On the relations between SLE, CLE, GFF and the consequences

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Dedicated to my first quarter-century

Résumé

Cette thèse porte sur les relations entre les processus SLE, les ensembles CLE et le champ libre Gaussien. Dans le chapitre 2, nous donnons une construction des processus $\mathrm{SLE}_\kappa(\rho)$ à partir des boucles des CLE_κ et d'échantillons de restriction chordale. Sheffield et Werner ont prouvé que les CLE_κ peuvent être construits à partir des processus d'exploration symétriques des $\mathrm{SLE}_\kappa(\kappa-6)$. Nous montrons dans le chapitre 3 que la configuration des boucles construites à partir du processus d'exploration asymétrique des $\mathrm{SLE}_\kappa(\kappa-6)$ donne la même loi CLE_κ. Le processus SLE_4 peut être considéré comme les lignes de niveau du champ libre Gaussien et l'ensemble CLE_4 correspond à la collection des lignes de niveau de ce champ libre Gaussien. Dans la deuxième partie du chapitre 3, nous définissons un paramètre de temps invariant conforme pour chaque boucle appartenant à CLE_4 et nous donnons ensuite dans le chapitre 4 un couplage entre le champ libre Gaussien et l'ensemble CLE_4 à l'aide du paramètre de temps. Les processus SLE_κ peuvent être considérés comme les lignes de flot du champ libre Gaussien. Nous explicitons la dimension de Hausdorff de l'intersection de deux lignes de flot du champ libre Gaussien. Cela nous permet d'obtenir la dimension de l'ensemble des points de coupure et des points doubles de la courbe SLE, voir le chapitre 5. Dans le chapitre 6, nous définissons la mesure de restriction radiale, prouvons la caractérisation de ces mesures, et montrons la condition nécessaire et suffisante de l'existence des mesures de restriction radiale.

Mots clés: SLE (Schramm Loewner Evolution), CLE (Conformal Loop Ensemble), champ libre Gaussien, invariance conforme, propriété de Markov.

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Abstract

This thesis focuses on various relations between SLE, CLE and GFF. In Chapter 2, we give a construction of $\mathrm{SLE}_\kappa(\rho)$ processes from CLE_κ loop configuration and chordal restriction samples. Sheffield and Werner has proved that CLE_κ can be constructed from symmetric $\mathrm{SLE}_\kappa(\kappa-6)$ exploration processes. We prove in Chapter 3 that the loop configuration constructed from the asymmetric $\mathrm{SLE}_\kappa(\kappa-6)$ exploration processes also give the same law CLE_κ. SLE_4 can be viewed as level lines of GFF and CLE_4 can be viewed as the collection of level lines of GFF. We define a conformally invariant time parameter for each loop in CLE_4 in the second part of Chapter 3 and then give a coupling between GFF and CLE_4 with time parameter in Chapter 4. SLE_κ can be viewed as flow lines of GFF. We derive the Hausdorff dimension of the intersection of two flow lines in GFF. Then, from there, we obtain the dimension of the cut and double point set of SLE curve in Chapter 5. In Chapter 6, we define the radial restriction measure, prove the characterization of these measures, and show the if and only if condition for the existence of radial restriction measure.

Key words: SLE (Schramm Loewner Evolution), CLE (Conformal Loop Ensemble), GFF (Gaussian Free Field), conformal invariance, domain Markov property.

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Contents

Chapter 1

General Introduction

1.1 General Introduction in English

1.1.1 Introduction

Statistical physicists and probabilists often try to understand the macroscopic behavior of systems consisting of many microscopic random inputs, which can give rise to interfaces between two phases at a critical temperature, such as water and ice at 0 degree celsius. This can be modeled via the scaling limit behavior (macroscopic behavior) of discrete lattice models (microscopic inputs). In most cases (i.e. ranges of the parameter of the model that can play the role of temperature), the limits of these discrete models become deterministic (in the spirit of Law of Large Number); and in some critical cases (i.e. at the critical temperature), the limits can remain random, which is of particular interest.

The simplest example is fair simple random walk, that behaves (in the appropriately rescaled way) like Brownian motion in the scaling limit (and one can observe that Brownian motion is in a way more universal than random walk, since it is the scaling limit whatever fair random walk one considers).

In planar discrete models (in dimension two), curves appear naturally as interfaces between phases, level lines of random surfaces etc. In fact, these curves often provide a way to fully describe the random configuration. It has been noted that Brownian motion is in general not a sufficient tool to describe the complexity of these interfaces in the scaling limit, when they are random, and it has been predicted by theoretical physicists – and since then proved in a number of occasions – that these curves should be conformally invariant in this continuous scaling limit (this is a way to formulate these curves in terms of some of the axioms of Conformal Field Theory, see for instance the book [Car10] and the references therein).

Oded Schramm's SLE (Stochastic Loewner Evolution) processes [Sch00] have led mathematicians and physicists to a clean and novel understanding of the scaling limits of discrete models in two dimensions. Oded Schramm has realized that Loewner's coding of planar curves via iterations of conformal maps were exactly suited to the domain Markov property corresponding to the fact that one can explore interfaces progressively and describe the conditional distribution of the remaining configurations. A chordal SLE is a random non-self-traversing curve in a simply connected domain, joining two prescribed boundary points of the domain. And it is the only one-parameter family (usually indexed by a positive real number κ) of random planar curves that satisfies conformal invariance and (curve-configuration's) domain Markov property (the precise meaning of these two properties will be given after we precisely introduce SLE curves). SLE processes have already been proved to be the scaling limits of many discrete models: SLE_2 is the limit of loop-erased random walk [LSW04], SLE_3 is the limit of the interface of critical Ising model [CS12, CDCH$^+$12], SLE_4 is the scaling limit of the level line of DGFF [SS09], $SLE_{16/3}$ is the limit of the interface of critical FK model [CS12, CDCH$^+$12], SLE_6 is the limit of the interface of critical percolation [CN07], and SLE_8 is the scaling limit of uniform spanning tree [LSW04].

CLE (Conformal Loop Ensemble) is the limit geometric object when one tries to consider the "entire" scaling limit of discrete model (in contrast with only one interface which turns out to be the SLE process). A simple CLE [She09, SW12] can be viewed as a random countable collection of disjoint simple loops in the unit disk that are non-nested. It is the only one-parameter family that satisfies conformal invariance and (loops-configuration's) domain Markov property. It is proved (or almost proved) that CLE_3 is the limit of the collection of interfaces in critical Ising model, that CLE_4 is the collection of level lines of GFF (see [MS13a]), and CLE_6 is the limit of the collection of interfaces in critical percolation (see [CN06], note that this collection of loops is not a "simple CLE" because the loops have double points and are not disjoint). As one can somehow expect, each loop in CLE is a loop whose geometry is a SLE-type loop, with the same parameter κ.

The GFF (Gaussian Free Field) is a natural two-dimensional time analog of Brownian motion [She07], that has been used extensively as a basic building block in Quantum Field Theories. Like Brownian motion, it is a simple random object of widespread application and great intrinsic beauty. It plays an important role in statistical physics, the theory of random surfaces, and quantum field theory. The geometry of the two-dimensional Gaussian Free Field, i.e. the fact that it was possible to describe geometric lines in this very irregular distribution, has been discovered recently [SS12, SS09, MS13a, Dub09b, She11], and let to a number of recent developments. The GFF also corresponds to the scaling limit of simple discrete models (for instance the height function of dimer models, see [Ken08]).

SLE, CLE and GFF are three important related planar random structures and the present thesis will explore aspects of these three objects and of the relation between them. The present introduction is structured as follows: In the next section, we recall in a little more detail the definitions of SLE, CLE and GFF as well as some recent results. Then, we describe very briefly our contributions.

The corresponding papers that form the main body of the present thesis will form the subsequent five chapters.

1.1.2 Background

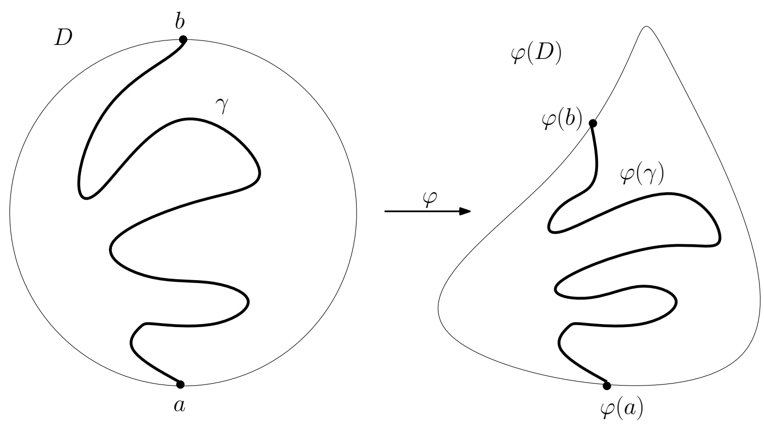
SLE

Suppose $(W_t, t \geq 0)$ is a real-valued continuous function. For each $z \in \overline{\mathbb{H}}$, define $g_t(z)$ as the solution to the chordal Loewner ODE:

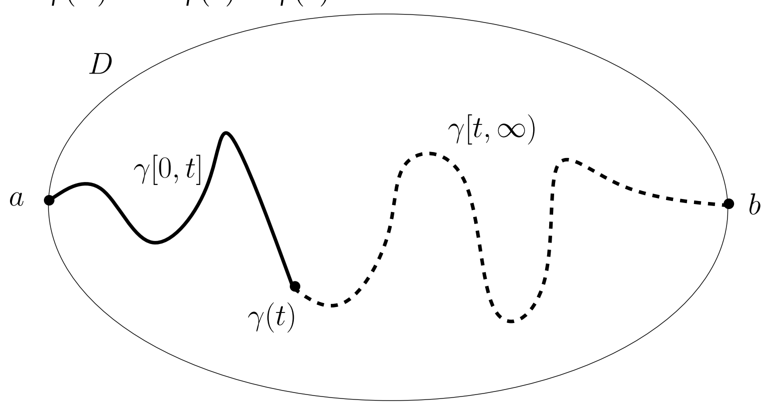
$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z. \tag{1.1.1}$$

Write $\tau(z) = \sup\{t \geq 0 : \inf_{s\in[0,t]} |g_s(z) - W_s| > 0\}$ and $K_t = \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$. Then g_t is the conformal map from $H_t := \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ such that $(g_t(z) - z)z \to 2t$ as $z \to \infty$. And $(g_t, t \geq 0)$ is called the chordal Loewner chain generated by the driving function $(W_t, t \geq 0)$.

A chordal SLE_κ is defined by the random family of chordal conformal maps g_t when $W = \sqrt{\kappa}B$ where B is a standard one-dimensional Brownian motion. It is proved that there exists a.s. a continuous curve γ in $\overline{\mathbb{H}}$ connecting 0 to ∞ such that for each $t \geq 0$, H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma([0,t])$ and g_t is the conformal map from H_t onto $\mathbb{H}$ (see [RS05]). SLE curves are the only curves that satisfy both conformal invariance–$(\Phi(\gamma(t)), t \geq 0)$ has the same law as γ for any Möbius transformation Φ of $\mathbb{H}$ that preserves 0 and ∞–and domain Markov property–for any $t > 0$, $f_t(\gamma([t,\infty)))$ has the same law as γ where $f_t = g_t - W_t$ (see Figure 1.1.1). From the conformal invariance, we can define SLE curves in any simply connected domain D with two distinct boundary points a, b : SLE curves in D from a to b are the image of γ in $\mathbb{H}$ from 0 to ∞ under any conformal map from $\mathbb{H}$ onto D that sends $0, \infty$ to a, b respectively.



(a) Conformal Invariance: γ is an SLE curve in D from a to b, φ is a conformal map, then $\varphi(\gamma)$ has the same law as an SLE curves in $\varphi(D)$ from $\varphi(a)$ to $\varphi(b)$.



(b) Domain Markov Property: γ is an SLE curve in D from a to b, given $\gamma([0,t])$, $\gamma([t,\infty))$ has the same law as SLE curves in $D \setminus K_t$ from $\gamma(t)$ to b.

Figure 1.1.1: Characterization of SLE.

A number of properties of SLE curves are now known (see [Law05]): When $\kappa \in [0,4]$, SLE$_\kappa$ curves are simple curves.When $\kappa \in (4,8)$, the curves are self-touching. And when $\kappa \geq 8$, the curves become space-filling. SLE$_\kappa$ curve has almost sure Hausdorff dimension $(1+\kappa/8) \wedge 2$ [Bef08]. When $\kappa \in (4,8)$, the SLE process hits the real line, and the intersection of the curve with this real line forms a Cantor set of dimension between 0 and 1. It is proved in [AS08] that this dimension is equal to $2 - 8/\kappa$.

It turns out to be rather difficult to derive rigorously the dimensions of the set of other natural simple subsets of SLE curves (such as the set of cut-points or of double-points) even if the conjectured relation between lattice models and SLE give a way to guess what these dimensions are (via the various arm exponents). Note also that the corresponding sets of corresponding times (in the

Loewner equation parametrization) are easier to study (see [Bef04]), but that this does not help in deriving the "spatial" dimensions of these sets of points.

For certain values of κ, SLE curves exhibit striking properties. For example, $\mathrm{SLE}_{8/3}$ satisfies "chordal restriction property" [LSW03] (we will recall this property in a few paragraphs), and this property leads to conjecture that $\mathrm{SLE}_{8/3}$ is the scaling limit of self-avoiding random walk which also satisfies chordal restriction property in the discrete setting (when properly defined). SLE_6 satisfies "locality property" which is the property satisfied by the interface in critical percolation [LSW01a]. SLE_2 satisfies the property that if we add to SLE_2 curve with an independent Poisson point process of Brownian loops, then the obtained set has the "same" law as a Brownian excursion. This property is reminiscent of loop erased random walk [LW04].

Two other important properties of SLE curves are the following: reversibility–SLE curves in D from a to b viewed as set has the same law as SLE curve in D from b to a–and duality–the outer boundary of $\mathrm{SLE}_{\kappa'}$ curve are variants of SLE_κ curves where $\kappa' \geq 4, \kappa = 16/\kappa' \leq 4$. These two properties are natural properties for discrete physics models and they are not obvious from the definition through Equation (1.1.1) [Zha08b, Dub09a, MS12a].

The chordal $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process is a variant of chordal SLE_κ in which one keeps track of multiple additional points, which we refer to as force points. Suppose $\underline{x}^L = (x^{l,L} < \cdots < x^{1,L} \leq 0)$ and $\underline{x}^R = (0 \leq x^{1,R} < \cdots < x^{r,R})$ are our force points. Associated with each force point $x^{i,q}, q \in \{L,R\}$, there is a weight $\rho^{i,q} \in \mathbb{R}, q \in \{L,R\}$. An $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process with force points $(\underline{x}^L;\underline{x}^R)$ is the random family of chordal conformal maps g_t with W_t replaced by the solution to the system of SDEs:

$$\begin{aligned} W_t &= \sqrt{\kappa}B_t + \sum_{i=1}^{l}\int_0^t \frac{\rho^{i,L}ds}{W_s - V_s^{i,L}} + \sum_{i=1}^{r}\int_0^t \frac{\rho^{i,R}ds}{W_s - V_s^{i,R}}, \\ V_t^{i,q} &= x^{i,q} + \int_0^t \frac{2ds}{V_s^{i,q} - W_s}, \quad i \in \mathbb{N}, \quad q \in \{L,R\}. \end{aligned} \tag{1.1.2}$$

For all $\kappa > 0$, there is a unique solution to (1.1.2) up until the *continuation threshold* is hit–the first time t for which either

$$\sum_{i:V_t^{i,L}=W_t} \rho^{i,L} \leq -2 \quad \text{or} \quad \sum_{i:V_t^{i,R}=W_t} \rho^{i,R} \leq -2.$$

For $\kappa > 0$, the compact hull associated to the process up to the continuation threshold is generated by a continuous curve (see [MS12a]).

The chordal SLE curves are curves in simply connected domain connecting two boundary points and it is also possible to define SLE curves connecting one boundary point to one interior point–the radial SLE curves. Suppose $(W_t, t \geq 0)$ is a real-valued continuous function. For each $z \in \overline{\mathbb{U}}$, define $g_t(z)$ as the solution to the radial Loewner ODE:

$$\partial_t g_t(z) = g_t(z)\frac{e^{iW_t} + g_t(z)}{e^{iW_t} - g_t(z)}, \quad g_0(z) = z. \tag{1.1.3}$$

Write $\tau(z) = \sup\{t \geq 0 : \inf_{s\in[0,t]} |g_s(z) - e^{iW_s}| > 0\}$ and $K_t = \{z \in \overline{\mathbb{U}} : \tau(z) \leq t\}$. Then g_t is the conformal map from $\mathbb{U} \setminus K_t$ onto $\mathbb{U}$ such that $g_t(0) = 0, g_t'(0) = e^t$. And $(g_t, t \geq 0)$ is called the radial Loewner chain generated by the driving function $(W_t, t \geq 0)$.

A radial SLE_κ is defined by the random family of radial conformal maps g_t when $W = \sqrt{\kappa}B$ where B is a standard one-dimensional Brownian motion. It is proved that there exists a.s. a continuous curve γ such that for each $t \geq 0$, $\mathbb{U} \setminus K_t$ is the connected component of $\mathbb{U} \setminus \gamma([0,t])$ containing the origin.

Radial SLE curves are "locally" the same as chordal SLE curves thus some of chordal SLE properties are also true for radial curves, say their Hausdorff dimension.

CLE

In [SW12], a CLE is a collection Γ of non-nested disjoint simple loops $(\gamma_j, j \in J)$ in $\mathbb{H}$ that possesses a particular conformal restriction property. In fact, this property that we will now recall, does characterize these simple CLEs (see Figure 1.1.2):

- (Conformal Invariance) For any Möbius transformation Φ of $\mathbb{H}$ onto itself, the laws of Γ and $\Phi(\Gamma)$ are the same. This makes it possible to define, for any simply connected domain D (that is not the entire plane – and can therefore be viewed as the conformal image of $\mathbb{H}$ via some map $\tilde{\Phi}$), the law of the CLE in D as the distribution of $\tilde{\Phi}(\Gamma)$ (because this distribution does then not depend on the actual choice of conformal map $\tilde{\Phi}$ from $\mathbb{H}$ onto D).

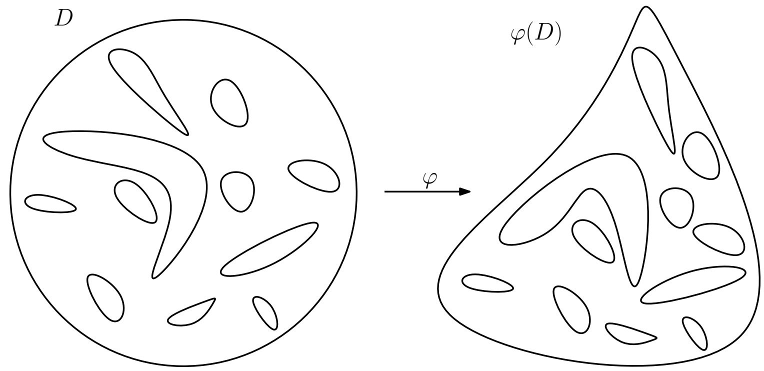
- (Domain Markov Propety) For any simply connected domain $H \subset \mathbb{H}$, define the set $\tilde{H} = \tilde{H}(H,\Gamma)$ obtained by removing from H all the loops (and their interiors) of Γ that do not entirely lie in H. Then, conditionally on $\tilde{H}$, and for each connected component U of $\tilde{H}$, the law of those loops of Γ that do stay in U is exactly that of a CLE in U.

It turns out that the loops in a given CLE are SLE$_\kappa$ type loops for some value of $\kappa \in (8/3, 4]$ (and they look locally like SLE$_\kappa$ curves). In fact for each such value of κ, there exists exactly one CLE distribution that has SLE$_\kappa$ type loops. As explained in [SW12], a construction of these particular families of loops can be given in terms of outer boundaries of outmost clusters of the Brownian loops in a Brownian loop-soup with subcritical intensity c (and each value of c corresponds to a value of κ).

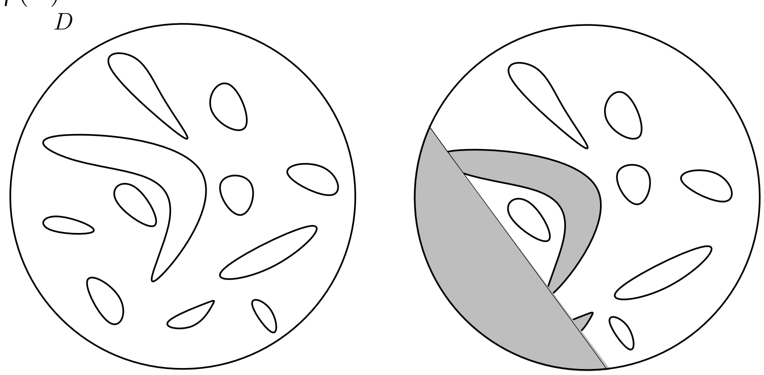
In the earlier paper [She09], Sheffield had pointed out a way to construct a number of random collections of loops, using variants of SLE$_\kappa(\kappa-6)$ processes. In particular, for any $\kappa \in [0,8]$, he has shown how to construct random collections of SLE$_\kappa$-type loops that should be the only possible candidates for the conformally invariant scaling limit of various discrete models, or of level lines of certain continuous models. Roughly speaking, one chooses some boundary point x on the unit circle ("the root") and launches from there a branching exploration tree of SLE$_\kappa(\kappa-6)$ processes that will trace some loops along the way, that one keeps track of. For each κ and x, there are in fact several ways to do this. One particular way is to impose certain "left-right" symmetry in the law of the exploration tree, but several other natural options are described in [She09]. Hence, for each κ, the exploration tree is defined via the choice of the root x and the exploration "strategy" that describes how "left-right" asymmetric the exploration is. These exploration strategies are particularly natural, because they are invariant under all conformal transformations that preserve x. So to sum up, once κ, x and a given strategy are chosen, the Loewner differential equation enables to construct a random family of quasi-loops in the unit disc (and the law of this family a priori depends on κ, on x and on the chosen strategy).

GFF

We will briefly recall the definition of GFF, in the Gaussian Hilbert space framework (as in [She07] for instance): Consider the space $H_s(D)$ of smooth, real-valued functions on $\mathbb{C}$ that are supported



(a) Conformal Invariance: $\Gamma = (\gamma_j, j \in J)$ is a CLE in D, φ is a conformal map, then $(\varphi(\gamma_j), j \in J)$ has the same law as CLE in $\varphi(D)$.



(b) Domain Markov Property: Γ is a CLE in D, U is a deterministic subset of D, then given the loops intersecting U, the remaining loops has the same law as CLE in the remaining domain.

Figure 1.1.2: Characterization of CLE.

on a compact subset of a domain $D \subset \mathbb{C}$ (so that, in particular, their first derivatives are in $L^2(D)$). This space can be endowed with a *Dirichlet inner product* defined by

$$(f_1, f_2)_\nabla = \int_D dx(\nabla f_1 \cdot \nabla f_2)$$

It is immediate to see that the Dirichlet inner product is invariant under conformal transformation. Denote by $H(D)$ the Hilbert space completion of $H_s(D)$. The quantity $(f, f)_\nabla$ is called the *Dirichlet energy* of f.

A *Gaussian Free Field* is any Gaussian Hilbert space $\mathscr{G}(D)$ of random variables denoted by "$(h, f)_\nabla$"—one variable for each $f \in H(D)$—that inherits the Dirichlet inner product structure of $H(D)$, i.e.,

$$\mathbb{E}[(h,a)_\nabla (h,b)_\nabla] = (a,b)_\nabla.$$

In other words, the map from f to the random variable $(h, f)_\nabla$ is an inner product preserving map from $H(D)$ to $\mathscr{G}(D)$. The reason for this notation is that it is possible to view h as a random linear operator, but we will not need this approach. We also view (h, ρ) as being well defined for all $\rho \in (-\triangle)H(D)$ (if $\rho = -\triangle f$ for some $f \in H(D)$, then we denote $(h, \rho) = (h, f)_\nabla$).

When ρ_1 and ρ_2 are in $H_s(D)$, the covariance of (h, ρ_1) and (h, ρ_2) can be written as

$$(-\triangle^{-1}\rho_1, -\triangle^{-1}\rho_2)_\nabla = (\rho_1, -\Delta^{-1}\rho_2) = (-\Delta^{-1}\rho_1, \rho_2).$$

Since $-\Delta^{-1}\rho$ can be written using the Green's function, we may also write:

$$\mathrm{Cov}[(h,\rho_1),(h,\rho_2)] = \frac{1}{2}\iint dxdy\, G_D(x,y)\rho_1(x)\rho_2(y).$$

Both the Dirichlet inner product and the Gaussian Free Field inherit naturally conformal invariance properties from the conformal invariance of the Green's function.

It turns out that the GFF is very closely related to SLE_4 and to CLE_4. Indeed, from [SS09, SS12, Dub09b], one can view SLE_4 as a level-line of the GFF. More precisely, let γ be an SLE_4 in $\mathbb{H}$ from 0 to ∞ and denote H_-, H_+ as the two connected components of $\mathbb{H}\setminus\gamma$. Sample GFF h_- (resp. h_+) in H_- (resp. H_+) with mean value $-\lambda$ (resp. $+\lambda$) where $\lambda = \pi/2$. The fields h_-, h_+ are sampled in the way that they are independent of γ and they are independent of each other given γ. Then the sum $h = h_- + h_+$ has the same law as a GFF in $\mathbb{H}$ and in this coupling (γ, h), γ is a deterministic function of h. From these facts, we say that SLE_4 is the level-line of the GFF.

Building on this fact and on the construction of CLE_4 (such for instance as provided by [SW12]), it is possible to couple the GFF with zero boundary condition with a CLE_4 as follows [MS13a]. Let $\mathscr{L}$ be a CLE_4 collection of loops. Given $\mathscr{L}$, we sample Bernoulli signs ε_L for each loop $L\in\mathscr{L}$ and all these signs are sampled in the way that they are independent of $\mathscr{L}$ and they are independent of each other given $\mathscr{L}$. For each loop $L\in\mathscr{L}$, sample a GFF h_L inside L with mean value $\varepsilon_L\times 2\lambda$. All these GFFs are sampled in the way that they are independent of $\mathscr{L}$ and they are independent of each other given $\mathscr{L}$. Then the sum $h=\sum_{L\in\mathscr{L}} h_L$ has the same law as a GFF with zero boundary value and in this coupling $(\mathscr{L}, h)$, the loop configuration $\mathscr{L}$ is a deterministic function of h.

More generally, one can associate a number of other SLE-type curves with a GFF ([SS09, SS12, MS12a]). Fix

$$\kappa\in(0,8),\quad \lambda=\frac{\pi}{\sqrt{\kappa}},\quad \chi=\frac{2}{\sqrt{\kappa}}-\frac{\sqrt{\kappa}}{2}$$

and let γ be the curve associated to $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process with force points $(\underline{x}^L;\underline{x}^R)$ and K be the corresponding compact hulls. Let h be the GFF on $\mathbb{H}$ with zero boundary value. There exists a coupling (γ,h) such that the following is true. Suppose τ is any finite stopping time less than the continuation threshold for γ. Let η_t^0 be the function which is harmonic in $\mathbb{H}$ with boundary values

$$\begin{cases} -\lambda(1+\overline{\rho}^{j,L}) & \text{if} \quad x\in(f_t(x^{j+1,L}), f_t(x^{j,L})) \\ \lambda(1+\overline{\rho}^{j,R}) & \text{if} \quad x\in(f_t(x^{j,R}), f_t(x^{j+1,R})) \end{cases}$$

Let

$$\eta_t(z)=\eta_t^0(f_t(z))-\chi\arg f_t'(z). \tag{1.1.4}$$

Then the conditional law of $h+\eta_0|_{\mathbb{H}\setminus K_\tau}$ given K_τ is equal to the law of $h\circ f_\tau+\eta_\tau$. In this coupling, the curve γ is almost surely determined by the field h, and we can view γ as the flow line (respectively level line) of $h+\eta_0$ if $\kappa\in(0,4)\cup(4,8)$ (respectively $\kappa=4$).

Chordal restriction

An important role in the present PhD thesis will also be played by the collection of random sets that satisfy a certain conformal restriction property, and have been studied and classified in [LSW03]. Let us now briefly recall the definition and some of the results of that paper. A chordal restriction sample is a closed random subset K of $\overline{\mathbb{H}}$ such that

- K is connected, $\mathbb{C}\setminus K$ is simply connected, $K\cap\mathbb{R}=\{0\}$, and K is unbounded.
- For any closed subset A of $\overline{\mathbb{H}}$ such that $A=\overline{\mathbb{H}\cap A}$, $\mathbb{H}\setminus A$ is simply connected, A is bounded and $0\notin A$, the law of $\Psi_A(K)$ conditioned on $(K\cap A=\emptyset)$ is equal to the law of K where Ψ_A is any conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ that preserves 0 and ∞.

It is proved in [LSW03] that there exists a one-parameter family $\mathbb{Q}_\beta$ of chordal restriction measures such that

$$\mathbb{Q}_\beta(K\cap A=\emptyset)=\Psi_A'(0)^\beta$$

where Ψ_A is the conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ that preserves 0 and $\Psi_A(z)/z\to 1$ as $z\to\infty$ (see [LSW03]). The chordal conformal restriction measure $\mathbb{Q}_\beta$ exists if and only if $\beta\geq 5/8$. Note that $\mathbb{Q}_1$ can be easily constructed by filling the loops of Brownian excursion in $\mathbb{H}$ from 0 to ∞.

A right-sided chordal restriction sample is a closed random subset K of $\overline{\mathbb{H}}$ such that

- K is connected, $\mathbb{C}\setminus K$ is connected, $K\cap\mathbb{R}=(-\infty,0]$.
- For any closed subset A of $\overline{\mathbb{H}}$ such that $A=\overline{\mathbb{H}\cap A}$, $\mathbb{H}\setminus A$ is simply connected, A is bounded and $A\cap\mathbb{R}\subset(0,\infty)$, the law of $\Psi_A(K)$ conditioned on $(K\cap A=\emptyset)$ is equal to the law of K where Ψ_A is any conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ that preserves 0 and ∞.

It is clear that the right boundary of chordal restriction sample is a right-sided restriction sample. In fact, there exists a one-parameter family $\mathbb{Q}^+_\beta$ such that

$$\mathbb{Q}^+_\beta(K\cap A=\emptyset)=\Psi_A'(0)^\beta$$

where Ψ_A is the conformal map from $\mathbb{H}\setminus A$ onto $\mathbb{H}$ that preserves 0 and $\Psi_A(z)/z\to 1$ as $z\to\infty$. $\mathbb{Q}^+_\beta$ exists if and only if $\beta\geq 0$. We usually ignore the trivial case $\beta=0$ where $K=\mathbb{R}_-$.

One example of right-sided restriction sample is given by $\mathrm{SLE}_{8/3}(\rho)$ process with force point 0^-. Fix $\rho>-2$. Let γ be an $\mathrm{SLE}_{8/3}(\rho)$ process in $\overline{\mathbb{H}}$ from 0 to ∞. Let K be the closure of the union of domains between γ and $\mathbb{R}_-$. Then K is a right-sided restriction sample with exponent $\beta=(\rho+2)(3\rho+10)/32$. Conversely, let K be a right-sided restriction sample with exponent $\beta>0$, then the right boundary of K is an $\mathrm{SLE}_{8/3}(\rho)$ process with $\rho=\rho(\beta)=2(\sqrt{24\beta+1}-1)/3-2$. From these properties, one sees that the outer boundary of Brownian excursion are variants of $\mathrm{SLE}_{8/3}$ curves.

1.1.3 Our contributions

From CLE to SLE

Chapter 2 (corresponding to the joint paper [WW13a] with Wendelin Werner) presents a way to construct $\mathrm{SLE}_\kappa(\rho)$ processes using a CLE and an independent restriction sample: Define independently, in a simply connected domain D with two marked boundary points a and b, the following two random objects: A CLE_κ (for some $\kappa\in(8/3,4]$) that we call Γ and a right-sided restriction path γ from a to b with restriction exponent α. Then, we define the set obtained by attaching to γ all the loops of Γ that it intersects. Finally, we take the right-most boundary of this set. This turns out to be again a simple curve from a to b in D that we call η (see Figure 1.1.3) and most of η will consist of parts of CLE loops. We prove in Chapter 2 that this curve η is in fact an $\mathrm{SLE}_\kappa(\rho)$ process where $\rho>-2$ and is related to α via

$$\alpha=\frac{(\rho+2)(\rho+6-\kappa)}{4\kappa}.$$

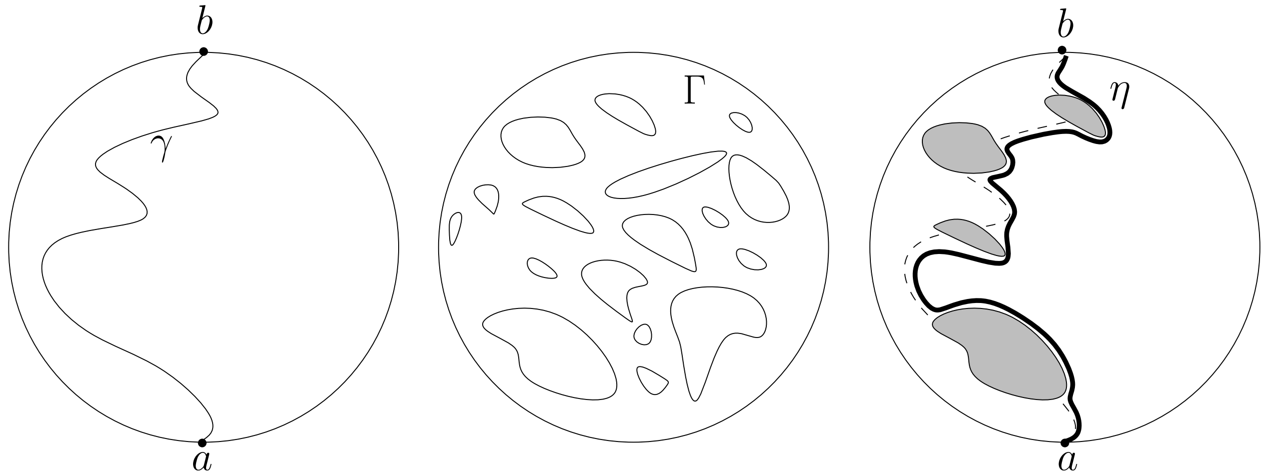


Figure 1.1.3: Construction of η out of γ and Γ.

From this construction of SLE processes, we have an interesting immediate consequence that provides alternative simple proofs of non-trivial results ([Zha08b, Zha10b, MS12b]): the reversibility of $\text{SLE}_\kappa(\rho)$ process with $\kappa \in [8/3,4], \rho > -2$. Another consequence is the two-point estimate of the intersection of $\text{SLE}_\kappa(\rho)$ with the boundary which in turn gives a simple proof of the Hausdorff dimension of this intersection.

From SLE to CLE

CLEs have first been defined by Scott Sheffield in [She09] via branching variants of SLE curves. In fact, in that paper, he provides several constructions of these branching SLE trees and conjectures that they should all correspond to the same random collection of loops. This has then been proved to hold for the symmetric exploration trees in [SW12]. The first main result of Chapter 3 (joint paper [WW13b] with Wendelin Werner) can be summarized as follows: *For each $\kappa \in (8/3,4]$, all the random collections of SLE_κ-type quasi-loops constructed via Sheffield's asymmetric exploration trees in [She09] have the same law. They all are the CLE_κ families of loops constructed in [SW12].* To this end, we first precisely define $\text{SLE}_\kappa(\kappa-6)$ process when $\kappa \in (8/3,4]$ (note that in this case $\kappa - 6 \leq -2$ which is no longer well-defined through Equation (1.1.2)) through Bessel processes. And then compare the asymmetric exploration structure to the symmetric case which has been studied in detail in [SW12].

Conformally invariant growing mechanism in CLE_4

The second main point of Chapter 3 is to highlight something specific to CLE_4 (recall that this is the CLE that is most directly related to the Gaussian Free Field, see [SS12, SS09, MS13a, Dub09b, She11]). In this particular case, it is possible to define a conformally invariant and unrooted (one does not need to even choose a starting point) growing mechanism of loops. Roughly speaking, the growth process that progressively discovers loops is growing "uniformly" from the boundary (even if it is a Poisson point process and each loop is discovered at once) and does not require to choose a root. The fact that such a conformally invariant non-local growth mechanism exists at all is quite surprising (and the fact that its time-parametrization as seen from different points does exactly coincide even more so).

From this growing mechanism in CLE_4, we can associate in a conformally invariant way a positive time parameter u_L to each loop L in the loop configuration $\mathscr{L}$, which is the time in the growing mechanism at which the loop has been discovered. We define a Markov process on domains $(D_u, u \geq 0)$: at time $u = 0$, it is the upper half plane; and at time $u > 0$, it is the remaining domain that we remove all the loops with time parameter smaller than u. From the conformally invariant growing mechanism, we know that $(D_u, u \geq 0)$ and $(\Phi(D_u), u \geq 0)$ are identically distributed for any conformal transformation Φ from $\mathbb{H}$ onto itself.

Coupling between GFF and CLE_4 with time parameter

As we have recalled above, SLE_4 curves are level lines of GFF and CLE_4 is a collection of level lines of GFF [MS13a], in the sense that one can couple a GFF h and a CLE_4 $\mathscr{L}$ in such a way that loops in $\mathscr{L}$ are outmost level lines of h of heights $\pm\lambda$ and the signs of the expected value of h inside the loop are give by i.i.d coin tosses independently of the CLE_4. In this coupling, one can actually prove that the CLE_4 loop configuration is deterministic function of h.

In Chapter 4, we provide a second and new coupling between GFF and CLE_4, making use of the time parameters defined in Chapter 3. More precisely, we couple a GFF h with zero boundary value with CLE_4 with time parameter $((L, u_L), L \in \mathscr{L})$ in such a way that for each loop L, it is a level line of h and the expected value of h inside L is $2\lambda - 2\lambda u_L$. In other words, the jump from the outside to the inside of a loop is always a positive jump of the GFF (as opposed to the previous case where one tosses a coin to decide if it was an upward or an downward jump). We further prove that, in this coupling, both the CLE_4 loop configuration and the time parameter are deterministic functions of h.

(a) First coupling (b) Second coupling

Figure 1.1.4: Relation between the two couplings.

These two couplings are reminiscent of the two constructions of one-dimensional Brownian motion from Brownian excursions: given a Poisson point process of Brownian excursions, the first way to construct a Brownian motion is to set each excursion positive or negative independently with equal probability $1/2$ and then to concatenate these signed excursions, the obtained process has the same law as a Brownian motion. The second way is to concatenate all the positive excursions (which we denote by $(Y_t, t \geq 0)$) and then define the local time L for Y, the process $(Y_t - L_t, t \geq 0)$ also has the same law as a Brownian motion. After recall these two constructions of Brownian motion and the fact that GFF is in fact a two-dimension analogue of Brownian motion, we see that our time parameter in CLE_4 in the second coupling is somewhat the GFF counterpart of the local time.

The results of Chapter 4 form part of a paper in preparation with Scott Sheffield and Sam Watson, where we also plan and hope to prove that, in $((L, u_L), L \in \mathscr{L})$, the time parameter is a deterministic function of the loop configuration $\mathscr{L}$.

From GFF to SLE and intersections of SLE paths

The main result of Chapter 5 is the determination of the Hausdorff dimension of the sets of double points and of the sets of cut points of SLE curves. Let us now state some of these results. Let $\kappa' \in (4,8)$ and γ' be an $\text{SLE}_{\kappa'}$ process. The cut point set of γ' is defined as $\mathscr{K} = \{\gamma'(t) : t \in (0,\infty), \gamma'(0,t) \cap \gamma'(t,\infty) = \emptyset\}$. We prove that, almost surely,

$$\dim_H(\mathscr{K}) = 3 - \frac{3\kappa'}{8}. \tag{1.1.5}$$

Let $\mathscr{D}$ denote the set of all double points of γ', we prove that, almost surely,

$$\dim_H(\mathscr{D}) = 2 - \frac{(12-\kappa')(4+\kappa')}{8\kappa'}. \tag{1.1.6}$$

We derive these two results building on the coupling between GFF and SLE that describes SLE curves as flow lines of GFF. On the way, we will in fact also derive further properties of $\text{SLE}_\kappa(\rho)$-type processes.

Let us very briefly describe the type of arguments and results we use and obtain: Fix

$$\kappa' > 4, \quad \kappa = \frac{16}{\kappa'} \in (0,4), \quad \lambda = \frac{\pi}{\sqrt{\kappa}}, \quad \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$

First, consider only one single flow line. For $\kappa \in (0,4), \rho > -2$, let γ be an $\text{SLE}_\kappa(\rho)$ process and it can be viewed as the flow line of GFF with boundary value $-\lambda$ on $\mathbb{R}_-$ and $\lambda(1+\rho)$ on $\mathbb{R}_+$. We get that, almost surely,

$$\dim_H(\gamma \cap \mathbb{R}) = 1 - \frac{1}{\kappa}(\rho+2)(\rho+4-\frac{\kappa}{2}). \tag{1.1.7}$$

Next, we consider two flow lines simultaneously. Let GFF have boundary value λ on $\mathbb{R}_+$ and $-\lambda$ on $\mathbb{R}_-$. Fix $\theta_2 > \theta_1$. Let γ_{θ_i} be the flow line of the field with angle θ_i for $i = 1,2$. From the Imaginary Geometry introduced in [MS12a], we know that γ_{θ_2} almost surely stays to the left of γ_{θ_1} (see Figure 1.1.5). And given γ_{θ_2}, the conditional law of γ_{θ_1} is $\text{SLE}_\kappa(\rho; \theta_1\chi/\lambda)$ where

$$\rho = \frac{(\theta_2 - \theta_1)\chi}{\lambda} - 2.$$

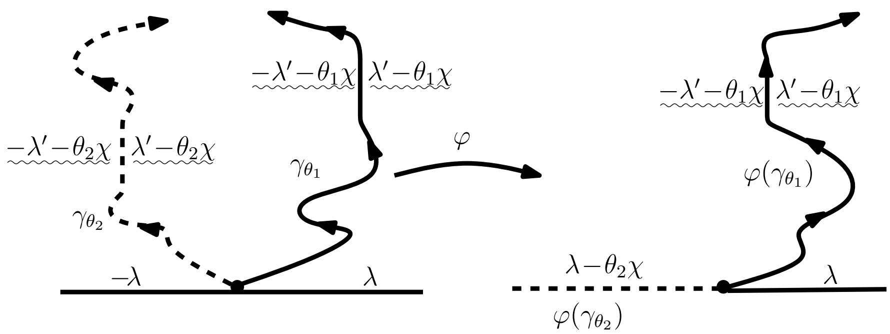


Figure 1.1.5: The flow line γ_{θ_2} almost surely stays to the left of γ_{θ_1}. Given γ_{θ_2}, the conditional law of γ_{θ_1} is $\mathrm{SLE}_\kappa((\theta_2-\theta_1)\chi/\lambda-2;\theta_1\chi/\lambda)$.

Combine this fact and (1.1.7), we get that, almost surely,

$$\dim_H(\gamma_{\theta_1}\cap\gamma_{\theta_2}\cap\mathbb{H}) = 2-\frac{1}{2\kappa}\left(\rho+\frac{\kappa}{2}+2\right)\left(\rho-\frac{\kappa}{2}+6\right) \tag{1.1.8}$$

From (1.1.8), we derive (1.1.5) by duality of SLE curves: Let γ' be an $\mathrm{SLE}_{\kappa'}$ process from ∞ to 0 in $\mathbb{H}$, then the left boundary and right boundary of γ' are flow lines of the field with angle $\pi/2$ and $-\pi/2$ respectively. Thus (1.1.5) is the case of (1.1.8) when the angle difference is $\theta_{\text{cut}}=\pi$.

To derive (1.1.6), we use the path decomposition of $\mathrm{SLE}_{\kappa'}(\kappa'/2-4;\kappa'/2-4)$ process introduced in [MS12c]. Then explain that the double points of $\mathrm{SLE}_{\kappa'}$ correspond to the intersection of two flow lines of the field with angle difference

$$\theta_{\text{double}} = \pi\left(\frac{\kappa-2}{2-\frac{\kappa}{2}}\right).$$

Radial conformal restriction

The final chapter is devoted to the study of the radial counterpart of the chordal restriction samples. These are random sets whose boundaries are all of $\mathrm{SLE}_{8/3}$ type. More precisely, consider the unit disc $\mathbb{U}$ and we fix a boundary point 1 and an interior point the origin. A radial restriction sample is a closed random subsets K of $\overline{\mathbb{U}}$ such that:

- K is connected, $\mathbb{C}\setminus K$ is connected, $K\cap\partial\mathbb{U}=\{1\}$, $0\in K$.
- For any closed subset A of $\overline{\mathbb{U}}$ such that $A=\overline{\mathbb{U}\cap A}$, $\mathbb{U}\setminus A$ is simply connected, contains the origin and has 1 on the boundary, the law of $\Phi_A(K)$ conditioned on $(K\cap A=\emptyset)$ is equal to the law of K where Φ_A is the conformal map from $\mathbb{U}\setminus A$ onto $\mathbb{U}$ that preserves 1 and the origin (see Figure 1.1.6).

We prove in Chapter 6 that the radial restriction measure is characterized by a pair of real numbers (α,β) such that

$$\mathbb{P}(K\cap A=\emptyset) = |\Phi_A'(0)|^\alpha \Phi_A'(1)^\beta$$

where A is any closed subset of $\overline{\mathbb{U}}$ such that $A=\overline{\mathbb{U}\cap A}$, $\mathbb{U}\setminus A$ is simply connected, contains the origin and has 1 on the boundary, and Φ_A is the conformal map from $\mathbb{U}\setminus A$ onto $\mathbb{U}$ that preserves 0

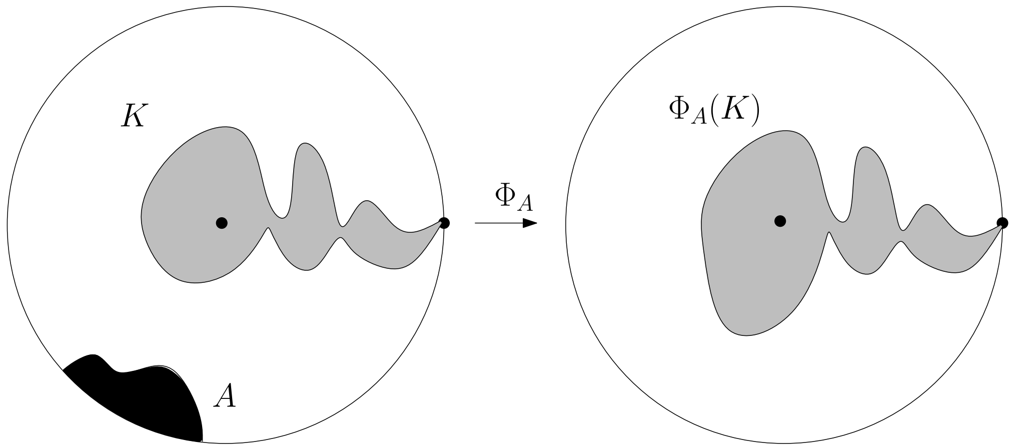


Figure 1.1.6: Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves 0 and 1. Conditioned on $(K \cap A = \emptyset)$, $\Phi_A(K)$ has the same law as K.

and 1. The corresponding radial restriction measure is denoted by $\mathbb{P}(\alpha,\beta)$. Then $\mathbb{P}(\alpha,\beta)$ exists if and only if

$$\beta \geq \frac{5}{8}, \quad \alpha \leq \xi(\beta) = \frac{1}{48}\left((\sqrt{24\beta+1}-1)^2-4\right).$$

For $\beta \geq 5/8$, if K is sampled according to $\mathbb{P}(\xi(\beta),\beta)$, then the right boundary of K is a radial $\mathrm{SLE}_{8/3}(\rho)$ with $\rho = \rho(\beta) = 2(\sqrt{24\beta+1}-1)/3-2$.

This is therefore the radial counterpart of the classification of chordal restriction samples in [LSW03].

1.2 Introduction Générale en français

Les physiciens-statisticiens et probabilistes essayent souvent de comprendre le comportement macroscopique de systèmes comprenant de nombreuses entrées aléatoires microscopiques, qui peuvent donner lieu à des interfaces entre deux phases à une température critique, comme par exemple l'eau et la glace à 0 degré Celsius. Cela peut être modélisé par le comportement à la limite d'échelle (comportement macroscopique) des modèles de réseaux discrets (entrées microscopiques). Dans la plupart des cas (i.e. dans le champ des valeurs du paramètre du modèle qui peuvent jouer le rôle de la température), les limites de ces modèles discrets deviennent déterministes (dans l'esprit de la loi des grands nombres) et, dans certains cas critiques (c'est à dire pour la température critique), les limites peuvent rester aléatoires, ce qui les rend particulièrement intéressantes.

L'exemple le plus simple est celui de la marche aléatoire simple qui se comporte (lorsqu'elle est correctement renormalisée) comme le mouvement brownien dans sa limite d'échelle. Notons que le mouvement brownien dans un certain sens est plus universel que la marche aléatoire, puisque c'est la limite d'échelle de n'importe quelle marche aléatoire simple.

Dans les modèles planaires (en dimension deux) discrets, certaines courbes apparaissent naturellement comme interfaces entre des phases, lignes de niveau de surfaces aléatoires etc. En fait, ces courbes permettent souvent de décrire la configuration aléatoire tout entière. Le mouvement brownien ne suffit pas en général à décrire la complexité de ces interfaces dans la limite d'échelle, lorsqu'elles sont aléatoires, et il a été prédit par les physiciens théoriques – et depuis prouvé dans plusieurs cas – que ces courbes devraient être conformément invariantes dans cette limite d'échelle continue.

Les processus SLE (Stochastic Loewner Evolution) d'Oded Schramm ont conduit les mathématiciens et les physiciens à une compréhension nouvelle et approfondie des limites d'échelle des modèles discrets en dimension deux. Oded Schramm a réalisé que le codage de Loewner des courbes planes par itérations de transformations conformes s'adaptait exactement à la propriété spatiale de Markov qui correspond au fait que l'on peut explorer les interfaces progressivement et décrire la loi conditionnelle des configurations restantes. Un SLE chordal est une courbe ne pouvant se croiser elle même dans un domaine simplement connexe, joignant deux points fixés du bord du domaine. Et c'est la seule famille à un paramètre (souvent indexée par un nombre réel positif κ) de courbes planes aléatoires qui satisfasse l'invariance conforme et la propriété spatiale de Markov. Il est maintenant prouvé que les processus SLE sont les limites d'échelle de nombreux modèles discrets: SLE_2 est la limite de la marche aléatoire à boucles effacées [LSW04]; SLE_3 est la limite de l'interface du modèle d'Ising critique [CS12, $CDCH^+12$]; SLE_4 est la limite de la ligne de niveau du champ libre Gaussien discret [SS09]; $SLE_{16/3}$ est la limite de l'interface du modèle FK critique [CS12, $CDCH^+12$]; SLE_6 est la limite de l'interface du modèle de la percolation critique [CN07]; SLE_8 est la limite de la courbe d'exploration d'un arbre couvrant uniforme [LSW04]. L'ensemble CLE (Conformal Loop Ensemble) est l'objet limite géométrique qui apparaît lorsque l'on considère la limite d'échelle du modèle discret tout entier (à la différence d'une seule interface, pour le processus SLE). Un CLE [She09, SW12] peut être considéré comme un ensemble dénombrable aléatoire de boucles simples disjointes dans le disque unité qui ne sont pas emboîtées. C'est la seule famille à un paramètre qui satisfait l'invariance conforme et la propriété spatiale de Markov (version de configuration des boucles). Il est prouvé (ou presque prouvé) que le CLE_3 est la limite de la collection des interfaces dans le modèle d'Ising critique; le CLE_4 est l'ensemble des lignes de niveau du champ libre Gaussien [MS13a], et le CLE_6 est la limite de la collection des interfaces dans la percolation critique [CN06]. Comme l'on peut s'y attendre, chaque boucle de CLE est une boucle de type SLE, avec le même paramètre κ.

Le champ libre Gaussien est l'analogue naturel du mouvement brownien ayant un temps deux-dimensionnel [She07], et il a été largement utilisé comme outil fondateur de la théorie quantique des champs. Tout comme le mouvement brownien, c'est un objet aléatoire simple qui a de très nombreuses applications et qui est d'une grande beauté intrinsèque. La géométrie du champ libre Gaussien, c'est à dire le fait qu'il soit possible de décrire des lignes géométriques dans cette distribution très irrégulière, a été découvert récemment [SS12, SS09, MS13a, Dub09b, She11], et a donné plusieurs développements récents. Le champ libre Gaussien correspond également à la limite d'échelle de modèles discrets simples (par exemple la fonction de hauteur des modèles de dimères [Ken08]).

Les SLEs, CLEs et le champ libre Gaussien sont trois importantes structures planaires aléatoires reliées et cette thèse va explorer les aspects de ces trois objets et des relations existant entre eux.

De l'ensemble CLE au processus SLE

Le chapitre 2 (article en collaboration avec Wendelin Werner [WW13a]) présente un moyen de construire les processus $SLE_\kappa(\rho)$ utilisant un CLE et un échantillon de restriction indépendant: Définissons de façon indépendante, dans un domaine simplement connexe D avec deux points du bord fixés a et b, les deux objets aléatoires suivants: Un CLE_κ ($\kappa \in (8/3,4]$) que nous appelons Γ et une courbe de restriction chordale γ de a à b avec l'exposant de restriction α. Ensuite, nous construisons l'ensemble obtenu en joignant à γ toutes les boucles de Γ qu'elle croise. Enfin, nous prenons la frontière la plus à droite de cet ensemble. Elle constitue encore une courbe simple de a

à b dans D que nous appelons η (voir Figure 1.1.3). Nous montrons dans le chapitre 2 que cette courbe η est un processus $\text{SLE}_\kappa(\rho)$ avec

$$\alpha = \frac{(\rho+2)(\rho+6-\kappa)}{4\kappa}.$$

De cette construction des SLEs, nous déduisons une conséquence immédiate intéressante qui fournit une preuve alternative simple d'un résultat non trivial ([Zha08b, Zha10b, MS12b]): la réversibilité des processus $\text{SLE}_\kappa(\rho)$ avec $\kappa \in [8/3,4], \rho > -2$. Une autre conséquence est l'estimée à deux points de l'intersection de $\text{SLE}_\kappa(\rho)$ avec la frontière qui à son tour donne une preuve simple de la dimension de Hausdorff de cette intersection.

De SLE à CLE

Les ensembles CLE ont d'abord été définis par Scott Sheffield dans [She09] via des variantes branchantes des courbes SLE. En effet, dans [She09], il fournit plusieurs constructions de ces arbres d'exploration des SLE et il conjecture qu'ils correspondent tous à la même collection aléatoire de boucles. Ceci a été ensuite prouvé dans le cas des arbres d'exploration symétriques dans [SW12]. Le premier résultat principal du chapitre 3 (article en collaboration avec Wendelin Werner [WW13b]) peut être résumé comme suit: Pour chaque $\kappa \in (8/3,4]$, toutes les collections aléatoires des SLE_κ de type quasi-boucles construites par les arbres d'exploration asymétriques de Sheffield dans [She09] ont la même loi. Ils correspondent tous à des CLE_κ, familles de boucles construites dans [SW12].

Mécanisme de croissance invariant conforme de CLE_4

Le deuxième résultat principal du chapitre 3 souligne une propriété spécifique de l'ensemble CLE_4 (notons qu'il correspond au CLE qui est le plus directement relié au champ libre Gaussien [SS12, SS09, MS13a, Dub09b, She11]). Dans ce cas particulier, il est possible de définir un mécanisme de croissance de boucles invariant conforme et non enraciné. Grosso modo, le processus de croissance qui découvre progressivement les boucles se forme "uniformément" sur la frontière et ne nécessite pas de choisir une racine. Le fait qu'un tel mécanisme de croissance non-local invariant conforme existe est assez surprenant (et le fait que ses paramétrisations temporelles vues à partir de différents points coïncident exactement l'est encore plus).

Grâce à ce mécanisme de croissance de l'ensemble CLE_4, nous pouvons associer d'une manière invariante conforme un paramètre de temps positif u_L à chaque boucle L dans la configuration des boucles $\mathscr{L}$, qui correspond au temps dans le mécanisme de croissance pour lequel la boucle a été découverte. Nous définissons un processus de Markov sur les domaines $(D_u, u \geq 0)$: à l'instant $u = 0$, c'est le demi-plan supérieur; et à un instant $u > 0$, c'est le domaine qui reste lorsque l'on retire toutes les boucles ayant un paramètre de temps inférieur à u. De ce mécanisme de croissance invariant conforme, on déduit que $(D_u, u \geq 0)$ et $(\Phi(D_u), u \geq 0)$ sont identiquement distribués pour toute transformation conforme Φ de $\mathbb{H}$ sur lui-même.

Couplage entre champ libre Gaussien et CLE_4 avec paramètre de temps

Notons que les courbes SLE_4 sont les lignes de niveau du champ libre Gaussien [SS09, SS12, Dub09b]. Et l'ensemble CLE_4 est la collection des lignes de niveau du champ libre Gaussien [MS13a], dans le sens où l'on peut coupler un champ libre Gaussien h avec un CLE_4 $\mathscr{L}$ de telle sorte que les boucles de $\mathscr{L}$ sont des lignes de niveau de h de hauteur $\pm\lambda$ ($\lambda = \pi/2$) et les signes de la valeur moyenne de h à l'intérieur de la boucle sont donnés par des variables de Bernoulli i.i.d indépendantes du CLE_4. Dans ce couplage, on peut effectivement prouver que la configuration des boucles du CLE_4 est une fonction déterministe de h.

Dans le chapitre 4, nous fournissons un deuxième nouveau couplage entre le champ libre Gaussien et l'ensemble CLE_4, en utilisant le paramètre de temps défini dans le chapitre 3. Plus précisément, nous couplons un champ libre Gaussien h (ayant une valeur zéro au bord) avec un CLE_4 (avec paramètre de temps) $((L,u_L), L \in \mathscr{L})$ en sorte que, pour chaque boucle L, il est la ligne de niveau de h et la valeur moyenne de h à l'intérieur de L est $2\lambda - 2\lambda u_L$. Nous montrons en plus que dans ce couplage la configuration des boucles de CLE_4 et le paramètre de temps sont des fonctions déterministes de h.

Du champ libre Gaussien aux processus SLE et aux intersections des courbes SLE

Les résultats principaux du chapitre 5 sont la détermination de la dimension de Hausdorff des ensembles de points doubles et des ensembles de points de coupure des courbes SLE. Soient $\kappa' \in (4,8)$ et γ' un processus $SLE_{\kappa'}$. L'ensemble de points de coupure de γ' est défini par $\mathscr{K} = \{\gamma'(t) : t \in (0,\infty), \gamma'(0,t) \cap \gamma'(t,\infty) = \emptyset\}$. Nous montrons que, presque sûrement,

$$\dim_H(\mathscr{K}) = 3 - \frac{3\kappa'}{8}.$$

Soit $\mathscr{D}$ l'ensemble de points doubles de γ'. Nous montrons que, presque sûrement,

$$\dim_H(\mathscr{D}) = 2 - \frac{(12-\kappa')(4+\kappa')}{8\kappa'}.$$

Restriction conforme: le cas radial

Le dernier chapitre est consacré à l'étude de l'analogue radial de l'échantillon de restriction chordale [LSW03]. Plus précisément, considérons le disque unité $\mathbb{U}$ et fixons un point du bord 1 et un point à l'intérieur 0. Un échantillon de restriction radiale est un sous-ensemble aléatoire fermé K de $\overline{\mathbb{U}}$ tel que:

- K est connexe, $\mathbb{C} \setminus K$ est connexe, $K \cap \partial\mathbb{U} = \{1\}$, $0 \in K$.
- Pour chaque sous-ensemble fermé A de $\overline{\mathbb{U}}$ tel que $A = \overline{\mathbb{U} \cap A}$, $\mathbb{U} \setminus A$ est simplement connexe, contient l'origine et le point 1 est sur sa frontière, la loi de $\Phi_A(K)$ conditionnée sur $(K \cap A = \emptyset)$ est égale à la loi de K où Φ_A est la transformation conforme de $\mathbb{U} \setminus A$ à $\mathbb{U}$ qui préserve 1 et l'origine (Figure 1.1.6).

Nous montrons dans le chapitre 6 que la mesure de restriction radiale est caractérisée par le couple de deux nombres réels (α, β) tel que

$$\mathbb{P}(K \cap A = \emptyset) = |\Phi_A'(0)|^{\alpha} \Phi_A'(1)^{\beta}$$

où A est un sous-ensemble fermé de $\overline{\mathbb{U}}$ tel que $A = \overline{\mathbb{U} \cap A}$, $\mathbb{U} \setminus A$ est simplement connexe, contient l'origine et 1 sur sa frontière, et Φ_A est la transformation conforme de $\mathbb{U} \setminus A$ à $\mathbb{U}$ qui préserve 0 et 1. La mesure de restriction radiale correspondante est notée $\mathbb{P}(\alpha, \beta)$. En plus, nous montrons que $\mathbb{P}(\alpha, \beta)$ existe si et seulement si

$$\beta \geq \frac{5}{8}, \quad \alpha \leq \xi(\beta) = \frac{1}{48}\left((\sqrt{24\beta+1}-1)^2 - 4\right).$$

Chapter 2

From CLE to SLE

The results in this chapter are contained in [WW13a].

2.1 Introduction

The goal of the present paper is to derive ways to construct samples of (chordal) SLE curves (or the related $\text{SLE}_\kappa(\rho)$ curves) out of the sample of a Conformal Loop Ensemble (CLE), using additional Brownian paths (or so-called restriction measure samples). In order to properly state a first version of our result, we need to briefly informally recall the definition of these three objects: SLE, CLE and the restriction measures.

- Recall that a chordal SLE (for Schramm-Loewner Evolution) in a simply connected domain D is a random curve that is joining two prescribed boundary points a and b of D. These curves have been first defined by Oded Schramm in 1999 [Sch00], who conjectured (and this conjecture was since then proved in several important cases) that they should be the scaling limit of particular random curves in two-dimensional critical statistical physics models when the mesh of the lattice goes to 0. More precisely, one has typically to consider the statistical physics model in a discrete lattice-approximation of D, with well-chosen boundary conditions, where (lattice-approximations of) the points a and b play a special role. When $\kappa \le 4$, these SLE_κ curves are random simple continuous curves that join a to b with fractal dimension is $1+\kappa/8$ (see for instance [Law05] and the references therein).
- CLEs (for Conformal Loop Ensembles) are closely related objects. A CLE is a random family of loops that is defined in a simply connected domain D. In the present paper, we will only discuss the CLEs that consist of simple loops. There are various equivalent definitions and constructions of these simple CLEs – see for instance the discussion in [SW12]. More precisely, one CLE sample is a collection of countably many disjoint simple loops in D, and it is conjectured to correspond to the scaling limit of the collection of all discrete (but macroscopic) interfaces in the corresponding lattice model from statistical physics. Here, the boundary conditions are "uniform" and involve no special marked points on the boundary of D (as opposed to the definition of chordal SLE that requires to choose the boundary points a and b). It is proved in [SW12] that there is exactly a one-dimensional family of simple CLEs, that is indexed by $\kappa \in (8/3, 4]$. Then, in a CLE_κ sample, the loops all locally look like SLE_κ type curves (and have fractal dimension $1+\kappa/8$). Note also that, even if any two loops are disjoint in CLE_κ sample, the Lebesgue measure of the set of points that are surrounded

by no loop is almost surely 0. This is therefore a random Cantor-like set, sometimes called the CLE carpet (its fractal dimension is actually proved in [SSW09, NW11] to be equal to $1+(2/\kappa)+3\kappa/32 \in [15/8,2))$. In the present paper, we will only discuss the CLEs for $\kappa \leq 4$, that consist of simple disjoint loops (there exists other CLEs for $\kappa \in (4,8]$).

- When a and b are two boundary points of a simply connected domain D as before, it is possible to define random simple curves from a to b that possess a certain "one-sided restriction" property, that is defined and discussed in [LSW03]. There is in fact a one-dimensional family of such random curves, that is parametrized by its restriction exponent, which can take any positive real value α. All these random restriction curves can be viewed as boundaries of certain Brownian-type paths (or like $\mathrm{SLE}_{8/3}$ curves). In particular, they all almost surely have a Hausdorff dimension that is equal to $4/3$.

Let us now state the main result that we prove in the present paper: Define independently, in a simply connected domain D with two marked boundary points a and b, the following two random objects: A CLE_κ (for some $\kappa \in (8/3,4]$) that we call Γ and a one-sided restriction path γ from a to b, with restriction exponent α. Finally, we define the set obtained by attaching to γ all the loops of Γ that it intersects. Then, we define the right-most boundary of this set. This turns out to be again a simple curve from a to b in D that we call η (see Figure 2.1.1). Note that in order to construct η, it is enough to know γ and the outermost loops of Γ.

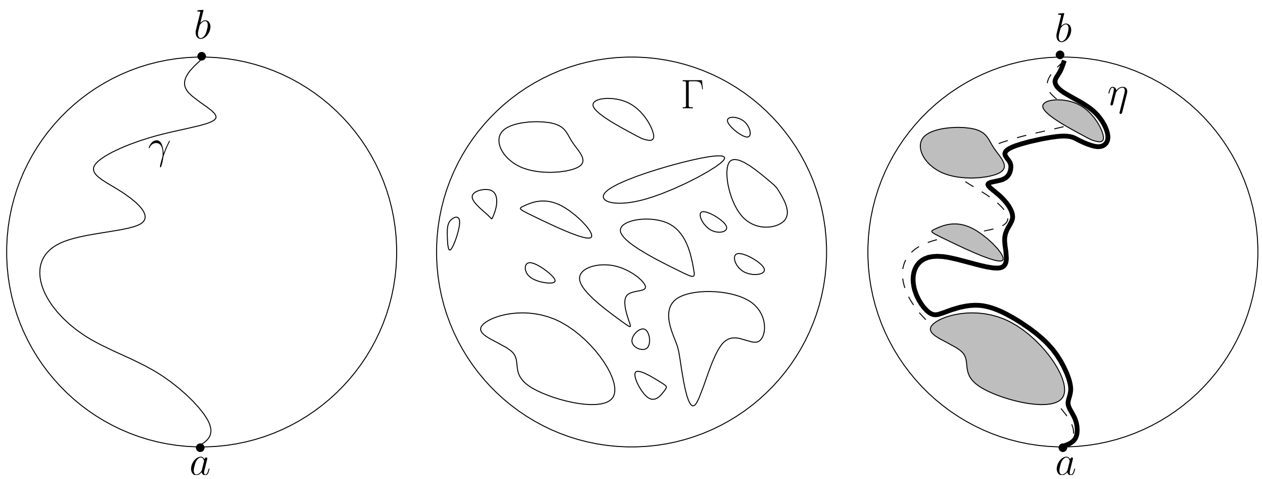


Figure 2.1.1: Construction of η out of γ and Γ.

Theorem 2.1.1. *When $\kappa \in (8/3,4]$ and $\alpha = (6-\kappa)/(2\kappa)$, then η is a chordal SLE_κ from a to b in D.*

In fact, for a given κ, the other choices of $\alpha > 0$ give rise to variants of SLE_κ, the so-called $\mathrm{SLE}_\kappa(\rho)$ curves, where ρ is related to κ and α by the relation $\alpha = (\rho+2)(\rho+6-\kappa)/(4\kappa)$. We will state this generalization of Theorem 2.1.1 in the next subsection, after having properly introduced these $\mathrm{SLE}_\kappa(\rho)$ processes.

To illustrate Theorem 2.1.1, let us give the following example for $\kappa = 3$, which corresponds to the scaling limit of the critical Ising model (see [CS12, CDCH$^+$12]). Consider a CLE_3 Γ in D which is the (soon-to-be proved) scaling limit of the collection of outermost critical Ising model

"− cluster" boundaries, when one considers the model with uniformly "+ boundary conditions". On the other hand, consider now the scaling limit of the critical Ising model with mixed boundary conditions, + between a and b (anti-clockwise) and − between b and a. This model defines loops as before, as well as the additional ± interface η joining a and b, which turns out to be a SLE_3 path (see [CDCH+12]). Now, our result shows that in order to construct a sample of η, one possibility is to take the right boundary of the union of a restriction measure with exponent $1/2$ together with all the loops in Γ that it intersects. It gives a way to see the "effect" of changing the boundary conditions (note that there are natural ways to couple the discrete Ising model with mixed boundary conditions to the model with uniform boundary conditions, it would be interesting to compare them with this coupling in the scaling limit).

We would like to make a few comments:

1. It is proved in [SW12] that CLEs can be constructed as outer boundaries of clusters of Poissonian clouds of Brownian loops in D (the "Brownian loop-soups" introduced in [LW04]) with intensity $c(\kappa)$. Hence, together with the construction of the restriction measure via clouds of Brownian excursions or reflected Brownian motions, this provides a "completely Brownian" construction of all these chordal SLE_κ curves and their $\mathrm{SLE}_\kappa(\rho)$ variants. This result was in fact announced in [Wer03], so that – combined with [SW12] – the present paper eventually completes the proof of that (not so recent) research announcement.

2. This Brownian construction of $\mathrm{SLE}_\kappa(\rho)$ paths turn out to be particularly useful and handy, when one has to derive "second moment estimates" for these SLE curves. We will illustrate this in the final section of the present paper by giving a short self-contained derivation of the Hausdorff dimension of the intersection of $\mathrm{SLE}_\kappa(\rho)$ (in the upper half-plane) with the real line.

3. A direct by-product of this construction of these chordal SLE_κ curves and their variants is that they are "reversible" simple paths (for instance, the SLE from a to b in D is a simple path has the same law as the SLE from b to a modulo reparametrization – in the case of $\mathrm{SLE}_\kappa(\rho)$ the statement is also clear, but the reversed $\mathrm{SLE}_\kappa(\rho)$ is then pushed/attracted from its right). This provides an alternative proof to the reversibility of these $\mathrm{SLE}_\kappa(\rho)$ curves that has been obtained thanks to their relation with the Gaussian Free Field in [MS12b] (see also [Zha08b, Zha10b, Dub09a] for earlier proofs of this result in the case $\rho = 0$ and then when the $\mathrm{SLE}_\kappa(\rho)$ curves do not hit the boundary of the domain i.e. when $\rho \geq (\kappa - 4)/2$). Note however that our approach does not yield any result for $\kappa \notin [8/3, 4]$.

4. The construction of the restriction measure via Poisson point processes of Brownian excursions, as explained in [Wer05], together with that of the CLE's via loop-soups, make it possible to define simultaneously in a fairly natural and "ordered way" (see the comments after the statement of Theorem 2.2.1), on a single probability space, all these $\mathrm{SLE}_\kappa(\rho)$'s in D from a to b, for all boundary points a and b, and for all $\kappa \in (8/3, 4]$ and all $\rho > -2$. This is of course reminiscent of the definitions of $\mathrm{SLE}_\kappa(\rho)$ processes within a Gaussian Free Field [MS12a]. It is interesting to see the similarities and differences between these two constructions.

2.2 Preliminaries

In this section, we will recall in a little more detail some definitions, notations and facts, and point to appropriate references for background. We then state our main result, Theorem 2.2.1 and make a couple of remarks.

2.2.1 Conformal restriction property

We first recall the definition and the basic properties of the paths satisfying conformal restriction (almost all the results that we shall describe have been derived in [LSW03], a survey as well as the construction of restriction samples from Brownian excursions can be found in [Wer05]).

Here and throughout the paper, we denote the upper half of the complex plane $\mathbb{C}$ by $\mathbb{H} := \{x+iy : x \in \mathbb{R}, y > 0\}$. Let $\mathscr{A}$ be the set of all bounded closed $A \subset \overline{\mathbb{H}}$ such that $\mathbb{R}_- \cap A = \emptyset$ and $H_A := \mathbb{H} \setminus A$ is simply connected.

For $A \in \mathscr{A}$, we define Φ_A to be the unique conformal map from H_A onto $\mathbb{H}$ such that $\Phi_A(z) \sim z$ as $z \to \infty$ and such that $\Phi_A(0) = 0$ (the fact that Φ_A can be extended analytically to a neighborhood of 0 follows easily from the Schwarz reflection principle).

We say that a random curve γ from 0 to infinity in $\overline{\mathbb{H}}$ does satisfy one-sided conformal restriction (to the right), if for any A, the law of $\Phi_A(\gamma)$ conditionally on $\gamma \cap A = \emptyset$ is in fact identical to the law of γ itself (see Figure 2.2.1).

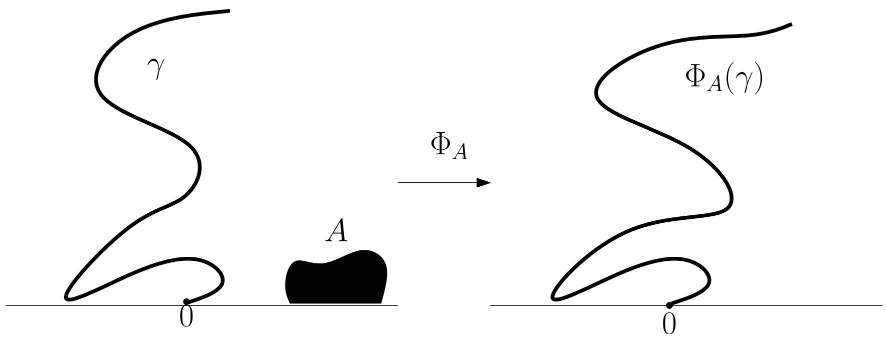


Figure 2.2.1: The law of $\Phi_A(\gamma)$ conditionally on $\gamma \cap A = \emptyset$ has the same law as γ itself.

It turns out that if this is the case, then there exists some non-negative α such that for all $A \in \mathscr{A}$,

$$P(\gamma \cap A = \emptyset) = \Phi_A'(0)^\alpha. \tag{2.2.1}$$

Conversely, for all non-negative α, there exists exactly one distribution for γ that fulfils (2.2.1) for all $A \in \mathscr{A}$. We call γ an one-sided restriction sample of exponent α. There exist several equivalent constructions of γ:

- As the right boundary of a certain Brownian motion from 0 to ∞, reflected on $(-\infty, 0]$ with a certain reflection angle $\theta(\alpha)$ and conditioned not to intersect $[0, \infty)$, see [LSW03].
- As the right boundary of a Poissonian cloud of Brownian excursions from $(-\infty, 0]$ in $\mathbb{H}$ (so it is the right boundary of the countable union of Brownian paths that start and end on the negative half-line, see [Wer05]). Note that if the Poissonian cloud of Brownian excursions has intensity α times the (appropriately normalized) Brownian excursion measure, then the right boundary of the union of all these excursions is sampled like the one-sided conformal restriction sample of exponent α.
- As an $\mathrm{SLE}_{8/3}(\rho)$ curve for some $\rho > -2$ (these processes will be defined in the next section), see [LSW03] for the relation between α and ρ. Note that this approach enables to show that γ does hit the negative half-line if and only if $\alpha < 1/3$.

We can note that the limiting case $\alpha = 0$ corresponds to the case where γ is the negative half-line, whereas the case $\alpha = 5/8$ corresponds to $\rho = 0$ i.e. to the $\mathrm{SLE}_{8/3}$ curve itself, which is left-right symmetric. Furthermore, the second construction shows immediately that for $\alpha < \alpha'$, it is possible to couple the corresponding restriction curves in such a way that γ' stays "to the right" of γ (with obvious notation). In other words, the larger α is, the more the restriction sample is "repelled" from the negative half-line.

In fact, we will be only using the second description in the present paper (and we will actually recall in Section 2.2.4 why this indeed constructs a random simple curve γ).

2.2.2 $\mathrm{SLE}_\kappa(\rho)$ process

The $\mathrm{SLE}_\kappa(\rho)$ processes are natural variants of SLE_κ processes that have been first introduced in [LSW03]. Recall first the definition of SLE_κ. Suppose $(W_t, t \geq 0)$ is a real-valued continuous function. For each $z \in \overline{\mathbb{H}}$, define $g_t(z)$ as the solution to the chordal Loewner ODE:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z. \tag{2.2.2}$$

We set $W_t = \sqrt{\kappa} B_t$ where $(B_t, t \geq 0)$ is a standard Brownian motion and $\kappa \leq 4$, then SLE_κ is the continuous simple random curve η in $\mathbb{H}$ from 0 to ∞ that, for each $t > 0$, g_t is the conformal map from $\mathbb{H} \setminus \eta[0,t]$ onto $\mathbb{H}$ with normalization $g_t(z) \sim z + o(1)$ when $z \to \infty$ (for the existence and uniqueness of such a continuous curve, see for instance [Law05]). Note that η is parametrized by its half-plane capacity (i.e. for any t, the conformal map g_t from $\mathbb{H} \setminus \eta[0,t]$ onto $\mathbb{H}$ in fact satisfies $g_t(z) - z \sim 2t/z$ as $z \to \infty$).

SLE_κ curves possess the following properties:

- The law of η is scale-invariant: For any positive λ, the traces of η and of $\lambda\eta$ have the same law.
- Let us suppose that η is parametrized by its half-plane capacity. For any positive time t, the distribution of $g_t(\eta[t,\infty)) - g_t(\eta_t)$ is identical to the distribution of η itself.

In fact, the SLE_κ curves are the only random curves with this property, which is what led Oded Schramm to this definition of these curves via Loewner differential equation driven by Brownian motion (see [Sch00]).

There exist variants of the SLE_κ curves that involve additional marked boundary points, and that are called the $\mathrm{SLE}_\kappa(\rho_1, \ldots, \rho_L)$ processes. Let us now describe the $\mathrm{SLE}_\kappa(\rho)$ processes that involve exactly one additional marked boundary point (see [LSW03, Dub05]). Consider g_t as the conformal maps generated by Loewner evolution (2.2.2) with W_t replaced by the solution to the system of SDEs:

$$dW_t = \sqrt{\kappa} dB_t + \frac{\rho}{W_t - O_t} dt, W_0 = 0; \quad dO_t = \frac{2}{O_t - W_t} dt, O_0 = x. \tag{2.2.3}$$

When $\kappa \leq 4, \rho > -2$, $\mathrm{SLE}_\kappa(\rho)$ in $\overline{\mathbb{H}}$ from 0 to ∞ with force point x is the increasing family of compact set (K_t) such that for each t, g_t is the conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with normalization $g_t(z) \sim z$ as $z \to \infty$. As we shall see, it turns out that these compact sets are almost surely a simple curve η, in other words $K_t = \eta[0,t]$ for each t. Note that when $\rho = 0$, the $\mathrm{SLE}_\kappa(\rho)$ is just the ordinary chordal SLE_κ (and the force point plays no role). When $\rho > 0$, the force point should be thought of as "repelling" while it is "attracting" when $\rho \in (-2, 0)$.

It turns out that these $\text{SLE}_\kappa(\rho)$ can also be characterized by a couple of properties. Let us now state the characterization that will be handy for our purposes: Suppose that the following four properties hold:

- η is a random simple curve from 0 to ∞ in $\overline{\mathbb{H}}$.
- The law of η is scale-invariant: For any positive λ, the traces of $\lambda\eta$ and η are identically distributed.
- $\eta \cap (0,\infty) = \emptyset$ and the Lebesgue measure of $\eta \cap (-\infty, 0]$ is almost surely equal to 0. Mind however that it is possible (and it will happen in a number of cases) that η hits the negative half-line.
- Suppose that η is parametrized by half-plane capacity as before. For any positive time t, define H_t as the unbounded connected component of $\mathbb{H} \setminus \eta[0,t]$ (if η intersects the negative half-line, it happens that $H_t \neq \mathbb{H} \setminus \eta[0,t]$) and o_t as the left-most point of the intersection $\eta[0,t] \cap \mathbb{R}_-$. Let f_t be the unique conformal map from H_t onto $\mathbb{H}$ that sends the triplet (o_t, η_t, ∞) onto $(0,1,\infty)$. Then, the distribution of $f_t(\eta[t,\infty))$ is independent of t (and of $\eta[0,t]$) (see Figure 2.2.2).

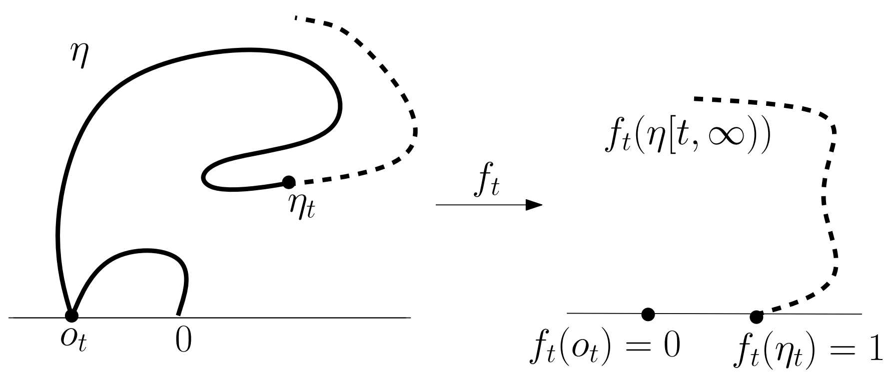


Figure 2.2.2: $f_t(\eta[t,\infty))$ is independent of $\eta[0,t]$.

Then, η is necessarily a $\text{SLE}_\kappa(\rho)$ for some $\kappa \in (0,4]$ and $\rho > -2$ (mind that the fact that this $\text{SLE}_\kappa(\rho)$ is almost surely a simple curve is then part of the conclusion; in fact in the present paper, we will never use the *a priori* fact that the $\text{SLE}_\kappa(\rho)$ processes are continuous simple paths).

This is very easy to see, using the Loewner chain description of the random simple curve η. If one parametrizes the curve η by its half-plane capacity (which is possible because its capacity is increasing continuously – this is due to the third property) and defines the usual conformal map g_t from H_t onto $\mathbb{H}$ normalized by $g_t(z) = z + o(1)$ near infinity, then one can define

$$W_t = g_t(\eta_t), O_t = g_t(o_t).$$

One observes that $X_t := W_t - O_t$ is a Markov process with the Brownian scaling property i.e., a multiple of a Bessel process. More precisely, one can first note that the first two items imply that for any given $t_0 > 0$, $\eta_{t_0} \notin (-\infty, 0)$ and therefore $u := X_{t_0} \neq 0$. The final property then implies readily that the law of $((X_{t_0+tu^2} - u)/u, t \geq 0)$ is independent of $(X_t, t \leq t_0)$. From this, it follows that at least up to the first time after t_0 at which X hits the origin, it does behave like a Bessel process. Then, one can notice that X is instantaneously reflecting away from 0 because the Lebesgue measure of the set of times at which it is at the origin is almost surely equal to 0. Hence, one gets that X is the multiple of some reflected Bessel process of positive dimension (see [RY94] for background on Bessel processes). From this, one can then recover the process $t \mapsto O_t$ (because of the Loewner equation $dO_t = 2dt/(O_t - W_t)$ when $X_t \neq 0$) and finally $t \mapsto W_t$. In particular, we get that

$$dW_t = \sqrt{\kappa} dB_t + \frac{\rho}{W_t - O_t} dt$$

for some $\rho > -2$ and $\kappa \le 4$ (the fact that $\rho > -2$ is a consequence of the fact that the dimension of the Bessel process X is positive; $\kappa \le 4$ is due to the fact that η does not hit the positive half-line). This characterizes the law of η, which is the same as $\mathrm{SLE}_\kappa(\rho)$.

Actually, it is possible to remove some items from this characterization of $\mathrm{SLE}_\kappa(\rho)$ curves; the first three items are slightly redundant, but since we do get these properties for free in our setting, the present presentation will be sufficient for our purposes (see for instance [SW05, MS12b] for a more general characterization).

Note that the $\mathrm{SLE}_\kappa(\rho)$ processes touch the negative half-line if and only if $\rho < (\kappa/2) - 2$ (as this corresponds to the fact that the Bessel process $(W_t - O_t)/\sqrt{\kappa}$ has dimension smaller than 2).

Let us point out that it is possible to make sense also of $\mathrm{SLE}_\kappa(\rho)$ processes for some values of $\rho \le -2$ by introducing either a symmetrization or a compensation procedure (see [Dub05, She09, WW13b]), some of which are very closely related to CLEs as well, but we will not discuss such generalized $\mathrm{SLE}_\kappa(\rho)$'s in the present paper.

2.2.3 Simple CLEs

Let us now briefly recall some features of the Conformal Loop Ensembles for $\kappa \in (8/3, 4]$ – we refer to [SW12] for details (and the proofs) of these statements. A CLE is a collection Γ of non-nested disjoint simple loops $(\gamma_j, j \in J)$ in $\mathbb{H}$ that possesses a particular conformal restriction property. In fact, this property that we will now recall, does characterize these simple CLEs:

- For any Möbius transformation Φ of $\mathbb{H}$ onto itself, the laws of Γ and $\Phi(\Gamma)$ are the same. This makes it possible to define, for any simply connected domain D (that is not the entire plane – and can therefore be viewed as the conformal image of $\mathbb{H}$ via some map $\tilde{\Phi}$), the law of the CLE in D as the distribution of $\tilde{\Phi}(\Gamma)$ (because this distribution does then not depend on the actual choice of conformal map $\tilde{\Phi}$ from $\mathbb{H}$ onto D).

- For any simply connected domain $H \subset \mathbb{H}$, define the set $\tilde{H} = \tilde{H}(H, \Gamma)$ obtained by removing from H all the loops (and their interiors) of Γ that do not entirely lie in H. Then, conditionally on $\tilde{H}$, and for each connected component U of $\tilde{H}$, the law of those loops of Γ that do stay in U is exactly that of a CLE in U.

It turns out that the loops in a given CLE are SLE_κ type loops for some value of $\kappa \in (8/3, 4]$ (and they look locally like SLE_κ curves). In fact for each such value of κ, there exists exactly one CLE distribution that has SLE_κ type loops. As explained in [SW12], a construction of these particular families of loops can be given in terms of outermost boundaries of clusters of the Brownian loops in a Brownian loop-soup with subcritical intensity c (and each value of c corresponds to a value of κ).

2.2.4 Main Statement

We can now state our main Theorem, that generalizes Theorem 2.1.1: Suppose that $\kappa \in (8/3, 4]$ is fixed (and it will remain fixed throughout the rest of the paper) and consider a CLE_κ in the upper half-plane. Independently, sample a restriction curve γ from 0 to ∞ in $\mathbb{H}$ with positive exponent α, and define η out of the CLE and γ just as in Theorem 2.1.1. Let $\tilde{\rho} := \tilde{\rho}(\kappa, \alpha)$ denote the unique real in $(-2, \infty)$ such that

$$\alpha = \frac{(\tilde{\rho}+2)(\tilde{\rho}+6-\kappa)}{4\kappa}$$

(we will use this notation throughout the paper). Then:

Theorem 2.2.1. *The curve η is a random simple curve which is an $\text{SLE}_\kappa(\tilde{\rho})$.*

Note that for a fixed $\kappa \in (8/3, 4]$, the function $\alpha \mapsto \tilde{\rho}$ is indeed an increasing bijection from $(0, \infty)$ onto $(-2, \infty)$. The limiting case $\rho = -2$ in fact can be interpreted as corresponding to the case where both γ and η are the negative half-line. Similarly, in the limiting case $\kappa = 8/3$, where the CLE is in fact empty, then Theorem 2.2.1 corresponds to the description of γ itself as an $\text{SLE}_{8/3}(\rho)$ curve.

Note that this construction shows that it is possible to couple an $\text{SLE}_\kappa(\rho)$ with an $\text{SLE}_{\kappa'}(\rho')$ in such a way that the former is almost surely "to the left" of the latter, when $8/3 < \kappa \le \kappa' \le 4$ and ρ and ρ' are chosen in such a way that

$$(\rho+2)(\frac{\rho+6}{\kappa}-1) \le (\rho'+2)(\frac{\rho'+6}{\kappa'}-1).$$

For example, an $\text{SLE}_\kappa(\rho)$ can be chosen to be to the left of an $\text{SLE}_\kappa(\rho')$ for $\rho \le \rho'$. Or an SLE_3 can be coupled to an $\text{SLE}_4(2\sqrt{2}-2)$ in such a way that it remains almost surely to its left. Such facts are seemingly difficult to derive directly from the Loewner equation definitions of these paths.

Similarly, it also shows that it is possible to couple an $\text{SLE}_\kappa(\rho)$ from 0 to ∞ with another $\text{SLE}_\kappa(\rho)$ from 1 to ∞, in such a way that the latter stays to the "right" of the former.

Let us recall that the definition of $\text{SLE}_\kappa(\rho)$ processes can be generalized to more than one marked boundary point. For instance, if one considers $x_1 < \ldots < x_n \le 0 \le x'_1 < x'_2 < \ldots < x'_l$, it is possible to define a $\text{SLE}_\kappa(\rho_1, \ldots, \rho_n; \rho'_1, \ldots, \rho'_l)$ from 0 to infinity in $\mathbb{H}$, with marked boundary points $x_1, \ldots, x'_l$ with corresponding weights. Several of these processes have also an interpretation in terms of conditioned $\text{SLE}_\kappa(\rho)$ processes (where the conditioning involves non-intersection with additional restriction samples) – see [Wer04a], so that they can also be interpreted via a CLE and restriction measures.

Let us now immediately explain why η is necessarily almost surely a continuous curve from 0 to ∞ in $\overline{\mathbb{H}}$. Let us first map all items (the CLE loops and the restriction sample) onto the unit disc, via the Moebius map Φ that maps 0, i and ∞ respectively onto -1, 0 and 1, and write $\tilde{\Gamma} = \Phi(\Gamma)$, $\tilde{\eta} = \Phi(\eta)$ and $\tilde{\gamma} = \Phi(\gamma)$.

Let us note that $\tilde{\gamma}$ is almost surely a continuous curve from -1 to 1 in the closed unit disc. One simple way to check this (but other justifications are possible) is to use the construction of $\tilde{\gamma}$ as the bottom boundary of the union of countably many excursions away from the top half-circle. More precisely, for each excursion e in this Poisson point process, one can define the loop $l(e)$ obtained by adding to this excursion the arc of the top half-circle that joins the endpoints of e. Then, one can construct a continuous path λ from -1 to 1 by moving from -1 to 1 on this top arc, and attaching all these loops $l(e)$ in the order in which one meets them (once one meets a loop, one travels around the loop before continuing at the point where the loop was encountered). As almost surely, for any $\varepsilon > 0$, there are only finitely many loops $l(e)$ of diameter greater than ε, there is a way to parametrize λ as a continuous function from $[0,1]$ into the closed disk. We then complete λ into a loop by adding the bottom half-circle. Then, we can interpret $\tilde{\gamma}$ as part of the boundary of a connected component of the complement of a continuous loop in the plane: It is therefore necessarily a continuous curve and it is easy to check that it is self-avoiding (because the Brownian excursions have no double cut-points).

We have detailed the previous argument, because it can be repeated in almost identical terms to explain why $\tilde{\eta}$ is a simple curve. Let us first recall from [SW12] that $\tilde{\Gamma}$ consists of a countable family of disjoint simple loops such that for any $\varepsilon > 0$, there exist only finitely many loops of diameter greater than ε. We now move along $\tilde{\gamma}$ and attach the loops of $\tilde{\Gamma}$ that it encounters, in their order of appearance (once one meets the loop, one travels around the loop before continuing). By an appropriate time-change, we can ensure that the obtained path that joins -1 to 1 in the closed disk is a continuous curve from $[0,1]$ into the closed unit disk. Then, just as above, we complete this curve into a loop by adding the bottom half-circle, and note that $\tilde{\eta}$ is a continuous curve from -1 to 1. It is then easy to conclude that it is self-avoiding, because almost surely, $\tilde{\gamma}$ does never hit a loop of $\tilde{\Gamma}$ at just one single point (this is due to the Markov property of Brownian motion: If one samples first the CLE and then the Brownian excursions that are used to construct γ, almost surely, a Brownian excursion will actually enter the inside of each individual loop of Γ that it hits).

2.3 Identification of ρ

The proof of Theorem 2.2.1 consists of the following two steps.

Lemma 2.3.1. *The random simple curve η is an $SLE_\kappa(\rho)$ curve for some $\rho > -2$.*

Lemma 2.3.2. *If η is an $SLE_\kappa(\rho)$ for some $\rho > -2$, then necessarily $\rho = \tilde{\rho}(\kappa, \alpha)$.*

The proof of Lemma 2.3.1 will be achieved in the next section by proving that it satisfies all the properties that characterize these curves (and that we have recalled in the previous subsection), which is the most demanding part of the paper. In the present section, we will prove Lemma 2.3.2. These ideas were already very briefly sketched in [Wer03].

Let us build on the loop-soup cluster construction of the CLE_κ as established in [SW12]. We therefore consider a Poisson point process of Brownian loops (as defined in [LW04]) in the upper-half plane with intensity $c(\kappa) \in (0,1]$ with

$$c(\kappa) = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

Then, we construct the CLE_κ as the collection of all outermost boundaries of clusters of Brownian loops (here, we say that two loops l, l' in the loop-soup are in the same cluster of loops if one find a finite chain of loops $l_0, ..., l_n$ in the loop-soup such that $l_0 = l, l_n = l$, and $l_j \cap l_{j-1} \neq \emptyset$ for $j \in \{1, ..., n\}$), as explained in [SW12].

We also sample the restriction sample γ with exponent α, via a Poisson point process of Brownian excursions attached to $\mathbb{R}_-$, as explained in [Wer05].

Suppose now that $A \in \mathscr{A}$, and define $H = H_A$ to be the unbounded connected component of $\mathbb{H} \setminus A$ as before. By definition of $\mathscr{A}$, the negative half-line still belongs to ∂H_A. If we restrict the loop-soup and the Poisson point process of Brownian excursions to those that stay in H_A, the restriction properties of the corresponding intensity measures imply immediately that one gets a sample of the Brownian loop-soup with intensity c in H_A, and a sample of the Poisson point process of Brownian excursions away from the negative half-line in H_A, with intensity α. In particular, because of the conformal invariance of these two underlying measures, it follows that these Poissonian samples have the same law as the image under Φ_A^{-1} of the original loop and excursion soups in $\mathbb{H}$.

Let us now first sample these items in H_A, and let η_A be the right-most boundary of a set defined in the same way as η but from the samples in H_A instead of in $\mathbb{H}$. Then, we sample those excursions and loops that do not stay in H_A, and we construct η itself. One can note that either $\eta \not\subset H_A$ or $\eta = \eta_A$. Indeed, the only way in which η can be different than η_A is because of these additional loops/excursions, that do force η to get out of H_A. Hence, the event $\eta \subset H_A$ holds if and only if on the one hand the curve γ stays in H_A (recall that this happens with probability $\Phi_A'(0)^\alpha$), and on the other hand, no loop in the loop-soup does intersect both η_A and A (see Figure 2.3.1). Let $P_{\mathbb{H}}$ and P_{H_A} be the laws of the processes η and η_A respectively. It follows immediately that for any $A \in \mathscr{A}$,

$$\frac{dP_{\mathbb{H}}}{dP_{H_A}}(\eta)1_{\eta\cap A=\emptyset} = \Phi_A'(0)^\alpha \exp(-cL(\mathbb{H};A,\eta))1_{\eta\cap A=\emptyset}$$

where $L(\mathbb{H};A,\eta)$ denotes the mass (according to the Brownian loop-measure in $\mathbb{H}$) of the set of loops that intersect both A and η.

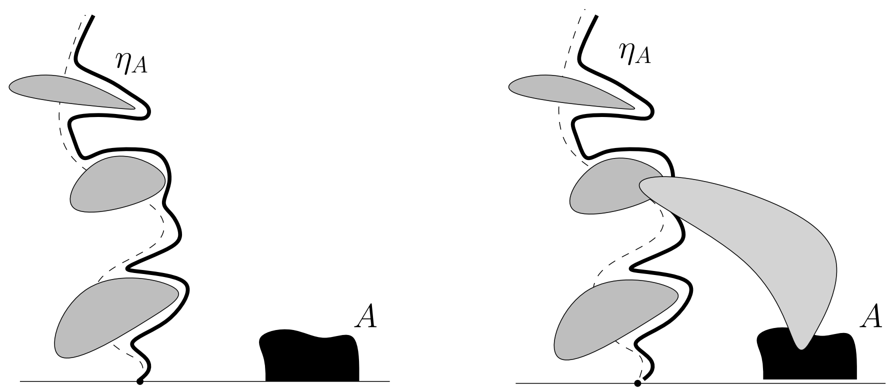


Figure 2.3.1: $\eta = \eta_A$ if and only if there is no loop in Γ that intersects η_A and A.

Equivalently,

$$\frac{dP_{H_A}}{dP_{\mathbb{H}}}(\eta)1_{\eta\cap A=\emptyset} = 1_{\eta\cap A=\emptyset}\Phi_A'(0)^{-\alpha}\exp(cL(\mathbb{H};A,\eta)). \tag{2.3.1}$$

Note that this implies that

$$E_{\mathbb{H}}\left(1_{\eta\cap A=\emptyset}\exp(cL(\mathbb{H};A,\eta))\right) = E_{H_A}(1_{\eta\cap A=\emptyset}\Phi_A'(0)^\alpha) = \Phi_A'(0)^\alpha \tag{2.3.2}$$

(and the present argument in fact shows that the expectation in the left-hand side is actually finite).

We now wish to compare (2.3.1) with features of $\mathrm{SLE}_\kappa(\rho)$ processes. Let us now suppose that the curve η is an $\mathrm{SLE}_{\bar\kappa}(\bar\rho)$ process for some $\bar\kappa \le 4$ and $\bar\rho > -2$. We keep the same notations as in Section 2.2.2. For $A \in \mathscr{A}$, let T be the (possibly infinite) first time at which η hits A. For $t < T$, write $h_t := \Phi_{g_t(A)}$. Then (see [Dub05, Lemma 1]), an Itô formula calculation shows that

$$M_t = h_t'(W_t)^{a_1} h_t'(O_t)^{a_2}\left(\frac{h_t(W_t)-h_t(O_t)}{W_t-O_t}\right)^{a_3}\exp(\bar cL(\mathbb{H};A,\eta[0,t]))$$

is a local martingale (for $t < T$) where $a_1 = (6-\bar\kappa)/(2\bar\kappa)$, $a_2 = \bar\rho(\bar\rho+4-\bar\kappa)/(4\bar\kappa)$, $a_3 = \bar\rho/\bar\kappa$ and and $\bar c = c(\bar\kappa) = (3\bar\kappa-8)(6-\bar\kappa)/(2\bar\kappa)$ (note that such martingale calculations have been used on several occasions in related contexts, see e.g. [Dub09a] and the references therein).

It can be furthermore noted that $M_0 = \Phi_A'(0)^{\bar\alpha}$ (and more generally, at those times when $O_t = W_t$, one puts $M_t = h_t'(W_t)^{\bar\alpha}\exp(\bar c L(\mathbb{H};A,\eta[0,t]))$, where

$$\bar\alpha = \alpha(\bar\kappa,\bar\rho) = a_1+a_2+a_3 = (\bar\rho+2)(\bar\rho+6-\bar\kappa)/(4\bar\kappa).$$

One has to be a little bit careful, because (as opposed to the case where $\bar\kappa < 8/3$), M_t is not bounded on $t < T$, so that we do not know if the local martingale stopped at T is uniformly integrable (indeed the term involving $L(\mathbb{H};A,\eta[0,t])$ actually does blow up when $t \to T-$ and $T < \infty$). However, even if some of the numbers a_2 and a_3 may be negative, one always has (see [Dub05], the proof of Lemma 2-(i))

$$0 \le h_t'(W_t)^{a_1} h_t'(O_t)^{a_2}\left(\frac{h_t(W_t)-h_t(O_t)}{W_t-O_t}\right)^{a_3} \le 1.$$

Furthermore (see again [Dub05]), when $\eta\cap A = \emptyset$, then when $t\to\infty$, then M_t converges to

$$M_\infty := \exp(\bar c L(\mathbb{H};A,\eta))$$

because each of the first three factors in the definition of M_t converge to 1.

Note also that $dM_t = M_t K_t \sqrt{\bar\kappa}\, dB_t$ where

$$K_t = a_1\frac{h_t''(W_t)}{h_t'(W_t)} + a_3\frac{h_t'(W_t)}{h_t(W_t)-h_t(O_t)} - a_3\frac{1}{W_t-O_t}.$$

Let T_n denote the first (possibly infinite) time that the distance between the curve and A reaches $1/n$. Then, for a fixed A, we see that $(M_{t\wedge T_n}, t\ge 0)$ is uniformly bounded by a finite constant. Hence, if $Q_{\mathbb{H}}$ is the probability measure under which W is the driving process of the SLE$_{\bar\kappa}(\bar\rho)$ η in $\mathbb{H}$, we can define the probability measure Q_n^* by $dQ_n^*/dQ_{\mathbb{H}} = M_{T_n}/M_0$. Under Q_n^*, we have

$$dB_t = dB_t^* + K_t dt, \quad dh_t(W_t) = \sqrt{\bar\kappa}h_t'(W_t)dB_t^* + \frac{\bar\rho}{h_t(W_t)-h_t(O_t)}h_t'(W_t)^2 dt.$$

This implies that Q_n^* is the law of a (time-changed) SLE$_{\bar\kappa}(\bar\rho)$ in H_A up to the time T_n, which happens to be the (possibly infinite) first time at which this curve gets to distance $1/n$ of A.

We can now note that by definition, the sequences Q_n^* are compatible in n, so that there exists a probability measure Q^* such that, under Q^*, and for each n, the curve, up to time T_n, is an SLE$_{\bar\kappa}(\bar\rho)$ in H_A up to the first time it is at distance $1/n$ of A. But we also know that an SLE$_{\bar\kappa}(\bar\rho)$ in H_A almost surely does not hit A. Hence, Q^* is just the law of SLE$_{\bar\kappa}(\bar\rho)$ in H_A.

By the definition of Q^*, we have that, for any n,

$$\frac{dQ^*}{dQ_{\mathbb{H}}}(\eta)1_{d(\eta,A)\ge 1/n} = \frac{M_{T_n}}{M_0}1_{d(\eta,A)\ge 1/n} = \frac{M_\infty}{M_0}1_{d(\eta,A)\ge 1/n}.$$

Hence, we finally see that

$$\frac{dQ^*}{dQ_{\mathbb{H}}}(\eta)1_{d(\eta,A)>0} = \frac{M_\infty}{M_0}1_{d(\eta,A)>0} = \Phi_A'(0)^{-\bar\alpha}\exp(\bar c L(\mathbb{H};A,\eta))1_{\eta\cap A=\emptyset}.$$

Comparing this with (2.3.1), we conclude that $\bar\kappa = \kappa$ and that $\bar\rho = \tilde\rho(\kappa,\alpha)$.

Note that a by-product of this proof (keeping in mind that (2.3.2) holds) is that in fact the stopped martingale $M_{t\wedge T}$ is indeed uniformly integrable: It is a positive martingale such that

$$E(M_T) = E(\lim_{t\to\infty} M_{t\wedge T}) \ge E(M_\infty 1_{T=\infty}) = \Phi_A'(0)^\alpha = E(M_0).$$

2.4 Proof of Lemma 2.3.1

We now describe the steps of the proof of Lemma 2.3.1. Quite a number of these steps are almost identical to ideas developed in [SW12]. We will therefore not always provide all details of those parts of the proof. Let us first note that the law of η is obviously scale-invariant, and that we already have seen that it is almost surely a simple curve. Furthermore, we know (for instance using the construction of γ via a Poisson point process of Brownian excursions, or via its $\mathrm{SLE}_{8/3}(\rho)$ description), that almost surely, the Lebesgue measure of $\gamma \cap (-\infty, 0)$ is zero. By construction (since $\eta \cap (-\infty, 0)$ is a subset of this set), the Lebesgue measure of $\eta \cap (-\infty, 0)$ is also 0. Hence, in order to prove the lemma, it only remains to check the "conformal Markov" property i.e. the last item in the characterization of $\mathrm{SLE}_\kappa(\rho)$ processes derived in Section 2.2.2.

2.4.1 Straight exploration and the pinned path

A first idea will be not to focus only on the curve η, but to also keep track of the CLE loops that lie to its right. In other words, we will consider half-plane configurations (η, Λ), where – as before – η is a curve in $\overline{\mathbb{H}}$ from 0 to ∞ that does not touch $(0, \infty)$ and Λ is a loop configuration in the connected component of $\mathbb{H} \setminus \eta$ that has $(0, \infty)$ on its boundary (we say that it is the connected component to the right of η). The conformal restriction property of the CLE shows that the following two constructions are equivalent:

- Construct η as in the statement (via a CLE Γ and a restriction path γ), and consider Λ to be the collection of loops in the CLE Γ (that one used to construct η) that lie to the right of η.
- First sample η, and then in the connected component H_η of $\mathbb{H} \setminus \eta$ that lies to the right of η, sample an independent CLE that we call Λ.

It turns out that the couple (η, Λ) does satisfy a simple "restriction-type" property, that one can sum up as follows: For a given $A \in \mathscr{A}$, let us condition on the event $\{\eta \cap A = \emptyset\}$. Then, one can define the collection $\tilde{\Lambda}_A$ of loops of Λ that intersect A, and the unbounded connected component $\tilde{H}_A$ of $\mathbb{H} \setminus (A \cup \tilde{\Lambda}_A)$. We also denote by Λ_A to be the collection of loops of Λ that stay in $\tilde{H}_A$. Let $\Psi = \Psi(\tilde{\Lambda}_A, A)$ denote the conformal map from $\tilde{H}_A$ onto $\mathbb{H}$ with $\Psi(0) = 0$ and $\Psi(z) \sim z$ when $z \to \infty$. Then, the conditional law of $(\Psi(\eta), \Psi(\Lambda_A))$ (conditionally on $\eta \cap A = \emptyset$) is identical to the original law of (η, Λ). This is a direct consequence of the construction of (η, Λ) and the restriction properties of γ and Γ.

This restriction property is of course reminiscent of the restriction property of CLEs themselves. In [SW12], the restriction property of CLE was exploited as follows: Fix one point in $\mathbb{H}$ (say the point i) and discover all loops of the CLE that lie on the segment $[0, i]$ (by moving upwards on this segment) until one discovers the loop that surrounds i (see Figure 2.4.1). This can be approximated by iterating discrete small cuts, discovering the loops that intersect these cuts and repeating the procedure. The outcome was a description of the law of the loop that surrounds i at the "moment" at which one discovers it (see Proposition 4.1 in [SW12]).

Here, we use the very same idea, except that the goal is to cut in the domain until one reaches the curve η (note that in the CLE case, the marked point i is an interior point of $\mathbb{H}$ and that here, the marked points 0 and ∞ on the boundary do also correspond to the choice of two degrees of freedom in the conformal map). We can for instance do this by moving upwards on the vertical half-line $L := 1 + i\mathbb{R}_+$; a simple 0-1 law argument shows that almost surely, the curve γ does intersect L, and that therefore $\eta \cap L \neq \emptyset$ too. Let η_T denote the point of $\eta \cap L$ with smallest y-coordinate. One way

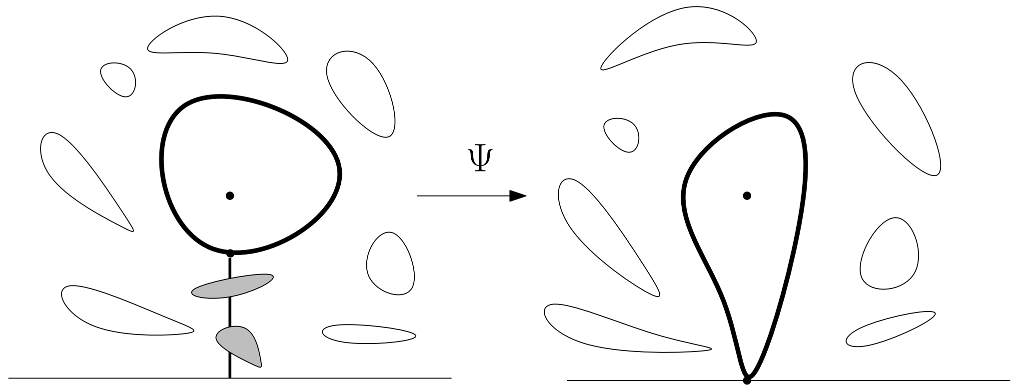


Figure 2.4.1: Discovering the loop that surrounds *i* in a CLE defines a pinned loop (see [SW12])

to find it, is to move on L upwards until one meets η for the first time. This can be approximated also by "exploration steps", in a way that is almost identical to the explorations of CLEs described in [SW12]. We refer to that paper for rather lengthy details, the arguments really just mimic those to that paper. The conclusion, analogous to Proposition 4.1 in [SW12] is that (see Figure 2.4.2):

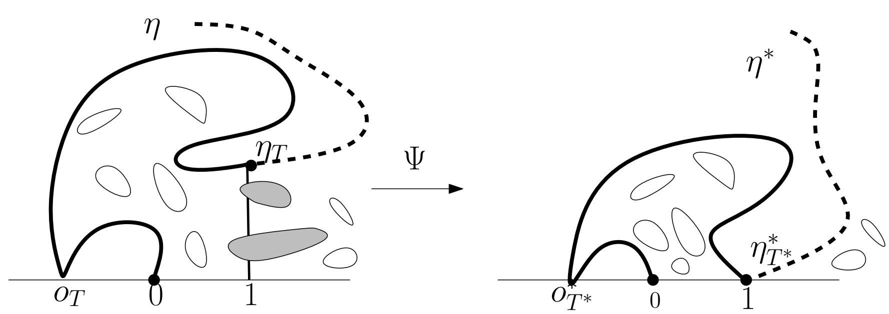


Figure 2.4.2: Discovering η in half-plane configuration defines a pinned path

Lemma 2.4.1. *The conditional law of η conditionally on the event that η passes through the ε-neighborhood of* 1*, converges as $\varepsilon \to 0$ to the distribution of $\eta^* := \Psi(\eta)$, where Ψ is the conformal map from $\tilde{H}_{[1,\eta_T]}$ onto $\mathbb{H}$ that maps the triplet $(0, \eta_T, \infty)$ onto $(0, 1, \infty)$.*

We will call η^* a "pinned" path, as in [SW12]. Note that this construction also shows that η^* is independent of Ψ.

2.4.2 Restriction property for the pinned path

When η^* is such a pinned path, then $\mathbb{H} \setminus \eta^*$ has several connected components, and we call U_0 the connected component with $(0,1)$ on its boundary and U_+ the one with $(1,\infty)$ on its boundary (see Figure 2.4.3). If one first samples η^* and then in U_0 and U_+ samples two independent CLE$_\kappa$'s , then one gets a "pinned configuration" (η^*, Λ^*).

This pinned configuration inherits the following restriction property from (η, Λ): Suppose that $A \in \mathscr{A}$ with $d(1,A) > 0$, and condition on $A \cap \eta^* = \emptyset$. Then, define H^*_A for (η^*, Λ^*) just as $\tilde{H}_A$ in the case of (η, Λ). Note that 0 and 1 are both boundary points of H^*_A so that it is possible to define the conformal transformation Φ^*_A from H^*_A onto $\mathbb{H}$ that fixes the three boundary points 0, 1 and ∞.

Then, the conditional law of $\Phi_A^*(\eta^*)$ (conditionally on the event that $\eta^* \cap A = \emptyset$) is equal to the initial (unconditioned) law of η^* itself. This result just follows by passing to the limit the restriction property of (η, Λ).

Let us define T^* the time at which $\eta^*_{T^*} = 1$, and $o^*_{T^*}$ as the leftmost point in $\eta^*[0,T^*] \cap \mathbb{R}_-$ (note that depending on the value of ρ, it may be the case that $o^*_{T^*} = 0$). Denote by φ^* the conformal map from the unbounded connected component of $\mathbb{H} \setminus \eta^*[0,T^*]$ onto $\mathbb{H}$, that maps the triplet $(o^*_{T^*}, 1, \infty)$ onto $(0,1,\infty)$ (see Figure 2.4.3). One can therefore note that φ^* is therefore a deterministic function of $\eta^*[0,T^*]$.

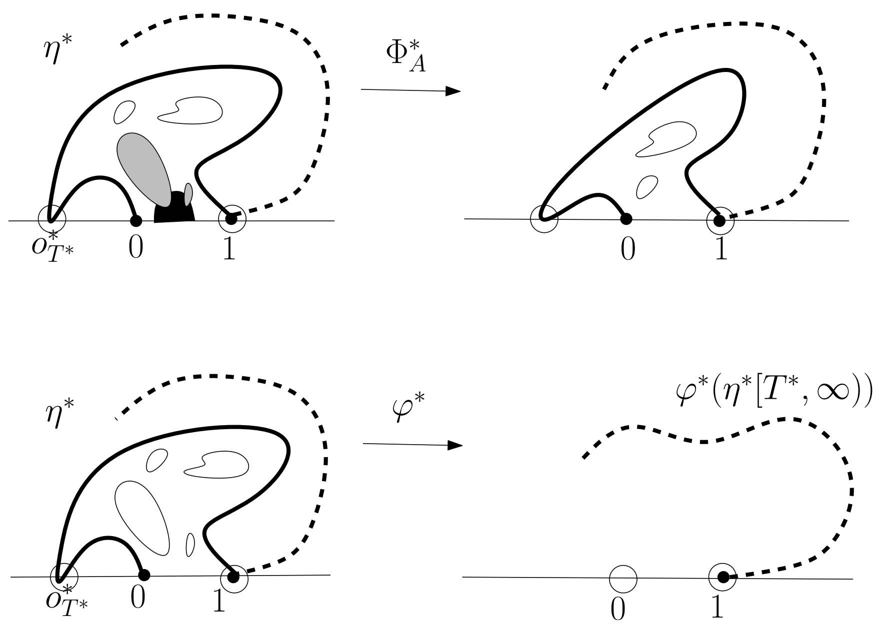


Figure 2.4.3: Definitions of Φ_A^* and φ^*

Let us now consider a set $A \in \mathscr{A}$ that is also at positive distance from $[1,\infty)$, i.e. that is attached to the segment $[0,1]$ (we call $\mathscr{A}_{[0,1]}$ this set of closed subsets of the plane). Then, the following restriction property will be inherited from the restriction property of (η^*, Λ^*):

Lemma 2.4.2. *The curve $\varphi^*(\eta^*[T^*,\infty))$ is independent of the event $\eta^*[0,T^*] \cap A = \emptyset$.*

Proof. Suppose that the event $\eta^*[0,T^*] \cap A = \emptyset$ holds (which is the same as $\eta^* \cap A = \emptyset$). Recall that the conditional distribution of $\Phi_A^*(\eta^*)$ is equal to the original (unconditioned) distribution of η^*.

Let us now define G the measurable transformation that allows to construct $\varphi^*(\eta^*[T^*,\infty))$ from the path η^* (as in the bottom line of Figure 2.4.3). When $\eta^*[0,T^*] \cap A = \emptyset$ holds, then we see that the same transformation G applied to $\Phi_A^*(\eta^*)$ (i.e. to the top right path in the figure) gives also $\varphi^*(\eta^*[T^*,\infty))$ i.e. that $G(\eta^*) = G(\Phi_A^*(\eta^*))$. Hence, the conditional distribution of $\varphi^*(\eta^*[T^*,\infty))$ given $\eta^*[0,T^*] \cap A = \emptyset$ is equal to the unconditional distribution of $\varphi^*(\eta^*[T^*,\infty))$, which proves the lemma. □

A direct consequence of the lemma is therefore that $\eta^*[0,T^*]$ and $\varphi^*(\eta^*[T^*,\infty))$ are independent. Indeed, the σ-field generated by the family of events of the type $\eta^*[0,T^*] \cap A = \emptyset$ when $A \in \mathscr{A}_{[0,1]}$ (which is stable by finite intersections) is exactly $\sigma(\eta^*[0,T^*])$.

2.4.3 General explorations and consequences

The rest of the proof mimics ideas from [SW12] that we now briefly describe.

In fact, just as in [SW12], it is easy to see that the argument that leads to Lemma 2.4.1 can be generalized to other curves than the straight line L. In particular, we choose L to be any oriented simple curve on the grid $\delta(\mathbb{Z}\times\mathbb{N})$ that starts on the positive half-line and disconnects 0 from infinity in $\mathbb{H}$, then define η_T to be the point of η that L meets first, and let $\tilde{L}$ denote the part of L until it hits η_T. If we parametrize L continously in some prescribed way, then $\eta_T = L_\sigma$ for some σ and $\tilde{L} = L[0,\sigma)$. We then define $\tilde{H}$ as the unbounded connected component of the set obtained by removing from $\mathbb{H}\setminus\tilde{L}$ all the loops of Λ that intersect $\tilde{L}$. Let Ψ denote the conformal map from $\tilde{H}$ onto $\mathbb{H}$ that sends the triplet $(0,\eta_T,\infty)$ onto $(0,1,\infty)$. Let $\hat{H}$ be the unbounded connected component of the set obtained by removing from $\mathbb{H}$ the union of $\eta[0,T],\tilde{L}$ and the loops in Λ that intersect $\tilde{L}$. Let $\hat{\Psi}_{\tilde{L}}$ denote the conformal map from $\hat{H}$ onto $\mathbb{H}$ that sends the triplet (o_T,η_T,∞) onto $(0,1,\infty)$ (see Figure 2.4.4).

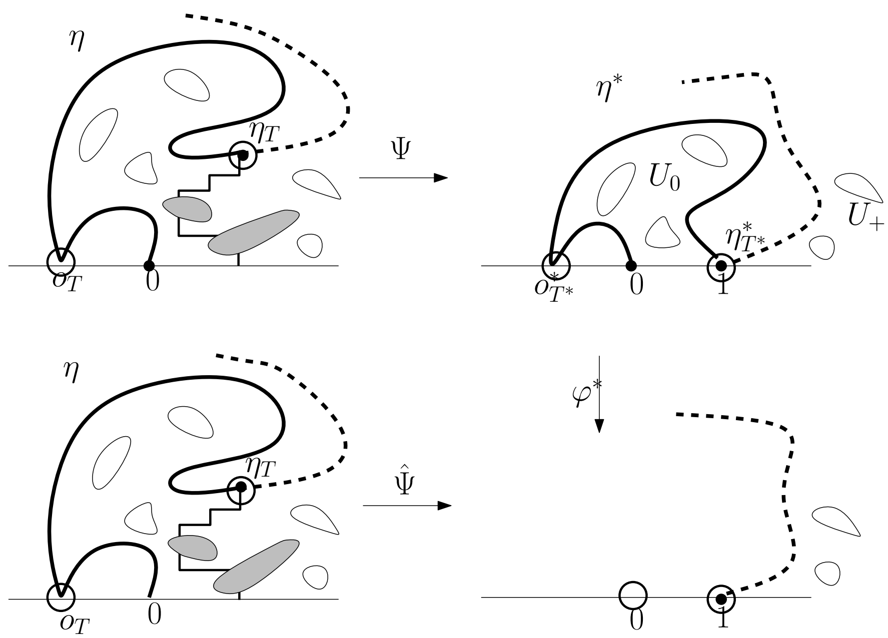


Figure 2.4.4: Ψ, φ^* and $\hat{\Psi} = \varphi^* \circ \Psi$.

Then the same arguments than the ones used to derive Lemma 2.4.1 imply that $\Psi(\eta)$ has the same law as pinned path η^*, and that it is independent from Ψ. From Lemma 2.4.2, using the fact that $\hat{\Psi}_{\tilde{L}}(\eta[T,\infty))$ can be viewed as the deterministic function G applied to $\Psi(\eta)$, we know that $\hat{\Psi}_{\tilde{L}}(\eta[T,\infty))$ is independent of $\Psi(\eta[0,T])$. Combining these two observations, we conclude that $\hat{\Psi}_{\tilde{L}}(\eta[T,\infty))$ is independent of $\eta[0,T]$.

Furthermore, it is also possible to condition on the position of L_σ. The previous results still hold when one considers the probability measure conditioned by $\sigma\in(s_1,s_2)$.

The next step of the proof is again almost identical to the corresponding one in [SW12]: Fix a time T and suppose that $\eta_T\notin\mathbb{R}$. Consider δ_n as a deterministic sequence converging to zero. Let β^n be an approximation of $\eta[0,T]$ from the right on the lattice $\delta_n(\mathbb{Z}\times\mathbb{N})$ such that the last edge is the only edge of β^n that crosses the curve η (see Figure 2.4.5). Here, one should view β^n as a deterministic given function of $\eta[0,T]$ (and there are a number of possibilities to choose such an approximation β^n). Let T^n be the first time that η hits β^n (note that of course, η_{T^n} is on the last edge of β^n).

Let us now consider a given deterministic linear path L such that the probability that $\beta^n = \tilde{L}$ is positive. For this event to happen, one in particularly requires that the curve η intersects $\tilde{L}$ only on its last edge (this corresponds to a conditioning of the type $\sigma \in (s_1, s_2]$). Furthermore, if this holds, in order to check whether $\beta^n = \tilde{L}$ or not, it is possible to define β^n in such a way that it can be read of from $\eta[0,T]$.

Hence, we can deduce from our previous considerations that conditionally on $\{\beta^n = \tilde{L}\}$, the path $\hat{\Psi}_{\tilde{L}}(\eta[T^n,\infty))$ is independent of $\eta[0,T^n]$. But for any given deterministic piecewise linear path L, on the event $\{\beta^n = \tilde{L}\}$, the probability that $\tilde{L}$ intersects some macroscopic loop in Λ is very small when n is large enough, so that $\hat{\Psi}_{\tilde{L}}(\eta[T^n,\infty))$ is very close to $f_T(\eta[T,\infty))$ on this event (recall that f_T is the conformal map from the unbounded connected component of $\mathbb{H}\setminus\eta[0,T]$ onto $\mathbb{H}$ that sends the triplet (o_T, η_T, ∞) onto $(0,1,\infty)$). Hence, by passing to the limit (as $n\to\infty$, possibly taking a subsequence), we conclude that $f_T(\eta[T,\infty))$ is independent of $\eta[0,T]$ as desired. This is exactly the conformal Markov property that was needed to conclude the proof of Lemma 2.3.1.

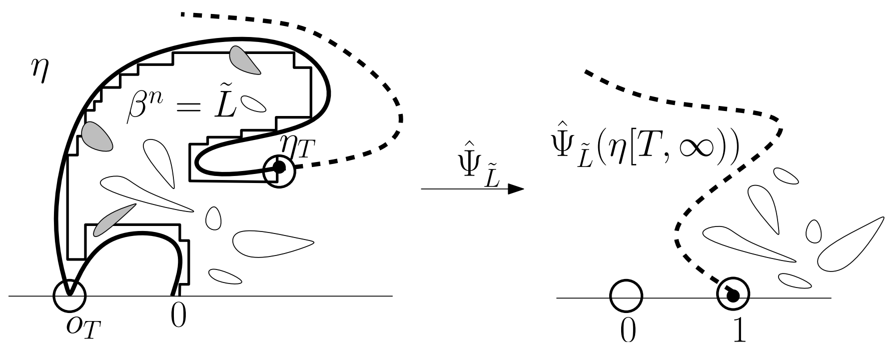


Figure 2.4.5: $\hat{\Psi}_{\tilde{L}}$ maps the triplet $(o_{T^n}, \eta_{T^n}, \infty)$ onto $(0,1,\infty)$.

2.5 Consequences for second-moment estimates

In order to illustrate how the present construction can be used in order to derive directly some properties of SLE$_\kappa(\rho)$ processes, we are going to derive in this section some information about the intersection of SLE$_\kappa(\rho)$ processes and the real line. Analogous ideas have been used in [NW11] to study the dimension of the CLE gasket, but the situation here is even more convenient.

Recall that from the definition, we know that the SLE$_\kappa(\rho)$ process η, from 0 to ∞ in $\mathbb{H}$ does not touch the positive half-line, but – as we already mentioned –, its definition via the Loewner equation and Bessel processes shows that it touches almost surely the negative half-line as soon as $\rho < (\kappa/2) - 2$. For instance, for $\kappa = 4$, this will happen for $\rho \in (-2,0)$, while for $\kappa = 3$, this will occur for $\rho \in (-2,-1/2)$. Here for obvious reasons, we will restrict ourselves to the case where $\kappa \in (8/3,4]$.

Proposition 2.5.1. *For $\kappa \in (8/3,4]$ and $\rho \in (-2,-2+\kappa/2)$, then the Hausdorff dimension of $\eta \cap \mathbb{R}_-$ is almost surely equal to $1-(\rho+2)(\rho+4-\kappa/2)/\kappa$.*

Note that this result is also derived in [MW13] for all $\kappa \in (0,8)$ and $\rho \in (-2,-2+(\kappa/2))$ using the properties of flow lines of GFF introduced in [MS12a].

Before turning our attention to the proof of this result, let us first focus on the following related question: Let us fix $c \in (0,1)$ and $\alpha > 0$. Consider on the one hand a Brownian loop-soup with intensity c in the upper half-plane, and its corresponding CLE_κ sample consisting of the outermost boundaries of the loop-soup clusters, as in [SW12].

On the other hand, consider a Poisson point process $(b_j, j \in J)$ of Brownian excursions away from the real line in $\mathbb{H}$, with intensity α. Each of these excursions b_j has a starting point S_j and an endpoint E_j that both lie on the real axis.

For each point x on the real line, for each $\varepsilon < r$, we define the semi-ring

$$A_x(\varepsilon, r) := \{z \in \overline{\mathbb{H}} \; : \; \varepsilon < |z - x| < r\}.$$

For each given ε and r, we can artificially restrict ourselves to those Brownian loops and excursions that stay in $A_x(\varepsilon, r)$. We define the event $E_x(\varepsilon, r)$ that the union of all these paths does not disconnect x from infinity in $\mathbb{H}$ (see Figure 2.5.1).

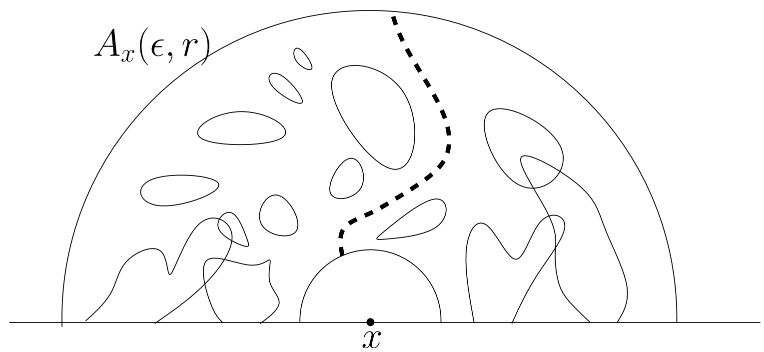


Figure 2.5.1: Event $E_x(\varepsilon, r)$: x is not disconnected from ∞ by the excursions and loops.

Clearly, the probability of $E_x(\varepsilon, r)$ is in fact a function of ε/r and does not depend on x. Let us denote this probability by $p(\varepsilon/r)$. It is elementary to see that for all $\varepsilon, \varepsilon' < 1$,

$$p(\varepsilon\varepsilon') \leq p(\varepsilon)p(\varepsilon').$$

Indeed, if one divides $A_0(\varepsilon\varepsilon', 1)$ into the two semi-annuli $A_0(\varepsilon\varepsilon', \varepsilon)$ and $A_0(\varepsilon, 1)$, one notices that

$$E_0(\varepsilon\varepsilon', 1) \subset E_0(\varepsilon\varepsilon', \varepsilon) \cap E_0(\varepsilon, 1)$$

and the latter two events are independent, due to their Poissonian definition.

On the other hand, for some universal constant C, we know that for all $\varepsilon, \varepsilon' < 1/4$,

$$p(8\varepsilon\varepsilon') \geq C p(\varepsilon)p(\varepsilon'). \tag{2.5.1}$$

Indeed, let us consider the following three events:

- U_1: No CLE loop touches both $\{z \; : \; |z| = 2\}$ and $\{z \; : \; |z| = 4\}$
- U_2: No Brownian excursion touches both $\{z \; : \; |z| = 1\}$ and $\{z \; : \; |z| = 2\}$.
- U_3: No Brownian excursion touches both $\{z \; : \; |z| = 4\}$ and $\{z \; : \; |z| = 8\}$.

All the events U_1, U_2, U_3, $E_0(8\varepsilon, 8)$ and $E_0(1, 1/\varepsilon')$ are decreasing events of the Poisson point processes of loops and excursions (i.e. if an event fails to be true, then adding an extra excursion or loop will not fix it). Hence, they are positively correlated. Furthermore, we have chosen these events in such a way that

$$\big(U_1 \cap U_2 \cap U_3 \cap E_0(8\varepsilon, 8) \cap E_0(1, 1/\varepsilon')\big) \subset E_0(8\varepsilon, 1/\varepsilon').$$

The fact that $c \leq 1$ ensures that the events U_1, U_2 and U_3 have a positive probability. Putting the pieces together, we get that

$$p(8\varepsilon\varepsilon') = P(E_0(8\varepsilon, 1/\varepsilon')) \geq P(U_1 \cap U_2 \cap U_3) p(\varepsilon) p(\varepsilon')$$

from which (2.5.1) follows. Hence, if we define $q(\varepsilon) := Cp(8\varepsilon)$, we get $q(\varepsilon\varepsilon') \geq q(\varepsilon)q(\varepsilon')$.

These properties of $p(\varepsilon)$ and $q(\varepsilon)$ ensure that there exists a positive finite β and a constant C' such that for all $\varepsilon < 1/8$,

$$\varepsilon^{\beta} \leq p(\varepsilon) \leq C'\varepsilon^{\beta}.$$

Let us now focus on the proof of the proposition. First, let us note that a simple 0-1 argument (because the studied property is invariant under scaling) shows that there exists D such that almost surely, the dimension of $\eta \cap \mathbb{R}_-$ is equal to D. Furthermore, we can use scale-invariance again to see that in order to prove that D is equal to some given value d, it suffices to prove that on the one hand, almost surely, the Hausdorff dimension of $\eta \cap [-2, -1]$ does not exceed d, and that on the other hand, with positive probability, the Hausdorff dimension of $\eta \cap [-2, -1]$ is equal to d.

Let us now note that if a point $x \in [-2, -1]$ belongs to the ε-neighborhood K_ε of η, then it implies that $E_x(\varepsilon, 1)$ holds. Hence, the first moment estimate implies readily that almost surely, the Minkovski dimension of $\eta \cap [-2, -1]$ is not greater than $1 - \beta$, and therefore that $D \leq 1 - \beta$.

In order to prove that with positive probability, the dimension of $\eta \cap [-2, -1]$ is actually equal to $1 - \beta$, we can make the following two observations.

- Suppose that $x \in [-2, -1]$ and that $E_x(\varepsilon/2, 8)$ holds. Suppose furthermore that no excursion in the Poisson point process of excursions attached to $(-\infty, -6)$ does intersect the ball of radius 4 around the origin, no excursion in the Poisson point process excursions attached to $(-2, 0)$ exits the ball of radius 4 around -2. Suppose furthermore that no loop in the CLE (in $\mathbb{H}$) intersects both the circle of radius 4 and 6 around the origin. Note that these two events have positive probability and are positively correlated to $E_x(\varepsilon/2, 8)$ (they are all decreasing events of the Poisson point processes of loops and excursions). Then, by construction, x is necessarily in the ε-neighborhood of η. It therefore follows that for some constant c', for all $x \in [-2, -1]$,
$$P(x \in K_\varepsilon) \geq c'\varepsilon^{\beta}.$$
- Suppose now that $-2 < x < y < -1$, that $y - x < 1/4$ and that $\varepsilon < (y - x)/4$. Clearly, if both x and y belong to K_ε, then it means that the three events $E_x(\varepsilon, (y - x)/2)$, $E_y(\varepsilon, (y - x)/2)$ and $E_x(2(y - x), 1/2)$ hold. These three events are independent, and the previous estimates therefore yield that there exists a constant c'' such that
$$P(x \in K_\varepsilon, y \in K_\varepsilon) \leq c'' \frac{\varepsilon^{2\beta}}{(y - x)^{\beta}}.$$

Standard arguments (see for instance [MP10]) then imply that with positive probability, the dimension of $\eta \cap [-2, -1]$ is not smaller than $1 - \beta$. This concludes the proof of the fact that almost surely, the Hausdorff dimension of $\eta \cap (-\infty, 0)$ is almost surely equal to $1 - \beta$.

In order to conclude, it remains to compute the actual value of β. A proof of this is provided in [MW13] using the framework of flow lines of the Gaussian Free Field. Let us give here an outline of how to compute β bypassing the use of the Gaussian Free Field, using the more classical direct way to derive the values of such exponents i.e. to exhibit a fairly simple martingales involving the derivatives of the conformal maps at a point, and then to use this to estimate the probability that the path ever reaches a small distance of this point: Consider the $\mathrm{SLE}_\kappa(\rho)$ process in $\mathbb{H}$ from 0 to ∞, and keep the same notations as in Section 2.2.2. First, one can note that for any real v,

$$M_t = g_t'(-1)^{v(\kappa v+4-\kappa)/4}(W_t - g_t(-1))^v (O_t - g_t(-1))^{v\rho/2}$$

is a local martingale. We then choose $v = (\kappa - 8 - 2\rho)/\kappa$, and define $\tilde{\beta} := (\rho+2)(\rho+4-\kappa/2)/\kappa$ as well as

$$\Upsilon_t = \frac{O_t - g_t(-1)}{g_t'(-1)}, \quad N_t = \frac{O_t - g_t(-1)}{W_t - g_t(-1)}, \quad \tau_\varepsilon = \inf\{t : \Upsilon_t = \varepsilon\}.$$

Then $M_t = \Upsilon_t^{-\tilde{\beta}} N_t^{-v}$. Furthermore, the probability that the curve gets within the ball centered at -1 of radius ε is comparable to $P(\tau_\varepsilon < \infty)$. But, one has

$$P(\tau_\varepsilon < \infty) = E(M_{\tau_\varepsilon} N_{\tau_\varepsilon}^v 1_{\tau_\varepsilon<\infty})\varepsilon^{\tilde{\beta}} = E^*(N_{\tau_\varepsilon}^v)\varepsilon^{\tilde{\beta}}$$

where P^* is the measure P weighted by the martingale M. Under P^*, we have that $\tau_\varepsilon < \infty$ almost surely and that $E^*(N_{\tau_\varepsilon}^v)$ is bounded both sides by universal constants independent of ε. It follows that indeed $\beta = \tilde{\beta}$.

We conclude with the following two remarks:

- Similar second-moment estimates can be performed for other questions related to $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (8/3, 4]$ and $\rho > -2$. For instance the boundary proximity estimates from Schramm and Zhou [SZ10] can be generalized/adapted to the $\mathrm{SLE}_\kappa(\rho)$ cases. We leave this to the interested reader.

- It is proved in [MS12a] that the left boundary of an $\mathrm{SLE}_{\kappa_0}(\rho_0)$ process for $\kappa_0 > 4$ and $\rho_0 > -2$ is an $\mathrm{SLE}_{\kappa_1}(\rho_1, \rho_2)$ process for $\kappa_1 = 16/\kappa_0$ with an explicit expression of ρ_1 and ρ_2 in terms of (κ_0, ρ_0) (this is the "generalized SLE duality"). Hence, it follows from Proposition 2.5.1 that its statement (i.e. the formula for the Hausdorff dimension) in fact holds true for all $\kappa \in (4, 6)$ as well. However, since the Gaussian Free Field approach is anyway used in the derivation of this generalized duality result, it is rather natural to use also the Gaussian Free Field in order to derive the second moments estimates, as done in [MW13]. The same remark applies to the intersection of the right boundary of an $\mathrm{SLE}_{\kappa_0}(\rho_0)$ when $\kappa_0 > 4$ and $\rho_0 \in (-2, 0)$; the Hausdorff dimension of the intersection of this right boundary with $\mathbb{R}_-$ then turns out to be
$$1 - \frac{(\rho_0+2)(\rho_0+(\kappa_0/2))}{\kappa_0} = -\rho_0\left(\frac{\rho_0+2}{\kappa_0} + \frac{1}{2}\right).$$

Chapter 3

From SLE to CLE

The results in this chapter are contained in [WW13b].

3.1 Introduction

The current paper will be devoted to the study of some properties of the Conformal Loop Ensembles (CLE) defined and studied by Scott Sheffield in [She09] and by Sheffield and Werner in [SW12].

A simple CLE can be viewed as a random countable collection $(\gamma_j, j \in J)$ of disjoint simple loops in the unit disk that are non-nested (almost surely no loop surrounds another loop in CLE). The paper [SW12] shows that there are several different ways to characterize them and to construct them. In that paper, a CLE is defined to be such a random family that also possesses two important properties: It is conformally invariant (more precisely, for any fixed conformal map Φ from the unit disk onto itself, the law of $(\Phi(\gamma_j), j \in J)$ is identical to that of $(\gamma_j, j \in J)$ – this allows to define the law of the CLE in any simply connected domain via conformal invariance) and it satisfies a certain natural restriction property that one would expect from interfaces in physical models. It is shown in [SW12] that there exists exactly a one-parameter family of such CLEs. Each CLE law corresponds exactly to some $\kappa \in (8/3, 4]$ in such a way that for this κ, the loops in the CLE are loop-variants of the SLE_κ processes (these are the Schramm-Loewner Evolutions with parameter κ – recall that an SLE_κ for $\kappa \le 4$ is a simple curve with Hausdorff dimension $1 + \kappa/8$), and that conversely, for each κ in that range, there exists exactly one corresponding CLE. Part of the arguments in the paper [SW12] are based on the analysis of discrete "exploration algorithms" of these loop ensembles, where one slices the CLE open from the boundary and their limits (roughly speaking when the step-size of explorations tends to zero).

In the earlier paper [She09], Sheffield had pointed out a way to construct a number of random collections of loops, using variants of SLE_κ processes. In particular, for any $\kappa \in (8/3, 4]$, he has shown how to construct random collections of SLE_κ-type loops (or rather "quasi-loops" that should turn out to be loops, we will come back to the precise definition later; for the time being the reader can think of these quasi-loops as boundaries of a bounded simply connected component that is a loop with at most one point of discontinuity) that should be the only possible candidates for the conformally invariant scaling limit of various discrete models, or of level lines of certain continuous models. Roughly speaking, one chooses some boundary point x on the unit circle ("the root") and launches from there a branching exploration tree of SLE processes (or rather target-independent variants of SLE_κ processes called the $\text{SLE}(\kappa, \kappa - 6)$ processes) that will trace some loops along the way, that one keeps track of. For each κ and x, there are in fact several ways to do this. One particular way, that we will refer to as *symmetric* in the current paper, is to impose certain "left-right" symmetry in the law of the exploration tree, but several other natural options are described in [She09]. Hence, for each κ, the exploration tree is defined via the choice of the root x and the exploration "strategy" that describes how "left-right" asymmetric the exploration is. These exploration strategies are particularly natural, because they are invariant under all conformal transformations that preserve x. Note also that it is conjectured that the exploration indeed traces a tree with continuous branches, but that this result is not yet proved (to our knowledge). Nevertheless, these processes indeed trace continuous quasi-loops along the way. So to sum up, once κ, x and a given strategy are chosen, the Loewner differential equation enables to construct a random family of quasi-loops in the unit disc (and the law of this family a priori depends on κ, on x and on the chosen strategy).

One can also note that the symmetric strategy is very natural for $\kappa = 4$ from the perspective of the Gaussian Free Field, but much less so from the perspective of interfaces of lattice models. For instance, in the case of $\kappa = 3$ viewed as the scaling limit of the Ising model (see [CS12]), the "totally asymmetric" procedure seems more natural.

Based on the conjectured or proved relation to discrete models and to the Gaussian Free Field, Sheffield conjectures in [She09] that for any given $\kappa \le 4$, all these random collections of loops traced by the various exploration trees have the same law.

One consequence of the results of Sheffield and Werner in [SW12] is that this conjecture is indeed true for all symmetric explorations: More precisely, for each $\kappa \in (8/3, 4]$, the law of the random collection of quasi-loops traced by a symmetric $\mathrm{SLE}(\kappa, \kappa-6)$ exploration tree rooted at x does not depend on x. In fact, their common law is proved to be that of "the CLE" with SLE_κ-type loops mentioned in the first paragraph of this introduction, and they can also be viewed (see [SW12]) as outer boundaries of clusters of Brownian loop-soups (which proves that the quasi-loops are in fact all loops). One main idea in [SW12] is to study the asymptotic behavior of the discrete explorations when the steps get smaller and smaller, and to prove that it converges to the above-mentioned symmetric $\mathrm{SLE}(\kappa, \kappa-6)$ process.

The first main result of the current paper can be summarized as follows: *For each $\kappa \in (8/3, 4]$, all the random collections of SLE_κ-type quasi-loops constructed via Sheffield's asymmetric exploration trees in [She09] have the same law. They all are the CLE_κ families of loops constructed in [SW12].* The proof of this fact will heavily rely on the results of [SW12], but we will try to make our paper as self-contained as possible.

Recall that when one works directly in the SLE-framework, certain questions turn out to be rather natural – it is for instance possible to derive rather directly the values of certain critical exponents, to compute explicitly probabilities of certain events, or to study questions related to conformal restriction – while the setting of the Loewner equation does not seem so naturally suited for some other questions. Proving reversibility of the SLE path (that the random curve defined by an SLE from a to b is the same as that defined by an SLE from b to a) turns out to be very tricky, see [Zha08b, MS12b, MS12c]. A by-product of [SW12] is that it provides another proof of this reversibility in the case where $\kappa \in (8/3, 4]$. In a way, our results are of a similar nature: One obtains results about these asymmetric branching SLE processes without going into fine Loewner chain technology.

The second main point of our paper is to highlight something specific to the case $\kappa = 4$ i.e. to CLE_4 (recall that this is the CLE that is most directly related to the Gaussian Free Field, see [SS12, SS09, Dub09b, She11]). In this particular case, it is possible to define a conformally invariant and unrooted (one does not need to even choose a starting point) growing mechanism of loops (the term "exploration" that is used in this paper is a little bit misleading, as it is not proved that the growth process is in fact a deterministic function of the CLE, we will discuss this at the end of the paper). Roughly speaking, the growth process that progressively discovers loops is growing "uniformly" from the boundary (even if it is a Poisson point process and each loop is discovered at once) and does not require to choose a root. The fact that such a conformally invariant non-local growth mechanism exists at all is quite surprising (and the fact that its time-parametrization as seen from different points does exactly coincide even more so). It also leads to a conformal invariant way to describe distances between loops in a CLE (where any two loops in a CLE are at a positive distance of each other) and to a new coupling of CLE with the Gaussian Free Field that will be studied in more detail in the subsequent work [SWW13], and to open questions that we will describe at the end of the paper.

3.2 Background and first main statement

In the present subsection, we recall some ideas, arguments and results from [She09, SW12], and set up the framework that will enable us to derive our main results in a rather simple way.

Bessel processes and principal values

Suppose throughout this subsection that $\delta \in (0,1]$. It is easy to define the *squared Bessel process* $(Z_t, t \geq 0)$ of dimension δ started from $Z_0 = z_0 \geq 0$ as the unique solution to the stochastic differential equation

$$dZ_t = 2\sqrt{Z_t} dB_t + \delta dt$$

where $(B_t, t \geq 0)$ is a Brownian motion (note that it is implicit that this solution is non-negative because one takes its square root).

The non-negative process $Y_t = \sqrt{Z_t}$ is then usually called the Bessel process of dimension δ started from $\sqrt{z_0}$. It is not formally the solution to the stochastic differential equation

$$dY_t = dB_t + (\delta - 1)\frac{dt}{2Y_t}$$

because it gets an (infinitesimal) upwards push whenever it hits the origin, so that $Y_t - (\delta - 1)\int_0^t ds/(2Y_s)$ is not a martingale (note for instance that when $\delta = 1$, the process Y_t is a reflected Brownian motion, which is clearly not the solution to $dY_t = dB_t$). This stochastic differential equation however describes well the evolution of Y while it is away from the origin, and if one adds the fact that Y is almost surely non-negative, continuous and that the Lebesgue measure of $\{t > 0 \ : \ Y_t = 0\}$ is almost surely equal to 0, then it does characterize Y uniquely. Note that the filtration generated by Y and by B do coincide (B can be recovered from Y).

Bessel processes have the same scaling property as Brownian motion: When $Y_0 = 0$, then for any given positive ρ, $(Y_t, t \geq 0)$ and $(\rho^{-1} Y_{\rho^2 t}, t \geq 0)$ have the same law (this is an immediate consequence of the definition of its squared process Z). Just as in the case of the Itô measure on Brownian excursions, it is possible to define an infinite measure λ on (positive) Bessel excursions of dimension δ. An excursion e is a continuous function $(e(t), t \in [0, \tau])$ defined on an interval of non-prescribed length $\tau = \tau(e)$ such that $e(0) = e(\tau) = 0$ and $e(t) > 0$ when $t \in (0, \tau)$. The measure λ is then characterized by the fact that for any x, the mass of the set of excursions

$$E_x := \{e \ : \ \sup_{s \leq \tau} e(s) \geq x\}$$

is finite, and that if one renormalizes λ in such a way that it is a probability measure on E_x, then the law of e on $[\tau_x, \tau]$ is that of a Bessel process of dimension δ, started from x and stopped at its first hitting time of the origin. The fact that $t \mapsto (Y_t)^{2-\delta}$ is a local martingale when Y is away from the origin (which follows immediately from the definition of the Bessel process Y and from Itô's formula) shows readily that

$$\lambda(E_x) = cx^{\delta-2}$$

for some constant c that can be chosen to be equal to one (this is the normalization choice of λ). Standard excursion theory shows that it is possible to define the process Y by gluing together a Poisson point process $(e_u, u \geq 0)$ of these Bessel excursions of dimension δ, chosen with intensity $\lambda \otimes du$.

Suppose that we are given a parameter $\beta \in [-1,1]$ and that for each excursion e of the Bessel process Y, one tosses an independent coin in order to choose $\varepsilon(e) = \varepsilon_\beta(e) \in \{-1,1\}$ in such a way that the probability that $\varepsilon(e) = +1$ (respectively $\varepsilon(e) = -1$) is $(1+\beta)/2$ (resp. $(1-\beta)/2$). Then, one can define a process $X^{(\beta)}$ by gluing together the excursions $(\varepsilon(e) \times e)$ instead of the excursions (e). Note that $|X^{(\beta)}| = X^{(1)} = Y$, but some of the excursions of $X^{(\beta)}$ are negative as soon as $\beta < 1$. The process $X^{(0)}$ is the *symmetrized* Bessel process such that $X^{(0)}$ and $-X^{(0)}$ have the same law. Note that (as opposed to $Y = X^{(1)}$) the process $X^{(\beta)}$ is not a deterministic function of the underlying driving Brownian motion B in the stochastic differential equation when $\beta \in (-1,1)$, because additional randomness is needed to choose the signs of the excursions. However, B is still a martingale with respect to the filtration generated by $X^{(\beta)}$ because the signs of the excursions are in a way independent of the excursions.

In the sequel, it will be useful to consider the processes $X^{(\beta)}$ for various values of β simultaneously. Clearly, it is easy to first define Y and then to couple all signs in such a way that for each excursion e of Y, $\varepsilon_\beta(e) \geq \varepsilon_{\beta'}(e)$ as soon as $\beta \geq \beta'$; we will implicitly always work with such a coupling.

In the context of SLE processes, it turns out to be essential to try to make sense of a quantity of the type $\int_0^t ds/X_s^{(\beta)}$. We define, for each excursion, the integral

$$i(e) := \int_0^\tau ds/e(s).$$

It is easy to check that

$$\lambda(i(e)1_{E_1}) < \infty,$$

from which it follows using scaling that $i(e) < \infty$ for λ almost all excursion e. Note also that the scaling shows that

$$\lambda(\{e \,:\, i(e) \geq x\}) = x^{\delta-2}\lambda(\{e \,:\, i(e) \geq 1\}).$$

It follows that typically, the number of excursions that occur before time 1 for which $i(e) \in [2^{-n}, 2^{-n+1})$ is of the order of $(2^{-n})^{\delta-2}$, so that their cumulative contribution to $\int_0^t ds/Y_s$ is of the order of $(2^{-n})^{\delta-1}$. If we sum this over n, one readily sees that when $t > 0$, then

$$\int_0^t ds/Y_s = \infty$$

almost surely as soon as $\delta \leq 1$ (and this argument can be easily made rigorous) due to the cumulative contributions of the many short excursions during the interval $[0,t]$. Hence, $\int_0^t ds/X_s^{(\beta)}$ can not be defined as a simple absolutely converging integral.

There are however ways to circumvent this difficulty. The first classical one works for all $\delta \in (0,1]$ but it is specific to the case where $\beta = 0$ i.e. to the symmetrized Bessel process $X^{(0)}$. In that case, when one formally evaluates the cumulative contribution to $\int_0^t ds/X_s^{(0)}$ of the excursions for which $i(e) \in [2^{-n}, 2^{-n+1})$, then the central limit theorem suggests that one will get a value of the order of $2^{-n} \times (2^{(2-\delta)n})^{1/2} = 2^{-\delta n/2}$; when one then sums over n, one gets an almost surely converging series. This heuristic can be easily be made rigorous, and this shows that one can define a process $I_t^{(0)}$ that one can informally interpret as $\int_0^t ds/X_s^{(0)}$ (even though this last integral does not converge absolutely). Another possible way to characterize this process is that it is the only process such that:

- $t \mapsto I_t^{(0)}$ is almost surely continuous and satisfies Brownian scaling.
- $dI_t^{(0)} - dt/X_t^{(0)}$ is zero on any time-interval where $X^{(0)}$ is non-zero.
- The process $I^{(0)}$ is a deterministic function of the process $X^{(0)}$.

Let us reformulate and detail our first approach to $I_t^{(0)}$ in a way that will be useful for our purposes. Suppose that $r > 0$ is given and small. We denote by J_r the set of times that belong to an excursion of Y away from the origin, that has time-length at least r^2 (we choose r^2 in order to have the same scaling properties as for the height and $i(e)$). Then, because the integral $\int ds/e(s)$ on each individual excursion is finite, we see that it is possible to define without any difficulty the absolutely converging integral

$$I_t^{(0,r)} := \int_0^t ds 1_{s \in J_r} / X_s^{(0)}.$$

Then, as $r \to 0$ the continuous process $I^{(0,r)}$ converges to the continuous process $I^{(0)}$. More rigorously

Lemma 3.2.1. *When $n \to \infty$, then on any compact time-interval, the sequence of continuous functions $I^{(0,1/2^n)}$ converges almost surely to a limiting continuous function $I^{(0)}$.*

Proof. Let $\tau = \tau(r_0)$ denote the end-time of first excursion that has time-length at least r_0^2 (here r_0 should be thought of as very large, so that this time, which is greater than r_0^2, is large too). It suffices to prove the almost sure convergence on the interval $[0,\tau]$ (as any given compact interval is inside some interval $[0,\tau]$ for large enough r_0).

Let us suppose that $n \geq m$. Notice that the process $t \mapsto I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)}$ is monotonous (i.e. non-increasing or non-decreasing) on each excursion of Y, so that

$$\sup_{t \leq \tau} (I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2 = \sup_{t \leq \tau, Y_t = 0} (I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2.$$

Next we define the σ-field $\mathscr{F}_0$ generated by the knowledge of all excursions $|e|$, but not their signs. If we condition on $\mathscr{F}_0$ and look at the value of $I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)}$ at the end-times of the excursions of length greater than 2^{-n}, we get a discrete martingale. From Doob's L^2 inequality, we therefore see that almost surely,

$$E\left(\sup_{t \leq \tau} (I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2 \mid \mathscr{F}_0\right) \leq 4E\left((I_\tau^{(0,1/2^n)} - I_\tau^{(0,1/2^m)})^2 \mid \mathscr{F}_0\right).$$

The right-hand side is in fact the mean of the square of a series of symmetric random variables of the type $\sum \varepsilon_j i_j$ for some given i_j and coin-tosses ε_j. Therefore, it is equal to $4\sum_e i(e)^2$ where the sum is over all excursions appearing before time τ, corresponding to times in $J_{1/2^n} \setminus J_{1/2^m}$. By simple scaling, the expectation of this quantity is equal to a constant times $2^{-m\delta} - 2^{-n\delta}$, so that finally

$$E\left(\sup_{t\leq\tau}(I_t^{(0,1/2^n)} - I_t^{(0,1/2^m)})^2\right) \leq C(2^{-m\delta} - 2^{-n\delta}) \leq C2^{-m\delta}.$$

It then follows easily (via Borel-Cantelli) that almost surely, the function $t \mapsto I_t^{(0,1/2^n)}$ converges uniformly as $n \to \infty$ on the time-interval $[0,\tau]$ (and that the limiting process $I^{(0)}$ is continuous). □

This construction of $I^{(0)}$ can not be directly extended to the case where $\beta \neq 0$. Indeed, the cumulative contributions of those excursions of $X^{(\beta)}$ for which $i(e) \in [2^{-n}, 2^{1-n})$ is then of the same order of magnitude than when $\beta = 1$ (the previously described case where one looks at the integral of $1/Y_s$). A solution when the dimension of the Bessel process is smaller than 1, is to compensate the explosion of this integral appropriately. Let us first describe this in the case where $\beta = 1$ (i.e. $X^{(\beta)} = Y$ is the non-negative Bessel process). As for instance explained in [She09, Section 3], it is possible to characterize the principal value $I_t = I_t^{(1)}$ of the integral of $1/Y_t$ as the unique process such that:

- $t \mapsto I_t$ is almost surely continuous.
- $dI_t - dt/Y_t$ is zero on any time-interval where Y is non-zero.
- (I_t, Y_t) is adapted to the filtration of Y and satisfies Brownian scaling.

Let us describe how to construct explicitly this process I_t. For any very small r, recall the definition of the time-set J_r, and define $N_r(t)$ as the number of excursions of time-length at least r^2 that Y has completed before time t. Simple scaling considerations show that (for fixed t), $N_r(t)$ will explode like (some random number times) $r^{\delta-2}$ as $r \to 0$.

Just as before, there is no problem to define the absolutely converging integral

$$\int_0^t \frac{1_{s\in J_r} ds}{Y_s}.$$

But, as we have already indicated, when $\beta \neq 0$ this quantity tends to ∞ when r tends to 0. One option is therefore to consider the quantity

$$K_t^r := \int_0^t \frac{1_{s\in J_r} ds}{Y_s} - CrN_r(t)$$

where

$$Cr := \frac{\lambda(i(e)1_{\tau(e)\geq r^2})}{\lambda(1_{\tau(e)\geq r^2})}$$

is the mean value of the integral of $1/e$ for an excursion conditioned to have length greater than r^2. Note that

$$C = \lambda(i(e)1_{\tau(e)\geq 1})/\lambda(1_{\tau(e)\geq 1})$$

is a constant that does not depend on r. When $r \to 0$, $rN_r(t)$ explodes like $r^{\delta-1}$, but nevertheless:

Lemma 3.2.2. *As $n \to \infty$, the process $K^{1/2^n} = (K_t^{1/2^n}, t \geq 0)$ does almost surely converge uniformly on any compact time-interval to some continuous limiting process $I^{(1)}$.*

Proof. The proof goes along similar lines as in the case $\beta = 0$, but there are some differences. Let us prove again almost sure convergence on each $[0, \tau]$ for each given r_0.

First, let us notice that for each $r > r'$, the quantity $K_t^r - K_t^{r'}$ is constant, except on the excursion-intervals of time-length between r' and r. If we follow the value of this quantity only at (the discrete set of) end-times of excursions of length greater than r', we get a discrete martingale. Doob's inequality and scaling, just as before, imply that

$$E\left(\sup_{t \le \tau : Y_t = 0} (K_t^r - K_t^{r'})^2\right) \le E\left((K_\tau^r - K_\tau^{r'})^2\right).$$

The right-hand side can be viewed as the sum of a geometric number of zero-mean random variables with bounded second moment, and it follows from scaling that it can be bounded by

$$c r_0^{2-\delta} r^{\delta}.$$

It follows that almost surely, the function $K^{1/2^n}$ converges uniformly as $n \to \infty$ on the set $\{t \le \tau : Y_t = 0\}$. The definition of K^r then yields that this almost sure uniform convergence takes place on all of $[0, \tau]$ (just because the supremum of the integral of $1/Y_s$ over all excursions of length greater than r before τ goes to 0 as $r \to 0$).

On the other hand, if we slightly modify K^r on each excursion interval by adding a linear function that makes it continuous on the closed support of the excursion in order to compensate the $-Cr$ jump of K^r and the end-time of the excursion, one obtains a continuous function $\tilde{K}^r$ such that $|K_t^r - \tilde{K}_t^r| \le Cr$ for each t. It follows that almost surely $\tilde{K}^{1/2^n}$ converges uniformly on any compact time-interval to the same limit as $K^{1/2^n}$. As the functions $\tilde{K}^r$ are continuous, it follows that this limit is almost surely a continuous function of time. □

For any β (and as long as $\delta \in (0, 1)$), the very same idea can be used to define a process $I^{(\beta)}$ associated to $X^{(\beta)}$ instead of Y, as the limit when $r \to 0$ of the process

$$\int_0^t \frac{1_{s \in J_r} ds}{X_s^{(\beta)}} - Cr\beta N_r(t).$$

Let us describe in more detail a variant of the previous construction that will be useful for our purposes. Suppose that one is working with the coupling of all processes $X^{(\beta)}$ (for fixed $\delta \in (0, 1)$). We then define the process

$$X_t^{(\beta, r)} := X_t^{(\beta)} 1_{t \in J_r} + X_t^{(0)} 1_{t \notin J_r}.$$

In other words, we replace $X^{(\beta)}$ by $X^{(0)}$ on all excursions of length smaller than r^2. Clearly, this makes it possible to make sense of the continuous process

$$\int_0^t \left(\frac{1}{X_s^{(\beta, r)}} - \frac{1}{X_s^{(0)}}\right) ds$$

(because only the times in J_r i.e. in the macroscopic excursions will contribute). We can therefore define the process

$$I_t^{(\beta, r)} := I_t^{(0)} + \int_0^t 1_{s \in J_r} \left(\frac{1}{X_s^{(\beta, r)}} - \frac{1}{X_s^{(0)}}\right) ds - \beta Cr N_r(t).$$

This process $I^{(\beta,r)}$ follows exactly the evolution of $I^{(0)}$ except that some excursions of length greater than r^2 are sign-changed (and on these excursions $I_t^{(\beta,r)} + I_t^{(0)}$ is constant), and that at the end of each of those excursions, it makes a small jump of $-\beta Cr$.

Then, almost surely, when $r \to 0$, the process $I^{(\beta,r)}$ converges uniformly on any compact time-interval to a process $I^{(\beta)}$ because the two processes

$$I_t^{(0)} - \int_0^t 1_{s\in J_r} ds/X_s^{(0)}$$

and

$$\int_0^t 1_{s\in J_r} ds/X_s^{(\beta)} - \beta Cr N_r(t) - I_t^{(\beta)}$$

do almost surely uniformly converge to zero on any given compact interval.

The previous definition of $I^{(\beta)}$ can not be directly adapted to the case $\delta = 1$. However, one notes that for any real μ, the process $I_t^{<\mu>} := I_t^{(0)} + \mu \ell_t$, where ℓ is the local time at 0 of X does also satisfy the Brownian scaling property and that $dI_t^{<\mu>} = dt/X_t$ on all intervals where X is non-zero. This process $I^{<\mu>}$ can in fact again be approximated via $N_r(t)$ (using the classical approximation of Brownian local time); more precisely, it is the limit as $r \to 0$ of the process

$$I_t^{<\mu,r>} := I_t^{(0)} + \mu r N_r(t).$$

From Bessel processes to ensembles of (quasi-)loops

We now recall (mostly from [She09]) how to use the previous considerations in order to give an SLE based construction of the loop ensembles. If we work in the upper half-plane $\mathbb{H}$, then Loewner's construction shows that as soon as one has defined a continuous real-valued function $(w_t, t \geq 0)$, one can define a two-dimensional "Loewner chain" (that in many cases turns out to correspond to a two-dimensional path) as follows: For any $z \in \overline{\mathbb{H}}$, define the solution $(Z_t = Z_t(z))$ to the ordinary differential equation

$$Z_t = z + \int_0^t \frac{2ds}{Z_s - w_s}.$$

This equation is well-defined up to a (possibly infinite) explosion/swallowing time $T(z) = \sup\{t \geq 0 \ : \ \inf\{|Z_s - w_s| \ : \ s \in [0,t)\} > 0\}$. For each given t, the map $g_t : z \mapsto Z_t(z)$ is a conformal map from some subset H_t of $\mathbb{H}$ onto $\mathbb{H}$ such that $g_t(z) - z = o(1)$ when $z \to \infty$. One defines $K_t = \overline{\mathbb{H} \setminus H_t}$; the Loewner chain usually means the chain $(K_t, t \geq 0)$.

When w_t is chosen to be equal to $\sqrt{\kappa} B_t$, where B is a standard real-valued Brownian motion, then this defines the SLE$_\kappa$ processes, that turn out to be simple curves as soon as $\kappa \leq 4$. The so-called SLE$(\kappa, \kappa - 6)$ processes are variants of SLE$_\kappa$ with a particular target independence property first pointed out in [SW05]. More precisely, suppose that one considers the joint evolution of two points (W_t, O_t) in $\mathbb{R}$ started from (W_0, O_0) with $W_0 \neq O_0$, and described by

$$dO_t = \frac{2dt}{O_t - W_t} \text{ and } dW_t = \sqrt{\kappa} dB_t + \frac{\kappa - 6}{W_t - O_t} dt \tag{3.2.1}$$

as long as $W_t \neq O_t$ (where B is a standard Brownian motion). Then, one can use the random function W as the driving function of our Loewner chain, which is this SLE$(\kappa, \kappa - 6)$. There is no difficulty in defining the process (W, O) as long as W_t does not hit O_t, but more is needed to understand what happens after such a meeting time.

Note that if one writes $X_t = (W_t - O_t)/\sqrt{\kappa}$, then

$$dX_t = dB_t + \frac{\kappa - 4}{\kappa X_t} dt$$

so that X evolves like a Bessel process of dimension

$$\delta = 3 - \frac{8}{\kappa} \in (0,1]$$

when $\kappa \in (8/3, 4]$, and that dO_t is a constant multiple of dt/X_t as long as $X_t \neq 0$. Furthermore, the knowledge of X_t and of O_t enables to recover $W_t = O_t + \sqrt{\kappa} X_t$.

This gives the following options to define a driving process $(W_t, t \geq 0)$ at all non-negative times (even for $W_0 = O_0 = 0$):

- When $\delta \in (0,1)$ (i.e., $\kappa \in (8/3, 4)$) and $\beta \in [-1,1]$: Define first one of the Bessel processes $X^{(\beta)}$ as before started at 0, and its corresponding process $I^{(\beta)}$. Then define $O_t^{(\beta)} = 2\sqrt{\kappa} I_t^{(\beta)}$ and
$$W_t^{(\beta)} = \sqrt{\kappa} X_t^{(\beta)} + O_t^{(\beta)} = \sqrt{\kappa} X_t^{(\beta)} + 2\sqrt{\kappa} I_t^{(\beta)}.$$
- When $\delta = 1$ (i.e., $\kappa = 4$) and $\mu \in \mathbb{R}$, then, define $O_t^{<\mu>} = 4 I_t^{<\mu>}$ and
$$W_t^{<\mu>} = 2B_t + O_t^{<\mu>} = 2B_t + 4 I_t^{<\mu>}$$
where B is standard one-dimensional Brownian motion.

In all these cases, one constructs a couple (W_t, O_t) that satisfies the Brownian scaling property and that evolves according to (3.2.1) when $W_t \neq O_t$. The process $(W_t, t \geq 0)$ defines a Loewner chain $(K_t, t \geq 0)$ from the origin to infinity in the upper half-plane. More precisely, for each t, $H_t := \mathbb{H} \setminus K_t$ is the preimage of $\mathbb{H}$ under the conformal map g_t characterized by the fact that for all $s \leq t$,

$$g_0(z) = z \quad \text{and} \quad \partial_s g_s(z) = \frac{2}{g_s(z) - W_s}$$

(see e.g. [Law05] for background). The Brownian scaling property shows that this Loewner chain is invariant (in law) under scaling (modulo time-parametrization). This makes it possible to also define (via conformal invariance) the law of the Loewner chain in $\mathbb{H}$ from 0 to some $u \in \mathbb{R}$ (and more generally from any boundary point to any other boundary point of a simply connected domain) by considering the conformal image of the previously defined chain from 0 to infinity under a conformal map from $\mathbb{H}$ onto itself that maps 0 onto itself, and ∞ onto u.

In the sequel, when we will refer to "a SLE($\kappa, \kappa - 6$) process", we will implicitly mean such a chordal chain, for some $\kappa \in (8/3, 4]$, and some choice of β (if $\kappa \in (8/3, 4)$) or μ (when $\kappa = 4$).

All these SLE($\kappa, \kappa - 6$) processes are of particular interest because of their target-independence property: Up to the first time at which the Loewner chain disconnects u from infinity, the two Loewner chains (from 0 to ∞, and from 0 to u) have the same law (modulo time-change). When $O_t - W_t$ is not equal to 0, the fact that the local evolution of chain is independent of the target point is derived (via Itô formula computations for (O_t, W_t)) in [SW05]. Note that the two evolutions match up to a time-change only, because time corresponds to the size of the Loewner chain seen from either infinity or from u. In order to check that target-independence remains valid at all times, one needs to check that the "local push" rule that is used in order to define $I_t^{(\beta)}$ (or $I_t^{<\mu>}$ when $\kappa = 4$) and then O_t is also the same (modulo the time-change) for both processes. This is basically explained in [She09, Section 7].

In fact, if we formally replace the Bessel process X by just one Bessel excursion e, the procedure defines an SLE "pinned loop" or SLE "bubble" i.e. a continuous simple curve in $\mathbb{H} \cup \{0\}$ that passes through the origin. More precisely, let us start with the excursion $(e(s), s \leq \tau)$ and use the driving function

$$w_t := \sqrt{\kappa} e(t) + 2\sqrt{\kappa} \int_0^t ds/e(s)$$

to generate the Loewner chain. It is shown in [SW12] that for almost all (positive) Bessel excursion e (according to the Bessel excursion measure that we have denoted by λ), this defines a such a simple loop $\gamma(e)$ in the upper half-plane, that starts and ends at the origin (and $H_t = \mathbb{H} \setminus \gamma(0,t]$). The time-length $\tau(e)$ of the loop corresponds to the half-plane capacity seen from infinity of γ. The infinite measure on loops γ that one obtains when starting from the infinite measure λ on Bessel excursions is referred to as the one-point pinned measure in [SW12].

Hence, for each excursion e of X, corresponding to the excursion interval $[t_-(e), t_+(e)]$ (with $t_+ - t_- = \tau(e)$), one can define the preimage of $\gamma(e)$ under the conformal map g_{t_-} of the Loewner chain. It is not clear at this point that this is a proper continuous loop in $\mathbb{H}$ because we do not know whether $g_{t_-}^{-1}$ extends continuously to the origin, but we already know that it is almost a loop: In particular, the preimage of $\gamma(e) \setminus \{0\}$ is a simple curve such that its closure disconnects some interior domain from an outer domain in H_t. In the sequel, we will refer to this as a quasi-loop (mind that in [She09], this is called a "conformal loop" and that the term quasi-simple loops is used for something different). One of the consequences of [SW12] is that *in the symmetric case*, all these quasi-loops are in fact loops (here one uses the alternative construction using Brownian loop-soup clusters). For the other cases, we shall see that it is also the case, but at this stage of the proof, we do not know it yet.

As explained in [She09], the target-independence makes it possible to define (for each version of the SLE$(\kappa, \kappa-6)$ that we have defined, and that we will implicitely keep fixed in the coming three paragraphs) a "branching SLE" structure starting from 0 and aiming at a dense set of points in the upper half-plane. Let us first describe the process targeting i, until it discovers a quasi-loop around i (this is the "radial" SLE$(\kappa, \kappa-6)$; we choose here not to introduce the radial Loewner equation but to explain this radial process via the chordal setting): Consider a chordal SLE$(\kappa, \kappa-6)$ from 0 to ∞ until time t_1, which is the first end-time of an excursion of X after the first moment at which the Loewner chain reaches the (half)-circle of radius $1/2$ away from the origin. One has two possibilities, either at t_1, the Loewner chain has traced a quasi-loop around i (and it can be only the quasi-loop that the chain had started to trace when reaching the circle, and t_1 is the end-time of the corresponding excursion) or not. In the latter case, it means that i is still in the remaining to be explored unbounded simply connected component H_{t_1} of the chain at time t_1. At this time, the chain is growing at a prime end (i.e. loosely speaking, a boundary point) of H_{t_1}. We can now consider the conformal transformation F_1 of H_{t_1} onto $\mathbb{H}$ that maps this prime end onto the origin, and keeps i fixed (another way to describe F_1 is to say that it is the composition of g_{t_1} with the Moebius transformation of the upper half-plane that maps $g_{t_1}(i)$ onto i and W_{t_1} onto 0). Then, we repeat the same procedure again: Grow an SLE$(\kappa, \kappa-6)$ from the origin in $\mathbb{H}$ until the first end-time of an quasi-loop that touches the circle of radius $1/2$ etc., and we look at its preimage under F_1 (this can be interpreting as a switch of chordal target at t_1, this pre-image chain now aims at $F_1^{-1}(\infty)$ instead of ∞). After a geometric number of iterations, one finds a chordal SLE$(\kappa, \kappa-6)$ in $\mathbb{H}$ that catches i via a quasi-loop that intersects the circle of radius $1/2$. The concatenation of the preimages of these Loewner chains is what is called the radial SLE$(\kappa, \kappa-6)$ targeting i, and the preimage of this quasi-loop that surrounds i is the quasi-loop $\gamma(i)$ defined by this radial SLE.

In a similar way, one can define for each given $z \in \mathbb{H}$, the radial SLE$(\kappa, \kappa-6)$ targeting z, and the quasi-loop $\gamma(z)$ that it discovers. The target-independence property of chordal SLE$(\kappa, \kappa-6)$ can be then used in order to see that for any $z_1, \ldots, z_n$, it is possible to couple all these radial explorations in such a way that for any $k \neq j$, the explorations targeting z_k and z_j coincide as long as z_k and z_j remain in the same connected component, and that they are conditionally independent after the moment at which z_k and z_j are disconnected from each other. Hence, one can define a process $(\gamma(z), z \in \mathbb{H})$ of quasi-loops via the law of its finite-dimensional marginals, that has the property that for all z, z', on the event where $\gamma(z)$ surrounds z', one has $\gamma(z') = \gamma(z)$. The countable collection of loops that is defined for instance via the loops that surround points with rational coordinates is the CLE constructed by Sheffield [She09] associated to this value of κ and β (or μ).

Let us repeat that the law of this family of quasi-loops is characterized by its finite-dimensional distributions (i.e. the laws of the families $(\gamma(z_1), \ldots, \gamma(z_k))$ for any finite set $\{z_1, \ldots, z_k\}$ in the upper half-plane). A convenient topology to use in order to define these random quasi-loops is, for the loop $\gamma(z)$, to use the Carathéodory topology for the inside of the quasi-loop as seen from z. In the sequel, when we will say that a sequence of CLE's converges in law to another CLE in the sense of finite-dimensional distributions, we will implicitly be using this topology.

Note that a quasi-loop will be traced clockwise or anti-clockwise, depending on the sign of the corresponding excursion. For instance, for $\beta = 1$, all quasi-loops are traced anti-clockwise.

Let us summarize what these procedures define:

- When $\kappa \in (8/3, 4)$, for each $\beta \in [-1, 1]$ and for each boundary point x (corresponding to our choice of starting point in the previous setting) a random family of quasi-loops that we can denote by $\mathrm{CLE}_\kappa^\beta(x)$.

- When $\kappa = 4$, for each $\mu \in \mathbb{R}$ and each boundary point x, a random family of quasi-loops that we denote by $\mathrm{CLE}_{4,\mu}(x)$.

The construction and the target-independence ensures that these CLE's are conformally-invariant (this follows basically from the target-independence property, see [She09]), i.e. for each given x, κ and β (or μ), and any Moebius transformation of the upper half-plane, the image of a $\mathrm{CLE}_\kappa^\beta(x)$ under Φ is distributed like a $\mathrm{CLE}_\kappa^\beta(\Phi(x))$.

In [SW12], it is proved that the law of a $\mathrm{CLE}_\kappa^0(x)$ does not depend on x, and that the law of $\mathrm{CLE}_{4,0}(x)$ does not depend on x (note that these are the families of quasi-loops constructed via the *symmetric* exploration procedure). Furthermore, for these ensembles, all quasi-loops are almost sure plain loops (the proof of these last facts are based on the fact that these loop ensembles are the only family of loops that satisfy some axiomatic properties, and that they can also be constructed as boundaries of clusters of Brownian loops).

3.3 The asymmetric explorations

The present subsection is devoted to the proof of the following proposition and to its analogue for $\kappa = 4$, Proposition 3.3.2):

Proposition 3.3.1. *For all given $\kappa \in (8/3, 4)$, the law of $\mathrm{CLE}_\kappa^\beta(x)$ does depend neither on x nor on β.*

Note that this also proves that in these ensembles, all quasi-loops are indeed loops because we already know that $\mathrm{CLE}^0_\kappa(x)$ almost surely consists of loops. Recall also that we know that this $\mathrm{CLE}^0_\kappa(x)$ does not depend on x (in the present subsection, we will simply refer to it as the CLE_κ). It is therefore enough to check that in the upper half-plane, the laws of $\mathrm{CLE}^\beta_\kappa(0)$ and of $\mathrm{CLE}^0_\kappa(0)$ coincide, ie., that for any given $z_1,\dots,z_n$ in the upper half-plane, the joint law of $(\gamma(z_1),\dots,\gamma(z_n))$ is the same for these two CLEs.

Let us first make the following observations:

1. A by-product of the characterization/uniqueness of CLE derived in [SW12] is the fact that the image measure μ^0 on pinned loops $\gamma(e)$ (i.e. the image measure of λ under $e \mapsto \gamma(e)$) is invariant under the symmetry with respect to the imaginary axis ($x+iy \mapsto -x+iy$). In other words, it is the same as the image measure on loops defined by $e \mapsto \gamma(-e)$ if one forgets about the time-parametrization of the loops (this is closely related to the reversibility of SLE paths first derived by Zhan [Zha08b] by other more direct means). Note that the time-length $\tau(e)$ is both the half-plane capacity of $\gamma(e)$ and of $\gamma(-e)$ for a given e.

2. Suppose now that we consider a *symmetric* SLE$(\kappa,\kappa-6)$ with driving function W, that we stop at the first time T_1 at which it completes an excursion of length greater than r^2. This defines a certain family of loops in the unit disc via the procedure described above. If we are given some sign $\varepsilon_1 \in \{-1,+1\}$ (independent of the process W), we can decide to modify the previous process W into another process $\hat{W}$, by just (maybe) changing the sign of the final excursion before T_1 into ε_1 (it may be that we do not have to change it in order to have it equal to ε_1). Then, clearly, the law of $(\hat{W},t \le T_1)$ is absolutely continuous with respect to that of $(W_t,t \le T_1)$, and it therefore also defines almost surely a family of loops in the unit disk. The previous item shows in fact that the collection of loops defined by these two processes up to T_1 have exactly the same law (because changing the sign of one final excursion does not change the distribution of the corresponding loop). Furthermore, the law of the collection of connected components of the complement of this symmetric SLE$(\kappa,\kappa-6)$ at that moment are clearly identical too.

3. Suppose that T is some stopping time for the driving function $(W_t,t \ge 0)$ of the symmetric SLE$(\kappa,\kappa-6)$ process started from the origin in the half-plane. Suppose furthermore that this stopping time is chosen in such a way that almost surely $W_T = O_T$. Then, we know that the process $(W_{T+t}-W_T,t \ge 0)$ is distributed exactly as W itself, and furthermore, it is independent of $(W_t,t \le T)$. This means that the conditional law given $(W_t,t \le T)$ of the non-yet explored loops is just a CLE_κ in the yet-to-be explored domain. This makes it possible to change the starting point of the upcoming evolution, because we know that the law of the loops defined by the branching symmetric SLE$(\kappa,\kappa-6)$ is independent of the chosen starting point. In particular, if we consider an increasing sequence of stopping times T_n (and $T_0=0$) such that $T_n \to \infty$ almost surely and $W_{T_n} = O_{T_n}$ for each n, and define the process
$$\hat{W}_t = W_t + cN_t \tag{3.3.1}$$
where c is some constant and $N_t = \max\{n \ge 0 \ : \ T_n \le t\}$, the planar loops associated with the excursions intervals of the Bessel process X will be distributed according to loops in a CLE_κ. More precisely, for each n, if one samples (conditionally on the process up to T_n) independent CLE_κ's in the remaining unexplored connected components created by the Loewner chain at time T_n, and considers the union of the obtained loops with the loops that have been discovered before T_n, then one gets a full CLE_κ sample.

Let us now combine the previous facts in the case where $\kappa < 4$ (i.e., $\delta < 1$). Suppose now that $r > 0$ is fixed. Consider a symmetrized Bessel process $X^{(0)}$ and the corresponding driving function $W^{(0)}$. Define the stopping times $T_n(r)$ as the end-time of the n-th excursion of length at least r^2 of $|X^{(0)}|$. Up to time $T_1(r)$, we perform the exploration using the driving function $W_t^{(0)}$ (that is defined using the symmetrized Bessel process) except that the sign of the last excursion may have been changed depending on the sign $\varepsilon_1(r)$. Note that we have just recalled that this procedure defines exactly loops of a CLE_κ. Note that $T_1(r)$ is the end-time of an excursion and corresponds exactly to the completion of a CLE loop. Then, we force a jump of $-\beta Cr2\sqrt{\kappa}$ of the driving function, and continue from there until time $T_2(r)$ by following the dynamics of $W_t^{(0)}$ (possibly changing the sign of the last excursion before $T_2(r)$). At $T_2(r)$, we again wake a jump of $-\beta Cr2\sqrt{\kappa}$. Combining the previous two items shows that for any n and r, the following procedure constructs a CLE sample:

- Sample the previous process until time $T_n(r)$. Keep all the loops corresponding to the excursions of X.

- In each connected components created along the way by this Loewner chain (one can view these connected components as those of the complement of the closure of the union of the interior of all the loops created by the chain – because these loops are "dense" in the created chain) as well as in the remaining unbounded component, sample independent CLE_κ's.

Note that the result still holds, if for some given r, one stops the process at a stopping time which is the end-time of some loop (not necessarily a $T_n(r)$).

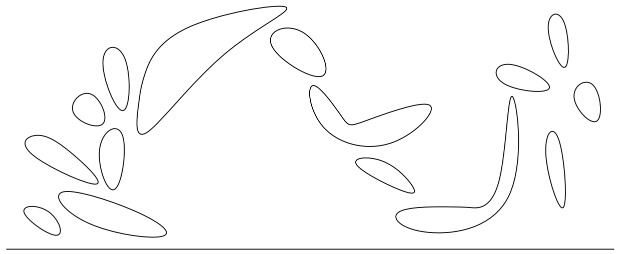
Let us now look at the driving function of the previously defined Loewner chain. It is exactly the one that would obtain if the signs of the excursions of length larger than r^2 are those given by $\varepsilon_1(r), \varepsilon_2(r)$ etc., and one also puts in the negative jumps at each time $T_n(r)$. This corresponds exactly to the driving function

$$W_t^{(\beta,r)} := \sqrt{\kappa} X_t^{(\beta,r)} + 2\sqrt{\kappa} I_t^{(\beta,r)}.$$

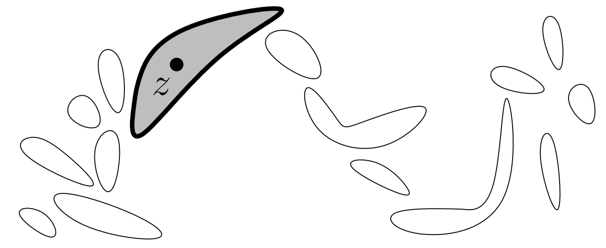
We now need to control what happens when $r \to 0$. A first important observation is that, as argued in the previous sections, the function $W^{(\beta,r)}$ converges uniformly towards $W^{(\beta)}$ on any compact interval. On the other hand, we have just described a relation between the Loewner chain/the loops generated by $W^{(\beta,r)}$ and CLE_κ. Our goal is now to deduce a similar statement for the Loewner chain generated by $W^{(\beta)}$. For that chain, let $H_t^{(\beta)}$ denote the complement of the chain at time t (using the usual notations, and the time-parametrization of $W^{(\beta)}$).

For simplicity, let us first focus only on the law of the loop that surrounds one fixed interior point $z \in \mathbb{H}$, in a $\text{CLE}_\kappa^\beta(0)$. Two scenarios can occur. Either for the Loewner chain (without branching, and targeting infinity) generated by $W^{(\beta)}$, z is at some point swallowed by a quasi-loop (that we then call $\gamma^{(\beta)}(z)$) or not. Let $A_1^{(\beta)}(z)$ and $A_2^{(\beta)}(z)$ denote these two events. If $A_2^{(\beta)}(z)$ holds, then let $\tau^{(\beta)}(z) = \sup\{t > 0 \ : \ z \in H_t^{(\beta)}\}$.

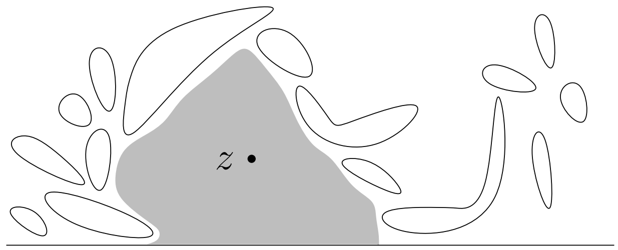
When $A_1^{(\beta)}(z)$ holds, the quasi-loop $\gamma^{(\beta)}(z)$ corresponds to a time-interval $(t_-(e), t_+(e))$ for some excursion e, that in turn corresponds (for each r) to some loop $\gamma^{(\beta,r,*)}$ traced by the Loewner chain driven by $W^{(\beta,r)}$. The almost sure convergence of $W^{(\beta,r)}$ to $W^{(\beta)}$ implies that (with full probability on the event that $\gamma^{(\beta)}(z)$ exists) in Carathéodory topology seen from z, the inside of the loop corresponding to $\gamma^{(\beta,r,*)}$ converges to the inside of $\gamma^{(\beta)}(z)$ as $r \to 0$ (note that the loops $\gamma^{(\beta,r,*)}$ will surround z for all small enough r).



(a) Loops generated by $W^{(\beta)}$.



(b) z is contained in a loop.



(c) The connected component containing z.

Figure 3.3.1: Two scenarios for z.

It is a little more tricky to handle the case where $A_2^{(\beta)}(z)$ holds. Note that then $\tau^{(\beta)}(z)$ cannot belong to some $(t_-(e), t_+(e)]$ (because this interval corresponds to a quasi-loop that is locally drawn like a simple slit). Furthermore, in this case, for any time t before $\tau^{(\beta)}(z)$, z belongs to the still to be explored unbounded simply connected domain $H_t^{(\beta)}$. Target-independence of the SLE$(\kappa, \kappa-6)$ processes yields that modulo reparametrization, one can view the evolution of the Loewner chain before $\tau^{(\beta)}(z)$ as a radial SLE$(\kappa, \kappa-6)$ targeting z as described before. In particular, this implies that the conformal radius of $H_t^{(\beta)}$ seen from z decreases continuously to some limit when t increases to $\tau^{(\beta)}(z)$. This limit is the conformal radius of the simply connected domain that we denote by $\Omega^{(\beta)}(z)$ and that is the decreasing limit (seen from z in Carathéodory topology) of $H_t^{(\beta)}$ as t increases to $\tau^{(\beta)}(z)$ (note that we have not excluded the – somewhat unlikely – case where $\tau^{(\beta)}(z) = \infty$ when $A_2^{(\beta)}(z)$ holds).

In the next couple of paragraphs, since z is fixed, we will omit to mention the dependence of $\tau^{(\beta)}(z)$, $\Omega^{(\beta)}(z)$ etc on z, and write $\tau^{(\beta)}$, $\Omega^{(\beta)}$ instead.

In the same way, we can define almost surely, for each r, $\gamma^{(\beta,r)} = \gamma^{(\beta,r)}(z)$ or $\Omega^{(\beta,r)} = \Omega^{(\beta,r)}(z)$ depending on whether the Loewner chain drawn by $W^{(\beta,r)}$ creates a quasi-loop around z or not (here, we use the target-independence of the symmetric SLE$(\kappa, \kappa-6)$). One problem to circumvent is that the fact that $W^{(\beta,r)}$ converges uniformly to $W^{(\beta)}$ on any compact interval is not enough to ensure that $\Omega^{(\beta,r)}$ converges to $\Omega^{(\beta)}$, for instance because just before the disconnection time, a very small fluctuation of the driving function can create a path that enters and exits this soon-to-be-cut-off domain. An additional problem is that it could be that, even for small r, $\tau^{(\beta,r)}(z)$ is much larger than $\tau^{(\beta)}(z)$.

One way around this is to introduce additional stopping times that approximate $\tau^{(\beta)}$ from below. For instance, for each ε, define $\hat{\tau}_\varepsilon^{(\beta)}$ to be the first time t at which it is possible to disconnect z from infinity in H_t by removing from H_t a ball of radius ε. Then, let $\tilde{\tau}_\varepsilon^{(\beta)} = \min(1/\varepsilon, \hat{\tau}_\varepsilon^{(\beta)})$ (this is just to take care of the possibility that $\tau^{(\beta)} = \infty$). Then finally, if this time belongs to some quasi-loop interval $(t_-(e), t_+(e))$, define $\tau_\varepsilon^{(\beta)}$ to be the end-time $t_+(e)$ of this quasi-loop, and otherwise let $\tau_\varepsilon^{(\beta)} = \tilde{\tau}_\varepsilon^{(\beta)}$. Note that $\tau_\varepsilon^{(\beta)}$ is a stopping time with respect to the filtration of $W^{(\beta)}$.

Clearly, for any small ε, $\tau_\varepsilon^{(\beta)} < \tau^{(\beta)}$ as soon as $\tau^{(\beta)}$ is finite (recall that $\tau^{(\beta)}$ cannot be equal to some $t_+(e)$). On the other hand, $\tau_\varepsilon^{(\beta)}$ cannot increase to anything else than $\tau^{(\beta)}$ as ε decreases to 0.

Furthermore, the convergence of $W^{(\beta,r)}$ to $W^{(\beta)}$ ensures that for each given ε, the domain $H^{(\beta,r)}_{\tau_\varepsilon^{(\beta)}}$ converges in Carathéodory topology (seen from z) to $H^{(\beta)}_{\tau_\varepsilon^{(\beta)}}$ when $r \to 0$. Hence, we conclude that for a well-chosen $r(\varepsilon)$ (small enough), as $\varepsilon \to 0$ along some well-chosen sequence, almost surely on the even $\tau^{(\beta)} < \infty$, the sequence of domains $H^{(\beta,r(\varepsilon))}_{\tau_\varepsilon^{(\beta)}}$ converges in Carathéodory topology to $\Omega^{(\beta)}$.

For each given ε, choose $r(\varepsilon)$ very small, we can construct a loop surrounding z as follows, using the Loewner chain associated to $W^{(\beta,r)}$. If this chain has traced a loop $\gamma^{(\beta,r,\varepsilon)}$ that surrounds z along the way before $\tau_\varepsilon^{(\beta)}$, then just keep this loop. If not, then sample in the domain $H^{(\beta,r(\varepsilon))}_{\tau_\varepsilon^{(\beta)}}$ an independent CLE$_\kappa$, look at the loop that surrounds z in this sample, and call it $\gamma^{(\beta,r,\varepsilon)}$. Our previous arguments relating the loops traced by $W^{(\beta,r)}$ to CLE$_\kappa$ (note that $\tau_\varepsilon^{(\beta)}$ is a stopping time at which all these Loewner chains complete a quasi-loop) show that for each fixed ε and r, the law of $\gamma^{(\beta,r,\varepsilon)}$ is the same as the law of the loop that surrounds z in an CLE$_\kappa$ in $\mathbb{H}$.

Note finally that in the case $A_1^{(\beta)}$ where a quasi-loop $\gamma^{(\beta)}$ is discovered by the Loewner chain driven by $W^{(\beta)}$, then (as we have argued before) the same holds true for the Loewner chain driven by $W^{(\beta,r)}$ for all small r, and that furthermore, for all small ε and r, $\gamma^{(\beta,r,\varepsilon)} = \gamma^{(\beta,r)}$.

Hence, we can conclude that for a well-chosen sequence $\varepsilon_n \to 0$ (with a well-chosen $r(\varepsilon_n)$), in the limit when $n \to \infty$, the constructed loop (which has always the law of $\gamma(z)$ in a CLE$_\kappa$ in $\mathbb{H}$ as we have just argued) is:

- Either $\gamma^{(\beta)}$ if $A_1^{(\beta)}$ holds
- Or otherwise, obtained by sampling a CLE$_\kappa$ in $\Omega^{(\beta)}$.

But now, because we know that this procedure defines a loop that is distributed like $\gamma(z)$ is a CLE$_\kappa$, one can use the branching idea, and iterate this result within $\Omega^{(\beta)}$. We then readily get that the law of the quasi-loop that surrounds z in a CLE$_\kappa^{(\beta)}(0)$ is identical to the law of the loop that surrounds z for a CLE$_\kappa$.

In fact, in order to describe the law of CLE$_\kappa^{(\beta)}(0)$, we have to see what happens when one looks at the joint distribution of the loops that surround n given points $z_1, \ldots, z_n$. The argument is almost identical: The main difference is that one first replaces $\tau^{(\beta)}(z)$ by $\tau^{(\beta)} := \tau^{(\beta)}(z_1, \ldots, z_n) = \min(\tau^{(\beta)}(z_1), \ldots, \tau^{(\beta)}(z_n))$ and also changes $\tau_\varepsilon^{(\beta)}(z)$ into $\tau_\varepsilon^{(\beta)} := \tau_\varepsilon^{(\beta)}(z_1, \ldots, z_n)$ which is associated to the first time at which adding some ball of radius ε to the Loewner chain disconnects at least one of these n points from infinity in $H_t^{(\beta)}$. The rest of the procedure is basically unchanged and readily leads to the fact that the joint distribution of the set of loops that surround these points in a $\text{CLE}_\kappa^{(\beta)}(x)$ or in a CLE_κ are identical, which completes the proof of Proposition 3.3.1. The only additional ingredient is to note that when one considers a decreasing sequence of open simply connected sets Ω_n, the law of the CLE_κ in Ω_n (as described via its finite-dimensional marginals) converges to the law obtained by sampling independent CLE_κ's in the different connected components of the interior of $\cap\Omega_n$ (which is a fact that follows for instance directly from the description of CLE_κ as loop-soup cluster boundaries). This ensures that in the limit when $n \to \infty$ (for well-chosen ε_n and r_n), on the event $\tau^{(\beta)}(z_1, \dots, z_n) < \infty$, sampling a CLE_κ in $H^{(\beta, r_n)}_{\tau^{(\beta)}_{\varepsilon_n}}$ converges in law (in the sense of finite-dimensional distributions) to sampling two independent CLE_κ's in $H^{(\beta)}_{\tau^{(\beta)}}$ and in the cut-out component $\Omega^{(\beta)}$ at time $\tau^{(\beta)}$.

For $\kappa = 4$ and $\mu \in \mathbb{R}$. In the same spirit, we choose the driving function

$$W_t^{<\mu,r>} := W_t + 4\mu r N_r(t) = 2B_t + 4(I_t^{(0)} + \mu r N_r(t)).$$

Then, the very same arguments as before show that the corresponding constructed loops are those of a CLE. And on the other hand, the driving function $W^{<\mu,r>}$ converges to $W^{<\mu>}$. This allows to complete the proof of the following fact:

Proposition 3.3.2. *When $\kappa = 4$, the law of $CLE_{4,\mu}(x)$ does depend neither on x nor on μ.*

3.4 The uniform exploration of CLE_4

Let us now modify the "symmetric" Loewner driving function $W^{(0)}$ by introducing some random jumps. Basically, at each time $T_n(r)$ (the end-times of the excursions of X of time-length at least r^2), we decide to resample the position of the driving function according to the uniform density in $[-m, m]$ – here m should be thought of as very large, we will then let it go to infinity.

Suppose for a while that m is fixed (we will omit to mention the dependence in m during the next paragraphs in order to avoid heavy notation). Let us associate to each excursion e of X a random variable $\xi(e)$ with this uniform distribution on $[-m, m]$, in such a way that conditionally on X, all these variables ξ are i.i.d. (for notational simplicity, we sometimes also write $\xi = \xi(T)$ when T is the time of X at which the corresponding excursion is finished). Then, we define the function $t \mapsto \hat{W}_t^r$ as follows: $T_0(r) = 0$ and for each $n \geq 0$,

- $\hat{W}^r(T_n) = \xi(T_{n+1})$.
- $\hat{W}^r - W^{(0)}$ is constant on each interval $[T_n(r), T_{n+1}(r))$.

The function $\hat{W}^r$ is piecewise continuous, and it is therefore the driving function of some Loewner chain. The very same arguments as before show that for each given r, it defines a family of loops distributed like loops in a CLE_κ (in the sense that, just as before, when one completes the picture with independent CLE's in the not-yet-filled parts, one obtains a CLE_κ sample (note that the jump distribution – i.e. the choice of the new point according to the uniform distribution – is in fact independent of the future behavior of X).

It is easy to understand what happens to this construction when r tends to 0. As before, we are going to look at the almost sure behavior of $\hat{W}^r$ when $r \to 0$, for a given sample of $W^{(0)}$ and ξ's .

Let us define the process $\hat{W}$ by the fact that for each excursion e corresponding to a time-interval (S,T),

$$\hat{W}_t = (W_t^{(0)} - W_S^{(0)}) + \xi(T)$$

for $t \in [S,T)$ (this defines $\hat{W}$ for all t, except on the zero-Lebesgue measure set of times that are not in the time-span of some excursion, for those times, we can choose $\hat{W}$ as we wish). Then, clearly, the fact that $t \mapsto W_t^{(0)}$ is continuous ensures that for each given excursion interval, $\hat{W}^r$ converges uniformly to $\hat{W}$ as $r \to 0$ on this time-interval, because for small enough r, $\hat{W}^r = \hat{W}$ on this excursion. It follows readily that the Loewner chain generated by $\hat{W}^r$ does (almost surely) converge (in Carathéodory topology) to the one generated by $\hat{W}$.

Hence, using the same arguments as above (the law of the traced loops is always that of loops in CLE, that are simple disjoint loops, the excursion-intervals correspond to the loops, and these intervals are the same for all r), we conclude that during each excursion time-interval, the driving process $\hat{W}$ does indeed trace a loop, and that the joint law of all these loops are those of loops in a CLE.

Let us now rephrase all the above construction in terms of the Poisson point process of Bessel excursions $(e_u, u \geq 0)$. As we have explained earlier, each Bessel excursion in fact corresponds (via Loewner's equation) to a two-dimensional loop in the upper-half plane, that touches the boundary only at the origin. Let us call γ_u the loop corresponding to e_u. To each excursion e_u of the Bessel process, we also associate a random position $x_u \in \mathbb{R}$ sampled according to the uniform measure on $[-m,m]$ (more precisely, conditionally on all Bessel excursions (e_{u_j}), the random variables (x_{u_j}) are i.i.d. with this distribution). Then, we define the loop $\hat{\gamma}_u$ by shifting γ_u horizontally by x_u (and so, the loop $\hat{\gamma}_u$ touches the real axis at x_u).

For each excursion e_u, we can now define the conformal transformation $\hat{f}_u$ from the connected component of $\mathbb{H} \setminus \hat{\gamma}_u$ that contains i onto $\mathbb{H}$ such that $\hat{f}_u(i) = i$ and $\hat{f}_u'(i) \in \mathbb{R}_+$. As this will be useful, we now reintroduce the dependence on m in the notation for these maps (and write $\hat{f}_u = \hat{f}_u^m$).

For a given m, we start with a Poisson point process $(e_u, \geq 0)$ of Bessel excursions defined under the measure $2m\lambda$ and we then associate to each excursion the uniform random variable $\xi(e_u)$. A cleaner equivalent way to describe the process $((e_u, \xi_u), u \geq 0)$ is to say that it is a Poisson point process with intensity $\lambda \otimes dx 1_{x \in (-m,m)}$.

Then, clearly, we get a Poisson point process $(\hat{f}_u^m, u \geq 0)$ of such conformal maps (because for each u, $\hat{f}_u^m$ is a deterministic function of the pair (e_u, x_u) and $((e_u, x_u), u \geq 0)$ is a Poisson point process).

For each $u > 0$, one can then define

$$\hat{F}_u^m = \circ_{v<u} \hat{f}_v^m$$

(where the composition is done in the order of appearance of the maps $\hat{f}_v$). Clearly, $\hat{F}_u^m$ corresponds to the Loewner map (generated by the driving function $\hat{W}$) at the time (in the Loewner time parametrization) corresponding to the completion of all loops $\hat{\gamma}_v^m$ for $v < u$. In other words, if $\tau(e_u)$ is the time-length of the excursion e_u, the Loewner time at which the loop corresponding to that excursion will start being traced is $\sum_{v<u} \tau(e_v)$.

Hence, the loops

$$\tilde{\gamma}_u^m := (\hat{F}_u^m)^{-1}(\hat{\gamma}_u^m)$$

are distributed like CLE loops. In particular, the loop that contains i will be the loop $(\hat{F}_\tau^m)^{-1}(\hat{\gamma}_\tau^m)$ where

$$\tau = \inf\{u \geq 0 \ : \ \hat{\gamma}_u^m \text{ surrounds } i\}.$$

Let us rephrase what we have done so far: For each m, we have seen that one can define CLE loops by considering the Loewner chain generated by $\hat{W}$, using the Poisson point process $(\hat{f}_u^m, u \geq 0)$, or equivalently, via the Poisson point process $\hat{\Gamma}^m := (\hat{\gamma}_u^m, u \geq 0)$ with intensity measure

$$M^m = \int_{-m}^{m} dx \mu^x$$

where μ^x denotes the measure on loops rooted at x (like in [SW12], we define this measure as the measure μ^0 on loops in the upper half-plane generated via a Bessel excursion defined under λ, and shifted horizontally by x).

Now, let us describe what happens when $m \to \infty$. Suppose now that we consider the Poisson point process $\hat{\Gamma} := (\hat{\gamma}_u, u \geq 0)$ with intensity

$$M := \int_{\mathbb{R}} dx \mu^x$$

and the corresponding iterations of maps $\hat{F}_u$. Even though this measure seems "even more infinite" than M^m, this iteration of conformal maps does not explode. This is due to the scaling properties of μ^x and to the fact that one normalizes always at i (so that loops rooted far away do not contribute much the derivative at i) – one can for instance justify this using Lemma 3.4.2 below.

Note also that if we keep only those loops in $\hat{\Gamma}$ that are rooted at a point in $[-m, m]$, we obtain a process with the same law as $\hat{\Gamma}^m$. The key observation is now to see that when $m \to \infty$, each map $\hat{F}_u^m$ converges uniformly in any compact subdomain of the closed upper half-plane to $\hat{F}_u$. This implies the following:

Lemma 3.4.1. *The loops $\tilde{\Gamma} = (\tilde{\gamma}_u := \hat{F}_u^{-1}(\hat{\gamma}_u), u \leq \tau)$ are also distributed like loops in a CLE.*

At this stage, everything we have said is still true if we replace the Lebesgue measure on $\mathbb{R}$ by (almost) any other given distribution on $\mathbb{R}$, and any $\kappa \in (8/3, 4]$. An important reason to choose this particular measure and to focus on the case where $\kappa = 4$ is that the following Lemma holds only in this case:

Lemma 3.4.2. *When $\kappa = 4$ the measure M is conformal invariant.*

Proof. Recall from [SW12] that when $\kappa = 4$ and if Φ is a conformal transformation of the half-plane onto itself,

$$\Phi \circ \mu^x = |\Phi'(x)| \mu^{\Phi(x)}$$

where the measure $\Phi \circ \mu^x$ is defined by

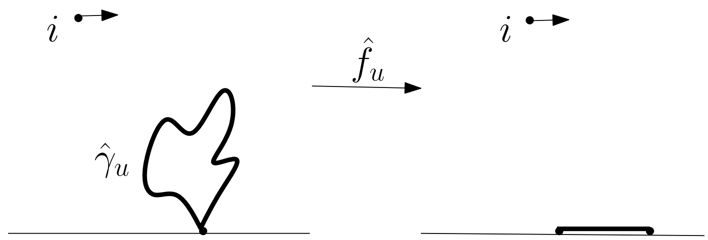
$$\Phi \circ \mu^x(A) = \mu^x\{\gamma : \Phi(\gamma) \in A\}.$$

Hence, it follows immediately that $\Phi \circ M = M$. □

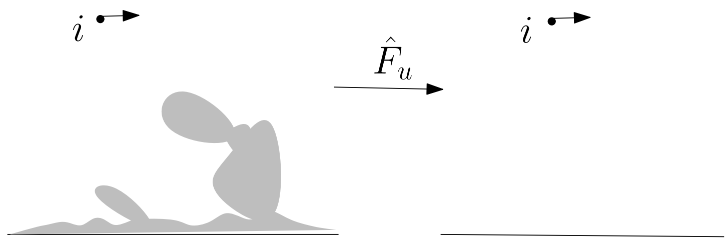
A direct consequence of this conformal invariance is that

Corollary 3.4.3. *When $\kappa = 4$, the law of $\tilde{\Gamma} = (\tilde{\gamma}_u, u \leq \tau)$ is invariant under any Moebius transformation Φ of the upper half-plane that preserves i.*

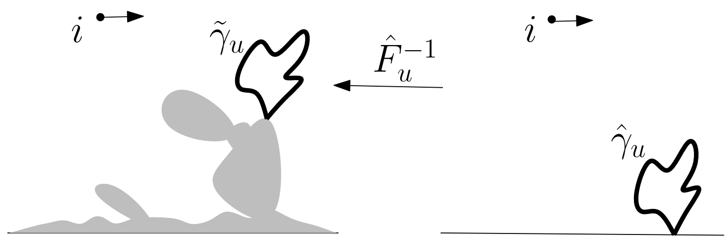
Note that there is no time-change involved. The law of $\Phi(\tilde{\gamma}_u) 1_{u \leq \tau}$ and $\tilde{\gamma}_u 1_{u \leq \tau}$ are for instance identical.



(a) $\hat{f}_u$ fixes i and $\hat{f}_u'(i) > 0$.



(b) $\hat{F}_u$ is the composition of $\hat{f}_v$ for $v < u$.



(c) $\tilde{\gamma}_u$ is the preimage of $\hat{\gamma}_u$ under $\hat{F}_u$.

Figure 3.4.1: construction of $\tilde{\gamma}_u$.

Proof. Let Φ be a Moebius transformation of the upper half-plane that preserves i, and $(\hat{\gamma}_u, u \geq 0)$ be a Poisson point process with intensity M. And define $\tau, \hat{f}_u$ for $u < \tau$, and $\hat{F}_u, \tilde{\gamma}_u$ for $u \leq \tau$ as described above.

Note that $(\bar{\gamma}_u := \Phi(\hat{\gamma}_u), u \geq 0)$ is a Poisson point process with intensity $M = \Phi \circ M$, and it has therefore the same distribution as $(\hat{\gamma}_u, u \geq 0)$. For $u < \tau$, let $\bar{f}_u$ be the conformal map from the connected component of $\mathbb{H} \setminus \bar{\gamma}_u$ that contains i onto $\mathbb{H}$ such that $\bar{f}_u(i) = i$ and $\bar{f}_u'(i) \in \mathbb{R}_+$. It is easy to see that

$$\bar{f}_u = \Phi \circ \hat{f}_u \circ \Phi^{-1}$$

and hence for $u \leq \tau$,

$$\bar{F}_u := \circ_{v<u} \bar{f}_v = \Phi \circ \hat{F}_u \circ \Phi^{-1}.$$

As a result, for $u \leq \tau$,

$$\Phi(\tilde{\gamma}_u) = \Phi(\hat{F}_u(\hat{\gamma}_u)) = \bar{F}_u(\bar{\gamma}_u).$$

Since $(\bar{\gamma}_u, u \geq 0)$ has the same distribution as $(\hat{\gamma}_u, u \geq 0)$, it follows that $(\Phi(\tilde{\gamma}_u), u \leq \tau)$ has the same distribution as $(\tilde{\gamma}_u, u \leq \tau)$. □

In fact, a stronger result holds. Let us now choose some other point z than i in the upper half-plane. Let σ denote the first moment if it exists at which the process $(\tilde{\gamma}_u, u \leq \tau)$ disconnects i from z. If the loop $\tilde{\gamma}_\tau$ surrounds both i and z, we simply set $\sigma = \tau$. Note that the event that $\sigma < \tau$ can happen when the process discovers a loop surrounding one of the two points and not the other, but at this stage, it is not excluded that it can disconnect two points strictly before discovering the loops that surround them, just like the symmetric SLE$(\kappa, \kappa-6)$ does, see [SW12].

Define the same process $(\tilde{\gamma}^z_u, u \leq \tau^z)$ as above, except that we choose to normalize "at z" instead of normalizing at i. One way to describe it would be to write $\tilde{\gamma}^z_u = \varphi(\tilde{\gamma}_u)$, where φ is the affine transformation from $\mathbb{H}$ onto itself such that $\varphi(i) = z$ (but we will use another way to describe it in terms of $\tilde{\gamma}_u$ in a moment). We then define σ^z to be the first moment at which it disconnects z from i or discovers the loop that surrounds both points.

Lemma 3.4.4. *When $\kappa = 4$, and for any $z \in \mathbb{H}$, the law of $(\tilde{\gamma}^z_u, u \leq \sigma^z)$ is identical to the law of $(\tilde{\gamma}_u, u \leq \sigma)$.*

We shall use the following classical result about Poisson point process (see for instance [Ber96, Section 0.5]:

Result. *Let $(a_u, u \geq 0)$ be a Poisson point process with some intensity ν (defined in some metric space A). Let $\mathfrak{F}_{u-} = \sigma(a_v, v < u)$. If $(\Phi_u, u \geq 0)$ is a process (with values on functions of A onto A) such that for any $u \geq 0$, Φ_u is $\mathfrak{F}_{u-}$-measurable, and that Φ_u preserves ν then $(\Phi_u(a_u), u \geq 0)$ is still a Poisson point process with intensity ν.*

We are now ready to prove the lemma.

Proof. Consider the process $(\tilde{\gamma}_u, u \leq \tau)$ defined from the Poisson point process $(\hat{\gamma}_u, u \geq 0)$ with intensity M as above (and keep the same definitions for τ, $\hat{f}_u$ with $u < \tau$, $\hat{F}_u, \tilde{\gamma}_u$ with $u \leq \tau$, where the latter are defined by normalizing the maps at i).

We denote $\mathfrak{F}_{u-} = \sigma(\hat{\gamma}_v, v < u)$. For $u < \sigma$, $\hat{F}_u^{-1}(\mathbb{H})$ is a simply connected domain of $\mathbb{H}$ containing z. Let G_u be the conformal map from $\hat{F}_u^{-1}(\mathbb{H})$ onto $\mathbb{H}$ normalized at z by $G_u(z) = z$ and $G'_u(z) \in \mathbb{R}_+$. Define for each $u < \sigma$

$$\Phi_u = G_u \circ \hat{F}_u^{-1}.$$

We also define $\Phi_\sigma = \lim_{u \to \sigma-} \Phi_u$, and we say that Φ_u is the identity map for all $u > \sigma$. It is clear that, for each positive u, the map Φ_u is a $\mathfrak{F}_{u-}$-measurable Moebius transformation from the upper half-plane onto itself. Hence, the process $(\bar{\gamma}_u := \Phi_u(\hat{\gamma}_u), u \geq 0)$ is also Poisson point process with intensity M.

If we use the point process $(\bar{\gamma}_u)$ to construct the process $(\tilde{\gamma}^z_u)$ normalized at z, we get a coupling of $(\tilde{\gamma}_u)$ and $(\tilde{\gamma}^z_u)$ in such a way that they coincide up to time σ: For all $u < \sigma$, $\tilde{\gamma}_u = \tilde{\gamma}^z_u$, and in addition,

$$\bar{\gamma}_\sigma = \Phi_\sigma(\hat{\gamma}_\sigma)$$

(if this $\hat{\gamma}_\sigma$ exists) so that $\tilde{\gamma}_\sigma = \tilde{\gamma}^z_\sigma$.

Hence, with this coupling, we see that $\sigma \leq \sigma^z$ almost surely. By symmetry (because there exists a conformal map interchanging these two points), it follows that $\sigma = \sigma^z$ almost surely. □

This means that it is possible to couple these two processes up to the first moment at which it disconnects i from z. By scaling, this shows that for any pair of points z and z', we can couple the two processes $\tilde{\gamma}^z$ and $\tilde{\gamma}^{z'}$ up to the first time at which they disconnect z from z'. Hence, it is possible to couple the processes $\tilde{\gamma}^z$ for *all* $z \in \mathbb{H}$ simultaneously in such a way that for any two points z and z', the previous statement holds.

If we now use such a coupling, we get a Markov process on domains $(D_u, u \geq 0)$: At time $u = 0$, the domain is the upper half-plane, and at time $u > 0$, it is the union of all the (disjoint) open sets corresponding to the evolution to all points z at time u. Existence of such a conformally invariant process is a rather striking feature, as it uses no reference point, and the time of the evolution is preserved through the conformal transformation. We can therefore sum up the properties of this process as follows.

Proposition 3.4.5. *The process $(D_u, u \geq 0)$ provides a way to construct CLE$_4$. Furthermore, the processes $(D_u, u \geq 0)$ and $(\Phi(D_u), u \geq 0)$ are identically distributed (with no time-change) for all Moebius transformations Φ.*

Note that this uniform exploration mechanism can also be viewed as the limit (in law) of the asymmetric CLE$_{4,\mu}$ construction of Proposition 3.3.2 in the limit when $\mu \to \infty$ (and the boundary points $+\infty$ and $-\infty$ of the upper half-plane are identified, alternatively, one can state this easily in the radial setting). We leave the details to the interested reader.

3.5 Comments and open questions

Some open questions.

We first mention some natural open questions that are closely related to the current paper:

It is proved in [SS09, Dub09b] that an SLE$_4$ can be deterministically drawn as the contour lines (or sometimes called level-lines or cliff-lines) in a Gaussian Free Field with appropriate boundary conditions. See also [MS13a] for the fact that the entire CLE$_4$ can be deterministically embedded in a Gaussian Free Field. Note that the symmetric exploration process looks a priori more naturally associated to the Gaussian Free Field than the asymmetric ones, because when one defines a Gaussian Free Field out of a CLE$_4$ with the coupling described in [MS13a], one has to toss an independent coin for each CLE$_4$ loop to decide an orientation, so that the symmetric SLE$(4,-2)$ (including the coin tosses) is defined via the randomness present in the GFF. However, as shown in [SWW13], the dynamic "uniform" construction of SLE$_4$ described in Proposition 3.4.5 actually also yields a rather natural construction of the GFF.

In general, the conformally invariant ways to construct a CLE that we described in the current paper via these branching Loewner chains induce additional information than just the CLE (which loop is discovered where, what is the starting and end-point of the loop when one uses a given exploration etc.). This leads naturally to the following open questions:

1. If we are given a CLE$_\kappa$ in a simply connected domain D and a starting point on the boundary of D, and a family of Bernoulli random variables $\varepsilon(\gamma)$ with parameter β (one for each CLE loop γ), is the asymmetric exploration process with parameter β deterministically defined?

2. In particular, is the totally asymmetric exploration process (when $\beta = 1$) sample a deterministic function of the CLE sample and of the starting point?

3. Is the uniform exploration process in fact a deterministic function of the CLE$_4$?

A positive answer to this last question would give rise to a conformally invariant distance between loops in a CLE$_4$. This will be investigated further in the forthcoming paper [SWW13].

Discrete explorations

Our proofs rely a lot on the fact that the symmetric Bessel explorations do indeed construct the loops in a CLE, which was derived in [SW12] using a discretization of the exploration procedure that was proved to converge to the symmetric Bessel construction. The other CLE constructions that we have studied in the current paper also have natural discrete counterparts that we now briefly describe. However, it turns out to be (seemingly) technically more unpleasant to control the convergence of these asymmetric discrete exploration procedures than the symmetric ones, so that it seemed simpler to derive our results building on the relation between CLE's and the symmetric construction.

We first recall the exploration procedures described in [SW12] to explore a CLE little by little (here a CLE is just a collection of loops that satisfy the CLE axioms defined in [SW12]). It will be easier to explain things in the radial setting i.e. in the unit disk instead of the half-plane.

Suppose that $\Gamma = (\gamma_j, j \in J)$ is a CLE in the unit disc $\mathbb{U}$ and that $\varepsilon > 0$ is given. We denote $\gamma(z)$ as the loop of Γ (when it exists) surrounding $z \in \mathbb{U}$. Throughout this subsection, $D(1,\varepsilon)$ will denote the image of the set $\{z \in \mathbb{H} : |z| < \varepsilon\}$ under the conformal map $\Psi : z \mapsto (i-z)/(i+z)$ from the upper half-plane $\mathbb{H}$ onto the unit disc such that $\Psi(i) = 0, \Psi(0) = 1$. Note that for small ε, this set is rather close to a small semi-disc centered at 1.

At the first step, we "explore" the small shape $D(1,\varepsilon)$ in $\mathbb{U}$, and we discover all the loops in Γ that intersect $D(1,\varepsilon)$. If $\gamma(0)$ has already been discovered during this first step, we define $N = 1$ and we stop. Otherwise, we let U_1 denote the connected component that contains the origin of the set obtained when removing from $U_1' = \mathbb{U} \setminus D(1,\varepsilon)$ all the loops that do not stay in U_1'. From the restriction property in the CLE axioms, the conditional law of Γ restricted to U_1 (given U_1) is that of a CLE in this domain.

We now choose some point x_1 on ∂U_1, and the conformal map φ_1^ε from U_1 onto $\mathbb{U}$ such that $\varphi_1^\varepsilon(0) = 0$ and $\varphi_1^\varepsilon(x_1) = 1$. Note that we allow here for different possible choices for x_1. It can be a deterministic function of U_1, but the choice of x_1 can also involve additional randomness (we can for instance choose it according to the harmonic measure at the origin etc.), but we impose the constraint that conditionally on U_1, the CLE restricted to U_1 and the point x_1 are conditionally independent (in other words, one is not allowed to use information about the loops in U_1 in order to choose x_1).

During the second step of the exploration, one discovers the loops of $\Gamma_1 := \varphi_1^\varepsilon(\Gamma \cap U_1)$ that intersect $D(1,\varepsilon)$. In other words, we consider the pushforward of Γ by φ_1^ε (which has the same law as Γ itself, due to the CLE axioms) and we repeat step 1. If we discover a loop that surrounds the origin at that step, then we stop and define $N = 2$. Otherwise, we define the connected component U_2 that contains the origin of the domain obtained when removing from $\mathbb{U} \setminus D(1,\varepsilon)$ the loops of Γ_1 that do not stay in this domain, and we define the conformal map φ_2^ε from U_2 onto $\mathbb{U}$ with $\varphi_2^\varepsilon(0) = 0$ and $\varphi_2^\varepsilon(x_2) = 1$, where x_2 is chosen in a conditionally independent way of $\Gamma_1 \cap U_2$, given U_1, U_2 and x_1.

We then explore $\Gamma_2 := \varphi_2^\varepsilon(\Gamma_1)$ and so on. We can iterate this procedure until the step N at which we eventually "discover" a loop that surrounds the origin. Note that $\gamma(0)$ (the loop in Γ that surrounds the origin) is the preimage of this loop (the loop in Γ_N that surrounds the origin and intersects $D(1,\varepsilon)$) under $\varphi_N^\varepsilon \circ \cdots \circ \varphi_1^\varepsilon$.

In this definition, the discrete exploration "strategy" is encoded by ε (the "step-size") and by the rule used to choose the x_n's. Since the probability to discover the loop at each given step n (conditionally on the fact that it has not been discovered before) is constant and positive, it follows that N is almost surely finite, that its law is geometric (regardless of the choice of x_n's).

In [SW12], it is shown that if a CLE (ie. satisfying the CLE axioms exist), then its loops are of SLE_κ-type for some $\kappa \in (8/3, 4]$ and that it is necessarily the one constructed via the symmetric Bessel construction (and therefore unique). Conversely (using a different argument involving loop-soups) it is shown that these CLE do exist. The strategy of one part of the proof is to control the behavior of certain natural discrete exploration strategies when ε tends to 0:

The first one is the "exploration normalized at the origin". Here, at each step, x_n and φ_n^ε are chosen according to the rule that $(\varphi_n^\varepsilon)'(0)$ is a positive real number. In other words, we choose φ_n^ε using the standard normalization at the origin.

Towards the end of the paper [SW12], it is shown that the symmetric exploration indeed constructs an axiomatic CLE, by using the following symmetric discrete exploration procedure: Define 1_ε^+ and 1_ε^- the two intersections of $\partial D(1, \varepsilon)$ with the unit circle. At each step, one tosses a (new) fair coin to decide which one of the two points gets mapped conformally onto 1. In other words, the maps φ_n^ε are i.i.d., x_1 is independent of U_1, and

$$P(x_1 = 1_\varepsilon^+) = P(x_1 = 1_\varepsilon^-) = 1/2.$$

The definition of the asymmetric discrete explorations is then natural: For a given β, we toss a $(1+\beta)/2$ vs. $(1-\beta)/2$ coin in order to chose which one of the two points 1_ε^+ or 1_ε^- to choose, but in order to compensate the created bias, we post-compose the obtained map $\tilde{\varphi}_n^\varepsilon$ with a deterministic rotation of some angle $\theta(\varepsilon)$ that vanishes as $\varepsilon \to 0$ (that corresponds to the jump in the approximation $I^{(\beta,r)}$ of $I^{(\beta)}$). However, we see that this rotation depends on the chosen base-point (here the origin); this is one reason for which this discrete approximation is a little harder to master than in the case $\beta = 0$.

The definition of uniform discrete approximations is also very natural: Just choose x_n at random on the boundary of U_n according to the harmonic measure seen from 0. Equivalently, choose any φ_n^ε and compose it with a uniformly chosen rotation. Again, this rule depends on the target point (the origin) – but this one is less tricky to control as $\varepsilon \to 0$. We leave it to the interested (and motivated) reader to check that these discrete explorations indeed converge in distribution to the continuous CLE constructions that we have studied in the current paper.

Chapter 4

Coupling between GFF and CLE_4

4.1 Introduction

Before describing the result of this section, we first recall a standard fact about Brownian motion. Consider $(B_t, t \geq 0)$ as an one-dimensional Brownian motion, and $(L_t, t \geq 0)$ as its local time process. $(|B_t|, t \geq 0)$ can be decomposed into countably many Brownian excursions and when we parameterize these excursions by the local time, the excursion process $(e_u, u \geq 0)$ is a Poisson point process. Conversely, given a Poisson point process $(e_u, u \geq 0)$ with intensity of Brownian excursion measure, there are two ways to construct a Brownian motion. For the first way, one sample iid coin tosses for each excursion so that the excursion is positive or negative with equal probability $1/2$. Then we concatenate these signed excursions, this process has the same law as a Brownian motion. For the second way, one concatenate the excursions and denote the process as $(Y_t, t \geq 0)$ (this process has the same law as $(|B_t|, t \geq 0)$). Define the local time process $(L_t, t \geq 0)$ for Y. Then $(Y_t - L_t, t \geq 0)$ has the same law as a Brownian motion. In the present section, we will discuss somewhat analogous pair of couplings for CLE_4 coupled with Gaussian Free Field on a planar domain with zero boundary condition.

A simple CLE (Conformal Loop Ensemble) is a random countable family $\mathscr{L} = (L_j, j \in J)$ of *simple, disjoint, non-nested* loops in simply connected domain D in $\mathbb{C}$. In [SW12], a CLE is defined to be such a random family that possesses two properties: conformal invariance (precisely, $(\Phi(L_j), j \in J)$ has the same law as $(L_j, j \in J)$ for any conformal map Φ from D onto itself) and domain Markov property (for any deterministic subset U of D, conditioned on the loops intersecting U, the collection of the other loops has the same law as CLE in the remaining domain). It is proved in [SW12] that there exists an exactly one-parameter family of such CLEs. Each CLE law corresponds to some $\kappa \in (8/3, 4]$ in the way that each CLE loop is a loop-variant of SLE_κ process. In the earlier work [She09], Sheffield has defined CLE via exploration tree and it is proved in [SW12] and [WW13b] that the two definitions give the same law of the loop configuration. We are interested in CLE with $\kappa = 4$. In [MS13a], Miller and Sheffield give the first coupling between CLE_4 $\mathscr{L} = (L_j, j \in J)$ and GFF h in the way that L_j's are the outmost level lines of h with heights $\pm\lambda$ ($\lambda = \pi/2$ is a special constant for the GFF) and the sign of the expected value of h inside L_j is given by iid coin tosses. They also proved that, in this first coupling, both the loop configuration and the signs are determined by the field h.

In [WW13b], the authors construct a time-parameter in CLE_4. Roughly speaking, they define a conformally invariant growing mechanism of SLE_4 loops and the loops are growing uniformly from the boundary. In this construction, the obtained loop configuration has the same law as CLE_4 and each loop inherits a time parameter from the growing mechanism: $((L, t_L), L \in \mathscr{L})$. The main result of the current section is the coupling between GFF and CLE$_4$ with time parameter which we call the second coupling between GFF and CLE$_4$.

Proposition 4.1.1. *Sample a CLE$_4$ with time parameter $((L,t_L),L\in\mathscr{L})$. Conditioned on $((L,t_L),L\in\mathscr{L})$, for each loop L sample a GFF h_L inside L with mean value $2\lambda-2\lambda t_L$ and all these $h_L,L\in\mathscr{L}$ are independent. Then the sum of these GFF's $h=\sum_{L\in\mathscr{L}}h_L$ has the same law as a GFF with mean zero.*

Just as the deterministic result in the first coupling, we also prove that

Proposition 4.1.2. *In the coupling between GFF h and CLE$_4$ with time parameter given by Proposition 4.1.1, both the loop configuration and the time parameter are deterministic functions of the field h.*

Outline. In Section 4.2, we will review the definition of SLE process, the coupling between GFF and SLE$_4$. In Section 4.3, we introduce the boundary exploration tree by use of which we prove Proposition 4.1.1. In Section 4.4, we define the downward height-varying level line and prove Proposition 4.1.2.

4.2 Preliminaries

SLE$_\kappa$ and SLE$_\kappa(\underline{\rho})$ processes. We work in the upper half-plane $\mathbb{H}$, the Loewner's construction shows that as soon as one has defined a continuous real-valued function $(W_t,t\geq 0)$, one can define a two-dimensional "Loewner-chain" as follows: for any $z\in\mathbb{H}$, define the solution $(Z_t=Z_t(z))$ to the ordinary differential equation

$$Z_t=z+\int_0^t\frac{2ds}{Z_s-W_s}.$$

This equation is well-defined up to a swallowing time $T(z)=\sup\{t\geq 0:\inf\{|Z_s-W_s|:s\in[0,t)\}>0\}$. For each given t, the map $g_t:z\mapsto Z_t(z)$ is a conformal map from some subset $\mathbb{H}\setminus K_t$ of $\mathbb{H}$ onto $\mathbb{H}$. For all $t\geq 0$, set K_t is a compact subset of $\overline{\mathbb{H}}$, and $\mathbb{H}\setminus K_t$ is simply connected. We also use the notation $f_t=g_t-W_t$.

When W_t is chosen to be equal to $\sqrt{\kappa}B_t$, where B is a standard real-valued Brownian motion, then this defines the SLE$_\kappa$ processes. For $\kappa>0$, the compact sets $(K_t,t\geq 0)$ are generated by continuous curves which we call SLE$_\kappa$ curves. When $\kappa\leq 4$, the curves are simple.

Generally, an SLE$_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process is a variant of SLE$_\kappa$ in which one keeps track of multiple additional points, which we refer to as force points. Suppose $\underline{x}^L=(x^{l,L}<\cdots<x^{1,L}\leq 0)$ and $\underline{x}^R=(0\leq x^{1,R}<\cdots<x^{r,R})$ are our force points. Associated with each force point $x^{i,q},q\in\{L,R\}$, there is a weight $\rho^{i,q}\in\mathbb{R},q\in\{L,R\}$. An SLE$_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process with force points $(\underline{x}^L;\underline{x}^R)$ is the measure on continuously growing compact hulls K_t generated by the Loewner chain with W_t replaced by the solution to the system of SDEs:

$$W_t=\sqrt{\kappa}B_t+\sum_{i=1}^{l}\int_0^t\frac{\rho^{i,L}ds}{W_s-V_s^{i,L}}+\sum_{i=1}^{r}\int_0^t\frac{\rho^{i,R}ds}{W_s-V_s^{i,R}}, \tag{4.2.1}$$

$$V_t^{i,q}=x^{i,q}+\int_0^t\frac{2ds}{V_s^{i,q}-W_s},i\in\mathbb{N},q\in\{L,R\}. \tag{4.2.2}$$

For all $\kappa > 0$, there is a unique solution to (4.2.1, 4.2.2) up until the *continuation threshold* is hit–the first time t for which either

$$\sum_{i:V_t^{i,L}=W_t} \rho^{i,L} \leq -2 \quad \text{or} \quad \sum_{i:V_t^{i,R}=W_t} \rho^{i,R} \leq -2.$$

For $\kappa > 0$, the compact subsets associated to the process are generated by a continuous curve up to the continuation threshold (see [MS12a]). We call this curve as the curve associated to the $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process with force points $(\underline{x}^L;\underline{x}^R)$.

Given a configuration

$$c = (D, z^0, \underline{z}^L, \underline{z}^R, z^\infty),$$

where D is a simply connected domain with $l+r+2$ marked points on the boundary, we can define an $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process in c by the conformal image of an $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process in $\mathbb{H}$ with force points $(\underline{x}^L;\underline{x}^R)$ under the conformal map Φ where Φ is from D onto $\mathbb{H}$, sends z^0 to 0, z^∞ to ∞, and $\underline{x}^L = \Phi(\underline{z}^L)$, $\underline{x}^R = \Phi(\underline{z}^R)$.

GFF and the coupling with SLE$_4$. Let γ be the curve associated to the $\mathrm{SLE}_4(\underline{\rho}^L;\underline{\rho}^R)$ process with force points $(\underline{x}^L;\underline{x}^R)$ and K is the corresponding compact hulls. Let h be a GFF on $\mathbb{H}$ with zero boundary value. There exists a coupling (γ, h) such that the following is true. Suppose τ is any finite stopping time less than the continuation threshold for γ.

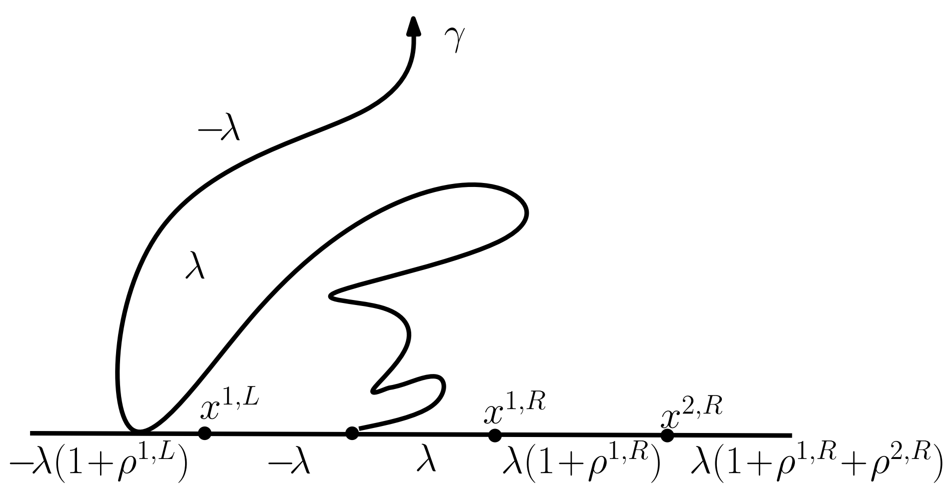


Figure 4.2.1: h is a GFF in $\mathbb{H}$ with boundary value depicted in the figure. γ is the level line of h which is in fact an $\mathrm{SLE}_4(\rho^{1,L};\rho^{1,R},\rho^{2,R})$ process with force points $(x^{1,L};x^{1,R},x^{2,R})$.

Let η_t be the function which is harmonic in $\mathbb{H}$ with boundary values (see an illustration in Figure 4.2.1)

$$\begin{cases} -\lambda(1+\overline{\rho}^{j,L}) & \text{if} \quad x \in (f_t(x^{j+1,L}), f_t(x^{j,L})) \\ \lambda(1+\overline{\rho}^{j,R}) & \text{if} \quad x \in (f_t(x^{j,R}), f_t(x^{j+1,R})) \end{cases}$$

where

$$\overline{\rho}^{j,q} = \sum_{i=1}^{j} \rho^{j,q}$$

for $q \in \{L,R\}, j \in \mathbb{N}$. Then the conditional law of $h + \eta_0|_{\mathbb{H}\setminus K_\tau}$ given K_τ is equal to the law of $h \circ f_\tau + \eta_\tau$. In this coupling, the curve γ is almost surely determined by the field h, and we can

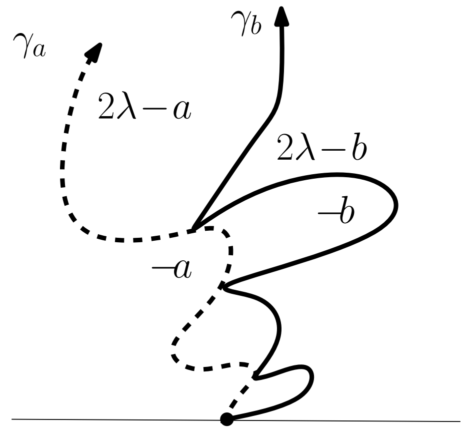


Figure 4.2.2: If $b < a$, then γ_a almost surely stays to the left of γ_b.

view γ as the level line of $h+\eta_0$ with height 0 (see [SS09, SS12]). For $a > 0$, we usually denote γ_a as the level line of $h+\eta_0-(\lambda-a)$. We call γ_a as the level line of $h+\eta_0$ with height $\lambda-a$.

Given real numbers a,b such that $0<b<a<2\lambda$, consider a GFF in $\mathbb{H}$ with mean zero. Let γ_a (resp. γ_b) be the level line of height $\lambda-a$ (resp. $\lambda-b$). From the results in [SS09, SS12], we have that γ_a is in fact an SLE$_4(-a/\lambda;-2+a/\lambda)$ and almost surely γ_a stays to the left of γ_b (see Figure 4.2.2).

4.3 Constructing the coupling

Boundary exploration tree

Given a GFF h on $\mathbb{H}$ with zero boundary value. For $a \in (0,2\lambda)$, we run a level line of height $\lambda-a$ starting at 0. This is the curve associated to SLE$_4(-a/\lambda;-2+a/\lambda)$ process with force points $(0^-;0^+)$. Note that,this curve is target-independent. That is, if we fix distinct target points $x,y \in \mathbb{R}\setminus\{0\}$, and denote γ^x (resp. γ^y) as the curve associated to SLE$_4(-a/\lambda;-2+a/\lambda)$ process in $\mathbb{H}$ from 0 to x (resp. y) with force points $(0^-;0^+)$. Then the law of γ^x — up to the time t that $y \in K_t^x$ —and the law of γ^y— up to the time t that $x \in K_t^y$ — are the same (up to a time reparameterization) (See [SW05]). This implies that γ^x and γ^y can be coupled in such a way that the corresponding hulls K_t^x and K_t^y agree (after a time reparameterization) up to the first time t that x and y are separated (i.e. $x \in K_t^y, y \in K_t^x$) and evolve independently of one another after that time.

In fact, we can define the boundary branching process SLE$_4(-a/\lambda;-2+a/\lambda)$ to be a coupling of SLE$_4(-a/\lambda;-2+a/\lambda)$ curves from 0 targeted at each point in a countable dense set of $\mathbb{R}$ such that for any two different target points x and y, the processes agree almost surely until the first time that x and y are separated, after which they evolve independently. We will call such a branching process as an a-**boundary exploration tree**. There are two observations for the boundary exploration tree. First, the law on the boundary exploration tree is independent of the countable dense set of target points. Second, the a-boundary exploration tree is conformally invariant: the image of an a-boundary exploration tree under any conformal transformation from $\mathbb{H}$ onto itself has the same law as an a-boundary exploration tree.

When coupled with zero-boundary GFF, any arc of a-boundary exploration tree is level line of the field with height $\lambda-a$. The boundary exploration tree divide the upper half plane into countably many domains–the field has mean height $2\lambda - a$ on some domains and mean height $-a$ on the other domains. We call the domains with mean height $2\lambda - a$ as "plateaus" and the domains with mean height $-a$ as "valleys". Fix a point $z \in \mathbb{H}$ and let p be the probability that z is inside a "plateau". By the conformal invariance of the boundary exploration tree, we know that p is independent of z. Consider the expected value of the field at point z, we have that

$$(2\lambda - a)p + (-a)(1-p) = 0, \quad p = \frac{a}{2\lambda}. \tag{4.3.1}$$

Discrete exploration process

Consider a GFF in $\mathbb{H}$ with zero boundary value. Take an $\varepsilon > 0$ small, and run an ε-boundary exploration tree. The tree divides the upper-half plane into plateaus and valleys. We continue to run ε-boundary exploration trees inside each valleys and these trees divide the domain into plateaus and valleys. And we repeat the same procedure inside each valley, and finally, every point is inside some plateaus. In this way, we explore the field step-by-step.

Precisely, we fix an interior point $i \in \mathbb{H}$. Let Υ_1 be an ε-boundary exploration tree. Then Υ_1 divides the upper-half plane into countably many domains: plateaus with boundary value $2\lambda - \varepsilon$ and valleys with boundary value ε. Υ_1 divides $\mathbb{H}$ into plateaus with boundary value $2\lambda - \varepsilon$ and valleys with boundary value ε. If i is inside a plateau, we stop. If not, we denote H_1 as the connected component of $\mathbb{H} \setminus \Upsilon_1$ containing i and let φ_1 be the conformal map from H_1 onto $\mathbb{H}$ such that $\varphi_1(i) = i, \varphi_1'(i) > 0$.

The image of the field in H_1 under the map φ_1 is a GFF on $\mathbb{H}$ with boundary value $-\varepsilon$. We run an ε-boundary exploration tree Υ_2 of this field. Note that this time Υ_2 divides $\mathbb{H}$ into plateaus with boundary value $2\lambda - 2\varepsilon$ and valleys with boundary value 2ε. If i is in the plateaus, we stop. If not, let H_2 be the connected component of $\mathbb{H} \setminus \Upsilon_2$ containing i and let φ_2 be the conformal map from H_2 onto $\mathbb{H}$ such that $\varphi_2(i) = i, \varphi_2'(i) > 0$. The image of the field in H_2 under φ_2 is a GFF on $\mathbb{H}$ with boundary value -2ε. Then we explore this field by ε-boundary exploration tree, and so on. We can iterate this procedure until finally i is inside some plateau. Clearly, this will happen after finitely many steps with probability one. We call $N(\varepsilon)$ the random finite step after which i is inside some plateau. It is important to note that at each step until $N(\varepsilon)$, we just repeat the same procedure as the previous step except that we are in the field lowered by $-\varepsilon$ as the previous step. We list some basic facts concerning this discrete procedure:

- $N(\varepsilon)$ is geometric with distribution

$$\mathbb{P}(N(\varepsilon) \geq k) = (1 - \frac{\varepsilon}{2\lambda})^k, \quad k = 1, 2, 3....$$

- Conditional on $N(\varepsilon)$, the random conformal maps $\varphi_1, ..., \varphi_{N(\varepsilon)}$ are i.i.d. and the random trees $\Upsilon_1, ..., \Upsilon_{N(\varepsilon)}$ are i.i.d.
- For $n \leq N(\varepsilon)$, in H_n, we have the GFF with boundary value $-n\varepsilon$.
- At the step $N(\varepsilon) + 1$, i is inside a plateau and this time the GFF in the connected component of $\mathbb{H} \setminus \Upsilon_{N(\varepsilon)+1}$ that contains i has the boundary value $2\lambda - (N(\varepsilon) + 1)\varepsilon$.

For $n \leq N(\varepsilon)$, we define

$$\Phi_n^\varepsilon = \varphi_n \circ \cdots \circ \varphi_1, \quad \Phi^\varepsilon = \varphi_{N(\varepsilon)} \circ \cdots \circ \varphi_1.$$

Continuous exploration process

We begin by reminding the reader of some definitions and results from [SW12] and [WW13b]. A **pinned loop** is a simple loop in $\bar{\mathbb{H}}$ intersecting the real line at a single point x. If we define the measure P_x^ε to be the law of SLE_4 from x to $x+\varepsilon$ in $\mathbb{H}$, then $\varepsilon^{-1}P_x^\varepsilon$ converges vaguely to an infinite measure μ^x on pinned loops [SW12, Lemma 6.7]. We can "symmetrize" by choosing $x \in \mathbb{R}$ according to Lebesgue measure, thus obtaining the measure

$$M = \int_{\mathbb{R}} \mu^x \, dx$$

on pinned loops. This measure is conformally invariant (i.e. $\Phi \circ M(\cdot) := M(\Phi^{-1}(\cdot))$ is the same as M for any conformal transformation Φ from $\mathbb{H}$ onto itself) and $M(\gamma : \gamma \text{ surrounds } i) = 1$.

In our discrete exploration process of GFF, fix $\varepsilon > 0$ and at each step, we denote P^ε as the law of the plateau that has the largest harmonic measure in $\mathbb{H}$ seen from i. From (4.3.1) we know that

$$P^\varepsilon(\gamma \text{ surrounds } i) = \frac{\varepsilon}{2\lambda}.$$

Then it is clear that

Lemma 4.3.1. *$2\lambda/\varepsilon \times P^\varepsilon$ converges weakly to M as ε goes to zero in Gromov-Hausdorff metric.*

Consider a Poisson point process $(\gamma_u, u \geq 0)$ with intensity M. Let τ be the first time u such that γ_u surrounds the origin. For $u < \tau$, let ψ_u be the conformal map from $\mathbb{H} \setminus \gamma_u$ onto $\mathbb{H}$ and normalizing at i: $\psi_u(i) = i, \psi_u'(i) > 0$. For any $r > 0$, let $u_1, ...u_j$ be the time u before τ that γ_u has harmonic measure in $\mathbb{H}$ seen from i larger than r. The composition

$$\Psi^r = \psi_{u_j} \circ \cdots \psi_{u_1}$$

converges to some conformal map Ψ in Carathéodory topology in $\mathbb{H}$ seen from i(see [SW12, WW13b]). And we formally define that, for $u \leq \tau$,

$$\Psi_u = \circ_{v<u} \psi_v, \quad \Psi = \circ_{v<\tau} \psi_v.$$

We collect several facts concerning this Poisson point process

- τ has exponential law with parameter 1.
- For $u \leq \tau$, define $D_u = \Psi_u^{-1}(\mathbb{H})$ which is a simply connected component of $\mathbb{H}$ containing i. $L_u^i := \Psi_u^{-1}(\gamma_u)$ is an SLE_4 bubble in D_u. We also define that $D_{u+} = D_u \setminus L_u^i$.
- The sequence of loops $(L_u^i, u \leq \tau)$ has the same law as loops of CLE_4 in $\mathbb{H}$, i.e. we sample independent CLE_4's in each connected component of $\mathbb{H} \setminus \cup_{u \leq \tau} L_u^i$, then the union of all these loops together with $(L_u^i, u \leq \tau)$ has the same law as CLE_4 in $\mathbb{H}$.

From Lemma 4.3.1, we can further prove that

Lemma 4.3.2. *Φ^ε converges in distribution to Ψ in Carathéodory topology in $\mathbb{H}$ seen from i.*

Proof. [SW12, Section 7.1] ☐

From this lemma, the discrete exploration process of GFF "converges" to the Poisson point process and thus

Corollary 4.3.3. *There exists a coupling between GFF h and $(L^i_u, u \le \tau)$ such that*

- *Given $(L^i_u, u \le \tau)$, for each $u \le \tau$, $h|_{L^i_u}$ has the same law as a GFF inside L^i_u with boundary value $2\lambda - 2\lambda u$.*
- *Let h_v be the expected value of h at i given $(L^i_u, u < v)$ and h_{v+} be the expected value of h given $(L^i_u, u \le v)$, then*
$$\begin{aligned} h_v &= -2\lambda v, \quad \text{if} \quad v \le \tau, \\ h_{\tau+} &= 2\lambda - 2\lambda\tau. \end{aligned}$$

Recall that we start from Poisson point process $(\gamma_u, u \ge 0)$ and construct a sequence of loops $(L^i_u, u \le \tau)$ in $\mathbb{H}$ by targeting at i. For any interior point $z \in \mathbb{H}$, we can also construct a sequence of loops $(L^z_u, u \le \tau^z)$ from $(\gamma_u, u \ge 0)$ by targeting at z, i.e. let τ^z be the first time that γ_u surrounds z and for each $u < \tau^z$, we let ψ_u be the conformal map from $\mathbb{H} \setminus \gamma_u$ onto $\mathbb{H}$ and normalizing at z: $\psi_u(z) = z, \psi'_u(z) > 0$. Then define Ψ_u, Ψ, L^z_u correspondingly.

Given two interior points $z, w \in \mathbb{H}$, for the sequence $(L^z_u, u \le \tau^z)$, let $\sigma^{z,w}$ be the time that z and w are separated, i.e. the first time u that $w \notin D_u$. For the sequence $(L^w_u, u \le \tau^w)$, let $\sigma^{w,z}$ be the time that z and w are separated. By the conformal invariance in M, we know that the process $(L^z_u, u < \sigma^{z,w})$ and the process $(L^w_u, u \le \sigma^{w,z})$ have the same law (see [WW13b]). Thus we can couple the two processes in the way that, up to the first time that z, w are separated, the two processes are identical and after the separating time, the two processes evolve independently towards their own target point.

We can also couple all processes $(L^z_u, u \le \tau^z)$ for all $z \in \mathbb{H}$ simultaneously such that for any two interior points z, w, the two processes are identical up to the disconnecting time and after which the two processes evolve independently. This is the **conformal invariant growing mechanism** of CLE_4 introduced in [WW13b]. This growing process gives us a CLE_4 with time parameter: $((L, t_L), L \in \mathscr{L})$. By Corollary 4.3.3, we see that, given $((L, t_L), L \in \mathscr{L})$, we sample independent GFF's $h_L$inside each $L \in \mathscr{L}$ with boundary value $2\lambda - 2\lambda t_L$, the sum of all these fields $h = \sum_L h_L$ has the same law as a GFF in $\mathbb{H}$ with boundary value zero. This completes the proof of Proposition 4.1.1. In this coupling, it is clear that, given $((L, t_L), L \in \mathscr{L})$, $h|_L$ has the same law as a GFF inside L with boundary value $2\lambda - 2\lambda t_L$.

4.4 Downward height-varying level line

Given a GFF h in $\mathbb{H}$ with zero boundary value, we fix a starting point $x \in \mathbb{R}$ and a target point $z \in \mathbb{H}$ and constants $a_1, \ldots, a_k$ taking values in $(0, 2\lambda)$. Let $\gamma^{x\to z}_{a_1}$ be the level line of h starting at x with height $\lambda - a_1$, and let τ_1 be a stopping time. Note that this $\gamma^{x\to z}_{a_1}$ is the curve associated to $\mathrm{SLE}_4(-a_1/\lambda; -2 + a_1/\lambda)$ process with force points $(x^-; x^+)$. The curve may touch the boundary and it is target-independent. Thus, we may require $\gamma^{x\to z}_{a_1}$ evolves in the way that after the curve separates the point z from ∞, the curve continues in the connected component containing z. For $2 \le j \le k$, we inductively let $\gamma^{x\to z}_{a_1\cdots a_j}$ be the level line of h conditioned on $\gamma^{x\to z}_{a_1\cdots a_{j-1}}[0, \tau_{j-1}]$ starting at $\gamma^{x\to z}_{a_1\cdots a_{j-1}}(\tau_{j-1})$ with height $\lambda - a_j$ and let τ_j be a stopping time. And we always require the curve to continue inside the connected component containing z. We call $\gamma^{x\to z}_{a_1\cdots a_k}$ an **height-varying level line** starting at x, targeting at z with heights $\lambda - a_1, \ldots, \lambda - a_k$ with respect to the stopping times $\tau_1, \ldots, \tau_k$. See Figure 4.4.1.

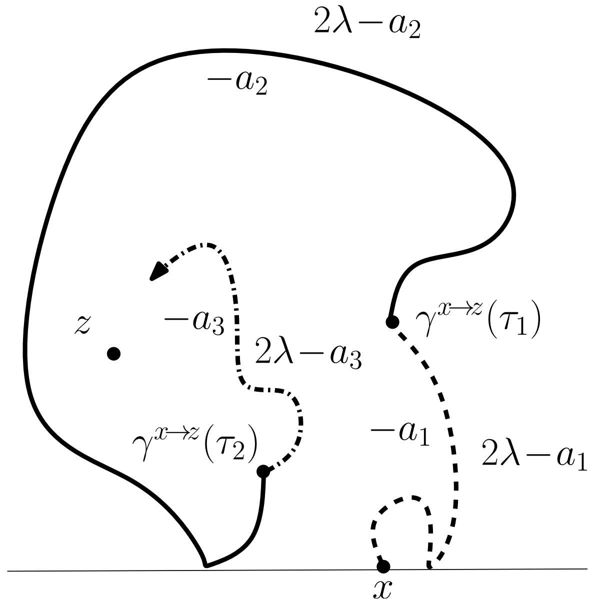


Figure 4.4.1: $\gamma^{x\to z}$ is the height-varying level line. The curve always continues inside the connected component containing z. And the field always has a height 2λ higher to the right side of the curve than the left side of the curve.

Lemma 4.4.1. *Let $\gamma = \gamma^{x\to z}_{a_1\cdots a_k}$ be an height-varying level line of h starting at x, targeting at z with heights $\lambda - a_1, ..., \lambda - a_k$ with respect to the stopping times $\tau_1, ..., \tau_k$. Then*

- *γ is almost surely a continuous curve in $\mathbb{H}$ from x to z;*
- *γ is almost surely determined by the field h.*

Proof. We prove by induction on k. For the first segment $\gamma^{x\to z}[0,\tau_1]$, it is just part of the curve associated to SLE$_4(-a_1/\lambda; -2+a_1/\lambda)$ with force points $(x^-; x^+)$. Thus it is continuous and is determined by the field (see [MS12a, Lemma 5.6, Lemma 6.2]). For $j \geq 2$, given $\gamma^{x\to z}[0,\tau_{j-1}]$, $\gamma^{x\to z}[\tau_{j-1},\tau_j]$ is part of the curve associated to some generalized SLE$_4$ process in the connected component of $\mathbb{H}\setminus\gamma^{x\to z}[0,\tau_{j-1}]$ containing z with some weights and some force points. Hence, it is continuous and determined by the field. □

In this subsection, we are more interested in the **downward height-varying level lines** which are the curves $\gamma^{x\to z}_{a_1\cdots a_k}$ with a decreasing sequence of heights: $\lambda - a_1 > \cdots > \lambda - a_k$ or $0 < a_1 < \cdots < a_k < 2\lambda$. The reason to consider this kind of level line is the following

Lemma 4.4.2. *In the coupling given by Proposition 4.1.1, any downward height-varying level line of the field cannot go inside loops of the CLE$_4$ loop configuration.*

Proof. Consider a nontrivial loop L in the loop configuration with time parameter t_L. Then the field has the height $2\lambda - 2\lambda t_L$ to the inside of L and the height $-2\lambda t_L$ to the outside of L. We know that two level lines can not cross each other, if some downward height-varying level line goes inside L, then it must hit L at its one stopping time τ_j and then, after τ_j, it goes inside L as a level line with height $\lambda - a_{j+1}$.

Recall the relation between two level lines explained in Figure 4.2.2, inside L, we must have $a_{j+1} < 2\lambda t_L$; and outside L, we must have $a_j \geq 2\lambda t_L$. This contradicts with the fact that, we are running a downward height-varying level line, and we have $a_{j+1} > a_j$. See Figure 4.4.2. □

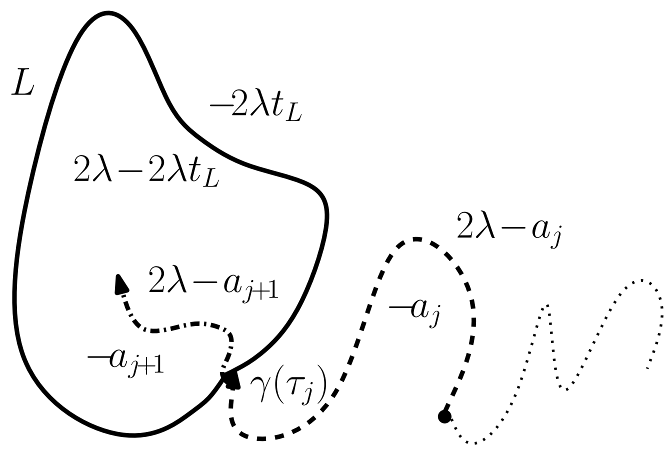


Figure 4.4.2: If a height-varying level line γ goes inside L, it should hit L at one stopping time τ_j and after τ_j, the line goes inside L. We can see that, L is to the left of $\gamma[\tau_j, \tau_{j+1}]$, thus we have $a_{j+1} < 2\lambda t$; L is to the right of $\gamma[\tau_{j-1}, \tau_j]$, thus $a_j \geq 2\lambda t$.

Then we are ready to complete the proof of Proposition 4.1.2.

Proof of Proposition 4.1.2. In the coupling given by Proposition 4.1.1, we denote $((L,t_L), L \in \mathscr{L})$ as the loop configuration with time parameter, and h as the GFF with zero-boundary value. Let K be the gasket of the loop configuration, i.e. the closure of the set of points that are not surrounded by any loop. For $z \in \mathbb{H}$, let $L(z)$ be the connected component of $\mathbb{H} \setminus K$ that contains z and let $t(z)$ be the time parameter such that h has boundary value $2\lambda - 2\lambda t(z)$ inside $L(z)$.

We fix a countable dense subset $\mathscr{S}$ of $\mathbb{R}$ and a countable dense subset $\mathscr{T}$ of $\mathbb{H}$. Define the sequence

$$\mathbf{a}^{n,i} = (\frac{1}{2^n}, \frac{2}{2^n}, ..., \frac{i}{2^n}), \quad \lambda - \mathbf{a}^{n,i} = (\lambda - \frac{1}{2^n}, ..., \lambda - \frac{i}{2^n}).$$

Let $\Upsilon^{n,i}$ be the closure of the set of points accessible by downward height-varying level lines of h starting from point in $\mathscr{S}$, targeting at point in $\mathscr{T}$, with heights $\lambda - \mathbf{a}^{n,i}$ and with positive rational height change times. It is clear that

$$\Upsilon^{n,i} \subset \Upsilon^{n,i+1},$$

thus we can define the closure of the union of this increasing sequence

$$\Upsilon^n = \overline{\bigcup_{i \in \mathbb{N}} \Upsilon^{n,i}}.$$

For any $z \in \mathbb{H}$, we can define $L^n(z)$ as the connected component of $\mathbb{H} \setminus \Upsilon^n$ containing z. And we define $N^n(z)$ be the integer that the field has the boundary value $2\lambda - 2^{-n} N^n(z)$ inside $L^n(z)$.

We define the closure of the union of the increasing sequence

$$K^\infty = \overline{\bigcup_{n \in \mathbb{N}} \Upsilon^n}.$$

We denote $L^\infty(z)$ as the connected component of $\mathbb{H} \setminus K^\infty$ that contains z. It is clear that $L^n(z)$ converges decreasingly to $L^\infty(z)$. From the discrete exploration procedure, we have that

$$N^{n+1}(z) = 2N^n(z) + B^n(z),$$

where $B^n(z)$ is a $\{0,1\}$-valued variable, thus

$$|2^{-n-1}N^{n+1}(z) - 2^{-n}N^n(z)| \leq 2^{-n-1},$$

and $2^{-n}N^n(z)$ will, almost surely, converge to some parameter $2\lambda t^\infty(z)$.

From Lemma 4.4.1, we know that K^∞ is deterministic function of h. From Lemma 4.4.2, K^∞ is contained in K. From the discrete exploration process of GFF, we can see that 2^{-n}-boundary exploration trees are contained in Υ^n and thus K has the same law as gasket of CLE$_4$. Combine these three facts, we have that $K = K^\infty$ almost surely and K is a deterministic function of h. Moreover, $L(z) = L^\infty(z)$ and thus $t(z) = t^\infty(z)$ which is also a deterministic function of h. This completes the proof. □

Chapter 5

Intersections of SLE Paths

The results in this chapter are contained in [MW13].

5.1 Introduction

5.1.1 Overview

The Schramm-Loewner evolution SLE_κ ($\kappa > 0$) is the canonical model for a conformally invariant probability measure on non-crossing, continuous paths in a proper simply connected domain D in $\mathbb{C}$. SLE_κ was introduced by Oded Schramm [Sch00] as the candidate for the scaling limit of loop-erased random walk and for the interfaces in critical percolation. Since its introduction, SLE has been proved to describe the limiting interfaces in many different models from statistical mechanics [LSW04, CN07, SS09, Mil11, CS12, CDCH$^+$12]. The purpose of this article is to study self-intersections of SLE paths as well as the intersection of multiple SLE paths when coupled together using the Gaussian free field (GFF). Our main results are Theorems 5.1.1–5.1.6 which give the dimension of the self-intersection and cut points of chordal, radial, and whole-plane SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes as well as the dimension of the intersection of such paths with the domain boundary. Theorems 5.1.1–5.1.4 are actually derived from Theorem 5.1.5 which gives the dimension of the intersection of two $\mathrm{SLE}_\kappa(\underline{\rho})$ processes coupled together as flow lines of a GFF [She, Dub09b, MS10, SS12, HBB10, IK10, She11, MS12a, MS12b, MS12c, MS13b] with different angles.

5.1.2 Main Results

Throughout, unless explicitly stated otherwise we shall assume that $\kappa' > 4$ and $\kappa = 16/\kappa' \in (0,4)$. The first result that we state is the double point dimension for chordal $\mathrm{SLE}_{\kappa'}$.

Theorem 5.1.1. *Let η be a chordal $\mathrm{SLE}_{\kappa'}$ process for $\kappa' > 4$ and let $\mathscr{D}$ be the set of double points of η. Almost surely,*

$$\dim_{\mathscr{H}}(\mathscr{D}) = \begin{cases} 2 - \frac{(12-\kappa')(4+\kappa')}{8\kappa'} & \text{for} \quad \kappa' \in (4,8) \\ 1 + \frac{2}{\kappa'} & \text{for} \quad \kappa' \geq 8. \end{cases} \tag{5.1.1}$$

In particular, when $\kappa' = 6$, $\dim_{\mathscr{H}}(\mathscr{D}) = \frac{3}{4}$.

Recall that chordal $\mathrm{SLE}_{\kappa'}$ is self-intersecting for $\kappa' > 4$ and space-filling for $\kappa' \geq 8$ [RS05]. The dimension in (5.1.1) for $\kappa' \in (4,8)$ was first predicted by Duplantier–Saleur [DS89] in the context of the contours of the FK model. The almost sure Hausdorff dimension of SLE_κ is $1+\frac{\kappa}{8}$ for $\kappa \in (0,8)$ and 2 for $\kappa \geq 8$ [Bef08] and, by SLE duality, the outer boundary of an $\mathrm{SLE}_{\kappa'}$ process for $\kappa' > 4$ stopped at a positive and finite time is described by a certain SLE_κ process [Zha08a, Zha10a, Dub09a, MS12a, MS12c, MS13b]. Thus (5.1.1) for $\kappa' \geq 8$ states that the double point dimension is equal to the dimension of the outer boundary of the path. We note that chordal $\mathrm{SLE}_{\kappa'}$ does not have triple points for $\kappa' \in (4,8)$ and the number of triple points is countable for $\kappa' \geq 8$; see Remark 5.5.3.

Our second main result is the dimension of the cut-set of chordal $\mathrm{SLE}_{\kappa'}$:

Theorem 5.1.2. *Let η be a chordal $\mathrm{SLE}_{\kappa'}$ process for $\kappa' > 4$ and let*

$$\mathscr{K} = \{\eta(t) : t \in (0,\infty),\ \eta(0,t) \cap \eta(t,\infty) = \emptyset\}$$

be the cut-set of η. Then, for $\kappa' \in (4,8)$, almost surely

$$\dim_{\mathscr{H}}(\mathscr{K}) = 3 - \frac{3\kappa'}{8}. \tag{5.1.2}$$

In particular, when $\kappa' = 6$, $\dim_{\mathscr{H}}(\mathscr{K}) = \frac{3}{4}$. For $\kappa' \geq 8$, almost surely $\mathscr{K} = \emptyset$.

The dimension (5.1.2) was conjectured in [Dup04] by Duplantier in the context of quantum field theory. Note that we recover the cut-set dimension for Brownian motion and SLE_6 established in the works of Lawler and Lawler-Schramm-Werner [Law96, LSW01a, LSW01b, LSW02]. The dimension of the *cut times* (with respect to the capacity parameterization for SLE), i.e. the set $\{t \in (0,\infty) : \eta(0,t) \cap \eta(t,\infty) = \emptyset\}$ is $2 - \frac{\kappa'}{4}$ for $\kappa' \in (4,8)$ and was computed by Beffara in [Bef04, Theorem 5].

Our next result gives the dimension of the self-intersection points of the radial and whole-plane $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4)$. Unlike chordal SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes, such processes can intersect themselves depending on the value of $\rho > -2$. The maximum number of times that such a process can hit any given point for $\kappa > 0$ is given by [MS13b, Proposition 3.31]:

$$\lceil J_{\kappa,\rho} \rceil \quad \text{where} \quad J_{\kappa,\rho} = \frac{\kappa}{2(2+\rho)}. \tag{5.1.3}$$

In particular, $J_{\kappa,\rho} \uparrow +\infty$ as $\rho \downarrow -2$ and $J_{\kappa,\rho} \downarrow 1$ as $\rho \uparrow \frac{\kappa}{2} - 2$. Recall that -2 is the lower threshold for an $\mathrm{SLE}_\kappa(\rho)$ process to be defined. For radial or whole-plane $\mathrm{SLE}_\kappa(\rho)$, the interval of ρ values in which such a process is self-intersecting is given by $(-2, \frac{\kappa}{2}-2)$ (see, e.g., [MS13b, Section 2.1]). (For chordal $\mathrm{SLE}_\kappa(\rho)$, this is the interval of ρ values in which such a process is boundary intersecting.) For $\rho \geq \frac{\kappa}{2} - 2$, such processes are almost surely simple.

Theorem 5.1.3. *Suppose that η is a radial $\mathrm{SLE}_\kappa(\rho)$ process in $\mathbb{D}$ for $\kappa \in (0,4)$ and $\rho \in \big(-2, \frac{\kappa}{2} - 2\big)$. Assume that η starts from 1 and has a single boundary force point of weight ρ located at 1^- (immediately to the left of 1 on $\partial\mathbb{D}$). For each $j \in \mathbb{N}$, let $\mathscr{I}_j$ denote the set of points in (the interior of) $\mathbb{D}$ that η hits exactly j times. For each $2 \leq j \leq \lceil J_{\kappa,\rho} \rceil$, where $J_{\kappa,\rho}$ is given by (5.1.3), we have that*

$$\dim_{\mathscr{H}}(\mathscr{I}_j) = \frac{1}{8\kappa}\Big(4+\kappa+2\rho-2j(2+\rho)\Big)\Big(4+\kappa-2\rho+2j(2+\rho)\Big) \tag{5.1.4}$$

almost surely. For $j > \lceil J_{\kappa,\rho} \rceil$, almost surely $\mathscr{I}_j = \emptyset$. These results similarly hold if η is a whole-plane $\mathrm{SLE}_\kappa(\rho)$ *process.*

Let $\mathscr{B}_j$ be the set of points in $\partial\mathbb{D}$ that η hits exactly j times. For each $1 \le j \le \lceil J_{\kappa,\rho} \rceil - 1$, we have that

$$\dim_{\mathscr{H}}(\mathscr{B}_j) = \frac{1}{2\kappa}\Big(\kappa - 2j(2+\rho)\Big)\Big(2 + j(2+\rho)\Big) \quad \text{almost surely on} \quad \{\mathscr{B}_j \neq \emptyset\}. \tag{5.1.5}$$

For each $j > \lceil J_{\kappa,\rho} \rceil - 1$, almost surely $\mathscr{B}_j = \emptyset$.

Note that $J_{\kappa,\rho} + 1$ is the value of j that makes the right side of (5.1.4) equal to zero. Similarly, $J_{\kappa,\rho}$ is the value of j that makes the right side of (5.1.5) equal to zero. Inserting $j = 1$ into (5.1.4) we recover the dimension formula for the range of an SLE_κ process [Bef08] (though we do not give an alternative proof of this result).

We next state the corresponding result for whole-plane and radial $\mathrm{SLE}_{\kappa'}(\rho)$ processes with $\kappa' > 4$. Such a process has two types of self-intersection points. Those which arise when the path wraps around its target point and intersects itself in either its left or right boundary (which are defined by lifting the path to the universal cover of the domain minus the target point of the path) and those which occur between the left and right boundaries. It is explained in [MS13b, Section 4.2] that these two self-intersection sets are almost surely disjoint and the dimension of the latter is almost surely given by the corresponding dimension for chordal $\mathrm{SLE}_{\kappa'}$ (Theorem 5.1.1). In fact, the set which consists of the multiple intersection points of the path where the path hits itself without wrapping around its target point and are also contained in its left and right boundaries is almost surely countable. The following gives the dimension of the former:

Theorem 5.1.4. *Suppose that η' is a radial $\mathrm{SLE}_{\kappa'}(\rho)$ process in $\mathbb{D}$ for $\kappa' > 4$ and $\rho \in (\frac{\kappa'}{2} - 4, \frac{\kappa'}{2} - 2)$. Assume that η' starts from 1 and has a single boundary force point of weight ρ located at 1^- (immediately to the left of 1 on $\partial\mathbb{D}$). For each $j \in \mathbb{N}$, let $\mathscr{I}'_j$ denote the set of points that η' hits exactly j times and which are also contained in its left and right boundaries. For each $2 \le j \le \lceil J_{\kappa',\rho} \rceil$ where $J_{\kappa',\rho}$ is given by (5.1.3), we have that*

$$\dim_{\mathscr{H}}(\mathscr{I}'_j) = \frac{1}{8\kappa'}\Big(4 + \kappa' + 2\rho - 2j(2+\rho)\Big)\Big(4 + \kappa' - 2\rho + 2j(2+\rho)\Big) \tag{5.1.6}$$

almost surely. For $j > \lceil J_{\kappa',\rho} \rceil$, almost surely $\mathscr{I}'_j = \emptyset$. These results similarly hold if η' is a whole-plane $\mathrm{SLE}_{\kappa'}(\rho)$ *process.*

Similarly, let $\mathscr{L}'_j$ (resp. $\mathscr{R}'_j$) be the set of points on $\partial\mathbb{D}$ which η' hits exactly j times while traveling in the clockwise (resp. counterclockwise) direction. Then

$$\dim_{\mathscr{H}}(\mathscr{L}'_j) = \frac{1}{2\kappa'}\Big(\kappa' - 2j(2+\rho)\Big)\Big(2 + j(2+\rho)\Big) \quad \text{almost surely on} \quad \{\mathscr{L}'_j \neq \emptyset\}. \tag{5.1.7}$$

and

$$\dim_{\mathscr{H}}(\mathscr{R}'_j) = \frac{1}{2\kappa'}\Big(\kappa' + 2\rho - 2j(2+\rho)\Big)\Big(2 - \rho + j(2+\rho)\Big) \quad \text{almost surely on} \quad \{\mathscr{R}'_j \neq \emptyset\}. \tag{5.1.8}$$

The reason that we restrict to the case that $\rho > \frac{\kappa'}{2} - 4$ is that for $\rho \leq \frac{\kappa'}{2} - 4$ such processes almost surely fill their own outer boundary. That is, for any time t, the outer boundary of the range of the path drawn up to time t is almost surely contained in $\eta'([t,\infty])$ and processes of this type fall outside of the framework described in [MS13b].

The proofs of Theorem 5.1.1 and Theorem 5.1.2 are based on using various forms of SLE duality which arises in the interpretation of the SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes for $\kappa \in (0,4)$ as flow lines of the vector field $e^{ih/\chi}$ where h is a GFF and $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ [Dub09a, Dub09b, MS12a, MS12c, MS13b]. We will refer to these paths simply as "GFF flow lines." (An overview of this theory is provided in Section 5.2.) The duality statement which is relevant for the cut-set (see Figure 5.2.5) is that the left (resp. right) boundary of an $\mathrm{SLE}_{\kappa'}$ process is given by an SLE_κ flow line of a GFF with angle $\frac{\pi}{2}$ (resp. $-\frac{\pi}{2}$). Thus the cut set dimension is given by the dimension of the intersection of two flow lines with an angle gap of

$$\theta_{\mathrm{cut}} = \pi. \tag{5.1.9}$$

Another form of duality which describes the boundary of the $\mathrm{SLE}_{\kappa'}$ before and after hitting a given boundary point and also arises in the GFF framework allows us to relate the double point dimension to the dimension of the intersection of GFF flow lines with an angle gap of [MS12c]

$$\theta_{\mathrm{double}} = \pi\left(\frac{\kappa-2}{2-\frac{\kappa}{2}}\right). \tag{5.1.10}$$

We will explain this in more detail in Section 5.5. The set of points which a whole-plane or radial $\mathrm{SLE}_\kappa(\rho)$ process for $\kappa \in (0,4)$ and $\rho \in (-2, \frac{\kappa}{2} - 2)$ hits j times (in the interior of the domain) is locally absolutely continuous with respect to the intersection of two flow lines with an angle gap of

$$\theta_j = 2\pi(j-1)\left(\frac{2+\rho}{4-\kappa}\right) \quad \text{for} \quad 2 \leq j \leq \lceil J_{\kappa,\rho} \rceil; \tag{5.1.11}$$

see [MS13b, Proposition 3.32]. The angle gap which gives the dimension of the self-intersection set contained in the interior of the domain for $\kappa' > 4$ and $\rho \in (\frac{\kappa'}{2} - 4, \frac{\kappa'}{2} - 2)$ is given by

$$\theta_j' = \pi\left(\frac{2j(2+\rho) - 2\rho - \kappa'}{\kappa' - 4}\right) \quad \text{for} \quad 2 \leq j \leq \lceil J_{\kappa',\rho} \rceil; \tag{5.1.12}$$

see [MS13b, Proposition 4.10]. Thus Theorems 5.1.1–5.1.4 follow from (with the exception of (5.1.5), (5.1.7), (5.1.8)):

Theorem 5.1.5. *Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary data. Fix $\kappa \in (0,4)$, angles*

$$\theta_1 < \theta_2 < \theta_1 + \left(\frac{\kappa\pi}{4-\kappa}\right),$$

and let

$$\rho = \frac{1}{\pi}(\theta_2 - \theta_1)\left(2 - \frac{\kappa}{2}\right) - 2.$$

For $i = 1,2$, let η_{θ_i} be the flow line of h starting from 0. We have that

$$\dim_{\mathscr{H}}(\eta_{\theta_1} \cap \eta_{\theta_2} \cap \mathbb{H}) = 2 - \frac{1}{2\kappa}\left(\rho + \frac{\kappa}{2} + 2\right)\left(\rho - \frac{\kappa}{2} + 6\right)$$

almost surely on the event $\{\eta_{\theta_1} \cap \eta_{\theta_2} \cap \mathbb{H} \neq \emptyset\}$.

Theorem 5.1.5 gives the dimension of the intersection of two flow lines in the bulk. The following result gives the dimension of the intersection of one path with the boundary.

Theorem 5.1.6. *Fix $\kappa > 0$ and $\rho \in ((-2) \vee (\frac{\kappa}{2} - 4), \frac{\kappa}{2} - 2)$. Let η be an $\mathrm{SLE}_\kappa(\rho)$ process with a single force point located at 0^+. Almost surely,*

$$\dim_{\mathscr{H}}(\eta \cap \mathbb{R}_+) = 1 - \frac{1}{\kappa}(\rho+2)\left(\rho + 4 - \frac{\kappa}{2}\right). \tag{5.1.13}$$

(Recall that $\frac{\kappa}{2} - 4$ is the threshold at which such processes become boundary filling and -2 is the threshold for these processes to be defined.) In the case that $\rho = \frac{\theta}{\pi}(2 - \frac{\kappa}{2}) - 2$ for $\theta > 0$ and $\kappa \in (0,4)$, we say that η intersects $\partial\mathbb{H}$ with an angle gap of θ. This comes from the interpretation of such an $\mathrm{SLE}_\kappa(\rho)$ process as a GFF flow line explained in Section 5.2. See, in particular, Figure 5.2.4. By [MS13b, Proposition 3.33], applying Theorem 5.1.6 with an angle gap of θ_{j+1} where θ_j is as in (5.1.11) gives (5.1.5) of Theorem 5.1.3. Similarly, by [MS13b, Proposition 4.11], applying Theorem 5.1.6 with an angle gap of

$$\phi_{j,L} = \pi\left(\frac{4 - \kappa' + 2j(2+\rho)}{\kappa' - 4}\right) \tag{5.1.14}$$

gives (5.1.7) and with an angle gap of

$$\phi_{j,R} = \pi\left(\frac{4 - \kappa' - 2\rho + 2j(2+\rho)}{\kappa' - 4}\right) \tag{5.1.15}$$

gives (5.1.8). Theorem 5.1.6 is proved first by computing the boundary intersection dimension for $\kappa \in (0,4)$ and then using SLE duality to extend to the case that $\kappa' > 4$. In particular, we obtain as a corollary (when $\rho = 0$) the following which was first proved in [AS08]. We remark that an alternative proof to the lower bound of Theorem 5.1.6 for $\kappa \in [8/3, 4]$ is proved in [WW13a] using the relationship between the $\mathrm{SLE}_\kappa(\rho)$ processes for these κ values and the Brownian loop soups.

Corollary 5.1.7. *Fix $\kappa' \in (4,8)$ and let η be an $\mathrm{SLE}_{\kappa'}$ process in $\mathbb{H}$ from 0 to ∞. Then, almost surely*

$$\dim_{\mathscr{H}}(\eta \cap \mathbb{R}) = 2 - \frac{8}{\kappa'}.$$

One of the main inputs in the proof of Theorem 5.1.5 and Theorem 5.1.6 is the following theorem, which gives the exponent for the probability that an $\mathrm{SLE}_\kappa(\underline{\rho})$ process gets very close to a given boundary point.

Theorem 5.1.8. *Fix $\kappa > 0$, $\rho_{1,R} > -2$, $\rho_{2,R} \in \mathbb{R}$ such that $\rho_{1,R} + \rho_{2,R} > \frac{\kappa}{2} - 4$. Let η be an $\mathrm{SLE}_\kappa(\rho_{1,R}, \rho_{2,R})$ process with force points $(0^+, 1)$. Let*

$$\alpha = \frac{1}{\kappa}(\rho_{1,R}+2)\left(\rho_{1,R} + \rho_{2,R} + 4 - \frac{\kappa}{2}\right). \tag{5.1.16}$$

For each $\varepsilon > 0$, we let $\tau_\varepsilon = \inf\{t \geq 0 : \eta(t) \in \partial B(1,\varepsilon)\}$. We have that

$$\mathbf{P}[\tau_\varepsilon < \infty] = \varepsilon^{\alpha + o(1)} \quad \text{as} \quad \varepsilon \to 0. \tag{5.1.17}$$

By taking $\rho = \rho_{1,R} \in ((-2) \vee (\frac{\kappa}{2} - 4), \frac{\kappa}{2} - 2)$ and $\rho_{2,R} = 0$, Theorem 5.1.8 gives the exponent for the probability that an $\mathrm{SLE}_\kappa(\rho)$ process gets close to a fixed point on the boundary. Theorem 5.1.8 is proved (in somewhat more generality) in Section 5.3.

Outline

The remainder of this article is structured as follows. In Section 5.2, we will review the definition and important properties of the SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes. We will also describe the coupling between SLE and the Gaussian free field. Next, in Section 5.3, we will compute the Hausdorff dimension of $\mathrm{SLE}_\kappa(\rho)$ intersected with the boundary. We will extend this to compute the dimension of the intersection of two GFF flow lines in Section 5.4. Finally, in Section 5.5 we will complete the proof of Theorem 5.1.1.

5.2 Preliminaries

We will give an overview of the SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes in Section 5.2.1. Next, in Section 5.2.2, we will give an overview of the SLE/GFF coupling and then use the coupling to establish several useful lemmas regarding the behavior of the SLE_κ and $\mathrm{SLE}_\kappa(\underline{\rho})$ processes. In Section 5.2.3, we will compute the Radon-Nikodym derivative associated with a change of domains and perturbation of force points for an $\mathrm{SLE}_\kappa(\underline{\rho})$ process. Finally, in Section 5.2.4 we will record some useful estimates for conformal maps. Throughout, we will make use of the following notation. Suppose that f,g are functions. We will write $f \asymp g$ if there exists a constant $C \geq 1$ such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for all x. We will write $f \lesssim g$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ and $f \gtrsim g$ if $g \lesssim f$.

5.2.1 SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes

We will now give a very brief introduction to SLE. More detailed introductions can be found in many excellent surveys of the subject, e.g., [Wer04b, Law05]. Chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ is defined by the random family of conformal maps (g_t) obtained by solving the Loewner ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z \tag{5.2.1}$$

with $W = \sqrt{\kappa}B$ and B a standard Brownian motion. Write $K_t := \{z \in \mathbb{H} : \tau(z) \leq t\}$ where $\tau(z)$ is the swallowing time of z defined by $\sup\{t \geq 0 : \min_{s\in[0,t]} |g_s(z) - W_s| > 0\}$. Then g_t is the unique conformal map from $\mathbb{H}_t := \mathbb{H} \setminus K_t$ to $\mathbb{H}$ satisfying $\lim_{|z|\to\infty} |g_t(z) - z| = 0$.

Rohde and Schramm showed that there almost surely exists a curve η (the so-called SLE *trace*) such that for each $t \geq 0$ the domain $\mathbb{H}_t$ of g_t is the unbounded connected component of $\mathbb{H} \setminus \eta([0,t])$, in which case the (necessarily simply connected and closed) set K_t is called the "filling" of $\eta([0,t])$ [RS05]. An SLE_κ connecting boundary points x and y of an arbitrary simply connected Jordan domain can be constructed as the image of an SLE_κ on $\mathbb{H}$ under a conformal transformation $\varphi \colon \mathbb{H} \to D$ sending 0 to x and ∞ to y. (The choice of φ does not affect the law of this image path, since the law of SLE_κ on $\mathbb{H}$ is scale invariant.) For $\kappa \in [0,4]$, SLE_κ is simple and, for $\kappa > 4$, SLE_κ is self-intersecting [RS05]. The dimension of the path is $1 + \frac{\kappa}{8}$ for $\kappa \in [0,8]$ and 2 for $\kappa > 8$ [Bef08].

An $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process is a generalization of SLE_κ in which one keeps track of additional marked points which are called *force points*. These processes were first introduced in [LSW03, Section 8.3]. Fix $\underline{x}_L = (x_{\ell,L} < \cdots < x_{1,L} \leq 0)$ and $\underline{x}_R = (0 \leq x_{1,R} < \cdots < x_{r,R})$. We associate with each $x_{i,q}$ for $q \in \{L,R\}$ a weight $\rho_{i,q} \in \mathbb{R}$. An $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process with force points $(\underline{x}_L;\underline{x}_R)$ is the measure on continuously growing compact hulls K_t generated by the Loewner chain with W_t

replaced by the solution to the system of SDEs:

$$dW_t = \sum_{i=1}^{\ell} \frac{\rho_{i,L}}{W_t - V_t^{i,L}} dt + \sum_{i=1}^{r} \frac{\rho_{i,R}}{W_t - V_t^{i,R}} dt + \sqrt{\kappa} dB_t,$$
$$dV_t^{i,q} = \frac{2}{V_t^{i,q} - W_t} dt, \quad V_0^{i,q} = x_{i,q}, \quad i \in \mathbb{N}, \quad q \in \{L,R\}. \tag{5.2.2}$$

It is explained in [MS12a, Section 2] that for all $\kappa > 0$, there is a unique solution to (5.2.2) up until the *continuation threshold* is hit — the first time t for which either

$$\sum_{i: V_t^{i,L} = W_t} \rho_{i,L} \leq -2 \quad \text{or} \quad \sum_{i: V_t^{i,R} = W_t} \rho_{i,R} \leq -2.$$

The almost sure continuity of the $\mathrm{SLE}_\kappa(\underline{\rho})$ processes is proved in [MS12a, Theorem 1.3]. Let

$$\overline{\rho}_{j,q} = \sum_{i=0}^{j} \rho_{i,q} \quad \text{for} \quad q \in \{L,R\} \quad \text{and} \quad j \in \mathbb{N} \tag{5.2.3}$$

with the convention that $\rho_{0,L} = \rho_{0,R} = 0$, $x_{0,L} = 0^-$, $x_{\ell+1,L} = -\infty$, $x_{0,R} = 0^+$, and $x_{r+1,R} = +\infty$. The value of $\overline{\rho}_{k,R}$ determines how the process interacts with the interval $(x_{k,R}, x_{k+1,R})$ (and likewise when R is replaced with L). In particular:

Lemma 5.2.1. *Suppose that η is an $\mathrm{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process in $\mathbb{H}$ from 0 to ∞ with force points located at $(\underline{x}_L; \underline{x}_R)$.*

1. *If $\overline{\rho}_{k,R} \geq \frac{\kappa}{2} - 2$, then η almost surely does not hit $(x_{k,R}, x_{k+1,R})$.*
2. *If $\kappa \in (0,4)$ and $\overline{\rho}_{k,R} \in (\frac{\kappa}{2} - 4, -2]$, then η can hit $(x_{k,R}, x_{k+1,R})$ but cannot be continued afterwards.*
3. *If $\kappa > 4$ and $\overline{\rho}_{k,R} \in (-2, \frac{\kappa}{2} - 4]$, then η can hit $(x_{k,R}, x_{k+1,R})$ and be continued afterwards. Moreover, $\eta \cap (x_{k,R}, x_{k+1,R})$ is almost surely an interval.*
4. *If $\overline{\rho}_{k,R} \in ((-2) \vee (\frac{\kappa}{2} - 4), \frac{\kappa}{2} - 2)$ then η can hit and bounce off of $(x_{k,R}, x_{k+1,R})$. Moreover, $\eta \cap (x_{k,R}, x_{k+1,R})$ has empty interior.*

Proof. See [MS12a, Remark 5.3 and Theorem 1.3] as well as [Dub09a, Lemma 15]. □

In this article, it will also be important for us to consider *radial* SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes. These are typically defined using the radial Loewner equation. On the unit disk $\mathbb{D}$, this is described by the ODE

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + W_t}{g_t(z) - W_t}, \quad g_0(z) = z \tag{5.2.4}$$

where W_t is a continuous function which takes values in $\partial\mathbb{D}$. For $w \in \partial\mathbb{D}$, radial SLE_κ starting from w is the growth process associated with (5.2.4) where $W_t = we^{i\sqrt{\kappa}B_t}$ and B is a standard Brownian motion. For $w, v \in \partial\mathbb{D}$, radial $\mathrm{SLE}_\kappa(\rho)$ with starting configuration (w,v) is the growth process associated with the solution of (5.2.4) where the driving function solves the SDE

$$dW_t = -\frac{\kappa}{2} W_t \, dt + i\sqrt{\kappa} W_t \, dB_t - \frac{\rho}{2} W_t \frac{W_t + V_t}{W_t - V_t} \, dt, \quad W_0 = w \tag{5.2.5}$$

with $V_t = g_t(v)$, the force point. The continuity of the radial $\mathrm{SLE}_\kappa(\rho)$ processes for $\rho > -2$ can be extracted from the continuity of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ processes given in [MS12a, Theorem 1.3]; this is explained in [MS13b, Section 2.1]. The value of ρ for a radial $\mathrm{SLE}_\kappa(\rho)$ process has the same interpretation as in the setting of chordal $\mathrm{SLE}_\kappa(\rho)$ explained in Lemma 5.2.1. That is, the processes are boundary filling for $\rho \in (-2, \frac{\kappa}{2} - 4]$ (for $\kappa > 4$), boundary hitting but not filling for $\rho \in ((-2) \vee (\frac{\kappa}{2} - 4), \frac{\kappa}{2} - 2)$, and boundary avoiding for $\rho \geq \frac{\kappa}{2} - 2$. In particular, by the conformal Markov property for radial $\mathrm{SLE}_\kappa(\rho)$, such processes are self-intersecting for $\rho \in (-2, \frac{\kappa}{2} - 2)$ and fill their own outer boundary for $\rho \in (-2, \frac{\kappa}{2} - 4]$ ($\kappa > 4$). The latter means that, for any time t, the outer boundary of the range of η up to time t is almost surely contained in $\eta([t, \infty))$.

Martingales

From the form of (5.2.2) and the Girsanov theorem, it follows that the law of an $\mathrm{SLE}_\kappa(\underline{\rho})$ process can be constructed by reweighting the law of an ordinary SLE_κ process by a certain local martingale, at least until the first time τ that W hits one of the force points $V^{i,q}$ [Wer04a]. It is shown in [SW05, Theorem 6 and Remark 7] that this local martingale can be expressed in the following more convenient form. Suppose $x_{1,L} < 0 < x_{1,R}$ and define

$$M_t = \prod_{i,q} \left|g_t'(x_{i,q})\right|^{\frac{(4-\kappa+\rho_{i,q})\rho_{i,q}}{4\kappa}} \times \prod_{i,q} \left|W_t - V_t^{i,q}\right|^{\frac{\rho_{i,q}}{\kappa}} \times \prod_{(i,q)\neq(i',q')} \left|V_t^{i,q} - V_t^{i',q'}\right|^{\frac{\rho_{i,q}\rho_{i',q'}}{2\kappa}}. \tag{5.2.6}$$

Then M_t is a local martingale, and, the law of a standard SLE_κ process weighted by M (up to time τ, as above) is equal to that of an $\mathrm{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process with force points $(\underline{x}_L; \underline{x}_R)$. We remark that there is an analogous martingale in the setting of radial $\mathrm{SLE}_\kappa(\rho)$ processes [SW05, Equation 9], a special case of which we will describe and make use of in Section 5.4.

One application of this that will be important for us is as follows. Suppose that η is an $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ process with only two force points $x_L < 0 < x_R$. If we weight the law of η by the local martingale

$$M_t^L = |W_t - V_t^L|^{\frac{\kappa-4-2\rho_L}{\kappa}} \times |V_t^L - V_t^R|^{\frac{(\kappa-4-2\rho_L)\rho_R}{2\kappa}} \tag{5.2.7}$$

then the law of the resulting process is that of an $\mathrm{SLE}_\kappa(\widehat{\rho}_L; \rho_R)$ process where $\widehat{\rho}_L = \kappa - 4 - \rho_L$. If $\rho_L < \frac{\kappa}{2} - 2$ so that $\widehat{\rho}_L > \frac{\kappa}{2} - 2$, Lemma 5.2.1 implies that the reweighted process almost surely does not hit $(-\infty, x_L)$.

5.2.2 SLE and the GFF

We are now going to give a brief overview of the coupling between SLE and the GFF. We refer the reader to [MS12a, Sections 1 and 2] as well as [MS12b, Section 2] for a more detailed overview. Throughout, we fix $\kappa \in (0,4)$ and $\kappa' = 16/\kappa > 4$.

Suppose that $D \subseteq \mathbb{C}$ is a given domain. The Sobolev space $H_0^1(D)$ is the Hilbert space closure of $C_0^\infty(D)$ with respect to the Dirichlet inner product

$$(f,g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) dx. \tag{5.2.8}$$

The zero-boundary Gaussian free field (GFF) h on D is given by

$$h = \sum_n \alpha_n f_n \tag{5.2.9}$$

where (α_n) is a sequence of i.i.d. $N(0,1)$ random variables and (f_n) is an orthonormal basis for $H_0^1(D)$. The sum (5.2.9) does not converge in $H_0^1(D)$ (or any space of functions) but rather in an appropriate space of distributions. The GFF h with boundary data f is given by taking the sum of the zero-boundary GFF on D and the function F in D which is harmonic and is equal to f on ∂D. See [She07] for a detailed introduction.

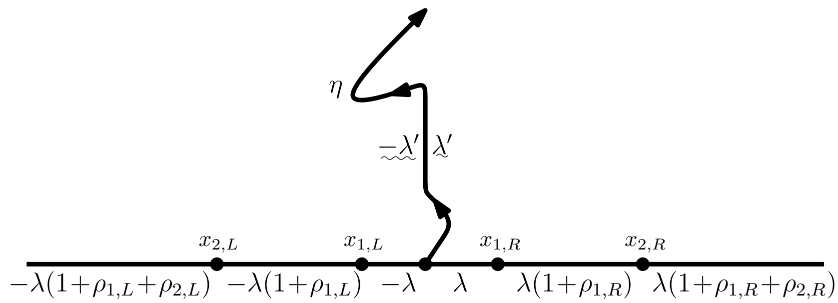


Figure 5.2.1: Suppose that h is a GFF on $\mathbb{H}$ whose boundary data is as indicated above. Then the flow line η of h starting from 0 is an $\mathrm{SLE}_\kappa(\rho_{2,L},\rho_{1,L};\rho_{1,R},\rho_{2,R})$ process ($\kappa \in (0,4)$) from 0 to ∞ with force points located at $x_{2,L} < x_{1,L} < 0 < x_{1,R} < x_{2,R}$. The conditional law of h given η (or η up to a stopping time) is that of a GFF off of η with the boundary data as illustrated on η; the notation $\underset{\sim}{x}$ is shorthand for $x + \chi \cdot \text{winding}$ and is explained in detail in [MS12a, Figures 1.9 and 1.10]. The boundary data for the coupling of $\mathrm{SLE}_\kappa(\underline{\rho})$ with many force points arises as the obvious generalization of the above.

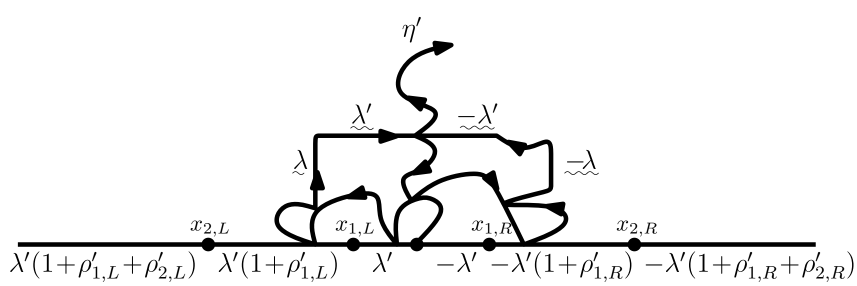
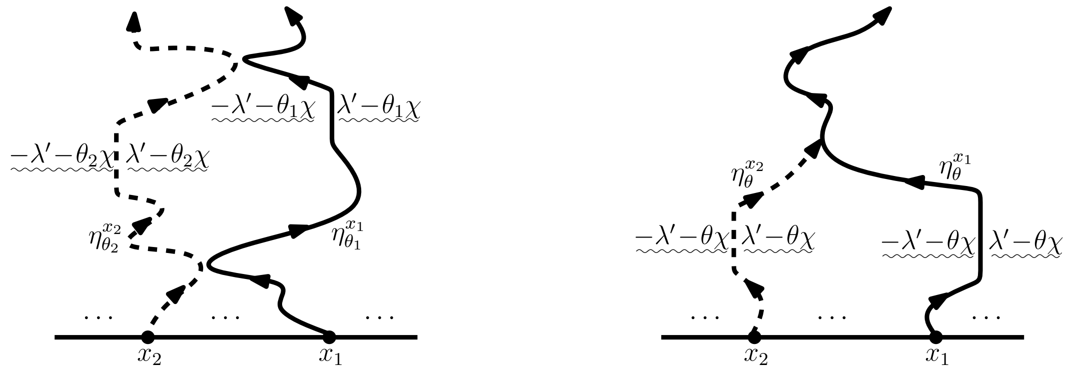


Figure 5.2.2: Suppose that h is a GFF on $\mathbb{H}$ whose boundary data is as indicated above. Then the counterflow line η' of h starting from 0 is an $\mathrm{SLE}_{\kappa'}(\rho'_{2,L},\rho'_{1,L};\rho'_{1,R},\rho'_{2,R})$ process ($\kappa' > 4$) from 0 to ∞ with force points located at $x_{2,L} < x_{1,L} < 0 < x_{1,R} < x_{2,R}$. The conditional law of h given η' (or η' up to a stopping time) is that of a GFF off of η' with the indicated boundary data; the notation $\underset{\sim}{x}$ is shorthand for $x + \chi \cdot \text{winding}$ and is explained in detail in [MS12a, Figures 1.9 and 1.10]. The boundary data for the coupling of $\mathrm{SLE}_{\kappa'}(\underline{\rho}')$ with many force points arises as the obvious generalization of the above.

Let

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}, \quad \lambda = \frac{\pi}{\sqrt{\kappa}}, \quad \text{and} \quad \lambda' = \frac{\pi}{\sqrt{\kappa'}} = \frac{\pi}{4}\sqrt{\kappa} = \lambda - \frac{\pi}{2}\chi. \tag{5.2.10}$$



(a) Monotonicity of flow lines. (b) Flow lines merging.

Figure 5.2.3: Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary data and $x_1, x_2 \in \partial\mathbb{H}$ with $x_2 \leq x_1$. Fix angles θ_1, θ_2 and, for $i = 1, 2$, let $\eta_{\theta_i}^{x_i}$ be the flow line of h with angle θ_i starting from x_i. If $\theta_2 > \theta_1$, then $\eta_{\theta_2}^{x_2}$ almost surely stays to the left of (but may bounce off of) $\eta_{\theta_1}^{x_1}$. If $\theta_1 = \theta_2 = \theta$, then $\eta_{\theta}^{x_1}$ merges with $\eta_{\theta}^{x_2}$ upon intersecting after which the paths never separate.

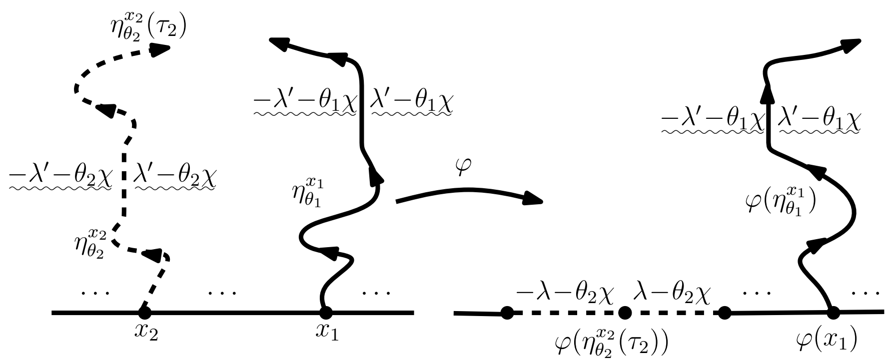


Figure 5.2.4: Assume that we have the same setup as in Figure 5.2.3 and that τ_2 is a stopping time for $\eta_{\theta_2}^{x_2}$. Then we can compute the conditional law of $\eta_{\theta_1}^{x_1}$ given $\eta_{\theta_2}^{x_2}|_{[0,\tau_2]}$. Let φ be a conformal map which takes the unbounded connected component of $\mathbb{H} \setminus \eta_{\theta_2}^{x_2}([0, \tau_2])$ to $\mathbb{H}$ and let $h_2 = h \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'$. Then $\varphi(\eta_{\theta_1}^{x_1})$ is the flow line of h_2 starting from $\varphi(x_1)$ with angle θ_1 and we can read off its conditional law from the boundary data of h_2 as in Figure 5.2.1.

Suppose that η is an $\mathrm{SLE}_\kappa(\underline{\rho}_L; \underline{\rho}_R)$ process in $\mathbb{H}$ from 0 to ∞ with force points $(\underline{x}_L; \underline{x}_R)$, let (g_t) be the associated Loewner flow, W its driving function, and $f_t = g_t - W_t$. Let h be a GFF on $\mathbb{H}$ with zero boundary values. It is shown in [She, Dub09b, MS10, SS12, HBB10, IK10, She11] that there exists a coupling (η, h) such that the following is true. Suppose τ is any stopping time for η. Let ϕ_t^0 be the function which is harmonic in $\mathbb{H}$ with boundary values (recall (5.2.3))

$$\begin{cases} -\lambda(1+\overline{\rho}_{j,L}) & \text{if} \quad x \in [f_t(x_{j+1,L}), f_t(x_{j,L})) \\ \lambda(1+\overline{\rho}_{j,R}) & \text{if} \quad x \in (f_t(x_{j,R}), f_t(x_{j+1,R})]. \end{cases}$$

Let

$$\phi_t(z) = \phi_t^0(f_t(z)) - \chi \arg f_t'(z).$$

Then the conditional law of $(h + \phi_0)|_{\mathbb{H}\setminus K_\tau}$ given K_τ is equal to the law of $h \circ f_\tau + \phi_\tau$. In this coupling, η is almost surely determined by h [SS12, Dub09b, MS12a]. For $\kappa \in (0, 4)$, η has the

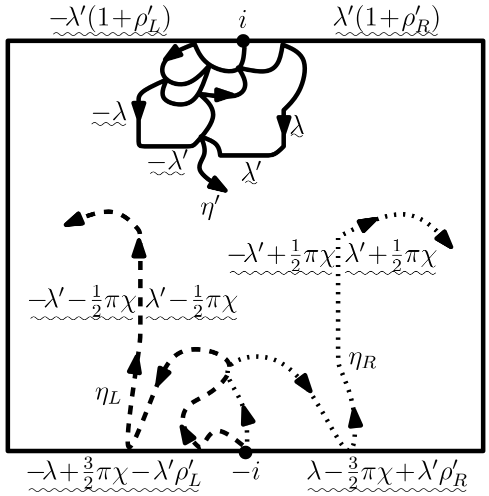


Figure 5.2.5: Let h be a GFF on $[-1,1]^2$ with the illustrated boundary data. Then the counterflow line η' of h from i to $-i$ is an $\mathrm{SLE}_{\kappa'}(\rho'_L;\rho'_R)$ process ($\kappa' > 4$) with force points located at $(i)^-,(i)^+$ (immediately to the left and right of i). The left (resp. right) boundary η_L (resp. η_R) of η' is given by the flow line of h with angle $\frac{\pi}{2}$ (resp. $-\frac{\pi}{2}$) starting from $-i$ and targeted at i; these paths can be drawn if $\rho'_L,\rho'_R \geq \frac{\kappa'}{2}-4$. Explicitly, η_L (resp. η_R) is an $\mathrm{SLE}_\kappa(\kappa-4+\frac{\kappa}{4}\rho'_L;\frac{\kappa}{2}-2+\frac{\kappa}{4}\rho'_R)$ (resp. $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-2+\frac{\kappa}{4}\rho'_L;\kappa-4+\frac{\kappa}{4}\rho'_R)$) process in $[-1,1]^2$ from $-i$ to i with force points located at $(-i)^-,(-i)^+$ ($\kappa = 16/\kappa' \in (0,4)$). The cut-set of η' is given by $\eta_L \cap \eta_R$ and $\eta' \cap \partial([-1,1]^2) = (\eta_L \cup \eta_R) \cap \partial([-1,1])^2$. The same holds if $[-1,1]^2$ is replaced by a proper, simply-connected domain and the boundary data of the GFF is transformed according to (5.2.11). Finally, if $\rho'_L,\rho'_R \geq \frac{\kappa'}{2}-4$, then conditional law of η' given η_L and η_R is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4;\frac{\kappa'}{2}-4)$ in each of the bubbles of $[-1,1]^2 \setminus (\eta_L \cup \eta_R)$ which lie to the right of η_L and to the left of η_R.

interpretation as being the flow line of the (formal) vector field $e^{i(h+\phi_0)/\chi}$ [She11] starting from 0; we will refer to η simply as a flow line of $h+\phi_0$. See Figure 5.2.1 for an illustration of the boundary data. The notation $\underset{\sim}{x}$ is used to indicate that the boundary data for the field is given by $x+\chi\cdot$winding where "winding" refers to the winding of the path or domain boundary. For curves or domain boundaries which are not smooth, it is not possible to make sense of the winding along the curve or domain boundary. However, the harmonic extension of the winding does make sense. This notation as well as this point are explained in detail in [MS12a, Figures 1.9 and 1.10]. When $\kappa = 4$, η has the interpretation of being the level line of $h+\phi_0$ [SS12]. Finally, when $\kappa' > 4$, η' has the interpretation of being a "tree of flow lines" which travel in the opposite direction of η' [MS12a, MS13b]. For this reason, η' is referred to as a *counterflow line* of $h+\phi_0$ in this case.

If h were a smooth function, η a flow line of the vector field $e^{ih/\chi}$, and φ a conformal map, then $\varphi(\eta)$ is a flow line of $e^{i\widetilde{h}/\chi}$ where

$$\widetilde{h} = h \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'; \tag{5.2.11}$$

see [MS12a, Figure 1.6]. The same is true when h is a GFF and this formula determines the boundary data for coupling the GFF with an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process on a domain other than $\mathbb{H}$. See also [MS12a, Figure 1.9]. SLE_κ flow lines and $\mathrm{SLE}_{\kappa'}$, $\kappa' = 16/\kappa \in (4,\infty)$, counterflow lines can be coupled with the same GFF. In order for both paths to transform in the correct way under the application of a conformal map, one thinks of the flow lines as being coupled with h as described above and the counterflow lines as being coupled with $-h$. This is because $\chi(\kappa') = -\chi(\kappa)$; see the discussion after the statement of [MS12a, Theorem 1.1]. This is why the signs of the boundary data in Figure 5.2.2 are reversed in comparison to that in Figure 5.2.1.

The theory of how the flow lines, level lines, and counterflow lines of the GFF interact with each other and the domain boundary is developed in [MS12a, MS13b]. See, in particular, [MS12a, Theorem 1.5]. The important facts for this article are as follows. Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary data. For each $\theta \in \mathbb{R}$ and $x \in \partial\mathbb{H}$, let η_θ^x be the flow line of h starting at x with angle θ (i.e., the flow line of $h + \theta\chi$ starting at x). If $\theta_1 < \theta_2$ and $x_1 \geq x_2$ then $\eta_{\theta_1}^{x_1}$ almost surely stays to the right of $\eta_{\theta_2}^{x_2}$. If $\theta_1 = \theta_2$, then $\eta_{\theta_1}^{x_1}$ may intersect $\eta_{\theta_2}^{x_2}$ and, upon intersecting, the two flow lines merge and never separate thereafter. See Figure 5.2.3. Finally, if $\theta_2 + \pi > \theta_1 > \theta_2$, then $\eta_{\theta_1}^{x_1}$ may intersect $\eta_{\theta_2}^{x_2}$ and, upon intersecting, crosses and possibly subsequently bounces off of $\eta_{\theta_2}^{x_2}$ but never crosses back. It is possible to compute the conditional law of one flow line given the realization of several others; see Figure 5.2.4. For simplicity, we use η_θ to indicate η_θ^x when $x = 0$. If η' is a counterflow line coupled with the GFF, then its outer boundary is described in terms of a pair of flow lines starting from the terminal point of η' [Dub09a, Dub09b, MS12a, MS13b]; see Figure 5.2.5.

We are now going to use the SLE/GFF coupling to collect several useful lemmas regarding the behavior of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes.

Lemma 5.2.2. *Fix $\kappa > 0$. Suppose that $(x_{n,L})$ (resp. $(x_{n,R})$) is a sequence of negative (resp. positive) real numbers converging to $x_L \leq 0^-$ (resp. $x_R \geq 0^+$) as $n \to \infty$. For each n, suppose that $(W^n, V^{n,L}, V^{n,R})$ is the driving triple for an $\mathrm{SLE}_\kappa(\rho_L;\rho_R)$ process in $\mathbb{H}$ with force points located at $(x_{n,L} \leq 0 \leq x_{n,R})$. Then $(W^{n,L}, V^{n,L}, V^{n,R})$ converges weakly in law with respect to the local uniform topology to the driving triple (W, V^L, V^R) of an $\mathrm{SLE}_\kappa(\rho_L;\rho_R)$ process with force points located at $(x_L \leq 0 \leq x_R)$ as $n \to \infty$. The same likewise holds in the setting of multi-force-point $\mathrm{SLE}_\kappa(\underline{\rho})$ processes.*

Proof. See [MS12a, Section 2]. □

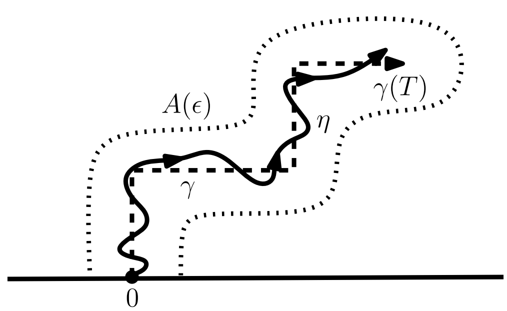


Figure 5.2.6: Suppose that η is an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process in $\mathbb{H}$ from 0 to ∞ with $x_{1,L} = 0^-$ and $x_{1,R} = 0^+$ with $\rho_{1,L}, \rho_{1,R} > -2$ and fix any deterministic curve $\gamma\colon [0,T] \to \mathbb{H}$. For each $\varepsilon > 0$, let $A(\varepsilon)$ be the ε neighborhood of γ. We show in Lemma 5.2.3 that with positive probability, η gets within distance ε of $\gamma(T)$ before leaving $A(\varepsilon)$.

Lemma 5.2.3. *Fix $\kappa > 0$. Suppose that η is an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process in $\mathbb{H}$ from 0 to ∞ with force points located at $(\underline{x}_L;\underline{x}_R)$ with $x_{1,L} = 0^-$ and $x_{1,R} = 0^+$ (possibly by taking $\rho_{1,q} = 0$ for $q \in \{L,R\}$). Assume that $\rho_{1,L}, \rho_{1,R} > -2$. Suppose that $\gamma\colon [0,T] \to \mathbb{R}$ is any deterministic simple curve in $\overline{\mathbb{H}}$*

starting from 0 *and otherwise does not hit* $\partial\mathbb{H}$. *Fix* $\varepsilon > 0$, *let* $A(\varepsilon)$ *be the* ε *neighborhood of* $\gamma([0,T])$, *and define stopping times*

$$\sigma_1 = \inf\{t \geq 0 : |\eta(t) - \gamma(T)| \leq \varepsilon\} \quad \textit{and} \quad \sigma_2 = \inf\{t \geq 0 : \eta(t) \notin A(\varepsilon)\}.$$

Then $\mathbf{P}[\sigma_1 < \sigma_2] > 0$.

Proof. See Figure 5.2.6 for an illustration. We will use the terminology "flow line," but the proof holds for $\kappa > 0$. By running η for a very small amount of time and using that $\mathbf{P}[W_t = V_t^{1,L}] = \mathbf{P}[W_t = V_t^{1,R}] = 0$ for all $t > 0$ before the continuation threshold is reached [MS12a, Section 2] and then conformally mapping back, we may assume without loss of generality that $\rho_{1,L} = \rho_{1,R} = 0$. Let U be a Jordan domain which contains $\gamma([0,T])$ and is contained in $A(\varepsilon)$. Assume, moreover, that $\partial U \cap [x_{2,L}, x_{2,R}]$ is an interval, say $[y_L, y_R]$, which contains 0. Suppose $\kappa \in (0,4)$ and let h be a GFF on $\mathbb{H}$ whose boundary data has been chosen so that its flow line η from 0 is an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process as in the statement of the lemma. Pick a point $x_0 \in \partial U$ with $|\gamma(T) - x_0| \leq \varepsilon$. Let $\widetilde{h}$ be a GFF on U whose boundary conditions are chosen so that its flow line $\widetilde{\eta}$ starting from 0 is an SLE_κ process from 0 to x_0. Let $\widetilde{\sigma}_1 = \inf\{t \geq 0 : |\widetilde{\eta}(t) - \gamma(T)| \leq \varepsilon\}$. Since $\widetilde{\eta}|_{(0,\widetilde{\sigma}_1]}$ almost surely does not hit ∂U, it follows that $\widetilde{X} \equiv \mathrm{dist}(\widetilde{\eta}|_{[0,\widetilde{\sigma}_1]}, \partial U \setminus [y_L, y_R]) > 0$ almost surely. For each $\delta > 0$, let $U_\delta = \{x \in U : \mathrm{dist}(x, \partial U \setminus [y_L, y_R]) > \delta\}$. Then the laws of $h|_{U_\delta}$ and $\widetilde{h}|_{U_\delta}$ are mutually absolutely continuous [MS12a, Proposition 3.2]. Thus the result follows since we can pick $\delta > 0$ sufficiently small so that $\mathbf{P}[\widetilde{X} > \delta] > 0$. This proves the result for $\kappa \in (0,4)$. For $\kappa' > 4$, one chooses the boundary data for $\widetilde{h}$ so that the counterflow line is an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2} - 2; \frac{\kappa'}{2} - 2)$ process (recall Lemma 5.2.1). □

Lemma 5.2.4. *Fix* $\kappa > 0$. *Suppose that* η *is an* $\mathrm{SLE}_\kappa(\rho_L;\rho_R)$ *process in* $\mathbb{H}$ *from* 0 *to* ∞ *with force points located at* $(x_L \leq 0 \leq x_R)$ *and with* $\rho_R > -2$. *Let* $\gamma\colon [0,1] \to \overline{\mathbb{H}}$ *be the unit segment connecting* 0 *to i. Fix* $\varepsilon > 0$ *and define stopping times* σ_1, σ_2 *as in Lemma 5.2.3. For each* $x_0^L < 0$ *there exists* $p_0 = p_0(x_0^L, \varepsilon) > 0$ *such that for every* $x_L \in (-\infty, x_0^L]$ *and* $x_R \geq 0$, *we have that*

$$\mathbf{P}[\sigma_1 < \sigma_2] \geq p_0. \tag{5.2.12}$$

If $\rho_L > -2$, *then there exists* $p_0 = p_0(\varepsilon)$ *such that* (5.2.12) *holds for* $x_0^L = 0^-$.

Proof. We know that this event has positive probability for each fixed choice of x_L, x_R as above by Lemma 5.2.3. Therefore the result follows from Lemma 5.2.2 and the results of [Law05, Section 4.7]. □

Lemma 5.2.5. *Fix* $\kappa > 0$. *Suppose that* η *is an* $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ *process in* $\mathbb{H}$ *from* 0 *to* ∞ *with force points located at* $(\underline{x}_L;\underline{x}_R)$ *with* $x_{1,L} = 0^-$ *and* $x_{1,R} = 0^+$ *(possibly by taking* $\rho_{1,q} = 0$ *for* $q \in \{L,R\}$*). Assume that* $\rho_{1,L}, \rho_{1,R} > -2$. *Fix* $k \in \mathbb{N}$ *such that* $\rho = \sum_{j=1}^k \rho_{j,R} \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$ *and* $\varepsilon > 0$. *There exists* $p_1 > 0$ *depending only on* $\kappa, \max_{i,q}|\rho_{i,q}|$, ρ, *and* ε *such that if* $|x_{2,q}| \geq \varepsilon$ *for* $q \in \{L,R\}$, $x_{k+1,R} - x_{k,R} \geq \varepsilon$, *and* $x_{k,R} \leq \varepsilon^{-1}$ *then the following is true. Suppose that* γ *is a simple curve starting from* 0, *terminating in* $[x_{k,R}, x_{k+1,R}]$, *and otherwise does not hit* $\partial\mathbb{H}$. *Let* $A(\varepsilon)$ *be the* ε *neighborhood of* $\gamma([0,T])$ *and let*

$$\sigma_1 = \inf\{t \geq 0 : \eta(t) \in (x_{k,R}, x_{k+1,R})\} \quad \textit{and} \quad \sigma_2 = \inf\{t \geq 0 : \eta(t) \notin A(\varepsilon)\}.$$

Then $\mathbf{P}[\sigma_1 < \sigma_2] \geq p_1$.

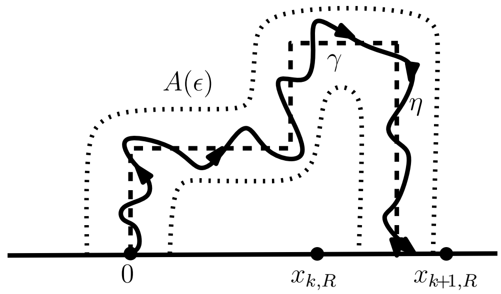


Figure 5.2.7: Suppose that η is an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process in $\mathbb{H}$ from 0 to ∞ with $x_{1,L}=0^-$ and $x_{1,R}=0^+$ with $\rho_{1,L},\rho_{1,R}>-2$ and fix any deterministic curve γ: $[0,T]\to\mathbb{H}$ which connects 0 to $[x_{k,R},x_{k+1,R}]$ where k is such that $\sum_{j=1}^k \rho_{j,R}\in(\frac{\kappa}{2}-4,\frac{\kappa}{2}-2)$. For each $\varepsilon>0$, let $A(\varepsilon)$ be the ε neighborhood of γ. We show in Lemma 5.2.5 that with positive probability, η hits $[x_{k,R},x_{k+1,R}]$ before leaving $A(\varepsilon)$.

Proof. See Figure 5.2.7 for an illustration. We will use the terminology "flow line," but the proof holds for $\kappa>0$. Arguing as in the proof of Lemma 5.2.3, we may assume without loss of generality that $\rho_{1,L}=\rho_{1,R}=0$. Let U be a Jordan domain which contains γ and is contained in $A(\varepsilon)$. Assume, moreover, that $\partial U\cap[x_{2,L},x_{2,R}]$ is an interval which contains 0 and $\partial U\cap[x_{k,R},x_{k+1,R}]$ is also an interval, say $[y_L,y_R]$. Suppose $\kappa\in(0,4)$. Let h be a GFF on $\mathbb{H}$ whose boundary data has been chosen so that its flow line η from 0 is an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process as in the statement of the lemma. Let $\widetilde{h}$ be a GFF on U whose boundary conditions are chosen so that its flow line $\widetilde{\eta}$ starting from 0 and targeted at y_R is an $\mathrm{SLE}_\kappa(\rho)$ process with a single force point located at y_L with ρ as in the statement of the lemma. Let $\widetilde{\sigma}_1$ be the first time that $\widetilde{\eta}$ hits $[y_L,y_R]$. Since $\widetilde{\eta}|_{(0,\widetilde{\sigma}_1]}$ almost surely does not hit $\partial U\setminus[y_L,y_R]$, it follows that

$$\mathrm{dist}(\widetilde{\eta}|_{[0,\widetilde{\tau}]},\partial U\setminus([x_{2,L},x_{2,R}]\cup[y_L,y_R]))>0$$

almost surely. Since $\widetilde{\eta}$ almost surely hits $[y_L,y_R]$, the assertion follows using the same absolute continuity argument for GFFs as in the proof of Lemma 5.2.3. As in the proof of Lemma 5.2.3, one proves the result for $\kappa'>4$ by taking the boundary conditions for $\widetilde{h}$ on U so that the counterflow line starting from 0 is an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-2;\frac{\kappa'}{2}-2,\rho-(\frac{\kappa'}{2}-2))$ process. □

Lemma 5.2.6. *Fix $\kappa>0$. Suppose that η is an $\mathrm{SLE}_\kappa(\rho_L;\rho_R)$ process in $\mathbb{H}$ from 0 to ∞ with force points located at $(x_L\le 0\le x_R)$ with $\rho_L\in(\frac{\kappa}{2}-4,\frac{\kappa}{2}-2)$ and $\rho_R>-2$. For each $x_0^L\in(-1,0)$ there exists $p_2=p_2(x_0^L)\in[0,1)$ such that the following is true. Fix $x_L\in[x_0^L,0]$ and define stopping times*

$$\sigma_1=\inf\{t\ge 0:|\eta(t)|=1\}\quad and\quad \tau_0^L=\inf\{t\ge 0:\eta(t)\in(-\infty,x_L]\}.$$

Then we have that

$$\mathbf{P}[\sigma_1\le\tau_0^L]\le p_2.$$

Proof. See Figure 5.2.8. Lemma 5.2.5 implies that this event has probability strictly smaller than 1 for each fixed choice of x_L,x_R as above. Therefore the result follows from Lemma 5.2.2. □

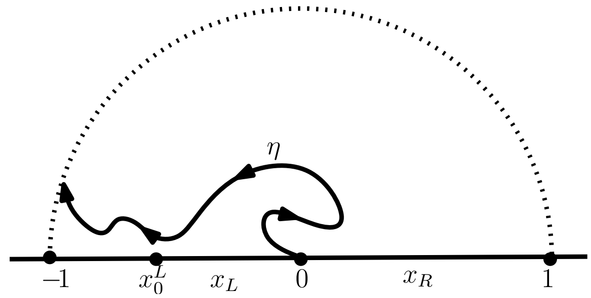


Figure 5.2.8: Suppose that η is an $\mathrm{SLE}_\kappa(\rho_L;\rho_R)$ process in $\mathbb{H}$ starting from 0 to ∞ with force points located at $x_L \leq 0 \leq x_R$ with $\rho_L \in (\frac{\kappa}{2}-4, \frac{\kappa}{2}-2)$ and $\rho_R > -2$. We show in Lemma 5.2.6 that for each choice of $x_0^L \in (-1,0)$ there exists $p_2 = p_2(x_0^L) \in [0,1)$ such that the probability that η hits $\partial B(0,1)$ before hitting $(-\infty, x_L]$ is at most p_2 uniformly in $x_L \in [x_0^L, 0]$.

5.2.3 Radon-Nikodym Derivative

Following [Dub09a, Lemma 13], we will now describe the Radon-Nikodym derivative between $\mathrm{SLE}_\kappa(\underline{\rho})$ processes arising from a change of domains and the locations of the force points. Let $c = (D, z_0, \underline{x}_L, \underline{x}_R, z_\infty)$ be a configuration consisting of a Jordan domain D in $\mathbb{C}$ with $\ell + r + 2$ marked points on ∂D. An $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process η with configuration c is given by the image of an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process $\widetilde{\eta}$ in $\mathbb{H}$ under a conformal transformation φ taking $\mathbb{H}$ to D with $\varphi(0) = z_0$, $\varphi(\infty) = z_\infty$, and which takes the force points of $\widetilde{\eta}$ to those of η.

Suppose that $c = (D, z_0, \underline{x}_L, \underline{x}_R, z_\infty)$ and $\widetilde{c} = (\widetilde{D}, z_0, \widetilde{\underline{x}}_L, \widetilde{\underline{x}}_R, \widetilde{z}_\infty)$ are two configurations such that $\widetilde{D}$ agrees with D in a neighborhood U of z_0. Let μ_c^U denote the law of an $\mathrm{SLE}_\kappa(\underline{\rho}_L;\underline{\rho}_R)$ process in c stopped at the first time τ that it exits U and define $\mu_{\widetilde{c}}^U$ analogously. Let

$$\rho_\infty = \kappa - 6 - \sum_{i,q} \rho_{i,q}$$

and

$$\begin{aligned} Z(c) = {} & \mathscr{H}_D(z_0, z_\infty)^{-\frac{\rho_\infty}{2\kappa}} \times \prod_{i,q} \mathscr{H}_D(z_0, x_{i,q})^{-\frac{\rho_{i,q}}{2\kappa}} \\ & \times \prod_{(i,q)\neq(i',q')} \mathscr{H}_D(x_{i,q}, x_{i',q'})^{-\frac{\rho_{i,q}\rho_{i',q'}}{8\kappa}} \times \prod_{i,q} \mathscr{H}_D(x_{i,q}, z_\infty)^{-\frac{\rho_{i,q}\rho_\infty}{4\kappa}} \end{aligned} \tag{5.2.13}$$

where $\mathscr{H}_D$ is the Poisson excursion kernel of the domain D. We also let

$$\xi = \frac{(6-\kappa)(8-3\kappa)}{2\kappa},$$

$$c_\tau = (D \setminus K_\tau, \eta(\tau), \underline{x}_L^\tau, \underline{x}_R^\tau, z_\infty),$$

$$m(D; K, K') = \mu^{\text{loop}}\left(\ell : \ell \subseteq D,\ \ell \cap K \neq \emptyset,\ \ell \cap K' \neq \emptyset\right),$$

with K_τ the compact hull associated with $\eta([0,\tau])$, $x_{i,q}^\tau$ the evolution of $x_{i,q}$ at time τ (precisely, if at time τ, $x_{i,q}$ is not swallowed, then $x_{i,q}^\tau = x_{i,q}$; if not, $x_{i,L}^\tau$ (resp. $x_{i,R}^\tau$)is the leftmost (resp. rightmost) point of $\partial K_\tau \cap \partial D$ in the clockwise (resp. counterclockwise) arc on ∂D from z_0 to z_∞), and μ^{loop} the Brownian loop measure on unrooted loops in $\mathbb{C}$ (see [LW04] for more on the Brownian loop measure).

The following result is proved in [Dub09a, Lemma 13] in the case that U is at a positive distance from the marked points of $c,\widetilde{c}$ other than z_0. We are now going to use the SLE/GFF coupling described in the previous section to extend the result to the case that U is at a positive distance from the marked points of $c,\widetilde{c}$ which are different.

Lemma 5.2.7. *Assume that we have the setup described just above. Suppose that U is at a positive distance from those marked points of $c,\widetilde{c}$ which differ. The probability measures $\mu_{\widetilde{c}}^U$ and μ_c^U are mutually absolutely continuous and*

$$\begin{aligned} &\frac{d\mu_{\widetilde{c}}^U}{d\mu_c^U}(\eta) \\ &= \left(\frac{Z(\widetilde{c}_\tau)/Z(\widetilde{c})}{Z(c_\tau)/Z(c)}\right)\exp\big(-\xi m(D;K_\tau,D\setminus\widetilde{D})+\xi m(\widetilde{D};K_\tau,\widetilde{D}\setminus D)\big) \end{aligned} \tag{5.2.14}$$

Proof. We are first going to prove the result in the case that $x_{1,L}\neq z_0\neq x_{1,R}$. We know that we can couple $\eta\sim\mu_c^U$ (resp. $\widetilde{\eta}\sim\mu_{\widetilde{c}}^U$) with a GFF h (resp. $\widetilde{h}$) on D (resp. $\widetilde{D}$) so that η (resp. $\widetilde{\eta}$) is the flow line of h (resp. $\widetilde{h}$) starting from z_0. By our hypotheses, the boundary data of h and $\widetilde{h}$ agree with each other in the boundary segments which are contained in ∂U. Consequently, the laws of $h|_U$ and $\widetilde{h}|_U$ are mutually absolutely continuous [MS12a, Proposition 3.2]. Since η (resp. $\widetilde{\eta}$) is almost surely determined by h (resp. $\widetilde{h}$) [MS12a, Theorem 1.2], it follows that μ_c^U and $\mu_{\widetilde{c}}^U$ are mutually absolutely continuous. Thus, to complete the proof, we just need to identify $f(\eta):=(d\mu_{\widetilde{c}}^U/d\mu_c^U)(\eta)$. By [Dub09a, Lemma 13], we know that $f(\eta)$ is equal to the right side of (5.2.14) for paths η which intersect the boundary only in the counterclockwise segment of ∂D from $x_{1,L}$ to $x_{1,R}$. Therefore, to complete the proof, we need to show that the same equality holds for paths η which intersect the other parts of the domain boundary. Note that the right hand side of (5.2.14) is a continuous function of η with respect to the uniform topology on paths. Therefore, to complete the proof, it suffices to show that the Radon-Nikodym derivative $f(\eta)$ is also continuous with respect to the same topology. Indeed, then the result follows since both functions are continuous and agree with each other on a dense set of paths. We are going to prove that this is the case using that $\eta,\widetilde{\eta}$ are coupled with $h,\widetilde{h}$, respectively.

Let ν_c^U (resp. $\nu_{\widetilde{c}}^U$) denote the joint law of $(\eta,h|_U)$ (resp. $(\widetilde{\eta},\widetilde{h}|_U)$). As explained above, ν_c^U and $\nu_{\widetilde{c}}^U$ are mutually absolutely continuous. Moreover, the Radon-Nikodym derivative $d\nu_{\widetilde{c}}^U/d\nu_c^U$ is a function of h alone since $h,\widetilde{h}$ almost surely determine $\eta,\widetilde{\eta}$, respectively. Let $\nu_c^U(\cdot\,|\,\cdot)$ (resp. $\nu_{\widetilde{c}}^U(\cdot\,|\,\cdot)$) denote the conditional law of $h|_U$ given η (resp. $\widetilde{h}|_U$ given $\widetilde{\eta}$). Note that

$$\eta\mapsto\frac{d\nu_{\widetilde{c}}^U(\cdot\,|\,\eta)}{d\nu_c^U(\cdot\,|\,\eta)}$$

is continuous in η with respect to the uniform topology on continuous paths. Let $\nu_{c,h}^U(\cdot)$ (resp. $\nu_{\widetilde{c},h}^U(\cdot)$) denote the law of $h|_U$ (resp. $\widetilde{h}|_U$). Then we have that

$$f(\eta)=\frac{d\nu_{\widetilde{c},h}^U(\cdot)}{d\nu_{c,h}^U(\cdot)}\cdot\frac{d\nu_c^U(\cdot\,|\,\eta)}{d\nu_{\widetilde{c}}^U(\cdot\,|\,\eta)}$$

(the right side does not depend on the choice of $\cdot$). This implies the desired result in the case that $x_{1,L}\neq z_0\neq x_{1,R}$ since the latter factor on the right side is continuous in η, as we remarked above. The result follows in the case that one or both of $x_{1,L},x_{1,R}$ agrees with z_0 since the laws converge as one or both of $x_{1,L},x_{1,R}$ converge to z_0 (Lemma 5.2.2). □

Lemma 5.2.8. *Assume that we have the same setup as in Lemma 5.2.7 with $D = \mathbb{H}$, $\widetilde{D} \subseteq \mathbb{H}$, $U \subseteq \mathbb{H}$ bounded, and $z_0 = 0$. Fix $\zeta > 0$ and suppose that the distance between U and $\mathbb{H} \setminus \widetilde{D}$ is at least ζ, the force points of $c, \widetilde{c}$ in $\overline{U}$ are identical, the corresponding weights are also equal, and the force points which are outside of U are at distance at least ζ from U. There exists a constant $C \geq 1$ depending on U, ζ, κ, and the weights of the force points such that*

$$\frac{1}{C} \leq \frac{d\mu_{\widetilde{c}}^U}{d\mu_c^U} \leq C.$$

Proof. Note that $0 \leq m(\mathbb{H}; K_\tau, \mathbb{H} \setminus \widetilde{D}) \leq m(\mathbb{H}; U, \mathbb{H} \setminus U^\zeta)$ where U^ζ is the ζ-neighborhood of U. Moreover, we have that $m(\mathbb{H}; U, \mathbb{H} \setminus U^\zeta)$ is bounded from above by a finite constant depending on U and ζ since, the mass according to μ^{loop} of the loops which are contained in $\mathbb{H}$, intersect U, and have diameter at least ζ is finite [Law, Corollary 4.6]. Consequently, by Lemma 5.2.7, we only need to bound the quantity $\frac{Z(\widetilde{c}_\tau)/Z(\widetilde{c})}{Z(c_\tau)/Z(c)}$.

Recall from (5.2.13) that the terms in $\frac{Z(\widetilde{c}_\tau)/Z(\widetilde{c})}{Z(c_\tau)/Z(c)}$ are ratios of terms of the form $\mathscr{H}_X(u,v)$ where X is one of $\mathbb{H}$, $\mathbb{H}_\tau$, $\widetilde{D}$, $\widetilde{D}_\tau$ and u, v are two marked points on the boundary of X. We will complete the proof by considering several cases depending on the location of the marked points.

Case 1. At least one marked point is outside of U^ζ. This is the case handled in the proof of [Dub09a, Lemma 14].

Case 2. Both marked points u, v are contained in $\overline{U}$ and $u \neq v$. It is enough to bound from above and below the ratios:

$$A = \frac{\mathscr{H}_{\widetilde{D}}(x,y)}{\mathscr{H}_{\mathbb{H}}(x,y)} \quad \text{and} \quad B = \frac{\mathscr{H}_{\widetilde{D}_\tau}(x^\tau,y^\tau)}{\mathscr{H}_{\mathbb{H}_\tau}(x^\tau,y^\tau)}$$

where $x, y \in \partial U \cap \mathbb{R}$ are distinct and $x^\tau, y^\tau \in \partial \mathbb{H}_\tau \cap \overline{U}$ are distinct.

We can bound A as follows. Let $\varphi \colon \widetilde{D} \to \mathbb{H}$ be the unique conformal transformation with $\varphi(x) = x$, $\varphi(y) = y$, and $\varphi'(x) = 1$. Then $A = |\varphi'(y)|$ which, by [LSW03, Proposition 4.1], is equal to the mass of those Brownian excursions in $\mathbb{H}$ connecting x and y which avoid $\mathbb{H} \setminus \widetilde{D}$. We will write $q(\mathbb{H}, x, y, \mathbb{H} \setminus \widetilde{D})$ for this quantity. Since this is given by a probability, we have that $|\varphi'(y)| \leq 1$ and it follows that $|\varphi'(y)|$ is bounded from below by $q(\mathbb{H}, x, y, U^\zeta) > 0$. This lower bound is a positive continuous function in $x, y \in \partial U \cap \partial \mathbb{H}$ hence yields a uniform lower bound. Consequently, A is bounded from both above and below.

Similarly, B is equal to the mass $q(\mathbb{H} \setminus K_\tau, x^\tau, y^\tau, \mathbb{H} \setminus \widetilde{D})$ of those Brownian excursions in $\mathbb{H} \setminus K_\tau$ which connect x^τ and y^τ and avoid $\mathbb{H} \setminus \widetilde{D}$. As before, this quantity is bounded from above by 1. We will now establish the lower bound. Let g be the conformal map from $\mathbb{H} \setminus K_\tau$ onto $\mathbb{H}$ which sends the triple (x^τ, y^τ, ∞) to $(0, 1, \infty)$. Note that g can be extended to $\mathbb{C} \setminus (K \cup \bar{K})$ by Schwarz reflection where $\bar{K} = \{z \in \mathbb{C} : \bar{z} \in K\}$. We will view g as such an extension. Then it is clear that

$$\begin{aligned} q(\mathbb{H} \setminus K_\tau, x^\tau, y^\tau, \mathbb{H} \setminus \widetilde{D}) &\geq q(\mathbb{H} \setminus K_\tau, x^\tau, y^\tau, \mathbb{H} \setminus U^\zeta) \\ &= q(\mathbb{H}, 0, 1, \mathbb{H} \setminus g(U^\zeta)). \end{aligned}$$

Note that $q(\mathbb{H}, 0, 1, \mathbb{H} \setminus g(U^\zeta))$ is a continuous functional on compact hulls K inside $\overline{U}$ equipped with the Hausdorff metric. Indeed, suppose that (K_n) is a sequence of compact hulls inside $\overline{U}$ converging towards K in Hausdorff metric and, for each n, let g_n be the corresponding conformal map. Then g_n converges to g uniformly away from $K \cup \bar{K}$. In particular, $g_n(U^\zeta)$ converges to $g(U^\zeta)$ in Hausdorff metric. Let ϕ_n (resp. ϕ) be the conformal map from $\mathbb{H} \setminus g_n(U^\zeta)$ (resp.

$\mathbb{H}\setminus g(U^\zeta))$ onto $\mathbb{H}$ which fixes 0, 1 and has derivative 1 at 1. Then $\phi_n'(0)$ converges to $\phi'(0)$. Thus $q(\mathbb{H},0,1,\mathbb{H}\setminus g_n(U^\zeta)) = \phi_n'(0)$ converges to $q(\mathbb{H},0,1,\mathbb{H}\setminus g(U^\zeta)) = \phi'(0)$ which explains the continuity of $q(\mathbb{H},0,1,\mathbb{H}\setminus g(U^\zeta))$ in K. Since the set of compact hulls inside $\overline{U}$ endowed with Hausdorff metric is compact. There exists $q_0 > 0$ depending on U and ζ such that

$$q(\mathbb{H}\setminus K_\tau, x^\tau, y^\tau, \mathbb{H}\setminus \widetilde{D}) \geq q(\mathbb{H},0,1,\mathbb{H}\setminus g(U^\zeta)) \geq q_0$$

for any compact hull.

Case 3. A single marked point u contained in $\overline{U}$. The ratios which involve terms of the form $\mathscr{H}_X(u,u)$ are interpreted using limits hence are uniformly bounded by Case 2.

□

5.2.4 Estimates for conformal maps

For a proper simply connected domain D and $w \in D$, let $\mathrm{CR}(w;D)$ denote the conformal radius of D with respect to w, i.e., $\mathrm{CR}(w;D) \equiv f'(0)$ for f the unique conformal map $\mathbb{D}\to D$ with $f(0) = w$ and $f'(0) > 0$. Let $\mathrm{rad}(w;D) \equiv \inf\{r : B_r(w) \supseteq D\}$ denote the out-radius of D with respect to w. By the Schwarz lemma and the Koebe one-quarter theorem,

$$\mathrm{dist}(w,\partial D) \leq \mathrm{CR}(w;D) \leq [4\,\mathrm{dist}(w,\partial D)] \wedge \mathrm{rad}(w;D). \tag{5.2.15}$$

Further (see e.g. [Pom92, Theorem 1.3])

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq \frac{|f(\zeta)-w|}{\mathrm{CR}(w;D)} \leq \frac{|\zeta|}{(1-|\zeta|)^2} \tag{5.2.16}$$

As a consequence,

$$\frac{|\zeta|}{4} \leq \frac{|f(\zeta)-w|}{\mathrm{CR}(w;D)} \leq 4|\zeta| \tag{5.2.17}$$

where the right-hand inequality above holds for $|\zeta| \leq 1/2$.

Finally, we state the Beurling estimate [Law05, Theorem 3.76] which we will frequently use in conjunction with the conformal invariance of Brownian motion.

Theorem 5.2.9 (Beurling Estimate)**.** *Suppose that B is a Brownian motion in $\mathbb{C}$ and $\tau_{\mathbb{D}} = \inf\{t \geq 0 : B(t) \notin \mathbb{D}\}$. There exists a constant $c < \infty$ such that if $\gamma\colon [0,1] \to \mathbb{C}$ is a curve with $\gamma(0) = 0$ and $|\gamma(1)| = 1$, $z \in \mathbb{D}$, and $\mathbf{P}^z$ is the law of B when started at z, then*

$$\mathbf{P}^z[B([0,\tau_{\mathbb{D}}]) \cap \gamma([0,1]) = \emptyset] \leq c|z|^{1/2}.$$

5.3 The intersection of $\mathrm{SLE}_\kappa(\rho)$ with the boundary

5.3.1 The upper bound

The main result of this section is the following theorem, which in turn implies Theorem 5.1.8.

Theorem 5.3.1. *Fix $\kappa > 0$, $\rho_{1,R} > -2$, and $\rho_{2,R} \in \mathbb{R}$ such that $\rho_{1,R} + \rho_{2,R} > \frac{\kappa}{2} - 4$. Fix $x_R \in [0^+,1)$ and let η be an $\mathrm{SLE}_\kappa(\rho_{1,R},\rho_{2,R})$ process with force points $(x_R,1)$. Let*

$$\alpha = \frac{1}{\kappa}(\rho_{1,R}+2)\left(\rho_{1,R}+\rho_{2,R}+4-\frac{\kappa}{2}\right). \tag{5.3.1}$$

For each $\varepsilon > 0$, let $\tau_\varepsilon = \inf\{t \geq 0 : \eta(t) \in \partial B(1,\varepsilon)\}$ and, for each $r > 0$, let $\sigma_r = \inf\{t \geq 0 : \eta(t) \in \partial(r\mathbb{D})\}$. For each $\delta \in [0,1)$ and $r \geq 2$ fixed, let

$$E_\varepsilon^{\delta,r} = \{\tau_\varepsilon < \sigma_r,\ \text{Im}(\eta(\tau_\varepsilon)) \geq \delta\varepsilon\}. \tag{5.3.2}$$

We have that

$$\mathbf{P}[E_\varepsilon^{\delta,r}] = \varepsilon^{\alpha+o(1)} \quad as \quad \varepsilon \to 0. \tag{5.3.3}$$

The $o(1)$ in the exponent of (5.3.3) tends to 0 as $\varepsilon \to 0$ and depends only on κ, δ, x_R, and the weights $\rho_{1,R}$, $\rho_{2,R}$. The $o(1)$, however, is uniform in $r \geq 2$. Taking $\rho_{1,R} > (-2) \vee (\frac{\kappa}{2} - 4)$ and $\rho_{2,R} = 0$, we have that

$$\alpha = \frac{1}{\kappa}(\rho+2)\left(\rho+4-\frac{\kappa}{2}\right). \tag{5.3.4}$$

Thus Theorem 5.3.1 leads to the upper bound of Theorem 5.1.6. We begin with the following lemma which contains the same statement as Theorem 5.3.1 except is restricted to the case that $\delta \in (0,1)$ and, in particular, is not applicable for $\delta = 0$.

Lemma 5.3.2. *Assume that we have the same setup and notation as in Theorem 5.3.1. Then for each $\delta \in (0,1)$ and $r \geq 2$ fixed, we have that*

$$\mathbf{P}[E_\varepsilon^{\delta,r}] \asymp \varepsilon^\alpha$$

where the constants in $\asymp$ depend only on κ, δ, x_R, and the weights $\rho_{1,R}$, $\rho_{2,R}$.

Proof. For η, the $\text{SLE}_\kappa(\rho_{1,R},\rho_{2,R})$ process with force points $(x_R, 1)$, let (g_t) be the associated Loewner evolution and let V_t^R denote the evolution of x_R. From (5.2.6) we know that

$$M_t = \left(\frac{g_t(1)-V_t^R}{g_t'(1)}\right)^{-\alpha}\left(\frac{g_t(1)-W_t}{g_t(1)-V_t^R}\right)^{-\frac{2}{\kappa}(\rho_{1,R}+\rho_{2,R}+4-\kappa/2)}$$

is a local martingale and the law of η reweighted by M is that of an $\text{SLE}_\kappa(\rho_{1,R},\widetilde{\rho}_{2,R})$ process where $\widetilde{\rho}_{2,R} = -2\rho_{1,R} - \rho_{2,R} - 8 + \kappa$. We write $K = K_{\tau_\varepsilon}$ and $\overline{K} = \{\overline{z} : z \in K\}$. Let G be the extension of g_{τ_ε} to $\mathbb{C} \setminus (K \cup \overline{K})$ which is obtained by Schwarz reflection. By (5.2.15), we have

$$G'(x)\text{dist}(x,K) \asymp \text{dist}(G(x), G(K \cup \overline{K})). \tag{5.3.5}$$

Observe that $G(K \cup \overline{K}) = [O_{\tau_\varepsilon}^L, O_{\tau_\varepsilon}^R]$ where O_t^L (resp. O_t^R) is the image of the leftmost (resp. rightmost) point of $K_t \cap \mathbb{R}$ under g_t. Note that (5.3.5) implies

$$\varepsilon g_{\tau_\varepsilon}'(1) \asymp g_{\tau_\varepsilon}(1) - O_{\tau_\varepsilon}^R.$$

It is clear that $g_t(1) - W_t \geq g_t(1) - O_t^R \geq g_t(1) - V_t^R$. On the event $E_\varepsilon^{\delta,r}$, we run a Brownian motion started from the midpoint of the line segment $[1, \eta(\tau_\varepsilon)]$. Then this Brownian motion has uniformly positive (though δ-dependent) probability to exit $\mathbb{H} \setminus K$ through each of the left side of K, the right side of K, the interval $[x_R, 1]$, and the interval $(1,\infty)$. Consequently, by the conformal invariance of Brownian motion,

$$g_{\tau_\varepsilon}(1) - W_{\tau_\varepsilon} \asymp g_{\tau_\varepsilon}(1) - O_{\tau_\varepsilon}^R \asymp g_{\tau_\varepsilon}(1) - V_{\tau_\varepsilon}^R \quad \text{on} \quad E_\varepsilon^{\delta,r}.$$

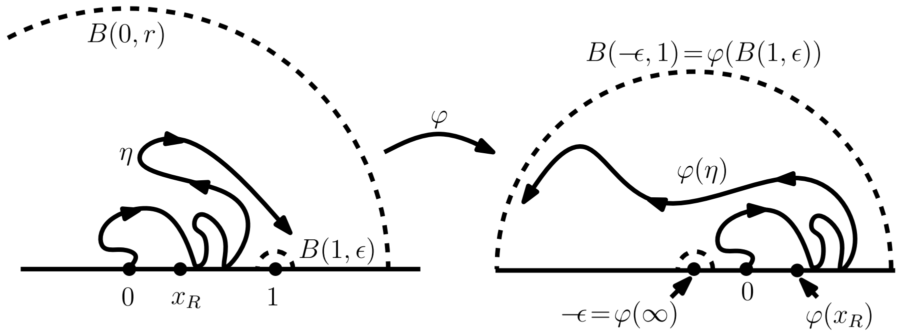


Figure 5.3.1: The image of an $\mathrm{SLE}_\kappa(\rho_{1,R},\rho_{2,R})$ process in $\mathbb{H}$ from 0 to ∞ with force points $(x_R,1)$ under $\varphi(z)=\varepsilon z/(1-z)$ has the same law as an $\mathrm{SLE}_\kappa(\rho_L;\rho_R)$ process in $\mathbb{H}$ from 0 to ∞ with force points $(-\varepsilon;\varepsilon x_R/(1-x_R))$ where $\rho_R=\rho_{1,R}$ and $\rho_L=\kappa-6-(\rho_{1,R}+\rho_{2,R})$.

These facts imply that $M_{\tau_\varepsilon}\asymp\varepsilon^{-\alpha}$ on $E_\varepsilon^{\delta,r}$ where the constants in $\asymp$ depend only on κ, δ, x_R, and the weights $\rho_{1,R}$, $\rho_{2,R}$. Thus

$$\mathbf{P}[E_\varepsilon^{\delta,r}]\asymp\varepsilon^{\alpha}\mathbb{E}[M_{\tau_\varepsilon}\mathbf{1}_{E_\varepsilon^{\delta,r}}]=\varepsilon^{\alpha}\mathbf{P}^\star[E_\varepsilon^{\delta,r}]$$

where $\mathbf{P}^\star$ is the law of η weighted by the martingale M. As we remarked earlier, $\mathbf{P}^\star$ is the law of an $\mathrm{SLE}_\kappa(\rho_{1,R},\widetilde{\rho}_{2,R})$ with force points $(x_R,1)$.

We now perform a coordinate change using the Möbius transformation $\varphi(z)=\varepsilon z/(1-z)$. Then the law of the image of a path distributed according to $\mathbf{P}^\star$ under φ is equal to that of an $\mathrm{SLE}_\kappa(2+\rho_{1,R}+\rho_{2,R};\rho_{1,R})$ process in $\mathbb{H}$ from 0 to ∞ with force points $(-\varepsilon;\varepsilon x_R/(1-x_R))$ (see Figure 5.3.1). Note that $2+\rho_{1,R}+\rho_{2,R}\geq\frac{\kappa}{2}-2$ by the hypotheses of the lemma. Let $\eta^\star$ be an $\mathrm{SLE}_\kappa(2+\rho_{1,R}+\rho_{2,R};\rho_{1,R})$ process in $\mathbb{H}$ from 0 to ∞ with force points $(-\varepsilon;\varepsilon x_R/(1-x_R))$. In particular, by Lemma 5.2.1, $\eta^\star$ almost surely does not hit $(-\infty,-\varepsilon)$. Under the coordinate change, the event $E_\varepsilon^{\delta,r}$ becomes $\{\sigma_{1,\varepsilon}^\star<\xi_{\varepsilon,r}^\star,\ \mathrm{Im}(\eta^\star(\sigma_{1,\varepsilon}^\star))\geq\delta\}$ where $\sigma_{1,\varepsilon}^\star$ is the first time that $\eta^\star$ hits $\partial B(-\varepsilon,1)$, $\xi_{\varepsilon,r}^\star$ is the first time that $\eta^\star$ hits $\partial B(-\varepsilon r^2/(r^2-1),\varepsilon r/(r^2-1))$. By Lemma 5.2.4, the probability of the event $\{\sigma_{1,\varepsilon}^\star<\xi_{\varepsilon,r}^\star,\ \mathrm{Im}(\eta^\star(\sigma_{1,\varepsilon}^\star))\geq\delta\}$ is bounded from below by a positive constant depending only on κ, δ, $\rho_{1,R}$, and $\rho_{2,R}$. Thus $\mathbf{P}^\star[E_\varepsilon^{\delta,r}]\asymp1$ which implies $\mathbf{P}[E_\varepsilon^{\delta,r}]\asymp\varepsilon^\alpha$ and the constants in $\asymp$ depend only on κ, δ, x_R, and the weights $\rho_{1,R}$, $\rho_{2,R}$. □

Corollary 5.3.3. *Fix $\kappa>0$, $\rho_L>-2,\rho_{1,R}>-2$ and $\rho_{2,R}\in\mathbb{R}$ such that $\rho_{1,R}+\rho_{2,R}>\frac{\kappa}{2}-4$. Fix $x_L\leq0,x_R\in[0^+,1)$ and let η be an $\mathrm{SLE}_\kappa(\rho_L;\rho_{1,R},\rho_{2,R})$ process with force points $(x_L;x_R,1)$. Let $E_\varepsilon^{\delta,r}$ be the event as in Theorem 5.3.1, then for each $\delta\in(0,1)$ and $r\geq2$ fixed, we have that*

$$\mathbf{P}[E_\varepsilon^{\delta,r}]\asymp\varepsilon^\alpha$$

where the constants in $\asymp$ depend only on κ, δ, r, x_L, x_R, and the weights $\rho_L,\rho_{1,R}$, $\rho_{2,R}$.

Proof. Let (g_t) be the Loewner evolution associated with η and let V_t^L,V_t^R denote the evolution of x_L,x_R, respectively, under g_t. From (5.2.6) we know that

$$M_t=\left(\frac{g_t(1)-V_t^R}{g_t'(1)}\right)^{-\alpha}\times\left(\frac{g_t(1)-W_t}{g_t(1)-V_t^R}\right)^{-\frac{2}{\kappa}(\rho_{1,R}+\rho_{2,R}+4-\kappa/2)}$$
$$\times\,(g_t(1)-V_t^L)^{-\frac{\rho_L}{\kappa}(\rho_{1,R}+\rho_{2,R}+4-\kappa/2)}$$

is a local martingale which yields that the law of η reweighted by M is that of an $\mathrm{SLE}_\kappa(\rho_L;\rho_{1,R},\widetilde{\rho}_{2,R})$ process where $\widetilde{\rho}_{2,R} = -2\rho_{1,R} - \rho_{2,R} - 8 + \kappa$. Note that, by similar analysis in Lemma 5.3.4, the term $g_{\tau_\varepsilon}(1) - V^L_{\tau_\varepsilon}$ is bounded both from below and above by positive finite constants depending only on r on the event $E^{\delta,r}_\varepsilon$. The rest of the analysis in the proof of Lemma 5.3.2 applies similarly in this setting. □

Throughout the rest of this subsubsection, we let:

$$\mathbb{T} = \mathbb{R} \times (0,1). \tag{5.3.6}$$

Lemma 5.3.4. *Let η be a continuous curve in $\overline{\mathbb{H}}$ starting from 0 with continuous Loewner driving function W and let (g_t) be the corresponding family of conformal maps. For each $t \geq 0$, let O^L_t (resp. O^R_t) be the leftmost (resp. rightmost) point of $g_t(\eta([0,t]))$ in $\mathbb{R}$. There exists a universal constant $C \geq 1$ such that the following is true. Fix $\vartheta > 0$ and let σ be the first time that η exits $\vartheta\mathbb{T}$. Then*

$$|W_\sigma - O^q_\sigma| \geq \frac{\vartheta}{C} \quad \text{for} \quad q \in \{L,R\}. \tag{5.3.7}$$

Let ζ be the first time that η exits $\mathbb{D}\cap\vartheta\mathbb{T}$. Then

$$|W_t - O^q_t| \leq C\vartheta \quad \text{for} \quad q \in \{L,R\} \quad \text{and all} \quad t \in [0,\zeta]. \tag{5.3.8}$$

Finally, if η exits $\mathbb{D}\cap\vartheta\mathbb{T}$ through the right side of $\partial\mathbb{D}\cap\vartheta\mathbb{T}$, then

$$|W_\zeta - O^L_\zeta| \geq \frac{1}{C}. \tag{5.3.9}$$

Proof. For $z \in \mathbb{C}$, we let $\mathbf{P}^z$ denote the law of a Brownian motion B in $\mathbb{C}$ started at z. By [Law05, Remark 3.50] we have that

$$|W_\sigma - O^L_\sigma| = \lim_{y\to\infty} y\mathbf{P}^{yi}\left[B \text{ exits } \mathbb{H}\setminus\eta[0,\sigma] \text{ on the left side of } \eta([0,\sigma])\right].$$

Let τ be the exit time of B from $\mathbb{H}\setminus\vartheta\mathbb{T}$ and let $I = [\eta(\sigma)-\vartheta, \eta(\sigma)]$. Then

$$\begin{aligned}|W_\sigma - O^L_\sigma| \geq &\lim_{y\to\infty} y\mathbf{P}^{yi}\left[B_\tau \in I\right] \\ &\times \mathbf{P}^{yi}\left[B \text{ exits } \mathbb{H}\setminus\eta([0,\sigma]) \text{ on the left side of } \eta([0,\sigma])\,|\,B_\tau \in I\right].\end{aligned} \tag{5.3.10}$$

We have,

$$\begin{aligned}\lim_{y\to\infty} y\mathbf{P}^{yi}\left[B_\tau \in I\right] &= \lim_{y\to\infty}\int_{I-\vartheta i}\frac{1}{\pi}\frac{y(y-\vartheta)}{w^2+(y-\vartheta)^2}dw \\ &= \int_{I-\vartheta i}\frac{1}{\pi}dw = \frac{\vartheta}{\pi}\end{aligned} \tag{5.3.11}$$

(recall the form of the Poisson kernel on $\mathbb{H}$, see e.g. [Law05, Exercise 2.23]). It is easy to see that there exists a universal constant $p_0 > 0$ such that for any $z \in I$,

$$\mathbf{P}^z\left[B \text{ exits } \mathbb{H}\setminus\eta[0,\sigma] \text{ on the left side of } \eta([0,\sigma])\right] \geq p_0. \tag{5.3.12}$$

Combining (5.3.10) with (5.3.11) and (5.3.12) gives (5.3.7). The bounds (5.3.8) and (5.3.9) are proved similarly. □

Lemma 5.3.5. *Fix $\kappa > 0$, $\rho_L \in (\frac{\kappa}{2}-4, \frac{\kappa}{2}-2)$, and $\rho_R > -2$. Let η be an $\mathrm{SLE}_\kappa(\rho_L;\rho_R)$ process with force points $(-\varepsilon; x_R)$ for $x_R \geq 0^+$ and $\varepsilon > 0$. Let $\sigma_1 = \inf\{t \geq 0 : \eta(t) \in \partial\mathbb{D}\}$. Define, for $u \geq 0$, $T_u^L = \inf\{t \geq 0 : W_t - V_t^L = u\}$, where V_t^L denotes the evolution of x^L. Let $p_2 = p_2(\frac{1}{2})$ be the constant from Lemma 5.2.6. There exists constants $\varepsilon_0 > 0$, $\vartheta_0 > 0$, and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\vartheta \in (0, \vartheta_0)$ we have*

$$\mathbf{P}[\sigma_1 < T_0^L \wedge T_\vartheta^L] \leq p_2^{1/(C\vartheta)}.$$

Proof. Let $E_\vartheta = \{\sigma_1 < T_0^L \wedge T_\vartheta^L\}$. By definition, we have that

$$|W_t - V_t^L| < \vartheta \quad \text{for all} \quad t \in [0, \sigma_1] \quad \text{on} \quad E_\vartheta. \tag{5.3.13}$$

By (5.3.7) of Lemma 5.3.4 there exists a constant $C_1 > 0$ such that $\eta([0,\sigma_1]) \subseteq C_1\vartheta\mathbb{T}$. Moreover, η exits $\mathbb{D} \cap (C_1\vartheta\mathbb{T})$ on its left side for all $\vartheta > 0$ small enough because a Brownian motion argument (analogous to (5.3.9)) implies there exists a constant $C_2 > 0$ such that $|W_{\sigma_1} - V_{\sigma_1}^L| \geq C_2$ on the event that η exits through the right side, contradicting (5.3.13).

Suppose $C > 0$; we will set its value later in the proof. For each $1 \leq k \leq \frac{1}{C\vartheta}$, we let

$$L_k = \{z \in \mathbb{H} : \mathrm{Re}(z) = -kC\vartheta\} \quad \text{and} \quad \zeta_k = \inf\{t \geq 0 : \eta(t) \in L_k\}.$$

On E_ϑ, we have that $\zeta_1 < \zeta_2 < \cdots < \sigma_1 < T_0^L$. For each k, let $F_k = \{\zeta_k < T_\vartheta^L\}$ and let $\mathscr{F}_k$ be the σ-algebra generated by $\eta|_{[0,\zeta_k]}$. To complete the proof, we will show that

$$\mathbf{P}[\zeta_{k+1} < T_0^L \,|\, \mathscr{F}_k]\mathbf{1}_{F_k} \leq p_2 \mathbf{1}_{F_k} \quad \text{for each} \quad 1 \leq k \leq \frac{1}{C\vartheta}$$

where $p_2 = p_2(\frac{1}{2})$ is the constant from Lemma 5.2.6. To see this, we just need to show that $g_{\zeta_k}(\eta|_{[\zeta_k,\zeta_{k+1}]})$ satisfies the hypotheses of Lemma 5.2.6 and that with

$$\widetilde{L}_{k+1} = \frac{g_{\zeta_k}(L_{k+1}) - W_{\zeta_k}}{W_{\zeta_k} - V_{\zeta_k}^L}$$

we have that $\widetilde{L}_{k+1} \cap 2\mathbb{D} = \emptyset$ on F_k. Therefore it suffices to prove

$$\frac{\mathrm{dist}(W_{\zeta_k}, g_{\zeta_k}(L_{k+1}))}{W_{\zeta_k} - V_{\zeta_k}^L} \to \infty \quad \text{on} \quad F_k \quad \text{as} \quad C \to \infty. \tag{5.3.14}$$

Let B be a Brownian motion starting from $z_k^\vartheta = \eta(\zeta_k) - \vartheta$ and let $H_{k+1} = \{z \in \mathbb{H} : \mathrm{Re}(z) \geq -(k+1)C\vartheta\}$ be the subset of $\mathbb{H}$ which is to the right of L_{k+1} (see Figure 5.3.2). The probability that B exits $H_{k+1} \setminus \eta([0,\zeta_k])$ through the right side of $\eta([0,\zeta_k])$ (blue) is $\gtrsim 1$, through $(-(k+1)C\vartheta, -kC\vartheta)$ (green) is $\gtrsim 1$, and through L_{k+1} (orange) is $\lesssim 1/C$ (since this probability is less than the probability that the Brownian motion exits $\{z \in \mathbb{C} : -(k+1)C\vartheta < \mathrm{Re}(z) < -kC\vartheta\}$ through L_{k+1} which is less than $1/C$). Let

$$\widetilde{z}_k^\vartheta \equiv \widetilde{x}_k^\vartheta + \widetilde{y}_k^\vartheta i \equiv \frac{g_{\zeta_k}(z_k^\vartheta) - W_{\zeta_k}}{W_{\zeta_k} - V_{\zeta_k}^L} \quad .$$

By the conformal invariance of Brownian motion, we have that

$$\frac{\mathrm{dist}(\widetilde{z}_k^\vartheta, \widetilde{L}_{k+1})}{\widetilde{y}_k^\vartheta} \gtrsim C. \tag{5.3.15}$$

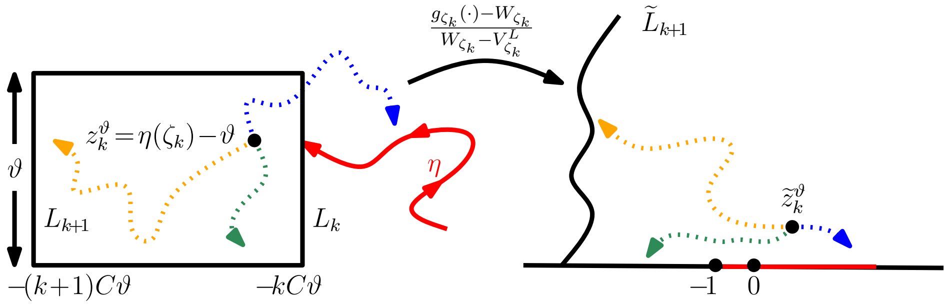


Figure 5.3.2: Illustration of the justification of (5.3.14) in the proof of Lemma 5.3.5.

Indeed, the probability of a Brownian motion started from $\widetilde{z}_k^\vartheta$ to exit $\widetilde{H}_{k+1} := (g_{\zeta_k}(H_{k+1}) - W_{\zeta_k})/(W_{\zeta_k} - V^L_{\zeta_k})$ through $\widetilde{L}_{k+1}$ is bounded from below by a positive universal constant times the probability that a Brownian motion starting from $\widetilde{z}_k^\vartheta$ exits $B(\widetilde{z}_k^\vartheta, \widetilde{d}) \cap \mathbb{H}$, $\widetilde{d} = \mathrm{dist}(\widetilde{z}_k^\vartheta, \widetilde{L}_{k+1})$, through $\partial B(\widetilde{z}_k^\vartheta, \widetilde{d}) \cap \mathbb{H}$. This latter probability is bounded from below by a positive universal constant times $\widetilde{y}_k^\vartheta/\widetilde{d}$. Thus $1/C \gtrsim \widetilde{y}_k^\vartheta/\widetilde{d}$, as desired.

The conformal invariance of Brownian motion and the estimates above also imply that $\sin(\arg(\widetilde{z}_k^\vartheta)) \asymp 1$, hence $|\widetilde{z}_k^\vartheta| \asymp |\widetilde{y}_k^\vartheta|$. Combining this with (5.3.15) implies that

$$\frac{\mathrm{dist}(\widetilde{z}_k^\vartheta, \widetilde{L}_{k+1})}{|\widetilde{z}_k^\vartheta|} \gtrsim C.$$

Thus, by the triangle inequality,

$$\mathrm{dist}(\widetilde{L}_{k+1}, 0) \gtrsim C|\widetilde{z}_k^\vartheta|$$

(provided C is large enough). Since $|\widetilde{z}_k^\vartheta| \asymp 1$, this proves (5.3.14), hence the lemma. □

Proof of Theorem 5.3.1. Lemma 5.3.2 implies the lower bound in (5.3.3) because we can take, e.g., $\delta = \frac{1}{2}$. In order to prove the upper bound, it is sufficient to show

$$\mathbf{P}[\tau_\varepsilon < \infty] \leq \varepsilon^{\alpha + o(1)} \quad \text{as} \quad \varepsilon \to 0.$$

We are first going to perform a change of coordinates. Let $\varphi \colon \mathbb{H} \to \mathbb{H}$ be the Möbius transformation $z \mapsto \varphi(z) := \varepsilon z/(1-z)$. Fix $\widetilde{x}^R \in [0^+, 1)$ and let $\widetilde{\eta}$ be an $\mathrm{SLE}_\kappa(\rho_{1,R}, \rho_{2,R})$ process with force points located at $(\widetilde{x}^R, 1)$ as in Theorem 5.3.1. Then the law of $\eta = \varphi(\widetilde{\eta})$ is that of an $\mathrm{SLE}_\kappa(\rho_L; \rho_R)$ process with force points $(-\varepsilon; x_R)$ where $x_R = \varepsilon \widetilde{x}^R/(1-\widetilde{x}^R)$ and

$$\rho_L = \kappa - 6 - \left(\rho_{1,R} + \rho_{2,R}\right) \quad \text{and} \quad \rho_R = \rho_{1,R}. \tag{5.3.16}$$

Let σ_1 be the first time that η hits $\partial \mathbb{D}$ and let V_t^L, V_t^R denote the evolution of x_L, x_R under g_t, respectively. For $u \geq 0$, define $T_u^L = \inf\{t \geq 0 : W_t - V_t^L = u\}$ (as in the statement of Lemma 5.3.5). Then it is sufficient to prove $\mathbf{P}[\sigma_1 < T_0^L] \leq \varepsilon^{\alpha + o(1)}$. Note that the exponent α comes from the sum of the exponent of $|V_t^L - V_t^R|$ and the exponent of $|W_t - V_t^L|$ in the left martingale M^L from (5.2.7) with these weights. For $u \geq 0$, define $\tau_u^L = \inf\{t \geq 0 : M_t^L = u\}$. Note that $\tau_0^L = T_0^L$. Fix $\beta \in (0,1)$ and set $\vartheta = \varepsilon^\beta$. For $u > 0$, we have the bound

$$\mathbf{P}[\sigma_1 < \tau_0^L] \leq \mathbf{P}[\tau_u^L < \tau_0^L] + \mathbf{P}[\sigma_1 < \tau_0^L < \tau_u^L]. \tag{5.3.17}$$

We claim that exists constants $C_1 > 0$ and $\gamma > 0$ depending only on ρ_L, ρ_R, and κ such that

$$|W_t - V_t^L|^\gamma \le C_1 M_t^L \quad \text{for all} \quad t \in [0,\sigma_1]. \tag{5.3.18}$$

Since $\rho_{1,R} + \rho_{2,R} > \frac{\kappa}{2} - 4$ it follows that $\rho_L < \frac{\kappa}{2} - 2$. Therefore the sign of the exponent of $|V_t^L - V_t^R|$ in the definition of M_t^L is the same as the sign of ρ_R. If $\rho_R \ge 0$, then the exponent has a positive sign. In this case, $M_t^L \ge |W_t - V_t^L|^\alpha$ so that we can take $\gamma = \alpha$. Now suppose that $\rho_R < 0$. By (5.3.8) of Lemma 5.3.4 we know that there exists a constant $C_2 > 0$ such that

$$|V_t^L - V_t^R| \le C_2 \quad \text{for all} \quad t \in [0,\sigma_1]. \tag{5.3.19}$$

Thus, in this case, there exists a constant $C_3 > 0$ such that $M_t^L \ge C_3 |W_t - V_t^L|^{(\kappa - 4 - 2\rho_L)/\kappa}$. Therefore we can take $\gamma = (\kappa - 4 - 2\rho_L)/\kappa$. This proves the claimed bound in (5.3.18).

Set $u = \vartheta^\gamma / C_1$. To bound the second term on the right side of (5.3.17), we first note by (5.3.18) that

$$\mathbf{P}[\sigma_1 < \tau_0^L < \tau_u^L] \le \mathbf{P}[\sigma_1 < T_0^L \wedge T_\vartheta^L]. \tag{5.3.20}$$

By Lemma 5.3.5, we know that

$$\mathbf{P}[\sigma_1 < T_0^L \wedge T_\vartheta^L] \le p_2^{1/(C\vartheta)}. \tag{5.3.21}$$

We will now bound the first term on the right side of (5.3.17). Since τ_0^L, τ_u^L are stopping times for the martingale M^L and $M_{\tau_0 \wedge \tau_u} = u\mathbf{P}[\tau_u^L < \tau_0^L]$, we have that

$$\mathbf{P}[\tau_u^L < \tau_0^L] = \frac{1}{u}\mathbb{E}[M_{\tau_0 \wedge \tau_u}^L] = \frac{M_0^L}{u} = \frac{\varepsilon^\alpha}{u(1 - \widetilde{x}^R)^{(\kappa - 4 - 2\rho_L)\rho_R/(2\kappa)}}. \tag{5.3.22}$$

Combining (5.3.17) with (5.3.21) and (5.3.22) we get that $\mathbf{P}[\sigma_1 < T_0^L] \le \varepsilon^{\alpha + o(1)}$, as desired. □

Recall that (see for example [MP10, Section 4]) the β-Hausdorff measure of a set $A \subseteq \mathbb{R}$ is defined as

$$\mathscr{H}^\beta(A) = \lim_{\varepsilon \to 0^+} \mathscr{H}_\varepsilon^\beta(A)$$

where

$$\mathscr{H}_\varepsilon^\beta(A) := \inf\left\{ \sum_j |I_j|^\beta \ : \ A \subseteq \cup_j I_j \ \text{ and } \ |I_j| \le \varepsilon \ \text{ for all } \ j \right\}.$$

Proof of Theorem 5.1.6 for $\kappa \in (0,4)$, upper bound. Fix $\kappa \in (0,4), \rho \in (-2, \frac{\kappa}{2} - 2)$. Let η be an $\text{SLE}_\kappa(\rho)$ process with a single force point located at 0^+. Let $\alpha \in (0,1)$ be as in (5.3.4). Fix $0 < x < y$. We are going to prove the result by showing that

$$\dim_{\mathscr{H}}(\eta \cap [x,y]) \le 1 - \alpha \quad \text{almost surely.} \tag{5.3.23}$$

For each $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ we let $I_{k,n} = [k2^{-n}, (k+1)2^{-n}]$ and let $z_{k,n}$ be the center of $I_{k,n}$. Let $\mathscr{I}_n$ be the set of k such that $I_{k,n} \subseteq [x/2, 2y]$ and let $E_{k,n}$ be the event that η gets within distance 2^{1-n} of $z_{k,n}$. Therefore there exists $n_0 = n_0(x,y)$ such that for every $n \ge n_0$ we have that $\{I_{k,n} : k \in \mathscr{I}_n,\ E_{k,n} \text{ occurs}\}$ is a cover of $\eta \cap [x,y]$.

Fix $\zeta > 0$. Theorem 5.3.1 implies that there exists a constant $C_1 > 0$ (independent of n) and $n_1 = n_1(\zeta)$ such that

$$\mathbf{P}[E_{k,n}] \le C_1 2^{-(\alpha - \zeta)n} \quad \text{for each} \quad n \ge n_1 \quad \text{and} \quad k \in \mathscr{I}_n.$$

Consequently, there exists a constant $C_2 > 0$ such that

$$\mathbb{E}\left[\mathscr{H}^{\beta}_{2^{-n}}(\eta\cap[x,y])\right] \leq \mathbb{E}\left[\sum_{k\in\mathscr{I}_n} 2^{-\beta n}\mathbf{1}_{E_{k,n}}\right] \leq C_2 2^{-\beta n}\times 2^n \times 2^{-(\alpha-\zeta)n}.$$

By Fatou's lemma,

$$\begin{aligned}\mathbb{E}\left[\mathscr{H}^{1-\alpha+2\zeta}(\eta\cap[x,y])\right] &\leq \liminf_n \mathbb{E}\left[\mathscr{H}^{1-\alpha+2\zeta}_{2^{-n}}(\eta\cap[x,y])\right]\\ &\leq \liminf_n C_2 2^{-n\zeta} = 0.\end{aligned}$$

This implies that $\mathscr{H}^{1-\alpha+2\zeta}(\eta\cap[x,y]) = 0$ almost surely. This proves (5.3.23) which completes the proof of the upper bound. □

5.3.2 The lower bound

Throughout, we fix $\kappa\in(0,4)$ and $\rho\in(-2,\frac{\kappa}{2}-2)$ and let h be a GFF on $\mathbb{H}$ with boundary data $-\lambda$ on $\mathbb{R}_-$ and $\lambda(1+\rho)$ on $\mathbb{R}_+$. (Recall the values in (5.2.10) as well as Figure 5.2.1.) For each $x\geq 0$, we let η^x be the flow line of h starting from x and let $\eta=\eta^0$. Note that η is an SLE$_\kappa(\rho)$ process in $\mathbb{H}$ from 0 to ∞ with a single force point located at 0^+, i.e., has configuration $(\mathbb{H},0,0^+,\infty)$ (recall the notation of Section 5.2.3). By Lemma 5.2.1, it follows that η can hit $(0,\infty)$. For each $x>0$, η^x is an SLE$_\kappa(2+\rho,-2-\rho;\rho)$ process with configuration $(\mathbb{H},x,(0,x^-),(x^+),\infty)$. By Lemma 5.2.1, it follows that η^x can hit (x,∞) and, if $\rho>-\kappa/2$, then η^x can also hit $(0,x)$. Fix $\delta\in(0,1)$, $a>\log 8$, and let

$$\epsilon_n = e^{-an} \quad \text{for each} \quad n\in\mathbb{N}.$$

We will eventually take limits as $a\to\infty$ and $\delta\to 0^+$. For $U\subseteq\mathbb{H}$, we let

$$\sigma^x(U) = \inf\{t\geq 0 : \eta^x(t)\in U\}. \tag{5.3.24}$$

We will omit the superscript in (5.3.24) if $x=0$. For $k\in\mathbb{N}$ and $x\in[1,\infty)$, we let

$$x_k = \begin{cases} x-\frac{1}{4}\epsilon_k & \text{if} \quad k\geq 2 \quad \text{and}\\ 0 & \text{if} \quad k=1.\end{cases}$$

We also let

$$\sigma^x_m = \sigma^{x_m}(B(x,\epsilon_{m+1})). \tag{5.3.25}$$

Let $E^1_k(x)$ be the event that

1. $\sigma^x_k<\infty$ and $\mathrm{Im}(\eta^{x_k}(\sigma^x_k))\geq\delta\epsilon_{k+1}$ and
2. η^{x_k} hits $B(x,\epsilon_{k+1})$ before exiting $B(x,\frac{1}{2}\epsilon_k)$.

We let $E^2_k(x)$ be the event that $\eta^{x_{k-1}}|_{[\sigma^x_{k-1},\infty)}$ merges with $\eta^{x_k}|_{[0,\sigma^x_k]}$ before exiting the annulus $B(x,\frac{1}{2}\epsilon_{k-1})\setminus B(x,\epsilon_{k+1})$ (see Figure 5.3.3). Finally, we let $E_k(x)=E^1_k(x)\cap E^2_k(x)$,

$$E^{m,n}(x) = E^1_{m+1}(x)\cap\bigcap_{k=m+2}^{n} E_k(x), \quad \text{and} \quad E^n(x) = E^{0,n}(x).$$

The following is the main input into the proof of the lower bound.

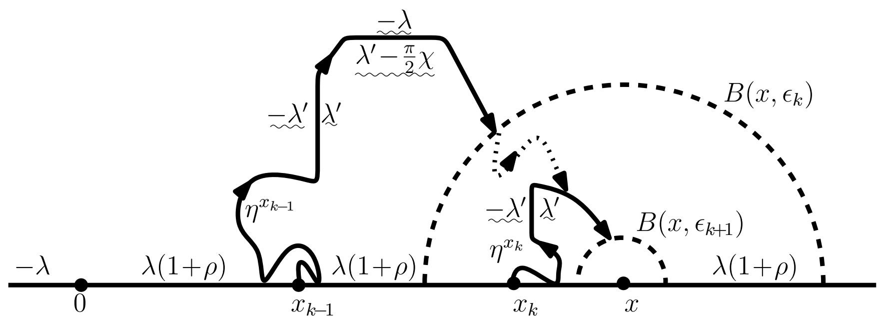


Figure 5.3.3: On $E^1_{k-1}(x)$, $\eta^{x_{k-1}}$ hits $B(x,\varepsilon_k)$ and does so for the first time above the horizontal line through $i\delta\varepsilon_k$. Given that $E^1_k(x)$ has occurred, $E^2_k(x)$ is the event that $\eta^{x_{k-1}}$ merges with η^{x_k} before the path leaves the annulus $B(x,\frac{1}{2}\varepsilon_{k-1})\setminus B(x,\varepsilon_{k+1})$. Also indicated is the boundary data for h along $\partial\mathbb{H}$ as well as along the paths $\eta^{x_{k-1}}$ and η^{x_k}.

Proposition 5.3.6. *For each $\delta\in(0,1)$, there exists a constant $c(\delta)>0$ such that for all $x,y\in[1,2]$ and $m\in\mathbb{N}$ such that $\frac{1}{2}\varepsilon_{m+1}\le|x-y|<\frac{1}{2}\varepsilon_m$ we have*

$$\mathbf{P}[E^n(x),E^n(y)]\le c(\delta)^{-m}\varepsilon_m^{-\alpha}\mathbf{P}[E^n(x)]\mathbf{P}[E^n(y)].$$

The main steps in the proof of Proposition 5.3.6 are contained in the following three lemmas.

Lemma 5.3.7. *For each $x\ge 1$ and $m,n\in\mathbb{N}$ with $m\le n$, we have that*

$$\mathbf{P}[E^{m,n}(x),E^m(x)]\asymp\mathbf{P}[E^{m,n}(x)]\mathbf{P}[E^m(x)] \tag{5.3.26}$$

If, moreover, $y\ge 1$ and $\frac{1}{2}\varepsilon_{m+2}<|x-y|\le\frac{1}{2}\varepsilon_{m+1}$, then we have that

$$\mathbf{P}[E^{m+1,n}(x),E^{m+1,n}(y),E^m(x)]\asymp\mathbf{P}[E^{m+1,n}(x)]\mathbf{P}[E^{m+1,n}(y)]\mathbf{P}[E^m(x)].$$

In each of the above, the constants in $\asymp$ depend only on δ, κ and ρ.

Proof. We begin by proving (5.3.26) which is equivalent to

$$\mathbf{P}[E^{m,n}(x)\,|\,E^m(x)]\asymp\mathbf{P}[E^{m,n}(x)].$$

Recall that $\eta^{x_{m+1}}$ is an $\mathrm{SLE}_\kappa(2+\rho,-2-\rho;\rho)$ process with configuration

$$c=(\mathbb{H},x_{m+1},(0,x^-_{m+1}),(x^+_{m+1}),\infty).$$

Let $\omega=\eta(\sigma(B(x,\varepsilon_m)))$, let H be the closure of the complement of the unbounded connected component of $\mathbb{H}\setminus\cup_{j=1}^m\eta^{x_j}([0,\sigma_j^x])$, and let v be the rightmost point of $H\cap\mathbb{R}$ (see Figure 5.3.4). The conditional law of $\eta^{x_{m+1}}$ given $\eta^{x_1}|_{[0,\sigma_1^x]},\ldots,\eta^{x_m}|_{[0,\sigma_m^x]}$ on $E^m(x)$ is that of an $\mathrm{SLE}_\kappa(2,\rho,-2-\rho;\rho)$ process in

$$\widetilde{c}=(\mathbb{H}\setminus H,x_{m+1},(\omega,v,x^-_{m+1}),(x^+_{m+1}),\infty)$$

(recall Figure 5.2.4.)

Let $U=B(x,\frac{1}{2}\varepsilon_{m+1})$, $\tau=\sigma^{x_{m+1}}(\mathbb{H}\setminus U)$, K be the closure of the complement of the unbounded connected component of $\mathbb{H}\setminus\eta^{x_{m+1}}([0,\tau])$, $\omega^{x_{m+1}}=\eta^{x_{m+1}}(\tau)$, and let u^-,u^+ be the leftmost (resp. rightmost) point of $K\cap\mathbb{R}$. By Lemma 5.2.7, we have that

$$\frac{d\mu^U_{\widetilde{c}}}{d\mu^U_c}=\frac{Z(\widetilde{c}_\tau)/Z(\widetilde{c})}{Z(c_\tau)/Z(c)}\exp(-\xi m(\mathbb{H};H,K))$$

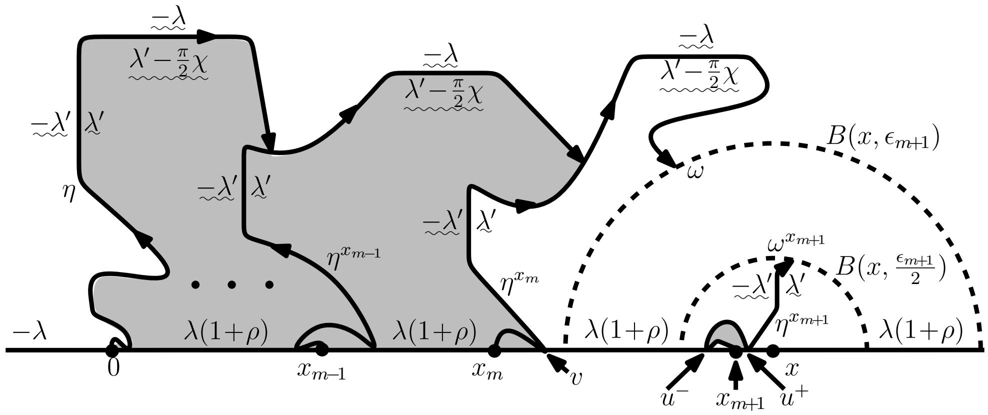


Figure 5.3.4: Let H (shown in red) be the closure of the complement of the unbounded connected component of $\mathbb{H}\setminus\cup_{j=1}^m \eta^{x_j}([0,\sigma_j^x])$ and let K (shown in blue) be the closure of the complement of the unbounded connected component of $\mathbb{H}\setminus\eta^{x_{m+1}}([0,\tau])$ where τ is the first time that $\eta^{x_{m+1}}$ leaves $U=B(x,\frac{\varepsilon_{m+1}}{2})$. Then $\mathrm{dist}(H,K)\gtrsim \mathrm{diam}(U)$.

where

$$c_\tau = (\mathbb{H}\setminus K, \omega^{x_{m+1}}, (0,u^-), (u^+), \infty),$$
$$\widetilde{c}_\tau = (\mathbb{H}\setminus (H\cup K), \omega^{x_{m+1}}, (\omega, v, u^-), (u^+), \infty).$$

Note that $H\subseteq \mathbb{H}\setminus B(x,\frac{3}{4}\varepsilon_{m+1})$, $K\subseteq \overline{B(x,\frac{1}{2}\varepsilon_{m+1})}$, and $\mathrm{diam}(U)=\varepsilon_{m+1}$. Consequently,

$$\frac{\mathrm{dist}(H,K)}{\mathrm{diam}(U)} \gtrsim 1.$$

Therefore Lemma 5.2.8 implies there exists $C_1\geq 1$ so that

$$\frac{1}{C_1} \leq \frac{d\mu^U_{\widetilde{c}}}{d\mu^U_c} \leq C_1. \tag{5.3.27}$$

This proves (5.3.26) in the case that $n=m+1$. We now suppose that $n\geq m+2$. Given $\eta^{x_{m+1}}|_{[0,\tau]}$, we similarly have that the Radon-Nikodym derivative between the conditional law of η^{x_n} stopped upon exiting the connected component of $B(x,\frac{1}{2}\varepsilon_n)\setminus\eta^{x_{m+1}}([0,\tau])$ with x_n on its boundary with respect to the law in which we additionally condition on H on $E_m(x)$ is bounded from above and below by C_1 and C_1^{-1}, respectively, possibly by increasing the value of $C_1>1$ (see Figure 5.3.5). Moreover, conditional on both of the paths $\eta^{x_{m+1}}|_{[0,\sigma^{x_{m+1}}(B(x,\varepsilon_{n+1}))]}$ and $\eta^{x_n}|_{[0,\sigma_n^x]}$ as well as the event that they have merged before exiting U, the joint law of $\eta^{x_j}|_{[0,\sigma_j^x]}$ for $j=m+2,\ldots,n-1$ is independent of $\eta^{x_k}|_{[0,\sigma_k^x]}$ for $k=1,\ldots,m$ (see Figure 5.3.5). This proves (5.3.26).

The second part of the lemma is proved similarly. □

Lemma 5.3.8. *For each $x\geq 1$ and $m,n\in\mathbb{N}$ with $m\leq n$ we have that*

$$\mathbf{P}[E^n(x)] \asymp \mathbf{P}[E^m(x)]\mathbf{P}[E^{m,n}(x)] \tag{5.3.28}$$

where the constants depend only on δ, κ, and ρ.

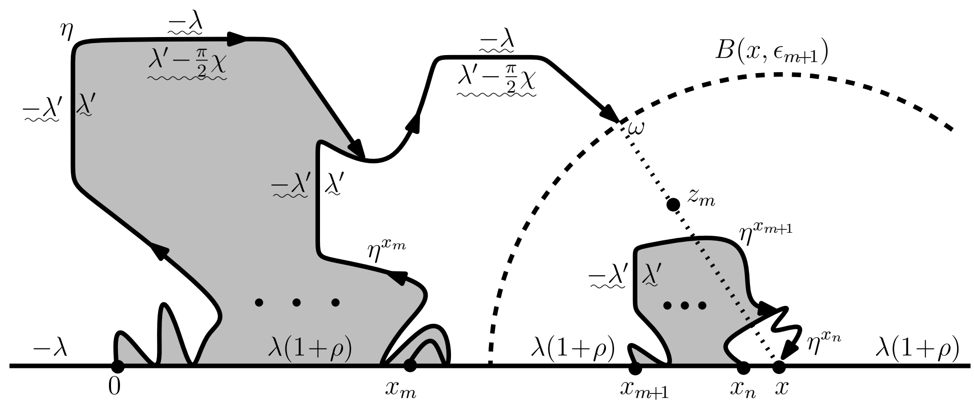


Figure 5.3.5: Assume that we are working on $E^m(x) \cap E^{m,n}(x)$. Let H (shown in red) be the closure of the complement of the unbounded connected component of $\mathbb{H} \setminus \cup_{j=1}^m \eta^{x_j}([0,\sigma_j^x])$ and let K (shown in blue) be the closure of the complement of the unbounded connected component of $\mathbb{H} \setminus \cup_{j=m+1}^n \eta^{x_j}([0,\sigma_j^x])$. Let z_m be the point that lies at distance $\delta\varepsilon_{m+1}$ from ω along the line connecting ω to x. Then a Brownian motion starting from z_m has positive probability to exit $\mathbb{H} \setminus (H \cup K)$ through each of the left side of H, the right side of H, and the left side of K.

Proof. The upper bound follows from (5.3.26) of Lemma 5.3.7. To complete the proof of the lemma, it suffices to show that

$$\mathbf{P}[E_{m+1}^2(x) \,|\, E^m(x), E^{m,n}(x)] \asymp 1.$$

Throughout, we assume that we are working on $E^m(x) \cap E^{m,n}(x)$. To see this, we let H (resp. K) be the closure of the complement of the unbounded connected component of $\mathbb{H} \setminus \cup_{j=1}^m \eta^{x_j}([0,\sigma_j^x])$ (resp. $\mathbb{H} \setminus \cup_{j=m+1}^n \eta^{x_j}([0,\sigma_j^x])$). Let $\omega = \eta^{x_m}(\sigma_m^x)$ and let z_m be the point which lies at distance $\delta\varepsilon_{m+1}$ from ω along the line segment connecting ω to x (see Figure 5.3.5). Note that the probability that a Brownian motion starting from z_m exits $\mathbb{H} \setminus (H \cup K)$ in the left (resp. right) side of H is $\asymp 1$ (though this probability decays as $\delta \downarrow 0$) and likewise for the left side of K. Let $\varphi \colon \mathbb{H} \setminus (H \cup K) \to \mathbb{H}$ be the conformal map which takes z_m to i and ω to 0. Let x_L (resp. x_R) be the image of the leftmost (resp. rightmost) point of $H \cap \mathbb{R}$ under φ. The conformal invariance of Brownian motion implies that there exists $\varepsilon > 0$ depending only on δ such that $|x_q| \geq \varepsilon$ for $q \in \{L,R\}$. Let y_L (resp. y) be the image of the leftmost point of $K \cap \mathbb{R}$ (resp. $\eta^{x_{m+1}}(\sigma_{m+1}^x)$) under φ. By shrinking $\varepsilon > 0$ if necessary (but still depending only on δ), it is likewise true that $y - y_L \geq \varepsilon$ and $y_L \leq \varepsilon^{-1}$. Consequently, it follows from Lemma 5.2.5 that $\eta^{x_m}|_{[\sigma_m^x,\infty)}$ has a positive chance (depending only on δ, κ, and ρ) of hitting (hence merging into) the left side of $\eta^{x_{m+1}}|_{[0,\sigma_{m+1}^x)}$ before leaving $B(x,\frac{1}{2}\varepsilon_m) \setminus B(x,\varepsilon_{m+2})$. $\square$

Lemma 5.3.9. *For each $\delta \in (0,1)$ there exists a constant $c(\delta) > 0$ such that the following is true. For each $x \geq 1$, we have that*

$$\mathbf{P}[E^m(x)] \geq c(\delta)^m \times \varepsilon_m^\alpha.$$

Proof. By (5.3.26) of Lemma 5.3.7, we know that

$$\mathbf{P}[E_k^1(x) \,|\, E^{k-1}(x)] \asymp \mathbf{P}[E_k^1(x)].$$

Therefore we just have to show that there exists a constant $c(\delta) > 0$ such that

$$\mathbf{P}[E_k^1(x)] \geq c(\delta)\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^{\alpha} = c(\delta)e^{-a\alpha} \quad \text{and} \tag{5.3.29}$$

$$\mathbf{P}[E_k^2(x)\,|\,E^{k-1}(x), E_k^1(x)] \asymp 1. \tag{5.3.30}$$

Note that (5.3.30) follows from Lemma 5.2.5 using the same argument as in the proof of Lemma 5.3.8. We know that η^{x_k} is an SLE$_\kappa(2+\rho, -2-\rho; \rho)$ process within the configuration $c = (\mathbb{H}, x_k, (0, x_k^-), (x_k^+), \infty)$. Consequently, (5.3.29) follows by combining Corollary 5.3.3 and Lemma 5.2.8. The latter is used to get that the Radon-Nikodym derivative between the law of an SLE$_\kappa(2+\rho, -2-\rho; \rho)$ process with configuration $(\mathbb{H}, x_k, (0, x_k^-), (x_k^+), \infty)$ and the law of an SLE$_\kappa(-2-\rho; \rho)$ process with configuration $(\mathbb{H}, x_k, (x_k^-), (x_k^+), \infty)$, where each path is stopped upon exiting $B(x, \frac{\varepsilon_k}{2})$, is bounded both from below and above by universal positive and finite constants. □

Proof of Proposition 5.3.6. We have that,

$$\begin{aligned}
\mathbf{P}[E^n(x), E^n(y)] &\leq \mathbf{P}[E^n(x), E^{m,n}(y)] \\
&\lesssim \mathbf{P}[E^m(x)]\mathbf{P}[E^{m+1,n}(x)]\mathbf{P}[E^{m+1,n}(y)] \qquad \text{(Lemma 5.3.7)} \\
&= \frac{\mathbf{P}[E^m(x)]\mathbf{P}[E^m(y)]}{\mathbf{P}[E^m(y)]}\mathbf{P}[E^{m+1,n}(x)]\mathbf{P}[E^{m+1,n}(y)] \\
&\lesssim \frac{\mathbf{P}[E^n(x)]\mathbf{P}[E^n(y)]}{c(\delta)^m \varepsilon_m^{\alpha}} \qquad \text{(Lemma 5.3.8 and Lemma 5.3.9)}
\end{aligned}$$

□

Proof of Theorem 5.1.6. We are first going to give the lower bound for $\kappa \in (0,4)$ and then explain how to extract the dimension result for $\kappa' > 4$ from the result for $\kappa \in (0,4)$. For each $\beta \in \mathbb{R}$ and Borel measure μ, let

$$I_\beta(\mu) := \int\int \frac{\mu(dz)\mu(dw)}{|z-w|^\beta}$$

be the β-energy of μ. To prove the lower bound, we will show that, for each $\zeta > 0$, there exists a nonzero Borel measure supported on $\eta \cap [1,2]$ that has finite $(1-\alpha-2\zeta)$-energy.

Fix $n \in \mathbb{N}$. We divide $[1,2]$ into ε_n^{-1} intervals of equal length ε_n and let $z_{j,n} = (j-\frac{1}{2})\varepsilon_n + 1$ be the center of the jth such interval for $j = 1, \ldots, \varepsilon_n^{-1}$. Let $\mathscr{C}_n$ be the subset of $\mathscr{D}_n = \{z_{j,n} : j = 1, \ldots, \varepsilon_n^{-1}\}$ for which $E^n(z)$ occurs. Let $I_n(z) = [z - \frac{\varepsilon_n}{2}, z + \frac{\varepsilon_n}{2}]$ be the interval with center z and length ε_n. Finally, we let

$$\mathscr{C} = \bigcap_{k\geq 1} \overline{\bigcup_{n\geq k} \bigcup_{z\in\mathscr{C}_n} I_n(z)}.$$

It is easy to see that

$$\mathscr{C} \subseteq \eta \bigcap \mathbb{R}_+.$$

Let μ_n be the measure on $[1,2]$ defined by

$$\mu_n(A) = \int_A \sum_{z\in\mathscr{D}_n} \frac{\mathbf{1}_{E^n(z)}}{\mathbf{P}[E^n(z)]} \mathbf{1}_{I_n(z)}(z')dz' \quad \text{for} \quad A \subseteq [1,2] \quad \text{Borel}.$$

Then $\mathbb{E}[\mu_n([1,2])] = 1$. Moreover, we have that

$$\begin{aligned}
\mathbb{E}[\mu_n([1,2])^2] &= \varepsilon_n^2 \sum_{z,w\in\mathscr{D}_n} \frac{\mathbf{P}[E^n(z)\cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \\
&= \varepsilon_n^2 \sum_{\substack{z,w\in\mathscr{D}_n \\ z\neq w}} \frac{\mathbf{P}[E^n(z)\cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} + \varepsilon_n^2 \sum_{z\in\mathscr{D}_n} \frac{1}{\mathbf{P}[E^n(z)]} \\
&\lesssim \varepsilon_n^2 \sum_{\substack{z,w\in\mathscr{D}_n \\ z\neq w}} |z-w|^{-\alpha-\zeta} + \varepsilon_n^2 \sum_{z\in\mathscr{D}_n} \varepsilon_n^{-1+\alpha-\zeta} \quad \text{(Proposition 5.3.6 and Lemma 5.3.9)} \\
&\lesssim 1
\end{aligned}$$

provided we choose n and a large enough. Set $\beta = 1-\alpha-2\zeta$. We also have that

$$\begin{aligned}
\mathbb{E}[I_\beta(\mu_n)] &= \sum_{\substack{z,w\in\mathscr{D}_n \\ z\neq w}} \frac{\mathbf{P}[E^n(z)\cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \iint_{I_n(z)\times I_n(w)} \frac{dz'dw'}{|z'-w'|^\beta} \\
&= \sum_{\substack{z,w\in\mathscr{D}_n \\ z\neq w}} \frac{\mathbf{P}[E^n(z)\cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \iint_{I_n(z)\times I_n(w)} \frac{dz'dw'}{|z'-w'|^\beta} \\
&\qquad + \sum_{z\in\mathscr{D}_n} \frac{1}{\mathbf{P}[E^n(z)]} \iint_{I_n(z)\times I_n(z)} \frac{dz'dw'}{|z'-w'|^\beta} \\
&\lesssim \sum_{\substack{z,w\in\mathscr{D}_n \\ z\neq w}} \frac{\mathbf{P}[E^n(z)\cap E^n(w)]}{\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)]} \frac{\varepsilon_n^2}{|z-w|^\beta} + \sum_{z\in\mathscr{D}_n} \frac{1}{\mathbf{P}[E^n(z)]} \varepsilon_n^{2-\beta} \\
&\lesssim \sum_{\substack{z,w\in\mathscr{D}_n \\ z\neq w}} |z-w|^{-\alpha-\zeta} \varepsilon_n^2 |z-w|^{-\beta} + \sum_{z\in\mathscr{D}_n} \varepsilon_n^{-1+\alpha-\zeta}\varepsilon_n^{2-\beta} \lesssim 1.
\end{aligned}$$

Consequently, the sequence (μ_n) has a subsequence (μ_{n_k}) that converges weakly to some nonzero measure μ. It is clear that μ is supported on $\mathscr{C}$ and has finite $(1-\alpha-2\zeta)$-energy. From [MP10, Theorem 4.27], we know that

$$\mathbf{P}\left[\dim_{\mathscr{H}}(\eta \bigcap \mathbb{R}_+) \geq 1-\alpha-2\zeta\right] > 0.$$

Since η is conformally invariant, by 0-1 law (see [Bef08]), we have that

$$\mathbf{P}\left[\dim_{\mathscr{H}}(\eta \bigcap \mathbb{R}_+) \geq 1-\alpha-2\zeta\right] = 1$$

for any $\zeta > 0$. This proves the lower bound for $\kappa \in (0,4)$.

It is left to prove the result for $\kappa' > 4$. Fix $\rho' \in (\frac{\kappa'}{2}-4, \frac{\kappa'}{2}-2)$. Consider a GFF h on $[-1,1]^2$ with the boundary values as depicted in Figure 5.2.5 with $\rho'_R = \rho'$ and $\rho'_L = 0$, and let η' be the counterflow line of h from i to $-i$. Then η' is an $\mathrm{SLE}_{\kappa'}(\rho')$ process with a single force point located at $(i)^+$, i.e., immediately to the right of i. As explained in Figure 5.2.5, the right boundary of η' is equal to the flow line η_R of h with angle $-\frac{\pi}{2}$ starting from $-i$. In particular, η_R is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-2; \kappa-4+\frac{\kappa}{4}\rho')$ process with force points $((-i)^-;(-i)^+)$ where $\kappa = \frac{16}{\kappa'} \in (0,4)$. The intersection of η' with the counterclockwise segment $\mathscr{S}$ of $\partial([-1,1]^2)$ from $-i$ to i coincides with $\eta_R \cap \mathscr{S}$. Consequently, it follows that the dimension of $\eta' \cap \mathscr{S}$ is given by

$$1 - \frac{1}{\kappa}\left(\kappa - 2 + \frac{\kappa}{4}\rho'\right)\left(\frac{\kappa}{2} + \frac{\kappa}{4}\rho'\right) = 1 - \frac{1}{\kappa'}\left(\rho' + 2\right)\left(\rho' + 4 - \frac{\kappa'}{2}\right).$$

□

5.4 The intersection of flow lines

In this section, we will prove Theorem 5.1.5. We begin in Section 5.4.1 by proving an estimate for the derivative of the Loewner map associated with an $\mathrm{SLE}_\kappa(\underline{\rho})$ process when it gets close to a given point. Next, in Section 5.4.2 we will prove the one point estimate which we will use in Section 5.4.3 to prove the upper bound. Finally in Section 5.4.4 we will complete the proof by establishing the lower bound.

5.4.1 Derivative estimate

Recall from Section 5.2.4 that for a point w in a simply connected domain U, $\mathrm{CR}(w;U)$ denotes the conformal radius of U as viewed from w. Fix $\kappa \in (0,4)$, let η be an ordinary SLE_κ process in $\mathbb{H}$ from 0 to ∞ and, for each t, let $\mathbb{H}_t$ denote the unbounded connected component of $\mathbb{H} \setminus \eta([0,t])$. We use the notation of [VL12, Section 6.1]. We let

$$Z_t = Z_t(z) = X_t + iY_t = g_t(z) - W_t.$$

For $z \in \mathbb{H}$, we let

$$\Delta_t = |g_t'(z)|, \quad \Upsilon_t = \frac{Y_t}{|g_t'(z)|}, \quad \Theta_t = \arg Z_t, \quad \text{and} \quad S_t = \sin \Theta_t. \tag{5.4.1}$$

We note that $\Upsilon_t = \frac{1}{2}\mathrm{CR}(z;\mathbb{H}_t) \asymp \mathrm{dist}(z,\partial\mathbb{H}_t)$. For each $r \in \mathbb{R}$, we also let

$$\nu = \nu(r) = \frac{r^2}{4}\kappa + r\left(1 - \frac{\kappa}{4}\right) \quad \text{and} \quad \xi = \xi(r) = \frac{r^2}{8}\kappa. \tag{5.4.2}$$

Then we have that [VL12, Proposition 6.1]:

$$M_t = M_t(z) = |Z_t|^r Y_t^{\xi} \Delta_t^{\nu} = S_t^{-r} \Upsilon_t^{\xi + r} \Delta_t^{\nu + r} \tag{5.4.3}$$

is a local martingale. This martingale also appears in [SW05, Theorem 6], though it is expressed there in a slightly different form. (The martingale in (5.2.6) is of the same type, though there we have not included the interior force points.) For each $\varepsilon > 0$ and $R > 0$, we let

$$\begin{aligned} \tau_\varepsilon &= \inf\{t \geq 0 : \Upsilon_t = \tfrac{1}{2}\varepsilon\} = \inf\{t \geq 0 : \mathrm{CR}(z;\mathbb{H}_t) = \varepsilon\} \quad \text{and} \\ \sigma_R &= \inf\{t \geq 0 : |\eta(t)| = R\}. \end{aligned} \tag{5.4.4}$$

Then we have that

Lemma 5.4.1. *Fix $r < \frac{1}{2} - \frac{4}{\kappa}$, $\delta \in (0, \frac{\pi}{2})$, and $z \in \mathbb{H}$ such that $\arg(z) \in (\delta, \pi - \delta)$. Let $\mathbf{P}^\star$ be the law of η weighted by M. We have that,*

$$\mathbf{P}^\star[\tau_\varepsilon < \infty] = 1 \tag{5.4.5}$$

and

$$\mathbb{E}^\star[S^r_{\tau_\varepsilon}] \asymp 1 \tag{5.4.6}$$

where the constants depend only on δ, κ, and r. We also have that

$$\mathbf{P}^\star[\Theta_{\tau_\varepsilon} \in (\delta, \pi - \delta)] \asymp 1 \tag{5.4.7}$$

where constants depend only on δ, κ, and r. Finally, we have that

$$\mathbf{P}^\star[\sigma_R \le \tau_\varepsilon] \to 0 \quad as \quad R \to \infty \tag{5.4.8}$$

uniformly over $\varepsilon > 0$.

Proof. Note that (5.4.5) and (5.4.6) are proved in [VL12, Equation (6.9)], so we will not repeat the arguments here. Following [VL12], we define the radial parametrization (i.e., by log conformal radius) $u(t)$ by

$$\widehat{\Upsilon}_t = \Upsilon_{u(t)} = e^{-2t}$$

and write $\widehat{\eta}(t) = \eta(u(t))$ and $\widehat{\Theta}_t = \Theta_{u(t)}$. Then $\widehat{\Theta}_t$ satisfies the SDE (see [VL12, Section 6.3])

$$d\widehat{\Theta}_t = \left(1 - \frac{4}{\kappa} - r\right)\cot\left(\widehat{\Theta}_t\right)dt + d\widehat{W}_t \tag{5.4.9}$$

where $\widehat{W}$ is a $\mathbf{P}^\star$-Brownian motion. The process $\widehat{\Theta}$ almost surely does not hit $\{0, \pi\}$ (see [Law05, Lemma 1.27]) and the density with respect to Lebesgue measure on $[0, \pi]$ for the stationary distribution for (5.4.9) is given by

$$f(\theta) = c(\sin\theta)^{2\left(1 - \frac{4}{\kappa} - r\right)}$$

where $c > 0$ is a normalizing constant (see [Law05, Lemma 1.28]). Moreover, as $t \to \infty$, the law of $\widehat{\Theta}_t$ converges to the stationary distribution with respect to the total variation norm.

We can use this to extract (5.4.7) as follows. Fix $0 < T < \infty$. We first note that by the Girsanov theorem the law of $\widehat{\Theta}|_{[0,T]}$ stopped upon leaving $(\frac{\delta}{2}, \pi - \frac{\delta}{2})$ is mutually absolutely continuous with respect to that of $B|_{[0,T]}$ where B is a Brownian motion starting from $\widehat{\Theta}_0$, also stopped upon leaving $(\frac{\delta}{2}, \pi - \frac{\delta}{2})$. Fix $0 \le t \le T$. Then a Brownian motion starting from $\widehat{\Theta}_0 \in [\delta, \pi - \delta]$ has a uniformly positive chance of staying in $[\frac{\delta}{2}, \pi - \frac{\delta}{2}]$ during the time interval $[0, t]$ and then being in $(\delta, \pi - \delta)$ at time t. Therefore it is easy to see that (5.4.7) holds for all $0 \le t \le T$.

The lower bound, however, that comes from this estimate decays as T increases. We are now going to explain how we make our choice of T as well as get a uniform lower bound for $t \ge T$. We suppose that $\widehat{\Theta}^1, \widehat{\Theta}^2$ are solutions of (5.4.9) where $\widehat{\Theta}^1_0 = \delta$ and $\widehat{\Theta}^2_0 = \pi - \delta$. We assume further that the Brownian motions driving $\widehat{\Theta}$, $\widehat{\Theta}^1$, and $\widehat{\Theta}^2$ are independent of each other until the time that any two of the processes meet, after which we take the Brownian motions for the pair to be the same. This gives us a coupling $(\widehat{\Theta}^1, \widehat{\Theta}, \widehat{\Theta}^2)$ such that $\widehat{\Theta}^1_t \le \widehat{\Theta}_t \le \widehat{\Theta}^2_t$ for all $t \ge 0$ almost surely. Note that after $\widehat{\Theta}^1$ first hits $\widehat{\Theta}^2$, all three processes stay together and never separate. Let $q_\delta > 0$ be the mass that the stationary distribution puts on $(\delta, \pi - \delta)$. We then take $T > 0$ sufficiently large so that:

1. For all $t \geq T$, the total variation distance between the law of $\widehat{\Theta}_t^1$ and the stationary distribution is at most $\frac{q_\delta}{2}$.

2. Let $\xi = \inf\{t \geq 0 : \widehat{\Theta}_t^1 = \widehat{\Theta}_t^2\}$. Then $\mathbf{P}[\xi \geq T] \leq \frac{q_\delta}{4}$.

With this particular choice of T, we have that

$$\begin{aligned}\mathbf{P}^\star[\widehat{\Theta}_t \in (\delta,\pi-\delta)] &\geq \mathbf{P}^\star[\widehat{\Theta}_t^1 \in (\delta,\pi-\delta)] - \mathbf{P}^\star[\xi \geq T] \\ &\geq \frac{q_\delta}{2} - \frac{q_\delta}{4} = \frac{q_\delta}{4} \quad \text{for all} \quad t \geq T.\end{aligned}$$

This proves (5.4.7).

For (5.4.8), note that, under $\mathbf{P}^\star$, $\widehat{\eta}$ has the same law as a radial $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ from 0 to z with a single boundary force point located at ∞ of weight $\rho = \kappa - 6 - r\kappa \geq \frac{\kappa}{2} - 2$ (see [SW05, Theorem 3 and Theorem 6]). Define $\widehat{\sigma}_R = \inf\{t \geq 0 : |\widehat{\eta}(t)| = R\}$. Then

$$\mathbf{P}^\star[\sigma_R < \tau_\varepsilon] \leq \mathbf{P}^\star[\widehat{\sigma}_R < \infty].$$

The endpoint continuity of the radial $\mathrm{SLE}_\kappa(\rho)$ processes with $\rho > -2$ [MS13b, Theorem 1.12] implies that $\mathbf{P}^\star[\widehat{\sigma}_R < \infty] \to 0$ as $R \to \infty$, as desired. □

We are now going to use Lemma 5.4.1 to estimate the moments of $g_t'(z)$ at times when η is close to z. We will actually prove this for general $\mathrm{SLE}_\kappa(\underline{\rho})$ processes which is why we truncate on various events in the estimates proved below.

Lemma 5.4.2. *Fix $r < \frac{1}{2} - \frac{4}{\kappa}$ and $\delta \in (0, \frac{\pi}{2})$. There exists $R_0 = R_0(r) > 0$ such that for all $R \geq R_0$ the following holds. Suppose $\eta \sim \mathrm{SLE}_\kappa(\underline{\rho})$ in $\mathbb{H}$ from 0 to ∞ where the force points lie outside of $2R\mathbb{D}$. Fix $z \in \mathbb{D} \cap \mathbb{H}$ with $\arg(z) \in (\delta, \pi - \delta)$. For each $\varepsilon > 0$ and $R > 0$ we let τ_ε and σ_R be as in (5.4.4). Then*

$$\mathbb{E}\left[\left|g_{\tau_\varepsilon}'(z)\right|^{\nu+r} \mathbf{1}_{\{\tau_\varepsilon < \sigma_R\}}\right] \asymp \varepsilon^{-\xi-r} \quad \textit{provided} \quad \mathrm{CR}(z;\mathbb{H}) \geq \varepsilon \tag{5.4.10}$$

where the constants depend only on δ, κ, and the weights $\underline{\rho}$ of the force points. Fix a constant $C > 1$ and suppose that ζ_ε is a stopping time for η such that $\tau_{C\varepsilon} \leq \zeta_\varepsilon \leq \tau_{\varepsilon/C}$. Let

$$E_{\varepsilon,R}^\delta = \{\zeta_\varepsilon < \sigma_R,\ \Theta_{\zeta_\varepsilon} \in (\delta, \pi - \delta)\}. \tag{5.4.11}$$

Then we have that

$$\mathbb{E}\left[\left|g_{\zeta_\varepsilon}'(z)\right|^{\nu+r} \mathbf{1}_{E_{\varepsilon,R}^\delta}\right] \asymp \varepsilon^{-\xi-r} \quad \textit{provided} \quad \mathrm{CR}(z;\mathbb{H}) \geq \varepsilon \tag{5.4.12}$$

where the constants depend only on C, δ, κ, and the weights $\underline{\rho}$ of the force points.

Proof. It suffices to prove the result for an ordinary SLE_κ process since it is clear from the form of (5.2.6) that the Radon-Nikodym derivative between the law of an SLE_κ and an $\mathrm{SLE}_\kappa(\underline{\rho})$ process whose force points lie outside of $2R\mathbb{D}$ stopped at time σ_R is bounded from above and below by finite and positive constants which depend only on the total (absolute) weight of the force points and κ.

We only need to prove the upper bound of (5.4.10) and the lower bound of (5.4.12). We have that,

$$\begin{aligned}\mathbb{E}\left[\left|g'_{\tau_\varepsilon}(z)\right|^{\nu+r}\mathbf{1}_{\{\tau_\varepsilon<\sigma_R\}}\right] &\le \mathbb{E}\left[\left|g'_{\tau_\varepsilon}(z)\right|^{\nu+r}\mathbf{1}_{\{\tau_\varepsilon<\infty\}}\right]\\ &\asymp \varepsilon^{-\xi-r}\mathbb{E}[M_{\tau_\varepsilon}S^r_{\tau_\varepsilon}\mathbf{1}_{\{\tau_\varepsilon<\infty\}}]\\ &= \varepsilon^{-\xi-r}M_0\mathbb{E}^\star[S^r_{\tau_\varepsilon}]\\ &\lesssim \varepsilon^{-\xi-r} \qquad \text{(by (5.4.6))}.\end{aligned}$$

This proves the upper bound of (5.4.10). We are first going to give the proof of the lower bound of (5.4.12) for $\zeta_\varepsilon = \tau_\varepsilon$. We compute

$$\begin{aligned}\mathbb{E}\left[\left|g'_{\tau_\varepsilon}(z)\right|^{\nu+r}\mathbf{1}_{E^\delta_{\varepsilon,R}}\right] &\asymp \varepsilon^{-\xi-r}\mathbb{E}\left[M_{\tau_\varepsilon}S^r_{\tau_\varepsilon}\mathbf{1}_{E^\delta_{\varepsilon,R}}\right]\\ &\ge \varepsilon^{-\xi-r}\mathbb{E}\left[M_{\tau_\varepsilon}\mathbf{1}_{E^\delta_{\varepsilon,R}}\right]\\ &= \varepsilon^{-\xi-r}M_0\mathbf{P}^\star[E^\delta_{\varepsilon,R}].\end{aligned}$$

To bound $\mathbf{P}^\star[E^\delta_{\varepsilon,R}]$, we have

$$\begin{aligned}\mathbf{P}^\star[E^\delta_{\varepsilon,R}] &= \mathbf{P}^\star[\tau_\varepsilon < \sigma_R,\ \Theta_{\tau_\varepsilon} \in (\delta,\pi-\delta)]\\ &\ge \mathbf{P}^\star[\Theta_{\tau_\varepsilon} \in (\delta,\pi-\delta)] - \mathbf{P}^\star[\sigma_R < \tau_\varepsilon].\end{aligned}$$

From (5.4.7), we know that $\mathbf{P}^\star[\Theta_{\tau_\varepsilon} \in (\delta,\pi-\delta)]$ is bounded from below uniformly in $\varepsilon > 0$. From (5.4.8), we know that $\mathbf{P}^\star[\sigma_R < \tau_\varepsilon]$ converges to zero as $R \to \infty$ uniformly over $\varepsilon > 0$. These show that $\mathbf{P}^\star[E^\delta_{\varepsilon,R}]$ is bounded from below which proves the lower bound for (5.4.12). The case in which we replace τ_ε with ζ_ε is proved similarly. In particular, it is not difficult to see that

$$\mathbf{P}^\star[\Theta_t \in (\delta,\pi-\delta) \text{ for all } t \in [\tau_{C\varepsilon},\tau_{\varepsilon/C}] \,|\, \Theta_{\tau_{C\varepsilon}} \in (\delta,\pi-\delta)] > 0$$

uniformly in $\varepsilon > 0$ and

$$\mathbf{P}^\star[\sigma_R \le \zeta_\varepsilon] \le \mathbf{P}^\star[\sigma_R \le \tau_{\varepsilon/C}] \to 0 \quad \text{as} \quad R \to \infty$$

uniformly in $\varepsilon > 0$. □

5.4.2 Hitting probabilities

Fix an angle $\theta \in (\pi - 2\lambda/\chi, 0)$. This is the range so that GFF flow lines with angles $0,\theta$ are able to intersect each other where the flow line with angle θ stays to the right of the flow line with angle 0 [MS12a, Theorem 1.5]. Let

$$A = \frac{1}{2\kappa}\left(\rho + \frac{\kappa}{2} + 2\right)\left(\rho - \frac{\kappa}{2} + 6\right) \quad \text{where} \quad \rho = -\frac{\theta\chi}{\lambda} - 2. \tag{5.4.13}$$

Lemma 5.4.3. *Fix $C > 2$, let $x_1 = 0$, and fix $x_2 \ge 2R_0$ where R_0 is the constant from Lemma 5.4.2 with*

$$r = -\frac{2}{\kappa}\left(\rho + 6 - \frac{\kappa}{2}\right).$$

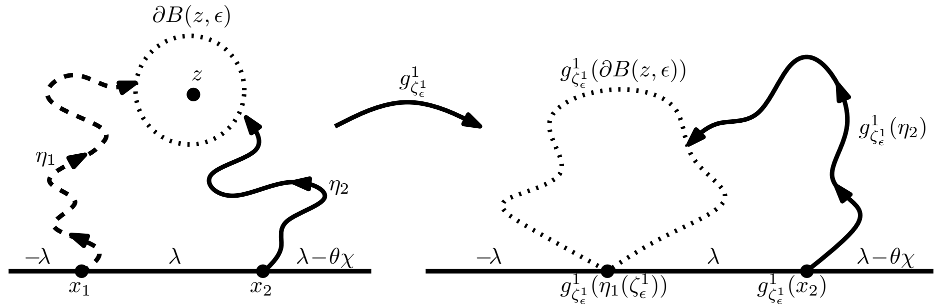


Figure 5.4.1: Illustration of the setup of Lemma 5.4.3, the one point estimate for the intersection dimension. On the left side, η_1 (resp. η_2) is a flow line of a GFF on $\mathbb{H}$ with the indicated boundary data with angle 0 (resp. $\theta \in (\pi - 2\lambda/\chi, 0)$) starting from x_1 (resp. $x_2 > x_1$). Note that η_1 (resp. η_2) is an $\mathrm{SLE}_\kappa(-\theta\chi/\lambda)$ (resp. $\mathrm{SLE}_\kappa(2, -\theta\chi/\lambda - 2)$) process. The force point for η_1 is located at x_2 and the force points for η_2 are located at x_1 and x_2^-. By Figure 5.2.4, the conditional law of η_2 given η_1 drawn up to any stopping time is also an $\mathrm{SLE}_\kappa(2, -\theta\chi/\lambda - 2)$ process. Shown is the event $G_\varepsilon^\delta(z)$ that η_1 hits $\partial B(z,\varepsilon)$, say for the first time at ζ_ε^1, before exiting $B(0,R_0)$ where $R_0 > 0$ is a large, fixed constant, the harmonic measure of the left (resp. right) side of η_1 stopped upon hitting $\partial B(z,\varepsilon)$ is not too small, and that η_2 also hits $\partial B(z,\varepsilon)$. We estimate the probability of $G_\varepsilon^\delta(z)$ by combining Lemma 5.4.2 with Theorem 5.3.1.

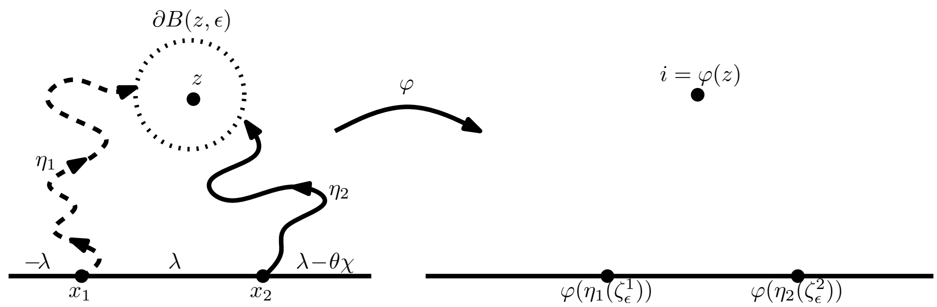


Figure 5.4.2: (Continuation of Figure 5.4.1.) Let $\zeta_\varepsilon^1, \zeta_\varepsilon^2$ be the times that η_1, η_2 hit $\partial B(z,\varepsilon)$, respectively, and let φ be the unique conformal map that uniformizes the unbounded connected component of $\mathbb{H} \setminus (\eta_1([0,\zeta_\varepsilon^1]) \cup \eta_2([0,\zeta_\varepsilon^2]))$ with z sent to i and ∞ fixed. For the lower bound of Theorem 5.1.5, we will also need to estimate the probability of the event $H_\varepsilon^\delta(z)$ that $G_\varepsilon^\delta(z)$ occurs (as described in Figure 5.4.1), that the diameter of $\eta_2([0,\zeta_\varepsilon^2])$ is not too large, and that the images of $\eta_i(\zeta_\varepsilon^i)$ for $i = 1,2$ under φ are not too far from i as illustrated on the right.

Let h be a GFF on $\mathbb{H}$ with boundary data as illustrated in Figure 5.4.1. That is,

$$h|_{(-\infty,0)} \equiv -\lambda, \quad h|_{[0,x_2]} \equiv \lambda, \quad \text{and} \quad h|_{(x_2,\infty)} \equiv \lambda - \theta\chi. \tag{5.4.14}$$

Let η_1 (resp. η_2) be the flow line of h starting from x_1 (resp. x_2) with angle 0 (resp. θ). Fix $\delta \in (0, \frac{\pi}{2})$ and let $z \in \mathbb{D} \cap \mathbb{H}$ with $\arg(z) \in (\delta, \pi - \delta)$. For $i = 1,2$, let ζ_ε^i be the first time that η_i hits $B(z,\varepsilon)$ and let Θ_t^1 be the process as in Lemma 5.4.2 for η_1.

1. *Let $G_\varepsilon^\delta(z)$ be the event that η_1 hits $\partial B(z,\varepsilon)$ before exiting $\partial B(0,R_0)$, $\Theta^1_{\zeta^1_\varepsilon} \in (\delta, \pi - \delta)$, and that η_2 hits $\partial B(z,\varepsilon)$. Then we have that*

$$\mathbf{P}[G_\varepsilon^\delta(z)] = \varepsilon^{A+o(1)} \tag{5.4.15}$$

where the $o(1)$ term depends only on δ, κ, θ, and x_2.

2. *On $G_\varepsilon^\delta(z)$, let φ be the unique conformal map which takes the unbounded connected component of $\mathbb{H} \setminus (\eta_1([0,\zeta_\varepsilon^1]) \cup \eta_2([0,\zeta_\varepsilon^2]))$ to $\mathbb{H}$ sending z to i and fixing ∞. There exists a constant $R_1 > 0$ such that with*

$$H_\varepsilon^\delta(z) = G_\varepsilon^\delta(z) \cap \{\max_{i=1,2} |\varphi(\eta_i(\zeta_\varepsilon^i))| \le R_1, \;\; \eta_2([0,\zeta_\varepsilon^2]) \subseteq B(0,10x_2)\}$$

we have that

$$\mathbf{P}[H_\varepsilon^\delta(z)] \gtrsim \varepsilon^A \tag{5.4.16}$$

where the constants depend only on δ, κ, θ, and x_2.

The same likewise holds if h is a GFF on $\mathbb{H}$ with piecewise constant boundary conditions which change values a finite number of times and in the interval $[-20x_2, 20x_2]$ takes the form in (5.4.14)*. In this case, the constants also depend on $\|h|_{\mathbb{R}}\|_\infty$.*

Proof. For each $t \ge 0$, let $\mathbb{H}_t^1$ be the unbounded connected component of $\mathbb{H} \setminus \eta_1([0,t])$, let $\tau_\varepsilon^1 = \inf\{t \ge 0 : \mathrm{CR}(z;\mathbb{H}_t^1) = \varepsilon\}$, $\sigma_{R_0}^1 = \inf\{t \ge 0 : \eta(t) \notin B(0,R_0)\}$, and let (g_t^1) be the Loewner evolution associated with η_1. By (5.2.17), note that $\tau_{4\varepsilon}^1 \le \zeta_\varepsilon^1$. It then follows from Theorem 5.3.1 that

$$\mathbf{P}[G_\varepsilon^\delta(z) \,|\, \eta_1|_{[0,\tau_{4\varepsilon}^1]}] \le |(g_{\tau_{4\varepsilon}^1}^1)'(z)\varepsilon|^{\alpha+o(1)}.$$

Note that $r < 1 - \frac{8}{\kappa} < \frac{1}{2} - \frac{4}{\kappa}$ since $\rho > -2$. With this choice of r, we have

$$\nu + r = \alpha \quad \text{and} \quad \nu - \xi = A.$$

Thus, by (5.4.10) of Lemma 5.4.2, we have that

$$\mathbf{P}[G_\varepsilon^\delta(z)] \le \mathbb{E}\left[|(g_{\tau_{4\varepsilon}^1}^1)'(z)\varepsilon|^{\alpha+o(1)} \mathbf{1}_{\{\tau_{4\varepsilon}^1 \le \sigma_{R_0}^1\}}\right] \le \varepsilon^{A+o(1)}.$$

This gives the upper bound for (5.4.15).

Let $E_{\varepsilon,R_0}^\delta = \{\zeta_\varepsilon^1 < \sigma_{R_0}^1,\ \Theta_{\zeta_\varepsilon^1}^1 \in (\delta, \pi-\delta)\}$. On $E_{\varepsilon,R_0}^\delta$ and $\{\zeta_\varepsilon^2 < \infty\}$, we let $w_\varepsilon = g_{\zeta_\varepsilon^1}^1(\eta_2(\zeta_\varepsilon^2))$ and $r_\varepsilon = |(g_{\zeta_\varepsilon^1}^1)'(z)|\varepsilon$. From Lemma 5.3.2, we have that

$$\mathbf{P}\left[G_\varepsilon^\delta(z) \,|\, \eta_1|_{[0,\zeta_\varepsilon^1]}\right] \mathbf{1}_{E_{\varepsilon,R_0}^\delta} \gtrsim r_\varepsilon^\alpha \mathbf{1}_{E_{\varepsilon,R_0}^\delta}.$$

We see from (5.4.12) of Lemma 5.4.2 that $\mathbf{P}[G_\varepsilon^\delta(z)] \gtrsim \varepsilon^A$.

We will now explain how to prove the result for $H_\varepsilon^\delta(z)$ in place of $G_\varepsilon^\delta(z)$. First of all, we note that on $E_{\varepsilon,R_0}^\delta$, it follows from [Law05, Corollary 3.44] that $|g_{\zeta_\varepsilon^1}^1(w) - w| \le 3R_0$ for all $w \in \mathbb{H}_{\zeta_\varepsilon^1}^1$. Consequently,

$$B(g_{\zeta_\varepsilon^1}^1(z), 10x_2 - 6R_0) \subseteq g_{\zeta_\varepsilon^1}^1(B(z,10x_2)); \tag{5.4.17}$$

recall that $10x_2 \ge 20R_0$. By Lemma 5.3.2 and (5.4.17), we have that,

$$\mathbf{P}\left[\zeta_\varepsilon^2 < \infty,\ \eta_2([0,\zeta_\varepsilon^2]) \subseteq B(z,10x_2),\ \mathrm{Im}(w_\varepsilon) \ge \delta r_\varepsilon \,|\, \eta_1|_{[0,\zeta_\varepsilon^1]}\right] \mathbf{1}_{E_{\varepsilon,R_0}^\delta} \gtrsim r_\varepsilon^\alpha \mathbf{1}_{E_{\varepsilon,R_0}^\delta}.$$

On the event in the probability above, a Brownian motion starting from z has a uniformly positive chance (depending on δ) of hitting both the left side of $\eta_1([0,\zeta_\varepsilon^1])$ and right side of $\eta_2([0,\zeta_\varepsilon^2])$. Consequently, the desired result follows by applying (5.4.12) from Lemma 5.4.2.

The final claim of the lemma follows from (5.2.6) to compare the case with extra force points to the case without considered above. □

In order for Lemma 5.4.3 to be useful, we need that as η_1 gets progressively closer to a given point z, it is unlikely that $\Theta^1 \notin (\delta, \pi-\delta)$ for some $\delta > 0$. This is the purpose of the following estimate.

Lemma 5.4.4. *Suppose that η is an SLE_κ process in $\mathbb{H}$ from 0 to ∞ with $\kappa \in (0,4)$. Fix $z \in \mathbb{H}$ and let $n_z = -\log_2 \mathrm{Im}(z)$ so that $n \geq n_z$ implies that $B(z,2^{-n}) \subseteq \mathbb{H}$. Let Θ be the process as in (5.4.1). For each n, let ζ_n be the first time that η hits $B(z,2^{-n})$ and, for each $\delta \in (0,\frac{\pi}{2})$, let $E_n^\delta = \{\zeta_n < \infty,\ \Theta_{\zeta_n} \notin (\delta,\pi-\delta)\}$. There exists a function $p\colon (0,1) \to [0,1]$ with $p \downarrow 0$ as $\delta \downarrow 0$ such that for each $r \geq n_z$ we have that*

$$\mathbf{P}[\cap_{m=n}^r E_m^\delta] \leq (p(\delta))^{r-n} \quad \textit{for all} \quad n_z \leq n \leq r.$$

Proof. Since the SLE_κ processes are scale-invariant in law, almost surely transient, and do not intersect the boundary for $\kappa \in (0,4)$ [RS05], it follows that

$$\lim_{s\to\infty} \mathbf{P}\big[\eta \text{ hits } [s,s+2]\times[0,2]\big] = \lim_{s\to\infty} \mathbf{P}\big[\eta \text{ hits } [1,1+\tfrac{2}{s}]\times[0,\tfrac{2}{s}]\big] = 0.$$

(For otherwise η would intersect the boundary with positive probability.) Consequently, it follows that there exists a function $q\colon (0,1) \to [0,1]$ with $q(\delta) \downarrow 0$ as $\delta \downarrow 0$ such that the following is true. If $z \in \mathbb{H}$ with $\mathrm{Im}(z) = 1$ and $\arg(z) \notin (\delta, \pi-\delta)$, then

$$\mathbf{P}[\eta \text{ hits } B(z,1)] \leq q(\delta). \tag{5.4.18}$$

For each $n \geq n_z$, on the event $\{\zeta_n < \infty\}$, let $\varphi_n\colon \mathbb{H}\setminus \eta([0,\zeta_n]) \to \mathbb{H}$ be the unique conformal map with $\varphi_n(\eta(\zeta_n)) = 0$, $\varphi_n(\infty) = \infty$, and satisfies $\mathrm{Im}(\varphi_n(z)) = 1$. Note that $\varphi_n(B(z,2^{-n-3})) \subseteq B(\varphi_n(z),1)$ by [Law05, Corollary 3.25]. Therefore it follows from (5.4.18) that

$$\mathbf{P}[E_{n+3}^\delta \,|\, \eta|_{[0,\zeta_n]}]\mathbf{1}_{E_n^\delta} \leq q(\delta)\mathbf{1}_{E_n^\delta}. \tag{5.4.19}$$

Iterating (5.4.19) and taking $p(\delta) = (q(\delta))^{1/3}$ proves the lemma. □

For each $n \in \mathbb{N}$, we let $\mathscr{D}_n$ be the set of squares with side length 2^{-n} which are contained in $\mathbb{H}$ and with corners in $2^{-n}\mathbb{Z}^2$. For each $Q \in \mathscr{D}_n$, let $z(Q)$ be the center of Q and let $\widetilde{Q}_n(Q) = B(z(Q),2^{1-n})$. For each $z \in \mathbb{H}$, let $Q_n(z)$ be the element of $\mathscr{D}_n$ which contains z and let $\widetilde{Q}_n(z) = \widetilde{Q}_n(Q_n(z))$. See Figure 5.4.3 for an illustration.

Lemma 5.4.5. *Suppose that η is an SLE_κ process in $\mathbb{H}$ from 0 to ∞ with $\kappa \in (0,4)$. For each $z \in \mathbb{H}$, let Θ^z be the process from (5.4.1) (with respect to z) and let $\zeta_{z,n} = \inf\{t \geq 0 : \eta(t) \in \widetilde{Q}_n(z)\}$. Let $\mathscr{S}_n^\delta$ be the set of points $z \in \mathbb{H}$ such that $E_{z,n}^\delta = \{\zeta_{z,n} < \infty,\ \Theta_{\zeta_{z,n}}^z \notin (\delta,\pi-\delta)\}$ occurs and let $\mathscr{S}^\delta = \cup_{n=1}^\infty \cap_{m=n}^\infty \mathscr{S}_m^\delta$. There exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0)$ we have that $\mathscr{S}^\delta = \emptyset$ almost surely.*

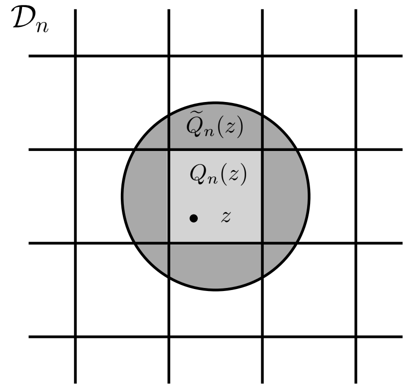


Figure 5.4.3: Shown in the illustration are $Q_n(z)$ and $\widetilde{Q}_n(z)$ for a given point $z \in \mathbb{H}$.

Proof. Fix $z \in \mathbb{H}$ and let $n_z = -\log_2 \operatorname{Im}(z)$. Note that $\widetilde{Q}_n(z) \subseteq B(z, 2^{2-n})$ so that $\widetilde{Q}_n(z) \subseteq \mathbb{H}$ provided $n \geq n_z + 2$. By Lemma 5.4.4, we have that

$$\mathbf{P}[\cap_{m=n}^r E_{z,m}^\delta] \leq \big(p(\delta)\big)^{r-n} \quad \text{for all} \quad n_z + 2 \leq n \leq r \tag{5.4.20}$$

(where $p(\delta)$ is as in the statement of Lemma 5.4.4).

Suppose that $Q \in \mathscr{D}_m$ and suppose that $n \in \mathbb{N}$ with $n \leq m$. Then the function $Q \to \mathbb{R}$ given by $w \mapsto \Theta_{\zeta_{w,n}}^w$ is positive and harmonic. Consequently, it follows from the Harnack inequality [Law05, Proposition 2.26] that there exists a universal constant $K \geq 1$ (independent of m, n) such that the following is true. If $E_{w,m}^\delta$ occurs for any $w \in Q$, then $E_{z(Q),m}^{K\delta}$ occurs. Thus letting $E_{Q,m}^\delta = \cup_{w \in Q} E_{w,m}^\delta$ we have that

$$\mathbf{P}[\cap_{m=n}^r E_{Q,m}^\delta] \leq \mathbf{P}[\cap_{m=n}^n E_{z(Q),m}^{K\delta}] \quad \text{for any} \quad n_{z(Q)} + 2 \leq n \leq r. \tag{5.4.21}$$

Combining this with Lemma 5.4.4 implies that

$$\mathbf{P}[\cap_{m=n}^r E_{Q,m}^\delta] \leq (p(K\delta))^{r-n} \quad \text{for any} \quad n_{z(Q)} + 2 \leq n \leq r. \tag{5.4.22}$$

Fix $\alpha \in (0,1)$ and let $n = -\log_2 \alpha$. For each $r \geq n$, let $\mathscr{V}_r^{\alpha,\delta}$ be the collection of squares Q in $\mathscr{D}_r$ with $Q \subseteq \{z \in \mathbb{H} : |z| < \frac{1}{\alpha}, \operatorname{Im}(z) \geq \alpha\}$ and for which $\cap_{m=n}^r E_{Q,m}^\delta$ occurs. Then (5.4.22) implies that there exists a constant $C > 0$ such that

$$\sum_{r=n}^\infty \mathbb{E}\big[|\mathscr{V}_r^{\alpha,\delta}|\big] \leq \frac{C}{\alpha^2} \sum_{r=n}^\infty 2^{2r} \big(p(K\delta)\big)^{r-n}. \tag{5.4.23}$$

Take $\delta_0 > 0$ so that $\delta \in (0, \delta_0)$ implies that $4p(K\delta) < 1$. Then for $\delta \in (0, \delta_0)$, the summation on the right side of (5.4.23) is finite. This implies that for every $\alpha \in (0,1)$, $\mathscr{V}_r^{\alpha,\delta} = \emptyset$ for all but finitely many r almost surely. This, in turn, implies the desired result since $\alpha > 0$ was arbitrary and $\mathscr{V}_r^{\alpha,\delta}$ increases as α decreases. □

5.4.3 The upper bound

Now that we have established Lemma 5.4.3 and Lemma 5.4.5, we can prove the upper bound in Theorem 5.1.5.

Proposition 5.4.6. *Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary conditions which change values a finite number of times. Let η_1 (resp. η_2) be the flow line of h starting from $x_1 = 0$ (resp. $x_2 > 0$) with angle 0 (resp. $\theta \in (\pi - 2\lambda/\chi, 0)$). We have that*

$$\dim_{\mathscr{H}}(\eta_1 \cap \eta_2 \cap \mathbb{H}) \leq 2 - A \quad \textit{almost surely}$$

where A is as in (5.4.13).

Proof. We are going to prove the proposition assuming that the boundary data is as in Lemma 5.4.3. This suffices by absolute continuity for GFFs. Fix $0 < \varepsilon < \delta < \frac{\pi}{4}$. For each $t > 0$, we let $\mathbb{H}_t^1$ be the unbounded connected component of $\mathbb{H} \setminus \eta_1([0,t])$. For each $z \in \mathbb{H}$, we let $\zeta_{z,\varepsilon}^1 = \inf\{t \geq 0 : \eta^1(t) \in \partial B(z, \varepsilon)\}$ and let $\Theta^{1,z}$ be the process as in (5.4.1) for η_1 and z. We let $I^{\varepsilon,\delta}$ consist of those $z \in \eta_1 \cap \eta_2 \cap B(0, \delta^{-1})$ such that

1. $\mathrm{Im}(z) \geq \delta$.
2. $\Theta_t^{1,z} \in (2\delta, \pi - 2\delta)$ for all $t \in [\zeta_{z,\varepsilon/2}^1, \zeta_{z,2\varepsilon}^1]$.
3. Let ζ_z^1 be the first time that η_1 hits z and $\sigma_{z,\delta}^1$ be the first time after $\zeta_{z,\varepsilon}^1$ that η_1 hits $\partial B(z, \delta)$. Then $\zeta_z^1 \leq \sigma_{z,\delta}^1$.

By the transience, continuity, and simplicity of the $\mathrm{SLE}_\kappa(\underline{\rho})$ processes for $\kappa \in (0,4)$ (which almost surely do not hit the continuation threshold) [MS12a, Theorem 1.3], we have that $\eta_1 \cap \eta_2 \cap \mathbb{H} \subseteq \cup_{\varepsilon \in \mathbb{Q}_+} \cup_{\delta \in \mathbb{Q}_+} I^{\varepsilon,\delta}$ almost surely. (If this were not true then we would be led to the contradiction that η_1 has double points with positive probability.) We are going to prove the result by showing that for every $\varepsilon, \delta > 0$,

$$\dim_{\mathscr{H}}(I^{\varepsilon,\delta}) \leq 2 - A \quad \text{almost surely.}$$

It in fact suffices to show that this is the case for $0 < \varepsilon < \delta < \delta_0$ where δ_0 is as in Lemma 5.4.5. Let $\mathscr{D}_n$ and $z(Q)$ be as before the statement of Lemma 5.4.5. We let $\mathscr{U}_n^{\varepsilon,\delta}$ consist of those $Q \in \mathscr{D}_n$ which are hit by both η_1 and η_2, contained in $B(0, \delta^{-1})$, and:

1. $\mathrm{Im}(z(Q)) \geq \delta$.
2. $\Theta_{\zeta_{z(Q),\varepsilon}^1}^{1,z(Q)} \in (\delta, \pi - \delta)$ and $\Theta_{\zeta(Q),2^{-n}}^{1,z(Q)} \in (\delta, \pi - \delta)$.
3. After $\zeta_{z(Q),\varepsilon}^1$, η_1 hits Q before $\sigma_{z(Q),\delta}^1$.

We are now going to show that, for every $n \in \mathbb{N}$, $\mathscr{W}_n^{\varepsilon,\delta} = \cup_{m \geq n} \mathscr{U}_m^{\varepsilon,\delta}$ is a cover of $I^{\varepsilon,\delta}$. To see this, we fix $z \in I^{\varepsilon,\delta}$ and let (Q_k) be a sequence of squares in $\cup_{m \geq n} \mathscr{D}_m$ such that $z \in Q_k$ for every k and $|Q_k| \to 0$ as $k \to \infty$. Let $z_k = z(Q_k)$. Since $\zeta_{z_k,\varepsilon}^1 \in [\zeta_{z,\varepsilon/2}^1, \zeta_{z,2\varepsilon}^1]$ for all k large enough, there exists $K_0 = K_0(z)$ such that for all $k \geq K_0$, we have that $\Theta_{\zeta_{z_k,\varepsilon}^1}^{1,z_k} \in (\delta, \pi - \delta)$. Since $z \in Q_k$, we have that η_1 hits Q_k. If there exists a subsequence (k_j) such that, for every j, η_1 hits $\partial B(z_{k_j}, \delta)$ after hitting $\partial B(z_{k_j}, \varepsilon)$ and before hitting Q_{k_j}, we get a contradiction that $z \in I^{\varepsilon,\delta}$. Therefore there exists $K_1 = K_1(z)$ such that for every $k \geq K_1$, we have that, after hitting $\partial B(z_k, \varepsilon)$, η_1 hits Q_k before hitting $\partial B(z_k, \delta)$. Combing this with Lemma 5.4.5 implies that there exists a sequence (k_j) such that $Q_{k_j} \in \mathscr{W}_n^{\varepsilon,\delta}$ for all j, which proves our claim.

By running η_1 until time $\zeta^1_{z,\varepsilon}$ and then conformally mapping back, Lemma 5.4.3 implies for $Q \in \mathscr{D}_m$ with $Q \subseteq B(0,\delta^{-1})$ and $\mathrm{Im}(z(Q)) \geq \delta$ that $\mathbf{P}[Q \in \mathscr{U}_m^{\varepsilon,\delta}] \leq 2^{-m(A+o(1))}$ provided m is large enough and $\varepsilon > 0$ is small enough relative to $\delta > 0$. (The purpose of choosing $\varepsilon > 0$ smaller than $\delta > 0$ is so that the force points of η_1 are mapped far away from $\eta_1(\zeta^1_{z,\varepsilon})$ relative to the distance of z.) Consequently, it follows that there exists $C = C(\varepsilon,\delta) > 0$ such that for each $\xi > 0$, we have

$$\mathbb{E}[\mathscr{H}^{2-A+2\xi}(I^{\varepsilon,\delta})] \leq C \sum_{m=n}^{\infty} 2^{2m} \times 2^{-m(A-\xi)} \times 2^{-m(2-A+2\xi)} < \infty.$$

Since the above holds for every n, we therefore have that $\mathscr{H}^{2-A+2\xi}(I^{\varepsilon,\delta}) = 0$ almost surely. Since $\xi > 0$ was arbitrary, we have that $\dim_{\mathscr{H}}(I^{\varepsilon,\delta}) \leq 2 - A$ almost surely, as desired. □

5.4.4 The lower bound

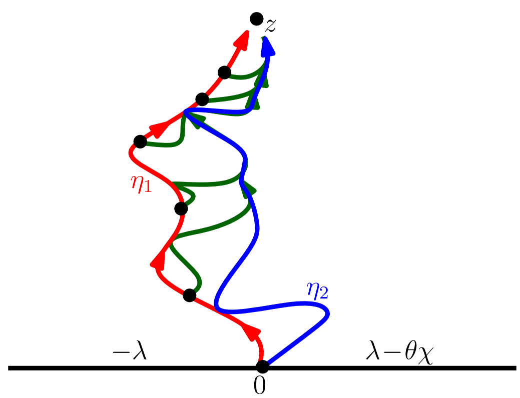


Figure 5.4.4: Suppose that h is a GFF on $\mathbb{H}$ with the illustrated boundary data. Let η_1 (resp. η_2) be the flow line of h starting from 0 with angle 0 (resp. $\theta \in (\pi - 2\lambda/\chi, 0)$). Shown is an illustration of the construction of the event that a given point, say $z \in \mathbb{H}$, is a "perfect point" for the intersection of η_1 and η_2. Each of the green flow lines has angle θ — the same as that of η_2 — and start at points along η_1 which get progressively closer to z. The reason that we introduce the auxiliary green flow lines is that this is what gives us the approximate independence necessary for the two point estimate, see e.g. Figure 5.4.7.

We are now going to prove the lower bound for Theorem 5.1.5. As in the proof of Theorem 5.1.6, we will accomplish this by introducing a special class of points, so-called "perfect points," which are contained in the intersection of two flow lines whose correlation structure is easy to control. Fix $\widetilde{\beta} > \beta^2 > \beta > 1$; we will eventually send $\widetilde{\beta} \to \infty$ but we will take β fixed and large.

Definition of the events

We are going to define the perfect points as follows.

Definition 5.4.7. *Suppose that γ_1 is a path in $\mathbb{H}$ starting from 0 and γ_2 is a path starting from $x_2 \in [0, e^{\beta}]$. Let $\widetilde{\zeta}_1$ be the first time that γ_1 hits $\partial B(i, e^{-\widetilde{\beta}})$ and suppose that $\widetilde{\gamma}_2$ is a path starting from $\gamma_1(\widetilde{\zeta}_1)$. Fix $u \in \mathbb{R} \setminus [0, x_2]$. We let $E_u^{\beta}(\gamma_1, \widetilde{\gamma}_2, \gamma_2)$ be the event that the following hold (see Figure 5.4.5 for an illustration):*

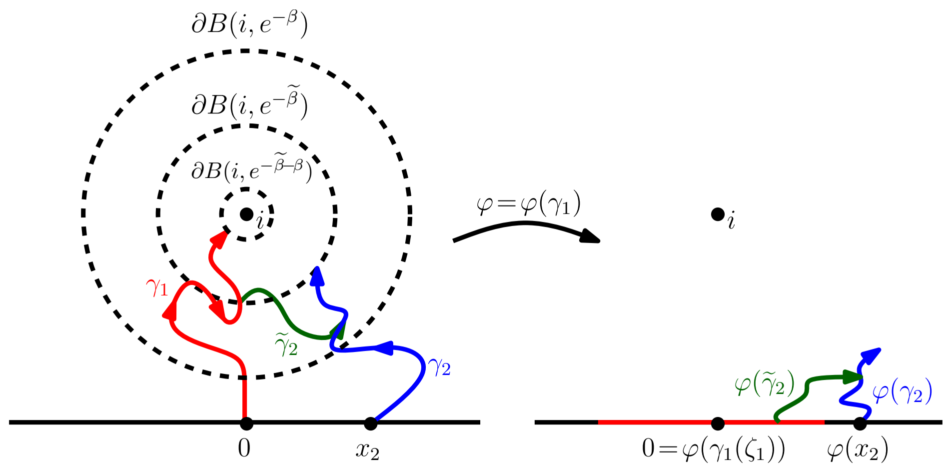


Figure 5.4.5: Suppose that γ_1, γ_2 are paths in $\mathbb{H}$ starting from $0, x_2 \in \mathbb{R}$, respectively, with $x_2 \in [0, e^{\beta}]$. Let $\widetilde{\zeta}_1$ be the first time that γ_1 hits $\partial B(i, e^{-\widetilde{\beta}})$ and let $\widetilde{\gamma}_2$ be a path starting from $\gamma_1(\widetilde{\zeta}_1)$. Fix $u \in \mathbb{R} \setminus [0, x_2]$. Then $E_u^{\beta,\widetilde{\beta}}(\gamma_1, \widetilde{\gamma}_2, \gamma_2)$ is the event that the following hold. First, γ_1 hits $\partial B(i, e^{-\beta})$ before leaving the $e^{-2\beta}$ neighborhood of $[0, i]$. Second, γ_1 (resp. γ_2) hits $\partial B(i, e^{-\widetilde{\beta}-\beta})$ (resp. $\partial B(i, e^{-\widetilde{\beta}})$) before leaving $B(i, e^{2\beta})$. Let ζ_1, ζ_2 be the first hitting times for γ_1, γ_2, respectively, for these small circles. Third, the first time $\widetilde{\zeta}_2$ that $\widetilde{\gamma}_2$ hits γ_2 is finite and $\widetilde{\gamma}_2([0, \widetilde{\zeta}_2])$ is disjoint from both $\partial B(i, \frac{1}{2} e^{-\widetilde{\beta}})$ and $\partial B(i, 2e^{-\widetilde{\beta}})$. Fourth, the three paths stopped at the aforementioned times do not separate i from u. Fifth, the probability that a Brownian motion starting from i exits $\mathbb{H} \setminus (\gamma_1([0, \zeta_1]) \cup \widetilde{\gamma}_2([0, \widetilde{\zeta}_2]) \cup \gamma_2([0, \zeta_2]))$ in the left (resp. right) side of γ_1 is at least $\frac{1}{2} - e^{-\beta/4}$ and in the left (resp. right) side of $\widetilde{\gamma}_2([0, \widetilde{\zeta}_2])$ (resp. $\gamma_2([0, \zeta_2])$) is at least $e^{-\beta}$. We take H to be the connected component of $\mathbb{H} \setminus \gamma_1([0, \zeta_1])$ with u on its boundary and let $\varphi = \varphi(\gamma_1)$ be the conformal transformation $H \to \mathbb{H}$ fixing i and with $\varphi(\gamma_1(\zeta_1)) = 0$. Then the image of (the right side of) $\gamma_1(\widetilde{\zeta}_1)$ under φ is contained in $[0, e^{\beta}]$ and $\varphi(\widetilde{\gamma}_2([0, \widetilde{\zeta}_2])) \subseteq B(i, e^{\beta})$.

1. *γ_1 hits $\partial B(i, e^{-\beta})$ before leaving the $e^{-2\beta}$ neighborhood of $[0, i]$,*

2. *The first time ζ_1 (resp. ζ_2) that γ_1 (resp. γ_2) hits $\partial B(i, e^{-\widetilde{\beta}-\beta})$ (resp. $\partial B(i, e^{-\widetilde{\beta}})$) is finite and $\gamma_i([0, \zeta_i]) \subseteq B(i, e^{2\beta})$ for $i = 1, 2$.*

3. *The first time $\widetilde{\zeta}_2$ that $\widetilde{\gamma}_2$ hits γ_2 is finite and $\widetilde{\gamma}_2([0, \widetilde{\zeta}_2])$ does not intersect either $\partial B(i, \frac{1}{2} e^{-\widetilde{\beta}})$ or $\partial B(i, 2e^{-\widetilde{\beta}})$.*

4. *The connected component of $\mathbb{H} \setminus (\gamma_1([0, \zeta_1]) \cup \widetilde{\gamma}_2([0, \widetilde{\zeta}_2]) \cup \gamma_2([0, \zeta_2]))$ which contains i also contains u on its boundary.*

5. *The probability that a Brownian motion starting from i exits $\mathbb{H} \setminus (\gamma_1([0, \zeta_1]) \cup \widetilde{\gamma}_2([0, \widetilde{\zeta}_2]) \cup \gamma_2([0, \zeta_2]))$ on the left (resp. right) side of $\gamma_1([0, \zeta_1])$ is at least $\frac{1}{2} - e^{-\beta/4}$ and the probability of exiting on the left (resp. right) side of $\widetilde{\gamma}_2([0, \widetilde{\zeta}_2])$ (resp. $\gamma_2([0, \zeta_2])$) is at least $e^{-\beta}$. We take H to be the connected component of $\mathbb{H} \setminus \gamma_1([0, \zeta_1])$ with u on its boundary and let $\varphi = \varphi(\gamma_1)$ be the conformal transformation $H \to \mathbb{H}$ which fixes i and with $\varphi(\gamma_1(\zeta_1)) = 0$. Finally, the image of (the right side of) $\gamma_1(\widetilde{\zeta}_1)$ under φ is contained in $[0, e^{\beta}]$ and $\varphi(\widetilde{\gamma}_2([0, \widetilde{\zeta}_2])) \subseteq B(i, e^{\beta})$.*

The purpose of Part 1 above is that, by drawing a path up until hitting $\partial B(i,e^{-\beta})$ and then conformally mapping back, the resulting configuration of paths satisfies the hypotheses of Lemma 5.4.3.

Lemma 5.4.8. *Suppose that we have the same setup described just above. There exists a constant $C_1 > 0$ such that the following is true. On the event $E_u^\beta(\gamma_1,\widetilde{\gamma}_2,\gamma_2)$ with $\varphi = \varphi(\gamma_1)$, for each $\alpha \in (0,1)$ we have that $B(i, C_1 e^{(1-\alpha)(\beta+\widetilde{\beta})/2}) \subseteq \varphi(B(i,e^{-\alpha(\beta+\widetilde{\beta})}))$.*

Proof. Throughout, we shall suppose that $E_u^\beta(\gamma_1,\widetilde{\gamma}_2,\gamma_2)$ occurs. Fix $\alpha \in (0,1)$. The probability that a Brownian motion starting from i hits $\partial B(i,e^{-\alpha(\beta+\widetilde{\beta})})$ before hitting $\partial \mathbb{H} \cup \gamma_1([0,\zeta_1])$ is $O(e^{-(1-\alpha)(\beta+\widetilde{\beta})/2})$ by Beurling estimate 5.2.9. By the conformal invariance of Brownian motion, the probability of the event X that a Brownian motion starting from i exits $\varphi(B(i,e^{-\alpha(\beta+\widetilde{\beta})}))$ in $\varphi(\partial B(i,e^{-\alpha(\beta+\widetilde{\beta})}))$ is also $O(e^{-(1-\alpha)(\beta+\widetilde{\beta})/2})$. Let

$$d = \mathrm{dist}(\varphi(\partial B(i,e^{-\alpha(\beta+\widetilde{\beta})})), i).$$

We claim $\mathbf{P}[X] \gtrsim d^{-1}$. Indeed, $X_1 \cap X_2 \subseteq X$ where X_1 is the event that the Brownian motion exits $\partial B(0,d)$ before hitting $\partial\mathbb{H}$ at a point with argument in $[\frac{\pi}{4},\frac{3\pi}{4}]$ and X_2 is the event that it hits $\varphi(\partial B(i,e^{-\alpha(\beta+\widetilde{\beta})}))$ after hitting $\partial B(0,d)$ before hitting $\partial\mathbb{H}$. It is easy to see that $\mathbf{P}[X_1] \gtrsim d^{-1}$ and $\mathbf{P}[X_2 \,|\, X_1] \gtrsim 1$. Consequently, $e^{-(1-\alpha)(\beta+\widetilde{\beta})/2} \gtrsim d^{-1}$ hence $d \gtrsim e^{(1-\alpha)(\beta+\widetilde{\beta})/2}$, as desired. □

Flow line estimates

Fix $\theta \in (\pi - 2\lambda/\chi, 0)$; recall that this is the range of angles so that a GFF flow line with angle θ can hit and bounce off of a GFF flow line with angle 0 on its right side. We will now use the events introduced in Definition 5.4.7 to define the perfect points. Suppose that h_1 is a GFF on $\mathbb{H}$ with the following boundary data: suppose $x_{1,1} = x_{1,2} = 0$ and $u_1 \in \mathbb{R}\setminus\{0\}$. If $u_1 < x_{1,1} = x_{1,2} = 0$, the boundary data is

$$h|_{(-\infty,u_1]} \equiv \lambda + (2\pi - \theta)\chi, \quad h|_{(u_1,0]} \equiv -\lambda, \quad \text{and} \quad h|_{(0,\infty)} \equiv \lambda - \theta\chi.$$

If $u_1 > x_{1,1} = x_{1,2} = 0$, then the boundary data is

$$h|_{(-\infty,0]} \equiv -\lambda, \quad h|_{(0,u_1]} \equiv \lambda - \theta\chi, \quad \text{and} \quad h|_{(u_1,\infty)} \equiv -\lambda - 2\pi\chi.$$

These two possibilities correspond to the boundary data that arises when one takes a GFF with boundary conditions as in Figure 5.4.1 and Figure 5.4.2 and then applies a change of coordinates which takes a given point $z \in \mathbb{H}$ to i. In either case, we let $\eta_{1,1}$ (resp. $\eta_{1,2}$) be the flow line of h_1 starting from $x_{1,1}$ (resp. $x_{1,2}$) of angle 0 (resp. θ). We also let $\widetilde{\zeta}_{1,1}$ be the first time that η_1 hits $\partial B(i,e^{-\widetilde{\beta}})$ and let $\widetilde{\eta}_{1,2}$ be the flow line of h_1 starting from (the right side of) $\eta_{1,1}(\widetilde{\zeta}_{1,1})$ with angle θ.

Let $E_1 = E_{u_1}^\beta(\eta_{1,1},\widetilde{\eta}_{1,2},\eta_{1,2})$. Let $\zeta_{1,1}$ (resp. $\zeta_{1,2}$) be the first time that $\eta_{1,1}$ (resp. $\eta_{1,2}$) hits $\partial B(i,e^{-\widetilde{\beta}-\beta})$ (resp. $\partial B(i,e^{-\widetilde{\beta}})$) and let $\widetilde{\zeta}_{1,2}$ be the first time that $\widetilde{\eta}_{1,2}$ hits $\eta_{1,2}$. Let φ_1 be the unique conformal map from the connected component of $\mathbb{H}\setminus\eta_{1,1}([0,\zeta_{1,1}])$ with u_1 on its boundary which fixes i and sends the tip $\eta_{1,1}(\zeta_{1,1})$ to 0.

Suppose that the events E_j have been defined as well as paths $\eta_{j,1},\widetilde{\eta}_{j,2},\eta_{j,2}$, GFFs h_j, and conformal transformations φ_j for $1 \le j \le k$. On the event that $\eta_{k,1}$ hits $\partial B(i,e^{-\beta-\widetilde{\beta}})$, we take

$\eta_{k+1,1} = \varphi_k(\eta_{k,1})$ and $\eta_{k+1,2} = \varphi_k(\widetilde{\eta}_{k,2})$. Note that $\eta_{k+1,1}$ is the flow line of the GFF $h_{k+1} = h_k \circ \varphi_k^{-1} - \chi \arg(\varphi_k^{-1})'$ starting from 0. Similarly, $\eta_{k+1,2}$ is the flow line of h_{k+1} starting from $x_{k+1,2} = \varphi_k(\eta_{k,1}(\widetilde{\zeta}_{k,1}))$ with angle θ. We let $\widetilde{\zeta}_{k+1,1}$ be the first time that $\eta_{k+1,1}$ hits $\partial B(i, e^{-\widetilde{\beta}})$ and let $\widetilde{\eta}_{k+1,2}$ be the flow line starting from (the right side of) $\eta_{k+1,1}(\widetilde{\zeta}_{k+1,1})$ with angle θ and let $u_{k+1} = \varphi_k(u_k)$.

On the event that $\eta_{k+1,1}$ hits $\partial B(i, e^{-\widetilde{\beta}-\beta})$, say for the first time at time $\zeta_{k+1,1}$, we let φ_{k+1} be the conformal transformation which uniformizes the connected component of $\mathbb{H} \setminus \eta_{k+1,1}([0, \zeta_{k+1,1}])$ with u_{k+1} on its boundary fixing i and with $\varphi_{k+1}(\eta_{k+1,1}(\zeta_{k+1,1})) = 0$. We then define the event E_{k+1} in terms of the paths $\eta_{k+1,1}$, $\widetilde{\eta}_{k+1,2}$, and $\eta_{k+1,2}$ analogously to E_1 as well as stopping times $\zeta_{k+1,2}, \widetilde{\zeta}_{k+1,2}$. For each $n \geq m$ we let

$$E^{m,n} = \cap_{k=m+1}^{n} E_k \quad \text{and} \quad E^n = E^{0,n}. \tag{5.4.24}$$

Remark 5.4.9.

1. *Note that $E^{m,n}$ for $n > m \geq 1$ can occur even if only a subset of (or none of) $E^1, \ldots, E^m$ occur.*

2. *The conformal maps φ_j are measurable with respect to $\eta_{1,1}$. Note that each of the paths $\widetilde{\eta}_{k,2}$ is given by the conformal image of a flow line which starts at a point in the range of $\eta_{1,1}$. The starting points of these flow lines are likewise measurable with respect to $\eta_{1,1}$. These facts will be important when we establish the two point estimate for the lower bound of Theorem 5.1.5 at the end of this subsection.*

We will now work towards proving the one point estimate for the perfect point i.

Proposition 5.4.10. *There exists $\beta_0 > 1$ such that for all $\widetilde{\beta} > \beta^2 > \beta \geq \beta_0$ we have*

$$\mathbf{P}[E^n] \asymp e^{-\widetilde{\beta}(1+O_\beta(1)o_{\widetilde{\beta}}(1))nA} \tag{5.4.25}$$

where A is the constant from (5.4.13) *and the constants in the $\asymp$ of* (5.4.25) *depend only on u_1, κ, and θ.*

In the statement of Proposition 5.4.10, we write $o_{\widetilde{\beta}}(1)$ to indicate a quantity which converges to 0 as $\widetilde{\beta} \to \infty$ and $O_\beta(1)$ for a term which is bounded by some constant which depends only on β. In particular, for β fixed, $O_\beta(1)o_{\widetilde{\beta}}(1) \to 0$ as $\widetilde{\beta} \to \infty$. The first step in the proof of Proposition 5.4.10 is Lemma 5.4.11. The second step, which allows one to iterate the estimate in (5.4.26), is Lemma 5.4.13 and is stated and proved below.

Lemma 5.4.11. *There exists $\beta_0 > 1$ such that for all $\widetilde{\beta} > \beta^2 > \beta \geq \beta_0$ we have*

$$\mathbf{P}[E_1] \asymp e^{-\widetilde{\beta}(1+O_\beta(1)o_{\widetilde{\beta}}(1))A} \tag{5.4.26}$$

where A is the constant from (5.4.13) *and the constants in the $\asymp$ of* (5.4.26) *depend only on u_1, κ, and θ.*

Proof. By Lemma 5.2.3, we know that $\eta_{1,1}$ has a positive chance of being uniformly close to $[0,i]$ before hitting $\partial B(i,e^{-\beta})$. Let τ be the first time that $\eta_{1,1}$ hits $\partial B(i,e^{-\beta})$ and let g be the conformal transformation from the connected component of $\mathbb{H}\setminus\eta_{1,1}([0,\tau])$ containing i which fixes i and sends $\eta_{1,1}(\tau)$ to 0. By choosing β_0 sufficiently large, it is clear that $g(\eta_{1,1})$ and $g(\eta_{1,2})$ satisfy the hypotheses of (5.4.16) of Lemma 5.4.3. From this, we deduce that the probability that $\eta_{1,1}$ and $\eta_{1,2}$ both hit $\partial B(i,2e^{-\widetilde{\beta}})$ before leaving $B(i,e^{2\beta})$ and such that the harmonic measure of the left (resp. right) side of each of the paths stopped at this time as viewed from i is bounded from below by some universal constant is equal to $e^{-\widetilde{\beta}(1+O_\beta(1)o_{\widetilde{\beta}}(1))A}$. The rest of the lemma follows from repeated applications of Lemma 5.2.3 and Lemma 5.2.5. □

For each $z\in\mathbb{H}$, we let ψ_z be the unique conformal transformation $\mathbb{H}\to\mathbb{H}$ taking z to i and fixing 0. For each $k\in\mathbb{N}$, we let $\eta_{k,i}^z$ for $i=1,2$ and $\widetilde{\eta}_{k,2}^z$ be the paths after applying the conformal map ψ_z and we let $\zeta_{k,i}^z$, $\widetilde{\zeta}_{k,i}^z$ be the corresponding stopping times. We define

$$\begin{aligned} E^{m,n}(z) &= E^{m,n}(\eta_{1,1}^z,\widetilde{\eta}_{1,2}^z,\eta_{1,2}^z) \quad \text{and} \\ E^n(z) &= E^{0,n}(z). \end{aligned} \tag{5.4.27}$$

In other words, $E^{m,n}(z)$ and $E^n(z)$ are the events corresponding to $E^{m,n}$ and E^n defined in (5.4.24) but with respect to the flow lines of the GFF $h_1\circ\psi_z^{-1}-\chi\arg(\psi_z^{-1})'$ starting from 0 and $\psi_z(x_{1,2})$. Let $\varphi_{k,z}$ be the corresponding conformal maps. We let

$$\varphi_z^{j,k}=\varphi_{j+1,z}\circ\cdots\circ\varphi_{k,z} \quad \text{for each} \quad 0\le j\le k \quad \text{and} \quad \varphi_z^k=\varphi_z^{0,k}. \tag{5.4.28}$$

We also let

$$V_n(z)=B(z,2^{8n+4}\mathrm{Im}(z)e^{-n(\beta+\widetilde{\beta})}) \quad \text{for each} \quad n\in\mathbb{N}.$$

Lemma 5.4.12. *There exists $\beta_0>1$ such that for all $\widetilde{\beta}>\beta^2>\beta\ge\beta_0$, the following is true. For each $m,n\in\mathbb{N}$ with $m\ge n+1$, on $E^m(z)$ we have that $\psi_z^{-1}\circ(\varphi_z^{m-1})^{-1}(\gamma)\subseteq V_n(z)$ for $\gamma=\eta_{m,i}^z([0,\zeta_{m,i}^z])$ for $i=1,2$ and $\gamma=\widetilde{\eta}_{m,2}^z([0,\widetilde{\zeta}_{m,2}^z])$.*

Proof. We are first going to give the proof in the case that $z=i$. Fix $m,n\in\mathbb{N}$ with $m\ge n+1$. Throughout, we shall assume that we are working on E^m. It follows from [Law05, Corollary 3.25] that if $r\in(0,\frac{1}{2})$ then

$$\varphi_k^{-1}(B(i,r))\subseteq B(i,16re^{-\widetilde{\beta}-\beta}) \quad \text{for} \quad 1\le k\le m. \tag{5.4.29}$$

Iterating (5.4.29) implies that

$$(\varphi^k)^{-1}(B(i,\tfrac{1}{2}))\subseteq B(i,2^{8k}e^{-k(\widetilde{\beta}+\beta)}) \quad \text{for} \quad 1\le k\le m \tag{5.4.30}$$

(provided we take β_0 large enough).

Note that $\eta_{m,i}([0,\zeta_{m,i}])\subseteq B(i,e^{2\beta})$ for $i=1,2$ by the definition of the events. Consequently, it follows from Lemma 5.4.8 that $\varphi_m^{-1}(\eta_{m,i}([0,\zeta_{m,i}]))\subseteq B(i,e^{-\widetilde{\beta}/4})$ for $i=1,2$ provided β_0 is large enough. We also assume that β_0 is sufficiently large so that $e^{-\widetilde{\beta}/4}<\frac{1}{2}$. Applying (5.4.30) proves the result for $\eta_{m,i}([0,\zeta_{m,i}])$ for $i=1,2$ and $\widetilde{\eta}_{m,2}([0,\widetilde{\zeta}_{m,2}])$. This proves the result for $z=i$. For the case that $z\neq i$, we note that applying [Law05, Corollary 3.25] again yields,

$$\psi_z^{-1}(B(i,r))\subseteq B(i,16r\mathrm{Im}(z)). \tag{5.4.31}$$

Combining (5.4.30) with (5.4.31) gives the desired result. □

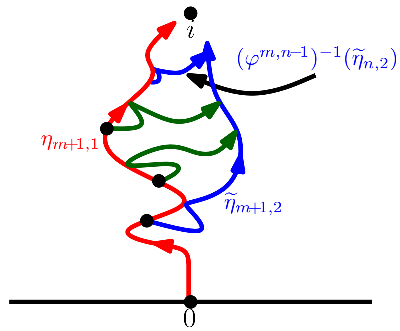


Figure 5.4.6: Illustration of the configuration of paths used in the proof of Lemma 5.4.13. On $E^{m,n}$, $\eta_{m+1,1}$, $\widetilde{\eta}_{m+1,2}$, and $(\varphi^{m,n-1})^{-1}(\widetilde{\eta}_{n,2})$ separate the paths $(\varphi^{m,j-1})^{-1}(\widetilde{\eta}_{j,2})$ for $m+2 \leq j \leq n-1$ (shown in green) stopped upon hitting $\widetilde{\eta}_{m+1,2}$ from i. Thus, once $\eta_{m+1,1}$, $\widetilde{\eta}_{m+1,2}$, and $(\varphi^{m,n-1})^{-1}(\widetilde{\eta}_{n,2})$ have been fixed, the conditional law of the remaining paths does not depend on the boundary data of h_{m+1} or on the other auxiliary paths.

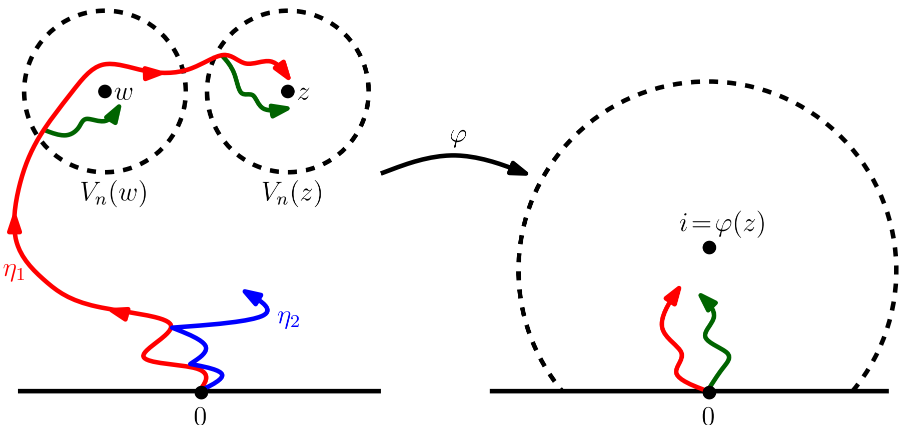


Figure 5.4.7: Illustration of the setup for the two point estimate (Lemma 5.4.13 and Lemma 5.4.15) in the case that η_1 gets close first to w and then to z. Conformally map back everything drawn before the paths hit the neighborhood of z. Then all of the auxiliary paths are outside of a large ball which is far from $i = \varphi(z)$, so we can apply the one point estimate for perfect points (Lemma 5.4.11) for this region as before. We can also apply the one point estimate for the paths near z. Finally, to complete the proof, we apply the one point estimate a final time for the paths up to when they hit a neighborhood containing both z and w.

For each $m \in \mathbb{N}$ and $z \in \mathbb{H}$, let $\mathscr{F}_m(z)$ be the σ-algebra generated by $\eta_{k,i}^z|_{[0,\zeta_{k,i}^z]}$ for $i = 1,2$ and $\widetilde{\eta}_{k,2}^z|_{[0,\widetilde{\zeta}_{k,2}^z]}$ for $1 \leq k \leq m$.

Lemma 5.4.13. *There exists $\beta_0 > 1$ such that for all $\widetilde{\beta} > \beta^2 > \beta \geq \beta_0$ the following is true. Fix $\delta \in (0, \frac{\pi}{2})$ and $z \in \mathbb{D} \cap \mathbb{H}$ with $\arg(z) \in (\delta, \pi - \delta)$. For each $m \in \mathbb{N}$ we have that*

$$\mathbf{P}[E^{m,n}(z) \,|\, \mathscr{F}_m(z)]\mathbf{1}_{E^m(z)} \asymp e^{O_\beta(1) o_{\widetilde{\beta}}(1) \widetilde{\beta}} \mathbf{P}[E^{n-m}]\mathbf{1}_{E^m(z)} \tag{5.4.32}$$

where the constants in $\asymp$ depend only on δ, κ, and θ.

Proof. By applying ψ_z, we may assume without loss of generality that $z=i$. Recall the definition of the GFF h_{m+1} as well as the paths $\eta_{k,i}$ for $i=1,2$ and $\widetilde{\eta}_{k,2}$ from just before Remark 5.4.9. By the definition of E^m and the conformal invariance of Brownian motion, we know that there exists a constant $c_1>0$ such that the boundary data for h_{m+1} in $(-c_1,0)$ (resp. $(0,c_1)$) is given by $-\lambda$ (resp. λ). The same is likewise true for h_1. Moreover, by Lemma 5.4.8, it follows that the auxiliary paths coupled with h_{m+1} are far away from i provided β_0 is large enough. Consequently, by Lemma 5.2.8, the laws of $\eta_{m+1,1}$ (given E^m) and $\eta_{1,1}$ stopped upon exiting the $\frac{c_1}{2}$ neighborhood of the line segment from 0 to i are mutually absolutely continuous with Radon-Nikodym derivative which is bounded from above and below by universal positive and finite constants which depend only on κ and θ.

On $E^{m,n}$, $\eta_{m+1,1}$ does not leave this tube before getting very close to i and neither does $\eta_{1,1}$ on E^{n-m}. For a given choice of η, by Lemma 5.2.8, we moreover have that the Radon-Nikodym derivative of the conditional law of $\widetilde{\eta}_{m+1,2}$ given $\eta_{m+1,1}=\eta$ stopped upon exiting the tube with respect to that of $\widetilde{\eta}_{1,2}$ given $\eta_{1,1}=\eta$ is bounded from above and below by universal finite and positive constants which do not depend on the specific choice of η. On this event, the same is also true for the Radon-Nikodym derivative of the conditional law of $(\varphi^{m,n-1})^{-1}(\widetilde{\eta}_{n,2})$ given $\eta_{m+1,1}=\eta$ and $\widetilde{\eta}_{m+1,2}=\widetilde{\eta}$ with respect to the conditional law of $(\varphi^{n-m-1})^{-1}(\widetilde{\eta}_{n-m,2})$ given $\eta_{1,1}=\eta$ and $\widetilde{\eta}_{1,2}=\widetilde{\eta}$. The conditional law of $(\varphi^{m,j-1})^{-1}(\widetilde{\eta}_{j,2})$ for $m+2\le j\le n-1$ stopped upon hitting $\widetilde{\eta}_{m+1,2}$ given $\eta_{m+1,1}$, $\widetilde{\eta}_{m+1,2}$, and $\widetilde{\eta}_{n,2}$ is independent of the boundary data of h_{m+1} (as well as the other auxiliary paths). (See Figure 5.4.6.) The same is likewise true for the conditional law of $(\varphi^{j-1})^{-1}(\widetilde{\eta}_{j,2})$ for $2\le j\le n-m-1$ stopped upon hitting $\widetilde{\eta}_{1,2}$ given $\eta_{1,1}$, $\widetilde{\eta}_{1,2}$, and $\widetilde{\eta}_{n-m,2}$.

Let K be the compact hull associated with these paths and let g be the conformal transformation $\mathbb{H}\setminus K\to\mathbb{H}$ with $g(z)\sim z$ as $z\to\infty$. Conditionally on all of these paths and the event that they are contained in $B(i,2e^{-\widetilde{\beta}})$, the probability that $\eta_{m+1,2}$ hits $\partial B(i,10e^{-\widetilde{\beta}})$ before leaving $B(i,e^{2\beta})$ is $\asymp |g'(i)e^{-\widetilde{\beta}}|^{\alpha+O_\beta(1)o_{\widetilde{\beta}}(1)}$ (as in the proof of Lemma 5.4.3; the extra force points only change the probability by a positive and finite factor by Lemma 5.2.8.) Given that $\eta_{m+1,2}$ has hit $\partial B(i,10e^{-\widetilde{\beta}})$, the conditional probability that it then merges with $\widetilde{\eta}_{m+1,2}$ before the latter has hit $\partial B(i,\frac{1}{2}e^{-\widetilde{\beta}})$ or $\partial B(i,2e^{-\widetilde{\beta}})$ is positive by Lemma 5.2.5. The same is true with $\eta_{1,2}$ in place of $\eta_{m+1,2}$, which completes the proof. □

Proof of Proposition 5.4.10. This follows by combining Lemma 5.4.11 with Lemma 5.4.13. □

Lemma 5.4.14. *Fix $\delta\in(0,\frac{\pi}{2})$ and $z,w\in\mathbb{D}\cap\mathbb{H}$ distinct with $\arg(z),\arg(w)\in(\delta,\pi-\delta)$ and let m be the smallest integer such that $V_m(z)\cap V_m(w)=\emptyset$. Let P_w be the event that $\eta_{1,1}$ hits $V_m(w)$ before hitting $V_m(z)$. There exists $\beta_0>1$ such that for every $\widetilde{\beta}>\beta^2>\beta\ge\beta_0$ we have that*

$$\mathbf{P}[E^{m,n}(z)\,|\,\mathscr{F}_k(w)]\mathbf{1}_{E^k(w),P_w}\le e^{O_\beta(1)\widetilde{\beta}}\mathbf{P}[E^{n-m}]\mathbf{1}_{E^k(w),P_w} \tag{5.4.33}$$

for all $k\ge m$.

Proof. We are going to extract (5.4.33) from (5.4.32) of Lemma 5.4.13. As before, by applying ψ_z, we may assume without loss of generality that $z=i$. Fix $k\ge m$. By Proposition 5.4.10, it suffices to prove

$$\mathbf{P}[E^{m+1,n}\,|\,E_{m+1},\mathscr{F}_k(w)]\mathbf{1}_{E^k(w),P_w}\lesssim\mathbf{P}[E^{n-m-1}]\mathbf{1}_{E^k(w),P_w} \tag{5.4.34}$$

in place of (5.4.33). By Lemma 5.4.12, we know that the paths involved in $E^{m,n}$ are disjoint from those involved in $E^k(z)$ due to the choice of m. Thus by conformally mapping back (see

Figure 5.4.7) and applying Lemma 5.2.8 as in the proof of Lemma 5.4.13, it is therefore not hard to see that

$$\mathbf{P}[E^{m+1,n}\,|\,E_{m+1},\mathscr{F}_k(w)]\mathbf{1}_{E^k(w),P_w} \asymp \mathbf{P}[E^{1,n-m}\,|\,E_1]\mathbf{1}_{E^k(w),P_w}.$$

Combining this with (5.4.32) completes the proof. □

Lemma 5.4.15. *For every $\varepsilon > 0$ and $\delta \in (0,\frac{\pi}{2})$ there exists $\beta_0 > 1$ such that for all $\widetilde{\beta} > \beta^2 > \beta \geq \beta_0$ there exists constants $C > 0$ and $n_0 \in \mathbb{N}$ such that the following is true. Fix $z,w \in \mathbb{D}\cap\mathbb{H}$ distinct with $\arg(z),\arg(w) \in (\delta,\pi-\delta)$. Let m be the smallest integer such that $V_m(z)\cap V_m(w) = \emptyset$. Then*

$$\mathbf{P}[E^n(z),E^n(w)] \leq Ce^{\widetilde{\beta}(1+\varepsilon)mA}\mathbf{P}[E^n(z)]\mathbf{P}[E^n(w)] \quad \textit{for all} \quad n \geq n_0.$$

Proof. Suppose that $z,w \in \mathbb{H}$ are as in the statement of the lemma. Let P_w be the event that η_1 hits $V_m(w)$ before hitting $V_m(z)$ and let P_z be the event in which the roles of z and w are swapped. We have that

$$\begin{aligned}\mathbf{P}[E^n(z),E^n(w)] &= \mathbf{P}[E^n(z),E^n(w),P_w] + \mathbf{P}[E^n(z),E^n(w),P_z]\\ &\leq \mathbf{P}[E^n(z)\,|\,E^n(w),P_w]\mathbf{P}[E^n(w)] + \mathbf{P}[E^n(w)\,|\,E^n(z),P_z]\mathbf{P}[E^n(z)].\end{aligned} \tag{5.4.35}$$

We are going to bound the first summand; the second is bounded analogously. We have,

$$\mathbf{P}[E^n(z)\,|\,E^n(w),P_w] \leq \mathbf{P}[E^{m,n}(z)\,|\,E^n(w),P_w]. \tag{5.4.36}$$

By (5.4.33) of Lemma 5.4.14, we have that

$$\mathbf{P}[E^{m,n}(z)\,|\,E^n(w),P_w] \leq e^{O_\beta(1)\widetilde{\beta}}\mathbf{P}[E^{n-m}]. \tag{5.4.37}$$

By (5.4.32) of Lemma 5.4.13 and Proposition 5.4.10, we have that

$$\mathbf{P}[E^{n-m}] \leq e^{\widetilde{\beta}(1+\varepsilon)mA}\mathbf{P}[E^n(z)] \tag{5.4.38}$$

(possibly increasing β_0). The same likewise holds when we swap the roles of P_w and P_z. Combining (5.4.35)–(5.4.38) gives the result. □

We can now complete the proof of Theorem 5.1.5.

Proof of Theorem 5.1.5. We first suppose that h is a GFF on $\mathbb{H}$ with boundary conditions

$$h|_{(-\infty,0]} \equiv -\lambda \quad \text{and} \quad h|_{(0,\infty)} \equiv \lambda - \theta\chi$$

and let η_1 (resp. η_2) be the flow line of h starting from 0 with angle 0 (resp. $\theta \in (\pi - 2\lambda/\chi, 0)$). We have already established the upper bound for $\dim_{\mathscr{H}}(\eta_1\cap\eta_2\cap\mathbb{H})$ in Proposition 5.4.6. We will now establish the lower bound. Once we have proved this, we get the corresponding dimension when h has general piecewise constant boundary data as described in the theorem statement by absolute continuity for GFFs.

The proof is completed in the same manner as the proof of Theorem 5.1.6. Indeed, we let $\varepsilon_n = 2^{8n+4}e^{-(\beta+\widetilde{\beta})n}$. We divide $[-1,1]\times[1,2]$ into $2\varepsilon_n^{-2}$ squares of equal side length ε_n and let z_j^n be the center of the jth such square for $j = 1,\ldots,2\varepsilon_n^{-2}$. Let $\mathscr{C}_n$ be the set of centers z of these squares for which $E^n(z)$ occurs. Let $S_n(z)$ be the square with center z and length ε_n. Finally, we let

$$\mathscr{C} = \bigcap_{k\geq 1}\overline{\bigcup_{n\geq k}\bigcup_{z\in\mathscr{C}_n} S_n(z)}.$$

It is easy to see that

$$\mathscr{C} \subseteq \eta_1 \cap \eta_2 \cap \mathbb{H}.$$

The argument of the proof of Theorem 5.1.6 combined with Lemma 5.4.15 implies, for each $\xi > 0$, that $\mathbf{P}[\dim_{\mathscr{H}}(\eta_1 \cap \eta_2) \geq 2 - A - \xi] > 0$. To finish the proof, we only need to explain the 0-1 argument: that for each $d \in [0,2]$, $\mathbf{P}[\dim_{\mathscr{H}}(\eta_1 \cap \eta_2 \cap \mathbb{H}) = d] \in \{0,1\}$. As explained above, by absolute continuity for GFFs, for the purposes of computing the intersection dimension we can make whichever choice of boundary data is most convenient for the proof. Here, the most convenient choice is to take the boundary data to be given by the constant value $c = \frac{\pi\chi}{2} - \lambda$ which guarantees that η_1, η_2 can be continued towards ∞. For $r > 0$, let $D_r = \dim_{\mathscr{H}}(\eta_1 \cap \eta_2 \cap B(0,r) \cap \mathbb{H})$. It is clear that $0 < r_1 < r_2$ implies $D_{r_1} \leq D_{r_2}$. By the scale invariance of the setup, we have that D_{r_1} has the same law as D_{r_2}. Thus $D_{r_1} = D_{r_2}$ almost surely for all $0 < r_1 < r_2$. In particular, $\mathbf{P}[D_\infty = D_r] = 1$ for all $r > 0$. Thus the events $\{D_\infty = d\}$ and $\{D_r = d\}$ are the same up to a set of probability zero. The latter is measurable with respect to the GFF restricted to $B(0,r)$. Letting $r \downarrow 0$, we see that this implies that the event $\{D_\infty = d\}$ is trivial, which completes the proof. □

5.5 Proof of Theorem 5.1.1

We will first work towards proving (5.1.1) for $\kappa' \in (4,8)$; let $\kappa = \frac{16}{\kappa'} \in (2,4)$. It suffices to compute the almost sure Hausdorff dimension of the double points of the chordal $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2} - 4; \frac{\kappa'}{2} - 4)$ processes. Indeed, this follows since the conditional law of an $\mathrm{SLE}_{\kappa'}$ process given its left and right boundaries is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2} - 4; \frac{\kappa'}{2} - 4)$ in each of the bubbles which lie between these boundaries (recall Figure 5.2.5). In order to establish this result, we are going to make use of the path decomposition developed in [MS12c] which was used to prove the reversibility of $\mathrm{SLE}_{\kappa'}$ for $\kappa' \in (4,8)$. This, in turn, makes use of the duality results established in [MS12a, Section 7]. For the convenience of the reader, we are going to review the path decomposition here.

Throughout, we suppose that h is a GFF on the horizontal strip $\mathbb{T} = \mathbb{R} \times (0,1)$ with boundary values given by $-\lambda + \frac{\pi}{2}\chi = -\lambda'$ on the lower boundary $\partial_L \mathbb{T} = \mathbb{R}$ of the strip and $\lambda - \frac{3\pi}{2}\chi = \lambda' - \pi\chi$ on the upper boundary $\partial_U \mathbb{T} = \mathbb{R} \times \{1\}$ of the strip. (See Figure 5.5.1 for an illustration of the setup and recall the identities from (5.2.10).) Let η' be the counterflow line of h from $+\infty$ to $-\infty$. Then η' is an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2} - 4; \frac{\kappa'}{2} - 4)$ process in $\mathbb{T}$ from $+\infty$ to $-\infty$ where the force points are located immediately to the left and right of the starting point of the path. Recall that $\frac{\kappa'}{2} - 4$ is the critical threshold at or below which an $\mathrm{SLE}_{\kappa'}(\rho)$ process fills the domain boundary. Fix $z \in \partial \mathbb{T}$ and let $t(z)$ be the first time t that η' hits z. Then $t(z) < \infty$ almost surely (and this holds for all boundary points simultaneously). Assume further that $z \in \partial_L \mathbb{T}$ and let η_z^1 be the outer boundary of $\eta'([0,t(z)])$. Explicitly, η_z^1 is equal to the flow line of h with angle $\frac{\pi}{2}$ starting from z stopped at time τ_z^1, the first time that it hits $\partial_U \mathbb{T}$ (see Figure 5.5.1). The conditional law of η' given $\eta_z^1([0,\tau_z^1])$ in each of the connected components C of $\mathbb{T} \setminus \eta_z^1([0,\tau_z^1])$ which lie to the right of $\eta_z^1([0,\tau_z^1])$ is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2} - 4; \frac{\kappa'}{2} - 4)$ process starting from the first point of $\overline{C}$ visited by η' and terminating at the last.

Let $w = \eta_z^1(\tau_z^1) \in \partial_U \mathbb{T}$. Since η' is boundary filling and cannot enter the loops it creates with itself or with the domain boundary, the first point on $\partial_U \mathbb{T}$ that η' hits after time $t(z)$ is w. Let η_z^2 be the outer boundary of $\eta'([t(z),\infty))$. Then η_z^2 is the flow line of h given $\eta_z^1([0,\tau_z^1])$ with angle $\frac{\pi}{2}$ starting from w and stopped at time τ_z^2, the first time the path hits z. Let $P(z)$ be the region which lies between $\eta_z^1([0,\tau_z^1])$ and $\eta_z^2([0,\tau_z^2])$. Then $P(z)$ separates the set of points that η' visits before and after hitting z. The right (resp. left) boundary of $P(z)$ is given by $\eta_z^1([0,\tau_z^1])$ (resp. $\eta_z^2([0,\tau_z^2])$).

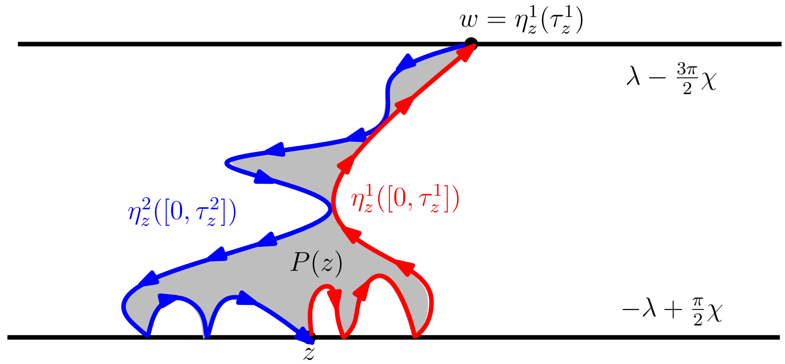


Figure 5.5.1: Suppose that h is a GFF on the horizontal strip $\mathbb{T} = \mathbb{R} \times (0,1)$ with the illustrated boundary data and let η' be the counterflow line of h starting from $+\infty$ and targeted at $-\infty$. Then η' is an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4; \frac{\kappa'}{2}-4)$ with force points located immediately to the left and right of the starting point of the path. Fix z in the lower boundary $\partial_L \mathbb{T} = \mathbb{R}$ of $\mathbb{T}$ and let $t(z)$ be the first time that η' hits z. Since η' is boundary filling, $t(z) < \infty$ almost surely. Let η_z^1 be the outer boundary of $\eta'([0,t(z)])$. Then η_z^1 is equal to the flow line of h with angle $\frac{\pi}{2}$ starting from z and stopped at time τ_z^1, the first time that it hits $\partial_U \mathbb{T}$. Let $w = \eta_z^1(\tau_z^1)$. Given $\eta_z^1([0,\tau_z^1])$, let η_z^2 be the outer boundary of $\eta'([t(z),\infty))$. Then η_z^2 is equal to the flow line of h given $\eta_z^1([0,\tau_z^1])$ with angle $\frac{\pi}{2}$ started from w stopped at time τ_z^2, the first time it hits z. Let $P(z)$ be the region between $\eta_z^1([0,\tau_z^1])$ and $\eta_z^2([0,\tau_z^2])$ (indicated in gray). Given $P(z)$, the conditional law of η' in each component C of $\mathbb{T} \setminus P(z)$ is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4; \frac{\kappa'}{2}-4)$ from the first point in $\overline{C}$ visited by η' to the last. The points $\eta_z^1([0,\tau_z^1]) \cap \eta_z^2([0,\tau_z^2])$ are double points of η'.

The conditional law of η' given $P(z)$ is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4; \frac{\kappa'}{2}-4)$ process in each of the components C of $\mathbb{T} \setminus P(z)$ starting from the first point of $\overline{C}$ hit by η' and terminating at the last — the same as that of η' up to a conformal transformation. This symmetry allows us to iterate this exploration procedure to eventually discover the entire path. Note that the intersection points $\eta_z^1([0,\tau_z^1]) \cap \eta_z^2([0,\tau_z^2])$ are double points of η'. If $z \in \partial_U \mathbb{T}$, then we can define the paths η_z^1, η_z^2 analogously except the angle $\frac{\pi}{2}$ is replaced with $-\frac{\pi}{2}$. This is because when η' hits $z \in \partial_U \mathbb{T}$, only its right boundary is visible from $-\infty$ which is contrast to the case when it hits $z \in \partial_L \mathbb{T}$ when only its left boundary is visible from $-\infty$.

The following lemma allows us to relate the dimension of the double points of η' to the intersection dimension of GFF flow lines given in Theorem 5.1.5. This immediately leads to the lower bound in Theorem 5.1.1 for $\kappa' \in (4,8)$. We will explain a bit later how to extract from this the upper bound as well.

Lemma 5.5.1. *Let $P_\cap(z) = \eta_z^1([0,\tau_z^1]) \cap \eta_z^2([0,\tau_z^2])$. We have that*

$$\dim_{\mathcal{H}}(P_\cap(z)) = 2 - \frac{(12-\kappa')(4+\kappa')}{8\kappa'} \quad \textit{almost surely.}$$

That is, $\dim_{\mathcal{H}}(P_\cap(z))$ is almost surely equal to the Hausdorff dimension of the intersection of two GFF flow lines with an angle gap of θ_{double} (recall (5.1.10)) as given in Theorem 5.1.5.

Proof. See Figure 5.5.2 for an illustration of the argument. We shall assume throughout for simplicity that $z \in \partial_L \mathbb{T}$. A similar argument gives the same result for $z \in \partial_U \mathbb{T}$. Suppose that $\widetilde{h}$ is a GFF on $\mathbb{H}$ with the boundary data as indicated in the left side of Figure 5.5.2. Let η_0^1 be the flow

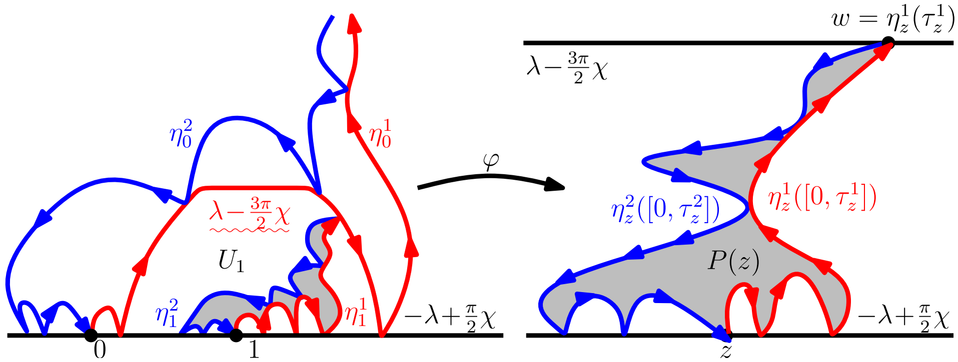


Figure 5.5.2: (Continuation of Figure 5.5.1.) Suppose that $\widetilde{h}$ is a GFF on $\mathbb{H}$ with the boundary data indicated on the left side. Let η_0^1 be the flow line of $\widetilde{h}$ from 0 to ∞ with angle $\frac{\pi}{2}$. Given η_0^1, let η_0^2 be the flow line of $\widetilde{h}$ given η_0^1 from ∞ with angle $\frac{\pi}{2}$ in the connected component of $\mathbb{H} \setminus \eta_0^1$ which is to the left of η_0^1. Then η_0^1 is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-2;-\frac{\kappa}{2})$ in $\mathbb{H}$ from 0 to ∞. Moreover, the conditional law of η_0^2 given η_0^1 is that of an $\mathrm{SLE}_\kappa(\kappa-4;-\frac{\kappa}{2})$ in the component of $\mathbb{H} \setminus \eta_0^1$ which is to the left of η_0^1 from ∞ to 0 (the $\kappa-4$ force point lies between the paths). Shown is the boundary data for the conditional law of $\widetilde{h}$ given (η_0^1, η_0^2) in the component U_1 of $\mathbb{H} \setminus (\eta_0^1 \cup \eta_0^2)$ which contains 1 on its boundary. Let $\varphi\colon U_1 \to \mathbb{H}$ be the conformal transformation with $\varphi(1) = z$ and which takes leftmost (resp. rightmost) point of $\partial U_1 \cap \partial \mathbb{H}$ to $-\infty$ (resp. $+\infty$). Then $\widetilde{h} \circ \varphi^{-1} - \chi \arg(\varphi^{-1})'$ has the boundary data shown on the right side. Let (η_1^1, η_1^2) be a pair of paths defined in the same way as (η_0^1, η_0^2) except starting from 1. Then the image of the region in U_1 between η_1^1 and η_1^2 under φ has the same law as $P(z)$ described in Figure 5.5.1. (See also [MS12c, Figure 3.2].)

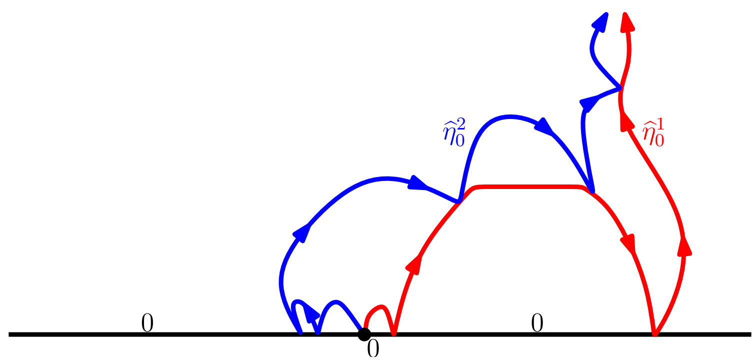


Figure 5.5.3: Suppose that $\widehat{h}$ is a GFF on $\mathbb{H}$ with zero boundary conditions as illustrated. Let $\widehat{\eta}_0^1$ (resp. $\widehat{\eta}_0^2$) be the flow line of $\widehat{h}$ starting from 0 with angle $-\frac{1}{2}\theta_{\mathrm{double}}$ (resp. $\frac{1}{2}\theta_{\mathrm{double}}$); recall (5.1.10). Then $\widehat{\eta}_0^1$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-2;-\frac{\kappa}{2})$ process in $\mathbb{H}$ from 0 to ∞ (Figure 5.2.1) and the conditional law of $\widehat{\eta}_0^2$ given $\widehat{\eta}_0^1$ in the connected component of $\mathbb{H} \setminus \widehat{\eta}_0^1$ which is to the left of $\widehat{\eta}_0^1$ is an $\mathrm{SLE}_\kappa(-\frac{\kappa}{2};\kappa-4)$ process from 0 to ∞ (Figure 5.2.4). Similarly, $\widehat{\eta}_0^2$ is an $\mathrm{SLE}_\kappa(-\frac{\kappa}{2};\frac{\kappa}{2}-2)$ process in $\mathbb{H}$ from 0 to ∞ (Figure 5.2.1) and the conditional law of $\widehat{\eta}_0^1$ given $\widehat{\eta}_0^2$ is an $\mathrm{SLE}_\kappa(\kappa-4;-\frac{\kappa}{2})$ process from 0 to ∞ in the component of $\mathbb{H} \setminus \widehat{\eta}_0^2$ which is to the right of $\widehat{\eta}_0^2$ (Figure 5.2.4). In particular, by the main result of [MS12b], the joint law of the ranges of $\widehat{\eta}_0^1$ and $\widehat{\eta}_0^2$ is equal to the joint law of the ranges of η_0^1 and η_0^2 from the left side of Figure 5.5.2. Consequently, we can use Theorem 5.1.5 to compute the almost sure dimension of the intersection of the latter.

line of $\widetilde{h}$ from 0 with angle $\frac{\pi}{2}$. Given η_0^1, let η_0^2 be the flow line of $\widetilde{h}$ with angle $\frac{\pi}{2}$ from ∞ in the component L of $\mathbb{H} \setminus \eta_0^1$ which is to the left of η_0^1. Note that η_0^1 is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-2;-\frac{\kappa}{2})$ process in $\mathbb{H}$ from 0 to ∞. Moreover, the conditional law of η_0^2 given η_0^1 is an $\mathrm{SLE}_\kappa(\kappa-4;-\frac{\kappa}{2})$ process in L from ∞ to 0; see [MS12c, Lemma 3.3]. (The $\kappa-4$ force point lies between η_0^1 and η_0^2.) By the main result of [MS12b], the time-reversal $\widetilde{\eta}_0^2$ of η_0^2 is an $\mathrm{SLE}_\kappa(-\frac{\kappa}{2};\kappa-4)$ process in L from 0 to ∞. As explained in Figure 5.5.3, it consequently follows from Theorem 5.1.5 that

$$\dim_{\mathscr{H}}(\eta_0^1 \cap \eta_0^2) = 2 - \frac{(12-\kappa')(4+\kappa')}{8\kappa'} \quad \text{almost surely} \tag{5.5.1}$$

since this is the almost sure dimension of $\widehat{\eta}_0^1 \cap \widehat{\eta}_0^2$ (using the notation of Figure 5.5.3). Thus to complete the proof, we just have to argue that $\dim_{\mathscr{H}}(P_\cap(z))$ is also given by this value.

Let U_1 be the component of $\mathbb{H} \setminus (\eta_0^1 \cup \eta_0^2)$ which contains 1 on its boundary. Let $\varphi \colon U_1 \to \mathbb{T}$ be the conformal transformation which takes 1 to z and the leftmost (resp. rightmost) point of $\partial U_1 \cap \mathbb{R}$ to $-\infty$ (resp. $+\infty$). Let (η_1^1, η_1^2) be a pair of paths constructed in exactly the same manner as (η_0^1, η_0^2) except starting from 1 rather than 0. We consequently have that the image under φ of the region between η_1^1 and η_1^2 is equal in distribution to $P(z)$ as described before the lemma statement. Since $\dim_{\mathscr{H}}(\eta_1^1 \cap \eta_1^2)$ is also almost surely given by the value in (5.5.1), the desired result follows. ☐

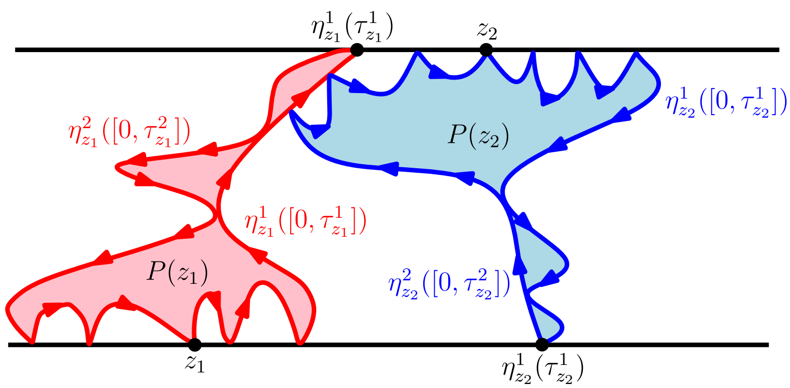


Figure 5.5.4: Suppose that we have the same setup as described in Figure 5.5.1. Shown is $P(z_1)$ where $z_1 \in \partial\mathbb{T}$ is fixed. The conditional law of η' given $P(z_1)$ is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4;\frac{\kappa'}{2}-4)$ in each of the components C of $\mathbb{T} \setminus P(z_1)$ starting from the first point of $\overline{C}$ hit by η' and exiting at the last. Fix z_2 on the boundary of a component C of $\mathbb{T} \setminus P(z_1)$. Then we can consequently form the set $P(z_2)$ which describes the interface between the set of points that η', viewed as a path in C, hits before and after hitting z_2. The intersection of the left and right boundaries of $P(z_2)$ consists of double points of η'. Moreover, the conditional law of η' given both $P(z_1)$ and $P(z_2)$ is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4;\frac{\kappa'}{2}-4)$ in each of the components of $\mathbb{T} \setminus (P(z_1) \cup P(z_2))$. Consequently, we can iterate this procedure to eventually explore the entire trajectory of η' (and, as we will explain in Lemma 5.5.2, the double points of η'). We will use this in Lemma 5.5.2 to reduce the double point dimension to computing the intersection dimension of GFF flow lines with an angle gap of θ_{double} (recall (5.1.10)).

Let $\mathscr{D}$ be the set of double points of η'. To complete the proof of Theorem 5.1.1, we will show that every double point of η' is in fact in some $P_\cap(z)$. To this end, we explore the trajectory of

η' as follows. Let $(d_j)_{j\in\mathbb{N}}$ be a sequence that traverses $\mathbb{N}\times\mathbb{N}$ in diagonal order, i.e. $d_1=(1,1)$, $d_2=(1,2)$, $d_3=(2,1)$, etc. Let $(z_{1,k})_{k\in\mathbb{N}}$ be a countable dense subset of $\partial\mathbb{T}$, and set $z_1=z_{d_1}$. Let $P(z_1)$ be the set which separates $\mathbb{T}$ into the set of points visited by η' before and after hitting z_1, as in Figure 5.5.1. We then let $(z_{2,k})_{k\in\mathbb{N}}$ be a countable dense subset of $\partial(\mathbb{T}\setminus P(z_1))$ and set $z_2=z_{d_2}$. Recall that the conditional law of η' given $P(z_1)$ is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4;\frac{\kappa'}{2}-4)$ process in each of the components of $\mathbb{T}\setminus P(z_1)$ — this is the same as the law of η' itself, up to conformal transformation. Consequently, once we have fixed $P(z_1)$, we define $P(z_2)$ analogously in terms of the segment of η' which traverses the component of $\mathbb{T}\setminus P(z_1)$ with z_2 on its boundary (see Figure 5.5.4). Generally, given $P(z_1),\ldots,P(z_n)$, we let $(z_{n+1,k})_{k\in\mathbb{N}}$ be a countable dense subset of $\partial(\mathbb{T}\setminus\cup_{j=1}^n P(z_j))$ and set $z_{n+1}=z_{d_{n+1}}$. The conditional law of η' given $P(z_1),\ldots,P(z_n)$ is independently that of an $\mathrm{SLE}_{\kappa'}(\frac{\kappa'}{2}-4;\frac{\kappa'}{2}-4)$ in each of the components of $\mathbb{T}\setminus\cup_{j=1}^n P(z_j)$. Thus given $P(z_1),\ldots,P(z_n)$, we define $P(z_{n+1})$ analogously in terms of the segment of η' which traverses the component which has z_{n+1} on its boundary. For each $n\in\mathbb{N}$, η' almost surely hits z_n only once at time $t(z_n)$. Moreover, from the construction, we have that $(t(z_n))_{n\in\mathbb{N}}$ is a dense set of times in $[0,\infty)$ (see [MS12c, Section 3.3]).

Lemma 5.5.2. *Almost surely,* $\mathscr{D}\subseteq\cup_{j=1}^\infty P_\cap(z_j)$.

Proof. For each $\omega\in\mathscr{D}$, let $t^f(\omega)$ and $t^\ell(\omega)$ be the first and last time that η' hits ω. For each $\delta>0$ we let $\mathscr{D}_\delta=\{\omega\in\mathscr{D}:t^\ell(\omega)-t^f(\omega)\geq\delta\}$. Clearly, the sets $\mathscr{D}_\delta$ increase as $\delta>0$ decreases and $\mathscr{D}=\cup_{\delta>0}\mathscr{D}_\delta$. Therefore it suffices to show that $\mathscr{D}_\delta\subseteq\cup_{n=1}^\infty P_\cap(z_n)$ for each $\delta>0$. Fix $\omega\in\mathscr{D}_\delta$ and consider $P(z_1)$. If $t^f(\omega)<t(z_1)<t^\ell(\omega)$, then $\omega\in P_\cap(z_1)$ and we stop the exploration. If $t(z_1)>t^\ell(\omega)$ or $t(z_1)<t^f(\omega)$, then ω is a double point of $\eta'|_{[0,t(z_1)]}$ or a double point of $\eta'|_{[t(z_1),\infty)}$, respectively. Consider $P(z_2)$. If $t^f(\omega)<t(z_2)<t^\ell(\omega)$, then $\omega\in P_\cap(z_2)$ and we stop the exploration. If $t(z_2)<t^f(\omega)$ or $t(z_2)>t^\ell(\omega)$, we continue the exploration. We continue to iterate this until the first k that $\omega\in P(z_k)$. To see that the exploration terminates after a finite number of steps, recall that $(t(z_n))_{n\in\mathbb{N}}$ is a dense set of times in $[0,\infty)$. In particular, letting

$$k=\min\left\{j\geq 1:t^f(\omega)<t(z_j)<t^\ell(\omega)\right\}$$

we have that $\omega\in P_\cap(z_k)$. □

We now have all of the ingredients to complete the proof of Theorem 5.1.1 for $\kappa'\in(4,8)$.

Proof of Theorem 5.1.1 for $\kappa'\in(4,8)$. Lemma 5.5.1 and Lemma 5.5.2 together imply that $\dim(\mathscr{D})=2-(12-\kappa')(4+\kappa')/(8\kappa')$ almost surely, as desired. □

We finish by proving Theorem 5.1.1 for $\kappa'\geq 8$.

Proof of Theorem 5.1.1 for $\kappa'\geq 8$. Fix $\kappa'\geq 8$ and let $\kappa=\frac{16}{\kappa'}\in(0,2]$. Let η' be an $\mathrm{SLE}_{\kappa'}$ process in $\mathbb{H}$ from 0 to ∞ and let $\mathscr{D}$ be the set of double points of η'. Then η' is space-filling [RS05]. For each point $z\in\mathbb{H}$, let $t(z)$ be the first time that η' hits z and let $\gamma(z)$ be the outer boundary of $\eta([0,t(z)])$. It follows from [MS13b, Theorem 1.1 and Theorem 1.13] and [Bef08] that the dimension of $\gamma(z)$ is equal to $1+\frac{\kappa}{8}=1+\frac{2}{\kappa'}$. Given $\gamma(z)$, $\eta'([t(z),\infty))$ is an $\mathrm{SLE}_{\kappa'}$ process in the remaining domain, and thus almost surely hits every point on $\gamma(z)$ except the point z. This implies that every point on $\gamma(z)$ except for z is contained in $\mathscr{D}$. This gives the lower bound for $\dim_{\mathscr{H}}(\mathscr{D})$.

Let $(z_k)_{k\in\mathbb{N}}$ be a countable dense set in $\mathbb{H}$. For the upper bound, we will show that every element of $\mathscr{D}$ is in fact on $\gamma(z_k)$ for some k. Note that $(t(z_k))_{k\in\mathbb{N}}$ is a dense set of times in $[0,\infty)$ because η'

is continuous. For each $\omega \in \mathscr{D}$, let $t^f(\omega)$ and $t^\ell(\omega)$ be the first and last times, respectively, that η' hits ω. For each $\delta > 0$, $\mathscr{D}_\delta = \{\omega \in \mathscr{D} : t^\ell(\omega) - t^f(\omega) \geq \delta\}$. Then $\mathscr{D} = \cup_{\delta>0}\mathscr{D}_\delta$. Since the sets $\mathscr{D}_\delta$ are increasing as $\delta > 0$ decreases, it suffices to show that $\mathscr{D}_\delta \subseteq \cup_k \gamma(z_k)$ for each $\delta > 0$. Fix $\delta > 0$ and $\omega \in \mathscr{D}_\delta$. Since $(t(z_k))_{k\in\mathbb{N}}$ is dense, we have that

$$k = \min\{j \geq 1 : t^\ell(\omega) > t(z_j) > t^f(\omega)\} < \infty.$$

Clearly, $\omega \in \gamma(z_k)$. This completes the proof for $\kappa' \geq 8$. □

Remark 5.5.3. *We note that* SLE'_κ *for* $\kappa' \in (4,8)$ *does not have triple points and, when* $\kappa' \geq 8$*, the set of triple points is countable. Indeed, to see this we note that if* z *is a triple point of an* SLE'_κ *process* η' *then there exists rational times* $t_1 < t_2$ *such that* z *is a single-point of and contained in the outer boundary of* $\eta'|_{[0,t_1]}$ *and a double point of and contained in the outer boundary of* $\eta'|_{[0,t_2]}$*. For each pair* $t_1 < t_2$ *there are precisely two points which satisfy these properties. The claim follows for* $\kappa' \in (4,8)$ *since* SLE'_κ *for* $\kappa' \in (4,8)$ *almost surely does not hit any given boundary point distinct from its starting point. The claim likewise follows for* $\kappa' \geq 8$ *because this describes a surjection from* $\mathbb{Q}_+ \times \mathbb{Q}_+$, $\mathbb{Q}_+ = (0,\infty) \cap \mathbb{Q}$*, to the set of triple points.*

Chapter 6

Radial Conformal Restriction

The results in this chapter are contained in [Wu13].

6.1 Introduction

The present paper is a write-up of the "radial" counterpart of some of the results derived in the "chordal" setting in the paper [LSW03] by Lawler, Schramm and Werner. The goal is to describe the laws of all random sets that satisfy a certain radial conformal restriction property.

Let us describe without further ado this property, and the main result of the present paper: Consider the unit disc $\mathbb{U}$ and we fix a boundary point 1 and an interior point the origin. We will study closed random subsets K of $\overline{\mathbb{U}}$ such that:

- K is connected, $\mathbb{C} \setminus K$ is connected, $K \cap \partial\mathbb{U} = \{1\}$, $0 \in K$.
- For any closed subset A of $\overline{\mathbb{U}}$ such that $A = \overline{\mathbb{U} \cap A}$, $\mathbb{U} \setminus A$ is simply connected, contains the origin and has 1 on the boundary, the law of $\Phi_A(K)$ conditioned on $(K \cap A = \emptyset)$ is equal to to law of K where Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves 1 and the origin (see Figure 6.1.1).

The law of such a set K is called a radial restriction measure, by analogy with the chordal restriction measures defined in [LSW03].

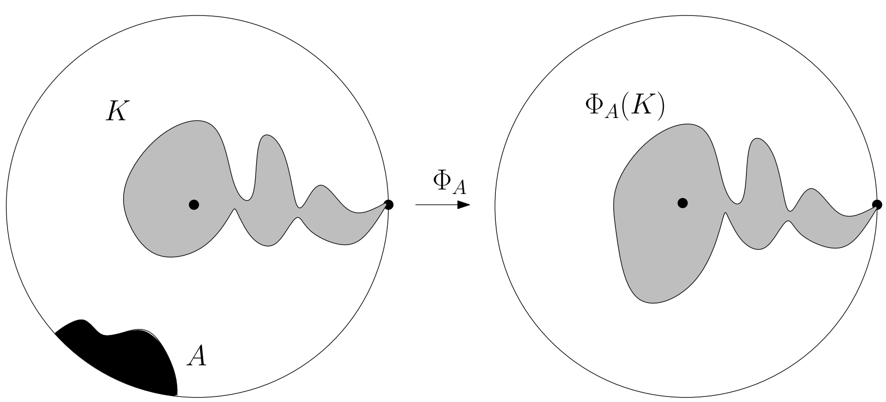


Figure 6.1.1: Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves 0 and 1. Conditioned on $(K \cap A = \emptyset)$, $\Phi_A(K)$ has the same law as K.

The main result of the present paper is the following classification and description of all radial restriction measures.

Theorem 6.1.1. *1. (Characterization). A radial restriction measure is fully characterized by a pair of real numbers* (α, β) *such that*

$$\mathbb{P}\left[K \cap A = \emptyset\right] = |\Phi_A'(0)|^{\alpha} \Phi_A'(1)^{\beta}$$

where A is any closed subset of $\overline{\mathbb{U}}$ *such that* $A = \overline{\mathbb{U} \cap A}$, $\mathbb{U} \setminus A$ *is simply connected, contains the origin and has 1 on the boundary, and* Φ_A *is the conformal map from* $\mathbb{U} \setminus A$ *onto* $\mathbb{U}$ *that preserves* 0 *and* 1. *We denote the corresponding radial restriction measure by* $\mathbb{P}(\alpha, \beta)$.

2. (Existence). The measure $\mathbb{P}(\alpha, \beta)$ *exists if and only if*

$$\beta \geq \frac{5}{8}, \quad \alpha \leq \xi(\beta) = \frac{1}{48}\left((\sqrt{24\beta+1}-1)^2 - 4\right).$$

We shall give an explicit construction of the measures $\mathbb{P}(\alpha, \beta)$ for all these admissible values of α and β. The function $\xi(\beta)$ is (as could be expected) the so-called disconnection exponent associated with the half-plane exponent β (see [LW00, LSW01a, LSW01b, LSW02]).

It is worth observing that $|\Phi_A'(0)| \geq 1$ and that $\Phi_A'(1) \leq 1$. In Theorem 6.1.1, we see that the value of β is necessarily positive (and that therefore $\Phi_A'(1)^{\beta} \leq 1$), but the value of α can be negative or positive (as long as $\alpha \leq \xi(\beta)$), so that $|\Phi_A'(0)|^{\alpha}$ can be greater than one (but of course, the product $|\Phi_A'(0)|^{\alpha}\Phi_A'(1)^{\beta}$ cannot be greater than one which is guaranteed by the condition $\alpha \leq \xi(\beta)$).

This theorem is the counterpart of the classification of chordal restriction measures in [LSW03] that we shall recall in the next section. It is worth noticing already that while the class of chordal conformal restriction measures was parametrized by a single parameter $\beta \geq 5/8$, the class of radial restriction samples is somewhat larger as it involves the additional parameter α. This can be rather easily explained by the fact that the radial restriction property is in a sense weaker than the chordal one. It involves an invariance property of the probability distribution under the action of the semi-group of conformal transformations that preserve both an inner point and a boundary point of the disc. In the chordal case, the semi-group of transformations were those maps that preserve two given boundary points (which is a larger family). Another way to see this is that the chordal restriction samples in the upper half-plane are scale-invariant, while the radial ones aren't. However, and this will be apparent in the latter part of the proof of Theorem 6.1.1, chordal restriction samples of parameter β can be viewed as limits of radial ones with parameters (α, β) (for all admissible α's), in the same way as chordal SLE can be viewed as the limit of radial SLE when the inner point converges to the boundary of the domain.

These results have been discussed and mentioned before, at least partially, in e-mail exchanges, lectures and discussions by a number of mathematicians, including of course Lawler, Schramm and Werner, and also Dubédat or Gruzberg. In fact, reference 31. in the paper [LSW03] written in 2003 by Lawler, Schramm and Werner is precisely a paper "in preparation" with the very same title as the present one. I wish to hereby thank Greg Lawler and Wendelin Werner for letting me write up the present paper and work out the details of the proofs.

6.2 Preliminaries

We now briefly recall some background material that will be needed in our proofs, concerning chordal or radial SLE and their $\mathrm{SLE}_\kappa(\rho)$ variants, Brownian loop-soups as well as chordal restriction measures. When K is a subset of $\mathbb{C}$ and $x \in \mathbb{C}$, we denote $x+K$ as the set $\{x+z : z \in K\}$ and xK as the set $\{xz : z \in K\}$.

6.2.1 Chordal Loewner chains and SLE

Suppose $(W_t, t \geq 0)$ is a real-valued continuous function. For each $z \in \overline{\mathbb{H}}$, define $g_t(z)$ as the solution to the chordal Loewner ODE:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

Write $\tau(z) = \sup\{t \geq 0 : \inf_{s\in[0,t]} |g_s(z) - W_s| > 0\}$ and $K_t = \{z \in \mathbb{H} : \tau(z) \leq t\}$. Then g_t is the unique conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ such that $|g_t(z) - z| \to 0$ as $z \to \infty$. And $(g_t, t \geq 0)$ is called the chordal Loewner chain generated by the driving function $(W_t, t \geq 0)$. In fact, we have $(g_t(z) - z)z \to 2t$ as $z \to \infty$.

SLE curves are introduced by Oded Schramm as candidates of scaling limits of discrete statistical physics models (see [Sch00]). A chordal SLE_κ is defined by the random family of chordal conformal maps g_t when $W = \sqrt{\kappa}B$ where B is a standard one-dimensional Brownian motion. It is proved that there exists a.s. a continuous curve η such that for each $t \geq 0$, $\mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \eta([0,t])$ (see [RS05]).

Chordal $\mathrm{SLE}_\kappa(\rho)$ processes are variants of SLE_κ process. For simplicity, we will here only describe the $\mathrm{SLE}_\kappa(\rho)$ processes with just one additional force point: It is the measure on the random family of conformal maps g_t generated by chordal Loewner chain with W_t replaced by the solution to the system of SDEs:

$$dW_t = \sqrt{\kappa} dB_t + \frac{\rho}{W_t - V_t} dt;$$
$$dV_t = \frac{2}{V_t - W_t} dt, \quad V_0 = x \neq 0, \quad (W_t - V_t)/(W_0 - V_0) \geq 0.$$

When $\kappa > 0, \rho > -2$, there is a unique solution to the above SDEs. The force point is repelling when ρ is positive while it is attracting when ρ is negative. There exists a.s. a continuous curve η in $\overline{\mathbb{H}}$ from 0 to ∞ associated to the $\mathrm{SLE}_\kappa(\rho)$ process (see [MS12a]).

In the limit when $x \to 0+$ (respectively $0-$), the process has a limit that is scale-invariant in distribution. This enables to define the corresponding $\mathrm{SLE}_\kappa(\rho)$ (referred to as $\mathrm{SLE}^R_\kappa(\rho)$ or $\mathrm{SLE}^L_\kappa(\rho)$ to indicate if the force-point is to the right or to the left of the driving point) from a boundary point of a simply connected domain to another by conformal invariance, just as for ordinary SLE_κ.

6.2.2 Chordal restriction samples

We now recall briefly some facts from [LSW03]. Consider the upper half plane $\mathbb{H}$ and we fix two boundary points 0 and ∞. A chordal restriction sample is a closed random subset of $\overline{\mathbb{H}}$ such that

- K is connected, $\mathbb{C} \setminus K$ is simply connected, $K \cap \mathbb{R} = \{0\}$, and K is unbounded.

- For any closed subset A of $\overline{\mathbb{H}}$ such that $A = \overline{\mathbb{H} \cap A}$, $\mathbb{H} \setminus A$ is simply connected, A is bounded and $0 \notin A$, the law of $\Psi_A(K)$ conditioned on $(K \cap A = \emptyset)$ is equal to the law of K where Ψ_A is any given conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ that preserves 0 and ∞.

Note that this second property in the case where $A = \emptyset$ shows that the law of K is scale-invariant (ie. that K and λK have the same distribution for any fixed positive λ). It is proved that the chordal restriction measures form a one-parameter family $(\mathbb{Q}_\beta)$, such that for all A as before,

$$\mathbb{Q}_\beta\left[K \cap A = \emptyset\right] = \Psi_A'(0)^\beta$$

where Ψ_A is the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ that preserves 0 and $\Psi_A(z)/z \to 1$ as $z \to \infty$ (see [LSW03]). In that paper, it is proved that the chordal conformal restriction measure $\mathbb{Q}_\beta$ exists if and only if $\beta \geq 5/8$.

We would like to make the following remarks that will be relevant for the present paper:

1. Chordal restriction samples can be defined in any simply connected domain $H \neq \mathbb{C}$ by conformal invariance (using the fact that their law in $\mathbb{H}$ is scale-invariant). For instance, if H is such a simply connected domain and z, w are two different boundary points, the chordal restriction sample in H connecting z and w is the image of chordal restriction sample in $\mathbb{H}$ under any given conformal map ϕ from $\mathbb{H}$ onto H that sends the double $(0, \infty)$ to (z, w).

2. In the proof of the construction of these (two-sided) chordal restriction samples, an important role is played by the related "right-sided chordal restriction samples", that we shall also use at some point in the present paper. These are a closed random subset K of $\overline{\mathbb{H}}$ such that

   - K is connected, $\mathbb{C} \setminus K$ is connected, $K \cap \mathbb{R} = (-\infty, 0]$.
   - For any closed subset A of $\overline{\mathbb{H}}$ such that $A = \overline{\mathbb{H} \cap A}$, $\mathbb{H} \setminus A$ is simply connected, A is bounded and $A \cap \mathbb{R} \subset (0, \infty)$, the law of $\Psi_A(K)$ conditioned on $(K \cap A = \emptyset)$ is equal to the law of K where Ψ_A is any conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ that preserves 0 and ∞.

   It is clear that the right boundary of chordal restriction sample is a right-sided restriction sample. In fact, there exists a one-parameter family $\mathbb{Q}_\beta^+$ such that

   $$\mathbb{Q}_\beta^+\left[K \cap A = \emptyset\right] = \Psi_A'(0)^\beta$$

   where Ψ_A is the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ that preserves 0 and $\Psi_A(z)/z \to 1$ as $z \to \infty$. $\mathbb{Q}_\beta^+$ exists if and only if $\beta \geq 0$. We usually ignore the trivial case $\beta = 0$ where $K = \mathbb{R}_-$.

   One example of right-sided restriction sample is given by $\mathrm{SLE}_{8/3}^L(\rho)$: Let η be such a process in $\overline{\mathbb{H}}$ from 0 to ∞. Let K be the closure of the union of domains between η and $\mathbb{R}_-$. Then K is a right-sided restriction sample with exponent $\beta = (\rho+2)(3\rho+10)/32$. Conversely, let K be a right-sided restriction sample with exponent $\beta > 0$, then the right boundary of K is an $\mathrm{SLE}_{8/3}^L(\rho)$ process with

   $$\rho = \rho(\beta) = \frac{2}{3}(\sqrt{24\beta + 1} - 1) - 2. \tag{6.2.1}$$

3. We have just seen the the right boundary of a two-sided restriction sample is an $\mathrm{SLE}^L_{8/3}(\rho)$ process. It is also possible to construct the conditional law of the left boundary given the right boundary: Denote L_r as the domain between $\mathbb{R}_-$ and the right boundary of K. Then, given this right boundary, the conditional law of the left boundary of K is an $\mathrm{SLE}^R_{8/3}(\rho-2)$ from 0 to ∞ in L_r (see [Wer04a]). In fact, we shall *construct* our radial restriction samples using the radial analogue of this recipe.

4. Let $C(K)$ be the cut point set of K i.e. the set of points x in K such that $K\setminus\{x\}$ is not connected. Note that $C(K)$ is the intersection of the right and left boundaries of K. It turns out that the right and left boundaries of K can be coupled with a Gaussian Free Field as two flow lines, which enables to prove (see [MW13, Theorem 1.5]) that the Hausdorff dimension of $C(K)$ is almost surely equal to $(25-u^2)/12$ where $u=\sqrt{24\beta+1}-1$, when $5/8\le\beta\le 35/24$, whereas $C(K)=\emptyset$ almost surely when $\beta>35/24$.

5. It is possible to describe the half-plane Brownian non-intersection exponents $\tilde{\xi}$ in terms of restriction measures. For instance, consider two independent chordal restriction samples K_1 and K_2 with exponent β_1,β_2 respectively. One can derive that, conditioned on $(K_1\cap K_2=\emptyset)$ (viewed as the limit of $K_1\cap(x+K_2)\cap B(0,R)=\emptyset$ as $x\to 0, R\to\infty$), the "inside" of $K_1\cup K_2$ has the same law as a chordal restriction sample of exponent $\tilde{\xi}(\beta_1,\beta_2)$.

6. It is possible to use restriction samples in order to describe the law of $\mathrm{SLE}_\kappa(\rho)$ processes as SLE_κ processes conditioned not to intersect a chordal restriction sample. For details, see [Wer04a, Equation (9),(10)].

6.2.3 Brownian loop soup

We now briefly recall some results from [LW04]. It is well known that Brownian motion in $\mathbb{C}$ is conformal invariant. Let us now define for all $t\ge 0$, the law $\mu_t(z,z)$ of the two-dimensional Brownian bridge of time-length t that starts and ends at t and define

$$\mu^{loop}=\int_{\mathbb{C}}\int_0^\infty dz\frac{dt}{t}\mu_t(z,z)$$

where dz is the Lebesgue measure in $\mathbb{C}$ that we view as a measure on *unrooted* loops modulo time-reparametrization (see [LW04]). Then, μ^{loop} inherits a striking conformal invariance property. More precisely, if for any subset $D\subset\mathbb{C}$, one defines the Brownian loop measure μ^{loop}_D in D as the restriction of μ^{loop} to the set of loops contained in D, then it is shown in [LW04]:

- For two domains $D'\subset D$, μ^{loop}_D restricted to the loops contained in D' is the same as $\mu^{loop}_{D'}$ (this is a trivial consequence of the definition of these measures).

- For two simply connected domains D_1,D_2, let Φ be a conformal map from D_1 onto D_2, then the image of $\mu^{loop}_{D_1}$ under Φ has the same law as $\mu^{loop}_{D_2}$ (this non-trivial fact is inherited from the conformal invariance of planar Brownian motion).

From these two properties, if we denote $\mu^0_{\mathbb{U}}$ as $\mu^{loop}_{\mathbb{U}}$ restricted to the loops surrounding the origin, then it is further noted in [Wer08] that

$$\mu^0_{\mathbb{U}}(\gamma\not\subset U)=\log\Phi'(0) \tag{6.2.2}$$

where U is any simply connected subset of $\mathbb{U}$ that contains the origin and Φ is the conformal map from U onto $\mathbb{U}$ that preserves the origin and $\Phi'(0) > 0$.

For $c > 0$, let $(\gamma_j, j \in J)$ be a Poisson point process with intensity $c\mu^0_{\mathbb{U}}$, then, from (6.2.2), we have that

$$\mathbb{P}\left[\gamma_j \subset U, \forall j \in J\right] = \exp\left(-c\mu^0_{\mathbb{U}}(\gamma \not\subset U)\right) = \Phi'(0)^{-c}$$

where U is any simply connected subset of $\mathbb{U}$ that contains the origin and Φ is the conformal map from U onto $\mathbb{U}$ that preserves the origin and $\Phi'(0) > 0$.

6.2.4 Radial Loewner chains and SLE

Suppose $(W_t, t \geq 0)$ is a real-valued continuous function. For each $z \in \overline{\mathbb{U}}$, define $g_t(z)$ as the solution to the radial Loewner ODE:

$$\partial_t g_t(z) = g_t(z)\frac{e^{iW_t} + g_t(z)}{e^{iW_t} - g_t(z)}, \quad g_0(z) = z.$$

Write $\tau(z) = \sup\{t \geq 0 : \inf_{s\in[0,t]} |g_s(z) - e^{iW_s}| > 0\}$ and $K_t = \{z \in \overline{\mathbb{U}} : \tau(z) \leq t\}$. Then g_t is the unique conformal map from $\mathbb{U} \setminus K_t$ onto $\mathbb{U}$ such that $g_t(0) = 0, g_t'(0) > 0$. And $(g_t, t \geq 0)$ is called the radial Loewner chain generated by the driving function $(W_t, t \geq 0)$. In fact, we have $g_t'(0) = e^t$.

Before introducing the radial SLE, let us first define some special Loewner chains that will be of use later on. We want to define a radial Loewner curves η such that, for any $t > 0$, the future part of the curve $\eta([t,\infty))$ under g_t is exactly η up to a rotation of the disc. Precisely, fix $\theta \in (0, 2\pi)$, define the driving function $W_t^\theta = \theta - t\cot\frac{\theta}{2}$. Let $(g_t, t \geq 0)$ be the radial Loewner chain generated by W^θ. And define $f_t(\cdot) = g_t(\cdot)/g_t(1)$. Then there exists a continuous curve η^θ started from $e^{i\theta}$ and ended at the origin such that g_t is the conformal map from $\mathbb{U} \setminus \eta^\theta([0,t])$ and $g_t(0) = 0, g_t'(0) = e^t$. From the radial Loewner ODE, we have that $g_t(1) = e^{i(W_t - \theta)}$, and $f_t(\eta^\theta(t)) = e^{i\theta}$. Further, for any $t, s > 0$, $f_t(\eta^\theta([t, t+s])) = \eta^\theta([0,s])$. We call η^θ as *perfect radial curve* started from $e^{i\theta}$. Note that

$$|f_t'(0)| = e^t, \quad f_t'(1) = \exp(-\frac{t}{1-\cos\theta}). \tag{6.2.3}$$

A radial SLE$_\kappa$ is defined by the random family of radial conformal maps g_t when $W = \sqrt{\kappa}B$ where B is a standard one-dimensional Brownian motion. It is proved that there exists a.s. a continuous curve η such that for each $t \geq 0$, $\mathbb{U} \setminus K_t$ is the connected component of $\mathbb{U} \setminus \eta([0,t])$ containing the origin (this is due to the absolute continuity relation between radial and chordal SLEs and the corresponding results for chordal SLEs).

Let us briefly focus on radial SLE$_{8/3}$. Let η be an SLE$_{8/3}$ in $\mathbb{U}$ from 1 to the origin. It is known (see [Law05, Section 6.5]) that

$$\mathbb{P}\left[\eta \cap A = \emptyset\right] = |\Phi_A'(0)|^{5/48}\Phi_A'(1)^{5/8} \tag{6.2.4}$$

where A is any closed subset of $\overline{\mathbb{U}}$ such that $A = \overline{\mathbb{U} \cap A}$, $\mathbb{U} \setminus A$ is simply connected, contains the origin and has 1 on the boundary; Φ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves the origin and the boundary point 1. This result follows from a standard martingale computation for radial SLE$_{8/3}$. This will ensure that the measure that we will call $\mathbb{P}(5/48, 5/8)$ does exist.

We will also make use of a radial version of SLE$_\kappa(\rho)$ processes. For simplicity, let us just define the radial SLE$_\kappa(\rho)$ process with only one force point. It is the measure on the random family of conformal maps g_t generated by radial Loewner chain with W_t replaced by the solution to the system of SDEs:

$$\begin{aligned} dW_t &= \sqrt{\kappa} dB_t + \frac{\rho}{2}\cot(\frac{W_t - V_t}{2})dt; \\ dV_t &= -\cot(\frac{W_t - V_t}{2})dt, \quad V_0 = x \in (0, 2\pi). \end{aligned} \tag{6.2.5}$$

When $\kappa > 0, \rho > -2$, there is a unique solution to the above SDEs. And there exists a.s. a continuous curve η in $\overline{\mathbb{U}}$ from 1 to 0 associated to the radial SLE$_\kappa(\rho)$ process [MS13b]. Note that, in the radial case, a right force point e^{ix} with $x \in (0, 2\pi)$ can also be viewed as a left force point $e^{i(2\pi - x)}$. Thus, different from the chordal case, we do not use the terminology of "left" and "right" force point for the radial case. Let $x \to 0+$ (resp. $x \to 2\pi-$), the process has a limit and we call this limit process as radial SLE$_\kappa(\rho)$ in $\overline{\mathbb{U}}$ from 1 to 0 with force point 1^+ (resp. 1^-).

6.3 Characterization

The present section will be devoted to the proof of the characterization part of our main theorem.

Let $\mathscr{A}^r$ be the set of all closed $A \subset \overline{\mathbb{U}}$ such that $A = \overline{A \cap \mathbb{U}}$, $\mathbb{U} \setminus A$ is simply connected, contains the origin and has 1 on the boundary. For any $A \in \mathscr{A}^r$, define Φ_A as the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ such that preserves 1 and the origin. We usually call $\log|\Phi'_A(0)|$ as the capacity of A in $\mathbb{U}$ seen from the origin. Generally, for any domain $U \subset \mathbb{C}$, a closed subset $A \subset \overline{U}$, and a point $z \in \mathbb{U} \setminus A$, the capacity of A in U seen from z is $\log \Phi'(z)$ where Φ is the conformal map from the connected component of $U \setminus A$ that contains z onto $\mathbb{U}$ and is normalized at $z : \Phi(z) = 0, \Phi'(z) > 0$.

Let Ω be the collection of closed subsets K of $\overline{\mathbb{U}}$ such that K is connected, $\mathbb{C} \setminus K$ is connected and $1 \in K, 0 \in K$. Endow Ω with the σ-field generated by the family of events of the type $\{K \in \Omega : K \cap A = \emptyset\}$ where $A \in \mathscr{A}^r$ (note that this σ-field coincides with the σ-field generated by Hausdorff metric on Ω, this is similar to the chordal case). It is clear that this family of events is closed under finite intersection, so that, just as in the chordal case, we know that:

Lemma 6.3.1. *If $\mathbb{P}$ and $\mathbb{P}'$ are two probability measures on Ω such that $\mathbb{P}[K \cap A = \emptyset] = \mathbb{P}'[K \cap A = \emptyset]$ for all $A \in \mathscr{A}^r$, then $\mathbb{P} = \mathbb{P}'$.*

It will be useful to use our perfect radial curves. The following fact is the analogue of the fact derived through [LSW03, Equation (3.1)]:

Lemma 6.3.2. *Fix $\theta \in (0, 2\pi)$ and let η^θ be the perfect radial curve started from $e^{i\theta}$. Let K be a radial restriction sample, then there exists $\nu(\theta) \in (0, \infty)$ such that, for all $t \geq 0$,*

$$\mathbb{P}[K \cap \eta^\theta([0,t]) = \emptyset] = \exp(-\nu(\theta)t).$$

Proof. (See Figure 6.3.1) Recall that f_t is the conformal map from $\mathbb{U} \setminus \eta^\theta([0,t])$ onto $\mathbb{U}$ such that $f_t(0) = 0, |f'_t(0)| = e^t, f_t(\eta^\theta(t)) = e^{i\theta}$ and we also have that $f_t(\eta^\theta([t, t+s])) = \eta^\theta([0,s])$ for any $t, s > 0$. Then, for any $t, s > 0$, by the property of radial restriction sample, we have that

$$\begin{aligned} &\mathbb{P}[K \cap \eta^\theta([0, t+s]) = \emptyset \,|\, K \cap \eta^\theta([0,t]) = \emptyset] \\ &= \mathbb{P}[K \cap f_t(\eta^\theta([t, t+s])) = \emptyset] = \mathbb{P}[K \cap \eta^\theta([0,s]) = \emptyset]. \end{aligned}$$

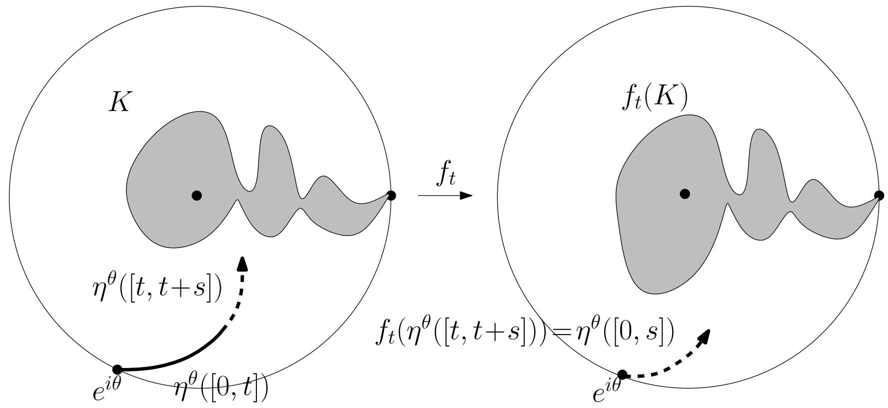


Figure 6.3.1: Conditioned on $(K\cap\eta^\theta([0,t])=\emptyset)$, $f_t(K)$ has the same law as K.

Thus, for any $t,s>0$, we have

$$\mathbb{P}\big[K\cap\eta^\theta([0,t+s])=\emptyset\big]=\mathbb{P}\big[K\cap\eta^\theta([0,t])=\emptyset\big]\times\mathbb{P}\big[K\cap\eta^\theta([0,s])=\emptyset\big].$$

This implies that

$$\mathbb{P}\big[K\cap\eta^\theta([0,t])\big]=\exp(-\nu(\theta)t)$$

for some $\nu(\theta)\in[0,\infty]$. If $\nu(\theta)=\infty$, then $K\cap\eta^\theta([0,t])\neq\emptyset$ a.s., for all $t>0$. However $\cap_{t>0}\eta^\theta([0,t])=\{e^{i\theta}\}$ and $e^{i\theta}\notin K$. This rules out the possibility of $\nu(\theta)=\infty$. If $\nu(\theta)=0$, then $K\cap\eta^\theta([0,\infty])=\emptyset$ a.s.. This is also impossible since $0\in K$ and η^θ ends at the origin. □

We would like to note at this point that in the chordal case, the analogous quantity was obviously constant because of scale-invariance of the chordal restriction measures in the upper half-plane. In the present radial case, this is not going to be the case. In particular, care will be needed to show that $\theta\mapsto\nu(\theta)$ is continuously differentiable.

We are now ready to prove the first part of Theorem 6.1.1 that we now state as a Proposition:

Proposition 6.3.3. *For any radial restriction sample K, there exist $\alpha,\beta\in\mathbb{R}$ such that*

$$\mathbb{P}\big[K\cap A=\emptyset\big]=|\Phi_A'(0)|^\alpha\Phi_A'(1)^\beta\quad \textit{for all}\quad A\in\mathscr{A}^r.$$

Note that Lemma 6.3.1 conversely shows that for any α and β, there exists at most one law (for K) that satisfies this property. When it exists, we call it $\mathbb{P}(\alpha,\beta)$. An example is provided by radial $\text{SLE}_{8/3}$ (see Equation (6.2.4)) that corresponds to $\mathbb{P}(5/48,5/8)$.

The main part of the proof of the proposition will be devoted to show that $\theta\mapsto\nu(\theta)$ is a continuously differentiable function. Once this will have been established, it will be possible to use "commutation relation ideas" inspired by the formal calculations in [LSW03] and by Dubédat's paper [Dub07].

In order to prove this proposition, it will in fact be a little easier to work in the upper half plane instead of the unit disc. Consider the conformal map $\varphi_0(z)=i(1-z)/(1+z)$ which maps $\mathbb{U}$ onto $\mathbb{H}$ and sends 1 to 0, 0 to i. A radial restriction sample in $\mathbb{H}$ (with specified points 0 and i) is just the image of radial restriction sample in $\mathbb{U}$ under the conformal map φ_0. For $x\in\mathbb{C},r>0$, We denote $B(x,r)$ as the disc centered at x with radius r.

Fix $x \in \mathbb{R} \setminus \{0\}$, let $0 < \varepsilon < |x|$. Then

$$g_{x,\varepsilon}(z) := z + \frac{\varepsilon^2}{z - x}$$

is a conformal map from $\mathbb{H} \setminus B(x,\varepsilon)$ onto $\mathbb{H}$. Define

$$f_{x,\varepsilon}(z) = b \frac{g_{x,\varepsilon}(z) - c}{b^2 + (c-a)(g_{x,\varepsilon}(z) - a)}$$

where $a = \Re(g_{x,\varepsilon}(i)), b = \Im(g_{x,\varepsilon}(i)), c = g_{x,\varepsilon}(0)$. Then $f_{x,\varepsilon}$ is the conformal map from $\mathbb{H} \setminus B(x,\varepsilon)$ onto $\mathbb{H}$ that preserves 0 and i.

Lemma 6.3.4. *Let K be a radial restriction sample in $\mathbb{H}$. For any $x \in \mathbb{R} \setminus \{0\}$, the following limit exists*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{P}\left[K \cap B(x,\varepsilon) \neq \emptyset\right].$$

We denote the limit as $\lambda(x)$, we have further that $\lambda(x) \in (0,\infty)$.

Proof. Fix $x \in (0,\infty)$ and let $\theta \in (0,\pi)$ such that $x = \sin\theta/(1+\cos\theta)$. Let η^x be the perfect radial curve in $\mathbb{H}$ started from x and ended at i which is the image of the perfect radial curve in $\mathbb{U}$ started from $e^{i\theta}$ and ended at the origin under the conformal map φ_0.

For $\varepsilon > 0$, define $N(\varepsilon) = \lceil \varepsilon^{-2} \rceil$. And $\varphi_1 = \cdots = \varphi_N = f_{x,\varepsilon}$. Let $\Phi_\varepsilon = \varphi_{N(\varepsilon)} \circ \cdots \circ \varphi_1$. Note that Φ_ε is a conformal map from $H := \varphi_1^{-1} \circ \cdots \circ \varphi_{N(\varepsilon)}^{-1}(\mathbb{H})$ onto $\mathbb{H}$ that preserves i and 0. Define $A_x(\varepsilon) = \overline{\mathbb{H} \setminus H}$ (see Figure 6.3.2). Then it is clear that,

$$A_x(\varepsilon) \supset \eta^x([0,t_x]), \quad \text{and} \quad A_x(\varepsilon) \to \eta^x([0,t_x]) \quad \text{as} \quad \varepsilon \to 0$$

where $t_x = (1+\cos\theta)^2$ by direct computation of the capacity of $A_x(\varepsilon)$ in $\mathbb{H}$ seen from i. And the convergence is under Hausdorff metric.

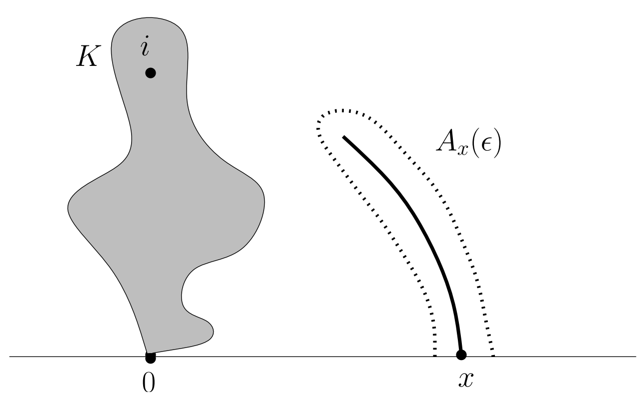


Figure 6.3.2: $A_x(\varepsilon)$ converges to $\eta^x([0,t_x])$ in Hausdorff metric.

Define $p_x(\varepsilon) = \mathbb{P}\left[K \cap B(x,\varepsilon) \neq \emptyset\right]$. On the one hand, from conformal restriction property, we know that

$$\mathbb{P}\left[K \cap A_x(\varepsilon) = \emptyset\right] = (1 - p_x(\varepsilon))^{N(\varepsilon)}.$$

On the other hand, we know that

$$\mathbb{P}\left[K \cap A_x(\varepsilon) = \emptyset\right] \to \mathbb{P}\left[K \cap \eta^x([0,t_x]) = \emptyset\right] = \exp(-\nu(\theta)t_x) \quad \text{as} \quad \varepsilon \to 0.$$

Compare these two relations, we have that

$$\lim_{\varepsilon\to 0} N(\varepsilon)\log(1-p_x(\varepsilon)) = -\nu(\theta)(1+\cos\theta)^2.$$

This completes the proof. And we further know that

$$\lambda(\frac{\sin\theta}{1+\cos\theta}) = \nu(\theta)(1+\cos\theta)^2. \quad (6.3.1)$$

□

Lemma 6.3.5. *The function λ defined in Lemma 6.3.4 is continuous for $x \in (-\infty,0)\cup(0,\infty)$.*

Proof. Suppose $0 < x_L < x_R < \infty$. It is enough to prove the continuity of λ in the compact interval $I := [x_L, x_R]$.

We first analyze the convergence in Lemma 6.3.4 and use the same notations as in the proof of Lemma 6.3.4. Since $A_x(\varepsilon) \supset \eta^x([0,t_x])$, we have that

$$\mathbb{P}\big[K\cap A_x(\varepsilon)=\emptyset\big] \le \mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset\big]$$

which implies that $(1-p_x(\varepsilon))^{N(\varepsilon)} \le e^{-\lambda(x)}$. We will show that there exists a universal upper bound for $e^{-\lambda(x)} - (1-p_x(\varepsilon))^{N(\varepsilon)}$. Note that

$$\begin{aligned}
& e^{-\lambda(x)} - (1-p_x(\varepsilon))^{N(\varepsilon)} \\
&= \mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset, K\cap A_x(\varepsilon)\neq\emptyset\big] \\
&= \mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset\big]\times\mathbb{P}\big[K\cap A_x(\varepsilon)\neq\emptyset \,|\, K\cap\eta^x([0,t_x])=\emptyset\big] \\
&= \mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset\big]\times\mathbb{P}\big[K\cap F_x(A_x(\varepsilon))\neq\emptyset\big] \\
&\le \mathbb{P}\big[K\cap F_x(A_x(\varepsilon))\neq\emptyset\big]
\end{aligned}$$

where F_x is the conformal map from $\mathbb{H}\setminus\eta^x([0,t_x])$ onto $\mathbb{H}$ that fixes 0 and i. Note that, the set $F_x(A_x(\varepsilon))$ is continuous in x and ε (in Hausdorff metric). There exist interval J and universal constant $c>0$ such that $F_x(A_x(\varepsilon)) \subset J^{c\varepsilon}$ for any $x\in\tilde{I} := [x_L/2, 2x^R]$, where J depends only on I, and $c>0$ depends on I, and J^δ denotes the δ neighborhood of J. Then we can see further that there exists $r(\varepsilon)>0$, depending on I and ε with the property that $r(\varepsilon)\to 0$ as $\varepsilon\to 0$, such that

$$\mathbb{P}\big[K\cap F_x(A_x(\varepsilon))\neq\emptyset\big] \le r(\varepsilon) \quad \text{for any} \quad x\in\tilde{I}.$$

So that we have the uniform bound for $x\in\tilde{I}$,

$$0 \le e^{-\lambda(x)} - (1-p_x(\varepsilon))^{N(\varepsilon)} \le r(\varepsilon),$$

or equivalently

$$\lambda(x) \le -N(\varepsilon)\log(1-p_x(\varepsilon)) \le -\log(e^{-\lambda(x)} - r(\varepsilon)). \quad (6.3.2)$$

For any $x\in I$, let $\rho\in(0,1)$, and $y\in(x-\rho\varepsilon, x+\rho\varepsilon)$, it is clear that

$$B(y,(1-\rho)\varepsilon) \subset B(x,\varepsilon) \subset B(y,(1+\rho)\varepsilon).$$

This implies that

$$p_y((1-\rho)\varepsilon) \le p_x(\varepsilon) \le p_y((1+\rho)\varepsilon),$$

or equivalently,

$$-N(\varepsilon)\log\left(-p_y((1-\rho)\varepsilon)\right)\le -N(\varepsilon)\log\left(1-p_x(\varepsilon)\right)\le -N(\varepsilon)\log\left(1-p_y((1+\rho)\varepsilon)\right).$$

By (6.3.2), we have that

$$\frac{N(\varepsilon)}{N((1-\rho)\varepsilon)}\lambda(y)\le -N(\varepsilon)\log\left(1-p_x(\varepsilon)\right)\le -\frac{N(\varepsilon)}{N((1+\rho)\varepsilon)}\log\left(e^{-\lambda(y)}-r(\varepsilon)\right).$$

Furthermore,

$$\frac{N(\varepsilon)}{N((1-\rho)\varepsilon)}\sup\{\lambda(y):|x-y|\le\rho\varepsilon\}\le -N(\varepsilon)\log\left(1-p_x(\varepsilon)\right)$$
$$\le \frac{N(\varepsilon)}{N((1+\rho)\varepsilon)}\inf\{-\log\left(e^{-\lambda(y)}-r(\varepsilon)\right):|x-y|\le\rho\varepsilon\}.$$

Let $\varepsilon\to 0$, we have that, for any $\rho\in(0,1)$,

$$(1-\rho)^2\lim_{\varepsilon\to 0}\sup\{\lambda(y):|x-y|\le\rho\varepsilon\}\le\lambda(x), \tag{6.3.3}$$

and

$$\lambda(x)\le(1+\rho)^2\liminf_{\varepsilon\to 0}\inf\{-\log\left(e^{-\lambda(y)}-r(\varepsilon)\right):|x-y|\le\rho\varepsilon\}. \tag{6.3.4}$$

Consider (6.3.3), since it is true for any $\rho\in(0,1)$, it particularly implies that

$$\lambda(x)\ge\limsup_{y\to x}\lambda(y). \tag{6.3.5}$$

If λ is not continuous at x, by (6.3.5), we know that there exist $\delta>0$ and a sequence $y_k\to x$ such that

$$\lambda(y_k)\le\lambda(x)-\delta.$$

However, this contradicts with (6.3.4).

□

Fix $x,y\in\mathbb{R}\setminus\{0\}$, Define

$$F(x,y)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}(f_{x,\varepsilon}(y)-y),\quad G(x,y)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}(f'_{x,\varepsilon}(y)-1).$$

By direct computation, we have that

$$F(x,y)=\frac{1+x^2+y^2+xy}{x(1+x^2)}+\frac{1}{y-x},\quad G(x,y)=\frac{x+2y}{x(1+x^2)}-\frac{1}{(y-x)^2}. \tag{6.3.6}$$

We use the notation $f\lesssim g$ to imply f/g is bounded by universal constant, $f\gtrsim g$ to imply $g\lesssim f$, and $f\asymp g$ to imply $f\lesssim g$ and $f\gtrsim g$.

Lemma 6.3.6. *The function λ defined in Lemma 6.3.4 is differentiable in $x\in(-\infty,0)\cup(0,\infty)$ and satisfies the following commutation relation: for any $x,y\in\mathbb{R}\setminus\{0\}$,*

$$\lambda'(y)F(x,y)+2\lambda(y)G(x,y)=\lambda'(x)F(y,x)+2\lambda(x)G(y,x). \tag{6.3.7}$$

Proof. Let $I\subset(-\infty,0)\cup(0,\infty)$ be a compact interval. By the continuity of λ and (6.3.2), we have that

$$p_x(\varepsilon):=\mathbb{P}\big[K\cap B(x,\varepsilon)\neq\emptyset\big]\asymp\varepsilon^2 \tag{6.3.8}$$

where the constant in $\asymp$ depends only on I.

We will first show that λ is Lipschitz continuous on I. For $x,y\in I$. Recall that

$$e^{-\lambda(x)}=\mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset\big].$$

So that

$$\begin{aligned}
&e^{-\lambda(x)}-e^{-\lambda(y)}\\
&= \mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset\big]-\mathbb{P}\big[K\cap\eta^y([0,t_y])=\emptyset\big]\\
&\le \mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset,K\cap\eta^y([0,t_y])\neq\emptyset\big]\\
&= \mathbb{P}\big[K\cap\eta^x([0,t_x])=\emptyset\big]\times\mathbb{P}\big[K\cap F_x(\eta^y([0,t_y]))\neq\emptyset\big]\\
&= e^{-\lambda(x)}\mathbb{P}\big[K\cap F_x(\eta^y([0,t_y]))\neq\emptyset\big]
\end{aligned}$$

where, recall that, F_x is the conformal map from $\mathbb{H}\setminus\eta^x([0,t_x])$ onto $\mathbb{H}$ that fixes 0 and i. So that

$$1-\exp(\lambda(x)-\lambda(y))\le\mathbb{P}\big[K\cap F_x(\eta^y([0,t_y]))\neq\emptyset\big]. \tag{6.3.9}$$

As we explained in the proof of Lemma 6.3.5, there exist a compact interval J depending on I and $c>0$ depending on I, such that $F_x(\eta^y([0,t_y]))\subset J^{c\varepsilon}$ as long as $|y-x|\le\varepsilon$. Consider $J^{c\varepsilon}$, there exists a number N depending on I such that $J^{c\varepsilon}$ can be covered by N/ε balls of radius $2c\varepsilon$. Together with (6.3.8), we have that

$$\mathbb{P}\big[K\cap F_x(\eta^y([0,t_y]))\neq\emptyset\big]\le\mathbb{P}\big[K\cap J^{c\varepsilon}\neq\emptyset\big]\lesssim\varepsilon\quad\text{when}\quad|x-y|\le\varepsilon \tag{6.3.10}$$

where the constant in $\lesssim$ depends only on I. Combine this relation and (6.3.9), we have that

$$|\lambda(x)-\lambda(y)|\lesssim|x-y|$$

where the constant in $\lesssim$ depends only on I.

Since λ is locally Lipschitz continuous in $\mathbb{R}\setminus\{0\}$, it is differentiable almost everywhere, i.e. it is differentiable except on a Lebesgue measure zero set. And there exists an integrable function ω such that, $\lambda'(x)=\omega(x)$ at the point x at which λ is differentiable, and, for any $x>y>0$ (or $y<x<0$),

$$\lambda(x)-\lambda(y)=\int_y^x\omega(u)du.$$

Consider two points x,y at which λ is differentiable. Let $\varepsilon>0,\delta>0$.

$$\begin{aligned}
&\mathbb{P}\big[K\cap B(x,\varepsilon)=\emptyset,K\cap B(y,\delta)=\emptyset\big]\\
&= \mathbb{P}\big[K\cap B(x,\varepsilon)=\emptyset\big]\times\mathbb{P}\big[K\cap f_{x,\varepsilon}(B(y,\delta))=\emptyset\big]\\
&= 1-p_x(\varepsilon)-\mathbb{P}\big[K\cap f_{x,\varepsilon}(B(y,\delta))\neq\emptyset\big]+p_x(\varepsilon)\mathbb{P}\big[K\cap f_{x,\varepsilon}(B(y,\delta))\neq\emptyset\big].
\end{aligned}$$

So that,

$$\begin{aligned}
&\mathbb{P}\big[K\cap B(x,\varepsilon)=\emptyset,K\cap B(y,\delta)=\emptyset\big]-1+p_x(\varepsilon)+p_y(\delta)\\
&= p_y(\delta)-\mathbb{P}\big[K\cap f_{x,\varepsilon}(B(y,\delta))\neq\emptyset\big]+p_x(\varepsilon)\mathbb{P}\big[K\cap f_{x,\varepsilon}(B(y,\delta))\neq\emptyset\big].
\end{aligned}$$

Divide by $\varepsilon^2\delta^2$ and take the limit, we have that

$$\begin{aligned}
&\lim_{\varepsilon,\delta\to 0}\frac{1}{\varepsilon^2\delta^2}\left(\mathbb{P}\left[K\cap B(x,\varepsilon)=\emptyset, K\cap B(y,\delta)=\emptyset\right]-1+p_x(\varepsilon)+p_y(\delta)\right)\\
=&\lim_{\varepsilon,\delta\to 0}\frac{1}{\varepsilon^2\delta^2}\left(p_y(\delta)-\mathbb{P}\left[K\cap f_{x,\varepsilon}(B(y,\delta))\neq\emptyset\right](1-p_x(\varepsilon))\right)\\
=&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left(\lambda(y)-\lambda(f_{x,\varepsilon}(y))|f'_{x,\varepsilon}(y)|^2(1-p_x(\varepsilon))\right)\\
=&\ \lambda(x)\lambda(y)-\lambda'(y)F(x,y)-2\lambda(y)G(x,y).
\end{aligned}$$

By the symmetry, we get (6.3.7) for the points x,y at which λ is differentiable.

Fix y in (6.3.7), we have

$$\lambda'(x)=(\lambda'(y)F(x,y)+2\lambda(y)G(x,y)-2\lambda(x)G(y,x))/F(y,x).$$

The right side is continuous in $x\in\mathbb{R}\setminus\{0,y\}$. Thus we can extend ω to $\mathbb{R}\setminus\{0,y\}$ by the right side. Then it is clear that ω is a continuous function in $\mathbb{R}\setminus\{0\}$ and in particular, this implies that λ is differentiable everywhere in $\mathbb{R}\setminus\{0\}$ and the derivative satisfies (6.3.7) for any points $x,y\in\mathbb{R}\setminus\{0\}$. □

Lemma 6.3.7. *There exist two constants* $c_0,c_2\geq 0$ *such that*

$$\lambda(x)=\frac{c_0+c_2x^2}{x^2(1+x^2)^2}\quad \textit{for}\quad x\in\mathbb{R}\setminus\{0\}.$$

Proof. From (6.3.7) and (6.3.6), we know that λ is smooth in $(-\infty,0)\cup(0,+\infty)$. In (6.3.7), fix $x\in\mathbb{R}\setminus\{0\}$, and let $y\to x$. Compare the coefficients of the two sides of the equation, we have that

$$\begin{aligned}
&x^2(1+x^2)^2\lambda'''(x)+6x(1+x^2)(1+3x^2)\lambda''(x)\\
&\qquad+6(1+12x^2+15x^4)\lambda'(x)+24x(2+5x^2)\lambda(x)=0.
\end{aligned}\tag{6.3.11}$$

Set $P(x)=x^2(1+x^2)^2\lambda(x)$, then (6.3.11) is equivalent to

$$P(x)'''=0.$$

Together with the symmetry in λ, we know that, there exist constants c_0,c_1,c_2 such that

$$\lambda(x)=\frac{c_0+c_1x+c_2x^2}{x^2(1+x^2)^2}\text{ for }x>0;\quad \lambda(x)=\frac{c_0-c_1x+c_2x^2}{x^2(1+x^2)^2}\text{ for }x<0.$$

Take $x>0>y$, by (6.3.7), we have that $c_1=0$. Since $\lambda(x)>0$ for all $x\in\mathbb{R}\setminus\{0\}$, we know that $c_0\geq 0, c_2\geq 0$. □

Proof of Proposition 6.3.3. Consider a radial restriction sample K in $\mathbb{U}$. Fix $\theta\in(0,\pi)$, let $\nu(\theta)$ be defined through Lemma 6.3.2. And let λ be defined through Lemma 6.3.4. From Lemma 6.3.7 and (6.3.1), we have that

$$\nu(\theta)=-\alpha+\frac{\beta}{1-\cos\theta}$$

where $\alpha=(c_0-c_2)/4, \beta=c_0/2$. Recall (6.2.3), we have that

$$\mathbb{P}\left[K\cap\eta^\theta([0,t])=\emptyset\right]=|f'_t(0)|^\alpha f'_t(1)^\beta.$$

Then the conclusion can be derived by similar explanation in the proof of [LSW03, Proposition 3.3]. □

6.4 Admissible range of (α,β)

6.4.1 Description of $\mathbb{P}(\alpha,\beta)$'s when $\beta \geq 5/8$

In order to complete the proof of our main theorem, it now remains to show for which values of α and β the previous measure exists. Note now that from the properties of Poisson point process of Brownian loops, we can deduce the following fact:

Lemma 6.4.1. *If the radial restriction measure $\mathbb{P}(\alpha_0,\beta_0)$ exists for some $\alpha_0,\beta_0 \in \mathbb{R}$, then for any $\alpha < \alpha_0$, $\mathbb{P}(\alpha,\beta_0)$ exists, and furthermore, almost surely for $\mathbb{P}(\alpha,\beta_0)$, the origin is not on the boundary of K.*

Proof. Let K_0 be a closed set sampled according to $\mathbb{P}(\alpha_0,\beta_0)$, and let $(\gamma_j, j\in J)$ be an independent Poisson Point Process with intensity $(\alpha_0-\alpha)\mu^0_{\mathbb{U}}$. We view each loop γ_j as the loop with the domain that it surrounds. Then let K be the closure of the union of K_0 and all loops in $(\gamma_j, j\in J)$. We have that, for any $A\in\mathscr{A}^r$,

$$\begin{aligned}
&\mathbb{P}\big[K\cap A=\emptyset\big]\\
&= \mathbb{P}\big[K_0\cap A=\emptyset\big]\times\mathbb{P}\big[\gamma_j\cap A=\emptyset,\forall j\in J\big]\\
&= |\Phi_A'(0)|^{\alpha_0}\Phi_A'(1)^{\beta_0}|\Phi_A'(0)|^{\alpha-\alpha_0}=|\Phi_A'(0)|^{\alpha}\Phi_A'(1)^{\beta_0}.
\end{aligned}$$

It is clear that K has the law of $\mathbb{P}(\alpha,\beta_0)$ and the $0\notin\partial K$. □

Hence, we have the following result:

Corollary 6.4.2. *Suppose that a radial restriction measure $\mathbb{P}(\alpha_0,\beta_0)$ exists for some $\alpha_0,\beta_0\in\mathbb{R}$, and that for this measure, $0\in\partial K$ almost surely. Then, $\mathbb{P}(\alpha,\beta_0)$ does exist if and only if $\alpha\le\alpha_0$.*

Proof. Suppose that $\mathbb{P}(\alpha,\beta_0)$ exists for some $\alpha>\alpha_0$, then the previous lemma implies that $\mathbb{P}(\alpha_0,\beta_0)$ almost surely, $0\notin\partial K$, which is a contradiction. On the other hand, the same lemma shows that $\mathbb{P}(\alpha,\beta_0)$ exists for all $\alpha<\alpha_0$. □

In (6.2.4), we already know the existence of $\mathbb{P}(\xi(\beta),\beta)$ for $\beta=5/8$. We will construct $\mathbb{P}(\xi(\beta),\beta)$ for $\beta>5/8$ in Proposition 6.4.4. Fix $\rho>0$. Let $(g_t,t\ge 0)$ be the radial Loewner chain $\mathrm{SLE}_{8/3}(\rho)$ generated by the driving function $(W_t,t\ge0)$, and η be the corresponding radial curve. Recall that W is the solution to the system of SDEs (6.2.5). To simplify notation, we denote $\theta_t=(W_t-V_t)/2$. For any $A\in\mathscr{A}^r$, let τ_A be the first time that η hits A. For any $t<\tau_A$, let h_t be the conformal map from $\mathbb{U}\setminus g_t(A)$ onto $\mathbb{U}$ such that $h_t(0)=0, h_t(e^{iW_t})=e^{iW_t}$. Then

Lemma 6.4.3.

$$M_t := \exp\left(\alpha(\int_0^t ds|h_s'(e^{iW_s})|^2-t)\right)\times|h_t'(e^{iW_t})|^{\frac{5}{8}}\times|h_t'(e^{iV_t})|^{\gamma}\times Z_t^{\frac{3}{8}\rho} \tag{6.4.1}$$

is a local martingale where

$$Z_t=\frac{\sin\vartheta_t}{\sin\theta_t},\quad \vartheta_t=\frac{1}{2}\arg(h_t(e^{iW_t})/h_t(e^{iV_t})),$$

$$\alpha=\frac{5}{48}+\frac{3}{64}\rho(\rho+4),\quad \gamma=\frac{1}{32}\rho(3\rho+4),\quad \beta=\frac{5}{8}+\gamma+\frac{3}{8}\rho=\frac{1}{32}(\rho+2)(3\rho+10).$$

Note that $\alpha=\xi(\beta)$.

Proof. Define $\phi_t(z) = -i\log h_t(e^{iz})$ where log denotes the branch of the logarithm such that $-i\log h_t(e^{iW_t}) = W_t$. Then

$$|h_t'(e^{iW_t})| = \phi_t'(W_t), \quad |h_t'(e^{iV_t})| = \phi_t'(V_t), \quad \vartheta_t = (\phi_t(W_t) - \phi_t(V_t))/2.$$

To simplify the notations, we set $X_1 = \phi_t'(W_t), X_2 = \phi_t''(W_t), Y_1 = \phi_t'(V_t)$. By Itô formula, we have that

$$\begin{aligned}
d\phi_t(W_t) &= \sqrt{8/3}X_1 dB_t + \left(-\frac{5}{3}X_2 + \frac{\rho}{2}X_1\cot\theta_t\right)dt,\\
d\phi_t(V_t) &= -X_1^2\cot\vartheta_t dt,\\
d\phi_t'(W_t) &= \sqrt{8/3}X_2 dB_t + \left(\frac{\rho}{2}X_2\cot\theta_t + \frac{X_2^2}{2X_1} + \frac{X_1 - X_1^3}{6}\right)dt,\\
d\phi_t'(V_t) &= \left(-\frac{1}{2}X_1^2 Y_1\frac{1}{\sin^2\vartheta_t} + \frac{1}{2}Y_1\frac{1}{\sin^2\theta_t}\right)dt,\\
d\theta_t &= \frac{\sqrt{8/3}}{2}dB_t + \frac{\rho+2}{4}\cot\theta_t dt,\\
d\vartheta_t &= \frac{\sqrt{8/3}}{2}X_1 dB_t + \left(-\frac{5}{6}X_2 + \frac{1}{2}X_1^2\cot\vartheta_t + \frac{\rho}{4}X_1\cot\theta_t\right)dt.
\end{aligned}$$

So that

$$dM_t = \frac{\sqrt{8/3}}{16}M_t\left(10\frac{X_2}{X_1} + 3\rho(X_1\cot\vartheta_t - \cot\theta_t)\right)dB_t.$$

□

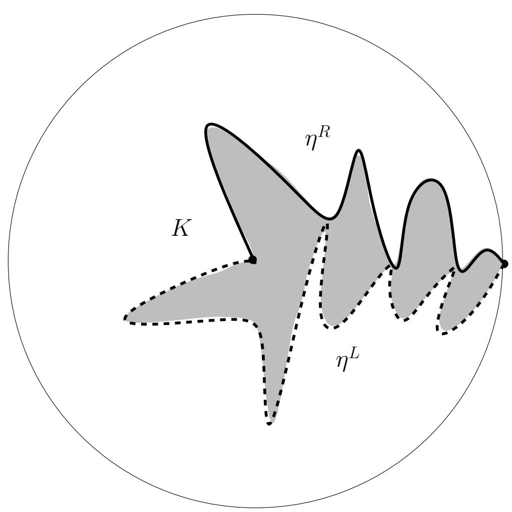


Figure 6.4.1: η^R is a radial $\mathrm{SLE}_{8/3}(\rho)$ in $\mathbb{U}$ from 1 to 0. Conditioned on η^R, η^L is a chordal $\mathrm{SLE}^R_{8/3}(\rho-2)$ in $\mathbb{U}\setminus\eta^R([0,\infty])$ from 1 to 0. K is the closure of the union of domains between the two curves.

Proposition 6.4.4. *For $\beta > 5/8$, let $\rho = \frac{2}{3}(\sqrt{24\beta+1}-1)-2 > 0$. Let η^R be a radial $SLE_{8/3}(\rho)$ in $\overline{\mathbb{U}}$ from 1 to 0 with force point 1^-. Given η^R, let η^L be an independent chordal $SLE^R_{8/3}(\rho-2)$*

in $\mathbb{U}\setminus\eta^R([0,\infty])$ from 1^- to 0. Define K as the closure of the union of the domains between η^R and η^L. Then the law of K is $\mathbb{P}(\xi(\beta),\beta)$ (that therefore exists) and under this probability measure, $0\in\partial K$ almost surely.

Hence, this proves that for $\beta\geq 5/8$, $\mathbb{P}(\alpha,\beta)$ exists if and only if $\alpha\leq\xi(\beta)$.

Proof. (See Figure 6.4.1) Let $(g_t,t\geq 0)$ be the radial Loewner chain for η^R. For any $A\in\mathscr{A}$, let τ_A be the first time that η^R hits A. For any $t<\tau_A$, define h_t as the conformal map from $\mathbb{U}\setminus g_t(A)$ onto $\mathbb{U}$ such that $h_t(0)=0, h_t(e^{iW_t})=e^{iW_t}$. Define the local martingale M as in (6.4.1). When $\rho>0$, M_t is positive and bounded by 1. Thus it is a real martingale. Note that

$$M_0=|\Phi_A'(0)|^{\xi(\beta)}\Phi_A'(1)^{\beta}.$$

If $\tau_A<\infty$, then there exists a sequence $t_n\to\tau_A$, such that $\lim_n M_{t_n}=0$.
If $\tau_A=\infty$, then there exists a sequence $t_n\to\infty$, such that (see [Wer04a, Section 5.2])

$$|h_{t_n}'(0)|\to 1,\quad |h_{t_n}'(e^{iW_{t_n}})|\to 1,\quad Z_{t_n}\to 1,\quad |h_{t_n}'(e^{iV_{t_n}})|^{\gamma}\to\mathbb{P}\big[K\cap A=\emptyset\,|\,\eta^R\big].$$

Thus, almost surely,

$$\lim_{t\to\tau_A}M_t=\mathbb{P}\big[K\cap A=\emptyset\,|\,\eta^R\big]1_{\tau_A=\infty}.$$

As a result

$$\mathbb{P}\big[K\cap A=\emptyset\big]=\mathbb{E}(M_{\tau_A})=M_0.$$

□

6.4.2 Why can β not be smaller than $5/8$?

It remains to show that if $\mathbb{P}(\alpha,\beta)$ exists, then $\beta\geq 5/8$. In the following we assume that $\mathbb{P}(\alpha,\beta)$ exists. We are going to show how to use this radial measure to construct a chordal restriction measure of exponent β, which will then imply that β cannot be smaller than $5/8$.

Let X be the collection of compact subsets K of $\overline{\mathbb{U}}$ such that K is connected and $\mathbb{C}\setminus K$ is connected. Let $\mathscr{A}$ be the collection of compact subset A of $\overline{\mathbb{U}}$ such that $A=\overline{\mathbb{U}\cap A}$, $\mathbb{U}\setminus A$ is simply connected. Endow X with the σ-field generated by the events $\mathscr{C}(A):=(K\in X: K\cap A=\emptyset)$ for $A\in\mathscr{A}$. This σ-field coincides with the σ-field generated by Hausdorff metric on X. In particular, X is compact since $\overline{\mathbb{U}}$ is compact.

Let K be a radial restriction sample of law $\mathbb{P}(\alpha,\beta)$. For any $\varepsilon>0$, define the probability measure μ_ε on X by

$$\mu_\varepsilon(\mathscr{C}(A))=\mathbb{P}\big[f_\varepsilon(K)\cap A=\emptyset\big]$$

where $A\in\mathscr{A}$ such that $+1\notin A,-1\notin A$ and f_ε is the conformal map from $\mathbb{U}$ onto itself such that $f_\varepsilon(0)=-1+\varepsilon, f_\varepsilon(1)=1$.

Since X is compact, the sequence $(\mu_\varepsilon,\varepsilon>0)$ is tight, thus there exists a subsequence $(\mu_{\varepsilon_k},k\in\mathbb{N})$ such that $\varepsilon_k\to 0$ and μ_{ε_k} converges weakly to some probability measure μ on X. There two observations:

- For any $A\in\mathscr{A}$ such that $+1\notin A,-1\notin A$,

$$\mu_\varepsilon(\mathscr{C}(A))=|\Phi_\varepsilon'(-1+\varepsilon)|^{\alpha}\Phi_\varepsilon'(1)^{\beta}\to\Psi_A'(1)^{\beta}\quad\text{as}\quad\varepsilon\to 0\tag{6.4.2}$$

  where Φ_ε is the conformal map from $\mathbb{U}\setminus A$ onto $\mathbb{U}$ that preserves $-1+\varepsilon$ and $+1$, Ψ_A is the conformal map from $\mathbb{U}\setminus A$ onto $\mathbb{U}$ that preserves ± 1 and $\Psi_A'(-1)=1$.

- For any $A \in \mathscr{A}$ such that $+1 \notin A, -1 \notin A$ and $\delta > 0$, define A_o^δ as the open δ-neighborhood of A and $A_i^\delta = \overline{\mathbb{U}} \setminus (\mathbb{U} \setminus A)_o^\delta$. Note that A_o^δ is open, A_i^δ is closed, $\mathscr{C}(A_o^\delta)$ is closed and $\mathscr{C}(A_i^\delta)$ is open. Thus

$$\mu(\mathscr{C}(A_i^\delta) \setminus \mathscr{C}(A_o^\delta)) \leq \lim_k \mu_{\varepsilon_k}(\mathscr{C}(A_i^\delta) \setminus \mathscr{C}(A_o^\delta)).$$

From (6.4.2), we know that there exists $g(\delta)$ goes to zero as δ goes to zero and is independent of ε such that

$$\mu_{\varepsilon_k}(\mathscr{C}(A_i^\delta) \setminus \mathscr{C}(A_o^\delta)) = \mu_{\varepsilon_k}(\mathscr{C}(A_i^\delta)) - \mu_{\varepsilon_k}(\mathscr{C}(A_o^\delta)) \leq g(\delta).$$

Thus we have that

$$\mu(\mathscr{C}(A_i^\delta) \setminus \mathscr{C}(A_o^\delta)) \leq g(\delta). \tag{6.4.3}$$

From (6.4.2) and (6.4.3), we have that

$$\mu(\mathscr{C}(A)) = \Psi_A'(1)^\beta$$

for any $A \in \mathscr{A}$ such that $\pm 1 \notin A$ and Ψ_A is the conformal map from $\mathbb{U} \setminus A$ onto $\mathbb{U}$ that preserves ± 1 and $\Psi_A'(-1) = 1$. Thus μ is the chordal restriction measure of exponent β, thus $\beta \geq 5/8$.

This concludes the proof of our main theorem.

6.4.3 Concluding remarks

We would just like to note that all the enumerated results on chordal restriction samples that we have briefly recalled in Section 6.2.2 do have a radial restriction counterpart: The dimension of cut-points is the same (and given by β only), the boundaries of radial restriction sample $\mathbb{P}(\xi(\beta),\beta)$ are radial $\mathrm{SLE}_{8/3}(\rho)$ processes, the full-plane Brownian intersection exponents describe the law of radial restriction samples conditioned not to intersect etc. We leave the precise statements and detailed proofs to the interested reader.

Bibliography

[AS08] Tom Alberts and Scott Sheffield. Hausdorff dimension of the SLE curve intersected with the real line. Electron. J. Probab., 13:no. 40, 1166–1188, 2008.

[Bef04] Vincent Beffara. Hausdorff dimensions for SLE_6. Ann. Probab., 32(3B):2606–2629, 2004.

[Bef08] Vincent Beffara. The dimension of the SLE curves. Ann. Probab., 36(4):1421–1452, 2008.

[Ber96] Jean Bertoin. Lévy processes, volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1996.

[Car10] J. Cardy. Conformal field theory and statistical mechanics. In Exact methods in low-dimensional statistical physics and quantum computing, pages 65–98. Oxford Univ. Press, Oxford, 2010.

[CDCH$^+$12] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of ising interfaces to sles. 2012.

[CN06] Federico Camia and Charles M. Newman. Two-dimensional critical percolation: the full scaling limit. Comm. Math. Phys., 268(1):1–38, 2006.

[CN07] Federico Camia and Charles M. Newman. Critical percolation exploration path and SLE_6: a proof of convergence. Probab. Theory Related Fields, 139(3-4):473–519, 2007.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. Invent. Math., 189(3):515–580, 2012.

[DS89] Bertrand Duplantier and Hubert Saleur. Exact fractal dimension of 2D Ising clusters. Phys. Rev. Lett., 63:2536, 1989.

[Dub05] Julien Dubédat. SLE(κ,ρ) martingales and duality. Ann. Probab., 33(1):223–243, 2005.

[Dub07] Julien Dubédat. Commutation relations for Schramm-Loewner evolutions. Comm. Pure Appl. Math., 60(12):1792–1847, 2007.

[Dub09a] Julien Dubédat. Duality of Schramm-Loewner evolutions. Ann. Sci. Éc. Norm. Supér. (4), 42(5):697–724, 2009.

[Dub09b] Julien Dubédat. SLE and the free field: partition functions and couplings. J. Amer. Math. Soc., 22(4):995–1054, 2009.

[Dup04] Bertrand Duplantier. Conformal fractal geometry & boundary quantum gravity. In Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2, volume 72 of Proc. Sympos. Pure Math., pages 365–482. Amer. Math. Soc., Providence, RI, 2004.

[HBB10] Christian Hagendorf, Denis Bernard, and Michel Bauer. The Gaussian free field and SLE_4 on doubly connected domains. J. Stat. Phys., 140(1):1–26, 2010.

[IK10] K. Izyurov and K. Kytölä. Hadamard's formula and couplings of SLEs with free field. ArXiv e-prints, June 2010.

[Ken08] Richard Kenyon. Height fluctuations in the honeycomb dimer model. Comm. Math. Phys., 281(3):675–709, 2008.

[Law] Gregory Lawler. A note on the Brownian loop measure.

[Law96] Gregory F. Lawler. Hausdorff dimension of cut points for Brownian motion. Electron. J. Probab., 1:no. 2, approx. 20 pp. (electronic), 1996.

[Law05] Gregory F. Lawler. Conformally invariant processes in the plane, volume 114 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.

[LSW01a] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. I. Half-plane exponents. Acta Math., 187(2):237–273, 2001.

[LSW01b] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. II. Plane exponents. Acta Math., 187(2):275–308, 2001.

[LSW02] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents. III. Two-sided exponents. Ann. Inst. H. Poincaré Probab. Statist., 38(1):109–123, 2002.

[LSW03] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: the chordal case. J. Amer. Math. Soc., 16(4):917–955 (electronic), 2003.

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. Ann. Probab., 32(1B):939–995, 2004.

[LW00] Gregory F. Lawler and Wendelin Werner. Universality for conformally invariant intersection exponents. J. Eur. Math. Soc. (JEMS), 2(4):291–328, 2000.

[LW04] Gregory F. Lawler and Wendelin Werner. The Brownian loop soup. Probab. Theory Related Fields, 128(4):565–588, 2004.

[Mil11] Jason Miller. Fluctuations for the Ginzburg-Landau $\nabla\phi$ interface model on a bounded domain. Comm. Math. Phys., 308(3):591–639, 2011.

[MP10] Peter Mörters and Yuval Peres. Brownian motion. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.

[MS10] Nikolai Makarov and Stanislav Smirnov. Off-critical lattice models and massive SLEs. pages 362–371, 2010.

[MS12a] Jason Miller and Scott Sheffield. Imaginary geometry i: Interacting sles. 2012.

[MS12b] Jason Miller and Scott Sheffield. Imaginary geometry ii: reversibility of $\mathrm{sle}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$. 2012.

[MS12c] Jason Miller and Scott Sheffield. Imaginary geometry iii: reversibility of sle_κ for $\kappa \in (4,8)$. 2012.

[MS13a] Jason Miller and Scott Sheffield. Cle(4) and gaussian free field. 2013.

[MS13b] Jason Miller and Scott Sheffield. Imaginary geometry iv: interior rays, whole-plane reversibility, and space-filling trees. 2013.

[MW13] Jason Miller and Hao Wu. Intersections of sle paths: the double and cut point dimension of sle, 2013.

[NW11] Şerban Nacu and Wendelin Werner. Random soups, carpets and fractal dimensions. J. Lond. Math. Soc. (2), 83(3):789–809, 2011.

[Pom92] Ch. Pommerenke. Boundary behaviour of conformal maps, volume 299 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992.

[RS05] Steffen Rohde and Oded Schramm. Basic properties of SLE. Ann. of Math. (2), 161(2):883–924, 2005.

[RY94] Daniel Revuz and Marc Yor. Continuous martingales and Brownian motion, volume 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 1994.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math., 118:221–288, 2000.

[She] S. Sheffield. Local sets of the gaussian free field: slides and audio. www.fields.utoronto.ca/0506/percolationsle/sheffield1, www.fields.utoronto.ca/audio/0506/percolationsle/sheffield2, www.fields.utoronto.ca/audio/0506/percolationsle/sheffield3.

[She07] Scott Sheffield. Gaussian free fields for mathematicians. Probab. Theory Related Fields, 139(3-4):521–541, 2007.

[She09] Scott Sheffield. Exploration trees and conformal loop ensembles. Duke Math. J., 147(1):79–129, 2009.

[She11] Scott Sheffield. Conformal weldings of random surfaces: Sle and the quantum gravity zipper. preprint, 2011.

[SS09] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. Acta Math., 202(1):21–137, 2009.

[SS12] Oded Schramm and Scott Sheffield. A contour line of the continuum gaussian free field. Probab. Theor. rel. Fields, page to appear, 2012.

[SSW09] Oded Schramm, Scott Sheffield, and David B. Wilson. Conformal radii for conformal loop ensembles. Comm. Math. Phys., 288(1):43–53, 2009.

[SW05] Oded Schramm and David B. Wilson. SLE coordinate changes. New York J. Math., 11:659–669 (electronic), 2005.

[SW12] Scott Sheffield and Wendelin Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. Ann. of Math. (2), 176(3):1827–1917, 2012.

[SWW13] Scott Sheffield, Samuel Watson, and Hao Wu. Couplings between gff and cle_4. 2013.

[SZ10] Oded Schramm and Wang Zhou. Boundary proximity of SLE. Probab. Theory Related Fields, 146(3-4):435–450, 2010.

[VL12] Fredrik Johansson Viklund and Gregory F. Lawler. Almost sure multifractal spectrum for the tip of an SLE curve. Acta Math., 209(2):265–322, 2012.

[Wer03] Wendelin Werner. SLEs as boundaries of clusters of Brownian loops. C. R. Math. Acad. Sci. Paris, 337(7):481–486, 2003.

[Wer04a] Wendelin Werner. Girsanov's transformation for SLE(κ,ρ) processes, intersection exponents and hiding exponents. Ann. Fac. Sci. Toulouse Math. (6), 13(1):121–147, 2004.

[Wer04b] Wendelin Werner. Random planar curves and Schramm-Loewner evolutions. In Lectures on probability theory and statistics, volume 1840 of Lecture Notes in Math., pages 107–195. Springer, Berlin, 2004.

[Wer05] Wendelin Werner. Conformal restriction and related questions. Probab. Surv., 2:145–190, 2005.

[Wer08] Wendelin Werner. The conformally invariant measure on self-avoiding loops. J. Amer. Math. Soc., 21(1):137–169, 2008.

[Wu13] Hao Wu. Conformal restriction: the radial case. arXiv:1304.5712, 2013.

[WW13a] Wendelin Werner and Hao Wu. From cle(\kappa) to sle(\kappa,\rho)'s. Electron. J. Probab., 18:no. 36, 1–20, 2013.

[WW13b] Wendelin Werner and Hao Wu. On Conformally Invariant CLE Explorations. Comm. Math. Phys., 320(3):637–661, 2013.

[Zha08a] Dapeng Zhan. Duality of chordal SLE. Invent. Math., 174(2):309–353, 2008.

[Zha08b] Dapeng Zhan. Reversibility of chordal SLE. Ann. Probab., 36(4):1472–1494, 2008.

[Zha10a] Dapeng Zhan. Duality of chordal SLE, II. Ann. Inst. Henri Poincaré Probab. Stat., 46(3):740–759, 2010.

[Zha10b] Dapeng Zhan. Reversibility of some chordal SLE$(\kappa;\rho)$ traces. J. Stat. Phys., 139(6):1013–1032, 2010.