Compliance optimization for thin elastic structures

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COMPLIANCE OPTIMIZATION FOR THIN ELASTIC STRUCTURES

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Italian Summary

L'argomento principale della Tesi è l'*ottimizzazione della compliance di strutture elastiche sottili*. Il problema consiste nel determinare le configurazioni più resistenti, quando una quantità infinitesimale di materiale elastico, sottoposta ad una forza fissata, viene confinata in un intorno sottile di un piano o di una retta. La resistenza al carico può essere misurata valutando un funzionale di forma, la *compliance*, sulla configurazione del materiale elastico.

In particolare, trattiamo il caso in cui la regione di disegno è un filo sottile, rappresentato da un cilindro con sezione trasversale infinitesima. Lo studio è motivato da problemi di carattere ingegneristico: la facilità di fabbricazione e trasporto legate al loro ridotto peso, rendono le strutture sottili molto convenienti per le applicazioni.

L'approccio che adottiamo trae ispirazione da alcuni recenti lavori in collaborazione tra G. Bouchitté, I. Fragalà e P. Seppecher, in cui gli autori affrontano il caso di piastre sottili [G. BOUCHITTÉ, I. FRAGALÀ, P. SEPPECHER: Structural optimization of thin plates: the three dimensional approach. *Arch. Rat. Mech. Anal.* (2011)]. Il caso di sbarre sottili non si presenta affatto come una variante tecnica del precedente, a causa della differenza sostanziale nei passaggi al limite *3d-1d* e *3d-2d*, cioè da 3 a 1 e da 3 a 2 dimensioni.

Lo studio di configurazioni ottimali ci ha condotto ad affrontare un altro interessante problema variazionale: in regime di pura torsione, stabilire se si verifichino o meno fenomeni di omogeneizzazione nei fili sottili risulta essere equivalente a risolvere un *problema non standard di frontiera libera* nel piano. Oltre al legame con il problema di ottimizzazione della compliance, questo problema variazionale ha un interesse matematico di per sé. Uno degli strumenti che possono essere usati per affrontare il problema è la teoria delle *derivate di forma per minimi di funzionali integrali*. La teoria delle derivate di forma è un campo molto studiato (vedi *e.g.* la monografia di A. Henrot e M.Pierre *Variations et Optimisation de Formes. Une Analyse Géométrique*, Springer Berlin (2005), e i riferimenti ivi contenute), ma l'approccio che proponiamo è nuovo e si basa su ipotesi più deboli di quelle classiche.

La Tesi è organizzata come segue.

Nella prima parte sono raccolte le nozioni prelimirari: nel Capitolo 1 richiamiamo i principali strumenti matematici di Analisi Convessa, Teoria Geometrica della Misura e Γ-convergenza che utilizziamo nella Tesi, successivamente presentiamo elementi della teoria dell'Elasticità Lineare, necessari per comprendere i termini del problema principale. La seconda parte (Capitoli

2 e 3) è dedicata allo studio della ottimizzazione della compliance in fli sottili. La terza parte (Capitoli 4 e 5) è dedicata ai già citati problemi correlati. Per una più agevole lettura della Tesi, le dimostrazioni tecniche, solitamente riguardanti lemmi ausiliari o semplici asserzioni, sono raccolte alla fne dei capitoli, nella Sezione Appendix. I problemi aperti e i possibili sviluppi sono presentati nella Sezione Perspectives.

I capitoli corrispondono ai seguenti articoli, che sono stati scritti durante questi tre anni di Dottorato:

- (Capitoli 2 e 3) G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI, P. SEPPECHER: Optimal Thin Torsion Rods and Cheeger Sets, *SIAM J. Math. Anal.*, **44**, 483-512, (2012).

- (Capitoli 2 e 3) I. LUCARDESI: Optimal design in thin rods, in preparazione.

- (Capitolo 4) J. ALIBERT, G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI: A non standard free boundary problem arising in shape optimization of thin torsion rods, in stampa su *Interfaces and Free Boundaries* (2012).

- (Capitolo 5) G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI: Shape derivatives for minima of integral functionals, in preparazione.

Illustriamo ora gli argomenti più nel dettaglio.

[Capitolo 2] Sia Q una regione di disegno in $\mathbb{R}^3$ sottoposta ad un carico esterno fssato $F \in H^{-1}(Q;\mathbb{R}^3)$. Dato un materiale elastico isotropo che occupa una certa regione $\Omega \subset Q$, la sua resistenza al carico, nell'ipotesi di piccoli spostamenti, può essere misurata calcolando un funzionale di forma, la *compliance*

$$\mathscr{C}(\Omega) := \sup\left\{ \langle F,u\rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\,dx \ :\ u\in H^1(Q;\mathbb{R}^3)\right\}\ , \tag{1}$$

dove $e(u)$ indica la parte simmetrica del gradiente ∇u, e $j:\mathbb{R}^{3\times 3}_{\rm sym}\to\mathbb{R}$ è il potenziale di strain, isotropo e strettamente convesso, defnito come

$$j(z) := \frac{\lambda}{2}\,\mathrm{tr}^2(z) + \eta\,|z|^2\ , \tag{2}$$

con $\lambda\,,\mu>0$ i coeffcienti di Lamé del materiale.
La compliance è proporzionale al lavoro fatto dal carico per portare la struttura all'equilibrio. In particolare, minore è la compliance e maggiore è la resistenza.
Chiaramente, affnchè $\mathscr{C}(\Omega)$ sia fnito, il carico deve avere supporto contenuto in $\overline{\Omega}$, inoltre deve essere *bilanciato*, *i.e.*

$$\langle F,u\rangle_{\mathbb{R}^3} = 0\ ,\quad \text{se } e(u)=0\ .$$

In queste ipotesi, uno spostamento ottimale $\overline{u}$ esiste e soddisfa $\mathscr{C}(\Omega)=\frac{1}{2}\langle F,\overline{u}\rangle_{\mathbb{R}^3}$.

Nella Tesi studiamo il problema di determinare le confgurazioni di materiale elastico più resistenti, cioè che minimizzino la compliance, quando la regione di disegno è un dominio sottile. Inoltre, allo stesso tempo, facciamo tendere a zero il rapporto tra il volume del materiale elastico e la regione di disegno.

Per struttura “sottile” intendiamo che una o due dimensioni spaziali sono molto più piccole delle altre. Nella Tesi consideriamo il caso di *f lo*, ovvero un corpo continuo approssimabile da un insieme unidimensionale.

Il modello matematico è il seguente: il f lo è un solido tridimensionale che occupa il volume generato da un dominio piano, chiamato *sezione trasversale*, che varia perpendicolarmente ad una curva, l'*asse*, a cui appartiene il suo baricentro; inoltre il diametro della sezione trasversale è molto più piccolo della lunghezza dell'asse. Ci restringiamo al caso particolare di *f li retti*, in cui l'asse è un segmento I e la sezione trasversale è un dominio limitato piano D, costante lungo l'asse: rappresentiamo una tale struttura come un cilindro della forma

$$Q_\delta := \delta\overline{D} \times I\,, \tag{3}$$

con $D \subset \mathbb{R}^2$ un dominio aperto limitato, I un intervallo chiuso e limitato e $\delta > 0$ un parametro inf nitesimo che rappresenta il rapporto tra il diametro della sezione e la lunghezza dell'asse.

La letteratura riguardo ai f li sottili è molto vasta: la teoria classica è stata sviluppata da Eulero, Bernoulli, Navier, Saint Venant, Timoshenko, Vlassov; negli ultimi anni, grazie a nuovi metodi numerici, questi problemi hanno incontrato un rinnovato interesse, e costituiscono un ampio campo della matematica applicata (ci limitiamo a citare [73, 78, 80, 86]). Per una trattazione completa, rimandiamo al libro di Trabucho e Viaño [95] e i riferimenti ivi contenuti.

Il problema che trattiamo, e di conseguenza l'approccio che adottiamo per la risoluzione, trae ispirazione da un lavoro recente di G. Bouchitté, I. Fragalà e P. Seppecher [19], in cui gli autori hanno studiato il problema di ottimizzazione della compliance nel caso in cui la regione di disegno può essere approssimata da un insieme bidimensionale, più precisamente il caso di piastre sottili, descritte da una famiglia di cilindri della forma

$$Q_\delta := \overline{D} \times \delta I\,,$$

con spessore inf nitesimo δ.

Nel seguito tratteremo il caso di insiemi sottili Q_δ def niti come in (3). Per trovare la conf gurazione più “leggera” e “robusta”, minimizziamo la compliance $\mathscr{C}(\Omega)$ al variare dei sottoinsiemi $\Omega \subset Q_\delta$ con volume f ssato m, cioè studiamo

$$\inf\left\{\mathscr{C}(\Omega)\ :\ \Omega \subset Q_\delta\,,\ \frac{|\Omega|}{|Q_\delta|} = m\right\} \tag{4}$$

nel doppio passaggio al limite per $\delta \to 0$ e $m \to 0$.
Se includiamo il vincolo di volume nel funzionale costo attraverso un moltiplicatore di Lagrange $k \in \mathbb{R}$, i problemi variazionali (4) assumono la forma

$$\phi^\delta(k) := \inf_{\Omega \subset Q_\delta}\left\{\mathscr{C}^\delta(\Omega) + k\frac{|\Omega|}{|Q_\delta|}\right\}\,, \tag{5}$$

con

$$\mathscr{C}^\delta(\Omega) := \sup\left\{\langle F^\delta, \tilde{u}\rangle_{\mathbb{R}^3} - \int_\Omega j(e(\tilde{u}))\,dx\ :\ \tilde{u} \in H^1(Q_\delta, \mathbb{R}^3)\right\}\,. \tag{6}$$

Qui F^δ è un riscalamento opportuno del carico F, scelto in modo tale che nel passaggio al limite l'estremo inferiore in (5) resti f nito. La scelta del riscalamento F^δ dipende dalle ipotesi fatte sul tipo di carichi applicati. In letteratura è consuetudine distinguere tra i casi di estensione, f essione e torsione: nella Tesi concentriamo la nostra attenzione sui carichi per i quali il contributo della f essione e della torsione possono essere disaccoppiati in modo opportuno, e scegliamo due diversi riscalamenti per le due componenti: grosso modo, consideriamo carichi F della forma $F = G + H$, in cui la f essione dipende solo dal carico verticale H. Il caso generale è diff cile da trattare, a causa dell'interazione tra queste componenti del carico.

Nel Capitolo 2 eseguiamo il primo passaggio al limite, studiando il comportamento asintotico del problema (5) per $\delta \to 0^+$: questo corrisponde a cercare le conf gurazioni più robuste in un f lo, quando il rapporto tra il volume del materiale e della regione di disegno sottile è f ssato. Il primo passo consiste nel riformulare i problemi variazionali $\phi^\delta(k)$ sul dominio f sso $Q := \overline{D} \times I$, invece che sui cilindri sottili Q_δ. Dopo un opportuno cambiamento di variabile per gli spostamenti e un opportuno riscalamento delle forze, possiamo riscrivere i problemi (5) come

$$\phi^\delta(k) = \inf_{\omega \subset Q} \left\{ \mathscr{C}^\delta(\omega) + k|\omega| \right\} , \tag{7}$$

con

$$\mathscr{C}^\delta(\omega) := \sup \left\{ \frac{1}{\delta} \langle G, u \rangle_{\mathbb{R}^3} + \langle H, u \rangle_{\mathbb{R}^3} - \int_\Omega j(e^\delta(u))\, dx \ : \ u \in H^1(Q_\delta, \mathbb{R}^3) \right\} , \tag{8}$$

dove $e^\delta : H^1(Q;\mathbb{R}^3) \to L^2(Q;\mathbb{R}^{3\times 3}_{\text{sym}})$ è l'operatore def nito da

$$e^\delta_{\alpha\beta}(u) := \delta^{-2} e_{\alpha\beta}(u) \ , \ \ e^\delta_{\alpha 3}(u) := \delta^{-1} e_{\alpha 3}(u) \ , \ \ e^\delta_{33}(u) := e_{33}(u) \ ,$$

come consuetudine nella letteratura della riduzione di dimensione $3d - 1d$. Lo studio asintotico di $\phi^\delta(k)$ si basa sul confronto con i "problemi f ttizi", cioè le loro formulazioni rilassate in $L^\infty(Q;[0,1])$. Infatti è ben noto che i problemi di minimo (7) sono in generale mal posti: l'esistenza di un dominio ottimale non è garantita, ma potrebbe verif carsi un fenomeno di omogeneizzazione (vedi [2]). È quindi necessario ampliare la classe di materiali ammissibili, passando dai materiali "reali", rappresentati di funzioni caratteristiche, ai materiali "compositi", rappresentati da dansità a valori in $[0,1]$. A questo scopo introduciamo la famiglia di problemi variazionali

$$\widetilde{\phi}^\delta(k) := \inf \left\{ \widetilde{\mathscr{C}}^\delta(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\} , \tag{9}$$

dove $\widetilde{\mathscr{C}}^\delta(\theta)$ indica l'estensione naturale della compliance $\mathscr{C}^\delta(\omega)$ a $L^\infty(Q;[0,1])$:

$$\widetilde{\mathscr{C}}^\delta(\theta) := \sup \left\{ \frac{1}{\delta} \langle G, u \rangle_{\mathbb{R}^3} + \langle H, u \rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u))\, \theta \, dx \ : \ u \in H^1(Q;\mathbb{R}^3) \right\} . \tag{10}$$

Sfruttando alcune delicate proprietà di compattezza ricavate da opportune varianti della disuguaglianza di Korn (vedi Sezione 1.5) e tecniche di Γ-convergenza (vedi Sezione 1.3), determiniamo il comportamento asintotico di $\widetilde{\phi}^\delta(k)$ nel limite per $\delta \to 0^+$: la successione di problemi f ttizi tende ad un problema variazionale posto sulle densità, avente la stessa struttura (termine di compliance con penalizzazione di volume). Più precisamente, il problema limite è dato da

$$\phi(k) := \inf \left\{ \mathscr{C}^{lim}(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\} , \tag{11}$$

dove $\mathscr{C}^{lim}$ è il Γ-limite della successione $\widetilde{\mathscr{C}^\delta}$ per $\delta \to 0^+$ rispetto alla convergenza debole * in $L^\infty(Q;[0,1])$. Sfruttando una stima cruciale per il funzionale di compliance rilassato, stabilita in [19, Proposition 2.8], deduciamo che $\phi^\delta(k)$ e $\widetilde{\phi}^\delta(k)$ hanno lo stesso comportamento asintotico, cioè $\lim_{\delta \to 0^+} \phi^\delta(k) = \phi(k)$.

Osserviamo che il processo di riduzione di dimensione è fatto senza alcuna ipotesi topologica sull'insieme Ω occupato dal materiale. Pertanto non rientra nella vasta letteratura sull'analisi $3d-1d$.

È naturale chiedersi se $\phi(k)$ ammetta o meno una soluzione classica (*i.e.* una densità a valori in $\{0,1\}$): questo corrisponde a chiedersi se il problema di compliance sotto vincolo di volume, in un filo, ammetta come soluzione un materiale reale piuttosto che un materiale composito. La riformulazione di $\phi(k)$ come problema variazionale sugli spostamenti e come problema variazionale sugli sforzi, ci permette di dare condizioni necessarie e sufficienti di ottimalità. Queste formulazioni alternative mostrano che il problema (11) può essere risolto sezione per sezione; inoltre, sfruttando le condizioni di ottimalità, la domanda circa l'esistenza di soluzioni reali può essere riformulata in modo più semplice, cioè può essere messa in relazione con l'esistenza di soluzioni "speciali" per un problema di frontiera libera. Una descrizione più dettagliata e un'analisi approfondita del problema sono rimandate al Capitolo 4.

Nel Capitolo 3 eseguiamo il secondo passaggio al limite, studiando il comportamento asintotico del problema (11) per $k \to +\infty$: ricordiamo che un valore grande per k corrisponde a considerare un piccolo riempimento relativo $\int_Q d\theta/|Q|$. Mostriamo che la successione $\phi(k)$ è asintoticamente equivalente a $\sqrt{2k}$, e determiniamo il limite $\overline{m}$ del rapporto $\phi(k)/\sqrt{2k}$. Tale limite è ancora un problema variazionale, il cui costo ha la stessa struttura (termine di compliance più termine di volume), ma è posto sullo spazio $\mathscr{M}^+(Q)$ delle misure positive su $\mathbb{R}^3$, con supporto compatto contenuto in Q:

$$\overline{m} := \inf\left\{ \mathscr{C}^{lim}(\mu) + \frac{1}{2}\int d\mu \ : \ \mu \in \mathscr{M}^+(Q) \right\} ; \tag{12}$$

con $\mathscr{C}^{lim}(\mu)$ l'estensione della compliance limite $\mathscr{C}^{lim}(\theta)$ alla classe $\mathscr{M}^+(Q)$. La nuova impostazione del problema sulle misure è abbastanza naturale: il problema limite descrive infatti i fenomeni di concentrazione che possono aver luogo in insiemi di dimensione inferiore.

Sfruttando la formulazione duale di $\overline{m}$, forniamo una caratterizzazione variazionale alternativa delle misure ottimali. In generale la soluzione di $\overline{m}$ non è unica ed è difficile da determinare esplicitamente. Tuttavia, se consideriamo dei carichi particolari, riusciamo a risolvere $\overline{m}$ completamente o, almeno, a dare un'informazione precisa sul supporto delle sue soluzioni. I fenomeni di concentrazione corrispondenti sono discussi nell'ultima parte del Capitolo 3, e possono essere riassunti come segue.

Se il carico è di pura torsione e D è convesso, la soluzione di $\overline{m}$ è unica e può essere determinata esplicitamente come una misura concentrata sezione per sezione sul bordo dell'*insieme di Cheeger* di D. Ricordiamo che, sotto l'ipotesi di D convesso, il suo insieme di Cheeger è l'unico punto di minimo per il rapporto perimetro/area tra tutti i sottinsiemi misurabili di D (vedi Sezione 1.4.3 nei Preliminaries):

$$\inf_{E\subset\overline{D},\, 1\!\!1_E\in BV(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2}|\nabla 1\!\!1_E|}{|E|} . \tag{13}$$

Il problema puramente geometrico (13) (che può essere formulato nel caso più generale in cui D sottoinsieme è un sottoinsieme connesso del piano) è noto come problema di Cheeger, e negli ultimi anni ha attratto l'attenzione di molti autori (vedi [3, 24, 30, 31, 56, 59, 60, 71]). Per quanto sappiamo, finora non esiste alcun risultato e dimostrazione di questa caratterizzazione di fili sottili in regime di torsione in termini di insiemi di Cheeger. Sottolineiamo che tale caratterizzazione è valida solo in regime di pura torsione.

Per carichi più generali, a causa dell'interazione tra energie di trazione, f essione e torsione, otteniamo un modello più complicato. Forniamo alcuni esempi per cui il problema variazionale (12) può essere risolto esplicitamente, e presentiamo alcuni casi in cui le sue soluzioni risultano essere collegate ad interessanti varianti del problema di Cheeger, di seguito riportate. È facile verif care che la formulazione rilassata del problema classico di Cheeger (13) è

$$\inf\left\{ \int_D |Du| \ : \ u \in BV_0(D)\,, \ \int_D u = 1 \right\} \ . \tag{14}$$

La prima variante che entra in gioco è una sorta di perturbazione con un termine di traslazione:

$$\inf\left\{ \int_D |Du+q| \ : \ u \in BV_0(D)\,, \ \int_D u = C(q) \right\} \ , \tag{15}$$

con q un campo vettoriale f ssato; la seconda variante è invece una versione pesata:

$$\inf\left\{ \int_D \alpha|Du| \ : \ u \in BV_0(D), \ \int_D u = 1 \right\} \ , \tag{16}$$

dove α è una funzione non negativa in D. Osserviamo che il problema (16) è stato affrontato qualche anno fa da Ionescu e Lachand-Robert: in [69] gli autori hanno studiato il caso in cui sia l'integrale da minimizzare che l'integrale nel vincolo sono pesati.

Nel Capitolo 4 affrontiamo la domanda naturale riguardo al problema variazionale $\phi(k)$ in (11):

Il problema $\phi(k)$ ammette una soluzione $\overline{\theta}$ a valori in $\{0,1\}$? (17)

Ricordiamo che $\phi(k)$ è stato ottenuto passando al limite per $\delta \to 0^+$ nel problema di ottimizzazione di compliance (7): poiché si tratta di un problema variazionale posto sulle densità in $L^\infty(Q;[0,1])$, una risposta affermativa alla domanda (17) signif cherebbe che il disegno ottimale in f li sottili ammette una soluzione classica, che può essere identif cata con un insieme, piuttosto che con un materiale composito.

Esaminiamo il caso di pura torsione: in questo caso la domanda può essere riformulata in modo più semplice. Poiché, come già osservato, $\phi(k)$ può essere risolto sezione per sezione, dopo un opportuno cambiamento di variabile e passaggio al problema duale, siamo portati a studiare il seguente problema variazionale:

$$m(s) := \inf\left\{ \int_{\mathbb{R}^2} \varphi(\nabla u) \ : \ u \in H^1_c(D)\,, \ \int_{\mathbb{R}^2} u = s \right\} , \tag{18}$$

dove s è un parametro reale proporzionale a k, φ è l'integrando convesso ma non strettamente convesso

$$\varphi(y) := \begin{cases} \frac{|y|^2}{2} + \frac{1}{2} & \text{se } |y| \geq 1 \\ |y| & \text{se } |y| < 1 \ , \end{cases}$$

e $H^1_c(D)$ è lo spazio delle funzioni H^1 che sono costanti fuori da D (se D è semplicemente connesso, $H^1_c(D)$ coincide con l'usuale spazio di Sobolev $H^1_0(D)$).
La domanda (17) risulta essere equivalente alla domanda seguente:

Il problema $m(s)$ ammette una soluzione $\overline{u}$ tale che $|\nabla \overline{u}| \in \{0\} \cup (1,+\infty)$ q.o. in D? (19)

Più precisamente, data una soluzione $\overline{u}$ per $m(s)$ e una soluzione $\overline{\theta}$ per $\phi(k)$, la regione in cui $\nabla\overline{u}$ è zero corrisponde all'assenza di materiale (cioè $\overline{\theta}=0$), la regione in cui $|\nabla\overline{u}|>1$ alla presenza di materiale (cioè $\overline{\theta}=1$) e la regione intermedia corrisponde alla regione di omogeneizzazione. In altre parole, lo studio di $m(s)$ consente di valutare, nel problema di ottimizzazione della compliance, l'inf uenza della geometria della sezione D e del riempimento relativo s sulla presenza di regioni di omogeneizzazione.

I risultati principali del Capitolo 4 riguardano lo studio della domanda (19) in relazione alla geometria del dominio D e riguardo al valore del parametro s.

Diciamo che u è una *soluzione speciale* per $m(s)$ se minimizza (18) e soddisfa $|\nabla u|\in\{0\}\cup(1,+\infty)$ q.o. in D, inoltre chiamiamo *plateau* di u, e lo indichiamo con $\Omega(u)$, l'insieme $\{\nabla u=0\}$ meno la componente connessa illimitata di $\mathbb{R}^2\setminus\overline{D}$ (in cui $u\equiv 0$).

Se D è un disco o un anello, attraverso conti espliciti e sfruttando le condizioni di ottimalità, mostriamo che $m(s)$ ammette una soluzione speciale, e non ci sono altre soluzioni.
Mostriamo inoltre che i dischi e gli anelli *non* sono gli unici domini per i quali $m(s)$ ammette una soluzione speciale. A questo proposito, è opportuno confrontare i nostri risultati con quelli ottenuti da Murat e Tartar in [F. MURAT, L. TARTAR: Calcul des variations et homogénéisation. Homogenization methods: theory and applications in physics (Bráu-sans-Nappe, 1983), 319-369, *Collect. Dir. Etudes Rech. Elec. France* **5**, Eyrolles, Paris (1985)], riguardo al problema di massimizzare la rigidità torsionale di una sbarra con sezione trasversale f ssata, e costituita da due materiali elastici in proporzioni f ssate. Il problema variazionale corrispondente è piuttosto simile al nostro, ma coinvolge un integrando *differenziabile*: risulta che non possono esistere soluzioni classiche (*i.e.* senza regioni di omogeneizzazione) a meno che la sezione trasversale D non sia un disco. Nel nostro caso l'integrando φ è non-differenziabile in zero e la conclusione va in una direzione del tutto opposta: dimostriamo che esiste un dominio non circolare D con bordo analitico tale che, per qualche s, il problema $m(s)$ ammette una soluzione speciale. Inoltre tale soluzione ha plateau convesso con bordo analitico. Per ottenere questo risultato, usiamo come strumento chiave la relazione tra $m(s)$ e il problema di Cheeger.

Osserviamo che il ruolo cruciale dell'insieme di Cheeger di D nello studio del problema asintotico di $m(s)$ in (18) per $s\to 0^+$ è già emerso nello studio asintotico del problema di ottimizzazione di compliance $\phi(k)$ in (11) per $k\to+\infty$: infatti il parametro s è proporzionale a $\frac{1}{\sqrt{k}}$, quindi piccoli valori di s corrispondono a grandi valori di k.
Mostriamo che se esiste una soluzione speciale u, allora essa risolve un problema non standard di frontiera libera con un ostacolo sul gradiente:

$$\begin{cases} -\Delta u=\lambda\ , \quad |\nabla u|>1 & \text{in } D\setminus\Omega(u) \\ |\nabla u|=1 & \text{su } \Gamma(u) \\ u=c_i & \text{su } \gamma_i\,, \end{cases} \tag{20}$$

dove $\Gamma(u):=\partial\Omega(u)\cap D$ è la frontiera libera e γ_i denotano le differenti componenti connesse di $\Gamma(u)$.
Supponendo che D sia semplicemente connesso e che esista una soluzione speciale u con frontiera libera regolare, dimostriamo alcune proprietà qualitative del plateau:

- ogni componente connessa di $D\setminus\Omega(u)$ deve toccare ∂D;
- sotto opportune ipotesi, il plateau $\Omega(u)$ deve essere convesso;
- se D non è insieme di Cheeger di se stesso, il plateau $\Omega(u)$ non può essere contenuto in modo compatto in D per coeff cienti di riempimento arbitrariamente piccoli.

La richiesta *a priori* di frontiera libera liscia è necessaria per poter applicare la teoria delle P-functions (cf. [93]). Lo studio della regolarità della frontiera libera rappresenta una prospettiva interessante e stimolante (vedi [27, 28, 85]): un primo risultato ottenuto in questa direzione riguarda la f nitezza del perimetro del plateau.

Ottenere una caratterizzazione completa dei domini D per i quali esiste una soluzione speciale per $m(s)$ è un problema molto complicato, che rimane al momento aperto: a questo proposito crediamo che, almeno nel caso in cui D sia convesso, l'esistenza di una soluzione speciale sia legata alla regolarità di ∂D, e al fatto che D coincida o meno con il suo insieme di Cheeger. Osserviamo che quest'ultimo criterio escluderebbe l'esistenza di soluzioni speciali nel caso in cui D sia un quadrato. Questa ipotesi sembra essere confermata da risultati numerici ottenuti in [72] per un problema molto simile, in cui sono osservate regioni di omogeneizzazione.
Più precisamente, se D non è insieme di Cheeger di se stesso, ci aspettiamo che qualche componente connessa $\Omega_0 \subset \{u \equiv 0\}$ del plateau tocchi il bordo in un intorno dei punti di curvatura più elevata, cioè gli angoli. Per provare la congettura, abbiamo cercato di sfruttare le derivate di forma. Infatti, se f ssiamo il parametro s, possiamo interpretare $m(s)$ come un funzionale di forma $J(D)$, che dipende dal dominio D come segue: includendo il vincolo di volume nel funzionale, risolvere $m(s)$ su D risulta essere equivalente a studiare

$$J(D) = -\inf\left\{ \int_D [\varphi(\nabla u) - \lambda u]\, dx \ : \ u \in H^1_0(D) \right\} , \tag{21}$$

con $\lambda = m'(s)$.
Chiaramente il funzionale di forma $J(\cdot)$ è stazionario sui domini $D' \subset D$ che contengono $D \setminus \Omega_0$. Inoltre, il segno della derivata di forma può fornire utili informazioni: se consideriamo piccole deformazioni interne di D, localizzate su qualche curva $\gamma \subset \partial D$, una derivata non nulla implicherebbe che Ω_0 non tocca tale porzione di bordo γ.

[Capitolo 5]
La teoria delle derivate di forma è un ambito molto studiato, con molte applicazioni in problemi variazionali e di disegno ottimo. La sua origine su può individuare nella prima metà del secolo scorso, con il lavoro pionieristico di Hadamard [66], seguito da Schiffer e Garabedian [61, 89]. Negli anni '70, sono stati compiuti alcuni importanti progressi da Céa, Murat, e Simon [32, 81, 90]. Dagli anni '90 in poi il rinnovato interesse per la teoria è stato in parte motivato dall'impulso dato dagli strumenti dell'Analisi Numerica nella ricerca di forme ottimali. Per una presentazione completa, facciamo riferimento alla recente monograf a [67] di Henrot e Pierre (vedi anche i libri [47, 91]) e, senza pretesa di completezza, ai lavori [23, 44, 46, 63, 64, 83].

A causa della mancanza di differenziabilità di φ nell'origine, il calcolo della derivata di forma del funzionalel J introdotto in (21) non rientra nella letteratura sopra citata. Abbiamo quindi cercato di sviluppare un nuovo metodo, che si applica a funzionali convessi più generali e in dimensione superiore. Consideriamo funzionali di forma del tipo

$$J(\Omega) := -\inf\left\{ \int_\Omega \left[f(\nabla u) + g(u)\right] dx \ : \ u \in W^{1,p}_0(\Omega) \right\} , \tag{22}$$

al variare di Ω tra i sottoinsiemi aperti limitati di $\mathbb{R}^n$ con frontiera Lipschitziana, con $f : \mathbb{R}^n \to \mathbb{R}$ e $g : \mathbb{R} \to \mathbb{R}$ due integrandi f ssati, che supponiamo essere continui, convessi e soddisfacenti una condizione di crescita, di ordine p e q rispettivamente. Analogamente consideriamo il problema di Neumann, in cui non si richiede nessuna condizione al contorno per le funzioni ammissibili in (22).

Dato un campo vettoriale V di classe $C^1(\mathbb{R}^n;\mathbb{R}^n)$, consideriamo la famiglia a un parametro di domini che sono ottenuti come deformazioni di Ω con velocità iniziale V, cioè poniamo

$$\Omega_\varepsilon := \Big\{ x + \varepsilon V(x) \ : \ x \in \Omega \Big\}, \qquad \varepsilon > 0 \,.$$

Per def nizione, la derivata di forma di J in Ω nella direzione V, se esiste, è data dal limite

$$J'(\Omega, V) := \lim_{\varepsilon \to 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} \,. \tag{23}$$

L'approccio che adottiamo per studiare la derivata di forma (23) è diverso da quello classicamente utilizzato in letteratura, e sembra avere un duplice interesse: da una parte permette di ottenere la derivata di forma per integrandi più generali f e g; d'altra parte conduce alla scoperta di una nuova condizione di ottimalità per le soluzioni del problema (22).

Prima di presentare i risultati, richiamiamo brevemente l'approccio abitualmente utilizzato per il calcolo di $J'(\Omega,V)$, per evidenziare i differenti punti di vista.

Classicamente, l'oggetto di studio nella teoria delle derivate di forma è la differenziabilità in $\varepsilon = 0^+$ di funzioni della forma

$$I(\varepsilon) := \int_{\Omega_\varepsilon} \psi(\varepsilon, x)\, dx \,, \tag{24}$$

con Ω_ε l'immagine di un insieme misurabile Ω attraverso una famiglia a un parametro di diffeomorf smi bi-Lipschitziani Ψ_ε. In particolare, la derivata di forma del minimo di un funzionale integrale può essere trattata come un caso particolare di (24): infatti, se si considera una soluzione u_ε del problema di minimo $J(\Omega_\varepsilon)$ e si pone

$$\psi(\varepsilon, x) := -\big[f(\nabla u_\varepsilon(x)) + g(u_\varepsilon(x)) \big] \,, \tag{25}$$

allora risulta $J(\Omega_\varepsilon) = I(\varepsilon)$.

La differenziabilità in $\varepsilon = 0^+$ della mappa $I(\varepsilon)$ e la formula della derivata destra sono dimostrate in [67] sotto opportune ipotesi di regolarità per l'integrando ψ. Più precisamente, si considerano i casi seguenti: $\psi(\varepsilon,\cdot)$ def nita su tutto $\mathbb{R}^n$ con

$$\psi(\varepsilon,\cdot) \in L^1(\mathbb{R}^n) \,, \qquad \varepsilon \mapsto \psi(\varepsilon,\cdot) \ \text{ derivabile in } 0 \,, \qquad \psi(0,\cdot) \in W^{1,1}(\mathbb{R}^n) \,, \tag{26}$$

oppure $\psi(\varepsilon,\cdot)$ def nita solo in Ω_ε con

$$\psi(\varepsilon, \Psi_\varepsilon(\cdot)) \in L^1(\Omega) \,, \qquad \varepsilon \mapsto \psi(\varepsilon, \Psi_\varepsilon(\cdot)) \ \text{ derivabile in } 0 \,, \qquad P(\psi(0,\cdot)) \in W^{1,1}(\mathbb{R}^n) \,, \tag{27}$$

con $P : L^1(\Omega) \to L^1(\mathbb{R}^n)$ un operatore di estensione lineare e continuo.

Al f ne di includere in questa impostazione i funzionali integrali di tipo (22), bisogna verif care che una delle condizioni (26) o (27) sia soddisfatta, con $\psi(\varepsilon,x)$ dato da (25). Questa verif ca deve essere fatta caso per caso, in quanto dipende dalla scelta di f e g. Inoltre, seguendo questo procedimento, è necessario calcolare la derivata

$$u' := \frac{d}{d\varepsilon} u_\varepsilon \big|_{\varepsilon = 0^+} \,, \tag{28}$$

che tipicamente richiede l'utilizzo dell'equazione di Eulero-Lagrange soddisfatta da u_ε.

Inf ne, per ottenere teoremi di struttura e risultati di rappresentazione per derivate di forma come integrali sul bordo $\partial\Omega$, sono necessarie ulteriori ipotesi di regolarità sull'integrando ψ, sul dominio Ω e sulle deformazioni Ψ_ε. Per una trattazione dettagliata facciamo riferimento a [67].

Il nostro approccio si basa invece sull'uso dell'Analisi Convessa, più precisamente sulla formulazione duale di $J(\Omega)$, che nel caso Dirichlet è data da

$$J^*(\Omega)=\inf\left\{\int_\Omega[f^*(\sigma)+g^*(\operatorname{div}\sigma)]dx\,:\,\sigma\in L^{p'}(\Omega;\mathbb{R}^n)\,,\ \operatorname{div}\sigma\in L^{q'}(\Omega)\right\}$$

dove f^* e g^* indicano le coniugate di Fenchel di f e g. Nel caso Neumann i campi ammissibili σ soddisfano la condizione aggiuntiva di traccia normale nulla $\sigma\cdot n=0$ su $\partial\Omega$.

La nostra strategia consiste nello sfruttare rispettivamente la formulazione primale $J(\Omega)$ e la formulazione duale $J^*(\Omega)$ per ottenere stime dal basso e dall'alto per il rapporto incrementale

$$q_\varepsilon(V):=\frac{J(\Omega_\varepsilon)-J(\Omega)}{\varepsilon}=\frac{J^*(\Omega_\varepsilon)-J^*(\Omega)}{\varepsilon}\,.$$

Tali stime dal basso e dall'alto assumono rispettivamente la forma

$$\liminf_{\varepsilon\to0^+}q_\varepsilon(V)\geq\inf_{\sigma\in\mathscr{S}^*}\sup_{u\in\mathscr{S}}\int_\Omega A(u,\sigma):DV\,dx \tag{29}$$

e

$$\limsup_{\varepsilon\to0^+}q_\varepsilon(V)\leq\sup_{u\in\mathscr{S}}\inf_{\sigma\in\mathscr{S}^*}\int_\Omega A(u,\sigma):DV\,dx \tag{30}$$

dove $\mathscr{S}$ e $\mathscr{S}^*$ indicano l'insieme di soluzioni di $J(\Omega)$ e $J^*(\Omega)$ rispettivamente, e $A(u,\sigma)$ è il campo di tensori definito sullo spazio prodotto $\mathscr{S}\times\mathscr{S}^*$ come

$$A(u,\sigma):=\nabla u\otimes\sigma-[f(\nabla u)+g(u)]I \tag{31}$$

(con I la matrice identità). Poiché l'inf-sup nel membro destro di (29) è maggiore o uguale al sup-inf nel membro sinistro di (30), concludiamo che essi coincidono, e quindi che il limite per $\varepsilon\to0^+$ di $q_\varepsilon(V)$, cioè la derivata di forma $J'(\Omega,V)$, esiste. Se indichiamo con $(u^\star,\sigma^\star)\in\mathscr{S}\times\mathscr{S}^*$ un elemento in cui il valore del sup-inf o dell'inf-sup è raggiunto, vale

$$J'(\Omega,V)=\int_\Omega A(u^\star,\sigma^\star):DV\,dx\,. \tag{32}$$

Sotto ulteriori ipotesi, la derivata di forma può essere riformulata come forma lineare in V, più precisamente come un integrale di bordo che dipende linearmente dalla componente normale di V su $\partial\Omega$. Le ipotesi aggiuntive di regolarità sono necessarie per affermare che una coppia ottimale $(u^\star,\sigma^\star)$ in (32) non dipende dal campo di deformazione V, e per poter applicare le formule di integrazione per parti che richiedono una nozione debole di traccia (vedi [6, 7, 34, 35]).

Sottolineiamo che, come conseguenza delle stime sopra descritte per $q_\varepsilon(V)$, otteniamo una nuova condizione necessaria di ottimalità per il problema variazionale $J(\Omega)$. Sorprendentemente, mediante variazioni orizzontali, sfruttando il fatto che i $q_\varepsilon(V)$ sono nulli per ogni $V\in C_0^1(\Omega,\mathbb{R}^n)$, le nostre stime danno come ulteriore risultato l'informazione che degli opportuni tensori della forma (31) sono a divergenza nulla. In particolare, se f è Gateaux-differenziabile eccetto al più nell'origine, risulta che per ogni $u\in\mathscr{S}$ vale

$$\operatorname{div}\Big(\nabla u\otimes\nabla f(\nabla u)-[f(\nabla u)+g(u)]I\Big)=0 \tag{33}$$

nel senso delle distribuzioni.

Sorprendentemente, per quanto ne sappiamo, la condizione (33) sembra essere f no ad ora sconosciuta, eccetto che nel caso scalare $n = 1$, in cui si riduce alla legge di conservazione (o integrale primo dell'equazione di Eulero) seguente, soddisfatta da estremali regolari di Lagrangiane lisce:

$$u' f'(u') - [f(u') + g(u)] = c\,,$$

(vedi *e.g.* [25, Proposition 1.13]).

Sottolineiamo che, seguendo questa strategia, non abbiamo mai utilizzato la derivata u' def nita in (28), in particolare non abbiamo bisogno della validità dell'equazione di Eulero-Lagrange per i punti di minimo (circa le condizioni necessarie per la sua validità, facciamo riferimento ai recenti lavori [10, 45], e i riferimenti ivi contenute). Possiamo quindi trattare anche funzionali integrali i cui punti di minimo soddisfano solo una disuguaglianza variazionale. Osserviamo che il problema (21), posto sulla sezione trasversale di un f lo, rientra in questa classe di problemi: la condizione di ottimalità soddisfatta dagli elementi di $\mathscr{S}$ non è un'equazione di Eulero-Lagrange, ma solo una disuguaglianza variazionale. Lo studio della derivata prima di forma per (21) non ci ha permesso di verif care la congettura riguardo al plateau, poiché abbiamo ottenuto derivata nulla.

I possibili sviluppi nell'ambito delle derivate di forma per minimi di funzionali integrali vanno in diverse direzioni. Il primo aspetto che sarebbe interessante approfondire è la linearità di J' rispetto al campo di deformazione V: abbiamo ottenuto condizioni suff cienti che garantiscono tale proprietà e vorremmo determinarne anche di necessarie. In generale, ad esempio nell'ambito di non unicità delle soluzioni, crediamo che la linearità sia una condizione troppo forte: più precisamente, la nostra congettura è che J' sia della forma

$$J'(\Omega,V) = \int_{\partial\Omega} \alpha(x)\,(V\cdot n)^{+}\,\mathscr{H}^{n-1}(x) + \int_{\partial\Omega} \beta(x)\,(V\cdot n)^{+}\,\mathscr{H}^{n-1}(x)\;,$$

con α, β due densità opportune in $L^{\infty}(\partial\Omega)$ che potrebbero dipendere dai dati del problema di minimo $J(\Omega)$, e $(V\cdot n)^{\pm}$ pari alla parte positiva e negativa del prodotto scalare $V\cdot n$ sul bordo. In particolare, ci aspettiamo che J' sia lineare rispetto a deformazioni puramente interne o puramente esterne.

Un altro problema interessante è studiare le derivate di forma di ordine superiore. In questa direzione abbiamo applicato lo stesso approccio per il calcolo della derivata seconda $J''(\Omega,V)$, supponendo maggiore regolarità sul dominio Ω e sugli integrandi f e g. Sfruttando ancora le formulazioni primale e duale di $J(\Omega)$, riusciamo a limitare dall'alto e dal basso il $\liminf$ e il $\limsup$ della successione

$$r_\varepsilon(V) := 2\,\frac{[J(\Omega_\varepsilon) - J(\Omega) - \varepsilon J'(\Omega,V)]}{\varepsilon^2}\;,\;\; \varepsilon > 0\,,$$

ed otteniamo la formula di rappresentazione

$$\begin{aligned} J''(\Omega,V) \;&= \int_{\partial\Omega} (V\cdot n)^2 \left[\frac{\partial \overline{u}}{\partial n}\,(\nabla^2\overline{u}\,\nabla^2 f(\nabla\overline{u})\,n)\cdot n + \big(\nabla\overline{u}\cdot\nabla f(\nabla\overline{u}) - f(\nabla\overline{u})\big)H_{\partial\Omega}\right] d\mathscr{H}^{n-1} + \\ &\quad - \inf_{\substack{v\in H^1(\Omega)\\ \mathrm{Tr}(v) = -V\cdot\nabla\overline{u}}} \left\{\int_{\Omega} (\nabla^2 f(\nabla\overline{u})\,\nabla v)\cdot\nabla v + g''(\overline{u})v^2\right\}, \end{aligned} \tag{34}$$

con $H_{\partial\Omega}$ la curvatura media di $\partial\Omega$.

Osserviamo che per il momento la (34) è stata ottenuta solo nel caso regolare e la sua estensione a integrandi più generali sarà oggetto dei prossimi studi. Prevediamo che i risultati riguardo alla derivata di forma di ordine 2 possano dare informazioni circa la curvatura del bordo del plateau.

French Summary

Le sujet principal de la Thèse est l' *optimisation de la compliance des structures élastiques minces*. Le problème consiste en déterminer la conf guration la plus résistante, lorsqu'une quantité inf nitésimale de matériau élastique est soumis à une force f xée et est conf née dans une région de volume inf nitésimal.

La résistance au chargement peut être mesurée en calculant une fonctionnelle de forme, la *compliance*, dans laquelle la forme représente le volume occupé par le matériau élastique. Donc nous sommes conduits à étudier un problème de minimisation d'une fonctionnelle de forme, sous une contrainte appropriée.

Nous nous intéresserons plus particulièrement au cas où la région de design est un f l f n, representé par un cylindre de section transverse inf nitésimale.

L'étude est motivée par des problèmes d'ingénierie: les structures minces sont très intéressantes d'un point de vue pratique.

La stratégie utilisée tire son inspiration des travaux récents par I. Fragalà, G. Bouchitté et P. Seppecher, dans lesquels les auteurs considèrent des plaques élastiques [G. BOUCHITTÉ, I. FRAGALÀ, P. SEPPECHER: Structural optimization of thin plates: the three dimensional approach. *Arch. Rat. Mech. Anal.* (2011)]. Cependant il faut souligner que le cas du cylindre est loin de se résumer à une variante technique du cas des plaques. Comme nous le verrons en effet, le modèle limite obtenu dans l'analyse asymptotique *3d-1d* est plus riche et subtile que celui correspondant à une analyse asymptotique *3d-2d*.

L'étude des conf gurations optimales pour le modèle limite obtenu nous a conduit à une problématique nouvelle: l'existence de vraies forme optimales (donc sans apparition de structures composites) pour une barre en régime de pure torsion est liée à l'existence de solutions pour un *problème non standard de frontière libre* dans le plan. Ce problème représente un challenge et nous nous contenterons de donner quelques premiers résultats et perspectives. Par ailleurs, en liaison avec ce problème, nous développerons une stratégie nouvelle pour caractériser *dérivée de forme pour le minimum d'une fonctionnelle intégrale*. La théorie de dérivées de forme est un sujet très largement étudié (voir *e.g.* la monographie de A. Henrot et M.Pierre *Variations et Optimisation de Formes. Une Analyse Géométrique*, Springer Berlin (2005), et les références qui sont y contenues), mais les techniques classiques qui y sont utilisées s'appuient sur des hypothèses de régularité non vérif ées dans notre cas.

L'organisation de la Thèse est la suivante.

Dans la première partie sont réunis les préliminaires: dans le Chapitre 1 nous rappellons les principaux outils mathématiques d'Analyse Convexe, Théorie Géometrique de la Mesure et Γ-convergence que nous utiliserons dans la Thèse; ensuite nous donnons un aperçu de la théorie de l'Élasticité Linéaire, qui motive l'étude du problème principal. La deuxième partie (Chapitres 2 et 3) est dédiée à l'étude de l'optimisation de la compliance lorsque le domaine de design est un cylindre de section inf nitésimale. Dans la troisième partie (Chapitres 4 et 5) nous dénveloppons les deux derniers problèmes mentionnés ci-dessus (problème à frontière libre et dérivée de forme). Lorsque nécessaire, les démonstrations les plus techniques sont renvoyées dans un Appendice en f n de Chapitre. Quelques problèmes ouverts et perspectives seront parfois présentés dans un paragraphe dédié.

La thèse englobe les résultats présentés dans les papiers suivants (écrits au cours de ces trois années de Doctorat):

- (Chapitres 2 et 3) G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI, P. SEPPECHER: Optimal Thin Torsion Rods and Cheeger Sets, *SIAM J. Math. Anal.*, **44**, 483-512, (2012).
- (Chapitres 2 et 3) I. LUCARDESI: Optimal design in thin rods, en préparation.
- (Chapitre 4) J. ALIBERT, G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI: A non standard free boundary problem arising in shape optimization of thin torsion rods, à apparaître en *Interfaces and Free Boundaries* (2012).
- (Chapitre 5) G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI: Shape derivatives for minima of integral functionals, en préparation.

Dans ce qui suit, nous décrivons plus en détail le contenu des différents chapitres.

[Chapitre 2] On considère une région de design Q dans $\mathbb{R}^3$ soumis à un chargement extérieur donné que nous représentons par une distribution vectorielle $F \in H^{-1}(Q;\mathbb{R}^3)$. La résistance à un tel chargement d' un matériau élastique isotrope occupant une certaine région $\Omega \subset Q$ peut être évaluée en calculant sa *compliance*. Sous l'hypothèse de petits déplacements, cette compliance est donnée par la fonctionnelle de forme suivante:

$$\mathscr{C}(\Omega) := \sup\left\{ \langle F,u\rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\,dx \ : \ u \in H^1(Q;\mathbb{R}^3) \right\} , \tag{1}$$

où, comme il est habituel en Éasticité Lineaire, $e(u)$ dénote la partie symétrique du gradient ∇u, et le potentiel de strain $j : \mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}$, supposé isotrope, est strictement convexe et est du type

$$j(z) := \frac{\lambda}{2}\,\mathrm{tr}^2(z) + \eta\,|z|^2 , \tag{2}$$

où $\lambda\,,\mu > 0$ sont les coéff cients de Lamé du matériau.
La valeur de $\mathscr{C}(\Omega)$ est proportionnelle au travail des forces extérieures nécessaire pour amener la structure jusqu'à l'équilibre. En particulier, plus la compliance est petite, plus la résistance est élevée.
Clairement, af n que $\mathscr{C}(\Omega)$ reste bornée, le chargement F doit avoir son support contenu dans $\overline{\Omega}$; de plus, en l'absence d'encastrement ou de déplacement imposé sur une partie de la structure, il est nécessaire que F soit *équilibrée*, *i.e.*

$$\langle F,u\rangle_{\mathbb{R}^3} = 0 \ , \quad \text{lorsque } e(u) = 0 \ .$$

Sous cette condition, un déplacement optimal $\overline{u}$ existe et satisfait l'égalité: $\mathscr{C}(\Omega) = \frac{1}{2}\langle F, \overline{u}\rangle_{\mathbb{R}^3}$.

Dans la Thèse nous étudions le problème de trouver les conf gurations de matériau élastique les plus robustes, *i.e.* qui minimisent la compliance, lorsque la région de design est un domaine mince. En outre, nous nous intéresserons au cas où le rapport entre le volume du matériau élastique et le volume de la région de design est inf nitésimal.

En disant "mince" nous entendons que une ou deux dimensions spatiales du corps sont beaucoup plus petites que les autres. Ces structures particulières sont très importantes dans les problèmes d'ingéniérie: leur petit poids et leur maniabilité les rendent très intéressantes pour les applications pratiques. Ici nous considérerons le cas d'une structure *f laire* dans lequel l'approximation choisie réxulte d'une analyse asymptotique 3d-1d.

La façon mathématique pour décrire un f l est la suivante: un f l est une structure tridimensionnelle qui occupe le volume engendré par un domaine connexe planaire, appelé *section transversale*, avec centroïd qui varie perpendiculairement à une courbe, appelée *axe*; de plus le diamètre de la section transversale est beaucoup plus petit que la longueur de l'axe. En particulier nous nous occupons des *f ls droits*, dans lequels l'axe est une ligne droite I et la section transversale est un domaine borné plain D, constant le long de l'axe: nous répresentons cettes structures comme des cylindres du type

$$Q_\delta := \delta\overline{D} \times I \,, \tag{3}$$

étant $D \subset \mathbb{R}^2$ un domaine ouvert borné, I un interval fermé borné et $\delta > 0$ un paramètre inf nitésimal qui répresent le petit rapport entre le diamètre de la section et la longueur.

La littérature sur les f ls minces est très vaste: la thèorie classique a été developpée par Euler, Bernoulli, Navier, Saint Venant, Timoshenko, Vlassov; dans les dernières années, grâce aux nouvelles techniques numériques, ces problèmes ont eu un regain d'intérêt et sont désormais un large domaine de mathématique appliquée (nous nous limitons à méntionner [73, 78, 80, 86]). Pour un aperçu complet sur ce sujet, nous renvoyons à l'ouvrage de référence par Trabucho et Viaño [95] et les références qui sont y contenues.

Le problème que nous traitons, et par conséquent l'approche que nous adoptons pour le résoudre, puise son inspiration dans un ouvrage récent de G. Bouchitté, I. Fragalà et P. Seppecher [19], dans lequel les auteurs ont étudié le problème d'optimisation de compliance lorsque la région de design peut être approchée par un ensemble à deux dimensions, plus précisément, ils ont considéré le cas de plaques minces, décrites par une famille de cylindres de la forme

$$Q_\delta := \overline{D} \times \delta I \,,$$

ayant une épaisseur inf nitésimale δ.

Dans ce qui suit nous traitons avec des ensembles minces Q_δ déf ni selon (3). Af n de trouver les conf gurations les plus légeres et plus robustes, nous minimisons la compliance $\mathscr{C}(\Omega)$ parmi les sous-ensembles Ω de Q_δ avec volume prescrit égal à m, *i.e.* nous étudions

$$\inf\left\{\mathscr{C}(\Omega) \ : \ \Omega \subset Q_\delta \,,\ \frac{|\Omega|}{|Q_\delta|} = m\right\} \,, \tag{4}$$

et nous procédons à la double limite pour $\delta \to 0$ et $m \to 0$.
Si nous rajoutons la contrainte de volume dans le coût au moyen d'un multiplicateur de Lagrange $k \in \mathbb{R}$, les problèmes variationnels (4) à l'étude prennent la forme

$$\phi^\delta(k) := \inf_{\Omega \subset Q_\delta} \left\{\mathscr{C}^\delta(\Omega) + k\frac{|\Omega|}{|Q_\delta|}\right\} \,, \tag{5}$$

étant

$$\mathscr{C}^\delta(\Omega) := \sup\left\{ \langle F^\delta, \tilde{u}\rangle_{\mathbb{R}^3} - \int_\Omega j(e(\tilde{u}))\,dx \ : \ \tilde{u} \in H^1(Q_\delta, \mathbb{R}^3) \right\} . \tag{6}$$

Ici F^δ est une mise à l'échelle appropriée du chargement F, choisie de telle sorte que dans le passage à la limite l'inf mum reste f ni. La choix de l'échelle F^δ dépend des hypothèses faites sur le genre de forces appliquées. Dans la littérature il est d'usage de distinguer entre les cas de extension, f exion et torsion: dans la Thèse nous concentrons notre attention sur les chargements pour lesquels les contributions de la f exion et de la torsion peuvent être découplées de façon appropriée, et nous choisissons deux échelles différentes pour ces deux composantes: grosso modo, nous considérons que les forces F de la forme $F = G + H$, lorsque la f exion ne dépend que de la composante verticale H. Le cas général est diff cile à traiter, en raison de l'interaction entre ces contributions de la force.

Dans le Chapitre 2 nous faisons le premier passage à la limite, *i.e.* nous étudions le comportement asymptotique de problème (5) lorsque $\delta \to 0^+$: il s'agit de chercher la conf guration la plus robuste dans un ensemble f laire, en gardant f xé le rapport entre le volume du matériau et le volume de la région mince de design.

La première étape de l'étude consiste à reformuler les problèmes variationnels $\phi^\delta(k)$ sur le domaine f xe $Q := \overline{D} \times I$, au lieu des cylindres minces Q_δ. Après un changement de variables approprié pour les déplacements et une mise à l'échelle adaptée au chargement, les problèmes (5) peuvent être réécrits comme

$$\phi^\delta(k) = \inf_{\omega \subset Q} \left\{ \mathscr{C}^\delta(\omega) + k|\omega| \right\} , \tag{7}$$

étant

$$\mathscr{C}^\delta(\omega) := \sup\left\{ \frac{1}{\delta}\,\langle G, u\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_\Omega j(e^\delta(u))\,dx \ : \ u \in H^1(Q_\delta, \mathbb{R}^3) \right\} , \tag{8}$$

où $e^\delta : H^1(Q;\mathbb{R}^3) \to L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ est l'opérateur déf ni par

$$e^\delta_{\alpha\beta}(u) := \delta^{-2} e_{\alpha\beta}(u) \ , \ \ e^\delta_{\alpha 3}(u) := \delta^{-1} e_{\alpha 3}(u) \ , \ \ e^\delta_{33}(u) := e_{33}(u) \ ,$$

comme il est d'usage dans la littérature sur la réduction de dimension $3d - 1d$. L'étude asymptotique de $\phi^\delta(k)$ est basée sur la comparaison avec les "problèmes f ctifs", c'est-à-dire les formulations rélaxées dans $L^\infty(Q;[0,1])$. En effet, il est bien connu que les problèmes de inf mum (7) sont en général mal posés, en raison de l'apparition de phénomènes d'homogénéisation qui empêchent l'existence d'un domaine optimal (voir [2]). Ainsi, nous avons besoin d'étendre la classe de matériaux admissibles, passant de "vrais" matériaux, représentés par des fonctions caractéristiques, aux matériaux "composites", représentés par des densités à valeurs dans $[0,1]$. Dans ce but, nous introduisons la famille des problèmes variationnels

$$\widetilde{\phi}^\delta(k) := \inf\left\{ \widetilde{\mathscr{C}}^\delta(\theta) + k \int_Q \theta\,dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\} , \tag{9}$$

oú $\widetilde{\mathscr{C}}^\delta(\theta)$ représente l'extension naturelle de la compliance $\mathscr{C}^\delta(\omega)$ à $L^\infty(Q;[0,1])$:

$$\widetilde{\mathscr{C}}^\delta(\theta) := \sup\left\{ \frac{1}{\delta}\langle G, u\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u))\,\theta\,dx \ : \ u \in H^1(Q;\mathbb{R}^3) \right\} . \tag{10}$$

En exploitant certaines propriétés délicates de compacité provenantes des variantes de l'inégalité de Korn (voir la Section 1.5) et techniques de Γ-convergence (voir la Section 1.3), nous déterminons le comportement limite de $\widetilde{\phi}^\delta(k)$ pour $\delta \to 0^+$: la suite des problèmes f ctifs tend vers une limite $\phi(k)$, ce qui est ecore un problème variationnel posé sur les densités, avec la même structure: un terme de compliance avec une pénalisation de volume. Plus précisément, $\phi(k)$ se lit

$$\phi(k) := \inf\left\{ \mathscr{C}^{lim}(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\} , \tag{11}$$

oú $\mathscr{C}^{lim}$ est la Γ-limite de la suite $\widetilde{\mathscr{C}}^\delta$ pour $\delta \to 0^+$ par rapport à la convergence faible * dans $L^\infty(Q;[0,1])$. Exploitant une estimation cruciale pour la fonctionnelle de compliance relaxée, établie dans [19, Proposition 2.8], on déduit que $\phi^\delta(k)$ et $\widetilde{\phi}^\delta(k)$ ont le même comportement asymptotique, c'est-à-dire $\lim_{\delta\to 0^+} \phi^\delta(k) = \phi(k)$.

Nous rappelons que le processus de réduction de dimension est effectué sans faire aucune hypothèse topologique sur l'ensemble Ω occupé par le matériau. Par conséquent, il n'est pas couvert par la littérature très vaste sur l'analyse $3d-1d$.

À ce point il est naturel de se demander si $\phi(k)$ admet une solution classique (*i.e.* une densité à valeurs dans $\{0,1\}$): cela correspond à se demander si le problème de compliance sous contrainte de volume, dans un ensemble f laire, admet comme solution un matériau réel plutôt qu'un composite. Fournir des reformulations de $\phi(k)$ en tant que un problème variationnel pour les déplacements, et comme un problème variationnel pour les tenseurs de stress, nous permet de donner des conditions d'optimalité nécessaires et suff santes. Ces formulations alternatives révèlent que le problème (11) peut être résolu section par section; en outre, en exploitant le système d'optimalité, la question de l'existence de vrais solutions peut être reformulée de manière plus aisée, et elle peut être reliée à l'existence de solutions "spéciaux" d'un certain problème à frontière libre. Une description plus détaillée et une analyse plus approfondie du problème est reportée au Chapitre 4.

Dans le Chapitre 3, nous faisons le deuxième passage à la limite, c'est-à-dire que nous étudions le comportement asymptotique du problème (11) quand $k \to +\infty$: nous rappelons que considérer grandes valeurs de k revient à considérer petits "taux de remplissage" $\int_Q d\theta/|Q|$. Nous montrons que la suite $\phi(k)$ est asymptotiquement équivalente à $\sqrt{2k}$, et nous déterminons la limite $\overline{m}$ du rapport $\phi(k)/\sqrt{2k}$. Cette limite est encore un problème variationnel, dont le coût a la même structure (terme de compliance plus terme de volume), mais il est posé dans l'espace $\mathscr{M}^+(Q)$ des mesures positives sur $\mathbb{R}^3$ à support compact dans Q:

$$\overline{m} := \inf\left\{ \mathscr{C}^{lim}(\mu) + \frac{1}{2}\int d\mu \ : \ \mu \in \mathscr{M}^+(Q) \right\} ; \tag{12}$$

où $\mathscr{C}^{lim}(\mu)$ est l'extension naturelle de la compliance limite $\mathscr{C}^{lim}(\theta)$ à la classe $\mathscr{M}^+(Q)$. Le nouveau cadre est tout à fait naturel, puisque le problème limite décrit des phénomènes de concentration qui peuvent survenir dans des parties de dimension inférieure.

En exploitant la formulation duale de $\overline{m}$ nous fournissons une caractérisation variationnelle alternative des mesures optimales. Il s'avère que, en général, la solution n'est pas unique et elle est diff cile à déterminer explicitement. Néanmoins, lors de l'examen des chargements particuliers, nous pouvons résoudre $\overline{m}$ complètement, ou, au moins, donner des informations précises sur le support de ses solutions. Les phénomènes de concentration correspondantes sont discutés dans la dernière partie du Chapitre 3, et peuvent être résumées comme suit.

Lorsque le chargement est purement de torsion et D est convexe, la solution de $\overline{m}$ se révèle être unique et peut être déterminée explicitement comme une mesure concentrée, section par section, sur le bord de ce qu'on appelle l'*ensemble de Cheeger de D*. On rappelle que, sous l'hypothèse de D convexe, son ensemble de Cheeger est l'unique minimum du quotient périmètre / surface parmi tous les sous-ensembles mesurables de D (voir la section 1.4.3 dans les Préliminaires):

$$\inf_{E \subset \overline{D},\, 1\!\!1_E \in BV(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla 1\!\!1_E|}{|E|} \,. \tag{13}$$

Le problème purement géométrique (13) (qui peut être déf ni de façon plus générale lorsque D est un sous-ensemble connexe du plan) est connu comme problème de Cheeger, et ces dernières années a attiré l'attention de nombreux auteurs (voir [3, 24, 30, 31, 56, 59, 60, 71]). Au meilleur de nos connaissances, jusqu'à présent, il n'y avait aucun énoncé et preuve de cette caractérisation géométrique de "barres de torsion" optimales légeres en termes d'ensembles de Cheeger. Nous soulignons que cette caractérisation est valable uniquement en régime de pure torsion.
Pour chargements plus généraux, en raison de l'interaction entre les énergies de f exion, de torsion et de stretching, nous obtenons un modèle plus compliqué. Nous donnons quelques exemples pour lesquels le problème variationnel (12) peut être résolu explicitement, et nous présentons quelques cas où ses solutions se révèlent être liées à des variantes intéressantes du problème de Cheeger. Il est facile de voir que la formulation relaxée du problème classique de Cheeger (13) est

$$\inf\left\{ \int_D |Du| \ : \ u \in BV_0(D) \,,\ \int_D u = 1 \right\} \,. \tag{14}$$

La première variante qui entre en jeu est une sorte de perturbation avec un terme de translation:

$$\inf\left\{ \int_D |Du + q| \ : \ u \in BV_0(D) \,,\ \int_D u = C(q) \right\} \,, \tag{15}$$

q étant un champ de vecteurs f xé; alors que la seconde variante est une version pondérée:

$$\inf\left\{ \int_D \alpha |Du| \ : \ u \in BV_0(D),\ \int_D u = 1 \right\} \,, \tag{16}$$

α étant une fonction non négative dans D. Nous rappelons que le problème (16) a été traité il y a quelques années par Ionescu et Lachand-Robert: dans [69], les auteurs, motivés par des applications à la modélisation des glissements de terrain, étudient le cas où l'integrande à minimiser ainsi que la contrainte sont pondérées.

Dans le Chapitre 4 nous affrontons la question naturelle suivante, qui vient du problème variationnel $\phi(k)$ en (11):

Est-ce que le problème $\phi(k)$ admet une solution $\overline{\theta}$ à valeurs dans $\{0,1\}$? (17)

Nous rappelons que $\phi(k)$ a été obtenu par passage à la limite lorsque $\delta \to 0^+$ dans le problème d'optimisation de compliance (7): puisqu'il s'agit d'un problème variationnel établi sur les densitées $L^\infty(Q,[0,1])$, une réponse aff rmative à la question (17) signif erait que le design optimal des f ls minces admet une solution classique, qui peut être identif ée avec un ensemble plutôt qu'un composite.

Nous concentrons notre attention sur le régime de pure torsion: dans ce cas, la question peut être reformulée de manière plus aisée. Comme on a déjà remarqué, $\phi(k)$ peut être résolu section par section, donc, après un changement de variable approprié et passage à un problème dual, nous sommes amenés à étudier le problème variationnel suivant sur le plan:

$$m(s) := \inf\left\{ \int_{\mathbb{R}^2} \varphi(\nabla u) \ : \ u \in H^1_c(D)\,,\ \int_{\mathbb{R}^2} u = s \right\}, \tag{18}$$

où s est un paramétre réel proportionnel à k, φ est l'intégrande convexe mais pas strictement-convexe suivante

$$\varphi(y) := \begin{cases} \frac{|y|^2}{2} + \frac{1}{2} & \text{if } |y| \geq 1 \\ |y| & \text{if } |y| < 1\,, \end{cases}$$

et $H^1_c(D)$ désigne l'espace des fonctions dans H^1 qui sont constantes en dehors de D (si D est simplement connexe, $H^1_c(D)$ coïncide avec l'habituel espace de Sobolev $H^1_0(D)$).
La question (17) est équivalente à la suivante:

Est-ce que le problème $m(s)$ admet une solution $\overline{u}$ telle que $|\nabla\overline{u}| \in \{0\} \cup (1, +\infty)$ p.p. dans D? (19)

Plus précisément, étant donné une solution $\overline{u}$ pour $m(s)$ et une solution $\overline{\theta}$ pour $\phi(k)$, la région où $\nabla\overline{u}$ s'annule correspond à l'absence de matériau (c'est-à-dire $\overline{\theta} = 0$), la région $|\nabla\overline{u}| > 1$ à la présence de matériau (c'est-à-dire $\overline{\theta} = 1$) et la région intermédiaire correspond à la région de homogénéisation. En d'autres termes, l'étude de $m(s)$ peut être appliquée à établir l'inf uence de la forme de la section et du taux de remplissage sur la présence des régions de homogénéisation dans les barres de torsion minces optimales.

Les résultats principaux du Chapitre 4 concernent l'étude de la question (19) en relation avec la géométrie du domaine D et aussi avec la valeur du paramètre s.

Nous disons que u est une *solution spéciale* pour $m(s)$ si elle minimise (18) et satisfait $|\nabla u| \in \{0\} \cup (1, +\infty)$ p.p. dans D, en outre nous appellons *plateau* de u, et on le note $\Omega(u)$, l'ensemble $\{\nabla u = 0\}$ moins la composante connexe illimitée de $\mathbb{R}^2 \setminus \overline{D}$ (où $u \equiv 0$).

Lorsque D est une boule ou un anneau, par des calculs explicites et exploitant les conditions d'optimalité, nous montrons que $m(s)$ admet une solution spéciale, et il n'y a pas d'autre solution.
Ensuite, on montre que les boules et les anneaux *ne sont pas* les seules domaines pour lesquels $m(s)$ admet une solution spéciale. À cet égard, il est intéressant de comparer nos résultats avec ceux obtenus par Murat et Tartar dans [F. MURAT, L. TARTAR: Calcul des variations et homogénéisation. Homogenization methods: theory and applications in physics (Bráu-sans-Nappe, 1983), 319-369, *Collect. Dir. Etudes Rech. Elec. France* **5**, Eyrolles, Paris (1985)], sur le problème de maximiser la rigidité de torsion d'une barre avec section donnée et composée par deux matériaux élastiques avec proportions f xes. Le problème variationnel correspondant est tout à fait similaire au nôtre, sauf qu'il s'agit d'une intégrande *différentiable*, et les solutions classiques (*i.e.* designs optimals sans région de homogénéisation) ne peuvent pas exister à moins que la section transversale D est un disque.

Dans notre cas, l'intégrande φ est non dérivable en zéro et la conclusion est dans une direction tout à fait opposée: nous prouvons qu'il existe un domaine non circulaire D avec bord analytique, tel que, pour certains s, le problème $m(s)$ admet une solution spéciale. En outre, cette solution présente un plateau convexe avec une frontière analytique. Pour atteindre ce résultat, nous utilisons comme outil essentiel la relation entre $m(s)$ et le problème de Cheeger.

On remarque que le rôle crucial joué par l'ensemble de Cheeger de D, dans l'étude du comportement asymptotique de $m(s)$ en (18) quand $s \to 0^+$, a déjà émergé dans l'asymptotique du problème de optimisation de la compliance $\phi(k)$ en (11) quand $k \to +\infty$: en fait le paramètre s est proportionnel à $\frac{1}{\sqrt{k}}$, de sorte que les petites valeurs de s correspondent aux grandes valeurs de k.

Après avoir donné quelques propriétés élémentaires sur le signe et le support de solutions généraux, nous obtenons des informations sur les propriétés qualitatives des solutions spéciaux, lorsque elles existent. Cela revient à étudier un problème non standard de frontière libre avec un obstacle sur le gradient:

$$\begin{cases} -\Delta u = \lambda \ , \quad |\nabla u| > 1 & \text{dans } D \setminus \Omega(u) \\ |\nabla u| = 1 & \text{sur } \Gamma(u) \\ u = c_i & \text{sur } \gamma_i \ , \end{cases} \tag{20}$$

où $\Gamma(u) := \partial\Omega(u) \cap D$ est la frontière libre et γ_i dénotent les différentes composantes connexes de $\Gamma(u)$.

Enf n, en supposant que D est simplement connexe, et qu'il existe une solution spéciale u avec une frontière libre lisse, nous démontrons des propriétés qualitatives du plateau:

- chaque composante connexe de $D \setminus \Omega(u)$ doit toucher le bord ∂D;
- sous des hypothèses convenables, le plateau $\Omega(u)$ doit être convexe;
- si D n'est pas ensemble de Cheeger de lui-même, le plateau $\Omega(u)$ ne peut pas être contenu de façon compacte dans D pour taux de remplissage arbitrairement petits.

Nous remarquons que la condition de frontière libre regulière que nous avons considérée *a priori* est nécessaire af n d'appliquer la théorie de P-fonctions (voir [93]). L'étude de la régularité de la frontière libre est une perspective intéressante et stimulante (voir [27, 28, 85]): un premier résultat obtenu dans cette direction concerne la f nitude du périmètre du plateau.

Nous soulignons que réussir à caractériser complètement les domaines D dans lesquels il existe une solution spéciale pour $m(s)$, semble être un problème très diff cile, qui reste à ce jour ouvert: à cet égard, nous croions que, au moins quand D est convexe, l'existence de solutions spéciaux est liée à la régularité de ∂D, et aussi au fait que D coïncide ou pas avec son ensemble de Cheeger. Notons que ce critère exclurait l'existence d'une solution spéciale dans le cas où D est un carré. Cette hypothése semble être conf rmée par les résultats numériques effectués dans [72] pour un problème très similaire, dans lequel des régions de homogénéisation sont observées.

Si D n'est pas un ensemble de Cheeger de lui-même, nous nous attendons que une certaine composante connexe $\Omega_0 \subset \{u \equiv 0\}$ du plateau touche la frontière dans un voisinage des points de courbure supérieur, c'est-à-dire des coins. Af n de prouver la conjecture, nous avons essayé d'exploiter les dérivés de forme. En fait, si nous f xons le paramètre s, nous pouvons interpréter $m(s)$ comme une fonctionnelle de forme $J(D)$, qui dépende du domaine D comme suit: en rajoutant la contrainte de volume dans la fonctionnelle, la résolution $m(s)$ sur D s'avère être équivalente à étudier

$$J(D) = -\inf\left\{ \int_D [\varphi(\nabla u) - \lambda u]\, dx \ : \ u \in H_0^1(D) \right\} \ , \tag{21}$$

avec $\lambda = m'(s)$.

Clairement la fonctionnelle de forme $J(\cdot)$ est stationnaire dans les domaines $D' \subset D$ qui contiennent $D \setminus \Omega_0$. Par ailleurs, le signe de la dérivée de forme peut donner des informations utiles:

si on considère des petites déformations internes de D, localisées sur une partie γ du bord, une dérivée de forme différente de zéro implique que Ω_0 ne touche pas cette partie γ.

[Chapitre 5]

La théorie des dérivées de forme est un sujet largement étudié, avec de nombreuses applications dans les problèmes variationnels et le design optimal. Sa origine peut être retracée à la première moitié du dernier siècle, avec le travail pionnier par Hadamard [66], suivi par Schiffer et Garabedian [61, 89]. Ensuite, des progrès importants ont été faits dans les années septante par Céa, Murat, et Simon [32, 81, 90]. Depuis les années nonante, les nombreuses contributions fournies par différents auteurs sont témoin d'un intérêt renouvelé, en partie motivé par l'impulsion donnée par le développement de l'analyse numérique dans la recherche de optimisation de forme. Nous nous référons à la récente monographie [67] par Henrot et Pierre comme un texte de référence (voir aussi les livres [47, 91]), et, sans aucune tentative d'exhaustivité, d'oeuvres représentatives [23, 44, 46, 63, 64, 83].

En raison de l'absence de différentiabilité de φ à l'origine, le calcul de la dérivée de forme de la fonctionnelle J en (21) n'est pas couvert par la littérature citée ci-dessus. Par conséquent, nous avons essayé de développer une nouvelle méthode, qui s'applique également aux fonctionnelles convexes plus généraux et en dimension supérieure. En fait, nous considérons les fonctionnelles de forme du type

$$J(\Omega) := -\inf\left\{ \int_\Omega \big[f(\nabla u) + g(u)\big]\, dx \ : \ u \in W_0^{1,p}(\Omega) \right\} . \tag{22}$$

Ici, Ω varie entre les sous-ensembles ouverts bornés de $\mathbb{R}^n$ avec bord Lipschitzien, $f : \mathbb{R}^n \to \mathbb{R}$ et $g : \mathbb{R} \to \mathbb{R}$ sont intégrandes données, qui sont supposées être continues, convexes, et satisfaisantes autres hypothèses de régularité et conditions de croissance, d'ordre p et q respectivement. De la même façon, nous pouvons traiter aussi le problème de Neumann, dans lequel aucune condition au bord est prescrite pour les fonctions admissibles dans (22).

Étant donné un champ de vecteurs V de classe $C^1(\mathbb{R}^n;\mathbb{R}^n)$, nous considérons la famille à un paramètre de domaines qui sont obtenus comme des déformations de Ω avec V comme vitesse initiale: nous posons

$$\Omega_\varepsilon := \Big\{ x + \varepsilon V(x) \ : \ x \in \Omega \Big\} , \qquad \varepsilon > 0 .$$

Par déf nition, la dérivée de forme de J en Ω dans la diréction V, si elle existe, est donnée par la limite

$$J'(\Omega, V) := \lim_{\varepsilon \to 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} . \tag{23}$$

L'approche que nous adoptons af n d'étudier la dérivée de forme (23) est différente de celle habituellement utilisée dans la littérature, et semble avoir un double intérêt: d'une part elle permet d'obtenir la dérivée de forme pour intégrandes f et g plus généraux, d'autre part, ainsi que la dérivée de forme, elle conduit à découvrir une nouvelle condition d'optimalité des solutions au problème (22).

Avant de décrire les résultats, rappelons brièvement l'approche habituelle pour le calcul de $J'(\Omega, V)$, af n d'éclairer la différence de point de vue.

Classiquement, l'objet d'étude dans la théorie des dérivées de forme est la différentiabilité en $\varepsilon = 0^+$ des fonctions du type

$$I(\varepsilon) := \int_{\Omega_\varepsilon} \psi(\varepsilon, x)\, dx , \tag{24}$$

étant Ω_ε l'image d'un ensemble mesurable Ω via une famille de difféomorphismes bi-Lipschitz Ψ_ε à un paramètre. En particulier, les dérivées de forme pour les minima de fonctionnelles intégrales peuvent être traitées comme un cas particulier de (24): si on prend une solution u_ε du problème de infmum $J(\Omega_\varepsilon)$ et on choisit

$$\psi(\varepsilon,x) := -\left[f(\nabla u_\varepsilon(x)) + g(u_\varepsilon(x))\right], \tag{25}$$

alors on a $J(\Omega_\varepsilon) = I(\varepsilon)$.

La différentiabilité en $\varepsilon = 0^+$ de la fonction $I(\varepsilon)$, avec la formule de la dérivée à droite, sont prouvées dans [67] en supposant des hypothèses de régularité appropriées sur l'integrande ψ. Plus précisément, les situations suivantes sont considerées: soit $\psi(\varepsilon,\cdot)$ est défnie sur $\mathbb{R}^n$ avec

$$\psi(\varepsilon,\cdot) \in L^1(\mathbb{R}^n), \qquad \varepsilon \mapsto \psi(\varepsilon,\cdot) \text{ derivable en } 0, \qquad \psi(0,\cdot) \in W^{1,1}(\mathbb{R}^n), \tag{26}$$

ou $\psi(\varepsilon,\cdot)$ est défnie seulement dans Ω_ε avec

$$\psi(\varepsilon,\Psi_\varepsilon(\cdot)) \in L^1(\Omega), \qquad \varepsilon \mapsto \psi(\varepsilon,\Psi_\varepsilon(\cdot)) \text{ derivable en } 0, \qquad P(\psi(0,\cdot)) \in W^{1,1}(\mathbb{R}^n), \tag{27}$$

étant $P : L^1(\Omega) \to L^1(\mathbb{R}^n)$ un opérateur d'extension linéaire et continu.

Afn d'inclure dans ce cadre les minima des fonctionnelles intégrales du type (22), il faut vérifer que l'une des conditions (26) ou (27) est valable, lorsque $\psi(\varepsilon,x)$ est pris comme en (25). Ce contrôle doit être fait cas par cas, en fonction du choix de f et g. En particulier, dans ce processus, on doit calculer la dérivée

$$u' := \frac{d}{d\varepsilon} u_\varepsilon\big|_{\varepsilon=0^+}, \tag{28}$$

ce qui généralement nécessite d'exploiter l'équation d'Euler-Lagrange satisfaite par u_ε.

Ensuite, d'autres hypothèses de régularité sur l'integrande ψ, sur le domaine Ω et sur les déformations Ψ_ε, sont nécessaires afn d'obtenir des théorèmes de structure et des résultats de représentation pour les dérivées de forme, qui conduisent à exprimer eux comme des intégrales sur le bord $\partial\Omega$. Pour une présentation détaillée, nous nous référons à [67].

En dépit, notre approche repose sur l'utilisation de l'Analyse Convexe, et plus spécifquement de la formulation duale de $J(\Omega)$, ce qui dans le cas Dirichlet est

$$J^*(\Omega) = \inf\left\{ \int_\Omega [f^*(\sigma) + g^*(\operatorname{div}\sigma)]dx \,:\, \sigma \in L^{p'}(\Omega;\mathbb{R}^n),\ \operatorname{div}\sigma \in L^{q'}(\Omega) \right\}$$

où f^* et g^* désignent les conjuguées de Fenchel de f et g. Dans le cas Neumann les champs admissibles σ satisfont à la condition supplémentaire de trace normale $\sigma \cdot n = 0$ sur $\partial\Omega$.

Notre stratégie consiste à exploiter respectivement la formulation primale $J(\Omega)$ et la formulation duale $J^*(\Omega)$ afn d'obtenir des minorants and majorants pour le quotient

$$q_\varepsilon(V) := \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} = \frac{J^*(\Omega_\varepsilon) - J^*(\Omega)}{\varepsilon}.$$

Ces bornes prennent respectivement la forme

$$\liminf_{\varepsilon\to 0^+} q_\varepsilon(V) \geq \inf_{\sigma\in\mathscr{S}^*} \sup_{u\in\mathscr{S}} \int_\Omega A(u,\sigma) : DV\, dx \tag{29}$$

et

$$\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) \leq \sup_{u\in\mathscr{S}} \inf_{\sigma\in\mathscr{S}^*} \int_\Omega A(u,\sigma) : DV\, dx \tag{30}$$

où $\mathscr{S}$ et $\mathscr{S}^*$ désignent l'ensemble des solutions de $J(\Omega)$ et $J^*(\Omega)$, et $A(u,\sigma)$ est le tenseur déf ni sur l'espace produit $\mathscr{S}\times\mathscr{S}^*$ par

$$A(u,\sigma) := \nabla u \otimes \sigma - [f(\nabla u)+g(u)]I \tag{31}$$

(étant I la matrice identité).

Comme le inf-sup dans le terme à droite de (29) est supérieure ou égale au sup-inf dans le terme à droite de (30), nous concluons qu'ils sont égaux et que la limite de $q_\varepsilon(V)$ pour $\varepsilon \to 0^+$, c'est-à-dire la dérivée de forme $J'(\Omega,V)$, existe. En désignant par $(u^\star,\sigma^\star)\in\mathscr{S}\times\mathscr{S}^*$ un élément où la valeur du sup-inf ou inf-sup est atteinte, on a

$$J'(\Omega,V) = \int_\Omega A(u^\star,\sigma^\star) : DV\,dx. \tag{32}$$

Sous des hypothèses de régularité supplémentaires, la dérivée de forme peut être reformulée aussi comme une forme linéaire de V, c'est-à-dire comme un intégral sur le bord qui dépend linéairement de la composante normale de V sur $\partial\Omega$. Les hypothèses de régularité supplémentaires sont nécessaires af n de dire qu'un couple optimal $(u^\star,\sigma^\star)$ dans (32) ne dépend pas du champ de déformation V, et pour réaliser les formules d'intégration par parties qui exigent des notions faibles de trace (voir [6, 7, 34, 35]).

Nous insistons sur le fait que, par conséquent aux bornes écrites ci-dessus pour $q_\varepsilon(V)$, on découvre une nouvelle condition nécessaire d'optimalité pour les problèmes variationnels classiques à l'étude. En fait, en faisant des variations horizontales (un peu dans le même esprit de [58]), c'est-à-dire en exploitant le fait que $q_\varepsilon(V)$ est zéro pour tout $V\in C^1_0(\Omega,\mathbb{R}^n)$, nos bornes donnent comme sous-produit l'information que des tenseurs appropriés du type (31) se révèlent à divergence nulle. En particulier, dans le cas où f est Gateaux-dérivable sauf au plus à l'origine, le résultat est simplement que l'égalité suivante est vérif ée dans le sens des distributions pour tout $u\in\mathscr{S}$:

$$\operatorname{div}\Big(\nabla u \otimes \nabla f(\nabla u) - [f(\nabla u)+g(u)]I\Big) = 0. \tag{33}$$

Dans une certaine mesure étonnamment, autant que nous le sachions, la condition (33) semble être jusqu'à présent inconnue, à l'exception du cas scalaire $n=1$, alors qu'elle se réduit à la loi de conservation (ou intégrale première de l'équation d'Euler) suivante, satisfaite par les extrémales lisses des Lagrangiens régulières:

$$u' f'(u') - [f(u')+g(u)] = c,$$

voir *e.g.* [25, Proposition 1.13].

Nous soulignons que, dans nôtre stratégie, nous n'avons jamais fait usage de la dérivée u' déf nie en (28), en particulier nous n'avons pas besoin de la validité de l'équation d'Euler-Lagrange pour les minimiseurs (sur les conditions requises pour sa validité, nous nous référons aux papiers récents [10, 45], et les références y citées). Ainsi nous pouvons traiter aussi des fonctionnelles intégrales dont les minima satisfont seulement une inéquation variationnelle. Nous rappelons que le problème (21), posé sur la section transversale, appartient à cette catégorie de problèmes: la condition d'optimalité satisfaite par des éléments de $\mathscr{S}$ n'est pas une équation d'Euler-Lagrange, mais simplement une inéquation variationnelle. L'étude de la dérivée de forme de premier ordre pour (21) ne nous permet pas d'obtenir la conjecture sur le plateau, puisque nous obtenons une dérivée nulle.

Les perspectives dans l'étude des dérivées de forme pour les minima de fonctionnelles intégrales vont dans des directions différentes. Le premier aspect à examiner est la linéarité de J' par rapport au champ de déformation V: nous avons fourni des conditions suff santes qui garantissent une telle propriété, et nous aimerions également de déterminer celles qui sont nécessaires. Nous croyons que, en général, par exemple dans le cadre des solutions non-uniques, la linéarité est une condition trop forte: plus précisément notre conjecture est que J' est de la forme

$$J'(\Omega,V) = \int_{\partial\Omega} \alpha(x)\,(V\cdot n)^+\,\mathscr{H}^{n-1}(x) + \int_{\partial\Omega} \beta(x)\,(V\cdot n)^+\,\mathscr{H}^{n-1}(x)\ ,$$

α,β étant deux densités appropriées dans $L^\infty(\partial\Omega)$ qui peuvent dépendre des données du problème de inf mum $J(\Omega)$, et $(V\cdot n)^\pm$ désignant la partie positive et négative du produit scalaire $V\cdot n$ sur le bord. En particulier, nous nous attendons que J' est linéaire par rapport aux déformations purement internes ou aux déformations purement extérieures.

Un autre problème intéressant est d'étudier les dérivées de forme d'ordre supérieur. Dans cette direction, nous avons appliqué la même approche pour calculer la dérivée seconde de forme $J''(\Omega,V)$, en supposant plus de régularité sur le domaine Ω et sur les intégrandes f et g. Encore une fois, en exploitant les formulations primale et duale de $J(\Omega)$, nous pouvons trouver un minorant et un majorant pour les $\liminf$ et $\limsup$ de la suite

$$r_\varepsilon(V) := 2\,\frac{[J(\Omega_\varepsilon)-J(\Omega)-\varepsilon J'(\Omega,V)]}{\varepsilon^2}\ ,\ \ \varepsilon>0\ ,$$

et nous arrivons à la formule de représentation

$$\begin{aligned} J''(\Omega,V) \ &= \int_{\partial\Omega}(V\cdot n)^2\left[\frac{\partial\overline{u}}{\partial n}\,(\nabla^2\overline{u}\,\nabla^2 f(\nabla\overline{u})\,n)\cdot n + \big(\nabla\overline{u}\cdot\nabla f(\nabla\overline{u}) - f(\nabla\overline{u})\big)H_{\partial\Omega}\right] d\mathscr{H}^{n-1} + \\ &\quad - \inf_{\substack{v\in H^1(\Omega)\\ \mathrm{Tr}(v)=-V\cdot\nabla\overline{u}}} \left\{\int_\Omega (\nabla^2 f(\nabla\overline{u})\,\nabla v)\cdot\nabla v + g''(\overline{u})v^2\right\}\ , \end{aligned} \tag{34}$$

où $H_{\partial\Omega}$ désigne la courbure moyenne de $\partial\Omega$.
Nous rappelons que, jusqu'à présent, la formule (34) a été obtenue seulement dans le cas lisse, et sa extension à une intégrande plus générale est un sujet délicat qui pourrait être développé ci-après. Nous prévoyons que les résultats concernants la dérivée seconde de forme peuvent donner quelques informations sur la courbure de la frontière du plateau.

Introduction

THE main topic of the Thesis is *optimization of the compliance of thin elastic structures*. The problem consists in finding the most robust configurations, when an infinitesimal amount of elastic material is subjected to a fixed force, and contained within a region having infinitesimal volume. The resistance to a load can be measured by computing a shape functional, the *compliance*, in which the shape represents the volume occupied by the elastic material. Thus we are led to study a minimization problem of a shape functional, under suitable constraints.

In particular, we treat the case in which the design region is a thin rod, represented by a cylinder with infinitesimal cross section. The study finds its motivation in engineering problems: thin structures are very convenient to be used in practical applications.

The approach we adopt draws inspiration from some recent works by I. Fragalà, G. Bouchitté and P. Seppecher, in which the authors deal with the case of thin elastic plates [G. BOUCHITTÉ, I. FRAGALÀ, P. SEPPECHER: Structural optimization of thin plates: the three dimensional approach. *Arch. Rat. Mech. Anal.* (2011)]. We point out that these two problems are not merely technical variants one of the other, due to the substantial difference between the limit passages *3d-1d* and *3d-2d*, namely from 3 to 1 and from 3 to 2 dimensions.

The study of optimal configurations led us to face another interesting variational problem: actually to establish whether homogenization phenomena occur in bars in pure torsion regime turns out to be equivalent to solve a *nonstandard free boundary problem* in the plane. This new problem is very challenging and, besides the link with torsion rods, it has mathematical interest in itself. One of the tools which can be employed to attack the problem is *shape derivative for minima of integral functionals*. The theory of shape derivatives is a widely studied topic (see *e.g.* the monograph by A. Henrot and M.Pierre *Variations et Optimisation de Formes. Une Analyse Géométrique*, Springer Berlin (2005), and the references therein), but the approach we propose in new and relies on assumptions which are weaker that the classical ones.

The Thesis is organized as follows. In the first part we gather the preliminaries: in Chapter 1 we recall the main mathematical tools of Convex Analysis, Geometric

Measure Theory and Γ-convergence that we use in the Thesis, then we summarize the theory of linear elasticity, which motivates the study of the main problem. The second part (Chapters 2 and 3) is devoted to the study of the compliance optimization problem in thin rods. The third part (Chapters 4 and 5) is dedicated to the above mentioned related problems. When needed, the more technical proofs, usually concerning auxiliary lemmas or easy to prove statements, are postponed to the end of the Chapters, in the Appendix. The open problems and the possible advances are gathered in the Perspectives.

The chapters correspond to the following papers that have been written in these three years of PhD:

- (Chapters 2 and 3) G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI, P. SEPPECHER: Optimal Thin Torsion Rods and Cheeger Sets, *SIAM J. Math. Anal.*, **44**, 483-512, (2012).
- (Chapters 2 and 3) I. LUCARDESI: Optimal design in thin rods, in preparation.
- (Chapter 4) J. ALIBERT, G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI: A non standard free boundary problem arising in shape optimization of thin torsion rods, to appear in *Interfaces and Free Boundaries* (2012).
- (Chapter 5) G. BOUCHITTÉ, I. FRAGALÀ, I. LUCARDESI: Shape derivatives for minima of integral functionals, in preparation.

Let us describe the contents more in details.

[Chapter 2] Let Q be a design region in $\mathbb{R}^3$ subject to a fixed external load $F \in H^{-1}(Q;\mathbb{R}^3)$. Given an isotropic elastic material that occupies a certain region $\Omega \subset Q$, its resistance to the load, in the framework of small displacements, can be measured by computing a shape functional, the *compliance*:

$$\mathscr{C}(\Omega) := \sup\left\{ \langle F,u\rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\,dx \ : \ u \in H^1(Q;\mathbb{R}^3)\right\} , \tag{1}$$

where, as usual in linear elasticity, $e(u)$ denotes the symmetric part of the gradient ∇u, and the strain potential $j : \mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}$, assumed to be isotropic, is strictly convex and has the form

$$j(z) := \frac{\lambda}{2}\,\mathrm{tr}^2(z) + \eta\,|z|^2 , \tag{2}$$

$\lambda\,,\mu > 0$ being the Lamé coefficients of the material.
The compliance is proportional to the work done by the load in order to bring the structure to equilibrium. In particular the smaller is the compliance, the higher is the resistance.
Clearly, in order that $\mathscr{C}(\Omega)$ remains finite, the load must have support contained into $\overline{\Omega}$, moreover it has to be *balanced*, *i.e.*

$$\langle F,u\rangle_{\mathbb{R}^3} = 0 , \quad \text{whenever } e(u) = 0 .$$

Under this condition, an optimal displacement $\overline{u}$ exists and satisfies $\mathscr{C}(\Omega) = \frac{1}{2}\langle F, \overline{u}\rangle_{\mathbb{R}^3}$.

In the Thesis we study the problem of finding the most robust configurations of elastic material, *i.e.* minimizing the compliance, when the design region is a thin domain. Moreover, at the same time, we let the ratio between the volume of the elastic material and the volume of the design region tend to zero.

By "thin" we mean that one or two spatial dimensions of the body are much smaller with respect to the others. This particular solids are very important in engineering problems: their small weight and ease of manufactoring and transport make them very convenient to be used in practical applications. Here we consider the case in which the continuum body can be approximated by a one dimensional set, namely it is a *rod*.

The mathematical way of describing a rod is the following one: a rod is a three-dimensional solid occupying the volume generated by a planar connected domain, called the *cross section*, with centroid varying perpendicularly to a spatial curve, the *axis*; moreover the diameter of the cross section is much smaller than the lenght of the axis. The particular case we deal with is a *straight rod*, in which the axis is a straight line segment I and the cross section is a planar bounded domain D, constant along the axis: we represent such a structure by a cylinder of the form

$$Q_\delta := \delta\overline{D} \times I \,, \tag{3}$$

with $D \subset \mathbb{R}^2$ an open bounded domain, I a closed bounded interval and $\delta > 0$ a vanishing parameter describing the small ratio between the diameter of the cross section and the length.

The literature about thin rods is very extensive: the classical theory has been developed by Euler, Bernoulli, Navier, Saint Venant, Timoshenko, Vlassov; in the last years, due to new numerical techniques, these problems have had renewed interest, and new design methods are now a wide field of applied mathematics (we limit ourselves to mention [73, 78, 80, 86]). For a complete overview about this topic, we refer to the reference book by Trabucho and Viaño [95] and the references therein.

The problem we treat, and consequently the approach we adopt to solve it, draws its inspiration from a recent work by G. Bouchitté, I. Fragalà and P. Seppecher [19], in which the authors studied the compliance optimization problem when the design region can be approximated by a two dimensional set, more precisely they studied the case of thin plates, described by a family of cylinders of the form

$$Q_\delta := \overline{D} \times \delta I \,,$$

having infinitesimal thickness δ.

In what follows we deal with thin sets Q_δ defined according to (3). In order to find the lightest and more robust configurations, we minimize the compliance $\mathscr{C}(\Omega)$ over the subsets $\Omega \subset Q_\delta$ with prescribed volume m, namely we study

$$\inf\left\{\mathscr{C}(\Omega) \;:\; \Omega \subset Q_\delta \,,\; \frac{|\Omega|}{|Q_\delta|} = m\right\} \,, \tag{4}$$

and then we perform the double limit as $\delta \to 0$ and $m \to 0$.

If we enclose the volume constraint in the cost through a Lagrange multiplier $k \in \mathbb{R}$, the variational problems (4) under study take the form

$$\phi^{\delta}(k) := \inf_{\Omega \subset Q_\delta} \left\{ \mathscr{C}^{\delta}(\Omega) + k \frac{|\Omega|}{|Q_\delta|} \right\} , \tag{5}$$

with

$$\mathscr{C}^{\delta}(\Omega) := \sup \left\{ \langle F^{\delta}, \tilde{u} \rangle_{\mathbb{R}^3} - \int_{\Omega} j(e(\tilde{u}))\, dx \; : \; \tilde{u} \in H^1(Q_\delta, \mathbb{R}^3) \right\} . \tag{6}$$

Here F^{δ} is a suitable scaling of the load F, chosen so that in the limit process the infimum remains finite. The choice of the scaling F^{δ} depends on the assumptions made on the type of applied loads. In the literature it is customary to distinguish between the stretching, bending and the torsion cases : in the Thesis we focus our attention on the loads for which the contribution of bending and torsion may be decoupled in a suitable way, and we choose two different scalings for these two components: roughly speaking, we consider loads F of the form $F = G + H$, where the bending depends only on a vertical load H. The general case is difficult to handle, due to the interplay between these contributions of the charging.

In Chapter 2 we perform the first passage to the limit, namely we study the asymptotic behavior of problem (5) as $\delta \to 0^+$: this corresponds to look for the most robust configuration in a rod-like set, keeping the ratio between the volume of material and the volume of the thin design region fixed. The first step in the study is to reformulate the variational problems $\phi^{\delta}(k)$ on the fixed domain $Q := \overline{D} \times I$, instead of the thin cylinders Q_δ. After a suitable change of variables for the displacements and a suitable scaling for the load, problems (5) can be rewritten as

$$\phi^{\delta}(k) = \inf_{\omega \subset Q} \left\{ \mathscr{C}^{\delta}(\omega) + k|\omega| \right\} , \tag{7}$$

with

$$\mathscr{C}^{\delta}(\omega) := \sup \left\{ \frac{1}{\delta} \langle G, u \rangle_{\mathbb{R}^3} + \langle H, u \rangle_{\mathbb{R}^3} - \int_{\Omega} j(e^{\delta}(u))\, dx \; : \; u \in H^1(Q_\delta, \mathbb{R}^3) \right\} , \tag{8}$$

where $e^{\delta} : H^1(Q; \mathbb{R}^3) \to L^2(Q; \mathbb{R}^{3\times 3}_{\text{sym}})$ is the operator defined by

$$e^{\delta}_{\alpha\beta}(u) := \delta^{-2} e_{\alpha\beta}(u) \, , \;\; e^{\delta}_{\alpha 3}(u) := \delta^{-1} e_{\alpha 3}(u) \, , \;\; e^{\delta}_{33}(u) := e_{33}(u) \, ,$$

as it is usual in the literature on $3d - 1d$ dimension reduction. The asymptotical study of $\phi^{\delta}(k)$ is based on the comparison with the "fictitious counterpart", namely their relaxed formulation in $L^\infty(Q; [0,1])$. Indeed it is well known that the infimum problems (7) are in general ill-posed, due to occurrence of homogenization phenomena which prevent the existence of an optimal domain (see [2]). Thus we need to enlarge the class of admissible materials, passing from "real" materials, represented by characteristic functions, to "composite" materials, represented by densities with values in $[0,1]$. To this aim we introduce the family of variational problems

$$\widetilde{\phi}^{\delta}(k) := \inf \left\{ \widetilde{\mathscr{C}}^{\delta}(\theta) + k \int_Q \theta \, dx \; : \; \theta \in L^\infty(Q; [0,1]) \right\} , \tag{9}$$

where $\widetilde{\mathscr{C}}^{\delta}(\theta)$ denotes the natural extension of the compliance $\mathscr{C}^{\delta}(\omega)$ to $L^\infty(Q;[0,1])$:

$$\widetilde{\mathscr{C}}^{\delta}(\theta) := \sup\left\{ \frac{1}{\delta}\langle G,u\rangle_{\mathbb{R}^3} + \langle H,u\rangle_{\mathbb{R}^3} - \int_Q j(e^{\delta}(u))\,\theta\,dx \;:\; u\in H^1(Q;\mathbb{R}^3)\right\}. \quad (10)$$

Exploiting some delicate compactness properties derived from variants of the Korn inequality (see Section 1.5) and Γ-convergence techniques (see Section 1.3), we determine the limit behavior of $\widetilde{\phi}^{\delta}(k)$ as $\delta\to 0^+$: the sequence of fictitious problems tends to a limit $\phi(k)$ which is still a variational problem posed on densities, with the same structure: a compliance term with a volume penalization. More precisely, $\phi(k)$ reads

$$\phi(k) := \inf\left\{ \mathscr{C}^{lim}(\theta) + k\int_Q \theta\,dx \;:\; \theta\in L^\infty(Q;[0,1])\right\}, \quad (11)$$

where $\mathscr{C}^{lim}$ is the Γ-limit of the sequence $\widetilde{\mathscr{C}}^{\delta}$ as $\delta\to 0^+$ with respect to the weak * convergence in $L^\infty(Q;[0,1])$. Exploiting a crucial bound for the relaxed functional of the compliance, established in [19, Proposition 2.8], we deduce that $\phi^{\delta}(k)$ and $\widetilde{\phi}^{\delta}(k)$ have the same asymptotic, that is $\lim_{\delta\to 0^+}\phi^{\delta}(k) = \phi(k)$.

We point out that the dimension reduction process is performed without making any topological assumption on the set Ω occupied by the material. Therefore, it is not covered by the very extensive literature on $3d-1d$ analysis.

At this point it is natural to ask whether $\phi(k)$ admits a classical solution (*i.e.* a density with values in $\{0,1\}$): this corresponds to ask whether the compliance problem under volume constraint, in a rod-like set, admits as solution a real material rather that a composite. Providing reformulations of $\phi(k)$ both as a variational problem for twist displacements fields, and as a variational problem for stress tensors, allows us to give necessary and sufficient optimality conditions. These alternative formulations reveal that problem (11) can be solved section by section; moreover, by exploiting the optimality system, the question about the existence of real solutions can be rephrased in an easier way, namely it can be related to the existence of "special" solutions to a certain free boundary problem. A more detailed description and a deeper analysis of the problem is postponed to Chapter 4.

In Chapter 3 we perform the second limit process, namely we study the asymptotic behavior of problem (11) as $k\to+\infty$: we point out that considering large values of k means considering small "filling ratios" $\int_Q d\theta/|Q|$. We show that the sequence $\phi(k)$ is asymptotically equivalent to $\sqrt{2k}$, and we determine that the limit $\overline{m}$ of the ratio $\phi(k)/\sqrt{2k}$. Such limit is again a variational problem, whose cost has the same structure (compliance term plus volume term), but it is set in the space $\mathscr{M}^+(Q)$ of positive measures on $\mathbb{R}^3$, compactly supported in Q:

$$\overline{m} := \inf\left\{ \mathscr{C}^{lim}(\mu) + \frac{1}{2}\int d\mu \;:\; \mu\in\mathscr{M}^+(Q)\right\}; \quad (12)$$

here $\mathscr{C}^{lim}(\mu)$ is the natural extension of the limit compliance $\mathscr{C}^{lim}(\theta)$ to the class $\mathscr{M}^+(Q)$. The new setting is quite natural, since the limit problem describes concentration phenomena that may occur in lower dimensional parts. Finding a solution $\overline{\mu}$

for problem $\overline{m}$ gives us relevant informations: indeed optimal measures describe the optimal configurations of material in a rod-like set in the vanishing volume limit.

By exploiting the dual formulation of $\overline{m}$, we provide an alternative variational characterization of such optimal measures. It turns out that in general the solution is not unique and it is difficult to be determined explicitly. Nevertheless, when considering particular loads, we are able to solve $\overline{m}$ completely or, at least, to have precise information about the support of its solutions. The corresponding concentration phenomena are discussed in the last part of Chapter 3, and can be summarized as follows.

When the load is purely torsional and D is convex, the solution of $\overline{m}$ turns out to be unique and can be explicitly determined as a measure concentrated section by section on the boundary of the so called *Cheeger set* of D. Let us recall that, under the assumption that D is convex, its Cheeger set is the unique minimizer for the quotient perimeter/area among all the measurable subsets of D (see Section 1.4.3 in the Preliminaries):

$$\inf_{E\subset\overline{D},\, 1\!\!1_E\in BV(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2}|\nabla 1\!\!1_E|}{|E|}\,. \tag{13}$$

The purely geometric problem (13) (which can be set more generally when D is any connected subset of the plane) is known as Cheeger problem, and in recent years has captured the attention of many authors (see [3, 24, 30, 31, 56, 59, 60, 71]). To the best of our knowledge, until now there was no rigorous statement and proof for this geometric characterization of optimal "light" torsion rods in terms of Cheeger sets. Let us emphasize that such characterization is valid only in pure torsion.
For more general loads, due to the interplay between the bending, twisting and stretching energies, we obtain a more complicated model. We provide some examples for which the variational problem (12) can be solved explicitly, and we present some cases in which its solutions turn out to be linked to interesting variants of the Cheeger problem. Moreover, some numerical experiences have been done in order to describe qualitatively such solutions. Let us mention these variants of the Cheeger problem. It is easy to see that the relaxed formulation of the classical Cheeger problem (13) reads

$$\inf\left\{\int_D |Du|\ :\ u\in BV_0(D)\,,\ \int_D u=1\right\}\,. \tag{14}$$

The first variant that comes into play is a sort of perturbation with a translation term:

$$\inf\left\{\int_D |Du+q|\ :\ u\in BV_0(D)\,,\ \int_D u=C(q)\right\}\,, \tag{15}$$

q being a fixed vector field; while the second variant is a weighted version:

$$\inf\left\{\int_D \alpha|Du|\ :\ u\in BV_0(D),\ \int_D u=1\right\}\,, \tag{16}$$

α being a non negative function in D. We point out that problem (16) has been treated some years ago by Ionescu and Lachand-Robert: in [69] the authors, motivated by applications to landslides modeling, study the case in which both the integral to minimize and the integral in the constraint are weighted.

In Chapter 4 we face the following natural question about the variational problem $\phi(k)$ in (11):

$$\textit{Does problem } \phi(k) \textit{ admit a solution } \overline{\theta} \textit{ taking values into } \{0,1\}? \tag{17}$$

We recall that $\phi(k)$ was obtained by passing to the limit as $\delta \to 0^+$ in the compliance optimization problem (7): since it is a variational problem set over densities in $L^\infty(Q;[0,1])$, an affirmative answer to question (17) would mean that the optimal design of thin rods admits a classical solution, which may be identified with a set rather than a composite.

We focus our attention to the pure torsion regime: in this case the question can be rephrased in an easier way. Since, as already noticed, $\phi(k)$ can be solved section by section, after a suitable change of variable and passage to a dual problem, we are led to study the following planar variational problem:

$$m(s) := \inf\left\{ \int_{\mathbb{R}^2} \varphi(\nabla u) \ : \ u \in H^1_c(D)\,,\ \int_{\mathbb{R}^2} u = s \right\}, \tag{18}$$

where s is a real parameter proportional to k, φ is the convex but non-strictly convex integrand

$$\varphi(y) := \begin{cases} \frac{|y|^2}{2} + \frac{1}{2} & \text{if } |y| \geq 1 \\ |y| & \text{if } |y| < 1\,, \end{cases}$$

and $H^1_c(D)$ denotes the space of H^1 functions that are constant outside D (if D is simply connected, it coincides with the usual Sobolev space $H^1_0(D)$).
It turns out that question (17) is equivalent to the following one:

$$\textit{Does problem } m(s) \textit{ admit a solution } \overline{u} \textit{ such that } |\nabla \overline{u}| \in \{0\} \cup (1,+\infty) \textit{ a.e. in } D? \tag{19}$$

More precisely, given a solution $\overline{u}$ for $m(s)$ and a solution $\overline{\theta}$ for $\phi(k)$, the region in which $\nabla\overline{u}$ vanishes corresponds to absence of material (namely $\overline{\theta} = 0$), the region $|\nabla\overline{u}| > 1$ to the presence of material (namely $\overline{\theta} = 1$) and the intermediate region corresponds to the homogenization region. In other words, the study of $m(s)$ can be applied to study the influence of the section's shape and of the filling ratio on the presence of homogenization regions in optimal thin torsion rods.

The main results of Chapter 4 concern the study of question (19) in relation with the geometry of the domain D and also with the value of the parameter s.

We say that u is a *special solution* to $m(s)$ if it minimizes (18) and satisfies $|\nabla u| \in \{0\} \cup (1,+\infty)$ a.e. in D, moreover we call the *plateau* of u, and we denote it by $\Omega(u)$, the set $\{\nabla u = 0\}$ minus the unbounded connected component of $\mathbb{R}^2 \setminus \overline{D}$ (where $u \equiv 0$).

When D is a ball or a ring, through explicit computations and exploiting the optimality conditions, we show that $m(s)$ admits a special solution, and there is no other solution. Next we show that balls and rings are *not* the unique domains for which $m(s)$ admits a special solution. In this respect, it is worth to compare our results with those obtained by Murat and Tartar in [F. MURAT, L. TARTAR: Calcul des variations et homogénéisation. Homogenization methods: theory and applications in physics (Bráu-sans-Nappe, 1983), 319-369, *Collect. Dir. Etudes Rech. Elec. France* **5**, Eyrolles, Paris (1985)], about the problem of maximizing the torsional rigidity of a bar with a given cross-section made by two linearly elastic materials in fixed proportions. The corresponding variational problem is quite similar to ours, except that it involves a *differentiable* integrand, and classical solutions (*i.e.* optimal designs with no homogenization regions) cannot exist unless the cross-section D is a disk. In our case the integrand φ is non-differentiable at zero and the conclusion goes in a quite opposite direction: we prove that there exists a non circular domain D with analytic boundary such that, for some s, problem $m(s)$ admits a special solution. Moreover this solution has a convex plateau with analytic boundary. To achieve this result, we use as a key tool the relationship between $m(s)$ and the Cheeger problem.

We remark that the crucial role played by the Cheeger set of D in the study of the asymptotic behavior of $m(s)$ in (18) as $s \to 0^+$ already emerged in the asymptotics of the optimal compliance problem $\phi(k)$ in (11) as $k \to +\infty$: indeed the parameter s is proportional $\frac{1}{\sqrt{k}}$, so that small values of s correspond to large values of k.
After providing some elementary properties on the sign and the support of generic solutions, we derive some information on qualitative properties of special solutions, when the latter exist. This amounts to study a nonstandard free boundary problem with an obstacle on the gradient:

$$\begin{cases} -\Delta u = \lambda \ , \quad |\nabla u| > 1 & \text{in } D \setminus \Omega(u) \\ |\nabla u| = 1 & \text{on } \Gamma(u) \\ u = c_i & \text{on } \gamma_i \ , \end{cases} \tag{20}$$

where $\Gamma(u) := \partial\Omega(u) \cap D$ is the free boundary and γ_i denote the different connected components of $\Gamma(u)$.
Finally, assuming that D is simply connected, and that there exists a special solution u with a smooth free boundary, we prove some qualitative properties of the plateau:

- each connected component of $D \setminus \Omega(u)$ must touch ∂D;
- under suitable assumptions, the plateau $\Omega(u)$ must be convex;
- when D is not Cheeger set of itself, the plateau $\Omega(u)$ cannot be compactly contained into D for arbitrarily small filling ratios.

We remark that the *a priori* requirement of smooth free boundary is needed in order to apply the theory of P-functions (see [93]). The study of the regularity of the free boundary is an interesting and challenging perspective (see [27, 28, 85]): a first result obtained in this direction concerns the finiteness of the perimeter of the plateau.

We point out that achieving a complete characterization of domains D where a special solution to $m(s)$ exists seems to be a very challenging problem, which remains by now open: in this respect we believe that, at least when D is convex, the existence of special solutions is likely related to the regularity of ∂D, and also to whether or not D coincides with its Cheeger set. Let us notice that the latter criterium would exclude the existence of a special solution in the case when D is a square. This guess seems to be confirmed by the numerical results performed in [72] for a very similar problem, in which homogenization regions are observed.

When D is not Cheeger set of itself, we expect that some connected component $\Omega_0 \subset \{u \equiv 0\}$ of the plateau touches the boundary in a neighborhood of the points of higher curvature, namely the corners. In order to prove the conjecture, we tried to exploit shape derivatives. Actually, if we fix the parameter s, we can interpret $m(s)$ as a shape functional $J(D)$, depending on the domain D as follows: enclosing the volume constraint in the functional, solving $m(s)$ over D turns out to be equivalent to study

$$J(D) = -\inf\left\{ \int_D [\varphi(\nabla u) - \lambda u]\, dx \;:\; u \in H_0^1(D) \right\}, \tag{21}$$

with $\lambda = m'(s)$.
Clearly the shape functional $J(\cdot)$ is stationary over the domains $D' \subset D$ containing $D \setminus \Omega_0$. Moreover, the sign of the shape derivative may give useful information: if we consider small inner deformations of D, localized on some part γ of the boundary, a nonzero shape derivative would imply that Ω_0 does not touch such portion γ.

[Chapter 5]

The theory of shape derivatives is a widely studied topic, with many applications in variational problems and optimal design. Its origin can be traced back to the first half of the last century, with the pioneering work by Hadamard [66], followed by Schiffer and Garabedian [61, 89]. Afterwards, some important advances came in the seventies by Céa, Murat, and Simon [32,81,90]. From the nineties forth, the many contributions given by different authors are witness of a renewed interest, partly motivated by the impulse given by the development of the field of numerical analysis in the research of optimal shapes. We refer to the recent monograph [67] by Henrot-Pierre as a reference text (see also the books [47, 91]), and, without any attempt of completeness, to the representative works [23,44,46,63,64,83].

Due to the lack of differentiability of φ at the origin, the computation of the shape derivative of the functional J in (21) is not covered by the above quoted literature. Therefore, we tried to develop a new method, which applies also to more general convex functionals and in higher dimension. In fact, we consider shape functionals of the kind

$$J(\Omega) := -\inf\left\{ \int_\Omega \big[f(\nabla u) + g(u)\big]\, dx \;:\; u \in W_0^{1,p}(\Omega) \right\}. \tag{22}$$

Here Ω varies among the open bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary, while $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given integrands, which are assumed to be continuous, convex, and to satisfy further regularity assumptions and growth conditions, of order p and q respectively. In a similar way, we can deal also with the Neumann problem, in which no boundary condition is prescribed for the admissible functions in (22).

Given a vector field V in $C^1(\mathbb{R}^n;\mathbb{R}^n)$, we consider the one-parameter family of domains which are obtained as deformations of Ω with V as initial velocity, that is we set

$$\Omega_\varepsilon := \Big\{ x + \varepsilon V(x) \;:\; x \in \Omega \Big\}, \qquad \varepsilon > 0\,.$$

By definition, the shape derivative of J at Ω in direction V, if it exists, is given by the limit

$$J'(\Omega, V) := \lim_{\varepsilon \to 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon}\,. \tag{23}$$

The approach we adopt in order to study the shape derivative (23) is different from the one usually employed in the literature, and seems to have a twofold interest: on one hand it allows to obtain the shape derivative for more general integrands f and g; on the other hand, along with the shape derivative, it leads to discover a new optimality condition for solutions to problem (22).

Before describing the results, let us briefly recall the habitual approach to the computation of $J'(\Omega,V)$, in order to enlighten the difference of perspective.

Classically, the object of study in theory of shape derivatives is the differentiability at $\varepsilon=0^+$ of functions of the form

$$I(\varepsilon) := \int_{\Omega_\varepsilon} \psi(\varepsilon,x)\,dx\,, \tag{24}$$

being Ω_ε the image of a measurable set Ω via a one-parameter family of bi-Lipschitz diffeomorphisms Ψ_ε. In particular, shape derivatives for minima of integral functionals can be dealt as a special case of (24): namely, letting u_ε be a solution to the infimum problem $J(\Omega_\varepsilon)$ and choosing

$$\psi(\varepsilon,x) := -\left[f(\nabla u_\varepsilon(x)) + g(u_\varepsilon(x))\right]\,, \tag{25}$$

there holds $J(\Omega_\varepsilon)=I(\varepsilon)$.

The differentiability at $\varepsilon=0^+$ of the map $I(\varepsilon)$, along with the formula for its right derivative, is proved in [67] assuming suitable regularity hypotheses on the integrand ψ. More precisely, the following situations are considered: either $\psi(\varepsilon,\cdot)$ is defined on the whole of $\mathbb{R}^n$ with

$$\psi(\varepsilon,\cdot)\in L^1(\mathbb{R}^n)\,, \qquad \varepsilon\mapsto\psi(\varepsilon,\cdot)\ \text{derivable at } 0\,, \qquad \psi(0,\cdot)\in W^{1,1}(\mathbb{R}^n)\,, \tag{26}$$

or $\psi(\varepsilon,\cdot)$ is defined just in Ω_ε with

$$\psi(\varepsilon,\Psi_\varepsilon(\cdot))\in L^1(\Omega)\,, \qquad \varepsilon\mapsto\psi(\varepsilon,\Psi_\varepsilon(\cdot))\ \text{derivable at } 0\,, \qquad P(\psi(0,\cdot))\in W^{1,1}(\mathbb{R}^n)\,, \tag{27}$$

being $P: L^1(\Omega)\to L^1(\mathbb{R}^n)$ some linear continuous extension operator .

In order to include into this setting minima of integral functionals like (22), one has to check that one of the conditions (26) or (27) hold true, when $\psi(\varepsilon,x)$ is taken as in (2.39). This check has to be done case by case, according to the choice of f and g. In particular, in this process one has to compute the derivative

$$u' := \frac{d}{d\varepsilon}u_\varepsilon\big|_{\varepsilon=0^+}\,, \tag{28}$$

which typically requires to exploit the Euler-Lagrange equation satisfied by u_ε.

Subsequently, further regularity assumptions on the integrand ψ, on the domain Ω and on the deformations Ψ_ε, are necessary in order to obtain structure theorems and representation results for shape derivatives, which lead to express them as boundary integrals over $\partial\Omega$. We refer to [67] for a detailed presentation.

In spite, our approach relies on the use of Convex Analysis, and more specifically of the dual formulation of $J(\Omega)$, which in the Dirichlet case reads

$$J^*(\Omega)=\inf\left\{\int_\Omega [f^*(\sigma)+g^*(\operatorname{div}\sigma)]dx\,:\,\sigma\in L^{p'}(\Omega;\mathbb{R}^n)\,,\ \operatorname{div}\sigma\in L^{q'}(\Omega)\right\}$$

where f^* and g^* denote the Fenchel conjugates of f and g. In the Neumann case the admissible fields σ satisfy the additional condition on the normal trace $\sigma \cdot n = 0$ on $\partial\Omega$.

Our strategy consists in exploiting respectively the primal formulation $J(\Omega)$ and the dual formulation $J^*(\Omega)$ in order to obtain lower and upper bounds for the quotient

$$q_\varepsilon(V) := \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} = \frac{J^*(\Omega_\varepsilon) - J^*(\Omega)}{\varepsilon}.$$

Such lower and upper bounds take respectively the form

$$\liminf_{\varepsilon \to 0^+} q_\varepsilon(V) \geq \inf_{\sigma \in \mathscr{S}^*} \sup_{u \in \mathscr{S}} \int_\Omega A(u, \sigma) : DV \, dx \tag{29}$$

and

$$\limsup_{\varepsilon \to 0^+} q_\varepsilon(V) \leq \sup_{u \in \mathscr{S}} \inf_{\sigma \in \mathscr{S}^*} \int_\Omega A(u, \sigma) : DV \, dx \tag{30}$$

where $\mathscr{S}$ and $\mathscr{S}^*$ denote the set of solutions to $J(\Omega)$ and $J^*(\Omega)$, and $A(u,\sigma)$ is the tensor defined on the product space $\mathscr{S} \times \mathscr{S}^*$ by

$$A(u, \sigma) := \nabla u \otimes \sigma - [f(\nabla u) + g(u)]\, I \tag{31}$$

(being I the identity matrix). Since the inf-sup at the r.h.s. of (29) is larger than or equal to the sup-inf at the r.h.s. of (30), we conclude that they agree, and that the limit as $\varepsilon \to 0^+$ of $q_\varepsilon(V)$, namely the shape derivative $J'(\Omega, V)$, exists. Denoting by $(u^\star, \sigma^\star) \in \mathscr{S} \times \mathscr{S}^*$ an element where the value of the sup-inf or inf-sup is attained, there holds

$$J'(\Omega, V) = \int_\Omega A(u^\star, \sigma^\star) : DV \, dx. \tag{32}$$

Under additional regularity assumptions, the shape derivative can be recast also as a linear form in V, namely as a boundary integral depending linearly on the normal component of V on $\partial\Omega$. The additional regularity assumptions are necessary in order to state that an optimal pair $(u^\star, \sigma^\star)$ in (32) does not depend on the deformation field V, and in order to perform integration by parts formulas involving weak notions of trace (see [6, 7, 34, 35]).

Let us emphasize that, as a consequence of the above described bounds for $q_\varepsilon(V)$, we discover a new necessary condition of optimality for the classical variational problems under study. Actually, by making horizontal variations (somewhat in the same spirit of [58]), namely by exploiting the vanishing of $q_\varepsilon(V)$ for all $V \in C^1_0(\Omega, \mathbb{R}^n)$, our bounds give as a by-product the information that suitable tensors of the type (31) turn out to be divergence-free. In particular, in case f is Gateaux-differentiable except at most at the origin, the outcome is simply that the following equality holds in the sense of distributions for every $u \in \mathscr{S}$:

$$\operatorname{div}\Big(\nabla u \otimes \nabla f(\nabla u) - [f(\nabla u) + g(u)]\, I\Big) = 0. \tag{33}$$

To some extent surprisingly, as far as we are aware, condition (33) seems to be until now undiscovered, except from the scalar case $n = 1$, when it reduces to the following conservation law or first integral of the Euler equation, satisfied by smooth extremals of smooth Lagrangians:

$$u' f'(u') - [f(u') + g(u)] = c\,,$$

see *e.g.* [25, Proposition 1.13].

Let us stress that, along this way, we never make use of the derivative function u' in (28), and in particular we do not need the validity of the Euler-Lagrange equation for minimizers (about the conditions required for its validity, we refer to the recent papers [10, 45], and references therein). Thus may deal also with integral functionals whose minima satisfy just a variational inequality. We point out that problem (21), settled on the bar cross-section, enters into this class of problems: for such a functional $J(\cdot)$, the shape derivative cannot be computed by using the classical approach, since the optimality condition satisfied by elements of $\mathscr{S}$ is not an Euler-Lagrange equation, but merely a variational inequality. Our approach allows to encompass this difficulty, as our existence and representation results for the shape derivative can be applied, despite the lackness of regularity of the integrand φ at the origin. The study of the first order shape derivative for (21) didn't allow us to obtain the conjecture about the plateau, since we obtain a zero derivative.

The perspectives in the study of shape derivatives for minima of integral functionals go in various directions. The first aspect to be investigated is the linearity of J' with respect to the deformation field V: we have provided sufficient conditions that ensure such a property, and we would like to determine also necessary ones. We believe that in general, for example in the framework of non uniqueness of solutions, linearity is a too strong requirement: more precisely our conjecture is that J' is of the form

$$J'(\Omega, V) = \int_{\partial\Omega} \alpha(x)\,(V\cdot n)^{+}\,\mathscr{H}^{n-1}(x) + \int_{\partial\Omega} \beta(x)\,(V\cdot n)^{+}\,\mathscr{H}^{n-1}(x)\ ,$$

α, β being two suitable densities in $L^\infty(\partial\Omega)$ that might depend on the data of the infimum problem $J(\Omega)$, and $(V\cdot n)^{\pm}$ denoting the positive and negative part of the scalar product $V\cdot n$ on the boundary. In particular, we expect J' to be linear with respect to purely inner deformations or purely outer deformations.

Another interesting problem is to study higher order shape derivatives. In this direction we have applied the same approach to compute the second order shape derivative $J''(\Omega, V)$, assuming higher regularity on the domain Ω and on the integrands f and g. Again exploiting the primal and dual formulations of $J(\Omega)$, we are able to bound from above and below the $\liminf$ and $\limsup$ of the sequence

$$r_\varepsilon(V) := 2\,\frac{[J(\Omega_\varepsilon) - J(\Omega) - \varepsilon J'(\Omega, V)]}{\varepsilon^2}\ ,\ \ \varepsilon > 0\,,$$

and we arrive at the representation formula

$$\begin{aligned} J''(\Omega, V) \ &= \int_{\partial\Omega} (V\cdot n)^2 \left[\frac{\partial \overline{u}}{\partial n}\,(\nabla^2\overline{u}\,\nabla^2 f(\nabla\overline{u})\,n)\cdot n + \big(\nabla\overline{u}\cdot\nabla f(\nabla\overline{u}) - f(\nabla\overline{u})\big)H_{\partial\Omega}\right] d\mathscr{H}^{n-1} + \\ &\quad - \inf_{\substack{v\in H^1(\Omega)\\ \mathrm{Tr}(v) = -V\cdot\nabla\overline{u}}} \left\{\int_\Omega (\nabla^2 f(\nabla\overline{u})\,\nabla v)\cdot\nabla v + g''(\overline{u})v^2\right\}\,, \end{aligned} \tag{34}$$

where $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$.
We point out that by now (34) has been obtained just in the smooth case, and its extension to more general integrand is a delicate topic which could be developed hereafter. We foresee that the results concerning the second order shape derivative might give some information about the curvature of the plateau's boundary.

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Notation

Throughout the Thesis we adopt the following conventions.

We let the Greek indices α and β run from 1 to 2, the Latin indices i and j run from 1 to 3, and as usual we omit to indicate the sum over repeated indices.

Given a matrix $M \in \mathbb{R}^{n\times n}$ with components M_{ij}, we denote by M^T the transpose matrix M_{ji}, moreover we decompose M as the sum of its symmetric part and skew-symmetric part:

$$M = M^{sym} + M^a , \quad \text{with } M^{sym} := \frac{1}{2}(M + M^T) , \; M^a := \frac{1}{2}(M - M^T) .$$

We denote by SO(3) the space of rotations in $\mathbb{R}^3$ and by Sym the space of second order symmetric tensors. Given two vectors a, b in $\mathbb{R}^n$ and two matrices B and C in $\mathbb{R}^{n\times n}$, we use the standard notation $a \cdot b$ and $A : B$ to denote their Euclidean scalar products, namely

$$a \cdot b := \sum_{i=1}^{n} a_i b_i , \qquad A : B := \mathrm{tr}(A^T B) = \sum_{i,j=1}^{n} A_{ij} B_{ij} ,$$

where tr denotes the trace. We denote by $a \otimes b$ the matrix $(a \otimes b)_{ij} := a_i b_j$, and by I the identity matrix. Given a tensor field $A \in C^1(\mathbb{R}^n; \mathbb{R}^{n\times n})$, by $\mathrm{div} A$ we mean its divergence with respect to lines, namely

$$(\mathrm{div} A)_i := \sum_{j=1}^{n} \partial_j A_{ij} .$$

For every measurable set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure, namely $|A| := \int_A 1 \, dx$.

Given a set $A \subset \mathbb{R}^n$, we define two characteristic functions: $1\!\!1_A$, which equals 1 in A and 0 outside, and χ_A, which equals 0 in A and $+\infty$ outside. We denote by $\mathrm{Int}(A)$ the interior of the set A, and by $\overline{A}$ its closure.

Given $a \in \mathbb{R}^n$ we denote by δ_a the Dirac mass at $x = a$.

We denote by C_b and by C_0 the space of continuous functions which are bounded or compactly supported respectively.

Given an open bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, we denote by $\mathrm{Lip}(\Omega)$ the space of Lipschitz functions on Ω.

Given $A \subset \mathbb{R}^n$, we recall that a function $f : A \to \mathbb{R}^n$ is called *locally Lipschitz* if for every compact subset $K \subset A$ there exists a positive constant C_k such that

$$|f(x) - f(y)| \le C_K |x-y| \quad \forall x, y \in K \,.$$

Given $1 \le p \le +\infty$ we denote by p' its conjugate exponent, defined as usual by the equality $1/p + 1/p' = 1$.

In the integrals, unless otherwise indicated, integration is made with respect to the n-dimensional Lebesgue measure. Furthermore, in all the circumstances when no confusion may arise, we omit to indicate the integration variable.

Whenever we consider L^p-spaces over Ω and over $\partial\Omega$, they are intended the former with respect to the n-dimensional Lebesgue measure over Ω, and the latter with respect to the $(n-1)$-dimensional Hausdorff measure over $\partial\Omega$.

We write any $x \in \mathbb{R}^3$ as $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$.

Derivation of functions depending only on x_3 will be denoted by a prime.

For every compact set $Q \subset \mathbb{R}^3$ of the form $Q = \overline{D} \times I$, where I is a closed bounded interval and D is an open, bounded, connected subset of $\mathbb{R}^2$ with a Lipschitz boundary, we denote by $\mathscr{D}'(Q)$ the subset of distributions on $\mathbb{R}^3$ whose support is contained in Q. These distributions are in duality with $\mathscr{C}^\infty(Q)$, the space of restrictions to Q of functions in $\mathscr{C}^\infty(\mathbb{R}^3)$, and $\langle T, \varphi \rangle_{\mathbb{R}^3}$ represents the duality bracket.

For any $T \in \mathscr{D}'(Q)$, we denote by $[[T]] \in \mathscr{D}'(\mathbb{R})$ the $1d$ distribution obtained by "averaging" T with respect to the cross section variable. It is defined by the identity

$$\langle [[T]], \varphi \rangle_{\mathbb{R}} := \langle T, \varphi \rangle_{\mathbb{R}^3} \qquad \forall \varphi = \varphi(x_3) \in \mathscr{C}_0^\infty(\mathbb{R}) \,.$$

In the following $H^1(Q)$ denotes the space of restrictions to Q of elements of the Sobolev space $H^1(\mathbb{R}^3)$, equipped with the usual norm $\|u\|^2_{H^1(Q)} = \int_Q (|u|^2 + |\nabla u|^2)$. Notice that, by the boundary regularity assumed on D, it coincides with the usual Sobolev space $H^1(D \times I)$. The dual space, denoted by $H^{-1}(Q)$, can be identified to the subspace of distributions $T \in \mathscr{D}'(Q)$ verifying the inequality $|\langle T, \varphi \rangle| \le C \|\varphi\|_{H^1(Q)}$ for every $\varphi \in H^1(Q)$, being C a suitable constant. Similar conventions will be adopted for functions or distributions on D or on I. It is easy to check that $[[T]]$ belongs to $H^{-1}(I)$ whenever $T \in H^{-1}(Q)$.

We recall that in dimension 1, a distribution $S \in \mathscr{D}'(I)$ satisfying $\langle S, 1 \rangle_{\mathbb{R}} = 0$ has a unique primitive belonging to $\mathscr{D}'(I)$, that we denote by $\mathscr{P}_0(S)$. In the general case, given $S \in \mathscr{D}'(I)$, we denote by $\mathscr{P}(S)$ the primitive $\mathscr{P}(S) := \mathscr{P}_0(S - \langle S, 1 \rangle_{\mathbb{R}})$.

When we add a subscript $_m$ to a functional space, we are considering the subspace of its elements which have zero integral mean.

In the particular case of $H^2_m(I)$ we require that also the distributional derivative has zero integral mean, i.e.

$$H^2_m(I) := \left\{ \zeta \in H^2(I) \;:\; \int_I \zeta = \int_I \zeta' = 0 \right\} \,. \tag{35}$$

Further notations will be specified if necessary throughout the Thesis.

CHAPTER 1

Preliminaries

This Chapter is devoted to recall the main mathematical tools used in this Thesis. In the first part we collect the usual techniques adopted in the Calculus of Variations, such as Convex Analysis, Γ-convergence and Geometric Measure Theory. Beside these classical fields, we introduce two other topics: Cheeger problem and Korn inequalities.

In the second part we summarize some facts in linear elasticity and introduce the compliance functional.

1.1 Convex Analysis and Duality Methods

Convex Analysis provides powerful techniques for studying optimization problems. Let us recall the definitions and the main results. For this Section we refer to [11, 12, 40, 41, 52].

Let us briefly fix some notations and recall elementary definitions.

Let X be a normed vector space and let X^* be its topological dual. We denote by $\langle \cdot, \cdot \rangle$ the duality product. We endow X with the strong topology induced by the norm, and X^* with the weak * topology. In case we consider different topologies, such as the weak one in X, it will be specified.

Given $F : X \to \mathbb{R} \cup \{+\infty\}$, we define the *domain* of F $\mathrm{dom} F$ as the set of points $x \in X$ where $F(x)$ is finite, namely

$$\mathrm{dom} F = \{x \in X \; : \; F(x) < +\infty\} \; ,$$

and the *epigraph* of F $\mathrm{epi}\,(F)$ as the subspace of $X \times \mathbb{R}$

$$\mathrm{epi}\,(F) = \{(x,t) \in X \times \mathbb{R} \; : \; F(x) \le t\} \; .$$

We say that F is

- *proper* if it is not identically $+\infty$;
- *convex* if epi(F) is a convex subset of $X \times \mathbb{R}$;
- *lower semicontinuous* if epi(F) is a closed subset of $X \times \mathbb{R}$;
- *upper semicontinuous* if $-F$ is lower semicontinuous.

It is well known that the convexity and lower semicontinuity can also be characterized as follows:

- F is convex if and only if

$$F(tx+(1-t)y)) \leq tF(x)+(1-t)F(y) , \qquad \forall x,y \in X , \; \forall t \in [0,1];$$

- F is lower semicontinuous if and only if for every $t \in \mathbb{R}$ the sublevel $\{F(x) \leq t\}$ is closed in X, or equivalently if and only if for every $x \in X$ and for every sequence $x_h \in X$ converging to x, there holds

$$F(x) \leq \liminf_{h\to\infty} F(x_h) .$$

We remark that lower semicontinuity can be defined also for the weak topology: in this case the sequential characterization is in general a weaker property, which becomes equivalent in case X, endowed with the weak topology, satisfies the First Axiom of Countability.

In the sequel we may write simply *l.s.c.* instead of lower semicontinuous and, similarly, *u.s.c.* instead of upper semicontinuous.

Proposition 1.1.1. *Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be convex and proper, then the following implications hold true:*

(i) if $\sup_U F < +\infty$ for some U open subset of X, then F is continuous and locally Lipschitz in all Int(dom$F$)*;*

(ii) F is l.s.c. with respect to the strong topology if and only if F is l.s.c. with respect to the weak topology.

1.1.1 Fenchel conjugate

Given a proper function $F : X \to \mathbb{R} \cup \{+\infty\}$, we define its *Fenchel conjugate* $F^* : X^* \to \mathbb{R} \cup \{+\infty\}$ as

$$F^*(x^*) := \sup_{x\in X} \{\langle x^*, x\rangle - F(x)\} .$$

Similarly, if F^* is proper, we introduce the *biconjugate* $F^{**} : X \to \mathbb{R} \cup \{+\infty\}$ as

$$F^{**}(x) := \sup_{x^*\in X^*} \{\langle x, x^*\rangle - F^*(x^*)\} .$$

Theorem 1.1.1. *Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then*

(i) *F^* is convex and lower semicontinuous with respect to the weak* topology of X^* ;*

(ii) *the Fenchel inequality holds true:*

$$F(x) + F^*(x^*) \geq \langle x^*, x\rangle \quad \forall x \in X\,,\ x^* \in X^*\,;$$

(iii) *$F \geq G \implies F^* \leq G^*$;*

(iv) *for every $\lambda > 0$ $\ (\lambda F)^*(\lambda x^*) = \lambda F^*(x^*/\lambda)$, for every $x^* \in X^*$;*

(v) *if F is convex, then F is l.s.c. in x_0 if and only if F^* is proper and $F^{**}(x_0) = F(x_0)$;*

(vi) *the biconjugate satisfies $F^{**} \leq F$, and equality holds true if and only if F is convex and lower semicontinuous.*

We can reformulate (vi) saying that F^{**} is the greatest convex and l.s.c. function majorized by F, in particular, when F is convex, it coincides with the l.s.c. envelope of F (see Definition 1.3.2).

Example 1.1.1. Let $F : X \to \mathbb{R}$ be defined as $F(x) = \frac{1}{p}\|x\|_X^p$. If $1 < p < +\infty$, then

$$F^*(X^*) = \frac{1}{p'}\,\|x^*\|_{X^*}^{p'}\,,$$

where p' satisfies $1/p + 1/p' = 1$. If instead $p = 1$ the Fenchel conjugate is given by

$$F^*(x^*) = \chi_{B^*}\,,$$

where $B^* := \{x^* \in X^*\ :\ \|x^*\|_{X^*} \leq 1\}$.

Proof. Let us consider the case $1 < p < 1$. For every $x^* \in X^*$, by definition there holds

$$\begin{aligned} F^*(x^*) &= \sup_{x\in X}\left\{\langle x^*, x\rangle - \frac{1}{p}\,\|x\|_X^p\right\} = \sup_{t\in\mathbb{R}}\ \sup_{\|x\|_X=1}\ \left\{t\langle x^*, x\rangle - \frac{|t|^p}{p}\right\} \\ &= \sup_{t\in\mathbb{R}^+}\left\{t\|x^*\|_{X^*} - \frac{t^p}{p}\right\} = \frac{1}{p'}\,\|x^*\|_{X^*}^{p'}\,. \end{aligned}$$

Let us now consider the case $p = 1$: for every $x^* \in X^*$ there holds

$$\begin{aligned} F^*(x^*) &= \sup_{x\in X}\{\langle x^*, x\rangle - \|x\|_X\} = \sup_{t\in\mathbb{R}^+}\ \sup_{\|x\|_X=1}\ \{t\langle x^*, x\rangle - t\} \\ &= \sup_{t\in\mathbb{R}^+} t(\|x^*\|_{X^*} - 1) = \begin{cases} 0 & \text{if } \|x^*\|_{X^*} \leq 1 \\ +\infty & \text{otherwise} \end{cases}\,. \end{aligned}$$

□

We say that F admits an *affine minorant* if there exists $x_0^* \in X^*$ and $c \in \mathbb{R}$ such that

$$\langle x_0^*, x\rangle - c \leq F(x) \quad \forall x \in X\,. \tag{1.1}$$

We remark that F satisfies (1.1) if and only if $F^*(x_0^*) \leq c$. As an immediate consequence we infer that a proper, l.s.c. and convex function always admits an affine minorant (see (v) in Theorem 1.1.1).

1.1.2 Subgradients

Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. We say that $x^* \in X^*$ is a *subgradient* for F at x if there holds

$$\langle x^*, y - x \rangle + F(x) \leq F(y) \quad \forall y \in X \, .$$

We denote by $\partial F(x)$ the possibly empty set of subgradients at x, and call it the *subdifferential* of F at x. If $\partial F(x) \neq \emptyset$ we say that F is *subdifferentiable* at the point x.

In the next theorems we collect the properties of subgradients and subdifferentials.

Theorem 1.1.2. *The following facts are equivalent:*

(i) $x^* \in \partial F(x)$ *;*

(ii) $x \in \partial F^*(x^*)$ *;*

(iii) the Fenchel equality holds true:

$$F(x) + F^*(x^*) = \langle x^*, x \rangle \, .$$

Example 1.1.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex radial function of the form $f(z) = \varphi(|z|)$. Then

$$f^*(z^*) = \varphi^*(|z^*|) \, ,$$

moreover

$$z^* \in \partial f(z) \quad \Longleftrightarrow \quad z^* = t\,\frac{z}{|z|} \, , \ \text{ with } t \in \partial \phi(|z|) \, .$$

Proof. Let us compute the Fenchel conjugate of f:

$$\begin{aligned} f^*(z^*) &= \sup_z \{\langle z^*, z\rangle - f(z)\} = \sup_z \{\langle z^*, z\rangle - \phi(|z|)\} = \sup_{t\in\mathbb{R}} \left\{ \left\langle z^*, t\,\frac{z^*}{|z^*|} \right\rangle - \phi(t) \right\} \\ &= \sup_{t\in\mathbb{R}} \{|z^*|\,t - \phi(t)\} = \phi^*(|z^*|) \, . \end{aligned}$$

In particular, if z realizes the supremum, namely $z \in \partial f^*(z^*)$, then it is of the form $z = t\,\frac{z^*}{|z^*|}$ with $t \in \partial\phi^*(|z^*|)$. Similarly there holds

$$z^* \in \partial f(z) \quad \Longleftrightarrow \quad z^* = t\,\frac{z}{|z|} \, , \ \text{ with } t \in \partial \phi(|z|) \, . \tag{1.2}$$

□

Theorem 1.1.3. *The subdifferential satisfies the following properties:*

(i) for every $x \in X$, $\partial F(x)$ *is convex and closed with respect to to the weak * topology in* X^* *;*

(ii) if F *is convex and continuous in* $\overline{x} \in X$, *then* $\partial F(\overline{x})$ *is a nonempty and weakly * compact subset of* X^* *;*

Proof. (i) Let us fix $x \in X$ such that $F(x) < +\infty$. In view of property (ii) of Theorem 1.1.1 and by definition of subdifferential, there holds

$$\begin{aligned}\partial F(x) &= \{x^* \in X^* \ : \ F(x) + F^*(x^*) \le \langle x^*, x\rangle\} \\ &= \{x^* \in X^* \ : \ F^*(x^*) - \langle x^*, x\rangle \le -F(x)\} \ .\end{aligned}$$

Since by Theorem 1.1.1 (i) the map $x^* \mapsto F^*(x^*) - \langle x^*, x\rangle$ is convex and l.s.c. with respect to the weak * topology of X^*, we infer that the sublevel corresponding to $-F(x)$ is convex and weak * closed.

(ii) (cf. [41, Proposition 2.1.2]). Since F is finite and continuous at $\bar{x}$, we infer that $\operatorname{Int}\operatorname{epi}(F) \subset X \times \mathbb{R}$ is a nonempty convex set. By the Hahn-Banach theorem, we can separate $\operatorname{Int}\operatorname{epi}(F)$ and $(\bar{x}, F(\bar{x})) \in X \times \mathbb{R}$ with a closed hyperplane: there exist $\xi \in X^*$ and $\lambda \in \mathbb{R}$ such that

$$\langle \xi, x\rangle + \lambda t < \langle \xi, \bar{x}\rangle + \lambda F(\bar{x}) \ , \ \ \forall (x,t) \in \operatorname{Int}\operatorname{epi}(F) \ .$$

Clearly $\lambda < 0$ (it is enough to evaluate the expression above in $(x,t) = (\bar{x}, F(\bar{x}) + 1)$). Since $\operatorname{epi}(F) \subset \overline{\operatorname{epi}(F)} = \overline{\operatorname{Int}\operatorname{epi}(F)}$ (this last equality follows by convexity of the set), the separation inequality above becomes weaker in all the epigraph:

$$\langle \xi, x\rangle - |\lambda| t \le \langle \xi, \bar{x}\rangle - |\lambda| F(\bar{x}) \ , \ \ \forall (x,t) \in \operatorname{epi}(F) \ . \tag{1.3}$$

If in (1.3) we consider $t = F(x)$ and we divide by $|\lambda|$ the expression, we obtain

$$\langle |\lambda|^{-1}\xi, x - \bar{x}\rangle \le F(x) - F(\bar{x}) \ , \ \ \forall x \in X \ ,$$

that is $|\lambda|^{-1}\xi \in \partial F(\bar{x})$.

Since F is finite, continuous in a neighborhood of $\bar{x}$ and convex, by Proposition 1.1.1 it is locally Lipschitz in all X, in particular there exist a constant $L > 0$ and a neighborhood $\mathscr{U}$ of $\bar{x}$ such that

$$|F(x) - F(\bar{x})| \le L\|x - \bar{x}\|_X \ , \ \ \forall x \in \mathscr{U} \ .$$

On the other hand, if x^* is an arbitrary element $\partial F(\bar{x})$, we obtain

$$|F(x) - F(\bar{x})| \ge F(x) - F(\bar{x}) \ge \langle |\lambda|^{-1} x^*, x - \bar{x}\rangle \ , \ \ \forall x \in \mathscr{U} \ ,$$

hence we conclude that $\|x^*\|_{X^*} \le L$, thus $\partial F(\bar{x})$ is bounded. By Banach-Alaoglu theorem, we conclude that $\partial F(\bar{x})$ is weakly* compact in X^*. □

Let us recall the definition of *one-sided directional derivative* and *Gâteaux differentiability*:

- for every $x \in X$ and every direction $v \in X$, the one-sided directional derivative of F at x in direction v is the following limit, if it exists:

$$F'_+(x, v) = \lim_{h \to 0^+} \frac{F(x + tv) - F(x)}{h} \ .$$

- we say that F is Gâteaux differentiabile at x if there exists $x^* \in X^*$ such that

$$\forall y \in X \quad \lim_{t\to 0} \frac{F(x+ty)-F(x)}{t} = \langle x^*, y\rangle \ ,$$

in this case x^* is called the Gâteaux derivative and is denoted by $F'_G(x)$.

Theorem 1.1.4. *Let $F : X \to \mathbb{R}\cup\{+\infty\}$ be a convex proper function, and let $x \in \operatorname{dom} F$. Then*

(i) for every $v \in X$

$$F'_+(x,v) = \sup_{x^*\in\partial F(x)} \langle x^*, v\rangle \ ;$$

(ii) if F is Gâteaux differentiabile at x, then $\partial F(x) = \{F'_G(x)\}$. Conversely, if F is continuous at x and $\partial F(x) = \{x^*\}$, then F is Gâteaux differentiabile at x and $F'_G(x) = x^*$.

1.1.3 Optimization problems

In this paragraph we recall two key results concerning optimization problems: in Proposition 1.1.2 we state a standard duality procedure and in Theorem 1.1.3 we recall a minimax principle.

Before stating the results, let us recall two useful lemmas that show the behavior of Fenchel conjugation when summing or composing functions.

Lemma 1.1.1. *Let $F, G : X \to \mathbb{R}\cup\{+\infty\}$ be two convex functions which admit a point in $\operatorname{dom} F \cap \operatorname{dom} G$ at which F is continuous. Then*

$$(F+G)^*(x^*) = \inf_{x_1^*+x_2^*=x^*} \{F^*(x_1^*) + G^*(x_2^*)\} \ .$$

Moreover, if the left and right hand sides are both finite, the infimum in the right hand side is achieved.

We recall the definition of *adjoint operator*: given $A : X \to Y$ a linear operator between normed spaces, the adjoint operator A^* is a linear function from X^* to Y^*, uniquely defined by

$$\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle \quad \forall x \in X\,,\ y^* \in Y^* \ .$$

Lemma 1.1.2. *Let X and Y be two Banach spaces and let be given the following functions:*

- *$A : X \to Y$ a linear operator with dense domain $D(A)$;*
- *$\Psi : Y \to \mathbb{R}\cup\{+\infty\}$ a convex l.s.c. function, which is continuous at a point $\sigma_0 := Au_0$, with $u_0 \in D(A)$;*
- *$F : X \to \mathbb{R}\cup\{+\infty\}$ the composition function given by*

$$F(u) := \begin{cases} \Psi(Au) & \text{if } u \in D(A) \\ +\infty & \text{otherwise} \end{cases} \ .$$

Then the Fenchel conjugate of the composition F is given by

$$F^*(f) = \inf\{\Psi^*(\sigma) \ : \ \sigma \in Y^*,\, A^*\sigma = f\} \ .$$

Moreover, if both sides are finite, the infimum in the right hand side is achieved.

By combining Lemma 1.1.1 and 1.1.2 one can easily prove the following duality result (cf. [12, Proposition 14]).

Proposition 1.1.2. *Let X and Y be two Banach spaces and let be given the following functions*

- $A : X \to Y$ *a linear operator with dense domain* $D(A)$ *;*
- $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ *a convex function;*
- $\Psi : Y \to \mathbb{R} \cup \{+\infty\}$ *a convex and l.s.c. function, which is continuous at some point* $A\,u_0$*, with* $u_0 \in D(A)$ *;*

then

$$-\inf_{u \in X} \Big\{ \Psi(A\,u) + \Phi(u) \Big\} = \inf_{\sigma \in Y^*} \Big\{ \Psi^*(\sigma) + \Phi^*(-A^*\,\sigma) \Big\} \,, \tag{1.4}$$

where the infimum on the right hand side is achieved.

Furthermore, a pair $(\overline{u}, \overline{\sigma})$ *is optimal for the left hand side and right hand side of (4.1.2) respectively, if and only if it satisfies the relations* $\overline{\sigma} \in \partial\Psi(A\overline{u})$ *and* $-A^*\overline{\sigma} \in \partial\Phi(\overline{u})$.

We now recall a minimax theorem (cf. [40]). A proof of this result can be found in [53].

Proposition 1.1.3. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be nonempty convex subsets of two locally convex topological vector spaces, and let* $\mathscr{B}$ *be compact. Assume that* $L : \mathscr{A} \times \mathscr{B} \to \mathbb{R}$ *is such that for every* $b \in \mathscr{B}$, $L(\cdot, b)$ *is convex, and for every* $a \in \mathscr{A}$, $L(a, \cdot)$ *is upper semicontinuous and concave. Then, if the quantity*

$$\gamma := \inf_{a \in \mathscr{A}} \sup_{b \in \mathscr{B}} L(a, b)$$

is finite, we have $\gamma = \sup_{b \in \mathscr{B}} \inf_{a \in \mathscr{A}} L(a, b)$*, and there exists* $b^\star \in \mathscr{B}$ *such that* $\inf_{a \in \mathscr{A}} L(a, b^\star) = \gamma$*. If in addition* $\mathscr{A}$ *is compact and, for every* $b \in \mathscr{B}$, $L(\cdot, b)$ *is lower semicontinuous, there exists* $a^\star \in \mathscr{A}$ *such that* $L(a^\star, b^\star) = \gamma$.

1.2 Functionals over L^p spaces

In this Section we collect the results concerning the well-posedness and the sequential lower semicontinuity of integral functionals defined over L^p spaces. More precisely, we consider functionals of the form

$$L^p(\Omega; \mathbb{R}^m) \ni z \ \mapsto \ \int_\Omega f(x, z(x))\, dx \,,$$

or

$$L^p(\Omega; \mathbb{R}^m) \times L^q(\Omega; \mathbb{R}^d) \ni (z, u) \ \mapsto \ \int_\Omega g(x, z(x), u(x))\, dx \,,$$

with $1 \leq p, q < +\infty$ and

$$f : \Omega \times \mathbb{R}^m \to [-\infty, +\infty] \; , \; \; g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [-\infty, +\infty]$$

two $\mathscr{L}^n \times \mathscr{B}$ measurable functions (namely measurable with respect to the σ-algebra generated by the products of subsets of Ω and Borel subsets of $\mathbb{R}^m$ or $\mathbb{R}^m \times d$ respectively). In this Section Ω will denote a Lebesgue measurable subset of $\mathbb{R}^n$ with finite measure.

For the proofs and for the analogous results for the case $p, q = +\infty$, we refer to the book [57, Chapters 6 and 7].

For brevity of notation, in what follows, we simply write I_f to denote the integral functional

$$L^p(\Omega;\mathbb{R}^m) \ni z \; \mapsto \; I_f(z) := \int_\Omega f(x, z(x))\, dx \; .$$

We say that I_f is well-posed in $L^p(\Omega;\mathbb{R}^m)$ if for every $v \in L^p(\Omega;\mathbb{R}^m)$

$$\int_\Omega f^-(x, v(x))\, dx < +\infty \; .$$

Analogous notations will be adopted for I_g.

1.2.1 Integral functionals with integrand $f(x,z)$

Theorem 1.2.1. ***(Well-posedness)*** *Let $1 \leq p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to [-\infty, +\infty]$ be a $\mathscr{L}^n \times \mathscr{B}$ measurable function. Then I_f is well-posed in $L^p(\Omega;\mathbb{R}^m)$ if and only if there exists a nonnegative function $\gamma \in L^1(\Omega)$ and a constant $C > 0$ such that*

$$f(x,z) \geq -C|z|^p - \gamma(x)$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^m$.

We say that two functions $f_1, f_2 : \Omega \times \mathbb{R}^m \to [-\infty, +\infty]$ are *equivalent integrands* in the L^p sense if for every $v \in C_b(\Omega;\mathbb{R}^m)$ we have

$$f_1(x, v(x)) = f_2(x, v(x)) \;\; \mathscr{L}^n - a.e. \; x \in \Omega \; .$$

Clearly, for equivalent integrands there holds

$$\int_\Omega f_1(x, v(x))\, dx = \int_\Omega f_2(x, v(x))\, dx$$

whenever the integrals are defined.

Theorem 1.2.2. ***(Strong lower semicontinuity)*** *Let $1 \leq p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to (-\infty, +\infty]$ be a $\mathscr{L}^n \times \mathscr{B}$ measurable function. Assume that the functional I_f is well-posed in $L^p(\Omega;\mathbb{R}^m)$. Then I_f is lower semicontinuous with respect to the strong topology in $L^p(\Omega;\mathbb{R}^m)$ if and only if (up to equivalent integrands) $f(x,\cdot)$ is lower semicontinuous in $\mathbb{R}^m$ for a.e. $x \in \Omega$.*

Corollary 1.2.1. ***(Strong continuity)*** *Let $1 \leq p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to [-\infty, +\infty]$ be a $\mathscr{L}^n \times \mathscr{B}$ measurable function. Assume that there exists a nonnegative function $\gamma \in L^1(\Omega)$ and a constant $C > 0$ such that*

$$|f(x,z)| \leq C|z|^p + \gamma(x)$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^m$, then the functional I_f is continuous with respect to the strong convergence in $L^p(\Omega;\mathbb{R}^m)$ if and only if $f(x,\cdot)$ is continuous in $\mathbb{R}^m$ for a.e. $x \in \Omega$.

Theorem 1.2.3. ***(Weak lower semicontinuity)*** *Let $1 \leq p < +\infty$ and let $f : \Omega \times \mathbb{R}^m \to (-\infty, +\infty]$ be a $\mathscr{L}^n \times \mathscr{B}$-measurable function.*

Assume that $f(x,\cdot)$ is lower semicontinuous in $\mathbb{R}^m$ for a.e. $x \in \Omega$ and assume that the functional I_f is well-posed in $L^p(\Omega;\mathbb{R}^m)$. Then I_f is sequentially lower semicontinuous with respect to the weak convergence in $L^p(\Omega;\mathbb{R}^m)$ if and only if the two following conditions hold true:

(i) $f(x,\cdot)$ is convex in $\mathbb{R}^m$ for a.e. $x \in \Omega$,

(ii) there exist two functions $a \in L^1(\Omega)$ and $b \in L^{p'}(\Omega;\mathbb{R}^m)$ such that

$$f(x,z) \geq a(x) + b(x) \cdot z \ ,$$

for a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^m$.

1.2.2 Integral functionals with integrand $g(x,u,z)$

Theorem 1.2.4. ***(Well-posedness)***
Let $1 \leq p,q < +\infty$ and let $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [-\infty, +\infty]$ be $\mathscr{L}^n \times \mathscr{B}$-measurable. Then I_g is well-posed in $L^p(\Omega;\mathbb{R}^m) \times L^q(\Omega;\mathbb{R}^d)$ if and only if there exist a nonnegative function $\omega \in L^1(\Omega)$ and a constant $C > 0$ such that

$$g(x,u,z) \geq -C(|u|^q + |z|^p) - \omega(x) \ ,$$

for a.e. $x \in \Omega$ and for every $(z,u) \in \mathbb{R}^m \times \mathbb{R}^d$.

Theorem 1.2.5. ***(Strong-strong lower semicontinuity)***
Let $1 \leq p,q < +\infty$ and let $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to (-\infty, +\infty]$ be $\mathscr{L}^n \times \mathscr{B}$-measurable. Assume that I_g is well-posed in $L^p(\Omega;\mathbb{R}^m) \times L^q(\Omega;\mathbb{R}^d)$. Then I_g is sequentially lower semicontinuous with respect to the (strong-strong) convergence in $L^p(\Omega;\mathbb{R}^m) \times L^q(\Omega;\mathbb{R}^d)$ if and only if (up to equivalent integrands) $f(x,\cdot,\cdot)$ is lower semicontinuous in $\mathbb{R}^m \times \mathbb{R}^d$ for a.e. $x \in \Omega$.

Theorem 1.2.6. ***(Weak-strong lower semicontinuity)***
Let $1 \leq p,q < +\infty$ and let $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to (-\infty, +\infty]$ be $\mathscr{L}^n \times \mathscr{B}$-measurable. Assume that $g(x,\cdot,\cdot)$ is lower semicontinuous in $\mathbb{R}^m \times \mathbb{R}^d$ for a.e. $x \in \Omega$, I_g is well-posed in $L^p(\Omega;\mathbb{R}^m) \times L^q(\Omega;\mathbb{R}^d)$ and there exists $z_0 \in L^p(\Omega;\mathbb{R}^m)$ such that

$$\int_\Omega g(x, z_0(x), u(x)) \, dx < +\infty$$

for every $u \in L^q(\Omega;\mathbb{R}^d)$.
Then I_g *is sequentially lower semicontinuous with respect to the (weak-strong) convergence in* $L^p(\Omega;\mathbb{R}^m) \times L^q(\Omega;\mathbb{R}^d)$ *if and only if (up to equivalent integrands) the following conditions hold true:*

(i) $g(x,\cdot,u)$ *is convex in* $\mathbb{R}^m$ *for a.e.* $x \in \Omega$ *and for every* $u \in \mathbb{R}^d$*;*

(ii) there exist a constant $C > 0$ *and two functions* $\alpha \in L^1(\Omega)$*,* $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^m$ $\mathscr{L}^n \times \mathscr{B}$ *measurable such that*

$$g(x,z,u) \geq \alpha(x) + \beta(x,u)\cdot z - C|u|^q ,$$

for a.e. $x \in \Omega$ *and for every* $(z,u) \in \mathbb{R}^m \times \mathbb{R}^d$*;*

(iii) there exist a constant $C_1 > 0$ *and a function* $b_1 \in L^1(\Omega)$ *such that*

$$|\beta(x,u)|^{p'} \leq C_1|u|^q + b_1(x)$$

for a.e. $x \in \Omega$ *and for every* $u \in L^q(\Omega;\mathbb{R}^d)$.

1.2.3 Fenchel conjugates for integral functionals

In this paragraph we present the results concerning the Fenchel transform in the class of integral functionals. In turns out that, under suitable assumptions, the Fenchel conjugate, the subgradients and the subdifferential are related to the Fenchel conjugate, the subgradients and the subdifferential of the integrand.

In what follows Ω denotes an open bounded domain of $\mathbb{R}^n$, $1 \leq p \leq +\infty$ and p' satisfies $1/p + 1/p' = 1$.

Proposition 1.2.1. *Let* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ *be l.s.c. and convex with respect to the second variable, and assume that there exist* $\overline{v}, \overline{v}^* \in L^\infty(\Omega;\mathbb{R}^m)$ *such that*

$$\int_\Omega |f(x,\overline{v}(x))|\,dx < +\infty\ , \quad \int_\Omega |f^*(x,\overline{v}^*(x))|\,dx < +\infty\ , \tag{1.5}$$

f^* *being the Fenchel conjugate with respect to the second variable.*

Then for every $v^* \in L^{p'}(\Omega;\mathbb{R}^m)$ *and* $v \in L^p(\Omega;\mathbb{R}^m)$ *there hold*

$$(I_f)^*(v^*) = \int_\Omega f^*(x,v^*(x))\,dx$$

and

$$\partial I_f(v) = \left\{ v^* \in L^{p'}(\Omega;\mathbb{R}^m)\ :\ v^*(x) \in \partial f(x,v(x))\ a.e.\ in\ \Omega \right\}\ . \tag{1.6}$$

The proof can be found in [52, Theorem 2 and Corollary 3 of Section 3 in Chapter II]. We remark that assumption 1.5 is readily satisfied when f does not depend on $x \in \Omega$ and is proper: indeed by lower semicontinuity and convexity of f, in view of (v) in Theorem 1.1.1, we infer that also f^* is proper.

Proposition 1.2.2. *Let* $f : \mathbb{R}^m \to \mathbb{R}$ *be a continuous, convex and finite function. If* $\mathrm{v} \in L^\infty(\Omega;\mathbb{R}^m)$*, then every vector field* $\mathrm{v}^* \in \partial f(\mathrm{v})$ *belongs to* $L^\infty(\Omega;\mathbb{R}^m)$.

Proof. Let $\mathrm{v} \in L^\infty(\Omega;\mathbb{R}^m)$ and let $\mathrm{v}^* \in \partial f(\mathrm{v})$. In view of characterization (1.6) we infer that for a.e. $x \in \Omega$

$$\mathrm{v}^*(x) \in \partial f(\mathrm{v}(x)) ,$$

moreover, without loss of generality, we may assume that $|\mathrm{v}(x)| \leq \|\mathrm{v}\|_\infty$.

For such points we deduce a uniform bound for $|\mathrm{v}^*(x)|$. We recall that, by definition of subdifferential, v^* satisfies

$$\mathrm{v}^*(x) \cdot (\xi - \mathrm{v}(x)) \leq f(\xi) - f(\mathrm{v}(x)) \quad \forall \xi \in \mathbb{R}^m . \tag{1.7}$$

Let us fix a real parameter $r > 0$. Exploiting the property (1.7) we conclude

$$\begin{aligned} |\mathrm{v}^*(x)| &= \sup_{|\eta|\leq 1} \mathrm{v}^*(x)\cdot\eta = \frac{1}{r}\sup_{|\eta|\leq r} \mathrm{v}^*(x)\cdot\eta \\ &= \frac{1}{r}\sup_{|\xi-\mathrm{v}(x)|\leq r} \mathrm{v}^*(x)\cdot(\xi-\mathrm{v}(x)) \leq \frac{1}{r}\left[f(\xi)-f(\mathrm{v}(x))\right] \\ &\leq \frac{2}{r}\|f\|_{L^\infty(B)} , \end{aligned}$$

with B the ball centered in the origin with radius $(\|\mathrm{v}\|_\infty + r)$. □

1.3 Γ-convergence

Γ-convergence theory was introduced by De Giorgi in the seventies, and it is a powerful and adaptable instrument in the Calculus of Variations: under suitable assumptions, it permits to characterize the asymptotic behavior of families of infimum problems, more precisely it lets to establish a link between the minima (minimizers) of a sequence of functionals and the minimum (resp. minimizers) of the limit functional.

The general theory develops in the framework of topological spaces, but here we present just the case of metric or metrizable spaces, such as L^p spaces or bounded subsets of Sobolev spaces, endowed with the weak convergence. For the proofs of the following results and for a complete description of the theory, we refer to the book by Dal Maso [43].

1.3.1 Definition and first properties

In what follows X will be a metric space.

Definition 1.3.1. *Given a sequence $F_h : X \to \mathbb{R}\cup\{+\infty\}$, we define its Γ-liminf and Γ-limsup as the following functions from X to $\mathbb{R}\cup\{+\infty\}$:*

$$\begin{aligned} (\Gamma\text{-}\liminf F_h)(x) &= \inf_{x_h\to x} \liminf_{h\to\infty} F_h(x_h) , \\ (\Gamma\text{-}\limsup F_h)(x) &= \inf_{x_h\to x} \limsup_{h\to\infty} F_h(x_h) . \end{aligned}$$

Moreover, if there exists $F : X \to \mathbb{R}\cup\{+\infty\}$ such that

$$F = (\Gamma\text{-}\liminf F_h) = (\Gamma\text{-}\limsup F_h) ,$$

we say that the sequence F_h Γ-converges to the Γ-limit F, and we write $F_h \stackrel{\Gamma}{\to} F$.

Clearly $\Gamma\text{-}\liminf F_h \leq \Gamma\text{-}\limsup F_h$, then the sequence F_h Γ-converges to F if and only if

$$\Gamma\text{-}\limsup F_h \leq F \leq \Gamma\text{-}\liminf F_h\,.$$

Given a sequence F_h, in order to verify that a certain function F is the Γ-limit one can exploit the following, more tractable, characterization.

Theorem 1.3.1. *The sequence F_h Γ-converges to F if and only if*

(i) for every $x \in X$ and for every sequence $x_h \in X$ converging to x there holds

$$F(x) \leq \liminf_h F_h(x_h)\,;$$

(ii) for every $x \in X$ there exists a sequence $x_h \in X$ (called recovery sequence) converging to x such that

$$F(x) \geq \limsup_h F_h(x_h)\,.$$

We remark that Theorem 1.3.1 is still valid if we replace (ii) by one of the following equivalent conditions:

(ii)$'$ for every $x \in X$ there exists a sequence x_h converging to x such that

$$F(x) = \lim_h F_h(x_h)\,.$$

(ii)$''$ for every $x \in X$ and for every $\varepsilon > 0$ there exists a sequence x_h converging to x such that

$$F(x) \geq \limsup_h F_h(x_h) - \varepsilon\,.$$

It is easy to show that Γ-convergence is independent from the other kinds of convergence, such as uniform or punctual convergence, moreover it is not stable under sum. Nevertheless it satisfies the following properties:

- the uniform convergence to a continuous function implies the Γ-convergence;
- the Γ-convergence is stable under perturbation by continuous funcions, namely there holds

$$F_h \xrightarrow{\Gamma} F\,,\quad G \text{ continuous} \quad \Longrightarrow (F_h + G) \xrightarrow{\Gamma} (F + G)\,.$$

- a sort of viceversa holds true: if a sequence of equicoercive functions F_h satisfies

$$\min(F_h + G) \longrightarrow \min(F + G)\,,\quad \text{for } h \to \infty\,,$$

  for every G continuous and bounded from below, then

$$F_h \xrightarrow{\Gamma} F\,.$$

1.3.2 Relaxation

The next results show that the natural framework of Γ-convergence theory is the class of lower semicontinuous functions.

We recall that a function $F : X \to \mathbb{R} \cup \{+\infty\}$ is (sequentially) *lower semicontinuous* if for every $x \in X$

$$F(x) \leq \liminf_{h\to\infty} F(x_h)$$

for every sequence $x_h \to x$ in X.

Definition 1.3.2. *For every function $F : X \to \mathbb{R} \cup \{+\infty\}$ we define the lower semicontinuous envelope (or relaxed function) $\mathrm{Sc}^- F$ of F as*

$$(\mathrm{Sc}^- F)(x) := \sup\{G(x) \mid G \; lsc \,, \; G \leq F\} \; .$$

This procedure is known as *relaxation*.

Since the supremum of l.s.c. functions is l.s.c., the function $\mathrm{Sc}^- F$ is l.s.c., moreover it is the greatest function with such a property among functions majorized by F.

Proposition 1.3.1. *The functions $\Gamma\text{-}\liminf_{h\to\infty} F_h$ and $\Gamma\text{-}\limsup_{h\to\infty} F_h$ are lsc on X. In particular, if the Γ-limit exists, it is lsc.*

Proposition 1.3.2. *The following equalities hold true:*

$$\Gamma\text{-}\liminf F_h = \Gamma\text{-}\liminf \mathrm{Sc}^- F_h \,, \quad \Gamma\text{-}\limsup F_h = \Gamma\text{-}\limsup \mathrm{Sc}^- F_h \,.$$

In particular, F_h Γ-converges to F if and only if $\mathrm{Sc}^- F_h$ Γ-converges to F.

The importance of lower semicontinuity enlightened in the next proposition.

Proposition 1.3.3. *Let $F : X \to \mathbb{R} \cup \{+\infty\}$, then*

(i) every cluster point of a minimizing sequence F (if it exists) is a minimizer for $\mathrm{Sc}^- f$;

(ii) if F is coercive, then $\mathrm{Sc}^- F$ admits a minimum and

$$\inf_X F = \min_X \mathrm{Sc}^- F \; .$$

1.3.3 Convergence of minima and minimizers

Under suitable assumptions of coercivity, the Γ-convergence of a sequence F_h to a limit F implies the convergence of minima of F_h to the minimum of F. Moreover, if F has only one minimum point, a sequence of minimizers for F_h converges to such point.

We recall that $F : X \to \mathbb{R} \cup \{+\infty\}$ is coercive if for every $t \in \mathbb{R}$ the sublevel set $\{F \leq t\}$ is precompact (namely its closure is compact in X).

We say that a sequence F_h is equicoercive in X if for every $t \in \mathbb{R}$ there exists a compact set $K_t \subset X$ such that $\{F_h \leq t\} \subseteq K_t$ for every $h \in \mathbb{N}$.

An interesting characterization is given in [43, Theorem 7.7]: a sequence F_h is equicoercive if and only if there exists a coercive and lsc function $\psi : X \to \mathbb{R} \cup \{+\infty\}$ such that $F_h \geq \psi$ in X for every $h \in \mathbb{N}$.

Let us recall the main results about the convergence of minima and minimizers of Γ-convergent sequences.

Theorem 1.3.2. *Assume that the sequence $F_h : X \to \mathbb{R} \cup \{+\infty\}$ Γ-converges to a function $F : X \to \mathbb{R} \cup \{+\infty\}$. For every $h \in \mathbb{N}$, let $x_h \in \mathrm{Argmin}(F_h)$. If $x_h \to \overline{x}$ in X, then $\overline{x} \in \mathrm{Argmin}(F)$, moreover*

$$F(\overline{x}) = \lim_{h\to\infty} F_h(x_h) .$$

We say that $x \in X$ is an ε-minimizer for a function f if $\varepsilon \geq 0$ and

$$f(x) \in [\inf_X f, \inf_X f + \varepsilon] .$$

Theorem 1.3.2 is still valid if we consider a sequence x_h of ε_h-minimizers, ε_h being an infinitesimal sequence of positive parameters.

Moreover the thesis holds true even if $\overline{x}$ is a cluster point of the sequence x_h: it is a minimizer for F and

$$F(\overline{x}) = \limsup_{h\to\infty} F_h(x_h) .$$

Theorem 1.3.3. *Let $F_h : X \to \mathbb{R} \cup \{+\infty\}$ be an equicoercive sequence Γ-converging to a function $F \not\equiv +\infty$. For every $h \in \mathbb{N}$, let x_h be a minimizer (or an ε_h-minimizer, with ε_h infinitesimal sequence of positive parameters) for F_h. Then*

$$\exists x_{h_k} \to \overline{x} , \ \ \overline{x} \in \mathrm{Argmin}(F) ,$$

$$F(\overline{x}) = \lim_{h\to\infty} F_h(x_h) .$$

Corollary 1.3.1. *Let $F_h : X \to \mathbb{R} \cup \{+\infty\}$ be an equicoercive sequence Γ-converging to a function $F \not\equiv +\infty$ having only one minimizer, namely $\mathrm{Argmin}(F) = \{x_0\}$. Then every sequence of minimizers (or ε_h-minimizers) for F_h verify*

$$x_h \to x_0 ,$$

$$F(x_0) = \lim_{h\to\infty} F_h(x_h) .$$

1.3.4 Compactness and metrizability

In order to complete the overview on Γ-convergence, we present two properties of compactness and metrizability of the functions from X to $\mathbb{R} \cup \{+\infty\}$, with respect to to Γ-convergence.

Proposition 1.3.4. *In a metric separable space every sequence F_h admits a Γ-converging subsequence.*

The Γ-convergence, considered over the family $\mathscr{S}(X)$ of lower semicontinuous functions, in general is not induced by a topology, unless the space X is locally compact. Such assumption can be removed if we consider the subclass $\mathscr{S}_\psi(X) := \{G : X \to \mathbb{R} \cup \{+\infty\} \mid G\ lsc\,,\ G \geq \psi\}$, where $\psi : X \to [0, +\infty]$ is a coercive and lsc function (*cf* [43, Corollary 10.23]).

1.3.5 Mosco-convergence

We conclude the Section by presenting an other variational convergence, introduced by Mosco in [79].

Definition 1.3.3. *Let X be a Banach space. We say that a sequence $F_h : X \to \mathbb{R} \cup \{+\infty\}$ Mosco-converges to a function $F : X \to \mathbb{R} \cup \{+\infty\}$ if and only if*

(i) for every $x \in X$ and for every sequence $x_h \in X$ weakly converging to x there holds

$$F(x) \leq \liminf_h F_h(x_h);$$

(ii) for every $x \in X$ there exists a sequence $x_h \in X$ strongly converging to x such that

$$F(x) \geq \limsup_h F_h(x_h).$$

In this case we write $F_h \stackrel{M}{\to} F$. Clearly, Mosco-convergence is weaker than Γ-convergence with respect to the weak topology: indeed condition (ii) provides the existence of a recovery sequence converging with respect to the strong topology on X, giving as a direct consequence the convergence with respect to the weak topology on X.

Moreover Mosco-convergence is stable when passing to the Fenchel conjugates (see *e.g.* [8, Theorem 1.3]).

Theorem 1.3.4. *Let X be a reflexive, separable, Banach space and let $F_h, F : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions. Then*

$$F_h \stackrel{M}{\to} F \quad \Longleftrightarrow \quad F_h^* \stackrel{M}{\to} F^*,$$

where F_h^* and F^* are the Fenchel conjugates of F_h and F respectively.

Another interesting characterization of Mosco-convergence is the following:

Proposition 1.3.5. *Let X be a reflexive, Banach space and let $F_h, F : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions. Then $F_h \stackrel{M}{\to} F$ if and only if*

(i) for every $x \in X$ and for every sequence $x_h \in X$ weakly converging to x there holds

$$F(x) \leq \liminf_h F_h(x_h);$$

(ii) for every $x^* \in X^*$ and for every sequence $x_h^* \in X^*$ weakly converging to x^* there holds

$$F^*(x^*) \leq \liminf_h F_h^*(x_h^*).$$

1.4 Some topics in Geometric Measure Theory

In the first part of the Section we recall some standard definitions and statements about the functions with bounded variations (see [5, 54]). Then we pass to present some weak notions of trace (see [6, 7, 34, 35]).

Finally we introduce a variational problem in the plane: the Cheeger problem.

1.4.1 Functions of Bounded Variations

Let Ω denote an open bounded subset of $\mathbb{R}^n$.

Definition 1.4.1. *A function $u \in L^1(\Omega)$ has bounded variation in Ω if its distributional derivative is a finite Radon measure in Ω, i.e. if*

$$\int_\Omega u \nabla\phi \, dx = -\int_\Omega \phi \, dDu \quad \forall \phi \in C_0^\infty(\Omega) \,,$$

for some $\mathbb{R}^n$-valued measure Du in Ω. The vector space of all functions with bounded variation in Ω is denoted by $BV(\Omega)$.

Given $u \in L^1_{loc}(\Omega)$ we define its *variation* as

$$V(u,\Omega) := \sup\left\{ \int_\Omega u \operatorname{div}\phi \, dx \ : \ \phi \in C_0^1(\Omega;\mathbb{R}^n) \,,\ \|\phi\|_\infty \le 1 \right\} .$$

Proposition 1.4.1. *A function $u \in L^1(\Omega)$ is in $BV(\Omega)$ if and only if $V(u,\Omega) < +\infty$. In addition $V(u,\Omega)$ coincides with the total variation $|Du|(\Omega)$ and the map $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega)$ with respect to the L^1_{loc} topology.*

In view of Proposition 1.4.1, sometimes we will call the total variation $|Du|$ simply variation. We notice that the space BV endowed with the norm

$$\|u\|_{BV} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

is a Banach space.

We now introduce a particular class of BV functions: the characteristic functions of sets of finite perimeter.

Definition 1.4.2. *An $\mathscr{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has finite perimeter in Ω if the characteristic function $1\!\!1_E$ is an element of $BV(\Omega)$. In this case the perimeter of E is defined as the variation of $1\!\!1_E$ as a BV function, namely*

$$\operatorname{Per}(E,\Omega) := \sup\left\{ \int_E \operatorname{div}\phi \, dx \ : \ \phi \in C_0^1(\Omega;\mathbb{R}^n) \,,\ \|\phi\|_\infty \le 1 \right\} .$$

For a complete overview on this topic we refer to [5, 54].

Let us conclude by recalling a BV version of the coarea formula.

Theorem 1.4.1. *For any open set $\Omega \subset \mathbb{R}^n$ and $u \in L^1_{loc}(\Omega)$ one has*

$$V(u,\Omega) = \int_{-\infty}^{+\infty} \operatorname{Per}(\{x \in \Omega \ : \ u(x) > t\},\Omega) \, dt \,.$$

In particular, if $u \in BV(\Omega)$, the superlevel set $\{u > t\}$ has finite perimeter in Ω for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$. Moreover, for every Borel set $B \subset \Omega$, there holds

$$|Du|(B) = \int_{-\infty}^{+\infty} |D1\!\!1_{\{u>t\}}|(B) \, dt \,, \quad Du(B) = \int_{-\infty}^{+\infty} D1\!\!1_{\{u>t\}}(B) \, dt \,.$$

1.4.2 Traces

In this paragraph Ω is an open bounded domain of $\mathbb{R}^n$ with Lipschitz boundary.

If $v \in W^{1,p}(\Omega)$, we denote by $\mathrm{Tr}\,(v)$ its trace on $\partial\Omega$. We recall that $v \mapsto \mathrm{Tr}\,(v)$ is a bounded linear operator from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$, and it can be characterized via the divergence theorem

$$\int_{\partial\Omega} \mathrm{Tr}\,(v)\varphi\, n_i\, \mathscr{H}^{n-1} = \int_{\Omega} v\,\partial_i\varphi\, dx + \int_{\Omega} \varphi\, d(D_i v) \qquad \forall \varphi \in C^1(\overline{\Omega}), \tag{1.8}$$

where n is the unit outward normal to the boundary $\partial\Omega$. Moreover, $\mathrm{Tr}\,(v)$ can be computed as

$$\mathrm{Tr}\,(v)(x_0) = \lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} v \qquad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x_0 \in \partial\Omega,$$

where $C^-_{r,\rho}(x_0)$ denotes the inner cylindrical neighborhood

$$C^-_{r,\rho}(x_0) := \{ y \in \Omega \ :\ y = x - t n(x_0)\,,\ x \in B_\rho(x_0) \cap \partial\Omega\,,\ t \in (0,r) \}\,. \tag{1.9}$$

In particular, in case $v \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, $\mathrm{Tr}\,(v)$ coincides with the restriction of v to $\partial\Omega$.

We remark that a similar notion of trace extends to functions $v \in BV(\Omega)$ (*cf.* [5]); in this case Tr is a bounded linear operator from $BV(\Omega)$ to $L^1(\partial\Omega)$.

Finally, let us recall the definition of normal trace for vector fields in the class

$$X_\infty(\Omega;\mathbb{R}^n) := \left\{ \Psi \in L^\infty(\Omega;\mathbb{R}^n) \ :\ \mathrm{div}\,\Psi \in L^\infty(\Omega) \right\}\,.$$

Equipped with the norm $\|\Psi\|_{X_\infty} := \|\Psi\|_\infty + \|\,\mathrm{div}\,\Psi\|_\infty$, $X_\infty(\Omega;\mathbb{R}^n)$ is a Banach space. The following definition and properties of the normal trace of elements belonging to X_∞ is in fact valid in the larger space of L^∞ vector fields whose divergence is a measure with finite total variation (*cf.* [6, 35]). For every $\Psi \in X_\infty(\Omega;\mathbb{R}^n)$, there exists a unique function $[\Psi\cdot n]_{\partial\Omega} \in L^\infty(\partial\Omega)$ such that

$$\int_{\partial\Omega} [\Psi\cdot n]_{\partial\Omega}\, \varphi\, d\mathscr{H}^{n-1} = \int_{\Omega} \Psi\cdot\nabla\varphi\, dx + \int_{\Omega} \varphi\,\mathrm{div}\,\Psi\, dx \qquad \forall\varphi \in C^1(\overline{\Omega})\,. \tag{1.10}$$

The normal trace operator $\Psi \mapsto [\Psi\cdot n]_{\partial\Omega}$ from $X_\infty(\Omega;\mathbb{R}^n)$ to $L^\infty(\partial\Omega)$ is linear and bounded. Moreover, we recall from [7, Proposition 2.2] that, if $\partial\Omega$ is piecewise C^1, $[\Psi\cdot n]_{\partial\Omega}$ can be computed as

$$[\Psi\cdot n]_{\partial\Omega}(x_0) = \lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} \Psi\cdot\tilde{n} \qquad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x_0 \in \partial\Omega\,, \tag{1.11}$$

being $\tilde{n}$ the following extension of n to $C^-_{r,\rho}(x_0)$

$$\tilde{n}(y) := n(x) \qquad \text{if } y = x - t n(x_0)\,. \tag{1.12}$$

In the next Lemma we collect some preliminary facts about boundary traces, concerning functions belonging to BV or X_∞.

Lemma 1.4.1. *Given an open bounded domain $\Omega \subset \mathbb{R}^n$ with boundary piecewise C^1, let v and Ψ be respectively a scalar function and a vector field defined defined on Ω which are both L^∞ and BV. Denote by $C^-_{r,\rho}$ and $\tilde{n}$ the cylinder and the extension of the unit outer normal defined in* (2.93) *and* (1.12)*. Then the following equalities hold true at $\mathscr{H}^{n-1}$-a.e. $x_0 \in \partial\Omega$:*

$$\mathrm{Tr}\,(v)(x_0)\, n(x_0) = [vn]_{\partial\Omega}(x_0)\,; \tag{1.13}$$

$$\lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} \Big|\Psi(x) - \mathrm{Tr}\,(\Psi)(x_0)\Big| = 0\,; \tag{1.14}$$

$$\lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} \Big|\Psi(x)\cdot\tilde{n}(x) - \mathrm{Tr}\,(\Psi)(x_0)\cdot n(x_0)\Big| = 0\,. \tag{1.15}$$

Proof. Let $v \in BV(\Omega)\cap L^\infty(\Omega)$. As an element of $BV(\Omega)$, v has a trace $\mathrm{Tr}\,(v) \in L^1(\partial\Omega)$ and the product $\mathrm{Tr}\,(v)\,n$ is characterized in a functional way by (1.8). On the other hand, as an element of $X_\infty(\Omega)$, v has a normal trace $[vn]_{\partial\Omega} \in L^\infty(\partial\Omega)$, which is characterized by (5.7). By comparing the two characterizations (1.8) and (5.7) we infer that, for every test function $\varphi \in C^1(\overline{\Omega})$, it holds

$$\int_{\partial\Omega} \mathrm{Tr}\,(v)\varphi\, n\, d\mathscr{H}^{n-1} = \int_{\partial\Omega} [vn]_{\partial\Omega}\varphi\, d\mathscr{H}^{n-1}\,,$$

which implies the validity of (1.13) $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$.

The proof of (1.14) can be found in [54, Section 5.3].

Finally, in order to prove (1.15), we claim that, if $x_0 \in \partial\Omega$ is a Lebesgue point for $n \in L^\infty(\partial\Omega)$, there holds

$$\lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} |\tilde{n}(x) - n(x_0)| = 0\,, \tag{1.16}$$

Once proved this claim, (1.15) follows easily. Indeed, by adding and subtracting suitable terms to the integrand in (1.15), we obtain:

$$\begin{aligned}
&\fint_{C^-_{r,\rho}(x_0)} \Big|\Psi(x)\cdot\tilde{n}(x) - \mathrm{Tr}\,(\Psi)(x_0)\cdot n(x_0)\Big| \\
&\le \fint_{C^-_{r,\rho}(x_0)} \Big|\Psi(x)\cdot\tilde{n}(x) - \Psi(x)\cdot n(x_0)\Big| + \fint_{C^-_{r,\rho}(x_0)} \Big|\Psi(x)\cdot n(x_0) - \mathrm{Tr}\,(\Psi)(x_0)\cdot n(x_0)\Big| \\
&\le \|\Psi\|_{L^\infty(\Omega;\mathbb{R}^n)} \fint_{C^-_{r,\rho}(x_0)} |\tilde{n}(x) - n(x_0)| + \fint_{C^-_{r,\rho}(x_0)} \Big|\Psi(x) - \mathrm{Tr}\,(\Psi)(x_0)\Big|
\end{aligned}$$

and the two integrals in the last line are infinitesimal as r,ρ tend to zero for $\mathscr{H}^{n-1}$-a.e. $x_0 \in \partial\Omega$, respectively thanks to (1.16) and (1.14).

Let us go back to the proof of (1.16). Without loss of generality, we may assume that $n(x_0) = (0,0,\dots,1)$ and that, in a neighborhood of x_0, the boundary $\partial\Omega$ is the graph of a C^1 function $h : A \to \mathbb{R}$, for some open set $A \subset \mathbb{R}^{n-1}$. More precisely, denoting by x' the first $n-1$ variables of a point $x \in \mathbb{R}^n$, we can write

$$B_\rho(x_0)\cap\partial\Omega = \{(x', h(x'))\ :\ x' \in A_\rho(x_0)\}$$

for some open set $A_\rho(x_0) \subset \mathbb{R}^{n-1}$, and

$$C^-_{r,\rho}(x_0) = \{(x', h(x') - t) \ : \ x' \in A_\rho(x_0)\,,\ t \in (0,r)\}\ .$$

We recall that by definition of the extension $\tilde{n}$, there holds

$$\tilde{n}(x', h(x') - t) = n(x', h(x')) \quad \forall x' \in A_\rho(x_0)\ .$$

Hence, by applying the area formula, we infer

$$\begin{aligned}
\fint_{C^-_{r,\rho}(x_0)} |\tilde{n}(x) - n(x_0)| &= \fint_{A_\rho(x_0)\times(0,r)} |\tilde{n}(x', h(x') - t) - n(x_0)|\, dx'\, dt \\
&= \fint_{A_\rho(x_0)\times(0,r)} |n(x', h(x')) - n(x_0)|\, dx'\, dt \\
&\le \fint_{A_\rho(x_0)\times(0,r)} |n(x', h(x')) - n(x_0)| \sqrt{1 + |Dh|^2(x')}\, dx'\, dt \\
&= \frac{\mathscr{H}^{n-1}(B_\rho(x_0)\cap\partial\Omega)}{\mathscr{L}^{n-1}(A_\rho(x_0))} \fint_{B_\rho(x_0)\cap\partial\Omega} |n(x) - n(x_0)|\, d\mathscr{H}^{n-1} \\
&\le C \fint_{B_\rho(x_0)\cap\partial\Omega} |n(x) - n(x_0)|\, d\mathscr{H}^{n-1}\,,
\end{aligned}$$

and the last integral is infinitesimal as $\rho \to 0^+$, since by assumption x_0 is a Lebesgue point for n.

□

In particular, in case $\Psi \in X_\infty(\Omega;\mathbb{R}^n) \cap C^0(\overline{\Omega};\mathbb{R}^n)$, the normal trace operator applied to Ψ agrees with the normal component of the pointwise trace:

$$[\Psi \cdot n]_{\partial\Omega}(x_0) = \Psi(x_0) \cdot n(x_0) \qquad \forall x_0 \in \partial\Omega\ .$$

In the sequel, we also use the notation $X_\infty(\Omega;\mathbb{R}^{n\times n})$ and $X_\infty(\Omega)$ to denote respectively the class of tensors A with rows in $X_\infty(\Omega;\mathbb{R}^n)$, and the class of scalar functions ψ with $\psi I \in X_\infty(\Omega;\mathbb{R}^{n\times n})$. Accordingly, we indicate by $[A\,n]_{\partial\Omega}$ and $[\psi\,n]_{\partial\Omega}$ the normal traces of A and ψI intended row by row as in (5.7).

1.4.3 Cheeger problem

Given a nonempty open bounded set E of $\mathbb{R}^2$, we call *Cheeger constant* the quantity

$$h_E := \min_{A\subset\overline{E}} \frac{|\partial A|}{|A|}\,, \tag{1.17}$$

where the minimum is taken over all the nonempty subsets of $\overline{E}$ with finite perimeter. A *Cheeger set* of E is any set A which minimizes (1.17). If E itself is a minimizer, we say that it is *Cheeger set of itself*.

Problem (1.17) can be relaxed as follows: the Cheeger constant can also be recast as

$$h_E = \inf\left\{ \int_E |\nabla v| \ : \ v \in BV_0(E)\,,\ \int_E v = 1 \right\}\ . \tag{1.18}$$

In the last years, such minimization problem has captured the attention of many authors (see for instance [3,4,24,30,31,56,59,60,87]). Here we limit ourselves to state the results that will be used later.

Finding a Cheeger set is in general a difficult task, moreover it might be not unique (see *e.g.* Figure 1.1).

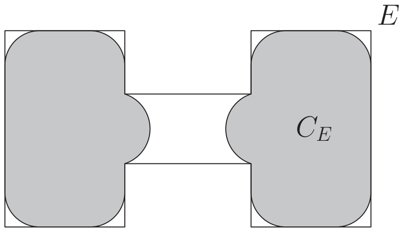


Figure 1.1: *An example of non uniqueness: the shaded sets and each component are Cheeger sets of E.*

In the particular case of convex sets, there exists only one Cheeger set, that can be characterized explicitly, hence we may speak of *the* Cheeger set of E and denote it by C_E.

It is well known that a Cheeger set touches the boundary, more precisely the contact is tangential (at the regular points of ∂E). Roughly speaking, a Cheeger set occupies almost all the set $\overline{E}$ (in order to maximize the denominator of (1.17)), avoiding the parts of the boundary where the curvature is higher (in order to minimize the numerator of (1.17)). For example, if E is a square C_E can be obtained from E "rounding the corners", see Figure 1.2.

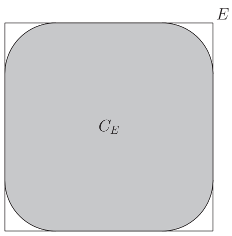


Figure 1.2: *The Cheeger set of the square.*

Let us now focus our attention to the study of Cheeger sets of themselves. A disk, an ellipse or an annulus are examples of such sets, namely that satisfy $E = C_E$.

An important notion for characterizing the Cheeger sets of themselves is *calibrability*. We say that E is *calibrable* if there exists a *calibration*, namely a field $\sigma \in L^2(E;\mathbb{R}^2)$ such that

$$-\operatorname{div}\sigma = h_E \quad \text{in } E\,, \qquad \|\sigma\|_{L^\infty(E)} \leq 1\,, \qquad [\sigma\cdot n_E] = -1 \quad \mathscr{H}^1\text{-a.e. on } \partial E\,.$$

Here $[\sigma \cdot n_E]$ is meant as the weak notion of the trace of the normal component of σ on ∂E, defined according to [7, Theorem 3.5] (see also [6, Theorem 1.2] for the same definition in case ∂E is Lipschitz).

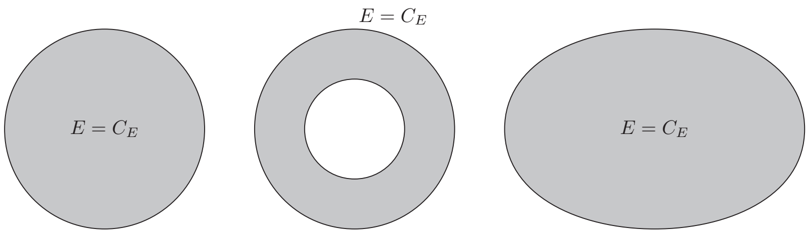


Figure 1.3: *Three examples of Cheeger sets of themselves.*

The easiest example of calibrable set is the disk. Let B_r denote the disk of radius r and centered in the origin, then the vector field $\sigma(x) = -\frac{x}{r}$ is a calibration: indeed the vector field belongs to $L^2(B_r;\mathbb{R}^2)$ and satisfies

$$\begin{aligned} -\operatorname{div}\sigma &= \frac{2}{r} = \frac{|\partial B_r|}{|B_r|} \quad \text{in } B_r \ , \\ \|\sigma\|_{L^\infty(B_r)} &\le 1 \quad \text{in } B_r \ , \\ \sigma \cdot n \llcorner \partial B_r &= -\frac{x}{r}\cdot\frac{x}{|x|}\llcorner \partial B_r = -1 \quad \text{on } \partial B_r \ . \end{aligned}$$

Proposition 1.4.2. *There holds*

$$E \text{ is calibrable} \quad \Longrightarrow \quad E \text{ is Cheeger set of itself.} \tag{1.19}$$

Proof. Let σ be a calibration associated to E. With an integration by parts we infer

$$h_E = \frac{1}{|E|}\int_E (-\operatorname{div}\sigma)\,dx = -\frac{1}{|E|}\int_{\partial E}[\sigma\cdot n_E]\,d\mathscr{H}^{n-1} = \frac{|\partial E|}{|E|} \ .$$

□

If in addition we require that E is convex the existence of a calibration turns out to be also a necessary condition, moreover calibrability can be characterized in two other equivalent ways.

Proposition 1.4.3. *Let E be a convex set. Then the following properties are equivalent:*

(i) E is calibrable ,

(ii) E is Cheeger set of itself,

(iii) the mean curvature $H_{\partial E}$ of the boundary ∂E satisfies

$$\|H_{\partial E}\|_{L^\infty(\partial E)} \le \frac{|\partial E|}{|E|} \ ,$$

iv) the function $u = 1\!\!1_E$ solves

$$-\operatorname{div}\left(\frac{Du}{|Du|}\right) = h_E u \quad \text{in } \mathbb{R}^2 \ .$$

Proof. See [71, Theorem 2] for the proof of the equivalence between (ii) and (iii), and [9, Theorem 4] and [4, Proposition 2] for the other implications. □

Remark 1.4.1. We point out that in Proposition 4.2.2 we will show that, also under the weaker assumption of E simply connected, (i) and (ii) of Proposition 1.19 are equivalent. Moreover we remark that condition (iv) is in general (namely without any assumption on E) stronger that property (i) of calibrability (see [9, Lemma 3]).

1.5 Korn inequalities

Korn inequality asserts the control of the L^2 norm of the gradient of a vector field by the L^2 norm just of the symmetric part of this gradient, under certain conditions. The literature about this topic is very rich: there are different proofs, generalizations to surfaces and estimates of the best constants. The result turns out to be very useful in many fields of Mechanics, as hydrodynamics, statistical physics (see for example [51]) and especially linearized elasticity. In this paragraph we list (and prove) Korn inequality and some interesting variants, that will be a key tool for the study of the compliance optimization problem.

Before stating the results, let us recall some standard inequalities. Let Ω be an open, bounded, connected subset of $\mathbb{R}^n$ with Lipschitz boundary Γ. Then

- $v \in L^2(\Omega) \;\Rightarrow\; v \in H^{-1}(\Omega)\,,\; \partial_i v \in H^{-1}(\Omega)\,,\; i \leq 1 \leq n\,.$
- (Lemma of Lions) If $v \in \mathscr{D}'(\Omega)$, then
$$v \in H^{-1}(\Omega)\,,\; \partial_i v \in H^{-1}(\Omega)\,,\; i \leq 1 \leq n \;\Rightarrow\; v \in L^2(\Omega)\,.$$
- (Poincaré-Wirtinger inequality) For every $1 \leq p \leq +\infty$ there exists a constant $C_p > 0$ such that
$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}\,, \qquad \text{where } \bar{u} := \fint_\Omega u\,.$$

Let us recall the definition of the symmetric gradient: given $v \in H^1(\Omega;\mathbb{R}^n)$, we denote by $e(v)$ the symmetric gradient of v, namely

$$e(v) := \frac{1}{2}\left(\nabla v + \nabla v^T\right)\,.$$

Clearly $e(v) \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. Let us recall the definition of L^2 norm of tensors:

$$\|e(v)\|_{L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} = \left(\sum_{i,j=1}^{n} \|e_{ij}(v)\|^2_{L^2(\Omega)}\right)^{1/2}\,.$$

When there is no ambiguity, we will simply write $\|\cdot\|_{H^1}$ and $\|\cdot\|^2_L$, omitting the domain Ω and the codomain $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}_{\mathrm{sym}}$.

We are now ready to enounce Korn theorem (cf. [37] and [50]). From now on we fix the dimension $n = 3$, nevertheless the following results can also be stated in higher dimension.

Theorem 1.5.1. *The following facts hold true:*
(a) ***Korn inequality without boundary conditions:*** *there exists a constant* $C = C(\Omega)$ *such that*

$$\|v\|^2_{H^1} \leq C\left(\|v\|^2_{L^2} + \|e(v)\|^2_{L^2}\right) \qquad \forall\, v \in H^1(\Omega;\mathbb{R}^3)\,;$$

(b) ***Korn inequality with boundary conditions:*** *let* Γ_0 *be a measurable subset of the boundary* Γ *such that* $H^2(\Gamma_0) > 0$, *then there exists a constant* $C = C(\Omega, \Gamma_0)$ *such that*

$$\|v\|^2_{H^1} \leq C\,\|e(v)\|^2_{L^2} \qquad \forall\, v \in H^1(\Omega;\mathbb{R}^3) \text{ vanishing on } \Gamma_0\,.$$

Proof. The proof will follow from several steps.

(i) The space

$$\mathbf{E}(\Omega) := \{v \in L^2(\Omega;\mathbb{R}^3)\ :\ e_{ij}(v) \in L^2(\Omega)\ \forall\, i,j\} \tag{1.20}$$

equipped with the norm

$$\|v\|^2_E := \|v\|^2_{L^2} + \|e(v)\|^2_{L^2}$$

is a Hilbert space.

The function $\|\cdot\|_E$ is a norm, since it is the sum of a norm and a seminorm. Furthermore it is induced by a scalar product. Then, to prove the claim, it is sufficient to show that **E** is complete, *i.e.* avery Cauchy sequence converges.
Let $\{v^k\}_k \subset \mathbf{E}(\Omega)$ be a Cauchy sequence, then

$$\forall\, \varepsilon > 0\ \exists\, \overline{k}\ :\ \forall\, h,k \geq \overline{k} \quad \|v^h - v^k\|_E < \varepsilon\,.$$

In particular $\{v^k_i\}_k$ and $\{e_{ij}(v^k)\}_k$ are Cauchy sequences in $L^2(\Omega)$, for all i,j. Since L^2 is complete, we obtain

$$v^k_i \stackrel{L^2}{\to} v_i\,, \quad e_{ij}(v^k) \stackrel{L^2}{\to} e_{ij}\,,$$

for $k \to \infty$, for some $v_i\,, e_{ij} \in L^2(\Omega)$.
We now show that $e_{ij}(v) = e_{ij}$, and conclude that $v^k \stackrel{E}{\to} v$. Let φ be a test function in $\mathscr{D}'(\Omega)$:

$$\int_\Omega e_{ij}(v^k)\,\varphi\,dx = -\frac{1}{2}\int_\Omega \left(v^k_i\,\partial_j\varphi + v^k_j\,\partial_i\varphi\right)\,dx\,.$$

Finally, considering the convergence of left/right hand sides, we get

$$\int_\Omega e_{ij}\,\varphi\,dx = -\frac{1}{2}\int_\Omega \left(v_i\,\partial_j\varphi + v_j\,\partial_i\varphi\right)\,dx\,,$$

that is $e_{ij} = e_{ij}(v)$.

(ii) The two spaces $\mathbf{E}(\Omega)$ *and* $H^1(\Omega;\mathbb{R}^3)$ *coincide.*

The inclusion $\supseteq$ is trivial.
Let now $v \in \mathbf{E}(\Omega)$, then $v \in L^2(\Omega;\mathbb{R}^3)$ and $\partial_i v_j \in H^{-1}(\Omega)$. Moreover $\partial_k\partial_i v_j \in H^{-1}(\Omega)$: in fact

$$\partial_k\partial_i v_j = \frac{\partial}{\partial_k}\,e_{ij}(v) + \frac{\partial}{\partial_j}\,e_{i,k}(v) - \frac{\partial}{\partial_i}\,e_{j,k}(v) \in H^{-1}(\Omega)\,,$$

since $e_{ij}(v) \in L^2(\Omega)$.
By Lions' Lemma, we conclude that $\partial_i v_j \in L^2(\Omega)$, then $v \in H^1(\Omega)$.

(iii) Korn inequality without boundary conditions.
We consider the inclusion map

$$\iota : \left(H^1(\Omega;\mathbb{R}^3)\,, \|\cdot\|_{H^1}\right) \longrightarrow \left(\mathbf{E}(\Omega)\,, \|\cdot\|_E\right).$$

By definition

$$\|v\|_E^2 = \|v\|_{L^2}^2 + \|e(v)\|_{L^2}^2 \le \|v\|_{L^2}^2 + \sum_{i,j=1}^{3} \|\partial_i v_j\|_{L^2}^2 + \|\partial_j v_i\|_{L^2}^2 \le 2\,\|v\|_{H^1}^2\,,$$

i.e. ι is (obviously linear) continuous and injective. Thanks to the previous point, we have also that ι is surjective. The open mapping theorem implies then that ι^{-1} is also continuous, implying *(a)*.

(iv) The seminorm $|\cdot|$ defined by

$$|v| := \|e(v)\|_{L^2}$$

is a norm over the space

$$\mathbf{V}(\Omega) := \{v \in H^1(\Omega;\mathbb{R}^3)\ :\ v = 0 \text{ on } \Gamma_0\} \quad \text{when } \mathscr{H}^2(\Gamma_0) > 0\,.$$

Let $v \in \mathbf{V}(\Omega)$ such that $|v| = 0$. We have to show that v is the zero field.
In the step (ii) we have shown that $\partial_j\partial_k v_i \in \mathscr{D}'(\Omega)$. Imposing the condition on the seminorm, we obtain that all these derivatives are 0 in $\mathscr{D}'$. Since Ω is a connected domain, each v_i is a polynomial of degree less or equal one:

$$\exists c_i\,, b_{ij} \in \mathbb{R}\ :\quad v_i(x) = c_i + \sum_{i=1}^{3} b_{ij}x_j \quad \forall j = 1,2,3\,,\ \forall x \in \Omega\,.$$

In other words, there exists a matrix B and a vector c such that $v(x) = c + Bx$. Moreover $e_{ij}(v) = 0$ implies that B is skewsymmetric, then there exists a vector b such that $v(x) = c + b \wedge x$. Such a vector field vanishes everywhere or (aut aut) on a negligible set. Since Γ_0 is not negligible, v has to be identically zero.
(v) Korn inequality with boundary conditions.
Assume on the contrary that the claim is false. Then there exists a sequence $\{v^k\}_k \in \mathbf{V}(\Omega)$ such that

$$\|v^k\|_{H^1} \equiv 1\ \forall k\,, \quad \|e(v^k)\| \to 0 \text{ as } k \to \infty\,.$$

Since $\{v^k\}_k$ is bounded in $L^2(\Omega;\mathbb{R}^3)$, there exists a subsequence (not relabeled) converging strongly L^2. This implies that $\{v^k\}_k$ is a Cauchy sequence in $\mathbf{E}(\Omega)$ and, by (iii), also in $H^1(\Omega;\mathbb{R}^3)$.
Since $\mathbf{V}(\Omega)$ is complete, as a closed subspace of $H^1(\Omega;\mathbb{R}^3)$, there exists $v \in \mathbf{V}(\Omega)$ such that $v^k \overset{H^1}{\to} v$ and $\|e(v)\| = \lim_k \|e(v^k)\| = 0$. By the step (iv) we conlude that $v = 0$, and this is absurd. □

Remark 1.5.1. In the proof of step (ii) we have shown that each partial derivative of ∇u is a linear combination of partial derivatives of $\nabla^{sym} u$, in other words we can say that $\nabla\,\nabla u$ are linear combinations of $\nabla\,\nabla^{sym} u$.

Remark 1.5.2. The key point in general Korn inequalities, for functions belonging to other subspaces of H^1, is to show the step (iv) for the subspace considered.

Remark 1.5.3. The inequalities remain true if we consider L^p norms, $1 < p < \infty$, instead of L^2 ones.

Let us recall the definition of the subspace $R(\Omega)$ of $H^1(\Omega;\mathbb{R}^3)$ of rigid motions:

$$R(\Omega) = \{u \in H^1(Q;\mathbb{R}^3) \,:\, e(u) = 0\} = \{r \in H^1(\Omega;\mathbb{R}^3) \,:\, \exists a, b \in \mathbb{R}^3 \,:\, r(x) = a + b \wedge x\}\,. \tag{1.21}$$

The next Theorem is a version of Korn inequality in the quotient space $H^1(\Omega;\mathbb{R}^3)/R(\Omega)$ (cf. [50] and [38]). For brevity of notation, just for this Section, we denote it by $\dot{H}^1$, and by $\dot{v}$ a generic element, namely a class of equivalence.

Theorem 1.5.2. *There exists a constant $\dot{C} > 0$ such that*

$$\|\dot{v}\|_{\dot{H}^1} \leq \dot{C}\,\|e(\dot{v})\|_{L^2} \quad \forall \dot{v} \in \dot{H}^1(\Omega;\mathbb{R}^3)\,.$$

Proof. The $\dot{H}^1$ norm is defined as usual as

$$\|\dot{v}\|_{\dot{H}^1} := \inf_{r \in R} \|v + r\|_{H^1}\,.$$

Thanks to the previous theorem, we have

$$\|\dot{v}\|_{\dot{H}^1} \leq C \left(\|e(v)\|_{L^2} + \inf_{r \in R} \|v + r\|_{L^2} \right)\,.$$

Therefore it's enough to show that

$$\inf_{r \in R} \|v + r\|_{L^2} \leq C\,\|e(v)\|_{L^2}\,,$$

for some constant C.
In the following we will denote by C a generic constant, that might be different from line to line.
Let P be the operator of orthogonal projection (orthogonal with respect to the L^2 scalar product) from $L^2(\Omega;\mathbb{R}^3)$ to $R(\Omega)$, then

$$\inf_{r \in R} \|v + r\|_{L^2}^2 = \|v - Pv\|_{L^2}^2\,.$$

Replacing v by $\frac{v}{\|v - Pv\|_{L^2}^2}$, we only have to show that

$$\|e(v)\|_{L^2} \geq C \quad \forall v \,:\, \|v - Pv\|_{L^2} = 1\,.$$

Assume on the contrary that the claim is false. Then we can find a sequence $\{v^k\}_k$ such that

$$\|e(v^k)\|_{L^2} \to \infty, \;\; \|v^k - Pv^k\| \equiv 1\,.$$

If we consider $w^k := v^k - Pv^k$ we easily find the absurd. □

Corollary 1.5.1. *There exists $C > 0$ such that for any $v \in H^1(\Omega;\mathbb{R}^3)$*

$$\int_\Omega |v - Pv|^2\,dx \le C \int_\Omega |e(v)|^2\,dx\,,$$

where P is the orthogonal projection operator from $L^2(\Omega;\mathbb{R}^3)$ to $R(\Omega)$.

Let us now pass to some non-standard variants of the Korn inequality.
The first one is a skew-symmetric version (cf. [86], [51] and [94]).
We recall that, given $u \in H^1(\Omega;\mathbb{R}^3)$, we denote with $\nabla^a u$ the skew symmetric part of the gradient, *i.e.*

$$\nabla^a u = \frac{1}{2}\left(\nabla u - \nabla u^T\right) = \nabla u - e(u)\,.$$

Proposition 1.5.1. *For any domain Ω of $\mathbb{R}^3$ there exists a constant $K = K(\Omega)$ such that, for all $u \in H^1(\Omega;\mathbb{R}^3)$,*

$$\left\| \nabla^a u - \left(⨍_\Omega \nabla^a u \right) \right\|_{L^2} \le K \|e(u)\|_{L^2}\,,$$

or equivalently

$$\left\| \operatorname{curl} u - \left(⨍_\Omega \operatorname{curl} u \right) \right\|_{L^2} \le K \|e(u)\|_{L^2}\,,$$

where $\operatorname{curl} u$ denotes the vector field associated to the skew symmetric part of the gradient $\nabla^a u$:

$$\nabla^a u \cdot a = 2 \operatorname{curl} u \wedge a \quad \forall a \in \mathbb{R}^3\,.$$

It turns out that the constant is

$$K(\Omega) = \sup\left\{ \|\nabla^a u\|_{L^2} \;:\; u \in H^1(\Omega;\mathbb{R}^3)\,,\; ⨍_\Omega \nabla^a u = 0\,,\; \|e(u)\|_{L^2} = 1 \right\}\,. \tag{1.22}$$

Proof. Let A be a tensor field in $L^2(\Omega;\mathbb{R}^{3\times 3})$, u be a vector field in $L^2(\Omega;\mathbb{R}^3)$, $\phi \in H^1(\Omega)$ and $\psi \in L^2(\Omega)$.
There hold the following estimates (see [94]):

(i) $\|\nabla u\|_{H^{-1}} \le \|u\|_{L^2}$;

(ii) $\|\operatorname{curl} A\|_{H^{-1}} \le \|A\|_{L^2}$;

(iii) $\|\phi - ⨍_\Omega \phi\|_{H^1} \le C_1 \|\nabla \phi\|_{L^2}$;

(iv) $\|\psi - ⨍_\Omega \psi\|_{L^2} \le C_1 \|\nabla \psi\|_{H^{-1}}$;

where $\operatorname{curl} A$ is defined as the unique tensor field such that $(\operatorname{curl} A)\cdot a = \operatorname{curl}(A^{\mathrm{T}} \cdot a)$ for all $a \in \mathbb{R}^3$. Consider $w := \operatorname{curl} u$, then $\nabla(\operatorname{curl} u)$ is a linear combination of the elements of $e(u)$ and $\operatorname{curl} e(u) = \nabla w$. Using the properties above we conclude that

$$\left\| w - ⨍_\Omega w \right\|_{L^2} \le C\|\nabla w\|_{H^{-1}} \le C\|\operatorname{curl} e(u)\|_{H^{-1}} \le C\|e(u)\|_{L^2}\,.$$

The proof of the representation formula (1.22) can be found in [86]. ☐

Corollary 1.5.2. *If* $\fint_\Omega \operatorname{curl} u = 0$ *then*

$$\|\nabla u\|_{L^2} \leq C\,\|e(u\|_{L^2}\ .$$

If in addition $\fint_\Omega u = 0$*, then*

$$\|u\|_{H^1} \leq C\,\|e(u)\|_{L^2}\,.$$

Proof. The proof follows by Theorem 1.5.1 and Poincaré-Wirtinger inequality. □

Proposition 1.5.2. *Let D be a bounded planar domain $D \subset \mathbb{R}^2$ with Lipschitz boundary, and let $\psi \in H^1_0(D)$ such that $\int_D \psi\, dx' \neq 0$. There exists positive constants $C = C(D)$ such that, for every $v \in H^1_m(D;\mathbb{R}^2)$, it holds*

$$\|v\|_{L^2(D;\mathbb{R}^2)} \leq C\Big(\|e(v)\|_{L^2(D;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \Big|\int_D (\nabla\psi\wedge v)\,dx'\Big|\Big) \tag{1.23}$$

$$\|\int_D (\nabla\psi\wedge v)\,dx' - \operatorname{curl} v\|_{L^2(D)} \leq C\|e(v)\|_{L^2(D;\mathbb{R}^{2\times 2}_{\mathrm{sym}})}\ . \tag{1.24}$$

Proof. To prove (4.25), we argue by contradiction: assume there exists a sequence $v_n \in H^1_m(D;\mathbb{R}^2)$, with

$$\int_D |v_n|^2\,dx' = 1\ \forall n\,, \qquad \lim_n \int_D |e(v_n)|^2\,dx' = 0\,, \qquad \lim_n \int_D (\nabla\psi\wedge v_n)\,dx' = 0\,.$$

By the first two conditions above and the Korn inequality on D, possibly passing to a subsequence, we deduce that v_n converges strongly in $L^2(D;\mathbb{R}^2)$. Its limit $\overline{v}$ is a rigid motion with zero integral mean, hence it is of the form $\overline{v} = \overline{\lambda}(-x_2, x_1)$ for some constant $\overline{\lambda} \in \mathbb{R}$. Then

$$0 = \lim_n \int_D (\nabla\psi\wedge v_n)\,dx' = \overline{\lambda}\int_D x'\cdot\nabla\psi\,dx' = -2\overline{\lambda}\int_D \psi\,dx'\ ,$$

where the last equality follows integrating by parts and recalling that $\psi \in H^1_0(D)$. Thus, since $\int_D \psi\,dx' \neq 0$, it must be $\overline{\lambda} = 0$. This implies $\overline{v} = 0$, that is $v_n \to 0$ strongly in $L^2(D;\mathbb{R}^2)$, against the assumption $\|v_n\|_{L^2(D;\mathbb{R}^2)} = 1$ for every n.

In order to show (1.24), up to replacing v by

$$v + \frac{\int_D (\nabla\psi\wedge v)\,dx'}{2\int_D \psi\,dx'}(-x_2, x_1)\ ,$$

it is not restrictive to assume that $\int_D (\nabla\psi\wedge v)\,dx' = 0$. Again by contradiction, let $v_n \in H^1_m(D;\mathbb{R}^2)$ be a sequence such that

$$\int_D |\operatorname{curl} v_n|^2\,dx' = 1\ \forall n\,, \qquad \lim_n \int_D |e(v_n)|^2\,dx' = 0\,, \qquad \int_D (\nabla\psi\wedge v_n)\,dx' = 0\ \forall n\,.$$

By (4.25) and Korn inequality, we infer that v_n converges strongly to 0 in $H^1_m(D;\mathbb{R}^2)$, which implies in particular that $\operatorname{curl} v_n$ converges strongly to 0 in $L^2(D)$, against the assumption $\|\operatorname{curl} v_n\|_{L^2(D)} = 1$ for every n. □

Finally let us state the last variant of Korn inequality, introduced by Monneau, Murat and Sili in [77].

Proposition 1.5.3. *Let Ω be a cylinder of the form $D \times I$, with D an open bounded planar domain with Lipshcitz boundary, and I a closed interval. Then, for any $z \in H^1(Q;\mathbb{R}^3)$, it holds*

$$\left\| z_3 - |D|^{-1} \left([[z_3]] - x_\alpha \, [[z_\alpha]]' \right) \right\|_{H^{-1}(I;L^2(D))} \le C \left(\|e_{\alpha\beta}(z)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \|e_{\alpha 3}(z)\|_{L^2(Q;\mathbb{R}^2)} \right) .$$

1.6 Linear Elasticity

In this Section we summarize some facts in linear elasticity, which lie beyond the formulation of the main problem.

In §1.6.1 and §1.6.2 we derive the origin of the compliance problem presented in (1), recalling the theory of linear elasticity for isotropic homogeneous elastic bodies. For these two sections we refer to the books [36, 65].

Then, in §1.6.3 we introduce thin elastic structures, the rods, that are object of the thesis. Finally, in §1.6.4, we gather the properties of the density j, introduced in (2), and present other densities that will play a crucial role in the dimension reduction process.

1.6.1 Linear elasticity

A continuum deformable body can take many different shapes or configurations depending on the loading applied to it. We choose one of these configurations to be the *reference configuration* of the body and label it Ω. The reference configuration provides a convenient fixed state of the body to which other configurations can be compared, in order to evaluate their deformation. We identify a particle with its position in Ω. The deformed configuration occupied by the body is described in terms of a deformation mapping function φ, that maps the reference position of every particle $x \in \Omega$ to its position y in the deformed configuration:

$$y = \varphi(x) .$$

The *displacement* of a particle from its initial position x to its final position y is given by

$$u(x) = \varphi(x) - x .$$

In order to satisfy the condition that particles are not destroyed or created, the deformation φ must be a one-to-one mapping. Moreover we require that the mapping preserves the local orientation, thus $\det \nabla\varphi > 0$.

The deformation gradient $\nabla\varphi$ provides a measure for the deformation of the neighborhood of the particle. In view of polar decomposition, the deformation gradient admits a unique decomposition $\nabla\varphi = RU = VR$, where R is a rotation, while U and V are symmetric positive-definite tensors, that represent the shape-change and are called, respectively, the right and left stretch tensor.

Another measure of strain is provided by the Cauchy-Green strain tensor $C := (\nabla\varphi)^T(\nabla\varphi)$. It is easy to verify that C, being the square of U, is a symmetric, positive definite tensor. For v_1 and v_2 unit vectors, the scalar product $(Cv_1)\cdot v_1$ gives the square of the stretch λ in the direction of v_1 in x, that is the per cent elongation of a line element that, prior deformation, was in the v_1 direction at x, while the scalar product $(Cv_2)\cdot v_1$ is related to the shear between the directions v_1 and v_2 at x, *i.e.* it measures the change in angle between two line elements directed, prior the deformation, as v_1 and v_2 respectively.

A further measure of strain is provided by the infinitesimal strain tensor $e(u)$, which is the symmetric part of ∇u

$$e(u) := \frac{1}{2}(\nabla u + \nabla u^T)\ .$$

The tensor $e(u)$ is related to the deformation gradient $\nabla\varphi$ and to the Cauchy-Green tensor C via the following equalities:

$$e(u) = \frac{1}{2}(\nabla\varphi + \nabla\varphi^T) - I = \frac{1}{2}(C - I) - \frac{1}{2}(\nabla u^T \nabla u)\ .$$

Notice that for small deformations

$$e(u) = \frac{1}{2}(C - I) + O(|\nabla u|^2)\ ,$$

so we can conclude that for small $|\nabla u|$, $e(u)$ differs from C by a constant factor and an offset.

The forces acting on a continuum can be divided into two kinds:

- body forces, which are forces acting at a distance, such as gravity and electromagnetic fields; they are given in terms of a density field f of body force per unit volume;
- surface forces, which result from the interaction of the body with its closest surroundings; they are defined in terms of a surface density field of force per unit area g, called the traction field.

The external load on the body in the configuration $\varphi(\Omega)$ is the couple (f, g), where f is the body force defined on $\varphi(\Omega)$ and g is the external traction field defined on $\partial(\varphi(\Omega))$.

If we consider a part $\mathscr{P}$ of the body and a point $y \in \partial\mathscr{P} \cap \mathrm{Int}(\varphi(\Omega))$ the surface force t depends on y and on the surface $\partial\mathscr{P}$. The theory of classical continuum mechanics is based on the assumption, known as Cauchy Postulate, that the traction field depends pointwise on the outward unit normal n to the surface in y. Cauchy's Theorem states that the dependence of the traction vector t on n is linear, that is there exists a second order tensor T such that

$$t(y, n) = T(y)n\ .$$

The balance laws for linear and angular momentum, written for an arbitrary open subset of $\varphi(\Omega)$, lead to the following local equilibrium equations

$$\begin{cases} \mathrm{div}\, T(y) + f(y) = 0 \\ T^T(y) = T(y) \end{cases} \qquad \forall y \in \varphi(\Omega)\ .$$

The set of balance equations must be accompanied by the constitutive relation on T that describes the response of the material to the deformation. A continuum body is said to be *elastic* if the Cauchy stress is a function

$$T(y) = \hat{T}(\nabla\varphi, x) \ . \tag{1.25}$$

The function $\hat{T}$ is called *response function*. A body is called *homogeneous* if the response function does not depend on x explicitly.

Constitutive relations cannot be arbitrarily chosen, in particular they must fulfill the *Principle of frame indifference*: a tensor must be the same physical object with respect to all frames of reference. This requirement for the response function $\hat{T}$ is equivalent to

$$\hat{T}(RB) = R\hat{T}(B)R^T \ .$$

for all R in SO(3) and for all B with $\det B > 0$.

An elastic body is said to be isotropic if for all R in SO(3)

$$\hat{T}(BR) = \hat{T}(B) \qquad \forall B \quad \text{with} \ \ \det B > 0 \ ,$$

which means that if the body is rotated and then undergoes a deformation, no experiment can reveal the prerotation.

It can be easily verified that frame indifference for an isotropic elastic body requires that the response function $\hat{T}$ must be isotropic, that is

$$\hat{T}(RBR^T) = R\hat{T}(B)R^T$$

for all R in SO(3) and for all B with $\det B > 0$.

The linearization of the constitutive equation $T = \hat{T}(\nabla\varphi)$ for small displacements from the reference configuration leads to

$$\hat{T}(\nabla\varphi) = \hat{T}(I + \nabla u) = \hat{T}(I) + D\hat{T}(I)[\nabla u] + o(\nabla u) \ . \tag{1.26}$$

The linear transformation

$$\mathbb{C} := D\hat{T}(I) \tag{1.27}$$

maps the space of second order tensors onto the space of symmetric tensors Sym, and the fourth order tensor $\mathbb{C}$ is called *elasticity tensor*.

Neglecting infinitesimal terms of higher order, by combining (1.25), (1.26) and (1.27), the constitutive relation can be written as

$$T = T_0 + \mathbb{C}[\nabla u] \ ,$$

where $T_0 = \hat{T}(I)$ represents the residual stress, *i.e.* the stress in the reference configuration. We suppose that the residual stress is zero.

If we decompose ∇u into the sum of its symmetric part $e(u)$ and its skew part $\nabla^a u$, for the linearity of $\mathbb{C}$, there holds

$$T = \mathbb{C}[\nabla u] = \mathbb{C}[e(u)] + \mathbb{C}[\nabla^a u] \ .$$

It can be proved that $\mathbb{C}$ is a linear function from Sym to Sym, hence $\mathbb{C}[\nabla u]$ depends only on $e(u)$, namely

$$T = \mathbb{C}[e(u)] \ ,$$

thus $\mathbb{C}$ is a linear function from Sym to Sym. It follows that the cartesian components of the elasticity tensor have the following index symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk} \ ,$$

which are known as minor symmetries.

For an isotropic linear elastic body it is easy to show that the elasticity tensor $\mathbb{C}$ is an isotropic fourth order tensor.

Then $\mathbb{C}$ admits the following representation,

$$\mathbb{C}[S] = \alpha_0(S)I + \alpha_1(S)S + \alpha_2(S)S^2 \ ,$$

for all $S \in$ Sym, where α_i are functions of the principal invariants of S.

Since $S \mapsto \mathbb{C}[S]$ is a linear function, the coefficient α_2 is zero, α_1 is constant and $\alpha_0 = c_0 \mathrm{tr} S + c_1$, for some constants c_i .

Thus, for all second order tensors $S \in$ Sym,

$$\mathbb{C}[S] = \lambda \mathrm{tr} S + 2\mu S \ .$$

The linearized Elastostatics equations are

$$\begin{cases} -\operatorname{div} \mathbb{C}[e(u)] = f & \text{in } \Omega \ , \\ \mathbb{C}[e(u)]n = g & \text{in } \Gamma_N \ , \\ u = u_0 & \text{in } \Gamma_D \ , \end{cases} \tag{1.28}$$

where Γ_D and Γ_N are subset of $\partial\Omega$ such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_D \cup \Gamma_N = \partial\Omega$, u_0 ang g are the prescribed fields on the boundary.

If $\mathscr{H}^2(\Gamma_D) = 0$ the problem is called *pure traction* problem, if $\mathscr{H}^2(\Gamma_N) = 0$ *pure displacement* problem.

1.6.2 The Compliance

Let us consider an homogeneous isotropic elastic isotropic occupying the volume $\Omega \subset \mathbb{R}^3$, subject to the external forces of volume and surface with densities f and g respectively. Assuming to be in the framework of small displacements, we are let to study the system (1.28). The problem can be solved in a weak sense as follows (see [36]).

Let us introduce the space of admissible displacements

$$\mathbf{V}(\Omega) := \{ u \in H^1(\Omega; \mathbb{R}^3) \ : \ u = u_0 \text{ on } \Gamma_D \} \ .$$

Clearly, in the case of pure traction, $\mathbf{V}(\Omega)$ is simply $H^1(\Omega;\mathbb{R}^3)$. Recalling that $\mathbb{C}[\nabla u] = \lambda tr(e(u))I + 2\mu e(u)$, a displacement $u \in \mathbf{V}(\Omega)$ is a solution to (1.28) if and only if

$$\int_\Omega (\lambda tr(e(u))\, tr(e(v)) + 2\mu e(u) : e(v))dx = \int_\Omega f \cdot v dx + \int_{\Gamma_N} g \cdot v d\mathscr{H}^{n-1} \qquad \forall v \in V(\Omega) \ . \tag{1.29}$$

In Propositions 1.6.1 and 1.6.2 we show that the weak solutions coincide with the minimizers of the energy functional

$$J(u) := \frac{1}{2} A(u,u) - L(u) \ ,$$

where

$$A(u,v) := \int_\Omega \lambda\,\mathrm{tr}\,(e(u))\,\mathrm{tr}\,(e(v)) + 2\mu\, e(u) : e(u)\, dx \tag{1.30}$$

is the "quadratic" part, and

$$L(u) := \int_\Omega f\cdot u\, dx + \int_{\Gamma_N} g\cdot u\, d\sigma \tag{1.31}$$

is the linear part.

Indeed the Euler-Lagrange equation of J turns out to be exactly (1.29) and, conversely, since J is convex, the critical points of J are minimizers.

In what follows we assume for simplicity that $u_0 = 0$.

Proposition 1.6.1. ***(displacement-traction case)***
If $\mathscr{H}^2(\Gamma_D) > 0$ the functional J admits a unique minimizer, and the linearized system (1.28) has a unique solution in $H^1(\Omega;\mathbb{R}^n)$ satisfying the boundary conditions.

Proof. Since $\lambda\,,\mu > 0$ we obtain

$$A(u,u) = \int_\Omega \lambda\,(\mathrm{tr}\,(e(u))^2)\, dx + 2\mu\,\|e(u)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} \geq 2\mu\,\|e(u)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\,.$$

Thanks to Korn inequality with boundary condition (see Theorem 1.5.1), we have that J is coercive. Moreover it is (strictly) convex and continuous, hence l.s.c. with respect to the weak topology of $H^1(\Omega;\mathbb{R}^n)$. Considering a minimizing sequence, being J proper, we may take a converging subsequence, and the limit point u turns out to be a minimizer. This minimizer is of course a critical point, then it satisfies Euler-Lagrange equation. Conversely, by convexity, every critical point of J must be a minimizer.
Finally let us prove uniqueness: assume that there exist $u_1\,,u_2$ minimizers for J, then the following implications hold true:

$$\begin{aligned} J(u_1) = J(u_2) \;&\Rightarrow\; A(u_1-u_2,v) = A(u_1,v) - A(u_2,v) = 0 \;\;\forall v\in V \\ &\Rightarrow\; A(u_1-u_2,u_1-u_2) = 0 \;\Rightarrow\; 0 \geq \|u_1-u_2\|^2_{H^1}\,, \end{aligned}$$

that is $u_1 = u_2$. □

We underline that the proof could also be done using Lax-Milgram theorem, since $A(\cdot,\cdot)$ is a bilinear, strictly coercive and continuous form.

In order to obtain the same result in the pure traction case, we have to impose an additional assumption on the loads: we have to require that they are *balanced*, namely

$$\int_\Omega f\cdot u\, dx + \int_{\partial\Omega} g\cdot u\, d\sigma = 0 \quad \text{whenever} \quad u\in R(\Omega)\,,$$

where $R(\Omega)$ denotes the space of rigid motions (see definition (1.21))

$$R(\Omega) := \{u\in H^1(\Omega;\mathbb{R}^3)\,:\, e(u)=0\} = \{u\in H^1(\Omega;\mathbb{R}^3)\,:\, \exists a,b\in\mathbb{R}^3 \text{ st } u = a + b\wedge x\}\,.$$

Proposition 1.6.2. ***(pure traction case)***
Assume $\Gamma_D = \emptyset$ and that $f\,,g$ are balanced. Then (1.28) admits a unique solution in the quotient space $H^1(\Omega;\mathbb{R}^n)/R(\Omega)$.

Proof. Thanks to the hypothesis of balanced forces, for every $u \in H^1(\Omega;\mathbb{R}^n)$ and $r \in R(\Omega)$ there holds

$$J(u) = J(u+r) .$$

Then

$$\inf_{H^1} J = \inf_{H^1/R} J .$$

Theorem 1.5.2 states that $\|e(\cdot)\|_{L^2}$ is an equivalent norm in $H^1(\Omega;\mathbb{R}^n)/R(\Omega)$. Following the same steps done for Proposition 1.6.1, we can conclude the proof. ☐

We remark that also the viceversa holds true: if the load is not balanced, it is easy to find a sequence of displacements u_n such that, for every n, $e(u_n) = 0$ and $J(u_n) \to +\infty$ as $n \to +\infty$.

Let us reformulate the minimization problem J, in the case of pure traction, in a more readable way. Enclosing the volume and surface contributions of the external loads in a unique term $F \in H^{-1}(\mathbb{R}^3;\mathbb{R}^3)$ (balanced load), the linear part (1.31) reads

$$L(u) = \langle F, u \rangle_{\mathbb{R}^3} .$$

Moreover, if we introduce the *strain potential* $j : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$

$$j(z) := \frac{\lambda}{2}(tr(z))^2 + \eta |z|^2 , \tag{1.32}$$

the quadratic term $A(u,u)$ reads

$$A(u,u) = \int_\Omega j(e(u))\, dx .$$

Hence we are led to study the minimization problem

$$\inf \left\{ -\langle F, u \rangle_{\mathbb{R}^3} + \int_\Omega j(e(u))\, dx \ : \ u \in H^1(\mathbb{R}^3;\mathbb{R}^3) \right\} ,$$

or, equivalently, its opposite, namely

$$\mathscr{C}(\Omega) := \sup \left\{ \langle F, u \rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\, dx \ : \ u \in H^1(\mathbb{R}^3;\mathbb{R}^3) \right\} . \tag{1.33}$$

The latter functional (2.1), corresponding to the opposite of the energy, is called the *compliance* of Ω.

In Chapters 2 and 3 we will deal with a problem involving the compliance under a volume constraint: the study will be confined in a fixed region Q of the space, called *design region*. If we consider Ω varying among the subsets of Q, we can consider as external loads the vector fields $F \in H^{-1}(Q;\mathbb{R}^3)$ such that

$$\langle F, u \rangle_{\mathbb{R}^3} = 0 \quad \text{whenever} \quad e(u) = 0 .$$

Under a suitable scaling, the design region represents the three dimensional approximation of a thin rod, as we illustrate in the next paragraph.

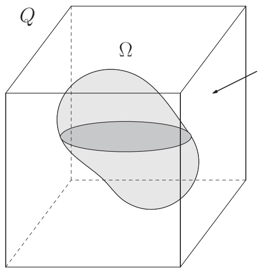


Figure 1.4: *A design region Q subject to an external load F and an elastic material occupying a subset $\Omega \subset Q$.*

1.6.3 Thin elastic structures: rods

In this paragraph we introduce the class of *thin* elastic structures, focusing our attention on rods.
The classical study of compliance optimization in thin structures has been developed by Euler, Bernoulli, Navier, Saint Venant, Timoshenko, Vlassov. Due to new numerical techniques, these problems have had in the last years renewed interest, and new design methods are now a wide field of applied mathematics. For a complete overview about this topic, we refer to the reference book by Trabucho and Viaño [95] and the references therein.

By "thin" we mean that one or two spatial dimensions of the body are much smaller with respect to the others. This particular solids are very important in engineering problems: their small weight and ease of manufactoring and transport make them very convenient to be used in practical applications. When the continuum body can be approximated by a two dimensional surface, the structure is a *plate* or a *shell*, if instead it can be approximated by a one dimensional set, it is a *rod*. In this second group, we include a great variety of structural elements commonly used in applications, such as usual rods, beams, bars, cables, axes, arches, pipelines, rails, antennae, and so on.

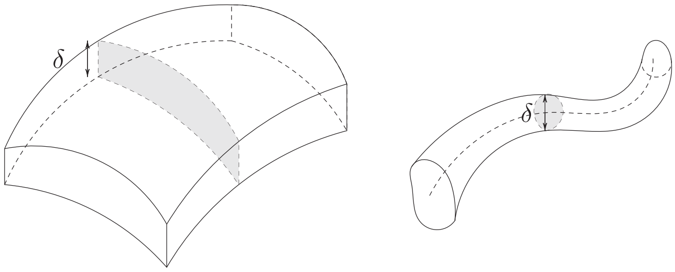


Figure 1.5: *Two examples of thin elastic structures: a plate and a rod, with infinitesimal thickness and cross section.*

The mathematical way of describing a rod is the following: it is a three-dimensional solid occupying the volume generated by a planar connected domain, called the *cross section*, with centroid varying perpendicularly to a spatial curve, the *axis*; moreover the diameter of the cross section is much smaller than the length of the axis. The particular case we deal with is a *straight rod*, in which the axis is a straight line segment I and the cross section is a planar bounded domain D, constant along the axis: we represent such a structure by a cylinder of the form

$$Q_\delta := \delta\overline{D} \times I \,,$$

with $D \subset \mathbb{R}^2$ an open bounded domain, I a closed bounded interval and $\delta > 0$ a vanishing parameter describing the small ratio between the diameter of the cross section and the length.

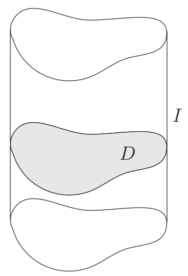


Figure 1.6: *A straight cylinder of the form* $Q = \overline{D} \times I$.

1.6.4 Strain potentials

We conclude the Section by gathering the properties of the strain potential j introduced in (1.32), and of other densities, which play a crucial role in the dimension reduction process.

Let us recall the definition: the strain potential $j : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$, assumed to be isotropic, has the form

$$j(z) := \frac{\lambda}{2}(tr(z))^2 + \eta |z|^2 \,, \tag{1.34}$$

where λ and μ are the Lamé coefficients, assumed to satisfy $\lambda, \eta > 0$, so that j is strictly convex, coercive and homogeneous of degree 2.

Let us introduce two other density energies, which come out in the asymptotics of the compliance problems in thin rods, in the dimension reduction process:

- the *reduced potential* $\overline{j} : \mathbb{R}^3 \to \mathbb{R}$

$$\overline{j}(y) := \inf_{A \in \mathbb{R}^{2\times 2}_{\text{sym}}} j \begin{pmatrix} & & y_1 \\ & A & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \; ; \tag{1.35}$$

- the *modified stored energy density* $j_0 : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$

$$j_0 := \sup\{z \cdot \xi - j^*(\xi) \; : \xi \in \mathbb{R}^{3\times 3}_{\text{sym}} \,,\; \det(\xi) = 0\} \,. \tag{1.36}$$

We denote by $\overline{j_0}$ the $2d$ reduced counterpart of j_0, defined as in (1.35) with j replaced by j_0.

In Lemma 1.6.1 and Lemma 1.6.2 we collect the properties of $\overline{j}$ and $j0$ respectively.

For every $\xi \in \mathbb{R}^3$, let us denote by $E_0\xi$ the symmetric matrix

$$E_0\xi := \frac{1}{2}\sum_{i=1}^{3}\xi_i(e_i \otimes e_3 + e_3 \otimes e_i) \ . \tag{1.37}$$

Lemma 1.6.1. *The reduced potential $\overline{j}$ is coercive, homogeneous of degree 2 and $\overline{j} = \overline{j}^{**}$. Moreover it satisfies the following properties*

(i) $\overline{j}(y) = 2\eta\sum_\alpha |y_\alpha|^2 + (Y/2)|y_3|^2$, *with* $Y := \eta\,\frac{3\lambda+2\eta}{\lambda+\eta}$*;*

(ii) $\overline{j}^*(\xi) = \frac{1}{8\eta}|\xi'|^2 + \frac{1}{2Y}\xi_3^2$;

(iii) $j^*(E_0\xi) = \overline{j}^*(\xi)$.

Proof. Let us prove (i). If we represent an arbitrary $A \in \mathbb{R}^{2\times 2}$ as

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ , \quad a,b,c \in \mathbb{R} \ ,$$

then $\overline{j}(y)$ reads

$$\inf\left\{ \ \frac{\lambda}{2}(a+c+y_3)^2 + \eta(a^2+2b^2+c^2+2y_\alpha^2+y_3^2) : \ a,b,c \in \mathbb{R} \right\} \ .$$

With an easy computation we infer that the infimum is attained for

$$a = c = -\frac{\lambda}{2(\lambda+\eta)}y_3 \ , \quad b = 0 \ ,$$

and $\overline{j}(y)$ agrees with the expression in (i).
In view of this formulation, it is clear that $\overline{j}$ is coercive, continuous, homogeneous of degree 2 and convex, in particular $\overline{j} = \overline{j}^{**}$.

Let us prove (ii). By definition of Fenchel transform, exploiting the representation (i), we infer

$$\begin{aligned}
\overline{j}^*(\xi) &= \sup_{y\in\mathbb{R}^3}\{y\cdot\xi - \overline{j}(y)\} \\
&= \sup_{y'\in\mathbb{R}^2}\left\{y'\cdot\xi' - 2\eta|y'|^2\right\} + \sup_{y_3\in\mathbb{R}}\left\{y_3\xi_3 - \frac{Y}{2}y_3^2\right\} \\
&= 4\eta\sup_{y'\in\mathbb{R}^2}\left\{y'\cdot\frac{\xi'}{4\eta} - \frac{|y'|^2}{2}\right\} + Y\sup_{y_3\in\mathbb{R}}\left\{y_3\frac{\xi_3}{Y} - \frac{y_3^2}{2}\right\} \\
&= \frac{1}{8\eta}|\xi'|^2 + \frac{1}{2Y}\xi_3^2 \ ,
\end{aligned}$$

where in the last equality we have used the fact $\left(|\cdot|^2/2\right)^* = |\cdot|^2/2$ (see Example 1.1.1).

Let us prove (iii). Let us represent the generic symmetric matrix B as

$$B = \begin{pmatrix} & & b_1 \\ & B' & b_2 \\ b_1 & b_2 & b_3 \end{pmatrix} ,$$

with $B' \in \mathbb{R}^{2\times 2}_{\text{sym}}$ and let us denote by b the vector with components b_i. Exploiting the definition of Fenchel transform, we infer

$$\begin{aligned} j^*(E_0\xi) &= \sup\left\{B : E_0\xi - j(B) \ : \ B \in \mathbb{R}^{3\times 3}_{\text{sym}}\right\} \\ &= \sup\left\{b\cdot\xi - (\lambda/2)(tr(B') + b_3)^2 - \eta(|B'|^2 + 2b_\alpha^2 + b_3) \ : \ B' \in \mathbb{R}^{2\times 2}_{\text{sym}}, b \in \mathbb{R}^3\right\} . \end{aligned}$$

It is clear that the latter supremum is attained for diagonal matrices B', hence, denoting by d_1 and d_2 the generic elements of a diagonal matrix, we obtain

$$\begin{aligned} j^*(E_0\xi) =& \sup\left\{b_\alpha\xi_\alpha - 2\eta b_\alpha^2 \ : \ b_\alpha \in \mathbb{R}\right\} \\ &+ \sup\left\{b_3\xi_3 - (\lambda/2)(d_1 + d_2 + b_3)^2 - \eta(d_1^2 + d_2^2 + b_3^2) \ : \ B' \in \mathbb{R}^{2\times 2}_{\text{sym}}, b \in \mathbb{R}^3\right\} . \end{aligned}$$

With an easy computation we infer that the supremum is attained for

$$b_\alpha = \frac{\xi_\alpha}{4\eta} , \quad d_1 = d_2 = -\frac{\lambda}{2(\lambda+\eta)} b_3 , \quad b_3 = \frac{\xi_3}{Y} ,$$

and $j^*(E_0\xi)$ equals $\overline{j}^*(\xi)$ in view of (ii). □

Lemma 1.6.2. *The function j_0 satisfies $j_0 \le j$, is coercive and homogeneous of degree 2. Moreover, the following algebraic identity holds*

$$\overline{j_0}(y) = \overline{j}(y) \qquad \forall y \in \mathbb{R}^3 . \tag{1.38}$$

Proof. Definition (1.36) implies immediately the inequality $j_0 \le j$ and also the 2-homogeneity of j_0, since j, and hence j^*, are 2-homogeneous.

We now prove the coercivity of j_0: for a fixed $z \in \mathbb{R}^{3\times 3}_{\text{sym}}$, we consider $\xi := \alpha\lambda_1(z)(e_z \otimes e_z)$, where $\lambda_1(z)$ is the largest (in modulus) eigenvalue of z, e_z is a corresponding eigenvector of norm 1 and α is an arbitrary constant. Since the tensor ξ is degenerate, by definition of j_0 it holds

$$j_0(z) \ge \sup_\alpha \{\alpha\lambda_1(z) z\cdot(e_z \otimes e_z) - j^*(\alpha\lambda_1(z) e_z \otimes e_z)\} .$$

Thanks to the 2-homogeneity of j^*, we obtain

$$j_0(z) \ge |\lambda_1(z)|^2 \sup_\alpha \{\alpha - \alpha^2 \sup_{\|e\|=1} j^*(e\otimes e)\} = \frac{|\lambda_1(z)|^2}{4c} \ge \frac{\|z\|^2}{12\,c} ,$$

where the constant $c := \sup_{\|e\|=1}\{j^*(e\otimes e)\}$ is clearly strictly positive and finite.

We finally prove (1.38). Applying the identity (iii) in Lemma 1.6.1 to j and to j_0 we infer, for every $y \in \mathbb{R}^3$:

$$\overline{j_0}(y) = \sup\left\{y\cdot\xi - j_0^*(E_0\xi) \ : \ \xi \in \mathbb{R}^3\right\} , \quad \overline{j}(y) = \sup\left\{y\cdot\xi - j^*(E_0\xi) \ : \ \xi \in \mathbb{R}^3\right\} ,$$

where $E_0\xi$ is defined in (1.37). Then (1.38) follows since $j_0^*(E_0\xi) = j^*(E_0\xi)$ for all $\xi \in \mathbb{R}^3$. Actually, j_0^* and j^* agree on the class of degenerated tensors, see [13, Lemma 3.1]. □

CHAPTER 2

Optimal design in thin rods: the small cross section limit

Given a design region $Q \subset \mathbb{R}^3$ subject to an external force $F \in H^{-1}(Q;\mathbb{R}^3)$, the resistance to the load of an isotropic elastic material that occupies a certain volume $\Omega \subset Q$, can be measured by computing a shape functional, the compliance (see §1.6.2 for more details):

$$\mathscr{C}(\Omega) := \sup\left\{ \langle F,u\rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\,dx \;:\; u \in H^1(Q;\mathbb{R}^3)\right\} . \tag{2.1}$$

Here, as usual in linear elasticity, $e(u)$ denotes the symmetric part of the gradient ∇u, and the strain potential $j : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$, assumed to be isotropic, is strictly convex and has the form

$$j(z) := \frac{\lambda}{2}\,\text{tr}^2(z) + \eta|z|^2 , \tag{2.2}$$

$\lambda\,,\mu > 0$ being the Lamé coefficients of the material.
Clearly, in order that $\mathscr{C}(\Omega)$ remains finite, the load must have support contained into $\overline{\Omega}$, moreover it has to be balanced, *i.e.*

$$\langle F,u\rangle_{\mathbb{R}^3} = 0 , \quad \text{whenever } e(u) = 0 . \tag{2.3}$$

Under this condition, an optimal displacement $\overline{u}$ exists and satisfies $\mathscr{C}(\Omega) = \frac{1}{2}\langle F,\overline{u}\rangle_{\mathbb{R}^3}$.
The compliance is proportional to the work done by the load F in order to bring the structure to equilibrium. In particular the smaller is the compliance, the higher is the resistance. Therefore finding the most robust configurations of a prescribed amount of material requires minimizing the shape functional $\mathscr{C}(\Omega)$ under a volume constraint:

$$\inf\{\mathscr{C}(\Omega) \;:\; \Omega \subset Q\,,\ |\Omega| = m\} . \tag{2.4}$$

It is well known that this variational problem is in general ill-posed due to the homogenization phenomena which prevent the existence of an optimal domain (see [2]), so that relaxed solutions must be sought under the form of densities with values in $[0,1]$.

In this Chapter we study the problem of finding the most robust configurations of elastic material, *i.e.* minimizing the compliance, when the design region tends to a thin set, keeping the ratio between the volume of material and the volume of the design region fixed.

Here we consider as thin structure the rods (see §1.6.3), that we represent as cylinders of the form

$$Q_\delta := \delta\overline{D} \times I\ , \tag{2.5}$$

with $D \subset \mathbb{R}^2$ an open bounded domain, I a closed bounded interval and $\delta > 0$ a vanishing parameter describing the small ratio between the diameter of the cross section and the length. The problem we treat, and consequently the approach we adopt to solve it, draws its inspiration from a recent work by G. Bouchitté, I. Fragalà and P. Seppecher [19], in which the authors studied the compliance optimization problem set in thin plates, described by a family of cylinders of the form

$$Q_\delta := \overline{D} \times \delta I\ ,$$

having infinitesimal thickness δ.
If we consider as design region the thin cylinder Q_δ in (2.5) and enclose the volume constraint in the cost through a Lagrange multiplier $k \in \mathbb{R}$, the variational problems (2.4) under study take the form

$$\phi^\delta(k) := \inf_{\Omega \subset Q_\delta} \left\{ \mathscr{C}^\delta(\Omega) + k\frac{|\Omega|}{|Q_\delta|} \right\}\ , \tag{2.6}$$

with

$$\mathscr{C}^\delta(\Omega) := \sup\left\{ \langle F^\delta, \tilde{u}\rangle_{\mathbb{R}^3} - \int_\Omega j(e(\tilde{u}))\,dx\ :\ \tilde{u} \in H^1(Q_\delta, \mathbb{R}^3) \right\}\ , \tag{2.7}$$

where F^δ is a suitable scaling of the load F, chosen so that in the limit process the infimum remains finite. The choice of the scaling F^δ depends on the assumptions made on the type of applied loads.

In this Chapter we study the asymptotic behavior of the sequence $\phi^\delta(k)$ defined in (2.6) as $\delta \to 0^+$, namely the compliance optimization problem in a rod-like set, under volume constraint. In Chapter 3 we will perform a second passage to the limit, as $k \to +\infty$: as already said in the Introduction, this corresponds to consider small filling ratios.

We point out that the dimension reduction process, from 3 to 1 dimension, is performed without any topological assumption on the set Ω occupied by the material. Therefore, it is not covered by the very extensive literature on $3d-1d$ analysis (we limit ourselves to mentioning [73, 78, 80, 86, 95] and references therein).

The Chapter is organized as follows.
In Section 2.1 we set up all the preliminaries, concerning in particular twist displacement fields, Bernoulli-Navier fields and the admissible loads under consideration.

In Section 2.2 we state the main result (Theorem 2.2.1): we determine the limit $\phi(k)$ as $\delta \to 0^+$ of the sequence $\phi^\delta(k)$ under the form of a convex, well-posed problem for material densities $\theta \in L^\infty(Q;[0,1])$.

In Section 2.3 we prove Theorem 2.2.1: the asymptotical study of $\phi^\delta(k)$ is based on the comparison with the "fictitious counterpart", namely their relaxed formulation in all $L^\infty(Q;[0,1])$. Indeed, as already observed, the the infimum problems (7) are in general ill-posed, so that we need to enlarge the class of admissible materials, passing from "real" materials, represented by characteristic functions, to "composite" materials, represented by densities with values in $[0,1]$. The proof is based upon some delicate compactness properties derived from variants of the Korn inequality (see the Section 1.5), Γ-convergence techniques (see Section 1.3) and a crucial bound for the relaxed functional of the compliance, established in [19, Proposition 2.8].

In Section 2.4 we give alternative formulations of $\phi(k)$ and a system of optimality conditions. Finally we face the question about the existence of a classical solution (*i.e.* a density with values in $\{0,1\}$) for $\phi(k)$: this corresponds to ask whether the compliance problem under volume constraint, in a rod-like set, admits as solution a real material rather that a composite. A deeper analysis of the problem is postponed to Chapter 4.

In the Appendix are gathered the proofs of two auxiliary lemmas.

2.1 Preliminaries

We recall that the design region under study is a right cylinder of the form $Q := \overline{D} \times I$, where $I = [-1/2, 1/2]$ and D is an open, bounded, connected subset of $\mathbb{R}^2$ with Lipschitz boundary. Without loss of generality we may assume $|D| = 1$, so that $|Q| = 1$. Finally, we chose the coordinate axes so that $\int_D x_\alpha \, dx' = 0$.

Let us now introduce the classes of displacement fields and fix the admissible loads.

2.1.1 Displacement fields

As usual, by *rigid motion* we mean the space

$$R(Q) := \left\{ r \in H^1(Q;\mathbb{R}^3) \;:\; e(r) = 0 \right\}$$

namely vector fields of the form $r(x) = a + b \wedge x$, with a, $b \in \mathbb{R}^3$.
We define the space of *Bernoulli-Navier fields*

$$BN(Q) := \left\{ u \in H^1(Q;\mathbb{R}^3) \;:\; e_{ij}(u) = 0 \quad \forall (i,j) \neq (3,3) \right\}$$

and the space

$$TW(Q) := \left\{ v = (v_\alpha, v_3) \in H^1(Q;\mathbb{R}^2) \times L^2(I; H^1_m(D)) \;:\; e_{\alpha\beta}(v) = 0 \quad \forall \alpha, \beta \in \{1,2\} \right\}.$$

It is easy to check that, up to subtracting a rigid motion, any $u \in BN(Q)$ admits the following representation:

$$u(x) = (\zeta_1(x_3), \zeta_2(x_3), \zeta_3(x_3) - x_\alpha \zeta'_\alpha(x_3)) \quad \text{for some } (\zeta_\alpha, \zeta_3) \in (H^2_m(I))^2 \times H^1_m(I)\,. \tag{2.8}$$

Similarly, up to subtracting a Bernoulli-Navier field, any $v \in TW(Q)$ can be written as a *twist field*, namely a displacement of the form

$$v(x) = (-x_2 c(x_3), x_1 c(x_3), w(x)) \text{ for some } c \in H^1_m(I)\ ,\ \ w \in L^2(I; H^1_m(D))\ . \tag{2.9}$$

We remark that, up to rigid motions, any $v \in TW(Q)$ can be written as $v = \overline{u} + \overline{v}$, with $\overline{u}$ as in (2.8) and $\overline{v}$ as in (2.9), and the decomposition is unique.

We notice that the third component w of a field belonging to $TW(Q)$ is not necessarily in $H^1(Q)$; nevertheless, using the representation (2.9), we see that

$$(e_{13}(v), e_{23}(v)) = \frac{1}{2}\left(c'(x_3)(-x_2, x_1) + \nabla_{x'} w\right) \in L^2(Q; \mathbb{R}^2)\ . \tag{2.10}$$

Finally, exploiting Korn inequality and Poincaré-Wirtinger inequality, it is easy to show that the quotients $BN(Q)/R(Q)$ and $TW(Q)/BN(Q)$, endowed with the norms $\|e_{33}(\cdot)\|_{L^2(\Omega)}$ and $(\|e_{1,3}(\cdot)\|^2_{L^2(Q)} + \|e_{2,3}(\cdot)\|^2_{L^2(Q)})^{1/2}$ respectively, are Banach spaces.

2.1.2 Admissible loads

In the literature it is customary to distinguish between the stretching, bending and the torsion loads. Let us recall the definitions and fix some notations.

Let $F \in H^{-1}(Q; \mathbb{R}^3)$ be an external load. In the asymptotic procedure, it turns out that the load F enters in the limit problem with its resultant and momentum averaged on the sections. In particular, the normal component of the load to the section gives the average axial load $[[F_3]]$, while the component lying on the section gives the average shear force $[[F_1]]e_1 + [[F_2]]e_2$ (for the definitions of the averages $[[\cdot]]$ we refer to §Notations). Similarly, the normal and planar components of the average of the momentum $[[x \wedge F]]$ give the torsion

$$m_F := [[x_1 F_2 - x_2 F_1]] \in H^{-1}(I; \mathbb{R}) \tag{2.11}$$

and the average bending moment

$$\underline{m}_F^{(b)} := ([[x_2 F_3 - x_3 F_2]]; [[-x_1 F_3 + x_3 F_1]]) \in H^{-1}(I; \mathbb{R}^2)\ , \tag{2.12}$$

respectively.

We now fix the type of exterior loads F we consider.

With any $\Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{sym})$, that we extend to zero over $\mathbb{R}^3 \setminus Q$, we associate the distribution $\operatorname{div}\Sigma$. As an element of $H^{-1}(Q; \mathbb{R}^3)$, it is characterized by

$$\langle \operatorname{div}\Sigma, u\rangle_{\mathbb{R}^3} = -\int_Q \Sigma \cdot \nabla u = -\int_Q \Sigma \cdot e(u) \quad \forall u \in H^1(Q; \mathbb{R}^3)\ . \tag{2.13}$$

Definition 2.1.1. *We say that $F \in H^{-1}(Q; \mathbb{R}^3)$ is an admissible load if it satisfies the following conditions:*

(h1) there exists $\Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{sym})$ such that $F = \operatorname{div}\Sigma$ in $\mathscr{D}'(\mathbb{R}^3; \mathbb{R}^3)$;

(h2) either $F_3 = \partial_1 \Sigma_{13} + \partial_2 \Sigma_{23}$ *or* $\begin{cases} F_3 = \partial_3 \Sigma_{33} \\ [[F_\alpha]] = 0 \end{cases}$;

(h3) the set $\{x \in Q \,:\, \mathrm{dist}(x, \mathrm{spt}(F)) < \delta\}$ *has vanishing Lebesgue measure as* $\delta \to 0$.

Remark 2.1.1. Assumption (h1) is equivalent to require that the load is balanced, namely it satisfies

$$\langle F, u\rangle_{\mathbb{R}^3} = 0 \quad \text{whenever } e(u) = 0 \,.$$

Indeed, as already observed in (2.3), this condition is necessary in order that the compliance remains finite.

Assumption (h3) is needed to ensure that the load can be supported by a small amount of material. From a technical point of view, (h3) enables us to apply Proposition 2.8 in [19]. This condition on the topological support of F is satisfied for instance when $\mathrm{spt}(F)$ is a 2-rectifiable set, and in particular in the standard case when F is applied at the boundary of Q.

In order to better understand the condition (h2), let us compute the resultant of the forces on the sections, the torque and the bending momentum for a load F that admits the divergence representation (h1).
Let $\Sigma \in L^2(Q;\mathbb{R}^{3\times3}_{sym})$ be associated to F. Hence, according to the definitions (2.11) and (2.12), we obtain that the components of the average resultant read

$$[[F_i]] = [[\partial_1\Sigma_{i1} + \partial_2\Sigma_{i2}]] + [[\partial_3\Sigma_{i3}]] = [[\partial_3\Sigma_{i3}]] \,, \tag{2.14}$$

the normal component of the average momentum $[[x \wedge F]]$ equals

$$\begin{aligned} m_F &= [[x_1F_2 - x_2F_1]] \\ &= [[x_1\partial_1\Sigma_{21} + \partial_2\Sigma_{22} - x_2(\partial_1\Sigma_{11} + \partial_2\Sigma_{12})]] + [[x_1\partial_3\Sigma_{23} - x_2\partial_3\Sigma_{13}]] \\ &= [[x_1\partial_1\Sigma_{21} - x_2\partial_2\Sigma_{12}]] + [[x_1\partial_2\Sigma_{22}]] - [[x_2\partial_1\Sigma_{11}]] + [[x_1\partial_3\Sigma_{23} - x_2\partial_3\Sigma_{13}]] \\ &= [[x_1\partial_3\Sigma_{23} - x_2\partial_3\Sigma_{13}]] \,, \end{aligned} \tag{2.15}$$

and the planar component of $[[x \wedge F]]$ is given by

$$\begin{aligned} \underline{m}^b_F &= ([[x_2F_3 - x_3F_2]];[[-x_1F_3 + x_3F_1]]) \\ &= (-x_3[[F_2]];x_3[[F_1]]) + ([[x_2F_3]];[[-x_1F_3]]) \\ &= (-x_3[[F_2]];x_3[[F_1]]) + ([[x_2(\partial_1\Sigma_{31} + \partial_2\Sigma_{32})]];[[-x_1(\partial_1\Sigma_{31} + \partial_2\Sigma_{32})]]) + \\ &\quad + ([[x_2\partial_3\Sigma_{33}]];-[[x_1\partial_3\Sigma_{33}]]) \,, \end{aligned} \tag{2.16}$$

where we have used the symmetry of the tensor Σ and the fact that, for every distribution $T \in \mathscr{D}'(Q)$, the average $[[T]]$, being an element of $\mathscr{D}'(I)$, satisfies

$$\langle [[\partial_\alpha T]], \varphi(x_3)\rangle_{\mathbb{R}} = -\langle T, \partial_\alpha\varphi(x_3)\rangle_{\mathbb{R}^3} = 0 \quad \forall \varphi \in \mathscr{D}(I) \,.$$

In view of (2.14)-(2.16), we see that all the quantities above do not depend on $\Sigma_{\alpha\beta}$; moreover we notice that imposing (h2) we require that either the bending moment *does not* depend on Σ_{33}, or it depends *only* on Σ_{33}.

We decompose an admissible load F as the sum $F = G + H$ of two loads belonging to $H^{-1}(Q;\mathbb{R}^3)$ defined as follows:

$$G := \mathrm{div}\,\Sigma_G \quad \text{with} \quad (\Sigma_G)_{ij} = \begin{cases} \Sigma_{ij} & \text{if } (i,j) \neq (3,3) \\ 0 & \text{if } (i,j) = (3,3) \end{cases} \tag{2.17}$$

$$H := \operatorname{div}\Sigma_H \quad \text{with} \quad (\Sigma_H)_{ij} = \begin{cases} 0 & \text{if } (i,j) \neq (3,3) \\ \Sigma_{33} & \text{if } (i,j) = (3,3) \end{cases}, \tag{2.18}$$

where $\Sigma \in L^2(Q;\mathbb{R}^{3\times 3}_{sym})$ is associated to F as in (h1).

In view of condition (h2), the admissible loads satisfy

$$\text{either} \quad \begin{cases} F &= G \\ H &= 0 \\ m_F &= m_G \\ \underline{m}_F^{(b)} &= \underline{m}_G^{(b)} \end{cases} \qquad \text{or} \quad \begin{cases} F_\alpha &= G_\alpha\,,\ [[G_\alpha]] = 0 \\ F_3 &= H_3 \\ m_F &= m_G \\ \underline{m}_F^{(b)} &= \underline{m}_H^{(b)} \end{cases}$$

In Section 2.2 we will consider two different scalings for the two components G and H. We point out that the first case, corresponding to $H = 0$, has been presented in the paper [17].

The properties of the action of an admissible load over the displacements are summarized in the next proposition.

Proposition 2.1.1. *Let F be a load satisfying (h1), and let $F = G + H$ with G and H defined in (2.17) and (2.18) respectively. Then the following facts hold:*

(i) the loads F, G and H are balanced, namely they do not act on rigid motions;

(ii) G does not act on Bernoulli-Navier displacements, whereas it acts on $TW(Q)$ being an element of $H^{-1}(Q;\mathbb{R}^2) \times L^2(I;H^{-1}(D))$. More precisely, for any $v \in TW(Q)$, there holds

$$\langle G, v\rangle_{\mathbb{R}^3} = \langle m_G, c\rangle_{\mathbb{R}} + \langle G_3, w\rangle_{\mathbb{R}^3}\,, \tag{2.19}$$

where c and w are associated to v according to (2.9)

(iii) the action of H on any Bernoulli-Navier displacement u is

$$\langle H, u\rangle_{\mathbb{R}^3} = -\langle \overline{H_\alpha}, \zeta'_\alpha\rangle_{\mathbb{R}} + \langle \overline{H_3}, \zeta_3\rangle_{\mathbb{R}}\,, \tag{2.20}$$

where ζ_i are associated to u according to (2.8), and $\overline{H_i} \in H^{-1}(I)$ are defined by

$$\overline{H_\alpha} := [[x_\alpha H_3]]\,, \qquad \overline{H_3} := [[H_3]]\,. \tag{2.21}$$

Proof. (i) By definition (h1), (2.17) and (2.18), F, G and H are defined as the divergence of suitable L^2 tensors in the sense of distributions. In view of (2.13) it is then clear that they vanish on rigid motions.

(ii) Let $\Sigma_G \in L^2(Q;\mathbb{R}^{3\times 3}_{sym})$ be associated to G according to (2.17). By (2.13), since $(\Sigma_G)_{33} = 0$, we infer that G vanishes on Bernoulli-Navier displacements. On the other hand, the action of G on $TW(Q)$ is well-defined through the equality

$$\langle G, v\rangle_{\mathbb{R}^3} = -2\,\langle \Sigma_{\alpha 3}, e_{\alpha 3}(v)\rangle_{\mathbb{R}^3} \tag{2.22}$$

for every $v \in TW(Q)$.

The right hand side of (2.22) makes sense as a scalar product in $L^2(Q;\mathbb{R}^2)$ thanks to (2.10). In particular, by taking $v=(0,0,v_3)$, one can see that $G_3 \in L^2(I;H^{-1}(D))$. Finally, writing v using the representation (2.9), equality (2.22) can be rewritten under the form (2.19).

(iii) Let u be a Bernoulli-Navier displacement. Since by (i) H vanishes on rigid motions, and by definition $H_\alpha = 0$, there holds

$$\langle H,u\rangle_{\mathbb{R}^3} = \langle H_3, \zeta_3 - x_\alpha \zeta'_\alpha\rangle_{\mathbb{R}^3} ,$$

where ζ_i are associated to u according to (2.8). By construction, the functions ζ_i depend only on x_3, then we infer

$$\langle H,u\rangle_{\mathbb{R}^3} = -\langle [[x_\alpha H_3]], \zeta'_\alpha\rangle_{\mathbb{R}} + \langle [[H_3]], \zeta_3\rangle_{\mathbb{R}} ,$$

that gives (2.20), thanks to definition (2.21) of $\overline{H_i}$. □

Remark 2.1.2. Notice that, from the definition (2.11) of m_G and the assumption (2.17) on G, it follows that $\langle m_G, 1\rangle_{\mathbb{R}} = 0$. Indeed,

$$\langle m_G,1\rangle_{\mathbb{R}} = \langle [[x_1G_2 - x_2G_1]],1\rangle_{\mathbb{R}} = \langle x_1G_2 - x_2G_1, 1\rangle_{\mathbb{R}^3} = \langle \partial_1\Sigma_{21}, x_1\rangle_{\mathbb{R}^3} - \langle \partial_2\Sigma_{12}, x_2\rangle_{\mathbb{R}^3} = 0 ,$$

where the last equality holds since Σ is symmetric.

Similarly, since $H_3 = \partial_3\Sigma_{33}$, there holds $\langle \overline{H}_3, 1\rangle_{\mathbb{R}} = 0$.

2.1.3 Examples of admissible loads

Let us introduce some examples of admissible loads: in the first ones we present some interesting choices for G, and in the last one a family of possible H.

In Chapter 3 we will analyze the behavior of optimal configurations when the design region is subject to these particular loads.

Example 2.1.1. *(Component G horizontal and concentrated on the "top and bottom faces" $D\times\{\pm 1/2\}$)*
For $\rho \in BV(I)$ and $\psi \in H^1_0(D)$, consider the horizontal load

$$(G_1,G_2) = \rho'(x_3)(-\partial_2\psi(x'), \partial_1\psi(x')) \ , \ G_3 = 0 . \tag{2.23}$$

Assumption (2.17) is readily satisfied by taking

$$\Sigma_{\alpha\beta} = \Sigma_{33} = 0 \quad \text{and} \quad (\Sigma_{13},\Sigma_{23}) = \rho(x_3)(-\partial_2\psi(x'), \partial_1\psi(x')) .$$

Hence G is an admissible load provided (h3) holds, which happens as soon as ρ is piecewise constant. In particular, the choice $\rho(x_3) = 1\!\!1_I(x_3)$ corresponds to applying a horizontal surface force on the top and bottom faces of the cylinder Q. Moreover, in this case, the average momentum of (G_1,G_2) is given by

$$m_G = \left(\int_D x'\cdot\nabla_{x'}\psi(x')\,dx'\right)(\delta_{-1/2} - \delta_{1/2})(x_3) .$$

By varying the choice of ψ, for every $c\in\mathbb{R}$ we can construct a load G such that

$$\begin{cases} m_G = c\,(\delta_{-1/2} - \delta_{1/2})(x_3) , \\ G_3 = 0 . \end{cases} \tag{2.24}$$

Let us give an explicit example (see Figure 2.1): if D is a circular disk of radius R and we take $\psi(x') = \frac{R^2-|x'|^2}{2}$, we obtain the classical boundary load in torsion problem, that is

$$(G_1, G_2) = (\delta_{1/2} - \delta_{-1/2})(x_3)(-x_2, x_1) ,$$

having average momentum

$$m_G = \frac{\pi R^4}{2}(\delta_{1/2} - \delta_{-1/2})(x_3) .$$

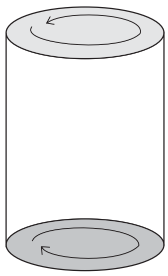


Figure 2.1: *Torsion load concentrated on the top and bottom faces of Q, with Q having circular section.*

Example 2.1.2. *(Component G horizontal and concentrated on the "lateral surface" $\partial D \times I$)*
Denote by $\tau_{\partial D}$ the unit tangent vector at ∂D. For any $\eta \in L^2_m(I)$, the following horizontal load supported on $\partial D \times I$ is admissible:

$$(G_1, G_2) = \eta(x_3)(-\partial_2 1\!\!1_D(x'), \partial_1 1\!\!1_D(x')) = \eta(x_3)\tau_{\partial D}(x')\mathscr{H}^1 \llcorner \partial D , \quad G_3 = 0 . \quad (2.25)$$

In order to check assumption (2.17), we choose $\psi \in H^1_0(D)$ such that $\int_D \psi = |D|$, and we decompose G as $G' + G''$, being

$$(G'_1, G'_2) := \eta(x_3)(-\partial_2\psi(x'), \partial_1\psi(x')) , \quad G'_3 = 0 ,$$

and $G'' := G - G'$. Since the class of loads satisfying (2.17) is a linear space, it is enough to show that both G' and G'' belong to it.

Since $\eta \in L^2_m(I)$ it admits a unique primitive $\rho \in H^1(I)$ such that $\eta = \rho'$ (see Notation). Hence G' can be written in the form (2.23) and, according to Example 2.1.1, is admissible.

Concerning G'', we may rewrite it as

$$(G''_1, G''_2) = \eta(x_3)(g_1(x'), g_2(x')) , \quad G''_3 = 0 ,$$

where $(g_1, g_2) := (-\partial_2(1\!\!1_D - \psi), \partial_1(1\!\!1_D - \psi))$. Since by construction (g_1, g_2) is a balanced load in $H^{-1}(D;\mathbb{R}^2)$, there exists a solution $\sigma \in L^2(D;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ to the equation $\operatorname{div}\sigma = (g_1, g_2)$. Then condition (2.17) is satisfied by taking

$$\Sigma_{\alpha\beta} = \eta(x_3)\sigma_{\alpha\beta}(x') \quad \text{and} \quad \Sigma_{\alpha 3} = 0 .$$

We notice that in this example the average momentum is absolutely continuous with respect to the Lebesgue measure, more precisely

$$m_G = -2|D|\eta(x_3)\ . \tag{2.26}$$

An example is given in Figure 2.2.

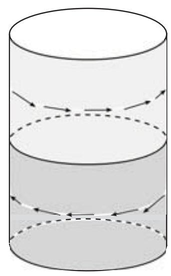


Figure 2.2: *An example of torsion load concentrated on the lateral surface of Q, defined according to Example 2.1.2, with* $\eta(x_3) := 1\!\!1_{[0,1/2]}(x_3) - 1\!\!1_{[-1/2,0]}(x_3)$.

Example 2.1.3. *(Component G concentrated on the whole boundary of Q)*
Let $h \in L^2_m(\partial D)$, and let $\psi \in H^1(D)$ be the solution of the two-dimensional Neumann problem

$$\begin{cases} \Delta\phi = 0 & \text{in } D, \\ \partial_\nu \phi = h & \text{on } \partial D. \end{cases}$$

The following load, which is supported on the whole boundary of Q and in particular is purely vertical on its lateral surface, is admissible:

$$(G_1, G_2) = (\delta_{-1/2} - \delta_{1/2})(x_3)\,\nabla_{x'}\phi(x')\ , \quad G_3 = -h\mathscr{H}^1 \llcorner \partial D\ . \tag{2.27}$$

Indeed, the condition (2.17) is satisfied by taking

$$\Sigma_{\alpha\beta} = 0 \quad \text{and} \quad \Sigma_{\alpha 3} = 1\!\!1_Q(x)\partial_\alpha\phi(x')\ .$$

In this case the average momentum of (G_1, G_2) is given by

$$m_G = \left(\int_D \nabla_{x'}\phi \cdot (-x_2, x_1)\, dx'\right)(\delta_{-1/2} - \delta_{1/2})(x_3)\ . \tag{2.28}$$

An example is represented in Figure 2.3.

Example 2.1.4. *(Component H concentrated on the top and bottom faces $D \times \{\pm 1/2\}$)*
Let $f \in L^2(\mathbb{R}^2; \mathbb{R}^2)$. Then the following load is admissible:

$$H_\alpha = 0\ , \quad H_3 = f(x')(\delta_{-1/2} - \delta_{1/2})(x_3)\ . \tag{2.29}$$

Indeed, condition (2.18) is satisfied by considering

$$\Sigma_{ij} = 0 \ \text{ if } (i,j) \neq (3,3) \quad \text{and} \quad \Sigma_{33} = f(x')\, 1\!\!1_Q\ .$$

The admissible load is vertical and concentrates on the top and bottom faces of the cylinder Q.

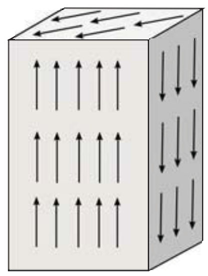


Figure 2.3: *An example of load concentrated on the whole boundary of Q, defined according to Example 2.1.3, with* $D=[0,1]^2$, $h=1$ *on* $\{1\}\times[0,1]$ *and* $[0,1]\times\{1\}$, *while* $h=-1$ *on* $\{0\}\times[0,1]$ *and* $[0,1]\times\{0\}$.

In particular, if we take

$$f(x')=\frac{a_\alpha}{\int_D x_\alpha^2}\,x_\alpha+\frac{b}{|D|}\;, \tag{2.30}$$

with a_α and b arbitrary real constants, we obtain a load F such that

$$\begin{cases} \overline{H_\alpha}=[[x_\alpha H_3]]=a_\alpha\,(\delta_{-1/2}-\delta_{1/2})(x_3)\;, \\ \overline{H_3}=[[F_3]]=b(\delta_{-1/2}-\delta_{1/2})(x_3)\;. \end{cases} \tag{2.31}$$

In Figure 2.4 we show an example of the trivial case in which $a_\alpha=0$.

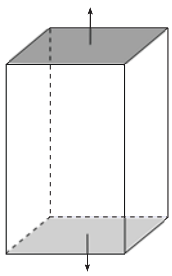


Figure 2.4: *An example of vertical load concentrated on the top and bottom faces of Q, defined according to Example 2.1.4, with* $a_\alpha=0$.

Other admissible loads can be obtained by combining G and H introduced above, provided that the resulting load $G+H$ satisfies assumption (h2).

2.2 The main result

In this Section we present the asymptotical analysis of the family of variational problems $\phi^\delta(k)$ introduced in (2.6), as $\delta\to 0^+$ and the parameter k is kept fix. The proofs are postponed to the next Section.

In order to perform the asymptotics as $\delta\to 0^+$, it is convenient to reformulate the variational problems (2.6) on the fixed domain Q instead of the thin cylinder $Q_\delta=$

$\delta\overline{D} \times I$. This operation corresponds to chose a suitable change of variables for the displacements and a suitable scaling for the loads.

Given an admissible load F and its decomposition $F = G + H$ defined according to (2.17) and (2.18), we consider the following scaling $F^\delta := G^\delta + H^\delta$: for every $x \in Q_\delta$ we set

$$G^\delta(x) := (\delta^{-1}G_1, \delta^{-1}G_2, \delta^{-2}G_3)(\delta^{-1}x', x_3)\,, \tag{2.32}$$

$$H^\delta(x) := (0, 0, \delta^{-1}H_3)(\delta^{-1}x', x_3)\,. \tag{2.33}$$

In what follows we deal with both the components G and H, implicitly meaning that one of the conditions of (h2) holds true. We remark that all the following results are valid also without such assumption.

Further, let us introduce the operator $e^\delta : H^1(Q;\mathbb{R}^3) \to L^2(Q;\mathbb{R}^{3\times 3}_{\text{sym}})$ defined by

$$e^\delta_{\alpha\beta}(u) := \delta^{-2}e_{\alpha\beta}(u)\,,\ \ e^\delta_{\alpha 3}(u) := \delta^{-1}e_{\alpha 3}(u)\,,\ \ e^\delta_{33}(u) := e_{33}(u)\,, \tag{2.34}$$

as it is usual in the literature on $3d-1d$ dimension reduction.

Lemma 2.2.1. *Let G^δ and H^δ be defined as in (2.32)-(2.33), then problem $\phi^\delta(k)$ defined in (2.6) reads*

$$\phi^\delta(k) = \inf\Big\{\mathscr{C}^\delta(\omega) + k|\omega|\ :\ \omega \subset Q\Big\}\,, \tag{2.35}$$

where

$$\mathscr{C}^\delta(\omega) := \sup\Big\{\frac{1}{\delta}\langle G,u\rangle_{\mathbb{R}^3} + \langle H,u\rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\,dx\ :\ u \in H^1(Q;\mathbb{R}^3)\Big\}\,. \tag{2.36}$$

Proof. In order to rewrite the variational problem $\mathscr{C}^\delta$ over the subsets of the fixed domain Q, we need to establish a suitable change of variables for the displacements: given $\tilde{u} \in H^1(Q_\delta;\mathbb{R}^3)$, we can rewrite it as

$$\widetilde{u}(x) = (\delta^{-2}u_1, \delta^{-2}u_2, \delta^{-1}u_3)(\delta^{-1}x', x_3) \qquad \text{in } Q_\delta \tag{2.37}$$

for some $u \in H^1(Q;\mathbb{R}^3)$.

The change of variables induces a 1-1 correspondence between the subsets of Q_δ and the subsets of Q: every $\Omega \subset Q^\delta$ is associated to a unique $\omega \subset Q$ such that

$$\Omega = \{(\delta x', x_3)\ :\ x' \in \omega\}\,, \tag{2.38}$$

e.g. see Figure 2.5.

Exploiting the scalings (2.32) and (2.33) and recalling that j is 2-homogeneous, it is easy to prove that for every $\tilde{u} \in H^1(Q_\delta;\mathbb{R}^3)$ there holds

$$\langle H^\delta, \tilde{u}\rangle_{\mathbb{R}^3} - \int_\Omega j(e(\tilde{u}))\,dx = \frac{1}{\delta}\langle G,u\rangle_{\mathbb{R}^3} + \langle H,u\rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\,dx\,, \tag{2.39}$$

where u associated to $\tilde{u}$ according to (2.37).

Hence we can rewrite the functional $\mathscr{C}^\delta(\Omega)$ as a shape functional posed on a subset of the fixed domain Q:

$$\mathscr{C}^\delta(\Omega) = \sup\left\{\frac{1}{\delta}\langle G,u\rangle_{\mathbb{R}^3} + \langle H,u\rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\,dx \,:\, u\in H^1(Q;\mathbb{R}^3)\right\}\,, \tag{2.40}$$

with $\omega\subset Q$ satisfying (2.38). Since ω is uniquely determined, we denote the expression above as $\mathscr{C}^\delta(\omega)$. Therefore we conclude that problem $\phi^\delta(k)$ defined in (2.6) reads

$$\phi^\delta(k) = \inf\left\{\mathscr{C}^\delta(\omega) + k|\omega| \,:\, \omega\subset Q\right\}\,,$$

since $|\Omega|/|Q_\delta| = |\omega|/|Q|$ and Q is assumed to have volume 1. □

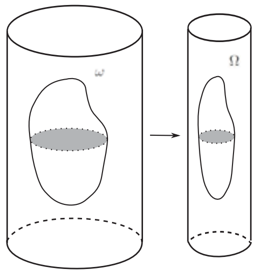


Figure 2.5: *For every $\Omega\subset Q_\delta$ there exists a unique $\omega\subset Q$ such that $\Omega = \{(\delta x', x_3)\,:\, x'\in\omega\}$.*

In order to write the limit problem as $\delta\to 0^+$, we need to introduce the reduced potential $\overline{j}:\mathbb{R}^3\to\mathbb{R}$, presented in (1.35). Let us recall the definition:

$$\overline{j}(y) := \inf_{A\in\mathbb{R}^{2\times 2}} j\begin{pmatrix} & & y_1 \\ & A & y_2 \\ y_1 & y_2 & y_3\end{pmatrix}\,. \tag{2.41}$$

In view of Lemma 1.6.1, there holds

$$\overline{j}(y) = 2\eta\sum_\alpha |y_\alpha|^2 + (Y/2)|z_3|^2\,,$$

where $Y = \eta\,\frac{3\lambda+2\eta}{\lambda+\eta}$ is the Young modulus.

The behavior of the optimal design problem (2.35) in the dimension reduction process, *i.e.* in the limit $\delta\to 0^+$, is described by the following Theorem.

Theorem 2.2.1. *For every $k\in\mathbb{R}$, as $\delta\to 0^+$ the sequence $\phi^\delta(k)$ in (2.35) converges to the following limit:*

$$\phi(k) := \inf\left\{\mathscr{C}^{lim}(\theta) + k\int_Q \theta\,dx \,:\, \theta\in L^\infty(Q;[0,1])\right\}\,, \tag{2.42}$$

where

$$\mathscr{C}^{lim}(\theta) := \sup\left\{\langle G,v\rangle_{\mathbb{R}^3} + \langle H,u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v), e_{23}(v), e_{33}(u))\,\theta\,dx : v\in TW,\, u\in BN\right\}\,. \tag{2.43}$$

Moreover, if ω^δ *is a sequence of minima for* $\phi^\delta(k)$ *then, up to subsequences,* $1\!\!1_{\omega^\delta} \overset{*}{\rightharpoonup} \overline{\theta}$ *and* $\overline{\theta}$ *is optimal for* $\phi(k)$*, i.e.* $\phi(k) = \mathscr{C}^{lim}(\overline{\theta}) + k\int \overline{\theta}\,dx$.

The next Section is entirely devoted to the proof of Theorem 2.2.1. It is based on the idea of considering the "fictitious counterpart" of problem (2.35), namely

$$\widetilde{\phi}^\delta(k) := \inf\left\{ \widetilde{\mathscr{C}}^\delta(\theta) + k\int_Q \theta\,dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\}, \tag{2.44}$$

where $\widetilde{\mathscr{C}}^\delta(\theta)$ denotes the natural extension of the compliance $\mathscr{C}^\delta(\omega)$ to $L^\infty(Q;[0,1])$:

$$\widetilde{\mathscr{C}}^\delta(\theta) := \sup\left\{ \frac{1}{\delta}\langle G,u\rangle_{\mathbb{R}^3} + \langle H,u\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u))\,\theta\,dx \ : \ u \in H^1(Q;\mathbb{R}^3) \right\}. \tag{2.45}$$

Indeed, it is well known that the variational problem (2.36), and hence (2.35), is in general ill-posed, due to homogenization phenomena that prevent the existence of minimizers, so that we need to enlarge the class of admissible materials, passing from "real" materials, represented by characteristic functions, to "composite" materials, represented by densities with values in $[0,1]$.

2.3 The proof of the main result

This Section is devoted to the proof of Theorem 2.2.1. For sake of clearness, we divide the proof in three parts. In Part I we establish some delicate compactness properties which are preliminary to Part II, where we show that the sequence $\widetilde{\phi}^\delta(k)$ converges to the limit problem $\phi(k)$ given by (2.42). We conclude by showing in Part III that the sequences $\phi^\delta(k)$ and $\widetilde{\phi}^\delta(k)$ have the same asymptotics.

2.3.1 Part I: compactness

We begin with a key lemma that enlightens the role of conditions (h1), (2.17) and (2.18), that we required for the admissibility of the load $F = G + H$.

Lemma 2.3.1. *Let* $\theta \in L^\infty(Q;[0,1])$ *such that* $\inf_Q \theta > 0$*, and let* $F = G + H$ *be an admissible load. Assume that* $u^\delta \in C^\infty(Q;\mathbb{R}^3)$ *is a sequence satisfying*

$$\inf_\delta \left\{ \frac{1}{\delta}\langle G,u^\delta\rangle_{\mathbb{R}^3} + \langle H,u^\delta\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u^\delta))\,\theta\,dx \right\} > -\infty, \tag{2.46}$$

then $e^\delta(u^\delta)$ *is bounded in* $L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$.

Proof. Since $F = G + H$ is an admissible load, G and H satisfy (2.17) and (2.18). Then, letting $\Sigma \in L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ be associated to F as in (h1), the following estimates hold true:

$$\frac{1}{\delta}\langle G,u^\delta\rangle_{\mathbb{R}^3} = -\delta\,\langle \Sigma_{\alpha\beta}, e^\delta_{\alpha\beta}(u^\delta)\rangle_{\mathbb{R}^3} - 2\langle \Sigma_{\alpha 3}, e^\delta_{\alpha 3}(u^\delta)\rangle_{\mathbb{R}^3} \le C_1 \|e^\delta(u^\delta)\|_{L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})},$$
$$\langle H,u^\delta\rangle_{\mathbb{R}^3} = -\langle \Sigma_{33}, e_{33}(u^\delta)\rangle_{\mathbb{R}^3} = -\langle \Sigma_{33}, e^\delta_{33}(u^\delta)\rangle_{\mathbb{R}^3} \le C_2 \|e^\delta(u^\delta)\|_{L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})},$$

for some positive constants C_1 and C_2.

On the other hand, since j is coercive and by assumption $\inf_Q \theta > 0$ we infer

$$\int_Q j(e^\delta(u^\delta))\theta\,dx \geq C_3\|e^\delta(u^\delta)\|^2_{L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})} ,$$

for some positive constant C_3. Finally, exploiting the hypothesis that the infimum (2.46) is a finite constant C_4, and combining the estimates found, we conclude that

$$\begin{aligned} C_3\|e^\delta(u^\delta)\|^2_{L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym}} &\leq \int_Q j(e^\delta(u^\delta))\theta\,dx \leq \frac{1}{\delta}\langle G,u^\delta\rangle_{\mathbb{R}^3} + \langle H,u^\delta\rangle_{\mathbb{R}^3} - C_4 \leq \\ &\leq (C_1+C_2)\|e^\delta(u^\delta)\|_{L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})} - C_4 \,. \end{aligned}$$

Hence $\|e^\delta(u^\delta)\|_{L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})}$ is uniformly bounded. □

In view of Lemma 2.3.1, we are led to establish compactness properties for sequences u^δ such that the L^2-norm of $e^\delta(u^\delta)$ is uniformly bounded.

Before stating these compactness properties, which are summarized in the next proposition, we need to introduce a shape potential ψ_D associated to the section D, defined as the unique solution of

$$-\Delta\psi_D = 2 \quad , \quad \psi_D \in H^1_0(D).$$

Some properties of this function, well known in classical torsion theory, are recalled in Lemma 2.3.3.

Lemma 2.3.2. *Let $u^\delta \in C^\infty(Q;\mathbb{R}^3)$ be a sequence such that*

$$\int_Q u^\delta\,dx = \int_Q \psi_D \,\mathrm{curl}\, u^\delta\,dx = 0 \qquad \forall\delta \,. \tag{2.47}$$

If $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$, then, up to subsequences,

(i) there exists $\overline{u} \in BN(Q)$ such that $\lim_{\delta\to 0} u^\delta = \overline{u}$ weakly in $L^2(Q;\mathbb{R}^3)$, moreover $\overline{u}$ is of the form (2.8);

(ii) setting

$$\begin{aligned} v^\delta_\alpha &:= \delta^{-1}(u^\delta-\overline{u})_\alpha - \delta^{-1}|D|^{-1}[[u^\delta-\overline{u}]]_\alpha \\ v^\delta_3 &:= \delta^{-1}(u^\delta-\overline{u})_3 - \delta^{-1}|D|^{-1}\Big([[u^\delta-\overline{u}]]_3 - x_\alpha[[u^\delta-\overline{u}]]'_\alpha\Big) \,, \end{aligned}$$

there exist $c \in H^1_m(I)$ and $w \in L^2(I;H^1_m(D))$ such that

$$\begin{aligned} &\lim_{\delta\to 0}(v^\delta_1,v^\delta_2) = c(x_3)(-x_2,x_1) \text{ weakly in } L^2(Q;\mathbb{R}^2) \\ &\lim_{\delta\to 0} v^\delta_3 = w \text{ weakly in } H^{-1}(I;L^2(D)) \,; \end{aligned}$$

(iii) the weak limits $\chi_i := \lim_{\delta\to 0} e^\delta_{i3}(u^\delta)$ in $L^2(Q)$ are given by

$$\begin{aligned} (\chi_1,\chi_2) &= \tfrac{1}{2}\,(c'(x_3)(-x_2,x_1) + \nabla_{x'}w) \\ \chi_3 &= e_{33}(\overline{u}) \,. \end{aligned}$$

For the proof of Proposition 2.3.2 we need some preliminary lemmas.

Lemma 2.3.3. *The potential ψ_D is positive in D. Moreover, setting*

$$\gamma := \int_D |\nabla \psi_D|^2 \, dx' = 2 \int_D \psi_D \, dx' \ , \tag{2.48}$$

there hold

$$\inf \Big\{ \int_D |\nabla \psi|^2 \, dx' \ : \ \psi \in \mathscr{C}_0^\infty(D) \ , \ \int_D \psi \, dx' = 1 \Big\} = 4\gamma^{-1} \tag{2.49}$$

and

$$\inf \Big\{ \int_D |(-x_2, x_1) + \nabla w|^2 \, dx' \ : \ w \in H^1(D) \Big\} = \gamma \, . \tag{2.50}$$

Proof. See Appendix.

Lemma 2.3.4. *Let u^δ be a sequence in $\mathscr{C}^\infty(Q;\mathbb{R}^3)$ with $e^\delta(u^\delta)$ bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ and such that, for every δ, it holds:*

$$\int_Q \psi_D \, \mathrm{curl}_{x'}(u_1^\delta, u_2^\delta) \, dx = 0 \, . \tag{2.51}$$

Then the sequence

$$c^\delta(x_3) := \frac{1}{2\delta} \int_D \big(\nabla \psi_D \wedge (u_1^\delta, u_2^\delta) \big) \, dx' \ , \tag{2.52}$$

turns out to be bounded in $H^1(I)$.

Proof. See Appendix.

We can now give the

Proof of Proposition 2.3.2

For convenience, the proof is divided into several steps.

Step 1. The sequence $\int_Q \mathrm{curl}\, u^\delta \, dx$ is bounded in $\mathbb{R}^3$.

A version of the Korn inequality (see (28) in [86]) states that the skew symmetric part $\nabla^a u$ of the gradient satisfies

$$\int_Q \Big| \nabla^a u - \Big(\frac{1}{|Q|} \int \nabla^a u \Big) \Big|^2 dx \le C \int_Q |e(u)|^2 \, dx \qquad \forall u \in H^1(Q;\mathbb{R}^3) \, . \tag{2.53}$$

We apply such inequality to

$$\tilde{u}^\delta := u^\delta - \frac{1}{2} b^\delta \wedge x, \ \text{ with } b^\delta := \frac{1}{|Q|} \int_Q \mathrm{curl}\, u^\delta \, dx.$$

By definition $\int_Q \mathrm{curl}\, \tilde{u}^\delta \, dx = 0$ and $e(\tilde{u}^\delta) = e(u^\delta)$, moreover by assumption $e(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$, then by (2.53) we deduce that

$$\| \mathrm{curl}\, \tilde{u}^\delta \|_{L^2(Q;\mathbb{R}^3)} \le C \, . \tag{2.54}$$

We now exploit the hypothesis (2.47): since $\operatorname{curl} u^\delta = \operatorname{curl} \tilde{u}^\delta + b^\delta$, for every δ we have

$$\int_Q \psi_D \operatorname{curl} \tilde{u}^\delta \, dx + b^\delta \int_D \psi_D \, dx' = 0 \,,$$

that is, recalling the definition (2.48) of γ,

$$\frac{\gamma}{2} b^\delta = -\int_Q \psi_D \operatorname{curl} \tilde{u}^\delta \, dx \,.$$

Thanks to (2.54) the right hand side is bounded , then we conclude that b^δ is bounded in $\mathbb{R}^3$.

Step 2. The sequence u^δ is bounded in $H^1(Q;\mathbb{R}^3)$ and any weak limit belongs to $BN(Q)$.

Applying the Korn inequality (2.53) to the sequence u^δ and taking into account that $\int_Q \operatorname{curl} u^\delta \, dx$ is bounded as shown in Step 1, we deduce that the L^2-norm of ∇u^δ remains bounded. Since we also know that $\int_Q u^\delta \, dx = 0$, the Poincaré-Wirtinger inequality ensures that the sequence u^δ is bounded, and hence weakly precompact, in $H^1(Q;\mathbb{R}^3)$. Again by the L^2-boundedness of $e^\delta(u^\delta)$, any weak L^2-limit $\overline{u}$ of u^δ satisfies $e_{ij}(\overline{u}) = 0$ for all $(i,j) \neq (3,3)$, and hence it belongs to $BN(Q)$. Moreover, we observe that the two integral conditions (2.47) hold also for the limit $\overline{u}$, then one can easily deduce that the Bernoulli-Navier field $\overline{u}$ is of the form (2.8).

Finally, taking the weak L^2-limit of the sequence $e_{33}(u^\delta)$, one obtains immediately that χ_3 it agrees with $e_{33}(\overline{u})$.

Step 3. The sequence v_α^δ is bounded in $L^2(Q;\mathbb{R}^2)$.

Let us apply Lemma 1.5.2 to the sequence $v_\alpha^\delta(\cdot,x_3)$ for fixed x_3 (notice that $v_\alpha^\delta(\cdot,x_3)$ is indeed in $H^1_m(D;\mathbb{R}^2)$). By taking into account that $e_{\alpha\beta}(\overline{u}) = 0$ and $\int_D (\nabla\psi_D \wedge (\overline{u}_1,\overline{u}_2)) \, dx' = 0$ (since $\overline{u}$ is of the form (2.8)), we deduce

$$\int_D |(v_1^\delta, v_2^\delta)|^2 \, dx' \leq C\Big[\frac{1}{\delta^2}\int_D |e_{\alpha\beta}(u^\delta)|^2 \, dx' + \Big|\frac{1}{\delta}\int_D \big(\nabla\psi_D \wedge (u_1^\delta,u_2^\delta)\big) \, dx'\Big|^2\Big] \text{ for a.e. } x_3 \in I \,.$$

Then, integrating with respect to x_3 over I, we get

$$\int_Q |(v_1^\delta, v_2^\delta)|^2 \, dx \leq C\Big[\delta^2 \int_Q |e^\delta_{\alpha\beta}(u^\delta)|^2 \, dx + \int_I |2c^\delta(x_3)|^2 \, dx_3\Big] \,,$$

where the sequence c^δ is associated to the sequence u^δ according to formula (2.52). Since the sequence u^δ satisfies $e^\delta(u^\delta)$ bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\text{sym}})$ and condition (2.51), Lemma 2.3.4 allows to conclude that v_α^δ is bounded in $L^2(Q;\mathbb{R}^2)$.

Step 4. Any weak limit of (v_1^δ, v_2^δ) is of the form $c(x_3)(-x_2,x_1)$, for some $c \in L^2_m(I)$.

Since $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\text{sym}})$, there exists a positive constant C such that $\|e_{\alpha\beta}(v^\delta)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\text{sym}})} \leq C\delta$. Therefore any weak limit $\overline{v} = (\overline{v}_1,\overline{v}_2)$ satisfies $e_{\alpha\beta}(\overline{v}) = 0$, and hence it is of the form $(\overline{v}_1,\overline{v}_2) = c(x_3)(-x_2,x_1) + (d_1(x_3),d_2(x_3))$ for some c and

d_α in $L^2(I)$. Since by their definition v_α^δ satisfy $[[v_\alpha^\delta]] = 0$, we have also $[[\overline{v}_\alpha]] = 0$, so that $d_\alpha = 0$. It remains to prove that c has zero integral mean. Set

$$\omega^\delta := \frac{1}{2\delta}\big(\partial_1 u_2^\delta - \partial_2 u_1^\delta\big) = \frac{1}{2}\big(\partial_1 v_2^\delta - \partial_2 v_1^\delta\big)\ .$$

We observe that, since by assumption $\int_Q \psi_D \operatorname{curl} u^\delta\, dx = 0$, the functions ω^δ satisfy

$$\int_Q \psi_D\, \omega^\delta\, dx = 0 \qquad \forall \delta\ . \tag{2.55}$$

Since $\lim_{\delta\to 0} \omega^\delta = c(x_3)$ in $\mathscr{D}'(Q)$, and since by definition the sequence ω^δ remains bounded in $L^2(I; H^{-1}(D))$, we have also $\lim_{\delta\to 0}\omega^\delta = c$ weakly in $L^2(I; H^{-1}(D))$. In particular, taking as a test function ψ_D, passing to the limit as $\delta \to 0$ in (2.55), we obtain $\int_I c(x_3)\, dx_3 = 0$.

Step 5. The distributional derivative of c is given by $c'(x_3) = \partial_1\chi_2 - \partial_2\chi_1$.
Since (v_1^δ, v_2^δ) converges to $(\overline{v}_1, \overline{v}_2)$ weakly in $L^2(Q;\mathbb{R}^2)$, it holds

$$\lim_{\delta\to 0} \partial_3\omega^\delta = \partial_3\big[\tfrac{1}{2}\big(\partial_1\overline{v}_2 - \partial_2\overline{v}_1\big)\big] = c'(x_3) \qquad \text{in } \mathscr{D}'(Q)\ .$$

On the other hand, since

$$\partial_3\omega^\delta = \frac{1}{\delta}\big[\partial_1 e_{23}(u^\delta) - \partial_2 e_{13}(u^\delta)\big] - \frac{1}{2\delta}\big(\partial_1\partial_2 u_3^\delta - \partial_2\partial_1 u_3^\delta\big) = \partial_1 e_{23}^\delta(u^\delta) - \partial_2 e_{13}^\delta(u^\delta)\ ,$$

it also holds

$$\lim_{\delta\to 0} \partial_3\omega^\delta = \partial_1\chi_2 - \partial_2\chi_1 \qquad \text{in } \mathscr{D}'(Q)\ .$$

It follows that $\partial_1\chi_2 - \partial_2\chi_1 = c'(x_3)$ in $\mathscr{D}'(Q)$.

Step 6. The function c belongs to $H^1_m(I)$.
Let us fix $\varphi \in \mathscr{C}_0^\infty(I)$, and $\psi \in \mathscr{C}_0^\infty(D)$ with $\int_D \psi\, dx' = 1$. We have

$$\langle \partial_1\chi_2 - \partial_2\chi_1, \varphi(x_3)\psi(x')\rangle_{\mathbb{R}^3} = \int_Q (\chi_1\partial_2\psi - \chi_2\partial_1\psi)\varphi\, dx \le \frac{1}{2}\Big(\int_Q |\chi|^2\, dx + \int_D |\nabla\psi|^2\, dx' \int_I \varphi^2\, dx_3\Big)\ . \tag{2.56}$$

By Step 5, we know that

$$\langle \partial_1\chi_2 - \partial_2\chi_1, \varphi(x_3)\psi(x')\rangle_{\mathbb{R}^3} = \int_I c'(x_3)\varphi(x_3)\, dx_3\ , \tag{2.57}$$

Combining (2.56) and (2.57), we obtain

$$\int_I c'(x_3)\varphi(x_3)\, dx_3 - \frac{1}{2}\int_D |\nabla\psi|^2\, dx' \int_I \varphi^2\, dx_3 \le \frac{1}{2}\int_Q |\chi|^2\, dx\ .$$

By the Fenchel inequality, this implies

$$\int_I |c'(x_3)|^2\, dx_3 \le \Big(\int_D |\nabla\psi|^2(x')\, dx'\Big)\Big(\int_Q |\chi|^2\, dx\Big)\ .$$

Passing to the infimum over all the functions ψ in $\mathscr{C}_0^\infty(D)$ with $\int_D \psi\,dx' = 1$, and applying (2.49) in Lemma 2.3.3, we obtain

$$\int_I |c'(x_3)|^2\,dx_3 \le 4\gamma^{-1}\int_Q |\chi|^2\,dx\,,$$

where γ is the positive constant defined in (2.48).

Step 7. The sequence v_3^δ converges weakly in $H^{-1}(I;L^2(D))$ to some limit w.

A partial Korn's inequality proved in [77] states that, for any $z \in H^1(Q;\mathbb{R}^3)$, it holds

$$\left\|z_3 - |D|^{-1}\left([[z_3]] - x_\alpha\,[[z_\alpha]]'\right)\right\|_{H^{-1}(I;L^2(D))} \le C\left(\|e_{\alpha\beta}(z)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})} + \|e_{\alpha 3}(z)\|_{L^2(Q;\mathbb{R}^2)}\right)\,.$$

Applying this inequality to the sequence $z^\delta := \delta^{-1}(u^\delta - \overline{u})$, since by assumption $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$ and $\overline{u} \in BN(Q)$, we deduce that v_3^δ is bounded in $H^{-1}(I;L^2(D))$. Therefore there exists w such that $\lim\limits_{\delta\to 0} v_3^\delta = w$ weakly in $H^{-1}(I;L^2(D))$. Notice that, since $\mathscr{D}(Q) \subset H_0^1(I;L^2(D))$, the convergence holds also in $\mathscr{D}'(Q)$.

Step 8. It holds $(\chi_1,\chi_2) = \frac{1}{2}\left(c'(x_3)(-x_2,x_1) + \nabla_{x'}w\right)$ in $L^2(Q;\mathbb{R}^2)$ and $w \in L^2(I;H_m^1(D))$.

Since

$$\begin{aligned} u_\alpha^\delta &= \overline{u}_\alpha + \delta v_\alpha^\delta + |D|^{-1}[[u^\delta - \overline{u}]]_\alpha \\ u_3^\delta &= \overline{u}_3 + \delta v_3^\delta + |D|^{-1}\left([[u^\delta - \overline{u}]]_3 - x_\alpha[[u^\delta - \overline{u}]]'_\alpha\right)\,, \end{aligned}$$

we have $e_{\alpha 3}^\delta(u^\delta) = e_{\alpha 3}(v^\delta)$. We know by assumption that $\lim_{\delta\to 0} e_{\alpha 3}^\delta(u^\delta) = \chi_\alpha$ weakly in $L^2(Q)$, and by Steps 4 and 7 that $\lim_{\delta\to 0}(e_{13}(v^\delta), e_{23}(v^\delta)) = \frac{1}{2}\left(c'(x_3)(-x_2,x_1) + \nabla_{x'}w\right)$ in $\mathscr{D}'(Q;\mathbb{R}^2)$. We infer that the equality $(\chi_1,\chi_2) = \frac{1}{2}\left(c'(x_3)(-x_2,x_1) + \nabla_{x'}w\right)$ holds in $\mathscr{D}'(Q;\mathbb{R}^2)$. This implies that $\nabla_{x'}w \in L^2(Q;\mathbb{R}^2)$ (because $\chi_\alpha \in L^2(Q)$ and by Step 6 $c' \in L^2(I)$), and that the same equality remains true in $L^2(Q;\mathbb{R}^2)$.

Finally we notice that by construction $[[v_3^\delta]] = 0$ for each δ, so that also $[[w]] = 0$. Therefore, applying Poincaré-Wirtinger inequality section by section we infer that $w \in L^2(I;H_m^1(D))$. □

2.3.2 Part II: asymptotics of fictitious problems

Theorem 2.3.1. *As $\delta \to 0^+$, the sequence $\widetilde{\mathscr{C}}^\delta$ defined in (2.45) Γ-converges, with respect to the weak * topology of $L^\infty(Q;[0,1])$, to the limit compliance $\mathscr{C}^{lim}$ defined in (2.43). In particular, for every fixed $k \in \mathbb{R}$, the sequence $\widetilde{\phi}^\delta(k)$ defined in (2.44) tends to the limit problem $\phi(k)$ given by (2.42), as $\delta \to 0^+$.*

Proof. By definition of Γ-convergence, the statement means that the so-called Γ-liminf and Γ-limsup inequalities hold:

$$\inf\left\{\liminf \widetilde{\mathscr{C}}^\delta(\theta^\delta)\ :\ \theta^\delta \overset{*}{\rightharpoonup} \theta\right\} \ge \mathscr{C}^{lim}(\theta) \tag{2.58}$$

$$\inf\left\{\limsup \widetilde{\mathscr{C}}^\delta(\theta^\delta)\ :\ \theta^\delta \overset{*}{\rightharpoonup} \theta\right\} \le \mathscr{C}^{lim}(\theta)\,. \tag{2.59}$$

Proof of (2.58). Consider an arbitrary sequence $\theta^\delta \overset{*}{\rightharpoonup} \theta$. Let $(v_k, u_k) \in TW(Q) \times BN(Q)$ be a sequence such that

$$\mathscr{C}^{lim}(\theta) = \lim_k \left\{ \langle G, v_k \rangle_{\mathbb{R}^3} + \langle H, u_k \rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k))\, \theta\, dx \right\}.$$

If we find a sequence $u_k^\delta \in H^1(Q;\mathbb{R}^3)$ such that, for every fixed k and as $\delta \to 0^+$,

$$\frac{1}{\delta} \langle G, u_k^\delta \rangle_{\mathbb{R}^3} \longrightarrow \langle G, v_k \rangle_{\mathbb{R}^3}, \quad \langle H, u_k^\delta \rangle_{\mathbb{R}^3} \longrightarrow \langle H, u_k \rangle_{\mathbb{R}^3}, \tag{2.60}$$

$$j(e^\delta(u_k^\delta)) \overset{L^1}{\longrightarrow} \overline{j}(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k)), \tag{2.61}$$

we have finished: indeed in this case there holds

$$\int_Q j(e^\delta(u_k^\delta))\theta^\delta\, dx \longrightarrow \int_Q \overline{j}(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k))\theta\, dx \quad \text{as } \delta \to 0^+$$

since it is the product of a sequence that converges strongly in L^1 and a sequence that converges weakly * in L^∞, hence we conclude that

$$C^{lim}(\theta) = \lim_k \lim_\delta \left\{ \frac{1}{\delta} \langle G, u_k^\delta \rangle_{\mathbb{R}^3} + \langle H, u_k^\delta \rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u_k^\delta))\, \theta^\delta\, dx \right\} \leq \liminf_\delta \widetilde{\mathscr{C}}^\delta(\theta^\delta).$$

We now build a suitable sequence u_k^δ, in terms of u_k and v_k. Set

$$u_k^\delta := u_k + \delta v_k + \delta^2 w_k,$$

with

$$w_k(x) := \begin{cases} C\,[x_1 \zeta'_{k,3}(x_3) + \frac{x_2^2 - x_1^2}{2} \quad \zeta''_{k,1}(x_3) - x_1 x_2 \zeta''_{k,2}(x_3)] \\ C\,[x_2 \zeta'_{k,3}(x_3) + \frac{x_1^2 - x_2^2}{2} \quad \zeta''_{k,2}(x_3) - x_1 x_2 \zeta''_{k,1}(x_3)] \\ 0 \end{cases},$$

where $C := -\frac{\lambda}{2(\lambda+\eta)}$ and $\zeta_{k,i}$ are associated to u_k according to (2.8).

For a fixed k, it is easy to prove that (2.60) holds true. We now pass to the property (2.61): since $v_k \in TW(Q)$, $u_k \in BN(Q)$ we obtain that

$$e^\delta(u_k^\delta) = \begin{pmatrix} e_{\alpha\beta}(w_k) & e_{\alpha 3}(v_k) + \delta e_{\alpha 3}(w_k) \\ e_{\alpha 3}(v_k) + \delta e_{\alpha 3}(w_k) & e_{33}(u_k) + \delta e_{33}(v_k) + \delta^2 e_{33}(w_k) \end{pmatrix},$$

moreover a direct computation gives

$$e_{\alpha\beta}(w_k) = \begin{pmatrix} Ce_{33}(u_k) & 0 \\ 0 & Ce_{33}(u_k) \end{pmatrix}.$$

Keeping k fixed and passing to the limit as $\delta \to 0$, we obtain

$$e^\delta(u_k^\delta) \to \begin{pmatrix} Ce_{33}(u_k) & 0 & e_{13}(v_k) \\ 0 & Ce_{33}(u_k) & e_{23}(v_k) \\ e_{13}(v_k) & e_{23}(v_k) & e_{33}(u_k) \end{pmatrix} \qquad \text{a.e. in } Q,$$

and then, by dominated convergence,

$$j(e^\delta(u_k^\delta)) \xrightarrow{L^1} j\begin{pmatrix} Ce_{33}(u_k) & 0 & e_{13}(v_k) \\ 0 & Ce_{33}(u_k) & e_{23}(v_k) \\ e_{13}(v_k) & e_{23}(v_k) & e_{33}(u_k) \end{pmatrix} . \tag{2.62}$$

Writing explicitly the limit in (2.62) we conclude the proof of (2.61): by definition of j, the limit equals

$$j\begin{pmatrix} 0 & 0 & e_{13}(v_k) \\ 0 & 0 & e_{23}(v_k) \\ e_{13}(v_k) & e_{23}(v_k) & 0 \end{pmatrix} + j\begin{pmatrix} Ce_{33}(u_k) & 0 & 0 \\ 0 & Ce_{33}(u_k) & 0 \\ 0 & 0 & e_{33}(u_k) \end{pmatrix}$$

$$= 2\eta(e_{\alpha 3}(v_k))^2 + \left[\frac{\lambda}{2}(2C+1)^2 + \eta(2C^2+1)\right](e_{33}(u_k))^2 = \overline{j}(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k)) ,$$

where the last equality follows from the choice of the constant C.

Proof of (2.59). For every fixed $\theta \in L^\infty(Q;[0,1])$, we have to find a recovery sequence $\theta^\delta \overset{*}{\rightharpoonup} \theta$ such that $\limsup_\delta \widetilde{\mathscr{C}}^\delta(\theta^\delta) \le \mathscr{C}^{lim}(\theta)$. Let us first show that, under the assumption $\inf_Q \theta > 0$, we are done simply by taking $\theta^\delta \equiv \theta$. Let u^δ be a sequence of functions satisfying

$$\limsup_\delta \widetilde{\mathscr{C}}^\delta(\theta) = \lim_{\delta\to 0}\left\{\frac{1}{\delta}\langle G, u^\delta\rangle_{\mathbb{R}^3} + \langle H, u^\delta\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u^\delta))\,\theta\,dx\right\} . \tag{2.63}$$

Since we may assume with no loss of generality that $\limsup_\delta \widetilde{\mathscr{C}}^\delta(\theta^\delta) > -\infty$, and since by assumption θ is bounded from below, we are in a position to apply Lemma 2.3.1 and deduce that the sequence $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$. Hence, up to subtracting from u^δ the rigid motion $a^\delta + b^\delta \wedge x$, with

$$a^\delta := \frac{1}{|Q|}\int_Q u^\delta\,dx , \qquad b^\delta := \frac{1}{2|Q|}\int_Q \psi_D \,\mathrm{curl}\, u^\delta\,dx ,$$

the sequence u^δ satisfies the hypotheses of Lemma 2.3.2.

Let c, w and $\overline{u}$ be associated to the sequence u^δ as in Lemma 2.3.2, and let $v := (-c(x_3)x_2, c(x_3)x_1, w) \in TW(Q)$. Then by applying the property (iii) of Lemma 2.3.2 we infer

$$\begin{aligned}\lim_{\delta\to 0}\frac{1}{\delta}\langle G, u^\delta\rangle_{\mathbb{R}^3} &= -\lim_{\delta\to 0}\delta\langle \Sigma_{\alpha\beta}, e^\delta_{\alpha\beta}(u^\delta)\rangle_{\mathbb{R}^3} - 2\lim_{\delta\to 0}\langle \Sigma_{\alpha 3}, e^\delta_{\alpha 3}(u^\delta)\rangle_{\mathbb{R}^3} \\ &= -2\langle \Sigma_{\alpha 3}, e_{\alpha 3}(v)\rangle_{\mathbb{R}^3} = \langle G, v\rangle_{\mathbb{R}^3} , &(2.64)\\ \lim_{\delta\to 0}\langle H, u^\delta\rangle_{\mathbb{R}^3} &= -\lim_{\delta\to 0}\langle \Sigma_{33}, e^\delta_{33}(u^\delta)\rangle_{\mathbb{R}^3} = -\langle \Sigma_{33}, e_{33}(\overline{u})\rangle_{\mathbb{R}^3} = \langle H, \overline{u}\rangle_{\mathbb{R}^3} . &(2.65)\end{aligned}$$

We now turn attention to estimate from below $\int_Q j(e^\delta(u^\delta))\,\theta\,dx$. We claim that

$$\liminf_{\delta\to 0}\int_Q j(e^\delta(u^\delta))\,\theta\,dx \ge \int_Q \overline{j}(e_{13}(v), e_{23}(v), e_{33}(\overline{u}))\,\theta\,dx . \tag{2.66}$$

Indeed, for every $\xi \in \mathbb{R}^3$, let us denote by $E_0\xi$ the symmetric matrix

$$E_0\xi := \frac{1}{2}\sum_{i=1}^{3} \xi_i(e_i \otimes e_3 + e_3 \otimes e_i) \,. \tag{2.67}$$

The Fenchel inequality and the weak L^2-convergence of $e^\delta_{i3}(u^\delta)$ expressed in (iii) of Lemma 2.3.2 yield

$$\begin{aligned}\liminf_\delta \int_Q j(e^\delta(u^\delta))\theta\, dx \; &\geq \liminf_\delta \Big\{ \int_Q e^\delta(u^\delta) : E_0\xi\, \theta\, dx - \int_Q j^*(E_0\xi)\,\theta\, dx \Big\} \\ &= \int_Q (e_{13}(v), e_{23}(v), e_{33}(\overline{u})) \cdot \xi\, \theta\, dx - \int_Q j^*(E_0\xi)\,\theta\, dx\end{aligned}$$

for every $\xi \in L^2(Q;\mathbb{R}^3)$.

By using the definition of $\overline{j}$, one can easily check (see Lemma 1.6.1) that

$$\overline{j}^*(\xi) = j^*(E_0\xi) \qquad \forall \xi \in \mathbb{R}^3 \,. \tag{2.68}$$

Such identity and the arbitrariness of $\xi \in L^2(Q;\mathbb{R}^3)$ in the previous inequality yield

$$\liminf_\delta \int_Q j(e^\delta(u^\delta))\,\theta\, dx \geq \sup_\xi \Big\{ \int_Q (e_{13}(v), e_{23}(v), e_{33}(\overline{u})) \cdot \xi\, \theta\, dx - \int_Q \overline{j}^*(\xi)\,\theta\, dx \Big\} \,.$$

By passing to the supremum over $\xi \in L^2(Q;\mathbb{R}^3)$ under the sign of integral (see *e.g.* [16, Lemma A.2]), and taking into account that $\overline{j} = \overline{j}^{**}$, we get the required inequality (2.66).

From (2.63), (2.64), (2.65) and (2.66), recalling the expression (2.43) of $\mathscr{C}^{lim}(\theta)$, it follows that $\limsup_\delta \widetilde{\mathscr{C}}^\delta(\theta^\delta) \leq \mathscr{C}^{lim}(\theta)$. It remains to get rid of the additional assumption $\inf_Q \theta > 0$. This can be done via a standard density argument. Indeed, for any θ we may find a sequence θ^h with $\inf_Q \theta^h > 0$ such that $\theta^h \stackrel{*}{\rightharpoonup} \theta$. Then, since the left hand side of (2.59) (usually called $\Gamma - \limsup \widetilde{\mathscr{C}}^\delta(\theta)$), is weakly * lower semicontinuous, and $\mathscr{C}^{lim}(\theta)$ is weakly * continuous, we obtain

$$(\Gamma - \limsup_\delta \widetilde{\mathscr{C}}^\delta)(\theta) \leq \liminf_h (\Gamma - \limsup_\delta \widetilde{\mathscr{C}}^\delta)(\theta^h) \leq \lim_h \mathscr{C}^{lim}(\theta^h) = \mathscr{C}^{lim}(\theta) \,.$$

The convergence of $\widetilde{\phi}^\delta(k)$ to $\phi(k)$ follows immediately by well-known properties of Γ-convergence.

□

2.3.3 Part III: back to the initial problems

In order to obtain the asymptotics of the original problem $\phi^\delta(k)$ defined in (2.35), we will bound them both from above and below in terms of suitable fictitious problems which admit the same limit. This technique has been used in [17, Section 3] and for the benefit of the reader we recall the main steps.

An important role in the proof is played by the *modified stored energy density* j_0 : $\mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$, introduced in (1.36). Let us recall the definition:

$$j_0(z) := \sup\{z\cdot\xi - j^*(\xi)\ :\ \xi \in \mathbb{R}^{3\times 3}_{\text{sym}}\ ,\ \det(\xi) = 0\}\ . \tag{2.69}$$

Heuristically, the condition $\det\xi = 0$ appearing in (2.69) corresponds to the degeneracy of stress tensors occurring when the material concentrates on low-dimensional sets (see [13, 15, 19] for more details, and also [2] for the explicit computation of j_0^*).

The main properties of j_0 are summarized in Lemma 1.6.2.

Proof of Theorem 2.2.1

Step 1: upper and lower bounds for $\phi^\delta(k)$

We first remark that, for every k, there holds

$$\phi^\delta(k) = \inf\left\{\overline{\mathscr{C}}^\delta(\theta) + k\int_Q \theta\ :\ \theta \in L^\infty(Q;[0,1])\right\}\ ,$$

$\overline{\mathscr{C}}^\delta$ being the lower semicontinuous envelope, in the weak * topology of $L^\infty(Q;[0,1])$, of the functional which is defined as in (2.36) if θ is the characteristic function of a set ω, and $+\infty$ otherwise. Then, by the weak * lower semicontinuity of the fictitious compliance defined in (2.45), we immediately obtain the inequality

$$\widetilde{\mathscr{C}}^\delta(\theta) \le \overline{\mathscr{C}}^\delta(\theta) \quad \forall\theta \in L^\infty(Q;[0,1])\ ,$$

and hence the following lower bound for $\phi^\delta(k)$:

$$\widetilde{\phi}^\delta(k) \le \phi^\delta(k)\ . \tag{2.70}$$

On the other hand, let us introduce another sequence of fictitious problems:

$$\widetilde{\phi}_0^\delta(k) := \inf\left\{\widetilde{\mathscr{C}}_0^\delta(\theta) + k\int_Q \theta\,dx\ :\ \theta \in L^\infty(Q;[0,1])\right\}\ . \tag{2.71}$$

associated to the fictitious compliance

$$\widetilde{\mathscr{C}}_0^\delta(\theta) := \sup\left\{\frac{1}{\delta}\langle G,u\rangle_{\mathbb{R}^3} + \langle F,u\rangle_{\mathbb{R}^3} - \int_Q j_0(e^\delta(u))\,\theta\,dx\ :\ u \in H^1(Q;\mathbb{R}^3)\right\}\ , \tag{2.72}$$

with j_0 defined in (2.69).

Under the assumption (h3) on the load, by applying [19, Proposition 2.8], we deduce the following crucial estimate:

$$\widetilde{\mathscr{C}}_0^\delta(\theta) \le \overline{\mathscr{C}}^\delta(\theta) \quad \forall\theta \in L^\infty(Q;[0,1])\ .$$

Consequently, as a counterpart to 2.70, we obtain the upper bound

$$\phi^\delta(k) \le \widetilde{\phi}_0^\delta(k)\ . \tag{2.73}$$

Step 2: limit of the upper bound $\widetilde{\phi}_0^\delta(k)$

We first prove that the sequence $\widetilde{\mathscr{C}}_0^{\delta}(\theta)$ defined in (2.72) Γ-converges, in the weak * topology of $L^\infty(Q;[0,1])$, to the limit compliance $\mathscr{C}^{lim}(\theta)$ defined by (2.43). Indeed the proof of Theorem 2.3.1 is still valid replacing j by j_0, and gives the Γ-convergence result for $\widetilde{\mathscr{C}}_0^{\delta}(\theta)$: the estimate $j_0 \leq j$ gives the Γ-liminf inequality, and the coercivity, 2-homogeneity and the equality $\overline{j} = \overline{j_0}$ ensure the Γ-limsup inequality. We recall that the properties of j_0 are gathered in Lemma 1.6.2.

As a consequence the fictitious problems $\widetilde{\phi}_0^{\delta}(k)$ defined in (2.71) converge to $\phi(k)$.

Step 3: limit of $\phi^{\delta}(k)$

By combining the estimates (2.70) and (2.73), and the convergence results obtained in Theorem 2.3.1 and Step 2, we infer that also the sequence $\phi^{\delta}(k)$ converges to $\phi(k)$ as $\delta \to 0^+$.

Let $\omega^{\delta} \subset Q$ be a sequence of domains such that $\phi^{\delta}(k) = \mathscr{C}^{\delta}(\omega^{\delta}) + k|\omega^{\delta}| + o(1)$. Since we know that the sequences $\widetilde{\phi}^{\delta}(k)$ and $\phi^{\delta}(k)$ have the same limit as $\delta \to 0$, we deduce that $\widetilde{\phi}^{\delta}(k) = \widetilde{\mathscr{C}}^{\delta}(1\!\!1_{\omega^{\delta}}) + k\int_Q 1\!\!1_{\omega^{\delta}}\,dx + o(1)$. Since by Theorem 2.3.1 the sequence $\widetilde{\mathscr{C}}^{\delta}(\theta) + k\int_Q \theta\,dx$ Γ-converges to $\mathscr{C}^{lim}(\theta) + k\int_Q \theta\,dx$ in the the weak * topology $L^\infty(Q;[0,1])$, any cluster □

2.4 Equivalent formulations of $\Phi(k)$ and optimality conditions

We conclude the Chapter by writing alternative formulations of $\mathscr{C}^{lim}(\theta)$ and $\phi(k)$, in a primal and dual form, see Proposition 2.4.1 and Theorem 2.4.1 respectively. The dual formulation is a key tool in the second passage to the limit, that we examine in Chapter 3.

The last subsection is devoted to the natural question whether the density formulation of $\phi(k)$ admits a classical solution (*i.e.* a density with values in $\{0,1\}$) or not: deriving a system of optimality conditions, the question can be rephrased in a very simple way. A more detailed description and a deeper analysis of the problem is postponed to Chapter 4.

2.4.1 Equivalent formulations

Exploiting the characterizations (2.8) and (2.9) of the spaces $BN(Q)$ and $TW(Q)$, and the action of G and H over these spaces (see Proposition 2.1.1) we obtain the following representation:

Proposition 2.4.1. *The limit compliance $\mathscr{C}^{lim}(\theta)$ defined in (2.43) and the limit func-*

tion $\phi(k)$ defined in (2.42) can be rewritten respectively as

$$\mathscr{C}^{lim}(\theta) = \sup\Big\{ \langle m_G, c\rangle_{\mathbb{R}} + \langle G_3, w\rangle_{\mathbb{R}^3} + \langle \overline{H_i}, \xi_i\rangle_{\mathbb{R}} + \\ - \int_Q \left[\frac{\eta}{2} \left| c'(x_3)(-x_2,x_1) + \nabla_{x'} w\right|^2 + \frac{Y}{2} \left|\xi_3'(x_3) + x_\alpha \xi_\alpha'(x_3)\right|^2 \right] \theta\, dx \Big\} , \tag{2.74}$$

$$\phi(k) = \sup\Big\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \left[\overline{j}(e_{13}(v), e_{23}(v), e_{33}(u)) - k\right]_+ dx \ : \\ v \in TW(Q)\,,\ u \in BN(Q) \Big\} \tag{2.75}$$

$$= \sup\Big\{ \langle m_G, c\rangle_{\mathbb{R}} + \langle G_3, w\rangle_{\mathbb{R}^3} + \langle \overline{H_i}, \xi_i\rangle_{\mathbb{R}} + \\ - \int_Q \left[\frac{\eta}{2} \left| c'(x_3)(-x_2,x_1) + \nabla_{x'} w\right|^2 + \frac{Y}{2} \left|\xi_3'(x_3) + x_\alpha \xi_\alpha'(x_3)\right|^2 - k \right]_+ dx \Big\} , \tag{2.76}$$

where m_G and $\overline{H}_i$ are defined in (2.11) and (2.21), and the supremum in (2.74) and (2.75) is taken over the set

$$\left\{ c \in H^1_m(I)\,, w \in L^2(I; H^1_m(D))\,,\ \xi_i \in H^1_m(I) \right\} .$$

Proof. We first prove (2.74).

As shown in Proposition 2.1.1 the action of H on any Bernoulli-Navier field u is given by

$$\langle H, u\rangle_{\mathbb{R}^3} = -\langle \overline{H_\alpha}, \zeta_\alpha' \rangle_{\mathbb{R}} + \langle \overline{H_3}, \zeta_3\rangle_{\mathbb{R}} \ , \tag{2.77}$$

with $\zeta_\alpha \in H^2_m(I)$ and $\zeta_3 \in H^1_m(I)$ associated to u as in (2.8). It is easy to prove that the space of derivatives of functions belonging to $H^2_m(I)$, defined in (35), equals $H^1_m(I)$. Then we can rewrite (2.77) as

$$\langle \overline{H_\alpha}, \xi_\alpha\rangle_{\mathbb{R}} + \langle \overline{H_3}, \zeta_3\rangle_{\mathbb{R}} \ ,$$

for some $\xi_\alpha \in H^1_m(I)$.

Similarly $e_{33}(u)$ reads

$$\zeta_3' - x_\alpha \zeta_\alpha'' = \zeta_3' + x_\alpha \xi_\alpha' \ .$$

By combining these results with the action of G over a field in $TW(Q)$, expressed in (2.19), and recalling the definition (2.43) of $\mathscr{C}^{lim}(\theta)$ we obtain (2.74).

Let us prove (2.75). Let $X = L^\infty(Q;[0,1])$ endowed with the weak * topology, and $Y = H^1(Q;\mathbb{R}^3) \times H^1(Q;\mathbb{R}^3)$ endowed with the weak topology. On the product space $X \times Y$ we consider, for a fixed $k \in \mathbb{R}$, the Lagrangian

$$\mathscr{L}_k(\theta,(v,u)) := \begin{cases} \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \left[\overline{j}\big(e_{13}(v), e_{23}(v), e_{33}(u)\big) - k\right] \theta\, dx \\ \qquad\qquad\qquad\qquad \text{if } (v,u) \in TW(Q) \times BN(Q) \\ -\infty \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} . \end{cases}$$

Since $\mathscr{L}_k(\theta,(v,u))$ is convex in θ on the compact space X and concave in (v,u) on Y, the equality $\inf_X \sup_Y \mathscr{L} = \sup_Y \inf_X \mathscr{L}$ holds by a standard commutation argument, see for instance [92, Proposition A.8], and gives (2.75).

Similarly, formulation (2.76) can be derived from (2.75) exploiting the formulation (2.74) of $\mathscr{C}^{lim}(\theta)$. □

Before stating Theorem 2.4.1, we need some preliminary definitions and results.

Lemma 2.4.1. *The Fenchel conjugate $[\overline{j}(\cdot)-k]_+^*(\xi)$ coincides with*

$$\psi_k(\xi) := \begin{cases} \overline{j}^*(\xi)+k & \text{if } \overline{j}^*(\xi) \geq k \\ 2\sqrt{k\,\overline{j}^*(\xi)} & \text{if } \overline{j}^*(\xi) \leq k \end{cases} . \tag{2.78}$$

Proof. Let us introduce the auxiliary function

$$g_k(\xi) := \begin{cases} \overline{j}^*(\xi)+k & if\ \xi \neq 0 \\ 0 & otherwise \end{cases} . \tag{2.79}$$

By definition of Fenchel conjugate, we have

$$\begin{aligned} g_k^*(y) &= \sup_{\xi\in\mathbb{R}^3} \{\xi\cdot y - g_k(\xi)\} = \sup\{\sup_{\xi\neq 0}\{\xi\cdot y - \overline{j}^*(\xi)\} - k, 0\} \\ &= \sup\{\overline{j}^{**}(y)-k,0\} = [\overline{j}^{**}(y)-k]_+ = [\overline{j}(y)-k]_+ , \end{aligned}$$

where the last equality follows by convexity of $\overline{j}$.

By applying again the Fenchel transform, we obtain

$$g_k^{**}(\xi) = [\overline{j}(\cdot)-k]_+^*(\xi) ,$$

namely $[\overline{j}(\cdot)-k]_+^*(\xi)$ coincides with the convex envelope of g_k.

A direct computation (*cf.* [19, Lemma 4.4]) gives $g_k^{**}(\xi) = \psi_k(\xi)$

□

We recall that, in view of Lemma 1.6.1, there holds

$$\overline{j}^*(\xi) = \frac{1}{8\eta}|\xi'|^2 + \frac{1}{2Y}\xi_3^2 . \tag{2.80}$$

Lemma 2.4.2. *Let $\sigma \in L^2(Q;\mathbb{R}^3)$, and $F = G+H$ be an admissible load. Then*

$$\int_Q \sigma\cdot(e_{1,3}(v),e_{2,3}(v),e_{3,3}(u))\,dx = \langle G,v\rangle_{\mathbb{R}^3} + \langle H,u\rangle_{\mathbb{R}} \quad \forall (v,u)\in TW(Q)\times BN(Q) \tag{2.81}$$

if and only if

$$\begin{cases} \partial_1\sigma_1 + \partial_2\sigma_2 = -2G_3 , \\ [[x_1\sigma_2 - x_2\sigma_1]] = -2\mathscr{P}_0(m_G) , \\ [[\sigma_3]] = -\mathscr{P}_0(\overline{H_3}) , \\ [[x_\alpha\sigma_3]] = -\mathscr{P}(\overline{H_\alpha}) , \end{cases} \tag{2.82}$$

in the sense of distributions.

Proof. The left and right hand side of (2.81) do not change if we subtract to $v \in TW(Q)$ any Bernoulli-Navier field and to $u \in BN(Q)$ any rigid motion (see Proposition 2.1.1), then without loss of generality we can consider v and u of the form (2.9) and (2.8) respectively. Exploiting this representation and Proposition 2.1.1, with a direct computation we obtain that (2.81) is equivalent to the following system:

$$\frac{1}{2}\langle(\sigma_1,\sigma_2),\nabla_{x'}w\rangle_{\mathbb{R}^3} = \langle G_3, w\rangle_{\mathbb{R}^3} \qquad \forall w \in L^2(I;H^1_m(D))\,, \tag{2.83}$$

$$\frac{1}{2}\langle[[-x_2\sigma_1+x_1\sigma_2]],c'\rangle_{\mathbb{R}} = \langle m_G, c\rangle_{\mathbb{R}} \qquad \forall c \in H^1_m(I)\,, \tag{2.84}$$

$$\langle[[\sigma_3]],\zeta'\rangle_{\mathbb{R}} = \langle\overline{H_3},\zeta\rangle_{\mathbb{R}} \qquad \forall \zeta \in H^1_m(I)\,, \tag{2.85}$$

$$\langle[[x_\alpha\sigma_3]],\eta''\rangle_{\mathbb{R}} = \langle\overline{H_\alpha},\eta'\rangle_{\mathbb{R}} \qquad \forall \eta \in H^2_m(I)\,. \tag{2.86}$$

It is easy to prove that (2.83) is equivalent to the first condition of (2.82) in $H^{-1}(Q)$.

As already observed in Remark 2.1.2, $\langle m_G, 1\rangle_{\mathbb{R}} = \langle\overline{H}_3, 1\rangle_{\mathbb{R}} = 0$, hence (see Notation) m_G and $\overline{H_3}$ admit a unique primitive in $\mathscr{D}'(I)$, denoted by $\mathscr{P}_0(m_G)$ and $\mathscr{P}_0(\overline{H_3})$ respectively. This implies that (2.84) and (2.85) are equivalent to the second and third conditions of (2.82).

It remains us to deal with (2.86). As already observed in the proof of Proposition 2.4.1, there holds

$$\left\{\eta' \;:\; \eta \in H^2_m(I)\right\} = H^1_m(I)\,,$$

then we can replace condition (2.86) by

$$\langle[[x_\alpha\sigma_3]],\xi'\rangle_{\mathbb{R}} = \langle\overline{H_\alpha},\xi\rangle_{\mathbb{R}} \quad \forall \xi \in H^1_m(I)\,. \tag{2.87}$$

By the arbitrariness of $\xi \in H^1_m(I)$ we obtain that $([[x_\alpha\sigma_3]])' + \overline{H_\alpha}$ is a constant distribution over I, *i.e.* there exists $\lambda \in \mathbb{R}$ such that

$$([[x_\alpha\sigma_3]])' + \overline{H_\alpha} = \lambda\, 1\!\!1_I\,.$$

If we test the distribution above against 1 we obtain that $\lambda = \langle\overline{H_\alpha},1\rangle_{\mathbb{R}}$, that is

$$([[x_\alpha\sigma_3]])' = -\left(\overline{H_\alpha} - \langle\overline{H_\alpha},1\rangle_{\mathbb{R}}\right)\,. \tag{2.88}$$

This concludes the proof, indeed (2.88) is equivalent to

$$[[x_\alpha\sigma_3]] = -\mathscr{P}_0\left(\overline{H_\alpha} - \langle\overline{H_\alpha},1\rangle_{\mathbb{R}}\right) = -\mathscr{P}(\overline{H_\alpha})\,.$$

□

We now write problems $\phi(k)$ and $\mathscr{C}^{lim}(\theta)$ in dual form.

Theorem 2.4.1. *For every $\theta \in L^\infty(Q;[0,1])$ and every $k \in \mathbb{R}$, problems (2.42) and (2.43) admit respectively the dual formulations*

$$\phi(k) = \inf_{\sigma \in L^2(Q;\mathbb{R}^3)}\left\{\int_Q \psi_k(\sigma)\,dx \;:\; \partial_1\sigma_1 + \partial_2\sigma_2 = -2G_3\,,\; [[x_1\sigma_2 - x_2\sigma_1]] = -2\mathscr{P}_0(m_G)\,,\right.$$
$$\left.[[\sigma_3]] = -\mathscr{P}_0(\overline{H_3})\,,\; [[x_\alpha\sigma_3]] = -\mathscr{P}(\overline{H_\alpha})\right\} \tag{2.89}$$

and

$$\mathscr{C}^{lim}(\theta) = \inf_{\sigma \in L^2(Q;\mathbb{R}^3)} \left\{ \int_Q \theta^{-1} \overline{j}^*(\sigma)\, dx \ : \partial_1 \sigma_1 + \partial_2 \sigma_2 = -2G_3 \,, \right.$$
$$[[x_1\sigma_2 - x_2\sigma_1]] = -2\mathscr{P}_0(m_G)\,,$$
$$\left. [[\sigma_3]] = -\mathscr{P}_0(\overline{H_3})\,,\ [[x_\alpha \sigma_3]] = -\mathscr{P}(\overline{H_\alpha}) \right\} . \tag{2.90}$$

Proof. The proof is based on a standard convex duality result, presented in the Introduction: Lemma 1.1.2.

Let us consider problem $\mathscr{C}^{lim}(\theta)$. Let $X = TW(Q) \times BN(Q)$, $Y = L^2(Q;\mathbb{R}^3)$, $A(v,u) = (e_{13}(v), e_{23}(v), e_{33}(u))$, $\Phi(v,u) = -\langle G, v\rangle_{\mathbb{R}^3} - \langle H, u\rangle_{\mathbb{R}^3}$, and $\Psi(y) = \int_Q \overline{j}(y)\,\theta\, dx$, then we are in a position to apply Lemma 1.1.2. The left hand side of (4.1.2) equals the primal formulation of $\mathscr{C}^{lim}(\theta)$, introduced in (2.43).

We now turn our attention to the right hand side of (4.1.2).

By definition of Fenchel conjugate, recalling that $\overline{j}$ is 2-homogeneous, we obtain

$$\Psi^*(\sigma) = \sup_{y \in L^2(Q;\mathbb{R}^3)} \left\{ \int_Q \sigma \cdot y - \int_Q \overline{j}(y)\,\theta\, dx \right\} = \int_Q \theta^{-1} \overline{j}^*(\sigma)\, dx\,.$$

On the other hand, since

$$\langle -A^*\sigma, (v,u)\rangle_{X^*,X} = -\langle \sigma, A(v,u)\rangle_{Y^*,Y} = -\langle \sigma, (e_{13}(v), e_{23}(v), e_{33}(u))\rangle_{\mathbb{R}^3}\,,$$

we infer

$$\phi^*(-A^*\sigma) = \sup_{(v,u)\in X} \{\langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \langle \sigma, (e_{13}(v), e_{23}(v), e_{33}(u))\rangle_{\mathbb{R}^3}\}$$
$$= \begin{cases} 0 & \text{if } \langle \sigma, (e_{13}(v), e_{23}(v), e_{33}(u))\rangle_{\mathbb{R}^3} = \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} \quad \forall (v,u) \in X \\ +\infty & \text{otherwise} \end{cases} .$$

Since we are interested in the lower value of $\phi^*(-A^*\sigma)$, we look for optimal fields σ such that

$$\int_Q \sigma \cdot (e_{13}(v), e_{23}(v), e_{33}(u)) = \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} \quad \forall (v,u) \in TW(Q) \times BN(Q)\,,$$

or equivalently, in view of Lemma 2.4.2, such that

$$\partial_1\sigma_1 + \partial_2\sigma_2 = -2G_3\,,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2\mathscr{P}_0(m_G)\,,\ [[\sigma_3]] = -\mathscr{P}_0(\overline{H_3})\,,\ [[x_\alpha\sigma_3]] = -\mathscr{P}(\overline{H_\alpha})\,,$$

in the sense of distributions. This concludes the proof of (2.90).

Similarly, by applying Lemma 1.1.2 with X, Y, A and Φ as above, and $\Psi(y) = \int_Q [\overline{j}(y) - k]_+\, dx$, one obtains the dual form (2.89), exploiting Lemma 2.4.1. □

2.4.2 Link with the classical torsion problem

The constraints that appear in the dual formulation (2.89) of $\phi(k)$ reveal that the limit optimization problem can be solved section by section. More precisely, $\phi(k)$ can be rewritten as

$$\phi(k) = \int_I \Lambda_k \left(G_3, \mathscr{P}_0(m_G), \mathscr{P}_0(\overline{H}_3), \mathscr{P}(\overline{H}_\alpha)\right) dx_3 \,, \tag{2.91}$$

where, for every $r \in H^{-1}(D)$, $t \in \mathbb{R}$, $s := (s_0, s_1, s_2) \in \mathbb{R}^3$, we set

$$\Lambda_k(r,t,s) := \inf_{\sigma \in L^2(D;\mathbb{R}^3)} \Big\{ \int_D \psi_k(\sigma)\, dx' \,:\, \mathrm{div}_{x'}(\sigma_1, \sigma_2) = -2r,\, [[x_1\sigma_2 - x_2\sigma_1]] = -2t, \\ [[\sigma_3]] = -s_0,\, [[x_\alpha \sigma_3]] = -s_\alpha \Big\} \,. \tag{2.92}$$

We recall that, in view of assumption (h2), we are just considering the case in which either $r = 0$ or $s_i = 0$.

This way of computing $\phi(k)$ enlightens the link with the classical torsion problem. Actually, the compliance of a cylindrical rod of section $E \subset D$ under a torque t is classically written as

$$\inf_{\sigma' \in L^2(D;\mathbb{R}^2)} \Big\{ \int_D \frac{1}{8\eta} |\sigma'|^2 \, dx' \,:\, \mathrm{div}_{x'} \sigma' = 0,\, [[x_1\sigma_2 - x_2\sigma_1]] = -2t \,,\, \mathrm{spt}\sigma' \subset \overline{E} \Big\} \,, \tag{2.93}$$

where we have used the notation $\sigma' := (\sigma_1, \sigma_2)$.

The optimization of such compliance with respect to the domain E under the volume constraint $|E| = m$ reads

$$\inf_{\sigma' \in L^2(D;\mathbb{R}^2)} \Big\{ \int_D \frac{1}{8\eta} |\sigma'|^2 \, dx' \,:\, \mathrm{div}_{x'} \sigma' = 0,\, [[x_1\sigma_2 - x_2\sigma_1]] = -2t \,,\, |\mathrm{spt}\sigma'| \le m \Big\} \,.$$

Introducing a Lagrange multiplier k, one is reduced to solve

$$\inf_{\sigma' \in L^2(D;\mathbb{R}^2)} \Big\{ \int_D \frac{1}{8\eta} |\sigma'|^2 \, dx' + k|\mathrm{spt}\sigma'| \,:\, \mathrm{div}_{x'} \sigma' = 0,\, [[x_1\sigma_2 - x_2\sigma_1]] = -2t \Big\}$$

$$= \inf_{\sigma' \in L^2(D;\mathbb{R}^2)} \Big\{ \int_D g_k(\sigma_1, \sigma_2, 0)\, dx' \,:\, \mathrm{div}_{x'}(\sigma_1, \sigma_2) = 0,\, [[x_1\sigma_2 - x_2\sigma_1]] = -2t \Big\} \,,$$

being g_k the function defined in (2.79). As already shown in the proof of Lemma 2.4.1, the convex envelope g_k^* equals $[\overline{j}(\cdot) - k]_+^*$, hence the relaxed formulation of the latter problem is nothing else than $\Lambda_k(0,t,0)$. This concordance is somehow surprising, since formulation (2.93) is valid only for cylindrical rods (or rods with slowly varying section) whereas, in the formulation (2.6) of our initial optimization problems $\phi^\delta(k)$, no topological constraint is imposed on the admissible domains $\Omega \subset \delta\overline{D} \times I$. What can be inferred from this comparison is that optimal thin torsion rods searched in a very large class without imposing any geometrical restriction are in fact not sensibly different from the nearly cylindrical ones treated in the classical theory.

2.4.3 Optimality conditions

We say that $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times BN(Q) \times L^2(Q;\mathbb{R}^3)$ is optimal for $\phi(k)$ if

- $(\cdot)$ $\overline{\theta}$ is optimal for $\phi(k)$ in its primal formulation (2.42);
- $(\cdot)$ the couple $(\overline{v}, \overline{u})$ is optimal for $\phi(k)$ and $\mathscr{C}^{lim}(\overline{\theta})$ in their primal formulations, given by (2.75) and (2.43) respectively;
- $(\cdot)$ $\overline{\sigma}$ is optimal for $\phi(k)$ and $\mathscr{C}^{lim}(\overline{\theta})$ in their dual formulations, given by (2.89) and (2.90) respectively.

By comparing the primal and dual formulations, we derive the following optimality conditions.

Theorem 2.4.2. *A vector $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times BN(Q) \times L^2(Q;\mathbb{R}^3)$ is optimal for $\phi(k)$ if and only if it satisfies the following system:*

$$\partial_1\overline{\sigma}_1 + \partial_2\overline{\sigma}_2 = -2G_3\,,\ [[x_1\overline{\sigma}_2 - x_2\overline{\sigma}_1]] = -2\mathscr{P}_0(m_G) \tag{2.94}$$

$$[[\overline{\sigma}_3]] = -\mathscr{P}_0(\overline{H_3})\,,\ [[x_\alpha\overline{\sigma}_3]] = -\mathscr{P}(\overline{H_\alpha}) \tag{2.95}$$

$$\overline{\sigma} = \overline{\theta}\ \overline{j}'(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) \tag{2.96}$$

$$\overline{\sigma} \in \partial[\overline{j} - k]_+(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) \tag{2.97}$$

$$\overline{\theta}\,[\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - k] = [\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - k]_+ \tag{2.98}$$

Proof. We recall that a vector field $\sigma \in L^2(Q;\mathbb{R}^3)$ is admissible for (2.89) if and only if

$$\int_Q \sigma \cdot (e_{13}(v), e_{23}(v), e_{33}(u)) = \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} \quad \forall (v,u) \in TW(Q) \times BN(Q)\,, \tag{2.99}$$

i.e., in view of Lemma 2.4.2, if and only if (2.94) and (2.95) hold true.

Assume that $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma})$ is optimal for $\phi(k)$. Since $\overline{\sigma}$ is optimal for the dual form (2.90) of $\mathscr{C}^{lim}(\overline{\theta})$, necessarily it must vanish on the set $\{\overline{\theta} = 0\}$. Then, using (2.99) and the equivalence between the primal and the dual forms (2.43) and (2.90) of $\mathscr{C}^{lim}(\overline{\theta})$, we obtain:

$$\begin{aligned} 0 &= \int_Q \Big\{ \overline{\sigma} \cdot (e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - \overline{\theta}\,\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - \overline{\theta}^{\,-1}\overline{j}^*(\overline{\sigma}) \Big\}\, dx \\ &= \int_{Q\cap\{\overline{\theta}>0\}} \Big\{ \overline{\theta}^{\,-1}\overline{\sigma} \cdot (e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - \overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - \overline{j}^*(\overline{\theta}^{\,-1}\overline{\sigma}) \Big\}\, \overline{\theta}\, dx\,, \end{aligned}$$

which yields (2.96) thanks to the Fenchel inequality.

Similarly, again using (2.99), the equivalence between (2.75) and (2.89) implies:

$$\int_Q \Big\{ \overline{\sigma} \cdot (e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - [\overline{j} - k]_+(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - [\overline{j} - k]_+^*(\overline{\sigma}) \Big\}\, dx = 0\,,$$

which yields (2.97) thanks to the Fenchel inequality.

Finally, the equivalence between (2.42) and (2.75) implies:

$$\int_Q \Big\{ \big(\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - k\big)\, \overline{\theta} - \big[\overline{j} - k\big]_+ (e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) \Big\}\, dx = 0 \,,$$

which yields (2.98) since the integrand is non positive.

Viceversa, assume that $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma})$ satisfy the optimality conditions (2.94)-(2.95)-(2.96)-(2.97)-(2.98).

By admissibility of $\overline{\sigma}$ for $\mathscr{C}^{lim}(\overline{\theta})$ in its dual form (2.90), we have

$$\begin{aligned}
&\langle G, \overline{v}\rangle_{\mathbb{R}^3} + \langle H, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u}))\, \overline{\theta}\, dx \\
&\leq \sup\Big\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v), e_{23}(v), e_{33}(u))\, \overline{\theta}\, dx \;:\; v \in TW(Q)\,,\; u \in BN(Q) \Big\} \\
&= \mathscr{C}^{lim}(\overline{\theta}) = \inf\Big\{ \int_Q \overline{\theta}^{-1} \overline{j}^*(\sigma)\, dx \;:\; \sigma \in L^2(Q;\mathbb{R}^2)\,,\ \text{such that (2.99)} \Big\} \\
&\leq \int_Q \overline{\theta}^{-1} \overline{j}^*(\overline{\sigma})\, dx \,.
\end{aligned}$$

Using (2.99) one sees that, thanks to (2.96), the first and the last term in the above chain of inequalities agree. Hence $(\overline{v}, \overline{u})$ and $\overline{\sigma}$ are optimal respectively for the primal and the dual forms (2.43) and (2.90) of $\mathscr{C}^{lim}(\overline{\theta})$.

Similarly, since $\overline{\sigma}$ is admissible for problem (2.89), we infer

$$\begin{aligned}
&\langle G, \overline{v}\rangle_{\mathbb{R}^3} + \langle H, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j} - k\big]_+ (e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u}))\, dx \\
&\leq \sup\Big\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j}(e_{13}(v), e_{23}(v), e_{33}(u)) - k\big]_+ dx \,:\, v \in TW, u \in BN \Big\} \\
&= \phi(k) = \inf\Big\{ \int_Q \psi_k(\sigma)\, dx \;:\; \sigma \in L^2(Q;\mathbb{R}^2)\,,\ \text{such that (2.99)} \Big\} \\
&\leq \int_Q \psi_k(\overline{\sigma})\, dx = \int_Q \big[\overline{j} - k\big]_+^* (\overline{\sigma})\, dx \,.
\end{aligned}$$

Using (2.99) one sees that, thanks to (2.97), the first and the last term in the above chain of inequalities agree. Hence $(\overline{v}, \overline{u})$ and $\overline{\sigma}$ are optimal respectively for problems (2.75) and (2.89).

It remains to check that $\overline{\theta}$ is optimal for problem (2.42). Indeed we have

$$\begin{aligned}
\mathscr{C}^{lim}(\overline{\theta}) + k \int_Q \overline{\theta}\, dx &= \langle G, \overline{v}\rangle_{\mathbb{R}^3} + \langle H, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \big(\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - k\big)\, \overline{\theta}\, dx \\
&= \langle G, \overline{v}\rangle_{\mathbb{R}^3} + \langle H, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j} - k\big]_+ (e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u}))\, dx = \phi(k) \,,
\end{aligned}$$

where in the first equality we have used the already proved optimality of $(\overline{v}, \overline{u})$ for the primal form (2.43) of $\mathscr{C}^{lim}(\overline{\theta})$, in the second equality the optimality condition (2.98), and finally in the third equality the already proved optimality of $(\overline{v}, \overline{u})$ for problem (2.75). □

Finally we rewrite the optimality condition (2.97), clarifying the relationship between an optimal $\overline{\sigma}$ and an optimal couple $(\overline{v}, \overline{u})$.

Lemma 2.4.3. *let $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma})$ be optimal for $\phi(k)$, then we can rewrite the optimality condition (2.97) as*

$$\overline{\sigma} = \begin{cases} 0 & \text{if } \overline{j}(\overline{e}) < k\,, \\ t(4\eta e', Ye_3)\,,\ \text{with } t \in [0,1] & \text{if } \overline{j}(\overline{e}) = k\,, \\ (4\eta e', Ye_3) & \text{if } \overline{j}(\overline{e}) > k\,, \end{cases} \tag{2.100}$$

with

$$\overline{e} := (e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u}))\,.$$

Proof. In view of (2.97), by definition of subdifferential, there holds

$$\langle \overline{\sigma}, \overline{e} \rangle_{\mathbb{R}^3} = [\overline{j}(\cdot) - k]_+(\overline{e}) + [\overline{j}(\cdot) - k]_+^*(\overline{\sigma})\,. \tag{2.101}$$

Letting $\tilde{\sigma} := \left(\sigma'/\sqrt{4\eta}, \sigma_3/\sqrt{Y}\right)$ and $\tilde{e} := \left(\sqrt{4\eta}e', \sqrt{Y}e_3\right)$ an easy computation gives

$$\langle \overline{\sigma}, \overline{e} \rangle_{\mathbb{R}^3} = \langle \tilde{\sigma}, \tilde{e} \rangle_{\mathbb{R}^3}\ ,\ \ \overline{j}(\overline{e}) = \frac{|\tilde{e}|^2}{2}\ ,\ \ [\overline{j}(\cdot) - k]_+^*(\overline{\sigma}) = \left[\frac{|\cdot|^2}{2} - k\right]_+^*(\tilde{\sigma})\,.$$

Hence equality (2.101) reads

$$\tilde{\sigma} \in \partial\left[\frac{|\cdot|^2}{2} - k\right]_+(\tilde{e})\,. \tag{2.102}$$

Let us introduce the radial function $\varphi(s) := \left[\frac{s^2}{2} - k\right]_+$. By combining Lemma (1.1.2) and (2.102), we infer

$$\tilde{\sigma} = t\,\frac{\tilde{e}}{|\tilde{e}|}\,, \quad \text{with } t \in \partial\varphi(|\tilde{e}|)\,.$$

The subdifferential of $\varphi(s)$ is almost everywhere a point, and for $s = \sqrt{2k}$ it is the interval $[0, \sqrt{2k}]$. An easy calculation shows that

$$\tilde{\sigma} = \begin{cases} 0 & \text{if } |\tilde{e}| < \sqrt{2k}\,, \\ t\tilde{e}\,,\ \text{with } t \in [0,1] & \text{if } |\tilde{e}| = \sqrt{2k}\,, \\ \tilde{e} & \text{if } |\tilde{e}| > \sqrt{2k}\,. \end{cases}$$

and recalling the definition of $\tilde{\sigma}$ and $\tilde{e}$ in terms of $\overline{\sigma}$ and $\overline{e}$ we obtain the thesis. □

2.4.4 Looking for classical solutions for $\Phi(k)$

It is interesting to ask whether, via the optimality system, it is possible to establish that problem $\phi(k)$ in the density formulation (2.42) admits a classical solution, represented by an optimal density with values into $\{0,1\}$.

If $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times BN(Q) \times L^2(Q;\mathbb{R}^2)$ is an optimal vector, the optimality condition (2.98) reveals that $\overline{\theta}$ is a characteristic function provided the level set

$$\{\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) = k\}$$

(or equivalently, in view of (2.78), when the set where $\psi_k(\overline{\sigma}) = 2(k\overline{j}^*(\overline{\sigma}))^{1/2}$), has zero Lebesgue measure.

If we consider the particular case of pure torsion regime, namely when $[[F_\alpha]] = F_3 = 0$ or equivalently $[[G_\alpha]] = G_3 = H = 0$, looking at problem (2.89) one sees that $\overline{\sigma}$ is optimal if and only if $\overline{\sigma} = (\overline{\sigma}_1, \overline{\sigma}_2, 0)$ and $\overline{\sigma}(\cdot, x_3)$ solves for a.e. x_3 the following section problem for $t = \mathscr{P}_0(m_G)$:

$$\alpha_k(t) := \inf\left\{ \int_D \psi_k(\sigma_1, \sigma_2, 0)\, dx' \ :\ \sigma' \in L^2(D;\mathbb{R}^2)\ ,\ \mathrm{div}_{x'} \sigma' = 0,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2t \right\}, \tag{2.103}$$

where we have used the notation $\sigma' := (\sigma_1, \sigma_2)$.

In view of (2.78) and (2.80), we remark that

$$\psi_k(\xi_1, \xi_2, 0) := \begin{cases} \frac{1}{8\eta}|\xi'|^2 + k & \text{if } |\xi'| \geq \sqrt{8\eta k} \\ \sqrt{\frac{k}{2\eta}}\,|\xi'| & \text{if } |\xi'| \leq \sqrt{8\eta k} \end{cases},$$

where we have used the notation $\xi' := (\xi_1, \xi_2)$. Moreover, we notice that $\alpha_k(t) = k\alpha_1\left(\frac{t}{\sqrt{k}}\right)$.

The constraint of divergence free in (2.103) implies that we can write any admissible σ' as a rotated gradient: since σ' vanished in $\mathbb{R}^2 \setminus \overline{D}$ and $\mathrm{div}_{x'} \sigma' = 0$, there exists $u \in H^1(\mathbb{R}^2)$ such that $\nabla u = 0$ in $\mathbb{R}^2 \setminus \overline{D}$ and $\sigma' = (-\partial_2 u, \partial_1 u)$. Let us introduce the space $H^1_c(D) := \{u \in H^1(\mathbb{R}^2)\ :\ \nabla u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D}\}$. Hence we are led to set

$$s := \frac{t}{\sqrt{k}} \tag{2.104}$$

and to study the solutions $\overline{u}$ of the following minimization problem

$$\alpha(s) := \inf\left\{ \int_D \psi_1(\nabla u, 0)\, dx'\ :\ u \in H^1_c(D)\ ,\ \int_{\mathbb{R}^2} u\, dx' = s \right\}. \tag{2.105}$$

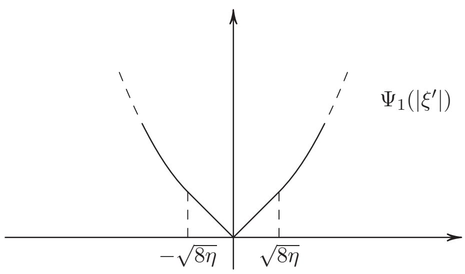


Figure 2.6: *The function ψ_1, evaluated in the points of the form $(\xi_1, \xi_2, 0)$, is a radial function depending on $|\xi'|$.*

In view of Lemma 2.4.3 (considering $k = 1$ and $|\overline{\sigma}| = |\nabla \overline{u}|$), we infer that the homogenization region corresponds to the set

$$\{0 < |\nabla \overline{u}| < \sqrt{8\eta}\}\ , \tag{2.106}$$

where the integrand ψ_1 is not strictly convex (see Figure 4.1). Hence asking whether $\phi(k)$ admits a classical solution or not, in the case of pure torsion loads, is equivalent to determine if there exists a solution $\overline{u}$ for which this set in (2.106) is Lebesgue negligible.

More precisely, in view of Lemma 2.4.3 and optimality condition (2.96), we infer that the optimal density $\overline{\theta}$ and a solution $\overline{u}$ for $\alpha(s)$ satisfy

$$\{\overline{\theta}=0\}=\{|\nabla\overline{u}|=0\}\,,\ \{\overline{\theta}=1\}=\{|\nabla\overline{u}|\geq\sqrt{8\eta}\}\,,\ \{\overline{\theta}\in(0,1)\}=\{|\nabla\overline{u}|\in(0,\sqrt{8\eta})\}\,,$$

as represented in Figure 2.7.

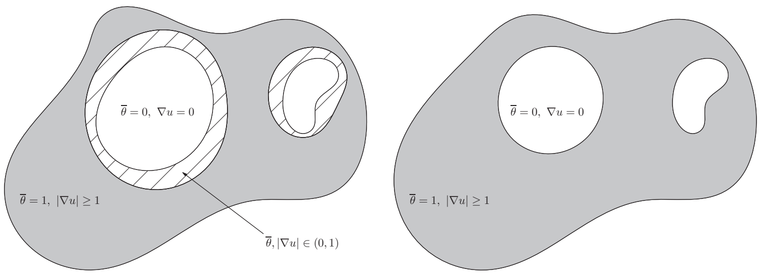


Figure 2.7: *On the left the case of presence of homogenization region, on the right the case of classic solution.*

We point out that, for a very similar problem, when D is a square, some numerical experiments seem to predict the existence of a homogenization region of nonzero measure [72].

This problem will be the subject of Chapter 4.

2.5 Appendix

Proof of Lemma 2.3.3.

The positivity of ψ_D is a consequence of the maximum principle.

A minimizing sequence ψ_n for the variational problem in (2.49) converges weakly in $H^1_0(D)$ to a function $\overline{\psi}\in H^1_0(D)$ which solves the Euler equation $-\Delta\overline{\psi}=2\lambda$ in D, for some $\lambda\in\mathbb{R}$. Thus $\overline{\psi}=\lambda\psi_D$, and

$$\int_D|\nabla\overline{\psi}|^2\,dx'=2\lambda\int_D\overline{\psi}\,dx'=2\lambda=2\Big(\int_D\psi_D\,dx'\Big)^{-1}=4\gamma^{-1}\,.$$

If $\overline{w}$ is a solution to (2.50), the Euler equation gives

$$\operatorname{div}\big(((-x_2,x_1)+\nabla\overline{w})1\!\!1_D\big)=0\quad\text{in }\mathscr{D}'(\mathbb{R}^2)\,.$$

Hence there exists a function $\psi\in H^1(\mathbb{R}^2)$ such that $((-x_2,x_1)+\nabla\overline{w})1\!\!1_D=(\partial_2\psi,-\partial_1\psi)$ in $\mathbb{R}^2$ and $\psi=0$ in $\mathbb{R}^2\setminus D$. This implies that ψ solves $-\Delta\psi=2$ in D and vanishes on ∂D, so that $\psi\llcorner D=\psi_D$. □

Proof of Lemma 2.3.4.

Let us first estimate the integral mean of c^δ defined in (2.52). Exploiting the hypothesis (2.51) and recalling that $\int_D \psi_D(x')\,dx' = \gamma/2$ (see (2.48)), we have:

$$\begin{aligned}\Big|\int_I c^\delta(x_3)\,dx_3\Big|^2 &= \Big|\frac{2}{\gamma}\int_Q \psi_D(x')c^\delta(x_3)\,dx - \frac{1}{\gamma\delta}\int_Q \psi_D(x')\,\mathrm{curl}_{x'}(u_1^\delta,u_2^\delta)\,dx\Big|^2\\ &= \frac{4}{\gamma^2}\Big|\int_Q \psi_D(x')\,[c^\delta(x_3) - \frac{1}{2\delta}\,\mathrm{curl}_{x'}(u_1^\delta,u_2^\delta)]\,dx\Big|^2\\ &\le C\int_Q \big|c^\delta(x_3) - \frac{1}{2\delta}\,\mathrm{curl}_{x'}(u_1^\delta,u_2^\delta)\big|^2\,dx\,, \qquad (2.107)\end{aligned}$$

where, in the last line, we have applied the Cauchy-Schwartz inequality. In order to estimate the integral (5.116), we now apply (1.24) in Lemma 1.5.2 with $\psi = \psi_D$ to the field $v = u_\alpha^\delta(\cdot,x_3) - [[u_\alpha^\delta]](x_3)$ (which belongs to $H^1_m(D;\mathbb{R}^2)$). Since subtracting from u_α^δ its mean $[[u_\alpha^\delta]]$ does not affect the expressions of the functions $c^\delta(x_3)$, $\mathrm{curl}_{x'}(u_1^\delta,u_2^\delta)$ and $e_{\alpha\beta}(u^\delta)$, we obtain

$$\int_Q \big|c^\delta(x_3) - \frac{1}{2\delta}\,\mathrm{curl}_{x'}(u_1^\delta,u_2^\delta)\big|^2\,dx \le \frac{C}{\delta^2}\int_Q \big|e_{\alpha\beta}(u_1^\delta,u_2^\delta)\big|^2\,dx. \qquad (2.108)$$

Combining (5.116) and (2.108), thanks to the L^2-boundedness of $e^\delta(u^\delta)$, we conclude

$$\Big|\int_I c^\delta(x_3)\,dx_3\Big|^2 \le C\delta^2\,. \qquad (2.109)$$

We now turn to estimate the derivative of c^δ. We have:

$$(c^\delta)'(x_3) = \int_D (\nabla\psi_D \wedge e^\delta_{\alpha 3}(u^\delta))\,dx' - \frac{1}{2\delta}\int_D (\nabla\psi_D \wedge \nabla_{x'}u_3^\delta)\,dx'\,.$$

Now we notice that the second integral vanishes: indeed, integrating by parts and taking into account that ψ_D vanishes on ∂D, we get

$$\int_D (\nabla\psi_D \wedge \nabla_{x'}u_3^\delta)\,dx' = 0\,.$$

Therefore

$$(c^\delta)'(x_3) = \int_D (\nabla\psi_D \wedge e^\delta_{\alpha 3}(u^\delta))\,dx'\,.$$

So we obtain the inequality

$$\big|(c^\delta)'(x_3)\big|^2 \le \int_D |\nabla\psi_D|^2\,dx' \int_D |e^\delta_{\alpha 3}(u^\delta)|^2\,dx'\,,$$

and, integrating over I,

$$\int_I \big|(c^\delta)'(x_3)\big|^2\,dx_3 \le \int_D |\nabla\psi_D|^2\,dx' \int_Q |e^\delta_{\alpha 3}(u^\delta)|^2\,dx\,. \qquad (2.110)$$

Combining (2.109) and (2.110), we conclude that c^δ is bounded in $H^1(I)$.

□

CHAPTER 3

Optimal design in thin rods: the small volume fraction limit

In this Chapter we investigate the behavior of optimal configurations of the compliance optimization problem introduced in (5), when the total amount of material becomes infinitesimal.

As said in the Introduction, this corresponds to study the asymptotics as $k \to +\infty$ of $\phi(k)$, where $\phi(k)$ has been introduced in (3.1), as the limit of the sequence $\phi^{\delta}(k)$ as $\delta \to 0^+$ (see Theorem 2.2.1).

Let us recall the definition:

$$\phi(k) := \inf\left\{\mathscr{C}^{lim}(\theta) + k\int_Q \theta\, dx \ : \ \theta \in L^\infty(Q;[0,1])\right\}\ , \tag{3.1}$$

where

$$\mathscr{C}^{lim}(\theta) := \sup\left\{\langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v), e_{23}(v), e_{33}(u))\,\theta\, dx : v \in TW,\, u \in BN\right\}. \tag{3.2}$$

The Chapter is divided in two parts. We first study the variational convergence, as $k \to +\infty$, of problems $\phi(k)$ suitably rescaled (see Theorem 3.1.1). Their limit takes the form of a minimization problem over the class of positive measures on Q. The optimal measures represent the limit of optimal density distributions for $\phi(k)$: they describe where it is convenient to put the material in an optimal way, when the amount of relative volume becomes infinitesimal.

In the second part of the Chapter, Section 3.3, we present some examples of concentration phenomena, described explicitly, which occur considering particular admissible loads, already introduced in paragraph 2.1.3.

3.1 The main results

Let us begin by extending the limit compliance $\mathscr{C}^{lim}(\theta)$ given by (3.2) to the class $\mathscr{M}^+(Q)$ of positive measures μ on $\mathbb{R}^3$ compactly supported in Q by setting

$$\begin{aligned}\mathscr{C}^{lim}(\mu) := \sup\Big\{\langle G,v\rangle_{\mathbb{R}^3}+\langle H,u\rangle_{\mathbb{R}^3}-\int_Q \overline{j}\big(e_{13}(v),e_{23}(v),e_{33}(u)\big)\,d\mu\ :\\ v\in TW(Q)\cap\mathscr{C}^\infty(Q;\mathbb{R}^3)\,,\ u\in BN(Q)\cap\mathscr{C}^\infty(Q;\mathbb{R}^3)\Big\}\,.\end{aligned}\tag{3.3}$$

We point out that in dual form $\mathscr{C}^{lim}(\mu)$ reads

$$\begin{aligned}\mathscr{C}^{lim}(\mu) = \inf_{\xi\in L^2_\mu(Q;\mathbb{R}^3)}\Big\{\textstyle\int_Q \overline{j}^*(\xi)\quad d\mu\ :\ \mathrm{div}_{x'}(\xi'\mu) = -2\,G_3\\ [[x_1(\xi_2\mu)-x_2(\xi_1\mu)]] = -2\mathscr{P}_0(m_G),\\ [[\xi_3\mu]] = -\mathscr{P}_0(\overline{H}_3)\,,\ [[x_\alpha(\xi_3\mu)]] = -\mathscr{P}(\overline{H}_\alpha)\ \Big\}\end{aligned}\tag{3.4}$$

(this follows by applying Lemma 1.1.2 in a similar way as repeatedly done in the previous Chapter).

Using definition (3.3), the limit problem $\phi(k)$ in (3.1) can be rewritten as

$$\begin{aligned}\phi(k)\ &= \inf\Big\{\mathscr{C}^{lim}(\mu)+k\int d\mu\ :\ \mu=\theta\,dx\,,\ \theta\in L^\infty(Q;[0,1])\Big\}\\ &= \sqrt{2k}\,\inf\Big\{\mathscr{C}^{lim}(\mu)+\frac{1}{2}\int d\mu\ :\ \mu=\theta\,dx\,,\ \theta\in L^\infty(Q;[0,\sqrt{2k}])\Big\}\,,\end{aligned}\tag{3.5}$$

where the second equality is obtained multiplying μ by $\sqrt{2k}$ (for $k>0$).

Thus, in view of (3.5), the natural candidate to be the limit problem of $\frac{\phi(k)}{\sqrt{2k}}$ as $k\to+\infty$ is the following minimization problem, set on $\mathscr{M}^+(Q)$:

$$\overline{m} := \inf\Big\{\mathscr{C}^{lim}(\mu)+\frac{1}{2}\int d\mu\ :\ \mu\in\mathscr{M}^+(Q)\Big\}\,.\tag{3.6}$$

In the next proposition, we give a useful reformulation of $\overline{m}$ as a maximization problem for a linear form under constraint, which in turn admits a pretty tractable dual form. Recall that $\eta>0$ is the second Lamé parameter in our elastic potential ($j(z) = (\lambda/2)(\mathrm{tr}z)^2+\eta|z|^2$).

Proposition 3.1.1. *Any optimal measure $\overline{\mu}$ in* (3.6) *satisfies*

$$\mathscr{C}^{lim}(\overline{\mu}) = \frac{1}{2}\int d\overline{\mu} = \frac{\overline{m}}{2}\,,\tag{3.7}$$

and $\overline{m}$ agrees with the following supremum:

$$\sup_{v\in TW(Q)}\Big\{\langle G,v\rangle_{\mathbb{R}^3}\ :\ \|\overline{j}(e_{13}(v),e_{23}(v),e_{33}(u))\|_{L^\infty(Q)}\le\frac{1}{2}\Big\}\,,\tag{3.8}$$

or alternatively with the minimum of the dual problem

$$\begin{aligned}\min_{\sigma\in\mathscr{M}(Q;\mathbb{R}^3)}\Big\{\int|\sigma|\ :\ \partial_1\sigma_1+\partial_2\sigma_2 = -\frac{G_3}{\sqrt{\eta}}\,,\ [[x_1\sigma_2-x_2\sigma_1]] = -\frac{\mathscr{P}_0(m_G)}{\sqrt{\eta}}\,,\\ [[\sigma_3]] = -\frac{\mathscr{P}_0(\overline{H}_3)}{\sqrt{Y}}\,,\ [[x_\alpha\sigma_3]] = -\frac{\mathscr{P}(\overline{H}_\alpha)}{\sqrt{Y}}\Big\}\,.\end{aligned}\tag{3.9}$$

We are now ready to establish that, as expected, $\overline{m}$ is the limit problem of $\frac{\phi(k)}{\sqrt{2k}}$ as $k \to +\infty$. Actually Theorem 3.1.1 below shows that such convergence holds true in the variational sense, namely not only for the values of the infima, but also for the corresponding solutions.

Theorem 3.1.1. *(i) For $k > 0$, the map $k \mapsto \frac{\phi(k)}{\sqrt{2k}}$ is nonincreasing and, as $k \to +\infty$, it converges decreasingly to $\overline{m}$.*

*(ii) if θ_k is a solution to the density formulation (3.1) of $\phi(k)$, up to subsequences θ_k converges weakly * in $L^\infty(Q;[0,1])$ to a solution $\overline{\mu}$ of problem* (3.6).

By the convergence statement (ii) in Theorem 3.1.1, in order to understand which kind of concentration phenomenon occurs for small amounts of material, one needs to answer the following question: what can be said about solutions $\overline{\mu}$ to problem (3.6)? We would like to write explicitly, or at least characterize, such solutions.

In this direction, let us first show that optimal measures $\overline{\mu}$ are strictly related to solutions $\overline{\sigma}$ to the dual problem (3.9). More precisely, we have:

Proposition 3.1.2. *If $\overline{\sigma}$ is optimal for problem* (3.9)*, then $\overline{\mu} := |\overline{\sigma}|$ is optimal for problem (3.6). Conversely, if $\overline{\mu}$ is optimal for problem (3.6), and $\overline{\xi}$ is optimal for the dual form (3.4) of $\mathscr{C}^{lim}(\overline{\mu})$, then $|\overline{\xi}| = 2\sqrt{\eta}\ \overline{\mu}$-a.e., and $\overline{\sigma} := \frac{\overline{\xi}}{2\sqrt{\eta}}\overline{\mu}$ is optimal for problem (3.9).*

Hence, if we determine a solution $\overline{\sigma}$ of the dual problem (3.9), we can solve the primal problem $\overline{m}$, simply by taking $\overline{\mu} := |\overline{\sigma}|$ (and viceversa). The advantage is that the dual formulation sometimes happens to be more tractable than the primal one.

We notice that the constraints imposed on the admissible measures σ in the minimization problem (3.9) of $\overline{m}$ only involve the behavior of $\sigma(\cdot,x_3)$ for each fixed $x_3 \in I$, this reveals that the problem can be solved section by section:

$$\overline{m} = \int_I m\left(-\frac{G_3}{\sqrt{\eta}}, -\frac{\mathscr{P}_0(m_G)}{\sqrt{\eta}}, -\frac{\mathscr{P}_0(\overline{H_3})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_1})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_2})}{\sqrt{Y}}\right) dx_3\,, \tag{3.10}$$

with m defined as

$$m(r(x'),t,s_0,s_1,s_2) := \inf_{\sigma \in \mathscr{M}(D;\mathbb{R}^3)} \left\{ \int_D |\sigma| \ :\ \operatorname{div}(\sigma_1,\sigma_2) = r(x')\,,\ \int_D (x_1\sigma_2 - x_2\sigma_1) = t\,, \right.$$
$$\left. \int_D \sigma_3 = s_0\,,\ \int_D x_\alpha \sigma_3 = s_\alpha \right\} \tag{3.11}$$

for any $r(x') \in H^{-1}(D)$ and t, s_i arbitrary real constants.

We recall that, in view of (h2), in our case there holds either $r = 0$ or $s_i = 0$.

In Section 3.3 we will deal with problem (3.11) considering the loads introduced in paragraph 2.1.3, and determine the concentration phenomena that occur for small amounts of material, characterizing the behavior on the sections of the design region.

3.2 The proofs of the main results

In this Section are gathered the proofs of the results stated in Section 3.1, which are quite technical.

Let us begin with the proof of of Proposition 3.1.1, useful for the demonstration of Theorem 3.1.1.

Proof of Proposition 3.1.1 Let m_0 denote the supremum in (3.8). For every $t \in \mathbb{R}^+$, by the definition (3.3) of $\mathscr{C}^{lim}(\mu)$ and the same inf-sup commutation argument already used in the proof of Proposition 2.4.1, we infer:

$$\begin{aligned}
&\inf_{\mu}\left\{ \mathscr{C}^{lim}(\mu) \ : \ \int d\mu \le t \right\} \\
&= \sup_{(v,u)} \inf_{\mu} \left\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v), e_{23}(v), e_{33}(u))\, d\mu \ : \ \int d\mu \le t \right\} \\
&= \sup_{(v,u)} \left\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - t \|\overline{j}(e_{13}(v), e_{23}(v), e_{33}(u))\|_{L^\infty(Q)} \right\} = \frac{m_0^2}{2t} ,
\end{aligned}$$

where the last equality follows by writing $v = sv_0$, with $s \in \mathbb{R}$ and v_0 admissible for problem (3.8), and optimizing in the real variable s.

Then, since by the definition (3.6) of $\overline{m}$ we have

$$\overline{m} = \inf_{t \in \mathbb{R}^+} \left\{ \mathscr{C}^{lim}(\mu) + \frac{t}{2} \ : \ \int d\mu \le t \right\} = \inf_{t \in \mathbb{R}^+} \left(\frac{m_0^2}{2t} + \frac{t}{2} \right) ,$$

and since the function $t \mapsto \left(\frac{m_0^2}{2t} + \frac{t}{2} \right)$ attains its minimum on $\mathbb{R}^+$ at $t = m_0$, we deduce that the equality $\overline{m} = m_0$ holds and that any optimal measure $\overline{\mu}$ satisfies (3.7).

The dual form (3.9) of problem (3.8) follows from Lemma 1.1.2, applied with $X := (TW \times BN) \cap (\mathscr{C}_0^\infty(Q;\mathbb{R}^3))^2$, $Y := \mathscr{C}_0(Q;\mathbb{R}^3)$, $A(v,u) := (e_{13}(v), e_{23}(v), e_{33}(u))$, $\Phi(v,u) := -\langle G, v\rangle_{\mathbb{R}^3} - \langle H, u\rangle_{\mathbb{R}^3}$, and $\Psi(y) = 0$ if $\|\overline{j}(y)\|_\infty \le 1/2$, and $+\infty$ otherwise. □

We can now give the proof of the main result: Theorem 3.1.1.

Proof of Theorem 3.1.1 We divide the proof into several steps.

Proof of (i)

The second equality in (3.5) shows that the map $k \mapsto \frac{\phi(k)}{\sqrt{2k}}$ is nonincreasing and satisfies the inequality $\frac{\phi(k)}{\sqrt{2k}} \ge \overline{m}$. In order to show that it converges to $\overline{m}$ as $k \to +\infty$, we exploit the formulation of $\phi(k)$ given in (2.75), in which we insert the change of variable $(\tilde{v}, \tilde{u}) = (v,u)/\sqrt{2k}$. We obtain

$$\frac{\phi(k)}{\sqrt{2k}} = \sup_{(v,u) \in TW \times BN} \left\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \sqrt{2k} \int_Q \left[\overline{j}\big(e_{13}(v), e_{23}(v), e_{33}(u)\big) - \frac{1}{2} \right]_+ dx \right\} .$$

Let (v_k, u_k) be fields in $(TW \times BN) \cap (\mathscr{C}^\infty(Q;\mathbb{R}^3))^2$ such that

$$\limsup_{k \to +\infty} \frac{\phi(k)}{\sqrt{2k}} = \lim_{k \to +\infty} \left\{ \langle G, v_k\rangle_{\mathbb{R}^3} + \langle H, u_k\rangle_{\mathbb{R}^3} - \sqrt{2k} \int_Q \left[\overline{j}\big(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k)\big) - \frac{1}{2} \right]_+ dx \right\} .$$

By using the coercivity of $[\bar{j}(z)-k]_+$, the inequality $\phi(k)\geq 0$, and the assumption that F and G are admissible loads, we may find positive constants C_1, C_2, C_3 and C_4 such that

$$\begin{aligned}
&C_1\|(e_{13}(v_k),e_{23}(v_k))\|^2_{L^2(Q;\mathbb{R}^2)}+C_2\|e_{33}(u_k)\|^2_{L^2(Q)}\\
&\leq \sqrt{2k}\int_Q [\bar{j}(e_{13}(v_k),e_{23}(v_k),e_{33}(u_k))-\frac{1}{2}]_+\,dx \leq \langle G,v_k\rangle_{\mathbb{R}^3}+\langle H,u_k\rangle_{\mathbb{R}^3}\\
&\leq C_3\|(e_{13}(v_k),e_{23}(v_k))\|_{L^2(Q;\mathbb{R}^2)}+C_4\|e_{33}(u_k)\|_{L^2(Q)}\,.
\end{aligned}$$

We deduce that $\|e_{\alpha 3}(v_k)\|_{L^2(Q)}$ and $\|e_{33}(u_k)\|_{L^2(Q)}$ are bounded. We claim that this property implies that, up to subsequences,

(a) there exists $v\in TW(Q)$ such that $\lim_k v_k=v$ weakly in $H^1(Q;\mathbb{R}^2)\times L^2(I;H^1_m(D))$ and $\lim_k e_{\alpha 3}(v_k)=e_{\alpha 3}(v)$ weakly in $L^2(Q)$;

(b) there exists $u\in BN(Q)$ such that $\lim_k u_k=u$ weakly in $H^1(Q;\mathbb{R}^3)$ and $\lim_k e_{33}(u_k)=e_{33}(u)$ weakly in $L^2(Q)$.

If we prove the claim we are done: by lower semicontinuity we obtain

$$\int_Q [\bar{j}(e_{13}(v),e_{23}(v),e_{33}(u))-1/2]_+\,dx \leq \liminf_k \int_Q [\bar{j}(e_{13}(v_k),e_{23}(v_k),e_{33}(u_k)-1/2]_+\,dx = 0\,,$$

and hence

$$\bar{j}(e_{13}(v),e_{23}(v),e_{33}(u))\leq \frac{1}{2} \qquad \text{a.e. in } Q,$$

i.e. the couple (v,u) is admissible in the definition (3.8) of m_0. Then we conclude that

$$\lim_{k\to+\infty}\frac{\phi(k)}{\sqrt{2k}} \leq \lim_{k\to+\infty}(\langle G,v_k\rangle_{\mathbb{R}^3}+\langle H,u_k\rangle_{\mathbb{R}^3})=\langle G,v\rangle_{\mathbb{R}^3}+\langle H,u\rangle_{\mathbb{R}^3}\leq m_0=\overline{m}\,.$$

Proof of (a)

Since $(e_{13}(v_k),e_{23}(v_k))$ is bounded in $L^2(Q;\mathbb{R}^2)$, there exists a positive constant C such that

$$\begin{aligned}
C &\geq \int_Q |c_k'(x_3)(-x_2,x_1)+\nabla_{x'}w_k|^2\,dx\\
&\geq \inf\Big\{\int_D |(-x_2,x_1)+\nabla w|^2\,dx' \,:\, w\in H^1(D)\Big\}\cdot\int_I |c_k'(x_3)|^2\,dx_3\\
&=\gamma\int_I |c_k'(x_3)|^2\,dx_3\,,
\end{aligned}$$

where c_k and w_k are associated to v_k according to (2.9), and γ is the positive constant introduced in Lemma 2.3.3. By applying the Poincaré-Wirtinger inequality, we obtain that c_k is uniformly bounded in $H^1_m(I)$.
By difference, it is also clear that $\nabla_{x'}w_k$ is uniformly bounded in $L^2(Q;\mathbb{R}^2)$, hence w_k is uniformly bounded in $L^2(I;H^1_m(D))$.
Let c and w be the weak limits of c_k and w_k in $H^1_m(I)$ and $L^2(I;H^1_m(D))$ respectively, and set $v:=(-c(x_3)x_2,c(x_3)x_1,w)$. Then $v\in TW(Q)$ and $\lim_k e_{\alpha 3}(v_k)=e_{\alpha 3}(v)$ weakly in $L^2(Q)$.

Proof of (b)
Using the representation (2.8) of $BN(Q)$, we can write

$$u_k(x) = \left(\zeta_{k,1}(x_3),\ \zeta_{k,2}(x_3),\ \zeta_{k,3}(x_3) - x_\alpha \zeta'_{k,\alpha}(x_3)\right)\ ,$$

for some $\zeta_{k,\alpha} \in H^2_m(I)$ and $\zeta_{k,3} \in H^1_m(I)$.
Exploiting the boundedness of $\|e_{33}(u_k)\|_{L^2(Q)}$, the Hölder inequality and the hypothesis $\int_D x_\alpha = 0$ we infer that

$$C_0\|\zeta'_{k,3}\|^2_{L^2(I)} + C_\alpha\|\zeta''_{k,\alpha}\|^2_{L^2(I)} - 2C_{12}\|\zeta''_{k,1}\|_{L^2(I)}\|\zeta''_{k,2}\|_{L^2(I)} \leq C\,, \tag{3.12}$$

where

$$C_0 := |D|\,,\ \ C_\alpha := \int_D x^2_\alpha\, dx'\,,\ \ C_{12} := \left|\int_D x_1 x_2\, dx'\right|$$

and C is a positive constant.
An immediate consequence is the estimate

$$C_1\|\zeta''_{k,1}\|^2_{L^2(I)} + C_2\|\zeta''_{k,2}\|^2_{L^2(I)} - 2C_{12}\|\zeta''_{k,1}\|_{L^2(I)}\|\zeta''_{k,2}\|_{L^2(I)} \leq C\,. \tag{3.13}$$

If we prove that $\|\zeta''_{k,\alpha}\|_{L^2(I)}$ are uniformly bounded we have finished: combining (3.12) and (3.13) we obtain that also $\|\zeta'_{k,3}\|_{L^2(I)}$ is uniformly bounded. Hence, since $(\zeta_{k,\alpha}, \zeta_{k,3}) \in (H^2_m(I))^2 \times H^1_m(I)$, by applying Poincaré-Wirtinger inequality, we infer that $\zeta_{k,i}$ weak converge in $H^1(Q)$, up to subsequences, to some ζ_i belonging to the same spaces. Moreover, if we call u the field associated to ζ_i according to (2.8), we have that $\lim_k e_{33}(u_k) = e_{33}(u)$ weakly in $L^2(Q)$.
From (3.13) is clear that either $\|\zeta''_{k,1}\|_{L^2(I)}$ and $\|\zeta''_{k,2}\|_{L^2(I)}$ are both uniformly bounded, or both unbounded. Assume by contradiction that there exists a subsequence (not relabeled) such that

$$\lim_k \|\zeta''_{k,1}\|_{L^2(I)} = +\infty,\ \ \lim_k \|\zeta''_{k,2}\|_{L^2(I)} = +\infty$$

and let

$$l_k := \frac{\|\zeta''_{k,1}\|_{L^2(I)}}{\|\zeta''_{k,2}\|_{L^2(I)}}\,.$$

Dividing (3.13) by $\|\zeta''_{k,2}\|^2_{L^2(I)}$ and passing to the limit as $k \to +\infty$, we obtain that $\limsup_k l_k < +\infty$. Let l be an accumulation point of the sequence l_k. Again, dividing (3.13) by $\|\zeta''_{k,2}\|^2_{L^2(I)}$ and passing to the limit as $k \to +\infty$ (subsequence not relabeled), we obtain

$$C_1 l^2 - 2C_{12} l + C_2 \leq 0\,,$$

that is absurd. Indeed the expression above is always strictly positive: it is continuous with respect to l and, since

$$|C_{12}|^2 = \left|\int_D x_1 x_2\, dx'\right|^2 \leq C\left(\int_D |x_1 x_2|\, dx'\right)^2 < \left(\int_D x_1^2\, dx'\right)\left(\int_D x_2^2\, dx'\right) = C_1 C_2$$

with C a constant depending on D, it has complex solutions.

Proof of (ii)
If θ_k is an optimal density for $\phi(k)$, setting $\mu_k := \sqrt{2k}\,\theta_k\,dx$ one has

$$\frac{\phi(k)}{\sqrt{2k}} = \mathscr{C}^{lim}(\mu_k) + \frac{1}{2}\int d\mu_k \,.$$

Since $\mathscr{C}^{lim}(\mu_k) \geq 0$ and since by monotonicity $\frac{\phi(k)}{\sqrt{2k}} \leq \phi(1)$, the above equation implies that the integral $\int d\mu_k$ remains uniformly bounded. Then up to a subsequence there exists $\overline{\mu}$ such that $\mu_k \stackrel{*}{\rightharpoonup} \overline{\mu}$. By using item (i) already proved, the weak * semicontinuity of the map $\mu \mapsto \mathscr{C}^{lim}(\mu)$, and the definition (3.6) of $\overline{m}$, we obtain

$$\overline{m} = \lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} = \lim_{k\to+\infty} \left\{ \mathscr{C}^{lim}(\mu_k) + \frac{1}{2}\int d\mu_k \right\} \geq \mathscr{C}^{lim}(\overline{\mu}) + \frac{1}{2}\int d\overline{\mu} \geq \overline{m}\,.$$

Hence $\overline{\mu}$ is a solution to problem (3.1.1). □

We conclude with the proof of Proposition 3.1.2, that enlightens the relationship between the solutions of primal and dual formulations of $\overline{m}$.

Proof of Proposition 3.1.2

Let $\overline{\sigma}$ be optimal for the dual problem (3.9), and set $\overline{\mu} := |\overline{\sigma}|$. Then we have $\left|\frac{d\overline{\sigma}}{d\overline{\mu}}\right| = 1$ $\overline{\mu}$-a.e. and

$$\int d\overline{\mu} = \overline{m}. \tag{3.14}$$

Moreover, since $\overline{\sigma}$ is admissible in (3.9) (see also Lemma 2.4.2), there hold

$$\langle G, v\rangle_{\mathbb{R}^3} = 2\sqrt{\eta}\langle \overline{\sigma}_\alpha, e_{\alpha 3}(v)\rangle_{\mathbb{R}^3} \quad \forall v \in TW(Q)\cap C^\infty(Q;\mathbb{R}^3) \tag{3.15}$$

and

$$\langle H, u\rangle_{\mathbb{R}^3} = \sqrt{Y}\langle \overline{\sigma}_3, e_{33}(u)\rangle_{\mathbb{R}^3} \quad \forall u \in BN(Q)\cap C^\infty(Q;\mathbb{R}^3)\,. \tag{3.16}$$

For brevity of notation, let us denote by x the vectorial function

$$x(v,u) := (2\sqrt{\eta}e_{\alpha 3}(v), \sqrt{Y}e_{33}(u))\,.$$

Then we get

$$\begin{aligned}
\mathscr{C}^{\mathrm{lim}}(\overline{\mu}) &= \sup\left\{ \langle\overline{\sigma}, x\rangle_{\mathbb{R}^3} - \frac{1}{2}\int_Q |x|^2\, d\overline{\mu} \ :\ x = x(v,u)\,(v,u) \in TW(Q)\times BN(Q)\right\} \\
&= \sup\left\{ \int_Q \left(\frac{d\overline{\sigma}}{d\overline{\mu}}\cdot x - \frac{1}{2}|x|^2\right) d\overline{\mu} \ :\ x = x(v,u)\,(v,u) \in TW(Q)\times BN(Q)\right\} \\
&\leq \frac{1}{2}\int_Q \left|\frac{d\overline{\sigma}}{d\overline{\mu}}\right|^2 d\overline{\mu} = \frac{1}{2}\int d\overline{\mu}\,.
\end{aligned} \tag{3.17}$$

In view of the inequalities (3.14) and (3.17) we conclude that

$$\mathscr{C}^{\mathrm{lim}}(\overline{\mu}) + \frac{1}{2}\int d\overline{\mu} \leq \int d\overline{\mu} \leq \overline{m}\,,$$

then $\overline{\mu}$ is optimal for the problem (3.1.1).

Conversely, assume that $\overline{\mu}$ is optimal for the problem (3.1.1), and let $\overline{\xi}$ be optimal for the dual form (3.4) of $\mathscr{C}^{\lim}(\overline{\mu})$, that is

$$\int_Q \overline{j}^*(\overline{\xi})\, d\overline{\mu} = \mathscr{C}^{\lim}(\overline{\mu})\,. \tag{3.18}$$

Set $\overline{\sigma} := \left(\frac{\overline{\xi}'\overline{\mu}}{2\sqrt{\eta}}\,,\, \frac{\overline{\xi}_3\overline{\mu}}{\sqrt{Y}}\right)$, and notice that it is admissible for problem (3.9). If we prove that

$$\left|\left(\frac{\overline{\xi}'}{2\sqrt{\eta}}\,,\, \frac{\overline{\xi}_3}{\sqrt{Y}}\right)\right| \leq 1 \quad \overline{\mu}-\text{a.e.} \tag{3.19}$$

we are done: indeed in this case $\overline{\sigma}$ is optimal for (3.9) because

$$\int |\overline{\sigma}| = \int \left|\frac{d\overline{\sigma}}{d\overline{\mu}}\right| d\overline{\mu} \leq \int d\overline{\mu} = \overline{m}\,.$$

Let us prove (3.19). By (3.18), if (v_k, u_k) is a minimizing sequence for $\mathscr{C}^{\lim}(\overline{\mu})$, one has

$$\int_Q \overline{j}^*(\overline{\xi})\, d\overline{\mu} = \mathscr{C}^{\lim}(\overline{\mu}) = \lim_k \left\{ \langle G, v_k\rangle_{\mathbb{R}^3} + \langle H, u_k\rangle_{\mathbb{R}^3} - \int_Q \overline{j}\big(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k)\big)\, d\overline{\mu} \right\}. \tag{3.20}$$

For every k, by (3.15), (3.16) and the definition of $\overline{\sigma}$, it holds

$$\langle G, v_k\rangle_{\mathbb{R}^3} + \langle H, u_k\rangle_{\mathbb{R}^3} = \int_Q \overline{\xi} \cdot (e_{13}(v_k), e_{23}(v_k), e_{33}(u_k))\, d\overline{\mu}\,. \tag{3.21}$$

Now, by arguing in a similar way as in the proof of Proposition 3.1.1 (see also [14, Corollary 2.4]) , we observe that the minimizing sequence (v_k, u_k) can be chosen so that $\overline{j}(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k)) \leq \frac{1}{2}$ on Q.
Denote by χ a cluster point of $\big(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k)\big)$ in $L^2_{\overline{\mu}}(Q;\mathbb{R}^3)$. Then we have

$$\overline{j}(\chi) \leq \frac{1}{2} \qquad \overline{\mu}\text{-a.e.} \tag{3.22}$$

and

$$\liminf_k \int_Q \overline{j}\big(e_{13}(v_k), e_{23}(v_k), e_{33}(u_k)\big)\, d\overline{\mu} \geq \int_Q \overline{j}(\chi)\, d\overline{\mu}\,. \tag{3.23}$$

By (3.20), (3.21) and (3.23), we obtain the following converse Fenchel inequality

$$\int_Q \overline{j}^*(\overline{\xi})\, d\overline{\mu} \leq \int_Q \overline{\xi}\cdot\chi\, d\overline{\mu} - \int_Q \overline{j}(\chi)\, d\overline{\mu}\,.$$

Hence

$$\overline{\xi} = \overline{j}'(\chi) = (4\eta\chi_1, 4\eta\chi_2, Y\chi_3)\,, \tag{3.24}$$

where the second equality follows by recalling the explicit form of $\overline{j}$.

In turn, (3.24) gives (3.19) in view of (3.22):

$$\left|\left(\frac{\overline{\xi}'}{2\sqrt{\eta}}\,,\, \frac{\overline{\xi}_3}{\sqrt{Y}}\right)\right|^2 = \left|(2\sqrt{\eta}\,\chi'\,,\, \sqrt{Y}\chi_3)\right|^2 = 2\overline{j}(\chi) \leq 1 \quad \overline{\mu}-\text{a.e.}$$

□

3.3 Concentration phenomena

We conclude the Chapter by studying the behavior of optimal configurations for small filling ratios, considering the loads introduced in paragraph 2.1.3.

Thanks to Proposition 3.1.1, in order to determine optimal measures $\overline{\mu}$ for problem (3.6), one is reduced to study the solutions $\overline{\sigma}$ to the dual problem (3.9), more precisely there holds $\overline{\mu} = |\overline{\sigma}|$. Moreover, in view of (3.11), $\overline{\sigma}$ solves, section by section, an infimum problem of the form

$$m(r(x'),t,s_0,s_1,s_2) := \inf_{\sigma \in \mathscr{M}(D;\mathbb{R}^3)} \left\{ \int_D |\sigma| \ : \ \operatorname{div}(\sigma_1,\sigma_2) = r(x')\,,\ \int_D (x_1\sigma_2 - x_2\sigma_1) = t\,, \right.$$
$$\left. \int_D \sigma_3 = s_0\,,\ \int_D x_\alpha \sigma_3 = s_\alpha \right\} \tag{3.25}$$

with $r(x') \in H^{-1}(D)$ and $t, s_i \in \mathbb{R}$ depending on the components G and H of the load.

For some choices of the loads, problem (3.25) is pretty tractable since some of the parameters $\{r, t, s_i\}$ vanish.

In what follows, when there is no ambiguity, we omit the parameters that vanish in the argument of m, and we assume for simplicity $\eta = 1$.

First, in §3.3.1, we present the case of pure torsion loads with null vertical component, namely the case $r = s_i = 0$ in (3.25): it turns out that, when the cross section D of the rod is a convex set, the material distribution tends to concentrate, section by section, near the boundary of its Cheeger set. Let us recall that, under the assumption that D is convex, its *Cheeger set* is the unique solution to the problem

$$\inf_{E \subset \overline{D},\, 1\!\!1_E \in BV(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla 1\!\!1_E|}{|E|} \tag{3.26}$$

(for more details about the Cheeger problem, see paragraph 1.4.3).

In §3.3.2 we deal with pure vertical loads H, namely the case $r = t = 0$ in (3.25): it turns out that the material distribution tends to concentrate, section by section, near some portions of the lateral surface of the design region.

Finally, in §3.3.3 and §3.3.4, we study the cases $s_i = 0$ and $r = 0$ respectively: here the optimal measures can't be characterized explicitly, nevertheless the study brings into play two interesting variants of the Cheeger problem for D.

3.3.1 The case $r = s_i = 0$

Theorem 3.3.1. *Assume that $G_3 = H = 0$ and that D is convex. Denote by C the Cheeger set of D. Then the unique solution to problem* (3.9) *is*

$$\overline{\sigma} := \frac{1}{2\sqrt{\eta}} M_G(x_3) \otimes \frac{1}{|C|} \tau_{\partial C}(x') \mathscr{H}^1 \llcorner \partial C\,, \tag{3.27}$$

and hence the unique solution $\overline{\mu}$ to problem (3.6) is

$$\overline{\mu} = \frac{1}{2} |\mathscr{P}_0(m_G)(x_3)| \otimes \frac{1}{|C|} \mathscr{H}^1 \lfloor_{\partial C}\,. \tag{3.28}$$

We remark that to the best of our knowledge, until now there was no rigorous statement and proof for this geometric characterization of optimal "light" torsion rods in terms of Cheeger sets. Let us emphasize that such characterization is valid only in pure torsion.

Proof of Theorem 3.3.1 By assumption $G_3 = H = 0$, then problem (3.9) reads

$$\min_{\sigma\in\mathscr{M}(Q;\mathbb{R}^2)} \left\{ \int_Q |(\sigma_1,\sigma_2)| \ :\ \partial_1\sigma_1 + \partial_2\sigma_2 = 0\,,\ [[x_1\sigma_2 - x_2\sigma_1]] = -\mathscr{P}_0(m_G) \right\} .$$

Since the constraints depend only on x_3, solutions can be searched under the form

$$\sigma = \gamma(x_3)\otimes\nu(x') \quad \text{with } \gamma\in\mathscr{M}(I) \text{ and } \nu\in\mathscr{M}(D;\mathbb{R}^2)\,.$$

It is easy to show that, up to constant multiples, the optimal measures $(\overline{\gamma},\overline{\nu})$ are uniquely determined respectively by

$$\overline{\gamma}(x_3) = \frac{\mathscr{P}_0(m_G)}{2}$$

and $\overline{\nu}$ solution of

$$m(0,-2,0,0,0) = \inf_{\nu\in\mathscr{M}(D;\mathbb{R}^2)} \left\{ \int_D |\nu| \ :\ \operatorname{div}\nu = 0\,,\ \int_D (x_1\,d\nu_2 - x_2\,d\nu_1) = -2 \right\} .$$

Since D is simply connected, the condition of zero divergence implies that any admissible ν is of the form $\nu = (-D_2u, D_1u)$, for some u in the space $BV_0(D)$ of bounded variation functions which vanish identically outside $\overline{D}$. So that problem $m(0,-2,0,0,0)$ can be rewritten as

$$m(0,-2,0,0,0) = \min\left\{ \int_D |Du| \ :\ u\in BV_0(D)\,,\ \int_D u = 1 \right\} .$$

This is precisely the relaxed formulation of problem (3.26) (see 1.18), and in the convex framework has a unique solution $\overline{u} = |C|^{-1}\,1\!\!1_C$, where C is the Cheeger set of D. Hence the unique solution $\overline{\sigma}$ of (3.9) is

$$\overline{\sigma} = \frac{1}{2}\,\mathscr{P}_0(m_G)(x_3)\otimes\frac{1}{|C|}\,\tau_{\partial C}(x')\,\mathscr{H}^1\lfloor_{\partial C}\,,$$

namely (3.27), in particular the optimal measure $\overline{\mu}$ is given by (3.28). □

In view of this result, if we consider G as in (2.24) of Example 2.1.1, with $c = 2|C|$, we obtain

$$\overline{\mu} = 1\!\!1_I(x_3)\otimes\mathscr{H}^1\lfloor_{\partial C}(x')\,, \tag{3.29}$$

and its support is represented in Figure 3.1.

3.3.2 The case $r = t = 0$

Let us now consider the case in which both G_3 and $\mathscr{P}_0(m_G)$ vanish, as it happens if $G\equiv 0$. As an example, let us consider a design region having as section D the square $[-1,1]^2$. A similar procedure can be performed also for general geometries.

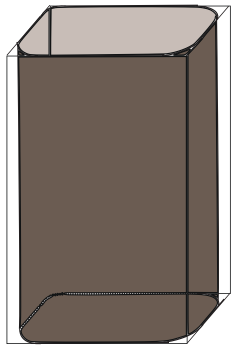


Figure 3.1: *The support of $\overline{\mu}$ in the case $G_3 = H = 0$, $m_G = c(\delta_{-1/2} - \delta_{1/2})$, the section D being the square.*

Before stating the result, let us introduce a family of subsets of the section D, depending on a triple of parameters $s = (s_0, s_1, s_2)$. For every $s \in \mathbb{R}^3 \setminus \{0\}$, we define the sets $M^+(s)$ and $M^-(s)$ according to the scheme (3.30): let us associate to the parameter s_i the symbol $+$ if $s_i = \max_j |s_j|$, the symbol $-$ if $-s_i = \max_j |s_j|$, and the symbol 0 if $|s_i| < \max_j |s_j|$.

The cases not included in (3.30), corresponding to the opposite signature of the triples (s_0, s_1, s_2), can be deduced by interchanging the role of M^+ and M^-.

We remark that, depending on the sign of s_i and whether they satisfy $\max |s_i|$ or not, the sets $M^{\pm}$ can be the empty set, the entire square D, one of the segments of ∂D, or one of the corners of the square $\{(i,j)\}_{i,j \in \{\pm 1\}}$.

s_0 s_1 s_2	M^+	M^-
$+$ 0 0	D	$\emptyset$
0 $+$ 0	$\{+1\}\times[-1,1]$	$\{-1\}\times[-1,1]$
0 0 $+$	$[-1,1]\times\{+1\}$	$[-1,1]\times\{-1\}$
$+$ $+$ 0	$\{+1\}\times[-1,1]$	$\emptyset$
$+$ $-$ 0	$\{-1\}\times[-1,1]$	$\emptyset$
$+$ 0 $+$	$[-1,1]\times\{+1\}$	$\emptyset$
$+$ 0 $-$	$[-1,1]\times\{-1\}$	$\emptyset$
0 $+$ $+$	$\{(1,1)\}$	$\{(-1,-1)\}$
0 $+$ $-$	$\{(-1,1)\}$	$\{(1,-1)\}$
$+$ $+$ $+$	$\{(1,1)\}$	$\emptyset$
$+$ $-$ $+$	$\{(-1,1)\}$	$\emptyset$
$+$ $-$ $-$	$\{(-1,-1)\}$	$\emptyset$
$+$ $+$ $-$	$\{(1,-1)\}$	$\emptyset$

(3.30)

Proposition 3.3.1. *Assume that $G \equiv 0$ and D is the square $[-1,1]^2$. Hence a solution $\overline{\mu}$ to problem (3.6) is of the form*

$$\overline{\mu} = |\rho_{opt}|(x', x_3) ,$$

where $\rho_{opt} \in \mathscr{M}(Q)$ satisfies, for a.e. $x_3 \in I$, the following system:

$$\begin{cases} \int_D |\rho_{opt}| = \max_{i=0\dots 2} |s_i(x_3)| , \\ \int_D \rho_{opt} = s_0(x_3) , \\ \int_D x_\alpha \rho_{opt} = s_\alpha(x_3) , \end{cases} \tag{3.31}$$

with

$$s(x_3) := \left(-\frac{\mathscr{P}_0(\overline{H_3})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_1})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_2})}{\sqrt{Y}} \right) .$$

Moreover the support of an optimal measure satisfies

$$\mathrm{spt}\rho_{opt}{}^{\pm} \subseteq M^{\pm}(s(x_3)) ,$$

$M^{\pm}(s)$ being the subsets of the section D introduced in (3.30).

Proof. Since $G \equiv 0$, problem (3.10) reads

$$\overline{m} = \int_I m \left(0, 0, -\frac{\mathscr{P}_0(\overline{H_3})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_1})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_2})}{\sqrt{Y}} \right) dx_3 .$$

It is easy to prove that $m(s_1, s_2, s_3)$ is an infimum problem over scalar measures:

$$m(s) = \inf_{\rho \in \mathscr{M}(D)} \left\{ \int_D |\rho| \;:\; \int_D \rho = s_0 , \int_D x_\alpha \rho = s_\alpha \right\} . \tag{3.32}$$

Hence every solution $\overline{\mu}$ for $\overline{m}$ is of the form

$$\overline{\mu} = |\rho_{opt}|(x', x_3) ,$$

with $\rho_{opt}(\cdot, x_3)$ optimal for $m \left(-\frac{\mathscr{P}_0(\overline{H_3})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_1})}{\sqrt{Y}}, -\frac{\mathscr{P}(\overline{H_2})}{\sqrt{Y}} \right)$.

Let us characterize the solutions ρ_{opt} of $m(s)$.
To this aim we write the dual problem:

$$\begin{aligned} m^*(s_0^*, s_1^*, s_2^*) &= \sup_{s \in \mathbb{R}^3} \{ s \cdot s^* - m(s) \} \\ &= -\inf_{s \in \mathbb{R}^3} \inf \left\{ \int |\rho| - s \cdot s^* \;:\; \rho \in \mathscr{M}(D) , \int \rho = s_0 , \int x_\alpha \rho = s_\alpha \right\} \\ &= -\inf_{\rho \in \mathscr{M}(D)} \left\{ \int_D (|\rho| - (s_0^* + s_\alpha^* x_\alpha)\rho) \right\} \\ &= \chi_{K(D)} , \end{aligned} \tag{3.33}$$

with $K(D)$ defined as the following convex set:

$$K(D) := \{ s^* \in \mathbb{R}^3 \;:\; |s_0^* + x_\alpha s_\alpha^*| \leq 1 \; \forall (x_1, x_2) \in D \} .$$

It is easy to prove that, in the case of the square $D = [-1,1]^2$, the set $K(D)$ is given by

$$\left\{s^* \in \mathbb{R}^3 \ : \ \sum |s_i^*| \le 1\right\} .$$

As a consequence we infer that m reads

$$m(s) = \sup_{s^*\in\mathbb{R}^3} \{s\cdot s^* - m^*(s^*)\} = \sup_{s^*\in K} \{s\cdot s^*\} = \max\{|s_i|\} , \tag{3.34}$$

in particular an optimal measure ρ_{opt} is characterized by (3.31).

In order to deduce information about the support of the positive and negative part of ρ_{opt} we compare m and m^*: by the Fenchel equality, formula (4.2) and formula (3.34), we infer that for every $s^* \in \partial m(s)$

$$\int_D |\rho_{opt}| = s\cdot s^* = \int_D (s_0^* + x_\alpha s_\alpha^*)\rho_{opt} ,$$

that is

$$\int_D |\rho_{opt}| - (s_0^* + x_\alpha s_\alpha^*)\rho_{opt} = 0 .$$

Then we have a precise information on the support of ρ_{opt}^+ and ρ_{opt}^-:

$$\begin{cases} \operatorname{spt}\rho_{opt}^+ \subseteq \{x' \in D \ : \ s_0^* + x_\alpha s_\alpha^* = 1\} , \\ \operatorname{spt}\rho_{opt}^- \subseteq \{x' \in D \ : \ s_0^* + x_\alpha s_\alpha^* = -1\} . \end{cases}$$

By the arbitrariness of $s^* \in \partial m(s)$ we obtain

$$\operatorname{spt}\rho_{opt}{}^{\pm} \subseteq M^\pm(s) := \cap_{s^*\in\partial m(s)}\{x' \in D \ : \ s_0^* + x_\alpha s_\alpha^* = \pm 1\} . \tag{3.35}$$

We conclude by characterizing the sets $M^\pm$. By definition, $\partial m(s)$ reads

$$\partial m(s) = \{s^* \in \mathbb{R}^3 \ : \ s\cdot s^* = \max\{|s_i|\} , \sum |s_i^*| \le 1\} ,$$

and it can be characterized explicitly. Since $\partial m(s)$ it is invariant under multiplication by positive constant, namely

$$\partial m(\alpha s) = \partial m(s) \quad \forall \alpha > 0 ,$$

we give its expression for s such that $m(s) = 1$. Let ξ and ζ denote two arbitrary constants with modulus less than 1, then

$$\begin{aligned} \partial m(\pm 1, \xi, \zeta) &= \{(\pm 1, 0, 0)\} , \\ \partial m(\pm 1, \pm 1, \xi) &= \{(\pm\alpha, \pm\beta, 0)\}_{\alpha+\beta=1,\ \alpha,\beta\ge 0} , \\ \partial m(\pm 1, \pm 1, \pm 1) &= \{(\pm\alpha, \pm\beta, \pm\gamma)\}_{\alpha+\beta+\gamma=1,\ \alpha,\beta,\gamma\ge 0} , \end{aligned}$$

and analogous expressions hold true exchanging the roles of s_i.
By combining these computations with (3.35) we obtain the representation (3.30) of $M^\pm(s)$: depending on the sign of s_i and whether they satisfy $\max|s_i|$ or not, the sets $M^\pm$ in which concentrates every optimal $\rho_{opt}{}^\pm$ can be the empty set, the entire square D, one of the segments of ∂D, or one of the corners of the square $\{(i,j)\}_{i,j\in\{\pm 1\}}$.

□

We remark that Example 2.1.4 falls into this category of problem. Let us consider the particular case introduced in (2.30): in view of (2.31) we obtain that

$$\begin{cases} \mathscr{P}_0(\overline{H_3}) = b 1\!\!1_I(x_3) \; , \\ \mathscr{P}(\overline{H_\alpha}) = a_\alpha 1\!\!1_I(x_3) \; . \end{cases}$$

Hence any optimal measure $\overline{\mu}$ is of the form

$$\overline{\mu} = 1\!\!1_I(x_3) \otimes |\rho_{opt}|(x')$$

with ρ_{opt} solution for $m(s)$ with

$$s_0 = -\frac{b}{\sqrt{Y}}, \;\; s_\alpha = -\frac{a_\alpha}{\sqrt{Y}} \; .$$

The subset of the boundary in which $\overline{\mu}$ concentrates depends on the values of the parameters a_α and b, according to the scheme in table (3.30). Some particular choices are represented in Figure 3.2 and 3.3.

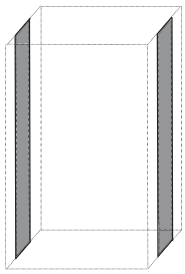


Figure 3.2: *In Example 2.1.4, considering the choice $a_2 = b = 0$ and $a_1 = -\sqrt{Y}$ in the definition (2.30) of f, a particular solution is $\overline{\mu} = \mathscr{H}^2 \lfloor_{\{\pm 1\}\times[-1/4,1/4]\times[-1/2,1/2]}$*

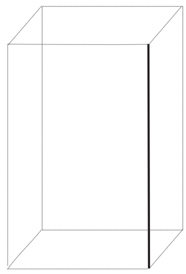


Figure 3.3: *In Example 2.1.4, considering the choice $a_1 = b = \sqrt{Y}$ and $a_2 = -\sqrt{Y}$ in the definition (2.30) of f, a particular solution is*
$\overline{\mu} = \mathscr{H}^1 \lfloor_{\{1\}\times\{-1\}\times[-1/2,1/2]}$

Remark 3.3.1. In general the solution $\overline{\mu}$ of problem (3.6) is not uniquely determined, unless its support is localized, section by section, in a single point of ∂D (see table (3.30)).

Moreover, we remark that in view of (3.30) it is clear that there is no superposition of solutions: a solution of problem $m(s)$ might not be the superposition of solutions of problems $m(s_0,0,0)$, $m(0,s_1,0)$ and $m(0,0,s_2)$, since its support must satisfy stricter constraints.

3.3.3 The case $s_i = 0$

Let us consider the case in which $H = 0$ and G is the admissible load introduced in Example 2.1.3: the a load concentrated on the whole boundary of Q and, according to (2.27) and (2.28), satisfies

$$\begin{cases} G_3 & = -h\mathscr{H}^1 \llcorner \partial D \\ m_G & = c_h(\delta_{-1/2} - \delta_{1/2})(x_3) \end{cases} \tag{3.36}$$

with $h \in L^2_m(\partial D)$ and c_h a constant depending on h (see (2.28) for the exact definition).

In the next proposition we see that the characterization of an optimal measure $\overline{\mu}$ for problem (3.9) brings into play a variant of the Cheeger problem.

Proposition 3.3.2. *Assume that G satisfies (3.36) and that D is simply connected. Hence a solution $\overline{\mu}$ for problem (3.9) is of the form*

$$\overline{\mu} = 1\!\!1_I(x_3) \otimes |Du+q|(x') \,, \tag{3.37}$$

where $q \in L^2(D)$ satisfies $q \cdot \tau \llcorner \partial D = h$ and u is a solution for

$$\min\left\{ \int |Du+q| \;:\; u \in BV_0(D)\,,\; \int_D u = C(q,h) \right\} \,, \tag{3.38}$$

$C(q,h)$ being a suitable constant depending just on q and h.

Proof. In view of (3.36) and since $H \equiv 0$, an optimal measure $\overline{\mu}$ is of the form

$$\overline{\mu} = 1\!\!1_I(x_3) \otimes |\overline{\sigma}|(x') \,,$$

with $\overline{\sigma} = (\overline{\sigma_1}, \overline{\sigma_2})$ solution of

$$\begin{aligned} m(h\mathscr{H}^1\lfloor_{\partial D}, -c_h) = \inf_{\sigma \in \mathscr{M}(D;\mathbb{R}^2)} \Bigg\{ \int_D |\sigma| \;:\; \partial_1\sigma_1 + \partial_2\sigma_2 = h\mathscr{H}\llcorner\partial D, \\ \int_D (x_1\, d\sigma_2 - x_2\, d\sigma_1) = -c_h \Bigg\} \,. \end{aligned} \tag{3.39}$$

Since D is simply connected, any admissible σ can be written ad $\sigma = (-\partial_2 u, \partial_1 u)$ for some u in the space $BV(D)$ of bounded variation functions such that

$$\begin{cases} \partial_\tau u = h & \text{on } \partial D\,, \\ \int_D x' \cdot Du = -c_h\,. \end{cases} \tag{3.40}$$

Hence we can reformulate problem (3.39) as

$$\inf\left\{ \int_D |Du| \;:\; u \in BV(D)\,,\; \partial_\tau u = h \text{ on } \partial D, \int_D x' \cdot Du = -c_h \right\} \,. \tag{3.41}$$

Let us fix a primitive P of h, that is

$$P \in H^1(D)\,,\; \nabla P \cdot \tau\lfloor_{\partial D} = h\,.$$

Let $u \in BV(D)$ be admissible for problem (3.41), then $\partial_\tau u = \partial_\tau P$ on ∂D, that is $(u-P)\lfloor_{\partial D} = c(u)$ for some constant $c(u) \in \mathbb{R}$. An easy computation leads

$$\begin{aligned} \int_D u &= \frac{1}{2}\int_{\partial D} u x' \cdot n - \frac{1}{2}\int_D x' \cdot Du = \frac{1}{2}\int_{\partial D} (P + c(u)) x' \cdot n + \frac{c_h}{2} \\ &= \frac{1}{2}\int_{\partial D} P(x' \cdot n) + \frac{c(u)}{2}\int_{\partial D} x' \cdot n + \frac{c_h}{2} \,. \end{aligned}$$

Let us now consider $\tilde{u} := u - P - c(u)$: exploiting the properties of u, P and the definition of $c(u)$ we infer

$$\begin{cases} \tilde{u} \in BV_0(D) \, , \\ D\tilde{u} = Du - \nabla P \, , \\ \int_D \tilde{u} = \left(\frac{1}{2}\int_{\partial D} P(x'\cdot n) - \int_D P\right) + \frac{c_h}{2} + c(u)\left(\frac{1}{2}\int_{\partial D} x'\cdot n - |D|\right) \, . \end{cases} \tag{3.42}$$

Integrating by parts it is easy to show that

$$\frac{1}{2}\int_{\partial D} x' \cdot n = |D| \, , \tag{3.43}$$

hence $\int_D \tilde{u}$ does not depend on the constant $c(u)$. In view of (3.42) and (3.43), we can rewrite problem (3.41) as follows:

$$\min\left\{ \int |Du + \nabla P| \; : \; u \in BV_0(D) \, , \; \int_D u = C(P,h) \right\} \, , \tag{3.44}$$

$$C(P,h) := \frac{1}{2}\int_{\partial D} P(x'\cdot n) - \int_D P + \frac{c_h}{2} \, .$$

Finally, denoting by $q := \nabla P$ we obtain the formulation (3.38). □

In this case problem (3.9) amounts to solve a "modified" Cheeger problem: in (3.38) the admissible functions are the same appearing in the classical version of Cheeger problem, while the functional to minimize is perturbed by the vector field q.

3.3.4 The case $r = 0$

In this last paragraph we focus our attention on the case in which the parameter r appearing in (3.9) vanishes. An explicit example is given by taking the component G as in Example (2.1.1) and the component H as in Example (2.1.4). Let us recall the data of such a problem:

$$\begin{cases} G_3 = 0 \\ m_G = c\,1\!\!1_I(x_3) \\ \mathscr{P}_0(\overline{H_3}) = b\,1\!\!1_I(x_3) \, , \\ \mathscr{P}(\overline{H_\alpha}) = a_\alpha 1\!\!1_I(x_3) \, . \end{cases}$$

As already noticed a solution $\overline{\mu}$ for problem (3.6) is of the form

$$\overline{\mu} = 1\!\!1_I(x_3) \otimes |\overline{\sigma}|(x') \, ,$$

with $\overline{\sigma}(x')$ solving problem $m(0,-c,-b,-a_1,-a_2)$.

In view of the properties found in §3.3.1 and §3.3.2, in which we studied $m(0,-c,0,0,0)$ and $m(0,0,-b,-a_a,-a_2)$ respectively, we expect that a solution $\overline{\mu}$ for problem (3.6), section by section, is linked with some variant of the Cheeger problem.

If in addition we consider D simply connected, exploiting the assumption $r = 0$ we infer that $\overline{\sigma} = (-D_2\overline{u}, D_1\overline{u}, \overline{\rho})$, with $(\overline{u}, \overline{\rho})$ optimal for problem

$$\inf\left\{ \int_D |(Du,\rho)| \; : \; u \in BV_0(D) \, , \; \int_D u = \frac{c}{2} \, , \; \rho \in \mathscr{M}(D) \, , \; \int_D \rho = b \, , \; \int_D x_\alpha \rho = a_\alpha \right\} \, .$$

In the Proposition 3.3.3 we characterize an optimal couple $(D\overline{u},\overline{\rho})$, enlightening the role of the Cheeger problem.

Before stating the result, let us introduce some useful notations and preliminary lemmas.

For every $t\in\mathbb{R}$, $s\in\mathbb{R}^3$ let us define $m(t,s)$ as the following infimum problem:

$$\inf\left\{\int_D |(Du,\rho)| \;:\; u\in BV_0(D)\,,\ \int_D u=t\,,\ \rho\in\mathscr{M}(D)\,,\ \int_D\rho=s_0\,,\ \int_D x_\alpha\rho=s_\alpha\right\}\,, \tag{3.45}$$

we underline that it is a reparametrization of the usual problem m introduced in (3.11), considering $t/2$ instead of t.

Let $K(D)$ be the convex subset of $\mathbb{R}^4$

$$K(D):=\{(\lambda,s^*)\in\mathbb{R}\times\mathbb{R}^3\;:\;\exists\sigma\in L^2(D;\mathbb{R}^2)\text{ st }-\operatorname{div}\sigma=\lambda\,,\ |\sigma|^2+|s_0^*+x_\alpha s_\alpha^*|^2\le 1\text{ in }D\}\,.$$

Finally, for every $s^*\in\mathbb{R}^3$ such that $|s_0^*+x_\alpha s_\alpha^*|\le 1$ we denote by α_{s^*} the positive function

$$\alpha_{s^*}(x'):=\sqrt{1-|s_0^*+x_\alpha s_\alpha^*|^2}\,, \tag{3.46}$$

defined in D.

Lemma 3.3.1. *For every $(\lambda,s^*)\in\mathbb{R}\times\mathbb{R}^3$, the dual problem $m^*(\lambda,s^*)$ reads*

$$m^*(\lambda,s^*)=\chi_{K(D)}\,.$$

Moreover the convex set $K(D)$ can be represented as the union of intervals as follows

$$K(D)=\bigcup_{\{s^*:|s_0^*+x_\alpha s_\alpha^*|\le 1\}}[-\overline{\lambda}(s^*),\overline{\lambda}(s^*)]\times\{s^*\}\,,$$

with $\overline{\lambda}(s^*)$ defined as

$$\overline{\lambda}(s^*):=\sup\{\lambda\;:\;\exists\sigma\in L^2(D;\mathbb{R}^2)\text{ st }-\operatorname{div}\sigma=\lambda\,,\ |\sigma|\le\alpha_{s^*}(x')\ a.e.\text{ in }D\}\,. \tag{3.47}$$

Proof. By definition of Fenchel transform, the dual problem of m reads

$$\begin{aligned} m^*(\lambda,s^*)&=\sup_{(t,s)}\{t\lambda+s\cdot s^*-m(t,s)\}\\ &=\sup_{(u,\rho)}\left\{\int_D u\lambda+\int_D(s_0^*+s_\alpha x_\alpha^*)\rho-\int_D|(Du,\rho)|\right\}\,.\end{aligned}$$

Hence $m^*(\lambda,s^*)=0$ if the couple (λ,s^*) satisfies

$$\int_D|(Du,\rho)|-\int_D[\lambda u+(s_o^*+x_\alpha s_\alpha^*)\rho]\ge 0\quad\forall(u,\rho)\in BV_0(D)\times\mathscr{M}(D) \tag{3.48}$$

and $+\infty$ otherwise. Exploiting the definition of total variation, it is easy to show that the set of (λ,s^*) satisfying (4.64) is given by $K(D)$. The alternative characterization of $K(D)$ can be easily deduced by considering the sections of such convex set for every fixed s^*. □

Lemma 3.3.2. *The variational problem (3.47) admits the following dual formulation:*

$$\inf\left\{\int_D \alpha_{s^*}|Dw| \;:\; w\in BV_0(D),\ \int_D w=1\right\}\,, \tag{3.49}$$

where α_{s^*} is the non negative function defined in (3.46).

Problem (3.49) is a version of the relaxed formulation of the Cheeger problem with a weigh α that varies in D. This variant of the Cheeger problem has been treated recently by Ionescu and Lachand-Robert in [69]: in the paper the authors present the case in which both the integral to minimize and the integral in the constraint are weighted; their study is motivated by applications to landslides modeling.

Proof of Lemma 3.3.2. For every $p\in\mathbb{R}$, let us consider the infimum problem

$$f(p):=\inf\left\{\int_D \varphi(Aw) \;:\; w\in BV_0(D),\ \int_D w=p\right\}\,.$$

with φ the convex function $\varphi(z):=\alpha_{s^*}|z|$ (we recall that α_{s^*} is assumed to be positive) and $Aw:=Dw$. In particular problem (3.47) equals $f(1)$. An easy computation gives

$$f^*(p^*)=\inf\left\{\int_D \varphi^*(\sigma) \;:\; -A^*\sigma=p^*\right\}\,.$$

Since $Aw=Dw$ we have $A^*\sigma=\operatorname{div}\sigma$, moreover $\varphi^*(z^*)=\chi_{|z^*|\le\alpha_{s^*}}$. Hence

$$f^*(p^*)=\begin{cases} 0 \ \text{ if } \exists\sigma \;:\; -\operatorname{div}\sigma=p^*,\ |\sigma|\le\alpha \\ +\infty \text{ otherwise}\end{cases}$$

By definition of Fenchel transform we infer

$$f(p)=\sup_{p^*}\{pp^*-f^*(p^*)\}=\sup\{pp^* \;:\; \exists\sigma \;:\; -\operatorname{div}\sigma=p^*,\ |\sigma|\le\alpha\}\,. \tag{3.50}$$

Recalling that (3.47) equals $f(1)$, formula (3.50) with $p=1$ gives (3.49). □

Proposition 3.3.3. *Let $(\lambda,s^*)\in\partial m(t,s)$. Then an optimal couple $(D\overline{u},\overline{\rho})$ for problem $m(t,s)$ defined in (3.45) satisfies*

(i) $-\operatorname{div}\left(\alpha_{s^*}(x')\frac{D\overline{u}}{|D\overline{u}|}\right)=\lambda$;

(ii) the singular part $\overline{\rho}_s$ of $\overline{\rho}$ with respect to $|D\overline{u}|$ concentrates on the straight lines $\{s_0^*+x_\alpha s_\alpha^*=\pm1\}$;

(iii) the absolutely continuous part $\overline{\rho}_a$ of $\overline{\rho}$ with resect to $|D\overline{u}|$ satisfies

$$\overline{\rho}_a=\frac{s_0^*+x_\alpha s_\alpha^*}{\alpha_{s^*}(x')}\,|D\overline{u}|\,.$$

Proof. It is easy to prove that $(\overline{u},\overline{\rho})$ is an optimal couple for problem $m(t,s)$ defined in (3.45) if and only if there exists $(\lambda,s^*)\in\partial m(t,s)$ such that

$$\int_D |(D\overline{u},\overline{\rho})| = \int_D \sigma\cdot dD\overline{u} + \int_D (s_0^* + x_\alpha s_\alpha^*)\, d\overline{\rho}\ , \tag{3.51}$$

with σ associated to λ according to the definition of $K(D)$, namely such that

$$\begin{cases} -\operatorname{div}\sigma = \lambda \text{ in } D \\ |\sigma| \le \alpha_{s^*}(x') \text{ in } D \end{cases} \tag{3.52}$$

Let us decompose the measures $|D\overline{u}|$ and $\overline{\rho}$ as follows:

$$\begin{aligned} \overline{\rho} &= \overline{\rho}_s + \overline{\rho}_a\ , \text{ with } \overline{\rho}_a << |D\overline{u}|\ ,\ \overline{\rho}_s \perp |D\overline{u}|\ . \\ \theta &:= \frac{d\overline{\rho}_a}{d|D\overline{u}|}\ , \\ v &:= \frac{dD\overline{u}}{d|D\overline{u}|}\ . \end{aligned}$$

In view of (3.51), it is clear that that $\overline{\rho}_s$ concentrates on the straight lines $\{s_0^*+x_\alpha s_\alpha^* = \pm 1\}$. Let us consider the absolutely continuous part. Using the notation above, the integrand in the left hand side of (3.51) reads

$$|(D\overline{u},\overline{\rho})| = \sqrt{1+\theta^2}\, d|D\overline{u}|\ . \tag{3.53}$$

Recalling that $|(\sigma, s_0^*+x_\alpha s_\alpha^*)|\le 1$, the condition (3.51) implies that

$$(\sigma, s_0^*+x_\alpha s_\alpha^*) = \frac{(D\overline{u},\overline{\rho})}{|(D\overline{u},\overline{\rho})|}\ .$$

In view of (3.53) we obtain

$$\sigma = \frac{v}{\sqrt{1+\theta^2}}\ ,\ \ s_0^*+x_\alpha s_\alpha^* = \frac{\theta}{\sqrt{1+\theta^2}}\ .$$

that is

$$\sigma = \alpha_{s^*}(x')\,\frac{dD\overline{u}}{|dD\overline{u}|}\ ,\ \ \theta = \frac{s_0^*+x_\alpha s_\alpha^*}{\alpha_{s^*}(x')}\ .$$

Hence, recalling that σ satisfies (3.52), we conclude that

$$-\operatorname{div}\left(\alpha_{s^*}(x')\frac{D\overline{u}}{|D\overline{u}|}\right) = \lambda\ .$$

□

CHAPTER 4

A nonstandard free boundary problem arising in the shape optimization of thin torsion rods

In this Chapter we face the question that raised in §2.4.4, about the existence of classical solutions for the compliance optimization problem in thin torsion rods:

$$\textit{Does problem } \phi(k) \textit{ admit a solution } \overline{\theta} \textit{ taking values into } \{0,1\} \text{ ?} \tag{4.1}$$

As we already noticed, problem $\phi(k)$ in (2.42) can be solved section by section (see (2.91)) and, in pure torsion regime, turns out to be linked to the following variational problem set in the plane.

Let D be a bounded and connected domain in $\mathbb{R}^2$, let s be a real parameter, and consider the variational problem

$$m(s) := \inf\left\{ \int_{\mathbb{R}^2} \varphi(\nabla u) \ : \ u \in H^1_c(D)\,,\ \int_{\mathbb{R}^2} u = s \right\}, \tag{4.2}$$

where

$$H^1_c(D) := \left\{ u \in H^1(\mathbb{R}^2) \ : \ \nabla u = 0 \text{ on } \mathbb{R}^2 \setminus \overline{D} \right\}$$

and the integrand $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is the following convex but non-strictly convex function:

$$\varphi(y) := \begin{cases} \frac{|y|^2}{2} + \frac{1}{2} & \text{if } |y| \geq 1 \\ |y| & \text{if } |y| < 1\,. \end{cases} \tag{4.3}$$

Notice that functions in $H^1_c(D)$ must vanish identically on the unique unbounded connected component of $\mathbb{R}^2 \setminus \overline{D}$; in particular, if D is simply connected, functions in $H^1_c(D)$ are elements of the usual Sobolev space $H^1_0(D)$, extended to zero out of D.

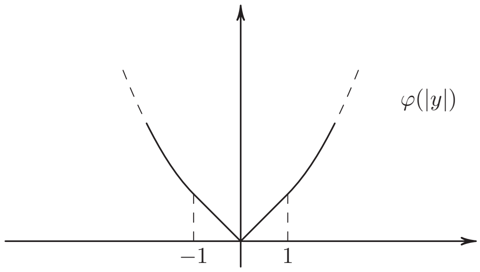


Figure 4.1: *The integrand φ is a radial function, convex but non-strictly convex in the ball of radius 1.*

More generally, if $D = D_0 \setminus \cup_{i=1}^{k} \overline{D_i}$, where D_i ($i = 0, 1, \dots, k$) are Jordan domains with mutually disjoint boundaries, functions in $H_c^1(D)$ are extensions to zero of elements of $H_0^1(D_0)$ which are constant on each D_i for $i = 1, \dots, k$.

Definition 4.0.1. *We say that u is a special solution to $m(s)$ if it minimizes (4.2) and satisfies the following constraint on the gradient:*

$$|\nabla u| \in \{0\} \cup (1, +\infty) \text{ a.e. in } D\,.$$

Below and throughout the Chapter, we adopt the following notation: if u is a special solution to problem $m(s)$, we call the *plateau* of u, and we denote it by $\Omega(u)$, the set $\{\nabla u = 0\}$ minus the unbounded connected component of $\mathbb{R}^2 \setminus \overline{D}$ (where $u \equiv 0$). The set $\Gamma(u) := \partial\Omega(u) \cap D$ will be called the *free boundary* of u (see Figure 4.2).

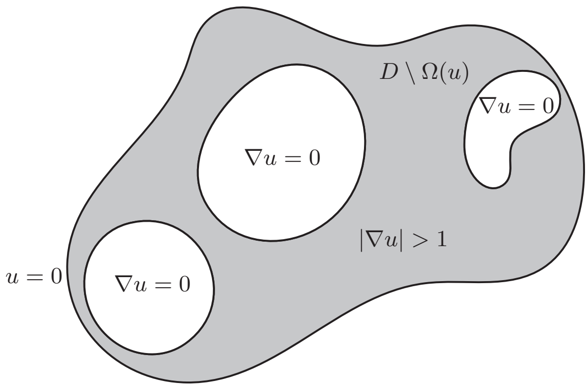


Figure 4.2: *Behavior of special solutions.*

In this framework, considering D as the cross section of a rod and s suitably chosen according to (2.104) (we recall that s is proportional to $1/\sqrt{k}$), question (4.1) is equivalent to ask

$$\text{Does problem } m(s) \text{ admit a special solution?} \tag{4.4}$$

More precisely, given $\overline{\theta}$ a solution for $\phi(k)$ on a fixed section and $\overline{u}$ a solution for $m(s)$, the following relations hold true:

$$\Omega(\overline{u}) = \{\overline{\theta} = 0\}\,, \quad \{|\nabla \overline{u}| > 1\} = \{\overline{\theta} = 1\}\,, \quad \{|\nabla \overline{u}| \in (0,1)\} = \{\overline{\theta} \in (0,1)\}\,,$$

in particular if $\overline{u}$ is not special, namely if $|\nabla\overline{u}|$ falls into the region of non-strict convexity of φ, homogenization phenomena occur.

In the light of the above discussion, the results presented in the next Sections can be applied to study the influence of the section's shape and of the filling ratio on the presence of homogenization regions in optimal thin torsion rods.

Let us emphasize that no precedent exists in this direction within the study of optimal thin plates. Indeed in that case the limit model obtained starting from three-dimensional elasticity through a $3d$-$2d$ dimension reduction process always admits classical "set" solutions, under the form of sandwich-like structures ([19], see also [14]).

The Chapter is organized as follows. In Section 4.1 we find necessary and sufficient optimality conditions, we deduce some consequences on the behavior of solutions to $m(s)$ (including a uniqueness criterion), and we study $m(s)$ as a function of s. In Section 4.2 we give some preliminary results about Cheeger sets.
In Section 4.3 we prove the existence and uniqueness of special solutions to $m(s)$ when D is a ball or a ring.
In Section 4.4 we prove the existence of special solutions to $m(s)$ for some domain D, different from those considered in Section 4.3.
In Section 4.5 we obtain qualitative properties of solutions and of special solutions.
In Section 4.6 we present some open problems and possible advances.

4.1 Existence, uniqueness, optimality conditions, and dependence on the parameter s.

The contents of this section are organized as follows: in §4.1.1 we study the minimization problem $m(s)$ in its primal formulation (4.2): we prove the existence of solutions, and a necessary and sufficient condition for optimality; in §4.1.2 we give the dual form of problem $m(s)$, we derive the corresponding optimality conditions and some of their consequences; in §4.1.3 we show some properties of $m(s)$ seen as a function of the parameter s.

4.1.1 Primal problem

We begin by establishing the existence of minimizers for $m(s)$, and their characterization as solutions to a variational inequality.

Proposition 4.1.1. *For every $s \in \mathbb{R}$, the infimum $m(s)$ is achieved in $H^1_c(D)$. A function $u \in H^1_c(D)$ is optimal if and only if*

$$\int_{\{\nabla u=0\}} |\nabla v| + \int_{\{\nabla u\neq 0\}} \langle \nabla\varphi(\nabla u), \nabla v\rangle \geq 0 \qquad \forall v \in H^1_c(D) \ : \ \int_{\mathbb{R}^2} v = 0\,.$$

Proof. We observe that, since functions in $H^1_c(D)$ vanish in the unbounded connected component of $\mathbb{R}^2 \setminus \overline{D}$, by Poincaré inequality there exists a positive constant C such that

$$\|u\|_{H^1(\mathbb{R}^2)} \leq C\,\|\nabla u\|_{L^2(\mathbb{R}^2)} \qquad \forall\, u \in H^1_c(D)\ .$$

Combined with the coercivity of φ (in fact $\varphi(y) \geq \frac{|y|^2}{2}$), this ensures that every minimizing sequence for problem $m(s)$ is weakly relatively compact in $H^1(\mathbb{R}^2)$. Clearly any cluster point belongs to $H^1_c(D)$. On the other hand, by the convexity of φ, the integral functional $I_\varphi(u) := \int_{\mathbb{R}^2} \varphi(\nabla u)$ is weakly lower semicontinuous on $H^1(\mathbb{R}^2)$ (see Theorem 1.2.3). Therefore the existence of at least one solution follows from the direct method of Calculus of Variations. Considering all the variations compatible with the integral constraint, it is straightforward to check that a minimizer u is characterized by the variational inequality $\delta I_\varphi(u,v) \geq 0$ for all $v \in H^1_c(D)$ such that $\int_{\mathbb{R}^2} v = 0$. Here the directional derivative $\delta I_\varphi(u,v)$ is given by

$$\delta I_\varphi(u,v) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\left[I_\varphi(u+\varepsilon v) - I_\varphi(u)\right] = \int_{\{\nabla u = 0\}} |\nabla v| + \int_{\{\nabla u \neq 0\}} \langle \nabla\varphi(\nabla u), \nabla v\rangle\,.$$

□

4.1.2 Dual problem

We are going to explicit the dual formulation of problem $m(s)$. An easy computation shows that the Fenchel conjugate of φ is given by

$$\varphi^*(\xi) = \begin{cases} \frac{1}{2}|\xi|^2 - \frac{1}{2} & \text{if } |\xi| > 1 \\ 0 & \text{if } |\xi| \leq 1\,. \end{cases} \tag{4.5}$$

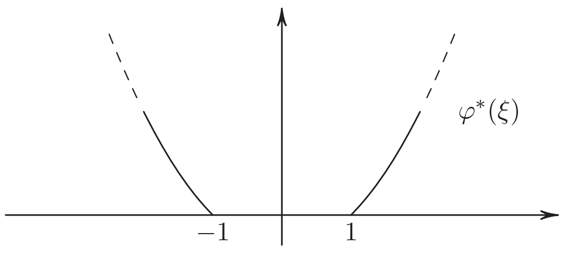


Figure 4.3: *The Fenchel conjugate φ is a radial function, convex but non-strictly convex in the ball of raius 1.*

Moreover let us introduce, for every $\lambda \in \mathbb{R}$, the class of vector fields

$$\mathscr{S}_\lambda(D) := \left\{\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2) \ :\ \text{spt}(\sigma) \subseteq \overline{D},\ \int_{\mathbb{R}^2} \sigma \cdot \nabla u = \lambda \int_{\mathbb{R}^2} u \quad \forall u \in H^1_c(D)\right\}. \tag{4.6}$$

By taking as test functions u in (4.6) elements of $H^1_0(D)$ extended to zero out of D, one can see that every $\sigma \in \mathscr{S}_\lambda(D)$ satisfies the condition $-\operatorname{div}\sigma = \lambda$ in D. In the special case when D is simply connected, all functions $u \in H^1_c(D)$ are of this type, so that $\mathscr{S}_\lambda(D)$ can be characterized as

$$\mathscr{S}_\lambda(D) = \left\{\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2) \ :\ \text{spt}(\sigma) \subseteq \overline{D},\ -\operatorname{div}\sigma = \lambda \text{ in } D\right\}. \tag{4.7}$$

More in general, if $D = D_0 \setminus \cup_{i=1}^k \overline{D_i}$, where D_i $(i = 0,1,\dots,k)$ are Jordan domains with mutually disjoint boundaries, one has

$$\mathscr{S}_\lambda(D) = \Big\{ .\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2) \ : \mathrm{spt}(\sigma) \subseteq \overline{D}, \qquad - \mathrm{div}\, \sigma = \lambda \text{ in } D, \int_{\partial D_i} \sigma \cdot n^{(i)} = -\lambda |D_i| \ \ \forall i = 1,\dots,k \Big\},$$

being $n^{(i)}$ the unit outer normal to ∂D_i.

Lemma 4.1.1. *The map $s \mapsto m(s)$ is a convex even function on $\mathbb{R}$, whose Fenchel conjugate is given by*

$$m^*(\lambda) = \min \Big\{ \int_{\mathbb{R}^2} \varphi^*(\sigma) \ : \ \sigma \in \mathscr{S}_\lambda(D) \Big\}\,. \tag{4.8}$$

Proof. Recalling definition (4.2), since the integrand φ is convex and even, we obtain immediately that the map $s \mapsto m(s)$ is a convex even function on $\mathbb{R}$. Its Fenchel conjugate is given by

$$m^*(\lambda) = \sup_{s \in \mathbb{R}} \Big\{ \lambda s - m(s) \Big\} = \sup_{u \in H^1_c(D)} \Big\{ \lambda \int_{\mathbb{R}^2} u - \int_{\mathbb{R}^2} \varphi(\nabla u) \Big\} \qquad \forall \lambda \in \mathbb{R}\,. \tag{4.9}$$

By seeing the constant λ as an element of the dual space of $H^1_c(D)$, we may rewrite (4.9) as the Fenchel conjugate of a composition:

$$m^*(\lambda) = \big(I_\varphi \circ A\big)^*(\lambda)\,,$$

where $I_\varphi : L^2(\mathbb{R}^2;\mathbb{R}^2) \to \mathbb{R}$ is the integral functional $I_\varphi(y) = \int_{\mathbb{R}^2} \varphi(y)$, and $A : H^1_c(D) \to L^2(\mathbb{R}^2;\mathbb{R}^2)$ is the gradient mapping $Au = \nabla u$. Then, since I_φ is convex continuous whereas A is a bounded linear operator, by Lemma 1.1.2, we have

$$m^*(\lambda) = \min \Big\{ (I_\varphi)^*(\sigma) \ : \ \sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)\,,\ \mathrm{spt}(\sigma) \subseteq \overline{D}\,,\ A^*\sigma = \lambda \Big\}\,.$$

The above equality entails (4.8), by taking into account that $(I_\varphi)^* = I_{\varphi^*}$ (see Proposition 1.2.1), and by observing that $A^*\sigma = \lambda$ holds if and only if σ belongs to the subset $\mathscr{S}_\lambda(D)$ given in (4.6).

□

We can now give the optimality conditions which characterize solutions to problems $m(s)$ and $m^*(\lambda)$.

Proposition 4.1.2. *Let $s, \lambda \in \mathbb{R}$, $u \in H^1_c(D)$, and $\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)$. There holds the following equivalence*

$$(i) \begin{cases} u \textit{ solution to } m(s) \\ \sigma \textit{ solution to } \ m^*(\lambda) \\ \lambda \in \partial m(s)\,. \end{cases} \iff (ii) \begin{cases} \int_{\mathbb{R}^2} u = s \\ \sigma \in \mathscr{S}_\lambda(D) \\ \sigma \in \partial\varphi(\nabla u) \textit{ a.e.} \end{cases}$$

Proof. [(i) $\Rightarrow$ (ii)] Let s, λ, u, σ satisfy (i). In particular, since by definition u and σ are admissible in problems (4.2) and (4.8) respectively, they satisfy $\int_{\mathbb{R}^2} u = s$ and $\sigma \in \mathscr{S}_\lambda(D)$. Thus we only have to show that $\sigma \in \partial\varphi(\nabla u)$ a.e. Since $\lambda \in \partial m(s)$, the Fenchel equality is satisfied

$$m(s) + m^*(\lambda) = s\lambda\,,$$

that is, thanks to the optimality of u and σ in (4.2) and (4.8),

$$\int_{\mathbb{R}^2} \varphi(\nabla u) + \int_{\mathbb{R}^2} \varphi^*(\sigma) = s\lambda = \lambda \int_{\mathbb{R}^2} u = \int_{\mathbb{R}^2} \nabla u \cdot \sigma\,, \tag{4.10}$$

which implies $\sigma \in \partial\varphi(\nabla u)$ a.e.

[(ii) $\Rightarrow$ (i)] Let s, λ, u, σ satisfy (ii). By the first two conditions in (ii), u and σ are admissible in problems (4.2) and (4.8) respectively. Moreover, the third condition $\sigma \in \partial\varphi(\nabla u)$ a.e. implies that (4.10) holds. Hence

$$\int_{\mathbb{R}^2} \varphi^*(\sigma) = \lambda s - \int_{\mathbb{R}^2} \varphi(\nabla u) \le \lambda s - m(s) \le m^*(\lambda)\,. \tag{4.11}$$

Therefore, σ is a solution to $m^*(\lambda)$, and all the inequalities in (4.11) hold with equality sign. This implies that u is a solution to $m(s)$ and that $\lambda \in \partial m(s)$. ☐

Let us examine more in detail the condition $\sigma \in \partial\varphi(\nabla u)$ a.e., appearing in Proposition 4.1.2. The convex integrand φ is differentiable at every $y \neq 0$, whereas its subdifferential at 0 is given by $\partial\varphi(0) = \{|y| \le 1\}$. Therefore, the inclusion $\sigma \in \partial\varphi(\nabla u)$ always holds true on $\mathbb{R}^2 \setminus \overline{D}$, where $\sigma = 0$ and $\nabla u = 0$. On the other hand, in view of the structure of the subdifferential of radial functions explained in Example 1.1.2, we infer that the same inclusion can be rewritten more explicitly on the different regions of D as

$$\begin{cases} \sigma = \nabla u & \text{on } \{x \in D : |\nabla u(x)| > 1\} \\ \sigma = \frac{\nabla u}{|\nabla u|} & \text{on } \{x \in D : 0 < |\nabla u(x)| \le 1\} \\ |\sigma| \le 1 & \text{on } \{x \in D : |\nabla u(x)| = 0\}\,. \end{cases} \tag{4.12}$$

These equalities have several implications, which are listed in the next corollaries. First of all, the region where solutions u to problem $m(s)$ satisfy the condition $|\nabla u| > 1$ turns out to be uniquely determined by s, together with the value of ∇u on it. More precisely we have:

Corollary 4.1.1. *There exist a measurable subset Q_s of D and a function $\psi_s \in L^2(Q_s;\mathbb{R}^2)$ such that, for any solution u to problem $m(s)$ and any solution σ to problem $m^*(\lambda)$, with $\lambda \in \partial m(s)$, it holds*

$$\{|\nabla u| > 1\} = \{|\sigma| > 1\} = Q_s \qquad \text{and} \qquad \nabla u = \sigma = \psi_s \text{ a.e. on } Q_s\,, \tag{4.13}$$

where the first equality is intended up to Lebesgue negligible sets.

Moreover, $Q_s = Q_t$ and $\psi_s = \psi_t$ whenever $\partial m(s) \cap \partial m(t) \neq \emptyset$.

Proof. It is enough to observe that the equalities in (4.55) hold true, choosing $\lambda \in \partial m(s)$, an arbitrary solution u to problem $m(s)$, and an arbitrary solution σ to problem $m^*(\lambda)$: it follows that the sets where $\{|\nabla u| > 1\}$ and $\{|\sigma| > 1\}$, and the values of ∇u and σ on them, only depend on s. Moreover, such sets and values agree as soon as there exists some $\lambda \in \partial m(s) \cap \partial m(t)$. □

From Corollary 4.1.1 we derive the following uniqueness criterion:

Corollary 4.1.2. *If there exists a special solution to problem $m(s)$, then there is no other solution.*

Proof. Let u be a special solution to $m(s)$, and let $\tilde{u}$ be another solution. From (4.13) we infer

$$\begin{aligned} m(s) &= \int_{\{0<|\nabla\tilde{u}|\le 1\}} |\nabla\tilde{u}| + \int_{\{|\nabla\tilde{u}|>1\}} \varphi(\nabla\tilde{u}) = \int_{\{0<|\nabla\tilde{u}|\le 1\}} |\nabla\tilde{u}| + \int_{Q_s} \varphi(\psi_s) \\ &= \int_{\{0<|\nabla\tilde{u}|\le 1\}} |\nabla\tilde{u}| + \int_{\mathbb{R}^2} \varphi(\nabla u) = \int_{\{0<|\nabla\tilde{u}|\le 1\}} |\nabla\tilde{u}| + m(s) \,, \end{aligned}$$

hence the set $\{0 < |\nabla\tilde{u}| \le 1\}$ is Lebesgue negligible. Then $\nabla\tilde{u} = \nabla u$ a.e., *i.e.* the two solutions u and $\tilde{u}$ coincide a.e. up to an additive constant. As elements of $H^1_c(D)$, they are both compactly supported, hence the additive constant is zero. □

As a further consequence of the equalities in (4.55), we get some information on the gradient of special solutions on their free boundary:

Corollary 4.1.3. *If u is a special solution for $m(s)$ with a smooth free boundary $\Gamma(u)$, it holds*

$$|\nabla u| = 1 \quad \text{on } \Gamma(u) \,. \tag{4.14}$$

Proof. If σ is a solution to problem $m^*(\lambda)$, with $\lambda \in \partial m(s)$, by Proposition 4.1.2 we know that $\sigma \in \mathscr{S}_\lambda(D)$ and $\sigma \in \partial\varphi(\nabla u)$ a.e. The former condition implies $-\operatorname{div}\sigma = \lambda$ in D, the latter implies that $|\sigma| > 1$ or $|\sigma| \le 1$ according to whether $|\nabla u| > 1$ or $\nabla u = 0$ (see (4.55) above). We deduce that

$$\left|\sigma \cdot n_{\Gamma(u)}\right| = 1 \qquad \text{on } \Gamma(u) \,,$$

where $n_{\Gamma(u)}$ denotes the unit normal to $\Gamma(u)$, pointing outside $\Omega(u)$. This implies (4.14) since

$$|\sigma \cdot n_{\Gamma(u)}| = |\nabla u \cdot n_{\Gamma(u)}| = |\nabla u| \qquad \text{on } \Gamma(u) \,.$$

□

4.1.3 Properties of the map $s \mapsto m(s)$

Below we derive several properties of $m(s)$, seen as a function of the real parameter s. Firstly we give some bounds on it, and we determine its asymptotic behavior as $s \to 0^+$ and $s \to +\infty$. To that aim we introduce two constants, τ_D and k_D, through the following variational problems set on the space $H^1_c(D)$:

$$\tau_D := \inf\Big\{ \int_{\mathbb{R}^2} |\nabla u|^2 \ : \ u \in H^1_c(D) \ , \ \int_{\mathbb{R}^2} u = 1 \Big\} \tag{4.15}$$

$$k_D := \inf\Big\{ \int_{\mathbb{R}^2} |\nabla u| \ : \ u \in H^1_c(D) \ , \ \int_{\mathbb{R}^2} u = 1 \Big\} \tag{4.16}$$

When D is simply connected, the constants τ_D and k_D are related to classical variational problems.

More precisely, solving problem (4.15) allows to determine the torsional rigidity of a cylinder with cross section D; indeed the Saint-Venant torsional stiffness of D, namely the Dirichlet energy of the unique solution $u \in H^1_0(D)$ to the equation $-\Delta u = 2$, is given precisely by $\frac{4}{\tau_D}$.

On the other hand, the relaxation of problem (4.16) in the space of BV functions, leads to the theory of Cheeger sets; the relationship between the constant k_D and the Cheeger constant of D will be discussed more in detail in Section 4.2.

Proposition 4.1.3. *The function $m(s)$ satisfies the following bounds:*

$$\max\Big\{ k_D\,|s|, \tau_D \frac{s^2}{2} \Big\} \ \le \ m(s) \ \le \frac{1}{2}\left(\tau_D\, s^2 + |D|\right) .$$

Furthermore, it holds

$$\lim_{s\to 0^+} \frac{m(s)}{s} \ = \ k_D\,, \qquad \lim_{s\to +\infty} \frac{m(s)}{s^2} \ = \ \frac{\tau_D}{2}\,. \tag{4.17}$$

Proof. The function φ defined by (4.3) satisfies the inequalities $\frac{1}{2}|y|^2 \le \varphi(y) \le \frac{1}{2}(|y|^2 + 1)$. Therefore, by homogeneity, we are led to:

$$\frac{1}{2}\,\tau_D\, s^2 \ \le \ m(s) \ \le \ \frac{1}{2}\left(\tau_D\, s^2 + |D|\right) ,$$

which implies the second equality in (4.17).

On the other hand, since $\varphi(y) \ge |y|$, it holds $m(s) \ge k_D|s|$, thus $\liminf_{s\to 0^+} \frac{m(s)}{s} \ge k_D$.

Let $u \in H^1_c(D)$ such that $\int_{\mathbb{R}^2} u = 1$ and $s > 0$. Since su is admissible for $m(s)$ and $\varphi(s\nabla u) \le s^2\,|\nabla u|^2$ on the set $\{|\nabla u| > \frac{1}{s}\}$, we have

$$\frac{m(s)}{s} \ \le \ \frac{1}{s}\int_{\mathbb{R}^2} \varphi(s\nabla u) \ \le \ \int_{\{|\nabla u| \le \frac{1}{s}\}} |\nabla u| + s \int_{\{|\nabla u| > \frac{1}{s}\}} |\nabla u|^2.$$

Thus $\limsup_{s\to 0^+} \frac{m(s)}{s} \ \le \ \int_{\mathbb{R}^2} |\nabla u|$ and the first equality in (4.17) follows by taking the infimum with respect to u over $H^1_c(D)$.

□

We now turn attention to the differentiability properties of $m(s)$. Proposition 4.1.4 below shows in particular that, for any $s>0$, the condition $\lambda \in \partial m(s)$ appearing in Proposition 4.1.2 turns out to determine λ uniquely, whereas this is not the case when $s=0$.

Proposition 4.1.4. *(i) At every $s>0$, $m(s)$ is differentiable, and*

$$m'(s)=\frac{1}{s}\Big[m(s)+\int_{Q_s}\big(\frac{1}{2}|\psi_s|^2-\frac{1}{2}\big)\Big]\ , \tag{4.18}$$

where Q_s and ψ_s are defined according to Corollary 4.1.1.

(ii) The subdifferential of m at the origin is given by

$$\partial m(0)=[-k_D,k_D]\ ,$$

where k_D is the constant defined in (4.16).

Remark 4.1.1. As a consequence of statement (ii) and of the convexity of m, we have that $m'(s)\geq k_D$ for all positive s and the map $s\mapsto m(s)$ is strictly increasing on $(0,+\infty)$.

Proof. (i) Let $s>0$ be fixed, and let $\lambda\in\partial m(s)$. If σ is a solution to $m^*(\lambda)$, by using the expressions of $m^*(\lambda)$ and φ^* given respectively by Lemma 4.1.1 and by (4.5), the Fenchel equality reads

$$\lambda s=m(s)+m^*(\lambda)=m(s)+\int_{\mathbb{R}^2}\varphi^*(\sigma)=m(s)+\int_{\{|\sigma|>1\}}\big(\frac{1}{2}|\sigma|^2-\frac{1}{2}\big)\ .$$

In view of Corollary 4.1.1, we conclude that λ is uniquely determined by the equality

$$\lambda s=m(s)+\int_{Q_s}\big(\frac{1}{2}|\psi_s|^2-\frac{1}{2}\big)\ .$$

Then $\partial m(s)=\{\lambda\}$, that is $m'(s)=\lambda$.

(ii) Since m is a convex even function, $\partial m(0)$ is a bounded closed interval of the form $[-c,c]$, for some positive constant c. Moreover, c agrees with the right derivative

$$m'_+(0):=\lim_{s\to0^+}\frac{m(s)-m(0)}{s}\ .$$

Since $m(0)=0$, by using the first equality in (4.17), we conclude that

$$m'_+(0)=\lim_{s\to0^+}\frac{m(s)}{s}=k_D\ .$$

□

Thanks to Proposition 4.1.4, we deduce that no special solutions can exist for s ranging in some open interval unless the map $s\mapsto m(s)$ is strictly convex on it.

Proposition 4.1.5. *Assume that the map $s \mapsto m(s)$ is affine on some interval $[a,b] \subset \{s \geq 0\}$. Then, for any $s \in (a,b]$, problem $m(s)$ does not admit a special solution. Moreover, if $a = 0$, for any $s \in [0,b]$ any solution u to $m(s)$ satisfies $|\nabla u| \leq 1$ a.e., and it holds $m(s) = k_D s$.*

Proof. We recall that the sets Q_s and ψ_s are defined as in Corollary 4.1.1.

Let us assume that for some $s \in [a,b]$ problem $m(s)$ admits a special solution, so that $m(s) = \int_{Q_s} \varphi(\psi_s)$, and let us show that necessarily $s = a$. By the assumption that m is affine on $[a,b]$, it follows that $m'(s) = m'(t)$ for any other $t \in [a,b]$. Therefore, in view of the last assertion of Corollary 4.1.1, for any $t \in [a,b]$ it holds $Q_t = Q_s$ and $\psi_t = \psi_s$. Thus, denoting by u_t a solution to $m(t)$, we have

$$\begin{aligned} m(t) &= \int_{\{|\nabla u_t| \leq 1\}} |\nabla u_t| + \int_{Q_t} \varphi(\psi_t) = \int_{\{|\nabla u_t| \leq 1\}} |\nabla u_t| + \int_{Q_s} \varphi(\psi_s) \\ &= \int_{\{|\nabla u_t| \leq 1\}} |\nabla u_t| + m(s) \ . \end{aligned}$$

In particular this implies $m(t) \geq m(s)$ and in turn, since m is strictly increasing, that $t \geq s$. By the arbitrariness of $t \in [a,b]$, we conclude that $s = a$.

In the special case when $a = 0$, we get $Q_s = Q_0$, for any $s \in [0,b]$. Clearly the equality $m(0) = 0$ implies $|Q_0| = 0$. Therefore it holds $|Q_s| = 0$ for any $s \in [0,b]$, which means that any solution u to problem $m(s)$ satisfies $|\nabla u| \leq 1$ and $\varphi(\nabla u) = |\nabla u|$ a.e., hence the conclusion.

□

4.2 Link with the Cheeger problem

Recall that the *Cheeger constant* of a bounded and connected domain D is defined by

$$h_D := \inf_{A \subset \overline{D}} \frac{|\partial A|}{|A|}, \tag{4.19}$$

where the infimum is taken over all the subsets A of $\overline{D}$ with finite perimeter (for a more detailed overview about the Cheeger problem see §1.4.3).

In this section we present some related properties which shed some light on the link between the Cheeger constant h_D and the minimization problem $m(s)$.

We point out that the fact that the Cheeger problem comes into play is not surprising: indeed the role of Cheeger sets already emerged in the study of the compliance optimization problem for vanishing filling ratios, in a rod-like set (see Theorem 3.3.1), namely in the limit of $\phi(k)$ as $k \to +\infty$, moreover, since s is proportional to $1/\sqrt{k}$ (see (2.104)), large values of k correspond to small values of s.

The first result in this direction is the relationship between h_D and the constant k_D defined in (4.16):

Proposition 4.2.1. *The constants h_D and k_D defined respectively in* (4.19) *and* (4.16) *satisfy the inequality $h_D \geq k_D$, with equality in case D is simply connected.*

Proof. The Cheeger constant introduced in (4.19) can also be recast as

$$h_D = \inf\left\{ \int_D |\nabla v| \ : \ v \in H^1_0(D)\ ,\ \int_{\mathbb{R}^2} v = 1 \right\}\ . \tag{4.20}$$

Then the statement follows by comparing (4.16) and (4.20). Indeed the space of extensions to zero of functions in $H^1_0(D)$ is included into $H^1_c(D)$, and coincides with it if D is simply connected. □

Remark 4.2.1. The above statement can be generalized to the case when $D = D_0 \setminus \cup_{i=1}^k \overline{D_i}$, being D_i ($i = 0, 1, \dots, k$) Jordan domains with mutually disjoint boundaries. Indeed, thanks to the inclusions $H^1_0(D) \subset H^1_c(D) \subset H^1_0(D_0)$, there holds $h_D \geq k_D \geq h_{D_0}$. Moreover, the equality $k_D = h_{D_0}$ holds as soon as there exists a Cheeger set C for D_0 such that

$$\partial C \cap \left(\bigcup_{i=1}^k D_i \right) = \emptyset\,, \tag{4.21}$$

and also the equality $h_D = k_D$ holds if in addition

$$\left(\bigcup_{i=1}^k D_i \right) \subset (D_0 \setminus C)\ . \tag{4.22}$$

Indeed, conditions (4.21) and (4.22) ensure respectively that the function $1\!\!1_C/|C|$ belongs not only to $H^1_0(D_0)$ but also to $H^1_c(D)$ and to $H^1_0(D)$. For instance, in Figure 4.4 below, the set D_0 is taken as a square, the grey region represents its Cheeger set, and conditions (4.21) and (4.22) are satisfied if the holes D_i are chosen respectively as in the left and in the right pictures.

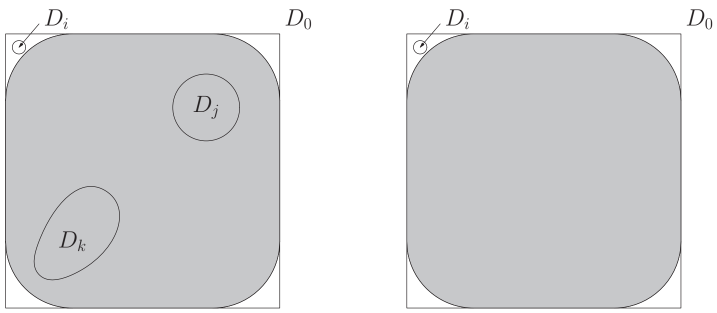


Figure 4.4: *About conditions* (4.21) *and* (4.22).

By combining Proposition 4.2.1 with Proposition 4.1.4 (ii) we obtain that, when D is simply connected, there holds

$$\partial m(0) = [-h_D, h_D]\,. \tag{4.23}$$

This identity allows to obtain Proposition 4.2.2 below, that will be exploited in Section 4.4. Though it is already known in the literature (see in particular [9, 62, 71]), we prefer to be self-contained and give a new proof of it, based on (4.23).

We need to introduce some definitions. Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected set with finite perimeter. We say that Ω is a *Cheeger set of itself* if

$$h_\Omega = \frac{|\partial\Omega|}{|\Omega|}.$$

Some examples of Cheeger sets of themselves are the ball, the ellipse and the annulus.

We say that Ω is *calibrable* if there exists a *calibration*, namely a field $\sigma \in L^2(\Omega;\mathbb{R}^2)$ such that

$$-\operatorname{div}\sigma = h_\Omega \quad \text{in } \Omega\,, \qquad \|\sigma\|_{L^\infty(\Omega)} \leq 1\,, \qquad [\sigma\cdot n_\Omega] = -1 \quad \mathscr{H}^1\text{-a.e. on } \partial\Omega\,.$$

Here $[\sigma\cdot n_\Omega]$ is meant as the weak notion of the trace of the normal component of σ on $\partial\Omega$, defined according to (5.7) (see also [7, Theorem 3.5] and [6, Theorem 1.2] for the same definition in case $\partial\Omega$ is Lipschitz).

Proposition 4.2.2. *Let Ω be a bounded and simply connected set with finite perimeter. Then*

$$\Omega \text{ is Cheeger set of itself} \iff \Omega \text{ is calibrable}\,. \tag{4.24}$$

Remark 4.2.2. Under the additional assumption that Ω is convex, it is known that each of the two equivalent conditions in (4.24) holds true if and only if the mean curvature of $\partial\Omega$ satisfies the uniform estimate $\|H_{\partial\Omega}\|_{L^\infty(\partial\Omega)} \leq \frac{|\partial\Omega|}{|\Omega|}$, see [71].

Proof. Assume that Ω is calibrable, and let σ be a calibration. Integrating over Ω the equality $-\operatorname{div}\sigma = h_\Omega$, by the generalized divergence theorem (5.7) (see also [7, Theorem 3.5]), since $[\sigma\cdot n_\Omega] = -1$ $\mathscr{H}^1$-a.e. on $\partial\Omega$, we get $h_\Omega = |\partial\Omega|/|\Omega|$.

Conversely, assume that $h_\Omega = |\partial\Omega|/|\Omega|$. For every $s\in\mathbb{R}$, let $m(s)$ be the variational problem defined as in (4.2), settled on the domain $D=\Omega$. Using the equality $m(0)=0$ and Lemma 4.1.1, we obtain

$$\partial m(0) = \Big\{\lambda \,:\, m^*(\lambda) = 0\Big\} = \Big\{\lambda \,:\, \exists \sigma \in \mathscr{S}_\lambda(\Omega)\,,\ \int_{\mathbb{R}^2} \varphi^*(\sigma) = 0\Big\}.$$

By recalling the expression of Fenchel conjugate of φ in (4.5), and the characterization of $\mathscr{S}_\lambda(\Omega)$ holding when Ω simply connected (*cf.* (4.7)), it follows

$$\partial m(0) = \Big\{\lambda \,:\, \exists \sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)\,,\ \operatorname{spt}(\sigma) \subseteq \overline{\Omega}\,,\ -\operatorname{div}\sigma = \lambda \text{ in } \Omega\,,\ \|\sigma\|_{L^\infty(\Omega)} \leq 1\Big\}\,. \tag{4.25}$$

On the other hand, by (4.23), we know that $\partial m(0) = [-h_\Omega, h_\Omega]$, that is

$$h_\Omega = \max\{\lambda \in \mathbb{R} \,:\, \lambda \in \partial m(0)\}\,. \tag{4.26}$$

By combining (4.25) and (4.26), we infer that there exists $\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$\operatorname{spt}(\sigma) \subseteq \overline{\Omega}\,, \qquad -\operatorname{div}\sigma = h_\Omega \text{ in } \Omega\,, \qquad \|\sigma\|_{L^\infty(\Omega)} \leq 1\,.$$

We claim that the restriction of such a field σ to Ω is a calibration for Ω (so that Ω is calibrable). We only have to show that $[\sigma\cdot n_\Omega] = -1$ $\mathscr{H}^1$-a.e. on $\partial\Omega$. By integrating again over Ω the equality $-\operatorname{div}\sigma = h_\Omega$, we obtain

$$\int_{\partial\Omega} [\sigma\cdot n_\Omega]\, d\mathscr{H}^1 = \int_\Omega \operatorname{div}\sigma = -h_\Omega|\Omega| = -|\partial\Omega|.$$

Since $\|\sigma\|_{L^\infty(\Omega)} \le 1$, the above equality implies $[\sigma \cdot n_\Omega] = -1$ $\mathscr{H}^1$-a.e. on $\partial\Omega$ as required.

□

4.3 Existence and uniqueness of special solutions on a ball or a ring

In this section we show that, when D is a ball or a ring, problem $m(s)$ has a unique solution, which is a special one and has a circular plateau.

Proposition 4.3.1. *Let $R > 0$ and let $D = B_R(0)$ be the ball of radius R centered at the origin. Then, for every $s \in \mathbb{R}$, problem $m(s)$ admits a unique solution $\overline{u}$, which is a special solution. More precisely: if $s = 0$ then $\overline{u} \equiv 0$; if $s > 0$, there exists $r \in (0,R)$, uniquely determined by the values of s and R, such that*

$$\overline{u}(x) = \begin{cases} \dfrac{R^2 - \left(|x|^2 \vee r^2\right)}{2r} & \text{if } |x| < R \\ 0 & \text{otherwise} . \end{cases} \tag{4.27}$$

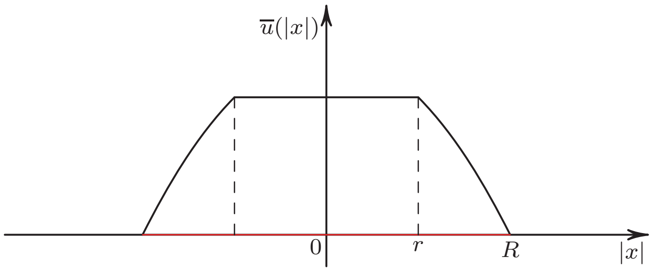


Figure 4.5: *The special solution $\overline{u}$ given by Proposition 4.3.1.*

Proof. If $s = 0$ the function $\overline{u} \equiv 0$ is clearly the unique solution to $m(0)$, and it is a special one. Assume $s > 0$. We begin by defining r as the unique number in the interval $(0,R)$ such that $f(r) = s$, where f is the map

$$f(t) := \frac{\pi}{4}\left(\frac{R^4}{t} - t^3\right)_+ \qquad \forall t \in (0,R) .$$

Notice that r is well-defined because f is strictly decreasing from $(0,R)$ onto $(0,+\infty)$. Using (4.27), the relation $f(r) = s$ and an integration by parts, it is straightforward to check that $\int_{\mathbb{R}^2} \overline{u} = s$. Moreover, $\overline{u}$ belongs to $H^1_c(D)$ since its gradient over $\mathbb{R}^2$ is given by

$$\nabla\overline{u}(x) = \begin{cases} -\dfrac{x}{r} & \text{if } |x| \in [r,R] \\ 0 & \text{otherwise} . \end{cases}$$

Hence $\overline{u}$ is admissible for problem $m(s)$. For every $v \in H^1_c(D)$ with $\int_{\mathbb{R}^2} v = 0$, it holds

$$\begin{aligned}\int_{\{\nabla\overline{u}=0\}} |\nabla v| + \int_{\{\nabla\overline{u}\neq 0\}} \langle \nabla\varphi(\nabla\overline{u}), \nabla v\rangle &\geq \int_{\{|x|<r\}} |\nabla v| - \int_{\{r<|x|<R\}} \langle \frac{x}{r}, \nabla v\rangle \\ &= \int_{\{|x|<r\}} \Big(|\nabla v| + \langle \frac{x}{r}, \nabla v\rangle\Big) \geq 0\,.\end{aligned}$$

Hence Proposition 4.1.1 implies that $\overline{u}$ is a solution to problem $m(s)$. It is a special solution as $|\nabla\overline{u}| = |\frac{x}{r}| > 1$ on the subset $\{r < |x| < R\}$. Finally, uniqueness follows from Corollary 4.1.2. ☐

Remark 4.3.1. With the same proof technique of Proposition 4.3.1, one can show that a similar result is valid also when $D = \bigcup_i B_i$ is the countable union of a family of pairwise disjoint balls B_i of radii R_i. Again, for every $s \in \mathbb{R}$ problem $m(s)$ admits a unique solution, which is a special one. More precisely: if $s = 0$ the solution is identically zero; if $s > 0$ there exists $r \in (0, \sup_i R_i)$, uniquely determined by the values of s and the radii R_i, such that on balls whose radius is smaller than r, the solution is identically zero, while on balls with a larger radius, it is of the form (4.27), with $R = R_i$. The critical radius r is the unique number in $(0, \sup_i R_i)$ such that $f(r) = s$, where

$$f(t) = \frac{\pi}{4} \sum_i \left(\frac{R_i^4}{t} - t^3 \right)_+ \qquad \forall t \in (0, \sup_i R_i)\,.$$

Proposition 4.3.2. *Let $R_2 > R_1 > 0$, and let $D := \{x \in \mathbb{R}^2 \;:\; R_1 < |x| < R_2\}$. Then, for every $s \in \mathbb{R}$, problem $m(s)$ admits a unique solution $\overline{u}$, which is a special solution. More precisely: if $s = 0$ then $\overline{u} \equiv 0$; if $s > 0$, there exists a unique $r \in (0, R_2)$, uniquely determined by the values of s and the radii R_1, R_2, such that*

$$\overline{u}(x) = \begin{cases} \dfrac{R_2^2 - \big(|x|^2 \vee (R_1 \vee r)^2\big)}{2r} & \text{if } |x| < R_2 \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.28}$$

Proof. If $s = 0$ the function $\overline{u} \equiv 0$ is clearly the unique solution to $m(0)$, and it is a special one. For $s > 0$, we define r as the unique number in the interval $(0, R)$ such that $f(r) = s$, where f is the map

$$f(t) := \begin{cases} \dfrac{\pi}{4} \left(\dfrac{R_2^4 - R_1^4}{t} \right) & \text{if } t \in (0, R_1) \\ \dfrac{\pi}{4} \left(\dfrac{R_2^4 - t^4}{t} \right) & \text{if } t \in [R_1, R_2)\,. \end{cases}$$

Notice that r is well-defined since the map f is strictly decreasing from $(0, R_2)$ onto $(0, +\infty)$. Using the definition of r and an integration by parts, it is straightforward

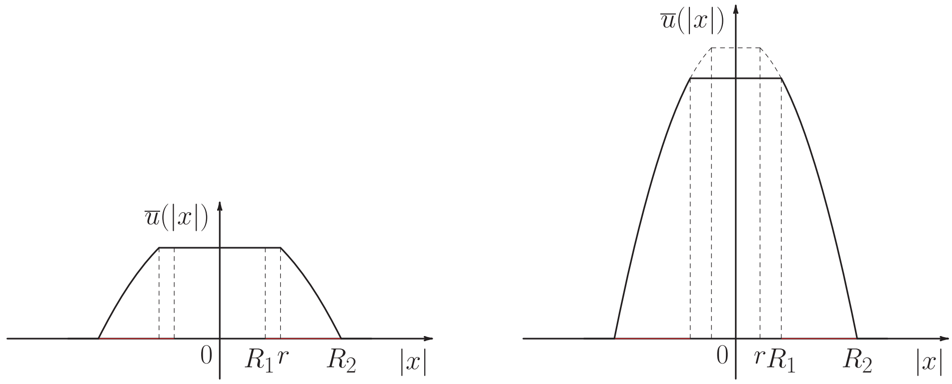


Figure 4.6: *The special solution $\overline{u}$ given by Proposition 4.3.2, respectively when $R_1 < r < R_2$ on the left, and when $0 < r < R_1$ on the right.*

to obtain that $\int_{\mathbb{R}^2} \overline{u} = s$. Moreover, $\overline{u}$ belongs to $H^1_c(D)$ since it is constant on each connected component of $\mathbb{R}^2 \setminus \overline{D}$:

$$\overline{u} \equiv \frac{R_2^2 - (R_1 \vee r)^2}{2r} \quad \text{if } |x| \leq R_1 \qquad \text{and} \qquad \overline{u} \equiv 0 \quad \text{if } |x| \geq R_2 \, .$$

Hence $\overline{u}$ is admissible for problem $m(s)$. Let us show that it is optimal. We distinguish the two cases when $s < f(R_1)$ or $s \geq f(R_1)$, which correspond respectively to $r \in (R_1, R_2)$ or $r \in (0, R_1]$.

If $r \in (R_1, R_2)$, $\overline{u}$ coincides with the function defined in (4.27), with $R = R_2$. The optimality of $\overline{u}$ for problem $m(s)$ set on the ball $B_{R_2}(0)$ implies the optimality also for problem $m(s)$ set on D, because of the inclusion $H^1_c(D) \subset H^1_c(B_{R_2}(0))$.

If $r \in (0, R_1]$, we apply Proposition 4.1.1: for every $v \in H^1_c(D)$ with $\int_{\mathbb{R}^2} v = 0$ it holds

$$\int_{\{\nabla \overline{u} = 0\}} |\nabla v| + \int_{\{\nabla \overline{u} \neq 0\}} \langle \nabla \varphi(\nabla \overline{u}), \nabla v \rangle = \int_{\mathbb{R}^2} \langle -\frac{x}{r}, \nabla v \rangle = \frac{2}{r} \int_{\mathbb{R}^2} v = 0 \, ,$$

where we have used the fact the gradient of $\overline{u}$ is given by

$$\nabla \overline{u}(x) = \begin{cases} -\dfrac{x}{r} & \text{if } |x| \in (R_1, R_2) \\ 0 & \text{otherwise} , \end{cases}$$

Thus $\overline{u}$ is a special solution, and uniqueness follows again from Corollary 4.1.2.

□

Remark 4.3.2. If in Proposition 4.3.2 we consider the case $s > f(R_1)$, when the solution $\overline{u}$ is given by (4.28) for a suitable $r \in (0, R_1)$ (see the above proof and Figure 4.6 at right), then the inequality $|\nabla \overline{u}(x)| > 1$ turns out to be strict up to $|x| = R_1$. This shows that, for a special solution, the equality (4.14) satisfied on the free boundary (lying in open set D) is in general false on $\partial \Omega(u) \cap \partial D$.

4.4 Existence of special solutions for some other domain D

By exploiting the results of Sections 4.1 and 4.2, we are going to prove that there exists some domain D, different from a ball, where problem $m(s)$ admits a special solution, see Theorem 4.4.1 below for a precise statement. To achieve this result, we use as a key tool the relationship between $m(s)$ and the Cheeger problem.

Let us remark that the proof of Theorem 4.4.1, and in particular the construction of the vector field σ therein, has some similarity with results contained in [81, Sections 4-5].

Theorem 4.4.1. *There exists an open bounded simply connected set D, different from a ball, such that problem $m(s)$ admits a special solution u for some $s \in \mathbb{R} \setminus \{0\}$. Moreover, both D and the plateau of u have analytic boundary, and the latter is convex.*

Proof. Let us construct an open bounded simply connected set D with analytic boundary, and

– a function $u \in H^1_0(D)$ with

$$\begin{cases} \int_D u = s\,, \text{ for some } s \in \mathbb{R}\setminus\{0\} \\ \nabla u = 0 \quad \text{in a convex set } \Omega \subset D \\ |\nabla u| > 1 \quad \text{in } D\setminus\Omega\,, \end{cases} \tag{4.29}$$

– a field $\sigma \in L^2(D;\mathbb{R}^2)$ with

$$\begin{cases} -\operatorname{div}\sigma = \lambda \quad \text{in } D\,, \text{ for some } \lambda \in \mathbb{R} \\ |\sigma| \le 1 \quad \text{in } \Omega \\ \sigma = \nabla u \quad \text{in } D\setminus\Omega\,. \end{cases} \tag{4.30}$$

We recall that, since D is simply connected, functions in $H^1_c(D)$ are extensions to zero of elements in $H^1_0(D)$, and $\mathscr{S}_\lambda(D)$ is given by (4.7). Then u and σ (extended to zero out of D), satisfy conditions (ii) in Proposition 4.1.2. Since $|\nabla u| \in \{0\}\cup(1,+\infty)$, we conclude that u is a special solution to problem $m(s)$ (settled on D).

The construction is divided into 3 steps.

STEP 1. We choose a bounded convex set Ω, with analytic boundary, whose curvature satisfies the *strict* inequality $\|H\|_{L^\infty(\partial\Omega)} < \frac{|\partial\Omega|}{|\Omega|}$. In view of Remark 4.2.2, there exists a calibration for Ω, namely a vector field $\sigma_1 \in L^2(\Omega;\mathbb{R}^2)$ such that

$$-\operatorname{div}\sigma_1 = h_\Omega \quad \text{in } \Omega\,, \qquad \|\sigma_1\|_{L^\infty(\Omega)} \le 1 \quad \text{in } \Omega\,, \qquad [\sigma_1 \cdot n_\Omega] = -1 \quad \mathscr{H}^1\text{–a.e. on } \partial\Omega\,.$$

STEP 2. Since $\partial\Omega$ is analytic, Cauchy-Kowalevskaya Theorem ensures the existence of an analytic solution v in a neighborhood $\mathscr{V}$ of $\partial\Omega$ to the initial value problem

$$\begin{cases} -\Delta v = h_\Omega \quad \text{in } \mathscr{V}\,, \\ v = 1\,,\ -v_n = 1 \quad \text{on } \partial\Omega\,, \end{cases}$$

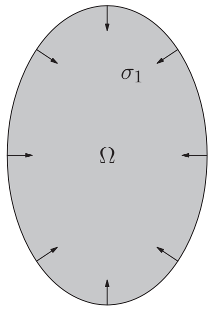


Figure 4.7: *Construction of the convex set Ω and the vector field σ_1 in Step 1.*

being n the unit outer normal to $\partial\Omega$. We claim that, up to choosing a smaller neighborhood $\mathscr{V}$, if we set $\mathscr{U} := \mathscr{V} \setminus \Omega$, it holds

$$v \leq 1 \quad \text{in } \mathscr{U} \tag{4.31}$$

and

$$|\nabla v| > 1 \quad \text{in } \mathscr{U} . \tag{4.32}$$

Indeed, (4.31) follows straightforward from the condition $v_n < 0$ on $\partial\Omega$. In order to prove (4.32), we exploit the equation $-\Delta v = h_{\Omega}$, which may be rewritten pointwise on $\partial\Omega$ as

$$-(H_{\partial\Omega} v_n + v_{nn}) = h_{\Omega} \qquad \text{on } \partial\Omega ,$$

being $H_{\partial\Omega}$ the (signed) curvature of $\partial\Omega$. By construction, we have

$$|\nabla v| = 1 , \quad v_n = -1, \quad |H_{\partial\Omega}| < h_{\Omega} \qquad \text{on } \partial\Omega .$$

Then (4.32) follows from the inequality

$$\partial_n(|\nabla v|^2) \;=\; 2v_n v_{nn} \;=\; -2v_{nn} \;=\; -2(H_{\partial\Omega} - h_{\Omega}) \;>\; 0 \qquad \text{on } \partial\Omega .$$

Next we choose $t_0 > 0$, independent of $y \in \partial\Omega$, such that the map

$$t \mapsto \phi_y(t) := v(y + tn(y))$$

is well-defined and satisfies the inequality $\phi_y'(t) < 0$ on $[0, t_0]$. Then, for some $\varepsilon_0 > 0$,

$$\max_{y \in \partial\Omega} \phi_y(t_0) = 1 - \varepsilon_0 < 1 .$$

Therefore, if we fix $\varepsilon \in (0, \varepsilon_0)$, it holds:

$$\forall y \in \partial\Omega , \; \exists t_y \in [0, t_0] \; : \; \phi_y(t_y) = 1 - \varepsilon .$$

We set $\gamma := \{y + t_y n(y) \; : \; y \in \partial\Omega\}$, so that $\gamma = \partial D$, with

$$D := \Omega \cup \big\{1 - \varepsilon \leq v \leq 1\big\} .$$

Finally we define

$$\overline{v} := v - (1 - \varepsilon) \quad \text{and} \quad \sigma_2 = \nabla \overline{v} \qquad \text{on } D \setminus \Omega .$$

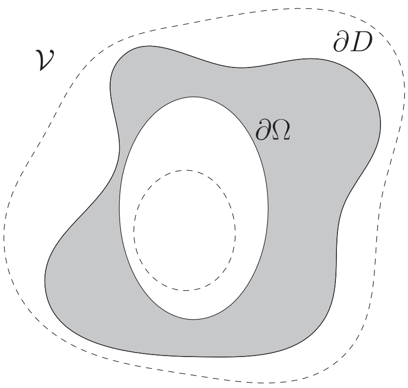


Figure 4.8: *Construction of the set D in Step 2.*

Notice in particular that σ_2 satisfies

$$-\operatorname{div}\sigma_2 = -\Delta v = h_\Omega \quad \text{in } D\setminus\Omega\,, \qquad [\sigma_2\cdot n_\Omega] = -1 \quad \mathscr{H}^1 - \text{a.e. on } \partial\Omega\,.$$

STEP 3. We set

$$u := \begin{cases} \varepsilon & \text{in } \Omega \\ \overline{v} & \text{in } D\setminus\Omega \end{cases}, \qquad \sigma := \begin{cases} \sigma_1 & \text{in } \Omega \\ \sigma_2 & \text{in } D\setminus\Omega \end{cases},$$

where Ω and σ_1 have been defined in Step 1, while D, $\overline{v}$ and σ_2 have been defined in Step 2.

It is easy to check that, by construction, u and σ verify respectively (4.29) and (4.30).

So, as claimed at the beginning of the proof, u is a special solution to $m(s)$. Moreover, the plateau Ω was chosen to be convex with analytic boundary. And also ∂D is analytic by the implicit function Theorem for analytic functions (see *e.g.* [96]): indeed, γ is a level set of an analytic function whose gradient is nonzero along γ (because of (4.32) and since $\gamma \subset \mathscr{U}$).

□

Remark 4.4.1. It is worth to compare our results with those obtained by Murat and Tartar in [82], about the problem of maximizing the torsional rigidity of a bar with a given cross-section made by two linearly elastic materials in fixed proportions. The corresponding variational problem is quite similar to ours, except that it involves a *differentiable* integrand, and classical solutions (*i.e.* optimal designs with no homogenization regions) cannot exist unless the cross-section D is a disk. In our case the integrand φ is non-differentiable at zero and the conclusion goes in a quite opposite direction.

4.5 Some qualitative properties of solutions and special solutions

In this Section we present some qualitative properties of solutions and special solutions for problem $m(s)$. Finally, in §4.5.1, we state a first result of regularity for the free boundary, more precisely concerning its perimeter.

We first state two results which concern arbitrary solutions to problem $m(s)$, and more precisely their sign (Proposition 4.5.1) and their support (Proposition 4.5.2).

Proposition 4.5.1. *For every $s \in \mathbb{R}^+$, any solution u to $m(s)$ satisfies the inequality $u \geq 0$ a.e.*

Proof. The unique solution to $m(0)$ is identically zero. Let $s > 0$ and let u be a solution to $m(s)$. We set $u_+ := \max\{u, 0\}$ and $\tilde{s} := \int_{\mathbb{R}^2} u_+$. Then

$$m(s) = \int_{\mathbb{R}^2} \varphi(\nabla u) \geq \int_{\mathbb{R}^2} \varphi(\nabla u_+) \geq m(\tilde{s}) .$$

Since $\tilde{s} \geq s$ and m is strictly increasing (recall Remark 4.1.1), we infer that $s = \tilde{s}$, and hence that the set $\{u < 0\}$ is Lebesgue negligible. □

Proposition 4.5.2. *Let s be positive and sufficiently small. Then any solution u to problem $m(s)$ satisfies*

$$\mathrm{spt}(u) \cap \partial D \neq \emptyset . \tag{4.33}$$

Proof. Assume that (4.33) is false for some $s > 0$. Then $\mathrm{spt}(u) \subset\subset D$ and, letting

$$u^\lambda(x) := \lambda\, u\big(\frac{x}{\lambda}\big) \qquad \forall x \in \mathbb{R}^2 ,\ \forall \lambda > 0 ,$$

by continuity we have also $\mathrm{spt}(u^\lambda) \subset\subset D$ for λ close to 1. Accordingly, the function u^λ is admissible for problem $m(\lambda^3 s)$, whence we deduce

$$m(\lambda^3 s) \leq \int_{\mathbb{R}^2} \varphi(\nabla u^\lambda) = \lambda^2 \int_{\mathbb{R}^2} \varphi(\nabla u) = \lambda^2 m(s) . \tag{4.34}$$

Therefore the function $g(\lambda) = m(\lambda^3 s) - \lambda^2 m(s)$ achieves a local maximum at $\lambda = 1$, and $g'(1) = 0$. It follows that $3m'(s) = 2m(s)$. Thus, by applying Remark 4.1.1, we find $m(s) \geq \frac{3}{2} k_D$, which is not possible for s small. □

We now turn our attention to investigate qualitative properties of *special* solutions, under the assumption that D is simply connected. The corresponding simplified formulation of $m(s)$, that we consider from now on, reads

$$m(s) = \inf\left\{ \int_D \varphi(\nabla u) \ :\ u \in H^1_0(D) ,\ \int_D u = s \right\} . \tag{4.35}$$

The search for special solutions to problem (4.35) leads to study a nonstandard free boundary value problem. Indeed, by Proposition 4.1.2, (4.55), and Corollary 4.1.3, if u is a special solution to $m(s)$ with plateau $\Omega(u)$ and free boundary $\Gamma(u)$, there exist constants $\lambda\, (= m'(s))$ and $c_i \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda\ , \quad |\nabla u| > 1 & \text{in } D \setminus \Omega(u) \\ |\nabla u| = 1 & \text{on } \Gamma(u) \\ u = c_i & \text{on } \gamma_i\, , \end{cases} \tag{4.36}$$

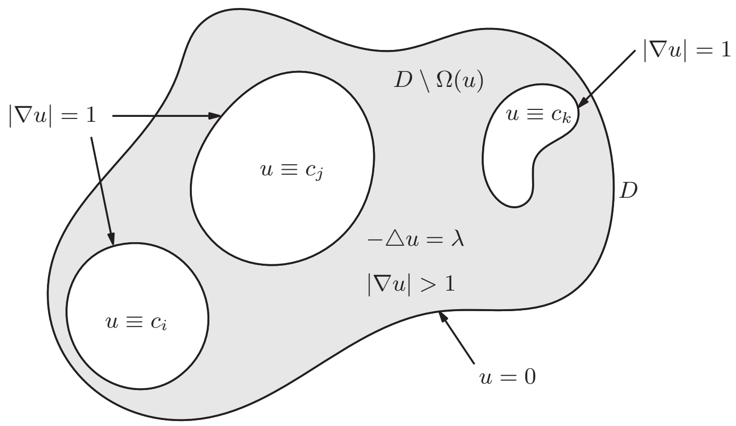


Figure 4.9: *The free boundary value problem* (4.36).

where γ_i denote the different connected components of $\Gamma(u)$ (see Figure 4.9).

A full understanding of problem (4.36) seems to be a quite challenging task. To the best of our knowledge, it is not directly covered by the extensive literature on free boundary problems (see [27–29, 85]). In particular, the available regularity results for free boundaries do not allow to obtain a priori the smoothness of $\Gamma(u)$. This is the reason why the results hereafter are stated under such smoothness assumption.

The key tool in the study of qualitative properties for special solution is the method of P-functions in dimension 2 (see [93] for the details). The main idea is the following: given a bounded planar domain A and a smooth solution u for the partial differential equation

$$\Delta u + f(u) = 0 \quad \text{in } A\,, \tag{4.37}$$

construct a suitable auxiliary function, called *P-function*

$$P = g(u)|\nabla u|^2 + h(u) \tag{4.38}$$

that satisfies a maximum principle. A direct computation leads to the following equality (cf. [93, formula (5.17)]):

$$\Delta P + L_\alpha \frac{\partial_\alpha P}{|\nabla u|^2} = g(\log g)''|\nabla u|^4 + [(h' - 2fg)' - fg']|\nabla u|^2 + \frac{1}{g}(h' - fg)(h' - 2fg)\,, \tag{4.39}$$

where L_α is defined as

$$L_\alpha = -\frac{1}{g}[\partial_\alpha P - 2(h' - fg)\partial_\alpha u]\,. \tag{4.40}$$

Let us recall the main statement, [93, Lemma 5.1].

Theorem 4.5.1. *Let u be a $C^3(A)$ solution of (4.37) in the planar domain A. If $g(u)$ and $h(u)$ are chosen so that the quadratic form in (4.39) is positive semidefinite, then the corresponding P-function defined in (4.38) is constant or assumes its maximum on ∂A or at a critical point of u.*

Remark 4.5.1. Let us recall that in our case, assuming to deal with a smooth special solution u, the function f is constant and equals λ, while A is an open bounded subset of $D\setminus\overline{\Omega}(u)$.

The easiest example of P-function satisfying Theorem 4.5.1 is

$$P = |\nabla u|^2 + 2\lambda u\ ,$$

indeed in this case the right hand side of (4.39) is identically zero.

Proposition 4.5.3. *Assume that problem $m(s)$ admits a special solution u, with $\Gamma(u)$ smooth. Then each connected component of $D\setminus\Omega(u)$ meets the boundary ∂D.*

Proof. Assume by contradiction that there exists a connected component A of $D\setminus\Omega(u)$ such that $A\subset\subset D$. Then $\partial A = \cup\gamma_i$, where γ_i are some of the connected components of $\Gamma(u)$. Then (*cf.* (4.36)), there exist constants $\lambda\,(=m'(s))$ and $c_i\in\mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda\ , \quad |\nabla u| > 1 & \text{in } A \\ u = c_i\ , \quad |\nabla u| = 1 & \text{on } \gamma_i\ . \end{cases}$$

By standard regularity theory, u is smooth enough in order to apply Theorem 4.5.1. By taking by taking therein $f(u)=\lambda$, $g(u)=1$, and $h(u)=0$, we obtain that the right hand side of (4.39) is positive and equals $2\lambda^2$, thus we deduce that the P-function

$$P(x) := |\nabla u|^2\ , \qquad x\in A\ ,$$

is either constant in A or it attains its maximum on ∂A. In both cases, since we know that $|\nabla u|\geq 1$ in A, we infer that

$$|\nabla u| \equiv 1 \qquad \text{in } A\ . \tag{4.41}$$

We now consider another P-function,

$$\widetilde{P}(x) := |\nabla u|^2 + \lambda u\ , \qquad x\in A\ .$$

From (4.41) we obtain

$$\Delta\widetilde{P} = -\lambda^2 = -(m'(s))^2 < 0 \tag{4.42}$$

(for the last inequality recall (4.18)). On the other hand, equality (4.39) applied now with $f(u)=\lambda$, $g(u)=1$ and $h(u)=\lambda u$, shows that

$$\Delta\widetilde{P} + L_\alpha \frac{\partial_\alpha P}{|\nabla u|^2} = 0$$

and recalling the definition (4.40) of L_α we infer

$$\Delta\widetilde{P} = -L_\alpha \frac{\partial_\alpha P}{|\nabla u|^2} = \frac{|\nabla\widetilde{P}|^2}{|\nabla u|^2} \geq 0\ ,$$

and this is in contradiction with (4.42). □

Proposition 4.5.4. *Let D be a convex set with a smooth boundary, and assume that problem $m(s)$ admits a special solution u, with $\Omega(u)$ connected, $\Omega(u)\subset\subset D$, and $\Gamma(u)$ smooth. Then Ω is convex.*

Proof. By applying Proposition 4.5.3, we obtain that $\Gamma(u)$ is connected (otherwise, some connected component of $D \setminus \Omega(u)$ would be compactly contained into D). Then (*cf.* (4.36)), there exist constants $\lambda (= m'(s))$ and $c \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda \,, \quad |\nabla u| > 1 & \text{in } D \setminus \Omega(u) \\ u = c \,, \quad |\nabla u| = 1 & \text{on } \Gamma(u) \,. \end{cases}$$

In order to prove that $\Omega(u)$ is convex, we follow the approach adopted in [70] (see also [68]): we consider the P-function

$$P(x) := |\nabla u|^2 + 2\lambda u \qquad \forall x \in D \setminus \Omega(u) \,.$$

As already noticed in Remark 4.5.1, the quadratic function defined in (4.39) is positive semidefinite. Moreover, by standard regularity theory, u is smooth enough in order to apply Theorem 4.5.1. Since by assumption u has no critical points in $D \setminus \Omega(u)$, we infer that one of the following facts occurs:

(a) P is constant;

(b) P attains its maximum on ∂D;

(c) P attains its maximum on $\Gamma(u)$.

Let us exclude the first two possibilities.

If P is constant, it holds

$$0 = P_n = 2(u_n u_{nn} + \lambda u_n) = -2(u_{nn} + \lambda) \qquad \text{on } \Gamma(u) \,. \tag{4.43}$$

On the other hand, since by assumption $\Gamma(u)$ is smooth, the equation $\Delta u + \lambda = 0$ can be rewritten pointwise on $\Gamma(u)$ as

$$H_\Gamma u_n + u_{nn} + \lambda = -H_\Gamma + u_{nn} + \lambda = 0 \qquad \text{on } \Gamma(u) \,, \tag{4.44}$$

where we have denoted by H_Γ the mean curvature of $\Gamma(u)$. Combining (4.43) and (4.44), we deduce that $H_\Gamma \equiv 0$ on $\Gamma(u)$, a contradiction.

If P attains its maximum at some point $x_0 \in \partial D$, since ∂D is smooth we may apply Hopf's boundary point lemma to infer that either P is constant or $P_n(x_0) > 0$ (here n stands for the unit outer normal to ∂D). Since we have already excluded the first possibility, let us show that also the second one leads to a contradiction. We have

$$0 < P_n(x_0) = 2u_n(x_0)u_{nn}(x_0) + 2\lambda u_n(x_0) = -2(u_n(x_0))^2 H_{\partial D}(x_0) \,, \tag{4.45}$$

where the last equality follows by exploiting the pde $\Delta u + \lambda = 0$ on ∂D. In particular, (4.45) implies $H_{\partial D}(x_0) < 0$, against the convexity of D.

We conclude that (c) holds true, namely P assumes its maximum on $\Gamma(u)$. Since P is constant on $\Gamma(u)$, every point of the free boundary is a maximum point. Then, thanks to the smoothness of $\Gamma(u)$, Hopf's lemma applies and yields

$$0 > P_n = 2u_n u_{nn} + 2\lambda u_n = -2H_\Gamma \qquad \text{on } \Gamma(u) \,.$$

Hence $\Omega(u)$ is convex. □

Proposition 4.5.5. *Assume that D is* not *Cheeger set of itself, and let s_ε be an infinitesimal sequence of positive numbers. Then problem $m(s_\varepsilon)$ cannot admit for every ε a special solution u_ε with $\Omega(u_\varepsilon) \subset\subset D$ and $\Gamma(u_\varepsilon)$ smooth.*

Proof. Set for brevity $\Omega_\varepsilon := \Omega(u_\varepsilon)$ and $\Gamma_\varepsilon := \Gamma(u_\varepsilon)$. Assume by contradiction $\Omega_\varepsilon \subset\subset D$ and Γ_ε smooth. We set $\lambda_\varepsilon = m'(s_\varepsilon)$, and we take an optimal field $\sigma_\varepsilon \in \mathscr{S}_{\lambda_\varepsilon}(D)$ for the dual problem $m^*(\lambda_\varepsilon)$. By Proposition 4.1.2 and (4.55), σ_ε satisfies

$$\begin{cases} -\operatorname{div}\sigma_\varepsilon = \lambda_\varepsilon & \text{in } D \\ |\sigma_\varepsilon| \le 1 & \text{in } \Omega_\varepsilon \\ \sigma_\varepsilon = \nabla u_\varepsilon & \text{in } D\setminus\Omega_\varepsilon\,. \end{cases}$$

By Corollary 4.1.3 and the regularity assumed on Γ_ε, we infer that $|\sigma_\varepsilon \cdot n_{\Gamma_\varepsilon}|$ equals 1 and has constant sign on Γ_ε. Integrating on Ω_ε the equation $-\operatorname{div}\sigma_\varepsilon = \lambda_\varepsilon$, we obtain $\sigma_\varepsilon \cdot n_{\Gamma_\varepsilon} = -\operatorname{sgn}(\lambda_\varepsilon) = -1$ and

$$\lambda_\varepsilon = \frac{|\Gamma_\varepsilon|}{|\Omega_\varepsilon|}\,. \tag{4.46}$$

Since $\lambda_\varepsilon = m'(s_\varepsilon)$, by using (5.48), the continuity from the right of the right derivative $s \mapsto m'_+(s)$ as $s \to 0^+$, Proposition 4.1.4 and Proposition 4.2.1, we get

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon = h_D. \tag{4.47}$$

Moreover, we have

$$|D\setminus\Omega_\varepsilon| \le \int_{D\setminus\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{D\setminus\Omega_\varepsilon} \nabla u_\varepsilon \cdot \sigma_\varepsilon = \int_D \nabla u_\varepsilon \cdot \sigma_\varepsilon = s_\varepsilon \cdot \lambda_\varepsilon\,.$$

In view of (4.47), we infer that $\lim_{\varepsilon\to 0} |D\setminus\Omega_\varepsilon| = 0$, which is equivalent to $\lim_{\varepsilon\to 0} 1\!\!1_{\Omega_\varepsilon} = 1\!\!1_D$ in $L^1(D)$. By using the lower semicontinuity of the perimeter with respect to the L^1-convergence (*cf.* [5, Proposition 3.38]), and (5.48), we obtain

$$|\partial D| \le \liminf_{\varepsilon\to 0} |\Gamma_\varepsilon| = \lim_{\varepsilon\to 0} \lambda_\varepsilon |\Omega_\varepsilon| = h_D |D|\,,$$

hence D is Cheeger set of itself, against the assumption. □

4.5.1 Regularity of the free boundary

Given a spacial solution u, we decompose its plateau $\Omega(u)$ as the union of two sets: $\Omega(u) = \Omega_0(u) \cup \Omega_+(u)$, with

$$\Omega_0(u) := \{x \in \Omega(u) \ : \ u(x) = 0\}\ ,\ \ \Omega_+(u) := \{x \in \Omega(u) \ : \ u(x) > 0\}\,. \tag{4.48}$$

In the next proposition we establish that some connected components of $\Gamma(u)$ have finite perimeter. This result is the first step in studying the regularity of the free boundary.

Proposition 4.5.6. *Let $s > 0$ and assume that problem $m(s)$ admits a special solution u. Then* *(i) if $\Omega_+(u)$ is connected, it has finite perimeter;*

(ii) $\Omega_0(u)$ has finite perimeter.

Proof. Let us first consider D simply connected. We notice that the volume constraint appearing in problem $m(s)$ can be enclosed in the functional through a Lagrange multiplier: more precisely, if u is optimal for $m(s)$ in the class $H^1_0(D)\cap\{v : \int_D v = s\}$, then it is a solution to the following minimization problem for $\lambda = m'(s)$:

$$\inf_{v\in H^1_0(D)} \int_D [\varphi(\nabla v) - \lambda v] \, . \tag{4.49}$$

We prove (i) and (ii) exploiting the optimality of u in (4.49), by comparing it with a suitable test function $u_\varepsilon \in H^1_0(D)$.

Proof of (i). We observe that, since $\Omega_+(u)$ is connected and u is a special solution, setting $M := \max_{\overline{D}} u$, there holds $\Omega_+(u) = \{x \in D : u(x) = M\}$.

For every $\varepsilon \in (0, M)$, we consider the function $u_\varepsilon := \min\{u, M-\varepsilon\}$. Since u_ε is admissible for problem (4.49), by the optimality of u we have

$$\int_D [\varphi(\nabla u) - \lambda u] \leq \int_D [\varphi(\nabla u_\varepsilon) - \lambda u_\varepsilon] \, ,$$

that is

$$\int_D [\varphi(\nabla u) - \varphi(\nabla u_\varepsilon)] \leq \lambda \int_D [u - u_\varepsilon]$$

or equivalently

$$\int_{\{M-\varepsilon<u\leq M\}} \varphi(\nabla u) \leq \lambda \int_{\{M-\varepsilon<u\leq M\}} [u-(M-\varepsilon)] \, . \tag{4.50}$$

The right hand side of (4.50) can be bounded above as

$$\lambda \int_{\{M-\varepsilon<u\leq M\}} [u-(M-\varepsilon)] \leq \lambda |D| \varepsilon \, .$$

The left hand side of (4.50), since u is a special solution, can be bounded below as

$$\int_{\{M-\varepsilon<u\leq M\}} \varphi(\nabla u) = \frac{1}{2}\int_{\{M-\varepsilon<u\leq M\}} |\nabla u|^2 \geq \frac{1}{2}\int_{\{M-\varepsilon<u\leq M\}} |\nabla u| \, .$$

Combining the two bounds above with (4.50), we obtain

$$\frac{1}{\varepsilon}\int_{\{M-\varepsilon<u\leq M\}} |\nabla u| \leq C \, , \tag{4.51}$$

for some positive constant C independent of ε. By the coarea formula (see Theorem 1.4.1), we can rewrite (5.54) as

$$\frac{1}{\varepsilon}\int_{M-\varepsilon}^{M} \mathrm{Per}(\{u>t\})\, dt \leq C \, .$$

Then there exists a sequence $t_n \nearrow M$ such that $\mathrm{Per}(\{u > t_n\}) \le C$. Since $1\!\mathrm{l}_{\{u>t_n\}}$ converge to $1\!\mathrm{l}_{\Omega_+(u)}$ in L^1, by the lower semicontinuity of the perimeter with respect to the L^1-convergence, we infer

$$\mathrm{Per}(\Omega_+(u)) \le C\,.$$

Proof of (ii). For every $\varepsilon \in (0,M)$, we consider the function $u_\varepsilon := \max\{u-\varepsilon, 0\}$. Since u_ε is admissible for problem (4.49), by the optimality of u we have

$$\int_D [\varphi(\nabla u) - \lambda u] \le \int_D [\varphi(\nabla u_\varepsilon) - \lambda u_\varepsilon]\,,$$

that is

$$\int_D [\varphi(\nabla u) - \varphi(\nabla u_\varepsilon)] \le \lambda \int_D [u - u_\varepsilon]$$

or equivalently

$$\int_{\{0\le u<\varepsilon\}} \varphi(\nabla u) \le \lambda \int_{\{0\le u<\varepsilon\}} u + \varepsilon\lambda|\{u \ge \varepsilon\}|\,. \tag{4.52}$$

The right hand side of (4.52) can be bounded above as

$$\varepsilon\lambda|\{0 \le u < \varepsilon\}| + \varepsilon\lambda|\{u \ge \varepsilon\}| \le C\varepsilon\,.$$

The left hand side of (4.52), since u is a special solution, can be bounded below as

$$\int_{\{0\le u<\varepsilon\}} \varphi(\nabla u) = \frac{1}{2}\int_{\{0\le u<\varepsilon\}} |\nabla u|^2 \ge \frac{1}{2}\int_{\{0\le u<\varepsilon\}} |\nabla u|\,.$$

Combining the two bounds above with (4.52), we obtain

$$\frac{1}{\varepsilon}\int_{\{0\le u<\varepsilon\}} |\nabla u| \le C\,, \tag{4.53}$$

for some positive constant C independent of ε.

As already done in the previous case, we can conclude the proof by applying the coarea formula: indeed we can rewrite (4.53) as

$$\frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{Per}(\{u > t\})\,dt \le C\,,$$

then there exists a sequence $t_n \searrow 0$ such that $\mathrm{Per}(\{u > t_n\}) \le C$. Finally, since $1\!\mathrm{l}_{\{u>t_n\}}$ converge to $1\!\mathrm{l}_{\Omega_0(u)}$ in L^1, by the lower semicontinuity of the perimeter with respect to the L^1-convergence, we infer

$$\mathrm{Per}(\Omega_0(u)) \le C\,.$$

In The general case, if D is connected but not simply connected, a similar proof applies: it is enough to replace D with

$$\widetilde{D} := \{A \subset \mathbb{R}^2 \;:\; A \supset D\,,\ A \text{ simply connected}\}\,,$$

and considering as test functions

$$v \in H^1_0(\widetilde{D}) \cap \{\nabla v = 0 \text{ in } \widetilde{D} \setminus D\}\,.$$

□

Remark 4.5.2. In view of Proposition 4.5.6 we infer that if $\Omega_+(u)$ is connected, then the free boundary $\Gamma(u) = \partial\Omega_0(u) \cup \partial\Omega_+(u)$ has finite perimeter. We point out that the statement (i) can be rephrased in a more general way: if we write the plateau as $\Omega(u) = \cup_l \{u \equiv c_l\}$ for some constants c_l, the connected component associated to $M := \max_l c_l$ (if it exists) has finite perimeter.

4.6 Perspectives and conjectures

We conclude the Chapter by presenting the open problems and the possible perspectives of the work.

The perspectives are the following:

- study the regularity of the free boundary and of a special solution;
- find examples of domains that do *not* admit a special solution;
- characterize the domains D that admit a special solution.

We point out that proving of disproving the existence of a special solution remains open even for simple geometries of D.

In this respect we believe that, at least when D is convex, the existence of special solutions is likely related to the regularity of ∂D, and also to whether or not D coincides with its Cheeger set.

For instance, when D is a square, in view of Propositions 4.5.3 and 4.5.4, it cannot happen that a special solution has the white regions in Figure 4.10 as plateau. Actually, having in mind Proposition 4.5.5, at least for small s the set where a solution u is constant may be expected to be shaped as the white region in Figure 4.11; but on its complement, the green region, it is difficult to guess whether $|\nabla u| > 1$, or some homogenization phenomenon occurs. Some numerical results performed in [72] for a very similar problem, in which homogenization regions are observed, seem to suggest that in the square there is no special solution.

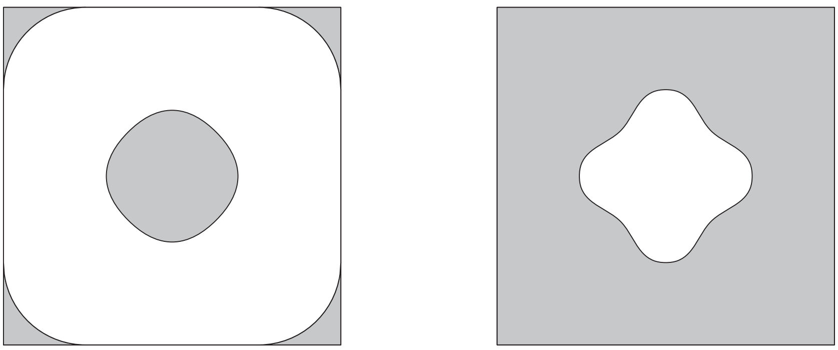


Figure 4.10: *Impossible plateaus for a special solution on the square.*

In order to better understand the behavior of solutions, for example in the square, we explored two different fields: numerics and shape derivatives. In the next paragraphs

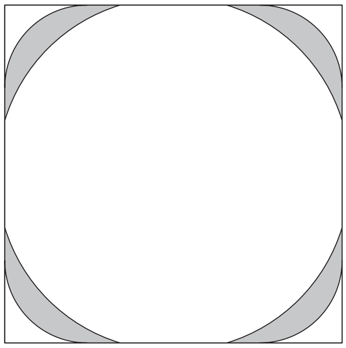


Figure 4.11: *A possible plateau for a special solution on the square.*

we are going to illustrate these techniques: we believe that they are interesting points that could be developed hereafter. We underline that in Chapter 5 we will face the shape derivative problem, in a more general framework.

4.6.1 Some numerics

For simplicity, leu us consider D simply connected.

In view of Theorem 1.1.2, we can rephrase the optimality conditions stated in Proposition 4.1.2, interchanging the role of the solutions of the primal problem $m(s)$ and the dual problem $m^*(\lambda)$: let $s,\lambda \in \mathbb{R}$, $u \in H_0^1(D)$, and $\sigma \in L^2(D;\mathbb{R}^2)$, then the following equivalence holds true:

$$(i)\begin{cases} u \text{ solution to } m(s) \\ \sigma \text{ solution to } m^*(\lambda) \\ \lambda \in \partial m(s)\,. \end{cases} \iff (ii)\begin{cases} \displaystyle\int_{\mathbb{R}^2} u = s \\ -\operatorname{div}\sigma = \lambda \text{ a.e. in } D \\ \nabla u \in \partial\varphi^*(\sigma) \text{ a.e. in } D \end{cases} \tag{4.54}$$

The Fenchel conjugate φ^* is differentiable at every ξ such that $|\xi| \neq 1$, whereas its subdifferential at $|\xi| = 1$ is a segment. More precisely the condition $\nabla u \in \partial\varphi^*(\sigma)$ can be rewritten more explicitly as

$$\nabla u = \begin{cases} 0 & \text{if } |\sigma| < 1 \\ \tau\sigma & \text{if } |\sigma| = 1\,, \text{ for some } \tau \in [0,1] \\ \sigma & \text{if } |\sigma| > 1\,. \end{cases} \tag{4.55}$$

Definition 4.6.1. *Given $\lambda > 0$ and $s \in \partial m^*(\lambda)$, we say that a couple $\overline{w} := (\overline{u},\overline{\sigma}) \in H_0^1(D) \times L^2(D;\mathbb{R}^2)$ is an optimal couple if $\overline{u}$ is optimal for $m(s)$ and $\overline{\sigma}$ is optimal for $m^*(\lambda)$.*

In view of (4.54), we infer that an optimal couple is characterized by

$$\begin{cases} 0 = \operatorname{div}\overline{\sigma} + \lambda \\ 0 \in \partial\varphi^*(\overline{\sigma}) - \nabla\overline{u}\,. \end{cases}$$

We remark that the right hand side of the second equation is a convex closed set, in general not a singleton:

$$\partial\varphi^*(\sigma) - \nabla u = \{\tau\sigma - \nabla u \,:\, \tau \in [0,1]\}\,. \tag{4.56}$$

We want to see the optimal couple as a stationary solution (as $t \to +\infty$) of the following evolution equation:

$$\begin{cases} \partial_t w \in Aw \\ w(0) = w_0 \ , \end{cases} \tag{4.57}$$

with $w := (u,\sigma) \in H^1_0(D) \times L^2(D;\mathbb{R}^2)$ and $A : D(A) \subset X \to X$ the operator defined as follows:

$$A : (u,\sigma) \mapsto (-\operatorname{div}\sigma - \lambda, \partial\varphi^*(\sigma) - \nabla u) \ , \tag{4.58}$$

with

$$D(A) := H^1_0(D) \times \{\sigma \in L^2(D;\mathbb{R}^2) \ : \ \operatorname{div}\sigma \in L^2(D)\} \ , \ \ X := L^2(D) \times L^2(D;\mathbb{R}^2) \ ,$$

X being endowed with the usual norm $\|(u,\sigma)\|^2_X := \|u\|^2_{L^2(D)} + \|\sigma\|^2_{L^2(D;\mathbb{R}^2)}$.

We remark that the operator A is a maximal monotone operator of X (see the Appendix for the definition of maximal monotone operator and the proof of the claim). Clearly, if problem (4.57) admits a stationary solution $w(t) = (u(t),\sigma(t)) \to \overline{w} = (\overline{u},\overline{v})$, then $(\overline{u},\overline{v})$ is an optimal couple, associated to the parameters s and λ satisfying $s \in \partial m^*(\lambda)$. Thus, a numerical search of stationary solutions for (4.57), may give information about the behavior of solutions for $m(s)$.

A possible choice in the numeric procedure is to consider instead of (4.57) the following evolution equation (see *e.g.* [22, 84, 88] for the motivations)

$$\begin{cases} \partial_t w = A_0 w \\ w(0) = w_0 \ , \end{cases} \tag{4.59}$$

where $A_0 w$ is the projection of $0 \in X$ over the closed convex set Aw. The element of minimal norm $A_0 w$ can be computed explicitly and problem (4.59) reads

$$\begin{cases} \partial_t u = -\operatorname{div}\sigma - \lambda \\ \partial_t \sigma = \tau(\nabla u,\sigma)\sigma - \nabla u \ , \end{cases} \tag{4.60}$$

where the right hand side of the second equation is the element of minimal norm in the set $\nabla u - \partial\varphi^*(\sigma)$ (see (4.56)). The coefficient τ is given by

$$\tau(\nabla u,\sigma) = \begin{cases} 0 & \text{if } |\sigma| < 1 \\ \alpha(\nabla u \cdot \sigma) & \text{if } |\sigma| = 1 \\ 1 & \text{if } |\sigma| > 1 \ , \end{cases} \tag{4.61}$$

with

$$\alpha(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } 0 < t < 1 \\ 1 & \text{if } t \ge 1 \ . \end{cases} \tag{4.62}$$

The equation under study should be a suitable approximation of system (4.60), because of the singularity of the subdifferential of φ^*.

The study in this direction is still a work in progress. Here we limit ourselves to list some possible approximations for the problem and some natural choices for the initial data:

- make φ^* smooth (at least C^2) in a neighborhood of 1 of the form $(1-\varepsilon, 1+\varepsilon)$, and let ε tend to 0;

- approximate the coefficient τ introduced in (4.61) with

$$\tau_\varepsilon(\nabla u, \sigma) = \begin{cases} 0 & \text{if } |\sigma| \leq 1-\varepsilon \\ \left(\frac{|\sigma|-(1-\varepsilon)}{2\varepsilon}\right) \alpha(\nabla u \cdot \sigma) & \text{if } 1-\varepsilon \leq |\sigma| \leq 1+\varepsilon \\ 1 & \text{if } |\sigma| \geq 1+\varepsilon \ , \end{cases}$$

  α being defined in (4.62), and then let ε tend to 0;

- consider as initial datum $(u_0, \sigma_0) := (0,0)$ or $(u_0, \sigma_0) := (\frac{\lambda}{2} u_D, \frac{\lambda}{2} \nabla u_D)$, with u_D the unique solution in $H_0^1(D)$ of

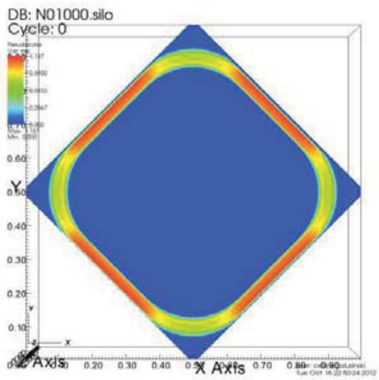
$$-\Delta u = 2 \quad \in D \ .$$

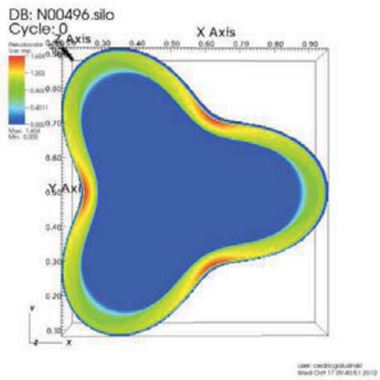
  For example, if D is the ball of radius 1 centered in the origin, then $u_D = \frac{1-|x|^2}{2}$.

We remark that we can exploit the properties found for the solutions in order to verify the compatibility of the computational results. For example, if we consider λ inferior to the Cheeger constant we know that the unique solution is the zero one (see (4.23)).

For completeness we present some results obtained by prof. Cédric Galusinski (Université du Sud Toulon et du Var, Laboratoire Imath), who implemented a similar model, trying to detect homogenization regions for different geometries of D. We gratefully acknowledge professor Galusinski for the permission to show the pictures above.

In the figures above the blue region region represents the plateau, the green one the homogenization region and the red one the region in which the gradient exceeds 1 in modulus. This seems to confirm the presence of homogenization phenomena, moreover it suggests that the modulus of the gradient of a solution is greater that 1 in a neighborhood of concave parts of ∂D, while it is smaller that 1 near the convex parts.





4.6.2 Shape derivatives

Let us consider the case D simply connected. If we fix the parameter s, we can interpret $m(s)$ as a shape functional, depending on the domain D as follows: enclosing the volume constraint in the functional, solving $m(s)$ over D turns out to be equivalent to study

$$J(D) = -\inf\left\{ \int_D [\varphi(\nabla u) - \lambda u]\, dx \;:\; u \in H_0^1(D) \right\} , \tag{4.63}$$

with $\lambda = m'(s)$.

Let assume that u is a solution for $J(D)$ and let us recall the definition of $\Omega_0(u)$ (already introduced in (4.48)): it is the connected component of the plateau associated to the zero level set, namely $\Omega_0(u) = \{x \in \Omega(u) \;:\; u(x) = 0\}$. Our conjecture is that $\Omega_0(u)$ contains the neighborhoods of the points of ∂D with higher curvature, namely the corners of the boundary (see for example Figure 4.11 for the square). Clearly, in this case the functional is stationary over the domains D' such that

$$D \setminus \Omega_0(u) \subset D' \subset D .$$

A qualitative example is shown in Figure 4.12. Thus we expect that the shape derivative (for the proper definition see Chapter 5), if it exists, equals zero with respect to small deformations in such direction. Moreover, the sign of the shape derivative may give useful information: if we consider small inner deformations of D, localized in a part of the boundary, a nonzero shape derivative implies that $\Omega_0(u)$ does not touch such portion of ∂D.

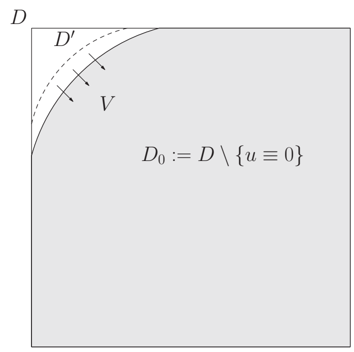


Figure 4.12: *If the white region represents the plateau, inner deformations in the direction V do not change the value of the shape functional $J(\cdot)$.*

The next Chapter is entirely devoted to the study of shape derivatives of minima of integral functionals of the form (4.63), with more general convex integrands and in higher dimension.

4.7 Appendix

Let us breafly recall the definition of maximal monotone operator (see [22, 88] for a complete reference).

Definition 4.7.1. *Let X be a Hilbert space. An operator is a multivalued mapping A from X into $\mathscr{P}(X)$. We call $D(A) := \{x \in X \ : \ A(x) \neq \emptyset\}$ the domain of A. We identify an operator with its graph $G(A) := \{(x,y) \ : \ x \in D(A)\,,\, y \in A(x)\} \subset X \times X$.*

Definition 4.7.2. *An operator A is monotone if for every $x_1\,, x_2 \in D(A)$*

$$\langle A(x_2) - A(x_1), x_2 - x_1 \rangle \geq 0 \ ,$$

or more precisely if for every $(x_1,y_1)\,, (x_2,y_2) \in G(A)$

$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0 \ .$$

An operator is maximal monotone if it is monotone and is maximal in the set of monotone operators of X, ordered by the inclusion of graphs in $X \times X$.

We cite an useful example of maximal monotone operator (see [22, Example 2.3.4.]):

Lemma 4.7.1. *Let ψ be a proper lower semicontinuous convex function on X, then $\partial\psi$ is a maximal monotone operator of X.*

Proposition 4.7.1. *The operator A in (4.58) is maximal monotone of X.*

Proof. Let $K : X \to \mathbb{R}$ be defined as

$$K(u,\sigma) := \int_D [\sigma \cdot \nabla u - \varphi^*(\sigma) - \lambda u] \ , \tag{4.64}$$

and let $H : D(H) \subset X \to \mathbb{R}$ be the following function

$$H(u,p) := \sup_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D p \cdot \sigma + K(u,\sigma) \right\} \ .$$

We remark that H is a proper convex lower semicontinuous function of X. Moreover, in view of the definition (4.64) of K, we can rewrite H as

$$\begin{aligned} H(u,p) &= \sup_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D p \cdot \sigma + \nabla u \cdot \sigma - \varphi^*(\sigma) - \lambda u \right\} \\ &= \sup_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D (p + \nabla u) \cdot \sigma - \varphi^*(\sigma) \right\} - \int_D \lambda u \\ &= \begin{cases} \int_D \varphi(\nabla u + p) - \lambda u & \text{if } u \in H^1_0(D) \\ +\infty & \text{otherwise} \ . \end{cases} \end{aligned}$$

Then the domain of H is $D(H) = H^1_0(D) \times L^2(D;\mathbb{R}^2) \subset X$.

We compute the Fenchel conjugate of H:

$$
\begin{aligned}
H^*(f,q) &= \sup_{(u,p)\in D(H)} \left\{ \int_D fu + p\cdot q - \varphi(\nabla u + p) + \lambda u \right\} \\
&= \sup_{u\in H_0^1(D)} \left\{ \sup_{p\in L^2(D;\mathbb{R}^2)} \int_D [(\nabla u + p)\cdot q - \varphi(\nabla u + p)]\,(\operatorname{div} q + f + \lambda)u \right\} \\
&= \int_D \varphi^*(q) + \sup_{u\in H_0^1(D)} \int_D (\operatorname{div} q + f + \lambda)u \\
&= \begin{cases} \int_D \varphi^*(q) & \text{if } \operatorname{div} q = -f-\lambda \\ +\infty & \text{otherwise .} \end{cases}
\end{aligned}
$$

In particular we obtain that the domain of H^* is $D(H^*) = L^2(D)\times L^2_{\mathrm{div}}(D;\mathbb{R}^2)$.

By comparing the explicit formulations of H and H^*, we obtain the following characterization of the subdifferential of H: $(f,q)\in\partial H(u,p)$ if and only if

$$
\begin{aligned}
& u\in H_0^1(D)\,,\ f = -\operatorname{div} q - \lambda\,,\ H(u,p) + H^*(f,q) = \int_D fu + p\cdot q \\
\Longleftrightarrow\quad & u\in H_0^1(D)\,,\ f = -\operatorname{div} q - \lambda\,,\ \int_D \varphi(\nabla u + p) - \lambda u + \varphi^*(q) = \int_D fu + p\cdot q \\
\Longleftrightarrow\quad & u\in H_0^1(D)\,,\ f = -\operatorname{div} q - \lambda\,,\ \int_D \varphi(\nabla u + p) + \varphi^*(q) = \int_D (\nabla u + p)\cdot q \\
\Longleftrightarrow\quad & u\in H_0^1(D)\,,\ f = -\operatorname{div} q - \lambda\,,\ \nabla u + p \in \partial\varphi^*(q) \\
\Longleftrightarrow\quad & u\in H_0^1(D)\,,\ f = -\operatorname{div} q - \lambda\,,\ p \in -\nabla u + \partial\varphi^*(q)\ .
\end{aligned}
$$

Then we can conclude that

$$
((u,q),(f,p))\in G(A) \iff ((u,p),(f,q))\in G(\partial H)\ , \tag{4.65}
$$

where $G(\cdot)$ denotes the graph of an operator, according to Definition 4.7.1.

By Lemma 4.7.1, since H is a proper lower semicontinuous convex function, we have that ∂H is a maximal monotone operator of X. Then we can easily conclude that also A is a maximal monotone operator of X. Let $l : X \to X$ be the map between $G(A)$ and $G(\partial H)$, given by the correspondence in (4.65): it is a bijection, more precisely it is an isometry. We begin by proving the monotonicity of A, verifying the definition: let $((u_i,q_i),(f_i,p_i)) =: (x_i,y_i)\in G(A)$, for $i = 1,2$, and $l(x_i,y_i) =: (\tilde{x}_i,\tilde{y}_i)\in G(\partial H)$, then

$$
\begin{aligned}
&\langle y_2 - y_1, x_2 - x_1\rangle \\
&= (f_2 - f_1)(u_2 - u_1) + (q_2 - q_1)\cdot(p_2 - p_1) \\
&= \langle \tilde{y}_2 - \tilde{y}_1, \tilde{x}_2 - \tilde{x}_1\rangle \geq 0\ ,
\end{aligned}
$$

where the last inequality follows by monotonicity of ∂H. We conclude showing maximality. Let B be another monotone operator of X such that $G(A)\subset G(B)$. Then there holds also the inclusion $l(G(A))\subset l(G(B))$. Since l is a bijection, its image is all the set $G(\partial H)$, then we have

$$
G(\partial H) = l(G(A)) \subset l(G(B))\ . \tag{4.66}
$$

We have shown that the relation l preserves monotonicity, than also the operator associated to $l(G(B))$ is monotone. By maximality of ∂H, we obtain that the inclusion in (4.66) is an equality. Hence, by applying l^{-1}, we conclude that $G(A) = G(B)$, that is A is maximal. $\square$

CHAPTER 5

Shape derivatives for minima of integral functionals

In this Chapter we deal with the shape derivative of functionals which are obtained by minimizing a classical integral of the Calculus of Variations, under Dirichlet or Neumann conditions. Namely, we consider the functionals of domain defined by

$$J_D(\Omega) := -\inf\left\{ \int_\Omega \big[f(\nabla u) + g(u)\big]\, dx \ : \ u \in W_0^{1,p}(\Omega) \right\} \tag{5.1}$$

$$J_N(\Omega) := -\inf\left\{ \int_\Omega \big[f(\nabla u) + g(u)\big]\, dx \ : \ u \in W^{1,p}(\Omega) \right\} . \tag{5.2}$$

Here Ω varies among the open bounded subsets of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and convex integrands, which satisfy growth conditions of order p and q respectively, specified later on.

In the sequel, the notation $J(\Omega)$ is adopted for brevity in all the statements which apply indistinctively in the Dirichlet and Neumann cases.

Given a vector field V in $C^1(\mathbb{R}^n;\mathbb{R}^n)$, we consider the one-parameter family of domains which are obtained as deformations of Ω with V as initial velocity, that is we set

$$\Omega_\varepsilon := \left\{ x + \varepsilon V(x) \ : \ x \in \Omega \right\}, \qquad \varepsilon > 0 . \tag{5.3}$$

By definition, the shape derivative of J at Ω in direction V, if it exists, is given by the limit

$$J'(\Omega, V) := \lim_{\varepsilon \to 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} . \tag{5.4}$$

The approach we adopt in order to study the shape derivative (5.4) is different from the one usually employed in the literature, and seems to have a twofold interest: on one hand it allows to obtain the shape derivative for more general integrands f and g (see Theorem 5.2.1); on the other hand, along with the shape derivative, it leads to discover a new optimality condition for solutions to problems (5.1)-(5.2) (see Theorem 5.2.2).

The classical approach is based upon the *a priori* knowledge, for every $\varepsilon > 0$, of a minimizer u_ε for problem $J(\Omega_\varepsilon)$, satisfying a suitable Euler-Lagrange equation (see the Introduction and the monograph [67] for more details about the classical approach, and see the recent papers [10, 45] and references therein about the conditions required for the validity of the Euler-Lagrange equation).

In spite, our approach relies on the use of Convex Analysis, and more specifically of the dual formulation of $J(\Omega)$, which in the Dirichlet and Neumann cases reads respectively

$$J_D^*(\Omega) = \inf\left\{ \int_\Omega [f^*(\sigma) + g^*(\operatorname{div}\sigma)]dx \,:\, \sigma \in L^{p'}(\Omega;\mathbb{R}^n)\,,\ \operatorname{div}\sigma \in L^{q'}(\Omega) \right\}\,, \quad (5.5)$$

$$J_N^*(\Omega) = \inf\left\{ \int_\Omega [f^*(\sigma) + g^*(\operatorname{div}\sigma)]dx \,:\, \sigma \in L^{p'}(\Omega;\mathbb{R}^n),\ \operatorname{div}\sigma \in L^{q'}(\Omega)\,, \right. \quad (5.6)$$
$$\left. \sigma\cdot n = 0 \text{ on } \partial\Omega \right\}$$

where f^* and g^* denote the Fenchel conjugates of f and g, while $\sigma\cdot n$ is the normal trace of σ on $\partial\Omega$ intended in the sense of distributions (see Lemma 5.1.2).

Our study is motivated by the problem that we introduced in §4.6.2, that raised in the shape optimization of thin rods in pure torsion regime, settled on the bar cross-section: we dealt with a variational problem of the form (5.1), with $\Omega \subset \mathbb{R}^2$ an open bounded simply connected domain, and f, g the following convex functions

$$f(y) := \begin{cases} \frac{|y|^2}{2} + \frac{1}{2} & \text{if } |y| \geq 1 \\ |y| & \text{if } |y| < 1 \end{cases}\,, \qquad g(y) = -\lambda y \quad (\lambda \in \mathbb{R}).$$

Due to the lackness of regularity of f at the origin, the solutions of the associated functional $J(\Omega)$ do not satisfy an Euler-Lagrange equation, but merely a variational inequality (see Proposition 4.1.1), hence the shape derivative cannot be computed by using the classical approach.

The Chapter is organized as follows.
In Section 5.1 we introduce the preliminary material: we fix the main notation, the standing assumptions, and the basic lemmata concerning the functionals under study.
In Section 5.2 we state our main results, which are proved in Section 5.3: we show the shape derivative $J'(\Omega,V)$ exists (see Theorem 5.2.1) and, under additional regularity assumptions, can be also recast as a linear form in V (see Corollary 5.2.1); moreover we discover a new necessary condition of optimality for the classical variational problems under study (Theorem 5.2.2).
In the Appendix of Section 5.3 we present an alternative proof of the existence of $J'(\Omega,V)$, for inner variations (that is $V\cdot n \leq 0$ on $\partial\Omega$).
Finally, in Section 5.4 we present the possible advances, with particular attention to the second order shape derivative.

5.1 Preliminaries

5.1.1 Notation

Firstly, let us recall that throughout the Chapter the notation $J(\Omega)$ is adopted each time it can be intended indistinctly as in (5.1) or as in (5.2). Similarly, $J^*(\Omega)$ is meant either as in (5.5) or as in (5.6).

Only when required, we shall distinguish between the Dirichlet and the Neumann cases, indicated respectively as (D) and (N) in the sequel.

For brevity, we denote by $W(\Omega)$ the domain of admissible functions for $J(\Omega)$ (namely $W_0^{1,p}(\Omega)$ in case (D) and $W^{1,p}(\Omega)$ in case (N)), and by $X(\Omega;\mathbb{R}^n)$ the domains of admissible vector fields for $J^*(\Omega)$ (namely the space of $L^{p'}$ vector fields with divergence in $L^{q'}$ in case (D), with the additional condition $\sigma \cdot n = 0$ on the boundary in case (N)).

Moreover, we define the subsets $\mathscr{S}$ of $W(\Omega)$ and $\mathscr{S}^*$ of $X(\Omega;\mathbb{R}^n)$ by

$$\mathscr{S} := \Big\{ \text{ solutions to } J(\Omega) \Big\} \qquad \text{and} \qquad \mathscr{S}^* := \Big\{ \text{ solutions to } J^*(\Omega) \Big\}.$$

Given $V \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ and $\varepsilon > 0$, we denote by $\Psi_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ the bi-Lipschitz diffeomorphism

$$\Psi_\varepsilon(x) := x + \varepsilon V(x) \, .$$

Finally we introduce the functional space

$$X_\infty(\Omega;\mathbb{R}^n) := \big\{ \Psi \in L^\infty(\Omega;\mathbb{R}^n) \; : \; \operatorname{div} \Psi \in L^\infty(\Omega) \big\} \, .$$

For every $\Psi \in X_\infty(\Omega;\mathbb{R}^n)$, the normal trace $[\Psi \cdot n]_{\partial\Omega}$ is well defined (*cf.* [6, 35]) in the following weak sense:

$$\int_{\partial\Omega} [\Psi \cdot n]_{\partial\Omega} \, \varphi \, d\mathscr{H}^{n-1} = \int_\Omega \Psi \cdot \nabla \varphi \, dx + \int_\Omega \varphi \operatorname{div} \Psi \, dx \qquad \forall \varphi \in C^1(\overline{\Omega}) \, , \tag{5.7}$$

moreover $[\Psi \cdot n]_{\partial\Omega} \in L^\infty(\partial\Omega)$. In the sequel, we also use the notation $X_\infty(\Omega;\mathbb{R}^{n\times n})$ and $X_\infty(\Omega)$ to denote respectively the class of tensors A with rows in $X_\infty(\Omega;\mathbb{R}^n)$, and the class of scalar functions ψ with $\psi I \in X_\infty(\Omega;\mathbb{R}^{n\times n})$. Accordingly, we indicate by $[A\,n]_{\partial\Omega}$ and $[\psi\,n]_{\partial\Omega}$ the normal traces of A and ψI intended row by row as in (5.7).

The properties the functional space X_∞ and of traces of its elements are gathered in §1.4.2.

5.1.2 Standing assumptions

Throughout the Chapter, we work under the following hypotheses, which will be referred to as standing assumptions:

(H1) Ω is an open bounded connected set with a Lipschitz boundary, with unit outer normal n;

(H2) V is a vector field in $C^1(\mathbb{R}^n;\mathbb{R}^n)$;

(H3) $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are convex, continuous functions such that $g(0) = 0$ and

$$\begin{cases} \alpha(|z|^p - 1) \leq f(z) \leq \beta(|z|^p + 1) & \forall z \in \mathbb{R}^n \\ \gamma(|v|^p - 1) \leq g(v) \leq \delta(|v|^q + 1) & \forall v \in \mathbb{R}\,. \end{cases} \tag{5.8}$$

Here α, β, γ are positive constants, while the exponents p, q are assumed to satisfy

$$1 < p < +\infty\,, \qquad \begin{cases} q = p^* := \frac{np}{n-p} & \text{if } p < n \\ 1 < q < +\infty & \text{if } p \geq n\,. \end{cases}$$

In the Dirichlet case, the lower bound for g in (5.8) can be relaxed to

$$-\gamma(|v| + 1) \leq g(v) \qquad \forall v \in \mathbb{R}\,. \tag{5.9}$$

Notice that a positive constant γ such that (5.9) holds true exists for any real valued continuous convex function g, as it admits an affine minorant (see §1.1.1).

When further assumptions on f and g are needed, they will be specified in each statement.

5.1.3 Basic lemmata on integral functionals

Lemma 5.1.1. *Under the standing assumptions on f and g, let I_f and I_g be defined respectively on $L^p(\Omega;\mathbb{R}^n)$ and $L^q(\Omega)$ by*

$$I_f(z) := \int_\Omega f(z) \qquad \text{and} \qquad I_g(v) := \int_\Omega g(v)\,. \tag{5.10}$$

Then:

(i) the functionals $I_f(z)$ and $I_g(u)$ are convex, finite, strongly continuous and weakly l.s.c. respectively on $L^p(\Omega;\mathbb{R}^n)$ and $L^q(\Omega)$;

(ii) the functional $I_f(\nabla u) + I_g(u)$ is convex, finite, weakly coercive and weakly l.s.c. on $W(\Omega)$;

(iii) the sets $\mathscr{S}$ and $\mathscr{S}^*$ of solutions to $J(\Omega)$ and $J^*(\Omega)$ are nonempty.

Proof. (i) Since f and g are convex and continuous they admit an affine minorant, namely there exist $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ such that

$$a + b \cdot z \leq f(z)\,, \quad \alpha + \beta u \leq g(u) \tag{5.11}$$

for every $z \in \mathbb{R}^m$, $u \in \mathbb{R}$. Recalling that $q > 1$, condition (5.11), together with the growth condition (H3) from above, implies that f and g satisfy

$$|f(z)| \leq C(|z|^p + 1)\,, \quad |g(u)| \leq C'(|u|^q + 1)\,. \tag{5.12}$$

By combining (5.11), (5.12) and the properties of continuity and convexity of the integrands f and g, in view of Corollary 1.2.1 and Theorem 1.2.3, we infer that I_f and I_g

are both convex, finite, continuous with respect to the strong topology and lower semicontinuous with respect to the weak topology in the functional spaces $L^p(\Omega;\mathbb{R}^n)$ and $L^q(\Omega)$ respectively.

(ii) Clearly, by convexity and growth assumption of the integrands, the functional $I_f + I_g$ is finite and convex in $L^p(\Omega;\mathbb{R}^n) \times L^q(\Omega)$.

Let us prove the coercivity: we have to show that every sublevel

$$K_t := \{u \in W(\Omega;\mathbb{R}^n) \ : \ I_f(\nabla u) + I_g(u) \leq t\}$$

is contained in a weakly compact set. In what follows C and C' denote positive constants, that may be different from line to line.

Let $u \in K_t$: in view of the growth conditions from below we infer

$$C'(\|u\|^p_{W^{1,p}} - 1) \leq C(\|\nabla u\|^p_{L^p} + \|u\|^p_{L^p} - 1) \leq I_f(\nabla u) + I_g(u) \leq t \quad \text{in case } (N)$$
$$C'(\|u\|^p_{W^{1,p_0}} - \|u\|_{W^{1,p}_0}) \leq C(\|\nabla u\|^p_{L^p} - \|u\|_{L^p} - 1) \leq I_f(\nabla u) + I_g(u) \leq t \quad \text{in case } (D)$$

where, for the Dirichlet case (D), we have used the fact that $\|\nabla u\|_{L^p}$ is an equivalent norm in $W^{1,p}_0(\Omega;\mathbb{R}^n)$.

Hence every sublevel K_t is contained in some bounded set of $W(\Omega)$, in particular its closure is weakly compact.

Exploiting the semicontinuity found in (i) for I_f and I_g separately found in (i) we obtain the lower semicontinuity in $W(\Omega)$: let $\{u_k\}_k$ be a weakly convergent sequence in $W(\Omega)$, that is

$$\nabla u_k \overset{L^p}{\rightharpoonup} \nabla u \ , \ \ u_k \overset{L^p}{\to} u \ ,$$

for some $u \in W(\Omega)$; then in view of (i) we conclude that

$$I_f(\nabla u) + I_g(u) \leq \liminf_k I_f(\nabla u_k) + \liminf_k I_g(u_k) \leq \liminf_k (I_f(\nabla u_k) + I_g(u_k)) \ ,$$

(iii) By combining the finiteness, the weak coercivity and weak lower semicontinuity of $I_f + I_g$ in $W(\Omega)$ we obtain the existence of a solution for problem $J(\Omega)$, namely $\mathscr{S} \neq \emptyset$. The existence of at least one solution for the dual problem follows by the equality $J(\Omega) = J^*(\Omega)$ (that we will prove in Lemma 5.1.2) and the duality Proposition 1.1.2. □

Lemma 5.1.2. *Under the standing assumptions, there holds*

$$J(\Omega) = J^*(\Omega) \ . \tag{5.13}$$

Moreover, if $u \in W(\Omega)$ *and* $\sigma \in X(\Omega;\mathbb{R}^n)$, *there holds the following equivalence:*

$$(i) \begin{cases} u \in \mathscr{S} \\ \sigma \in \mathscr{S}^* \end{cases} \iff (ii) \begin{cases} \sigma \in \partial f(\nabla u) \ \ a.e.\ in\ \Omega \\ \operatorname{div}\sigma \in \partial g(u) \ \ a.e.\ in\ \Omega \end{cases} .$$

Proof. In order to prove the equality (5.13), we apply a standard Convex Analysis result, which is enclosed in §1.1.3 for convenience of the reader (*cf.* Proposition 1.1.2). Introducing the Banach spaces $X := W(\Omega)$, $Y := L^p(\Omega;\mathbb{R}^n) \times L^q(\Omega)$, the function $\Phi : X \to \mathbb{R}$ identically zero, the function $\Psi : Y \to \mathbb{R}$ defined by $\Psi(z,u) := I_f(z) + I_g(u)$, and the linear operator $A : X \to Y$ defined by $A(u) := (\nabla u, u)$, we can rewrite the shape functional $J(\Omega)$ as

$$J(\Omega) = -\inf_{u \in X} \{\Psi(Au) + \Phi(u)\} .$$

In view of Lemma 5.1.1 (i), we infer that Ψ is convex, finite and sequentially continuous on Y. Finally, if $u_0 \equiv 0$, it holds $\Phi(u_0) < +\infty$ and Ψ is continuous at $A(u_0)$. Then Proposition 1.1.2 applies and gives

$$J(\Omega) = \inf_{(\sigma,\tau) \in Y^*} \{\Psi^*(\sigma,\tau) + \Phi^*(-A^*(\sigma,\tau))\} . \tag{5.14}$$

Let us compute the Fenchel conjugates Ψ^*, Φ^* and the adjoint operator A^*.

From Proposition 5.3.4 and Proposition 1.2.1 we obtain that, for every $(\sigma,\tau) \in Y^* = L^{p'}(\Omega;\mathbb{R}^n) \times L^{q'}(\Omega)$, there holds

$$\Psi^*(\sigma,\tau) = (I_f)^*(\sigma) + (I_g)^*(\tau) = I_{f^*}(\sigma) + I_{g^*}(\tau) .$$

Since $\Phi \equiv 0$, its Fenchel conjugate Φ^* is 0 at 0 and $+\infty$ otherwise.

As an element of X^*, $A^*(\sigma,\tau)$ is characterized by its action on the elements of X: since

$$\begin{aligned} \langle A^*(\sigma,\tau), u\rangle_{X^*,X} = \langle(\sigma,\tau), A(u)\rangle_{Y^*,Y} &= \langle \tau, u\rangle_{L^{q'},L^q} + \langle \sigma, \nabla u\rangle_{L^{p'},L^p} \\ &= \int_\Omega \tau u + \sigma \cdot \nabla u , \end{aligned}$$

we infer that $A^*(\sigma,\tau) = 0$ if and only if $\tau = \operatorname{div}\sigma$ (with the additional condition $\sigma \cdot n = 0$ in the sense of distributions on the boundary in case (N)). Hence we can rewrite (5.14) as

$$\inf\left\{ \int_\Omega f^*(\sigma) + g^*(\operatorname{div}\sigma) \ : \ \sigma \in L^{p'}(\Omega;\mathbb{R}^n) ,\ \operatorname{div}\sigma \in L^{q'}(\Omega) \right\} ,$$

in case (D) and

$$\inf\left\{ \int_\Omega f^*(\sigma) + g^*(\operatorname{div}\sigma) \ : \ \sigma \in L^{p'}(\Omega;\mathbb{R}^n) ,\ \operatorname{div}\sigma \in L^{q'}(\Omega) ,\ \sigma \cdot n = 0 \right\} ,$$

in case (N). We conclude that (5.13) holds.

Finally, let us check the equivalence between conditions (i) and (ii). By Proposition 1.1.2, we infer that condition (i) holds true if and only if $(\operatorname{div}\sigma, \sigma) \in \partial\Psi(A(u))$. In view of Proposition 5.3.4, there holds

$$\partial\Psi(A(u)) = \partial\Psi(u, \nabla u) = \partial I_f(\nabla u) \times \partial I_g(u) ,$$

and hence, by Proposition 1.2.1, we have $(\operatorname{div}\sigma, \sigma) \in \partial\Psi(A(u))$ if and only if condition (ii) holds true.

□

We now endow $W(\Omega)$ and $X(\Omega;\mathbb{R}^n)$ respectively with the following convergence, which in both cases will be simply called *weak convergence*:

$$u_\varepsilon \overset{W^{1,p}}{\rightharpoonup} u_0 , \tag{5.15}$$

$$\sigma_\varepsilon \overset{L^{p'}}{\rightharpoonup} \sigma_0 , \qquad \operatorname{div}\sigma_\varepsilon \overset{L^{q'}}{\rightharpoonup} \operatorname{div}\sigma_0 . \tag{5.16}$$

Lemma 5.1.3. *Under the standing assumptions, the sets $\mathscr{S}$ and $\mathscr{S}^*$ are weakly compact respectively in $W(\Omega)$ and $X(\Omega;\mathbb{R}^n)$.*

Proof. Let u_k be a sequence of elements in $\mathscr{S}$. By the coercivity statement in Lemma 5.1.1 (ii), the sequence is bounded in $W(\Omega)$, hence it admits a subsequence which converges in the weak $W^{1,p}$-topology to some $u \in W(\Omega)$. By the l.s.c. statement in Lemma 5.1.1 (ii), we infer that also the limit function u belongs to $\mathscr{S}$.

Let us consider the set $\mathscr{S}^*$. In view of Lemma 5.1.2, we can write $\mathscr{S}^*$ as

$$\mathscr{S}^* = \{\sigma \in X(\Omega;\mathbb{R}^n) \ : \ \sigma \in \partial f(u_0) \ a.e., \ \operatorname{div}\sigma \in \partial g(u_0) \ a.e.\} , \tag{5.17}$$

with u_0 arbitrarily chosen in $\mathscr{S}$.

Recalling that the functionals I_f and I_g defined in (5.10) are convex and strongly continuous on $L^p(\Omega;\mathbb{R}^n)$ and $L^q(\Omega;\mathbb{R}^n)$ (see Lemma 5.1.1 (i)), we can apply Theorem 1.1.3 (ii) and infer that $\partial f(\nabla u_0)$ and $\partial g(u_0)$ are weakly compact sets respectively in $L^{p'}(\Omega;\mathbb{R}^n)$ and in $L^{q'}(\Omega)$. Hence, exploiting the characterization (5.17) and taking into account that the constraint $\tau = \operatorname{div}\sigma$ is weakly closed, we conclude that $\mathscr{S}^*$ is weakly compact in $X(\Omega;\mathbb{R}^n)$.

□

5.2 Main results

We begin by introducing the following crucial

Definition 5.2.1. For any $(u,\sigma) \in \mathscr{S} \times \mathscr{S}^*$, we set

$$A(u,\sigma) := \nabla u \otimes \sigma - [f(\nabla u) + g(u)]\, I \ ,$$

Remark 5.2.1. (i) Thanks to the growth conditions (5.8) satisfied by f and g, it is easy to check that $A(u,\sigma) \in L^1(\Omega;\mathbb{R}^{n\times n})$.

(ii) By using the Fenchel equality satisfied by u and σ (*cf.* Lemma 5.1.2), the tensor $A(u,\sigma)$ can be rewritten as

$$A(u,\sigma) := \nabla u \otimes \sigma + [f^*(\sigma) + g^*(\operatorname{div}\sigma) - \nabla u \cdot \sigma - u \operatorname{div}\sigma]\, I \ .$$

(iii) In case f is Gateaux differentiable except at most in the origin, the optimality condition $\sigma \in \partial f(\nabla u)$ holding for all $(u,\sigma) \in \mathscr{S} \times \mathscr{S}^*$ determines uniquely σ (as $\nabla f(\nabla u)$) in the set $\{\nabla u \neq 0\}$. Therefore in this case the tensor $A(u,\sigma)$ turns out to be independent of σ, and as such it will be simply denoted by $A(u)$. Namely, when f is Gateaux differentiable except at most in the origin, for any $u \in \mathscr{S}$ we set

$$A(u) := \nabla u \otimes \nabla f(\nabla u) - [f(\nabla u) + g(u)]\, I. \tag{5.18}$$

We are now in a position to state our main results.

Theorem 5.2.1. (existence of the shape derivative)
Under the standing assumptions, the shape derivative of the functional $J(\cdot)$ at Ω in direction V defined according to (5.4) exists. Actually, for every $V \in C^1(\mathbb{R}^n;\mathbb{R}^n)$, the following inf-sup and sup-inf agree and are equal to $J'(\Omega,V)$:

$$J'(\Omega,V) = \sup_{u\in\mathscr{S}} \inf_{\sigma\in\mathscr{S}^*} \int_\Omega A(u,\sigma) : DV = \inf_{\sigma\in\mathscr{S}^*} \sup_{u\in\mathscr{S}} \int_\Omega A(u,\sigma) : DV\,. \tag{5.19}$$

Moreover, there exists a saddle point $(u^\star,\sigma^\star) \in \mathscr{S}\times\mathscr{S}^*$ at which the inf-sup and sup-inf above are attained.

Remark 5.2.2. In general, equality (5.19) does not allow to conclude that $V \mapsto J'(\Omega,V)$ is a linear form, since *a priori* the pair $(u^\star,\sigma^\star)$ depends on V. However, the linearity of the shape derivative in V can be asserted in one of the following situations:

– when both primal and dual problems have a unique solution (as in this case both $\mathscr{S}$ and $\mathscr{S}^*$ are singletons);

– when the primal problem has a unique solution $\overline{u}$, and f is Gateaux differentiable except at most at the origin (as in this case $\mathscr{S}$ is a singleton, and the tensor A depends only on $\overline{u}$).

In particular, in the latter case we are going to see that, under some additional regularity assumptions on $\overline{u}$, the shape derivative can also be recast as a boundary integral depending linearly on the normal component of V on the boundary, see Corollary 5.2.1 below.

As a by-product of Theorem 5.2.1, we obtain the following result of independent interest:

Theorem 5.2.2. (necessary conditions for optimality)
Under the standing assumptions, there holds:

(i) For every $\overline{u}\in\mathscr{S}$, there exists $\widehat{\sigma} = \widehat{\sigma}(\overline{u}) \in \mathscr{S}^*$ such that

$$\operatorname{div} A(\overline{u},\widehat{\sigma}) = 0 \qquad \text{in } \mathscr{D}'(\Omega;\mathbb{R}^n)\,. \tag{5.20}$$

In particular, in case f is Gateaux differentiable except at most at the origin, for every $\overline{u}\in\mathscr{S}$ there holds

$$\operatorname{div} A(\overline{u}) = 0 \qquad \text{in } \mathscr{D}'(\Omega;\mathbb{R}^n)\,. \tag{5.21}$$

(ii) For every $\overline{\sigma}\in\mathscr{S}^*$, there exists $\widehat{u} = \widehat{u}(\overline{\sigma}) \in \mathscr{S}$ such that

$$\operatorname{div} A(\widehat{u},\overline{\sigma}) = 0 \qquad \text{in } \mathscr{D}'(\Omega;\mathbb{R}^n)\,. \tag{5.22}$$

Remark 5.2.3. We underline that in case f is Gateaux-differentiable except at most at the origin, in view of (5.18) in Remark 5.2.1 (iii), the property (5.20) implies that for every $u\in\mathscr{S}$

$$\operatorname{div}\Big(\nabla u \otimes \nabla f(\nabla u) - [f(\nabla u) + g(u)]\,I\Big) = 0 \tag{5.23}$$

in the sense of distributions. To some extent surprisingly, as far as we are aware, condition (5.23) seems to be until now undiscovered, except from the scalar case $n = 1$, when it reduces to the following conservation law or first integral of the Euler equation, satisfied by smooth extremals of smooth Lagrangians:

$$u' f'(u') - [f(u') + g(u)] = c\,,$$

see *e.g.* [25, Proposition 1.13].

Thanks to equality (5.21) in Theorem 5.2.2, when $\bar{u}$ satisfies suitable regularity assumptions as specified below, the associated tensor $A(\overline{u})$ turns out to admit a normal trace on the boundary, which can also be characterized in terms of the energy density:

Proposition 5.2.1. (boundary trace of the tensor A)
Under the standing assumptions, suppose in addition that f is Gateaux differentiable except at most at the origin, and let $\overline{u} \in \mathscr{S}$.

If $\overline{u}$ belongs to $\mathrm{Lip}(\Omega)$*, then $A(\overline{u})$ belongs to $X_\infty(\Omega;\mathbb{R}^{n\times n})$, and as such it admits a normal trace on the boundary $[A(\overline{u})\,n]_{\partial\Omega} \in L^\infty(\partial\Omega;\mathbb{R}^n)$.*

If in addition $\partial\Omega$ is piecewise C^1, $\nabla\overline{u} \in BV(\Omega)$, and the field $\nabla f(\nabla\overline{u})$, which is defined on the set $\Omega_{\overline{u}} := \{\nabla\overline{u} \neq 0\}$, can be extended to a field $\zeta(\overline{u})$ defined on Ω, such that

$$\begin{cases} \zeta(\overline{u}) \in BV(\Omega;\mathbb{R}^n) & \textit{in case } (D) \\ \zeta(\overline{u}) \in BV(\Omega;\mathbb{R}^n),\ \textit{with } \mathrm{Tr}(\zeta(\overline{u}))\cdot n = 0 \textit{ on } \partial\Omega & \textit{in case } (N)\,, \end{cases} \tag{5.24}$$

then

$$[A(\overline{u})\,n]_{\partial\Omega} = \begin{cases} \mathrm{Tr}\big(\nabla\overline{u}\cdot\zeta(\overline{u}) - f(\nabla\overline{u})\big)n = \mathrm{Tr}\big(f^*(\zeta(\overline{u}))\big)n & \textit{in case } (D) \\ \mathrm{Tr}\big(f(\nabla\overline{u}) + g(\overline{u})\big)n & \textit{in case } (N)\,. \end{cases} \tag{5.25}$$

Remark 5.2.4. We point out that the existence of a vector field $\zeta(\overline{u})$ satisfying (5.24) is guaranteed for instance in one of the following situations:

$$\partial\Omega_{\overline{u}} \text{ is Lipschitz and } \nabla f(\nabla\overline{u}) \in BV(\Omega_{\overline{u}}) \tag{5.26}$$

$$\text{there exists } \overline{\sigma} \in \mathscr{S}^* \cap BV(\Omega;\mathbb{R}^n)\,. \tag{5.27}$$

Indeed, in case (5.26) it is enough to define $\zeta(\overline{u}) = 0$ outside $\Omega_{\overline{u}}$ (see [5, Corollary 3.89]), whereas in case (5.27) one can take $\zeta(\overline{u}) = \overline{\sigma}$ (see Remark 5.2.1 (iii)).

As a consequence of Theorem 5.2.1 and Proposition 5.2.1, we obtain:

Corollary 5.2.1. (shape derivative as a linear form on the boundary)
Under the standing assumptions, suppose in addition that f is Gateaux differentiable except at most at the origin, and assume that problem $J(\Omega)$ admits a unique solution $\overline{u}$, with $\overline{u} \in$ $\mathrm{Lip}(\Omega)$. *Then the shape derivative given by* (5.19) *can be identified with the following linear form*

$$J'(\Omega,V) = \int_{\partial\Omega} [A(\overline{u})\,n]_{\partial\Omega}\cdot V\,d\mathscr{H}^{n-1}\,. \tag{5.28}$$

If in addition $\partial\Omega$ *is piecewise* C^1, $\nabla\overline{u} \in BV(\Omega;\mathbb{R}^n)$, *and the field* $\nabla f(\nabla\overline{u})$ *can be extended to a field* $\zeta(\overline{u})$ *as in* (5.24)*, then*

$$J'(\Omega,V) = \begin{cases} \displaystyle\int_{\partial\Omega} \mathrm{Tr}\big(f^*(\zeta(\overline{u}))\big)(V\cdot n)\, d\mathscr{H}^{n-1} & \text{in case } (D) \\ \displaystyle\int_{\partial\Omega} \mathrm{Tr}\big(f(\nabla\overline{u})+g(\overline{u})\big)(V\cdot n)\, d\mathscr{H}^{n-1} & \text{in case } (N)\,. \end{cases}$$

Remark 5.2.5. Let us point out that the Lipschitz regularity of solutions to classical problems in the Calculus of Variations is a delicate matter which is currently object of investigation by several Authors. It is beyond of the scopes of our present study to discuss the conditions on f and g which yield Lipschitz solutions to $J(\Omega)$ as assumed in Proposition 5.2.1 and Corollary 5.2.1: without any attempt of giving a complete bibliographical list, we refer the interested reader to the papers [33, 55, 74, 76] (and references therein) for both local and global regularity results.

Remark 5.2.6. We remark that the study of the first order shape derivative doesn't allow us to obtain any further information about the behavior of the solutions for the problem $m(s)$ introduced in Chapter 4. As explained in §4.6.2, given a solution u for the primal problem, in order to better understand its behavior we can study the first order shape derivative of the functional in (4.63) with respect to inner deformations of the set $D\setminus\{|\nabla u|=0\}$. In view of (4.55), a solution σ for the dual problem satisfies

$$|\sigma| = 1 \quad \text{on } \partial\{|\nabla u|=0\}\cap D\,, \;\; \text{on } \partial\{0<|\nabla u|\leq 1\}\cap D\,, \;\; \text{and on } \partial\{|\nabla u|>1\}\cap D\,.$$

In case u is a special solution, the second set above is the empty set. The integrand φ in (4.3) is Gateaux differentiable except at the origin, moreover we assume that the free boundary is smooth, then we are in a position to apply Corollary 5.2.1. Recalling that the Fenchel conjugate φ^* in in (4.5) vanishes in the ball of radius 1 , we infer that the first order shape derivative in (5.28) vanishes. Thus we cannot exclude or confirm the presence of homogenization regions.

5.3 Proofs

In order to prove the results stated in the previous section, we carry over a thorough analysis of the asymptotic behavior as $\varepsilon \to 0^+$ of the sequence

$$q_\varepsilon(V) := \frac{J(\Omega_\varepsilon)-J(\Omega)}{\varepsilon} = \frac{J^*(\Omega_\varepsilon)-J^*(\Omega)}{\varepsilon}\,, \tag{5.29}$$

where the domains Ω_ε are deformations of Ω through V as in (5.3), and the second equality is due to Lemma 5.1.2.

More precisely, we proceed as follows: first we rewrite $J(\Omega_\varepsilon)$ and $J^*(\Omega_\varepsilon)$ as minimum problems for integral functionals set over the fixed domain Ω (Lemma 5.3.1) and we study the asymptotic behavior of their solutions (Proposition 5.3.1); as a consequence, we are able to provide a lower bound and an upper bound for $q_\varepsilon(V)$ (Propositions 5.4.1 and 5.3.3); afterwards, by exploiting these bounds, we prove Theorem 5.2.1 and, finally, all the other results stated in Section 5.2.

The proofs of the upper and lower bound are based on a Γ-convergence technique. In the Appendix we show an alternative proof: it requires additional assumptions on the growth condition (H3), nevertheless the more direct approach can be better applied for second order shape derivatives (see §5.4.1).

For every $\varepsilon > 0$, let E_ε and H_ε be the functionals defined respectively on $W(\Omega)$ and $X(\Omega;\mathbb{R}^n)$ by

$$E_\varepsilon(u) := \int_\Omega [f(D\Psi_\varepsilon^{-T}\nabla u) + g(u)]\,|\det D\Psi_\varepsilon|\,, \tag{5.30}$$

$$H_\varepsilon(\sigma) := \int_\Omega [f^*(|\det D\Psi_\varepsilon|^{-1} D\Psi_\varepsilon \sigma) + g^*(|\det D\Psi_\varepsilon|^{-1}\operatorname{div}\sigma)]\,|\det D\Psi_\varepsilon| \tag{5.31}$$

In terms of these functionals, we can rewrite the minimization problems under study as follows:

Lemma 5.3.1. *Under the standing assumptions, for every $\varepsilon > 0$ there holds*

$$J(\Omega_\varepsilon) = -\inf\{E_\varepsilon(u)\ :\ u \in W(\Omega)\}\,, \tag{5.32}$$

$$J^*(\Omega_\varepsilon) = \inf\{H_\varepsilon(\sigma)\ :\ \sigma \in X(\Omega;\mathbb{R}^n)\}\,. \tag{5.33}$$

Proof. Let $\varepsilon > 0$ be fixed. Functions $\tilde{u} \in W(\Omega_\varepsilon)$ are in 1-1 correspondence with functions $u \in W(\Omega)$ through the equality $\tilde{u} = u \circ \Psi_\varepsilon^{-1}$ in Ω_ε; moreover, via change of variables, there holds

$$\int_{\Omega_\varepsilon} [f(\nabla\tilde{u}) + g(\tilde{u})] = \int_\Omega [f(D\Psi_\varepsilon^{-T}\nabla u) + g(u)]|\det D\Psi_\varepsilon|\,. \tag{5.34}$$

Passing to the infimum over $\tilde{u} \in W(\Omega_\varepsilon)$ in the l.h.s. and over $u \in W(\Omega)$ at the r.h.s., we obtain (5.32).

For brevity, in the remaining of the proof we set

$$\beta_\varepsilon := |\det D\Psi_\varepsilon| \qquad f_\varepsilon(x,z) := f(D\Psi_\varepsilon^{-T} z)\beta_\varepsilon\,, \qquad g_\varepsilon(x,v) := g(v)\beta_\varepsilon\,.$$

Then, in view of (5.30) and (5.32), the functional $J(\Omega_\varepsilon)$ reads

$$J(\Omega_\varepsilon) = -\inf\Big\{\int_\Omega [f_\varepsilon(x,\nabla u) + g_\varepsilon(x,u)]\ :\ u \in W(\Omega)\Big\}\,.$$

Moreover, by arguing in the same way as already done in the proof of Lemma 5.1.2, we obtain that the dual form of $J(\Omega_\varepsilon)$ is given by

$$J^*(\Omega_\varepsilon) = \inf\left\{\int_\Omega [f_\varepsilon^*(x,\sigma) + g_\varepsilon^*(x,\operatorname{div}\sigma)]\ :\ \sigma \in X(\Omega;\mathbb{R}^n)\right\}\,. \tag{5.35}$$

Here f_ε^* and g_ε^* denote the Fenchel conjugates of f and g, performed with respect to the second variable. Their computation gives:

$$f_\varepsilon^*(x,z^*) = \sup_{z\in\mathbb{R}^n}\{z\cdot z^* - f(D\Psi_\varepsilon^{-T} z)\beta_\varepsilon\} = \beta_\varepsilon f^*(\beta_\varepsilon^{-1} D\Psi_\varepsilon z^*)\,,$$

$$g_\varepsilon^*(x,v^*) = \sup_{v\in\mathbb{R}}\{v v^* - g(v)\beta_\varepsilon\} = \beta_\varepsilon g^*(\beta_\varepsilon^{-1} v^*)\,.$$

Hence, recalling definition (5.31), we obtain (5.33). □

Now, we study the asymptotic behavior as $\varepsilon \to 0$ of solutions to problems (5.32) and (5.33). To that aim, we introduce the limit functionals defined on $W(\Omega)$ and $X(\Omega;\mathbb{R}^n)$ by

$$E(u) := \int_\Omega [f(\nabla u) + g(u)] \,, \tag{5.36}$$

$$H(\sigma) := \int_\Omega [f^*(\sigma) + g^*(\operatorname{div}\sigma)] \,. \tag{5.37}$$

We recall that $W(\Omega)$ and $X(\Omega;\mathbb{R}^n)$ are endowed with the weak convergence defined in (5.15)-(5.16).

Proposition 5.3.1. (asymptotic behavior of minimizers)

(i) On the space $W(\Omega)$ endowed with the weak convergence, the sequence E_ε in (5.30) is equicoercive and, as $\varepsilon \to 0$, it Γ-converges to the functional E in (5.36). In particular, every sequence u_ε such that $u_\varepsilon \in \operatorname{Argmin}(E_\varepsilon)$ admits a subsequence which converges weakly in $W(\Omega)$ to some $u_0 \in \operatorname{Argmin}(E)$.

(ii) On the space $X(\Omega;\mathbb{R}^n)$ endowed with the weak convergence, the sequence H_ε in (5.31) is equicoercive and, as $\varepsilon \to 0$, it Γ-converges to the functional H in (5.37). In particular, every sequence σ_ε such that $\sigma_\varepsilon \in \operatorname{Argmin}(H_\varepsilon)$ admits a subsequence which converges weakly in $X(\Omega;\mathbb{R}^n)$ to some $\sigma_0 \in \operatorname{Argmin}(H)$.

Proof. (i) The equicoercivity of the family of functionals E_ε can be easily obtained by exploiting the growth assumptions (H3) on f and g, the uniform boundedness from below of the positive coefficients β_ε, and the uniform control on the L^∞ norm of the tensors $D\Psi_\varepsilon^{-T}$. Let us prove the Γ-convergence statement for E_ε. By definition of Γ-convergence, we have to show that the so-called Γ-liminf and Γ-limsup inequalities hold, namely:

$$\inf\left\{ \liminf E_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \overset{W^{1,p}}{\rightharpoonup} u \right\} \geq E(u) \,, \tag{5.38}$$

$$\inf\left\{ \limsup E_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \overset{W^{1,p}}{\rightharpoonup} u \right\} \leq E(u) \,. \tag{5.39}$$

Let us prove (5.38). Consider an arbitrary sequence u_ε which converges weakly to u in $W(\Omega)$. We observe that the sequence $D\Psi_\varepsilon^{-T}\nabla u_\varepsilon$ converges to ∇u weakly in $L^p(\Omega;\mathbb{R}^n)$, and that β_ε converges to 1 uniformly in Ω. Hence, exploiting the weak lower semicontinuity of I_f and I_g in $L^p(\Omega;\mathbb{R}^n)$ and $L^q(\Omega)$ respectively (*cf.* Lemma 5.1.1 (i)), we infer that

$$E(u) \leq \liminf_\varepsilon \int_\Omega f(D\Psi_\varepsilon^{-T}\nabla u_\varepsilon)\beta_\varepsilon + \liminf_\varepsilon \int_\Omega g(u_\varepsilon)\beta_\varepsilon \, dx \leq \liminf_\varepsilon E_\varepsilon(u_\varepsilon) \,,$$

which implies (5.38).

Let us prove (5.39). For every fixed $u \in W(\Omega)$ we have to find a recovery sequence, namely a sequence u_ε which converges weakly to u in $W(\Omega)$ and satisfies

$$E(u) \geq \limsup_\varepsilon E_\varepsilon(u_\varepsilon) \,. \tag{5.40}$$

We claim that the sequence $u_\varepsilon \equiv u$ for every $\varepsilon > 0$ satisfies (5.40). Indeed, since $D\Psi_\varepsilon^{-T}\nabla u$ converges strongly to ∇u in $L^p(\Omega;\mathbb{R}^n)$, by exploiting Lemma 5.1.1 (i) we obtain :

$$E(u) = \lim_{\varepsilon\to 0}\int_\Omega f(D\Psi_\varepsilon^{-T}\nabla u)\beta_\varepsilon + \lim_{\varepsilon\to 0}\int_\Omega g(u)\beta_\varepsilon \geq \limsup_\varepsilon E_\varepsilon(u) \ .$$

Finally, the compactness of a minimizing sequence is a well-known consequence of Γ-convergence (see Theorem 1.3.3).

(ii) The equicoercivity of the sequence H_ε can be easily obtained by exploiting the uniform boundedness from below of the positive coefficients β_ε, the uniform control on the L^∞ norm of the tensors $D\Psi_\varepsilon^{-T}$, and the following growth conditions, holding for some positive constants a, b as a consequence of the standing assumption (H3):

$$\begin{aligned} f^*(z^*) &\geq a(|z^*|^{p'} - 1) \qquad \forall z^* \in \mathbb{R}^n, \\ g^*(v^*) &\geq b(|v^*|^{q'} - 1) \qquad \forall v^* \in \mathbb{R}. \end{aligned}$$

Let us prove the Γ-convergence statement for H_ε. We observe that the Γ-convergence of the functionals E_ε to E proved at item (i) above can be restated on the product space

$$\big\{(u,\nabla u) \, : \, u \in L^q(\Omega)\,, \nabla u \in L^p(\Omega;\mathbb{R}^n)\big\} \tag{5.41}$$

(endowed with the product of the weak L^q and weak L^p convergences), which in fact can be identified with the space of functions u in $W(\Omega)$ (endowed with the weak convergence defined in (5.15)). Moreover, such Γ-convergence can be strengthened into a Mosco-convergence (see [79]), because we have exhibited a recovery sequence which converges in the strong topology. Since the Mosco-convergence is stable when passing to the Fenchel conjugates (see §1.3.5), we deduce that the functionals E_ε^* Mosco-converge (and hence Γ-converge) to the functional E^*. We conclude by noticing that the dual of the product space in (5.41) (endowed with the product of the weak $L^{q'}$ and weak $L^{p'}$ convergences) is precisely $X(\Omega;\mathbb{R}^n)$ (endowed with the weak convergence in (5.16)), and on such space H_ε and H agree respectively with the Fenchel conjugates E_ε^* and E^*. Finally, the compactness of a minimizing sequence follows again as a consequence of Γ-convergence.

□

Proposition 5.3.2. (lower bound) *Under the standing assumptions, it holds*

$$\liminf_{\varepsilon\to 0^+} q_\varepsilon(V) \geq \inf_{\sigma\in\mathscr{S}^*}\sup_{u\in\mathscr{S}}\int_\Omega A(u,\sigma) : DV \ . \tag{5.42}$$

Proof. In order to prove (5.42) it is enough to show that

$$\liminf_{\varepsilon\to 0^+} q_\varepsilon(V) \geq \sup_{u\in\mathscr{S}}\int_\Omega A(u,\sigma_0) : DV \tag{5.43}$$

for some $\sigma_0 \in \mathscr{S}^*$.

In order to bound $q_\varepsilon(V)$ from below, we observe that, by Lemma 5.1.2 and Lemma 5.3.1, there holds

$$q_\varepsilon(V) = \frac{J^*(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} = \frac{\inf H_\varepsilon + \inf E}{\varepsilon}\ , \tag{5.44}$$

being H_ε and E the functionals defined respectively in (5.31) and (5.36). In view of (5.44), letting $\sigma_\varepsilon \in \mathrm{Argmin}(H_\varepsilon)$ and $u \in \mathrm{Argmin}(E) = \mathscr{S}$, $q_\varepsilon(V)$ reads

$$q_\varepsilon(V) = \frac{1}{\varepsilon}\left[\int_\Omega [f^*(\beta_\varepsilon^{-1} D\Psi_\varepsilon \sigma_\varepsilon) + g^*(\beta_\varepsilon^{-1}\operatorname{div}\sigma_\varepsilon)]\beta_\varepsilon + \int_\Omega [f(\nabla u) + g(u)]\right] .$$

Taking into account that the coefficient β_ε is (strictly) positive everywhere, by applying the Fenchel inequality we obtain

$$q_\varepsilon(V) \geq \frac{1}{\varepsilon}\left[\int_\Omega [(D\Psi_\varepsilon \sigma_\varepsilon)\cdot \nabla u + u \operatorname{div}\sigma_\varepsilon] - \int_\Omega [f(\nabla u) + g(u)](\beta_\varepsilon - 1)\right] . \tag{5.45}$$

Recalling that $D\Psi_\varepsilon = I + \varepsilon DV$, an integration by parts gives

$$\int_\Omega [(D\Psi_\varepsilon \sigma_\varepsilon)\cdot \nabla u + u \operatorname{div}\sigma_\varepsilon] = \varepsilon \int_\Omega (DV \sigma_\varepsilon)\cdot \nabla u\ . \tag{5.46}$$

By combining (5.45) and (5.46), we obtain

$$q_\varepsilon(V) \geq \int_\Omega [(DV\sigma_\varepsilon)\cdot \nabla u] - \int_\Omega [f(\nabla u) + g(u)]\frac{(\beta_\varepsilon - 1)}{\varepsilon}\ . \tag{5.47}$$

We recall that $\beta_\varepsilon = 1 + \varepsilon \operatorname{div} V + \varepsilon^2 m_\varepsilon$ for some $m_\varepsilon \in C^0(\Omega)$ such that $\sup_\varepsilon \|m_\varepsilon\|_{L^\infty(\Omega)} \leq C$ for some positive constant C. Therefore,

$$\frac{\beta_\varepsilon - 1}{\varepsilon} \to \operatorname{div} V \qquad \text{uniformly}\ . \tag{5.48}$$

In view of Proposition 5.3.1 (ii), up to subsequences there holds

$$\sigma_\varepsilon \overset{L^{p'}}{\rightharpoonup} \sigma_0 \tag{5.49}$$

for some $\sigma_0 \in \mathrm{Argmin}(H) = \mathscr{S}^*$. We remark that *a priori* σ_ε and σ_0 depend on V. Thanks to (5.48) and (5.49), by passing to the limit as $\varepsilon \to 0^+$ in (5.47), we conclude that

$$\liminf_{\varepsilon\to 0^+} q_\varepsilon(V) \geq \int_\Omega [(DV\sigma_0)\cdot \nabla u] - \int_\Omega [f(\nabla u) + g(u)] \operatorname{div} V = \int_\Omega A(u, \sigma_0) : DV\ .$$

Finally, by the arbitrariness of $u \in \mathscr{S}$, we obtain (5.43).

□

Proposition 5.3.3. (upper bound)
Under the standing assumptions, it holds

$$\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) \leq \sup_{u\in S}\ \inf_{\sigma\in\mathscr{S}^*} \int_\Omega A(u,\sigma) : DV\ . \tag{5.50}$$

Proof. In order to prove (5.93), it is enough to show that

$$\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) \le \inf_{\sigma\in\mathscr{S}^*}\int_\Omega A(u_0,\sigma):DV\ , \tag{5.51}$$

for some $u_0\in\mathscr{S}$.

In order to bound $q_\varepsilon(V)$ from above, we observe that, by Lemma 5.1.2 and Lemma 5.3.1, there holds

$$q_\varepsilon(V) = \frac{J(\Omega_\varepsilon)-J^*(\Omega)}{\varepsilon} = \frac{-\inf E_\varepsilon - \inf H}{\varepsilon}\,, \tag{5.52}$$

being E_ε and H the functionals defined respectively in (5.30) and (5.37). In view of (5.52), letting $u_\varepsilon\in \mathrm{Argmin}(E_\varepsilon)$ and $\sigma\in\mathrm{Argmin}(H)=\mathscr{S}^*$, $q_\varepsilon(V)$ reads

$$q_\varepsilon(V) = \frac{1}{\varepsilon}\left[-\int_\Omega\left[f(D\psi_\varepsilon^{-T}\nabla u_\varepsilon)+g(u_\varepsilon)\right]\beta_\varepsilon - \int_\Omega [f^*(\sigma)+g^*(\operatorname{div}\sigma)]\right]\ . \tag{5.53}$$

Taking into account that the coefficient β_ε is (strictly) positive everywhere, by applying the Fenchel inequality we obtain

$$\begin{aligned}\int_\Omega\left[f(D\psi_\varepsilon^{-T}\nabla u_\varepsilon)\ +g(u_\varepsilon)\right]\beta_\varepsilon &\ge \int_\Omega\left[(D\Psi_\varepsilon^{-1}\sigma)\cdot\nabla u_\varepsilon+\operatorname{div}\sigma u_\varepsilon - f^*(\sigma)-g^*(\operatorname{div}\sigma)\right]\beta_\varepsilon\\ &= \int_\Omega[\sigma\cdot\nabla u_\varepsilon + u_\varepsilon\operatorname{div}\sigma - f^*(\sigma)-g^*(\operatorname{div}\sigma)]+\\ &+\varepsilon\int_\Omega[(\sigma\cdot\nabla u_\varepsilon+u_\varepsilon\operatorname{div}\sigma - f^*(\sigma)-g^*(\operatorname{div}\sigma))\operatorname{div}V-(DV\sigma)\cdot\nabla u_\varepsilon]+\\ &+\varepsilon^2\int_\Omega[a_\varepsilon\cdot\nabla u_\varepsilon+\alpha_\varepsilon u_\varepsilon - m_\varepsilon(f^*(\sigma)+g^*(\operatorname{div}\sigma))]\end{aligned} \tag{5.54}$$

where we have used the following asymptotic expansions of β_ε and $D\Psi_\varepsilon^{-1}$ in terms of ε

$$\beta_\varepsilon = 1+\varepsilon\operatorname{div}V+\varepsilon^2 m_\varepsilon\ ,\ \ D\Psi_\varepsilon^{-1} = I-\varepsilon DV+\varepsilon^2 M_\varepsilon\ ,$$

for some $m_\varepsilon\in C(\mathbb{R})$ and $M_\varepsilon\in C(\mathbb{R}^n;\mathbb{R}^{n\times n})$, and we have denoted by a_ε and α_ε the the vector field and the scalar function defined by

$$\begin{aligned}a_\varepsilon &:= m_\varepsilon\sigma - \operatorname{div}V\,DV\,\sigma + M_\varepsilon\sigma - \varepsilon m_\varepsilon DV\sigma + \varepsilon\operatorname{div}V\,M_\varepsilon\sigma+\varepsilon^2 m_\varepsilon M_\varepsilon\sigma\ ,\\ \alpha_\varepsilon &:= m_\varepsilon\operatorname{div}\sigma\ .\end{aligned}$$

We remark that

$$\sup_\varepsilon\|m_\varepsilon\|_{L^\infty(\Omega)}\le C\ ,\ \sup_\varepsilon\|M_\varepsilon\|_{L^\infty(\Omega)}\le C\ ,\ \sup_\varepsilon\|a_\varepsilon\|_{L^{p'}(\Omega;\mathbb{R}^n)}\le C\ ,\ \sup_\varepsilon\|\alpha_\varepsilon\|_{L^{q'}(\Omega)}\le C\ . \tag{5.55}$$

Then, by combining (5.53) and (5.54), and recalling that

$$\int_\Omega \sigma\cdot\nabla u_\varepsilon + u_\varepsilon\operatorname{div}\sigma = 0\ ,$$

we infer

$$q_\varepsilon(V)\le\int_\Omega[f^*(\sigma)+g^*(\operatorname{div}\sigma)]\operatorname{div}V+\int_\Omega[(DV\,\sigma)\cdot\nabla u_\varepsilon-(\sigma\cdot\nabla u_\varepsilon+u_\varepsilon\operatorname{div}\sigma)\operatorname{div}V]-\varepsilon C_\varepsilon\ , \tag{5.56}$$

being

$$C_\varepsilon := \int_\Omega [a_\varepsilon \cdot \nabla u_\varepsilon + \alpha_\varepsilon u_\varepsilon - m_\varepsilon (f^*(\sigma) + g^*(\operatorname{div} \sigma))] \ .$$

By Proposition 5.3.1 (i), up to subsequences there holds

$$u_\varepsilon \overset{W^{1,p}(\Omega)}{\rightharpoonup} u_0 \tag{5.57}$$

for some $u_0 \in \operatorname{Argmin}(E) = \mathscr{S}$. We remark that *a priori* u_0 and u_ε depend on V.

Exploiting (5.55) and (5.57), we infer that the sequence C_ε is bounded. Then, passing to the limit as $\varepsilon \to 0^+$ in (5.56), we conclude that

$$\limsup_{\varepsilon \to 0^+} q_\varepsilon(V) \le \int_\Omega [f^*(\sigma) + g^*(\operatorname{div} \sigma)] \operatorname{div} V + \int_\Omega [(DV\sigma) \cdot \nabla u_0 - (\sigma \cdot \nabla u_0 + u_0 \operatorname{div} \sigma) \operatorname{div} V] \ . \tag{5.58}$$

Since $u_0 \in \mathscr{S}$ and $\sigma \in \mathscr{S}^*$, we can rewrite the r.h.s. of (5.58) as

$$\int_\Omega [(DV\sigma) \cdot \nabla u_0 - (f(\nabla u_0) + g(u_0)) \operatorname{div} V] \, dx = \int_\Omega A(u_0, \sigma) : DV \ .$$

Finally, by the arbitrariness of $\sigma \in \mathscr{S}^*$, we obtain (5.51).

□

Proof of Theorem 5.2.1.

By combining the lower and upper bounds obtained in Propositions 5.4.1 and 5.3.3, we infer

$$\inf_{\sigma \in \mathscr{S}^*} \sup_{u \in \mathscr{S}} \int_\Omega A(u, \sigma) : DV \le \liminf_{\varepsilon \to 0^+} q_\varepsilon(V) \le \limsup_{\varepsilon \to 0^+} q_\varepsilon(V) \le \sup_{u \in \mathscr{S}} \inf_{\sigma \in \mathscr{S}^*} \int_\Omega A(u, \sigma) : DV \ . \tag{5.59}$$

Since the sup-inf at the r.h.s. of (5.59) is always lower than or equal to the inf-sup at the l.h.s., we infer that all the inequalities in (5.59) are actually equalities; in particular, the sequence $q_\varepsilon(V)$ converges as $\varepsilon \to 0^+$, and its limit provides the shape derivative $J'(\Omega, V)$, namely

$$J'(\Omega, V) = \sup_{u \in \mathscr{S}} \inf_{\sigma \in \mathscr{S}^*} \int_\Omega A(u, \sigma) : DV = \inf_{\sigma \in \mathscr{S}^*} \sup_{u \in \mathscr{S}} \int_\Omega A(u, \sigma) : DV \ .$$

Now, we observe that the functionals $\sigma \mapsto \int_\Omega A(u, \sigma) : DV$ and $u \mapsto \int_\Omega A(u, \sigma) : DV$ are linearly affine (see respectively Definition 5.2.1 and Remark 5.2.1 (ii)), and hence weakly continuous respectively on $W(\Omega)$ and $X(\Omega; \mathbb{R}^n)$. Moreover, the sets $\mathscr{S} \subset W(\Omega)$ and $\mathscr{S}^* \subset X(\Omega; \mathbb{R}^n)$ are weakly compact (see Lemma 5.1.3). Therefore, by Proposition 1.1.3, the sup-inf or inf-sup above is attained at some optimal pair $(u^\star, \sigma^\star) \in \mathscr{S} \times \mathscr{S}^*$, which *a priori* depends on V.

□

Proof of Theorem 5.2.2.

(i) Let V be a deformation field in $C_0^\infty(\Omega;\mathbb{R}^n)$. Clearly, since V is compactly supported into Ω, for every ε small enough the deformed domain Ω_ε in (5.3) coincides with Ω, so that

$$J'(\Omega,V) = 0 \,. \tag{5.60}$$

Let us fix $\overline{u} \in \mathscr{S}$ and define $L_{\overline{u}} : C_0^\infty(\Omega;\mathbb{R}^n) \times \mathscr{S}^* \to \mathbb{R}$ as

$$L_{\overline{u}}(V,\sigma) := -\int_\Omega A(\overline{u},\sigma) : DV \,. \tag{5.61}$$

By (5.60) and Theorem 5.2.1, we get

$$\sup_{\sigma\in\mathscr{S}^*} L_{\overline{u}}(V,\sigma) = -\inf_{\sigma\in\mathscr{S}^*}\int_\Omega A(\overline{u},\sigma) : DV \geq -\sup_{u\in\mathscr{S}}\inf_{\sigma\in\mathscr{S}^*}\int_\Omega A(\overline{u},\sigma) : DV = -J'(\Omega,V) = 0\,;$$

then, by the arbitrariness of $V \in C_0^\infty(\Omega;\mathbb{R}^n)$, we deduce that

$$\inf_{V\in C_0^\infty(\Omega;\mathbb{R}^n)}\sup_{\sigma\in\mathscr{S}^*} L_{\overline{u}}(V,\sigma) \geq 0\,.$$

By taking $V \equiv 0$, we see that the inf-sup above is in fact zero.

Since by Lemma 5.1.3 the set $\mathscr{S}^*$ is convex and weakly compact in $X(\Omega;\mathbb{R}^n)$, $L_{\overline{u}}(\cdot,\sigma)$ is convex, $L_{\overline{u}}(V,\cdot)$ is concave and weakly upper semicontinuous, Proposition 1.1.3 applies and gives

$$0 = \inf_{V\in C_0^\infty(\Omega;\mathbb{R}^n)}\sup_{\sigma\in\mathscr{S}^*} L_{\overline{u}}(V,\sigma) = \sup_{\sigma\in\mathscr{S}^*}\inf_{V\in C_0^\infty} L_{\overline{u}}(V,\sigma)\,;$$

moreover, there exists $\widehat{\sigma} \in \mathscr{S}^*$, depending on $\overline{u}$, such that

$$\inf_{V\in C_0^\infty(\Omega;\mathbb{R}^n)} L_{\overline{u}}(V,\widehat{\sigma}) = 0\,. \tag{5.62}$$

Since $L_{\overline{u}}(\cdot,\widehat{\sigma})$ is linear, the equality (5.62) implies

$$L_{\overline{u}}(V,\widehat{\sigma}) = 0 \qquad \forall V \in C_0^\infty(\Omega;\mathbb{R}^n)\,. \tag{5.63}$$

Finally, recalling the definition (5.61) of $L_{\overline{u}}$, condition (5.63) can be written as

$$\int_\Omega A(\overline{u},\widehat{\sigma}) : DV = 0 \qquad \forall V \in C_0^\infty(\Omega;\mathbb{R}^n)\,,$$

namely $\widehat{\sigma}$ satisfies (5.20).

(ii) We argue in a similar way as already done for the proof of statement (i). Let us fix $\overline{\sigma} \in \mathscr{S}^*$ and define $L_{\overline{\sigma}} : C_0^\infty(\Omega;\mathbb{R}^n) \times \mathscr{S} \to \mathbb{R}$ as

$$L_{\overline{\sigma}}(V,u) := \int_\Omega A(u,\overline{\sigma}) : DV \,. \tag{5.64}$$

As above, considering deformations V compactly supported into Ω gives $J'(\Omega,V) = 0$, so that by applying Theorem 5.2.1 we obtain

$$\sup_{u\in\mathscr{S}} L_{\overline{\sigma}}(V,u) \geq \inf_{\sigma\in\mathscr{S}^*}\sup_{u\in\mathscr{S}}\int_\Omega A(u,\sigma) : DV = J'(\Omega,V) = 0\,.$$

By the arbitrariness of $V \in C_0^\infty(\Omega;\mathbb{R}^n)$ we infer

$$\inf_{V\in C_0^\infty(\Omega;\mathbb{R}^n)} \sup_{u\in\mathscr{S}} L_{\overline{\sigma}}(V,u) \geq 0 \ .$$

By taking $V \equiv 0$, we see that the inf-sup above is in fact zero.

Since by Lemma 5.1.3 the set $\mathscr{S}$ is convex and weakly compact in $W(\Omega)$, $L_{\overline{\sigma}}(\cdot,u)$ is convex, $L_{\overline{\sigma}}(V,\cdot)$ is concave and weakly upper semicontinuous, Proposition 1.1.3 applies and gives

$$0 = \inf_{V\in C_0^\infty(\Omega;\mathbb{R}^n)} \sup_{u\in\mathscr{S}} L_{\overline{\sigma}}(V,u) = \sup_{u\in\mathscr{S}} \inf_{V\in C_0^\infty} L_{\overline{\sigma}}(V,u) \ ;$$

moreover, there exists $\widehat{u} \in \mathscr{S}$, depending on $\overline{\sigma}$, such that

$$\inf_{V\in C_0^\infty(\Omega;\mathbb{R}^n)} L(V,\widehat{u}) = 0 \ . \tag{5.65}$$

Since $L_{\overline{\sigma}}(\cdot,\widehat{u})$ is linear, the equality (5.65) implies

$$L_{\overline{\sigma}}(V,\widehat{u}) = 0 \qquad \forall V \in C_0^\infty(\Omega;\mathbb{R}^n) \ . \tag{5.66}$$

Finally, recalling the definition (5.64) of $L_{\overline{\sigma}}$, condition (5.66) can be written as

$$\int_\Omega A(\widehat{u},\overline{\sigma}) : DV = 0 \qquad \forall V \in C_0^\infty(\Omega;\mathbb{R}^n) \ ,$$

namely $\widehat{u}$ satisfies (5.22).

□

We now turn attention to the proof of Proposition 5.2.1.

Proof of Proposition 5.2.1.

Let us assume that $\overline{u} \in \mathrm{Lip}(\Omega)$. Using also the growth conditions (5.8) satisfied by f and g, we see that $A(\overline{u})$ is in $L^\infty(\Omega;\mathbb{R}^{n\times n})$. Taking into account (5.21), we infer that $A(\overline{u})$ belongs to $X_\infty(\Omega;\mathbb{R}^{n\times n})$. As such, it admits a normal trace $[A(\overline{u})\,n]_{\partial\Omega} \in L^\infty(\partial\Omega;\mathbb{R}^n)$.

Now, assume that $\partial\Omega$ is piecewise C^1, that $\nabla\overline{u} \in BV(\Omega)$, and that $\nabla f(\nabla\overline{u})$ can be extended to a field $\zeta(\overline{u})$ as in (5.24). Let us define

$$\begin{aligned} a_D(\overline{u}) &:= \nabla\overline{u}\cdot\nabla f(\overline{u}) - f(\nabla\overline{u}) = \nabla\overline{u}\cdot\zeta(\overline{u}) - f(\nabla\overline{u}) \ , \\ a_N(\overline{u}) &:= -f(\nabla\overline{u}) - g(\overline{u}) \ . \end{aligned}$$

We remark that, by the Fenchel equality, $a_D(\overline{u}) = f^*(\zeta(\overline{u}))$ in Ω.

In the sequel, the notation $a(\overline{u})$ is adopted for brevity in all the assertions which apply indistinctly for $a_D(\overline{u})$ and $a_N(\overline{u})$.

From the assumption $\overline{u} \in \mathrm{Lip}(\Omega)$ and the growth conditions (5.8), we see that $a(\overline{u}) \in L^\infty(\Omega)$. We claim that $a(\overline{u}) \in BV(\Omega)$. Indeed, under the standing assumptions f and g are locally Lipschitz, and the composition of a locally Lipschitz with a BV function is still BV, so that $f(\nabla\overline{u})$ and $g(\overline{u})$ are in BV. Moreover, the product of two functions which are in $L^\infty \cap BV$ remains in $L^\infty \cap BV$, so that the scalar product $\nabla\overline{u}\cdot\zeta(\overline{u})$ is in BV. Then the claim is proved. In particular, the tensor $a(\overline{u})I$ is an element of $X_\infty(\Omega;\mathbb{R}^{n\times n})$, and consequently its normal trace $[a(\overline{u})I\,n]_{\partial\Omega}$ is well defined. Moreover, according to equality (1.13) in Lemma 1.4.1, it can be identified with the trace of $a(\overline{u})$ as a BV function, namely

$$\mathrm{Tr}\,(a(\overline{u}))n = [a(\overline{u})I\,n]_{\partial\Omega} \qquad \mathscr{H}^{n-1}\text{-a.e. on } \partial\Omega\,. \tag{5.67}$$

In view of (5.67), in order to obtain (5.25) it is enough to show that

$$\begin{aligned} &[A(\overline{u})\,n - a_D(\overline{u})I\,n]_{\partial\Omega} = 0 \quad \text{in case } (D)\,, \\ &[A(\overline{u})\,n - a_N(\overline{u})I\,n]_{\partial\Omega} = 0 \quad \text{in case } (N)\,, \end{aligned}$$

namely

$$\big[\big(\nabla\overline{u}\otimes\nabla f(\nabla\overline{u}) - g(\overline{u})I - \nabla\overline{u}\cdot\nabla f(\nabla\overline{u})I\big)\,n\big]_{\partial\Omega} = 0 \quad \text{in case } (D)\,, \tag{5.68}$$

$$[\nabla\overline{u}\otimes\nabla f(\nabla\overline{u})\,n]_{\partial\Omega} = 0 \quad \text{in case } (N)\,. \tag{5.69}$$

Let us first treat the Dirichlet case. Since by assumption $\partial\Omega$ is piecewise C^1, we can exploit the pointwise characterization (1.11) of the normal trace and rewrite (5.68) as

$$\lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} \big[(\nabla f(\nabla\overline{u})\cdot\tilde{n})\nabla\overline{u} - g(\overline{u})\tilde{n} - (\nabla\overline{u}\cdot\nabla f(\nabla\overline{u}))\tilde{n}\big] = 0 \;\text{ for } \mathscr{H}^{n-1}\text{-a.e. } x_0\in\partial\Omega\,.$$

Recalling that $g(\overline{u})$ is a continuous function which vanishes on $\partial\Omega$, we have

$$\lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} [g(\overline{u})\,\tilde{n}] = 0 \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x_0\in\partial\Omega\,,$$

and we are finally reduced to check that

$$\lim_{r,\rho\to 0^+} \fint_{C^-_{r,\rho}(x_0)} \big[(\nabla f(\nabla\overline{u})\cdot\tilde{n})\nabla\overline{u} - (\nabla\overline{u}\cdot\nabla f(\nabla\overline{u}))\tilde{n}\big] = 0 \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x_0\in\partial\Omega\,.$$

Setting $P_{\tilde{n}}(\nabla\overline{u}) := \nabla\overline{u} - (\nabla\overline{u}\cdot\tilde{n})\tilde{n}$, we have

$$\begin{aligned} &\left|\fint_{C^-_{r,\rho}(x_0)} \big[(\nabla f(\nabla\overline{u})\cdot\tilde{n})\nabla\overline{u} - (\nabla\overline{u}\cdot\nabla f(\nabla\overline{u}))\tilde{n}\big]\right| \\ = &\left|\fint_{C^-_{r,\rho}(x_0)} \big[(\nabla f(\nabla\overline{u})\cdot\tilde{n})P_{\tilde{n}}(\nabla\overline{u}) - (P_{\tilde{n}}(\nabla\overline{u})\cdot\nabla f(\nabla\overline{u}))\tilde{n}\big]\right| \\ \le\ & 2\|\sigma\|_{L^\infty} \fint_{C^-_{r,\rho}(x_0)} |P_{\tilde{n}}(\nabla\overline{u})| \end{aligned}$$

where σ is any element of $\mathscr{S}^*$ (notice that $\sigma = \nabla f(\nabla\overline{u})$ on $\{\nabla u \neq 0\}$ and σ is in $L^\infty(\Omega;\mathbb{R}^n)$; the latter assertion follows straightforward from the assumption $\nabla\overline{u}\in L^\infty(\Omega;\mathbb{R}^n)$ and Lemma 5.1.2 (ii)).

Now we observe that, since by assumption $\overline{u} = 0$ on $\partial\Omega$ and $\nabla\overline{u}\in BV(\Omega;\mathbb{R}^n)$, the trace $\mathrm{Tr}\,(\nabla\overline{u})$ is normal to $\partial\Omega$, that is

$$\mathrm{Tr}\,(\nabla\overline{u}) = (\mathrm{Tr}\,(\nabla\overline{u})\cdot n)\,n \quad \mathscr{H}^{n-1} - \text{a.e. on } \partial\Omega\,. \tag{5.70}$$

In fact, thanks to the assumption that $\partial\Omega$ is piecewise C^1, the equality (5.70) can be proved by an approximation argument, which can be found for instance in [48, Proposition 1.4 and Section 2] (see also [49, Theorem 2.3], where the same result is proved in a more general framework, allowing in particular piecewise C^1 boundaries).

In view of (5.70), for $\mathscr{H}^{n-1}-$a.e. $x_0 \in \partial\Omega$ we have

$$\fint_{C^-_{r,\rho}(x_0)} |P_{\tilde n}(\nabla\overline{u})| \leq \fint_{C^-_{r,\rho}(x_0)} \big|\nabla\overline{u}(x) - \mathrm{Tr}(\nabla\overline{u})(x_0)\big| + \big|(\mathrm{Tr}(\nabla\overline{u})(x_0)\cdot n(x_0))\,n(x_0) - (\nabla\overline{u}(x)\cdot\tilde n(x))\tilde n(x)\big|$$
$$\leq \fint_{C^-_{r,\rho}(x_0)} \big|\nabla\overline{u}(x) - \mathrm{Tr}(\nabla\overline{u})(x_0)\big| + \big|\mathrm{Tr}(\nabla\overline{u})(x_0)\cdot n(x_0) - \nabla\overline{u}(x)\cdot\tilde n(x)\big| + \|\nabla\overline{u}\|_{L^\infty}\big|n(x_0)-\tilde n(x)\big|\,.$$

By Lemma 1.4.1 (precisely, using (1.14), (1.15) and (1.16)), we infer that

$$\lim_{r,\rho\to 0^+}\fint_{C^-_{r,\rho}(x_0)} |P_{\tilde n}(\nabla\overline{u})| = 0\ ,$$

and the proof of (5.68) is achieved.

Let us now consider the Neumann case. In view of the pointwise characterization (1.11) of the normal trace, we can rewrite (5.69) as

$$\lim_{r,\rho\to 0^+}\fint_{C^-_{r,\rho}(x_0)} (\nabla f(\nabla\overline{u})\cdot\tilde n)\nabla\overline{u} = 0 \qquad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x_0\in\partial\Omega\,.$$

Exploiting the boundedness of $\nabla\overline{u}$ we infer

$$\left|\lim_{r,\rho\to 0^+}\fint_{C^-_{r,\rho}(x_0)} (\nabla f(\nabla\overline{u})\cdot\tilde n)\nabla\overline{u}\right| \leq \|\nabla\overline{u}\|_{L^\infty}\lim_{r,\rho\to 0^+}\fint_{C^-_{r,\rho}(x_0)} |\zeta(\overline{u})\cdot\tilde n| \qquad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x_0\in\partial\Omega\,.$$

Since by assumption $\zeta(\overline{u})$ satisfies

$$\mathrm{Tr}(\zeta(\overline{u}))\cdot n = 0 \quad \mathscr{H}^{n-1}\text{-a.e. on } \partial\Omega\,, \tag{5.71}$$

using (5.71) and (1.15), we conclude that

$$\lim_{r,\rho\to 0^+}\fint_{C^-_{r,\rho}(x_0)} |\zeta(\overline{u})\cdot\tilde n| = \lim_{r,\rho\to 0^+}\fint_{C^-_{r,\rho}(x_0)} |\zeta(\overline{u})\cdot\tilde n - \mathrm{Tr}(\zeta(\overline{u}))(x_0)\cdot n(x_0)| = 0\ ,$$

and the proof of (5.69) is achieved.

□

Proof of Corollary 5.2.1. Let $\overline{u}\in \mathrm{Lip}(\Omega)$ be the unique solution to problem $J(\Omega)$. Since f is Gateaux differentiable except at most at the origin, the tensor $A(\overline{u})$ is uniquely determined as in (5.18). By applying Theorem 5.2.1 and recalling that $\mathscr{S}$ is a singleton, we infer that

$$J'(\Omega,V) = \int_\Omega A(\overline{u}) : DV\ .$$

By Proposition 5.2.1, $A(\overline{u})$ belongs to $X_\infty(\Omega;\mathbb{R}^{n\times n})$. More precisely, in view of (5.21) in Theorem 5.2.2, $A(\overline{u})$ is divergence free in the sense of distributions. Hence, by applying the generalized divergence theorem introduced in formula (5.7), we obtain (5.28), namely

$$J'(\Omega,V) = \int_{\partial\Omega} [A(\overline{u})\,n]_{\partial\Omega} \cdot V \, d\mathscr{H}^{n-1} .$$

Finally, if in addition $\partial\Omega$ is piecewise C^1, $\nabla\overline{u} \in BV(\Omega)$, and $\nabla f(\nabla\overline{u})$ can be extended to a field $\zeta(\overline{u})$ as in (5.24), we are in a position to apply the second part of Proposition 5.2.1, and the explicit expressions (5.25) of the normal traces give the result. □

5.3.1 Appendix

Proposition 5.3.4. *Let X and Y be two Banach spaces and let $h : X \times Y \to \mathbb{R}$ be a finite function of the form*

$$h(x,y) = h_1(x) + h_2(y) .$$

For every $(x^*,y^*) \in X^* \times Y^*$, the Fenchel conjugate of h is

$$h^*(x^*,y^*) = h_1^*(x^*) + h_2^*(y^*) , \tag{5.72}$$

and for every $(x,y) \in X \times Y$ the subdifferential reads

$$\partial h(x,y) = \partial h_1(x) \times \partial h_2(y) , \tag{5.73}$$

as a subset of $X^* \times Y^*$.

Proof. The assertion (5.72) follows easily by the definition of h, indeed

$$\begin{aligned} h^*(x^*,y^*) &= \sup_{(x,y)\in X\times Y} \{(x^*,y^*)\cdot(x,y) - h(x,y)\} = \sup_{(x,y)\in X\times Y} \{x^*\cdot x + y^*\cdot y - h_1(x) - h_2(y)\} \\ &= \sup_{x\in X}\{x^*\cdot x - h_1(x)\} + \sup_{y\in Y}\{y^*\cdot y - h_2(y)\} = h_1^*(x^*) + h_2^*(y^*) . \end{aligned}$$

Let us now compute the subdifferential:

$$\begin{aligned} \partial h(x,y) &= \{(x^*,y^*) \in X^*\times Y^* \ : \ h^*(x^*,y^*) + h(x,y) = (x^*,y^*)\cdot(x,y)\} \\ &= \{(x^*,y^*) \in X^*\times Y^* \ : \ [h_1^*(x^*) + h_1(x) - x^*\cdot x] + [h_2^*(y^*) + h_2(y) - y^*\cdot y] = 0\} . \end{aligned} \tag{5.74}$$

This set clearly contains the product $\partial h_1(x) \times \partial h_2(y)$. Conversely, by the Fenchel inequality, for every $(x,y) \in X \times Y$ and $(x^*,y^*) \in X^* \times Y^*$, there hold

$$h_1^*(x^*) + h_1(x) - x^*\cdot x \geq 0 , \tag{5.75}$$

$$h_2^*(y^*) + h_2(y) - y^*\cdot y \geq 0 . \tag{5.76}$$

Hence a couple $(x^*,y^*) \in X^* \times Y^*$ belongs to the set (5.74) if and only if it satisfies (5.75) and (5.76) with the equality sign, that is $x^* \in \partial h_1(x)$ and $y^* \in \partial h_2(y)$. □

We now give two alternative proofs of the lower and upper bounds for the sequence $q_\varepsilon(V)$, stated in Propositions 5.4.1 and 5.3.3 respectively. We underline that the proof of 5.3.3 requires the assumption on the growth condition (5.8) for both the Dirichlet and Neumann case, nevertheless the more direct approach can be better applied for second order shape derivatives (see §5.4.1).

In what follows f'_+ denotes the one sided directional derivative, introduced in §1.1.2.

Proposition 5.3.5. *Under the standing assumptions, let $u \in \mathscr{S}$, and let w be an arbitrary admissible function for $J(\Omega)$. Then*

$$\liminf_{\varepsilon\to 0^+} q_\varepsilon(V) \geq -\int_\Omega \Big[(f(\nabla u)+g(u))\,\mathrm{div}V + f'_+(\nabla u, -DV^T\nabla u+\nabla w) + g'_+(u,w) \Big]\,. \tag{5.77}$$

As a consequence, it holds

$$\liminf_{\varepsilon\to 0^+} q_\varepsilon(V) \geq \sup_{u\in\mathscr{S}} \inf_{\sigma\in S^*} \int_\Omega A(u,\sigma) : DV\,. \tag{5.78}$$

Proof. Since u is a solution to $J(\Omega)$, we may rewrite $J(\Omega)$ as

$$J(\Omega) = -\int_\Omega [f(\nabla u)+g(u)]\,dx\,. \tag{5.79}$$

In order to bound $J(\Omega_\varepsilon)$ from below, let us fix an arbitrary function $w \in W(\Omega)$, and let us set $\Psi_\varepsilon(x) := x+\varepsilon V(x)$. Then the function $u_\varepsilon := (u+\varepsilon w)\circ\Psi_\varepsilon^{-1}$ belongs to $W(\Omega_\varepsilon)$ and is admissible for problem $J(\Omega_\varepsilon)$. Hence,

$$J(\Omega_\varepsilon) \geq -\int_{\Omega_\varepsilon} [f(\nabla u_\varepsilon)+g(u_\varepsilon)]\,. \tag{5.80}$$

Via the change of variables $y = \Psi_\varepsilon(x)$, we get

$$-\int_{\Omega_\varepsilon} [f(\nabla u_\varepsilon)+g(u_\varepsilon)] = -\int_\Omega [f(D\Psi_\varepsilon^{-T}(\nabla u+\varepsilon\nabla w)))+g(u+\varepsilon w)]|\det D\Psi_\varepsilon|\,. \tag{5.81}$$

We now study the asymptotics as $\varepsilon\to 0^+$ of the r.h.s. of (5.81). To that aim, we claim that $D\Psi_\varepsilon^{-T}$ and $\det D\Psi_\varepsilon$ can be expanded respectively as

$$D\Psi_\varepsilon^{-T} = I - \varepsilon(DV^T) + \varepsilon^2 M_\varepsilon\,, \tag{5.82}$$

$$\det(D\Psi_\varepsilon) = 1+\varepsilon(\mathrm{div}V)+\varepsilon^2 m_\varepsilon\,, \tag{5.83}$$

where the functions $M_\varepsilon \in C(\mathbb{R}^n;\mathbb{R}^{n\times n})$ and $m_\varepsilon \in C(\mathbb{R}^n)$ are uniformly bounded in L^∞ (that is, they satisfy the estimates $\sup_\varepsilon \|M_\varepsilon\|_{L^\infty} \leq C$ and $\sup_\varepsilon \|m_\varepsilon\|_{L^\infty} \leq C$ for some constants $C>0$ independent of ε). In particular, for ε small, (5.83) yields $|\det D\Psi_\varepsilon| = \det D\Psi_\varepsilon$.

To prove the claim, we write

$$\begin{aligned} D\Psi_\varepsilon^{-T} &= (I+\varepsilon DV)^{-T} = (I-\varepsilon DV+\varepsilon^2 DV^2-\varepsilon^3 DV^3+\ldots)^T \\ &= I-\varepsilon DV^T+\varepsilon^2(DV^T)^2\left(\sum_{k=0}^\infty (-\varepsilon)^k (DV^T)^k\right)\,. \end{aligned}$$

Since V is assumed to be of class C^1, and ε is a small parameter, the series of matrices appearing in the last term converges, and we obtain (5.82).

By writing the characteristic polynomial of a matrix B as

$$\det(B-\lambda I) = (-\lambda)^n + a_1(B)(-\lambda)^{n-1}+\ldots+a_n(B)\,,$$

where the first invariant a_1 is the trace, we infer

$$\det(I+\varepsilon DV) = 1 + a_1(DV)\varepsilon + \ldots + a_n(DV)\varepsilon^n = 1 + (\operatorname{div} V)\varepsilon + \left(\sum_{k=2}^{n} \varepsilon^{k-2} a_k(DV) \right) \varepsilon^2 .$$

Since V is assumed to be of class C^1, and the coefficients $a_k(DV)$ are sum and products of elements of DV, we obtain (5.83).

In view of (5.82) and (5.83), we can rewrite the r.h.s. of (5.81) as

$$-\int_\Omega \left[f(\nabla u + \varepsilon(-DV^T \nabla u + \nabla w) + \varepsilon^2 z_\varepsilon) + g(u+\varepsilon w) \right] (1 + \varepsilon \operatorname{div} V + \varepsilon^2 m_\varepsilon) , \quad (5.84)$$

where

$$z_\varepsilon := M_\varepsilon \nabla u - DV^T \nabla w + \varepsilon M_\varepsilon \nabla w .$$

From (5.79), (5.80), (5.81), and (5.84), it follows that the quotient $q_\varepsilon(V)$ satisfies the lower bound

$$q_\varepsilon(V) \geq \sum_{h=1}^{7} I_\varepsilon^h , \quad (5.85)$$

where the seven integral terms I_ε^h are given by:

$$\begin{aligned}
I_\varepsilon^1 &= -\varepsilon \int_\Omega \left[f(\nabla u + \varepsilon(-DV^T \nabla u + \nabla w) + \varepsilon^2 z_\varepsilon) \, m_\varepsilon \right] , \\
I_\varepsilon^2 &= -\varepsilon \int_\Omega \left[g(u+\varepsilon w) \, m_\varepsilon \right] , \\
I_\varepsilon^3 &= -\int_\Omega \left[f(\nabla u - \varepsilon DV^T \nabla u + \varepsilon^2 z_\varepsilon) \operatorname{div} V \right] , \\
I_\varepsilon^4 &= -\int_\Omega \left[g(u+\varepsilon w) \operatorname{div} V \right] , \\
I_\varepsilon^5 &= -\varepsilon^{-1} \int_\Omega \left[f(\nabla u + \varepsilon(-DV^T \nabla u + \nabla w)) - f(\nabla u) \right] , \\
I_\varepsilon^6 &= -\varepsilon^{-1} \int_\Omega \left[f(\nabla u + \varepsilon(-DV^T \nabla u + \nabla w) + \varepsilon^2 z_\varepsilon) - f(\nabla u + \varepsilon(-DV^T \nabla u + \nabla w)) \right] , \\
I_\varepsilon^7 &= -\varepsilon^{-1} \int_\Omega \left[g(u+\varepsilon w) - g(u) \right] .
\end{aligned}$$

Let us study separately the asymptotics as $\varepsilon \to 0^+$ of each term I_ε^h. We first observe that

$$\lim_{\varepsilon \to 0^+} I_\varepsilon^1 = \lim_{\varepsilon \to 0^+} I_\varepsilon^2 = 0 . \quad (5.86)$$

Indeed, by exploiting the growth conditions (5.8), and the uniform boundedness of m_ε and M_ε, we obtain

$$\left| \int_\Omega f(\nabla u + \varepsilon(-DV^T \nabla u + \varepsilon \nabla w) + \varepsilon^2 z_\varepsilon) \, m_\varepsilon \right| \leq C \left(1 + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^n)} \right) \leq C$$

and

$$\left| \int_\Omega g(u+\varepsilon w) \, m_\varepsilon \right| \leq C \int_\Omega |g(u+\varepsilon w)| \leq C(1 + \|u\|^q_{L^q(\Omega)}) \leq C(1 + \|\nabla u\|^q_{L^p(\Omega;\mathbb{R}^n)}) \leq C ,$$

where C denotes a positive constant that may be different in each inequality.

Next, we consider terms I_ε^3 and I_ε^4. Under the standing assumptions, the functionals

$$L^p(\Omega;\mathbb{R}^n) \ni z \mapsto \int_\Omega \left[f(z)\,\mathrm{div}\,V\right] \qquad \text{and} \qquad L^p(\Omega) \ni v \mapsto \int_\Omega \left[g(v)\,\mathrm{div}\,V\right]$$

are sequentially continuous with respect to the strong topology (*cf.* [57, Theorem 6.51]). By the uniform boundedness of m_ε and M_ε, we have

$$\nabla u - \varepsilon DV^T \nabla u + \varepsilon^2 z_\varepsilon \xrightarrow{L^p} \nabla u \qquad \text{and} \qquad u + \varepsilon w \xrightarrow{L^p} \overline{u}\,.$$

Hence we obtain

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^3 = -\int_\Omega \left[f(\nabla u)\,\mathrm{div}\,V\right] \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} I_\varepsilon^4 = -\int_\Omega \left[g(u)\,\mathrm{div}\,V\right]\,. \tag{5.87}$$

Finally, in order to deal with the terms I_ε^5, I_ε^6, and I_ε^7, we recall that, under the standing assumptions, there exists a constant $C > 0$ such that

$$|f(z_1) - f(z_2)| \le C\left(1 + |z_1|^{p-1} + |z_2|^{p-1}\right)|z_1 - z_2| \qquad \forall z_1, z_2 \in \mathbb{R}^n$$
$$|g(v_1) - g(v_2)| \le C\left(1 + |v_1|^{q-1} + |v_2|^{q-1}\right)|v_1 - v_2| \qquad \forall u_1, u_2 \in \mathbb{R}$$

(see [57, Proposition 4.64]). Then we can apply the dominated convergence Theorem and pass to the limit under the integral sign, which yields

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^5 = -\int_\Omega f'_+(\nabla u, -DV^T\,\nabla u + \nabla w) \tag{5.88}$$

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^6 = -\int_\Omega g'_+(u, w) \tag{5.89}$$

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^7 = 0\,. \tag{5.90}$$

By combining (5.85) with (5.86), (5.87), (5.88), (5.89), and (5.90), we obtain (5.77).

Let us now prove (5.78). By the arbitrariness of $u \in \mathscr{S}$, it is enough to show that

$$\liminf_{\varepsilon\to 0^+} q_\varepsilon(V) \ge \inf_{\sigma\in\mathscr{S}^*} \int_\Omega A(u,\sigma) : DV\,. \tag{5.91}$$

Hence, if we denote by $\mathscr{B} := \partial f(\nabla u) \times \partial g(u)$, we can rewrite (5.77) as

$$\liminf_{\varepsilon\to 0} q_\varepsilon(V) \ge \sup_{w\in W(\Omega)} \int_\Omega \inf_{(\sigma,\tau)\in\mathscr{B}} [\sigma\cdot(DV^T\nabla u - \nabla w) - \tau w - (f(\nabla u) + g(u))I : DV]\,.$$

Now we observe that, since $\mathscr{B}$ is convex, the infimum over $(\sigma, \tau) \in \mathscr{B}$ can be passed outside the sign of integral [21, Theorem 1]. Next, by Proposition 1.1.3, such infimum can be exchanged with the supremum over $w \in W(\Omega)$, thus obtaining an inf-sup in which the infimum is attained at some (σ_0, τ_0), thanks to the weak compactness of $\mathscr{B}$

in $L^{p'}(\Omega;\mathbb{R}^n)\times L^{q'}(\Omega)$ (by Theorem 1.1.3 (ii), Proposition 5.3.4 and Proposition 1.2.1). Thus we get:

$$\begin{aligned}\liminf_{\varepsilon\to 0} q_\varepsilon(V) &\ge \inf_{(\sigma,\tau)\in\mathscr{B}} \sup_{w\in W(\Omega)} \int_\Omega [\sigma\cdot(DV^T\nabla u-\nabla w)-\tau w-(f(\nabla u)+g(u))I : DV] .\\ &= \sup_{w\in W(\Omega)} \int_\Omega [\sigma_0\cdot(DV^T\nabla u-\nabla w)-\tau_0 w-(f(\nabla u)+g(u))I : DV] .\end{aligned}$$

At this point we observe that a necessary condition for the finiteness of the last supremum over $w\in W(\Omega)$ is the condition $\operatorname{div}\sigma_0=\tau_0$, and we end up with

$$\begin{aligned}\liminf_{\varepsilon\to 0} q_\varepsilon(V) &\ge \int_\Omega [\sigma_0\otimes\nabla u-(f(\nabla u)+g(u))I] : DV\\ &= \int_\Omega A(u,\sigma_0) : DV \ \ge \inf_{\sigma\in\mathscr{S}^*}\int_\Omega A(u,\sigma) : DV ,\end{aligned}$$

which proves (5.91). □

Proposition 5.3.6. *Let assume that f and g satisfy (H1)-(H3) in §5.1.2. Let $\overline{\sigma}\in\mathscr{S}^*$, and let η be an arbitrary admissible vector field for $J^*(\Omega)$. Then*

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) \le &\int_\Omega [(f^*(\overline{\sigma})+g^*(\overline{\sigma}))\operatorname{div} V]+\\ &+\int_\Omega [(f^*)'_+(\overline{\sigma},DV\overline{\sigma}-\operatorname{div} V\overline{\sigma}+\eta)+(g^*)'_+(\operatorname{div}\overline{\sigma},\operatorname{div}\eta-\operatorname{div} V\operatorname{div}\overline{\sigma})] .\end{aligned} \tag{5.92}$$

As a consequence, it holds

$$\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) \le \inf_{\sigma\in S^*}\sup_{u\in S}\int_\Omega A(u,\sigma) : DV . \tag{5.93}$$

Proof. Since $\overline{\sigma}\in\mathscr{S}^*$, in view of Lemma 5.1.2 we may rewrite $J(\Omega)$ in dual form as

$$J^*(\Omega)=\int_\Omega [f^*(\overline{\sigma})+g^*(\operatorname{div}\overline{\sigma})] . \tag{5.94}$$

In order to bound $J(\Omega_\varepsilon)$ from above, we first rewrite it as a variational problem in which the domain of integration is kept fix and the integrand depends on the parameter ε. Namely, with a change of variables we get:

$$J(\Omega_\varepsilon)=-\inf\Big\{\int_\Omega f_\varepsilon(x,\nabla u)+g_\varepsilon(x,u)] \ :\ u\in W^{1,p}(\Omega)\Big\},$$

with

$$f_\varepsilon(x,z):=f(D\Psi_\varepsilon^{-T}z)|\det D\Psi_\varepsilon|(x) , \tag{5.95}$$
$$g_\varepsilon(x,u):=g(u)|\det D\Psi_\varepsilon|(x) . \tag{5.96}$$

Then, by arguing in the same way as already done in the proof of Lemma 5.1.2, we obtain that the dual form of $J(\Omega_\varepsilon)$ is given in case (D) by

$$J^*(\Omega_\varepsilon) = \inf\left\{ \int_\Omega [f_\varepsilon^*(x,\sigma) + g_\varepsilon^*(x, \operatorname{div}\sigma)]\, dx \ :\ \sigma \in L^{p'}(\Omega;\mathbb{R}^n)\,,\ \operatorname{div}\sigma \in L^{q'}(\Omega)) \right\}\,, \tag{5.97}$$

and in case (N) by the same expression with the addition of the condition $\sigma \cdot n = 0$ on $\partial\Omega$. Here f_ε^* and g_ε^* denote the Fenchel conjugates of f and g, performed with respect to the second variable. Their computation gives:

$$f_\varepsilon^*(x,z^*) = \sup_{z\in\mathbb{R}^n} \left\{ z\cdot z^* - f(D\Psi_\varepsilon^{-T} z)|\det D\Psi_\varepsilon| \right\} = |\det D\Psi_\varepsilon| f^*(|\det D\Psi_\varepsilon|^{-1} D\Psi_\varepsilon z^*)\,,$$
$$g_\varepsilon^*(x,u^*) = \sup_{u\in\mathbb{R}} \{ uu^* - g(u)|\det D\Psi_\varepsilon| \} = |\det D\Psi_\varepsilon| g^*(|\det D\Psi_\varepsilon|^{-1} u^*)\,.$$

Recalling that $D\Psi_\varepsilon = I + \varepsilon DV$, we have

$$|\det D\Psi_\varepsilon| = 1 + \varepsilon \operatorname{div} V + \varepsilon^2 m_\varepsilon\,,\quad |\det D\Psi_\varepsilon|^{-1} = 1 - \varepsilon \operatorname{div} V + \varepsilon^2 \tilde m_\varepsilon\,,$$

where $m_\varepsilon, \tilde m_\varepsilon \in C(\mathbb{R}^n)$ are such that $\sup_\varepsilon \|m_\varepsilon\|_{L^\infty} \le C$ and $\sup_\varepsilon \|\tilde m_\varepsilon\|_{L^\infty} \le C$ for some constant $C > 0$.

Let now η be admissible for $J^*(\Omega)$. Then, since the vector fields $\sigma_\varepsilon := \overline{\sigma} + \varepsilon\eta$ are admissible for the dual form $J^*(\Omega_\varepsilon)$ in (5.97), we obtain the following estimate:

$$\begin{aligned} q_\varepsilon(V) &\le \frac{1}{\varepsilon}\left[\int_\Omega [f_\varepsilon^*(\overline{\sigma} + \varepsilon\eta) + g_\varepsilon^*(\operatorname{div}\overline{\sigma} + \varepsilon \operatorname{div}\eta)]\, dx - \int_\Omega [f^*(\overline{\sigma}) + g^*(\operatorname{div}\overline{\sigma})]\, dx \right] \\ &= \sum_{h=1}^{8} I_\varepsilon^{*h}\,, \end{aligned} \tag{5.98}$$

where the eight terms I_ε^{*h} are given by:

$$\begin{aligned}
I_\varepsilon^{*1} &= \varepsilon \int_\Omega f^*(|\det D\Psi_\varepsilon|^{-1} D\Psi_\varepsilon(\overline{\sigma} + \varepsilon\eta))\, m_\varepsilon\,,\\
I_\varepsilon^{*2} &= \varepsilon \int_\Omega g^*(|\det D\Psi_\varepsilon|^{-1}(\operatorname{div}\overline{\sigma} + \operatorname{div}\varepsilon\eta))\, m_\varepsilon\,,\\
I_\varepsilon^{*3} &= \int_\Omega f^*(|\det D\Psi_\varepsilon|^{-1} D\Psi_\varepsilon(\overline{\sigma} + \varepsilon\eta))\, \operatorname{div} V\,,\\
I_\varepsilon^{*4} &= \int_\Omega g^*(|\det D\Psi_\varepsilon|^{-1}(\operatorname{div}\overline{\sigma} + \varepsilon \operatorname{div}\eta))\, \operatorname{div} V\,,\\
I_\varepsilon^{*5} &= \varepsilon^{-1} \int_\Omega [f^*(\overline{\sigma} + \varepsilon(DV\overline{\sigma} - \operatorname{div} V\overline{\sigma} + \eta)) - f^*(\overline{\sigma})]\,,\\
I_\varepsilon^{*6} &= \varepsilon^{-1} \int_\Omega [f^*(|\det D\Psi_\varepsilon|^{-1} D\Psi_\varepsilon(\overline{\sigma} + \varepsilon\eta)) - f^*(\overline{\sigma} + \varepsilon(\eta - \operatorname{div} V\overline{\sigma}))],\\
I_\varepsilon^{*7} &= \varepsilon^{-1} \int_\Omega [g^*(\operatorname{div}\overline{\sigma} + \varepsilon(\operatorname{div}\eta - \operatorname{div} V \operatorname{div}\overline{\sigma})) - g^*(\operatorname{div}\overline{\sigma})]\,,\\
I_\varepsilon^{*8} &= \varepsilon^{-1} \int_\Omega [g^*(|\det D\Psi_\varepsilon|^{-1}(\operatorname{div}\overline{\sigma} + \varepsilon \operatorname{div}\eta)) - g^*(\operatorname{div}\overline{\sigma} + \varepsilon(\operatorname{div}\eta - \operatorname{div} V \operatorname{div}\overline{\sigma}))]\,.
\end{aligned}$$

Let us study separately the asymptotics as $\varepsilon \to 0^+$ of each term I_ε^{*h}.

Exploiting the growth assumptions (5.8) for f and g we deduce that

$$\begin{cases} c_1|z^*|^{p'} - c_2 \le f^*(z^*) \le c_3(1+|z^*|^{p'}) \\ c_4|u^*|^{q_2'} - c_5 \le g^*(u^*) \le c_6(1+|u^*|^{q_1'}) \end{cases}, \tag{5.99}$$

for every $z^* \in \mathbb{R}^n$, $u^* \in \mathbb{R}$, and for some constants $c_i > 0$. Moreover, by convexity, we also infer that

$$\begin{array}{ll} |f^*(z_1^*) - f^*(z_2^*)| \le C(1+|z_1^*|^{p'-1} + |z_2^*|^{p'-1})|z_1^* - z_2^*| & \forall z_1^*, z_2^* \in \mathbb{R}^n \\ |g^*(u_1^*) - g^*(u_2^*)| \le C(1+|u_1^*|^{q'-1} + |u_2^*|^{q'-1})|u_1^* - u_2^*| & \forall u_1^*, u_2^* \in \mathbb{R} \end{array}, \tag{5.100}$$

for some constant $C > 0$ (see [57, Proposition 4.64]).

In view of (5.99) and (5.100), as already done for the lower bound, we obtain

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^{*1} = \lim_{\varepsilon\to 0^+} I_\varepsilon^{*2} = 0\,, \tag{5.101}$$

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^{*3} = \int_\Omega f^*(\overline{\sigma})\,\mathrm{div}V\,, \tag{5.102}$$

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^{*4} = \int_\Omega g^*(\mathrm{div}\,\overline{\sigma})\,\mathrm{div}V\,, \tag{5.103}$$

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^{*5} = \int_\Omega (f^*)'_+(\overline{\sigma}, DV\overline{\sigma} - \mathrm{div}\,V\overline{\sigma} + \eta)\,, \tag{5.104}$$

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^{*7} = \int_\Omega (g^*)'_+(\mathrm{div}\,\overline{\sigma}, \mathrm{div}\,\eta - \mathrm{div}\,V\,\mathrm{div}\,\overline{\sigma})\,, \tag{5.105}$$

$$\lim_{\varepsilon\to 0^+} I_\varepsilon^{*6} = \lim_{\varepsilon\to 0^+} I_\varepsilon^{*8} = 0\,. \tag{5.106}$$

Hence, by combining (5.98) with (5.101)-(5.106), we conclude that (5.92) holds.

Let us now prove (5.93). By the arbitrariness of $\overline{\sigma} \in \mathscr{S}^*$, it is enough to show that

$$\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) \le \sup_{u\in\mathscr{S}} \int_\Omega A(u,\overline{\sigma}) : DV\,. \tag{5.107}$$

We set $\mathscr{B}^* := \partial f^*(\overline{\sigma}) \times \partial g^*(\mathrm{div}\,\overline{\sigma})$. Then we can rewrite (5.92) as

$$\begin{aligned} \limsup_{\varepsilon\to 0^+} q_\varepsilon(V) \le \inf_{\eta\in X(\Omega;\mathbb{R}^n)} \int_\Omega \sup_{(z,u)\in\mathscr{B}^*} \big[&(f^*(\overline{\sigma}) + g^*(\overline{\sigma}))I : DV \\ &+ z\cdot(DV\overline{\sigma} - \mathrm{div}\,V\overline{\sigma} + \eta) + u(\mathrm{div}\,\eta - \mathrm{div}\,V\,\mathrm{div}\,\overline{\sigma})\big]\,. \end{aligned}$$

Then, we proceed in a similar way as already done to prove statement (i). Namely we observe that, by the convexity of $\mathscr{B}^*$, the supremum over $(z,u) \in \mathscr{B}^*$ can be passed outside the sign of integral [21, Theorem 1]. Moreover, by Proposition 1.1.3, such supremum can be exchanged with the infimum over $\eta \in X(\Omega;\mathbb{R}^n)$, thus obtaining sup-inf in which the supremum is attained at some (z_0,u_0), thanks to the weak compactness of $\mathscr{B}^*$ in $L^p(\Omega;\mathbb{R}^n)\times L^q(\Omega)$. Thus we get:

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) &\le \sup_{(z,u)\in\mathscr{B}^*}\inf_{\eta\in X(\Omega;\mathbb{R}^n)*}\int_\Omega \big[(f^*(\overline{\sigma})+g^*(\overline{\sigma}))I : DV \\ &\qquad + z\cdot(DV\overline{\sigma}-\operatorname{div}V\overline{\sigma}+\eta)+u(\operatorname{div}\eta-\operatorname{div}V\operatorname{div}\overline{\sigma})\big] \\ &= \inf_{\eta\in X(\Omega;\mathbb{R}^n)}\int_\Omega \big[(f^*(\overline{\sigma})+g^*(\overline{\sigma}))I : DV \\ &\qquad + z_0\cdot(DV\overline{\sigma}-\operatorname{div}V\overline{\sigma}+\eta)+u_0(\operatorname{div}\eta-\operatorname{div}V\operatorname{div}\overline{\sigma})\big]\,.\end{aligned}$$

Since a necessary condition for the finiteness of the last infimum over $\eta\in X(\Omega;\mathbb{R}^n)$ is the condition $\nabla u_0 = z_0$, we end up with

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} q_\varepsilon(V) &\le \int_\Omega \big[(f^*(\overline{\sigma})+g^*(\overline{\sigma}))I : DV + \nabla u_0\cdot(DV\overline{\sigma}-\operatorname{div}V\overline{\sigma}) - u_0\operatorname{div}V\operatorname{div}\overline{\sigma}\big] \\ &= \int_\Omega \big[\overline{\sigma}\otimes\nabla u_0 + \big(f^*(\overline{\sigma})+g^*(\overline{\sigma})-\nabla u_0\cdot\overline{\sigma}-u_0\operatorname{div}\overline{\sigma}\big)I\big] : DV \\ &= \int_\Omega A(u_0,\overline{\sigma}) : DV \;\le\; \sup_{u\in\mathscr{S}}\int_\Omega A(u,\overline{\sigma}) : DV\,,\end{aligned}$$

which proves (5.107). □

5.4 Perspectives

The perspectives in the study of shape derivatives for minima of integral functionals go in various directions. The first aspect to be investigated is the linearity of J' with respect to the deformation field V, as already pointed out in Remark 5.2.2: we have provided sufficient conditions that ensure such a property (see 5.2.1), and we would like to determine also necessary ones. We believe that in general, for example in the framework of non uniqueness of solutions, linearity is a too strong requirement: more precisely our conjecture is that J' is of the form

$$J'(\Omega,V) = \int_{\partial\Omega}\alpha(x)\,(V\cdot n)^+\,\mathscr{H}^{n-1}(x) + \int_{\partial\Omega}\beta(x)\,(V\cdot n)^+\,\mathscr{H}^{n-1}(x)\ ,$$

α,β being two suitable densities in $L^\infty(\partial\Omega)$ that might depend on the data of the infimum problem $J(\Omega)$, and $(V\cdot n)^\pm$ denoting the positive and negative part of the scalar product $V\cdot n$ on the boundary. In particular, we expect J' to be linear with respect to purely inner deformations or purely outer deformations.

Another interesting problem is to study higher order shape derivatives. In this direction we have applied the same approach to compute the second order shape derivative $J''(\Omega,V)$, assuming higher regularity on the domain Ω and on the integrands f and g, and considering inner variations. Again exploiting the primal and dual formulations of $J(\Omega)$, we are able to bound from above and below the lim inf and lim sup of the sequence

$$r_\varepsilon(V) := 2\,\frac{[J(\Omega_\varepsilon)-J(\Omega)-\varepsilon J'(\Omega,V)]}{\varepsilon^2}\ ,\ \ \varepsilon>0\,. \tag{5.108}$$

We point out that by now the study is carried out in the smooth case, for inner variations and Dirichlet boundary conditions. The result we obtain agrees with the classical result (see *e.g.* the examples in [67, paragraph 5.9.6]), but the approach is new. Its extension to more general integrand is a delicate topic which could be developed hereafter.

5.4.1 Second order shape derivative

For this paragraph we consider the Dirichlet problem

$$J(\Omega) = -\inf\left\{ \int_\Omega f(\nabla u) + g(u)\,dx \ :\ u \in H_0^1(D) \right\} ,$$

over a domain Ω with boundary $\partial\Omega \in C^2$, with regular integrand.

Here we consider *inner deformations*, namely obtained via diffeomorphisms associated to a vector field V such that

$$V \cdot n \leq 0 \quad \text{on} \partial\Omega , \tag{5.109}$$

moreover we assume

$$V = V_n n \quad \text{on } \partial\Omega . \tag{5.110}$$

The second order shape derivative, it if exists, coincides with

$$J''(\Omega, V) = \lim_{\varepsilon\to 0^+} r_\varepsilon(V) . \tag{5.111}$$

Theorem 5.4.1. *Under the standing assumptions, suppose in addition that $\partial\Omega \in C^2$, f and g are strictly convex of class C^2, and that both primal problem (5.1) and dual problem (5.5) admit a unique solution of class C^2 $\overline{u}$ and $\overline{\sigma}$ respectively. Then the second order shape derivative of the functional $J(\cdot)$ at Ω in direction V satisfying (5.109)-(5.110) exists, and is given by*

$$\begin{aligned}
&= \int_{\partial\Omega} (V\cdot n)^2 \left[\frac{\partial \overline{u}}{\partial n} (\nabla^2 \overline{u} \nabla^2 f(\nabla\overline{u})n)\cdot n + \big(\nabla\overline{u}\cdot\nabla f(\nabla\overline{u}) - f(\nabla\overline{u})\big) H_{\partial\Omega} \right] + \\
&- \inf_{\substack{v\in H^1(\Omega)\\ \mathrm{Tr}(v) = -V\cdot\nabla\overline{u}}} \left\{ \int_\Omega (\nabla^2 f(\nabla\overline{u})\nabla v)\cdot\nabla v + g''(\overline{u})v^2 \right\} \\
&= \int_{\partial\Omega} (V\cdot n)^2 \left[\frac{\partial \overline{u}}{\partial n} (\nabla^2 \overline{u} \nabla^2 f(\nabla\overline{u})n)\cdot n + f^*(\overline{\sigma}) H_{\partial\Omega} \right] + \\
&+ \inf_{\zeta\in L^2(\mathbb{R}^n;\mathbb{R}^n)} \left\{ \int_\Omega (\nabla^2 f^*(\overline{\sigma})\zeta)\cdot\zeta + (g^*)''(\operatorname{div}\overline{\sigma})(\operatorname{div}\zeta)^2 + 2\int_{\partial\Omega} \frac{\partial\overline{u}}{\partial n}(V\cdot n)(\zeta\cdot n) \right\},
\end{aligned}$$

where $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$, and $\overline{\sigma}$ is the unique solution to the dual problem $J^*(\Omega)$ in (5.5).

Using respectively the first and the second expression above for $r_\varepsilon(V)$, we find a lower bound for $\liminf_{\varepsilon\to 0^+} r_\varepsilon(V)$ and an upper bound for $\limsup_{\varepsilon\to 0^+} r_\varepsilon(V)$. As these bounds agree, the result follows.

Proposition 5.4.1. (lower bound)
Under the same assumptions of Theorem 5.4.1, it holds

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} r_\varepsilon(V) &\geq \int_{\partial\Omega}(V\cdot n)^2\left[\frac{\partial\overline{u}}{\partial n}(\nabla^2\overline{u}\nabla^2 f(\nabla\overline{u})\,n)\cdot n+\big(\nabla\overline{u}\cdot\nabla f(\nabla\overline{u})-f(\nabla\overline{u})\big)H_{\partial\Omega}\right]+\\ &- \inf_{\substack{v\in H^1(\Omega)\\ \mathrm{Tr}(v)=-V\cdot\nabla\overline{u}}}\left\{\int_\Omega(\nabla^2 f(\nabla\overline{u})\,\nabla v)\cdot\nabla v+g''(\overline{u})v^2\right\}.\end{aligned} \tag{5.112}$$

Proof. Let $\overline{u}\in H^1_0(\Omega)$ be optimal for $J(\Omega)$, and let v be an arbitrary element of $H^1_0(\Omega)$. Then we have

$$J(\Omega)=-\int_\Omega[f(\nabla\overline{u})+g(\overline{u})]\,,$$

and, setting as usual $\Psi_\varepsilon(x)=x+\varepsilon V(x)$,

$$J(\Omega_\varepsilon)\geq-\int_\Omega\left[f(D\Psi_\varepsilon^{-T}(\nabla\overline{u}+\varepsilon\nabla v))+g(\overline{u}+\varepsilon v)\right]|\det D\Psi_\varepsilon|\,.$$

Moreover, we recall that

$$\begin{aligned}J'(\Omega,V) &= -\int_\Omega \mathrm{div}\big(A(\overline{u})V\big)=-\int_\Omega A(\overline{u}):Dv\\ &=\int_\Omega-[f(\nabla\overline{u})+g(\overline{u})]\,\mathrm{div}\,V+(DV\,\nabla f(\nabla\overline{u}))\cdot\nabla\overline{u}\,.\end{aligned}$$

Therefore, by exploiting the first expression for $r_\varepsilon(V)$ in (5.108), we have

$$\begin{aligned}\liminf_{\varepsilon\to0^+} r_\varepsilon(V)\geq\liminf_{\varepsilon\to0^+}\frac{2}{\varepsilon^2}\Bigg[&-\int_\Omega\left[f(D\Psi_\varepsilon^{-T}(\nabla\overline{u}+\varepsilon\nabla v))+g(\overline{u}+\varepsilon v)\right]|\det D\Psi_\varepsilon|+\\ &+\int_\Omega f(\nabla\overline{u})+g(\overline{u})\,dx+\varepsilon\int_\Omega[f(\nabla\overline{u})+g(\overline{u})]\,\mathrm{div}\,V-(DV\,\nabla f(\nabla\overline{u}))\cdot\nabla\overline{u}\Bigg]\,.\end{aligned}$$

Similarly as already done for the first derivative, we can write $D\psi_\varepsilon^{-T}$ and $\det D\psi_\varepsilon$ as

$$\begin{aligned}D\Psi_\varepsilon^{-T}&=I-\varepsilon DV^T+\varepsilon^2(DV^T)^2+\varepsilon^3 M_\varepsilon\,,\\ \det D\Psi_\varepsilon&=1+\varepsilon\,\mathrm{div}\,V+\varepsilon^2 a_2(DV)+\varepsilon^3 m_\varepsilon\,,\end{aligned}$$

where the functions $M_\varepsilon\in C(\mathbb{R}^n;\mathbb{R}^{n\times n})$ and $m_\varepsilon\in C(\mathbb{R}^n)$ are uniformly bounded in L^∞, and the scalar function $a_2(DV)$ is the second invariant of the matrix DV:

$$a_2(DV)=\frac{1}{2}(\mathrm{div}\,VI-DV^T):DV\,, \tag{5.113}$$

In particular, for ε small enough, $\det D\Psi_\varepsilon>0$, so that $|\det D\Psi_\varepsilon|=\det D\Psi_\varepsilon$.

By using the above expansions $D\Psi_\varepsilon^{-T}$ and $\det D\Psi_\varepsilon$, exploiting the regularity hypothesis made on f and g, and recalling that, since $\nabla f(\nabla\overline{u})=\overline{\sigma}$ and $\mathrm{div}\,\overline{\sigma}=g'(\overline{u})$,

$$\int_\Omega\nabla f(\nabla\overline{u})\cdot\nabla v=-\int_\Omega g'(\overline{u})v\,,$$

we obtain

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0^+} r_\varepsilon(V)(\Omega) \geq \\
&\qquad \liminf_{\varepsilon\to 0^+} \frac{2}{\varepsilon^2}\Bigg\{-\int_\Omega \Big[f(\nabla\overline{u}) + \varepsilon\overline{\sigma}\cdot(\nabla v - DV^T\nabla\overline{u}) + \varepsilon^2\overline{\sigma}\cdot((DV^T)^2\nabla\overline{u} - DV^T\nabla v) + \\
&\qquad\qquad + \frac{\varepsilon^2}{2}(\nabla^2 f(\nabla\overline{u})(\nabla v - DV^T\nabla\overline{u}))\cdot(\nabla v - DV^T\nabla\overline{u}) + \\
&\qquad\qquad + g(\overline{u}) + \varepsilon g'(\overline{u})v + \frac{\varepsilon^2}{2}g''(\overline{u})v^2\Big](1 + \operatorname{div} V\varepsilon + a_2(DV)\varepsilon^2 + m_\varepsilon\varepsilon^3) + \\
&\qquad\quad + \int_\Omega [f(\nabla\overline{u}) + g(\overline{u})](1 + \varepsilon\operatorname{div} V) - \varepsilon\int_\Omega (DV\,\overline{\sigma})\cdot\nabla\overline{u}\Bigg\}\,. \\
&\quad = -\int_\Omega 2[f(\nabla\overline{u}) + g(\overline{u})]\,a_2(DV) + 2\operatorname{div} V\overline{\sigma}\cdot(\nabla v - DV^T\nabla\overline{u}) + 2\operatorname{div} V g'(\overline{u})v + \\
&\qquad - \int 2\overline{\sigma}\cdot((DV^T)^2\nabla\overline{u} - DV^T\nabla v) + \\
&\qquad - \int (\nabla^2 f(\nabla\overline{u})(\nabla v - DV^T\nabla\overline{u}))\cdot(\nabla v - DV^T\nabla\overline{u}) + g''(\overline{u})v^2\,.
\end{aligned}
$$

We remark that a minimizing sequence for the infimum problem over $v \in H^1_0(\Omega)$ which appears in the lower bound for $J''_V(\Omega)$ is uniformly bounded in $H^1_0(\Omega)$.

We have thus obtained

$$
\liminf_{\varepsilon\to 0^+} r_\varepsilon(V) \geq (I) + (II)\,, \tag{5.114}
$$

with

$$
(I) = \int_\Omega -2[f(\nabla\overline{u}) + g(\overline{u})]\,a_2(DV) + 2((\operatorname{div} V DV - DV^2)\,\overline{\sigma})\cdot\nabla\overline{u} - (DV\,F\,DV^T\nabla\overline{u})\cdot\nabla\overline{u}\,;
$$

$$
(II) = \int_\Omega 2(DV\,\overline{\sigma} + F\,DV^T\nabla\overline{u})\cdot\nabla v - (F\,\nabla v)\cdot\nabla v - g''(\overline{u})v^2 - 2\operatorname{div} V\overline{\sigma}\cdot\nabla v - 2\operatorname{div} V g'(\overline{u})v\,.
$$

Here and below we have set for brevity $F := \nabla^2 f(\nabla\overline{u})$.

We now study separately terms (I) and (II).

Recalling the expression (5.113) of $a_2(DV)$, the integrand of term (I) can be rewritten as

$$
-(f(\nabla\overline{u}) + g(\overline{u}))(\operatorname{div} V I - DV^T) : DV + 2\,[\nabla\overline{u}\otimes(\operatorname{div} V - DV)\overline{\sigma}] : DV - \left[\nabla\overline{u}\otimes(F\,DV^T\nabla\overline{u})\right] : DV\,.
$$

Then, integrating by parts, we infer

$$
(I) = \int_\Omega B(\overline{u}) : DV = \int_{\partial\Omega} (B(\overline{u})\,n)\cdot V - \int_\Omega \operatorname{div} B(\overline{u})\cdot V\,, \tag{5.115}
$$

with

$$
B(\overline{u}) := -(f(\nabla\overline{u}) + g(\overline{u}))(\operatorname{div} V I - DV^T) + 2\nabla\overline{u}\otimes(\operatorname{div} V - DV)\overline{\sigma} - \nabla\overline{u}\otimes(F\,DV^T\nabla\overline{u})\,.
$$

Let us compute the boundary term in (5.115). By using the identity

$$\operatorname{div} V - (DV\, n)\cdot n = \operatorname{div}_{\partial\Omega} V = H_{\partial\Omega} V_n \, .$$

and the assumption $V = V_n n$ on $\partial\Omega$, we get that the following equality holds true on $\partial\Omega$: :

$$B(\overline{u})\, n\cdot V = -f(\nabla\overline{u})V_n^2 H + 2\frac{\partial\overline{u}}{\partial n}V_n(\operatorname{div} V\overline{\sigma}\cdot n - (DV\,\overline{\sigma})\cdot n - V_n\left(\frac{\partial\overline{u}}{\partial n}\right)^2 (F\,DV^T n)\cdot n \, ,$$

Let us now compute the divergence term in (5.115). By using the elementary identities (where α is a scalar, a,b vector fields and C a matrix),

$$\begin{aligned} &\operatorname{div}(\alpha C) = \alpha \operatorname{div} C + C\nabla\alpha \, , \\ &\operatorname{div}(a\otimes b) = a \operatorname{div} b + Da^T b \, , \\ &\operatorname{div}(Ca) = \operatorname{div}(C^T)\cdot a + C : (Da)^T \, . \end{aligned}$$

and the equalities

$$\begin{aligned} &\nabla(g(\overline{u})) = g'(\overline{u})\nabla\overline{u} = (\operatorname{div}\overline{\sigma})\nabla\overline{u} \\ &\operatorname{div}(\operatorname{div} V I - DV^T) = 0 \, , \\ &\nabla^2\overline{u} F = D\overline{\sigma}^T \, . \end{aligned}$$

we obtain

$$\begin{aligned} \operatorname{div} B(\overline{u}) = &\operatorname{div}\overline{\sigma}\, \operatorname{div} V\nabla\overline{u} + \operatorname{div}\overline{\sigma} DV^T\nabla\overline{u} - 2\nabla\overline{u}\,(DV : D\overline{\sigma}^T) - \nabla\overline{u}\operatorname{div}(F\,DV^T\nabla\overline{u}) + \\ &+ \operatorname{div} V\nabla^2\overline{u}\,\overline{\sigma} + DV^T\nabla^2\overline{u}\,\overline{\sigma} - 2\nabla^2\overline{u}DV\,\overline{\sigma} - D\overline{\sigma}^T DV^T\nabla\overline{u} \, . \end{aligned}$$

Then we have

$$\begin{aligned} (I) = &\int_{\partial\Omega}\Big[-f(\nabla\overline{u})V_n^2 H + 2\frac{\partial\overline{u}}{\partial n}V_n(\operatorname{div} V\overline{\sigma}\cdot n - (DV\,\overline{\sigma})\cdot n) - V_n\left(\frac{\partial\overline{u}}{\partial n}\right)^2 (F\,DV^T n)\cdot n\Big] + \\ &-\int_\Omega \big[\operatorname{div}\overline{\sigma}\, \operatorname{div} V\nabla\overline{u} + \operatorname{div}\overline{\sigma} DV^T\nabla\overline{u} - 2\nabla\overline{u}\,(DV : D\overline{\sigma}^T) - \nabla\overline{u}\operatorname{div}(F\,DV^T\nabla\overline{u})\big]\cdot V + \\ &-\int_\Omega \big[\operatorname{div} V\nabla^2\overline{u}\,\overline{\sigma} + DV^T\nabla^2\overline{u}\,\overline{\sigma} - 2\nabla^2\overline{u}DV\,\overline{\sigma} - D\overline{\sigma}^T DV^T\nabla\overline{u}\big]\cdot V \, . \end{aligned}$$

Let us now pass to term (II). Integrating by parts and recalling that an admissible function v vanishes on the boundary $\partial\Omega$, we infer

$$\begin{aligned} &\int_\Omega [\operatorname{div} V \operatorname{div}\overline{\sigma}\, v + \operatorname{div} V\overline{\sigma}\cdot\nabla v] = -\int_\Omega v\nabla(\operatorname{div} V)\cdot\overline{\sigma} \, , \\ &\int_\Omega (DV\,\overline{\sigma})\cdot\nabla v = -\int_\Omega \big[\operatorname{div}(DV^T)\cdot\overline{\sigma} + DV : D\overline{\sigma}^T\big]\, v \, , \\ &\int_\Omega (F\,DV^T\,\nabla\overline{u})\cdot\nabla v = -\int_\Omega v \operatorname{div}(F\,DV^T\nabla\overline{u}) \, . \end{aligned}$$

By combining these results and recalling that $\nabla(\operatorname{div}\overline{\sigma}) = \operatorname{div}(D\overline{\sigma}^T)$, we can rewrite (II) as

$$(II) = -\int_\Omega (F\,\nabla v)\cdot\nabla v - \int_\Omega g''(\overline{u})v^2 - 2\int_\Omega \big[DV : D\overline{\sigma}^T + \operatorname{div}(F\,DV^T\nabla\overline{u})\big]\, v \, .$$

Now we can write every admissible function $v \in H^1_0(\Omega)$ as

$$v = \tilde{v} + V \cdot \nabla \overline{u} ,$$

with $\tilde{v} \in H^1(\Omega)$ such that $T(\tilde{v}) = -V \cdot \nabla \overline{u}$. After this substitution, term (II) reads

$$(II) = (a) + (b) + (c) ,$$

with

$$(a) := -\int_\Omega (F \nabla \tilde{v}) \cdot \nabla \tilde{v} - \int_\Omega g''(\overline{u}) \tilde{v}^2 ,$$

$$\begin{aligned}(b) := &-\int_\Omega \Big[F \nabla (V \cdot \nabla \overline{u})\Big] \cdot \nabla (V \cdot \nabla \overline{u}) - \int_\Omega g''(\overline{u})(V \cdot \nabla \overline{u})^2 + \\ &-2 \int_\Omega \Big[DV : D\overline{\sigma}^T + \operatorname{div}(F\, DV^T \nabla \overline{u})\Big] (V \cdot \nabla \overline{u}) ,\end{aligned}$$

$$(c) := -2 \int_\Omega \big[DV : D\overline{\sigma}^T + \operatorname{div}(F\, DV^T \nabla \overline{u})\big]\, \tilde{v} - 2 \int_\Omega (F \nabla \tilde{v}) \cdot \nabla (V \cdot \nabla \overline{u}) - 2 \int_\Omega g''(\overline{u}) \tilde{v} (V \cdot \nabla \overline{u}) .$$

Integrating by parts we can simplify some of the integrals appearing in (b) and (c):

$$\begin{aligned}\int_\Omega \big[F \nabla (V \cdot \nabla \overline{u})\big] \cdot \nabla (V \cdot \nabla \overline{u}) = &\int_{\partial\Omega} \Big[V_n^2 \frac{\partial \overline{u}}{\partial n} (F \nabla^2 \overline{u}\, n) \cdot n + V_n \left(\frac{\partial \overline{u}}{\partial n}\right)^2 (F\, DV^T n) \cdot n\Big] + \\ &- \int_\Omega \big[\operatorname{div}(D\overline{\sigma}^T) \cdot V + DV : D\overline{\sigma}^T + \operatorname{div}(F\, DV^T \nabla \overline{u})\big] (V \cdot \nabla \overline{u}) , \\ \int_\Omega g''(\overline{u})(V \cdot \nabla \overline{u})^2 = &\int_\Omega (V \cdot \nabla \overline{u}) V \cdot \nabla (g'(\overline{u})) = \int_\Omega (V \cdot \nabla \overline{u}) V \cdot \nabla (\operatorname{div} \overline{\sigma}) , \\ \int_\Omega (F \nabla \tilde{v}) \cdot \nabla (V \cdot \nabla \overline{u}) = &-\int_{\partial\Omega} \Big[V_n^2 \frac{\partial \overline{u}}{\partial n} (F \nabla^2 \overline{u}\, n) \cdot n + V_n \left(\frac{\partial \overline{u}}{\partial n}\right)^2 (F\, DV^T n) \cdot n\Big] + \\ &- \int_\Omega \tilde{v} \big[\operatorname{div}(F\, DV^T \nabla \overline{u}) + V \cdot \operatorname{div}(D\overline{\sigma}^T) + D\overline{\sigma}^T : DV\big] , \\ \int_\Omega g''(\overline{u})(V \cdot \nabla \overline{u}) \tilde{v} = &\int_\Omega \tilde{v} V \cdot \nabla (\operatorname{div} \overline{\sigma}) .\end{aligned}$$

By the above computations and exploiting the identity $\nabla(\operatorname{div} \overline{\sigma}) = \operatorname{div}(D\overline{\sigma}^T)$, we obtain

$$\begin{aligned}(II) = &(a) + \int_{\partial\Omega} \left[V_n^2 \frac{\partial \overline{u}}{\partial n} (F \nabla^2 \overline{u}\, n) \cdot n + V_n \left(\frac{\partial \overline{u}}{\partial n}\right)^2 (F\, DV^T n) \cdot n\right] \\ &- \int_\Omega \big[DV : D\overline{\sigma}^T + \operatorname{div}(F\, DV^T \nabla \overline{u})\big]\, (V \cdot \nabla \overline{u}) .\end{aligned}$$

We conclude that

$$
\begin{aligned}
(I)+(II) = & \int_{\partial\Omega} V_n^2 \Big[\frac{\partial \overline{u}}{\partial n}(F\,\nabla^2 \overline{u} n)\cdot n - f(\nabla \overline{u})H\Big] + 2\frac{\partial \overline{u}}{\partial n} V_n \,[\operatorname{div} V \overline{\sigma}\cdot n - (DV\,\overline{\sigma})\cdot n] \\
& - \int_\Omega (F\,\nabla \tilde{v})\cdot \nabla \tilde{v} - \int_\Omega g''(\overline{u})\tilde{v}^2 \\
& + \int_\Omega (DV : D\overline{\sigma}^T)\,(V\cdot \nabla \overline{u}) - \int_\Omega \operatorname{div}\overline{\sigma}\,\operatorname{div} V\,(\nabla \overline{u}\cdot V) - \int_\Omega \operatorname{div}\overline{\sigma}(DV^T\,\nabla \overline{u})\cdot V \\
& - \int_\Omega (\nabla^2 \overline{u}\,\overline{\sigma})\cdot (\operatorname{div} V\,V) - \int_\Omega (\nabla^2 \overline{u}\,\overline{\sigma})\cdot (DV\,V) \\
& + 2\int_\Omega (\nabla^2 \overline{u} V)\cdot (DV\,\overline{\sigma}) + \int_\Omega (D\overline{\sigma}^T\,DV^T \nabla \overline{u})\cdot V\ .
\end{aligned}
$$

Finally, by adding up and subtracting the boundary integral

$$
\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n \overline{\sigma}\cdot n\,[\operatorname{div} V - (DV\,n)\cdot n]\; d\mathscr{H}^{n-1}\,,
$$

we can rewrite $(I)+(II)$ as

$$
(I)+(II) = \int_{\partial\Omega} V_n^2 \left[\frac{\partial \overline{u}}{\partial n}(F\,\nabla^2 \overline{u}\, n)\cdot n - f(\nabla \overline{u})H\right] - \int_\Omega (F\,\nabla \tilde{v})\cdot \nabla \tilde{v} - \int_\Omega g''(\overline{u})\tilde{v}^2 + \mathscr{R}\,, \tag{5.116}
$$

with

$$
\begin{aligned}
\mathscr{R} := & -\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n \overline{\sigma}\cdot n\,[\operatorname{div} V - (DV\,n)\cdot n] + 2\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n\,[\operatorname{div} V \overline{\sigma}\cdot n - (DV\,\overline{\sigma})\cdot n] \\
& + \int_\Omega (DV : D\overline{\sigma}^T)\,(V\cdot \nabla \overline{u}) - \int_\Omega \operatorname{div}\overline{\sigma}\,\operatorname{div} V\,(\nabla \overline{u}\cdot V) - \int_\Omega \operatorname{div}\overline{\sigma}\,(DV^T \nabla \overline{u})\cdot V \\
& - \int_\Omega (\nabla^2 \overline{u}\,\overline{\sigma})\cdot (\operatorname{div} V\,V) - \int_\Omega (\nabla^2 \overline{u}\,\overline{\sigma})\cdot (DV\,V) \\
& + 2\int_\Omega (\nabla^2 \overline{u} V)\cdot (DV\,\overline{\sigma}) + \int_\Omega (D\overline{\sigma} V)\cdot (DV^T\,\nabla \overline{u})\ .
\end{aligned}
$$

We claim that

$$
\mathscr{R} = 0\ . \tag{5.117}
$$

Once proved the claim, the proof is achieved: it is enough to combine (5.114), (5.116) and (5.117), and recall that $\tilde{v}$ is an arbitrary element of $H^1(\Omega)$ with $T(\tilde{v}) = -V\cdot \nabla \overline{u}$.

We now prove (5.117). In order to simplify the expression of $\mathscr{R}$ we carry over suitable integration by parts: adopting for the sake of clearness the notation of sum over repeated indices, we have

$$
\begin{aligned}
\int_\Omega (\nabla^2 \overline{u}\,\overline{\sigma}) \cdot (\operatorname{div} V\, V) =& \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n \overline{\sigma} \cdot n \operatorname{div} V - \int_\Omega \operatorname{div} \overline{\sigma} \operatorname{div} V (\nabla \overline{u} \cdot V) + \\
& - \int_\Omega (\nabla \overline{u} \cdot V) \overline{\sigma} \cdot \nabla(\operatorname{div} V) - \int_\Omega \operatorname{div} V\, (DV\, \overline{\sigma}) \cdot \nabla \overline{u}\ , \\
\int_\Omega (\nabla^2 \overline{u}\,\overline{\sigma}) \cdot (DV\, V) =& \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n (\overline{\sigma} \cdot n)\, (DV\, n) \cdot n - \int_\Omega \operatorname{div} \overline{\sigma}\, (DV\, V) \cdot \nabla \overline{u} + \\
& - \int_\Omega \partial_i \overline{u}\, \overline{\sigma}_j (\partial_j \partial_k V_i) V_k - \int_\Omega (DV^T\, \nabla \overline{u}) \cdot (DV\, \overline{\sigma})\ , \\
\int_\Omega (\nabla^2 \overline{u}\, V) \cdot (DV\, \overline{\sigma}) =& \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n (DV\, \overline{\sigma}) \cdot n - \int_\Omega (DV^T\, \nabla \overline{u}) \cdot (DV\, \overline{\sigma}) + \\
& - \int_\Omega (\nabla \overline{u} \cdot V) \overline{\sigma} \cdot \nabla(\operatorname{div} V) - \int_\Omega (\nabla \overline{u} \cdot V)(D\overline{\sigma} : DV^T)\ , \\
\int_\Omega (D\overline{\sigma}\, V) \cdot (DV^T\, \nabla \overline{u}) =& \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n (DV\, \overline{\sigma}) \cdot n - \int_\Omega \operatorname{div} V\ \overline{\sigma} \cdot (DV^T\, \nabla \overline{u}) + \\
& - \int_\Omega (\partial_j \partial_i V_k) \partial_k \overline{u} V_j \overline{\sigma}_i - \int_\Omega (\nabla^2 \overline{u}\, V) \cdot (DV\, \overline{\sigma})\ .
\end{aligned}
$$

Exploiting these computations we conclude that

$$
\begin{aligned}
\mathscr{R} =& - \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n \overline{\sigma} \cdot n [\operatorname{div} V - (DV\, n) \cdot n] + 2 \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n [\operatorname{div} V \overline{\sigma} \cdot n - (DV\, \overline{\sigma}) \cdot n] + \\
& + \int_\Omega (DV : D\overline{\sigma}^T)\, (V \cdot \nabla \overline{u}) - \int_\Omega \operatorname{div} \overline{\sigma} \operatorname{div} V (\nabla \overline{u} \cdot V) - \int_\Omega \operatorname{div} \overline{\sigma} (DV^T \nabla \overline{u}) \cdot V + \\
& - \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n \overline{\sigma} \cdot n \operatorname{div} V + \int_\Omega \operatorname{div} \overline{\sigma} \operatorname{div} V (\nabla \overline{u} \cdot V) + \\
& + \int_\Omega (\nabla \overline{u} \cdot V) \overline{\sigma} \cdot \nabla(\operatorname{div} V) + \int_\Omega \operatorname{div} V\, (DV\, \overline{\sigma}) \cdot \nabla \overline{u} + \\
& - \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n \overline{\sigma} \cdot n\, (DV\, n) \cdot n + \int_\Omega \operatorname{div} \overline{\sigma}\, (DV\, V) \cdot \nabla \overline{u} + \\
& + \int_\Omega \partial_i \overline{u}\, \overline{\sigma}_j (\partial_j \partial_k V_i) V_k + \int_\Omega (DV^T\, \nabla \overline{u}) \cdot (DV\, \overline{\sigma}) + \\
& + \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n (DV\, \overline{\sigma}) \cdot n - \int_\Omega (DV^T\, \nabla \overline{u}) \cdot (DV\, \overline{\sigma}) + \\
& - \int_\Omega (\nabla \overline{u} \cdot V) \overline{\sigma} \cdot \nabla(\operatorname{div} V) - \int_\Omega (\nabla \overline{u} \cdot V)(D\overline{\sigma} : DV^T) + \\
& + \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} V_n (DV\, \overline{\sigma}) \cdot n - \int_\Omega \operatorname{div} V\ \overline{\sigma} \cdot (DV^T\, \nabla \overline{u}) - \int_\Omega (\partial_j \partial_i V_k) \partial_k \overline{u} V_j \overline{\sigma}_i \\
=& 0\ .
\end{aligned}
$$

□

Proposition 5.4.2. (upper bound)

Under the same assumptions of Theorem 5.4.1, it holds

$$\limsup_{\varepsilon\to 0^+} r_\varepsilon(V) \le \int_{\partial\Omega} (V\cdot n)^2 \left[\frac{\partial \overline{u}}{\partial n}(\nabla^2\overline{u}\nabla^2 f(\nabla\overline{u})n)\cdot n + f^*(\overline{\sigma})H_{\partial\Omega}\right] +$$
$$+ \inf_{\zeta\in L^2(\mathbb{R}^n;\mathbb{R}^n)} \left\{\int_\Omega (\nabla^2 f^*(\overline{\sigma})\,\zeta)\cdot\zeta + (g^*)''(\operatorname{div}\overline{\sigma})(\operatorname{div}\zeta)^2 + 2\int_{\partial\Omega}\frac{\partial\overline{u}}{\partial n}(V\cdot n)(\zeta\cdot n)\right\}. \tag{5.118}$$

Prior to the proof of Proposition 5.4.2, we recall from [67, Chapter 5] the following results:

Lemma 5.4.1. *Assume that $\partial\Omega$ is Lipschitz, and $h\in W^{1,1}(\mathbb{R}^n)$. Then*

$$\frac{d}{d\varepsilon}\int_{\Omega_\varepsilon} h\,\Big|_{\varepsilon=0^+} = \int_{\partial\Omega} hV\cdot n\, d\mathscr{H}^{n-1}\,.$$

Lemma 5.4.2. *Assume that $\partial\Omega\in C^2$, and $h\in W^{2,1}(\mathbb{R}^n)$. Then*

$$\frac{d^2}{d\varepsilon^2}\int_{\Omega_\varepsilon} h\,\Big|_{\varepsilon=0^+} = \int_{\partial\Omega}(V\cdot n)^2\left(\partial_n h + hH_{\partial\Omega}\right)d\mathscr{H}^{n-1}\,.$$

Proof of Proposition 5.4.2.
Let $\overline{\sigma}$ be an optimal field for $J^*(\Omega)$, and let ζ be an arbitrary element of $L^2(\mathbb{R}^n;\mathbb{R}^n)$. Since for every $\varepsilon>0$ the vector field $\overline{\sigma}+\varepsilon\zeta$ is admissible for $J^*(\Omega_\varepsilon)$, by using the second expression for $r_\varepsilon(V)$ in (5.108) we get

$$r_\varepsilon(V) \le \frac{2}{\varepsilon^2}\Big[\int_{\Omega_\varepsilon}\left[f^*(\overline{\sigma}+\varepsilon\zeta)+g^*(\operatorname{div}\overline{\sigma}+\varepsilon\operatorname{div}\zeta)\right] - \int_\Omega\left[f^*(\overline{\sigma})+g^*(\operatorname{div}\overline{\sigma})\right]+$$
$$-\varepsilon\int_{\partial\Omega} f^*(\overline{\sigma})V\cdot n\Big]\,.$$

Exploiting the Gateaux differentiability of f^*, the convexity of f^* and the L^2-boundedness of ζ, we infer that $\limsup_{\varepsilon\to 0^+} r_\varepsilon(V)$ is majorized by

$$\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon^2}\left[\int_{\Omega_\varepsilon}(f^*(\overline{\sigma})+g^*(\operatorname{div}\overline{\sigma})) - \int_\Omega (f^*(\overline{\sigma})+g^*(\operatorname{div}\overline{\sigma})) - \varepsilon\int_{\partial\Omega} f^*(\overline{\sigma})V\cdot n\,d\mathscr{H}^{n-1}\right]$$
$$+2\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\left[\nabla f^*(\overline{\sigma})\cdot\zeta + (g^*)'\,(\operatorname{div}\overline{\sigma})\operatorname{div}\zeta\right]$$
$$+\limsup_{\varepsilon\to 0^+}\int_{\Omega_\varepsilon}\left[(\nabla^2 f^*(\overline{\sigma})\,\zeta)\cdot\zeta + (g^*)''\,(\operatorname{div}\overline{\sigma})(\operatorname{div}\zeta)^2\right]\,.$$

We are now going to study separately each of the three limits in the r.h.s. of the above inequality, that we denote for brevity by L_1, L_2 and L_3. We recall that the following equalities hold in Ω:

$$f^*(\overline{\sigma}) = \nabla\overline{u}\cdot\nabla f(\nabla\overline{u}) - f(\nabla\overline{u}) \qquad \text{and} \qquad g^*(\operatorname{div}\sigma) = \overline{u}g'(\overline{u}) - g(\overline{u}) \tag{5.119}$$

In particular, since $g^*(\operatorname{div}\overline{\sigma}) = 0$ on $\partial\Omega$, we have

$$\int_{\partial\Omega} g^*(\operatorname{div}\overline{\sigma}) V \cdot n \, d\mathcal{H}^{n-1} = 0\,.$$

By subtracting from L_1 the above vanishing integral, we see that L_1 is the second order derivative of the shape functional $\int_{\Omega_\varepsilon} h$, with $h = (f^*(\overline{\sigma}) + g^*(\operatorname{div}\overline{\sigma}))$. Thus, by applying Lemma 5.4.2, we obtain

$$L_1 = \int_{\partial\Omega} (V\cdot n)^2 \left[\partial_n (f^*(\overline{\sigma}) + g^*(\operatorname{div}\overline{\sigma})) + (f^*(\overline{\sigma}) + g^*(\operatorname{div}\overline{\sigma}))H\right] d\mathcal{H}^{n-1}\,.$$

By exploiting (5.119), it is straightforward to check that the following equalities hold on $\partial\Omega$:

$$\begin{aligned}
&\nabla(f^*(\overline{\sigma})) = \nabla^2\overline{u}\,\nabla^2 f(\nabla\overline{u})\nabla\overline{u}\\
&\nabla(g^*(\operatorname{div}\overline{\sigma})) = 0\\
&g^*(\operatorname{div}\overline{\sigma}) = 0\,.
\end{aligned}$$

Hence

$$L_1 = \int_{\partial\Omega} (V\cdot n)^2 \Big[\frac{\partial\overline{u}}{\partial n}(\nabla^2\overline{u}\,\nabla^2 f(\nabla\overline{u})\,n)\cdot n + f^*(\overline{\sigma})H_{\partial\Omega}\Big]\, d\mathcal{H}^{n-1}\,.$$

Let us consider L_2. Since $\nabla f^*(\overline{\sigma}) = \nabla\overline{u}$ and $(g^*)'(\operatorname{div}\overline{\sigma}) = \overline{u}$ in Ω, recalling that $\overline{u} = 0$ on $\partial\Omega$, we get

$$\int_{\Omega} \big[\nabla f^*(\overline{\sigma})\cdot\zeta + (g^*)'(\operatorname{div}\overline{\sigma})\operatorname{div}\zeta\big] = 0\,,$$

By subtracting from L_2 the above vanishing integral, we see that L_2 the first order derivative of the shape functional $\int_{\Omega_\varepsilon} h$, with $h = \nabla f^*(\overline{\sigma})\cdot\zeta + (g^*)'(\operatorname{div}\overline{\sigma})\operatorname{div}\zeta$. Thus, by Lemma 5.4.1, we obtain

$$L_2 = 2\int_{\partial\Omega} \frac{\partial\overline{u}}{\partial n}(V\cdot n)(\zeta\cdot n)\, d\mathcal{H}^{n-1}\,.$$

Finally, passing to the limit as $\varepsilon \to 0$ in the third term, we have

$$L_3 = \int_{\Omega} \big[(\nabla^2 f^*(\overline{\sigma})\,\zeta)\cdot\zeta + (g^*)''\,(\operatorname{div}\overline{\sigma})(\operatorname{div}\zeta)^2\big]\,.$$

Combining the the results obtained for L_1, L_2, and L_3, and recalling the arbitrariness of ζ in $L^2(\mathbb{R}^n;\mathbb{R}^n)$, the result follows. ☐

Proof of Theorem 5.4.1 By combining the results of Proposition 5.4.2 and Proposition 5.4.1, we obtain a lower and upper bound for the sequence $r_\varepsilon(V)$ in (5.108). If we show that they coincide, the proof is achieved.

In view of the Fenchel equality, since $\overline{\sigma} \in \partial f(\nabla\overline{u})$, we infer that the first term of (5.112) and the first term of (5.118) agree.

By applying the standard duality Proposition 1.1.2 with $X = H^1(\Omega)$, $Y = L^2(\Omega)$, A the gradient operator,

$$\Psi(z) = \big(\nabla^2 f(\nabla\overline{u})z\big)\cdot z\,,\quad \Phi(v) = \begin{cases} g''(\overline{u})v^2 & \text{if } \operatorname{Tr}(v) = -V\cdot\nabla\overline{u}\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

we obtain

$$-\inf_{\substack{v\in H^1(\Omega)\\ \mathrm{Tr}(v)=-V\cdot\nabla\overline{u}}}\left\{\int_\Omega (\nabla^2 f(\nabla\overline{u})\,\nabla v)\cdot\nabla v+g''(\overline{u})v^2\right\}$$
$$=\inf_{\zeta\in L^2(\mathbb{R}^n;\mathbb{R}^n)}\left\{\frac{1}{4}\int_\Omega((\nabla^2 f(\nabla\overline{u}))^{-1}\,\zeta)\cdot\zeta+(g'')^{-1}(\overline{u})(\operatorname{div}\zeta)^2+\int_{\partial\Omega}\frac{\partial\overline{u}}{\partial n}\,(V\cdot n)\,(\zeta\cdot n)\right\}, \tag{5.120}$$

Finally, recalling that for strictly convex functions there hold (see [52, Proposition 10, Chapter II,2]

$$(\nabla^2 f(\nabla\overline{u}))^{-1}=\nabla^2 f^*(\overline{\sigma})\ ,$$
$$(g'')^{-1}(\overline{u})=(g^*)''(\operatorname{div}\overline{\sigma})\ ,$$

and considering 2ζ instead of ζ as admissible functions in the right hand side of (5.120), we conclude that the minus infimum appearing in (5.112) coincide with the infimum appearing in (5.118), and this concludes the proof. □

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