z_{20}) \\
& +\cos^2\theta\, G^{(0,2)}(z_{10},z_{20}))/G(z_{10},z_{20})).
\end{aligned} \tag{5.116}$$

For the given example, the function ϕ_0 can be easily found by substituting the value of z_{10} and z_{20} in eq. (5.114),

$$\begin{aligned}
\phi_0(\theta) = \quad & -\frac{1}{2}\arg[(-0.11977 + 0.992802i)(-(86.7918\ + 561.191i)\sin^2\theta \\
& +(12.9862\ - 750.997i)\sin\theta\cos\theta + (-90.8047 - 615.223i)\cos^2\theta)].
\end{aligned} \tag{5.117}$$

The function F_s can be found by substituting the value of z_{10} and z_{20} in eq. (5.116),

$$F_s(\theta) = \frac{F_{s,N}(\theta)}{F_{s,D}(\theta)}, \tag{5.118}$$

where

$$\begin{aligned}
F_{s,N}(\theta) = \quad & \Re[(-0.00778181 + 0.0645055i)e^{3i\phi_0}(-(1500.41\ + 536.665i)\sin^3\theta \\
& -(1959.62\ - 1584.56i)\sin\theta\cos^2\theta - (1959.62\ - 1584.56i)\sin^2\theta\cos\theta \\
& +(-1611 - 816.12i)\cos^3\theta)],
\end{aligned} \tag{5.119}$$

$$\begin{aligned}
F_{s,D}(\theta) = \quad & 6\Re[(-0.00778181 + 0.0645055i)e^{i\phi_0}(-(86.7918\ + 561.191i)\sin\theta \\
& +(12.9862\ - 750.997i)\cos\theta - (90.8047\ + 615.223i)\cos\theta\cot\theta)].
\end{aligned} \tag{5.120}$$

At last, one will notice that unlike in one-dimensional integral case, I did not consider taking $F_s(\theta)$ to be a complex function. If we solve for $\Re F_s(\theta) \in \mathcal{R}$ and $\Im F_s(\theta) \in \mathcal{R}$ by considering the higher power series coefficient constraints, then $\Re F_s(\theta)$ and $\Im F_s(\theta)$ are not continuous function of θ (their denominators may vanish for some θ).

For $z_1 \gg 1$ and $z_2 \gg 1$ or $t \gg 1$, we perform the asymptotic replacements (5.21), and the integrand $G(z_1,\ z_2)$ becomes,

$$R_{G(z_1\to\infty,z_2\to\infty)}\exp(i\arg(G(z_1\to\infty,z_2\to\infty))), \tag{5.121}$$

where

$$R_{G(z_1\to\infty,z_2\to\infty)} \sim |G(z_1\to\infty,z_2\to\infty)|. \tag{5.122}$$

For the given example, $\arg(G(z_1 \to \infty, z_2 \to \infty))$ is

$$\begin{aligned}\arg(G(z_1 \to \infty, z_2 \to \infty)) = \quad & \Im(z_1) \ln\left(4\left|\frac{u_1(z_1+z_2)}{z_1}\right|\right) + \Im(z_2)\ln\left(\left|\frac{u_2(z_1+z_2)}{z_2}\right|\right) \\ & + \Re(z_1)(\arg(u_1) + 2\pi \text{sign}(\Im(z_1+z_2)) + \arg(z_1+z_2) - \arg(-z_1)) \\ & + \Re(z_2)(\arg(u_2) + 2\pi \text{sign}(\Im(z_1+z_2)) + \arg(z_1+z_2) - \arg(-z_2)) \\ & + 3\pi \text{sign}(\Im(z_1+z_2)) - \frac{1}{2}\arg(-z_1) - \frac{\arg(-z_2)}{2} - \frac{\arg(z_2)}{2}.\end{aligned} \tag{5.123}$$

We will use the same ansatz in Euclidean case (5.83). Then we expand $\arg(G(z_1 \to \infty, z_2 \to \infty))$ in a series about $t = \infty$. We extract the coefficient of t in this series and we require it to vanish so that $\arg(G(z_1 \to \infty, z_2 \to \infty))$ is not proportional to t. This ansatz permits us to write this constraint as a linear equation in $F_l(\theta)$ and also permits us to solve for $F_l(\theta)$ explicitly as a function of θ. For the given example, the solution for F_l is as follows,

$$F_l(\theta) = \frac{F_{l,N}}{F_{l,D}}, \tag{5.124}$$

where

$$F_{l,N} = \ln(25)\sin\theta + \cos\theta \ln\left(\frac{25\cos^2\theta}{49(\sin\theta+\cos\theta)^2}\right) - \sin\theta \ln\left(16\left(2\cot\theta + \csc^2\theta\right)\right), \tag{5.125}$$

$$\begin{aligned}F_{l,D} = \quad & -2\tan^{-1}(-1, -\sin\theta) - 2\cot\theta\tan^{-1}(-\cot\theta, -\cos\theta) \\ & -4\tan^{-1}(-\cot\theta - 1, -\sin\theta - \cos\theta) + 6\tan^{-1}(\cot\theta + 1, \sin\theta + \cos\theta) \\ & -4\cot\theta\tan^{-1}(-\cot\theta - 1, -\sin\theta - \cos\theta) + 6\cot\theta\tan^{-1}(\cot\theta + 1, \sin\theta + \cos\theta) \\ & +2\pi\cot\theta + 2\pi,.\end{aligned} \tag{5.126}$$

Once we've solved for $\phi_0(\theta)$, $F_s(\theta)$, and $F_l(\theta)$, we can write down the ansatz for this integral:

$$\begin{aligned}z_1(\theta,t) &= z_{10} + \frac{t^4 F_l(\theta) + t^2 F_s(\theta)}{t^3+1} + \frac{ite^{i\phi_0(\theta)}\sin\theta\left(t^3 e^{-i\phi_0(\theta)} + 1\right)}{t^3+1}, \\ z_2(\theta,t) &= z_{20} + \frac{\cot\theta\left(t^4 F_l(\theta) + t^2 F_s(\theta)\right)}{t^3+1} + \frac{ite^{i\phi_0(\theta)}\cos\theta\left(t^3 e^{-i\phi_0(\theta)} + 1\right)}{t^3+1}\end{aligned} \tag{5.127}$$

The relation between $(\Im z_1, \Im z_2)$ and (t, θ) is not simple any more, so we will not solve for (t, θ) as a function of $(\Im z_1, \Im z_2)$.

The functions $\phi_0(\theta)$, $F_s(\theta)$ and $F_l(\theta)$ are well-defined and finite everywhere in the interval $\theta \in (0, 2\pi)$. If we calculate the intersection of our contour with the real slice, we can see that it is still in the fundamental region, so we do not need to shift the integral. We can evaluate the integral on this contour numerically, and find that the result agrees with the evaluation on the naive contour. In fig. 5.15, fig. 5.16 and fig. 5.17, we show the behavior of the integrand on the surface we found.

In the next section, we will talk about the second kind of Minkowski integral case. If we try to apply the ansatz in this section, we will find around θ like $3\pi/4$, F_l goes to $-\infty$.

5.6 A Second Minkowski Integral

For other arguments to the integral considered in the previous section, the integration fails to converge numerically along the naive contour. Considering the example (5.106) in the Minkowski region, $u_1 = 1/5 > 0$ and $u_2 = -7/5 < 0$. We will set $u_2 = -7/5 + \delta i$ in eq. (5.106), where $0 < \delta \ll 1$ and the intersection

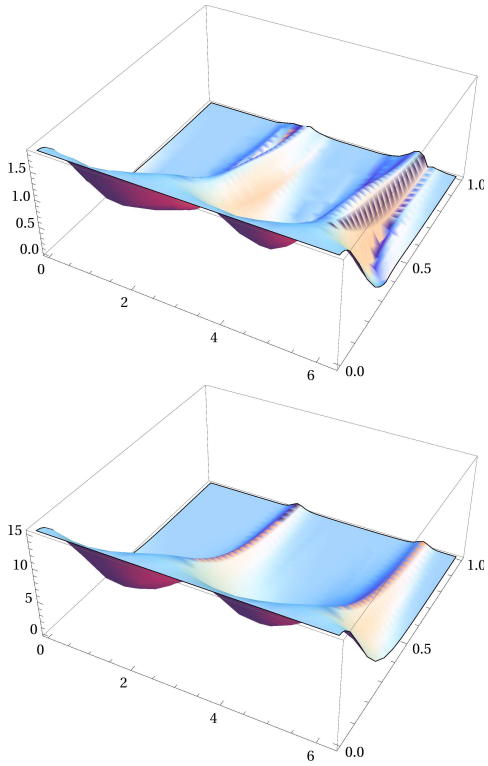

Figure 5.15: From up to down: the absolute value of real and imaginary part of the first Minkowski integrand on the approximated surface of stationary phase we found, $0 < \theta < 2\pi$, $0 < t < 1$

of the contour with the real slice is at $C_1 = -1/3$, $C_2 = -1/4$. The fundamental region surrounds this intersection.

In this case, we want to use a unified ansatz, which is general enough for both Euclidean integrals and Minkowski ones. For all these cases, our ansatz is

$$\begin{aligned} z_1 &= z_{10} + i\,t\,\sin\theta \frac{\exp i\phi_0(\theta)\ (1 + t^p \exp i(\phi_\infty(\theta) - \phi_0(\theta)))}{1+t^p} + \frac{F_s(\theta)t^2}{1+t^q}\,, \\ z_2 &= z_{20} + i\,t\,\cos\theta \frac{\exp i\phi_0(\theta)\ (1 + t^p \exp i(\phi_\infty(\theta) - \phi_0(\theta)))}{1+t^p} + \frac{\cot\theta\ F_s(\theta)t^2}{1+t^q}\,, \end{aligned} \tag{5.128}$$

where $p > 1$ and $q \geq 3$. We apply the ansatz to the example in this section. As usual, minimizing the function (5.54), we find

$$z_{10} = -0.2908925 + 0.08522724i, z_{20} = -0.2311106 - 0.12499303i, \tag{5.129}$$

Substitute the ansatz into the integrand and expand it in a power series in t near 0 up to third order. This yields the series expansion eq. (5.72). We demand that the phase of every term in (5.72) be the same as the value of the phase of the integrand at the stationary point. Then we can solve for $\phi_0(\theta)$ using the eq. (5.113). The expression for $\phi_0(\theta)$ was previously given above in eq. (5.114). Also, we can solve for $F_s(\theta)$ using eq. (5.116) just as in the first Minkowski case explained in the previous section. In the region where $|z_1| \gg 1$ and $|z_2| \gg 1$ or $t \gg 1$, the ansatz (5.128) can be expanded around $t = \infty$. The expansion yields,

$$\begin{aligned} z_1 &= z_{10} + i\,t\,\sin\theta \exp i\phi_\infty(\theta)\,, \\ z_2 &= z_{20} + i\,t\,\cos\theta \exp i\phi_\infty(\theta)\,. \end{aligned} \tag{5.130}$$

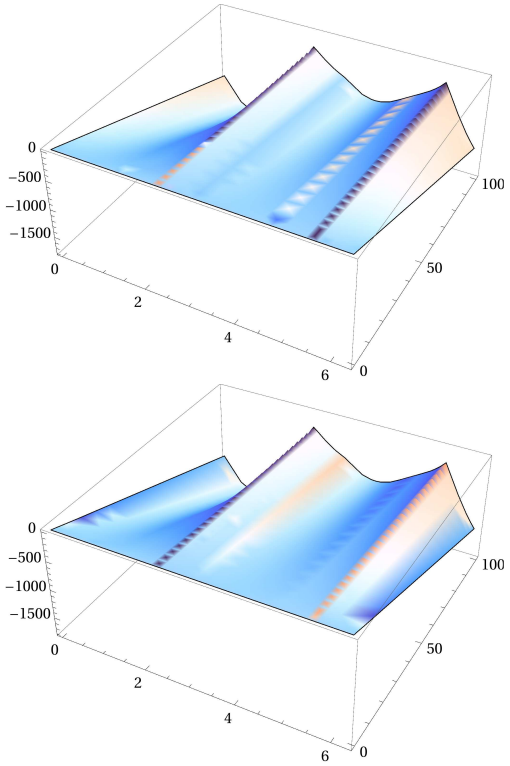

Figure 5.16: From up to down: the logarithmic absolute value of real and imaginary part of the first Minkowski integrand on the approximated surface of stationary phase we found, $0 < \theta < 2\pi$, $0 < t < 100$

We perform the asymptotic replacements (5.21), and the integrand $G(z_1,\ z_2)$ becomes,

$$R_{G(z_1\to\infty,z_2\to\infty)}\exp(i\arg(G(z_1\to\infty,z_2\to\infty))), \tag{5.131}$$

where

$$R_{G(z_1\to\infty,z_2\to\infty)} \sim |G(z_1\to\infty,z_2\to\infty)|. \tag{5.132}$$

The expression of $\arg(G(z_1\to\infty,z_2\to\infty))$ was previously given in eq. (5.123). Then we expand $\arg(G(z_1\to\infty,z_2\to\infty))$ as a series about $t=\infty$. We extract the coefficient of t in this series and we require it to vanish so that $\arg(G(z_1\to\infty,z_2\to\infty))$ is not proportional to t. In fact, if we use the ansatz in $t\gg 1$ region of the section eq. (5.83), $F_l(\theta)$ is not finite for $\theta\to 3\pi/4$. To stabilize the numerical evaluation, we use the expression $i\,t\,\sin\theta\exp(i\phi_\infty(\theta))$ to represent a straight line in the complex space. The function $\phi_\infty(\theta)$ can be found using from the constraints that the series expansion coefficient of t in $\arg(G(z_1\to\infty,z_2\to\infty))$ is 0. Since the equation for $\exp\phi_\infty(\theta)$ is linear, it is straight forward to obtain the solution $\phi_\infty(\theta)$. In the given example:

$$\phi_\infty(\theta) = -\arg[\cos\theta f_{\cos}(\theta)+\sin\theta f_{\sin}(\theta)], \tag{5.133}$$

where

$$\begin{aligned} f_{\cos}(\theta) = \quad & -2\tan^{-1}(\cos\theta,-\cos\theta)+6\tan^{-1}(-\sin\theta-\cos\theta,\sin\theta+\cos\theta) \\ & -4\tan^{-1}(\sin\theta+\cos\theta,-\sin\theta-\cos\theta)-i\ln\left(\frac{49}{25}(\tan\theta+1)^2\right)+2\pi, \end{aligned} \tag{5.134}$$

$$\begin{aligned} f_{\sin}(\theta) = \quad & -2\tan^{-1}(\sin\theta,-\sin\theta)+6\tan^{-1}(-\sin\theta-\cos\theta,\sin\theta+\cos\theta) \\ & -4\tan^{-1}(\sin\theta+\cos\theta,-\sin\theta-\cos\theta)-i\ln(\sin(2\theta)+1)-i\ln\left(\frac{16\csc^2\theta}{25}\right). \end{aligned} \tag{5.135}$$

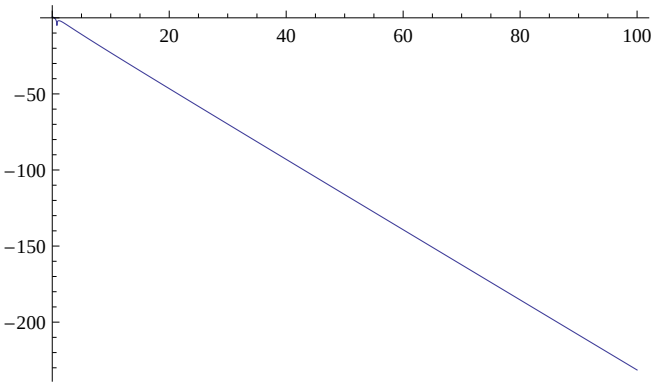

Figure 5.17: The logarithmic absolute value of real part of the first Minkowski integrand on the we found, $\theta = 3/4\pi$ one of the slowest convergent direction, $0 < t < 100$.

Again, $\phi_0(\theta)$, $\phi_\infty(\theta)$ and $F_s(\theta)$ are finite and continuous everywhere when θ varies between 0 and 2π. On the contour (5.128), the phase is not really a constant but changes slowly and is asymptotically free of t. For the example we've given in this section, the plots of $\phi_0(\theta)$, $F_s(\theta)$ and $\phi_\infty(\theta)$ (see fig. 5.18): in the plot of $\phi_\infty(\theta)$, $\lim_{\theta\to 3\pi/4}\phi_\infty(\theta) = -\pi/2$. It means that when $\theta \to 3\pi/4$, the contour is gradually parallel to the real axes of z_1 and z_2. As always, we should calculate the intersection of the obtained contour with the real slice to decide if we need to calculate some residues. For this given example, the intersection of the contour and the real slice is still in the original fundamental region. Then we can use the contour to evaluate the integral numerically. In fig. 5.19, fig. 5.20 and fig. 5.21, we present the values of the integrand on the surface we found.

With the contour given by eq. (5.128), we can evaluate the given integral numerically. We need another way to evaluate the same integral to check the obtained result. Indeed, for the given two-dimensional integral, I could not find a simple, separable contour ($\Re z_1 = \Re z_1(t)$, $\Im z_1 = \Im z_1(t)$, $\Re z_2 = \Re z_2(v)$, $\Im z_2 = \Im z_2(v)$ such as naive contour, etc.) which gave a numerically convergent result. The evaluation of this integration was performed in the following way. We replace the integration variable z_1 by $z_1' - z_2$, then the integrand (5.106) with $u_1 = 1/5$ and $u_2 = -7/5 + \delta i$ becomes

$$(\frac{1}{5})^{z_1'}(-7+\delta i)^{z_2}\frac{\Gamma^2(-z_1'-1)\Gamma(z_1'+2)\Gamma(2z_1'+2)\Gamma(-z_2)\Gamma^2(z_2+1)\Gamma(z_2-z_1')}{\Gamma(2z_2+2)}. \tag{5.136}$$

The contour changes to $C_1' = C_1 + C_2 = -7/12$. Next, calculate the sum of all the residues z_2 at $0, 1, \cdots$, which are all the residues of z_2 on the right-hand side of the contour. We obtain,

$$-(\frac{1}{5})^{z_1'}\Gamma^2(-z_1'-1)\Gamma(-z_1')\Gamma(2(z_1'+1))\Gamma(z_1'+2)\,{}_2F_1\left(1, -z_1', \frac{3}{2}, -\frac{-7+\delta i}{4}\right). \tag{5.137}$$

The minus sign comes from the fact that we sum the residues on the right-hand side of the contour. The positive number δ keeps the function on the correct side of the branch-cut, since this integral is already a Euclidean integral, then we can use the naive contour to evaluate this one-dimensional integral. We can check that the result obtained by integrating over the two-dimensional contour (5.128) agrees with the result we obtained here.

In the rest of this chapter, we try to combine the ansatz of stationary phase (we define

$$F_s'(\theta) = \frac{F_s(\theta)}{\sin\theta} \tag{5.138}$$

so that the new ansatz has a totally symmetric structure for the complex variables z_1 and z_2). The

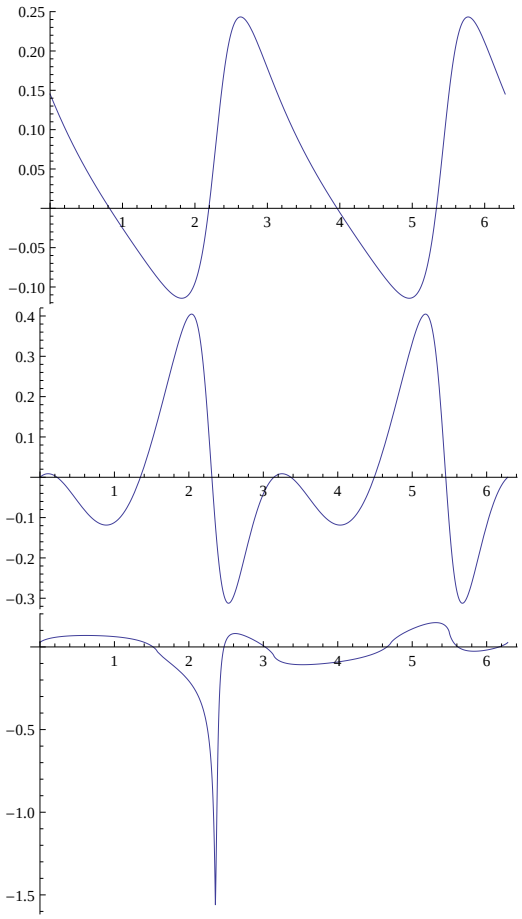

Figure 5.18: From up to down, the plots of $\phi_0(\theta)$, $F_s(\theta)$ and $\phi_\infty(\theta)$ for second Minkowski integrand

ansatz (5.128) becomes:

$$\begin{aligned} z_1(\theta,t) &= z_{10} + \sin\theta f(\theta,t), \\ z_2(\theta,t) &= z_{20} + \cos\theta f(\theta,t), \\ f(\theta,t) &= it\frac{\exp i\phi_0(\theta)\ (1+t^p \exp i(\phi_\infty(\theta)-\phi_0(\theta)))}{1+t^p} + \frac{F_s'(\theta)t^2}{1+t^q} \end{aligned} \tag{5.139}$$

to fix the stationary surface in two-complex dimensions. It seems that this is a possible way to generalize the one-dimensional case to higher dimension case for the case that there is only one stationary point in the fundamental region. The ansatz (5.139) implied a constraint over the variables $\Re z_1$, $\Im z_1$. $\Re z_2$, $\Im z_2$. The constraint reads:

$$\frac{\Re(z_1(\theta,t)-z_{10})}{\Re(z_2(\theta,t)-z_{20})} = \frac{\Im(z_1(\theta,t)-z_{10})}{\Im(z_2(\theta,t)-z_{20})}, \tag{5.140}$$

where (z_{10}, z_{20}) is a stationary phase point. For the Euclidean case, we can find only one stationary point on the real slice, $\Im(z_{10}) = \Im(z_{20}) = 0$, as explained in eq. (5.63), we obtain a contour from the ansatz (5.139) can definitely avoid the singularities of the integrand.

For Minkowski case, in general, we can find that the stationary point is complex, $\Im(z_{10}) \neq 0$, $\Im(z_{20}) \neq 0$.

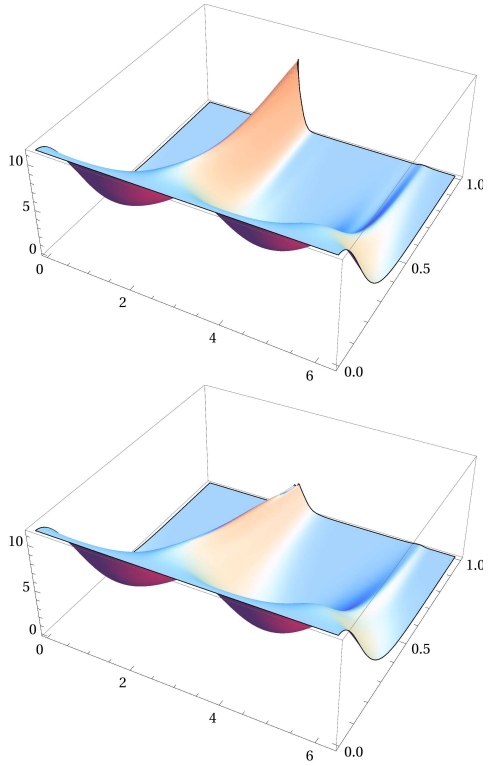

Figure 5.19: From up to down: the absolute value of real and imaginary part of the integrand in Minkowski type 2 on the approximated surface of stationary phase we found, $0 < \theta < 2\pi$, $0 < t < 1$

The constraint (5.140) is unchanged. For a gamma function $\Gamma(az_1 + bz_2 + c)$, if

$$\Im(az_1 + bz_2 + c) = (a\sin\theta + b\cos\theta)\Im(f(\theta,t)) + \Im(az_{10} + bz_{20}) = 0 \text{ where } \Im(f(\theta,t)) = 0 \Leftrightarrow t = 0$$

$$\not\Downarrow$$

$$\Re(az_1 + bz_2 + c) = (a\sin\theta + b\cos\theta)\Re(f(\theta,t)) + \Re(az_{10} + bz_{20} + c) \neq \pm k, \tag{5.141}$$

where k is an integer. It is theoretically possible that the contour could pass through a singularity of the Γ function. But in practice, for all the examples we investigated in this chapter, the contour does avoid all singularities of the integrand and also leads to the right result. So in this chapter, for two-dimensional integrals, we will still apply this ansatz (5.139).

5.7 The Integrand with One PolyGamma Functions with Two Stationary Points in the Fundamental Region

In this section, the integrand (5.106) is multiplied by a polygamma function. One example is:

$$G(z) = \frac{u_1^{z_1}u_2^{z_2}\Gamma(-z_2)\Gamma^2(z_2+1)\Gamma^2(-2z_1-z_2-1)\Gamma^2(z_1+2z_2+2)\Gamma(2z_1-z_2+2)\psi^{(0)}(z_1+2z_2)}{\Gamma(2z_2+2)}, \tag{5.142}$$

The fundamental region surrounds the intersection $\Re z_1 = -1/7$, $\Re z_2 = -1/3$. One way to evaluate the above integral is that we can try to shift the contours to the region $k_1 < \Re z_1 < k_1 + 1$, $k_2 < \Re z_2 < k_2 + 1$,

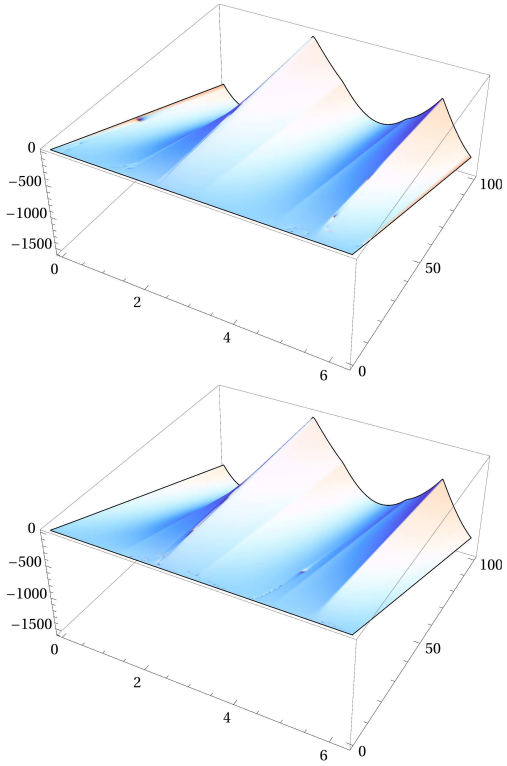

Figure 5.20: From up to down: the logarithmic absolute value of real and imaginary part of the integrand in Minkowski type 2 on the approximated surface of stationary phase we found, $0 < \theta < 2\pi$, $0 < t < 100$

where k_1 and k_2 are two integers so that $\psi^{(0)}(z_1 + 2z_2)$ does not vanish in that region. Then we can apply the ansatz (5.139) to find the stationary phase contour.

In the following text, we will try to find the contour of stationary phase without shifting the contour. Let's first set $u_1 = 1$ and $u_2 = 1$, so the integral is Euclidean. We may still try to find the minimum of the function eq. (5.54). We found two stationary points which are complex conjugate of each other:

$$\begin{aligned} z_{10,1} &= -0.155855 - 0.0362709i, z_{20,1} = -0.187776 + 0.141556i; \\ z_{10,2} &= -0.155855 + 0.0362709i, z_{20,2} = -0.187776 - 0.141556i. \end{aligned} \tag{5.143}$$

To each stationary point, we can apply the ansatz (5.139). We will obtain two independent contours,

$$\begin{aligned} z_{1,j}(\theta, t) &= z_{10,j} + \sin\theta f_j(\theta, t), \\ z_{2,j}(\theta, t) &= z_{20,j} + \cos\theta f_j(\theta, t), \\ f_j(\theta, t) &= it\frac{\exp i\phi_{0,j}(\theta)\ (1 + t^p \exp i(\phi_{\infty,j}(\theta) - \phi_{0,j}(\theta)))}{1 + t^p} + \frac{F'_{s,j}(\theta)t^2}{1 + t^q} \end{aligned} \tag{5.144}$$

where $j = 1, 2$. We can use the contour $(z_{1,1}(\theta, t), z_{2,1}(\theta, t))$ or the contour $(z_{1,2}(\theta, t), z_{2,2}(\theta, t))$ to evaluate the integral. Simply speaking, in this thesis, for high dimensional Mellin–Barnes integrals, we looked for the contour passing one but not several stationary points.

When we set $u_1 = -1$ and $u_2 = 1$ in our integrand, it becomes a Minkowski integral. The only thing which changes is that we will find two stationary points which are not complex conjugates of each other. We can use the same ansatz (5.139) to produce two contours to yield the right integration result.

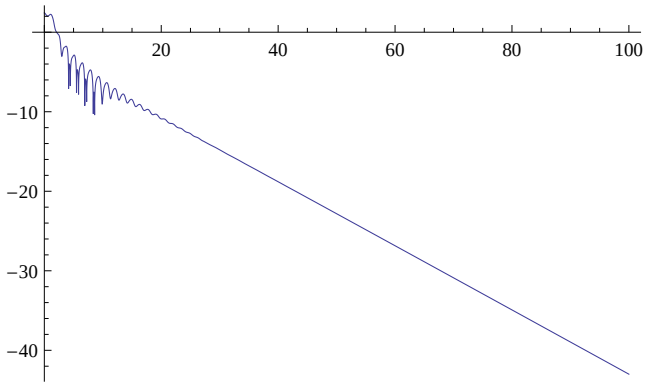

Figure 5.21: The logarithmic absolute value of real part of the integrand in Minkowski type 2 on the surface we found, $\theta = 3/4\pi$ one of the slowest convergent direction, $0 < t < 100$

5.8 The integrand with several stationary points in the fundamental region

At last, in this section, the integrand (5.106) is multiplied by several PolyGamma functions to see if anything changes. The answer is yes. Let's consider the following integrand,

$$\begin{aligned} G(z_1, z_2) = \quad & u_1^{z_1} u_2^{z_2} \frac{\psi^{(0)}(1-z_1)\Gamma(-z_2)\Gamma^2(z_2+1)\psi^{(0)}(-2z_2)\Gamma^2(-2z_1-z_2-1)\Gamma(2z_1-z_2+2)}{\Gamma(2z_2+2)} \\ & \times\Gamma^2(z_1+2z_2+2)\psi^{(0)}(z_1+2z_2)\psi^{(0)}(z_1+2z_2)\psi^{(0)}(-z_1-2z_2). \end{aligned} \tag{5.145}$$

The fundamental region surrounds the intersection $\Re z_1 = -1/4$, $\Re z_2 = -13/20$. When we set $u_1 = 1$ and $u_2 = 1$, the integral is Euclidean integral, we can find several stationary points are in the fundamental region (for complex stationary points, also there are their complex conjugates) by looking for the minimum of eq. (5.54),

$$\begin{aligned} & z_{10,1} = -0.639531 \mp 0.180776i, z_{20,1} = -0.445299 \pm 0.305451i; \\ & z_{10,2} = -0.559247 \mp 0.199474i, z_{20,2} = -0.47018 \pm 0.0993646i; \\ & z_{10,3} = -0.507114 \mp 0.185171i, z_{20,3} = -0.526822 \mp 0.114109i; \\ & z_{10,4} = -0.425605, z_{20,4} = -0.716548; \\ & z_{10,5} = -0.1807, z_{20,5} = -0.719721. \end{aligned} \tag{5.146}$$

Among these stationary points, there are only several points which are suitable base points for our ansatz (5.139). I find a numerical way to select the suitable base points in the set of stationary points. For example, When we solve for ϕ_0, in eq. (5.116) from the series expansion for a complex stationary point $(z_{10,j}, z_{20,j})$, we can substitute the following ansatz for the stationary point into the integrand $G(z_1, z_2)$:

$$\begin{aligned} z_1(\theta, t) &= z_{10,j} + i\, t\, \sin\theta\, e^{i\phi_{0,j}}, \\ z_2(\theta, t) &= z_{20,j} + i\, t\, \cos\theta\, e^{i\phi_{0,j}}. \end{aligned} \tag{5.147}$$

Then we can draw the plot of $\Re(\Im)G(z_1(\theta,0), z_2(\theta,0))$ and $\Re(\Im)G(z_1(\theta,t), z_2(\theta,t))$ where $0 \le t < 2\pi$ and t is set to be a small positive number, for example 10^{-5} or even less than 10^{-5}.

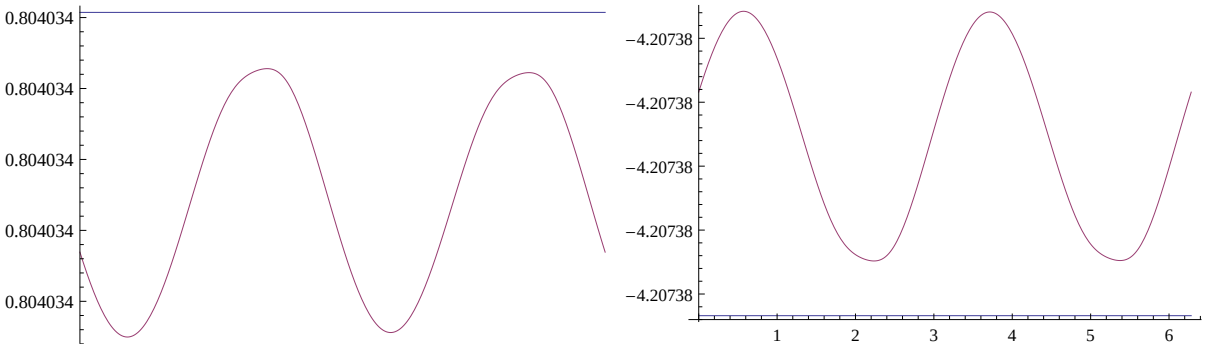

Figure 5.22: From left to right, the plots of $\Re G(z_1(\theta,0),z_2(\theta,0))$ represented by the straight line and $\Re G(z_1(\theta,10^{-6}),z_2(\theta,10^{-6}))$ and of $\Im G(z_1(\theta,0),z_2(\theta,0))$ and $\Im G(z_1(\theta,10^{-2}),z_2(\theta,10^{-2}))$ of the suitable base point $z_{10,1}=-0.639531-0.180776i$, $z_{20,2}=-0.445299+0.305451i$.

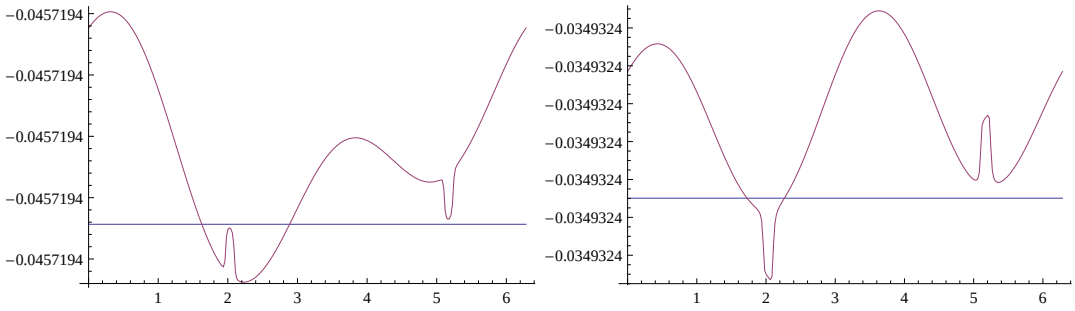

Figure 5.23: From left to right, the plots of $\Re G(z_1(\theta,0),z_2(\theta,0))$ and $\Re G(z_1(\theta,10^{-6}),z_2(\theta,10^{-6}))$ and of $\Im G(z_1(\theta,0),z_2(\theta,0))$ and $\Im G(z_1(\theta,10^{-6}),z_2(\theta,10^{-6}))$ of the non-suitable base point $z_{10,2}=-0.559247-0.199474i$, $z_{20,2}=-0.47018+0.0993646i$ because it exists some intersections between $\Re G(z_1(\theta,0),z_2(\theta,0))$ and $\Re G(z_1(\theta,10^{-6}),z_2(\theta,10^{-6}))$.

We expect that for the suitable base point, there is no intersection between the plot of $\Re(\Im)G(z_1(\theta,10^{-6}),z_2(\theta,10^{-6}))$ and the plot of $\Re(\Im)G(z_1(\theta,0),z_2(\theta,0))$ (see fig. 5.22). But for the non-suitable points, there are some intersections (see fig. 5.23). Then We can notice that $z_{10,1}=-0.639531-0.180776i$, $z_{20,1}=-0.445299+0.305451i$ is a suitable base point but $z_{10,2}=-0.559247-0.199474i$, $z_{20,2}=-0.47018+0.0993646i$ is a non-suitable base point. When the stationary point is on the real slice, we substitute the ansatz into the integrand $G(z_1,z_2)$:

$$\begin{aligned} z_1(\theta,t) &= z_{10,j}+i\,t\,\sin\theta,\\ z_2(\theta,t) &= z_{20,j}+i\,t\,\cos\theta. \end{aligned} \tag{5.148}$$

We can plot of $\Re G(z_1(\theta,0),z_2(\theta,0))$ and $\Re G(z_1(\theta,t),z_2(\theta,t))$ where $0\le\theta<2\pi$ and t is set to be a small positive number, for example 10^{-5} or even less than 10^{-5}. We expect that for the suitable base point, there is no intersection between the plot of $\Re G(z_1(\theta,10^{-5}),z_2(\theta,10^{-5}))$ the one of $\Re G(z_1(\theta,0),z_2(\theta,0))$. With the above criteria, we choose the following stationary points as suitable base points:

$$\begin{aligned} &z_{10,1}=-0.639531\mp0.180776i, z_{20,1}=-0.445299\pm0.305451i;\\ &z_{10,3}=-0.507114\mp0.185171i, z_{20,3}=-0.526822\mp0.114109i;\\ &z_{10,4}=-0.425605, z_{20,4}=-0.716548, \end{aligned} \tag{5.149}$$

For the complex stationary points, the phase of the integrand on these suitable base points are all different. We can apply the ansatz eq. (5.139) to each suitable base point to obtain the contours and use these contours to evaluate the integral.

When we set $u_1 = -5$ and $u_2 = 1$, the integral is Minkowski integral, we can find several stationary points are in the fundamental region by looking for the minimum of eq. (5.54):

$$\begin{aligned}
&z_{10,1} = -0.585258 + 0.099671i, z_{20,1} = -0.48104 - 0.263322i;\\
&z_{10,2} = -0.583931 - 0.347355i, z_{20,2} = -0.484734 - 0.0310352i;\\
&z_{10,3} = -0.519057 + 0.110009i, z_{20,3} = -0.520096 + 0.152334i;\\
&z_{10,4} = -0.423953 + 0.0069649i, z_{20,4} = -0.716389 + 0.00058606i;\\
&z_{10,5} = -0.289902 - 0.302508i, z_{20,5} = -0.666893 + 0.014548i;\\
&z_{10,6} = -0.182285 - 0.00888299i, z_{20,6} = -0.71953 + 0.000952542i.
\end{aligned} \tag{5.150}$$

With the criteria for classifying the stationary points, we find that the following stationary points are suitable base points for ansatz (5.139):

$$\begin{aligned}
&z_{10,1} = -0.585258 + 0.099671i, z_{20,1} = -0.48104 - 0.263322i;\\
&z_{10,2} = -0.583931 - 0.347355i, z_{20,2} = -0.484734 - 0.0310352i;\\
&z_{10,3} = -0.519057 + 0.110009i, z_{20,3} = -0.520096 + 0.152334i.
\end{aligned} \tag{5.151}$$

For each suitable base point, we can apply the ansatz (5.139) in last example to obtain a contour which yields the right integral result.

5.9 Higher Dimension Case for both Euclidean and Minkowski integrals

For three or even higher dimensional Mellin–Barnes integrals, if the integrand is a product of gamma functions and Polygamma functions and the arguments of these functions are $a+\sum_i b_i z_i$ where b_i are some integers. We can use the criteria posed in the last section to select suitable base points among stationary points. And apply the generalization of the ansatz eq. (5.139) to each suitable base points. For example, the generalization of ansatz (5.139) for three dimensional Mellin–Barnes integrals is:

$$\begin{aligned}
z_1(\theta_1,\theta_2,t) &= z_{10} + \sin\theta_1\ \sin\theta_2\, f(\theta_1,\theta_2,t),\\
z_2(\theta_1,\theta_2,t) &= z_{20} + \sin\theta_1\ \cos\theta_2\, f(\theta_1,\theta_2,t),\\
z_3(\theta_1,\theta_2,t) &= z_{30} + \cos\theta_1\, f(\theta_1,\theta_2,t),
\end{aligned} \tag{5.152}$$

where (z_{10}, z_{20}, z_{30}) is a suitable base point, $0 < \theta_1 < \pi$, $0 < \theta_2 < 2\pi$,

$$f(\theta_1,\theta_2,t) = it\frac{\exp i\phi_0(\theta_1,\theta_2)\ (1 + t^p \exp i(\phi_\infty(\theta_1,\theta_2) - \phi_0(\theta_1,\theta_2)))}{1+t^p} + \frac{F_s\, t^2}{1+t^q}, \tag{5.153}$$

where $a \geq 3$. Always following the same procedure to solve the unknown functions: $\phi_0(\theta_1,\theta_2)$, $\phi_\infty(\theta_1,\theta_2)$, $F_s(\theta_1,\theta_2)$. The functions $\phi_0(\theta_1,\theta_2)$ and $F_s(\theta_1,\theta_2)$ can be determined from that we demand every term of

the series expansion of the integrand around $t = 0$ up to the third order has the same phase. The function $\phi_\infty(\theta, \phi)$ can be solved from that we demand the argument of the asymptotic form of the integrand have no term not proportional to t.

At least for the examples we studied of Euclidean and Minkowski integrals, we can solve for $\phi_0(\theta_1, \theta_2)$, $\phi_\infty(\theta_1, \theta_2)$ and $F_s(\theta_1, \theta_2)$ as continuous functions of θ_1, θ_2. Maybe our ansatz can be considered as a candidate for evaluating high dimensional Mellin–Barnes integrals. I examined the ansatz for several integrals like

$$\frac{1}{(2\pi i)^3}\int_{C_1-i\infty}^{C_1+i\infty} dz_1 \int_{C_2-i\infty}^{C_2+i\infty} dz_2 \int_{C_3-i\infty}^{C_3+i\infty} dz_3 (-1-0.1i)^{z_3}\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_1+z_2+z_3+1),$$

where $C_1 = -0.1258,\ C_2 = -0.4192,\ C_3 = -0.4371,$

and

$$\frac{1}{(2\pi i)^3}\int_{C_1-i\infty}^{C_1+i\infty} dz_1 \int_{C_2-i\infty}^{C_2+i\infty} dz_2 \int_{C_3-i\infty}^{C_3+i\infty} dz_3 (-1-0.1i)^{z_3}\Gamma^2(-z_1)\Gamma^2(-z_2)\Gamma^2(-z_3)\Gamma^2(z_1+z_2+z_3+1),$$

where $C_1 = -0.1258,\ C_2 = -0.4192,\ C_3 = -0.4371.$

(5.154)

along with other integrals. The integration over the contour (5.152) yields the right result.

In this chapter, stationary phase condition and another constraint which can be generalized easily to n-dimensional integrals to find n-dimensional approximations to contours of stationary phase for Mellin–Barnes integrands, if the Mellin–Barnes integrand is a product of gamma and polygamma functions. The ansatzes I have considered here are just one of many options.

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